Monge

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> I've come to terms with Monge existing- for there to be good theorems there also have to be bad ones, and you need to accept them.
>
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Credits. This lecture was originally given at G2 2024 and rewritten with much help from Patrick Suwanich.

1 Motivation

The reason for making this lecture is because I find Monge problems overrated in difficulty.
The most-used characteristic of the similicenter is that it lies on the line of centers. This allows one to connect otherwise unrelated lines. Plus, don't you think it's such a cool theorem?

2 Theory

Depth limited, therefore this is just a list of fun facts apart from the first two results.

Fancy geometry parlance
Point P is a **similicenter** of two shapes $\mathcal{A}, \mathcal{B}$ if there's a homothety at P sending $\mathcal{A} \to \mathcal{B}$. It is said to be an **ex**similicenter or **in**similicenter according to whether said homothety has positive or negative scale factor.

Usually, the shapes in question are circles or polygons with ≤ 4 sides.

Theorem (Monge)
For any three circles $\omega_1, \omega_2, \omega_3$, pairwise exsimilicenters are collinear. Also, the insimilicenter of two pairs of circles and the exsimilicenter of the third are collinear as well concur.

Lemma: characterise similicenter
Let circles ω_1, ω_2 intersect at A, B. Point X is a similicenter iff $XA = XB$ and $\measuredangle AXB = \frac{1}{2}\left(\widehat{AB}_{\omega_1} - \widehat{AB}_{\omega_2}\right)$.

> **Theorem (Pitot)**
> Quadrilateral $ABCD$ has an incircle iff $AB + CD = AD + BC$, and has an excircle iff $AB + BC = AD + DC$ or $BA + AD = BC + CD$.

3 Problems

Because Monge is either easy or hard to see, problems involving this are always IMO1 or ≥IMO2.5 difficulty.

Problem 1 (Iran TST 2020). Let ABC be an isosceles triangle with $AB = AC$ and incenter I. Circle ω passes through C and I and is tangent to AI. Circle ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent. *

Problem 2 (ARML 2024/I10). Circles $\omega_A, \omega_B, \gamma$ (center O) respectively have radii 2, 3, 9. ω_A, ω_B internally touch γ at A, B respectively. The common external tangents of ω_A and ω_B meet at T. If $TO = 2AB$, what is AB?

Problem 3 (USA TST 2023/2). In acute triangle ABC, let M be the midpoint of BC and let E and F be the feet of the altitudes from B and C, respectively. Let K be the intersection of the common external tangents of (BME) and (CMF). Show that if $K \in (ABC)$,then $\overline{AK} \perp \overline{BC}$.

Problem 4 (EGMO 2016/4). Congruent circles ω_1 and ω_2 intersect at points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that $\overline{X_1T_1} \cap \overline{X_2T_2} \in \omega$.

Problem 5 (ISL 2007/G8). Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AK} \cap \overline{BL}$. Prove that points E, I, and F are collinear.

Problem 6 (IMO 2008/6). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC, ADC by ω_1, ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and BC beyond C, as well as to lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 7 (ELMO SL 2024/G4, by me). In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90^\circ$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 8 (RMM 2010/3). Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $1 \leq i \leq 4$, define ω_i as the circle tangent to the interior of $\overline{A_iA_{i+1}}$ and the extensions of $\overline{A_{i-1}A_i}$, $\overline{A_{i+1}A_{i+2}}$ (indices considered modulo 4). Let $T_i = \omega_i \cap \overline{A_iA_{i+1}}$. Prove that $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{T_2T_4}$ concur if and only if $\overline{A_2A_3}$, $\overline{A_4A_1}$, $\overline{T_1T_3}$ do.

Problem 9 (ISL 2017/G7). Quadrilateral $ABCD$ has incenter I. Let I_a, I_b, I_c and I_d be the respective incenters of triangles DAB, ABC, BCD and CDA, Suppose that the common external tangents of (AI_bI_d) and (CI_bI_d) meet at X, and those of the (BI_aI_c) and (DI_aI_c) meet at Y. Prove that $\angle XIY = 90^\circ$.

Problem 10 (ISL 2020/G5). Points K, L, M, N are chosen on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ of cyclic quadrilateral so that $KLMN$ is a rhombus with $KL \parallel AC$, $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the respective incircles of triangles ANK, BKL, CLM, DMN. Prove that the common internal tangents to (ω_A, ω_C) and (ω_B, ω_D) are concurrent.

Problem 11 (ISL 2015/G7). (difficulty warning) Points P, Q, R, S are on sides AB, BC, CD, DA of convex quadrilateral $ABCD$, respectively. Let $O = \overline{PR} \cap \overline{QS}$. Given that $APOS$, $BQOP$, $CROQ$, $DSOR$ each have an incircle, show that $\overline{AC}$, $\overline{PQ}$, $\overline{RS}$ concur.

* Haruka Kimura found an absolutely brilliant radical axis solution so there are multiple nice ways to interpret the midpoints.