Basics of opto-electronics

Philippe Velha

Qui custodiet custodes?

- Philippe Velha, Engineering degree in Electronics
- MsC in Integrated Electronics
- PhD in Physics
- 10 years of experience in nanofabrication in cleanroom
- More than 12 years of experience in research in the field of opto-electronics in France, UK and Italy

Welcome to this course

- Teacher
  - Philippe Velha
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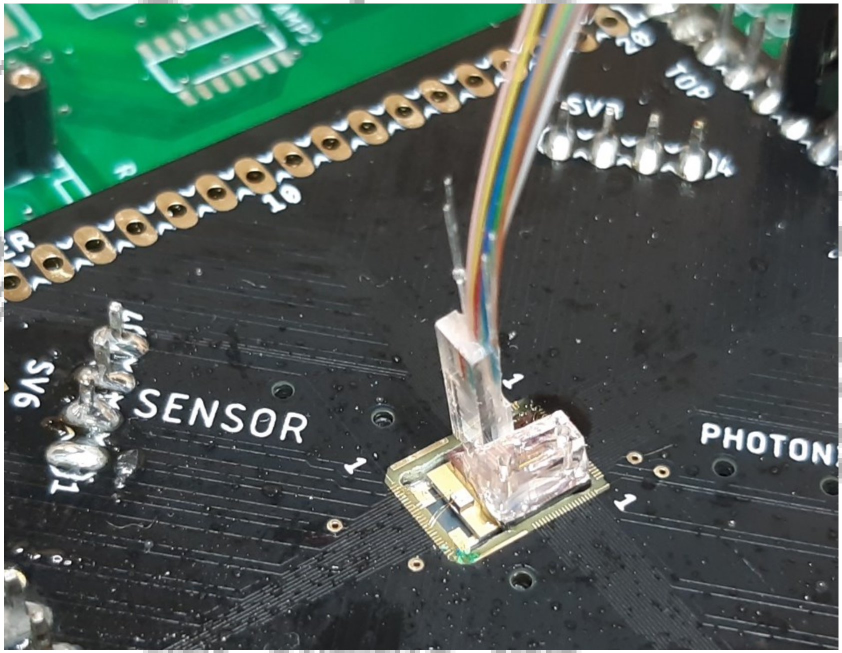
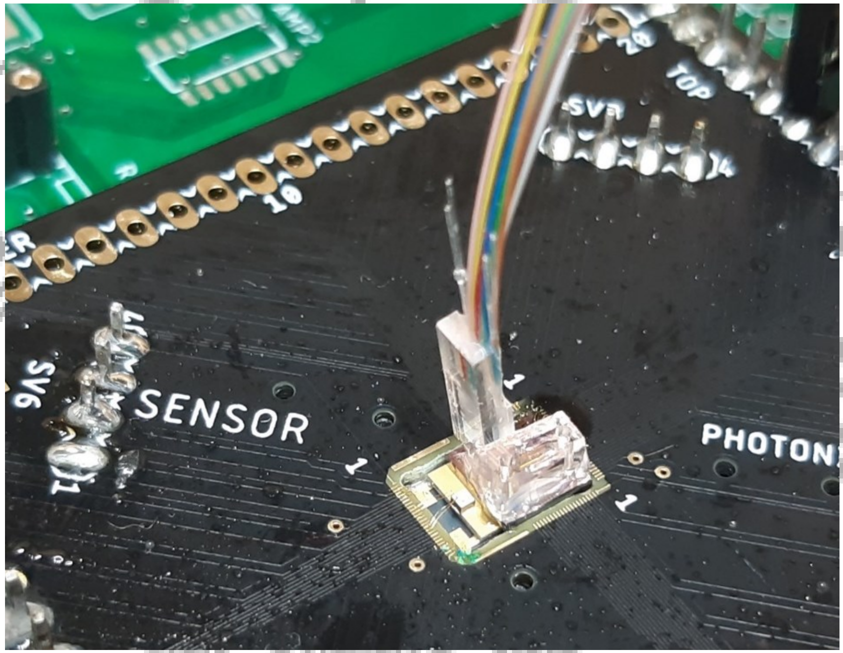
    - E-mail: **Philippe.Velha@unitn.it**
    - First floor, Valley side, **Povo 1**
    - Please take an appointement before poping up



Welcome to this course

- Basics of opto-electronics
  - 48 hours, 6 credits



Timetable

	Monday	Tuesday	Wednesday	Thursday	Friday
8:30 – 9:30					
9:30 – 10:30					
10:30 – 11:30			**A108**		
11:30 – 12:30			**A108**		
12:30 – 13:30					
13:30 – 14:30					A221
14:30 – 15:30					A221
15:30 – 16:30					
16:30 – 17:30					
17:30 – 18:30					

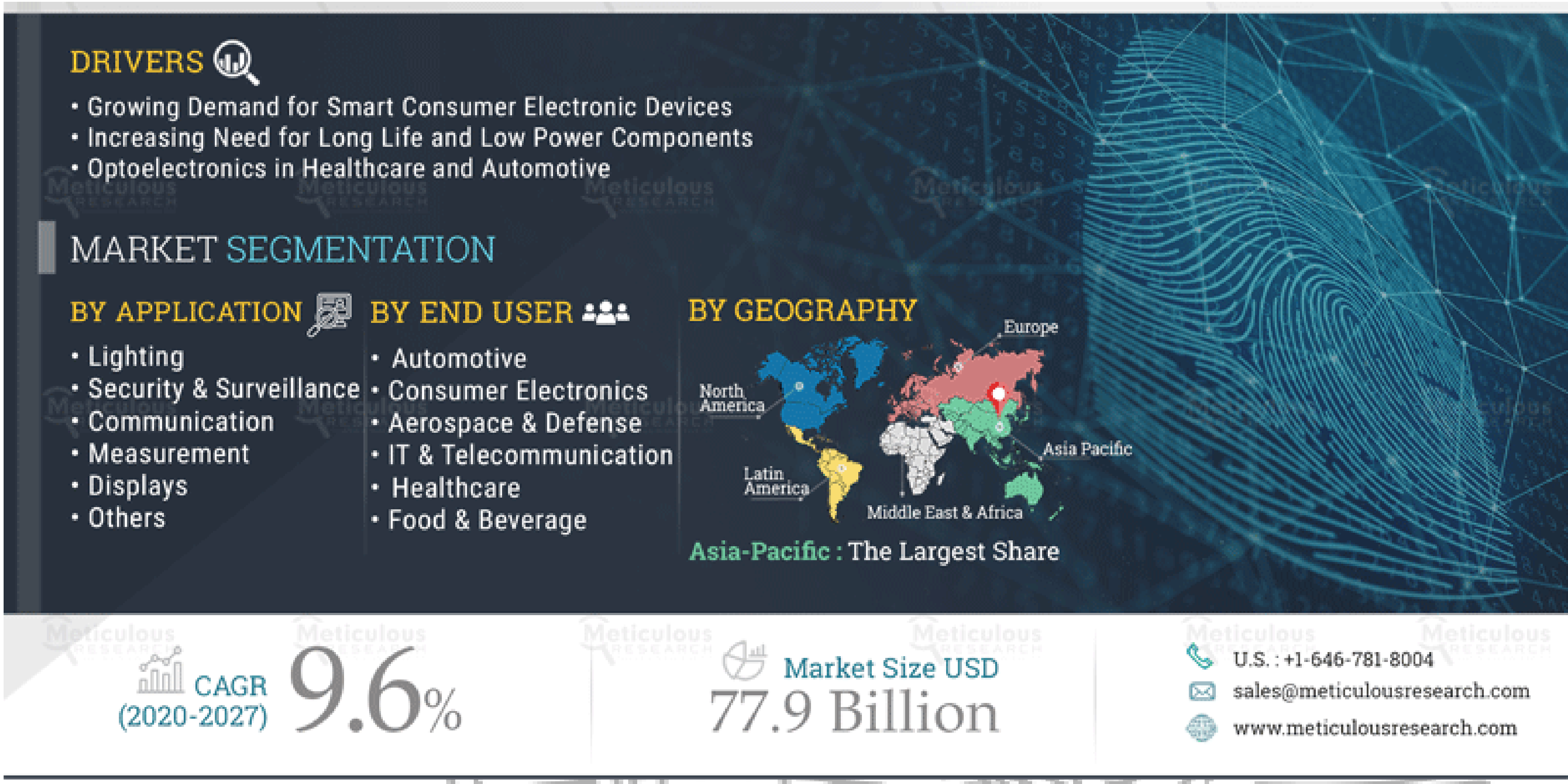
Evaluation

- Written esam
  - MANDATORY
  - Theoretical questions and exercises
  - Graded between 0 and 30
- Lab reports (Mandatory)
  - pass, no pass

Introduction

- Optoelectronics (or optronics or opto-electronics) is the study and application of electronic devices and systems that **source, detect and control light**, usually considered a sub-field of **photonics**.
- In this context, light often includes invisible forms of radiation such as gamma rays, X-rays, ultraviolet and infrared, in addition to visible light. Optoelectronic devices are electrical-to-optical or optical-to-electrical transducers, or instruments that use such devices in their operation. Electro-optics is often erroneously used as a synonym, but is a wider branch of physics that concerns all interactions between light and electric fields, whether or not they form part of an electronic device.
- The term optoelectronics is a specific discipline of electronics that focuses on light-emitting or light-detecting devices.
- Light-emitting devices use voltage and current to produce electromagnetic radiation (i.e., light). Such light-emitting devices are commonly used for purposes of illumination or as indicator lights.
- In contrast, light-detecting devices, such as phototransistors, are designed to convert received electromagnetic energy into electric current or voltage. Light-detecting devices can be used for light sensing and communication. Examples of these include darkness-activated switches and remote controls. In general terms, light-detecting devices work by using photons to liberate bound electrons within semiconductor materials.
- **Source: Wikipedia**

Optoelectronics market



OUTLINE of the course

- Introduction to the applications of opto-electronics
- Chapter 1: The wave properties of light
- Chapter 2: Wave guides and optical fibers
- Chapter 3: Introduction to semiconductor and LED science
- Chapter 4: Optical and laser amplifiers
- Chapter 5: Photo detectors and image sensors
- Chapter 6: Polarization and light modulation

Chapter 1 - Wave Nature of Light

- Electromagnetic wave
- Gaussian beams
- Refractive index
- Snell's Law
- Fresnel's law
- Polarized light
- Bragg mirrors
- Coherence
- Interference
- Optical resonator
- Diffraction
- Scattering

Physicists use the wave theory on Mondays, Wednesdays and Fridays and the particle theory on Tuesdays, Thursdays and Saturdays.
—Sir William Henry Bragg[1]

Augustin Jean Fresnel (1788–1827) was a French physicist and a civil engineer for the French government who was one of the principal proponents of the wave theory of light. He made a number of distinct contributions to optics including the well-known Fresnel lens that was used in lighthouses in the nineteenth century. He fell out with Napoleon in 1815 and was subsequently put under house arrest until the end of Napoleon's reign. During his enforced leisure time he formulated his wave ideas of light into a mathematical theory. *(© INTERFOTO/Alamy.)*

If you cannot saw with a file or file with a saw, then you will be no good as an experimentalist.

What are the quantities in Maxwell Equation's

QUANTITIES		SI or MKS
$\overline{E}$	is the electric field intensity	Volt/m
$\overline{H}$	is the magnetic field intensity	Amp/m
$\overline{D}$	is the electric flux density *(also called displacement vector or electric induction)*	Coul/m^2
$\overline{B}$	is the magnetic flux density	Wb/m^2 (Tesla)
$\overline{J}$	is the electric current density	A/m^2
ρ	is the electric charge density	Coul/m^3

Maxwell's Equations

Differential form

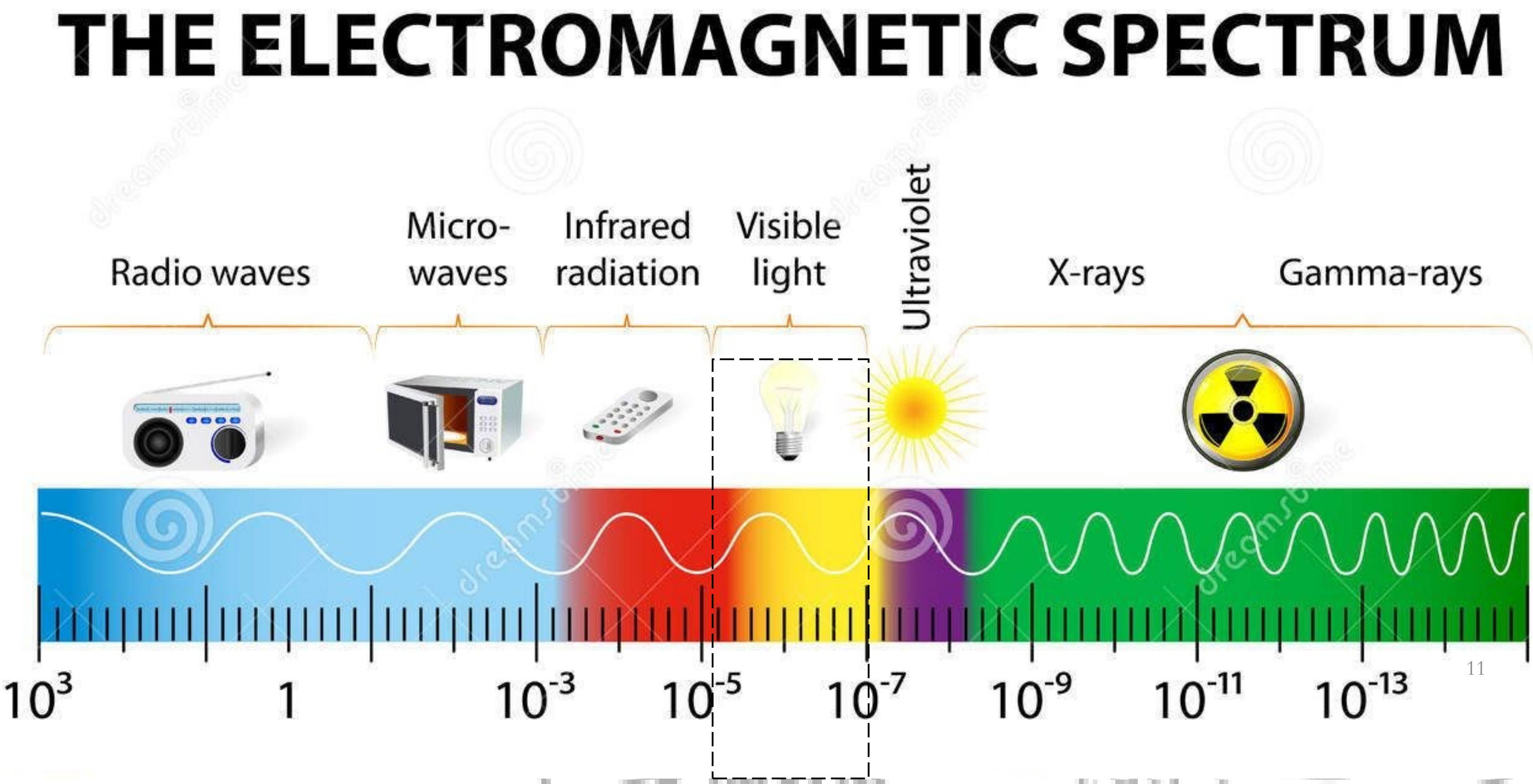
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

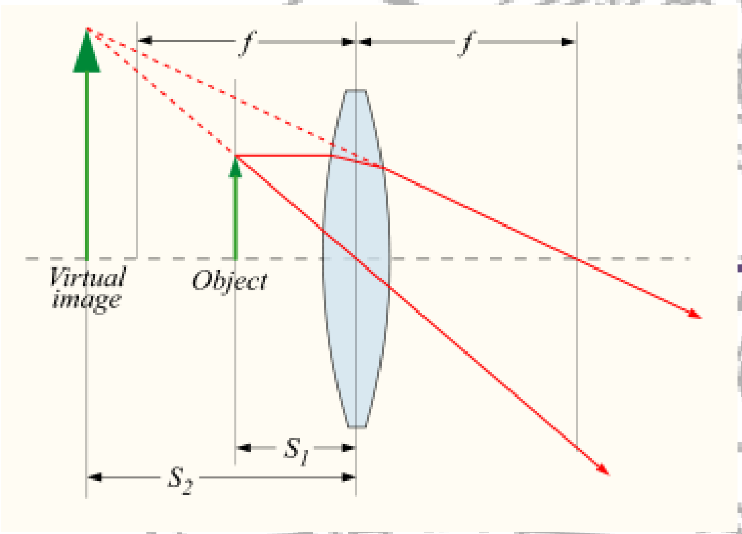
The electromagnetic spectrum



Different scale for optics

Subwavelength domain ~1 Resonant domain ~10 Scalar domain >100 Ray optics

a/λ

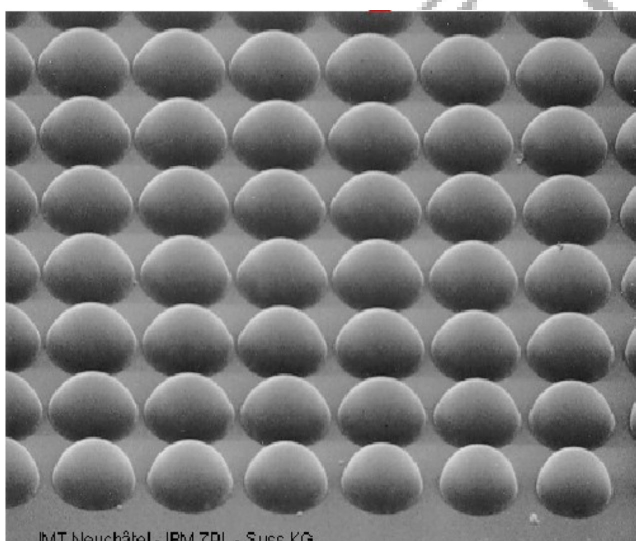


Paraxial approximation

Different scale for optics

Subwavelength domain
~1
Resonant domain
~10
Scalar domain
>100
Ray optics

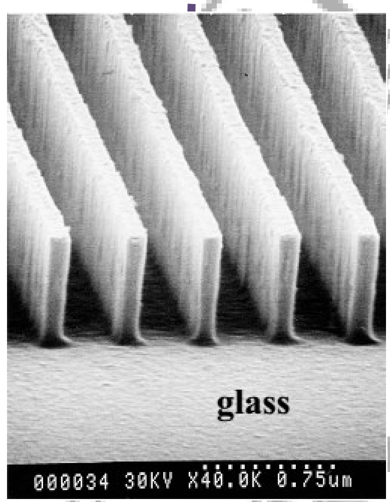
a/λ



Fresnel approximation
or
the Frauhhofer approximation

$a = 30\ \lambda$

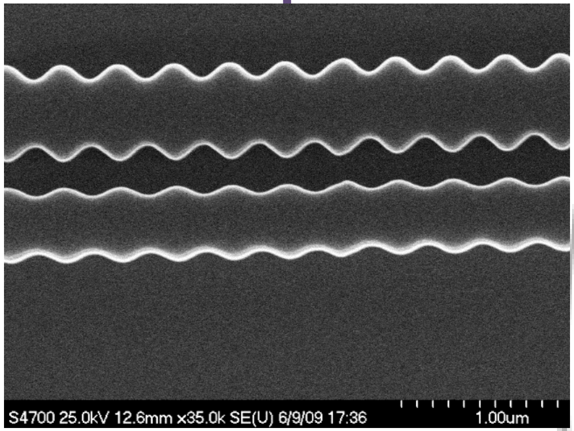
Different scale for optics

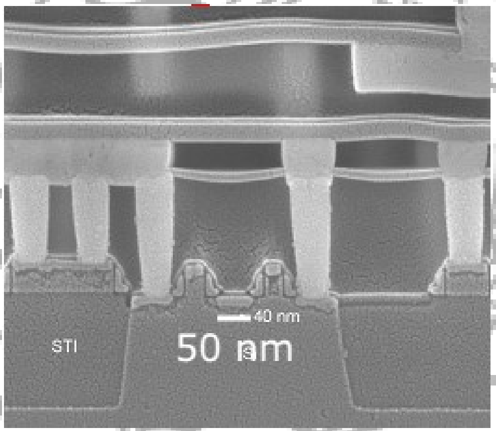


Beam splitter at 633 nm with an extinction ratio >100 and an efficiency >85%

Different scale for optics

Silicon photonics filter

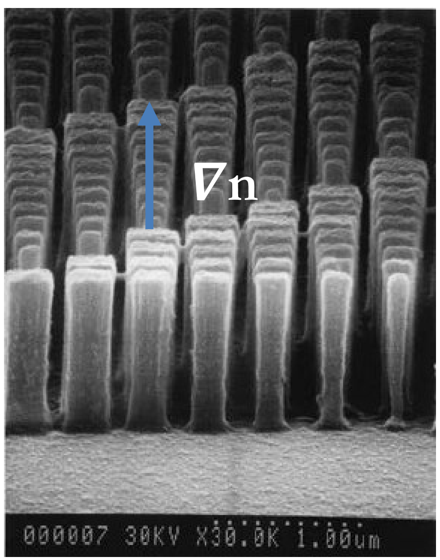


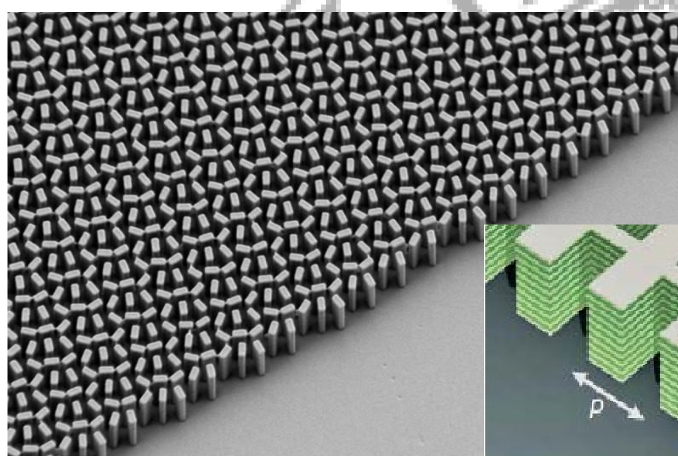


Merging optics and electronics requires nanoscale optics

Different scale for optics

Subwavelength domain | ~1 | Resonant domain | ~10 | Scalar domain | >100 | Ray optics | a/λ

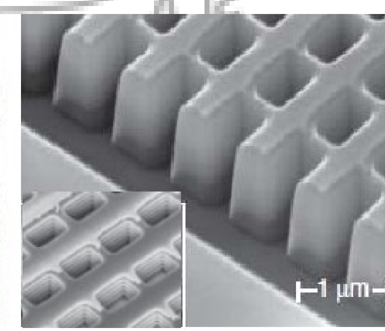




Metamaterials:

- blazed diffraction gratings
- Super lens
- Negative refraction material

p = 860 nm ~ $\lambda/2$

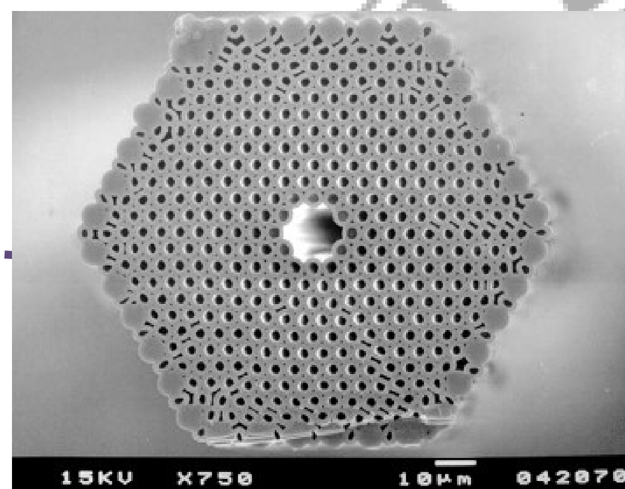


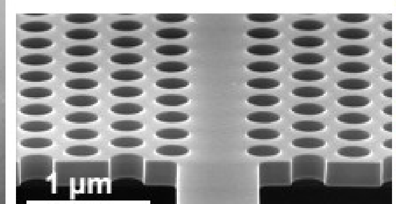
Different scale for optics

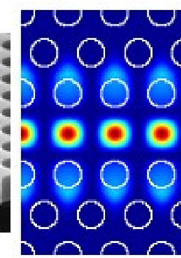
Subwavelength domain ~1 Resonant domain ~10 Scalar domain >100 Ray optics

a/λ

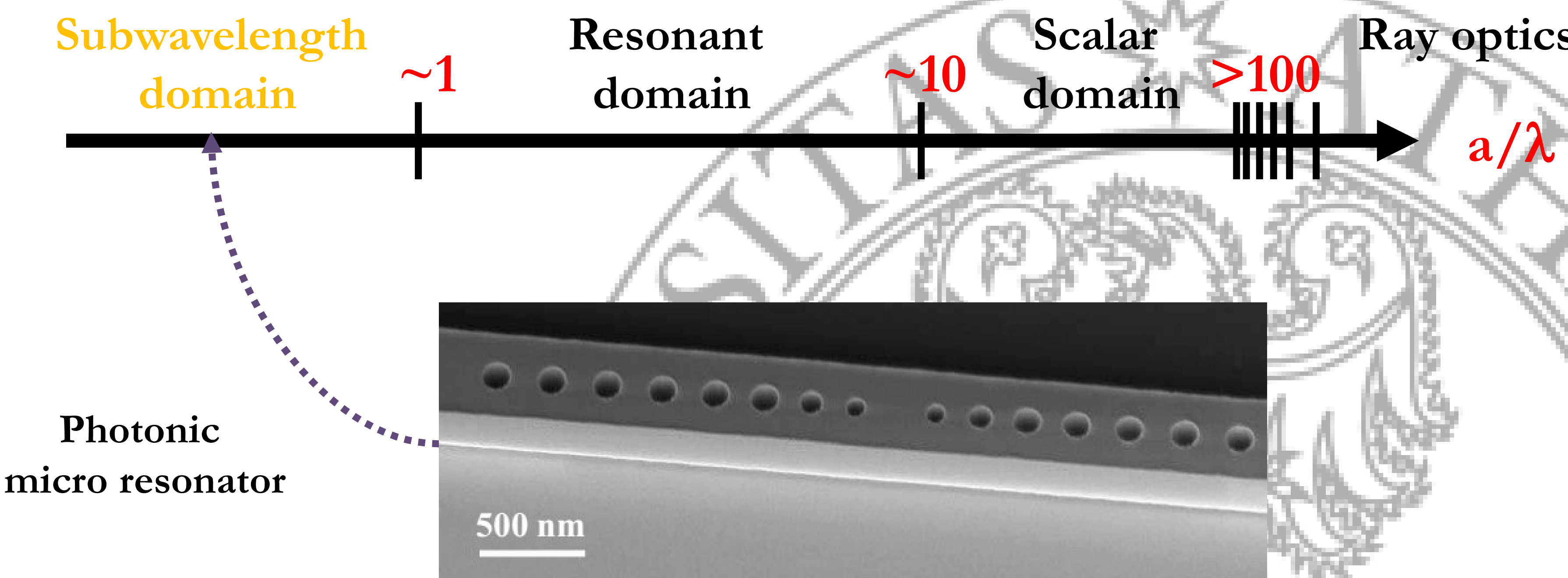
Photonic crystal waveguides





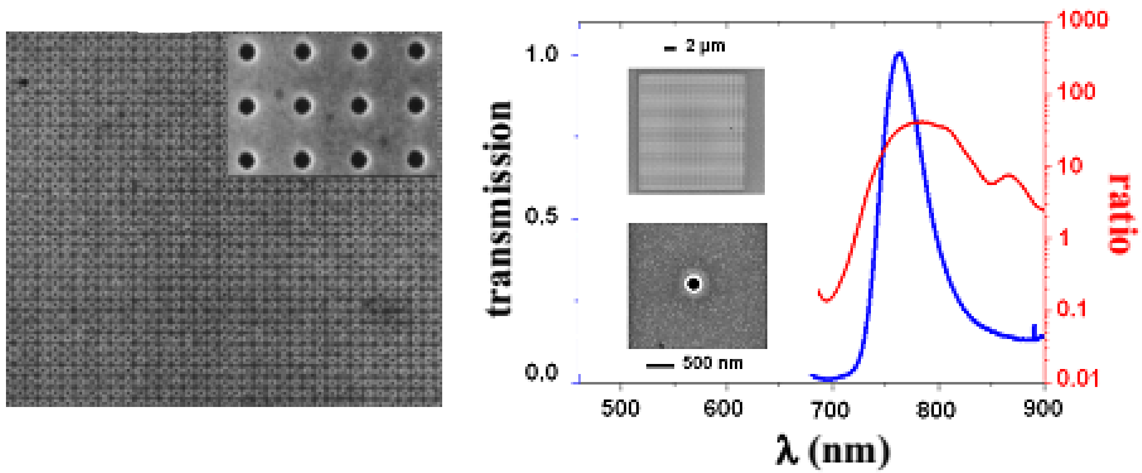


Different scale for optics



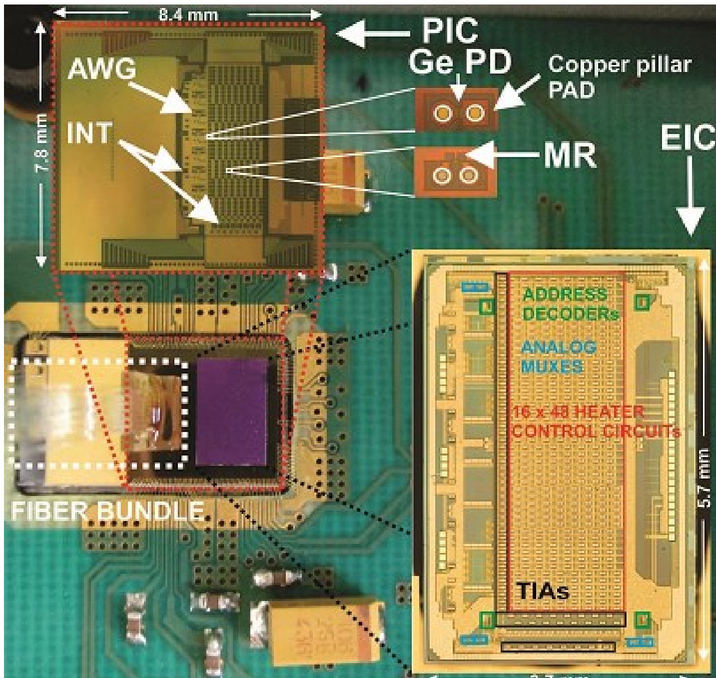
Different scale for optics

Plasmonic:
how to manipulate
light with metals

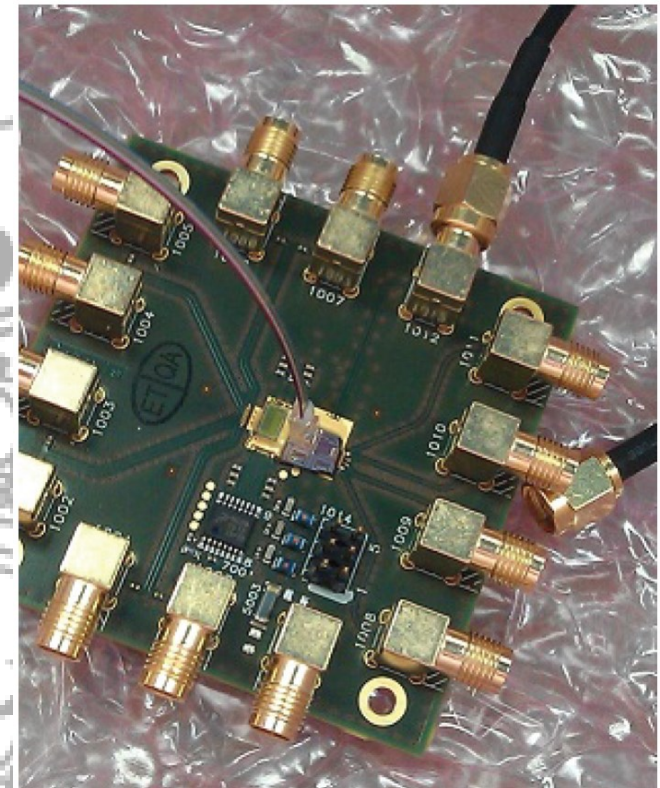


Plasmonics : extraordinary transmission

Nano-photonics applications: research



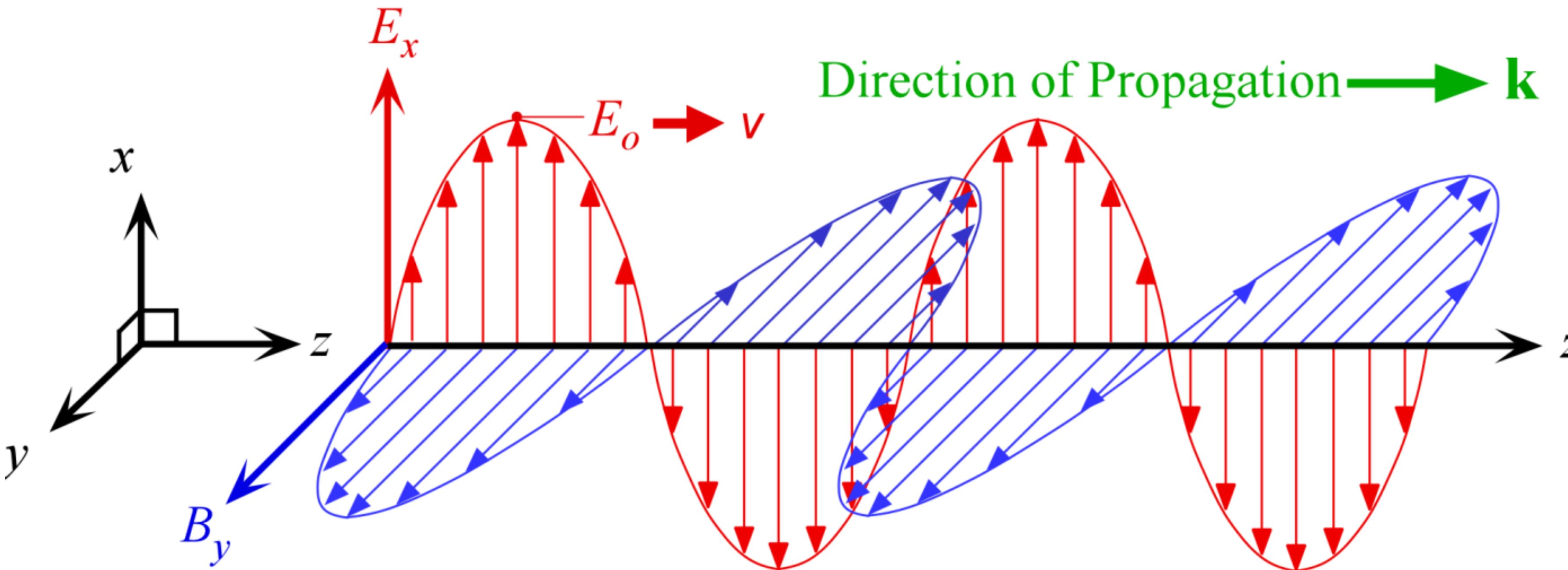
WDM transponder aggregator



High speed transceiver

[Links]

Light is an electromagnetic wave



An electromagnetic wave is a traveling wave that has time-varying electric and magnetic fields that are perpendicular to each other and the direction of propagation *z*.

Maxwell's Wave Equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \varepsilon_o \varepsilon_r \mu_o \frac{\partial^2 E}{\partial t^2} = 0$$

A plane wave is a solution of Maxwell's wave equation

$$E_x = E_o \cos(\omega t - kz + \phi_o)$$

Substitute into Maxwell's Equation to show that this is a solution.

$$E_x = E_o \cos(\omega t - kz + \phi_o)$$

E_x = Electric field along x at position z at time t

k = **Propagation constant** = $2\pi/\lambda$

λ = Wavelength

ω = Angular frequency = $2\pi\upsilon$ (υ = frequency)

E_o = Amplitude of the wave

ϕ_o = Phase constant; at $t = 0$ and $z = 0$, E_x may or may not necessarily be zero depending on the choice of origin.

$(\omega t - kz + \phi_o) = \phi$ = **Phase of the wave**

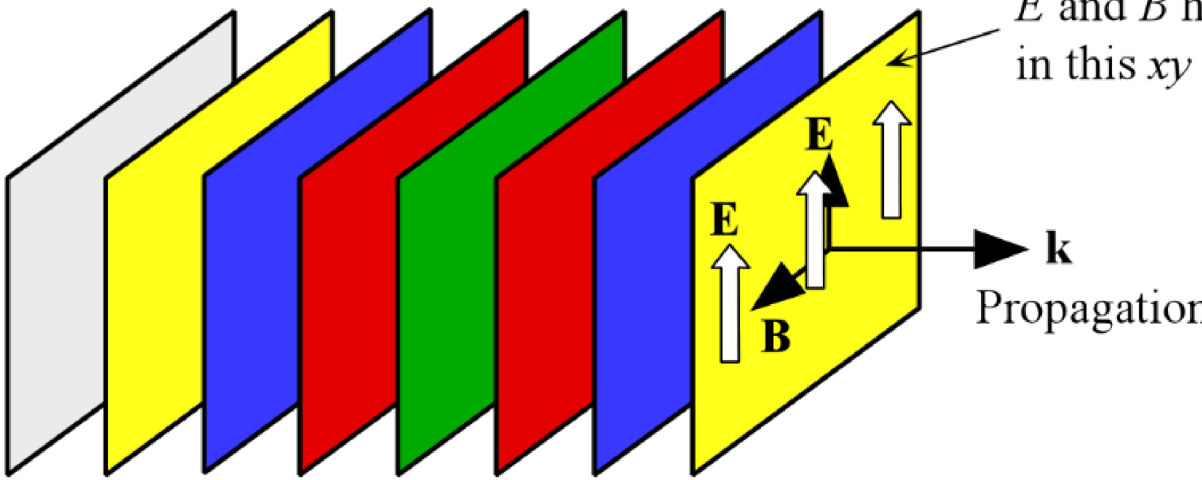
This is a **monochromatic plane wave** of *infinite extent* traveling in the positive z direction.

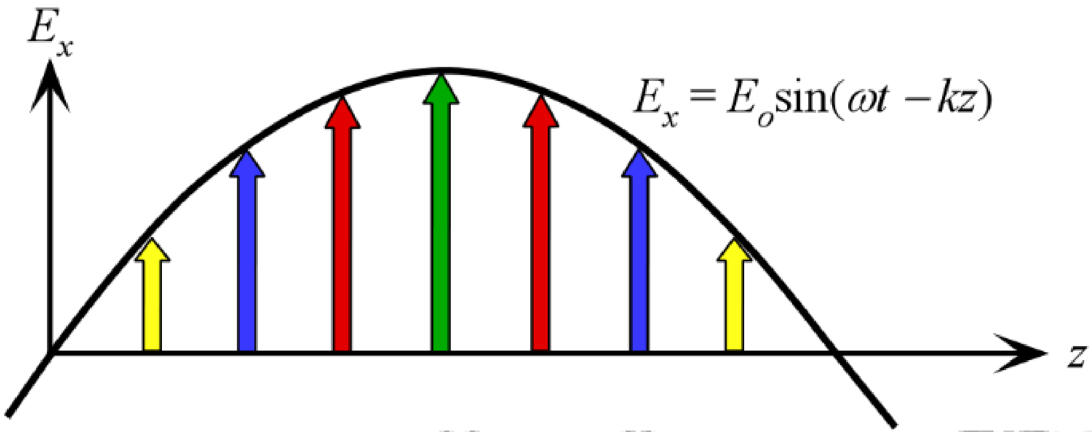
A wavefront of a plane wave is a plane perpendicular to the direction of propagation

The interaction of a light wave with a nonconducting medium (conductivity = 0) uses the electric field component E_x rather than B_y.

Optical field refers to the electric field E_x.

A Plane EM Wave





A plane EM wave traveling along z, has the same E_x (or B_y) at any point in a given xy plane. All electric field vectors in a given xy plane are therefore in phase. The xy planes are of infinite extent in the x and y directions.

Phase Velocity

The time and space evolution of a given phase ϕ, for example that corresponding to a maximum field is described by

$$\phi = \omega t - kz + \phi_o = \text{constant}$$

During a time interval δt, this constant phase (and hence the maximum field) moves a distance δz. The phase velocity of this wave is therefore $\delta z/\delta t$. The phase velocity v is

$$v = \frac{\delta z}{\delta t} = \frac{\omega}{k} = \upsilon\lambda$$

Phase change over a distance Δz

$$\phi = \omega t - kz + \phi_o$$

$$\Delta\phi = k\Delta z$$

The phase difference between two points separated by Δz is simply $k\Delta z$ since ωt is the same for each point

If this phase difference is 0 or multiples of 2π then the two points are in phase. Thus, the phase difference $\Delta\phi$ can be expressed as $k\Delta z$ or $2\pi\Delta z/\lambda$

Exponential Notation

Recall that

$$\cos\phi = \mathrm{Re}[\exp(j\phi)]$$

where Re refers to the real part. We then need to take the real part of any complex result at the end of calculations. Thus,

$$E_x(z,t) = \mathrm{Re}[E_o\exp(j\phi_o)\exp j(\omega t - kz)]$$

or

$$E_x(z,t) = \mathrm{Re}[E_c\exp j(\omega t - kz)]$$

where $E_c = E_o\exp(j\phi_o)$ is a complex number that represents the amplitude of the wave and includes the constant phase information ϕ_o.

Wave Vector or Propagation Vector

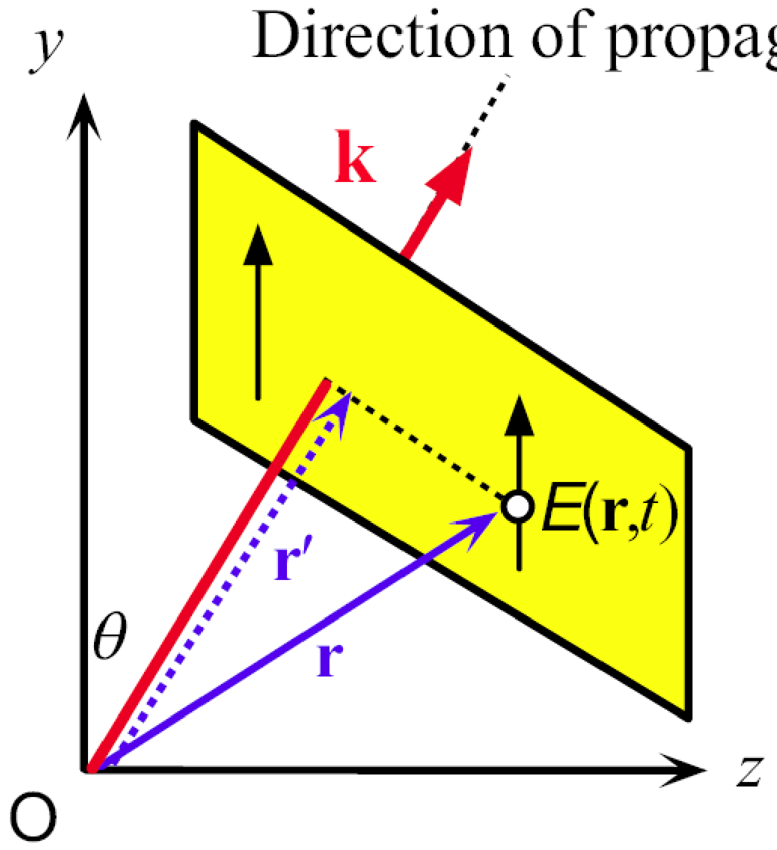
Direction of propagation is indicated with a vector k, called the wave vector, whose magnitude is the *propagation constant*, $k = 2\pi/\lambda$. k is *perpendicular* to constant phase planes.

When the electromagnetic (EM) wave is propagating along some arbitrary direction k, then the electric field $E(\mathbf{r},t)$ at a point r on a plane perpendicular to k is

$$E(\mathbf{r},t) = E_o \cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi_o)$$

If propagation is along *z*, k·r becomes k_z. In general, if k has components k_x, k_y and k_z along *x*, *y* and *z*, then from the definition of the dot product, $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$.

Wave Vector k

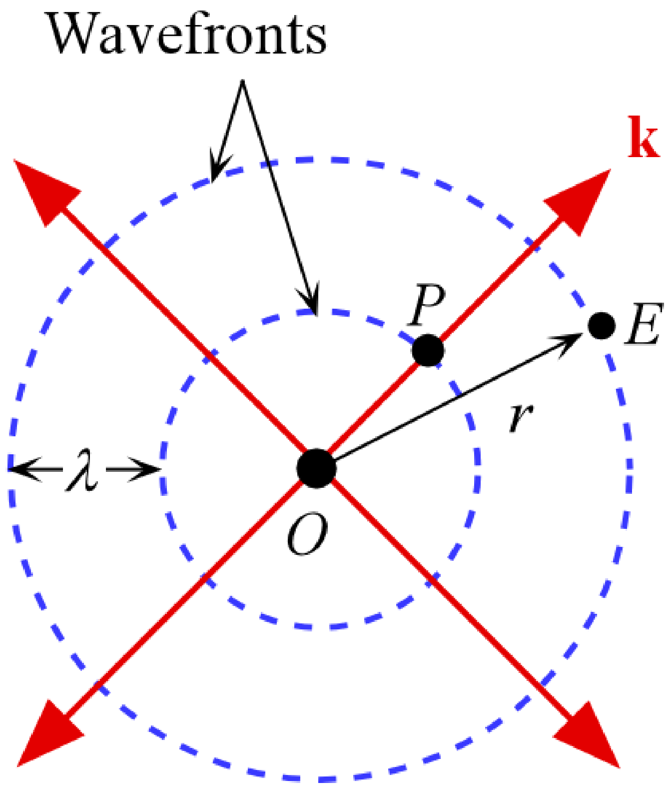


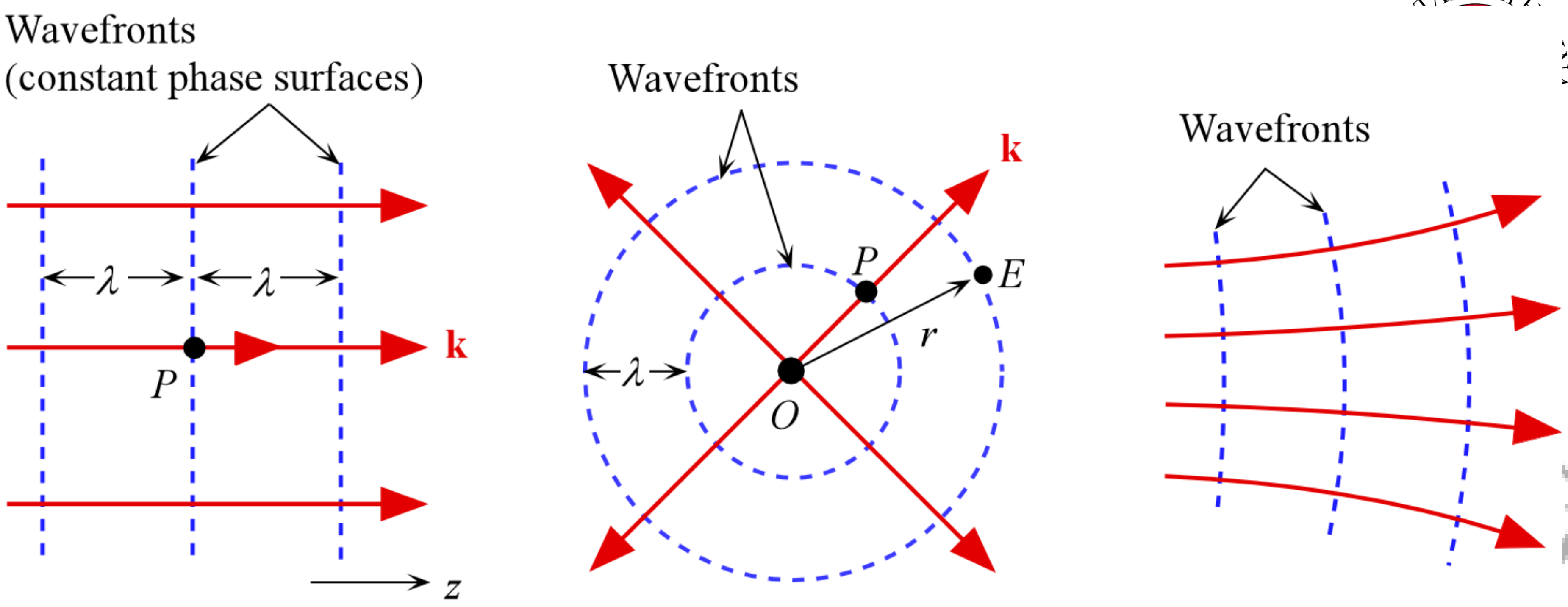
$$E(\mathbf{r},t) = E_o \cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi_o)$$

A traveling plane EM wave along a direction **k**

Spherical Wave

$$E = \frac{A}{r}\cos(\omega t - kr)$$





A perfect plane wave

(a)

A perfect spherical wave

(b)

A divergent beam

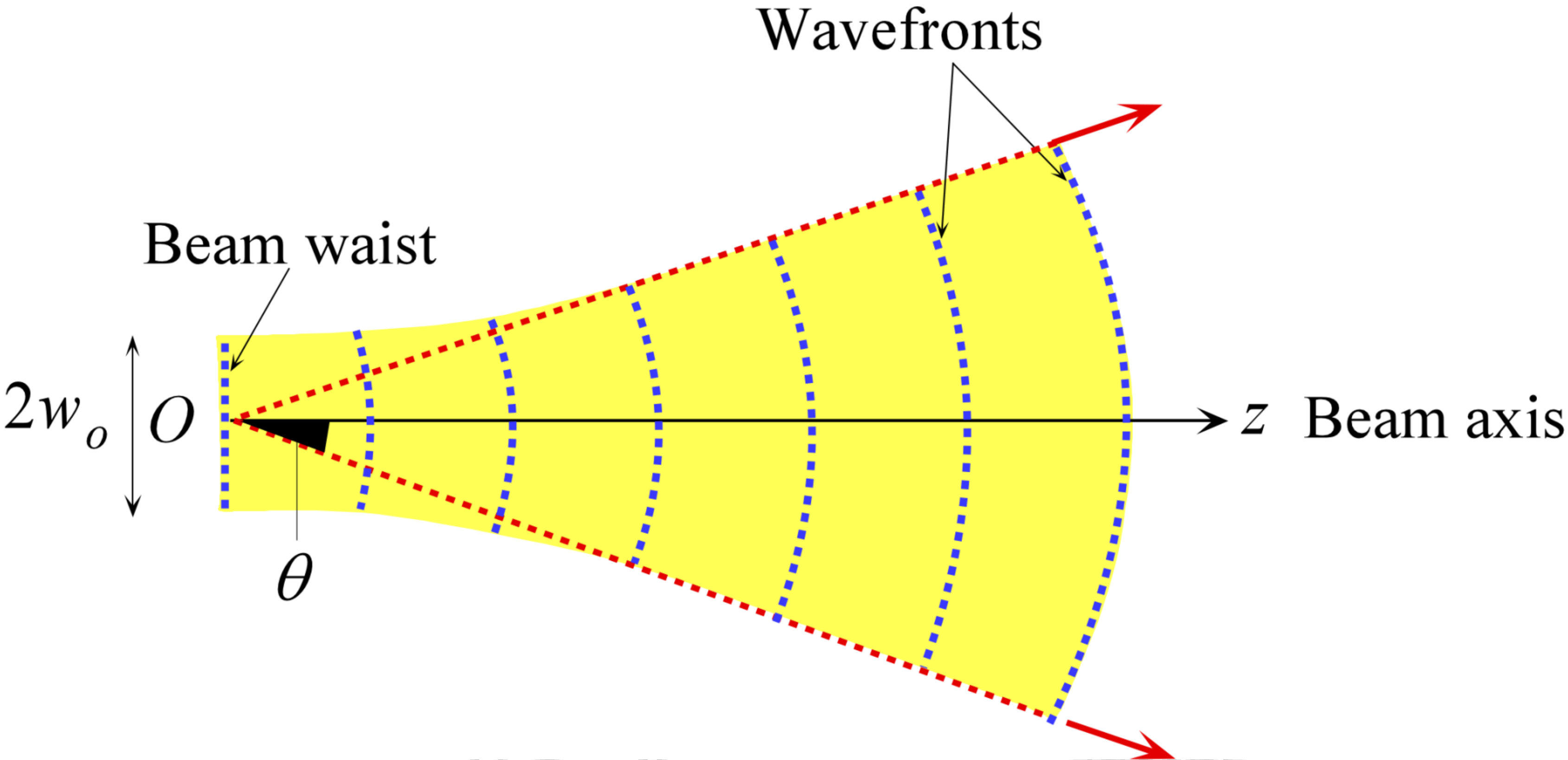
(c)

Examples of possible EM waves

Optical divergence refers to the angular separation of wave vectors on a given wavefront.

Gaussian Beam

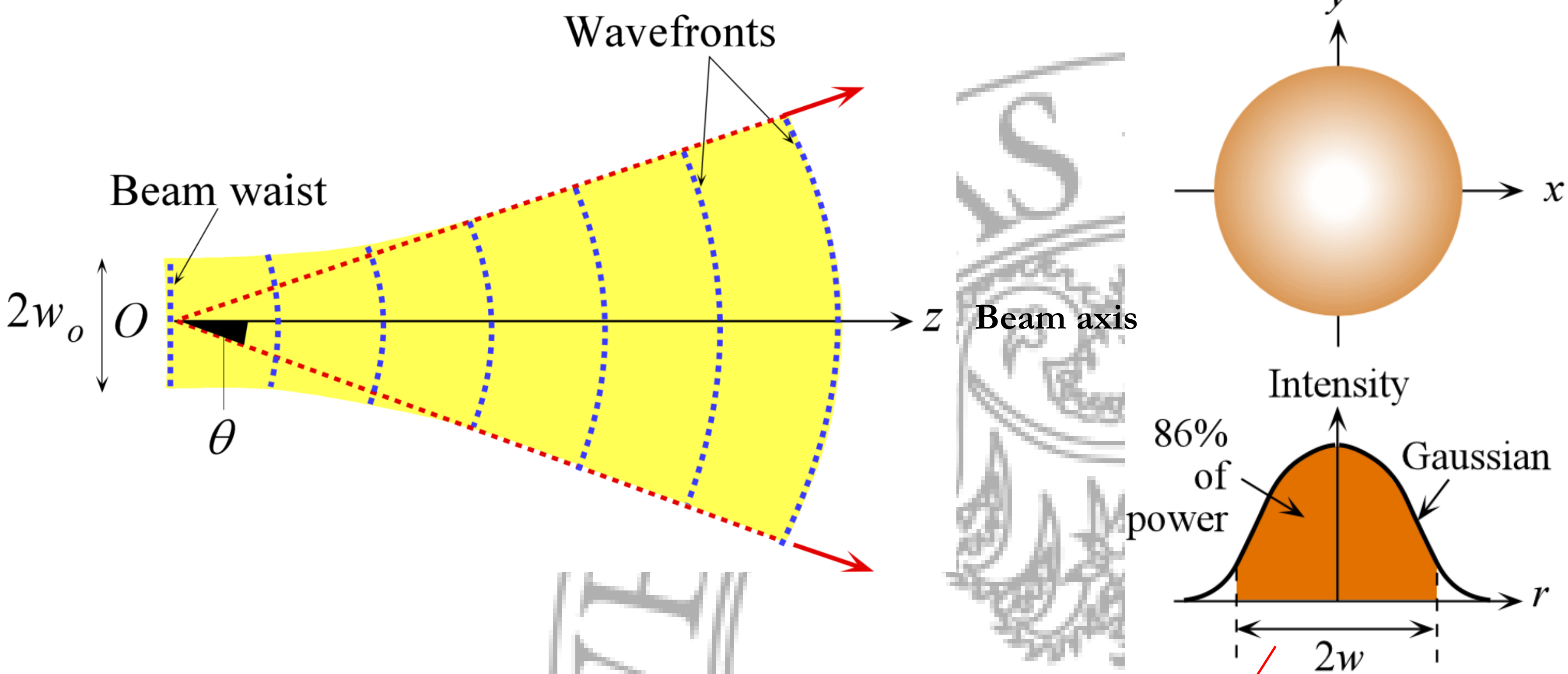
The radiation emitted from a laser can be approximated by a Gaussian beam. Gaussian beam approximations are widely used in photonics.



Wavefronts of a Gaussian light beam

Gaussian Beam

The intensity across the beam follows a Gaussian distribution

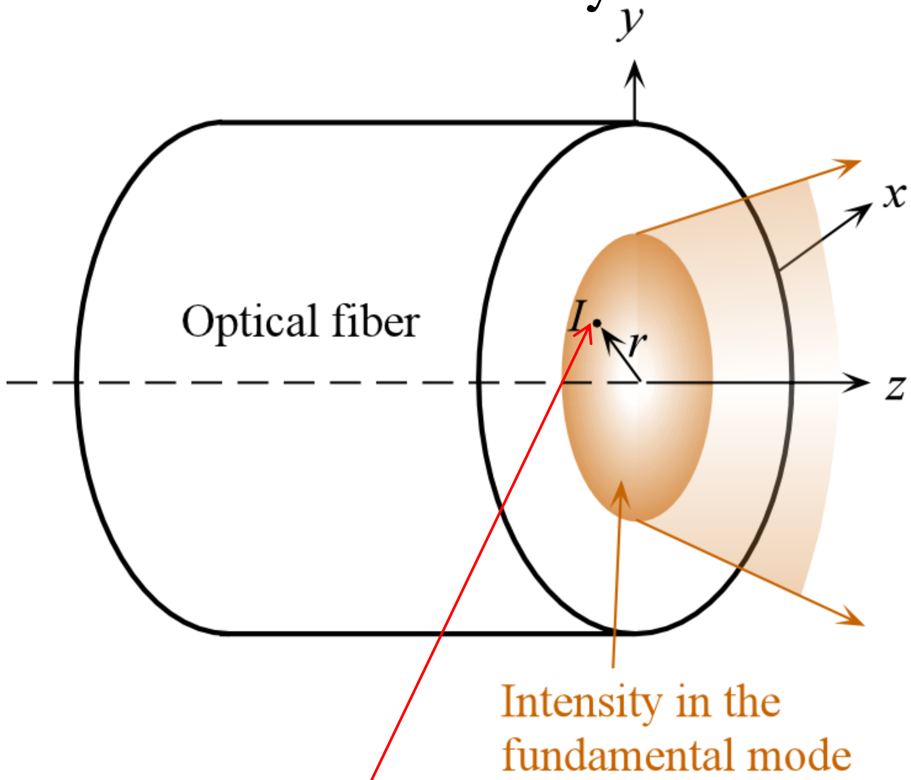


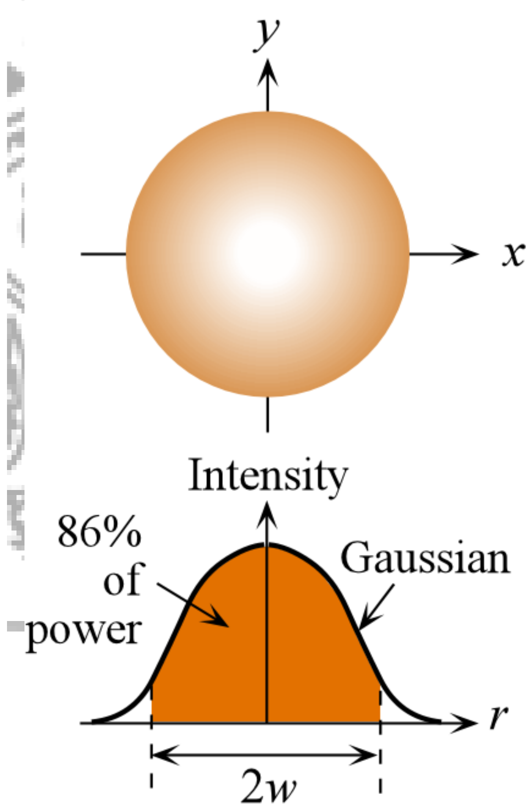
Intensity = $I(r,z) = [2P/(\pi w^2)]\exp(-2r^2/w^2)$

$\theta = w/z = \lambda/(\pi w_o)$ $\quad 2\theta$ = **Far field divergence**

The Gaussian Intensity Distribution is Not Unusual

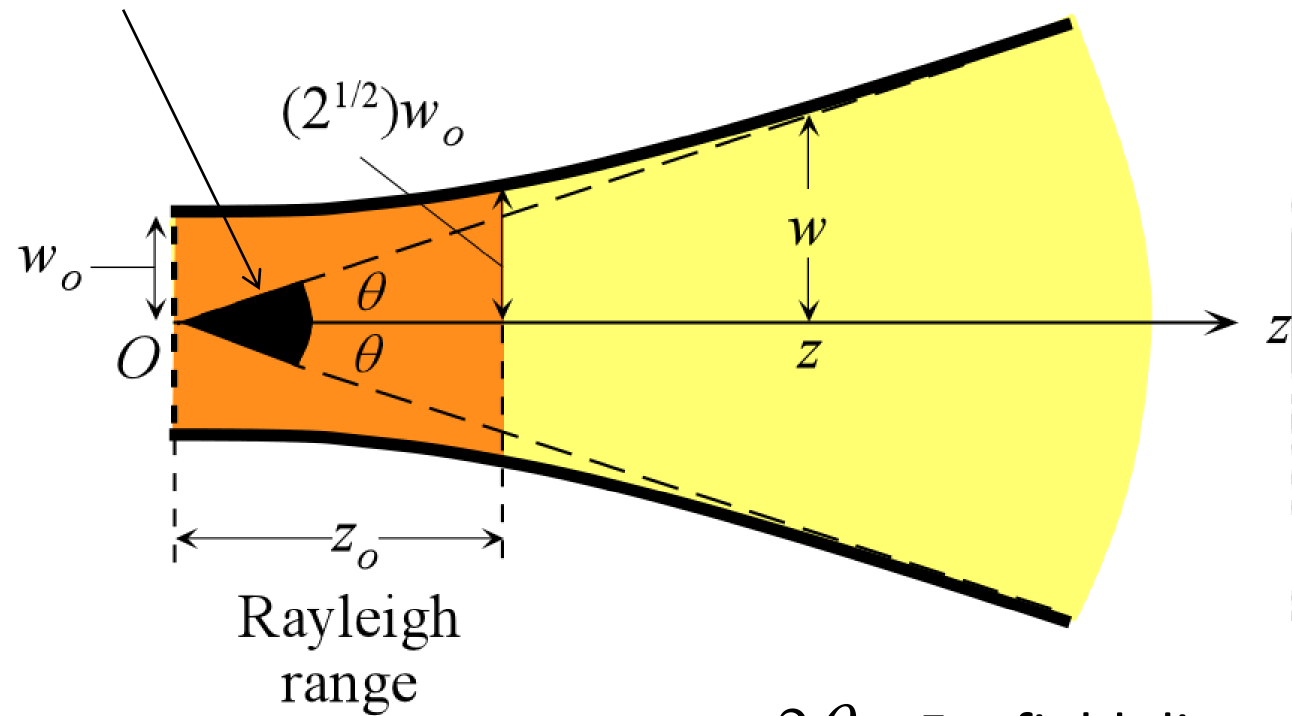
The Gaussian intensity distribution is also used in fiber optics
The fundamental mode in single mode fibers can be approximated with a Gaussian intensity distribution across the fiber core





$$I(r) = I(0)\exp(-2r^2 / w^2)$$

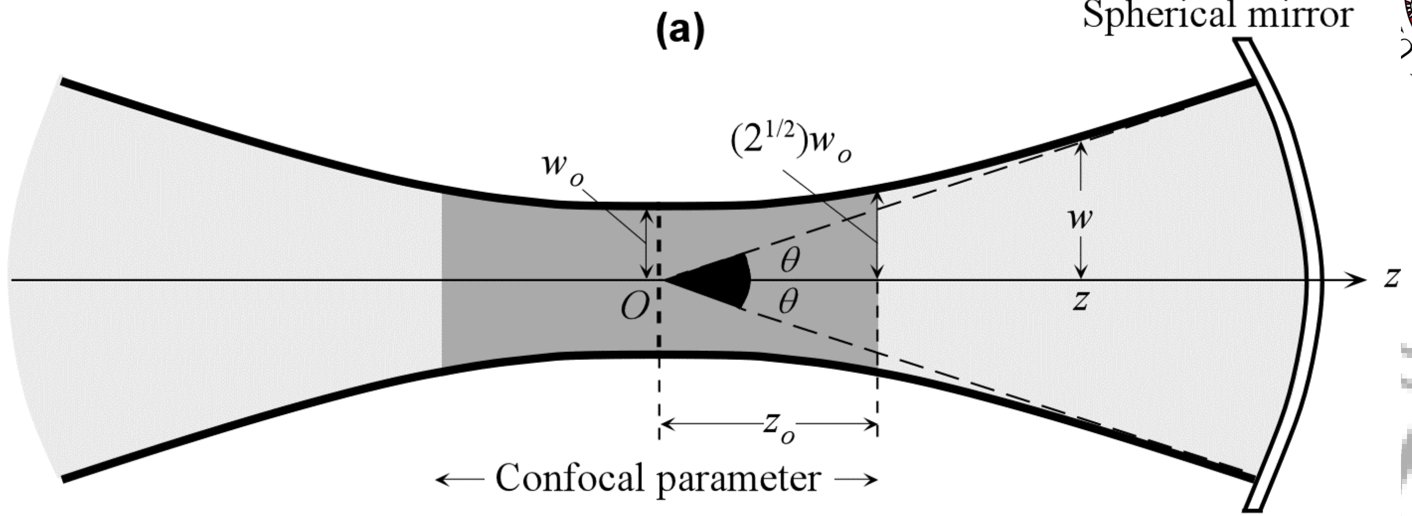
Gaussian Beam (Rayleigh range)



2θ = Far field divergence

$$z_o = \pi w_o^{\,2}/\lambda$$

Gaussian Beam

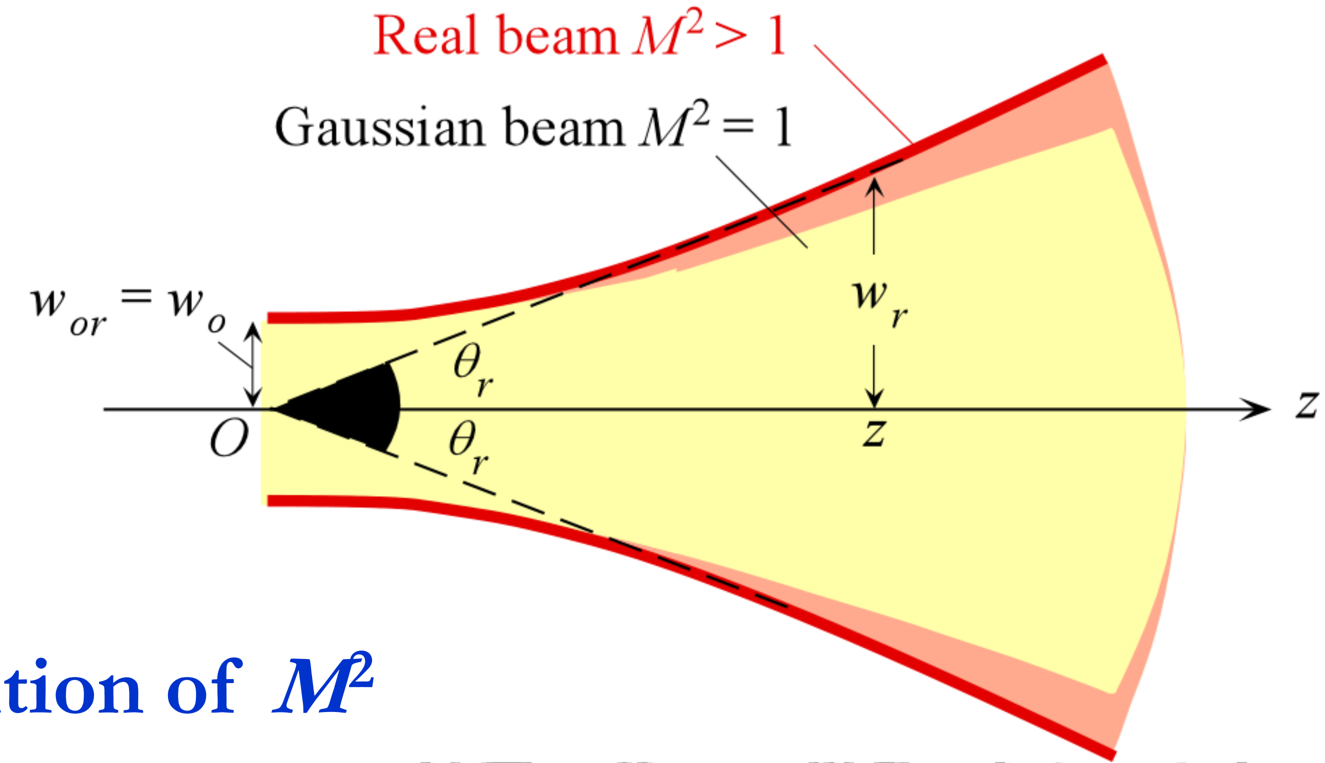


Rayleigh range

$$z_o = \frac{\pi w_o^2}{\lambda}$$

$$2w = 2w_o\left[1+\left(\frac{z}{z_o}\right)^2\right]^{1/2}$$

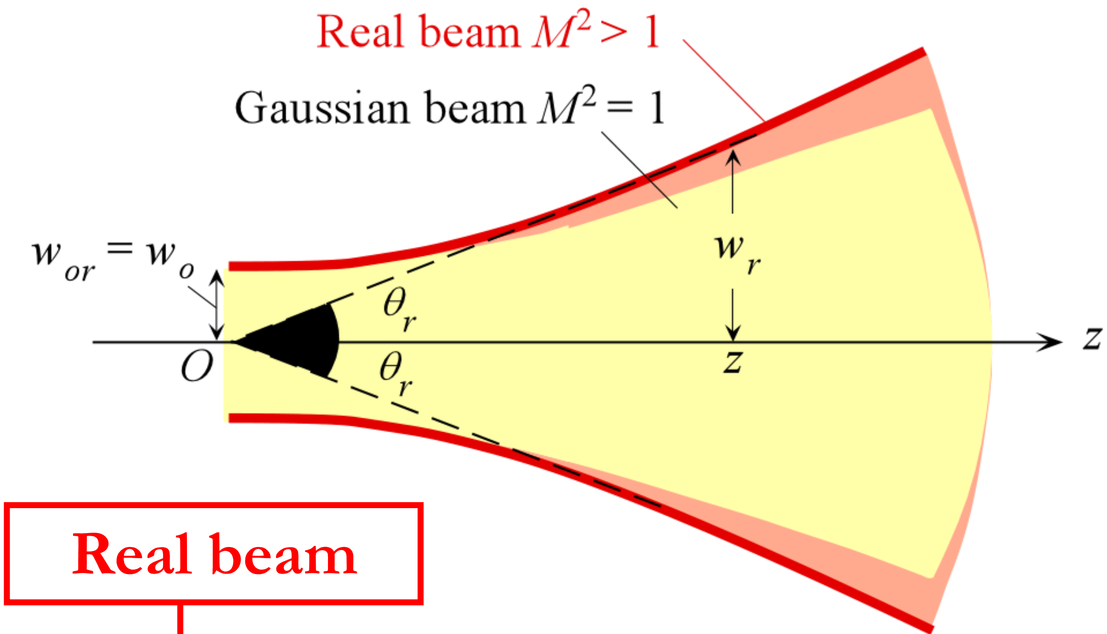
$$2w = 2w_o\left[1+\left(\frac{z\lambda}{\pi w_o^2}\right)^2\right]^{1/2}$$



Definition of M^2

$$M^2 = \frac{w_{or}\theta_r}{w_o\theta} = \frac{w_{or}\theta_r}{(\lambda/\pi)}$$

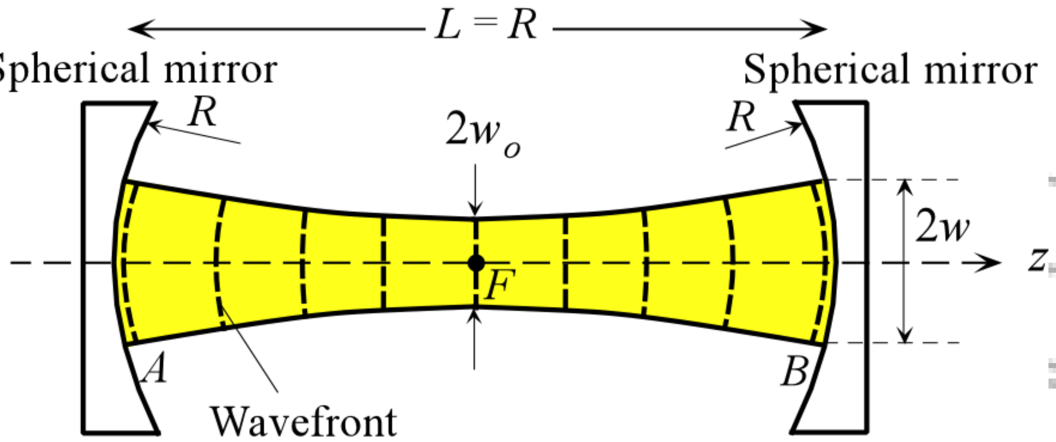
$$2w_r = 2w_{or}\left[1+\left(\frac{z\lambda M^2}{\pi w_{or}^2}\right)^2\right]^{1/2}$$



Real beam

$$2w_r = 2w_{or}\left[1+\left(\frac{z\lambda M^2}{\pi w_{or}^2}\right)^2\right]^{1/2}$$

Gaussian Beam in an Optical Cavity



Optical cavity

Two spherical mirrors reflect waves to and from each other. The optical cavity contains a Gaussian beam. This particular optical cavity is symmetric and confocal; the two focal points coincide at F.

