Monge

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> I've come to terms with Monge existing- for there to be good theorems there also have to be bad ones, and you need to accept them.
>
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Remark. This lecture was originally given at G2 2024; since then, the first sections have been rewritten and the topic broadened.

This is actually about circle homotheties in general; as will be seen, these two concepts are inseparable.

1 Motivation

The single most important thing about the similicenter is that it lies on the line of centers. This allows one to connect otherwise unrelated lines. Plus, don't you think it's such a cool theorem?

2 Theory

Depth limited, therefore this is jsut a list of fun facts apart from the first two results.

Fancy geometry parlance
Point P is a **similicenter** of two shapes $\mathcal{A}$, $\mathcal{B}$ if there's a homothety at P sending $\mathcal{A} \to \mathcal{B}$. It is said to be an **ex**similicenter or **in**similicenter according to whether said homothety has positive or negative scale factor.

Usually, the shapes in question are circles or polygons with ≤ 4 sides.

Theorem (Monge)
For any three circles $\omega_1, \omega_2, \omega_3$, pairwise exsimilicenters are collinear. Also, the insimilicenter of two pairs of circles and the exsimilicenter of the third are collinear as well. concur.

Lemma: characterise exsimilicenter
Let circles ω_1, ω_2 intersect at A, B and have exsimilicenter X. Then $\measuredangle AXB = \frac{1}{2}\left(\widehat{AB}_{\omega_1} - \widehat{AB}_{\omega_2}\right)$.

Pitot
Quadrilateral $ABCD$ has an incircle iff $AB + CD = AD + BC$, and has an excircle iff $AB + BC = AD + DC$ or $BA + AD = BC + CD$.

3 Problems

Problem 1 (Iran TST 2020). Let ABC be an isosceles triangle with $AB = AC$ and incenter I. Circle ω passes through C and I and is tangent to AI. Circle ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent. *

Problem 2 (ARML 2024/I10). Circle γ has center O and radius 9. Circle ω_A has radius 2 and is internally tangent to γ at point A. Circle ω_B has radius 3 and is internally tangent to γ at point B. The common external tangents of ω_A and ω_B meet at T. Given that $TO = 2AB$, compute AB.

Problem 3 (USA TST 2023/2). In acute triangle ABC, let M be the midpoint of BC and let E and F be the feet of the altitudes from B and C, respectively. Let K be the intersection of the common external tangents of (BME) and (CMF). Show that if $K \in (ABC)$,then $\overline{AK} \perp \overline{BC}$.

Problem 4 (EGMO 2016/4). Congruent circles ω_1 and ω_2 intersect at points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that $\overline{X_1T_1} \cap \overline{X_2T_2} \in \omega$.

Problem 5 (ISL 2007/G8). Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AK} \cap \overline{BL}$. Prove that points E, I, and F are collinear.

Problem 6 (ELMO SL 2024/G4, by me). In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90°$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 7 (Serbia 2017/6). Let k be the circumcircle of $\triangle ABC$ and let k_a be the A-excircle. Let the two common tangents of k, k_a intersect BC at P, Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

Problem 8 (ISL 2017/G7). Quadrilateral $ABCD$ has incenter I. Let I_a, I_b, I_c and I_d be the respective incenters of triangles DAB, ABC, BCD and CDA, Suppose that the common external tangents of (AI_bI_d) and (CI_bI_d) meet at X, and those of the (BI_aI_c) and (DI_aI_c) meet at Y. Prove that $\angle XIY = 90°$.

Problem 9 (ISL 2020/G5). Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the respective incircles of triangles ANK, BKL, CLM, DMN.

Prove that the common internal tangents to (ω_A, ω_C) and (ω_B, ω_D) are concurrent.

* Haruka Kimura found an absolutely brilliant radical axis solution so there are multiple nice ways to interpret the midpoints.