

On $(p, q; \alpha, \beta, l)$ -deformed oscillators and their oscillator algebras

I. M. Burban

Institute for Theoretical Physics, Kiev 03143, Ukraine

Abstract

We present a description of a new kind of the deformed canonical commutation relations, their representations and generated by them Heisenberg–Weyl algebra. This deformed algebra allows us to define operations of the Hopf algebra structure: comultiplication, counit and antipode. We discuss properties of a discrete spectrum of the Hamiltonian of the deformed harmonic oscillator corresponding to this oscillator-like system.

1. Introduction

Attempts to deform the canonical commutation relations are repeatedly undertaken in physical theories [1].

With the emergence of the quantum groups (quantum multi-parameter deformed universal enveloping algebras of Lie algebras) became evident their important role for theoretical and mathematical physics. From the physical point of view, the interest to quantum deformations, in particular to the quantum (p, q) -deformations of Lie algebras, is connected with the possible applications them in the quantum field theory (the conformal, topological field theories, etc.). The description of the two-parameter quantum groups and their representations has been started in the works [2], [3], [4].

As in the classical case, the problem of realization of q -deformed algebras by the one-parameter deformed creation and annihilation operators (the Jordan-Schwinger construction) [5], [6] is important for representation theory of quantum groups.

This problem remains important for (p, q) -deformed cases as well. The further exploration of these deformations led to investigation of (p, q) -deformed canonical commutation relations [7], [8], [9].

At the study of the quantum groups and algebras became evident their connection with the noncommutative geometry, special functions of q -analysis and others branches of mathematics.

In the framework of this program the problems of q -analysis and q -special functions have got natural extension to the (p, q) -case. Already in the paper [7] and in further works [10], [11] the (p, q) -exponential and other (p, q) -deformed functions have been introduced and their properties have been investigated.

In the paper [10] the definition of the basic (p, q) - hypergeometric series ${}_r\Psi_{r-1}$ was given and their properties were investigated. In [11] the general (p, q) -hypergeometric series are defined and various q -identities are converted into their (p, q) -analogs. In this framework the (p, q) -differentiation, and the (p, q) -Jackson integration has been defined and their main properties has been studied [12], [10].

The problem of extension of these results to the generalized one- and two-parameter deformed cases naturally arise. An example of such generalized q -deformed algebra with the Hopf algebra structure has been studied in [13], [14].

In this paper we define a new kind of the deformed canonical commutation relations and connected with them Heisenberg–Weyl algebra. We study representations of this algebra. This generalized deformed algebra allows us to define the operations of the comultiplication, antipode and counit which satisfy the axioms of the Hopf algebra structure. This generalized $(p, q; \alpha, \beta, l)$ -deformed system includes as a particular case the systems of [13], [14]. We discuss properties of discrete spectrum of the Hamiltonian of the deformed harmonic oscillator corresponding to this oscillator-like system.

2. The $(p, q; \alpha, \beta, l)$ -deformed oscillator algebra and its representations

A deformed Heisenberg–Weyl algebra is defined as the associative algebras generated by the operators $\{\mathbf{1}, a, a^+, N\}$ and defining relations

$$[N, a] = -a, \quad [N, a^+] = a^+, \quad (1)$$

$$a^+a = f(N), \quad aa^+ = f(N+1), \quad (2)$$

where structure function $f(x)$ is a positive analytic function. Instead of (2) one can consider the relation

$$[a, a^+] = f(N+1) - f(N), \quad (3)$$

although the algebras in this two cases in general are not isomorphic. We define the generalized deformed Heisenberg–Weyl algebra as an associative algebra generated by generators $\mathbf{1}, a, a^+, N$ satisfying the defining relations

$$[N, a] = -la, \quad [N, a^+] = la^+, \quad (4)$$

$$[a, a^+]_A = f(N+l) - A f(N), \quad (5)$$

where $[a, a^+]_A = aa^+ - A a^+ a$, and $A, l \in \mathbb{R}$.