EXAMPLE 1.1.1 A diverging laser beam

Consider a He-Ne laser beam at 633 nm with a spot size of 1 mm. Assuming a Gaussian beam, what is the divergence of the beam? What are the Rayleigh range and the beam width at 25 m?

Solution

Using Eq. (1.1.7), we find

$$2\theta = \frac{4\lambda}{\pi(2w_o)} = \frac{4(633 \times 10^{-9}\ \mathrm{m})}{\pi(1 \times 10^{-3}\ \mathrm{m})} = 8.06 \times 10^{-4}\ \mathrm{rad} = 0.046°$$

The Rayleigh range is

$$z_o = \frac{\pi w_o^2}{\lambda} = \frac{\pi\left[(1 \times 10^{-3}\ \mathrm{m})/2\right]^2}{(633 \times 10^{-9}\ \mathrm{m})} = 1.24\ \mathrm{m}$$

The beam width at a distance of 25 m is

$$2w = 2w_o[1 + (z/z_o)^2]^{1/2} = (1 \times 10^{-3}\ \mathrm{m})\{1 + [(25\ \mathrm{m})/(1.24\ \mathrm{m})]^2\}^{1/2}$$

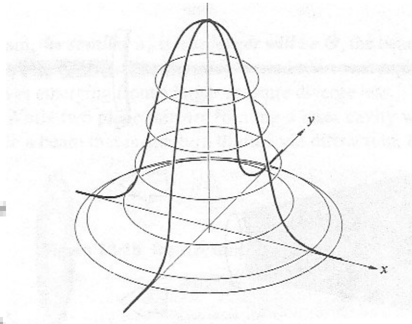
$$= 0.0202\ \mathrm{m} \quad \text{or} \quad 20\ \mathrm{mm}.$$

$$\boxed{2w = 2w_o\left[1+\left(\frac{z}{z_o}\right)^2\right]^{1/2} \approx 2w_o\frac{z}{z_o} = (1\,\mathrm{mm})\frac{25\,\mathrm{m}}{1.24\,\mathrm{m}} = 20\,\mathrm{mm}}$$

Gaussian beams: recap

- Zero order mode is Gaussian
- Intensity profile:
- beam waist: w_0

$$I = I_0 e^{-2r^2/w^2}$$



$$w = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}$$

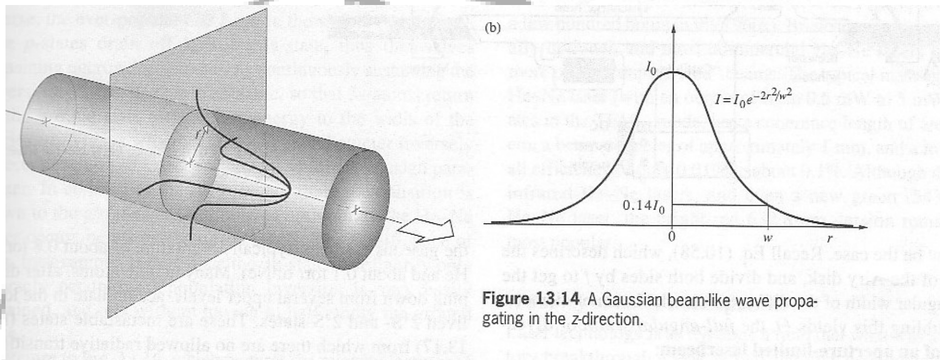


Figure 13.14 A Gaussian beam-like wave propagating in the z-direction.

- confocal parameter: z

$$z_R = \frac{\pi w_0^2}{\lambda}$$

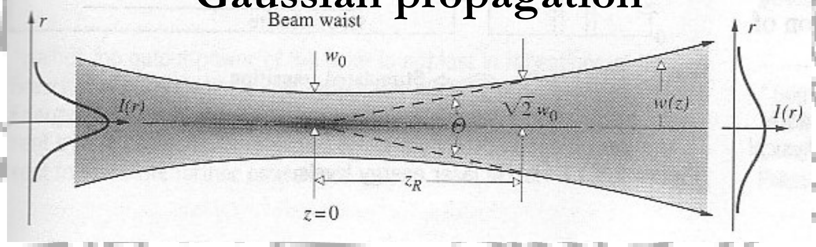
- far from waist

$$w \rightarrow \frac{\lambda z}{\pi w_0}$$

- divergence angle

$$\Theta = \frac{2\lambda}{\pi w_0} = 0.637 \frac{\lambda}{w_0}$$

Gaussian propagation



Refractive Index

When an EM wave is traveling in a dielectric medium, the oscillating electric field **polarizes** the molecules of the medium at the frequency of the wave

The **stronger** is the interaction between the field and the dipoles, the **slower** is the propagation of the wave

Refractive Index

TABLE 1.1 Low-frequency (LF) relative permittivity ε_r(LF) and refractive index n

Material	ε_r(LF)	$[\varepsilon_r(\mathrm{LF})]^{1/2}$	n (at λ)	Comment
Si	11.9	3.44	3.45 (at 2.15 μm)	Electronic bond polarization up to optical frequencies
Diamond	5.7	2.39	2.41 (at 590 nm)	Electronic bond polarization up to UV light
GaAs	13.1	3.62	3.30 (at 5 μm)	Ionic polarization contributes to ε_r(LF)
SiO_2	3.84	2.00	1.46 (at 600 nm)	Ionic polarization contributes to ε_r(LF)
Water	80	8.9	1.33 (at 600 nm)	Dipolar polarization contributes to ε_r(LF), which is large

Maxwell's Wave Equation in an isotropic medium

$$\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2}-\varepsilon_o\varepsilon_r\mu_o\frac{\partial^2 E}{\partial t^2}=0$$

A plane wave is a solution of Maxwell's wave equation

$$E_x = E_o\cos(\omega t - kz + \phi_o)$$

The phase velocity of this plane wave in the medium is given by

$$\boxed{v=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon_o\varepsilon_r\mu_o}}}$$

The phase velocity in vacuum is

$$\boxed{c=\frac{\omega}{k_o}=\frac{1}{\sqrt{\varepsilon_o\mu_o}}}$$

Phase Velocity and ε_r

The relative permittivity ε_r measures the ease with which the medium becomes polarized and hence it indicates the extent of interaction between the field and the induced dipoles.

For an EM wave traveling in a nonmagnetic dielectric medium of relative permittivity ε_r, the phase velocity v is given by

$$v = \frac{1}{\sqrt{\varepsilon_r \varepsilon_o \mu_o}}$$

Refractive Index n

Phase Velocity and ε_r

$$V = \frac{1}{\sqrt{\varepsilon_r \varepsilon_o \mu_o}}$$

Refractive index n definition

$$n = \frac{c}{V} = \sqrt{\varepsilon_r}$$

Optical frequencies

Typical frequencies that are involved in optoelectronic devices are in the infrared (including far infrared), visible, and UV, and we generically refer to these frequencies as **optical frequencies**

Somewhat arbitrary range:

Roughly 10^{12} Hz to 10^{16} Hz

Some typical values

Low frequency (LF) relative permittivity ε_r(LF) and refractive index n.

TABLE 1.1 Low-frequency (LF) relative permittivity ε_r(LF) and refractive index n

Material	ε_r(LF)	$[\varepsilon_r(\mathrm{LF})]^{1/2}$	n (at λ)	Comment
Si	11.9	3.44	3.45 (at 2.15 μm)	Electronic bond polarization up to optical frequencies
Diamond	5.7	2.39	2.41 (at 590 nm)	Electronic bond polarization up to UV light
GaAs	13.1	3.62	3.30 (at 5 μm)	Ionic polarization contributes to ε_r(LF)
SiO_2	3.84	2.00	1.46 (at 600 nm)	Ionic polarization contributes to ε_r(LF)
Water	80	8.9	1.33 (at 600 nm)	Dipolar polarization contributes to ε_r(LF), which is large

Refractive Index and Propagation Constant

k_o	**Free-space propagation constant (wave vector)**
k_o	$2\pi/\lambda_o$
λ_o	**Free-space wavelength**
k	**Propagation constant (vave vector) in the medium**
λ	**Wavelength in the medium**

$$n = \frac{k}{k_o}$$

In noncrystalline materials such as glasses and liquids, the material structure is the same in all directions and n does not depend on the direction. The refractive index is then **isotropic**

Refractive Index and Wavelength

It is customary to drop the subscript *o* on *k* and *λ* to indicate free space

$$k_{\text{medium}} = nk$$

$$\lambda_{\text{medium}} = \lambda / n$$

In free space

Refractive Index and Isotropy

Crystals, in general, have non-isotropic, or anisotropic, properties

Typically non-crystalline solids such as glasses and liquids, and cubic crystals are optically isotropic; they possess only one refractive index for all directions

n depends on the wavelength λ

Dispersion relation: $n = n(\lambda)$

The simplest electronic polarization gives

$$n^2 = 1 + \left(\frac{N_{\mathrm{at}} Z e^2}{\varepsilon_o m_e}\right)\left(\frac{\lambda_o}{2\pi c}\right)^2 \frac{\lambda^2}{\lambda^2 - \lambda_o^2}$$

N_{at} =Number of atoms per unit volume
Z = Number of electrons in the atom (atomic number)

λ_o = A "resonant frequency"

Sellmeier Equation

$$n^2 = 1 + \frac{A_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3 \lambda^2}{\lambda^2 - \lambda_3^2}$$

n depends on the wavelength λ

Cauchy dispersion relation

$$n = n(\upsilon)$$

$$n = n_{-2}(h\upsilon)^{-2} + n_0 + n_2(h\upsilon)^2 + n_4(h\upsilon)^4$$

n depends on the wavelength λ

TABLE 1.2 **Sellmeier and Cauchy coefficients**

Sellmeier	A_1	A_2	A_3	λ_1 (μm)	λ_2 (μm)	λ_3 (μm)
SiO_2 (fused silica)	0.696749	0.408218	0.890815	0.0690660	0.115662	9.900559
86.5%SiO_2-13.5%GeO_2	0.711040	0.451885	0.704048	0.0642700	0.129408	9.425478
GeO_2	0.80686642	0.71815848	0.85416831	0.068972606	0.15396605	11.841931
Sapphire	1.023798	1.058264	5.280792	0.0614482	0.110700	17.92656
Diamond	0.3306	4.3356	–	0.1750	0.1060	–

Cauchy	**Range of** $h\nu$ **(eV)**		n_{-2} (eV^2)	n_0	n_2 (eV^{-2})	n_4 (eV^{-4})
Diamond	0.05–5.47		-1.07×10^{-5}	2.378	8.01×10^{-3}	1.04×10^{-4}
Silicon	0.002–1.08		-2.04×10^{-8}	3.4189	8.15×10^{-2}	1.25×10^{-2}
Germanium	0.002–0.75		-1.0×10^{-8}	4.003	2.2×10^{-1}	1.4×10^{-1}

Source: Sellmeier coefficients combined from various sources. Cauchy coefficients from D. Y. Smith *et al.*, *J. Phys. CM*, *13*, 3883, 2001.

Example

EXAMPLE 1.2.1 Sellmeier equation and diamond

Using the Sellmeier coefficients for diamond in Table 1.2, calculate its refractive index at 610 nm (red light) and compare with the experimental quoted value of 2.415 to three decimal places.

Solution

The Sellmeier dispersion relation for diamond is

$$n^2 = 1 + \frac{0.3306\lambda^2}{\lambda^2 - 175\ \text{nm}^2} + \frac{4.3356\lambda^2}{\lambda^2 - 106\ \text{nm}^2}$$

$$n^2 = 1 + \frac{0.3306(610\ \text{nm})^2}{(610\ \text{nm})^2 - (175\ \text{nm})^2} + \frac{4.3356(610\ \text{nm})^2}{(610\ \text{nm})^2 - (106\ \text{nm})^2} = 5.8308$$

So that

$$n = 2.4147$$

which is 2.415 to three decimal places and matches the experimental value.

Example

EXAMPLE 1.2.2 Cauchy equation and diamond

Using the Cauchy coefficients for diamond in Table 1.2, calculate the refractive index at 610 nm.

Solution

At $\lambda = 610$ nm, the photon energy is

$$h\upsilon = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34}\,\text{J s})(2.998 \times 10^{8}\,\text{m s}^{-1})}{(610 \times 10^{-9}\,\text{m})} \times \frac{1}{1.602 \times 10^{-19}\,\text{J eV}^{-1}} = 2.0325\,\text{eV}$$

Using the Cauchy dispersion relation for diamond with coefficients from Table 1.2,

$$\begin{aligned} n &= n_{-2}(h\upsilon)^{-2} + n_0 + n_2(h\upsilon)^2 + n_4(h\upsilon)^4 \\ &= (-1.07 \times 10^{-5})(2.0325)^{-2} + 2.378 + (8.01 \times 10^{-3})(2.0325)^2 \\ &\quad + (1.04 \times 10^{-4})(2.0325)^4 \\ &= 2.4140 \end{aligned}$$

which is slightly different than the value calculated in Example 1.2.1; one reason for the discrepancy is due to the Cauchy coefficients quoted in Table 1.2 being applicable over a wider wavelength range at the expense of some accuracy. Although both dispersion relations have four parameters, A_1, A_2, λ_1, λ_2 for Sellmeier and n_{-2}, n_0, n_2, n_4 for Cauchy, the functional forms are different.

Back to the Wave equation

In uniform isotropic linear media, the wave equation is:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

They are satisfied by plane wave

$$\psi = A\, e^{\,i(\mathbf{k}\ \mathbf{r} - \omega t)} \qquad k = |\mathbf{k}| = \omega\sqrt{\mu\varepsilon}$$

ψ can be any Cartesian components of **E** and **H**

The phase velocity of plane wave travels in the direction of **k** is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$\nabla \times$

Back to the Wave equation

We can define the index of refraction as

$$n = \frac{v}{c} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Most media are nonmagnetic and have a magnetic permeability $\mu = \mu_0$, in this case

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

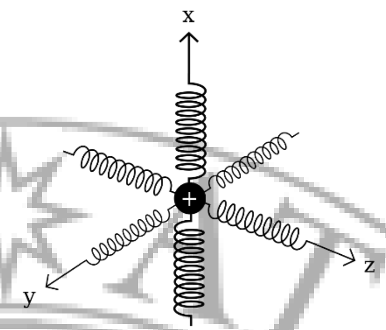
In most media, n is a function of frequency.

Origin of the refractive index

Let the electric field of optical wave in an atom be

$$\mathbf{E}=\mathbf{E_0}e^{-i\omega t}$$

the electron obeys the following equation of motion



$$m\frac{d^2}{dt^2}R + m\gamma\frac{d}{dt}R + m{\omega_0}^2R = -e\mathbf{E}$$

R is the position of the electron relative to the atom

m is the mass of the electron

ω_0 is the resonant frequency of the electron motion

γ is the damping coefficient

Origin of the refractive index

The solution is
$$R = \frac{-e\mathbf{E_0}}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)} e^{-i\omega t}$$

The induced dipole moment is

$$\mathbf{p} = -eR = \frac{e^2}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}\mathbf{E} = \alpha\mathbf{E}$$

α is atomic polarizability

$$\alpha = \frac{e^2}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}$$

The dielectric constant of a medium depends on the manner in which the atoms are assembled. Let N be the number of atoms per unit volume. Then the polarization can be written approximately as

$$\mathbf{P} = \mathrm{N}\,\mathbf{p} = \mathrm{N}\,\alpha\,\mathbf{E} = \varepsilon_0\,\chi\,\mathbf{E}$$

Origin of the refractive index

The dielectric constant of the medium is given by

$$\varepsilon = \varepsilon_0 (1+\chi) = \varepsilon_0 (1+N\alpha/ \varepsilon_0)$$

If the medium is nonmagnetic, the index of refraction is

$$n= (\varepsilon /\varepsilon_0)^{1/2} = (1+N\alpha/ \varepsilon_0)^{1/2}$$

$$n^2 = \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0^2 - \omega^2 - i\omega\gamma)}$$

If the second term is small enough then

$$\boxed{n = 1 + \frac{Ne^2}{2\varepsilon_0 m(\omega_0^2 - \omega^2 - i\omega\gamma)}}$$

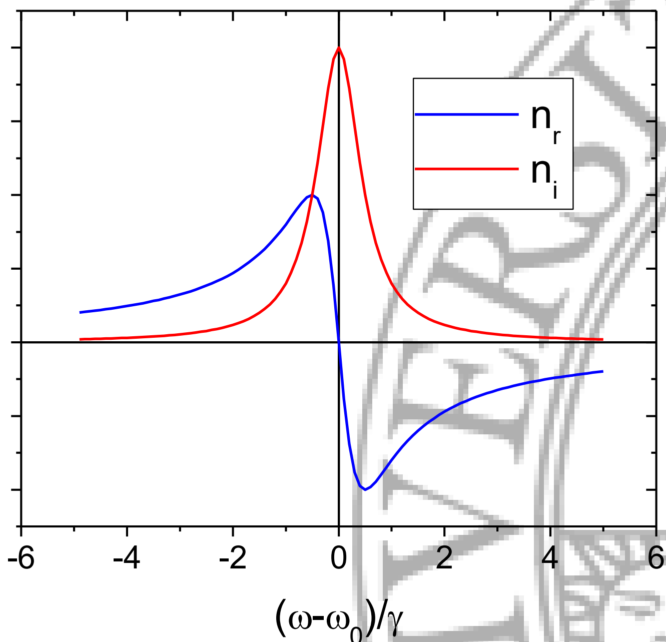
Origin of the refractive index

The complex refractive index is

$$n \rightarrow n_r + i n_i = 1 + \frac{Ne^2(\omega_0{}^2 - \omega^2)}{2\varepsilon_0 m[(\omega_0{}^2 - \omega^2)^2 + \omega^2\gamma^2]} + i\frac{Ne^2\gamma\omega}{2\varepsilon_0 m[(\omega_0{}^2 - \omega^2)^2 + \omega^2\gamma^2]}$$

at $\omega \sim \omega_0$,

$$n_r + i n_i = 1 + \frac{Ne^2(\omega_0 - \omega)}{4\varepsilon_0 m\omega_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]} + i\frac{Ne^2\gamma}{8\varepsilon_0 m\omega_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]}$$



Normalized plot of n-1 and k versus $\omega - \omega_0$

Origin of the refractive index

For more than one resonance frequencies for each atom,

$$n^2 = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{(\omega_j{}^2 - \omega^2 - i\omega\gamma_j)} \qquad \sum_j f_j = Z$$

Classical Electron Model (Drude model)

If we set ω_0=0, the Lorentz model become Drude model. This model can be used in free electron metals

$$n^2 = 1 - \frac{Ne^2}{\varepsilon_0 m(\omega^2 + i\omega\gamma)}$$

Defintion

By definition,

$$n^2 = \frac{\varepsilon}{\varepsilon_0}$$

$$n = n_r + i n_i$$

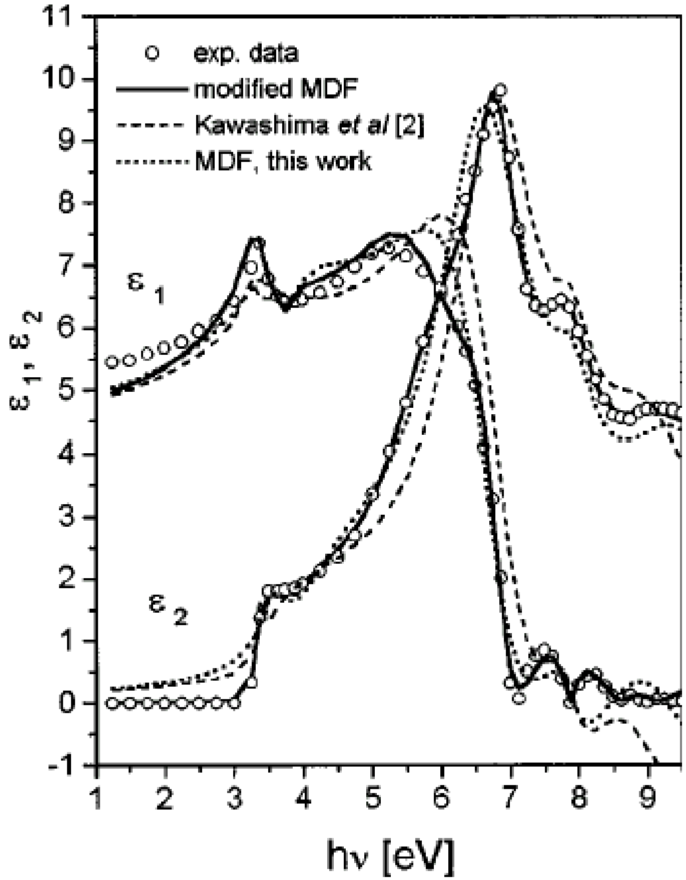
$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

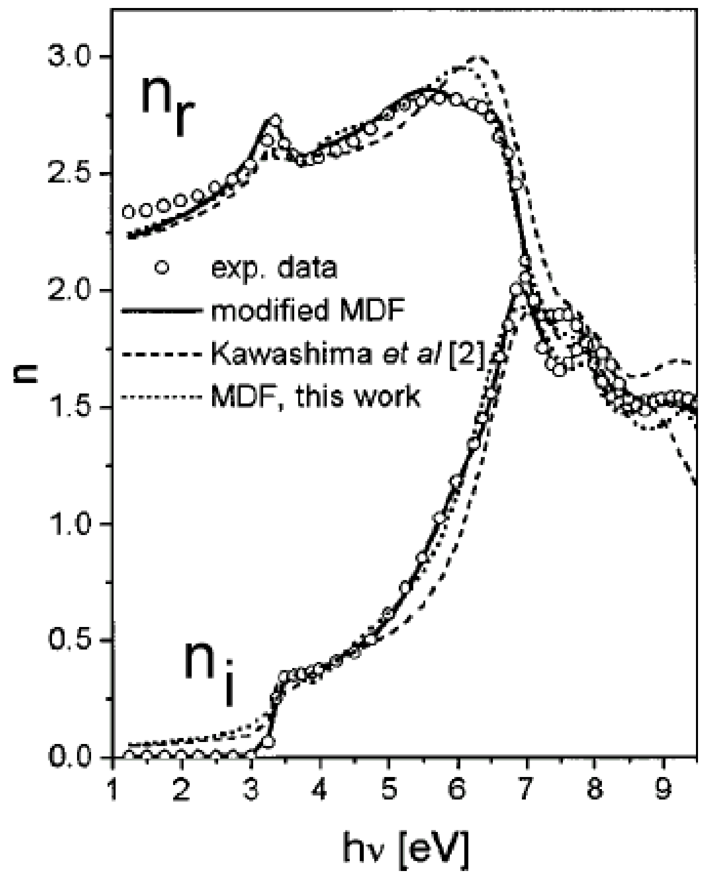
We can easily get:

$$n_r = \{\frac{1}{2}[(\varepsilon_1{}^2 + \varepsilon_2{}^2)^{1/2} + \varepsilon_1]\}^{1/2} / \varepsilon_0$$

$$n_i = \{\frac{1}{2}[(\varepsilon_1{}^2 + \varepsilon_2{}^2)^{1/2} - \varepsilon_1]\}^{1/2} / \varepsilon_0$$

Real and imaginary part





$$\epsilon(E)=\sum_{\alpha=A,B,C}\left(\sum_{n=1}^{\infty}\frac{A_{0\alpha}^{\mathrm{ex}}}{n^3}\frac{1}{E_{0\alpha}-(G_{0\alpha}^{3\mathrm{D}}/n^2)-E-i\Gamma}\right)$$

Real and imaginary part of the index of refraction of GaN vs. energy;

Real and imaginary part

The real part and imaginary part of the complex dielectric function ε (ω) are not independent. they can be connected by Kramers-Kronig relations:

$$\varepsilon_1(\omega) = \varepsilon_0 + \frac{2}{\pi} P\int_0^\infty \frac{\varepsilon_2(\omega')\omega'}{\omega'^2 - \omega^2} d\omega'$$

$$\varepsilon_2(\omega) = \frac{2\omega}{\pi} P\int_0^\infty \frac{\varepsilon_1(\omega') - \varepsilon_0}{\omega'^2 - \omega^2} d\omega'$$

P indicates that the integral is a principal value integral.

K-K relation can also be written in other form, like

$$n(\lambda) = \frac{1}{\pi} P\int_0^\infty \frac{\alpha(\lambda')}{1 - (\lambda'/\lambda)^2} d\lambda'$$

Group Velocity and Group Index

There are no perfect monochromatic waves

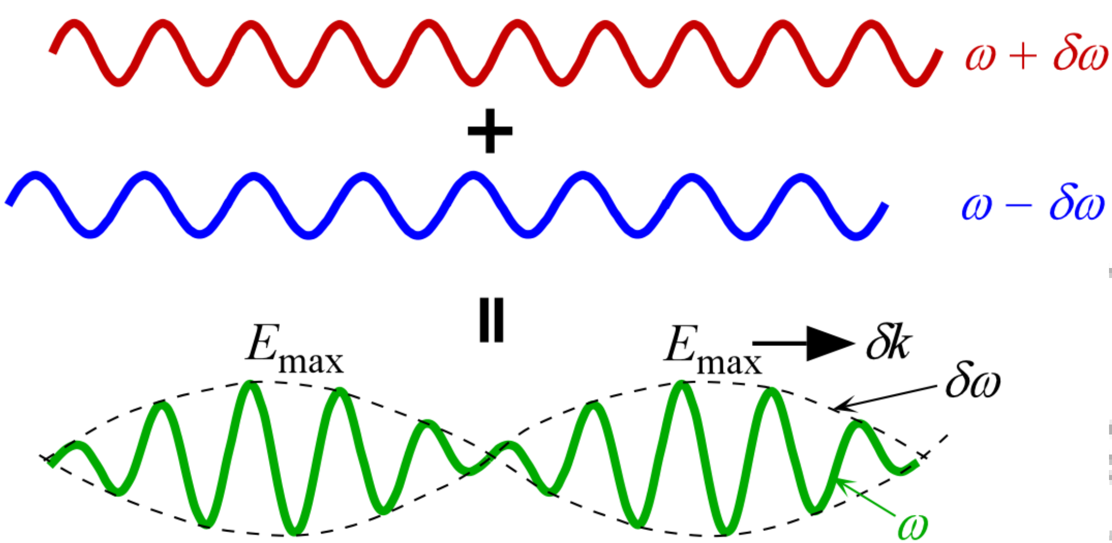
We have to consider the way in which a group of waves differing slightly in wavelength travel along the z-direction

Group Velocity and Group Index

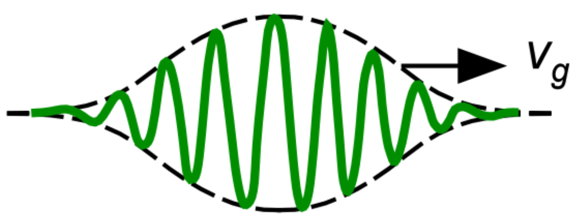
When two perfectly harmonic waves of frequencies $\omega - \delta\omega$ and $\omega + \delta\omega$ and wavevectors $k - \delta k$ and $k + \delta k$ interfere, they generate a wave packet which contains an oscillating field at the mean frequency ω that is amplitude modulated by a slowly varying field of frequency $\delta\omega$. The maximum amplitude moves with a wavevector δk and thus with a group velocity that is given by

$$v_g = \frac{d\omega}{dk}$$

Group Velocity

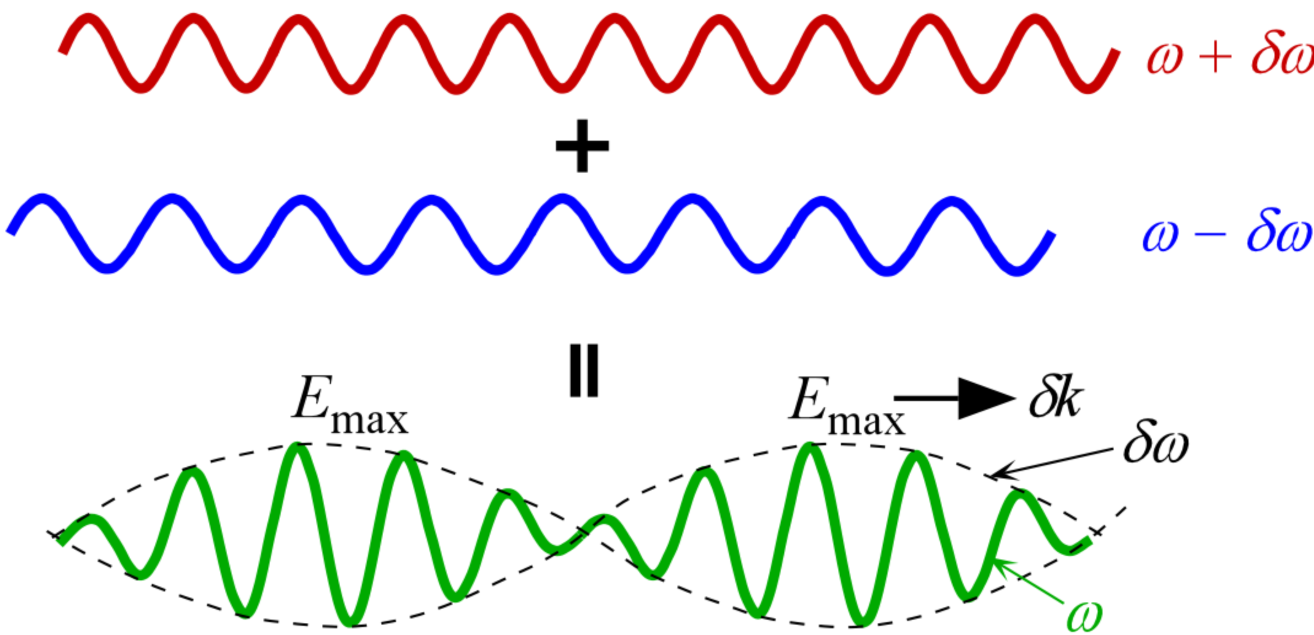


A periodic wave packet

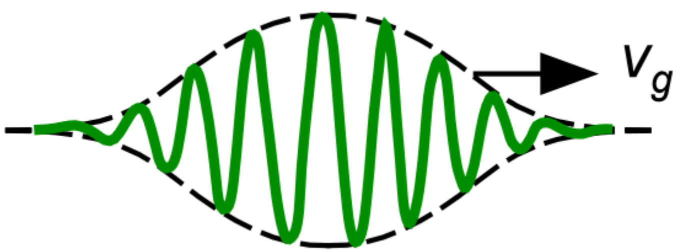


A single wave packet

Two slightly different wavelength waves traveling in the same direction result in a wave packet that has an amplitude variation that travels at the group velocity.



A periodic wave packet



A single wave packet

$$v_g = \frac{d\omega}{dk}$$

Group Velocity

Consider two sinusoidal waves that are close in frequency, that is, they have frequencies $\omega - \delta\omega$ and $\omega + \delta\omega$. Their wavevectors will be $k - \delta k$ and $k + \delta k$. The resultant wave is

$$\boldsymbol{E_x(z,t) = E_o\cos[(\omega - \delta\omega)t - (k - \delta k)z]}$$
$$\boldsymbol{+ E_o\cos[(\omega + \delta\omega)t - (k + \delta k)z]}$$

By using the trigonometric identity

$\cos A + \cos B = 2\cos[{}^1/_2(A - B)]\cos[{}^1/_2(A + B)]$
we arrive at

$$\boldsymbol{E_x(z,t) = 2E_o\cos[(\delta\omega)t - (\delta k)z][\cos(\omega t - kz)]}$$

amplitude modulated by a very slowly varying sinusoidal of

$E_x(z,t) =$

$2E_o\cos[(\delta\omega)t-(\delta k)z][\cos(\omega t-kz)]$

travels along z at a speed determined by the modulating term, $\cos[(\delta\omega)t-(\delta k)z]$. The maximum in the field occurs when $[(\delta\omega)t-(\delta k)z] = 2m\pi =$ constant (m is an integer), which travels with a velocity

$$\frac{dz}{dt} = \frac{\delta\omega}{\delta k} \quad \text{or} \quad \boxed{v_g = \frac{d\omega}{dk}}$$

This is the **group velocity** of the waves because it determines the speed of propagation of the maximum electric field along *z*.

The **group velocity** therefore defines the speed with which energy or information is propagated.

$$\boxed{v_g = \frac{d\omega}{dk}}$$

$\omega = 2\pi c/\lambda_o$ and $k = 2\pi n/\lambda_o$, λ_o is the free space wavelength.

Differentiate the above equations in red

$$d\omega = -(2\pi c/\lambda_o^2)d\lambda_o$$

$$dk = 2\pi n(-1/\lambda_o^2)d\lambda_o + (2\pi/\lambda_o)\left(\frac{dn}{d\lambda_o}\right)d\lambda_o$$

$$dk = -(2\pi/\lambda_o^2)\left(n - \lambda_o\frac{dn}{d\lambda_o}\right)d\lambda_o$$

$$\therefore \quad v_g = \frac{d\omega}{dk} = \frac{-(2\pi c/\lambda_o^2)d\lambda_o}{-(2\pi/\lambda_o^2)\left(n - \lambda_o\frac{dn}{d\lambda_o}\right)d\lambda_o} = \boxed{\frac{c}{n - \lambda_o\frac{dn}{d\lambda_o}}}$$

Group Velocity and Group Index

velocity v_g in a medium is given by,

$$v_g(\text{medium}) = \frac{d\omega}{dk} = \frac{c}{n - \lambda \dfrac{dn}{d\lambda}}$$

This can be written as

$$v_g(\text{medium}) = \frac{c}{N_g}$$

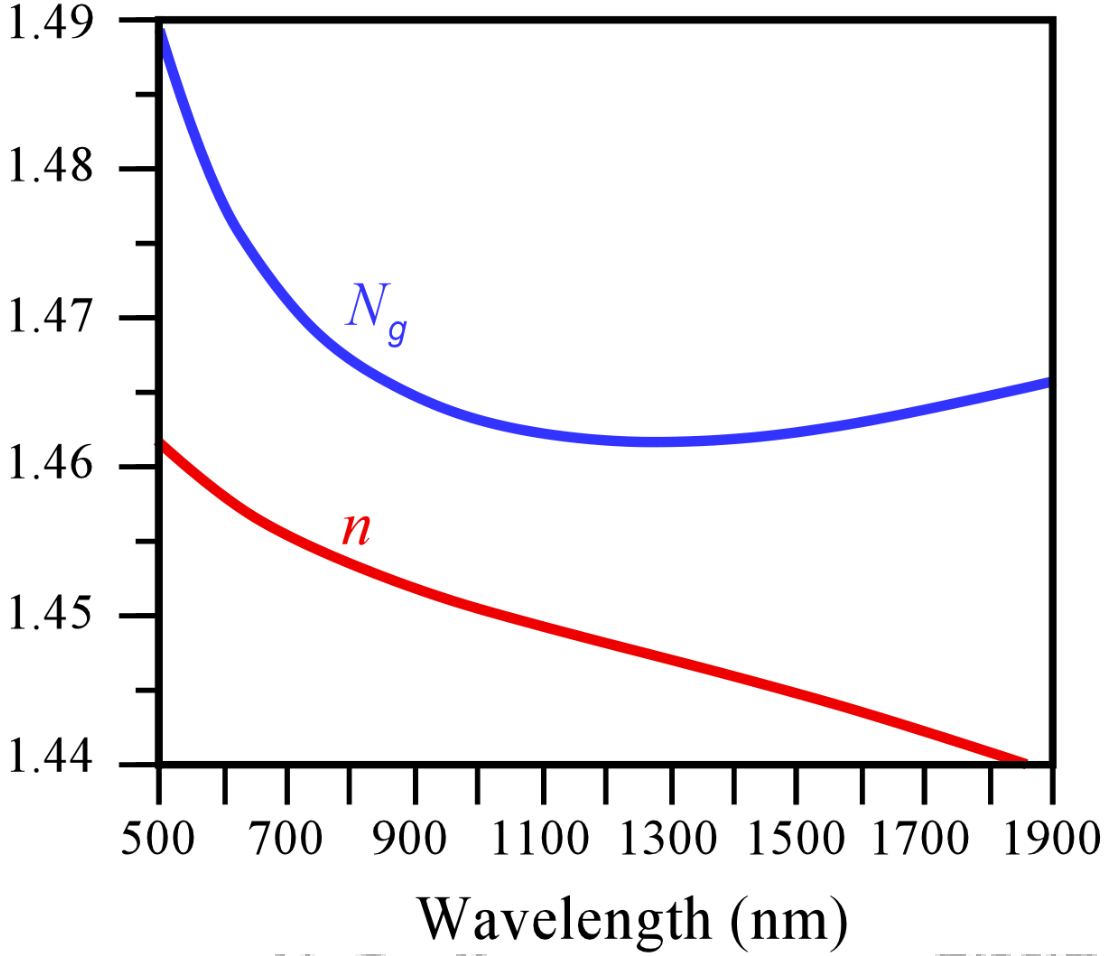
Group Index

$$N_g = n - \lambda \frac{dn}{d\lambda}$$

is defined as the **group index of the medium**

In general, for many materials the refractive index n and hence the group index N_g depend on the wavelength of light. Such materials are called **dispersive**

Refractive Index and Group Index



Refractive index n and the group index N_g of pure SiO_2 (silica) glass as a function of wavelength.

Magnetic Field, Irradiance and Poynting Vector

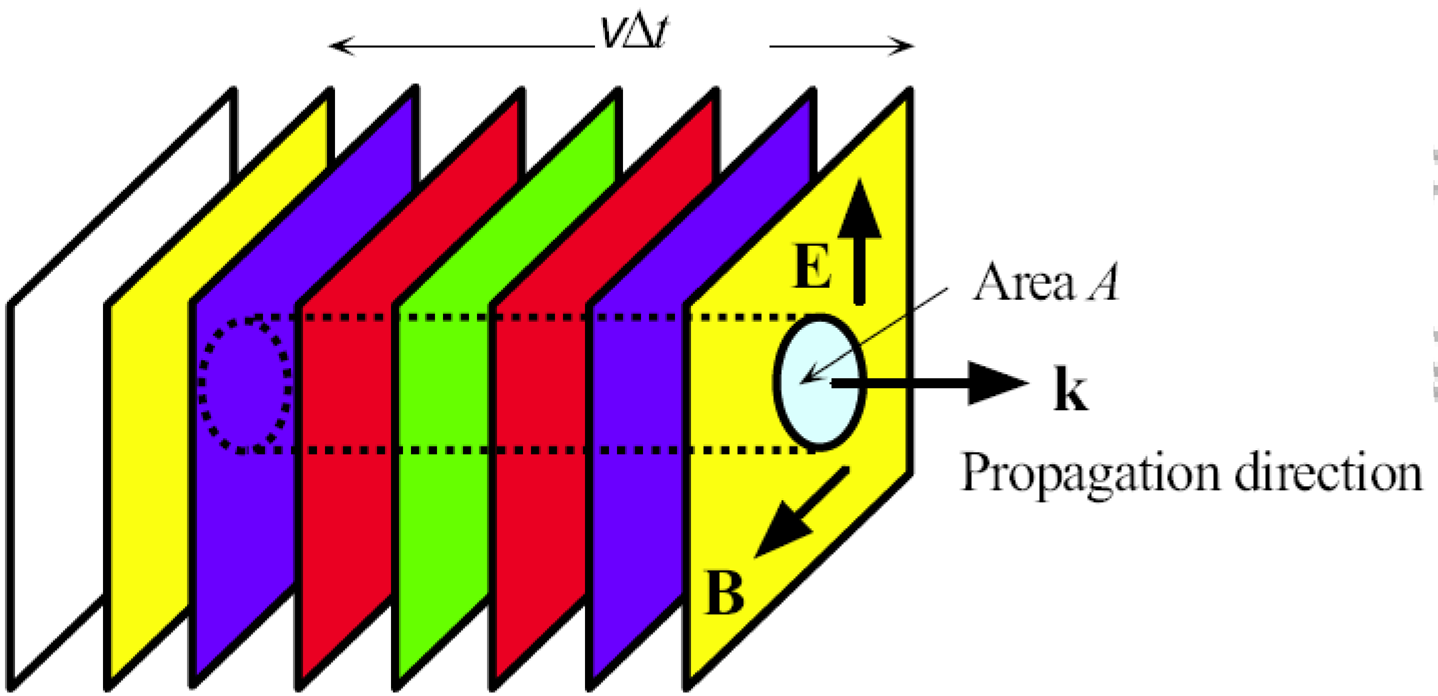
The magnetic field (magnetic induction) component B_y always accompanies E_x in an EM wave propagation.

If v is the **phase velocity** of an EM wave in an isotropic dielectric medium and n is the refractive index, then

$$E_x = vB_y = \frac{c}{n} B_y$$

where $v = (\varepsilon_o \varepsilon_r \mu_o)^{-1/2}$ and $n = \varepsilon^{1/2}$

EM wave carries energy along the direction of propagation k. What is the radiation power flow per unit area?



A plane EM wave traveling along **k** crosses an area A at right angles to the direction of propagation. In time ?t, the energy in the cylindrical volume A??t (shown dashed) flows through A.

Energy Density in an EM Wave

As the EM wave propagates in the direction of the wavevector **k**, there is an energy flow in this direction. The wave brings with it electromagnetic energy.

The energy densities in the E_x and B_y fields are the same,

$$\frac{1}{2}\varepsilon_o\varepsilon_r E_x^2 = \frac{1}{2\mu_o}B_y^2$$

The total energy density in the wave is therefore $\varepsilon_o\varepsilon_r E_x{}^2$.

Poynting Vector and EM Power Flow

If *S* is the EM power flow per unit area,

***S* = Energy flow per unit time per unit area**

$$S = \frac{(Av\Delta t)(\varepsilon_o \varepsilon_r E_x^2)}{A\Delta t} = v\varepsilon_o \varepsilon_r E_x^2 = v^2 \varepsilon_o \varepsilon_r E_x B_y$$

In an isotropic medium, the energy flow is in the direction of wave propagation. If we use the vectors **E** and **B** to represent the electric and magnetic fields in the EM wave, then the EM power flow per unit area can be written as

$$\mathbf{S} = v^2 \varepsilon_o \varepsilon_r \mathbf{E} \times \mathbf{B}$$

Poynting Vector and Intensity

where S, called the Poynting vector, represents the energy flow per unit time per unit area in a direction determined by E×B (direction of propagation). Its magnitude, power flow per unit area, is called the irradiance (instantaneous irradiance, or intensity).

The average irradiance is

$$\boxed{I = S_{\text{average}} = \tfrac{1}{2} v \varepsilon_o \varepsilon_r E_o^2}$$

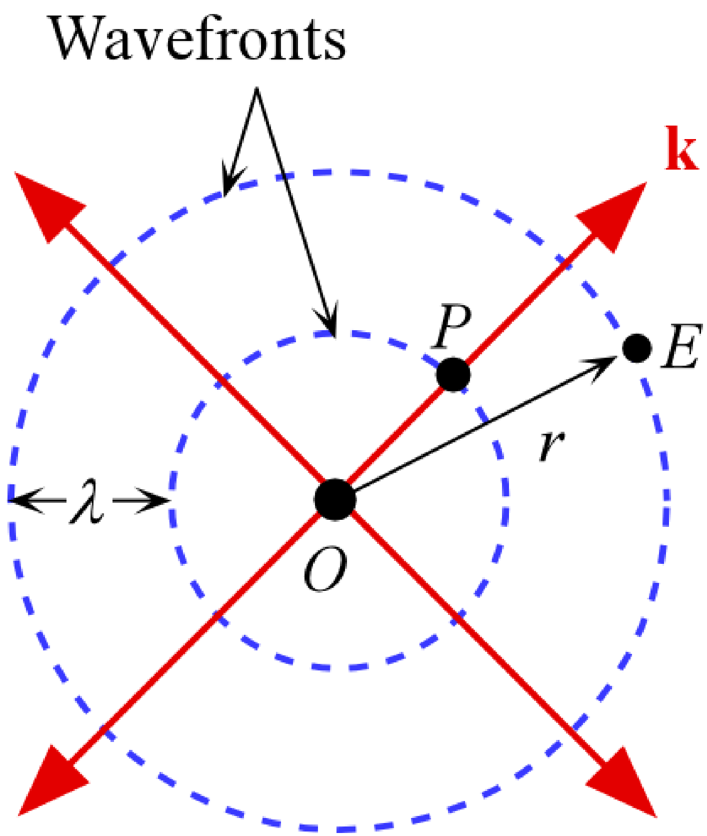
Average Irradiance or Intensity

Since $v = c/n$ and $\varepsilon_r = n^2$ we can write

$$I = S_{\text{average}} = \tfrac{1}{2} c \varepsilon_o n E_o^2 = (1.33 \times 10^{-3}) n E_o^2$$

The instantaneous irradiance can only be measured if the power meter can respond more quickly than the oscillations of the electric field. Since this is in the optical frequencies range, **all practical measurements yield the average irradiance** because all detectors have a response rate much slower than the frequency of the wave.

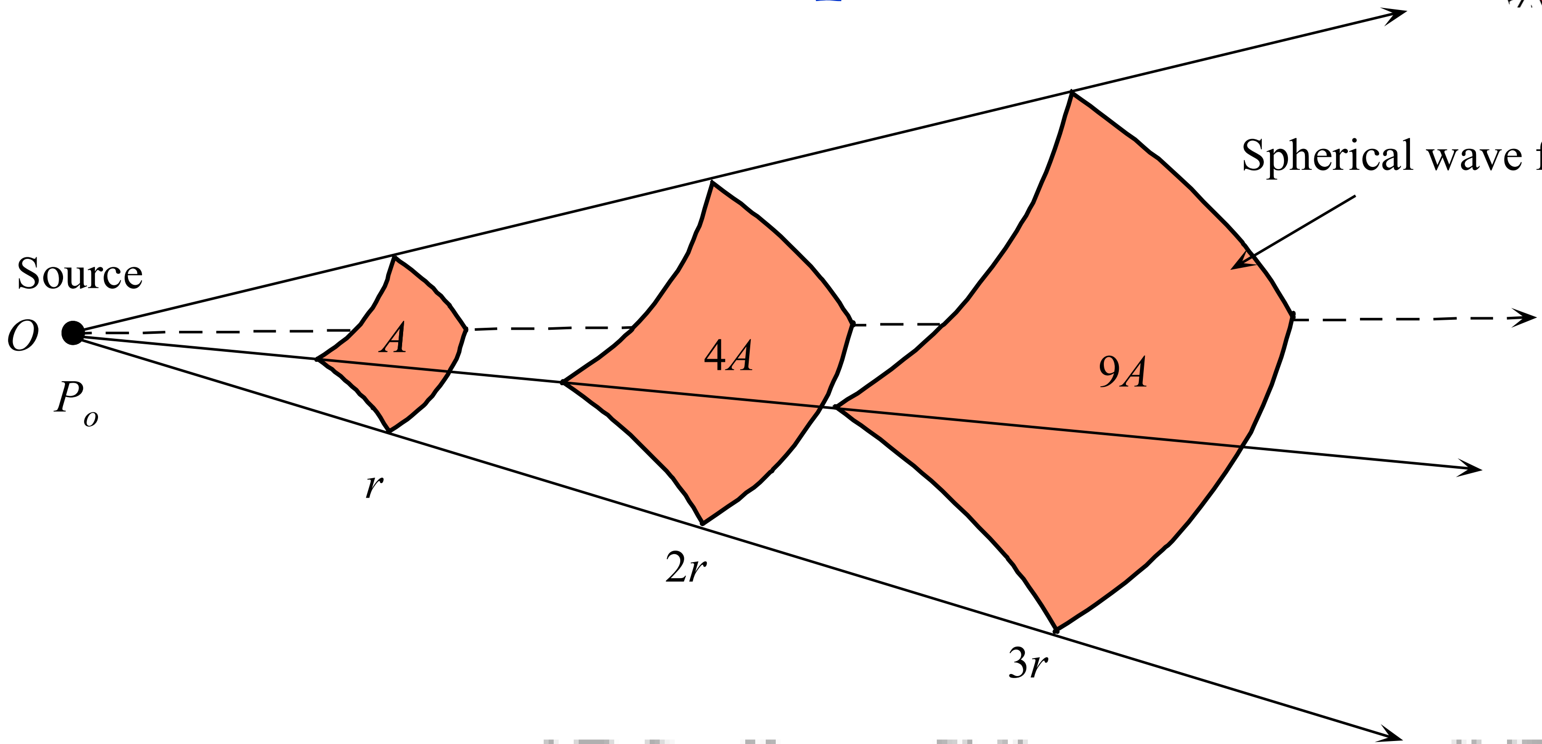
Irradiance of a Spherical Wave



Perfect spherical wave

$$I = \frac{P_o}{4\pi r^2}$$

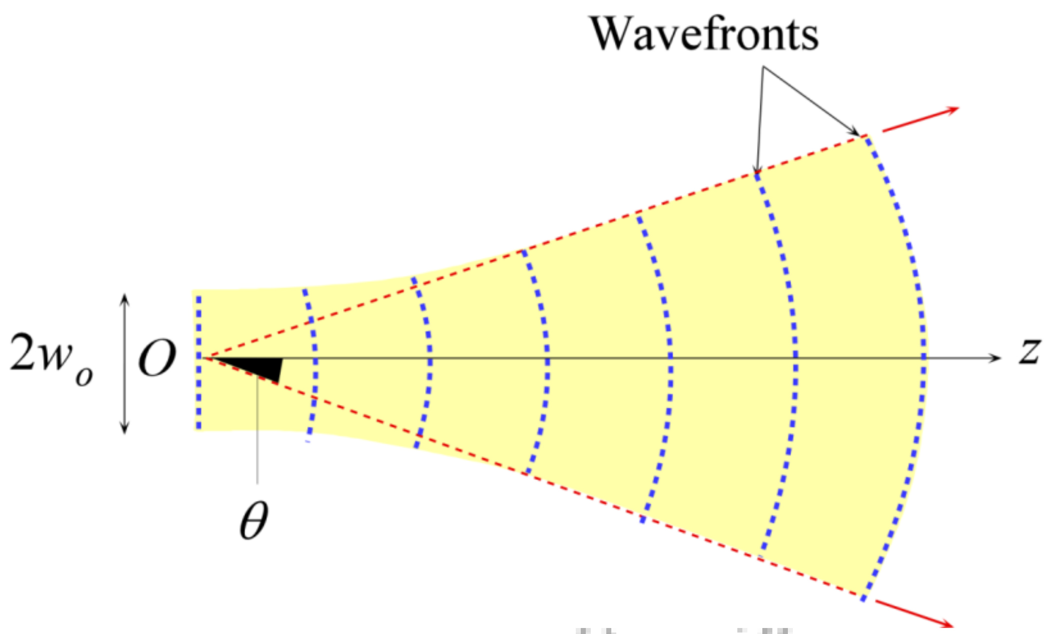
Irradiance of a Spherical Wave

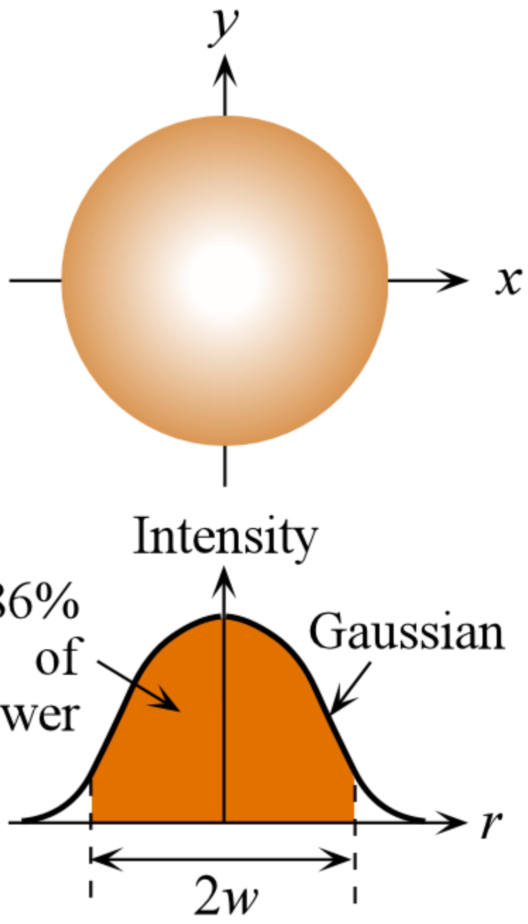


$$I = \frac{P_o}{4\pi r^2}$$

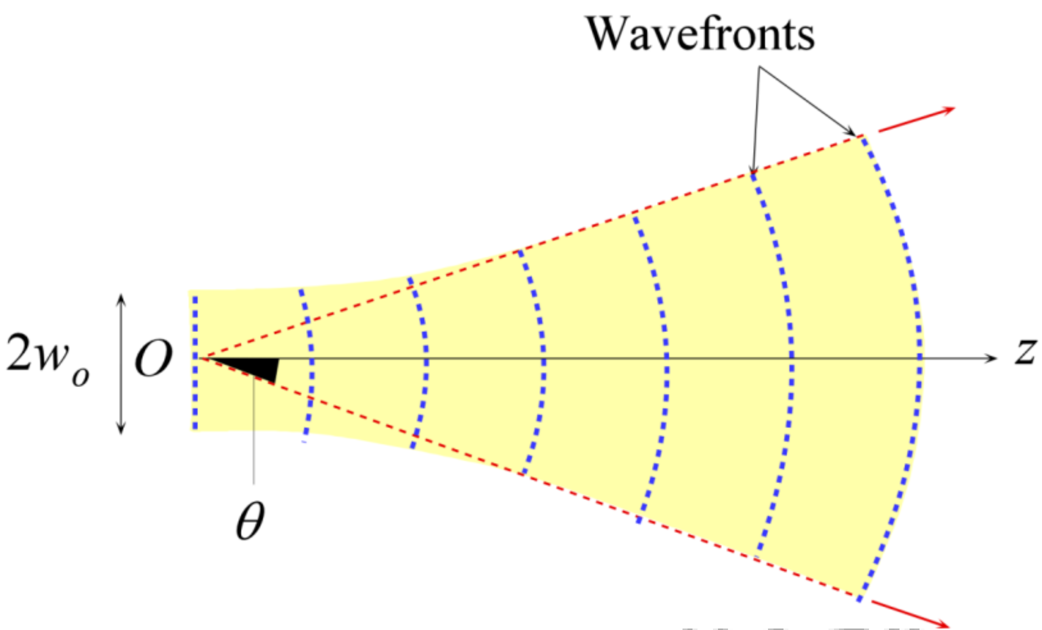
A Gaussian Beam

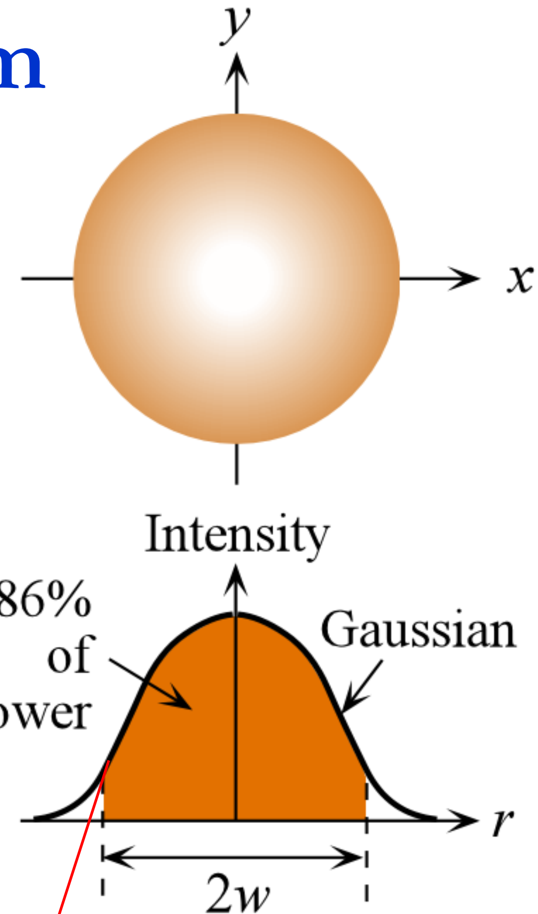
$I(r, z) = I_{max}\exp(-2r^2 / w^2)$





A Gaussian Beam





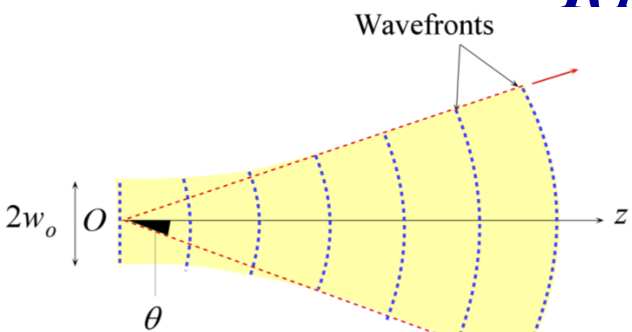
$$I(r,z) = \overbrace{[2P_o/(\pi w^2)]}^{I_{max}}\exp(-2r^2/w^2)$$

$$\theta = w/z = \lambda/(\pi w_o) \qquad 2\theta = \text{Far field divergence}$$

✷Corrected

A Gaussian Beam

$$I(r,z) = I_{\max}\exp(-2r^2/w^2)$$



$$I_{\max} = \frac{P_o}{\frac{1}{2}\pi w^2}$$

I_o = Maximum irradiance at the center $r = 0$ at the waist

$$I(z,r) = I_o\left(\frac{w_o^2}{w^2}\right)\exp\left(-\frac{2r^2}{w^2}\right)$$

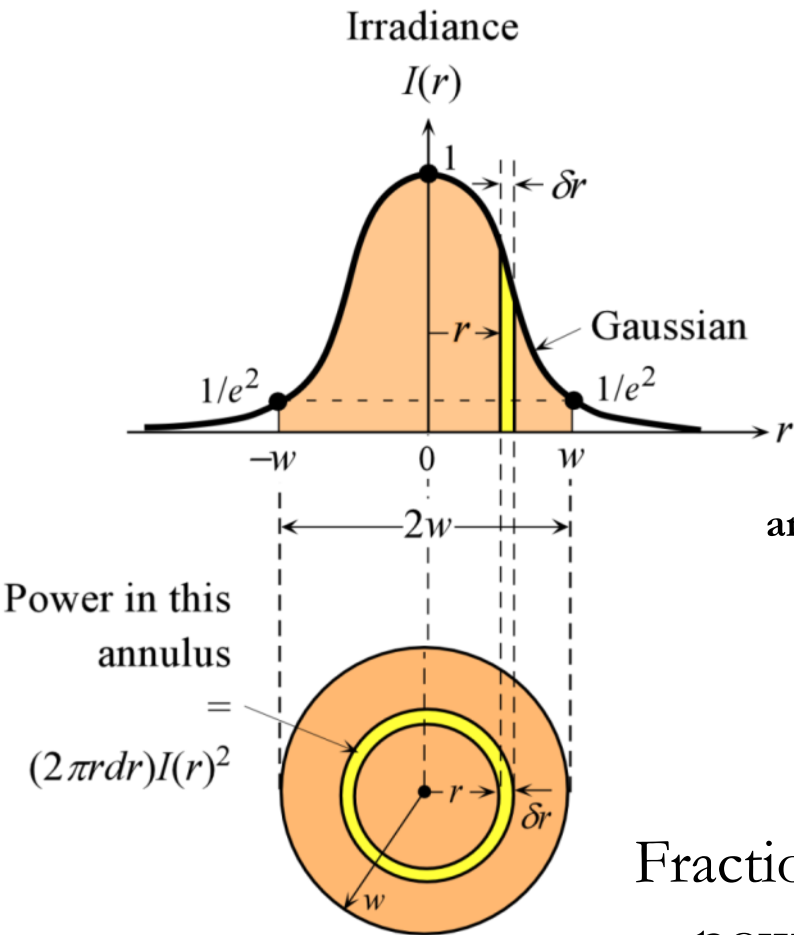
Beyond the Rayleigh range

$z > z_o$

$$2w = 2w_o\left(\frac{z}{z_o}\right)$$

$$I(z,0) = I_{\max} = I_o\frac{w_o^2}{w^2} = I_o\frac{z_o^2}{z^2}$$

Power in a Gaussian Beam



$$I(r)^2 = I(0)^2 \exp[-2(r/w)^2]$$

and

Area of a circular thin strip (annulus) with radius r is $2\pi r dr$. Power passing through this strip is proportional to $I(r)\,(2\pi r)\,dr$

$$\text{Fraction of optical power within } 2w = \frac{\int_0^w I(r)\boxed{2\pi r dr}}{\int_0^\infty I(r) 2\pi r dr} = 0.865$$

Gaussian Beam: example

Corrected

I_o = Maximum irradiance at the center $r = 0$ at the waist $\longrightarrow$ $$I_o = \frac{P_o}{\frac{1}{2}\pi w_o^2}$$

Example 1.4.2 Power and irradiance of a Gaussian beam

Consider a 5 mW HeNe laser that is operating at 633 nm, and has a spot size that is 1 mm. Find the maximum irradiance of the beam and the axial (maximum) irradiance at 25 m from the laser.

Solution

The 5 mW rating refers to the total optical power P_o available, and 633 nm is the free space output wavelength λ. Apply

$$P_o = I_o(\tfrac{1}{2}\pi w_o^2)$$

$$\therefore \quad 5\times10^{-3}\ \text{W} = I_o[\tfrac{1}{2}\pi(0.5\times10^{-3}\ \text{m})^2]$$

$$I_o = 1.273\ \text{W cm}^{-2}$$

Gaussian Beam: example

$$I(z,0) = I_{\max} = I_o \frac{w_o^2}{w^2} = I_o \frac{z_o^2}{z^2}$$

The Rayleigh range z_o was calculated previously, but we can recalculate

$$z_o = \pi w_o^2/\lambda = \pi(0.5\times10^{-3}\text{ m})^2/(633\times10^{-9}\text{ m}) = 1.24\text{ m}.$$

The beam width at 25 m is

$$2w = 2w_o[1 + (z/z_o)]^{1/2} = 20\text{ mm}$$

The irradiance at the beam axis is

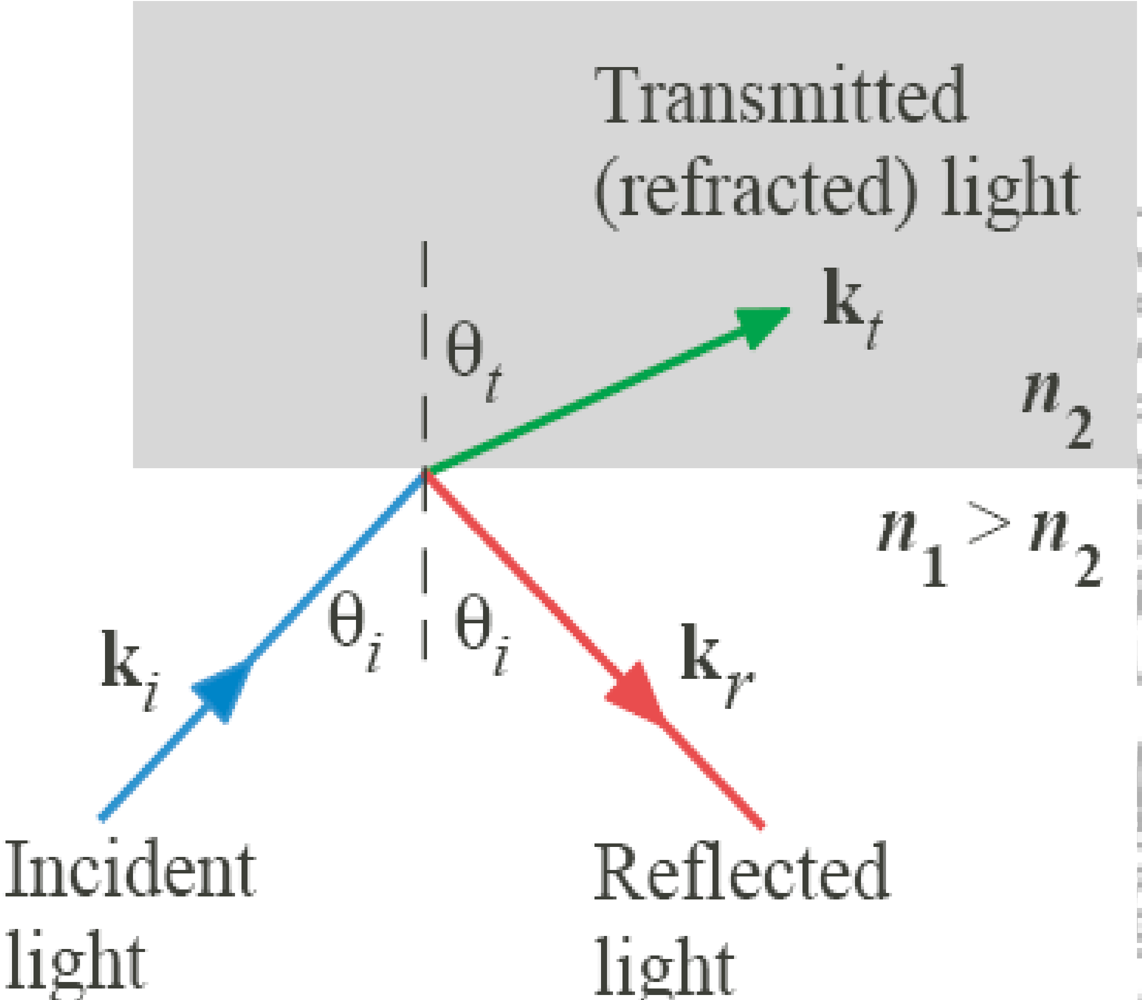
$$I_{\text{axis}} = I_o \frac{z_o^2}{z^2} = (1.273\text{ W cm}^{-2})\frac{(1.24\text{ m})^2}{(25\text{ m})^2} = 3.14\text{ mW cm}^{-2}$$

Snell's Law or Descartes's Law?

