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On the Possibility of a World with Constant Negative Curvature of Space†

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§1. 1. In our Notice "On the curvature of space"[1] we have considered those solutions of the *Einstein* world equations, which lead to world types that possess a positive constant curvature as a common feature; we have discussed all such possible cases. The possibility of deriving from the world equations a world of constant positive spatial curvature stands, however, in close relation with the question of the finiteness of space. For this reason it may be of interest to investigate whether one can obtain from the same world equations a world of constant negative curvature, the finiteness of which (even under some supplementary assumptions) can hardly be argued for.

In the present Notice it will be shown that it actually is possible to derive from the *Einstein* world equations a world with constant negative curvature of space. As in the cited work, so here too we have to distinguish two cases, namely 1. the case of a stationary world, whose curvature is constant in time, and 2. the case of a non-stationary world, whose curvature is spatially constant, but varies, however, in the course of time. There is an essential difference between the stationary worlds of constant negative and those of constant positive spatial curvature. The worlds of stationary negative curvature namely do not allow for positive density of

† Originally published in *Zeitschrift für Physik* **21**, 326-332 (1924), with the title "Uber die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes". See footnote to the first paper's title.

[1] ZS. f. Phys. **10**, 377, 1922, Heft 6.

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matter; this is either zero or negative. The physically possible stationary worlds (i.e. those with non-negative density of matter) therefore find their analogue in the *de Sitter*, but not in the *Einstein* world.[2]

At the end of this Notice we will touch upon the question of whether on the grounds of the curvature of space one is allowed at all to judge on its finiteness or infinitude.

2. We turn ourselves to our general assumptions, that we think of as grouped into the same two classes as in the cited Notice; thereby we maintain our earlier notation. The assumptions of the first class consist of laying down as the *Einstein* world equations the equations (A), (B), (C) of the mentioned work. The assumptions of the second class will now be different from the earlier ones. Assuming that one of the world coordinates, x_4, may be designated as time coordinate, we can thus (for the considered case of the world with negative constant curvature of space) express the assumptions of the second class by demanding that the interval ds shall be of the form:

$$ds^2 = \frac{R^2({dx_1}^2 + {dx_2}^2 + {dx_3}^2)}{{x_3}^2} + M^2 {dx_4}^2, \tag{D$'$}$$

where R denotes a function of time and M a function of all four world coordinates. The constant negative curvature of space of our world is hereby proportional to $-\frac{1}{R^2}$.[3]

Taking into account that for our world ds^2 is an indefinite form, we can, by changing the notation, rewrite the formula (D$'$) according to:

$$d\tau^2 = -\frac{R^2}{c^2}\frac{({dx_1}^2 + {dx_2}^2 + {dx_3}^2)}{{x_3}^2} + M^2 {dx_4}^2. \tag{D$''$}$$

Of course the spatial curvature of our world remains negative and proportional to $-\frac{1}{R^2}$.

The problem that lies before us consists of the determination of two functions R and M, which shall obey the *Einstein* world equations, i.e. the equations (A), (B), (C) of the mentioned Notice.

Setting $i = 1, 2, 3$, $k = 4$ in (A), we obtain the following three equations:

$$R'(x_4)\frac{\partial M}{\partial x_1} = R'(x_4)\frac{\partial M}{\partial x_2} = R'(x_4)\frac{\partial M}{\partial x_3} = 0\,.$$

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[2] I was made aware by my friend Prof. Dr. *Tamarkine* [Correctly, 'Tamarkin' — *Eds.*] of the necessity for a special investigation of the possibility of a world with negative curvature measure for space.

[3] With respect to the line element ds^2 see e.g. *Bianchi*, Lezioni di geometria differenziale **1**, 345.

These equations show that the considered worlds can belong to one of the two types:

1st type. Stationary worlds, $R' = 0$, R is constant in time.

2nd type. Non-stationary worlds, $R' \neq 0$, M depends only on time.

We consider first the case of the stationary world; the case of the non-stationary world offers a great similarity to that of the non-stationary world of constant positive spatial curvature; for this reason we will deal with this second case only briefly.

§2. 1. The equations (A) yield for the indices $i, k = 1, 2, 3$:*

$$\frac{\partial^2 M}{\partial x_1\, \partial x_2} = 0,$$

$$\frac{\partial^2 M}{\partial x_2\, \partial x_3} + \frac{1}{x_3}\frac{\partial M}{\partial x_2} = 0, \qquad \frac{\partial^2 M}{\partial x_1\, \partial x_3} + \frac{1}{x_3}\frac{\partial M}{\partial x_1} = 0.$$

The integration of these equations gives:

$$M = \frac{P(x_1, x_4) + Q(x_2, x_4)}{x_3} + L(x_3, x_4), \tag{1}$$

where P, Q and L for the moment are arbitrary functions of their arguments.

The equations (A) serve us for the determination of P, Q and L, where one has to set $i, k = 1, 2, 3$.