The structure functions $f(x)$ in (3) and (5) characterize the deformation scheme. For various known deformations of the harmonic oscillator they are given:

$$f(n) = \frac{1}{2}n \quad \text{for the oscillator of standard quantum}$$

mechanics;

$$f(n) = [n] \quad \text{and} \quad f(n) = q^{\alpha n + \beta} [n], \quad \text{where} \quad [n] = \frac{1 - q^n}{1 - q},$$

define the Arik–Coon and its generalization;

$$f(n) = [n] \quad \text{and} \quad f(n) = [\alpha n + \beta], \quad \text{where} \quad [n] = \frac{q^{-n} - q^n}{q^{-1} - q},$$

define the Biedenharn–Makfarlane and its symmetric generalization;

$$f(n) = [n] \quad \text{and} \quad f(n) = [\alpha n + \beta], \quad \text{where} \quad [n] = \frac{p^{-n} - q^n}{p^{-l} - q^l},$$

define the two-parameter deformation and its symmetric generalization.

In comparison with the one-parameter deformed commutation relations the multi-parameter deformation are less understood [2], [3], [4].

Two-parameter analogs of the one-parameter symmetric deformation [7], [9] of the oscillator algebra are defined as an associative algebra generated by the operators $\mathbf{1}, a, a^+, N$ and defining relations

$$aa^+ - qa^+a = p^{-N}, \quad (6)$$

$$aa^+ - p^{-1}a^+a = q^N, \quad (7)$$

$$[N, a] = -a, \quad [N, a^+] = a^+. \quad (8)$$

The generalized the Biedenharn–Macfarlane q -oscillator algebra with defining relations

$$aa^+ - qa^+a = q^{-\alpha N - \beta}, \quad (9)$$

$$aa^+ - q^{-1}a^+a = q^{\alpha N + \beta}, \quad (10)$$

$$[N, a] = -a, \quad [N, a^+] = a^+ \quad (11)$$

and its Hopf algebra structure have been studied in the papers [13] and [14]. The properties of this algebra and of the corresponding deformed oscillator it were studied in [16].

By analogy with the deformation of [13] and [14] we introduce *the corresponding $(p, q; \alpha, \beta, l)$ -deformed canonical commutation relations*. The $(p, q; \alpha, \beta, l)$ -deformed oscillator algebra is given by the generators $\mathbf{1}, a, a^+, N$ and the commutation relations

$$aa^+ - q^l a^+a = p^{-\alpha N - \beta}, \quad (12)$$

$$aa^+ - p^{-l}a^+a = q^{\alpha N + \beta}, \quad (13)$$

$$[N, a] = -la, \quad [N, a^+] = la^+, \quad (14)$$

where the function $f(n)$ has the form

$$f(n) = \left(\frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \right) \quad (15)$$

with $\alpha, \beta, l \in \mathbb{R}$.

Instead of the relations (12) and (13) we can consider the relations

$$aa^+ = \frac{p^{-\alpha N - \beta - l} - q^{\alpha N + \beta + l}}{p^{-l} - q^l}, \quad a^+a = \frac{p^{-\alpha N + \beta} - q^{\alpha N + \beta}}{p^{-l} - q^l}, \quad (16)$$

which together with relations (14) define an oscillator algebra which in general is not isomorphic to the algebra defined above. The difficulties to supply it with a Hopf algebra structure are the same as in [14].

Nevertheless, if we will replace the relations (16) by

$$[a, a^+]_A = \frac{p^{-\alpha N - \beta - l} - q^{\alpha N + \beta + l}}{p^{-l} - q^l} - A \frac{p^{-\alpha N - \beta} - q^{\alpha N + \beta}}{p^{-l} - q^l}, \quad (17)$$

then we obtain a new algebra which can be considered as $(p, q; \alpha, \beta, l)$ -deformed Heisenberg–Weyl algebra. This algebra generated relations (14) and (17), as we shall show in next section, admits a Hopf algebra structure for a properly chosen constant A .