Willebrord Snellius (Willebrord Snel van Royen, 1580–1626) was a Dutch astronomer and a mathematician, who was a professor at the University of Leiden. He discovered his law of refraction in 1621 which was published by Réne Descartes in France 1637; it is not known whether Descartes knew of Snell's law or formulated it independently. *(Courtesy of AIP Emilio Segre Visual Archives, Brittle Books Collection.)*

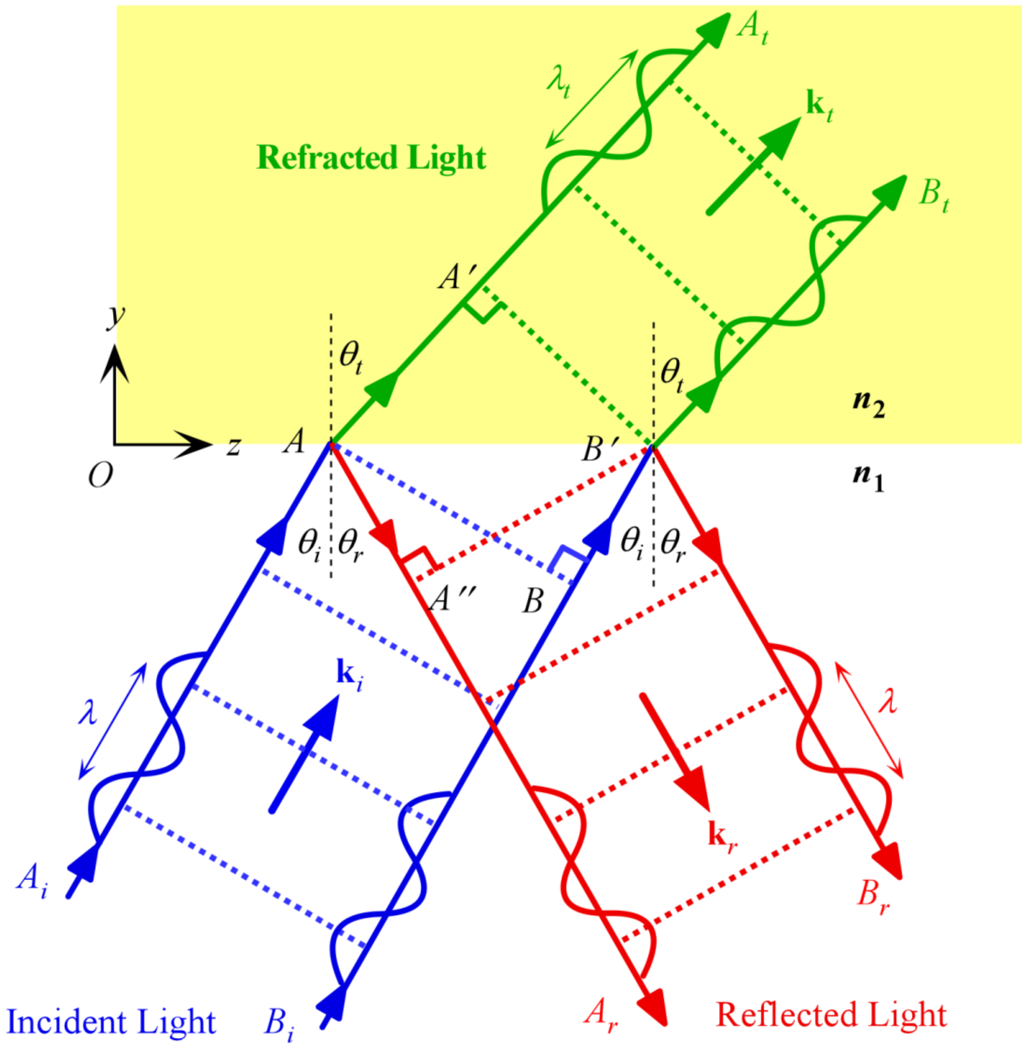
René Descartes (1596–1650) was a French philosopher who was also involved with mathematics and sciences. He has been called the "Father of Modern Philosophy." Descartes was responsible for the development of Cartesian coordinates and analytical geometry. He also made significant contributions to optics, including reflection and refraction. *(Courtesy of Georgios Kollidas/Shutterstock.com.)*

Snell's Law



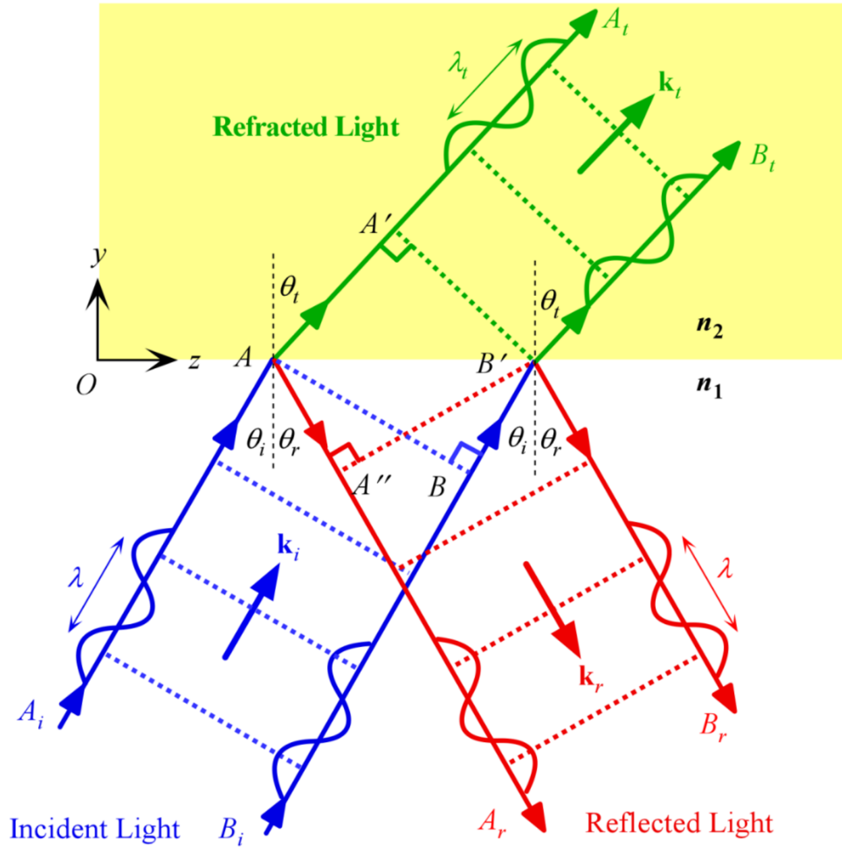
$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$$

Derivation of Descartes' Law



A light wave traveling in a medium with a greater refractive index ($n_1 > n_2$) suffers reflection and refraction at the boundary. (Notice that λ_t is slightly longer than λ.)

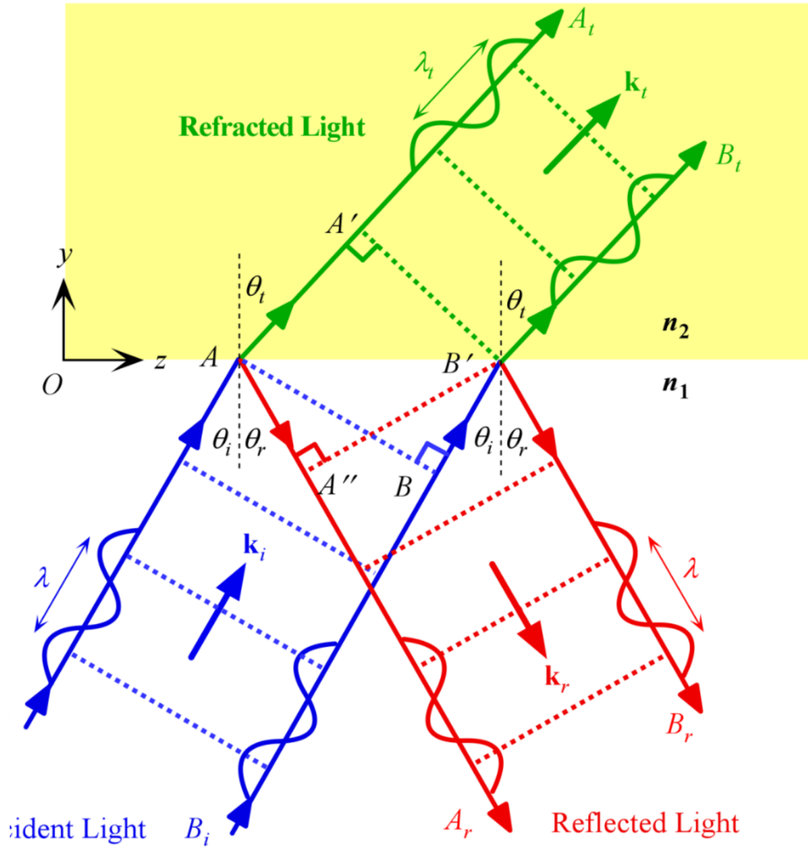
Snell's Law



We can use *constructive interference* to show that there can only be one reflected wave which occurs at an angle equal to the incidence angle. The two waves along A_i and B_i are in phase.

When these waves are reflected to become waves A_r and B_r then they must still be in phase, otherwise they will interfere destructively and destroy each other. The only way the two waves can stay in phase is if $\theta_r = \theta_i$. All other angles lead to the waves A_r and B_r being out of phase and interfering destructively.

Snell's Law



Unless the two waves at A' and B' still have the same phase, there will be no transmitted wave. A' and B' points on the front are only in phase for one particular transmitted angle, θ_t.

It takes time t for the phase at B on wave B_i to reach B'

$$\boldsymbol{BB'} = \boldsymbol{v_1 t = ct/n_1}$$

During this time t, the phase A has progressed to A'

$$\boldsymbol{AA'} = \boldsymbol{v_2 t = ct/n_2}$$

A' and B' belong to the same front just like A and B so that AB is perpendicular to $\mathbf{k}_i$ in medium 1 and $A'B'$ is perpendicular to $\mathbf{k}_t$ in medium 2. From geometrical considerations,

$\boldsymbol{AB' = BB'/\sin\theta_i}$ and $\boldsymbol{AB' = AA'/\sin\theta_t}$ so that

$$AB' = \frac{v_1 t}{\sin\theta_i} = \frac{v_2 t}{\sin\theta_t}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

or

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$n \sin\theta = \text{constant}$$

This is **Snell's law** which relates the angles of incidence and refraction to the refractive indices of the media.

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

When $n_1 > n_2$ then obviously the transmitted angle is greater than the incidence angle as apparent in the figure. When the refraction angle θ_t reaches 90°, the incidence angle is called the **critical angle** θ_c which is given by

$$\sin\theta_c = \frac{n_2}{n_1}$$

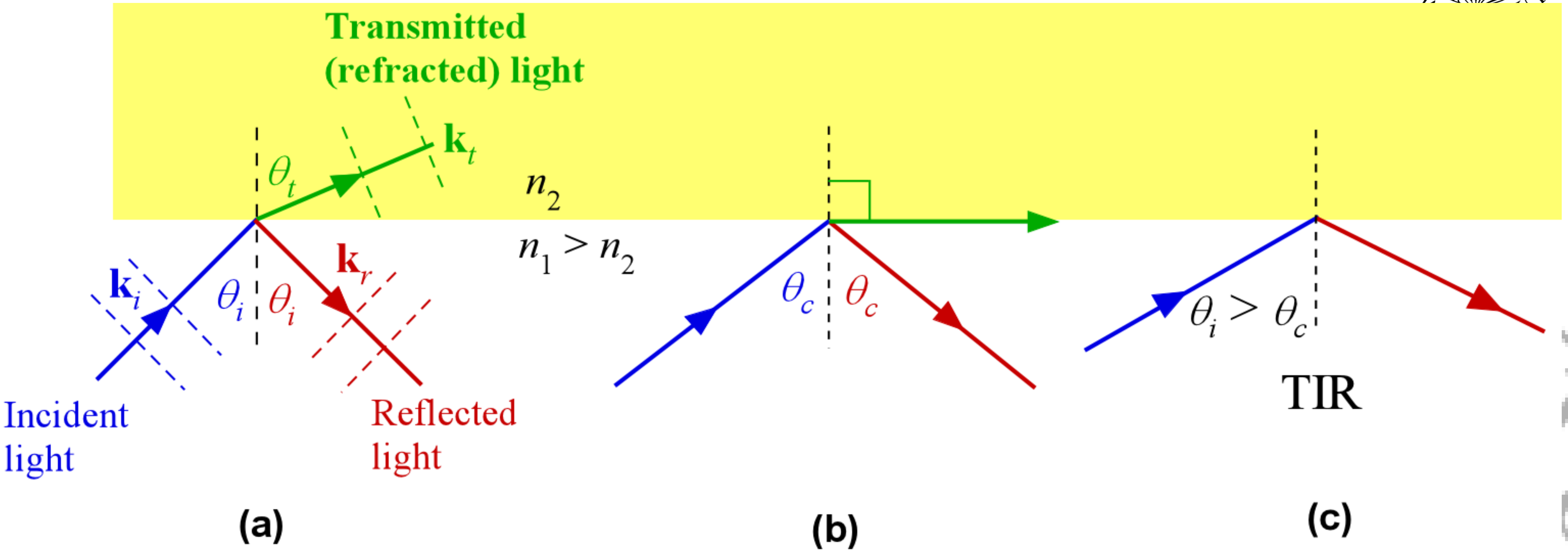
Snell's Law

dielectric medium … medium of smaller refractive index as i… les, *e.g.* optical fibers.

$$\sin\theta_c = \frac{n_2}{n_1}$$

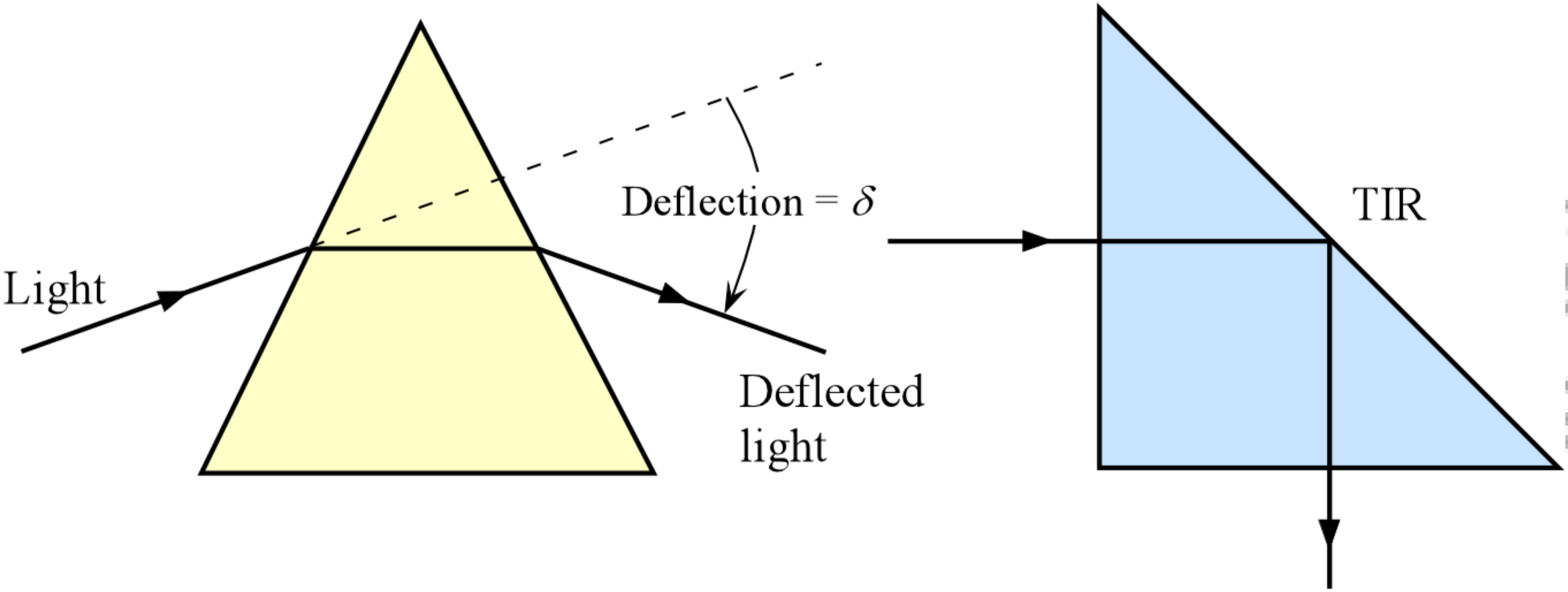
Although Snell's law … hat $\sin\theta_t > 1$ and hence θ_t is an "imaginary" … n, there is however an attenuated wave called the evanescent wave.

Total Internal Reflection



Light wave traveling in a more dense medium strikes a less dense medium. Depending on the incidence angle with respect to θ_c, which is determined by the ratio of the refractive indices, the wave may be transmitted (refracted) or reflected. (a) $\theta_i < \theta_c$ (b) $\theta_i = \theta_c$ (c) $\theta_i > \theta_c$ and total internal reflection (TIR).

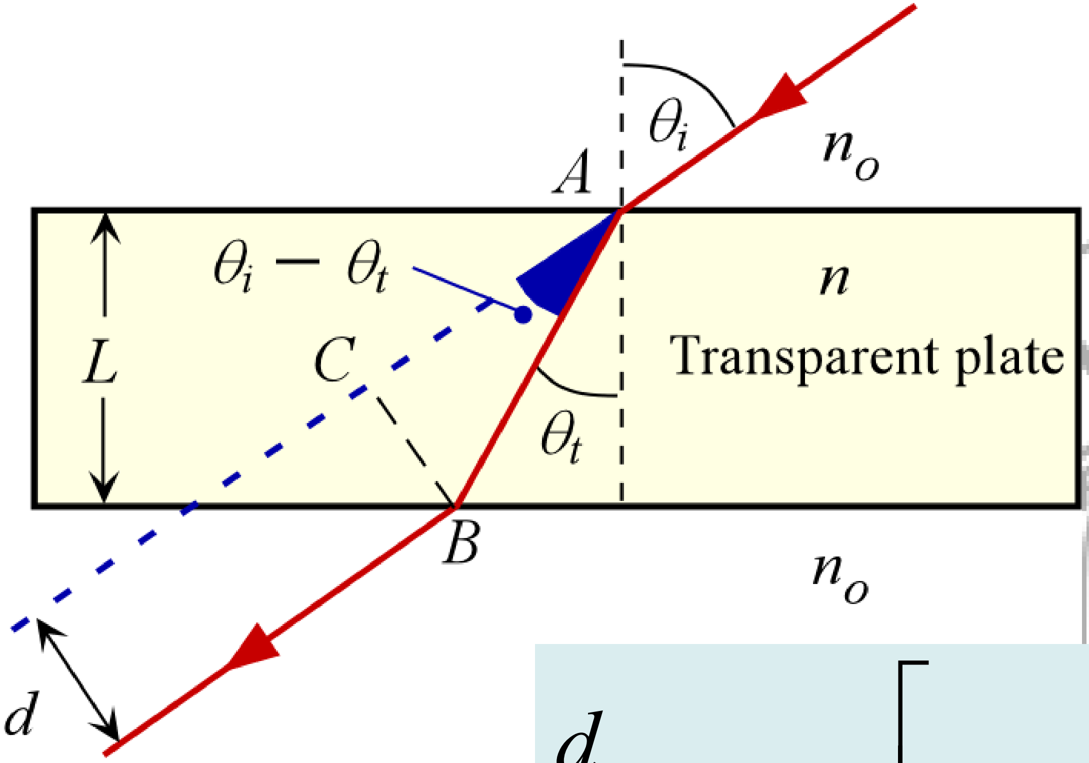
Prisms



Refracting prism

Reflecting prism

Lateral Displacement



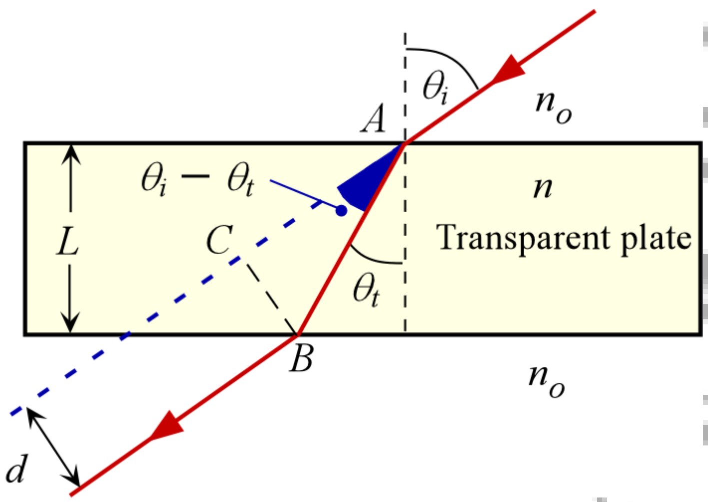
$$\frac{d}{L} = \sin\theta_i \left[1 - \frac{\cos\theta_i}{\sqrt{(n/n_o)^2 - \sin^2\theta_i}} \right]$$

Example: Lateral Displacement

Lateral displacement of light, or, **beam displacement,** occurs when a beam if light passes obliquely through a plate of transparent material, such as a glass plate. When a light beam is incident on a plate of transparent material of refractive index *n*, it emerges from the other side traveling parallel to the incident light but displaced from it by a distance *d*, called *lateral displacement*. Find the displacement *d* in terms of the incidence angle the plate thickness *L*. What is *d* for a glass of $n = 1.600$, $L = 10$ mm if the incidence angle is 45°

Solution

The displacement $d = BC = AB\sin(\theta_i - \theta_t)$. Further, $L/AB = \cos\theta_t$ so that combining these two equation we find

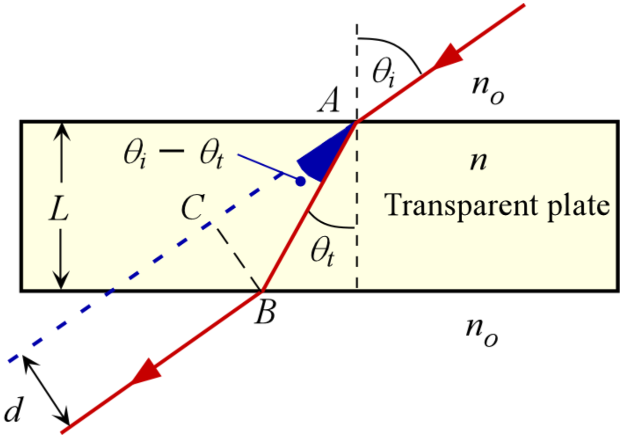


$$d = L\left[\frac{\sin(\theta_i - \theta_t)}{\cos\theta_t}\right]$$

Example: Lateral Displacement (Continued)

Solution (Continued)

Expand $\sin(\theta_i - \theta_t)$ and eliminate $\sin\theta_t$ and $\sin\theta_t$



$$d = L\left[\frac{\sin(\theta_i - \theta_t)}{\cos\theta_t}\right]$$

$$\boxed{\sin(\theta_i - \theta_t) = \sin\theta_i\cos\theta_t - \cos\theta_i\sin\theta_t}$$

$$\boxed{\cos\theta_t = \sqrt{1 - \sin^2\theta_t}}$$

Snell's law $n\sin\theta_t = n_o\sin\theta_i$

$$\frac{d}{L} = \sin\theta_i\left[1 - \frac{\cos\theta_i}{\sqrt{(n/n_o)^2 - \sin^2\theta_i}}\right]$$

Example: Lateral Displacement (Continued)

Solution (Continued)

$$\frac{d}{L} = \sin\theta_i \left[1 - \frac{\cos\theta_i}{\sqrt{(n/n_o)^2 - \sin^2\theta_i}} \right]$$

$L = 10$ mm

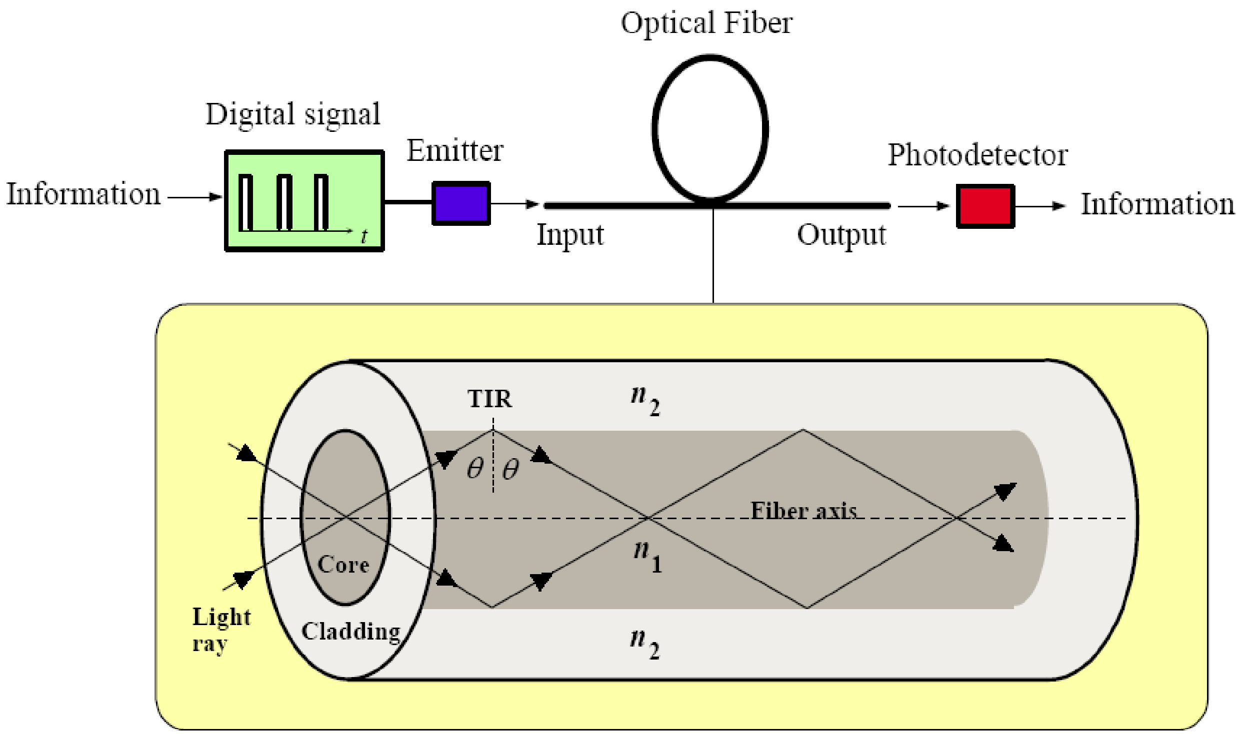
$\theta_i = 45°$

$n = 1.600$

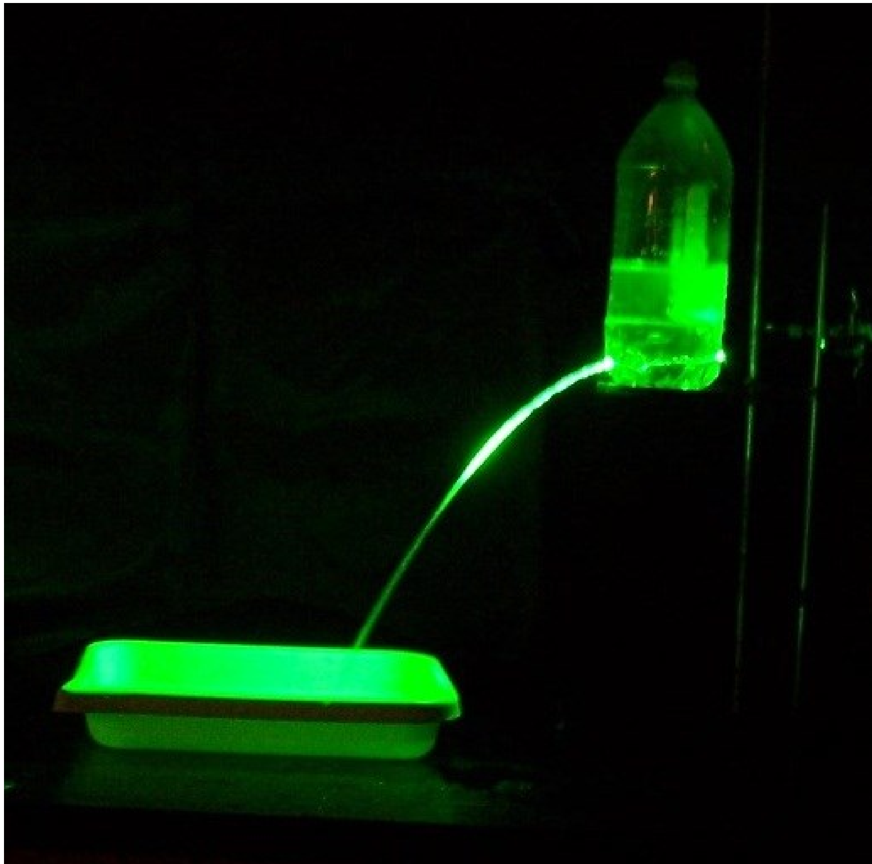
$n_o = 1$

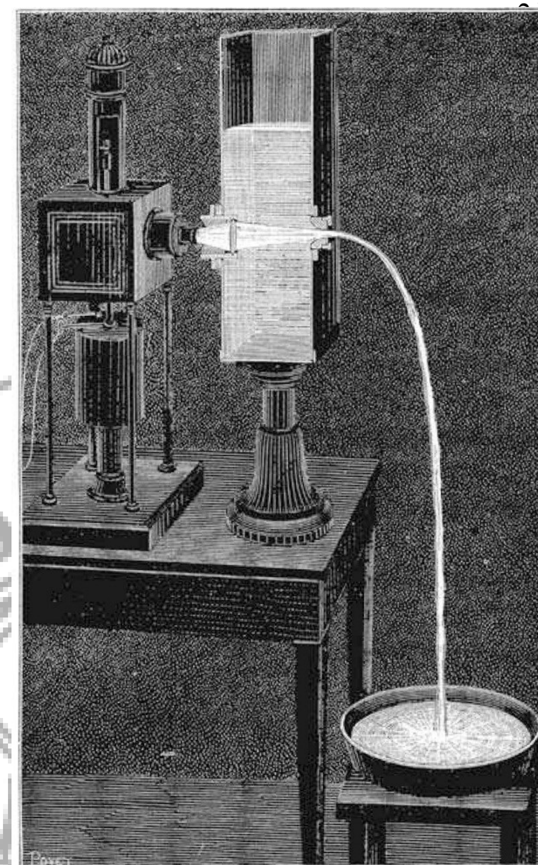
$$\boldsymbol{d = 3.587 \text{ mm}}$$

Light travels by total internal reflection in optical fibers



An optical fiber link for transmitting digital information in communications. The fiber core has a higher refractive index so that the light travels along the fiber inside the fiber core by total internal reflection at the core-cladding interface.





A small hole is made in a plastic bottle full of water to generate a water jet. When the hole is illuminated with a laser beam (from a green laser pointer), the light is guided by total internal reflections along the jet to the tray. The light guiding by a water jet was first demonstrated by Jean-Daniel Colladan, a Swiss scientist (Water with air bubbles was used to increase the visibility of light. Air bubbles scatter light.) [Left: Copyright: S.O. Kasap, 2005] [Right: *Comptes Rendes*, *15*, 800–802, October 24, 1842; *Cnum, Conservatoire Numérique des Arts et Métiers, France*

Physicists use the wave theory on Mondays, Wednesdays and Fridays and the particle theory on Tuesdays, Thursdays and Saturdays.

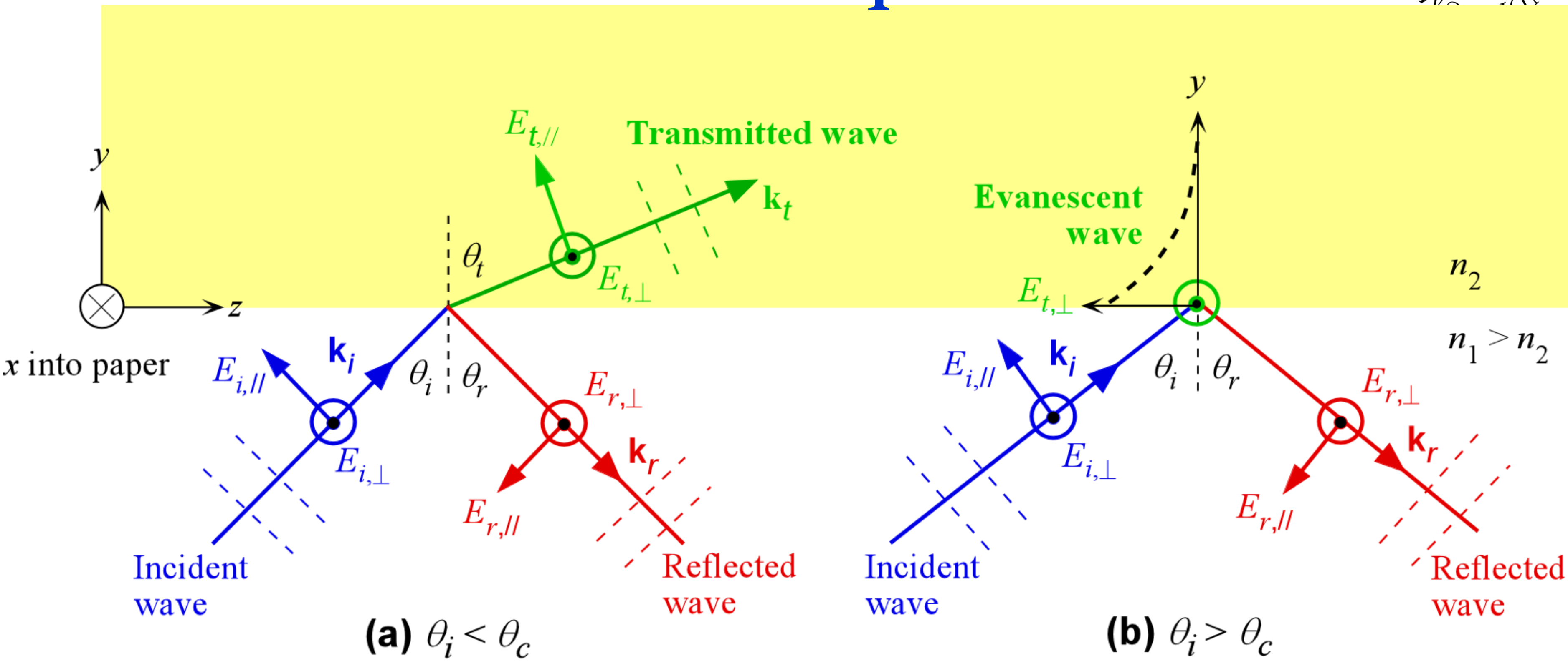
—Sir William Henry Bragg[1]

Augustin Jean Fresnel (1788–1827) was a French physicist and a civil engineer for the French government who was one of the principal proponents of the wave theory of light. He made a number of distinct contributions to optics including the well-known Fresnel lens that was used in lighthouses in the nineteenth century. He fell out with Napoleon in 1815 and was subsequently put under house arrest until the end of Napoleon's reign. During his enforced leisure time he formulated his wave ideas of light into a mathematical theory. *(© INTERFOTO/Alamy.)*

If you cannot saw with a file or file with a saw, then you will be no good as an experimentalist.

—Attributed to Augustin Fresnel

Fresnel's Equations



Light wave traveling in a more dense medium strikes a less dense medium. The plane of incidence is the plane of the paper and is perpendicular to the flat interface between the two media. The electric field is normal to the direction of propagation. It can be resolved into perpendicular and parallel components.

Fresnel's Equations

Describe the incident, reflected and refracted waves by the exponential representation of a traveling plane wave, *i.e.*

$$E_i = E_{io} \exp j(\omega t - \mathrm{k}_i \cdot \mathrm{r}) \qquad \text{Incident wave}$$

$$E_r = E_{ro} \exp j(\omega t - \mathrm{k}_r \cdot \mathrm{r}) \qquad \text{Reflected wave}$$

$$E_t = E_{to} \exp j(\omega t - \mathrm{k}_t \cdot \mathrm{r}) \qquad \text{Transmitted wave}$$

These are traveling plane waves

Fresnel's Equations

where r is the position vector, the wave vectors k_i, k_r and k_t describe the directions of the incident, reflected and transmitted waves and E_{io}, E_{ro} and E_{to} are the respective amplitudes.

Any phase changes such as ϕ_r and ϕ_t in the reflected and transmitted waves with respect to the phase of the incident wave are incorporated into the complex amplitudes, E_{ro} and E_{to}. Our objective is to find E_{ro} and E_{to} with respect to E_{io}.

Fresnel's Equations

The electric and magnetic fields anywhere on the wave must be perpendicular to each other as a requirement of electromagnetic wave theory. This means that with $E_{//}$ in the EM wave we have a magnetic field $B_{\perp}$ associated with it such that, $B_{\perp} = (n/c)E_{//}$. Similarly $E_{\perp}$ will have a magnetic field $B_{//}$ associated with it such that $B_{//} = (n/c)E_{\perp}$.

We use **boundary conditions**

$$E_{\text{tangential}}(1) = E_{\text{tangential}}(2)$$

Fresnel's Equations

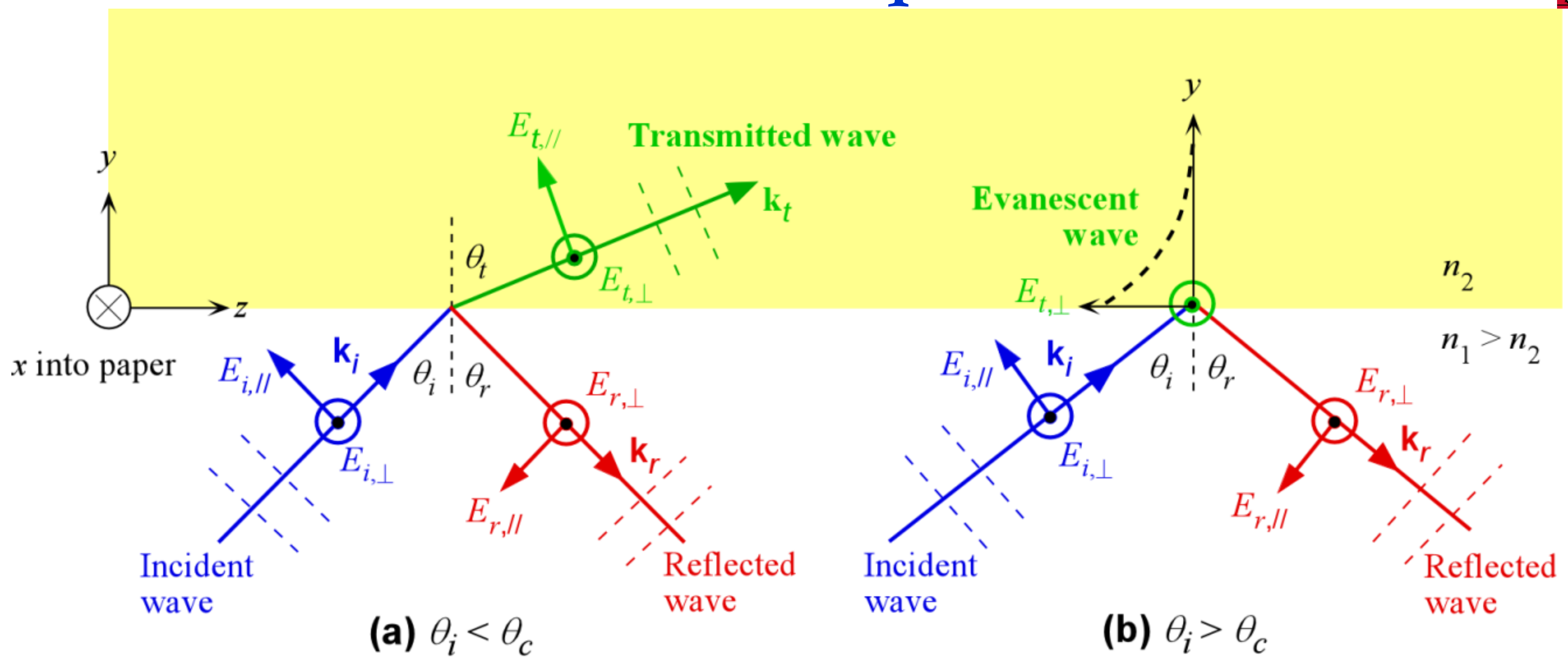
Non-magnetic media (relative permeability, $\mu_r = 1$),

$$B_{\text{tangential}}(1) = B_{\text{tangential}}(2)$$

Using the above boundary conditions for the fields at $y = 0$, and the relationship between the electric and magnetic fields, we can find the reflected and transmitted waves in terms of the incident wave.

The boundary conditions can only be satisfied if the reflection and incidence angles are equal, $\theta_r = \theta_i$ and the angles for the transmitted and incident wave obey Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Fresnel's Equations



Incident wave $E_i = E_{io}\exp j(\omega t - \mathbf{k}_i \cdot \mathbf{r})$

Reflected wave $E_r = E_{ro}\exp j(\omega t - \mathbf{k}_r \cdot \mathbf{r})$

Transmitted wave $E_t = E_{to}\exp j(\omega t - \mathbf{k}_t \cdot \mathbf{r})$

Fresnel's Equations

Applying the boundary conditions to the EM wave going from medium 1 to 2, the amplitudes of the reflected and transmitted waves can be readily obtained in terms of n_1, n_2 and the incidence angle θ_i alone. These relationships are called Fresnel's equations. If we define $n = n_2/n_1$, as the relative refractive index of medium 2 to that of 1, then the reflection and transmission coefficients for $E_\perp$ are,

$$r_\perp = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos\theta_i - \left[n^2 - \sin^2\theta_i\right]^{1/2}}{\cos\theta_i + \left[n^2 - \sin^2\theta_i\right]^{1/2}}$$

Fresnel's Equations

$$t_{\perp} = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2\cos\theta_i}{\cos\theta_i + \left[n^2 - \sin^2\theta_i\right]^{1/2}}$$

There are corresponding coefficients for the $E_{//}$ fields with corresponding **reflection and transmission coefficients**, $r_{//}$ and $t_{//}$,

$$r_{//} = \frac{E_{r0,//}}{E_{i0,//}} = \frac{\left[n^2 - \sin^2\theta_i\right]^{1/2} - n^2\cos\theta_i}{\left[n^2 - \sin^2\theta_i\right]^{1/2} + n^2\cos\theta_i}$$

$$t_{//} = \frac{E_{t0,//}}{E_{i0,//}} = \frac{2n\cos\theta_i}{n^2\cos\theta_i + \left[n^2 - \sin^2\theta_i\right]^{1/2}}$$

Fresnel's Equations

Further, the above coefficients are related by

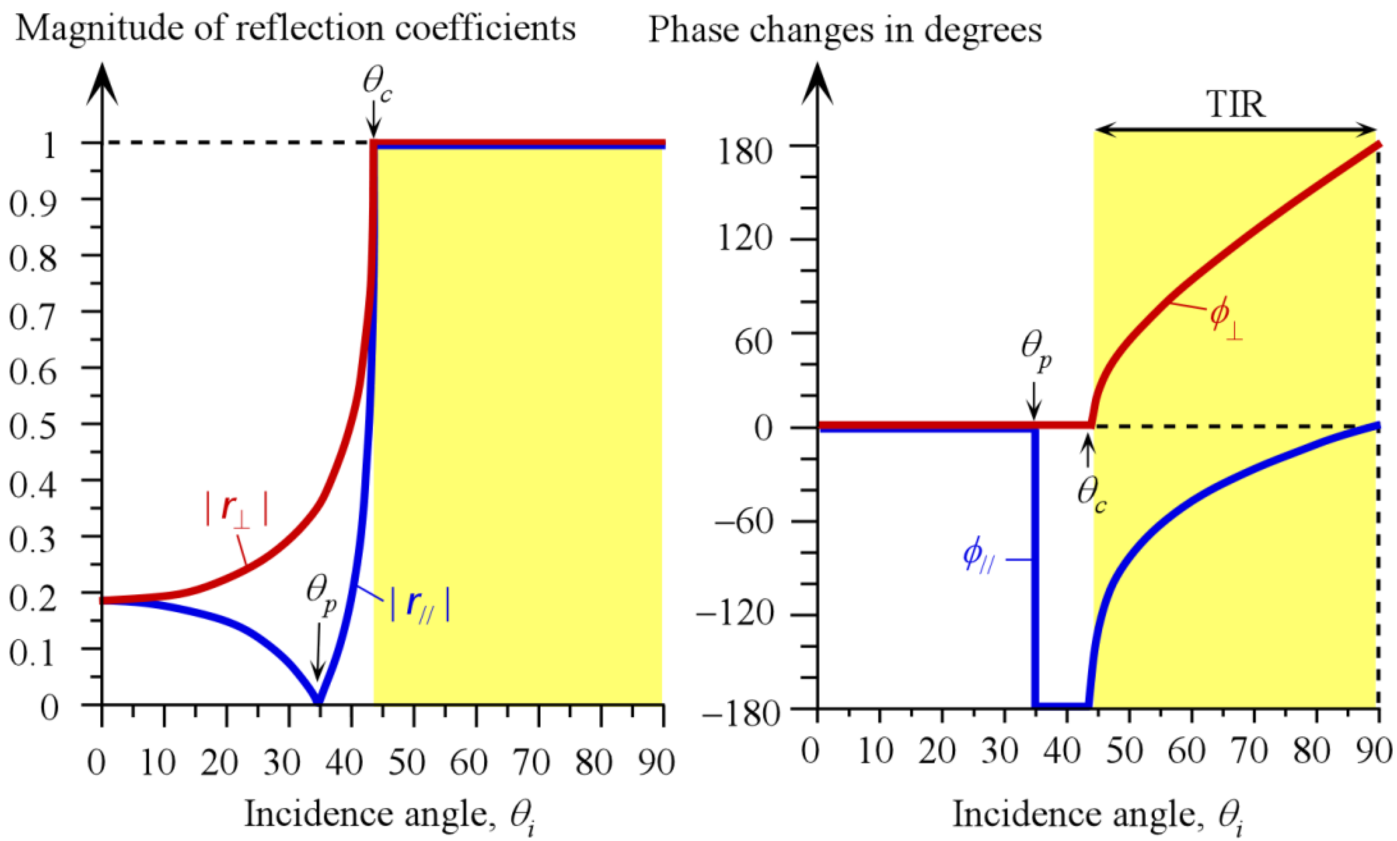
$$r_{//} + nt_{//} = 1 \quad \text{and} \quad r_{\perp} + 1 = t_{\perp}$$

For convenience we take E_{io} to be a real number so that phase angles of $r_{\perp}$ and $t_{\perp}$ correspond to the phase changes measured with respect to the incident wave.

For normal incidence ($\theta_i = 0$) into Fresnel's equations we find,

$$r_{//} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2}$$

Fresnel reflection



Internal reflection **(a)** **(b)**

(a) Magnitude of the reflection coefficients $r_{//}$ and $r_\perp$ vs. angle of incidence θ_i for $n_1 = 1.44$ and $n_2 = 1.00$. The critical angle is 44°.

(b) The corresponding changes $\phi_{//}$ and $\phi_\perp$ vs. incidence angle.

Reflection and Polarization Angle

We find a special incidence angle, labeled as θ_p, by solving the Fresnel equation for $r_{//} = 0$. The field in the reflected wave is then always perpendicular to the plane of incidence and hence well-defined. This special angle is called the polarization angle or Brewster's angle,

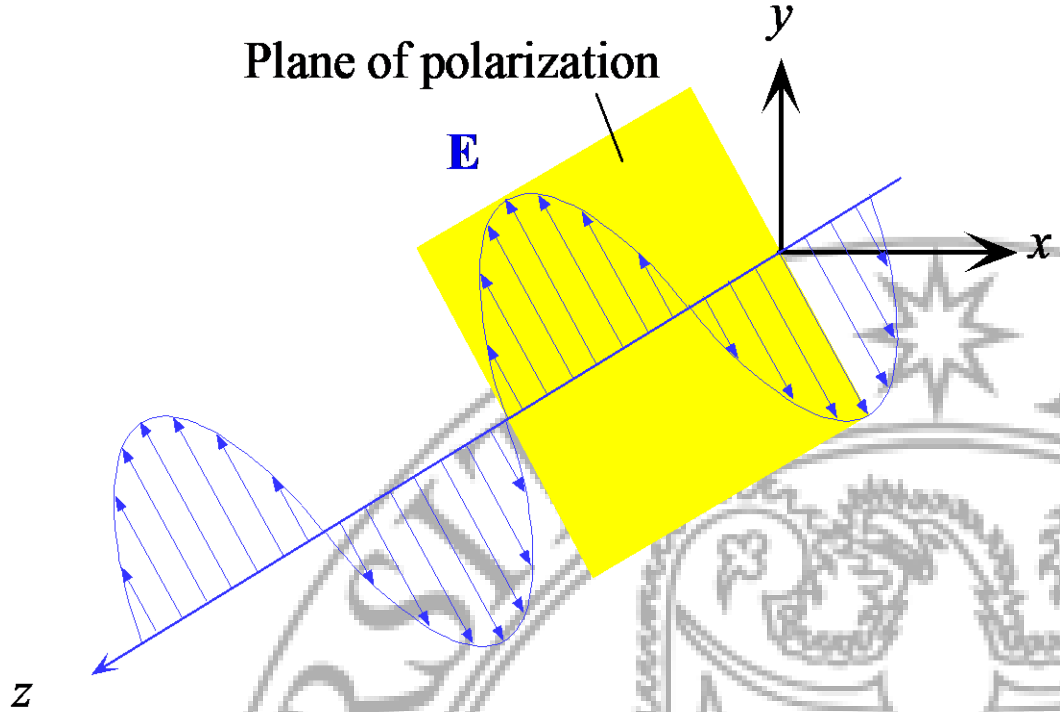
$$r_{//} = \frac{E_{r0,//}}{E_{i0,//}} = \frac{\left[n^2 - \sin^2\theta_i\right]^{1/2} - n^2\cos\theta_i}{\left[n^2 - \sin^2\theta_i\right]^{1/2} + n^2\cos\theta_i} = 0$$

$$\Downarrow$$

$$\tan\theta_p = \frac{n_2}{n_1}$$

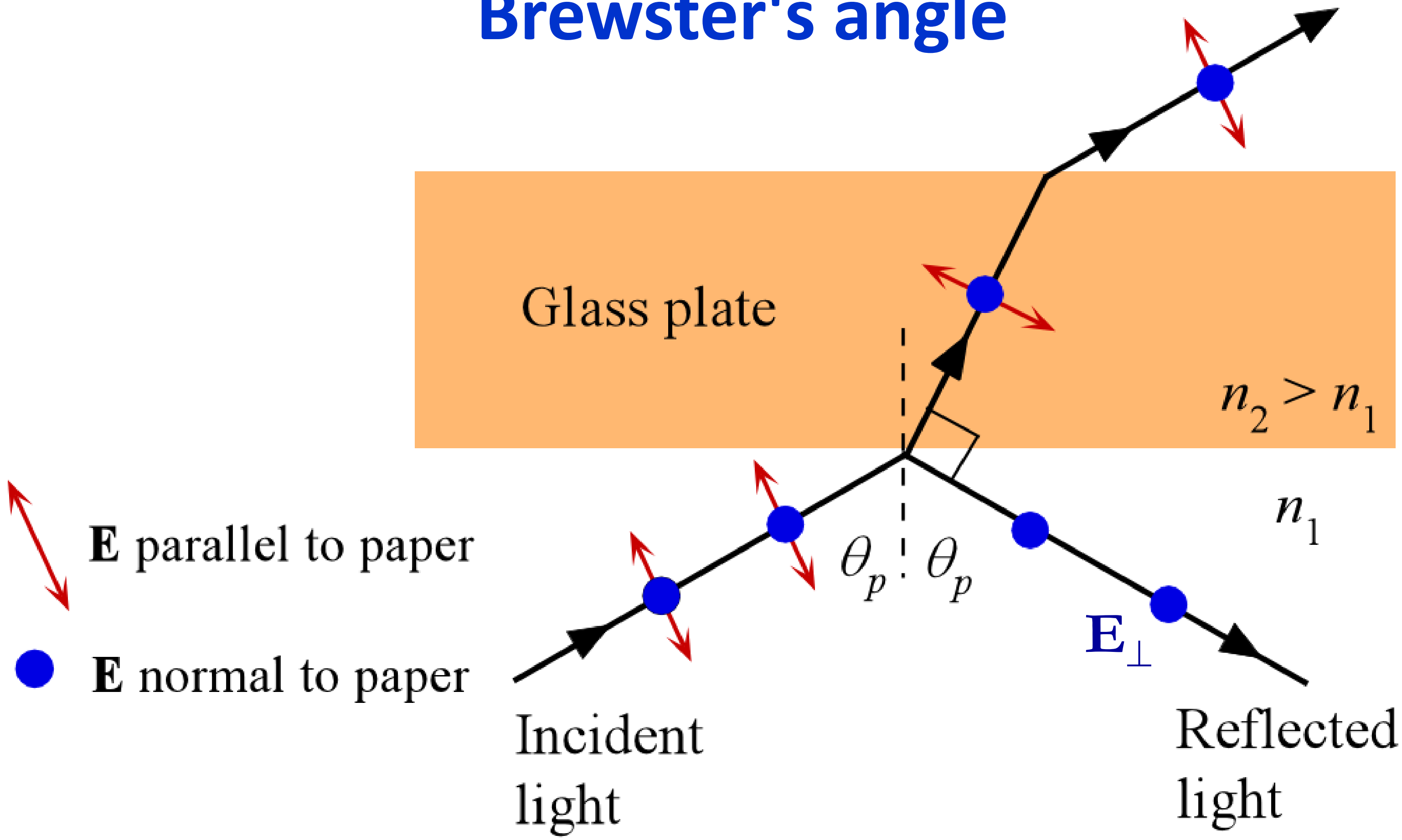
For both $n_1 > n_2$ or $n_1 < n_2$.

Polarized Light



A **linearly polarized** wave has its electric field oscillations defined along a line perpendicular to the direction of propagation, z. The field vector **E** and z define a **plane of polarization**.

Brewster's angle



Reflected light at $\theta_i = \theta_p$ has only $\mathbf{E}_\perp$

for both $n_1 > n_2$ or $n_1 < n_2$.

Total Internal Reflection

In linearly polarized light, however, the field oscillations are contained within a well defined plane. Light emitted from many light sources such as a tungsten light bulb or an LED diode is unpolarized and the field is randomly oriented in a direction that is perpendicular to the direction of propagation.

At the critical angle and beyond (past 44° in the figure), *i.e.* when $\theta_i \geq \theta_c$, the magnitudes of both $r_{//}$ and $r_{\perp}$ go to unity so that the reflected wave has the same amplitude as the incident wave. The incident wave has suffered total internal reflection, TIR.

Phase change upon total internal reflection

When $\theta_i > \theta_c$, in the presence of TIR, the reflection coefficients become complex quantities of the type

$r_\perp = 1 \cdot \exp(-j\phi_\perp)$ and $r_{//} = 1 \cdot \exp(-j\phi_{//})$

with the phase angles $\phi_\perp$ and $\phi_{//}$ being other than zero or 180°. The reflected wave therefore suffers phase changes, $\phi_\perp$ and $\phi_{//}$, in the components $E_\perp$ and $E_{//}$. These phase changes depend on the incidence angle, and on n_1 and n_2.

The phase change $\phi_\perp$ is given by

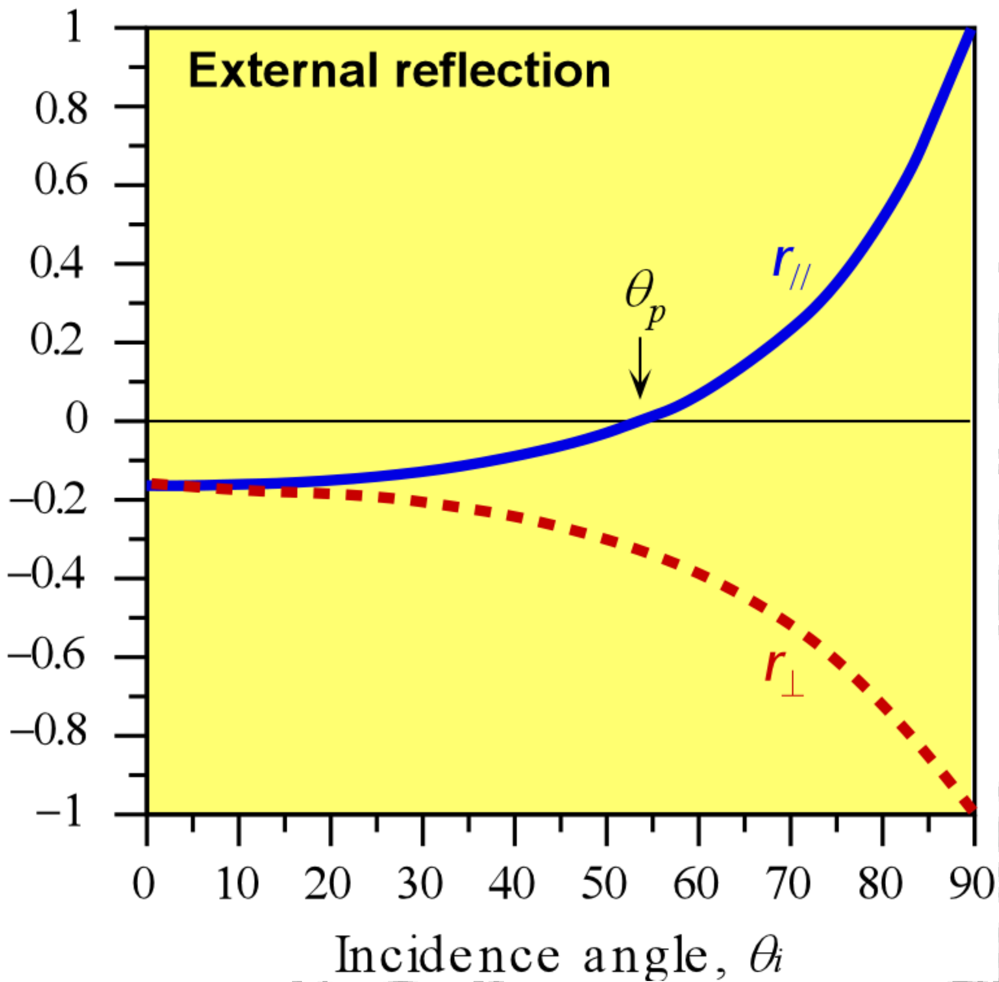
$$\tan\left(\frac{1}{2}\phi_\perp\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i}$$

Phase change upon total internal reflection

For the $E_{//}$ component, the phase change $\phi_{//}$ is given by

$$\tan\left(\tfrac{1}{2}\phi_{//}+\tfrac{1}{2}\pi\right)=\frac{\left[\sin^2\theta_i-n^2\right]^{1/2}}{n^2\cos\theta_i}$$

External Reflection



The reflection coefficients $r_{//}$ and $r_\perp$ versus angle of incidence θ_i for n_1 = 1.00 and n_2 = 1.44.

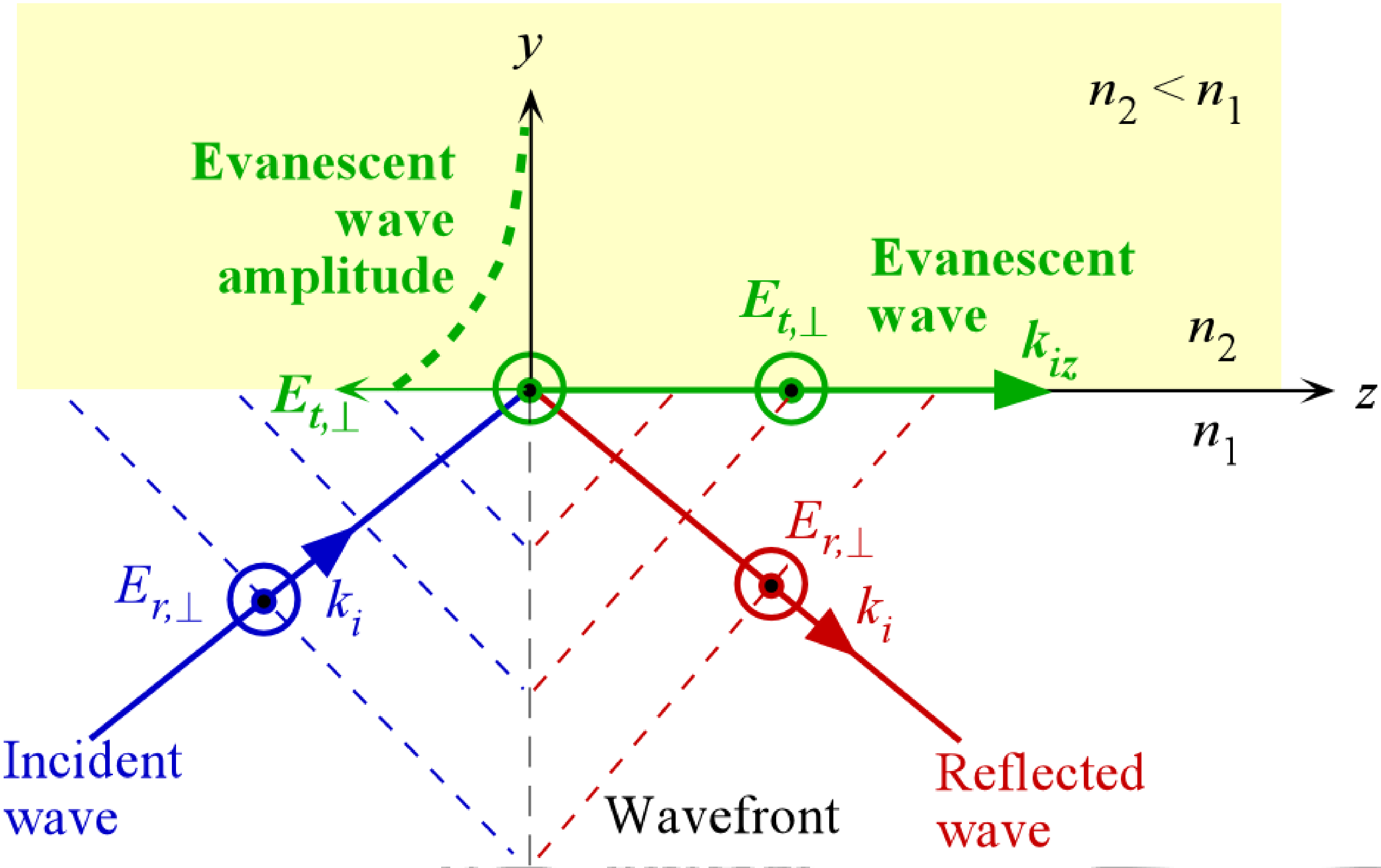
Evanescent Wave

In internal reflection ($n_1 > n_2$), the amplitude of the reflected wave from TIR is equal to the amplitude of the incident wave but its phase has shifted.

What happens to the transmitted wave when $\theta_i > \theta_c$?

According to the boundary conditions, there must still be an electric field in medium 2, otherwise, the boundary conditions cannot be satisfied. When $\theta_i > \theta_c$, the field in medium 2 is attenuated (decreases with y, and is called the evanescent wave.

Evanescent Wave



When $\theta_i > \theta_c$, for a plane wave that is reflected, there is an evanescent wave at the boundary propagating along z.

Evanescent wave when plane waves are incident and reflected

$$E_{t,\perp}(y,z,t) \propto \boldsymbol{e}^{-\alpha_2 y} \exp j(\omega t - k_{iz} z)$$

where $k_{iz} = k_i \sin\theta_i$ is the wavevector of the incident wave along the z-axis, and α_2 is an **attenuation coefficient** for the electric field penetrating into medium 2

$$\alpha_2 = \frac{2\pi n_2}{\lambda}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

Penetration depth of evanescent wave

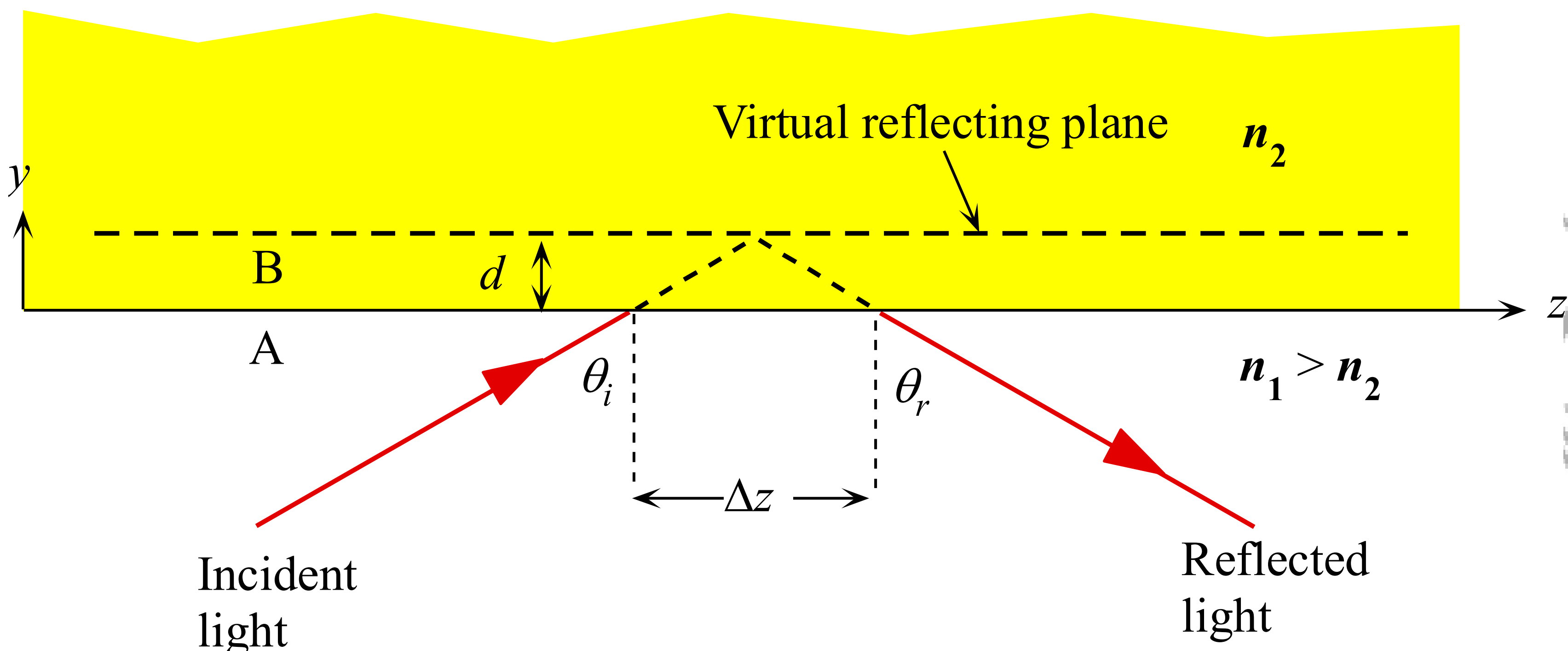
$\alpha_2 =$ **Attenuation coefficient** for the electric field penetrating into medium 2

$$\alpha_2 = \frac{2\pi n_2}{\lambda}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

The field of the evanescent wave is e^{-1} in medium 2 when

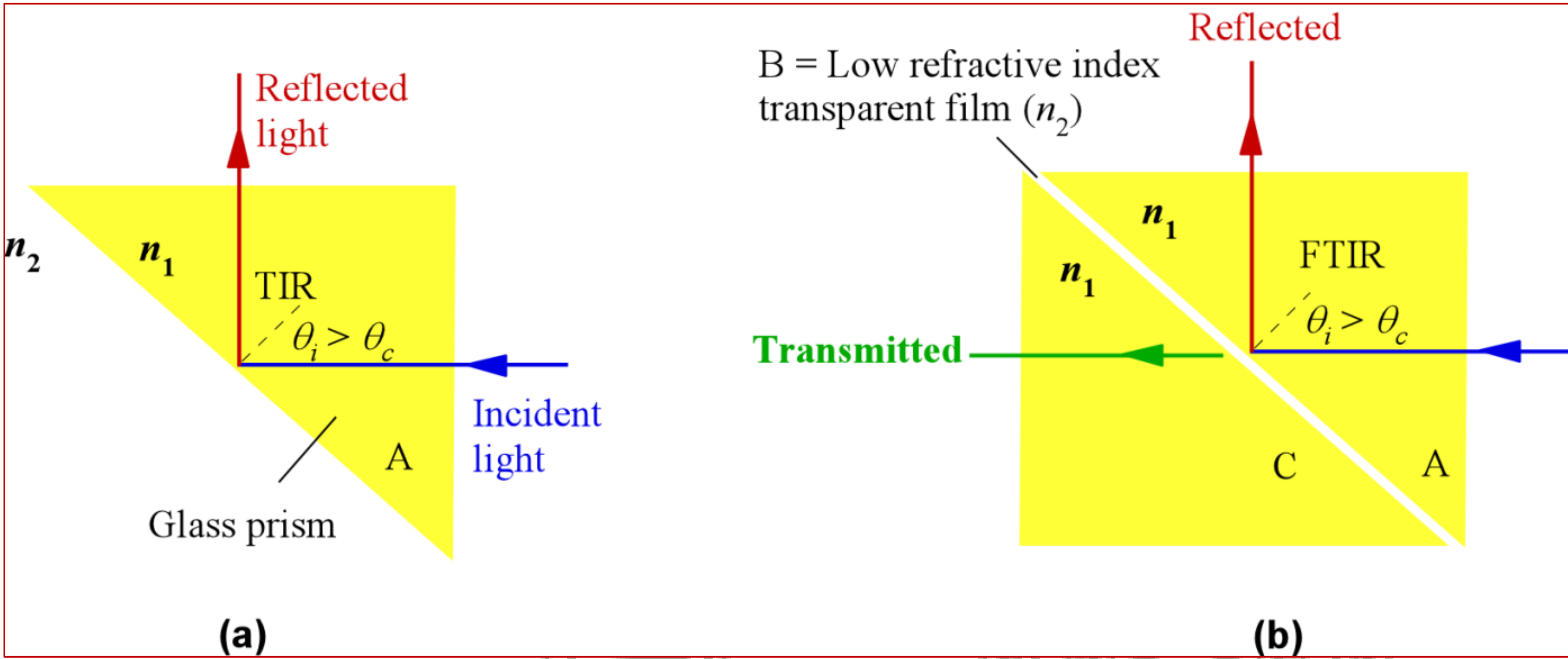
$y = 1/\alpha_2 = \delta =$ **Penetration depth**

Goos-Hänchen Shift



$$\boxed{\Delta z = 2\delta \tan\theta_i}$$

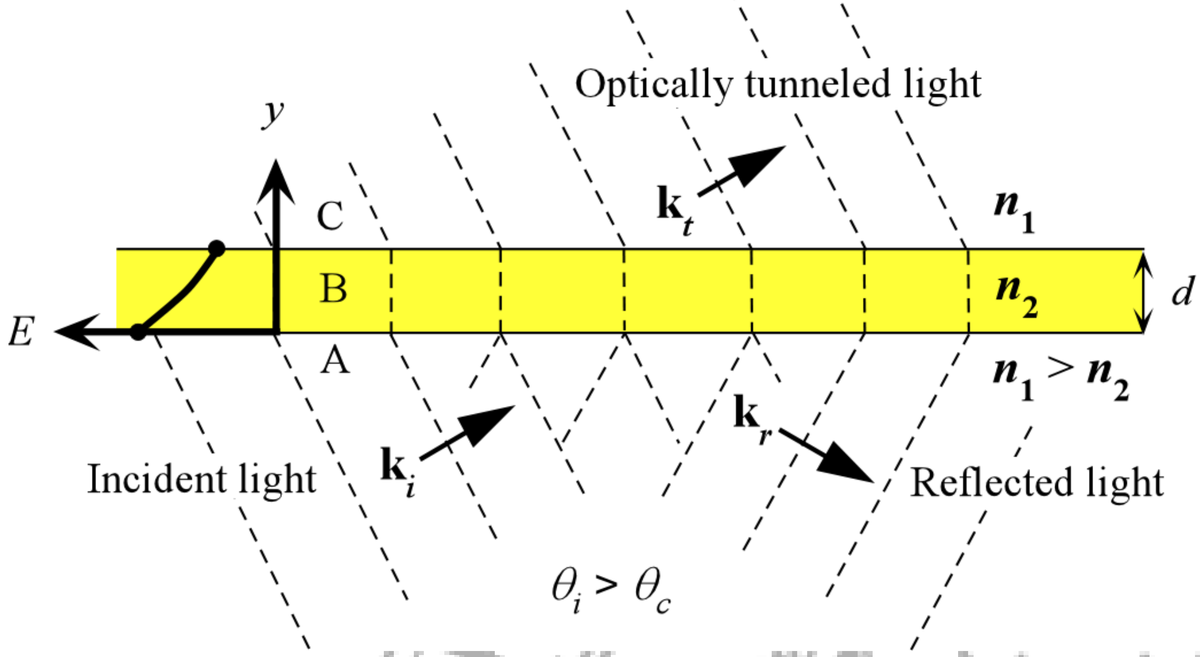
Beam Splitters
Frustrated Total Internal Reflection (FTIR)



(a) A light incident at the long face of a glass prism suffers TIR; the prism deflects the light.

