Derivation of the Fresnel Equations

Consider an interface at $z = 0$ which divides a material ($z < 0$) with a permitivity ε_i and a permeability μ_i from a material ($z > 0$) with ε_t and μ_t (see figure 1). Light hits the interface from the $-z$ direction. The incoming wave is described by wave vector $\vec{k}_\mathrm{i}$, the angle between the surface normal and the incoming wave is θ_i. The reflected beam is at angle $\theta_\mathrm{r} = \theta_\mathrm{i}$, with wave vector $\vec{k}_\mathrm{r}$. The transmitted beam is described by wave vector $\vec{k}_\mathrm{t}$. The angle θ_t of this beam can be calculated by Snell's law:

$$n_1 \sin\theta_\mathrm{i} = n_2 \sin\theta_\mathrm{t}\,, \tag{1}$$

where n_i are the refractive indices of the materials:

$$n_\alpha = \frac{c}{c_\alpha} = \frac{\sqrt{\varepsilon_\alpha \mu_\alpha}}{\sqrt{\varepsilon_0 \mu_0}}. \tag{2}$$

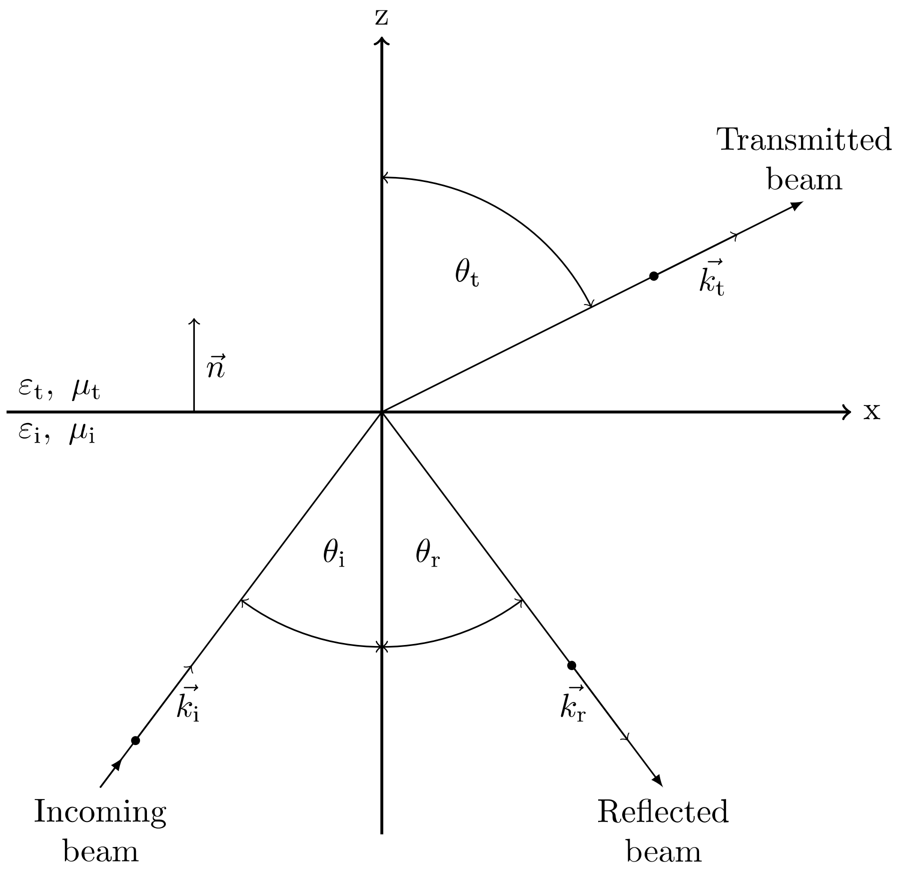

Figure 1: Light hitting the interface between material one (bottom) and material two (top)