The representation of the creation and annihilation operators a, a^+ and the operator of number particles N of the relations (12), (13), (14) in the Hilbert space \mathcal{H} with the basis $\{|n\rangle\}, n = 0, 1, 2 \dots$ are defined as follows

$$a|n\rangle = \left(\frac{p^{-\alpha-\beta} - q^{\alpha+\beta}}{p^{-l} - q^l} \right)^{1/2} |n-l\rangle, \quad a^+|n\rangle = \left(\frac{p^{-\alpha-\beta-l} - q^{\alpha+\beta+l}}{p^{-l} - q^l} \right)^{1/2} |n+l\rangle, \quad (18)$$

$$N|n\rangle = n|n\rangle. \quad (19)$$

In the space of functions (analytic if l/α is integer number) we can define the difference derivative

$$Df(z) = \frac{f(p^{-\alpha}z)p^{-\beta} - f(q^\alpha z)q^\beta}{(p^{-l} - q^l)z^{l/\alpha}}. \quad (20)$$

It follows

$$Dz^n = \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} z^{n-l/\alpha} = \frac{z^n}{z^{l/\alpha}} \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \frac{1}{(n)!} \frac{d^n z^n}{dz^n}$$

and (if l/α is integer number)

$$Df(z) = \sum_{n=0}^{\infty} a_n Dz^n = \sum_{n=1}^{\infty} \frac{z^n}{z^{l/\alpha}} \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \frac{1}{n!} \frac{d^n}{dz^n} f(z) \quad (21)$$

for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Now we can give a realization of the relations (12), (13), (14) in this space by the operators

$$a : f \rightarrow Df, \quad (22)$$

$$a^+ : f \rightarrow z^{l/\alpha} f, \quad (23)$$

$$N : f \rightarrow \alpha z \frac{d}{dz}, \quad (24)$$

$$q^N : f \rightarrow q^{z \frac{d}{dz}} f = f(qz), \quad (25)$$

$$p^{-N} : f \rightarrow p^{-z \frac{d}{dz}} f = f(p^{-1}z). \quad (26)$$

Indeed, from (22) and (24) we obtain

$$Na^+f(z) = \alpha z \frac{d}{dz} (z^{l/\alpha} f(z)) = lz^{l/\alpha} f + \alpha z^{1+l/\alpha} \frac{d}{dz} f(z)$$

and

$$a^+Nf(z) = \alpha z^{1+l/\alpha} \frac{d}{dz} f(z).$$

It follows that

$$[N, a^+]f = l a^+ f. \quad (27)$$

Analogously, from (23) and (24) we get

$$Naf = -l \frac{(f(p^{-\alpha}z)p^{-\beta} - f(q^\alpha z)q^\beta)}{z^{l/\alpha}(p^{-l} - q^l)} + \alpha \frac{zp^{-\alpha-\beta} f'(p^{-\alpha}z) - zq^{\alpha+\beta} f'(q^\alpha z)}{z^{l/\alpha}(p^{-l} - q^l)}$$

and

$$aNf = \alpha \frac{zp^{-\alpha-\beta} f'(p^{-\alpha}z) - zq^{\alpha+\beta} f'(q^\alpha z)}{z^{l/\alpha}(p^{-l} - q^l)}.$$

It follows that

$$[N, a] = -l a. \quad (28)$$

In a similar way, from (22), (23) we have

$$a^+ a f(z) = \frac{f(p^{-1}z)p^{-\beta} - f(q^\beta z)q^\beta}{p^{-l} - q^l}$$

and

$$a a^+ f(z) = \frac{f(p^{-\alpha}z)p^{-l-\beta} - f(q^\alpha z)q^{l+\beta}}{p^{-l} - q^l}.$$

Therefore,

$$a a^+ - q^l a^+ a = p^{-\alpha N - \beta}, \quad a a^+ - p^{-l} a^+ a = q^{\alpha N + \beta}. \quad (29)$$