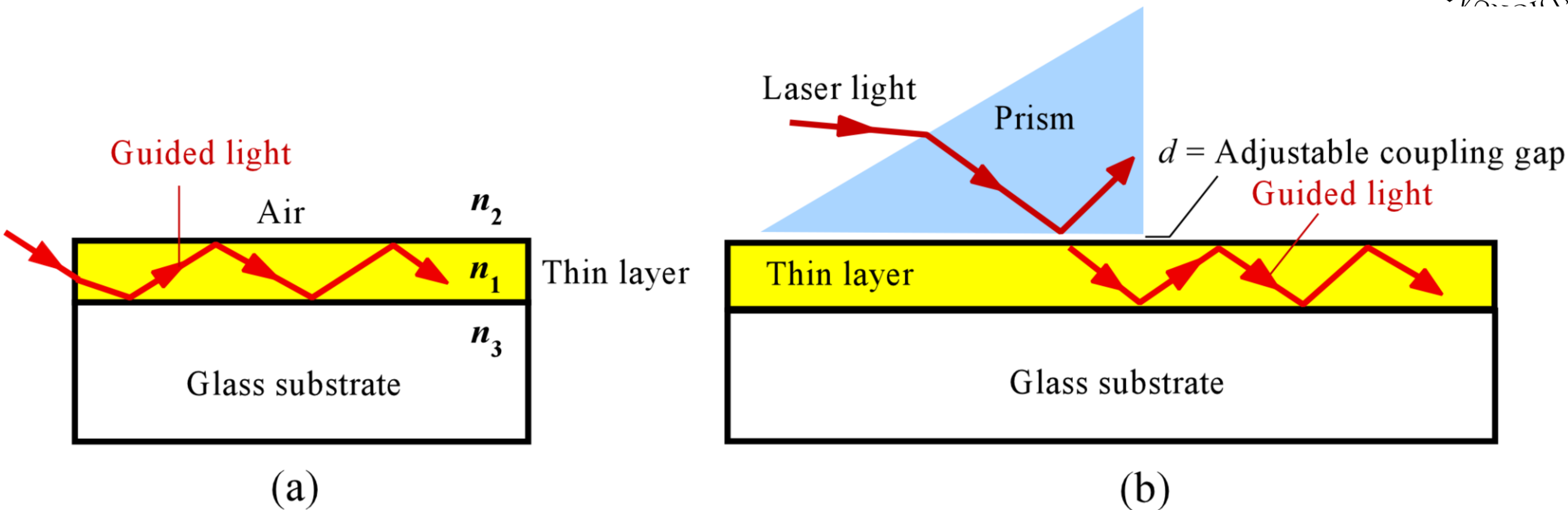
(b) Two prisms separated by a thin low refractive index film forming a beam-splitter cube. The incident beam is split into two beams by FTIR.

Optical Tunneling



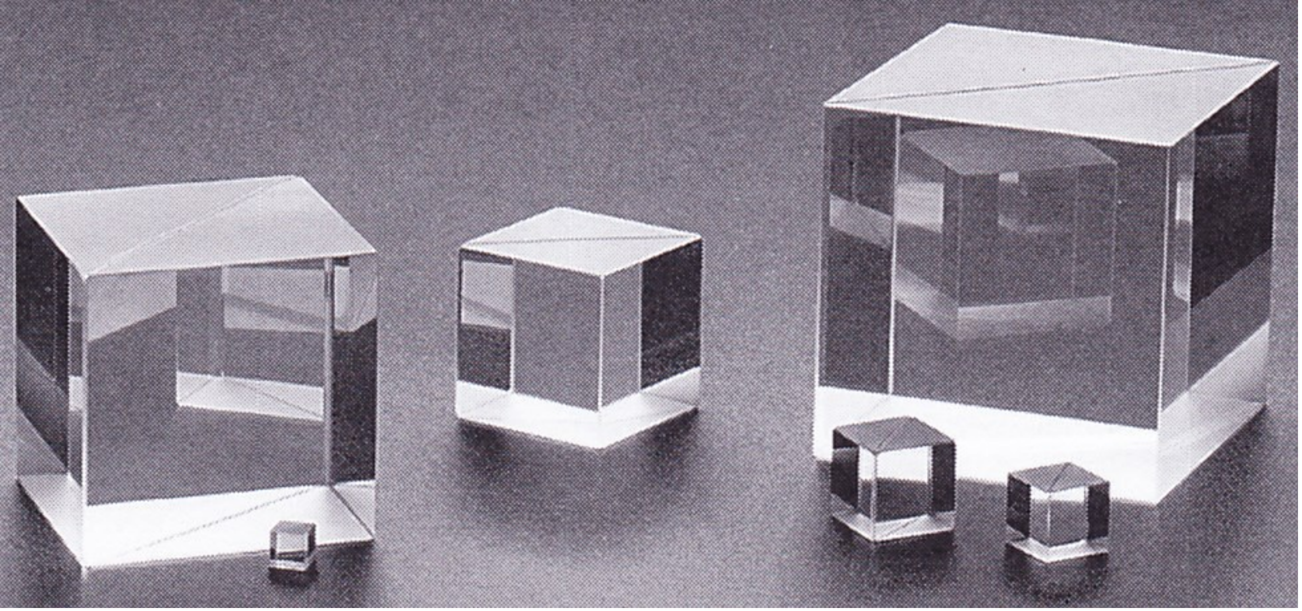
When medium B is thin (thickness d is small), the field penetrates from the AB interface into medium B and reaches BC interface, and gives rise to a transmitted wave in medium C. The effect is the tunneling of the incident beam in A through B to C. The maximum field E_{max} of the evanescent wave in B decays in B along y and but is finite at the BC boundary and excites the transmitted wave.

Optical Tunneling

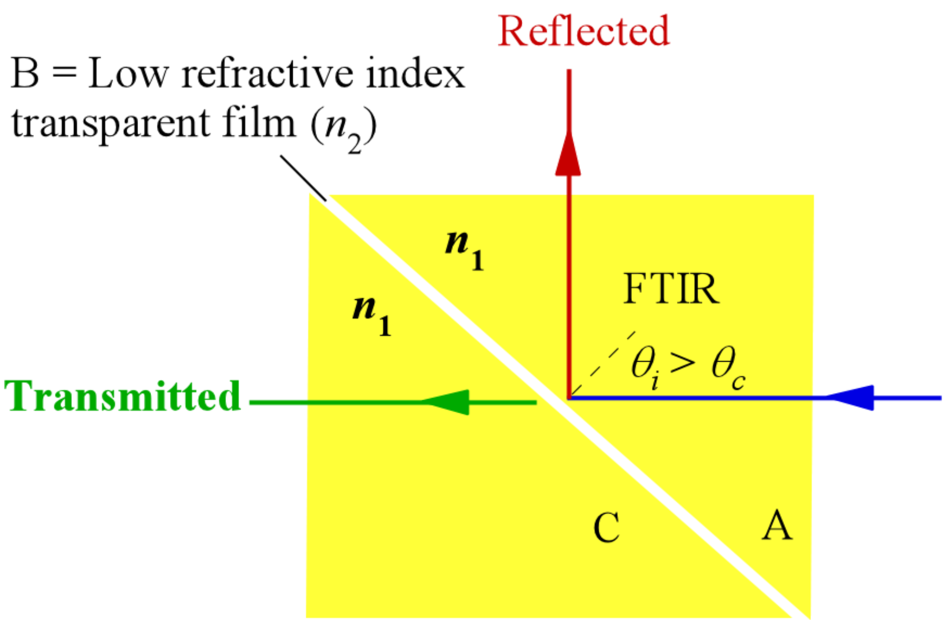


Light propagation along an optical guide by total internal reflections

Coupling of laser light into a thin layer - optical guide - using a prism. The light propagates along the thin layer.



Beam splitter cubes (Courtesy of CVI Melles Griot)



Two prisms separated by a thin low refractive index film forming a beam-splitter cube. The incident beam is split into two beams by FTIR.

External Reflection

Light approaches the boundary from the lower index side,

$$n_1 < n_2$$

This is external reflection.

Light becomes reflected by the surface of an optically denser (higher refractive index) medium.

$r_\perp$ and $r_{//}$ depend on the incidence angle θ_i. At normal incidence, $r_\perp$ and $r_{//}$ are negative. In external reflection at normal incidence there is a phase shift of 180°. $r_{//}$ goes through zero at the Brewster angle, θ_p. At θ_p, the reflected wave is polarized in the $E_\perp$ component only.

Transmitted light in both internal reflection (when $\theta_i < \theta_c$) and external reflection does not experience a phase shift.

Intensity, Reflectance and Transmittance

Reflectance R measures the intensity of the reflected light with respect to that of the incident light and can be defined separately for electric field components parallel and perpendicular to the plane of incidence. The reflectances $R_\perp$ and $R_{//}$ are defined by

$$R_\perp = \frac{\left|E_{ro,\perp}\right|^2}{\left|E_{io,\perp}\right|^2} = \left|r_\perp\right|^2$$

and

$$R_{//} = \frac{\left|E_{ro,//}\right|^2}{\left|E_{io,//}\right|^2} = \left|r_{//}\right|^2$$

At normal incidence

$$R = R_{\perp} = R_{//} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

Since a glass medium has a refractive index of around 1.5 this means that typically 4% of the incident radiation on an air-glass surface will be reflected back.

Example: Reflection at normal incidence. Internal and external reflection

Consider the reflection of light at normal incidence on a boundary between a glass medium of refractive index 1.5 and air of refractive index 1.

(a) If light is traveling from air to glass, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

(b) If light is traveling from glass to air, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

(c) What is the polarization angle in the external reflection in **a** above? How would you make a polaroid from this?

Solution

(a) The light travels in air and becomes partially reflected at the surface of the glass which corresponds to external reflection. Thus $n_1 = 1$ and $n_2 = 1.5$. Then,

$$r_{//} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2$$

This is negative which means that there is a 180° phase shift. The reflectance (R), which gives the fractional reflected power, is

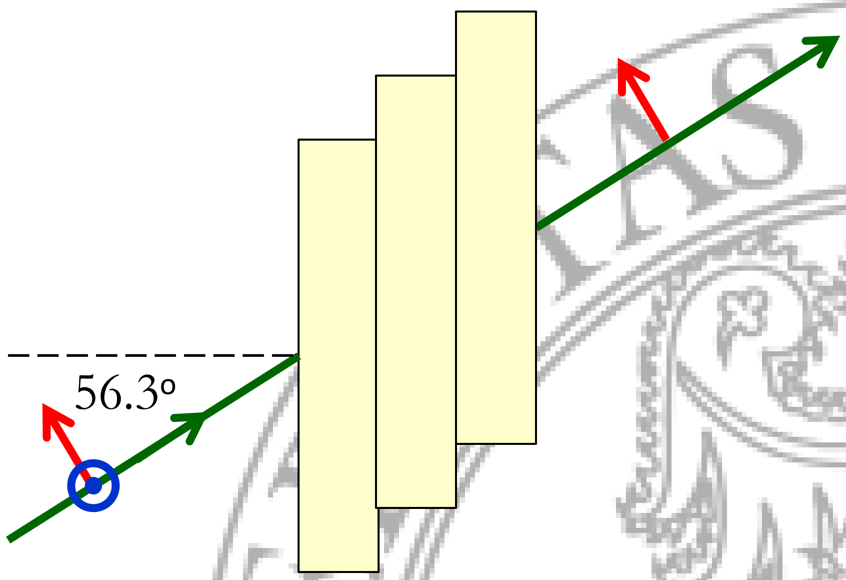
$$R = r_{//}{}^2 = 0.04 \text{ or } 4\%.$$

(b) The light travels in glass and becomes partially reflected at the glass-air interface which corresponds to internal reflection. $n_1 = 1.5$ and $n_2 = 1$. Then,

$$r_{//} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.5 - 1}{1.5 + 1} = 0.2$$

There is no phase shift. The reflectance is again 0.04 or 4%. In both cases **(a)** and **(b)** the amount of reflected light is the same.

(c) Light is traveling in air and is incident on the glass surface at the polarization angle. Here $n_1 = 1$, $n_2 = 1.5$ and $\tan\theta_p = (n_2/n_1) = 1.5$ so that $\theta_p = 56.3°$.



This type of *pile-of-plates polarizer* was invented by Dominique F.J. Arago in 1812

Transmittance

Transmittance T relates the intensity of the transmitted wave to that of the incident wave in a similar fashion to the reflectance.

However the transmitted wave is in a different medium and further its direction with respect to the boundary is also different due to refraction.

For normal incidence, the incident and transmitted beams are normal so that the equations are simple:

Transmittance

$$T_{\perp} = \frac{n_2 \left| E_{to,\perp} \right|^2}{n_1 \left| E_{io,\perp} \right|^2} = \left(\frac{n_2}{n_1} \right) \left| t_{\perp} \right|^2$$

$$T_{//} = \frac{n_2 \left| E_{to,//} \right|^2}{n_1 \left| E_{io,//} \right|^2} = \left(\frac{n_2}{n_1} \right) \left| t_{//} \right|^2$$

or

$$T = T_{\perp} = T_{//} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

Further, the fraction of light reflected and fraction transmitted must add to unity. Thus $R + T = 1$.

Reflection and Transmission – An Example

Question A light beam traveling in air is incident on a glass plate of refractive index 1.50 . What is the Brester or polarization angle? What are the relative intensities of the reflected and transmitted light for the polarization perpendicular and parallel to the plane of incidence at the Brestwer angle of incidence?

Solution Light is traveling in air and is incident on the glass surface at the polarization angle θ_p. Here $n_1 = 1$, $n_2 = 1.5$ and $\tan\theta_p = (n_2/n_1) = 1.5$ so that $\theta_p = 56.31°$. We now have to use Fresnel's equations to find the reflected and transmitted amplitudes. For the perpendicular polarization

$$r_\perp = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos\theta_i - [n^2 - \sin^2\theta_i]^{1/2}}{\cos\theta_i + [n^2 - \sin^2\theta_i]^{1/2}}$$

$$r_\perp = \frac{\cos(56.31^\circ) - [1.5^2 - \sin^2(56.31^\circ)]^{1/2}}{\cos(56.31^\circ) + [1.5^2 - \sin^2(56.31^\circ)]^{1/2}} = -0.385$$

On the other hand, $r_{//} = 0$. The reflectances $R_\perp = |r_\perp|^2 = 0.148$ and $R_{//} = |r_{//}|^2 = 0$ so that $R = 0.074$, and has no parallel polarization in the plane of incidence. Notice the negative sign in $r_\perp$, which indicates a phase change of π.

Reflection and Transmission – An Example

$$t_{\perp} = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2\cos\theta_i}{\cos\theta_i + [n^2 - \sin^2\theta_i]^{1/2}}$$

$$t_{\perp} = \frac{2\cos(56.31^{\circ})}{\cos(56.31^{\circ}) + [1.5^2 - \sin^2(56.31^{\circ})]^{1/2}} = 0.615$$

$$t_{//} = \frac{E_{t0,//}}{E_{i0,//}} = \frac{2n\cos\theta_i}{n^2\cos\theta_i + [n^2 - \sin^2\theta_i]^{1/2}}$$

$$t_{//} = \frac{2(1.5)\cos(56.31^{\circ})}{(1.5)^2\cos(56.31^{\circ}) + [1.5^2 - \sin^2(56.31^{\circ})]^{1/2}} = 0.667$$

Notice that $r_{//} + nt_{//} = 1$ and $r_{\perp} + 1 = t_{\perp}$, as we expect.

Reflection and Transmission – An Example

To find the transmittance for each polarization, we need the refraction angle θ_t. From Snell's law, $n_1 \sin\theta_i = n_t \sin\theta_t$ *i.e.* (1)sin(56.31°) = (1.5)sinθ_t, we find **θ_t = 33.69°**.

$$T_{//} = \frac{n_2\left|E_{to,//}\right|^2}{n_1\left|E_{io,//}\right|^2} = \left(\frac{n_2}{n_1}\right)\left|t_{//}\right|^2$$

$$T_{\perp} = \frac{n_2\left|E_{to,\perp}\right|^2}{n_1\left|E_{io,\perp}\right|^2} = \left(\frac{n_2}{n_1}\right)\left|t_{\perp}\right|^2$$

$$T_{//} = \left[\frac{(1.5)\cos(33.69^o)}{(1)\cos(56.31^o)}\right](0.667)^2 = 1$$

$$T_{\perp} = \left[\frac{(1.5)\cos(33.69^o)}{(1)\cos(56.31^o)}\right](0.615)^2 = 0.852$$

Clearly, light with polarization parallel to the plane of incidence has greater intensity.

If we were to reflect light from a glass plate, keeping the angle of incidence at 56.3°, then the reflected light will be polarized with an electric field component perpendicular to the plane of incidence. The transmitted light will have the field greater in the plane of incidence, that is, it will be partially polarized. By using a stack of glass plates one can increase the polarization of the transmitted light. (This type of *pile-of-plates polarizer* was invented by Dominique F.J. **Arago** in 1812.)

Example: Reflection of light from a less dense medium (internal reflection)

A ray of light which is traveling in a glass medium of refractive index $n_1 = 1.460$ becomes incident on a less dense glass medium of refractive index $n_2 = 1.440$. The free space wavelength (λ) of the light ray is 1300 nm.

(a) What should be the minimum incidence angle for TIR?

(b) What is the phase change in the reflected wave when $\theta_i = 87°$ and when $\theta_i = 90°$?

(c) What is the penetration depth of the evanescent wave into medium 2 when $\theta_i = 87°$ and when $\theta_i = 90°$?

Solution

(a) The critical angle θ_c for TIR is given by

$\sin\theta_c = n_2/n_1 = 1.440/1.460$ so that $\theta_c = 80.51°$

(b) Since the incidence angle $\theta_i > \theta_c$ there is a phase shift in the reflected wave. The phase change in $E_{r,\perp}$ is given by $\phi_\perp$.

Using $n_1 = 1.460$, $n_2 = 1.440$ and $\theta_i = 87°$,

$$= 2.989 = \tan[{}^{1}/_{2}(143.0°)]$$

$$\tan\left(\tfrac{1}{2}\phi_\perp\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i} = \frac{\left[\sin^2(87^\circ) - \left(\dfrac{1.440}{1.460}\right)^2\right]^{1/2}}{\cos(87^\circ)}$$

so that the phase change $\phi_\perp = 143°$.

For the $E_{r,//}$ component, the phase change is

$$\tan\left(\tfrac{1}{2}\phi_{//} + \tfrac{1}{2}\pi\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{n^2\cos\theta_i} = \frac{1}{n^2}\tan\left(\tfrac{1}{2}\phi_\perp\right)$$

so that

$$\tan(^1/_2\phi_{//} + {}^1/_2\pi) = (n_1/n_2)^2\tan(\phi_\perp/2) =$$

$$(1.460/1.440)^2\tan(^1/_2 143°)$$

which gives $\phi_{//} = 143.95° - 180°$ or $-36.05°$

Repeat with $\theta_i = 90°$ to find $\phi_\perp = 180°$ and $\phi_{//} = 0°$.

Note that as long as $\theta_i > \theta_c$, the magnitude of the reflection coefficients are unity. Only the phase changes.

$$E_{t,\perp}(y,t) \propto E_{to,\perp}\exp(-\alpha_2 y)$$

The field strength drops to e^{-1} when $y = 1/\alpha_2 = \delta$, which is called the penetration depth. The attenuation constant α_2 is

$$\alpha_2 = \frac{2\pi n_2}{\lambda}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

i.e.

$$\alpha_2 = \frac{2\pi(1.440)}{(1300\times10^{-9}\ \text{m})}\left[\left(\frac{1.460}{1.440}\right)^2 \sin^2(87^\circ) - 1\right]^{1/2}$$

$= 1.10\times10^6\ \text{m}^{-1}$.

The penetration depth is,

$\delta = 1/\alpha_2 = 1/(1.104\times10^6\ \text{m}) = 9.06\times10^{-7}$ m, or **0.906 μm**

For 90°, repeating the calculation, $\alpha_2 = 1.164\times10^6\ \text{m}^{-1}$, so that

$\delta = 1/\alpha_2 =$ **0.859 μm**

The penetration is greater for smaller incidence angles

Example: Antireflection coatings on solar cells

When light is incident on the surface of a semiconductor it becomes partially reflected. Partial reflection is an important energy loss in solar cells.

The refractive index of Si is about 3.5 at wavelengths around 700 - 800 nm. Reflectance with n_1(air) = 1 and n_2(Si) ≈ 3.5 is

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left(\frac{1 - 3.5}{1 + 3.5} \right)^2 = 0.309$$

Anti-reflection coating

30% of the light is reflected and is not available for conversion to electrical energy; a considerable reduction in the efficiency of the solar cell.

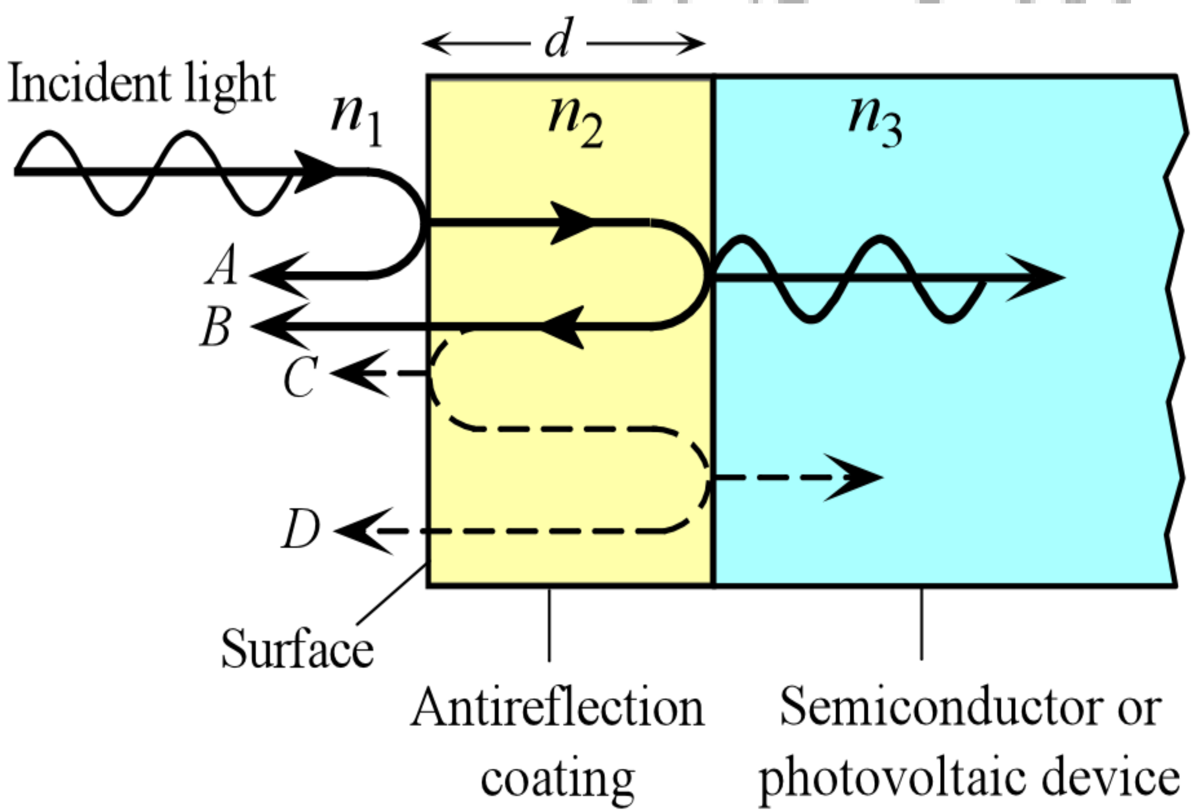
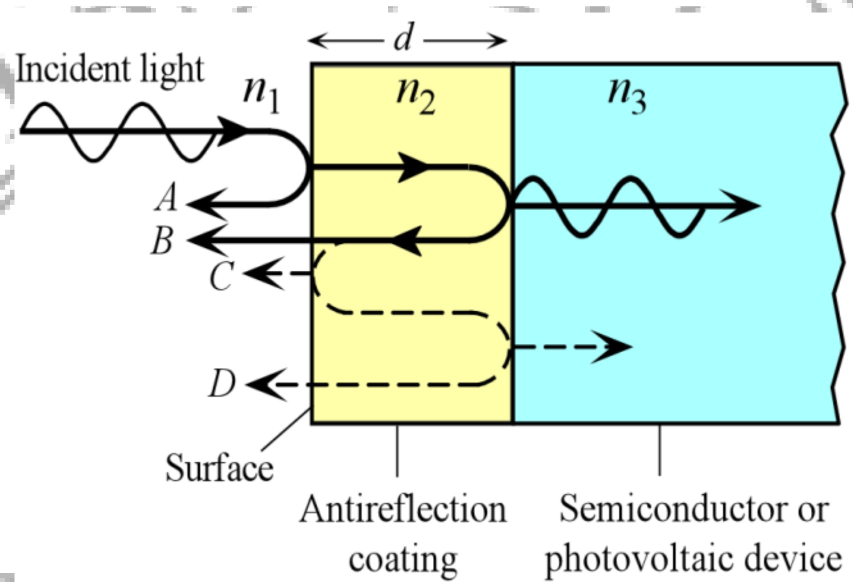


Illustration of how an antireflection coating reduces the reflected light intensity

Light is first incident on the air/coating surface. Some of it becomes reflected as *A* in the figure. Wave *A* has experienced a 180° phase change on reflection because this is an external reflection. The wave that enters and travels in the coating then becomes reflected at the coating/semiconductor surface.

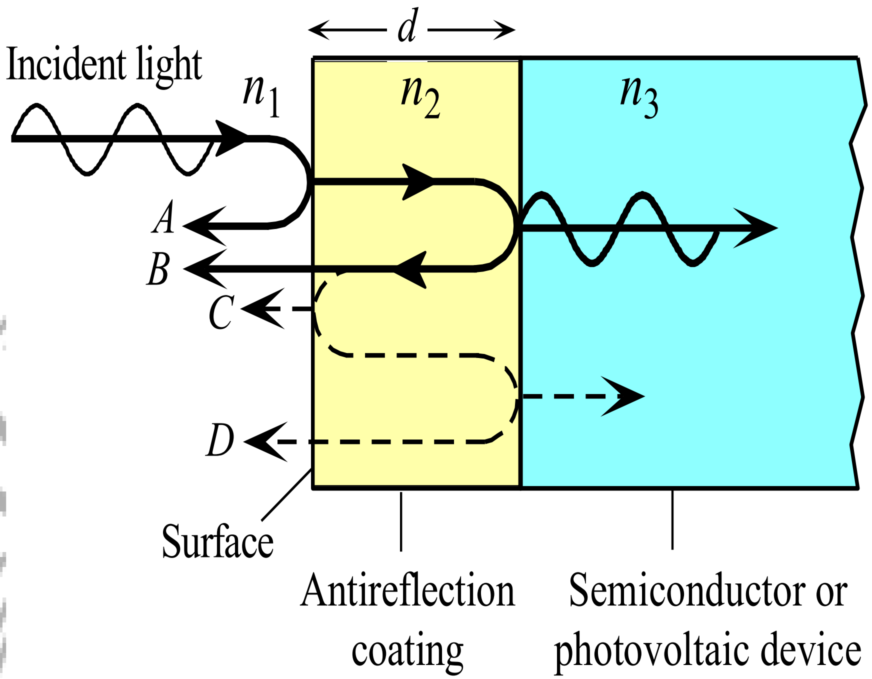
We can coat the surface of the semiconductor device with a thin layer of a dielectric material, *e.g.* Si_3N_4 (silicon nitride) that has an intermediate refractive index.



n_1(air) = 1, n_2(coating) ≈ 1.9 and n_3(Si) = 3.5

Anti-reflection coating

This reflected wave B, also suffers a 180° phase change since $n_3 > n_2$.



When B reaches A, it has suffered a total delay of traversing the thickness d of the coating twice. The **phase difference** is equivalent to $k_c(2d)$ where $k_c = 2\pi/\lambda_c$ is the propagation constant in the coating, *i.e.* $k_c = 2\pi/\lambda_c$ where λ_c is the wavelength in the coating.

Since $\lambda_c = \lambda / n_2$, where λ is the free-space wavelength, the phase difference $\Delta\phi$ between A and B is $(2\pi n_2/\lambda)(2d)$. To reduce the reflected light, A and B **must interfere destructively.** This requires the **phase difference to be π or odd-multiples of π, $m\pi$ where m = 1,3,5**,… is an odd-integer. Thus or

$$\left(\frac{2\pi n_2}{\lambda}\right)2d = m\pi$$

$$d = m\left(\frac{\lambda}{4n_2}\right)$$

The thickness of the coating must be odd-multiples of the quarter wavelength in the coating and depends on the wavelength.

$$R_{\min} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right)^2$$

$$d = m\left(\frac{\lambda}{4n_2}\right)$$

To obtain good destructive interference between waves *A* and *B*, the two amplitudes must be comparable. We need $n_2 = \sqrt{(n_1 n_3)}$. When $n_2 = \sqrt{(n_1 n_3)}$ then the reflection coefficient between the air and coating is equal to that between the coating and the semiconductor. For a Si solar cell, √(3.5) or 1.87. Thus, Si_3N_4 is a good choice as an antireflection coating material on Si solar cells.

Taking the wavelength to be 700 nm,

d **= (700 nm)/[4 (1.9)] = 92.1 nm or odd-multiples of *d*.**

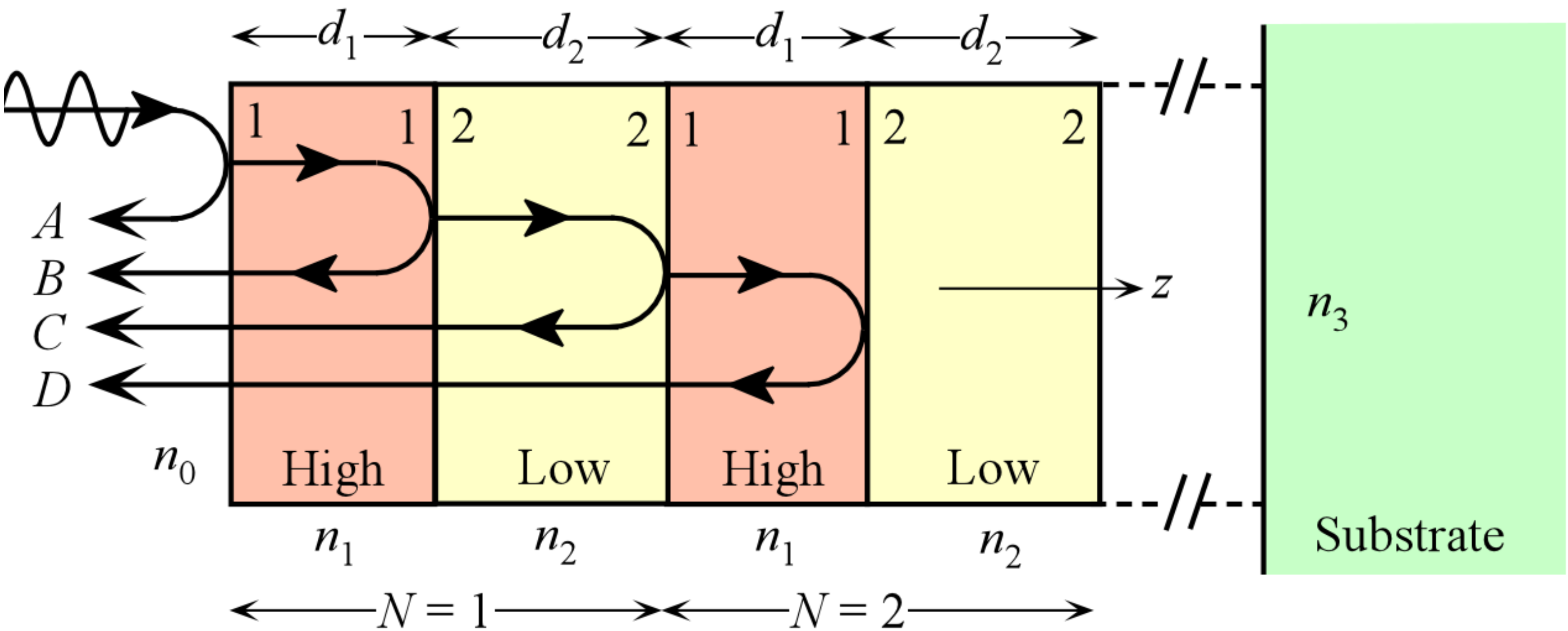
$$R_{\min} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3} \right)^2$$

$$R_{\min} = \left(\frac{1.9^2 - (1)(3.5)}{1.9^2 + (1)(3.5)} \right)^2 = 0.00024 \, or \; 0.24\%$$

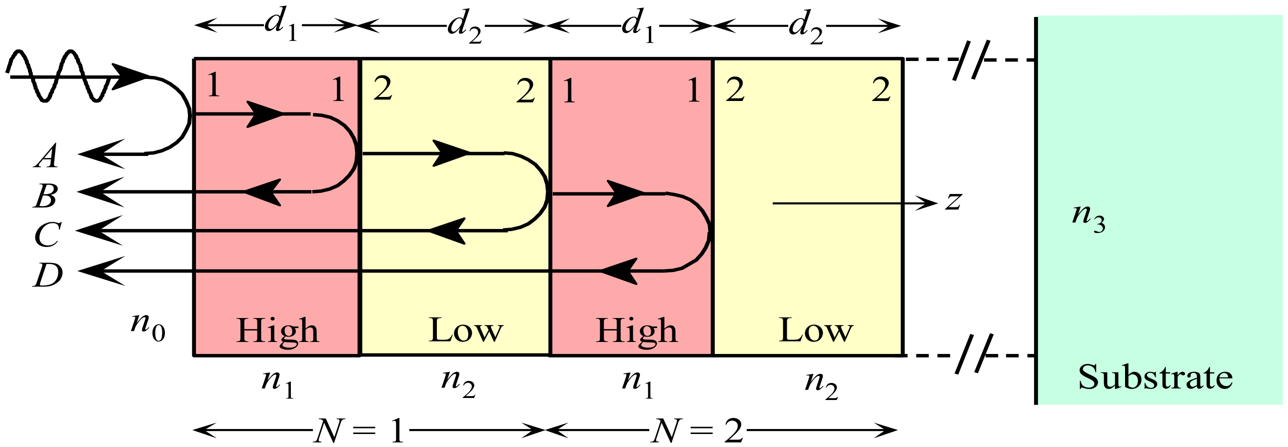
Reflection is almost entirely extinguished
However, only at 700 nm.

Without Anti reflection With Anti reflection

Dielectric Mirror or Bragg Reflector



Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers



r_{12} for light in layer 1 being reflected at the 1-2 boundary is $r_{12} = (n_1 - n_2)/(n_1 + n_2)$ and is a positive number indicating no phase change.

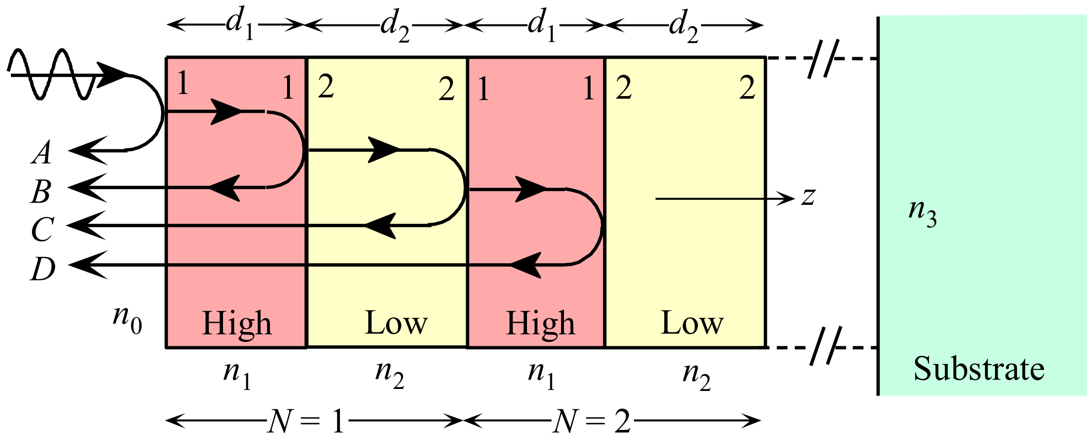
r_{21} for light in layer 2 being reflected at the 2-1 boundary is $r_{21} = (n_2 - n_1)/(n_2 + n_1)$ which is $-r_{12}$ (negative) indicating a π phase change. The reflection coefficient alternates in sign through the mirror

The phase difference between *A* and *B* is

$$0 + \pi + 2k_1 d_1 = 0 + \pi + 2(2\pi n_1/\lambda_o)(\lambda_o/4n_1) = 2\pi.$$

Thus, waves *A* and *B* are in phase and interfere constructively.

Dielectric mirrors are widely used in modern vertical cavity surface emitting semiconductor lasers.

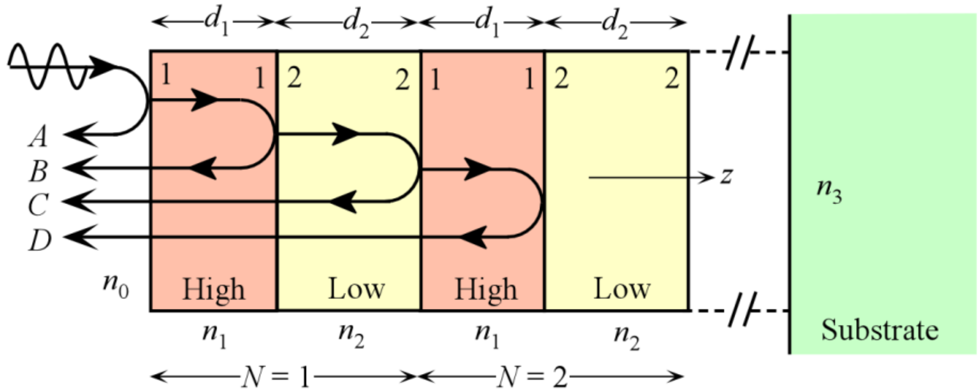


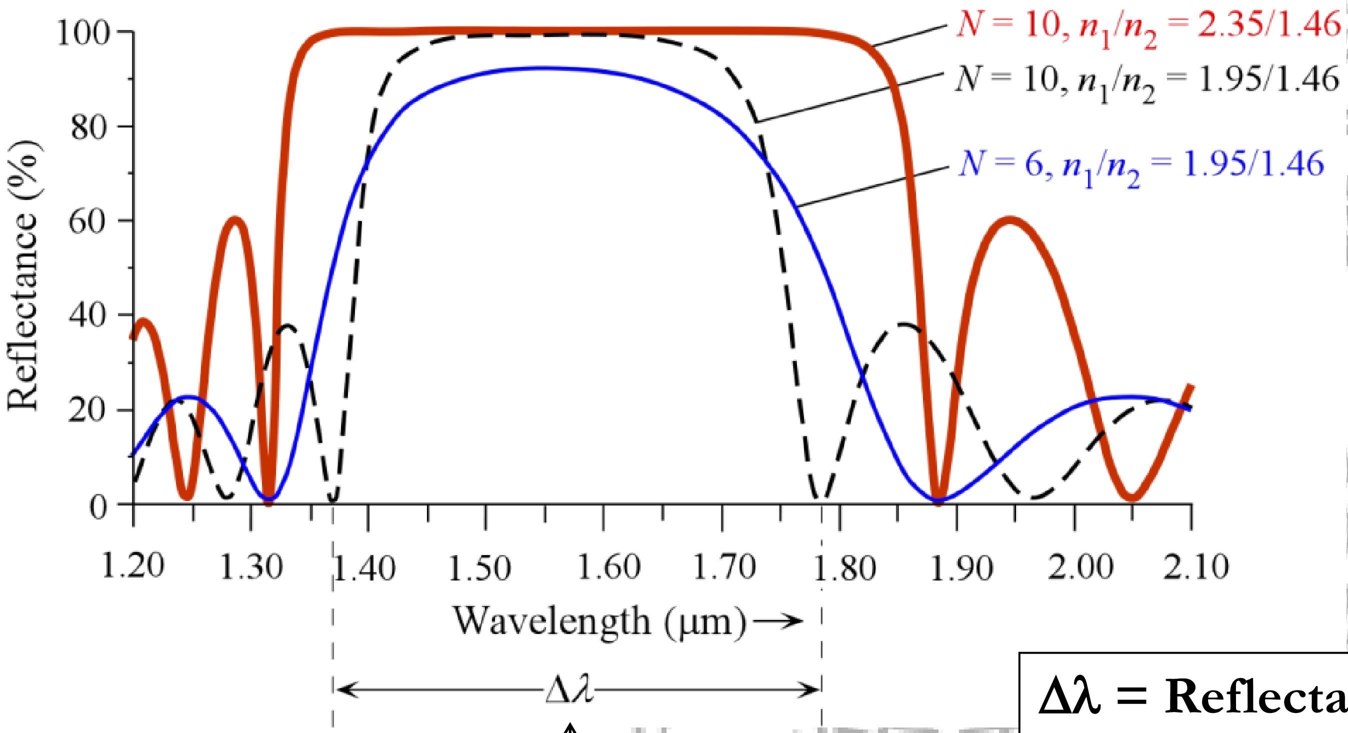
A dielectric mirror has a stack of dielectric layers of alternating refractive indices. Let n_1 (= n_H) > n_2 (= n_L)

Layer thickness d = Quarter of wavelength or $\lambda_{\text{layer}}/4$
$\lambda_{\text{layer}} = \lambda_o/n$; λ_o is the free space wavelength at which the mirror is required to reflect the incident light, n = refractive index of layer.

Reflected waves from the interfaces interfere constructively and give rise to a substantial reflected light. If there are sufficient number of layers, the reflectance can approach unity at λ_o.

Dielectric Mirror or Bragg Reflector

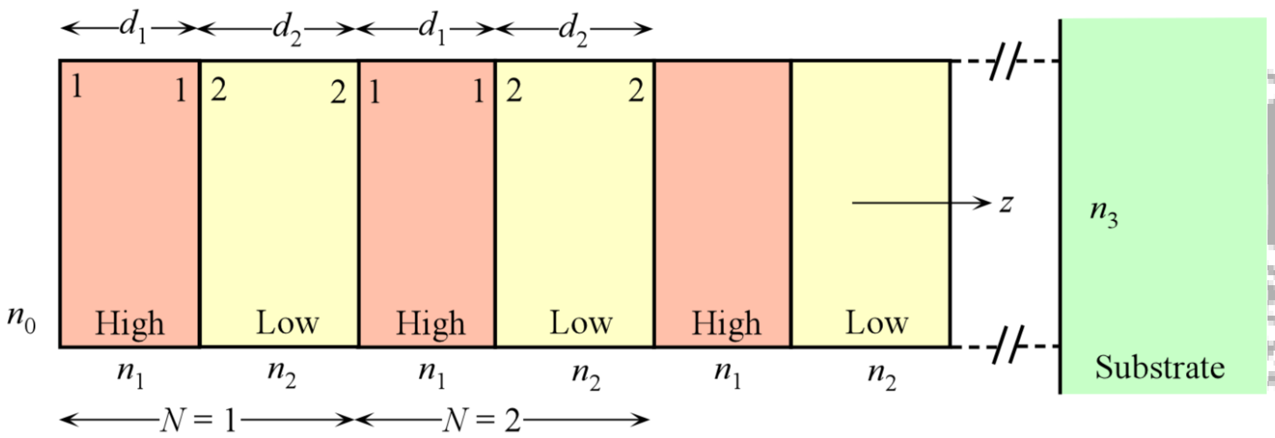


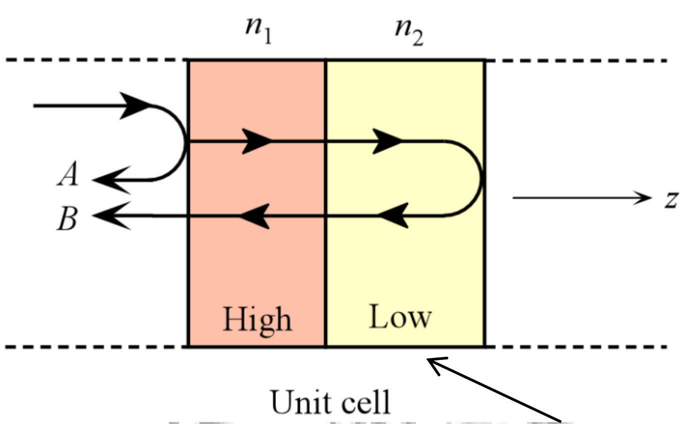


Δλ = Reflectance bandwidth
(**Stop-band** for transmittance)

Dielectric Mirror or Bragg Reflector

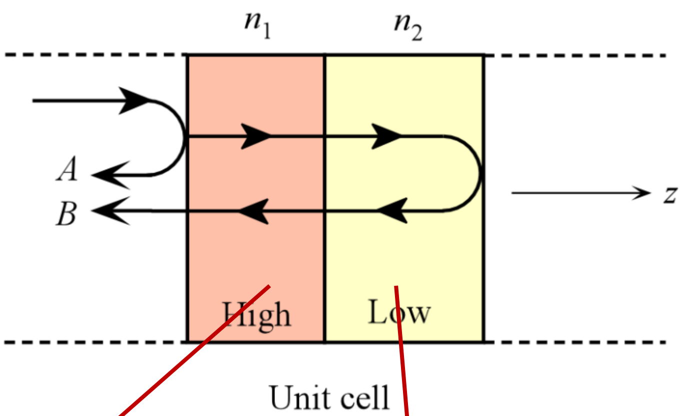
Consider an "infinite stack"





This is a "unit cell"

Dielectric Mirror or Bragg Reflector



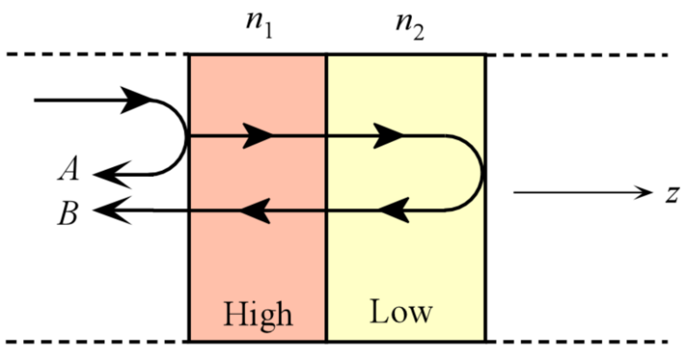
For reflection, the **phase difference between *A* and *B*** must be

$$2k_1d_1 + 2k_2d_2 = m(2\pi)$$

$$2(2\pi n_1/\lambda)d_1 + 2(2\pi n_2/\lambda)d_2 = m(2\pi)$$

$$\boxed{n_1d_1 + n_2d_2 = m\lambda/2}$$

Dielectric Mirror or Bragg Reflector



Unit cell

$$n_1 d_1 + n_2 d_2 = \lambda/2$$

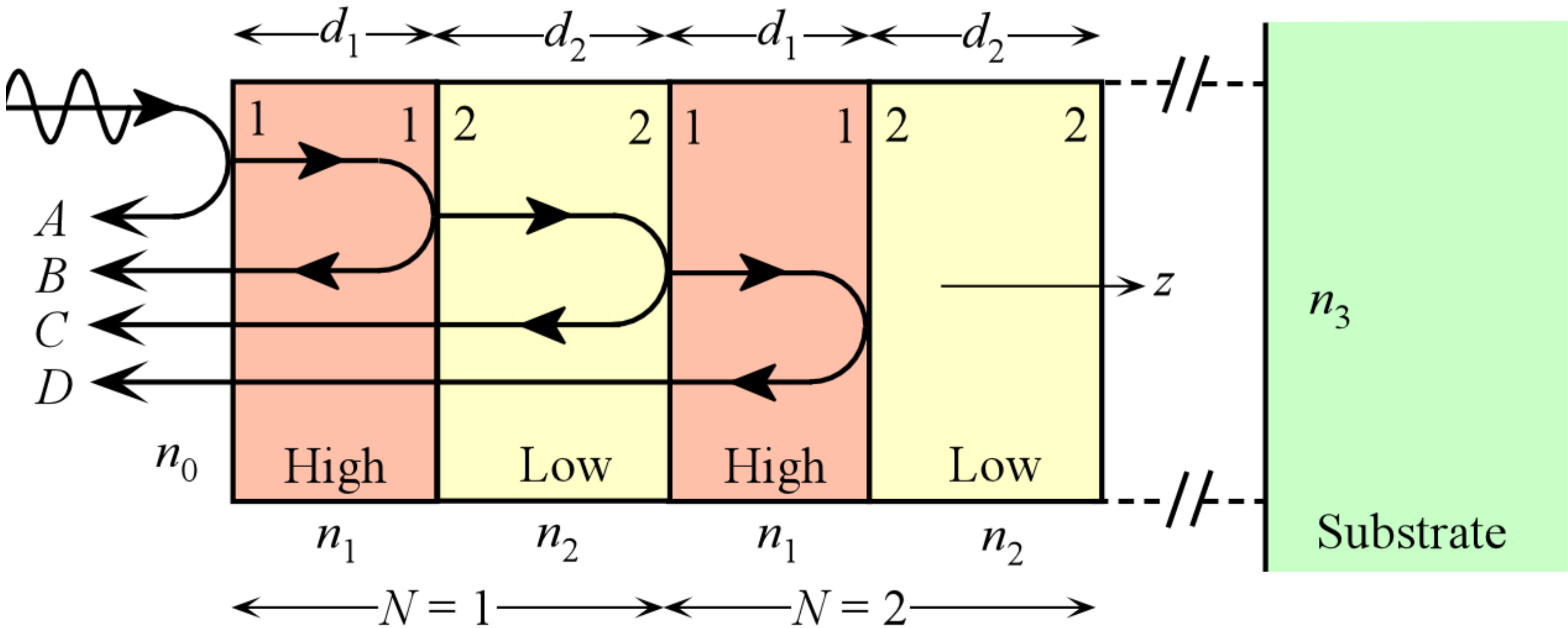
$$\boxed{d_1 = \lambda/4n_1}$$

$$\boxed{d_2 = \lambda/4n_2}$$

Quarter-Wave Stack

$d_1 = \lambda/4n_1$ **and** $d_2 = \lambda/4n_2$

Dielectric Mirror or Bragg Reflector

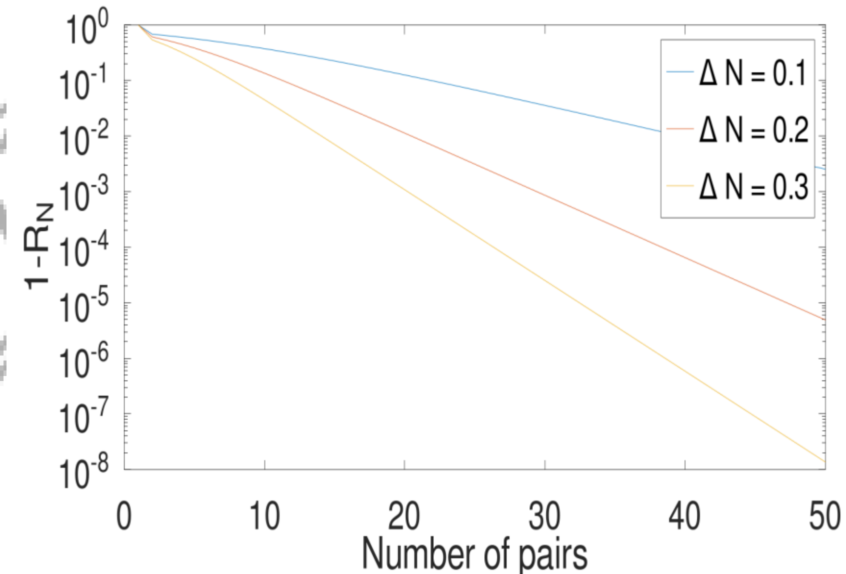


$$R_N = \left[\frac{n_1^{2N} - (n_0 / n_3) n_2^{2N}}{n_1^{2N} + (n_0 / n_3) n_2^{2N}} \right]^2$$

$$\frac{\Delta\lambda}{\lambda_o} \approx (4/\pi) \arcsin\left(\frac{n_1 - n_2}{n_1 + n_2} \right)$$

Periodic stack reflexion

- R increases rapidly with the number of pairs
- And depends also greatly with the index difference between stacks



Example: Dielectric Mirror

A dielectric mirror has quarter wave layers consisting of Ta_2O_5 with $n_H = 1.78$ and SiO_2 with $n_L = 1.55$ both at 850 nm, the central wavelength at which the mirror reflects light. The substrate is Pyrex glass with an index $n_s = 1.47$ and the outside medium is air with $n_0 = 1$. Calculate the maximum reflectance of the mirror when the number N of double layers is 4 and 12. What would happen if you use TiO_2 with $n_H = 2.49$, instead of Ta_2O_5? Consider the $N = 12$ mirror. What is the bandwidth and what happens to the reflectance if you interchange the high and low index layers? Suppose we use a Si wafer as the substrate, what happens to the maximum reflectance?

Solution

$n_0 = 1$ for air, $n_1 = n_H = 1.78$, $n_2 = n_L = 1.55$, $n_3 = n_s = 1.47$, $N = 4$. For 4 pairs of layers, the maximum reflectance R_4 is

$$R_4 = \left[\frac{(1.78)^{2(4)} - (1/1.47)(1.55)^{2(4)}}{(1.78)^{2(4)} + (1/1.47)(1.55)^{2(4)}}\right]^2 = 0.4 \text{ or } 40\%$$

Solution

$N = 12$. For 12 pairs of layers, the maximum reflectance R_{12} is

$$R_{12} = \left[\frac{(1.78)^{2(12)} - (1/1.47)(1.55)^{2(12)}}{(1.78)^{2(12)} + (1/1.47)(1.55)^{2(12)}} \right]^2 = 0.906 \text{ or } 90.6\%$$

Now use TiO_2 for the high-n layer with $n_1 = n_H = 2.49$,

$\boldsymbol{R_4} = \mathbf{94.0\%}$ and $\boldsymbol{R_{12}} = \mathbf{100\%}$ (to two decimal places).

The refractive index contrast is **important**. For the TiO_2-SiO_2 stack we only need 4 double layers to get roughly the same reflectance as from 12 pairs of layers of Ta_2O_5-SiO_2. If we interchange n_H and n_L in the 12-pair stack, *i.e.* $n_1 = n_L$ and $n_2 = n_H$, the Ta_2O_5-SiO_2 reflectance falls to 80.8% but the TiO_2-SiO_2 stack is unaffected since it is already reflecting nearly all the light.

Solution

We can only compare bandwidths $\Delta\lambda$ for "infinite" stacks (those with $R \approx$ 100%) For the TiO_2-SiO_2 stack

$$\Delta\lambda \approx \lambda_o (4/\pi)\arcsin\left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

$$\Delta\lambda \approx (850\ \text{nm})(4/\pi)\arcsin\left(\frac{2.49 - 1.55}{2.49 + 1.55}\right) = 254\,\text{nm}$$

For the Ta_2O_5-SiO_2 infinite stack, we get $\Delta\lambda$ =74.8 nm

As expected $\Delta\lambda$ is narrower for the smaller contrast stack

Thin film Stack: Penetration depth

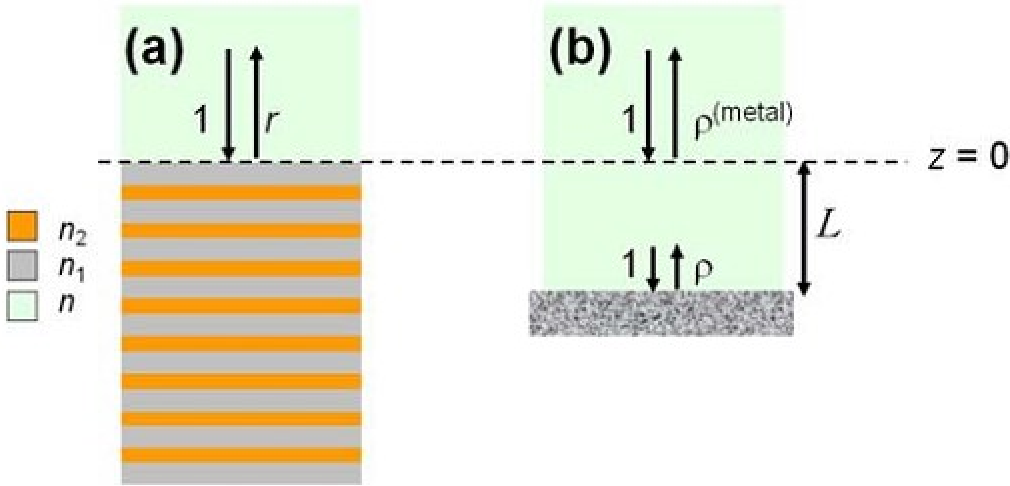
The reflectivity of a dielectric varies quickly with the wavelength as compared to a metallic mirror.
It is convenient to define the reflectivity as:

$$r(k_0) = |r|\exp(j\varphi)$$

Let's consider a metalic mirror with reflectivity ρ independent of the wavelength at a distance z =L

$$r^{(metal)} = \varrho\,\exp(2\varphi knL)$$

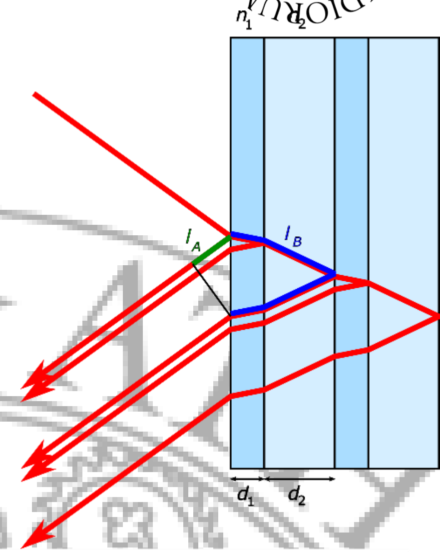
The exponential takes into account the delay due to the round trip to the mirror.



It is possible to derive that the 2 mirror are equivalent if:
And this distance L is called the **penetration depth**

$$L = \frac{\frac{d\phi}{d\lambda}}{2n}$$

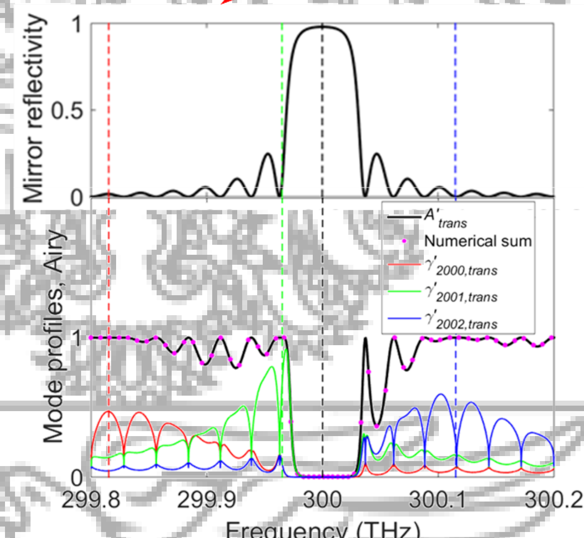
The Bragg mirror or a finite band gap



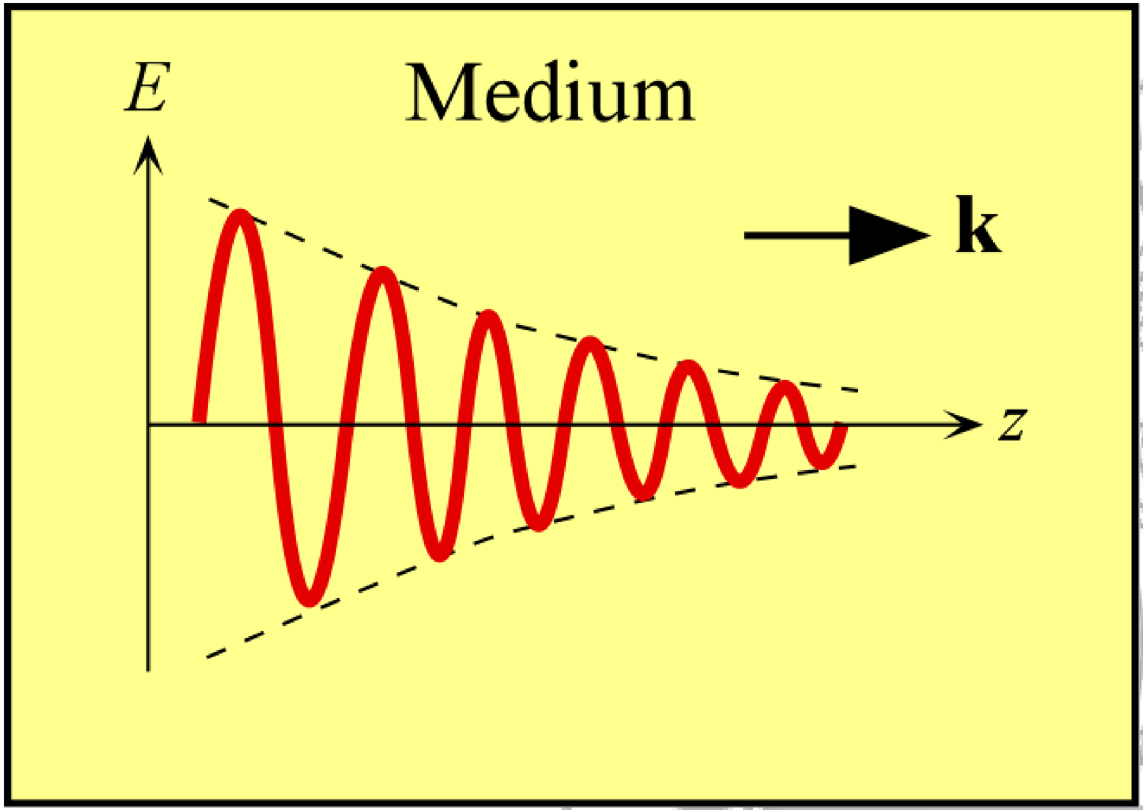
- Each interface between the two materials contributes with a Fresnel reflection.
- For the design wavelength, the optical path length difference between reflections from subsequent interfaces is half the wavelength;

all reflected components from the interfaces interfere constructively, which results in a **strong reflection (99,99%) compared to metals around 95 %.**

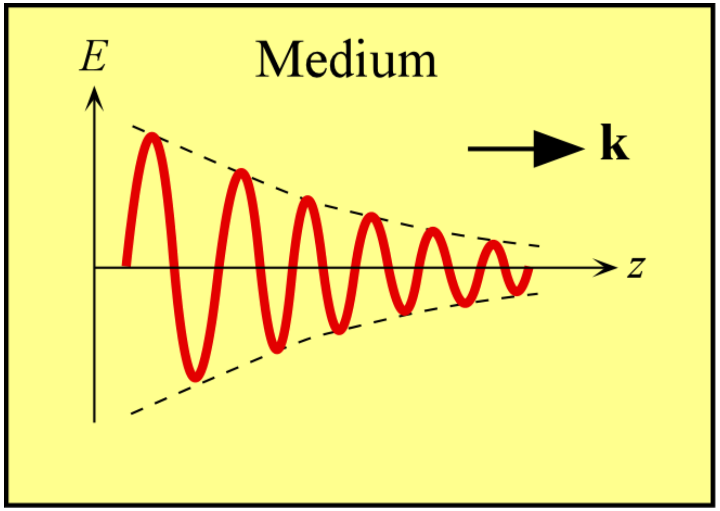
- The reflectivity achieved is determined by the number of layer pairs and by the refractive index contrast between the layer materials. (penetration depth)
- The reflection bandwidth is determined mainly by the index contrast.



