Monge

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MOSP 2025

> Honestly though I've come to terms with Monge existing- for there to be good theorems there also have to be bad ones, and you need to accept them.
>
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Remark. This lecture was originally given at G2 2024; since then, the first sections have been rewritten and the topic broadened.

This is actually about circle homotheties in general; as will be seen, these two concepts are inseparable.

1 Motivation

The single most important thing about the similicenter is that it lies on the line of centers. This allows one to connect otherwise unrelated lines. Plus, don't you think it's such a cool theorem?

2 Theory

Fancy geometry parlance
Point P is a **similicenter** of two shapes $\mathcal{A}$, $\mathcal{B}$ if there's a homothety at P sending $\mathcal{A} \to \mathcal{B}$. It is said to be an **ex**similicenter or **in**similicenter according to whether said homothety has positive or negative scale factor.

Usually, the shapes in question are circles or polygons with ≤ 4 sides.

Theorem (Monge)
For any three circles $\omega_1, \omega_2, \omega_3$, pairwise exsimilicenters are collinear. Also, the insimilicenter of two pairs of circles and the exsimilicenter of the third are collinear as well. concur.

Pitot
Quadrilateral $ABCD$ has an incircle iff $AB + CD = AD + BC$, and has an excircle iff $AB + BC = AD + DC$ or $BA + AD = BC + CD$.

Theorem
If $ABCD$ is tangential (has an incircle), then the incircles of triangle ABC and ADC are tangent on $\overline{AC}$ and similarly for BCD, BAD, $\overline{BD}$.

3 Problems

Problem 1 (Iran TST). Let ABC be an isosceles triangle with $AB = AC$ and incenter I. Circle ω passes through C and I and is tangent to AI. Circle ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent. [*]

Problem 2 (me). Variable triangles ABC and DEF share a fixed incircle ω and circmcircle Ω. Let ω_a be the A-mixtilinear incircle of $\triangle ABC$ and define ω_d with respect to $\triangle DEF$ similarly. Determine the locus of the intersection of the common external tangents of these two circles. [†]

Problem 3 (ARML 2024/I10). Circle γ has center O and radius 9. Circle ω_A has radius 2 and is internally tangent to γ at point A. Circle ω_B has radius 3 and is internally tangent to γ at point B. The common external tangents of ω_A and ω_B meet at T. Given that $TO = 2AB$, compute AB.

Problem 4 (EGMO 2016/4). Congruent circles ω_1 and ω_2 intersect at points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that $\overline{X_1T_1} \cap \overline{X_2T_2} \in \omega$.

Problem 5 (ISL 2007/G8). Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

Problem 6 (ELMO SL 2024/G4, by me). In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90^\circ$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 7 (Serbia 2017/6). Let k be the circumcircle of $\triangle ABC$ and let k_a be the A-excircle. Let the two common tangents of k, k_a intersect BC at P, Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

Problem 8 (ISL 2017/G7). Quadrilateral $ABCD$ has incenter I. Let I_a, I_b, I_c and I_d be the respective incenters of triangles DAB, ABC, BCD and CDA, Suppose that the common external tangents of (AI_bI_d) and (CI_bI_d) meet at X, and those of the (BI_aI_c) and (DI_aI_c) meet at Y. Prove that $\angle XIY = 90^\circ$.

Problem 9 (ISL 2020/G5). Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let ω_A, ω_B, ω_C, ω_D be the respective incircles of triangles ANK, BKL, CLM, DMN.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

[*] Haruka Kimura found an absolutely brilliant radical axis solution so there are multiple nice ways to interpret the midpoints.

[†] A-mixtilinear incircle of $\triangle ABC$ is inside (ABC) and tangent to $\overline{AB}$, $\overline{AC}$, and (ABC). This configuration has many nice properties, actually!