3. Spectrum of Hamiltonian of $(p, q; \alpha, \beta, l)$ -deformed oscillator

The Hamiltonian of the $(p, q; \alpha, \beta, l)$ -deformed oscillator system is defined in the same way as in case of the q -deformed oscillator. From the relations

$$a a^+ - q^{-l} a^+ a = p^{\alpha N + \beta}, \quad a a^+ - p^l a^+ a = q^{-\alpha N - \beta} \quad (30)$$

we have

$$a a^+ |n\rangle = \frac{p^{-\alpha N - \beta - l} - q^{\alpha N + \beta + l}}{p^{-l} - q^l} |n\rangle, \quad a^+ a |n\rangle = \frac{p^{-\alpha N - \beta} - q^{\alpha N + \beta}}{p^{-l} - q^l} |n\rangle. \quad (31)$$

The Hamiltonian

$$H = a^+ a + a a^+ \quad (32)$$

of this $(p, q; \alpha, \beta, l)$ -deformed oscillator has a diagonal form in the basis $\{|n\rangle\}$:

$$H |n\rangle = \lambda_n |n\rangle, \quad (33)$$

where

$$\lambda_n = \frac{p^{-\alpha n - \beta - l} - q^{\alpha n + \beta + l}}{p^{-l} - q^l} + \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l}. \quad (34)$$

Because of the identity

$$\begin{aligned} \frac{p^{-\alpha n - \beta - l} - q^{\alpha n + \beta + l}}{p^{-l} - q^l} &= \frac{p^{-\alpha n - \beta - l} - p^{-\alpha n - \beta} q^l + p^{-\alpha n - \beta} q^l - q^{\alpha n + \beta + l}}{p^{-l} - q^l} \\ &= \frac{p^{-\alpha n - \beta} (p^{-l} - q^l) + (p^{-\alpha n - \beta} - q^{\alpha n + \beta}) q^l}{p^{-l} - q^l} = p^{-\alpha n - \beta} + q^l \left(\frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \right) \end{aligned} \quad (35)$$

the relation (34) can be rewritten as

$$\lambda_n = p^{-\alpha n - \beta} + (q^l + 1) \left(\frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \right). \quad (36)$$

On the other hand

$$\frac{p^{-\alpha n - \beta - l} - q^{\alpha n + \beta + l}}{p^{-1} - q^l} = \frac{p^{-\alpha n - \beta - l} - p^{-l} q^{\alpha n + \beta} + p^{-l} q^{\alpha n + \beta} - q^{\alpha n + \beta + l}}{p^{-1} - q^l}$$

$$= p^{-l} \left(\frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \right) + q^{\alpha n + \beta}, \quad (37)$$

that is

$$\lambda_n = q^{\alpha n + \beta} + (p^{-l} + 1) \left(\frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \right). \quad (38)$$

It follows from (36) and (38) that spectrum of the Hamiltonian (32) is symmetric under the change of parameter $q \rightarrow p^{-1}$, $p \rightarrow q^{-1}$.

4. Hopf algebra structure of $(p, q; \alpha, \beta, l)$ -deformed oscillator algebra

It would be desirable to show that the generalized Heisenberg-Weyl algebra, generated by the generators defined above and the relations (4) and (5) carries a Hopf algebra structure.

Remind, the associative algebra C is a Hopf algebra if it admits operations of homomorphisms of a coproduct Δ , a counit ϵ and an anti-homomorphism of an antipode S :

$$\Delta : C \rightarrow C \otimes C, \quad \Delta(a b) = \Delta(a)\Delta(b), \quad (39)$$

$$\epsilon : C \rightarrow C, \quad \epsilon(ab) = \epsilon(a)\epsilon(b) \quad (40)$$

$$S(a b) = S(b) S(a). \quad (41)$$

which satisfy properties

$$(id \otimes \Delta)\Delta(h) = (\Delta \otimes id)\Delta(h), \quad (42)$$

$$(id \otimes \epsilon)\Delta(h) = (\epsilon \otimes id)\Delta(h), \quad (43)$$

$$m(id \otimes S)\Delta(h) = m(S \otimes id)\Delta(h) = \epsilon(h)\mathbf{1} \quad (44)$$

for all $h \in C$.