Continuity Conditions

For light hitting an interface there are certain continuity conditions for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ respectively for the electric displacement field $\vec{D}$ and $\vec{H}$. They can be found e. g. in [1, p. 317]:

$$\vec{n} \times \left(\overrightarrow{E_2} - \overrightarrow{E_1} \right) = 0\,, \tag{3}$$

$$\vec{n} \cdot \left(\overrightarrow{D_2} - \overrightarrow{D_1} \right) = 0\,, \tag{4}$$

$$\vec{n} \times \left(\overrightarrow{H_2} - \overrightarrow{H_1} \right) = 0\,, \tag{5}$$

$$\vec{n} \cdot \left(\overrightarrow{B_2} - \overrightarrow{B_1} \right) = 0\,, \tag{6}$$

where $\vec{B} = \mu\vec{H}$ and $\vec{D} = \varepsilon\vec{E}$.

From figure 1 we can deduce the needed quantites for equations 3 to 6:

$$\overrightarrow{E_1} = \overrightarrow{E_\mathrm{i}} + \overrightarrow{E_\mathrm{r}}\,, \quad \overrightarrow{E_2} = \overrightarrow{E_\mathrm{t}}\,, \tag{7}$$

$$\overrightarrow{D_1} = \varepsilon_\mathrm{i}\left(\overrightarrow{E_\mathrm{i}} + \overrightarrow{E_\mathrm{r}} \right)\,, \quad \overrightarrow{D_2} = \varepsilon_\mathrm{t}\overrightarrow{E_\mathrm{t}}\,, \tag{8}$$

$$\overrightarrow{H_1} = \frac{1}{\mu_\mathrm{i}}\left(\overrightarrow{B_\mathrm{i}} + \overrightarrow{B_\mathrm{r}} \right)\,, \quad \overrightarrow{H_2} = \frac{1}{\mu_\mathrm{t}}\overrightarrow{B_\mathrm{t}}\,, \tag{9}$$

$$\overrightarrow{B_1} = \overrightarrow{B_\mathrm{i}} + \overrightarrow{B_\mathrm{r}}\,, \quad \overrightarrow{B_2} = \overrightarrow{B_\mathrm{t}}\,. \tag{10}$$

s-polarization

We know that in the case of s-polarization, the electric field $\vec{E}$ is polarized perpendicularly to the interface. $\vec{k}$, $\vec{E}$ and $\vec{B}$ are always perpendicular to each other.

This results in the drawing shown in figure 2.

