

# Standard Model Muon Magnetic Dipole Moment

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The most recent high-precision determination of the hadronic leading order contribution to the muon magnetic dipole moment within the Standard Model of particle physics has revealed a five standard deviation discrepancy with the previous determination with the highest precision. A systematic effect of the luminous volume created during the measurements leading to the determinations seems to be the source of the discrepancy. Correcting for the luminous volume effect allows a consistent determination of the Standard Model muon magnetic dipole moment with precision that is comparable to the latest world average from muon spin precession experiments.

*Introduction*—Within the Standard Model of particle physics, the shift of the magnetic dipole moment of the muon  $a_\mu = g_\mu/2 - 1$  [1] from the relativistic quantum mechanical value  $g_\mu^{\text{RQM}}/2 = 1$  [2] is dominated by the leading order quantum electrodynamic contribution  $a_\mu^{\text{QED,LO}} = \alpha/(2\pi) \approx 1.164 \times 10^{-3}$  [3, 4], where  $\alpha$  is the electromagnetic “fine-structure” coupling constant [5]. Yet the dominant source of the uncertainty in the current consensus value  $a_\mu(\text{SM, WP20}) = (116\,599\,1810 \pm 43) \times 10^{-11}$  [6] is neither higher-order quantum electrodynamic contributions [7, 8] nor the contributions from the weak nuclear force [9, 10]. Instead, the leading order contribution from hadronic vacuum polarization  $a_\mu^{\text{HLO}} \approx 700 \times 10^{-10}$  [11–16] is the main source of uncertainty followed by uncertainties in the hadronic light-by-light contribution [17–25] and higher-order contributions from hadronic vacuum polarization [26].

In this Article, we resolve the long-standing tension between the determinations by the KLOE [27–30] and BABAR [31, 32] experiments of the resonant two-pion part  $a_\mu^{\pi\pi} \approx 380 \times 10^{-10}$  [33] of the leading order hadronic vacuum polarization contribution. The measurements of

the charged pion  $\pi^+\pi^-(\gamma)$  pair production cross-section from electron-positron annihilation at the neutral  $\rho^0$  vector meson resonance that determine  $a_\mu^{\pi\pi}$  [34] are shifted below the Standard Model value by an amount that scales with the luminous volume created by the colliding electron and positron bunches within the detector (See Table I). Correcting for the luminous volume effect resolves the long-standing KLOE-BABAR tension which is the main source of uncertainty in the current consensus value for the muon magnetic dipole moment within the Standard Model of particle physics [6].

*Luminous volume effect*—The luminous volume  $V_\rho = 8\pi\sigma_x\sigma_y\sigma_z$  created by the colliding electron and positron bunches at the interaction region within the detector combines with the time  $T_\rho = 2\sigma_z/c$  it takes the bunches to cross and the energy  $E_\rho$  of the  $\rho^0$  vector meson in the lab frame to give the shift [35]

$$\Delta a_\mu^{\pi\pi} = (19.6 \pm 1.2) \times 10^{-11} \left( \frac{E_\rho}{1000 \text{ MeV}} \right) \left( \frac{\sigma_x}{100 \mu\text{m}} \right)^{1/2} \times \left( \frac{\sigma_y}{10 \mu\text{m}} \right)^{1/2} \left( \frac{\sigma_z}{1 \text{ cm}} \right), \quad (1)$$

where,  $\sigma_x$  is the bunch width,  $\sigma_y$  is the bunch height, and  $\sigma_z$  is the bunch length (See Table II for  $E_\rho$  and Table III for  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ ).

*Resolving resonant  $\rho^0$  vector meson contribution*—The differences in beam parameters for KLOE and CMD-3 give them a relative shift [36]

$$\Delta a_\mu^{\pi\pi}(\text{KLOE}) - \Delta a_\mu^{\pi\pi}(\text{CMD3}) = (167 \pm 10) \times 10^{-11}. \quad (2)$$

The relative shift roughly resolves the  $5.1 \sigma$  discrepancy between KLOE and CMD-3 [33]

$$\begin{aligned} a_\mu^{\pi\pi}(\text{CMD3}) - a_\mu^{\pi\pi}(\text{KLOE}) &= ((3793 \pm 30) - (3606 \pm 21)) \times 10^{-11} \\ &= (187 \pm 37) \times 10^{-11} \end{aligned} \quad (3)$$