Propagation in Complex Refractive Index



$$\alpha = -\frac{dI}{Idz}$$



Complex Refractive Index

Consider $k = k' - jk''$

$E = E_o \exp(-k''z) \exp j(\omega t - k'z)$

$I \propto |E|^2 \propto \exp(-2k''z)$

We know from EM wave theory
$\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ and $\mathrm{N} = \varepsilon_r^{1/2}$

$\mathrm{N} = n - jK = k/k_o = (1/k_o)[k' - jk'']$

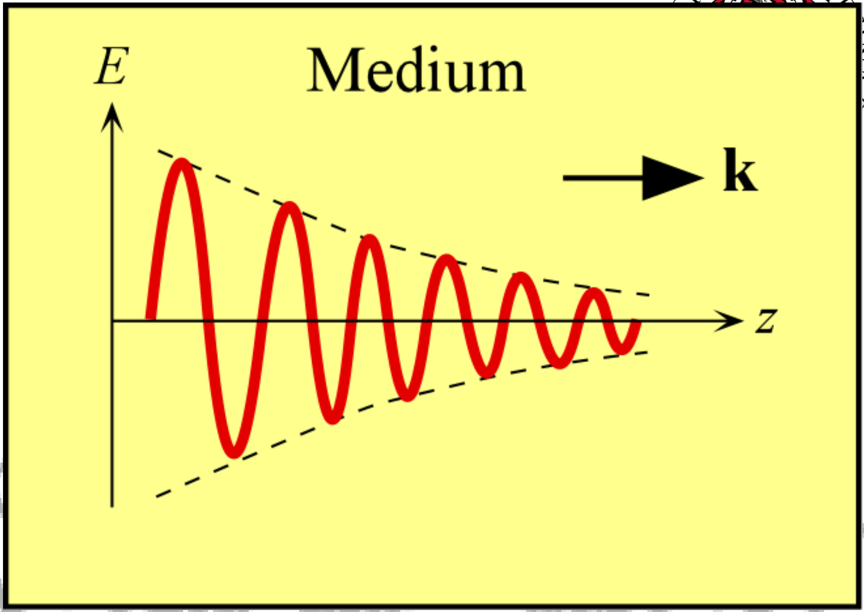
$$\mathrm{N} = n - jK = \sqrt{\varepsilon_r} = \sqrt{\varepsilon_r' - j\varepsilon_r''}$$

Reflectance

$\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ and $\mathrm{N} = \varepsilon_r^{1/2}$

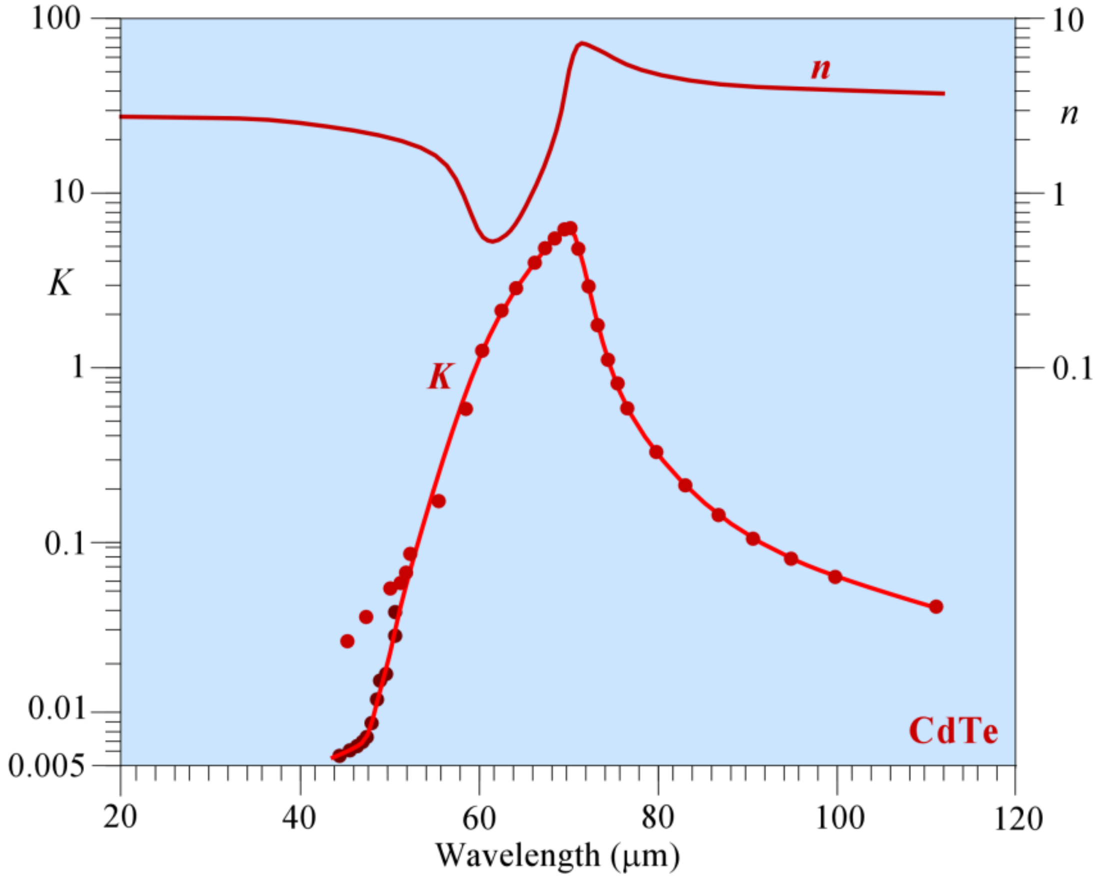
$\mathrm{N} = n - jK$

$\boldsymbol{n^2 - K^2 = \varepsilon_r'}$ and $\boldsymbol{2nK = \varepsilon_r''}$



$$R = \left| \frac{n - jK - 1}{n - jK + 1} \right|^2 = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2}$$

Complex Refractive Index for CdTe

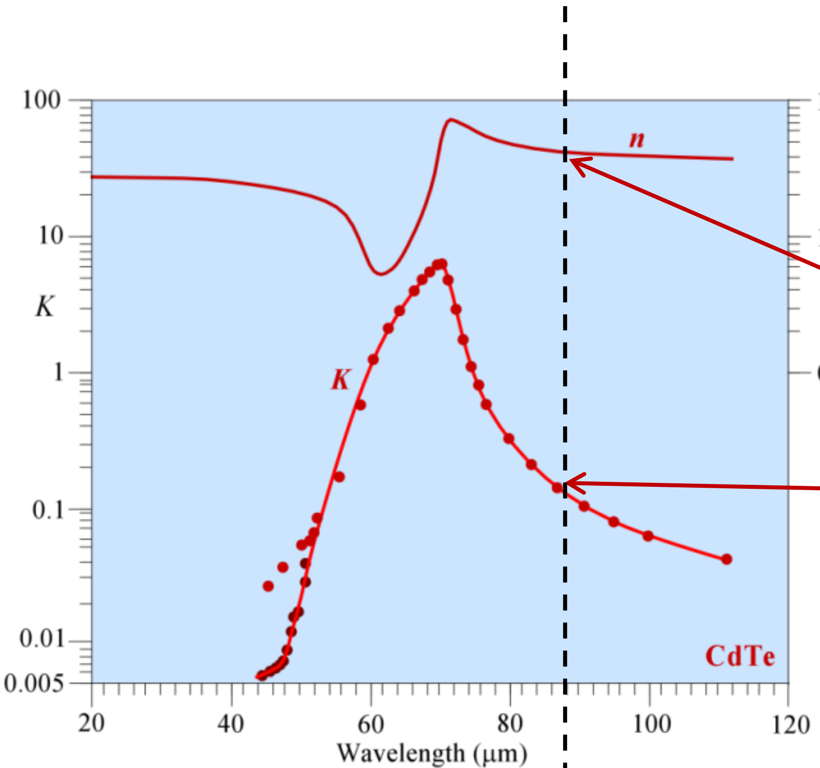


CdTe is used in various applications such as lenses, wedges, prisms, beam splitters, antireflection coatings, windows *etc* operating typically in the infrared region up to 25 μm. It is used as an optical material for low power CO_2 laser applications.

Complex Refractive Index

$$N = n - jK = \sqrt{\varepsilon_r} = \sqrt{\varepsilon_r' - j\varepsilon_r''}$$

$$n^2 - K^2 = \varepsilon_r' \quad \text{and} \quad 2nK = \varepsilon_r''$$



88 μm

$$R = \left| \frac{n - jK - 1}{n - jK + 1} \right|^2 = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2}$$

Example: Complex Refractive Index for CdTe

Calculate the absorption coefficient α and the reflectance R of CdTe at the Reststrahlen peak, and also at 50 μm. What is your conclusion?

Solution: At the Reststrahlen peak, $\lambda \approx 70$ μm, $K \approx 6$, and $n \approx 4$. The free-space propagation constant is

$$k_o = 2\pi/\lambda = 2\pi/(70\times10^{-6}\ \mathrm{m}) = 9.0\times10^{4}\ \mathrm{m}^{-1}$$

The absorption coefficient α is $2k$,

$$\alpha = 2k'' = 2k_oK = 2(9.0\times10^{4}\ \mathrm{m}^{-1})(6) = \mathbf{1.08\times10^{6}\ m^{-1}}$$

which corresponds to an **absorption depth** $1/\alpha$ of about 0.93 micron.

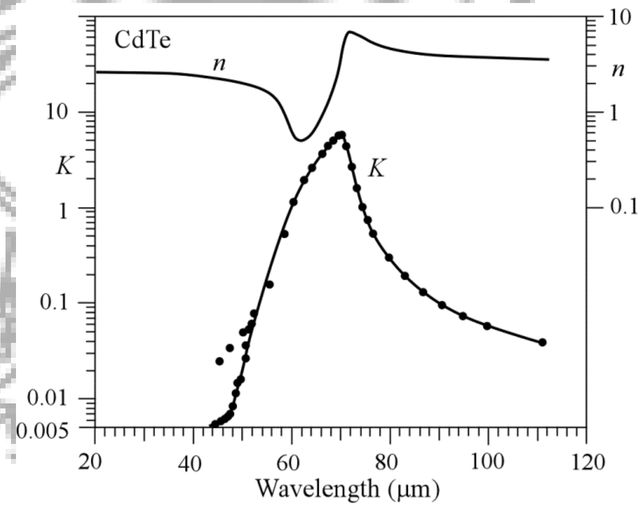
Solution continued: At the Reststrahlen peak, $\lambda \approx 70$ µm, $K \approx 6$, and $n \approx 4$, so that

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(4-1)^2 + 6^2}{(4+1)^2 + 6^2} \approx 0.74 \text{ or } 74\%$$

At $\lambda = 50$ µm, $K \approx 0.02$, and $n \approx 2$. Repeating the above calculations we get

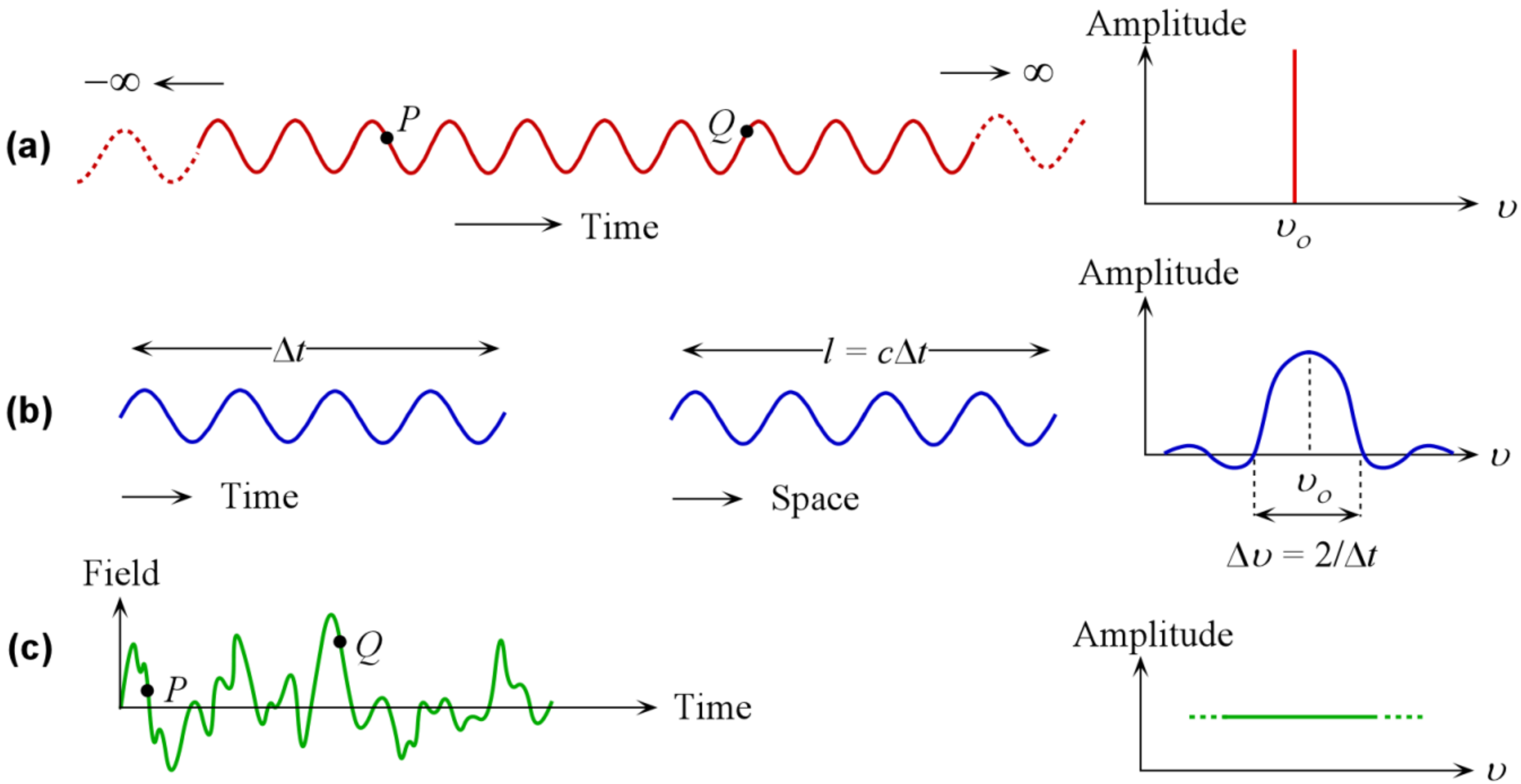
$$\boldsymbol{\alpha} = 5.0 \times 10^3 \text{ m}^{-1}$$

$$\boldsymbol{R} = 0.11 \text{ or } 11\ \%$$



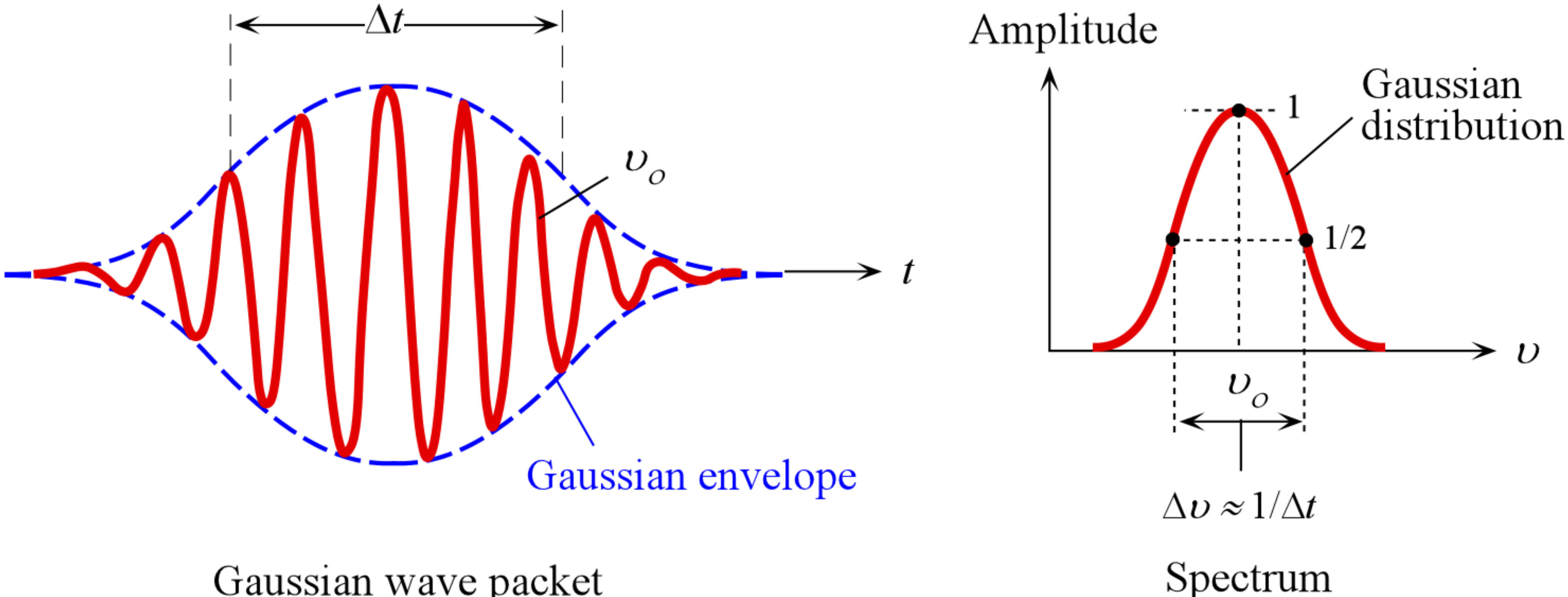
There is a sharp increase in the reflectance from 11 to 72% as we approach the Reststrahlen peak

Temporal and Spatial Coherence



(a) A sine wave is perfectly coherent and contains a well-defined frequency υ_o. (b) A finite wave train lasts for a duration Δt and has a length l. Its frequency spectrum extends over $\Delta\upsilon = 2/\Delta t$. It has a coherence time Δt and a coherence length λ. (c) White light exhibits practically no coherence.

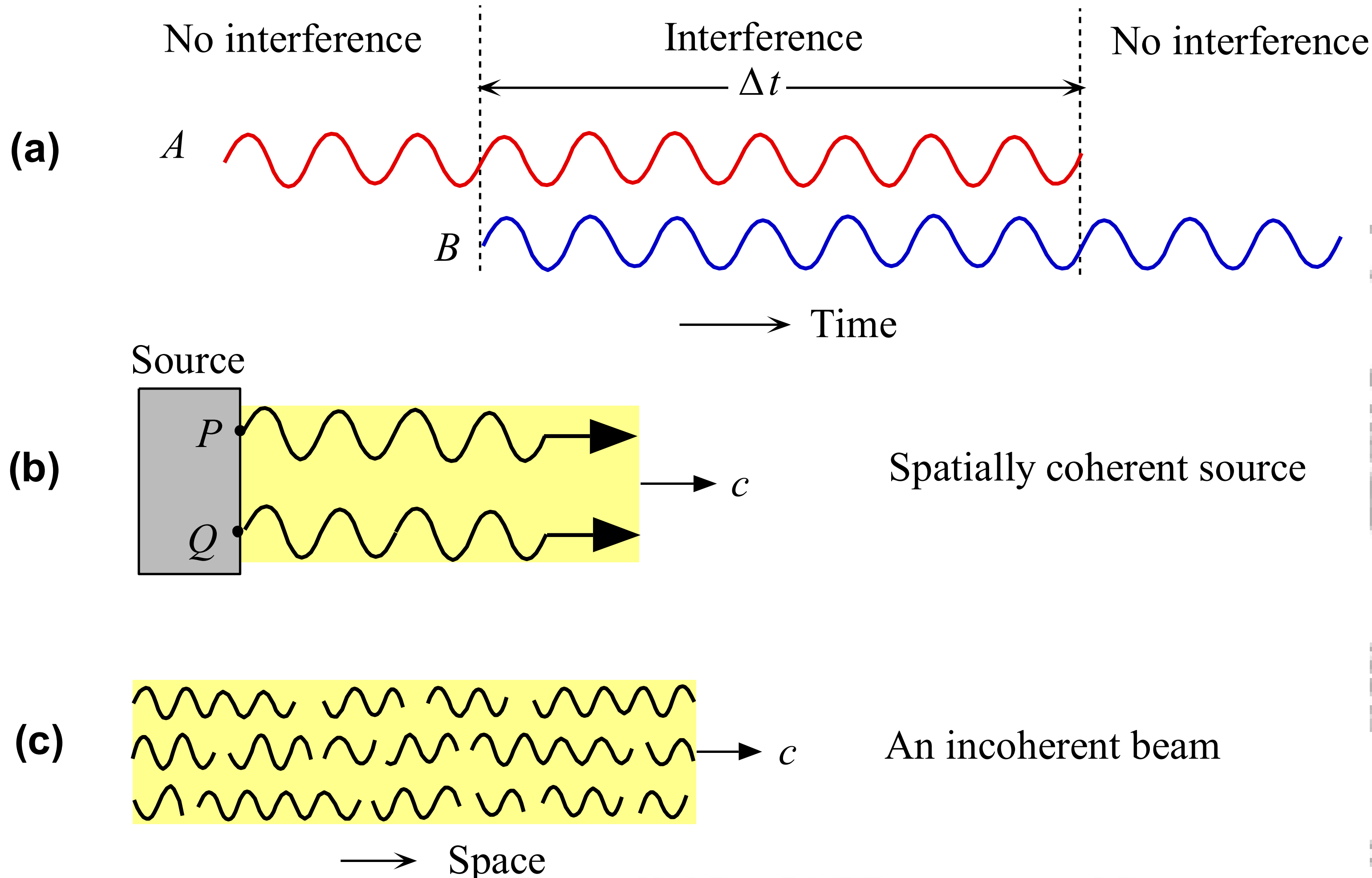
Temporal and Spatial Coherence



$$\Delta \upsilon \approx \frac{1}{\Delta t}$$

FWHM spreads

Temporal and Spatial Coherence



(a) Two waves can only interfere over the time interval Δt. (b) Spatial coherence involves comparing the coherence of waves emitted from different locations on the source. (c) An incoherent beam

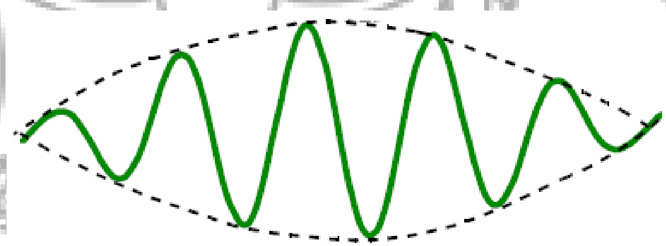
Temporal and Spatial Coherence

Δt = **coherence time**

$l = c\Delta t$ = **coherence length**

For a Gaussian light pulse

$$\Delta \upsilon \approx \frac{1}{\Delta t}$$



Spectral width

Pulse duration

Temporal and Spatial Coherence

Δt = **coherence time**

$l = c\Delta t$ = **coherence length**

$$\Delta\upsilon \approx \frac{1}{\Delta t}$$

Na lamp, orange radiation at 589 nm has spectral width $\Delta\upsilon \approx 5\times10^{11}$ Hz.

$\Delta t \approx 1/\ \Delta\upsilon = 2\times10^{-12}$ s or 2 ps,

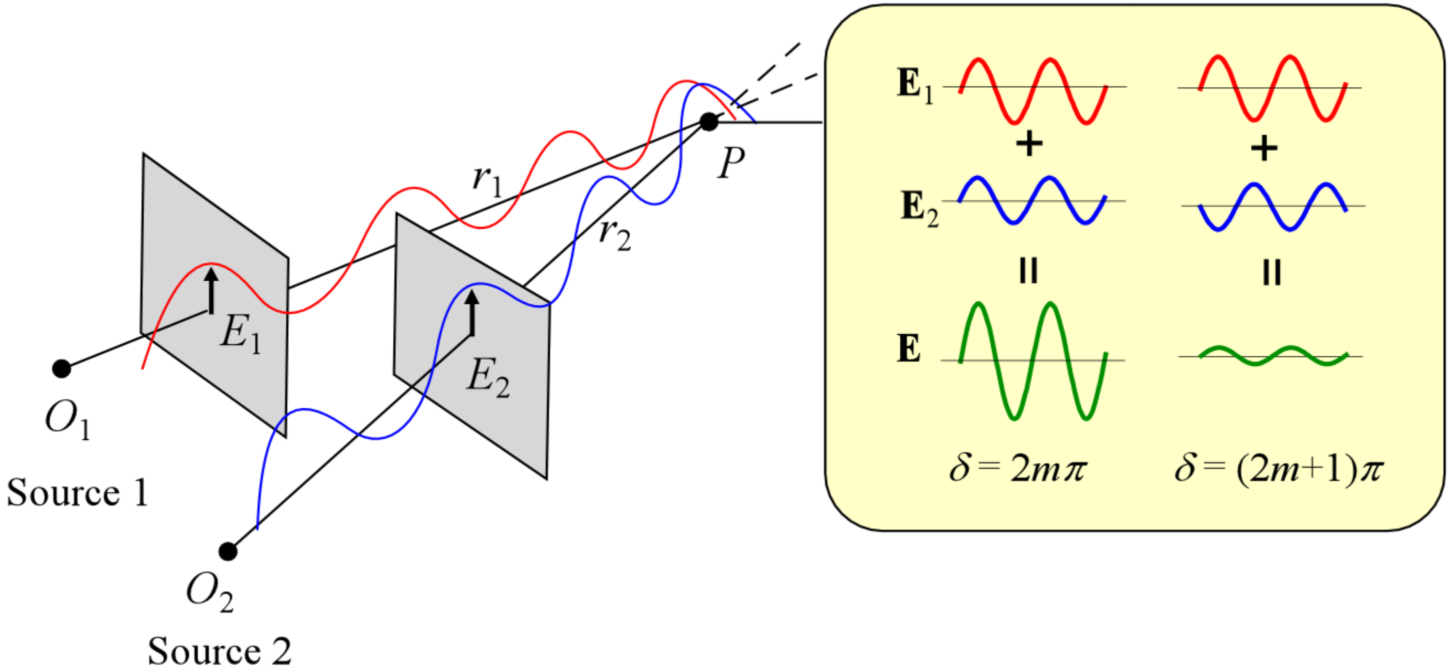
and its coherence length $l = c\Delta t$,

$l = 6\times10^{-4}$ m or 0.60 mm.

He-Ne laser operating in multimode has a spectral width around 1.5×10^{9} Hz, $\Delta t \approx 1/\Delta\upsilon = 1/1.5\times10^{9}$ s or 0.67 ns

$l = c\Delta t = 0.20$ m or 200 mm.

Interference

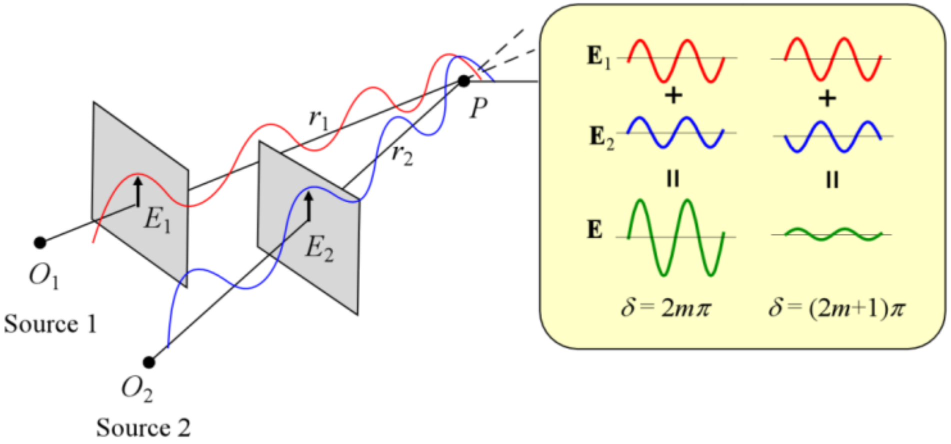


$\mathbf{E}_1 = \mathbf{E}_{o1}\sin(\boldsymbol{\omega t} - kr_1 - \phi_1)$ and $\mathbf{E}_2 = \mathbf{E}_{o2}\sin(\boldsymbol{\omega t} - kr_2 - \phi_2)$

Interference results in $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$

$$\overline{\mathbf{E}\cdot\mathbf{E}} = \overline{(\mathbf{E}_1 + \mathbf{E}_2)\cdot(\mathbf{E}_1 + \mathbf{E}_2)} = \overline{\mathbf{E}_1^2} + \overline{\mathbf{E}_2^2} + 2\overline{\mathbf{E}_1\mathbf{E}_2}$$

Interference



Resultant intensity I is

$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos\delta$$

$$\delta = \boxed{k(r_2 - r_1)} + (\phi_2 - \phi_1)$$

Phase difference due to optical path difference

Constructive interference

$$I_{max} = I_1 + I_2 + 2(I_1 I_2)^{1/2}$$

and

Destructive interference

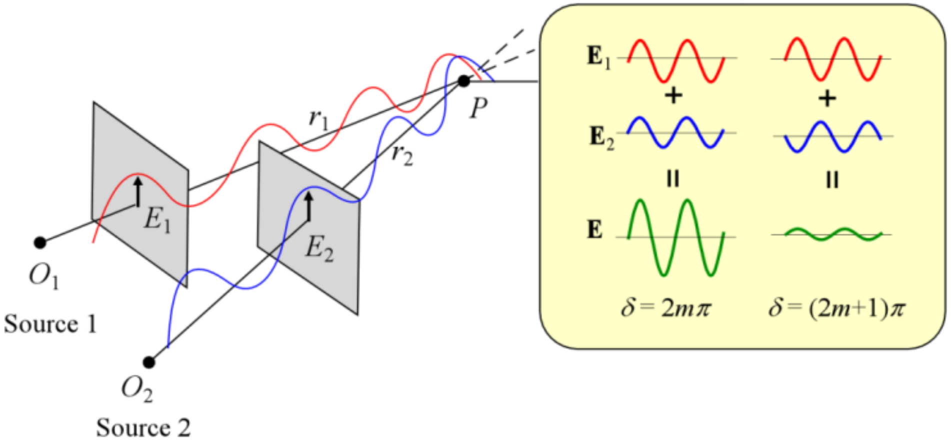
$$I_{min} = I_1 + I_2 - 2(I_1 I_2)^{1/2}$$

If the interfering beams have equal irradiances, then

$$I_{max} = 4I_1$$

$$I_{min} = 0$$

Interference



Resultant intensity I is

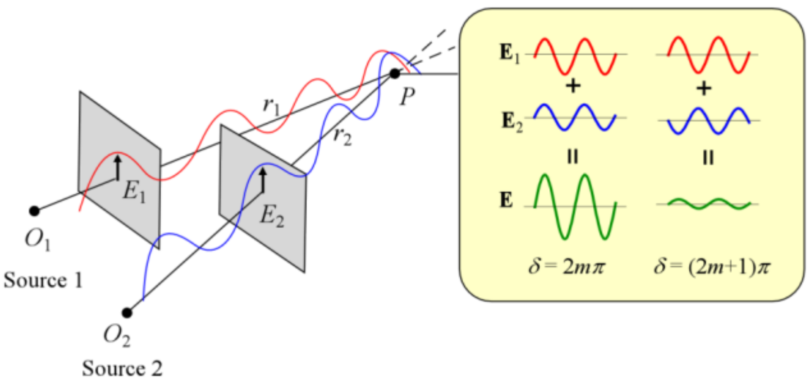
$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos\delta$$

$$\delta = k(r_2 - r_1) + (\phi_2 - \phi_1)$$

Interference between incoherent waves

$$I = I_1 + I_2$$

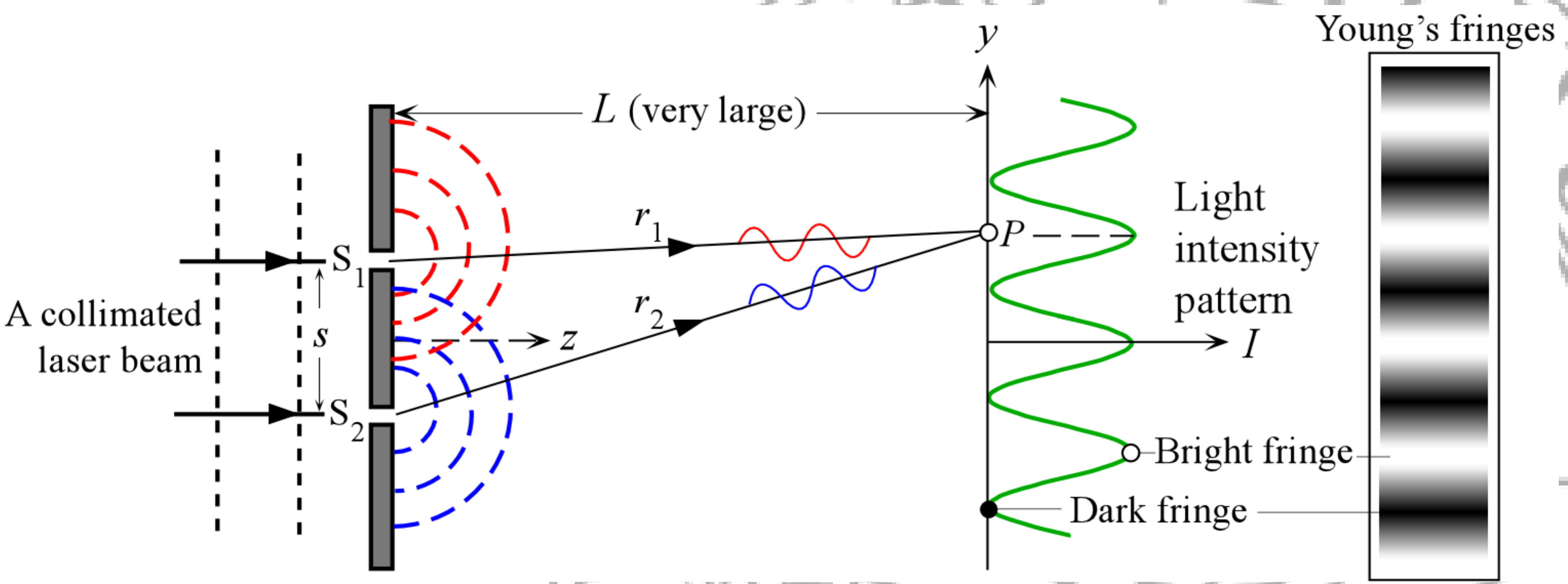
Interference



Resultant intensity I is

$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos\delta$$

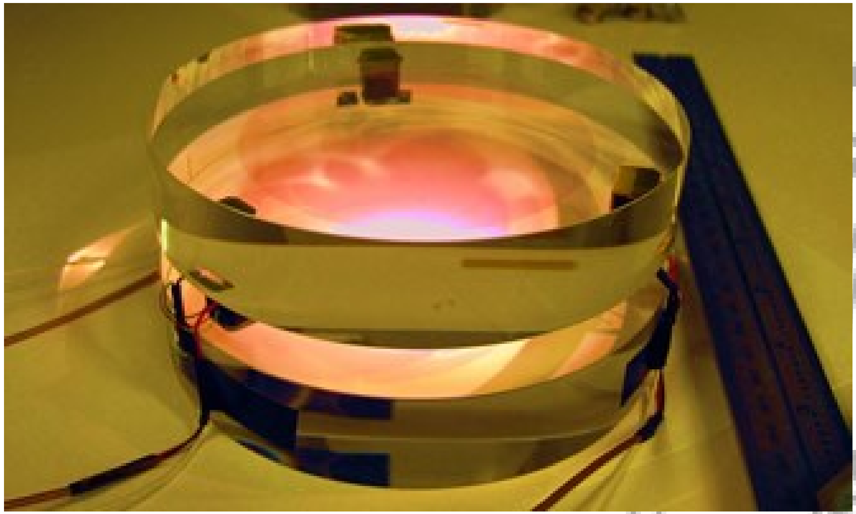
$$\delta = k(r_2 - r_1) + (\phi_2 - \phi_1)$$



Optical Resonator

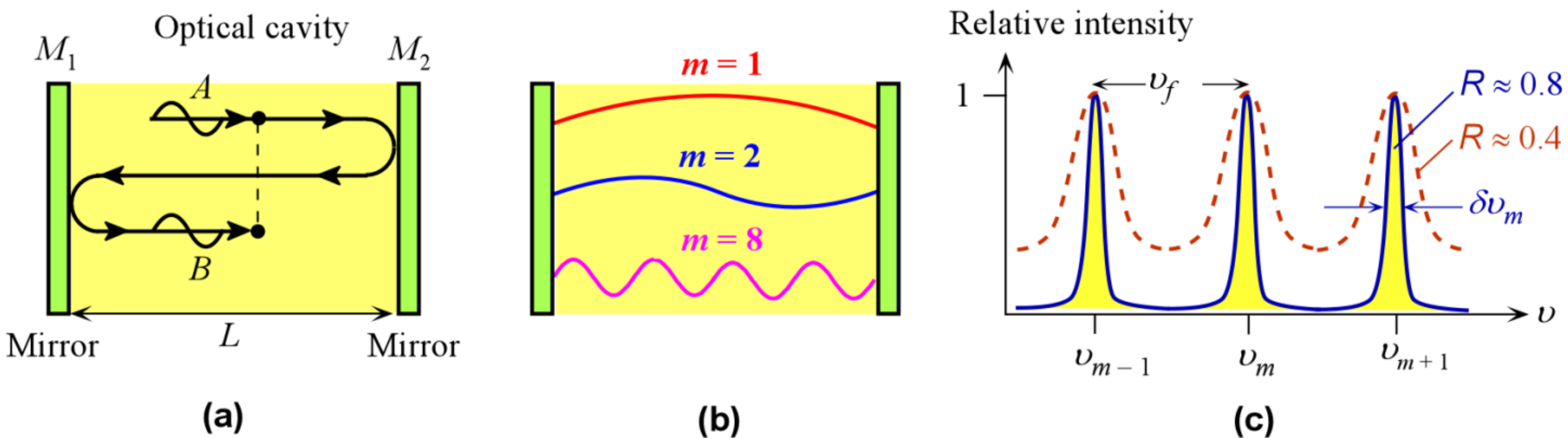
Fabry-Perot Optical Cavity

Charles Fabry (1867–1945), left, and Alfred Perot (1863–1925), right, were the first French physicists to construct an optical cavity for interferometry. *(Perot:* The Astrophysical Journal, *Vol. 64, November 1926, p. 208, courtesy of the American Astronomical Society. Fabry: Courtesy of Library of Congress Prints and Photographs Division, Washington, DC 20540, USA.)*



This is a tunable large aperture (80 mm) etalon with two end plates that act as reflectors. The end plates have been machined to be flat to $\lambda/110$. There are three piezoelectric transducers that can tilt the end plates and hence obtain perfect alignment. (Courtesy of Light Machinery)

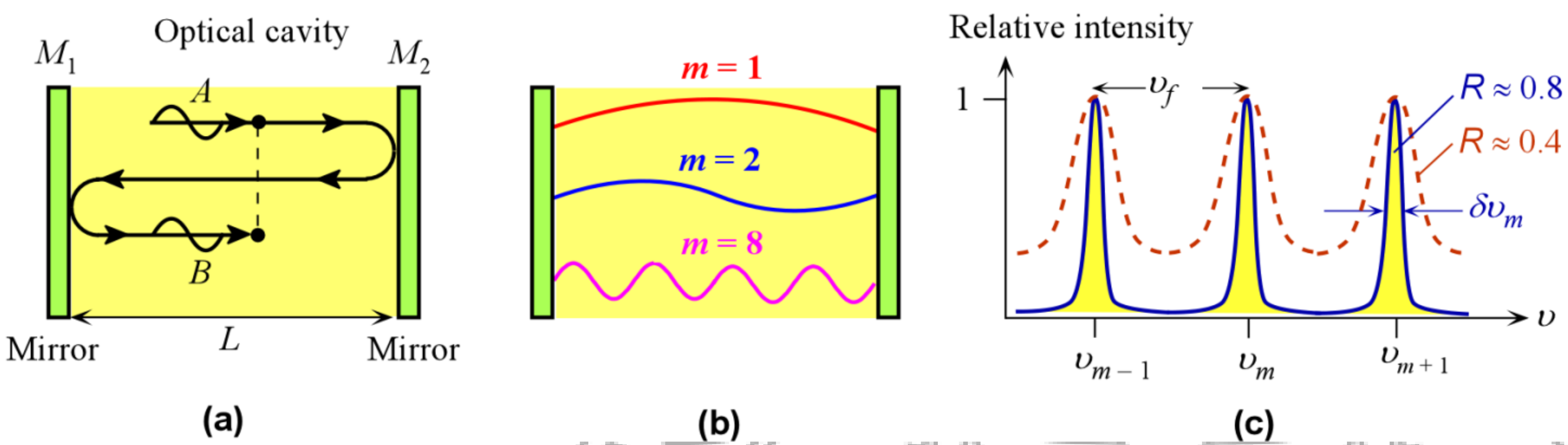
Optical Resonator
Fabry-Perot Optical Cavity



Schematic illustration of the Fabry-Perot optical cavity and its properties. (a) Reflected waves interfere. (b) Only standing EM waves, modes, of certain wavelengths are allowed in the cavity. (c) Intensity vs. frequency for various modes. *R* is mirror reflectance and lower *R* means higher loss from the cavity.

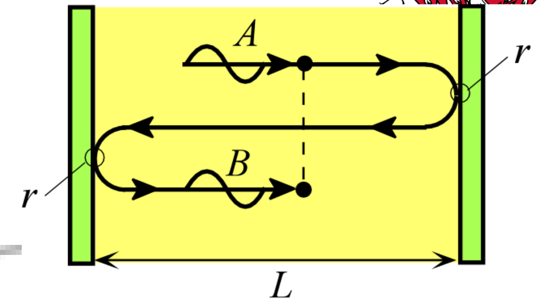
Note: The two curves are sketched so that the maximum intensity is unity

Optical Resonator
Fabry-Perot Optical Cavity



Each allowed EM oscillation is a cavity mode

Optical Resonator
Fabry-Perot Optical Cavity



$A + B = A + Ar^2\exp(-j2kL)$

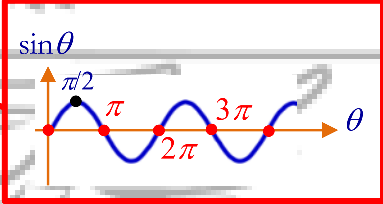
$E_{\text{cavity}} = A + B + \ldots = A + Ar^2\exp(-j2kL) + Ar^4\exp(-j4kL) + Ar^6\exp(-j6kL) + \ldots$

$$E_{\text{cavity}} = \frac{A}{1 - r^2\exp(-j2kL)}$$

$$I_{\text{cavity}} = \frac{I_o}{(1-R)^2 + 4R\sin^2(kL)}$$

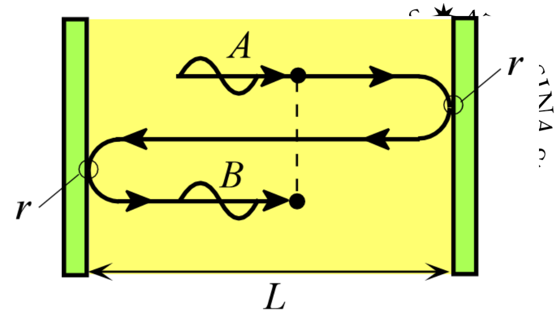
$$I_{\max} = \frac{I_o}{(1-R)^2}$$

Maxima at $k_m L = m\pi$



$m = 1,2,3,\ldots$integer

Optical Resonator
Fabry-Perot Optical Cavity



$$I_{\text{cavity}} = \frac{I_o}{(1-R)^2 + 4R\sin^2(kL)}$$

$$I_{\max} = \frac{I_o}{(1-R)^2}$$

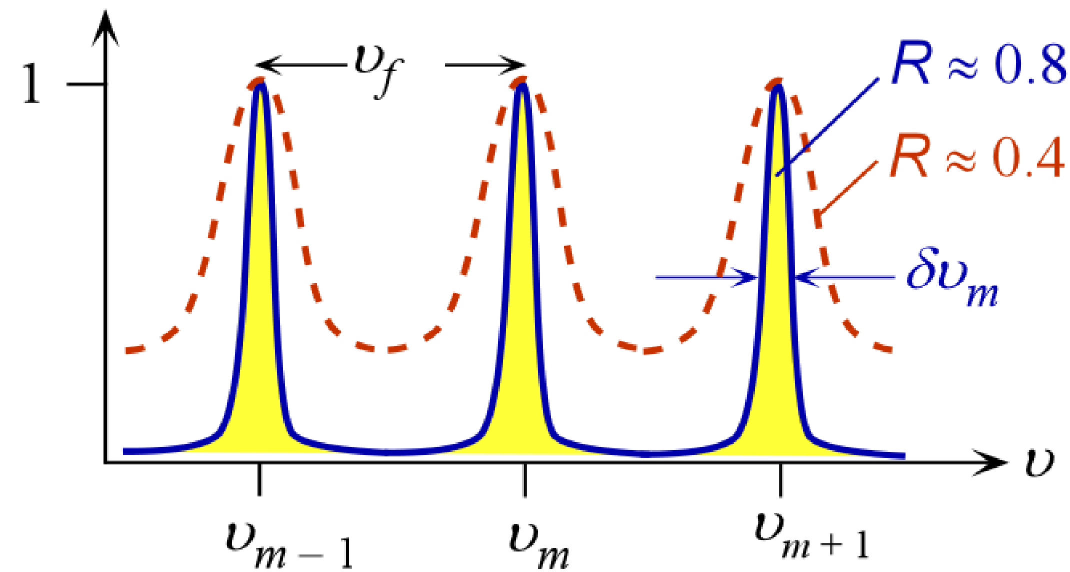
Maxima at $k_m L = m\pi$

$m = 1,2,3,\ldots$integer

$$(2\pi/\lambda_m)L = m\pi$$

$$\boxed{m(\lambda_m/2) = L}$$

Optical Resonator
Fabry-Perot Optical Cavity



$\upsilon_m = m(c/2L) = m\upsilon_f$ = Mode frequency

m = integer, 1,2,…

υ_f =free spectral range = $c/2L$ = Separation of modes

$$\delta\upsilon_m = \frac{\upsilon_f}{F}$$

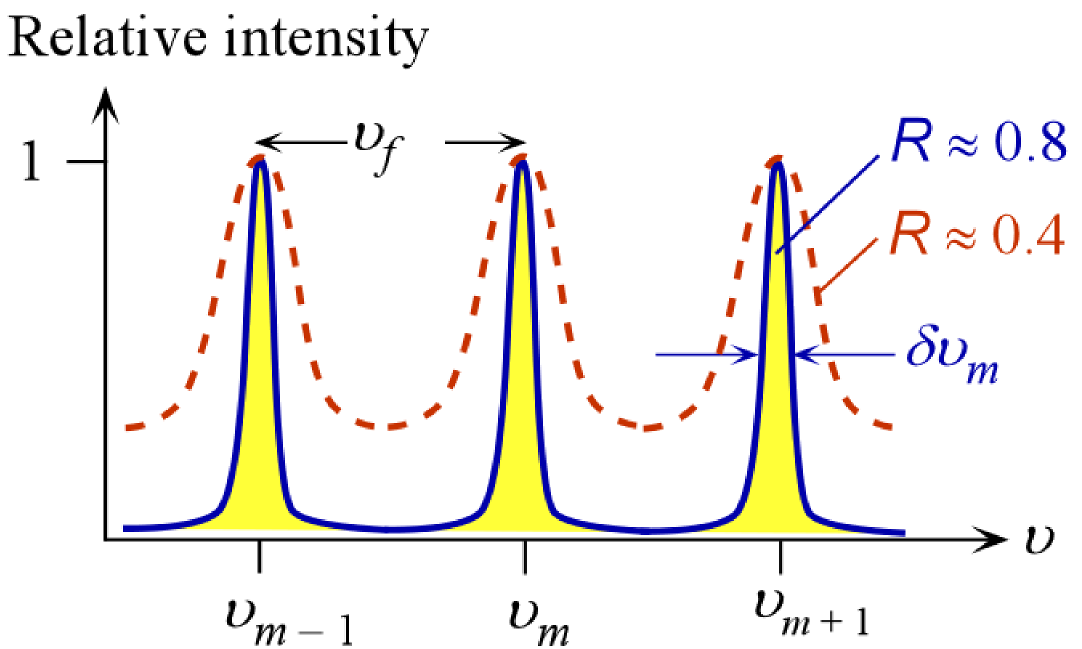
$$F = \frac{\pi R^{1/2}}{1-R}$$

Q: quality factor

F = Finesse

R = Reflectance (R > 0.6)

Optical Resonator
Fabry-Perot Optical Cavity



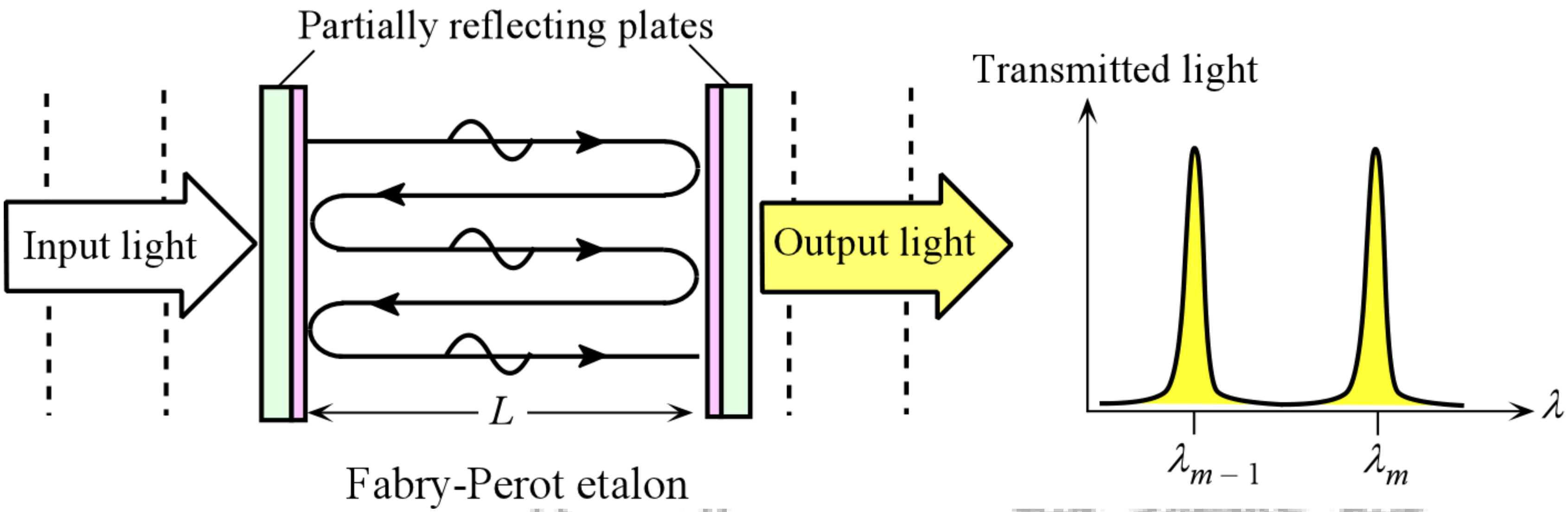
Quality factor *Q* is similar to the Finesse *F*

$$Q = \frac{\text{Resonant frequency}}{\text{Spectral width}} = \frac{\upsilon_m}{\delta\upsilon_m} = mF$$

Optical Resonator
Fabry-Perot Optical Cavity

Optical Resonator is also an optical filter

Only certain wavelengths (cavity modes) are transmitted

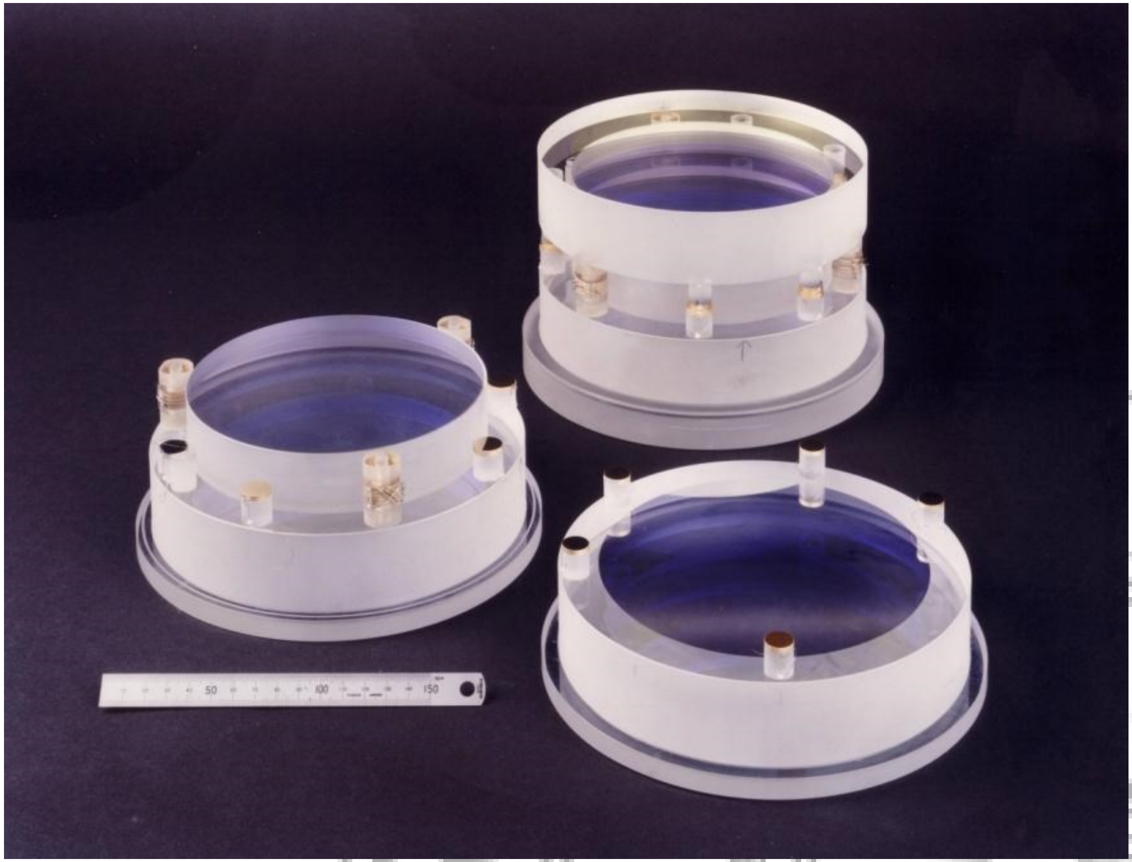


$$I_{\text{transmitted}} = I_{\text{incident}} \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(kL)}$$

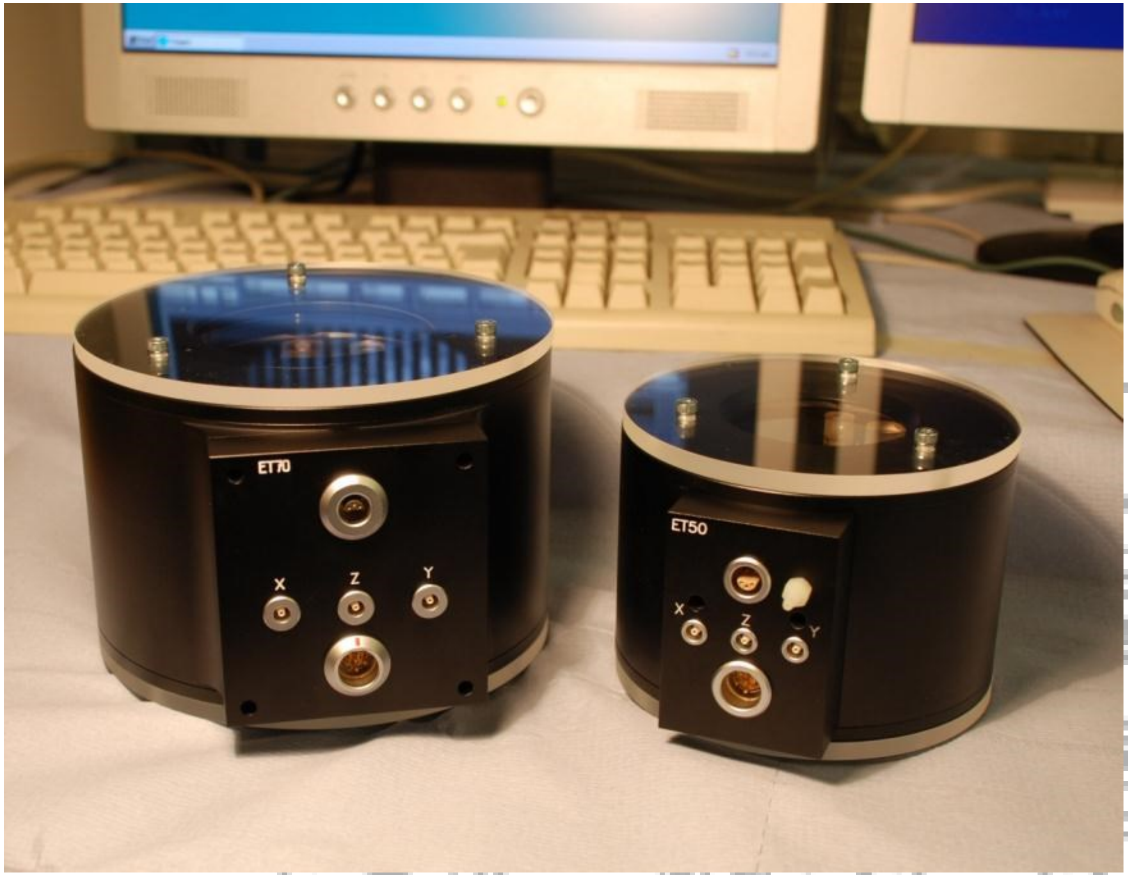
Optical Resonator
Fabry-Perot Optical Cavity

Fused silica etalon
(Courtesy of Light Machinery)

A 10 GHz air spaced etalon with 3 zerodur spacers. (Courtesy of Light Machinery)



Fabry-Perot etalons can be made to operate from UV to IR wavelengths with optical cavity spacings from a few microns to many centimeters
(Courtesy of IC Optical Systems Ltd.)



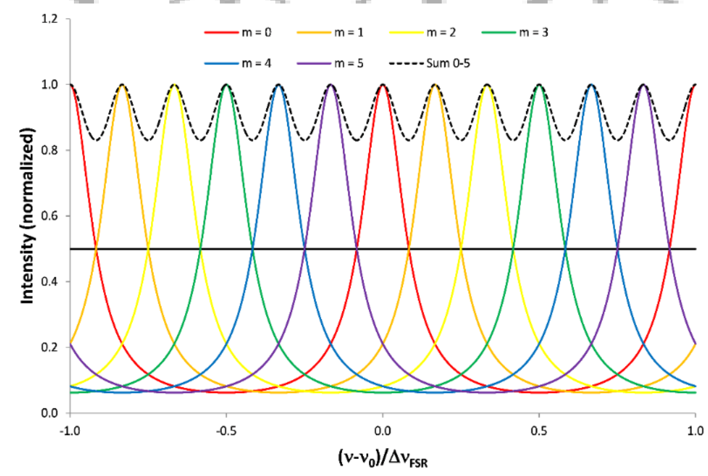
Piezoelectric transducer controlled Fabry-Perot etalons. Left has a 70 mm and the right has 50 mm clear aperture. The piezoelectric controller maintains the reflecting plates parallel while the cavity separation is scanned. (The left etalon has a reflection of interference fringes that are on the adjacent computer display)
(Courtesy of IC Optical Systems Ltd.)

A scanning Fabry-Perot interferometer (Model SA200), used as a spectrum analyzer, that has a free spectral range of 1.5 GHz, a typical finesse of **250**, spectral width (resolution) of 7.5 MHz. The cavity length is 5 cm. It uses two concave mirrors instead of two planar mirrors to form the optical cavity. A piezoelectric transducer is used to change the cavity length and hence the resonant frequencies. A voltage ramp is applied through the coaxial cable to the piezoelectric transducer to scan frequencies. (Courtesy of Thorlabs)

Scanning Fabry-Perot

- When the Fabry-Pérot resonator is used as a scanning interferometer, i.e., at varying resonator length (or angle of incidence), one can spectroscopically distinguish spectral lines at different frequencies within one FSR.
- Several Airy distributions , each one created by an individual spectral line, must be resolved. Therefore, the Airy distribution becomes the underlying fundamental function and the measurement delivers a sum of Airy distributions.
- The parameters that properly quantify this situation are the Airy linewidth and the Airy finesse .
- The FWHM linewidth of the Airy distribution is

$$\Delta\nu_{Airy} = \Delta\nu_{FSR}\frac{2}{\pi}\arcsin\left(\frac{1-\sqrt{R_1R_2}}{2\sqrt[4]{R_1R_2}}\right)$$



Fabry-Perot: Reflection needed

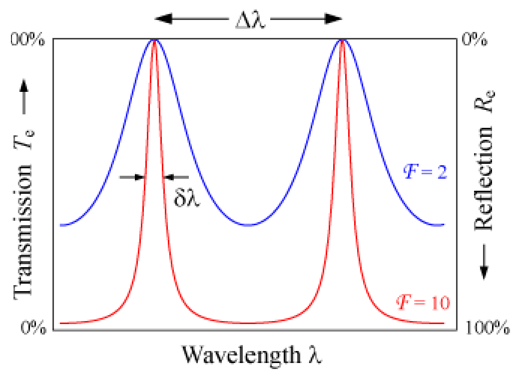
What are the requirements for the reflection:

If $R_1 = R_2 = R$ ➔ $\mathcal{F} = \frac{4R}{(1-R)^2}$

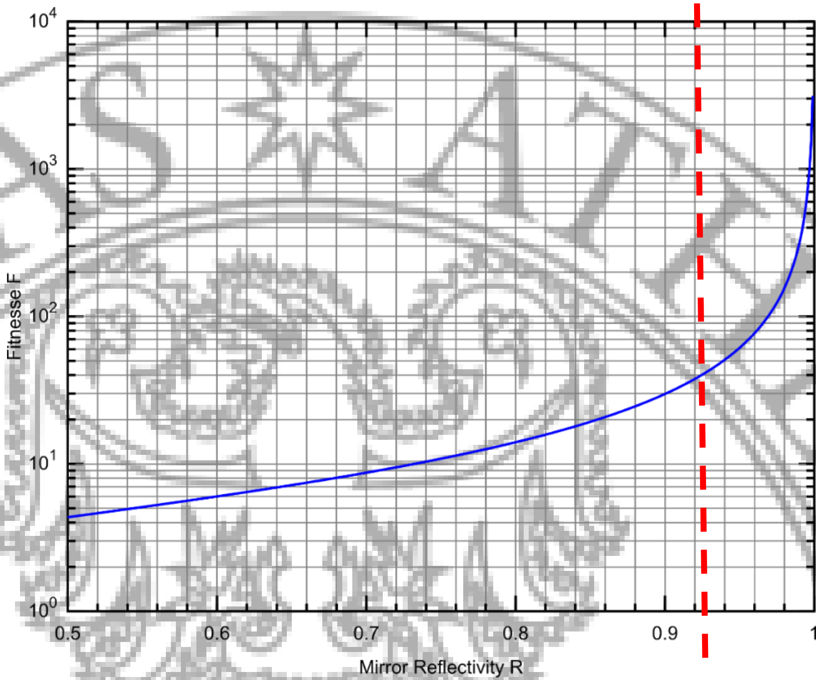
For finesse above 100 only multilayer dielectric mirrors can be used

$$\Delta\lambda = \frac{\lambda_0^2}{2n_g \ell \cos\theta + \lambda_0} \approx \frac{\lambda_0^2}{2n_g \ell \cos\theta}$$

$$T(\lambda) = \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2\left(\frac{2\pi n \ell}{\lambda}\right)}$$



Metal mirror limit



Consider a Fabry-Perot optical cavity in air of length 100 microns with mirrors that have a reflectance of 0.90. Calculate the cavity mode nearest to the wavelength 900 nm, and corresponding wavelength. Calculate the separation of the modes, the finesse, the spectral width of each mode and the *Q*-factor

Solution

Find the mode number m corresponding to 900 nm and then take the integer

$$m = \frac{2L}{\lambda} = \frac{2(100\times10^{-6})}{(900\times10^{-9})} = 222.2 \qquad \lambda_m = \frac{2L}{m} = \frac{2(100\times10^{-6})}{(222)} = 900.9 \text{ nm}$$

Thus, $m = 222$ (must be an integer)

$$\lambda_m = 900.90 \text{ nm} \approx 900 \text{ nm (very close)}$$

The frequency corresponding to λ_m is

$$\upsilon_m = c/\lambda_m = (3\times10^8)/(900.9\times10^{-9}) = 3.33\times10^{14}\text{ Hz}$$

Solution: Continued

Example: An Optical Resonator in Air

$\upsilon_f = c/2L =$ separation of modes

$= (3\times10^8) / [2(100\times10^{-6})] = 1.5\times10^{12}$ Hz.

$$F = \frac{\pi R^{1/2}}{1-R} = \frac{\pi 0.90^{1/2}}{1-0.90} = 29.8 \qquad \delta\upsilon_m = \frac{\upsilon_f}{F} = \frac{1.5\times10^{12}}{29.8} = 50.3\,\text{GHz}$$

$$\delta\lambda_m = \left|\delta\left(\frac{c}{\upsilon_m}\right)\right| = \left|-\frac{c}{\upsilon_m^2}\right|\delta\upsilon_m = \frac{(3\times10^8)}{(3.33\times10^{14})^2}(5.03\times10^{10}) = 0.136\,\text{nm}$$

The Q-factor is

$Q = mF = (222)(29.8) = 6.6\times10^3$

Consider a Fabry-Perot optical cavity of a semiconductor material of length 250 microns with mirrors, each with a reflectance of 0.90. Calculate the cavity mode nearest to 1310 nm. Calculate the separation of the modes, finesse, the spectral width of each mode, and the Q-factor. Take n = 3.6 for the semiconductor medium.