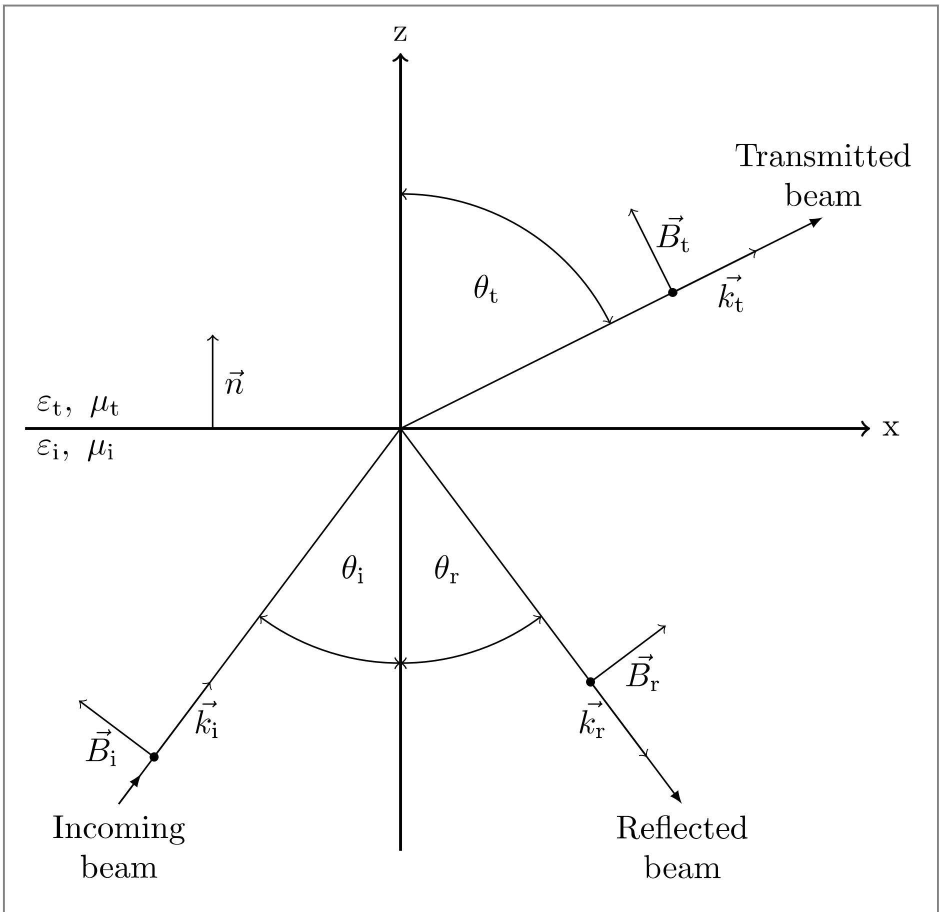

Figure 2: Figure 1 completed for the case of s-polarization: The electric fields $\overrightarrow{E_\mathrm{i}}$, $\overrightarrow{E_\mathrm{r}}$ and $\overrightarrow{E_\mathrm{t}}$ are pointing in the y-direction (into the screen).

Now we want to solve for $E_{\mathrm{r}}/E_{\mathrm{i}}$. Using equation 9 in equation 5 and $\vec{B} = \vec{k} \times \vec{E}$ results in:

$$\vec{n} \times \left[\frac{1}{\mu_{\mathrm{t}}}\left(\vec{k}_{\mathrm{t}} \times \overrightarrow{E_{\mathrm{t}}}\right) - \frac{1}{\mu_{\mathrm{i}}}\left(\vec{k}_{\mathrm{i}} \times \overrightarrow{E_{\mathrm{i}}} + \vec{k}_{\mathrm{r}} \times \overrightarrow{E_{\mathrm{r}}}\right)\right] = 0 \tag{11}$$

Using $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\left(\vec{a} \cdot \vec{b}\right)$ gives:

$$\frac{1}{\mu_{\mathrm{t}}}\left[\vec{k}_{\mathrm{t}}\left(\vec{n} \cdot \overrightarrow{E_{\mathrm{t}}}\right) - \overrightarrow{E_{\mathrm{t}}}\left(\vec{n} \cdot \overrightarrow{k_{\mathrm{t}}}\right)\right] - \frac{1}{\mu_{\mathrm{i}}}\left[\vec{k}_{\mathrm{i}}\left(\vec{n} \cdot \overrightarrow{E_{\mathrm{i}}}\right) - \overrightarrow{E_{\mathrm{i}}}\left(\vec{n} \cdot \vec{k}_{\mathrm{i}}\right) + \vec{k}_{\mathrm{r}}\left(\vec{n} \cdot \overrightarrow{E_{\mathrm{r}}}\right) - \overrightarrow{E_{\mathrm{r}}}\left(\vec{n} \cdot \vec{k}_{\mathrm{r}}\right)\right] = 0 \tag{12}$$

The dot products of the normal vector $\vec{n}$ and the electric fields are always zero because they are othogonal to each other. The dot products of the normal and wave vectors can be rewritten using $\vec{a} \cdot \vec{b} = ab\cos\alpha$, where α is the angle between $\vec{a}$ and $\vec{b}$:

$$-\frac{1}{\mu_{\mathrm{t}}}E_{\mathrm{t}}k_{\mathrm{t}}\cos\theta_{\mathrm{t}} + \frac{1}{\mu_{\mathrm{i}}}E_{\mathrm{i}}k_{\mathrm{i}}\cos\theta_{\mathrm{i}} - \frac{1}{\mu_{\mathrm{i}}}E_{\mathrm{r}}k_{\mathrm{r}}\cos\theta_{\mathrm{i}} = 0 \tag{13}$$