In our case the algebra is generated by $\mathbf{1}, a^+, a, N$, satisfying the relations

$$[N, a] = -la, \quad [N, a^+] = la^+, \quad (45)$$

$$[a, a^+]_A = \frac{p^{-\alpha N - \beta_1 - l} - q^{\alpha N - \beta_1}}{p^{-l} - q^l} - A \frac{p^{-\alpha N - \beta_2} - q^{\alpha N - \beta_2}}{p^{-l} - q^l}, \quad (46)$$

and a constant A will be determined later on.

In particular, for $\beta_1 - \beta_2 = l$ we obtain the relation (17) and at $p = q, l = 1$ this algebra reduced to the one of [14].

We define an action of coproduct Δ , counit ϵ , and antipode S on the generators of the algebra as

$$\Delta(a^+) = c_1 a^+ \otimes p^{-\alpha_1 N} + c_2 q^{\alpha_2 N} \otimes a^+, \quad (47)$$

$$\Delta(a) = c_3 a \otimes p^{-\alpha_3 N} + c_4 q^{\alpha_4 N} \otimes a, \quad (48)$$

$$\Delta(N) = c_5 N \otimes \mathbf{1} + c_6 \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1}, \quad (49)$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \quad (50)$$

$$\epsilon(a^+) = c_7, \quad \epsilon(a) = c_8, \quad (51)$$

$$\epsilon(N) = c_9, \quad \epsilon(\mathbf{1}) = 1, \quad (52)$$

$$S(a^+) = -c_{10}a^+, \quad S(a) = -c_{11}a, \quad (53)$$

$$S(N) = -c_{12}N + c_{13}\mathbf{1}, \quad S(\mathbf{1}) = \mathbf{1}, \quad (54)$$

where $c_i, i = 1 \dots 13$, and γ are unknown coefficients which must be determined by means of the rules of Hopf algebra structure.

Using the relations

$$a^+ r^{\alpha N} = r^{-\alpha l} r^{\alpha N} a^+, \quad a r^{\alpha N} = r^{\alpha l} r^{\alpha N} a, \quad (55)$$

where $r = p, q$ and

$$q^{\alpha \Delta(N)} = q^{\alpha \gamma} q^{\alpha N} \otimes q^{\alpha N}, \quad (56)$$

we shall verify the axiom (42) of the Hopf algebra structure for $h = a^+, a, N$. The condition (42) for $h = a^+$ gives

$$(id \otimes \Delta)\Delta a^+ = c_1 p^{-\alpha_1 \gamma} a^+ \otimes p^{-\alpha_1 N} \otimes p^{-\alpha_1 N} \\ + c_1 c_2 q^{\alpha_2 N} \otimes a^+ \otimes p^{-\alpha_1 N} + c_2 c_2 q^{\alpha_2 N} \otimes q^{\alpha_2 N} \otimes a^+ \quad (57)$$

and

$$(\Delta \otimes id)a^+ = c_1 c_1 a^+ \otimes p^{-\alpha_1 N} \otimes p^{\alpha_1 N} \\ + c_1 c_2 q^{\alpha_2 N} \otimes a^+ \otimes p^{-\alpha_1 N} + c_2 q^{\alpha_2 \gamma} q^{\alpha_2 N} \otimes q^{\alpha N} \otimes a^+. \quad (58)$$

From (57) and (58) it follows

$$c_1 = p^{-\alpha_1 \gamma}, \quad c_2 = q^{\alpha_2 \gamma}. \quad (59)$$

The condition (42) for $h = a, N$ gives

$$c_3 = p^{-\alpha_3 \gamma}, \quad c_4 = q^{\alpha_4 \gamma}, \quad c_5 = 1, \quad c_6 = 1. \quad (60)$$

