Monge

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> Honestly though I've come to terms with Monge existing- for there to be good theorems there also have to be bad ones, and you need to accept them.
>
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Remark. This lecture was originally given at G2 2024; since then, difficulty adjustment has been made.

This is actually about circle homotheties in general; as will be seen, these two concepts are inseparable.

1 Motivation

This is primarily a cool, incompletely explored piece of math that can turn up unexpectedly in problems. The 'cool' part becomes more apparent if you try to use it in problem proposals.

2 Theory

Fancy geometry parlance

Point P is a **similicenter** of two shapes $\mathcal{A}$, $\mathcal{B}$ if there's a homothety at P sending $\mathcal{A} \to \mathcal{B}$. It is said to be an **ex**similicenter or **in**similicenter according to whether said homothety has positive or negative scale factor.

Usually, the shapes in question are circles or polygons with ≤ 4 sides.

Theorem (Monge)

For any three circles $\omega_1, \omega_2, \omega_3$, pairwise exsimilicenters are collinear. Also, the insimilicenter of two pairs of circles and the exsimilicenter of the third are collinear as well. concur.

Pitot

Quadrilateral $ABCD$ has an incircle iff $AB + CD = AD + BC$, and has an excircle iff $AB + BC = AD + DC$ or $BA + AD = BC + CD$.

> **Theorem**
> If $ABCD$ is tangential (has an incircle), then the incircles of triangle ABC and ADC are tangent on $\overline{AC}$ and similarly for BCD, BAD, $\overline{BD}$.

3 Problems

Problem 1 (Iran TST). Let ABC be an isosceles triangle with $AB = AC$ and incenter I. Circle ω passes through C and I and is tangent to AI. Circle ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent. *

Problem 2 (me). Variable triangles ABC and DEF share a fixed incircle ω and circmcircle Ω. Let ω_a be the A-mixtilinear incircle of $\triangle ABC$ and define ω_d with respect to $\triangle DEF$ similarly. Determine the locus of the intersection of the common external tangents of these two circles. †

Problem 3 (ARML 2024/I10). Circle γ has center O and radius 9. Circle ω_A has radius 2 and is internally tangent to γ at point A. Circle ω_B has radius 3 and is internally tangent to γ at point B. The common external tangents of ω_A and ω_B meet at T. Given that $TO = 2AB$, compute AB.

Problem 4 (Part of ISL 2022/G6). Let $ZBPC$ be a quadrilateral. Let ω_A and ω_D be the incircles and A and D the incenters of $\triangle ZBC$ and $\triangle PBC$, respectively. If ω_A and ω_D are tangent at a point H on $\overline{BC}$, then $Q' = \overline{ZP} \cap \overline{AD}$ satisfies $(AD; HQ') = -1$.

Problem 5 (EGMO 2016/4). Congruent circles ω_1 and ω_2 intersect at points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that $\overline{X_1T_1} \cap \overline{X_2T_2} \in \omega$.

Problem 6 (ISL 2007/G8). Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

Problem 7 (ELMO SL 2024/G4, by me). In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90^\circ$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 8 (Serbia 2017/6). Let k be the circumcircle of $\triangle ABC$ and let k_a be the A-excircle. Let the two common tangents of k, k_a intersect BC at P, Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

Problem 9 (ISL 2017/G7). Quadrilateral $ABCD$ has incenter I. Let I_a, I_b, I_c and I_d be the respective incenters of triangles DAB, ABC, BCD and CDA, Suppose that the common external tangents of (AI_bI_d) and (CI_bI_d) meet at X, and those of the (BI_aI_c) and (DI_aI_c) meet at Y. Prove that $\angle XIY = 90^\circ$.

Problem 10 (ISL 2020/G5). Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the respective incircles of triangles ANK, BKL, CLM, DMN.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are

*Haruka Kimura found an absolutely brilliant radical axis solution so there are multiple nice ways to interpret the midpoints.

†A-mixtilinear incircle of $\triangle ABC$ is inside (ABC) and tangent to $\overline{AB}$, $\overline{AC}$, and (ABC). This configuration has many nice properties, actually!

concurrent.