The calculation gives:

$$\left.\begin{aligned} -\frac{1}{M}\left(\frac{\partial^2 M}{\partial x_2{}^2} + \frac{\partial^2 M}{\partial x_3{}^2}\right) &= \frac{1-\lambda R^2}{x_3{}^2}, \\ -\frac{1}{M}\left(\frac{\partial^2 M}{\partial x_1{}^2} + \frac{\partial^2 M}{\partial x_3{}^2}\right) &= \frac{1-\lambda R^2}{x_3{}^2}, \\ -\frac{1}{M}\left(\frac{\partial^2 M}{\partial x_1{}^2} + \frac{\partial^2 M}{\partial x_2{}^2}\right) + \frac{2}{x_3}\frac{1}{M}\frac{\partial M}{\partial x_3} &= \frac{1-\lambda R^2}{x_3{}^2}. \end{aligned}\right\} \tag{2}$$

If one subtracts the first equation of this system from the second, one has:

$$\frac{\partial^2 P}{\partial x_1{}^2} = \frac{\partial^2 Q}{\partial x_2{}^2}.$$

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* Additionally the condition i not $= k$ should have been stated; cf. §2. 1. of the first paper — *Eds.*

From this follows:

$$\left.\begin{aligned} P &= n(x_4)x_1{}^2 + a_1(x_4) + b_1(x_4), \\ Q &= n(x_4)x_2{}^2 + a_2(x_4) + b_2(x_4). \end{aligned}\right\} \tag{3}$$

If one looks at (1) and (3), the last of equations (2) allows itself to be written in the form:

$$-\frac{3-\lambda R^2}{x_3{}^2}(P+Q) = \frac{4n}{x_3} + \frac{1-\lambda R^2}{x_3{}^2} - \frac{2}{x_3}\frac{\partial L}{\partial x_3}. \tag{4}$$

If therefore $P + Q$ really contains one of the quantities x_1 or x_2, i.e. if one of the coefficients n, a_1, a_2 is different from zero, the factor of $P + Q$ in the equation (4) must vanish; for the right-hand side of this equation depends neither on x_1 nor on x_2. The case when all three quantities n, a_1, a_2 vanish must be separately considered.

In cases when the quantities n, a_1, a_2 do not all three vanish, the relation:*

$$\lambda R^3 = 3 \tag{5}$$

must hold between λ and the curvature of space.

Considering Eq. (5), the equations (2) reduce to a single equation which determines the function L, namely:

$$\frac{\partial L}{\partial x_3} + \frac{L}{x_3} = 2n. \tag{6}$$

2. In the following we must distinguish two cases: 1. $n \neq 0$, 2. $n = 0$. In the first case the formulae (D′), (1) and (3) show us that the quantity n may without restriction of generality be assumed to be equal to 1; namely one can always attain that $n = 1$ by a suitable substitution $\overline{x_4} = \varphi(x_4)$. Taking this into account, Eq. (6) gives:

$$L = \frac{L_0(x_4)}{x_3} + x_3. \tag{7}$$

To determine ρ, we set in the equations (A) $i = k = 4$; a simple calculation shows that in our case ρ becomes equal to zero. The first case is thus characterised by the vanishing density of matter and by the interval:

$$ds^2 = \frac{R^2}{x_3{}^2}(dx_1{}^2 + dx_2{}^2 + dx_3{}^2) + \left(\frac{x_1{}^2 + x_2{}^2 + a_1(x_4)x_1 + a_2(x_4)x_2 + a_3(x_4) + x_3{}^2}{x_3}\right)^2 dx_4{}^2. \tag{D$'_1$}$$

* This is a typo in the paper. It must be R^2; see Eq. (4) — *Eds.*

If we go over to the second case ($n = 0$), we find for L the equation:

$$L = \frac{L_0(x_4)}{x_3}. \tag{8}$$

Also in this case the calculation for ρ gives the value zero. Hence, the second case likewise characterises itself by the vanishing density of matter and by the interval:

$$ds^2 = \frac{R^2}{{x_3}^2}({dx_1}^2 + {dx_2}^2 + {dx_3}^2) + \left[\frac{a_1(x_4)x_1 + a_2(x_4)x_2 + a_3(x_4)}{x_3}\right]^2 {dx_4}^2. \tag{D$_2'$}$$

We finally consider the case when all three coefficients n, a_1, a_2 vanish, so that M does not depend on x_1 and x_2.

During the integration of Eq. (2) we again run into two cases:

$$1.\ \lambda R^2 = 3, \qquad M = \frac{M_0(x_4)}{x_3},$$

$$2.\ \lambda R^2 = 1, \qquad M = M(x_4),$$

where M_0 and M are arbitrary functions of their arguments.

The first case proves to be a special case of the interval determined by the formula (D$_2'$); an easy calculation shows that the density ρ of matter here vanishes.

The second case[4] leads, as one can easily persuade oneself, to a density of matter different from zero. To decide if here the density comes out positive or negative, we must draw on that form of the interval which corresponds to an indefinite quadratic form and is expressed by the formula (D$''$). Calculating with the gravitational potentials of the formula (D$''$), one finds that in the considered case M is a function of x_4 alone; accordingly we can, without harming generality, set $M = 1$ [to do so one only has to introduce instead of x_4 the coordinate $\overline{x_4} = \varphi(x_4)$]. Calculating under this assumption the density ρ, one finds:

$$\lambda = -\frac{c^2}{R^2}, \qquad \rho = -\frac{2}{\kappa R^2},$$

$$d\tau^2 = -\frac{R^2}{c^2}\frac{{dx_1}^2 + {dx_2}^2 + {dx_3}^2}{{x_3}^2} + {dx_4}^2. \tag{D$_3''$}$$

[4] Of the possibility of this case I was made aware by Dr. W. *Fock*.