Similarly, the differences in beam parameters at KLOE and BABAR give them a relative shift (See Table I)

$$\Delta a_\mu(\text{KLOE}) - \Delta a_\mu(\text{BABAR})$$

Experiment	$a_\mu^{\pi\pi}(10^{-11})$	$\Delta a_\mu^{\pi\pi}(10^{-11})$	$a_\mu^{\pi\pi}(\text{SM})(10^{-11})$
CMD-3	3793 (30)	0	3793 (30)(0)
BABAR	3701 (27)	55 (3)	3756 (27)(3)
CMD-2	3665 (34)	71 (4)	3736 (34)(4)
BES-III	3618 (36)	58 (4)	3676 (36)(4)
KLOE	3606 (21)	167 (10)	3773 (21)(10)

TABLE I. Leading order hadronic contribution  $a_\mu^{\pi\pi}$  to the muon magnetic dipole moment determined by pion pair production  $\pi^+\pi^-(\gamma)$  from electron-positron annihilation with center-of-mass energy in the range  $0.6 \leq \sqrt{s} \leq 0.88$  GeV at various experiments compiled by Ref. [33]. The luminous volume created by the colliding electron and positron bunches produces the shift  $\Delta a_\mu^{\pi\pi}$  given by Eq. (1). The Standard Model value  $a_\mu^{\pi\pi}(\text{SM}) = a_\mu^{\pi\pi} + \Delta a_\mu$  corrects for the luminous volume effect.

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Experiment	Collider	$E$ (MeV)	$E_\rho$ (MeV)
KLOE	DAΦNE	510	805
BES-III	BEPC-2	1885	1966
CMD-2	VEPP-2M	388	775
BABAR	PEP-II	5282	3131

TABLE II. Experiments at electron-positron colliders to detect  $\pi^+\pi^-(\gamma)$  production from the neutral  $\rho^0$  vector meson resonance at  $m_\rho = 775$  MeV [52]. The beam has energy  $E$  and the  $\rho^0$  meson has energy  $E_\rho = \gamma\sqrt{p_\rho^2 + m_\rho^2} - \gamma\beta p_\rho$  in the lab frame [53]. Here,  $p_\rho = E - m_\rho^2/(4E)$  is the momentum of the  $\rho^0$  meson in the center-of-mass frame. The time-dilation factor is  $\gamma \approx 1.00$  and the speed is  $\beta c \approx 0.00 c$  for the center-of-mass frame of the electron-positron collision with respect to the lab frame for all colliders except BABAR. For BABAR the effective beam energy  $E = \sqrt{E_+E_-} = 5282$  MeV is shown in the table, where,  $E_+ = 3100$  MeV was the positron beam energy and  $E_- = 9000$  MeV was the electron beam energy giving a time-dilation factor  $\gamma = 1.15$  and speed  $\beta c = 0.49 c$  [51]. The source for all beam parameters was Ref. [49] except VEPP-2M which came from Ref. [50].

Collider	$\sigma_x$ ( $\mu\text{m}$ )	$\sigma_y$ ( $\mu\text{m}$ )	$\sigma_z$ (cm)
DAΦNE [48]	2000	20	3.0
DAΦNE [49]	260	4.8	2.0
BEPC-2 [49]	347	4.5	1.2
VEPP-2M [50]	400	10	3.0
VEPP-2M [50]	35	35	3.0
PEP-II [49]	157	4.7	1.05

TABLE III. Beam parameters of high-energy electron-positron colliders. Bunch width  $\sigma_x$ , height  $\sigma_y$ , and length  $\sigma_z$  refer to the size of the bunch at the interaction point within the detectors. For colliders with more than one set of beam parameters, the shift  $\Delta a_\mu^{\pi\pi}$  was computed separately for each set of beam parameters using Eq. (1) and the equal-weight average of the shifts was then taken.

$$\begin{aligned}
&= ((167 \pm 10) - (55 \pm 3)) \times 10^{-11} \\
&= (112 \pm 7) \times 10^{-11}. \quad (4)
\end{aligned}$$

This relative shift resolves the long-standing  $2.8 \sigma$  tension between KLOE and BABAR [33]

$$a_\mu^{\pi\pi}(\text{BABAR}) - a_\mu^{\pi\pi}(\text{KLOE}) = (95 \pm 34) \times 10^{-11} \quad (5)$$

*Resonant  $\rho^0$  vector meson contribution*—Combining the volume-corrected determinations of the leading order hadronic vacuum polarization contribution to muon spin precession from charged pion pair  $\pi^+\pi^-(\gamma)$  production at the neutral  $\rho^0$  vector meson resonance gives the Standard Model (SM) result [37]

$$\begin{aligned}
a_\mu^{\pi\pi}(\text{SM}) &= 3756 (13)_\rho(5)_A \times 10^{-11} \\
&= (3756 \pm 14) \times 10^{-11} \quad (6)
\end{aligned}$$

where, the experimental uncertainty ( $\rho$ ) dominates over the uncertainty ( $A$ ) in the parameters entering the volume correction.