It is easy to see that

$$\Delta(a)\Delta(a^+) = c_1 c_3 a a^+ \otimes p^{-(\alpha_1 + \alpha_3)N} + c_2 c_4 q^{(\alpha_2 + \alpha_4)N} \otimes a a^+ \\ + c_2 c_3 q^{\alpha_2 l} q^{\alpha_2 N} a \otimes p^{-\alpha_3 N} a^+ + c_1 c_4 p^{-\alpha_1 l} q^{\alpha_4 N} a^+ \otimes p^{-\alpha_1 N} a \quad (61)$$

and

$$\Delta(a^+)\Delta(a) = c_1 c_3 a^+ a \otimes p^{-(\alpha_1 + \alpha_3)N} + c_2 c_4 q^{(\alpha_2 + \alpha_4)N} \otimes a^+ a \\ + c_2 c_3 p^{\alpha_3 l} q^{\alpha_2 N} a \otimes p^{-\alpha_3 N} a^+ + c_1 c_4 q^{-\alpha_4 l} a^+ p^{-\alpha_4 N} a^+ \otimes p^{-\alpha_1 N} a. \quad (62)$$

The action of the operation Δ on the left hand side of (46) gives

$$\Delta(a)\Delta(a^+) - A \Delta(a^+)\Delta(a) = c_1 c_3 [a, a^+]_A \otimes p^{-(\alpha_1 + \alpha_3)N} + c_2 c_4 q^{(\alpha_2 + \alpha_4)N} \otimes [a, a^+]_A, \quad (63)$$

if

$$q^{\alpha_2 l} - A p^{\alpha_3 l} = 0 \quad \text{and} \quad p^{-\alpha_1 l} - A q^{\alpha_4 l} = 0.$$

It leads to

$$A = p^{-\alpha_3 l} q^{\alpha_2 l}, \quad A = p^{-\alpha_1 l} q^{\alpha_4 l}$$

or $\alpha_1 = \alpha_3$, and $\alpha_2 = \alpha_4$.

Using the relation (46)

$$\begin{aligned} [a, a^+]_A &= \frac{p^{-\alpha N - \beta_1} - q^{\alpha N + \beta_1}}{p^{-l} - q^l} - A \frac{p^{-\alpha N - \beta_2} - q^{\alpha N + \beta_2}}{p^{-l} - q^l} \\ &= \frac{(p^{-\beta_1} - A p^{-\beta_2}) p^{-\alpha N} - (q^{\beta_1} - A q^{\beta_2}) q^{\alpha N}}{p^{-l} - q^l} \end{aligned}$$

one can represent the expression (63) in the form

$$\begin{aligned} &\Delta(a) \Delta(a^+) - A \Delta(a^+) \Delta(a) \\ &= c_1 c_3 \frac{(p^{-\beta_1} - A p^{-\beta_2}) p^{-\alpha N} - (q^{\beta_1} - A q^{\beta_2}) q^{\alpha N}}{p^{-l} - q^l} \otimes p^{-(\alpha_1 + \alpha_3)N} \\ &\quad + c_2 c_4 q^{(\alpha_2 + \alpha_4)N} \otimes \frac{(p^{-\beta_1} - A p^{-\beta_2}) p^{-\alpha N} - (q^{\beta_1} - A q^{\beta_2}) q^{\alpha N}}{p^{-l} - q^l} \\ &= c_1 c_3 \frac{(p^{-\beta_1} - A p^{-\beta_2}) p^{-\alpha N} \otimes p^{-(\alpha_1 + \alpha_3)N} - (q^{\beta_1} - A q^{\beta_2}) q^{\alpha N} \otimes p^{-(\alpha_1 + \alpha_3)N}}{p^{-l} - q^l} \\ &\quad + c_2 c_4 \frac{(p^{-\beta_1} - A p^{-\beta_2}) q^{(\alpha_2 + \alpha_4)N} \otimes p^{-\alpha N} - (q^{\beta_1} - A q^{\beta_2}) q^{(\alpha_2 + \alpha_4)N} \otimes q^{\alpha N}}{p^{-l} - q^l}. \end{aligned} \quad (64)$$