This case thus gives a negative value for ρ.

Summarising, we can say that *the stationary world with constant negative curvature of space is only possible for vanishing or negative density of matter; the interval corresponding to this world is expressed through the formulae* (D_1'), (D_2') *and* (D_3'') *given above*.

3. We now turn ourselves to the case of the non-stationary world. We remark first of all, that here M is a function of x_4 alone; the considerations used by us several times before show that one may assume that M is equal to unity. Under the stated presuppositions we find without trouble that the equations (A) for $i = 1, 2, 3$; $k = 4$; and for $i, k = 1, 2, 3$ are identically satisfied.* If we set $i = k = 1, 2, 3$ in here, we obtain the differential equation of second order that serves to determine the function $R(x_4)$, namely:**

$$\frac{R'^2}{R^2} + \frac{2RR''}{R^2} + \frac{1}{R^2} - \lambda = 0. \tag{9}$$

This equation is completely analogous to our earlier equation [Eq. (4) of the cited Notice]; the latter goes over precisely into Eq. (9), if one sets there $c = 1$. We can thus carry over the entire discussion of the Eq. (4), l.c., for the equation just written down.† For this reason we will not go into it, but only calculate the density of matter ρ for the non-stationary world.

If we write for the case of the non-stationary world the interval in the form (D''), we thus obtain for R the differential equation:

$$\frac{R'^2}{R^2} + \frac{2RR''}{R^2} - \frac{c^2}{R^2} - \lambda = 0.$$

The integration of this equation yields us the relation:

$$\frac{R'^2}{c^2} = \frac{A + R + \frac{\lambda}{3c^2}R^3}{R},$$

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* Again, the condition i not $= k$ should have been stated for the latter index combination; cf. §2. 2. of the first paper — *Eds.*

** This equation has the wrong sign in the curvature term for a $k = -1$ expanding universe; cf. the correct following equation. This is apparently because these are the field equations for the positive definite metric form (D') rather than the indefinite metric form (D'') — *Eds.*

† It is here that the author fails to distinguish adequately between the dynamics of $k = +1$ and $k = -1$ universes. Presumably he would have made this distinction if he had paid more attention to the following equation — *Eds.*

where A is an arbitrary constant. If one calculates the density ρ, there results:

$$\rho = \frac{3A}{\kappa R^3}. \tag{10}$$

The formula (10) shows that for a positive A the density of matter is likewise positive.

From this follows *the possibility of the non-stationary worlds with constant negative curvature of space and with positive density of matter*.

§3. 1. We turn ourselves to the discussion of the physical meaning of the result obtained in the preceding paragraphs. We have convinced ourselves that the *Einstein* world equations possess solutions that correspond to a world with constant negative curvature of space. This fact points out that the world equations taken alone are not sufficient to decide the question of the finiteness of our world. Knowledge of the curvature of space gives us still no immediate hint on its finiteness or infinitude. To arrive at a definite conclusion on the finiteness of space, one needs some supplementary agreements. Indeed we designate a space as finite, if the distance between two arbitrary non-coincident points of this space does not exceed a certain constant number, whatever pairs of points we might like to take. Consequently we must, before we tackle the problem of the finiteness of space, come to an agreement as to which points of this space we regard as different. If we e.g. understand a sphere to be a surface of the three-dimensional Euclidean space, we count the points that lie on the same circle of lattitude and whose longitudes are different by just 360° as coincident; if we had in contrast considered these points as different, we would obtain a multiply-leafed spherical surface in Euclidean space. The distance between arbitrary [two] points on a sphere does not exceed a finite number; if we, however, conceive of this sphere as an infinitely-many leafed surface, we can (by associating the points with different leaves in an appropriate manner) make this distance arbitrarily large. From this it becomes clear that before one goes into the considerations on the finiteness of space, one must make precise as to which points will be regarded as coincident and which as different.

2. As a criterion for the distinctness of points could serve, amongst others, the principle of "phantom anxiety". By this we mean the axiom, that between every two different points one could draw only one straight (geodesic) line. If one accepts this principle, then one cannot regard two points which can be joined by more than one straight line as different. According to this principle e.g. the two end points of the same diameter of a sphere are not different from each other. Of course this principle excludes the possibility of phantoms, for the phantom appears at the same

point as the image itself that is generating it.

The just discussed formulation of the concept of coincident and non-coincident points can lead to the picture that spaces with positive constant curvature are finite. However, the mentioned criterion does not allow us to conclude on the finiteness of spaces of negative constant curvature. This is the reason therefore that, according to our opinion, *Einstein* 's world equations without supplementary assumptions are not yet sufficient to draw a conclusion on the finiteness of our world.

St. Petersburg, November 1923.