Solution

$$\textit{Given},\ \ L = 250\times10^{-6}\ \text{m},\ \ n = 3.6,\ R = 0.90$$

$$\Delta\upsilon_m = \upsilon_f \ = c/2nL = \text{Separation of modes} = 1.67\times10^{11}\ \text{Hz}$$

$$F = \frac{\pi R^{1/2}}{1-R} = \frac{\pi 0.9^{1/2}}{1-0.9} = 29.8$$

$$\delta\upsilon_m = \frac{\upsilon_f}{F} = \frac{1.67\times10^{11}}{29.8} = 5.59\,\text{GHz}$$

Example: Semiconductor Optical Cavity

Solution: Continued

Mode number m corresponding to 1310 nm is

$$m = \frac{2nL}{\lambda} = \frac{2(3.6)(250\times10^{-6})}{(1310\times10^{-9})} = 1374.05$$

which must be an integer (1374) so that the actual mode wavelength is

$$\lambda_m = \frac{2nL}{m} = \frac{2(3.6)(250\times10^{-6})}{(1374)} = 1310.04 \text{ nm}$$

For all practical purposes the mode wavelength is 1310 nm

Mode frequency is

$$\upsilon_m = \frac{c}{\lambda_m} = \frac{(3\times10^{8})}{(1310\times10^{-9})} = 2.3\times10^{14} \text{ Hz}$$

Example: Semiconductor Optical Cavity

Solution: Continued

Spectral width of a mode in wavelength is

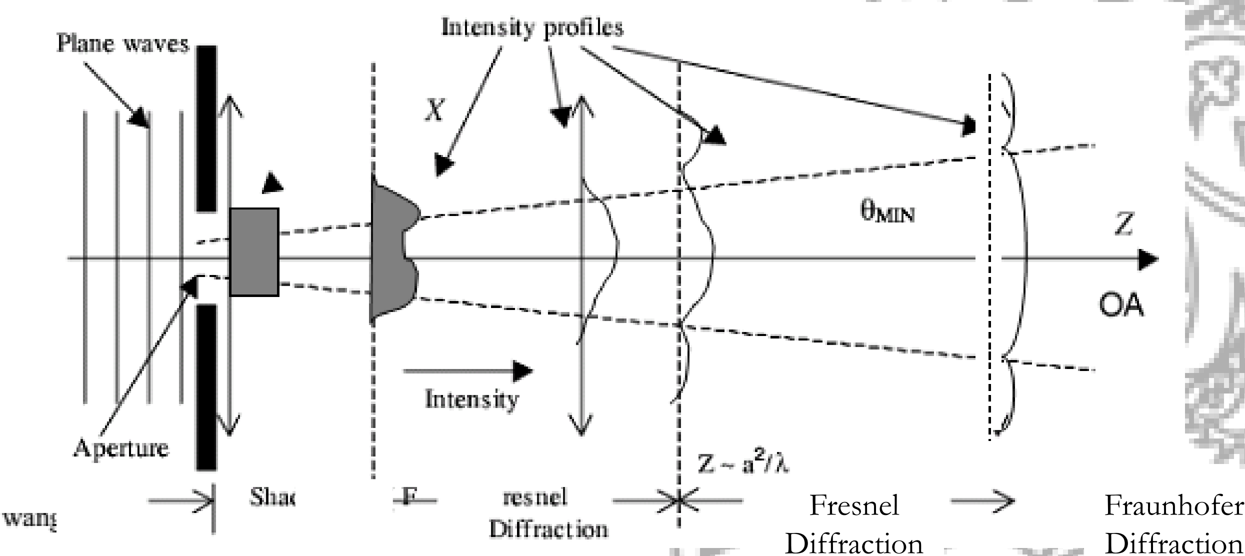
$$\delta\lambda_m = \left|\delta\left(\frac{c}{\upsilon_m}\right)\right| = \left|-\frac{c}{\upsilon_m^2}\right|\delta\upsilon_m = \frac{(3\times 10^8)}{(3.33\times 10^{14})^2}(5.03\times 10^{10}) = 0.136\text{ nm}$$

The Q-factor is

$$Q = mF = (1374)(29.8) = 4.1\times 10^4$$

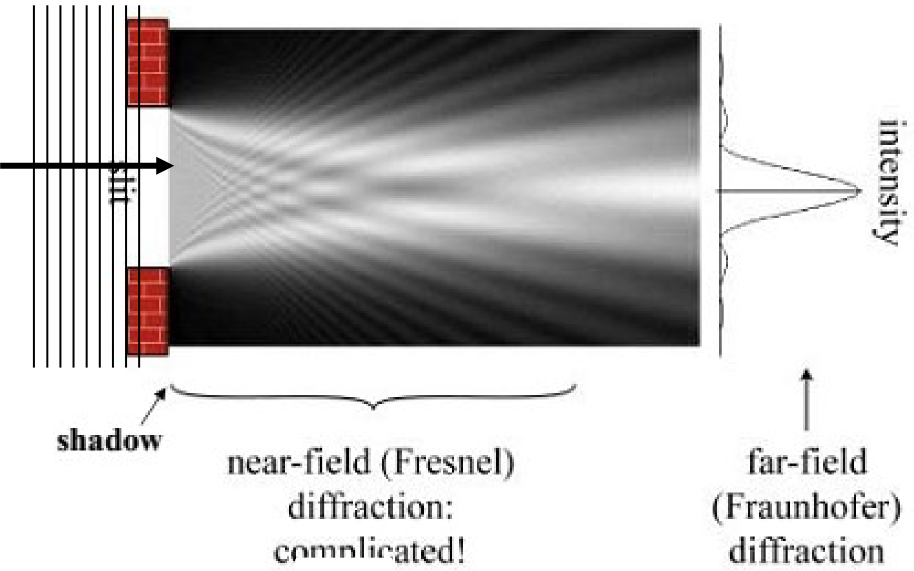
Basics of diffraction

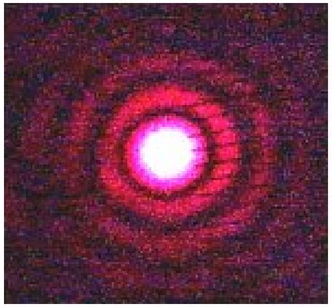
Diffraction refers to the spread of waves and appearance of fringes that occur when a wave front is constricted by an opaque obstacle.
The light pattern changes as you move away from the obstacle, being characterized in general by three regions



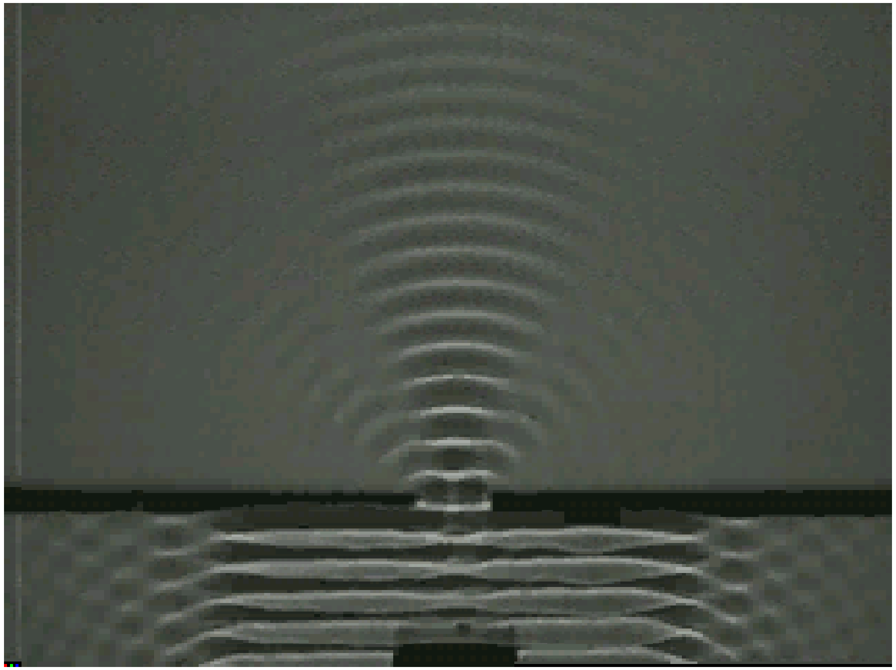
Basics of diffraction

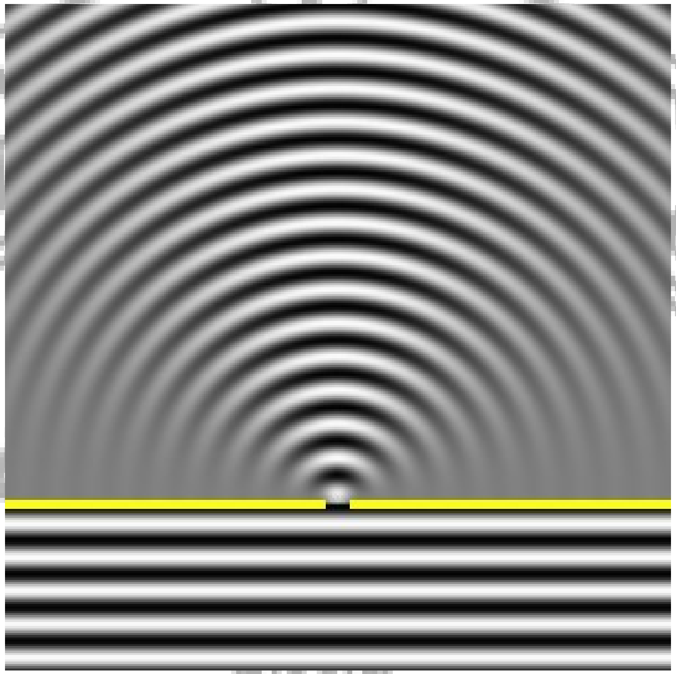
The intensity pattern behind a narrow single slit under uniform monochromatic illumination looks something like:



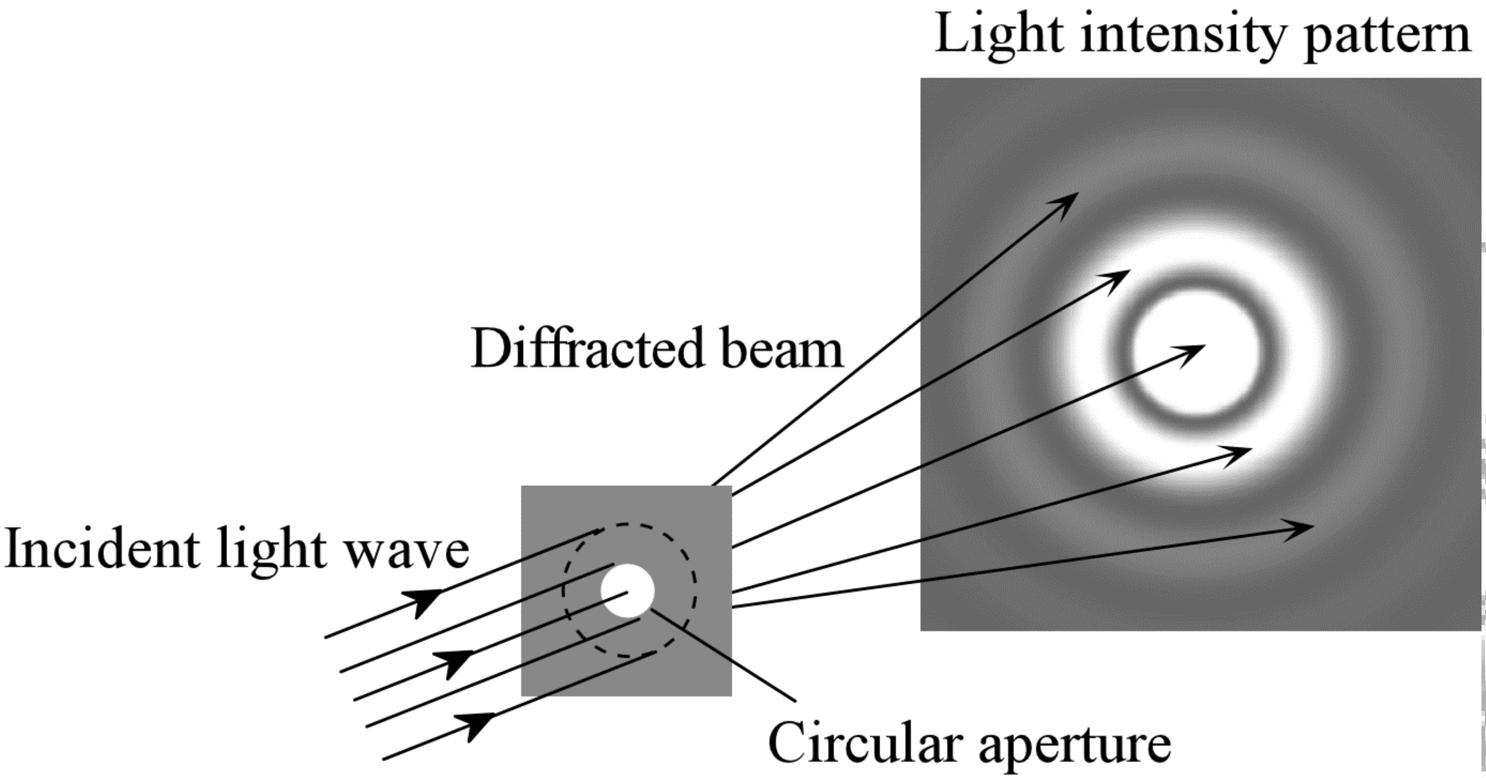


The Tank experiment





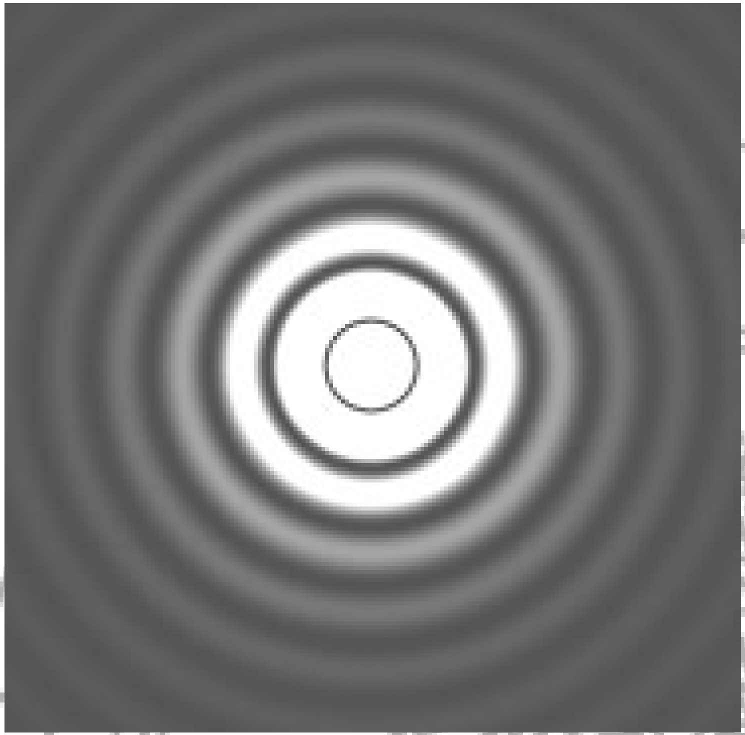
Basics of diffraction



A light beam incident on a small circular aperture becomes diffracted and its light intensity pattern after passing through the aperture is a diffraction pattern with circular bright rings (called Airy rings). If the screen is far away from the aperture, this would be a Fraunhofer diffraction pattern. (Diffraction image obtained by SK)

Basics of diffraction

Diffraction from a Circular Aperture



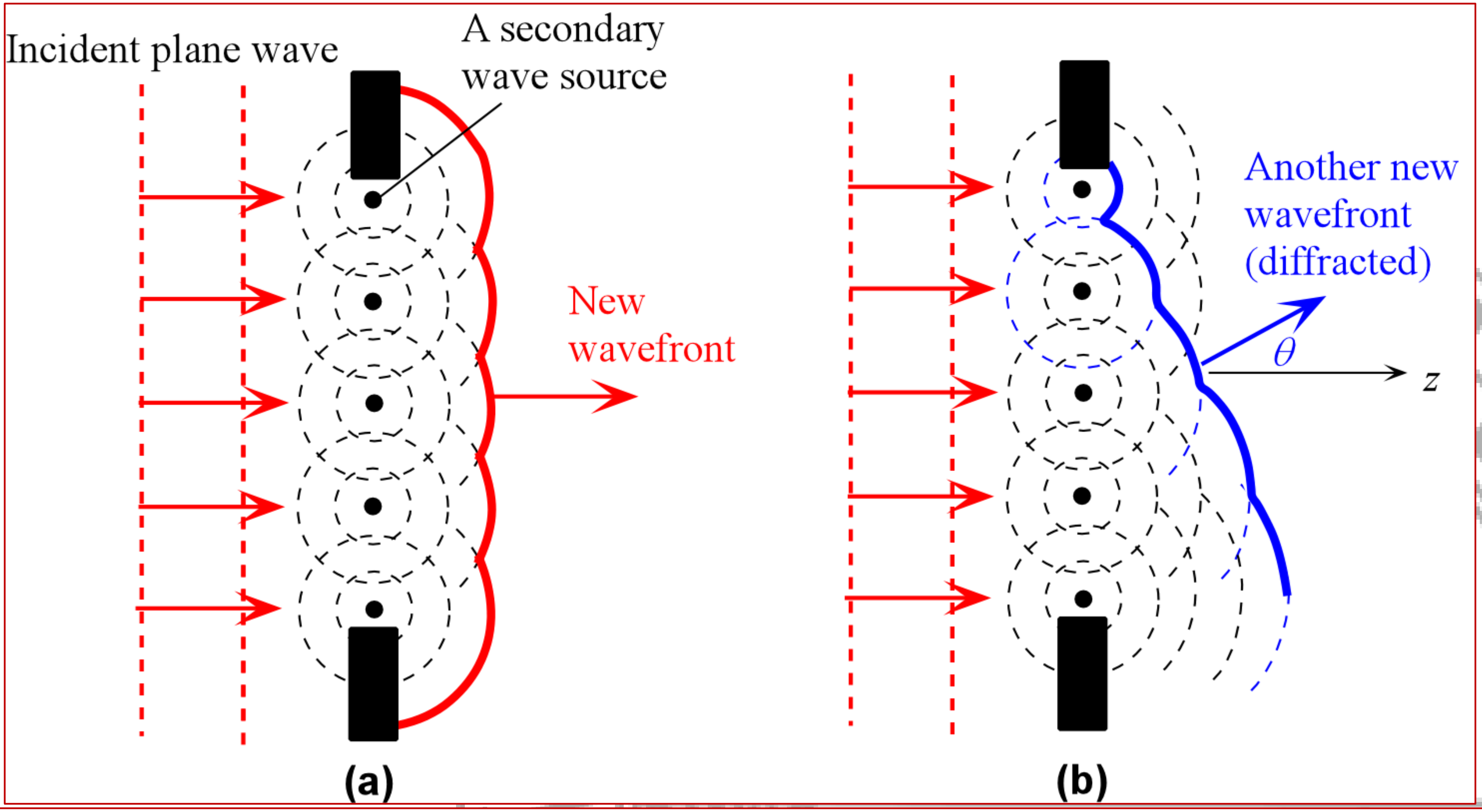
A light beam incident on a small circular aperture becomes diffracted and its light intensity pattern after passing through the aperture is a diffraction pattern with circular bright rings (called Airy rings). If the screen is far away from the aperture, this would be a Fraunhofer diffraction pattern. (Image obtained by SK. Overexposed to highlight the outer rings)

Basics of diffraction

Huygens-Fresnel principle

Every unobstructed point of a wavefront, at a given instant in time, serves as a source of spherical secondary waves (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases)

Basics of diffraction



(a) Huygens-Fresnel principle states that each point in the aperture becomes a source of secondary waves (spherical waves). The spherical wavefronts are separated by λ. The new wavefront is the envelope of the all these spherical wavefronts. (b) Another possible wavefront occurs at an angle θ to the z-direction which is a diffracted wave.

Superposition principle

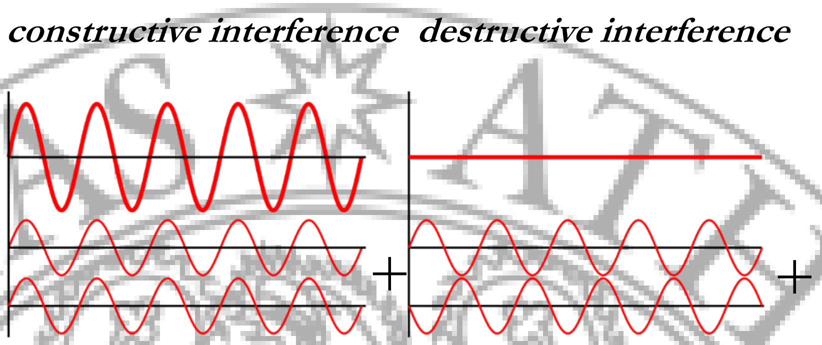
When two or more waves traverse the same space, the net amplitude at each point is the sum of the complex amplitudes of the individual waves.

In some cases, such as in noise-cancelling headphones, the summed variation has a smaller amplitude than the component variations;

- this is called ***destructive interference***.

In other cases, such as in a line array, the summed variation will have a bigger amplitude than any of the components individually;

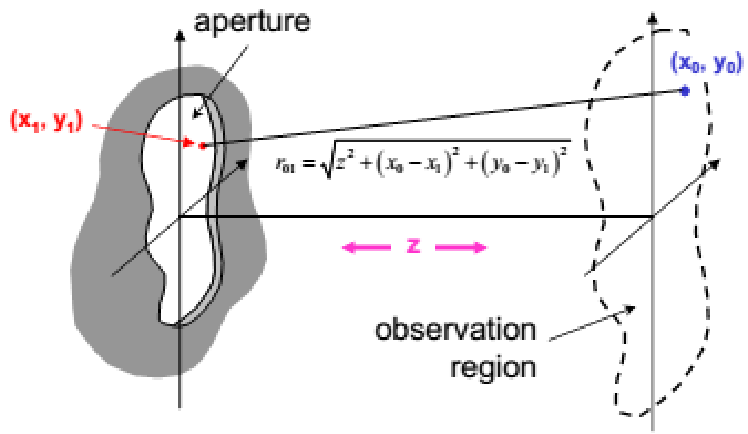
- this is called ***constructive interference***.



Kirchoff-Fresnel Diffraction Integral

Coordinates:

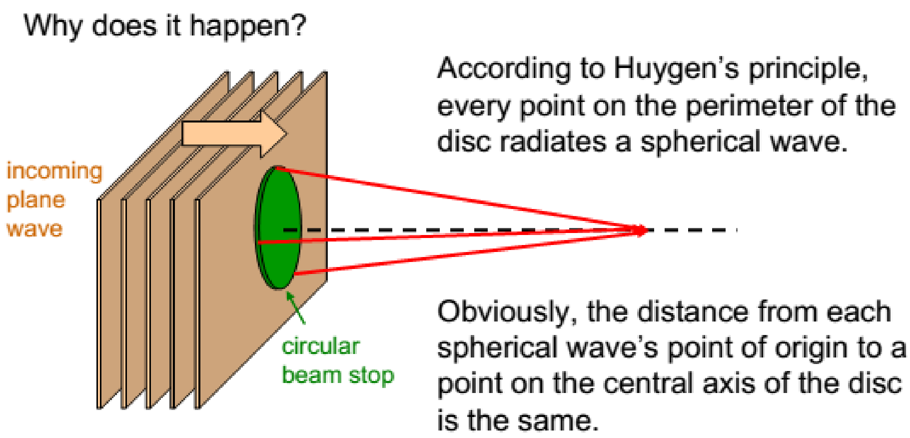
- the plane of the aperture: x_1, y_1
- the plane of observation: x_0, y_0
  (a distance z downstream)



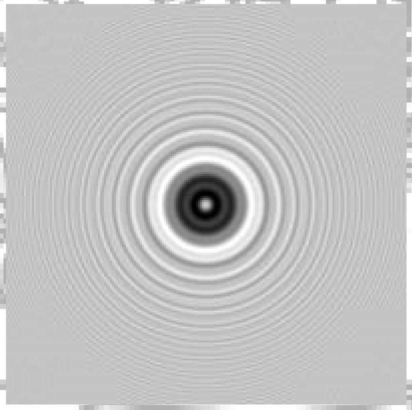
$$E(x_0, y_0) \propto \iint \exp\left\{ jk \left[\frac{(x_1^2 - 2x_0 x_1)}{2z} + \frac{(y_1^2 - 2y_0 y_1)}{2z} \right] \right\} Aperture(x_1, y_1) E(x_1, y_1) dx_1 dy_1$$

Quadratics in the exponent: a messy integral

One peculiar example: the Arago Spot



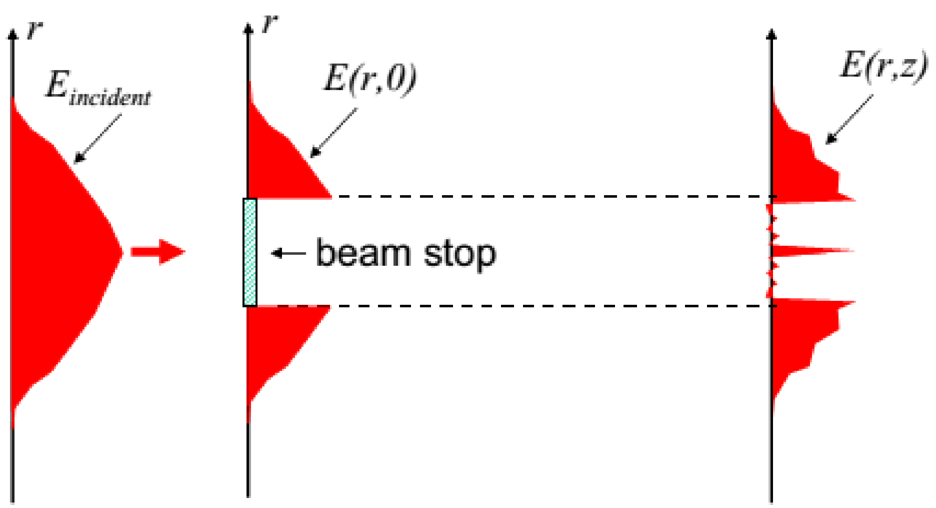
- In 1818, Poisson used Fresnel's theory to predict this phenomenon. He regarded this as proof that Fresnel's wave theory was nonsense, and that light must be a particle and not a wave. But almost immediately, Arago experimentally verified Poisson's prediction.



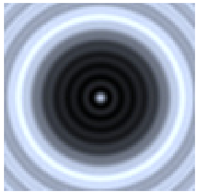
Experimental demonstration

One peculiar example: the Arago Spot

If a beam encounters a circular “stop”, it develops a hole, which fills in as it propagates and diffracts:



Interestingly, the hole fills in from the center first!



Simplification of the Fresnel Integral

$$E(x_0, y_0) \propto \iint \exp\left\{ jk\left[\frac{(-2x_0x_1 - 2y_0y_1)}{2z} + \frac{(x_1^2 + y_1^2)}{2z} \right] \right\} Aperture(x_1, y_1)\, dx_1 dy_1$$

Let D be the largest dimension of the aperture: $D^2 = \max(x_1^2 + y_1^2)$.

Our first step, which allowed us to obtain the Fresnel result, was the paraxial approximation: $z \gg D$ or $D/z \ll 1$

Note that this approximation does not contain the wavelength.

A more severe approximation is suggested by noticing that the integral simplifies A LOT if only we could neglect the quadratic terms x_1^2 and y_1^2.

If $kD^2/2z \ll 1$, then we could do that…

This new approximation, which depends on the wavelength enforcing that the aperture must be small compared to the wavelength and the plan of view far away from the aperture

Fraunhofer Diffraction: far field results

As in Fresnel diffraction, we'll typically assume a plane wave incident field, we'll neglect the phase factors, and we'll explicitly write the aperture function in the integral:

- In this case, the quadratic terms are tiny, so we can ignore them and obtain:

$$E(x_0, y_0) \propto \iint \exp\left\{-\frac{jk}{z}\left(x_0 x_1 + y_0 y_1\right)\right\} Aperture(x_1, y_1)\, dx_1 dy_1$$

- How far away is far enough? We must have both $z >> D$ and $z >> \pi D^2/\lambda$

Green light (λ = **500nm**) , D = 1mm ➔ z>> 6.3 m; D = 10 µm ➔ z>> 630 µm

Fraunhofer diffraction is a Fourier transform

$$E(x_0, y_0) \propto \iint \exp\left\{-\frac{jk}{z}(x_0 x_1 + y_0 y_1)\right\} Aperture(x_1, y_1)\, dx_1 dy_1$$

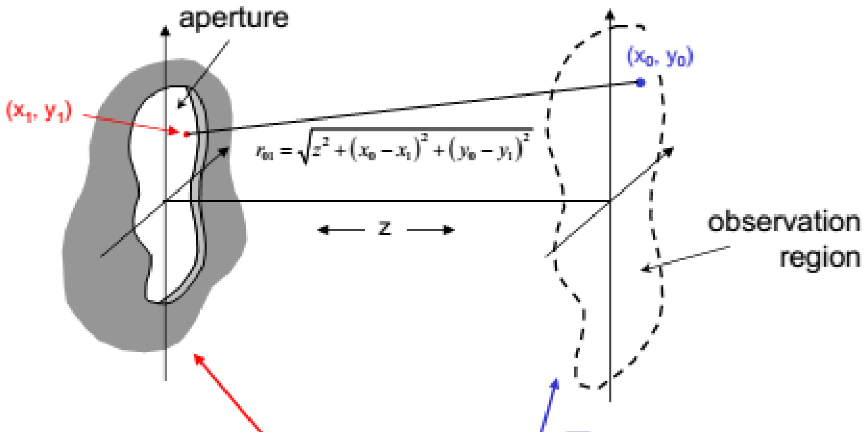
- This is just a Fourier Transform! (actually, two of them, in two variables)
- Interestingly, it's a Fourier Transform from position, x_1, to another position variable, x_0 (in

another plane, i.e., a different z position).

- Usually, the Fourier "conjugate variables" have reciprocal units (e.g., t and w, or x and k).
- The conjugate variables here are really x_1 and kx_0/z, which do have reciprocal units.

Fraunhofer diffraction is a Fourier transform

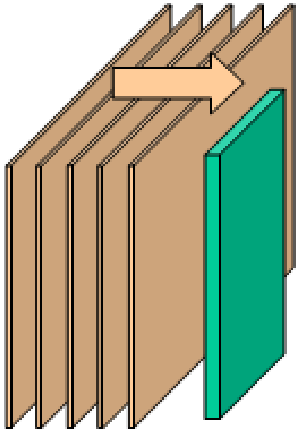
In one dimension: $E(x_0) \propto \int \exp\left\{-j\left(\frac{kx_0}{z}\right)x_1\right\} Aperture(x_1)dx_1$



So, the light in the Fraunhofer regime (the "far field") is simply the Fourier Transform of the apertured field!

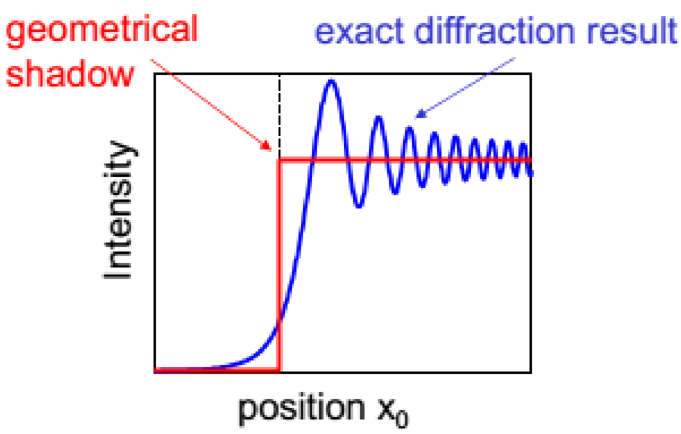
Knowing this makes the calculations **a lot** easier…

Fraunhofer diffraction regime needs to be met

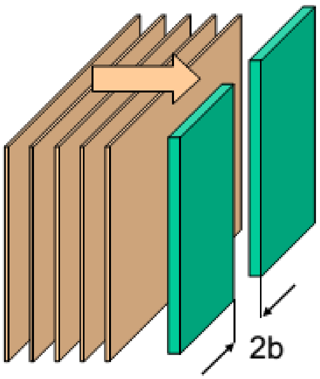


Example: light passing by an edge

In this case, the effective "width" of the slit, D, is infinite. It is impossible to reach the Fraunhofer regime of $z >> \pi D^2/\lambda$.



Single Slit Diffraction Intensity



In this case, the problem is a single Fourier transform (in x), rather than two of them (in x and y):

$$\boxed{E(x_0) \propto \int \exp\left\{-\frac{jk}{z}(x_0 x_1)\right\} Aperture(x_1)\,dx_1}$$

The aperture function is simple:

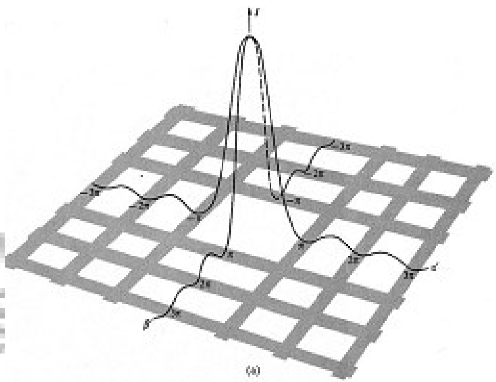
$$Aperture(x_1) = \begin{cases} 1 & -b < x_1 < b \\ 0 & \text{otherwise} \end{cases}$$

A square aperture (edge length = 2b) just gives the product of two sinc functions in x and in y. Just as if it were

But we know that the Fourier transform of a rectangle function (of width $2b$) is a sinc function:

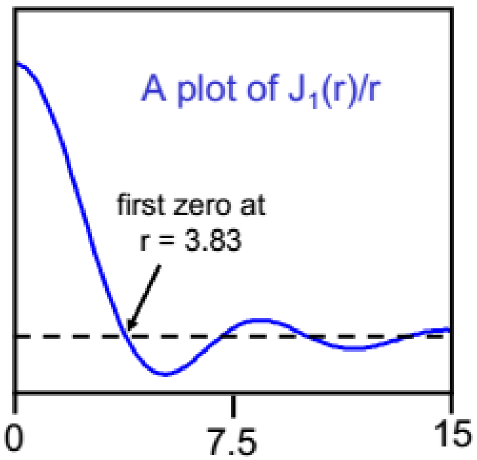
written here in terms of the Fresnel number $N = b^2/\lambda z$

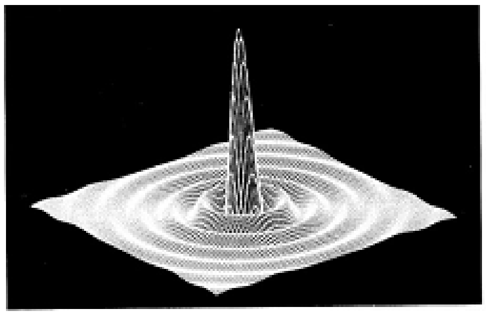
$$FT\left[Aperture(x_1)\right] \propto \frac{\sin(kx_0 b/z)}{kx_0 b/z} = \frac{\sin(2\pi N x_0/b)}{2\pi N x_0/b}$$



$z \gg D$ and $z \gg \pi D^2/\lambda$

Airy pattern



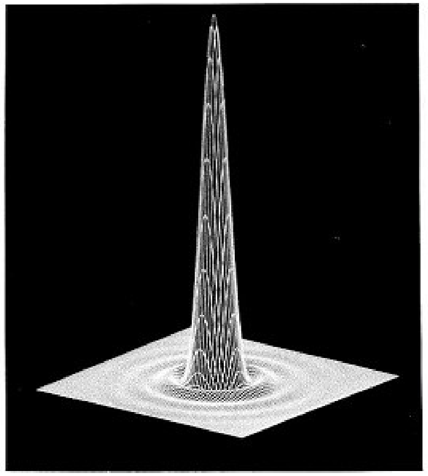


Diffracted E-field plotted in 2D

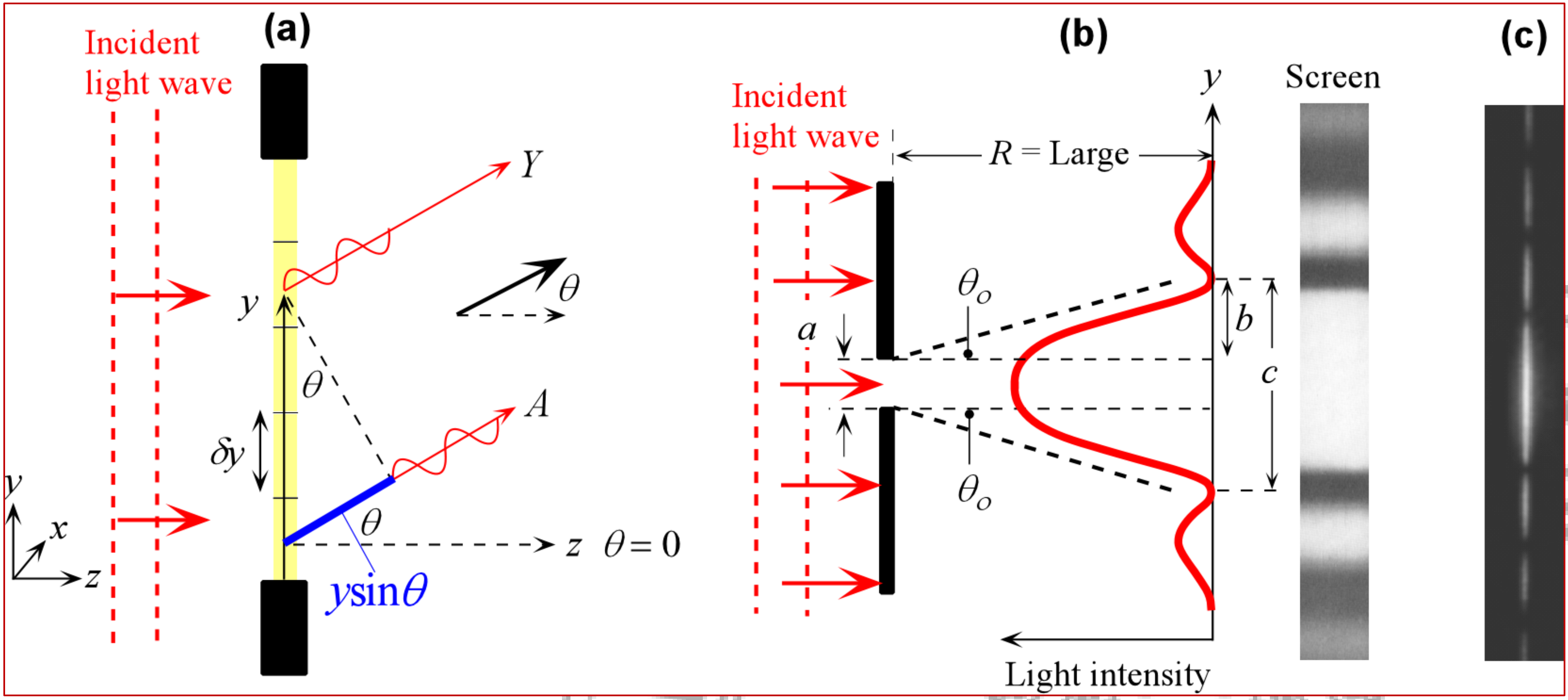
- The 2D Fourier transform of a circular aperture, radius = b, is given by a Bessel function of the first kind:

$$FT\left[Circular\ aperture(x_1, y_1)\right] \propto \frac{J_1(k\rho b/z)}{k\rho b/z}$$

This pattern is known as the Airy pattern or Airy disc

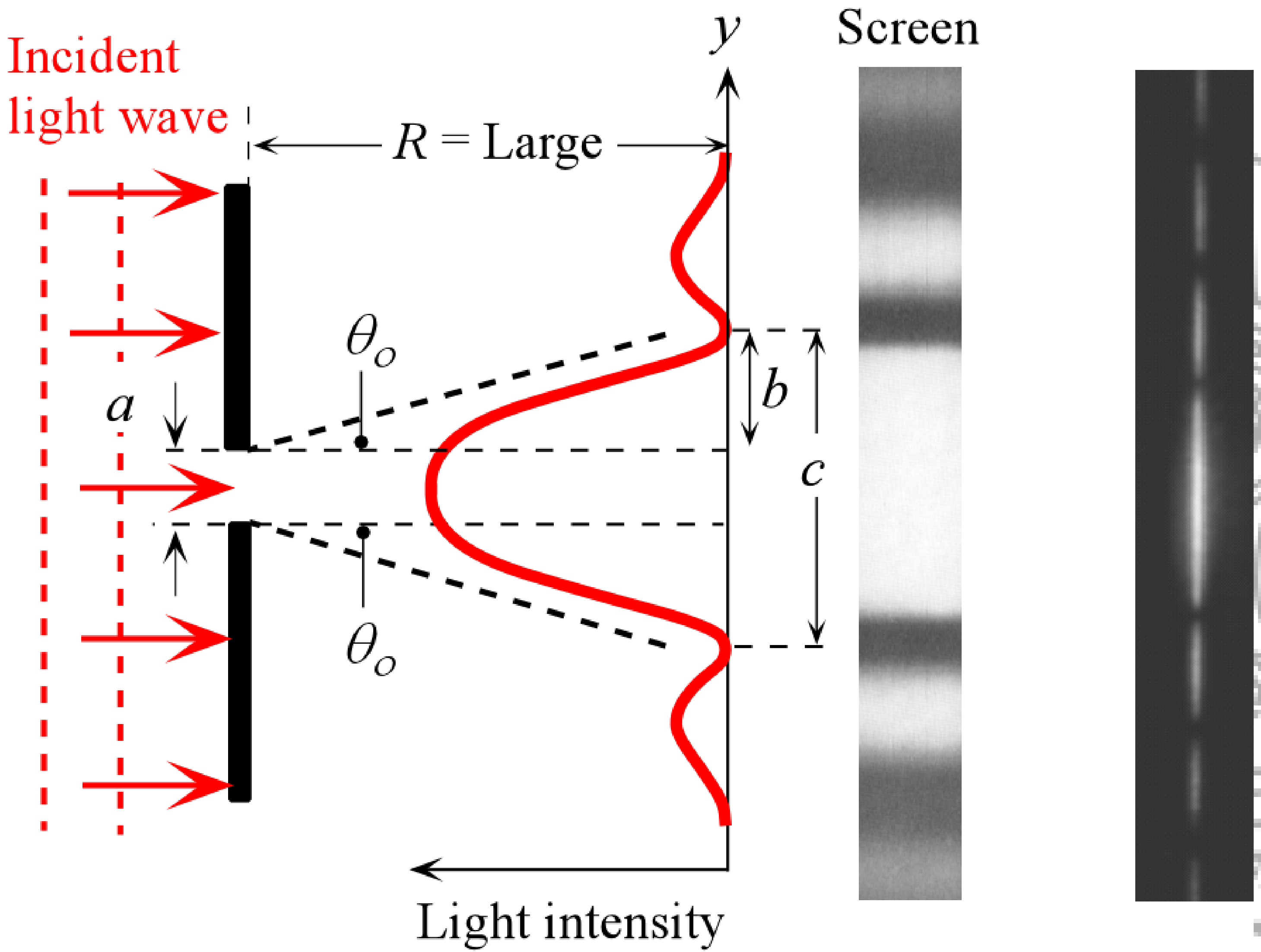


Diffraction from a Single Slit



(a) The aperture has a finite width *a* along *y*, but it is very long along *x* so that it is a one-dimensional slit. The aperture is divided into *N* number of point sources each occupying δy with amplitude proportional to δy since the slit is excited by a plane electromagnetic wave. (b) The intensity distribution in the received light at the screen *far away* from the aperture: *the diffraction pattern*. Note that the slit is very long along *x* and there is no diffraction along this dimension. (c) Diffraction patter obtained by using a laser beam from a pointer incident on a single slit.

Diffraction from a Single Slit



Diffraction from a Single Slit

$$\delta E \propto (\delta y)\exp(-jky\sin\theta)$$

$$E(\theta) = C\int_{y=0}^{y=a} \delta y \exp(-jky\sin\theta)$$

$$E(\theta) = \frac{Ce^{-j\frac{1}{2}ka\sin\theta}\, a\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}$$

$$I(\theta) = \left[\frac{C'a\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}\right]^2 = I(0)\mathrm{sinc}^2(\beta)$$

$$\beta = \frac{1}{2}ka\sin\theta$$

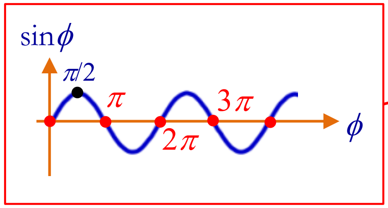
Diffraction from a Single Slit

$$I(\theta)=I(0)\left[\frac{\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}\right]^2=I(0)\left[\frac{\sin\beta}{\beta}\right]^2=I(0)\mathrm{sinc}^2(\beta)$$

$$\beta=\frac{1}{2}ka\sin\theta$$

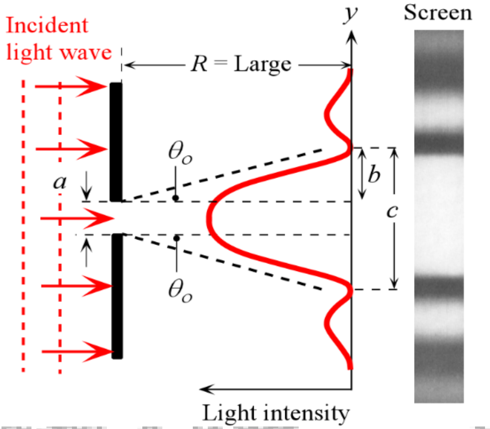
Zero intensity when I(θ) = 0

$$\sin\beta=\sin(\tfrac{1}{2}ka\sin\theta)=0$$



$$\frac{1}{2}ka\sin\theta_m=m\pi$$

$$\sin\theta=\frac{m\lambda}{a}$$

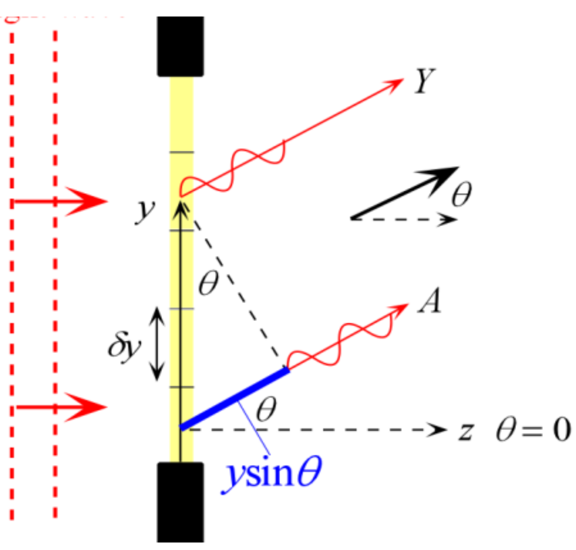


Diffraction from a Single Slit

$$I(\theta) = I(0)\left[\frac{\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}\right]^2 = I(0)\left[\frac{\sin\beta}{\beta}\right]^2 = I(0)\mathrm{sinc}^2(\beta)$$

$$\beta = \tfrac{1}{2}ka\sin\theta$$

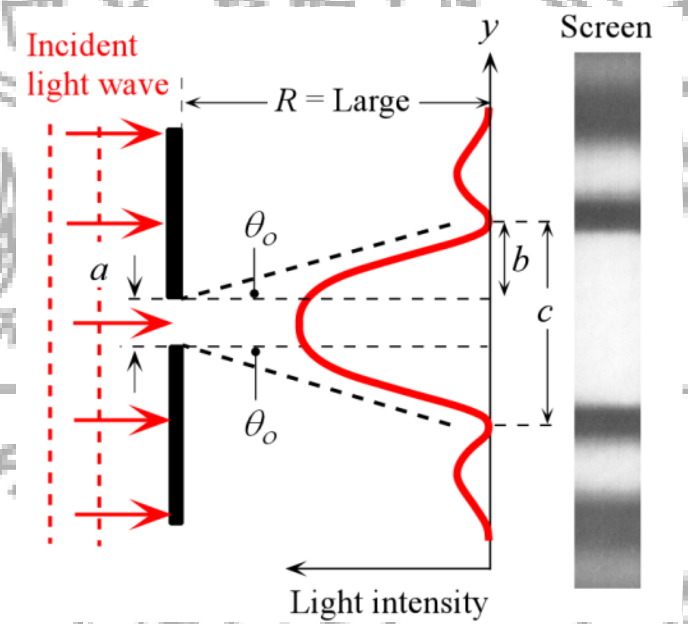
Zero intensity when $I(\theta) = 0$



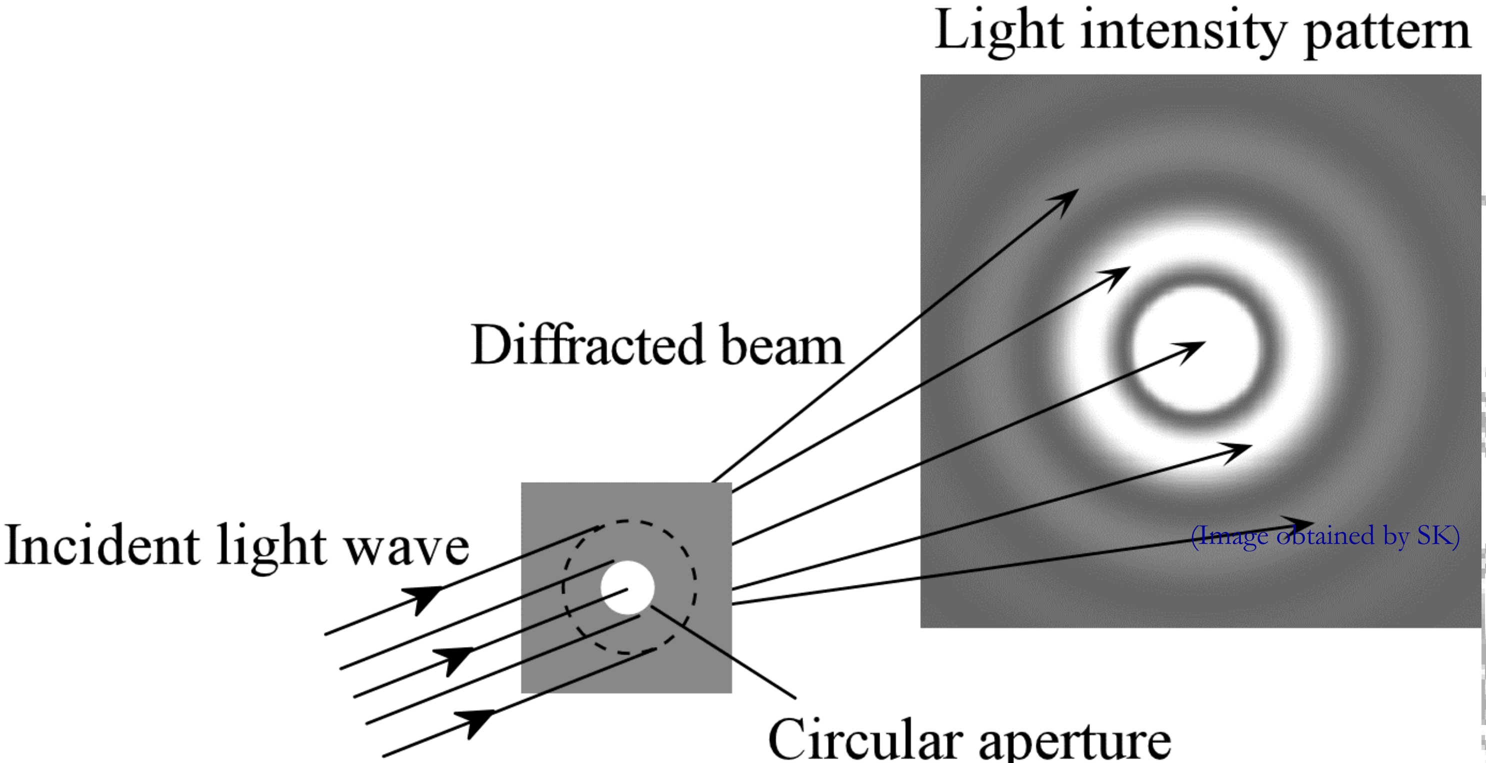
$$\sin\theta = \frac{m\lambda}{a}$$

Divergence

$$\Delta\theta = 2\theta_o \approx \frac{2\lambda}{a}$$



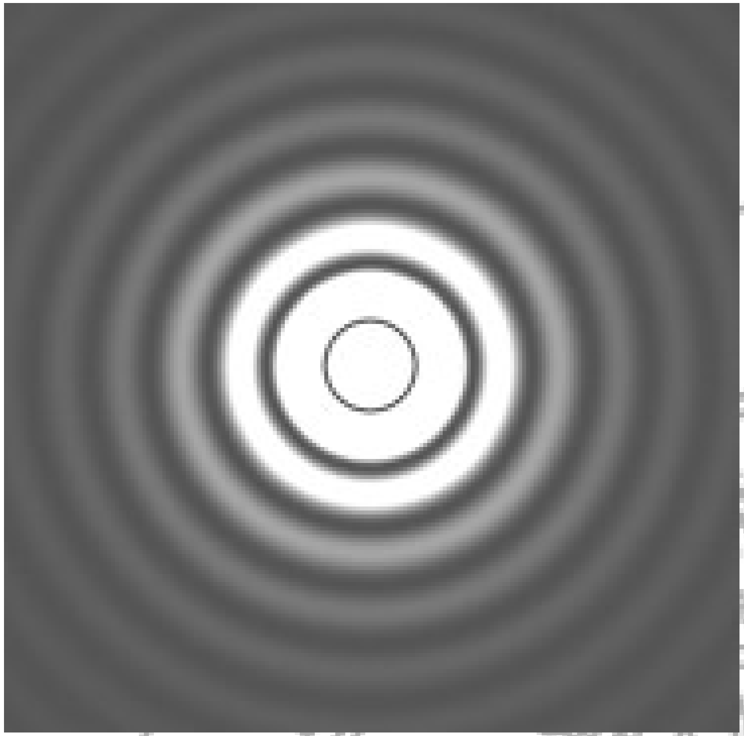
Diffraction from a circular aperture



$$\sin\theta_o = 1.22\,\frac{\lambda}{D}$$

Diameter of aperture

Diffraction from a Circular Aperture



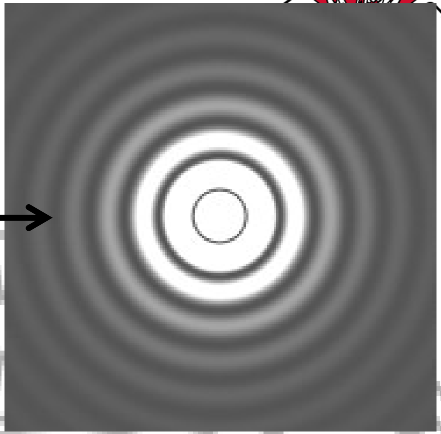
Diffraction pattern far away from a circular aperture. The image has been overexposed to capture the faint outer rings (By SK.)

Diffraction from a circular aperture

George Bidell Airy (1801–1892, England). George Airy was a professor of astronomy at Cambridge and then the Astronomer Royal at the Royal Observatory in Greenwich, England. *(© Mary Evans Picture Library/Alamy.)*

Intensity distribution

$$I(\gamma) = I_o\left(\frac{2J_1(\gamma)}{\gamma}\right)^2 \longrightarrow$$



(By SK)

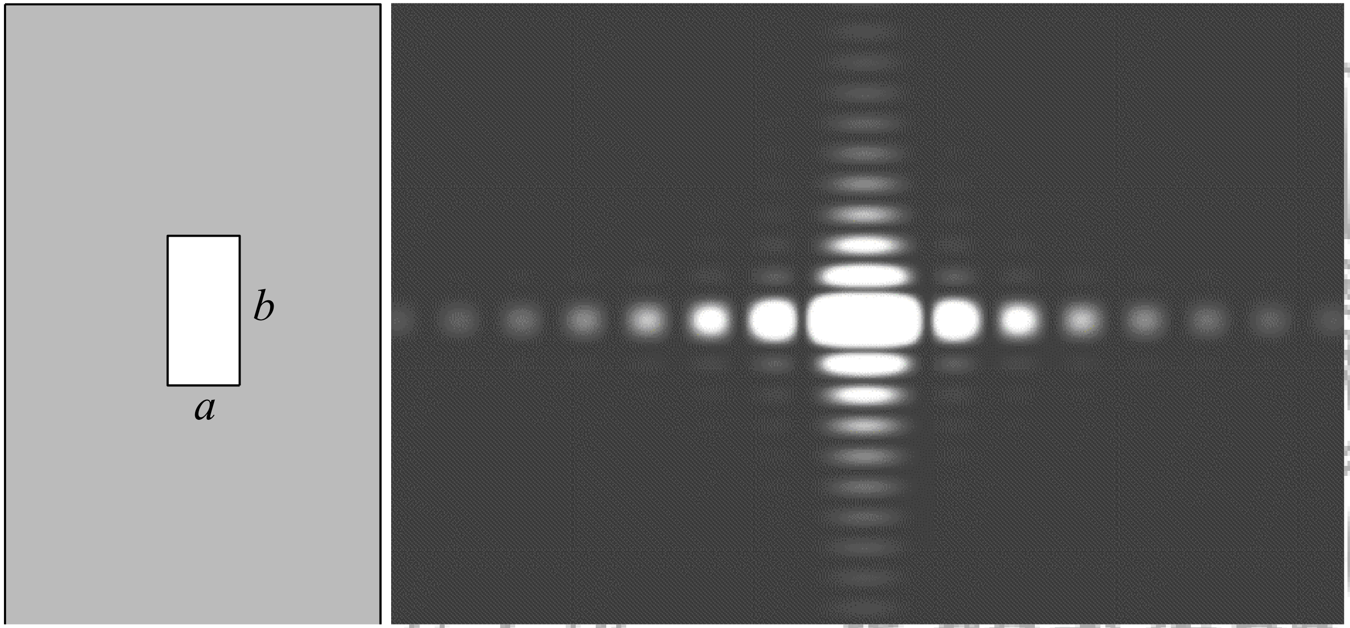
$$\gamma = (1/2)kD\sin\theta$$

$$k = 2\pi/\lambda$$

$$J_1(\gamma) = \frac{1}{\pi}\int_0^{\pi}\cos(\alpha - \gamma\sin\alpha)d\alpha$$

Bessel function (first kind, first order)

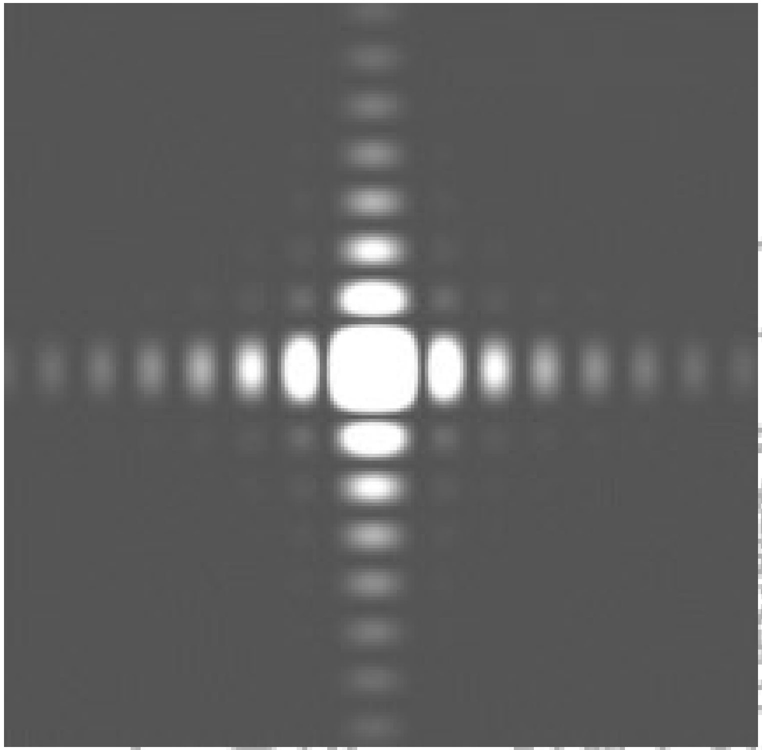
Diffraction from a Rectangular Aperture



The rectangular aperture of dimensions $a \times b$ on the left gives the diffraction pattern on the right. (b is twice a)

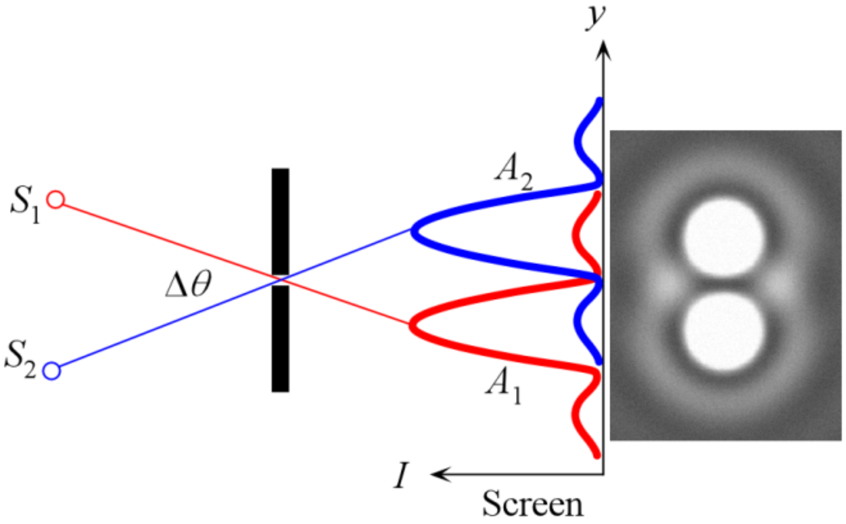
(Image obtained by SK. Overexposed to highlight the higher order lobes.)

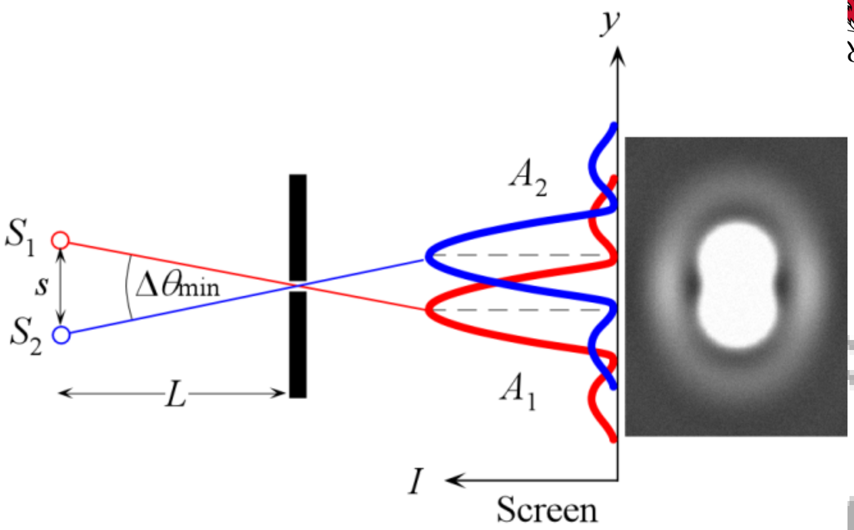
Diffraction from a Rectangular Aperture



Diffraction pattern far away from a square aperture. The image has been overexposed to capture the faint side lobes (Image obtained by SK)

Rayleigh Criterion





Resolution of imaging systems is limited by diffraction effects. As points S_1 and S_2 get closer, eventually the Airy patterns overlap so much that the resolution is lost. The Rayleigh criterion allows the minimum angular separation two of the point sources be determined. (Schematic illustration inasmuch as the side lobes are actually much smaller the center peak.)

$$\sin(\Delta\theta_{min}) = 1.22\frac{\lambda}{D}$$

Rayleigh Criterion

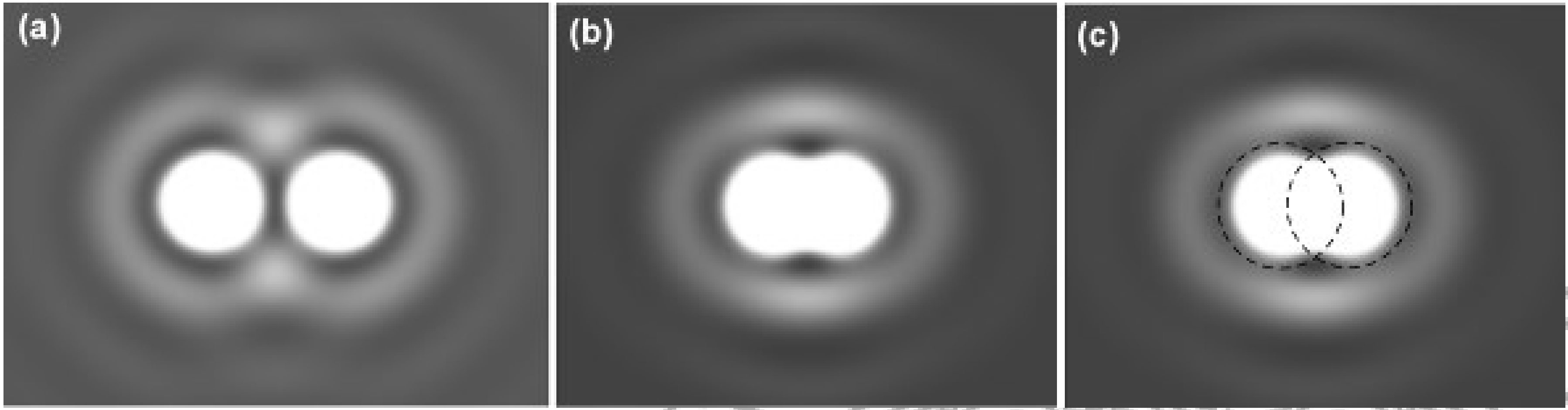
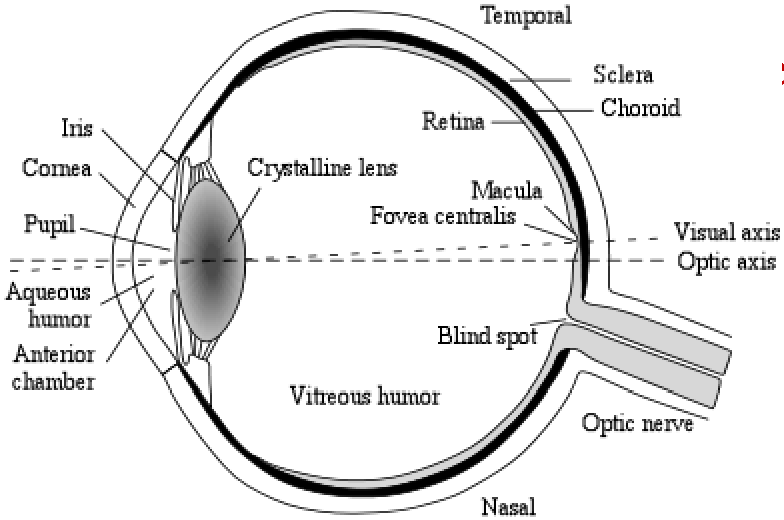


Image of two point sources captured through a small circular aperture. (a) The two points are fully resolved since the diffraction patterns of the two sources are sufficiently separated. (b) The two images near the Rayleigh limit of resolution. (c) The first dark ring through the center of the bright Airy disk of the other pattern. (Approximate.) (Images by SK)

:solution of the Human Eye



The human eye has a pupil diameter of about 2 mm. What would be the minimum angular separation of two points under a green light of 550 nm and their minimum separation if the two objects are 30 cm from the eye?

The image will be diffraction pattern in the eye, and is a result of waves in this medium. If the refractive index $n \approx 1.33$ (water) in the eye, then

$$\sin(\Delta\theta_{\min}) = 1.22\frac{\lambda}{nD} = 1.22\frac{(550\times 10^{-9}\text{ m})}{(1.33)(2\times 10^{-3}\text{ m})}$$

$$\Delta\theta_{\min} = 0.0145°$$

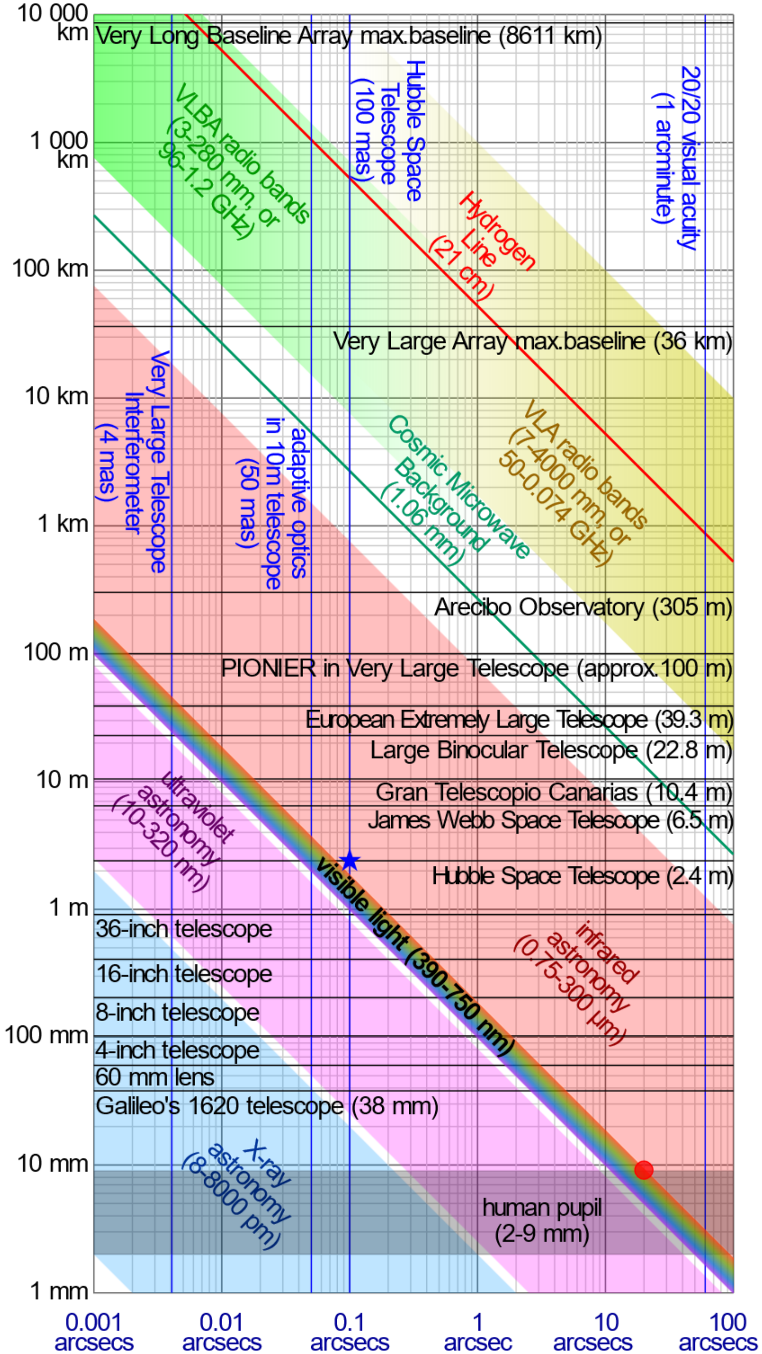
Their minimum separation s would be

$$s = 2L\tan(\Delta\theta_{\min}/2) = 2(300\text{ mm})\tan(0.0145°/2)$$

$$= 0.076\text{ mm} = 76\text{ micron}$$

which is about the thickness of a human hair (or this page).

Telescopes?

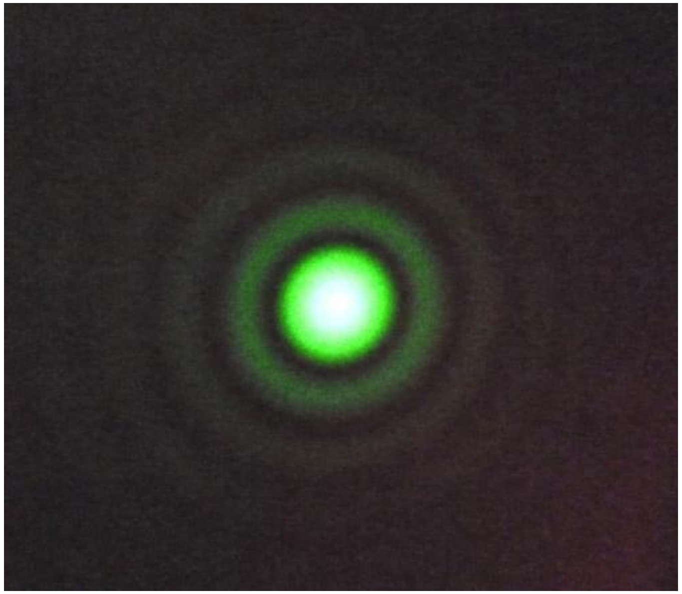


10 000 km
1 000 km
100 km
10 km
1 km
100 m
10 m
1 m
100 mm
10 mm
1 mm
0.001 arcsecs
0.01 arcsecs
0.1 arcsecs
1 arcsec
10 arcsecs
100 arcsecs
Very Long Baseline Array max.baseline (8611 km)
VLBA radio bands (3-280 mm, or 96-1.2 GHz)
Hubble Space Telescope (100 mas)
Hydrogen Line (21 cm)
20/20 visual acuity (1 arcminute)
Very Large Array max.baseline (36 km)
Very Large Telescope Interferometer (4 mas)
adaptive optics in 10m telescope (50 mas)
Cosmic Microwave Background (1.06 mm)
VLA radio bands (7-4000 mm, or 50-0.074 GHz)
Arecibo Observatory (305 m)
PIONIER in Very Large Telescope (approx.100 m)
European Extremely Large Telescope (39.3 m)
Large Binocular Telescope (22.8 m)
Gran Telescopio Canarias (10.4 m)
James Webb Space Telescope (6.5 m)
Hubble Space Telescope (2.4 m)
ultraviolet astronomy (10-320 nm)
visible light (390-750 nm)
infrared astronomy (0.75-300 µm)
36-inch telescope
16-inch telescope
8-inch telescope
4-inch telescope
60 mm lens
Galileo's 1620 telescope (38 mm)
X-ray astronomy (8-8000 pm)
human pupil (2-9 mm)

Experimental Diffraction Patterns

Diffraction from a circular aperture with a diameter of 30 μm

Green laser pointer used at a wavelength of 532 nm

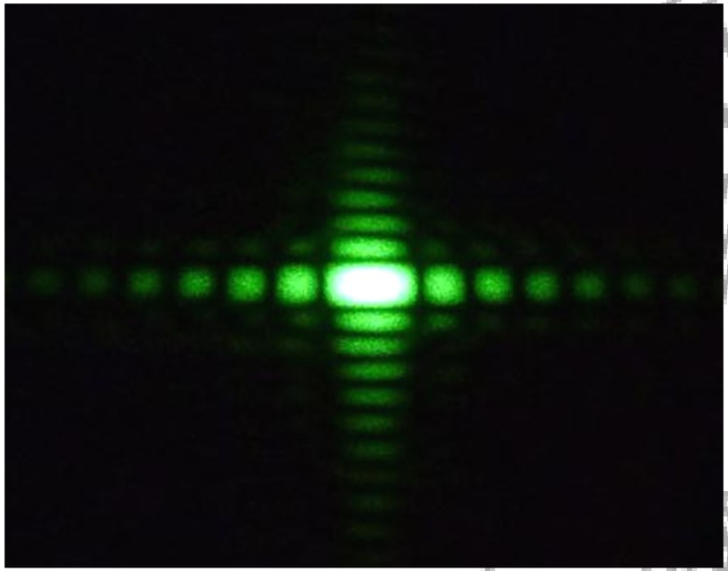


(Overexposed photo by SK)

Experimental Diffraction Patterns

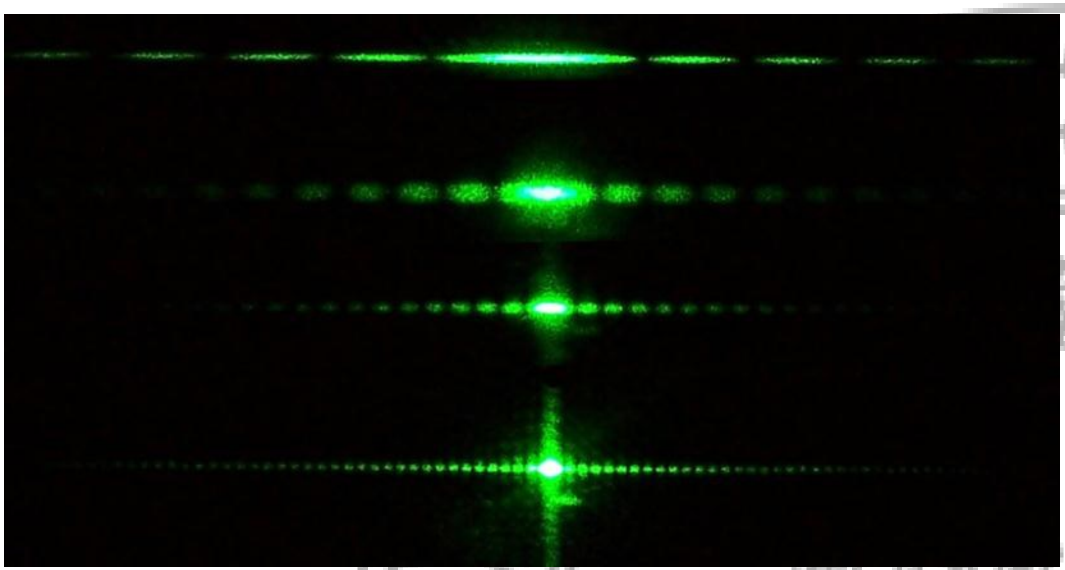
Crossed slits 200 x 100 μm

Green laser pointer used at a wavelength of 532 nm



(Overexposed photo by SK)

Experimental Diffraction Patterns



a = 20 μm

a = 40 μm

a = 80 μm

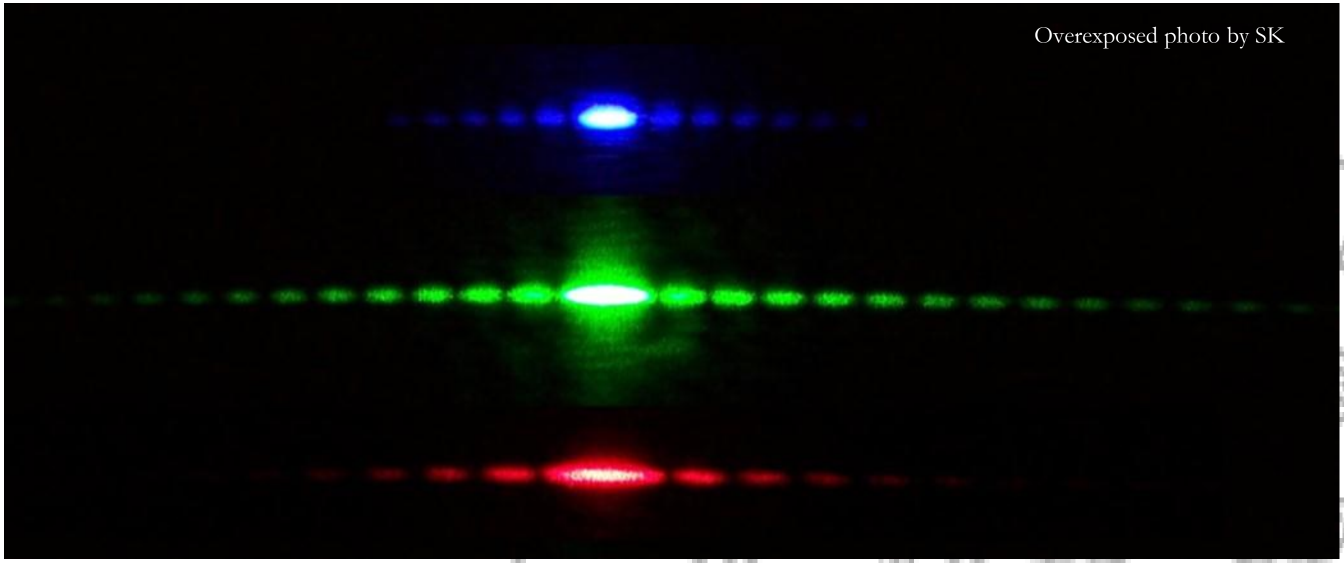
a = 160 μm

(Overexposed photo by SK)

Single slit with a width *a*

Green laser pointer used at a wavelength of 532 nm

Experimental Diffraction Patterns



Single slit with a width 100 μm

Blue = 402 nm

Green = 532 nm

Red = 670 nm

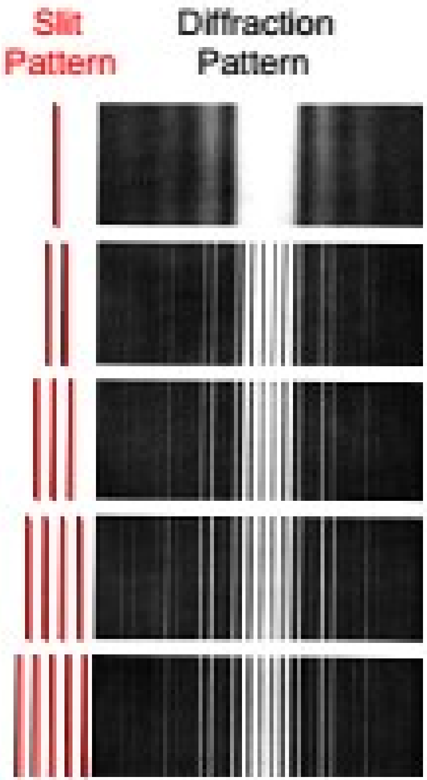
Why does the central bright lobe get larger with increasing wavelength?