On the other hand, the action of the Δ on the right hand side of (46) gives

$$\begin{aligned} &\Delta \left(\frac{p^{-\alpha N - \beta_1} - q^{\alpha N + \beta_1} - A(p^{-\alpha N - \beta_2} + q^{\alpha N + \beta_2})}{p^{-l} - q^l} \right) \\ &= \frac{p^{-\alpha \gamma} (p^{-\beta_1} - A p^{-\beta_2}) p^{-\alpha N} \otimes p^{-\alpha N} - q^{\alpha \gamma} (q^{\beta_1} - A q^{\beta_2}) q^{\alpha N} \otimes q^{\alpha N}}{p^{-l} - q^l}. \end{aligned} \quad (65)$$

From (64) and (65) we have

$$-c_1 c_3 (q^{\beta_1} - A q^{\beta_2}) + c_2 c_4 (p^{-\beta_1} - A p^{-\beta_2}) = 0.$$

If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha/2$, then

$$A = (p^{-1} q)^{\alpha l/2}, \quad c_1 c_3 = p^{-\alpha \gamma}, \quad c_2 c_4 = q^{\alpha \gamma}. \quad (66)$$

It follows

$$p^{-\alpha \gamma} (q^{\beta_1} - A q^{\beta_2}) - q^{\alpha \gamma} (p^{-\beta_1} - A p^{-\beta_2}) = 0,$$

$$(pq)^{\alpha \gamma} = \frac{q^{\beta_1} - A q^{\beta_2}}{p^{-\beta_1} - A p^{-\beta_2}}. \quad (67)$$

The last equation (67) defines the parameter γ in the equation (49) for the Hopf algebra structure.

Comparing the right-hand sides of the relations

$$(id \otimes \epsilon) \Delta(a^+) = c_1 a^+ \otimes p^{-\alpha_1 c_9} + c_2 q^{\alpha_2 N} \otimes c_7 + \gamma \mathbf{1} \otimes \mathbf{1}, \quad (68)$$

and

$$(\epsilon \otimes id)\Delta(a^+) = c_1 c_7 \mathbf{1} \otimes p^{-\alpha_1 N} + c_2 q^{\alpha_2 c_9} \mathbf{1} \otimes a^+ + \gamma \mathbf{1} \otimes \mathbf{1} \quad (69)$$

and using the axiom (43) for the generator a^+ , we obtain (take into account that $c_1 = p^{-\alpha_1 \gamma}$, $c_2 = q^{\alpha_2 \gamma}$)

$$c_1 p^{-\alpha_1 c_9} = c_2 q^{\alpha_2 c_9}, \quad -\alpha_1 \gamma - \alpha_1 c_9 = 0,$$

hence

$$c_9 = -\gamma. \quad (70)$$

An easy calculation gives

$$m(id \otimes S)\Delta(a^+) = c_1 p^{-\alpha_1 c_{13}} a^+ p^{\alpha_1 c_{12} N} - c_2 c_{10} q^{\alpha_2 N} a^+ \quad (71)$$

and

$$m(S \otimes id)\Delta(a^+) = -c_1 c_{10} a^+ p^{-\alpha_1 N} + c_2 q^{\alpha_2 c_{13}} q^{-\alpha_2 c_{13} N} a^+. \quad (72)$$

From these relations and from the axiom (44) for a^+ we obtain

$$p^{-\alpha_1 c_{13}} a^+ p^{\alpha_1 c_{12} N} = -c_{10} a^+ p^{-\alpha_1 N}, \quad -c_{10} q^{\alpha_2 N} = q^{\alpha_2 N} q^{-\alpha_2 c_{13} N}.$$

Then

$$c_{10} = -1, c_{12} = -1, c_{13} = 0. \quad (73)$$

The same calculations for a give $c_{11} = -1$.

A fulfillment of the remainder relations of the algebra under the action of the Hopf algebra operations can be easily verified.

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