We can rewrite this equation using the dispersion relation $\omega = ck$ and equation 2:

$$k_{\mathrm{i}} = \frac{\omega}{c_i} = \frac{\omega}{c}n_1 = \frac{\omega}{c}\sqrt{\mu_{\mathrm{i}}\varepsilon_{\mathrm{i}}} = k_{\mathrm{r}}, \tag{14}$$

$$k_{\mathrm{t}} = \frac{\omega}{c_t} = \frac{\omega}{c}n_2 = \frac{\omega}{c}\sqrt{\mu_{\mathrm{t}}\varepsilon_{\mathrm{t}}}. \tag{15}$$

and get

$$\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}\left(E_{0\mathrm{i}} - E_{0\mathrm{r}}\right)\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}E_{0\mathrm{t}}\cos\theta_{\mathrm{t}} = 0 \tag{16}$$

To get a second equation we use equation 7 in equation 3, which yields,

$$\vec{n} \times \left[\overrightarrow{E_{\mathrm{t}}} - \left(\overrightarrow{E_{\mathrm{i}}} + \overrightarrow{E_{\mathrm{r}}}\right)\right] = 0 \tag{17}$$

This can be rewritten as

$$\overrightarrow{E_{\mathrm{t}}} - \left(\overrightarrow{E_{\mathrm{i}}} + \overrightarrow{E_{\mathrm{r}}}\right) = 0\,, \tag{18}$$

which leads to

$$E_{0\mathrm{t}} - \left(E_{0\mathrm{i}} + E_{0\mathrm{r}}\right) = 0. \tag{19}$$

From this result we can express $E_{0\mathrm{t}} = E_{0\mathrm{i}} + E_{0\mathrm{r}}$ and put it in equation 16:

$$\sqrt{\frac{\varepsilon_{0\mathrm{i}}}{\mu_{0\mathrm{i}}}}\left(E_{0\mathrm{i}} - E_{0\mathrm{r}}\right)\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}\left(E_{0\mathrm{i}} + E_{0\mathrm{r}}\right)\cos\theta_{\mathrm{t}} = 0 \tag{20}$$

$$\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}E_{0\mathrm{i}}\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}E_{0\mathrm{r}}\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}E_{0\mathrm{i}}\cos\theta_{\mathrm{t}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}E_{0\mathrm{r}}\cos\theta_{\mathrm{t}} = 0 \tag{21}$$

$$E_{0\mathrm{i}}\left(\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}\cos\theta_{\mathrm{t}}\right) = E_{0\mathrm{r}}\left(\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}\cos\theta_{\mathrm{i}} + \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}\cos\theta_{\mathrm{t}}\right) \tag{22}$$

$$\frac{E_{0\mathrm{r}}}{E_{0\mathrm{i}}} = \frac{\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}\cos\theta_{\mathrm{i}} - \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}\cos\theta_{\mathrm{t}}}{\sqrt{\frac{\varepsilon_{\mathrm{i}}}{\mu_{\mathrm{i}}}}\cos\theta_{\mathrm{i}} + \sqrt{\frac{\varepsilon_{\mathrm{t}}}{\mu_{\mathrm{t}}}}\cos\theta_{\mathrm{t}}} \tag{23}$$

Often, the permeabilites are the same in both materials ($\mu_{\mathrm{i}} = \mu_{\mathrm{t}} = 1$), which simplifies the last equation to the result given in [3, equation 7]:

$$\frac{E_{r,\,s}}{E_{i,\,s}} = \frac{\sqrt{\varepsilon_\mathrm{i}}\cos\theta_\mathrm{i} - \sqrt{\varepsilon_\mathrm{t}}\cos\theta_\mathrm{t}}{\sqrt{\varepsilon_\mathrm{i}}\cos\theta_\mathrm{i} + \sqrt{\varepsilon_\mathrm{t}}\cos\theta_\mathrm{t}} \tag{24}$$

p-polarization

In the case of p-polarization, the electric field $\vec{E}$ is polarized linearly in the plane of incidence. As above, we can set up a figure (figure 3).