*Full  $\rho^0$  vector meson contribution*—The other leading order hadronic vacuum polarization contributions determined from pion production channels with no net charge in the final state combine with this resonant contribution to give the Standard Model leading order hadronic vacuum polarization contribution [38]:

$$\begin{aligned}
a_\mu^{\text{HLO}}(\rho^0) &= 3756 (13)_\rho(5)_A \times 10^{-11} \\
&+ 3225 (29)_{\text{off}\rho} \times 10^{-11} \\
&= 6981 (29)_{\text{off}\rho}(13)_\rho(5)_A \times 10^{-11}, \\
&= (6981 \pm 32) \times 10^{-11}, \quad (7)
\end{aligned}$$

where, the dominant uncertainty (off  $\rho$ ) is from experiments away from the  $\rho^0$  resonance.

*Standard Model hadronic leading order contribution*—The neutral determination  $a_\mu^{\text{HLO}}(\rho^0)$  from Eq. (7) combines with the charged result  $a_\mu^{\text{HLO}}(\rho^\pm) = (7030 \pm 44) \times 10^{-11}$  [39–41] centered on the  $\rho^\pm$  resonance and with the lattice quantum chromodynamic estimate  $a_\mu^{\text{HLO}}(\text{QCD}) = (7075 \pm 55) \times 10^{-11}$  [42] to give the Standard Model value of the hadronic leading order contribution to muon spin precession [43]

$$\begin{aligned}
a_\mu^{\text{HLO}}(\text{SM}) &= 7033 (16)_{\text{off}\rho}(12)_{\rho^\pm}(10)_{\text{QCD}}(7)_{\rho^0}(3)_A \times 10^{-11} \\
&= (7033 \pm 23) \times 10^{-11}, \quad (8)
\end{aligned}$$

where, the uncertainty from the charged measurement ( $\rho^\pm$ ) slightly exceeds that of the lattice (QCD).

*Tension between Standard Model and Experiment*—Using the hadronic leading order contribution determined from the full range of experimental and theoretical inputs including lattice quantum chromodynamic simulations and measurements at both the neutral  $\rho^0$  and charged  $\rho^\pm$  vector meson resonances, one can then update the tension between the muon spin precession experiments (Exp) and the Standard Model (SM) value for the muon magnetic dipole moment [44].

$$\begin{aligned}
a_\mu(\text{Exp}) - a_\mu(\text{SM}) &= 146 (22)_\mu(23)_{\text{HLO}}(18)_{\text{HLbL}} \times 10^{-11} \\
&= 146 (22)_\mu(29)_{\text{SM}} \\
&= (146 \pm 36) \times 10^{-11}, \quad (9)
\end{aligned}$$

where, the recently reduced uncertainty from the muon measurement ( $\mu$ ) is comparable to the Standard Model uncertainty (SM). The tension has  $4.1\sigma$  significance that falls just below the conventional threshold for the discovery of a new phenomenon.

*Resolving tension between Standard Model and experiment*—A similar volume effect shifts the muon spin precession frequency to create the tension with the Standard Model muon magnetic dipole moment [45]

$$\begin{aligned}
\Delta a_\mu &= (165 \pm 10) \times 10^{-11} \left( \frac{N_\mu}{1.93 \times 10^3} \right)^{1/2} \\
&\times \left( \frac{V_\mu}{3.12 \times 10^4 \text{ cm}^3} \right)^{1/2} \left( \frac{T_\mu}{64.4 \text{ } \mu\text{s}} \right)^{1/2} \\
&= (165 \pm 10) \times 10^{-11}, \quad (10)
\end{aligned}$$

where,  $N_\mu$  is the average number of muons in the beam during the measurement,  $V_\mu$  is the volume of the muon beam within the storage ring, and  $T_\mu$  is the life-time of the muon in the lab frame [46]. In the last line we average the shift for the Brookhaven and Fermilab experiments beam parameters weighted by the precision of their measurements [47]. Comparing the tension from Eq. (9) with shift shown in Eq. (10), we find that the shift resolves the  $4.1\sigma$  tension between the muon spin precession measurements and the Standard Model muon magnetic dipole moment.