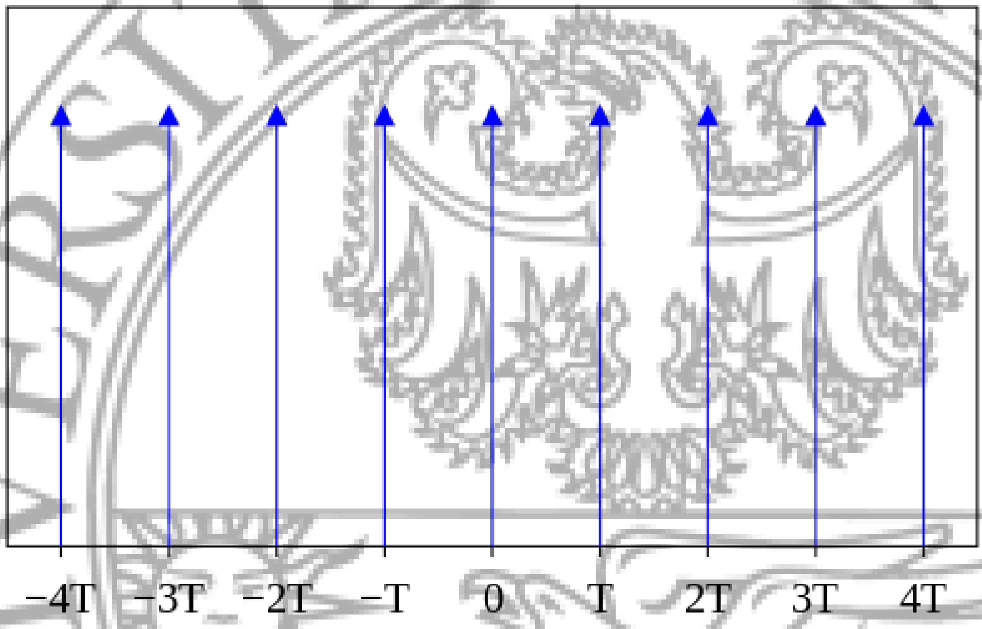
Answer

$$\Delta\theta = 2\theta_o \approx \frac{2\lambda}{a}$$

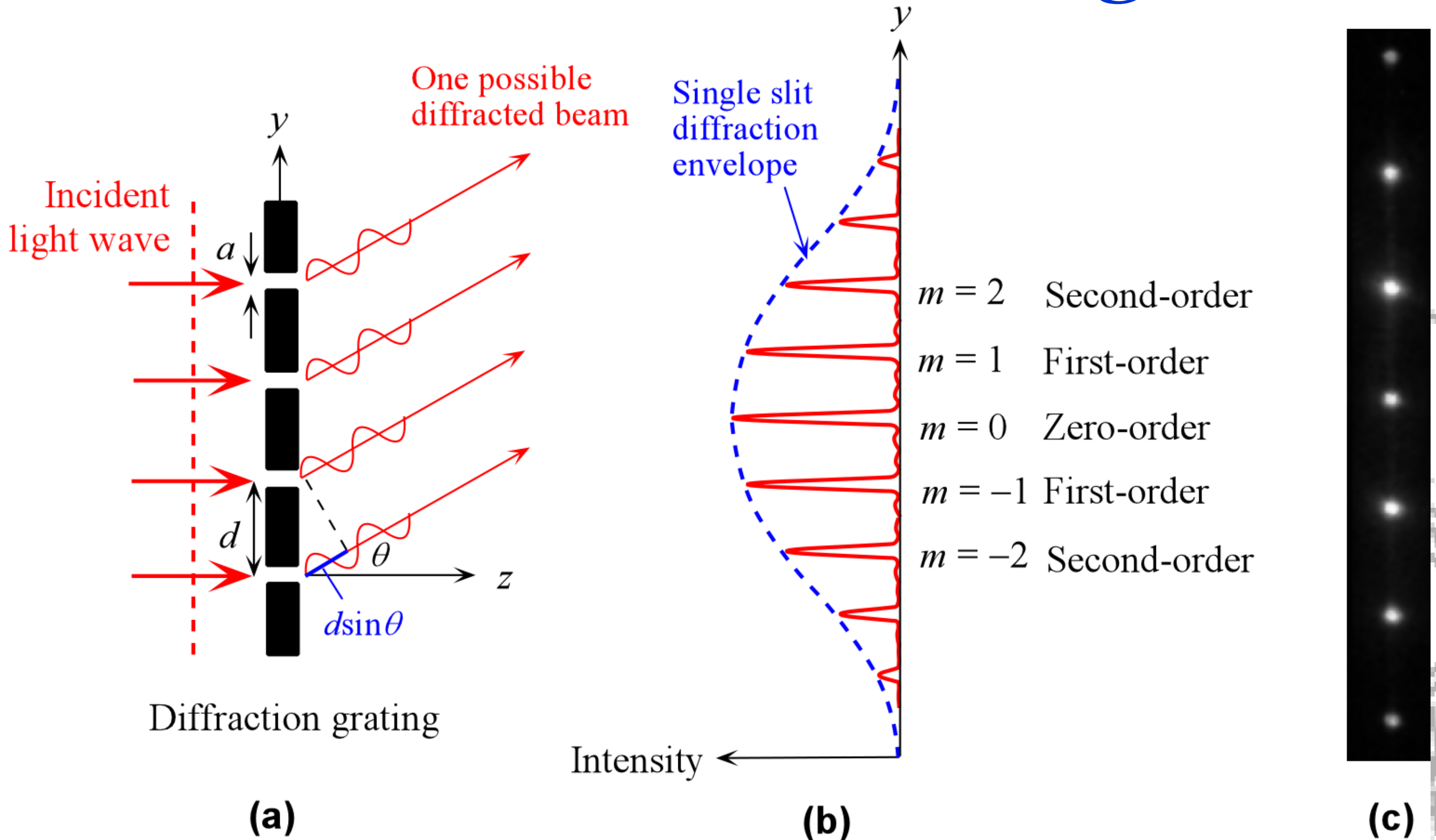
Diffraction from multi slit

- Infinitely many equally spaced slits (a Dirac Comb function!) yields a far-field pattern which is the Fourier transform; that is, a Dirac Comb function.





Diffraction Grating



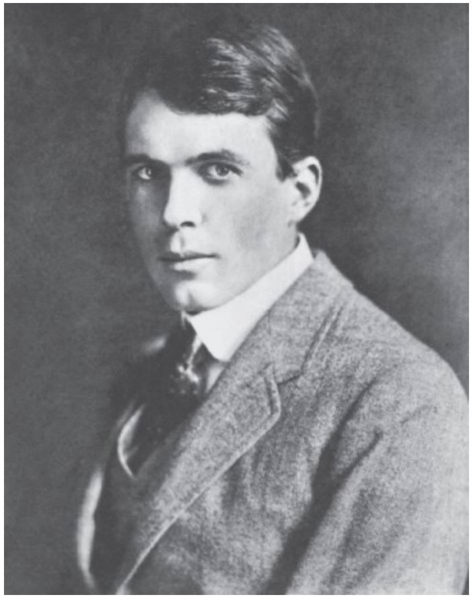
(a) A diffraction grating with N slits in an opaque screen. Slit periodicity is d and slit width is a; $a << d$. (b) The far-field diffracted light pattern. There are distinct, that is diffracted, beams in certain directions (schematic). (c) Diffraction pattern obtained by shining a beam from a red laser pointer onto a diffraction grating. The finite size of the laser beam results in the dot pattern. (The wavelength was 670 nm, red, and the grating has 2000 lines per inch.)

Diffraction Grating

Bragg diffraction condition

Normal incidence

$$d\sin\theta = m\lambda \;\; ; m = 0, \pm1, \pm2, \ldots$$



William Lawrence Bragg (1890-1971), Australian-born British physicist, won the Nobel prize with his father William Henry Bragg for his "famous equation" when he was only 25 years old (Courtesy of SSPL via Getty Images)

"The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them."

Diffraction Grating

Bragg diffraction condition

Normal incidence

$$d\sin\theta = m\lambda \;\; ; m = 0, \pm 1, \pm 2, \ldots$$

Oblique incidence

$$d(\sin\theta_m - \sin\theta_i) = m\lambda \;\; ; m = 0, \pm 1, \pm 2,$$

$$I(y) = I_o \left[\frac{\sin(\frac{1}{2} k_y a)}{\frac{1}{2} k_y a} \right]^2 \left[\frac{\sin(\frac{1}{2} N k_y d)}{N \sin(\frac{1}{2} k_y d)} \right]^2$$

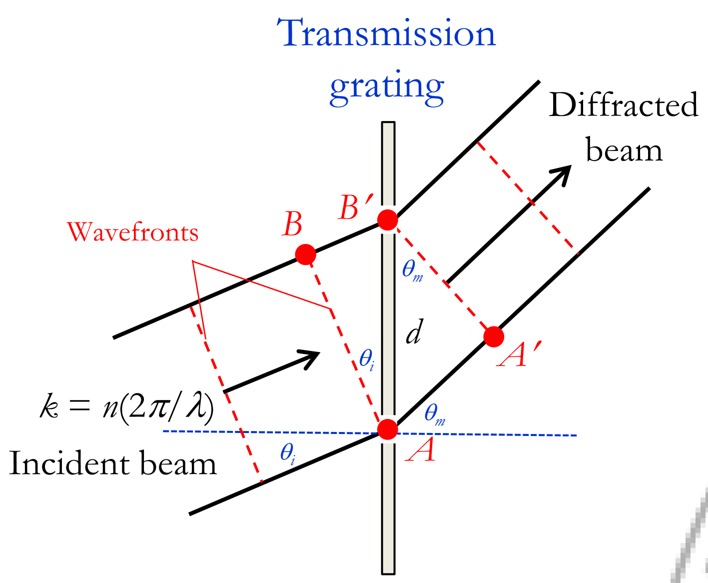
Diffraction from *a single slit*

$$k_y = (2\pi/\lambda)\sin\theta$$

Diffraction from N slits

Oblique Incidence on a Transmission Diffraction Grating

Periodicity $= d = AB'$



AB is a wavefront on the incident wave

A'B' is a wavefront on the diffracted wave

Point *A* on the wavefront *AB* progresses to *A'*.

Point *B* on the wavefront *AB* progresses to *B'*.

A' and *B'* on the diffracted wave must be in phase

if they are on the same wavefront.

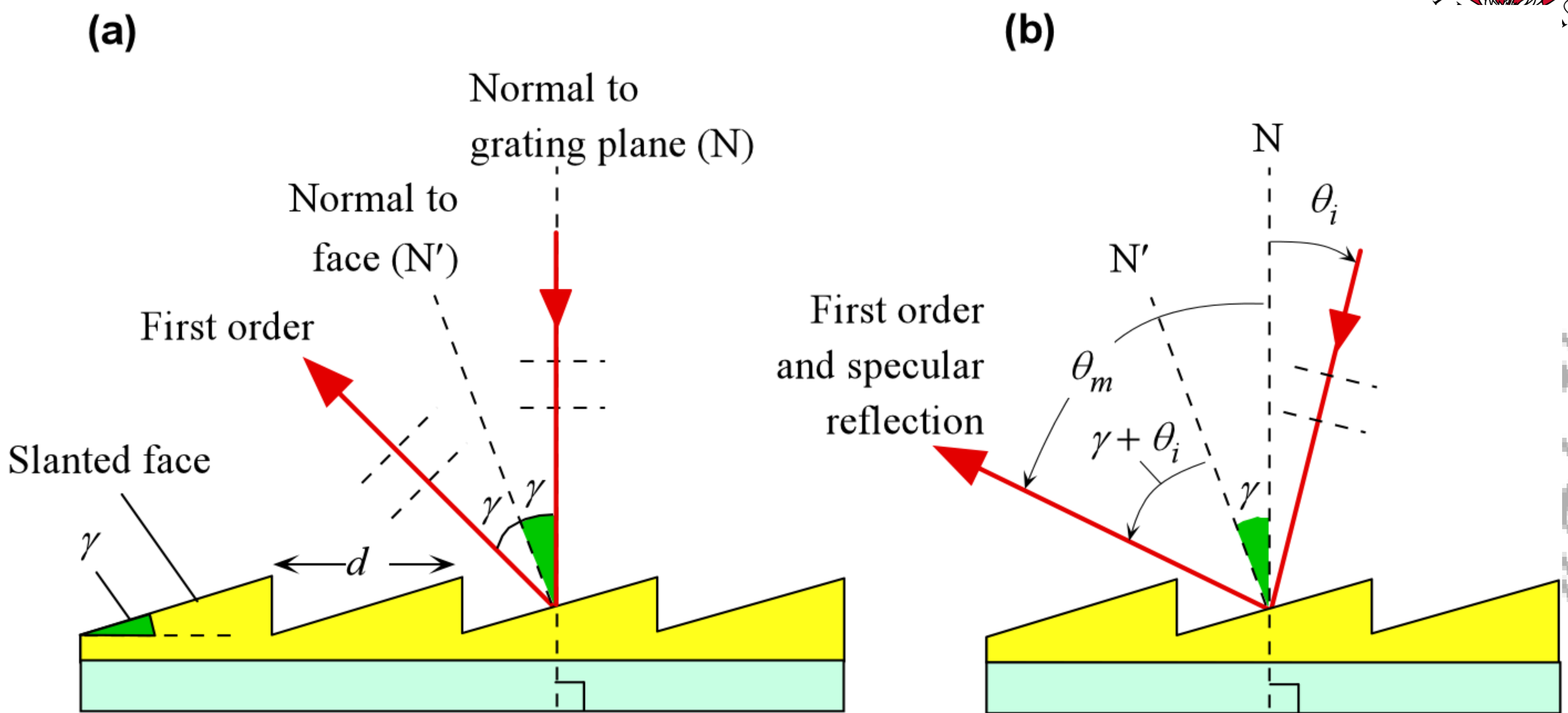
$\Delta\phi =$ Phase difference between the paths AA' and BB',

$$\Delta\phi = k(AA' - BB') = k(d\sin\theta_m - d\sin\theta_i) = m(2\pi)$$

$\boxed{k = n(2\pi/\lambda)}$

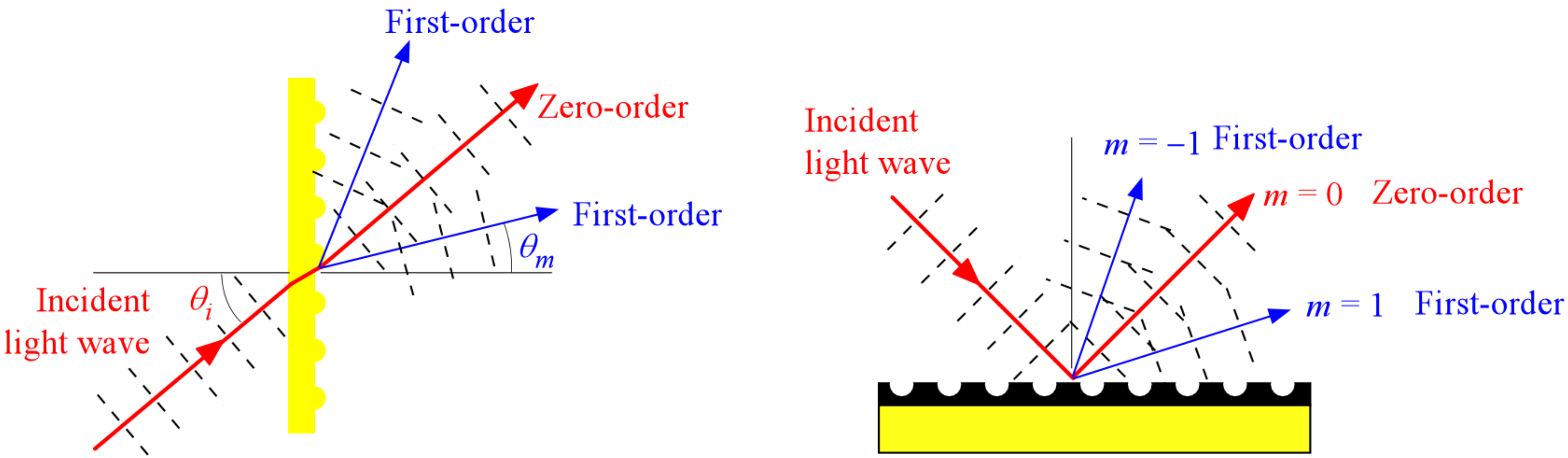
$$\boxed{d(\sin\theta_m - \sin\theta_i) = m\lambda \;\; ; m = 0, \pm 1, \pm 2,}$$

Diffraction Grating



(a) A blazed grating. Triangular grooves have been cut into the surface with a periodicity d. The side of a triangular groove make an angle γ to the plane of the diffraction angle. For normal incidence, the angle of diffraction must be 2γ to place the specular reflection on the diffracted beam. (b) When the incident beam is not normal, the specular reflection will coincides with the diffracted beam, when $(\gamma + \theta_i) + \gamma = \theta_m$

Diffraction Grating



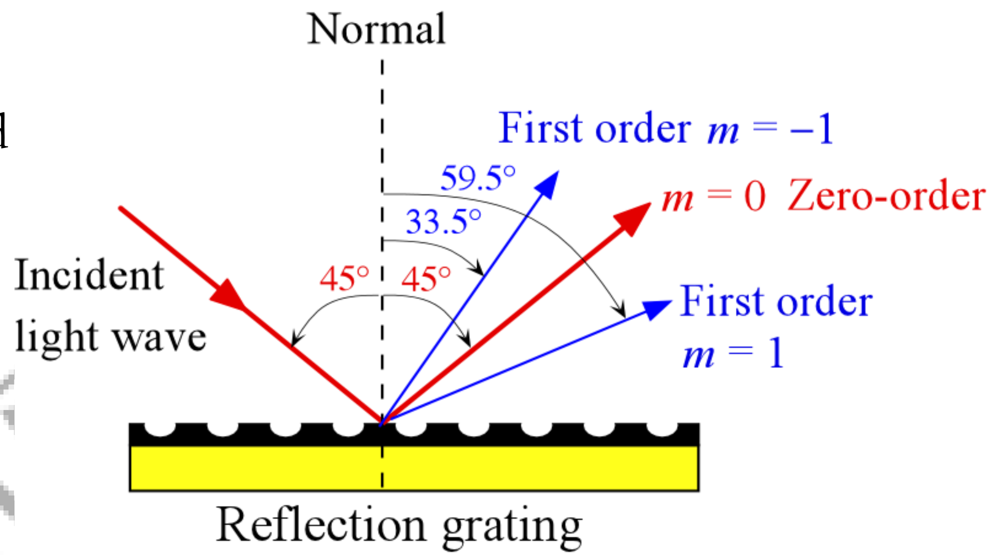
(a) Transmission grating **(b)** Reflection grating

(a) Ruled periodic parallel scratches on a glass serve as a *transmission grating*. (The glass plate is assumed to be very thin.) (b) A *reflection grating*. An incident light beam results in various "diffracted" beams. The zero-order diffracted beam is the normal reflected beam with an angle of reflection equal to the angle of incidence.

Diffraction Grating

Example: A reflection grating

Consider a reflection grating with a period d that is 10 μm. Find the diffracted beams if a collimated light wave of wavelength 1550 nm is incident on the grating at an angle of 45° to its normal. What should be the blazing angle γ if we were to use the blazed grating with the same periodicity? What happens to the diffracted beams if the periodicity is reduced to 2 μm?



Solution: Put, $m = 0$ to find the zero-order diffraction, $\theta_0 = 45°$, as expected. The general Bragg diffraction condition is

$$d(\sin\theta_m - \sin\theta_i) = m\lambda.$$

$$\therefore\ (10\ \mu\text{m})(\sin\theta_m - \sin(45°)) = (+1)(1.55\ \mu\text{m})$$

$$\therefore\ (10\ \mu\text{m})(\sin\theta_m - \sin(45°)) = (-1)(1.55\ \mu\text{m})$$

Solving these two equations, we find

$$\theta_m = 59.6° \text{ for } m = 1 \text{ and } \theta_m = 33.5° \text{ for } m = -1$$

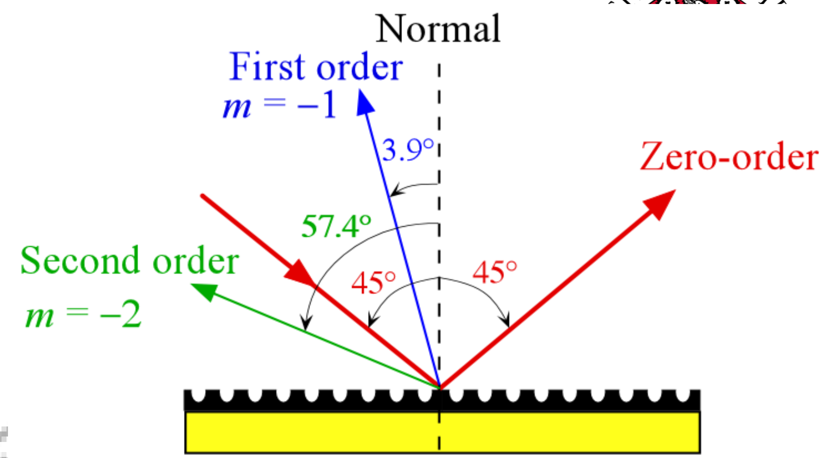
Suppose that we reduce d to 2 μm

Recalculating the above we find

$\theta_m = -3.9°$ for $m = -1$

and imaginary for $m = +1$.

Further, for $m = -2$, there is a second order diffraction beam at $\theta_m = -57.4°$.



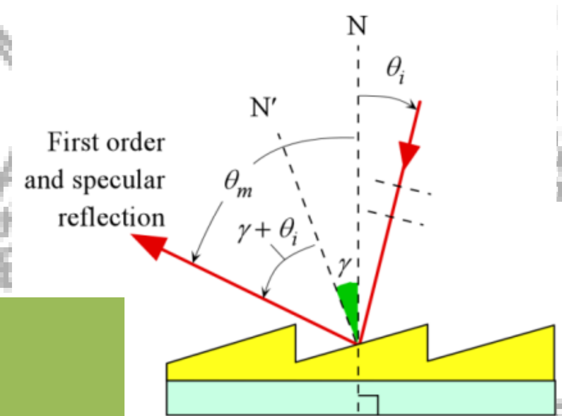
If we increase the angle of incidence, for example, $\theta_i = 85°$ on the first grating, the diffraction angle for $m = -1$ increases from 33.5° to 57.2° and the other diffraction peak ($m = 1$) disappears

Example: A reflection grating

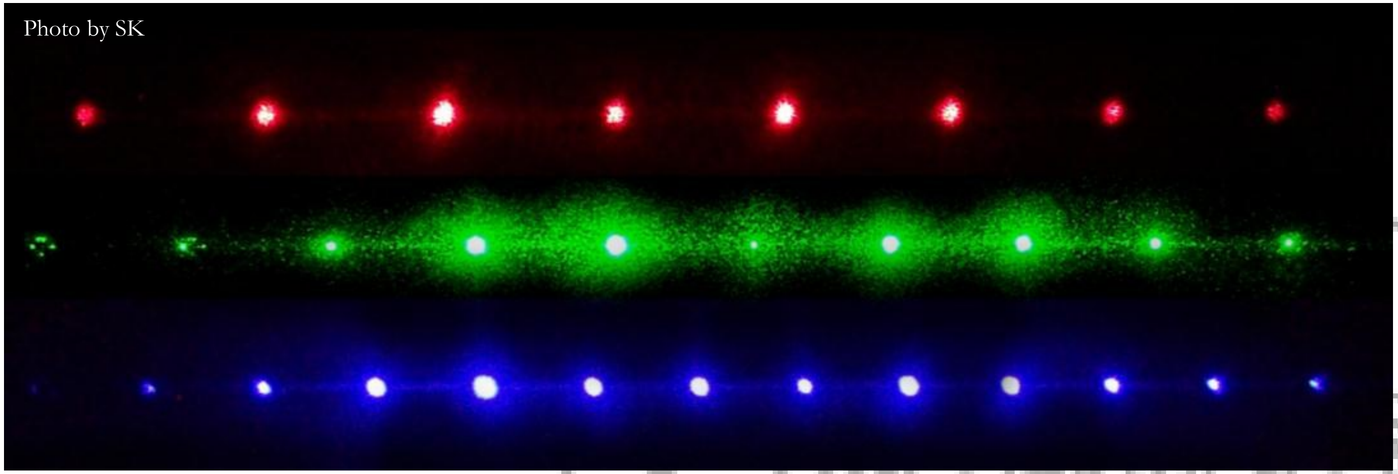
The secular reflection from the grooved surface coincides with the m_{th} order diffraction when

$$2\gamma = \theta_m - \theta_i$$

$$\therefore \qquad \gamma = (1/2)(\theta_m - \theta_i) = (1/2)(59.6° - 45°) = 7.3°$$



Experiments with Diffraction Gratings



Diffraction grating

(2000 lines/inch)

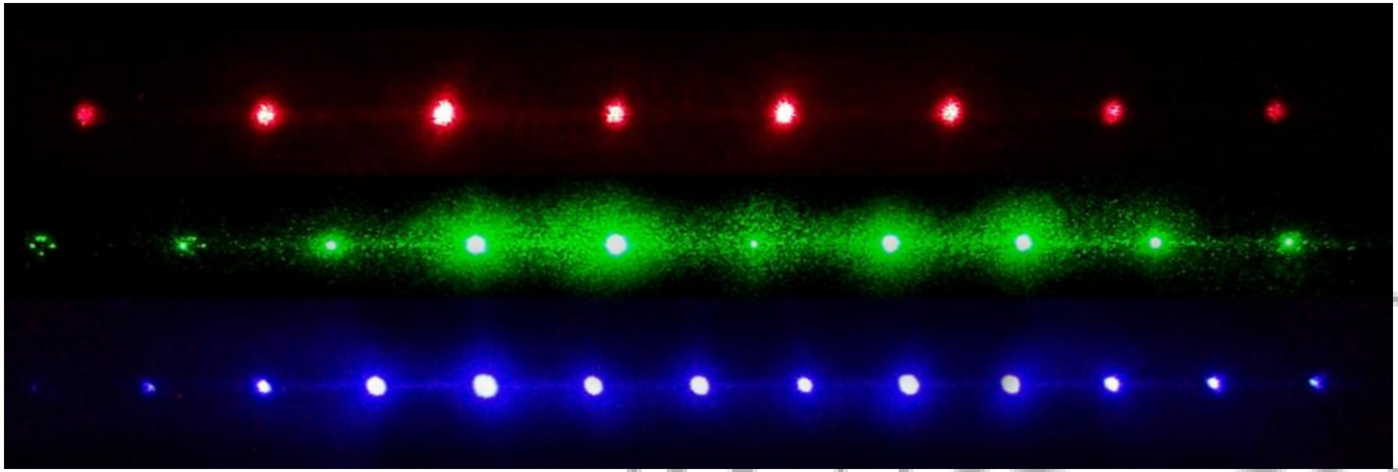
Blue = 402 nm

Green = 532 nm

Red = 670 nm

Why do the diffraction spots become further separated as you increase the wavelength?

ANSWER



$$d\sin\theta_1 = m\lambda$$

$$d\sin\theta_2 = (m+1)\lambda$$

$$d(\sin\theta_2 - \sin\theta_1) = \lambda$$

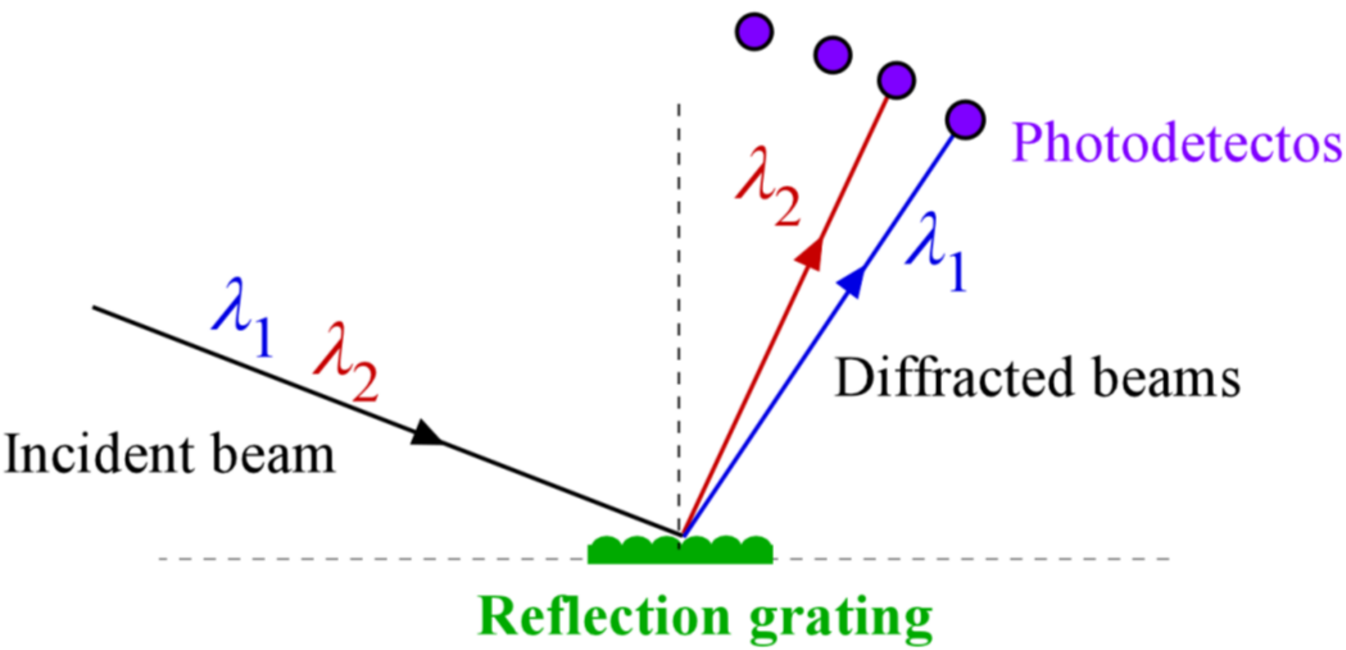
$$\sin\theta \approx \theta$$

Angular separation of spots

$$\boxed{\Delta\theta \approx \theta_2 - \theta_1 = \frac{\lambda}{d}}$$

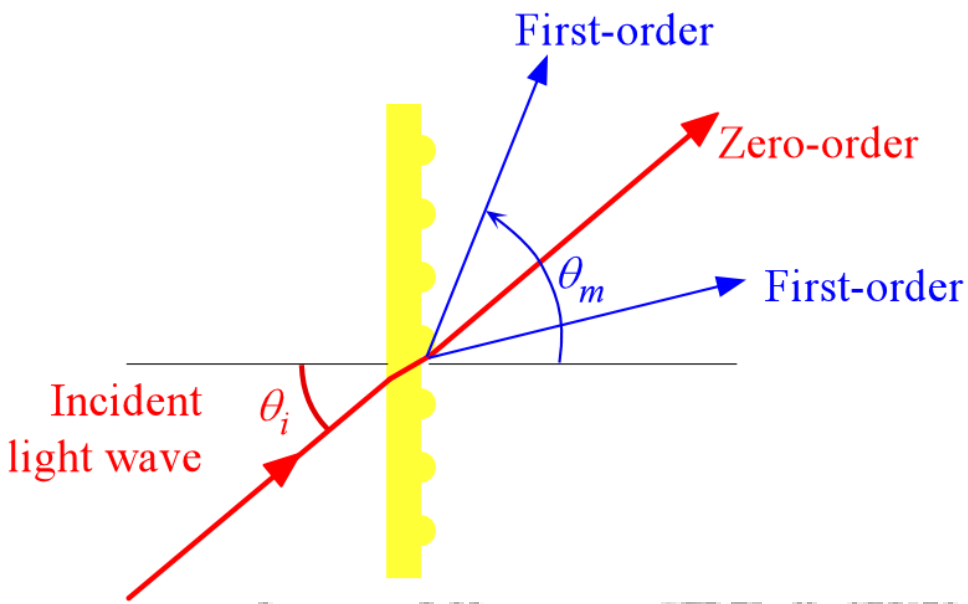
We can separate wavelengths by using a diffraction grating

Useful in Wavelength Division Multiplexing



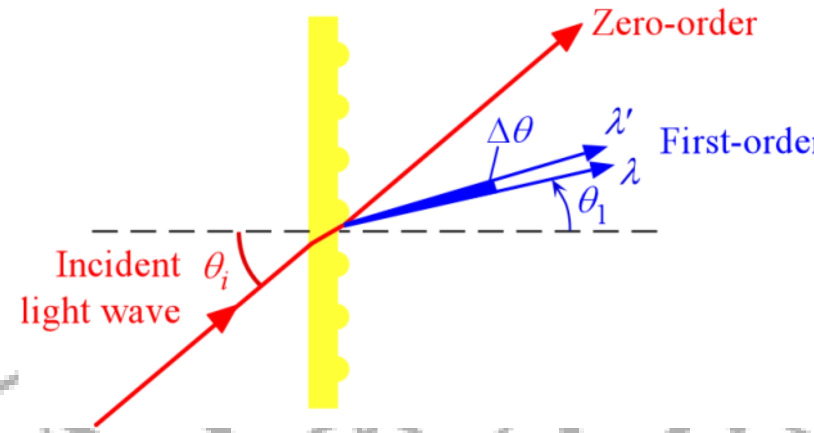
Example on Wavelength Separation by Diffraction

A transmission diffraction grating has a periodicity of 3 μm. The angle of incidence is 30° with respect to the normal to the diffraction grating. What is the angular separation of the two wavelength component s at 1550 nm and 1540 nm, separated by 10 nm?



$$d(\sin\theta_m - \sin\theta_i) = m\lambda$$

Example on Wavelength Separation



$\theta_i = 45°$. Periodicity = d = 3 μm

$$d(\sin\theta_m - \sin\theta_i) = m\lambda.$$

Substitute d = 3 μm, λ = 1.550 μm, $\theta_i = 45°$,

and calculate the diffraction angle θ_m for $m = -1$

$$(3\ \mu\text{m})[\sin\theta_1 - \sin(45°)] = (-1)(1.550\ \mu\text{m})$$

$\therefore$ $\quad \theta_1 = 10.978°$

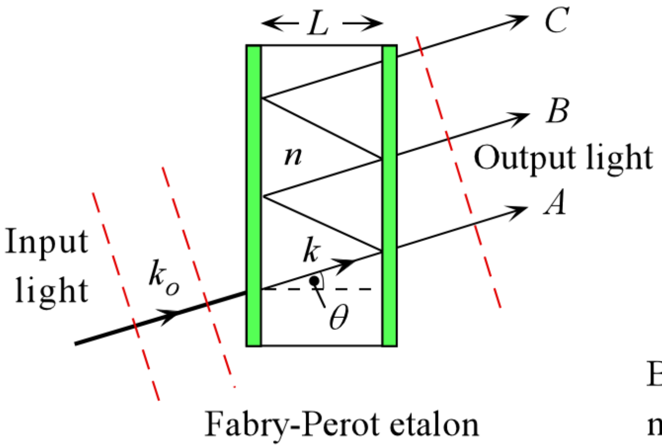
A $\lambda' = 1.540$ μm, examining the same order, $m = -1$, we find $\theta_1' = 11.173°$

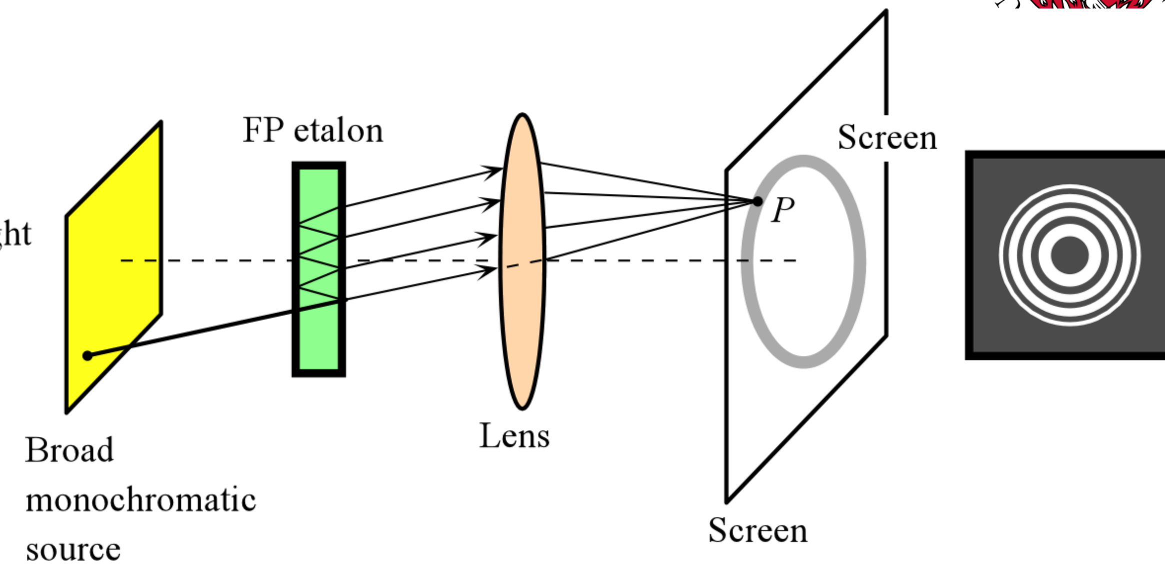
$\therefore$ $\quad \Delta\theta_1 = 11.173° - 10.978° = 0.20°$

Note, $m = 1$ gives a complex angle and should be neglected.

The problem is that the optical power in the zero-order beam is wasted. The zero-order does not separate wavelengths. Volume Phase Gratings (VPN) overcome this problem.

Fabry-Perot Interferometer





Bright rings

$$2nL\cos\theta = m\lambda;\; m = 1, 2, 3\ \ldots$$

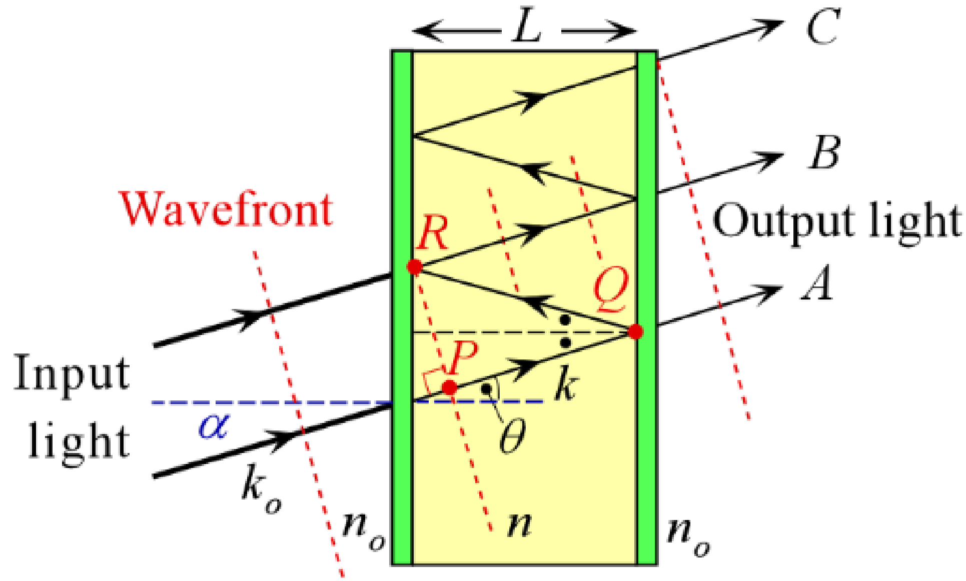
Oblique Incidence on a Fabry-Perot Cavity

Assume external reflection within the cavity.

The point P on wave A propagates to the right reflector at Q where it is reflected.

At Q, the wave experiences a phase change π.

Then it propagates to R, and gets reflected again and experiences another π phase change.



Fabry-Perot cavity

Right after reflection, the phase at R must be the same as that at the start point P; R and P are on the same wavefront.

$$\Delta\phi = \text{Phase difference from } P \text{ to } Q \text{ to } R = m(2\pi)$$

Oblique Incidence on a Fabry-Perot Cavity

Right after reflection, the phase at R must be the same as that at the start point P; R and P are on the same wavefront.

$$\Delta\phi = \text{Total phase difference from } P \text{ to } Q \text{ to } R = m(2\pi)$$

$$\Delta\phi = kPQ + \pi + kQR + \pi = k(PQ + QR) + 2\pi$$

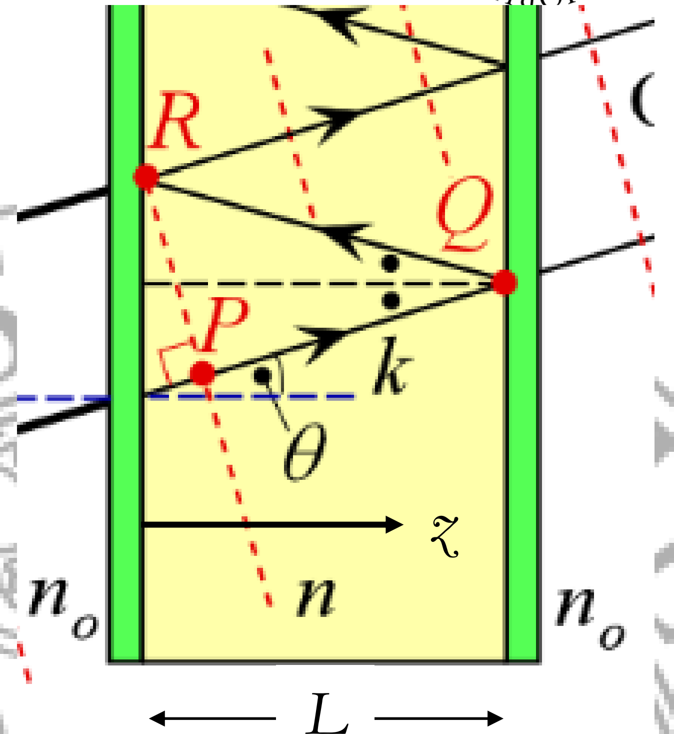
$$PQ = QR\cos 2\theta$$

$$\boxed{PQ = QR(2\cos^2\theta - 1)} \qquad \boxed{QR = L/\cos\theta}$$

$$\Delta\phi = k(PQ + QR) + 2\pi = kQR[2\cos^2\theta - 1 + 1] + 2\pi$$

$$\Delta\phi = k(L/\cos\theta)(2\cos^2\theta) + 2\pi = m(2\pi)$$

$$\Delta\phi = k\cos\theta(2L) = m(2\pi)$$



The result is as if we had resolved k along z, that is we had taken $k\cos\theta$ and considered the phase change $k\cos\theta \times (2L)$ along z

$$\boxed{k\cos\theta(2L) = m(2\pi)}$$

Oblique Incidence on a Fabry-Perot Cavity

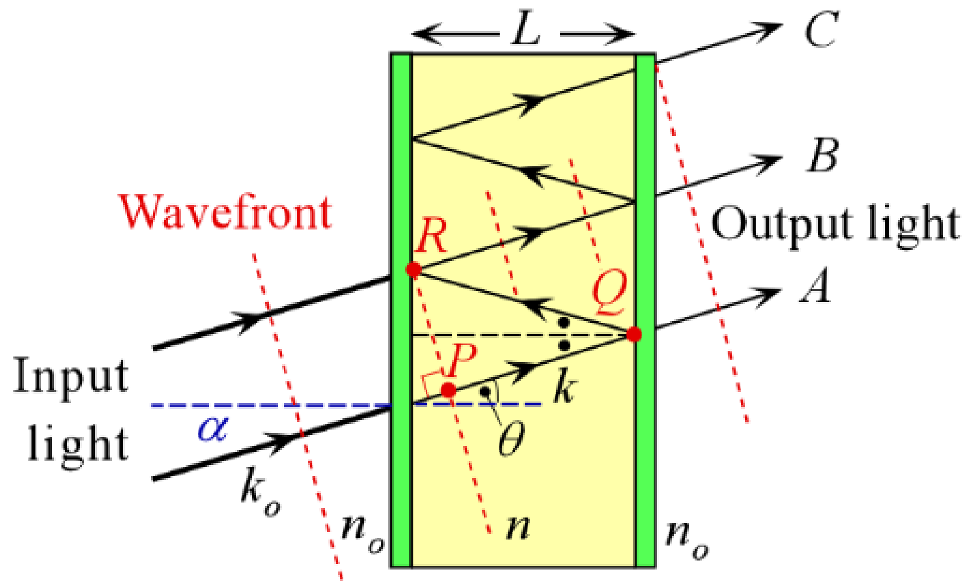
$k\cos\theta(2L) = m(2\pi)$

Substitute for $k = 2\pi n/\lambda_o$

$$L\cos\theta = m(\lambda/2) \quad \therefore \quad nL\cos\theta = m\frac{\lambda_o}{2}$$

$$L\sqrt{n^2 - n^2\sin^2\theta} = L\sqrt{n^2 - n_o^2\sin^2\alpha} = m\frac{\lambda_o}{2}$$

This is at an angle α to the etalon normal so that λ_o is $\lambda_o(\alpha)$



Fabry-Perot cavity

$$L\sqrt{n^2 - n_o^2\sin^2\alpha} = m\frac{\lambda_o(\alpha)}{2}$$

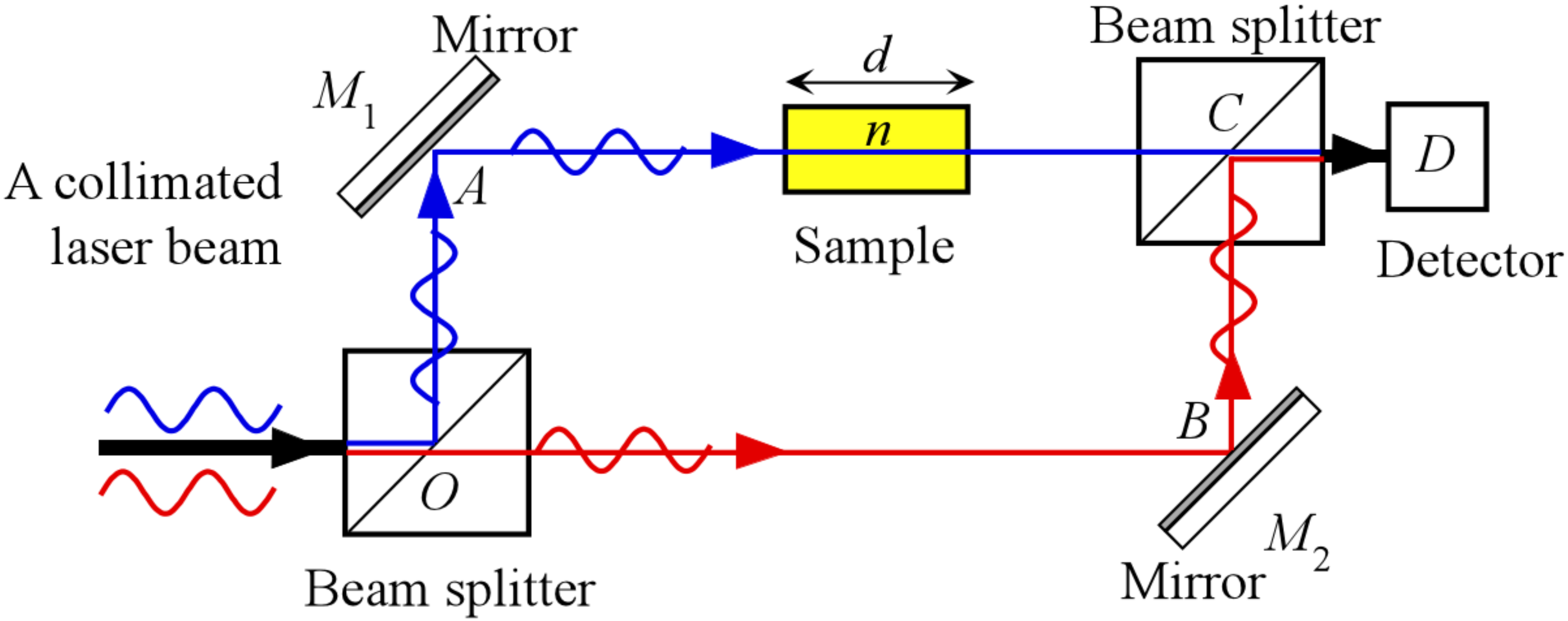
For normal incidence

$$Ln = m\frac{\lambda_o(0)}{2}$$

Take the ratio

$$\lambda_o(\alpha) = \lambda_o(0)\left[1-(n_o/n)^2\sin^2\alpha\right]^{1/2}$$

Mach-Zehnder Interferometer



$$\Phi = k_o(\overline{OAC} - d) + k\ d - k_o\overline{OBC}$$

$$\Delta\Phi = \Delta(kd) = \frac{2\pi}{\lambda}\Delta(nd)$$

Mach-Zehnder Interferometer

Resultant field at $C = A\sin(\omega t) + B\sin(\omega t - \Phi)$

Assume $A = B$ (equal splitting)

Resultant field at $C = 2A\sin(\omega t - \tfrac{1}{2}\Phi) + 2A\cos(\tfrac{1}{2}\Phi)$

Average field at $C = 2A\cos(\tfrac{1}{2}\Phi)$

Average intensity at $C = A'\cos^2(\tfrac{1}{2}\Phi)$

$$I = A'\cos^2\left(\tfrac{1}{2}\Phi\right)$$

$$\Phi = \frac{2\pi d\Delta n}{\lambda}$$

Largest change in I occurs at $\Phi = 90°$

Intensity periodically goes through maxima and minima

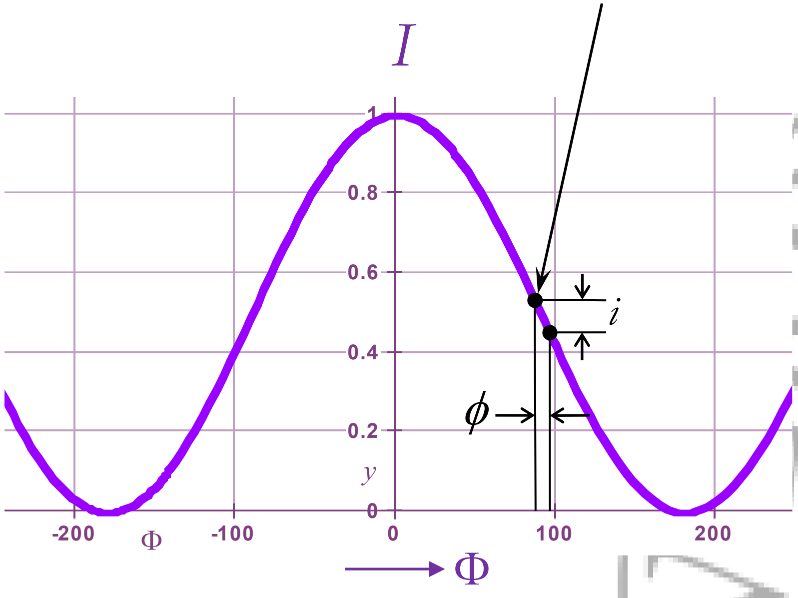
Modulating Φ modulates the intensity

Mach-Zehnder Interferometer

$$I = A' \cos^2\left(\tfrac{1}{2}\Phi\right)$$

$$\Phi = \frac{2\pi d \Delta n}{\lambda}$$

Largest change in I occurs at $\Phi = 90°$

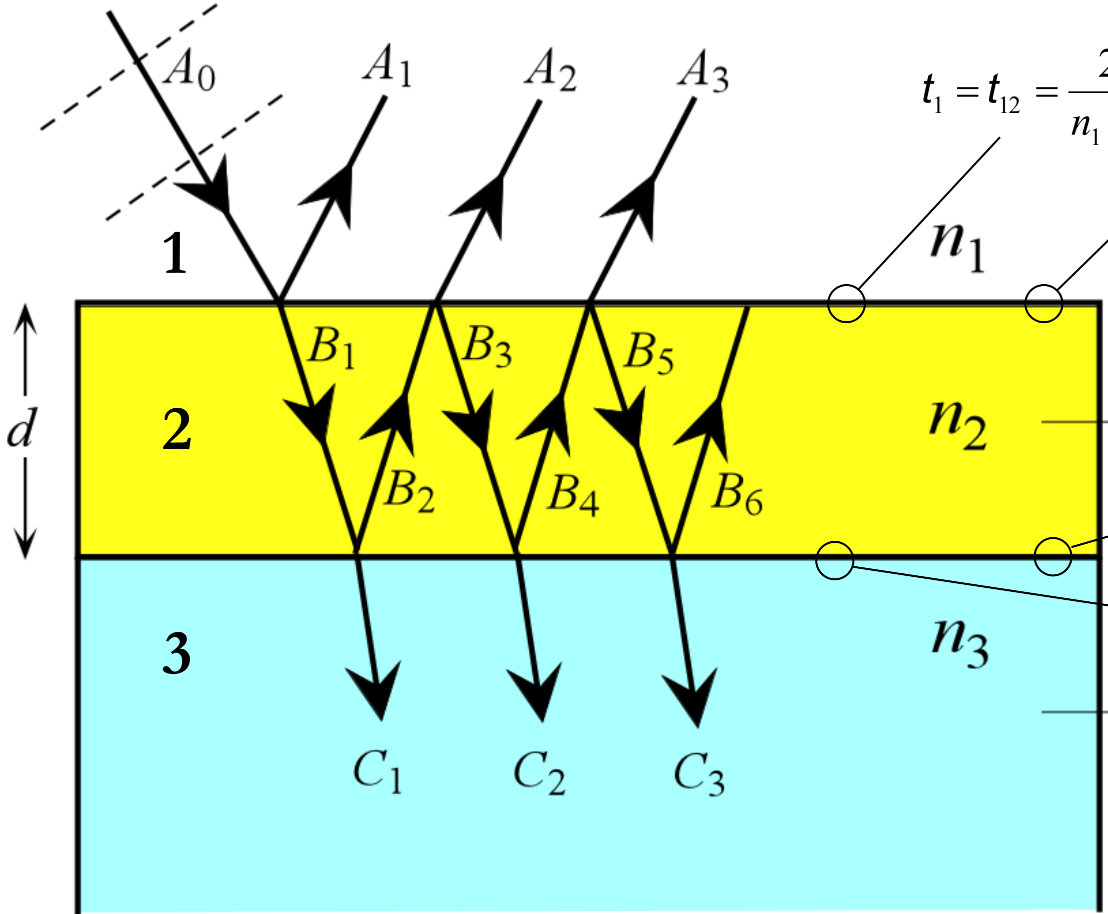


Let $\phi = \delta\Phi$ = small change in Φ at 90°

Let $i = \delta I$ = small change in I when Φ changes from 90°. We can differentiate this and find the gradient at $\Phi = 90°$. The result is

$$i = -\tfrac{1}{2} A' \phi$$

Thin Films Optics



$$t_1 = t_{12} = \frac{2n_1}{n_1 + n_2} \qquad t_1' = t_{21} = \frac{2n_2}{n_1 + n_2} \qquad 1 - t_1 t_1' = r_1^2$$

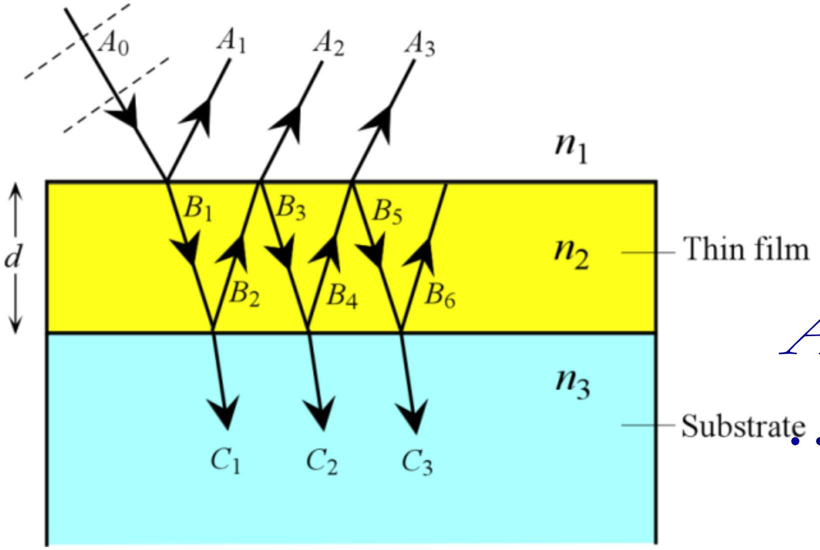
$$r_1 = r_{12} = \frac{n_1 - n_2}{n_1 + n_2} = -r_{21}$$

$$r_2 = r_{23} = \frac{n_2 - n_3}{n_2 + n_3}$$

$$t_2 = t_{23} = \frac{2n_2}{n_2 + n_3}$$

Assume normal incidence

$$\phi = 2 \times (2\pi/\lambda) n_2 d$$



$$A_{\text{reflected}} = A_1 + A_2 + A_3 + A_4 + \ldots$$

$\phi = 2\times(2\pi/\lambda)n_2 d$

$$A_{\text{reflected}}/A_0 = r_1 + t_1 t'_1 r_2 e^{-j\phi} - t_1 t'_1 r_1 r_2{}^2 e^{-j2\phi} + t_1 t'_1 r_1{}^2 r_2{}^3 e^{-j3\phi} + \ldots$$

Assume normal incidence

$$\phi = 2\times(2\pi/\lambda)n_2 d$$

Thin Films Optics

Corrected

$$r = \frac{r_1 + r_2 e^{-j\phi}}{1 + r_1 r_2 e^{-j\phi}}$$

$$t = \frac{t_1 t_2 e^{-j\phi/2}}{1 + r_1 r_2 e^{-j\phi}}$$

$$t_1 = t_{12} = \frac{2n_1}{n_1 + n_2} \qquad t_2 = t_{21} = \frac{2n_2}{n_1 + n_2} \qquad t_3 = t_{23} = \frac{2n_3}{n_2 + n_3}$$

Reflection Coefficient

$r_1 = r_2$

$$r = \frac{r_1 + r_2 e^{-j\phi}}{1 + r_1 r_2 e^{-j\phi}}$$

$\exp(-j\phi) = -1$

$\phi = 2\times(2\pi/\lambda)n_2 d = m\pi$

m is an odd integer

$$d = \frac{m\lambda}{4n_2}$$

$$n_2 = (n_1 n_3)^{1/2}$$

Choose

$n_2 = (n_1 n_3)^{1/2}$

$\therefore r_1 = r_2$

$$r = 0$$

Corrected

Transmission Coefficient

$$t = \frac{t_1 t_2 e^{-j\phi/2}}{1 + r_1 r_2 e^{-j\phi}}$$

$\exp(-j\phi) = -1$

$\phi = 2\times(2\pi/\lambda) n_2 d = m\pi$

m is an odd integer

$$d = \frac{m\lambda}{4n_2}$$

$t =$ Maximum

Minimum and Maximum Reflectance

$n_1 < n_2 < n_3$

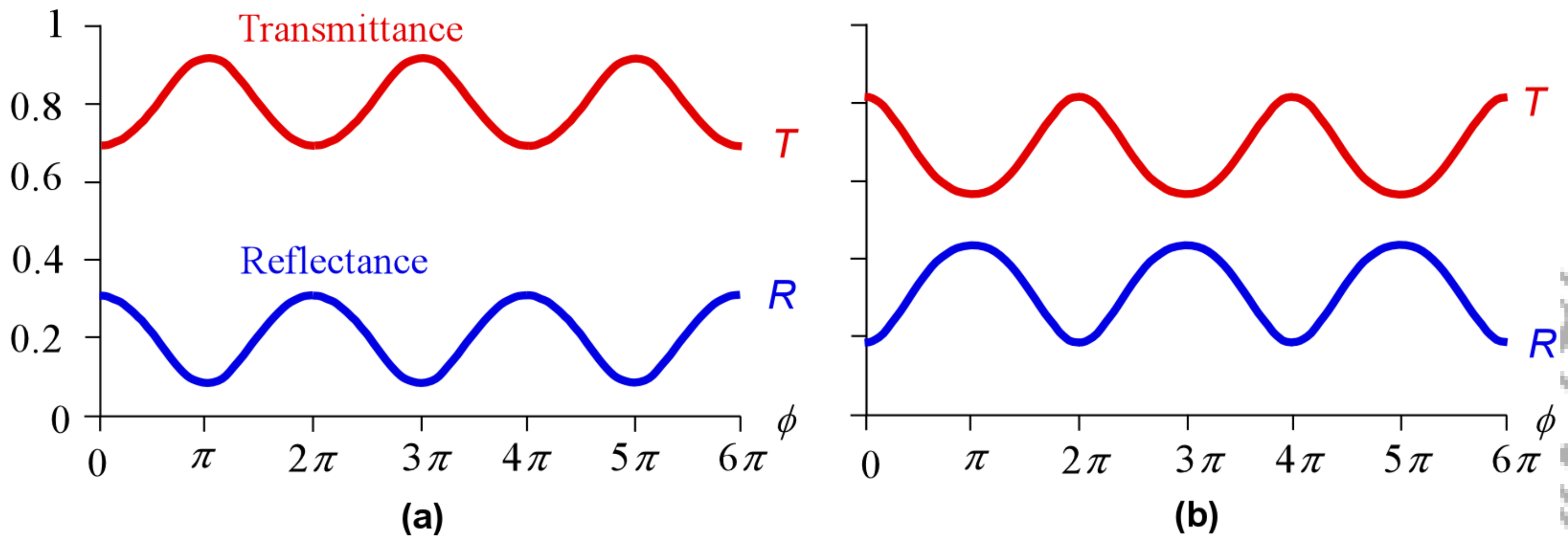
$$R_{\min} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3} \right)^2$$

$$R_{\max} = \left(\frac{n_3 - n_1}{n_3 + n_1} \right)^2$$

$n_1 < n_3 < n_2$ then $R_{\min}$ and $R_{\max}$ equations are interchanged

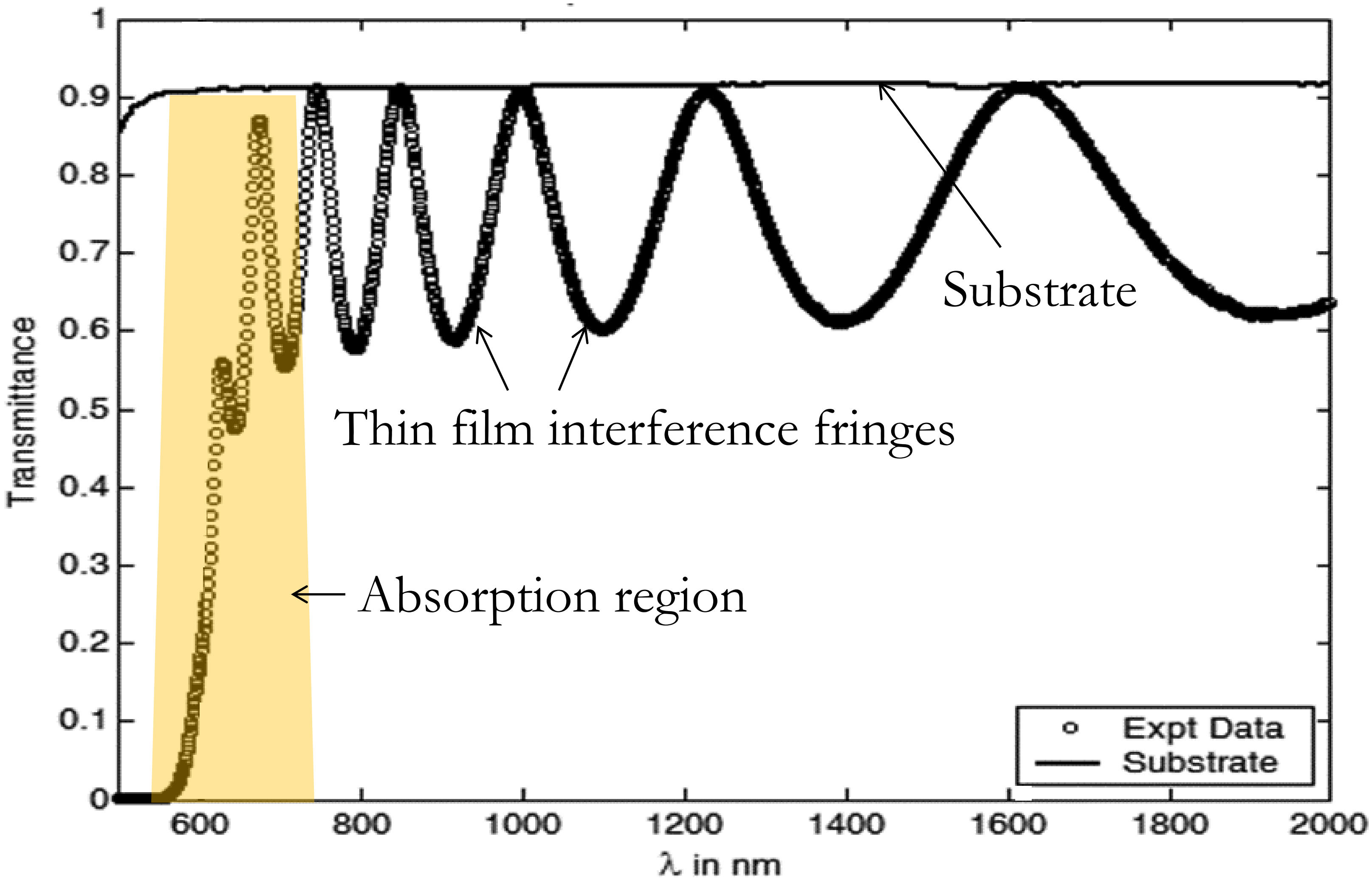
While $R_{\max}$ appears to be independent from n_2, the index n_2 is nonetheless still involved in determining maximum reflection inasmuch as R reaches $R_{\max}$ when $\phi = 2(2\pi/\lambda)n_2 d = \pi(2m)$; when $\phi = \pi \times (\text{even number})$

Reflectance and Transmittance of a Thin Film Coating

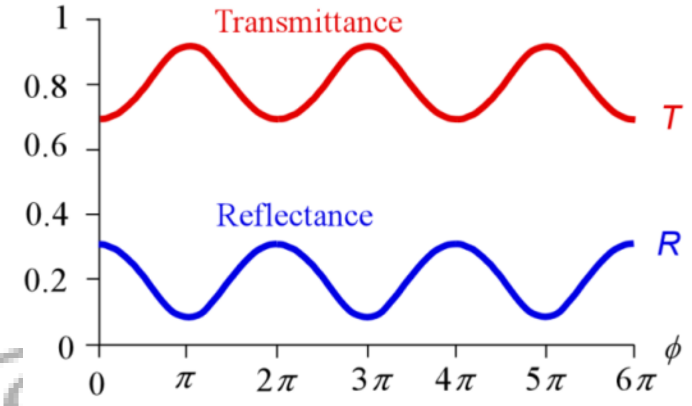


(a) Reflectance R and transmittance T vs. $\phi = 2\times(2\pi/\lambda)n_2d$, for a thin film on a substrate where $n_1 = 1$ (air), $n_2 = 2.5$ and $n_3 = 3.5$, and $n_1 < n_2 < n_3$. (b) R and T vs. ϕ for a thin film on a substrate where $n_1 = 1$ (air), $n_2 = 3.5$ and $n_3 = 2.5$, and $n_2 > n_3 > n_1$

EXAMPLE: Transmission spectra through a thin film (a-Se) on a glass



Example: Thin Film Optics



Consider a semiconductor device with $n_3 = 3.5$ that has been coated with a transparent optical film (a dielectric film) with $n_2 = 2.5$, $n_1 = 1$ (air). If the film thickness is 160 nm, find the minimum and maximum reflectances and transmittances and their corresponding wavelengths in the visible range. (Assume normal incidence.)