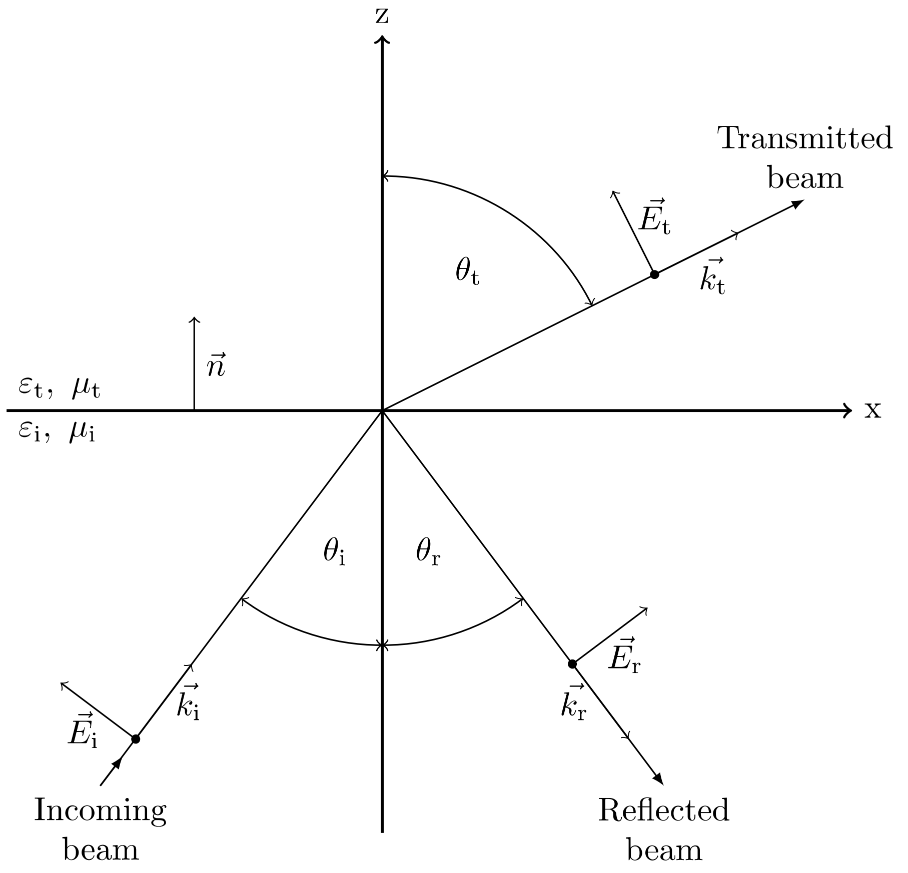

Figure 3: Figure 1 completed for the case of p-polarization: The magnetic fields $\vec{B}_\mathrm{i}$, $\vec{B}_\mathrm{r}$ and $\vec{B}_\mathrm{t}$ are pointing in the $-y$-direction (out of the screen).

We now use equation 8 in equation 4, leading to:

$$\vec{n}\cdot\left(\varepsilon_\mathrm{t}\vec{E}_2 - \varepsilon_\mathrm{i}\vec{E}_1\right) = 0 \tag{25}$$

$$\vec{n}\cdot\left[\varepsilon_\mathrm{t}\vec{E}_\mathrm{t} - \varepsilon_\mathrm{i}\left(\vec{E}_\mathrm{i} + \vec{E}_\mathrm{r}\right)\right] = 0 \tag{26}$$

Again, the dot product can be rewritten using $\vec{a}\cdot\vec{b} = ab\cos\alpha$, where α is the angle between $\vec{a}$ and $\vec{b}$:

$$\varepsilon_\mathrm{t}E_{0\mathrm{t}}\cos\left(\frac{\pi}{2}-\theta_\mathrm{t}\right) - \varepsilon_\mathrm{i}\left[E_{0\mathrm{i}}\cos\left(\frac{\pi}{2}-\theta_\mathrm{i}\right) + E_{0\mathrm{r}}\cos\left(\frac{\pi}{2}-\theta_\mathrm{i}\right)\right] = 0 \tag{27}$$

And because of $\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta$ we get:

$$\varepsilon_\mathrm{t}E_{0\mathrm{t}}\sin\theta_\mathrm{t} - \varepsilon_\mathrm{i}\left(E_{0\mathrm{i}}\sin\theta_\mathrm{i} + E_{0\mathrm{r}}\sin\theta_\mathrm{i}\right) = 0 \tag{28}$$

To get a second equation we use equation 7 in equation 3:

$$\vec{n}\times\left[\vec{E}_\mathrm{t} - \left(\vec{E}_\mathrm{i} + \vec{E}_\mathrm{r}\right)\right] = 0 \tag{29}$$

Using the definition of the cross product $\left(\vec{a}\times\vec{b}=ab\vec{n}\sin\alpha\right)$ in the first line and $\cos\left(\frac{\pi}{2}-\beta\right)=\sin\beta$ in the second we get:

$$E_{0\mathrm{t}}\sin\left(\frac{\pi}{2}-\theta_\mathrm{t}\right)-\left[E_{0\mathrm{i}}\sin\left(\frac{\pi}{2}-\theta_\mathrm{i}\right)-E_{0\mathrm{r}}\sin\left(\frac{\pi}{2}-\theta_\mathrm{i}\right)\right]=0 \tag{30}$$

$$E_{0\mathrm{t}}\cos\theta_\mathrm{t}-E_{0\mathrm{i}}\cos\theta_\mathrm{i}+E_{0\mathrm{r}}\cos\theta_\mathrm{i}=0 \tag{31}$$

We now solve both equations 28 and 31 for $E_{0\mathrm{t}}$ and set them equal to each other:

$$E_{0\mathrm{t}}=\frac{\varepsilon_\mathrm{i}E_{0\mathrm{i}}\sin\theta_\mathrm{i}+\varepsilon_\mathrm{i}E_{0\mathrm{r}}\sin\theta_\mathrm{i}}{\varepsilon_\mathrm{t}\sin\theta_\mathrm{t}}=\frac{E_{0\mathrm{i}}\cos\theta_\mathrm{i}-E_{0\mathrm{r}}\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}} \tag{32}$$

We now solve for $\frac{E_{0\mathrm{r}}}{E_{0\mathrm{i}}}$:

$$\varepsilon_\mathrm{i}E_{0\mathrm{i}}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}+\varepsilon_\mathrm{i}E_{0\mathrm{r}}\sin\theta_\mathrm{i}\cos\theta_t=\varepsilon_\mathrm{t}E_{0\mathrm{i}}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t}-\varepsilon_\mathrm{t}E_{0\mathrm{r}}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t} \tag{33}$$

$$E_{0\mathrm{i}}\left(\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}-\varepsilon_\mathrm{t}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t}\right)=-E_{0\mathrm{r}}\left(\varepsilon_\mathrm{t}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t}+\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}\right) \tag{34}$$

$$\Rightarrow\frac{E_{0\mathrm{r}}}{E_{0\mathrm{i}}}=\frac{\varepsilon_\mathrm{t}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t}-\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}}{\varepsilon_\mathrm{t}\cos\theta_\mathrm{i}\sin\theta_\mathrm{t}+\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}} \tag{35}$$