Solution: We have $n_1 < n_2 < n_3$. $R_{\min}$ occurs at $\phi = \pi$ or odd multiple of π, and maximum reflectance $R_{\max}$ at $\phi = 2\pi$ or an integer multiple of 2π.

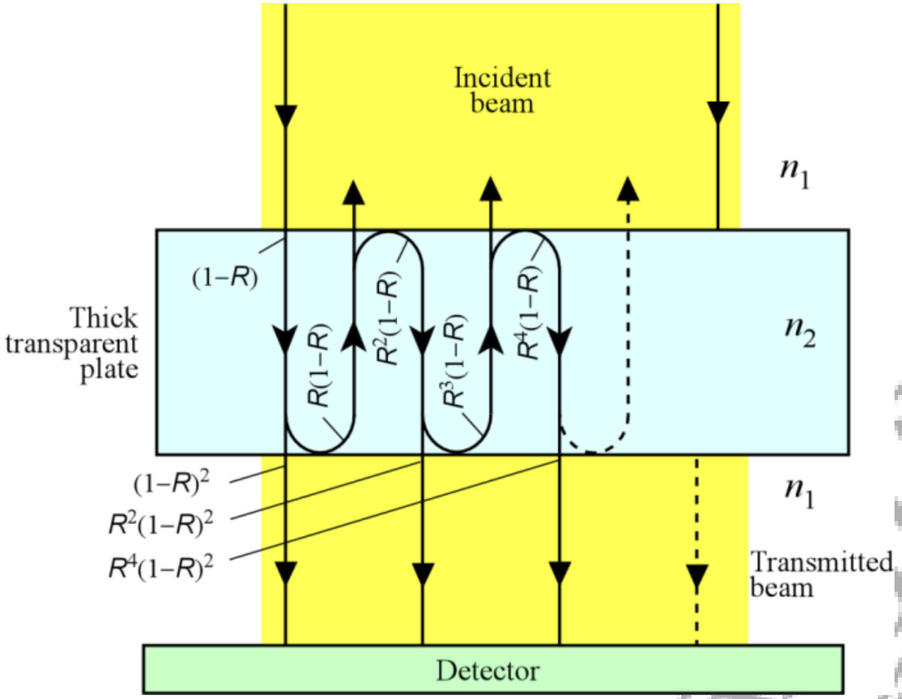
$$R_{\min} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right)^2 = \left(\frac{2.5^2 - (1)(3.5)}{2.5^2 + (1)(3.5)}\right)^2 = 0.080 \text{ or } 8.0\%$$

$$T_{\max} = 1 - R_{\min} = 0.92 \text{ or } 92\%$$

$$R_{\max} = \left(\frac{n_3 - n_1}{n_3 + n_1}\right)^2 = \left(\frac{3.5 - 1}{3.5 + 1}\right)^2 = 0.31 \text{ or } 31\%$$

$$T_{\min} = 1 - R_{\max} = 0.69 \text{ or } 69\%$$

Multiple Reflections in Plates and Incoherent Waves



$$T_{\text{plate}} = (1-R)^2 + R^2(1-R)^2 + R^4(1-R)^2 + \ldots$$

$$T_{\text{plate}} = (1-R)^2[1 + R^2 + R^4 + \ldots]$$

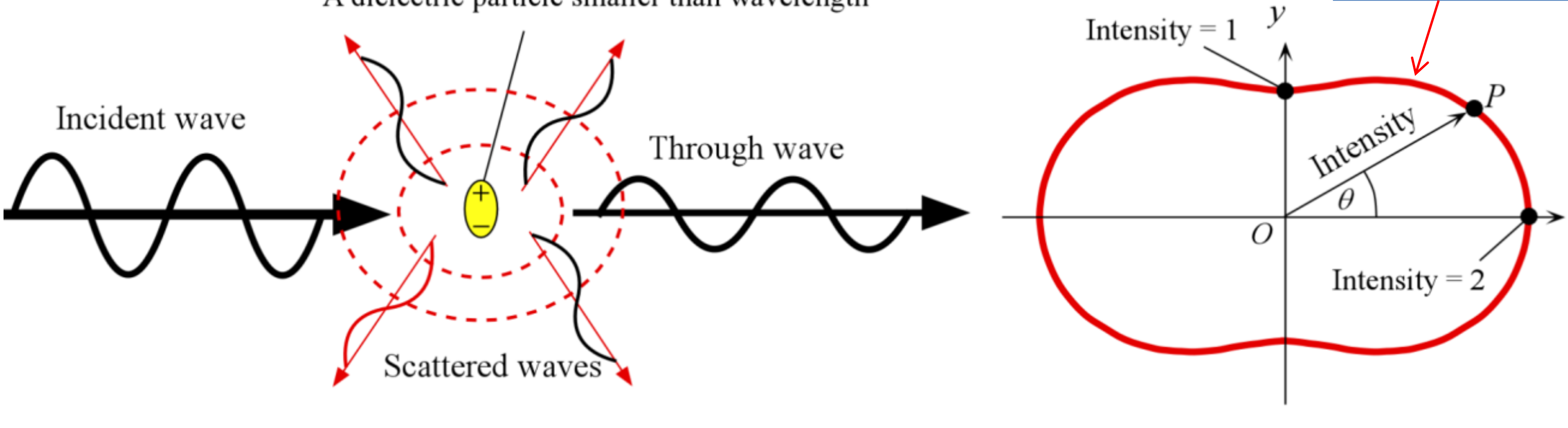
$$T_{\text{plate}} = \frac{(1-R)^2}{1-R^2}$$

$$T_{\text{plate}} = \frac{4n_1 n_2}{n_1^2 + n_2^2}$$

$$R_{\text{plate}} = \frac{(n_1 - n_2)^2}{n_1^2 + n_2^2}$$

Rayleigh Scattering

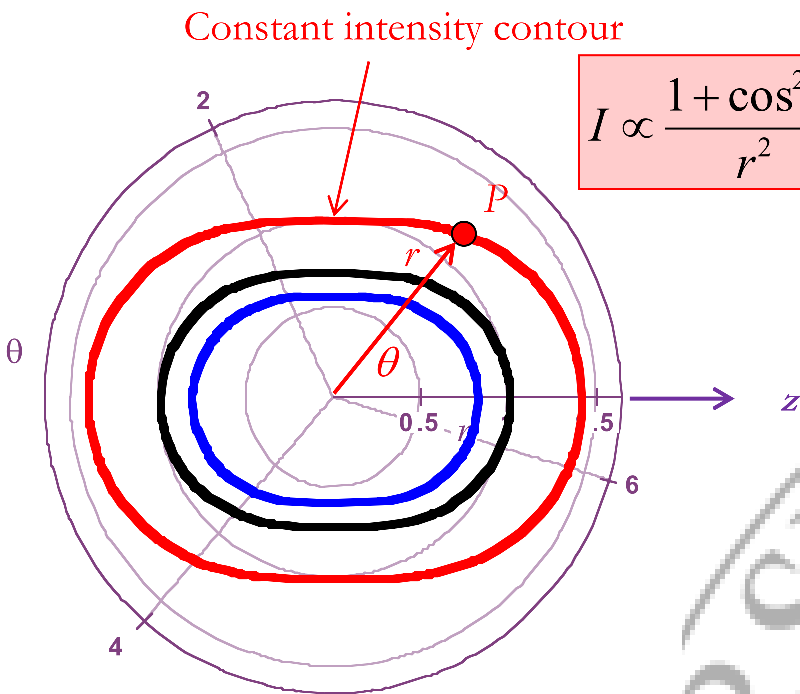
$$I \propto 1 + \cos^2 \theta$$



(a) (b)

(a) Rayleigh scattering involves the polarization of a small dielectric particle or a region that is much smaller than the light wavelength. The field forces dipole oscillations in the particle (by polarizing it) which leads to the emission of EM waves in "many" directions so that a portion of the light energy is directed away from the incident beam. (b) A polar plot of the dependence of the intensity of the scattered light on the angular direction θ with respect to the direction of propagation, x in Rayleigh scattering. (In a polar plot, the radial distance OP is the intensity.)

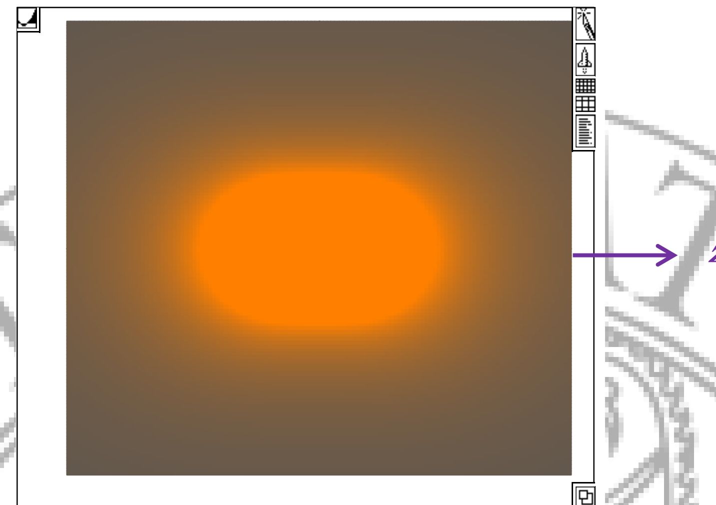
Rayleigh Scattering



$$I \propto \frac{1+\cos^2\theta}{r^2}$$

Scattered intensity contours. Each curve corresponds to a constant scattered intensity. The intensity at any location such as P on a given contour is the same. (Arbitrary units. Relative scattered intensities in arbitrary units are: blue = 1, black = 2 and red = 3)

(Generated on LiveMath)

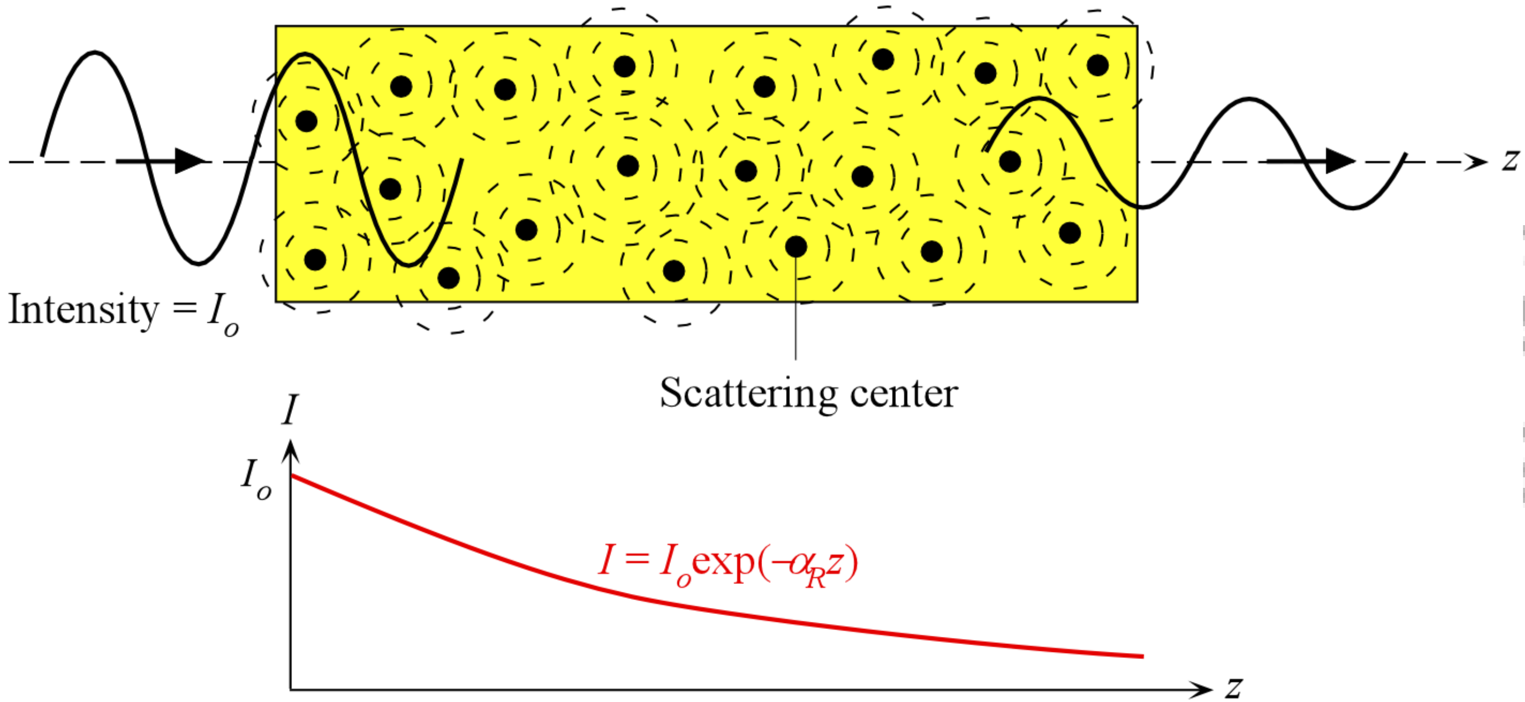


A density plot where the brightness represents the intensity of the scattered light at a given point r, θ

[Generated on LiveMath (SK)]

Rayleigh Scattering

$$I = I_o \exp(-a_R z)$$



When a light beam propagates through a medium in which there are small particles, it becomes scattered as it propagates and losses power in the direction of propagation. The light becomes attenuated.

Rayleigh Scattering

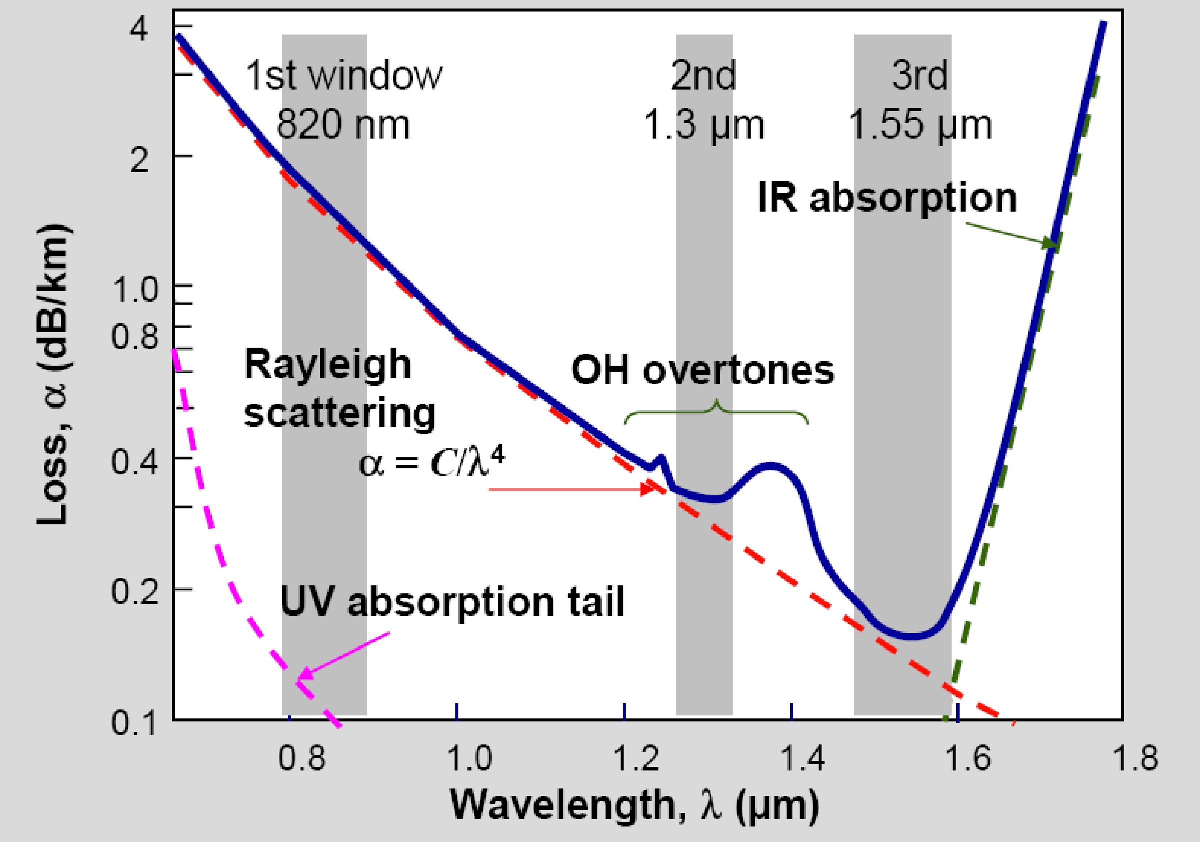
$$I = I_o \exp(-a_R z)$$

Rayleigh attenuation coefficient

$$\alpha_R \propto N \cdot a^6 \cdot \frac{1}{\lambda^4} \cdot \left(\frac{n^2 - n_o^2}{n^2 + n_o^2} \right)^2$$

Lord Rayleigh (John William Strutt) was an English physicist (1877–1919) and a Nobel Laureate (1904) who made a number of contributions to wave physics of sound and optics. He formulated the theory of scattering of light by small particles and the dependence of scattering on $1/\lambda^4$ circa 1871. Then, in a paper in 1899 he provided a clear explanation on why the sky is blue. Ludvig Lorentz, around the same time, and independently, also formulated the scattering of waves from a small dielectric particle, though it was published in Danish (1890).40 *(© Mary Evans Picture Library/Alamy.)*

Fiber Attenuation and Scattering



Attenuation due to:

- Material absorption
- Bending losses
- Rayleigh scattering

$$\alpha = \alpha_R + \alpha_M + \alpha_{MB} + \alpha_{mB}$$

Evolution of optical power along fiber length z:

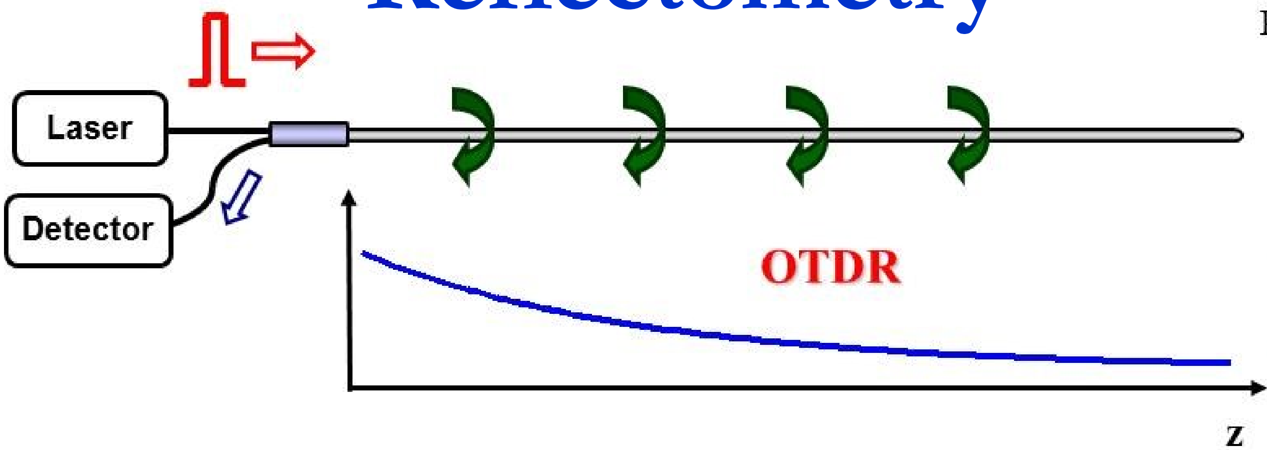
$$P(z) = P_0 \cdot e^{-\alpha_\lambda z}$$

In dB: $$P_{\mathrm{dBm}}(z) = P_{0\_\mathrm{dBm}} - \alpha_{\mathrm{dB/km}} \cdot z$$

Loss coefficient

$$\boxed{\alpha_{\mathrm{dB/km}} = \alpha \cdot 10 \log_{10} e = 4.343 \cdot \alpha}$$

Optical Time Domain Reflectometry



Pulse width determines the spatial resolution (e.g. 10 ns   1 m sp. res.)

Dynamic range limited by receiver sensitivity

Low SNR at long distances

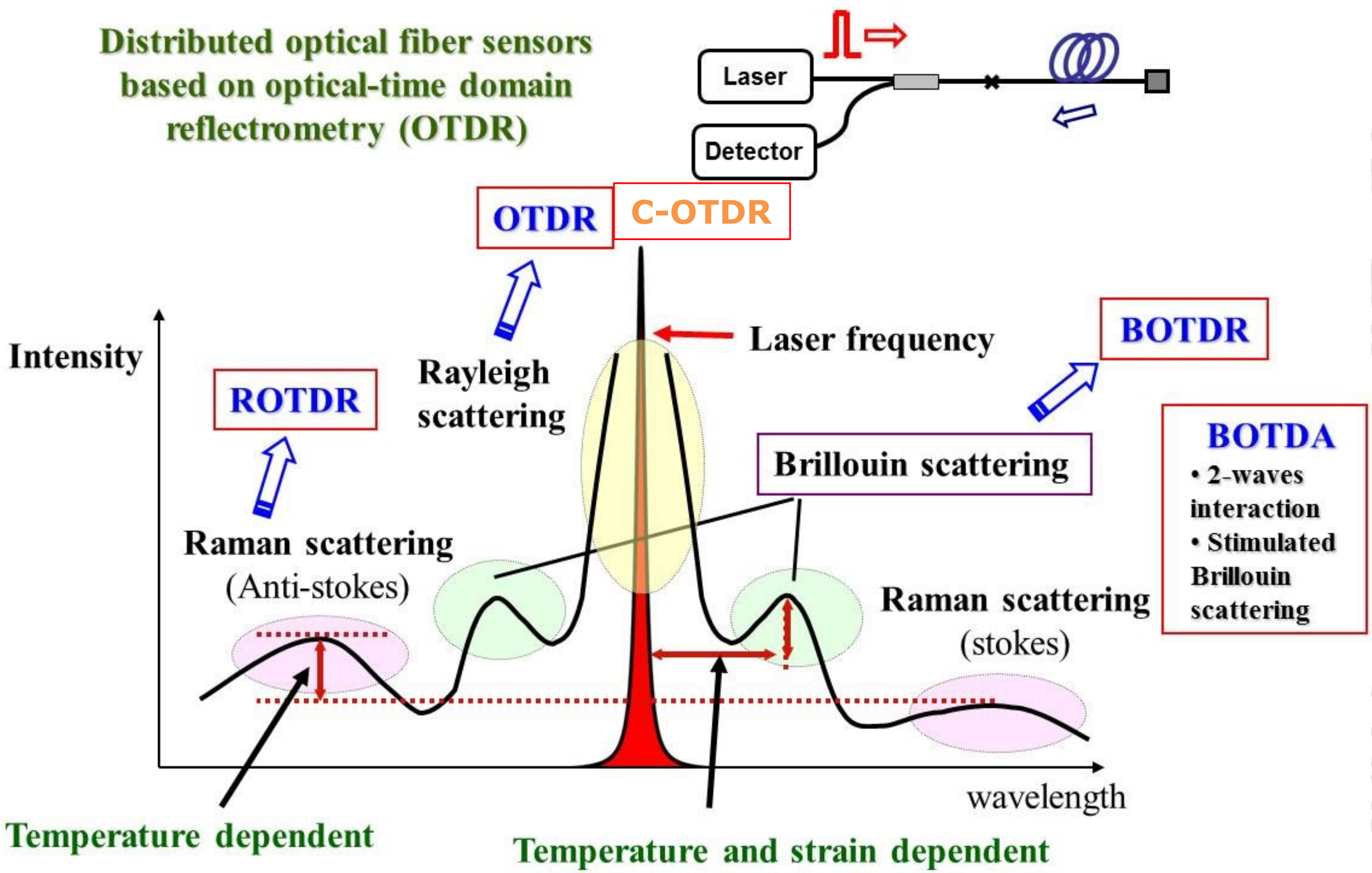
- To increase SNR at the receiver:
  - Use higher peak power → limited by nonlinear effects
  - Use longer pulses → degrades the spatial resolution

Trade off between distance range and spatial resolution

- To overcome the limitations
  - Use receiver with higher sensitivity (e.g. coherent detectors)
  - Use of optical pulse coding

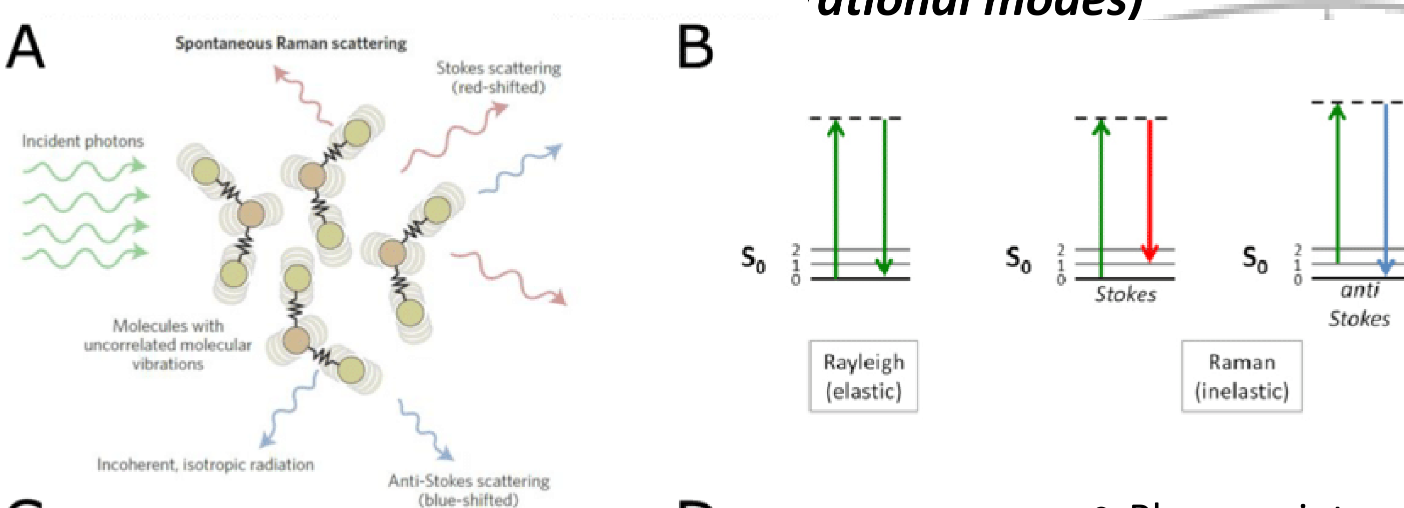
Coded-OTDR techniques:
- Spreading signal in time domain
- More optical input power
- It avoids to use high peak power pulses (nonlinearities)
- It allows to improve the SNR with no impact on the spatial resolution

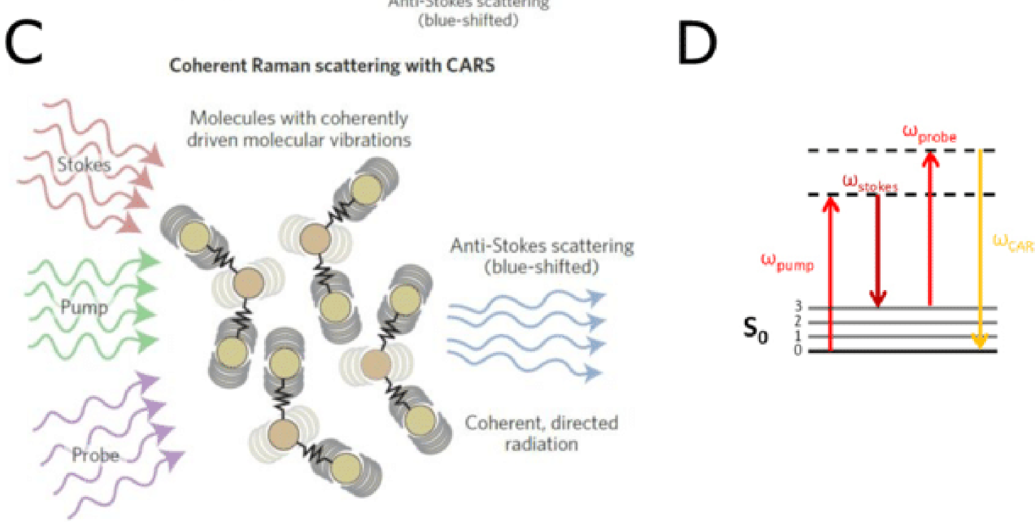
Scattering Phenomena in optical fibers



Spontaneous Raman Scattering

Raman scattering is generated by light interaction with
resonant modes of the molecules in the medium
rational modes)





- Phonons interact with photons in inelastic scattering
- Stokes line → photon energy is given to phonon
- Anti-Stokes line → phonon gives energy to photon

Spontaneous Raman Scattering

- *The high energy of vibrational modes induces a large Raman frequency shift (~13 THz in silica fibers)*

- *For each molecular vibration two Raman components are observed:*

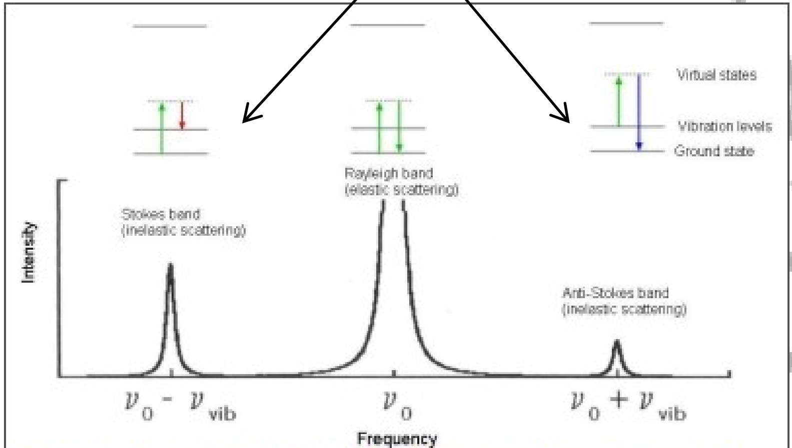
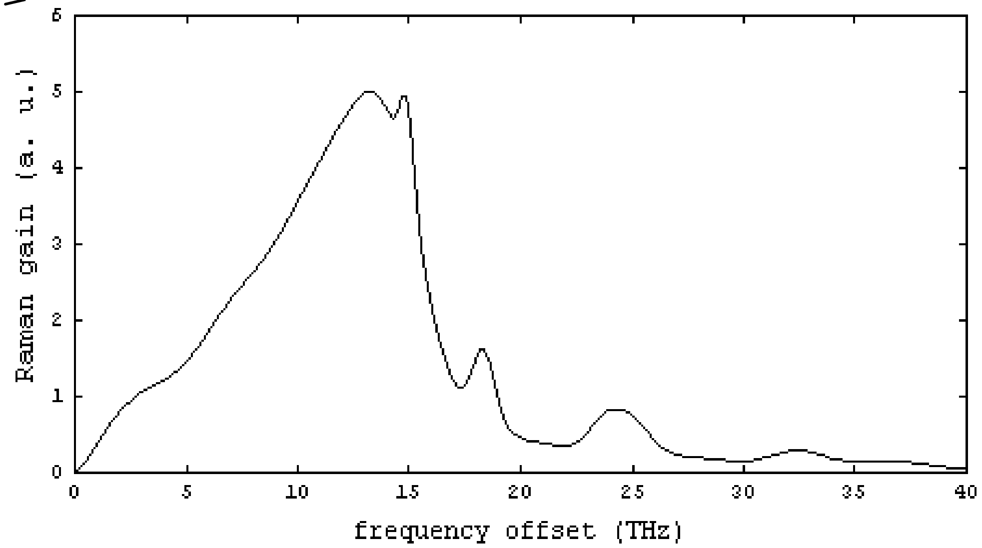


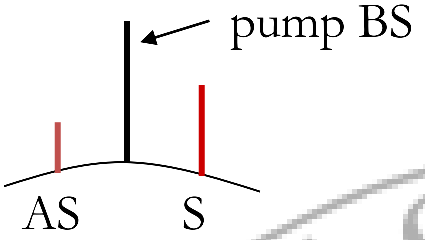
Fig. 4.1.2: Spectrum of photon transition from elemental ground level to virtual states.

© Mascarós et al., in: "Handbook on the Use of Lasers in Conservation and Conservation Science", 2008.



- *At different increasing temperature thermal excitation increases both Raman S and AS (asymmetry between Raman S and AS !)*

Spontaneous Raman Scattering

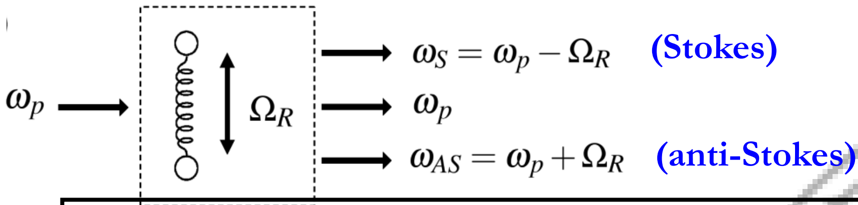


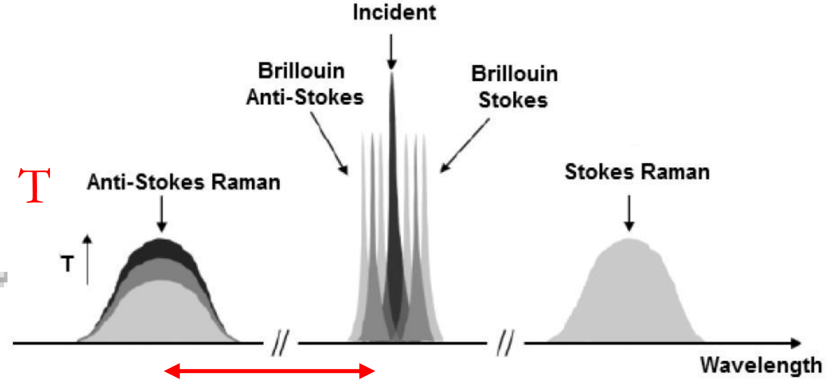
T dependence of phonon population → T dependence of SRS light

Ratio R(T), P_{AS}/P_S (or P_{AS}/P_{BS}) is usually used

Raman temperature sensitivity: 0.8% K^{-1}

Spontaneous Raman Scattering

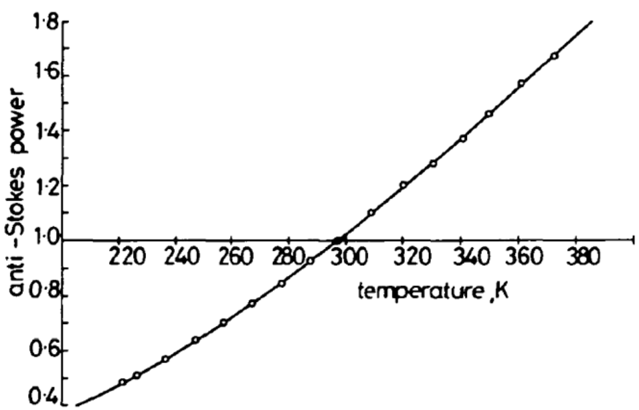




Suitable for Temperature Sensing only

12.5 THz
(100 nm @1550 nm)

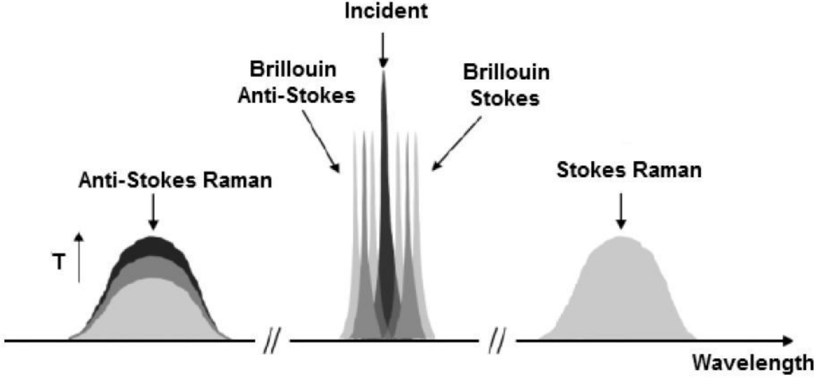
$$R(T,z) = C_R\, exp\left(-\frac{h\Delta\nu_R}{k_B T(z)}\right)$$

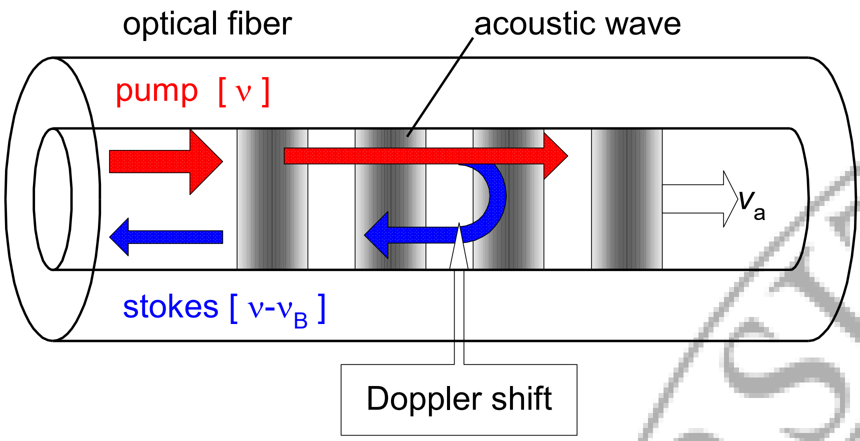


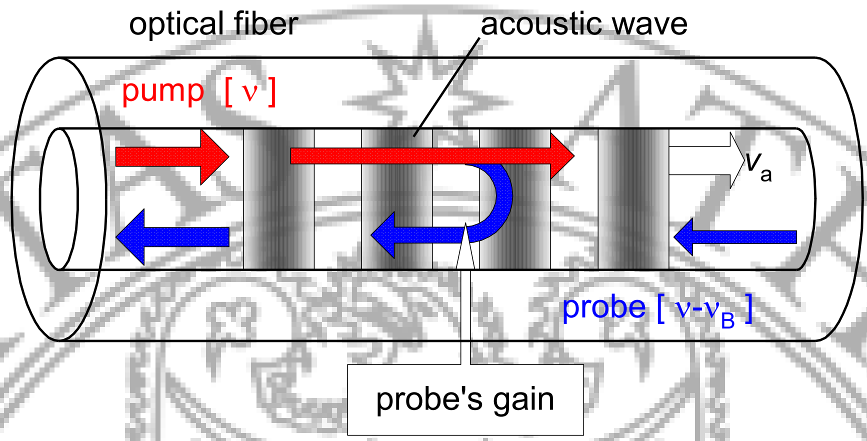
Sensitivity: 0.8 % K^{-1}

Advantages	***Disadvantages***
Easy detection	Low backscattered power
High sensitivity	High input power required

Brillouin Scattering





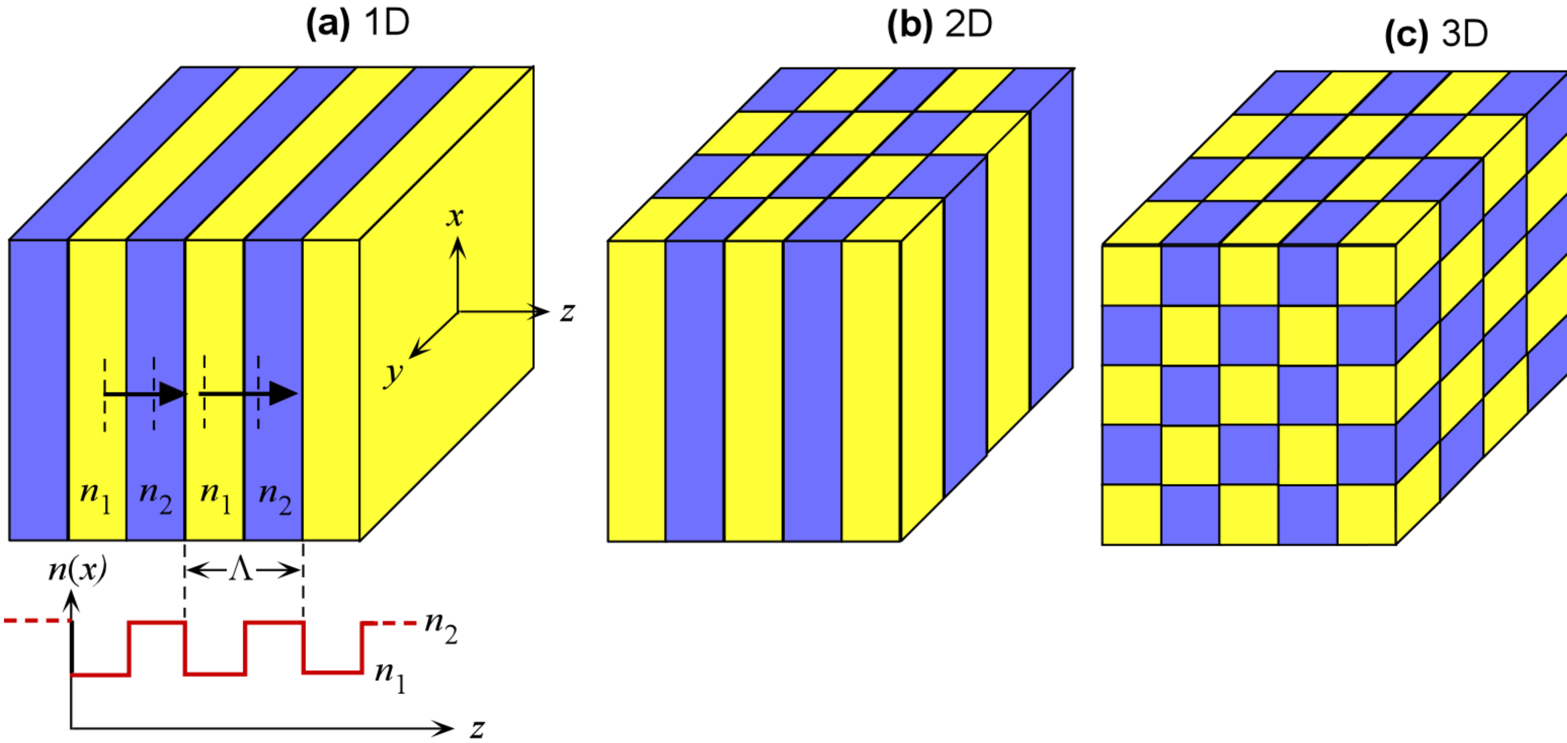


Spontaneous Brillouin Scattering

Stimulated Brillouin Scattering

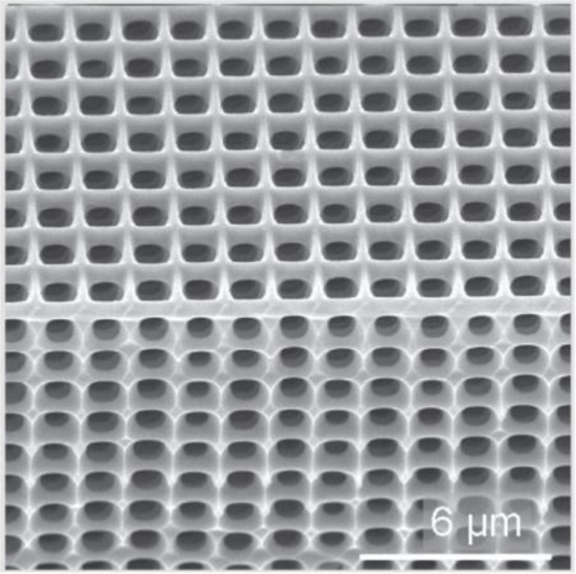
- Acoustic wave works as a diffraction grating
- Stokes wave’s frequency is down-shifted by Doppler shift
- Probe’s gain profile is called Brillouin gain spectrum (BGS)

Photonic Crystals

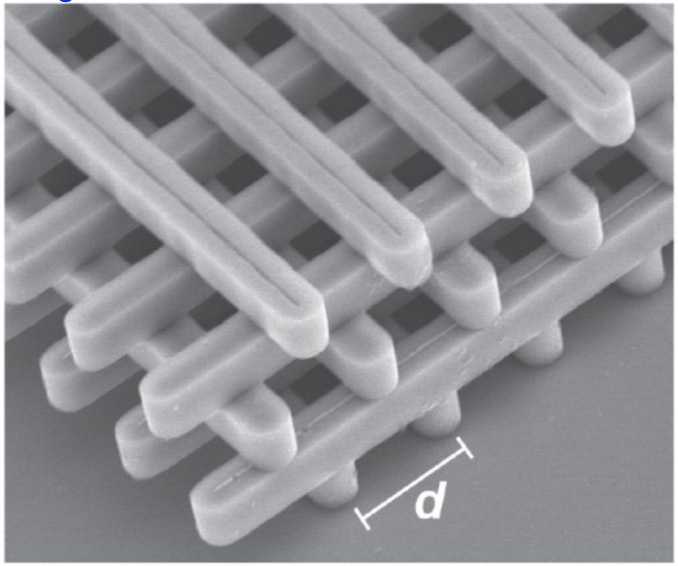


Photonic crystals in (a) 1D, (b) 2D and (c) 3D, D being the dimension. Grey and white regions have different refractive indices and may not necessarily be the same size. Λ is the periodicity. The 1D photonic crystal in (a) is the well-known Bragg reflector, a dielectric stack.

Photonic Crystals



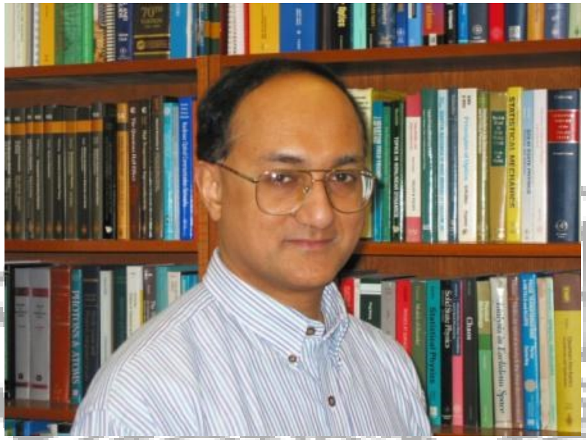
An SEM image of a 3D photonic crystal made from porous silicon in which the lattice structure is close to being simple cubic. The silicon squares, the unit cells, are connected at the edges to produce a cubic lattice. This 3D PC has a photonic bandgap centered at 5 μm and about 1.9 μm wide. *(Courtesy of Max-Planck Institute for Microstructure Physics.)*



An SEM image of a 3D photonic crystal that is based on the *wood pile* structure. The rods are polycrystalline silicon. Although 5 layers are shown, the unit cell has 4 layers *e.g.*, the fours layers starting from the bottom layer. Typical dimensions are in microns. In one similar structure with rod-to-rod pitch d = 0.65 μm with only a few layers, the Sandia researchers were able to produce a photonic bandgap of 0.8 μm centered around 1.6 μm within the telecommunications band. *(Courtesy of Sandia National Laboratories.)*

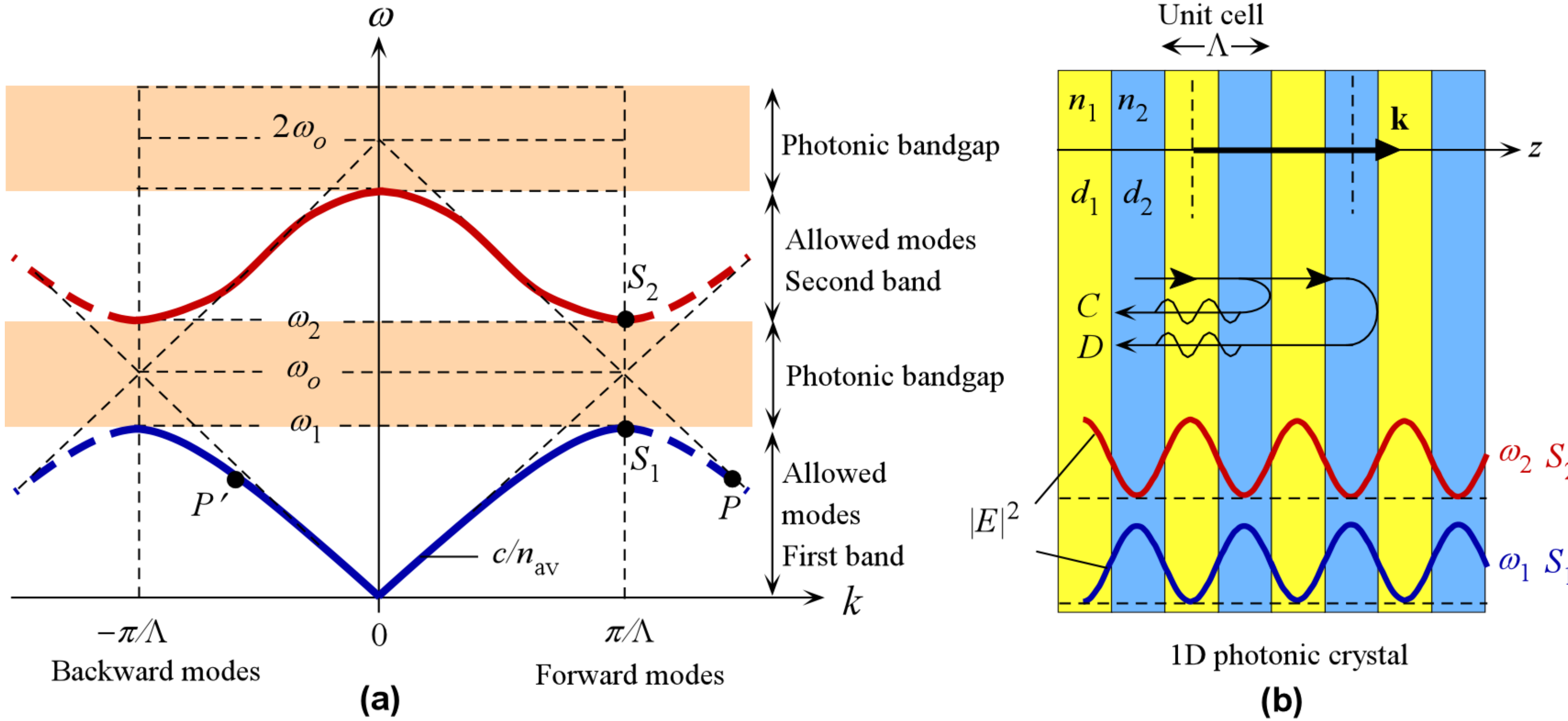
Photonic Crystals

Eli Yablonovitch (left) at the University of California at Berkeley, and Sajeev John (below) at the University of Toronto, carried out the initial pioneering work on photonic crystals. Eli Yablonovitch has suggested that the name "photonic crystal" should apply to 2D and 3D periodic structures with a large dielectric (refractive index) difference. (E. Yablonovitch, "Photonic crystals: what's in a name?", *Opt. Photon. News*, **18**, 12, 2007.) Their original papers were published in the same volume of Physical Review Letters in 1987. According to Eli Yablonovitch, "Photonic Crystals are semiconductors for light." (Courtesy of Eli Yablonovitch)



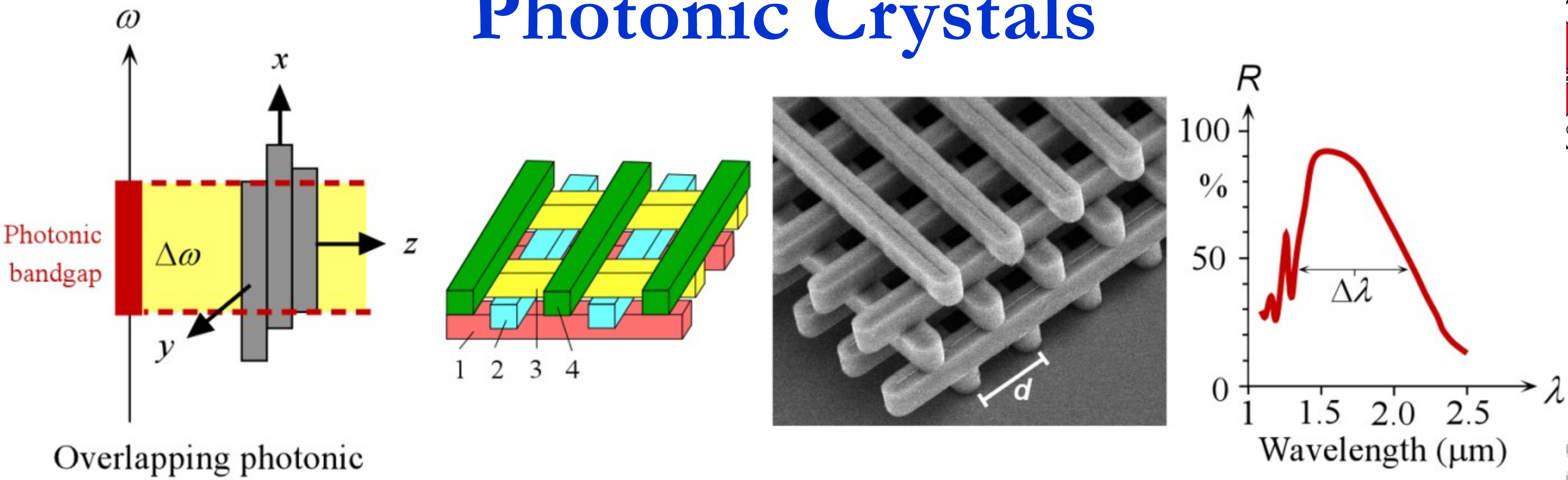
Sajeev John (right), at the University of Toronto, along with Eli Yablonovitch (above) carried out the initial pioneering work in the development of the field of photonics crystals. Sajeev John was able to show that it is possible to trap light in a similar way the electron is captured, that is localized, by a trap in a semiconductor. Defects in photonic crystals can confine or localize electromagnetic waves; such effects have important applications in quantum computing and integrated photonics. (Courtesy of Sajeev John)

Photonic Crystals



Dispersion relation, ω vs k, for waves in a 1D PC along the z-axis. There are allowed modes and forbidden modes. Forbidden modes occur in a band of frequencies called a photonic bandgap. (b) The 1D photonic crystal corresponding to (a), and the corresponding points S_1 and S_2 with their stationary wave profiles at ω_1 and ω_2.

Photonic Crystals

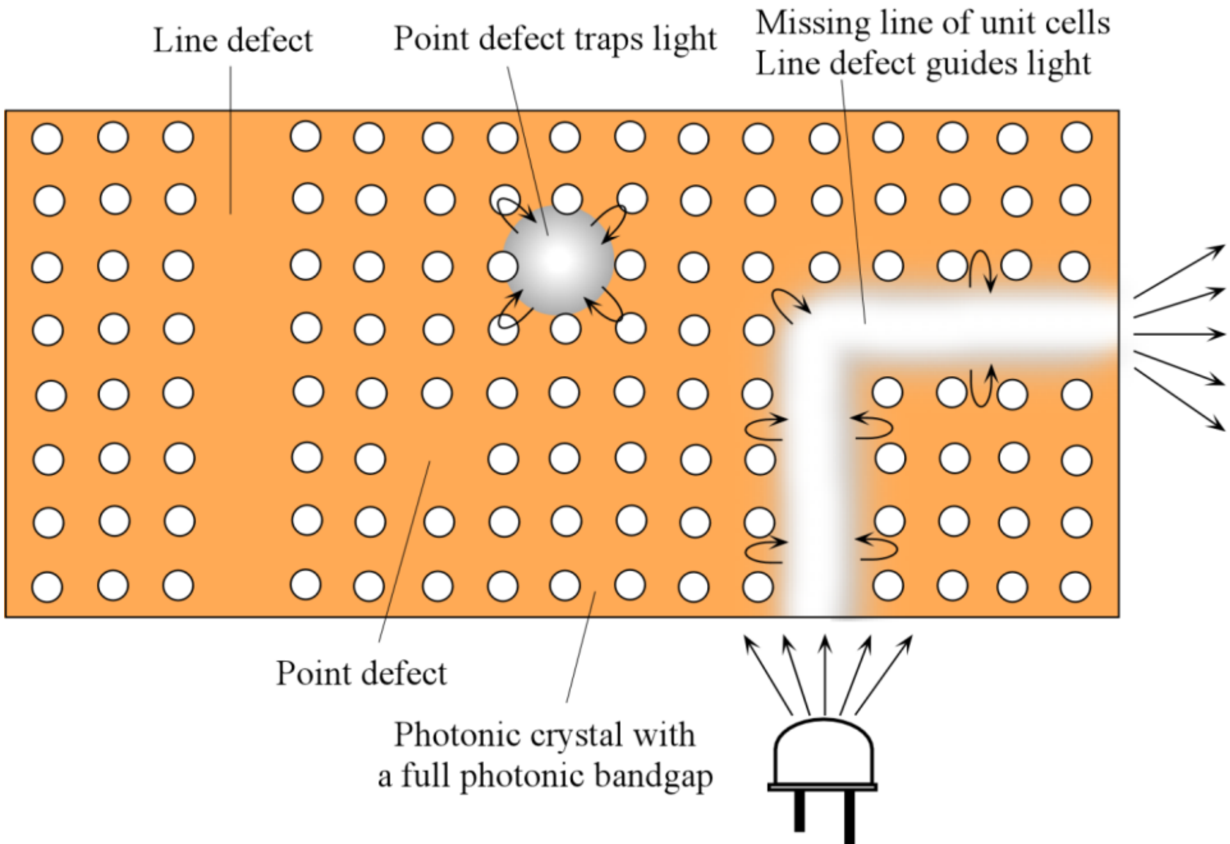


Overlapping photonic bandgaps | Woodpile unit cell | 3D photonic crystal | Reflectance spectrum

(a) **(b)** **(c)** **(d)**

The photonic bandgaps along *x, y* and *z* overlap for all polarizations of the field, which results in a full photonic bandgap $\Delta\omega$. (An intuitive illustration.) (b) The unit cell of a *woodpile* photonic crystal. There are 4 layers, labeled 1-4 in the figure, with each later having parallel "rods". The layers are at right angles to each other. Notice that layer 3 is shifted with respect to 1, and 4 with respect to 2. (c) An SEM image of a 3D photonic crystal that is based on the *wood pile* structure. The rods are polycrystalline silicon. Although 5 layers are shown, the unit cell has 4 layers *e.g.*, the fours layers starting from the bottom layer. (*Courtesy of Sandia National Laboratories.*) (d) The optical reflectance of a woodpile photonic crystal showing a photonic bandgap between 1.5 and 2 μm. The photonic crystal is similar to that in (c) with five layers and $d \approx 0.65$ μm. (Source: The reflectance spectrum was plotted using the data appearing in Fig. 3 in S-Y. Lin and J.G. Fleming, *J. Light Wave Technol.*, **17**, 1944, 1999.)

Photonic Crystals for Light Manipulation

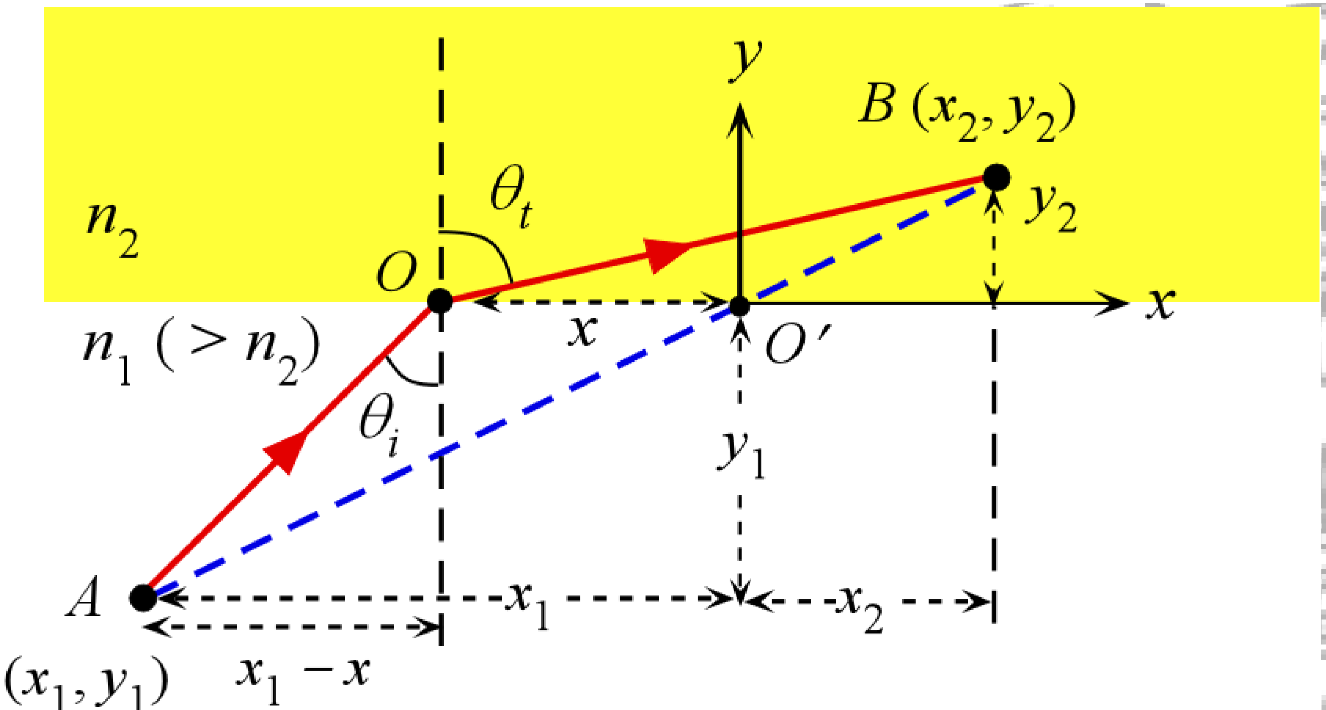


Schematic illustration of point and line defects in a photonic crystal. A point defect acts as an optical cavity, trapping the radiation. Line defects allow the light to propagate along the defect line. The light is prevented from dispersing into the bulk of the crystal since the structure has a full photonic bandgap. The frequency of the propagating light is in the bandgap, that is in the stop-band.

Slides on Questions and Problems

Fermat's principle of least time

Fermat's principle of least time in simple terms states that *when light travels from one point to another it takes a path that has the shortest time*. In going from a point A in some medium with a refractive index n_1 to a point B in a neighboring medium with refractive index n_2, the light path is AOB that involves refraction at O and satisfies Snell's law. The time it takes to travel from A to B is minimum only for the path AOB such that the incidence and refraction angles θ_i and θ_t satisfy Snell's law.



Consider a light wave traveling from point A (x_1, y_1) to B (x_1, y_2) through an arbitrary point O at a distance x from O'. The principle of least time from A to B requires that O is such that the incidence and refraction angles obey Snell's law.

Fermat's principle of least time

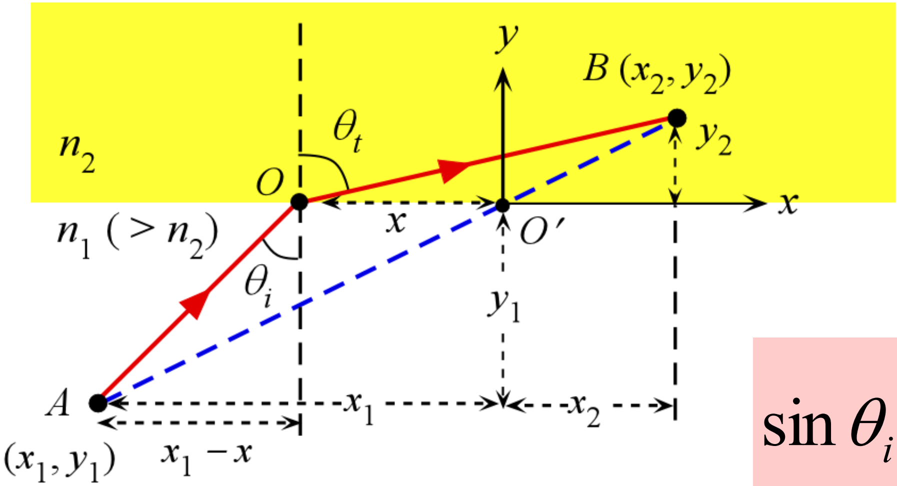
Fermat's principle of least time in simple terms states that:

When light travels from one point to another it takes a path that has the shortest time.

Pierre de Fermat (1601–1665) was a French mathematician who made many significant contributions to modern calculus, number theory, analytical geometry, and probability.
(Courtesy of Mary Evans Picture Library/Alamy.)

Fermat's principle of least time

Let's draw a straight line from A to B cutting the x-axes at O'. The line $AO'B$ will be our reference line and we will place the origin of x and y coordinates at O'. Without invoking Snell's law, we will vary point O along the x-axis (hence OO' is a variable labeled x), until the time it takes to travel AOB is minimum, and thereby derive Snell's law. The time t it takes for light to travel from A to B through O is

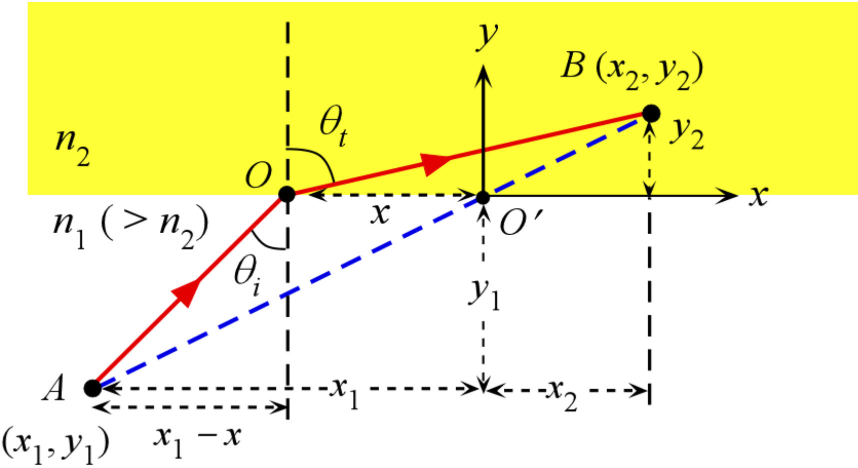


$$t = \frac{AO}{c/n_1} + \frac{OB}{c/n_2}$$

$$= \frac{[(x_1 - x)^2 + y_1^2]^{1/2}}{c/n_1} + \frac{[(x_2 + x)^2 + y_2^2]^{1/2}}{c/n_2}$$

$$\sin\theta_i = \frac{x_1 - x}{[(x_1 - x)^2 + y_1^2]^{1/2}}$$

$$\sin\theta_t = \frac{(x_2 + x)}{[(x_2 + x)^2 + y_2^2]^{1/2}}$$



$$t = \frac{AO}{c/n_1} + \frac{OB}{c/n_2}$$

$$= \frac{[(x_1 - x)^2 + y_1^2]^{1/2}}{c/n_1} + \frac{[(x_2 + x)^2 + y_2^2]^{1/2}}{c/n_2}$$

$$\frac{dt}{dx} = \frac{-1/2 \times 2(x_1 - x)[(x_1 - x)^2 + y_1^2]^{-1/2}}{c/n_1} + \frac{1/2 \times 2(x_2 + x)[(x_2 + x)^2 + y_2^2]^{-1/2}}{c/n_2}$$

The time should be minimum so $\dfrac{dt}{dx} = 0$

$$\frac{-(x_1 - x)[(x_1 - x)^2 + y_1^2]^{-1/2}}{c/n_1} + \frac{(x_2 + x)[(x_2 + x)^2 + y_2^2]^{-1/2}}{c/n_2} = 0$$

$$\frac{(x_1 - x)}{c/n_1[(x_1 - x)^2 + y_1^2]^{1/2}} = \frac{(x_2 + x)}{c/n_2[(x_1 - x)^2 + y_1^2]^{1/2}} \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$$

Snell's Law