This can be simplified by substituting the $\sin\theta_\mathrm{t}$ terms by $\frac{n_1}{n_2}\sin\theta_\mathrm{i}$ (which can be derived from Snell's Law (equation 1)):

$$\frac{E_{0\mathrm{r}}}{E_{0\mathrm{i}}}=\frac{\varepsilon_\mathrm{t}\frac{n_1}{n_2}\cos\theta_\mathrm{i}\sin\theta_\mathrm{i}-\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}}{\varepsilon_\mathrm{t}\frac{n_1}{n_2}\cos\theta_\mathrm{i}\sin\theta_\mathrm{i}+\varepsilon_\mathrm{i}\sin\theta_\mathrm{i}\cos\theta_\mathrm{t}} \tag{36}$$

$$=\frac{\varepsilon_\mathrm{t}n_1\cos\theta_\mathrm{i}-\varepsilon_\mathrm{i}n_2\cos\theta_\mathrm{t}}{\varepsilon_\mathrm{t}n_1\cos\theta_\mathrm{i}+\varepsilon_\mathrm{i}n_2\cos\theta_\mathrm{t}} \tag{37}$$

We now use the definition of the refractive index (equation 2):

$$\frac{E_{0\mathrm{r}}}{E_{0\mathrm{i}}}=\frac{\varepsilon_\mathrm{t}\sqrt{\varepsilon_\mathrm{i}\mu_\mathrm{i}}\cos\theta_\mathrm{i}-\varepsilon_\mathrm{i}\sqrt{\varepsilon_\mathrm{t}\mu_\mathrm{t}}\cos\theta_\mathrm{t}}{\varepsilon_\mathrm{t}\sqrt{\varepsilon_\mathrm{i}\mu_\mathrm{i}}\cos\theta_\mathrm{i}+\varepsilon_\mathrm{i}\sqrt{\varepsilon_\mathrm{t}\mu_\mathrm{t}}\cos\theta_\mathrm{t}} \tag{38}$$

$$=\frac{\sqrt{\varepsilon_\mathrm{t}\mu_\mathrm{i}}\cos\theta_\mathrm{i}-\sqrt{\varepsilon_\mathrm{i}\mu_\mathrm{t}}\cos\theta_\mathrm{t}}{\sqrt{\varepsilon_\mathrm{t}\mu_\mathrm{i}}\cos\theta_\mathrm{i}+\sqrt{\varepsilon_\mathrm{i}\mu_\mathrm{t}}\cos\theta_\mathrm{t}} \tag{39}$$

Often, the permeabilites are the same in both materials ($\mu_\mathrm{i}=\mu_\mathrm{t}=1$), which simplifies the last equation to the result given in [3, equation 6]:

$$\frac{E_{r,\,p}}{E_{i,\,p}}=\frac{\sqrt{\varepsilon_\mathrm{t}}\cos\theta_\mathrm{i}-\sqrt{\varepsilon_\mathrm{i}}\cos\theta_\mathrm{t}}{\sqrt{\varepsilon_\mathrm{t}}\cos\theta_\mathrm{i}+\sqrt{\varepsilon_\mathrm{i}}\cos\theta_\mathrm{t}} \tag{40}$$

Refraction and total reflection is due to the wave interference. Below are two videos of waves striking an interface at $y=125\ \mu$m between two materials. The frequency f of the oscillations are the same everywhere but wavelength is longer in the material at the bottom of the video so the speed of waves $c=\lambda f$ is higher in that material. The upper video shows conditions where refraction takes place and the lower video show conditions where total internal reflection takes place.

Both videos were created with this MATLAB script.

Sources

[1] Nolting, W. (2013). Grundkurs Theoretische Physik 3: Elektrodynamik (10. Aufl. 2013 ed.). Berlin, Heidelberg: Imprint: Springer Spektrum.
[2] Arrigoni, E. (2011). Skriptum zur Vorlesung "Elektromagnetische Felder und Elektrodynamik".
[3] Hadley, P. (2015). Optical Properties of Different Materials.