*Conclusion*—The Standard Model value for the muon magnetic dipole moment was determined using the full range of experimental and theoretical input for the first time. The result has comparable precision to the recently improved experimental world average from muon spin precession experiments. The tension between the Standard Model and experiment has  $4.1\sigma$  significance that falls just below the threshold for the discovery of a new phenomenon.

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- [1] The magnetic moment  $g_\mu/2$  is expressed in units of the muon magneton  $e\hbar/(2m_\mu) \approx 4.48 \times 10^{-26}$  J/T [5].
  - [2] F. Jegerlehner, *The Anomalous Magnetic Moment of the Muon*, 2nd ed., Springer-Verlag, Berlin (2017).
  - [3] J. Schwinger, Phys. Rev. **73**, 416 (1948).
  - [4] J. Schwinger, Phys. Rev. **82**, 664 (1951). See Appendix B.
  - [5] D. B. Newell *et al.* (CODATA), Metrologia **55**, L13 (2018).
  - [6] T. Aoyama *et al.*, The anomalous magnetic moment of the muon in the Standard Model, Phys. Rep. **887**, 1 (2020).
  - [7] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Complete Tenth-Order QED Contribution to the Muon  $g-2$ , Phys. Rev. Lett. **109**, 111808 (2012).
  - [8] T. Aoyama, T. Kinoshita, and M. Nio, Theory of the Anomalous Magnetic Moment of the Electron, Atoms **7**, 28 (2019).
  - [9] A. Czarnecki, W. J. Marciano, and A. Vainshtein, Refinements in electroweak contributions to the muon anomalous magnetic moment, Phys. Rev. D **67**, 073006 (2003), [Erratum: Phys. Rev. D **73**, 119901 (2006)].
  - [10] C. Gnendiger, D. Stockinger, and H. Stockinger-Kim, The electroweak contributions to  $(g-2)_\mu$  after the Higgs boson mass measurement, Phys. Rev. D **88**, 053005 (2013).
  - [11] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon  $g-2$  and  $\alpha(m_Z^2)$  using newest hadronic cross-section data, Eur. Phys. J. C **77**, 827 (2017).
  - [12] A. Keshavarzi, D. Nomura, and T. Teubner, Muon  $g-2$  and  $\alpha(m_Z^2)$ : a new data-based analysis, Phys. Rev. D **97**, 114025 (2018).
  - [13] G. Colangelo, M. Hoferichter, and P. Stoffer, Two-pion contribution to hadronic vacuum polarization, J. High Energy Phys. **02**, 006 (2019).
  - [14] M. Hoferichter, B.-L. Hoid, and B. Kubis, Three-pion contribution to hadronic vacuum polarization, J. High Energy Phys. **08**, 137 (2019).
  - [15] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to  $\alpha(m_Z^2)$ , Eur. Phys. J. C **80**, 241 (2020), [Erratum: Eur. Phys. J. C **80**, 410 (2020)].
  - [16] A. Keshavarzi, D. Nomura, and T. Teubner, The  $g-2$  of charged leptons,  $\alpha(m_Z^2)$  and the hyperfine splitting of muonium, Phys. Rev. D **101**, 014029 (2020).
  - [17] K. Melnikov and A. Vainshtein, Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment revisited, Phys. Rev. D **70**, 113006 (2004).
  - [18] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera, and P. Stoffer, Remarks on higher-order hadronic corrections to the muon  $g-2$ , Phys. Lett. B **735**, 90 (2014).
  - [19] P. Masjuan and P. Sanchez-Puertas, Pseudoscalar-pole contribution to the  $(g_\mu-2)$ : A rational approach, Phys. Rev. D **95**, 054026 (2017).
  - [20] G. Colangelo, M. Hoferichter, M. Procura, and P. Stoffer, Dispersion relation for hadronic light-by-light scattering: two-pion contributions, J. High Energy Phys. **04**, 161 (2017).
  - [21] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold, and S. P. Schneider, Dispersion relation for hadronic light-by-light scattering: Pion pole, J. High Energy Phys. **10**, 141 (2018).
  - [22] A. Gerardin, H. B. Meyer, and A. Nyffeler, Lattice calculation of the pion transition form factor with  $N_f = 2+1$  Wilson quarks, Phys. Rev. D **100**, 034520 (2019).
  - [23] J. Bijnens, N. Hermansson-Truedsson, and A. Rodriguez-Sanchez, Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment, Phys. Lett. B **798**, 134994 (2019).
  - [24] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, and P. Stoffer, Longitudinal short-distance constraints for the hadronic light-by-light contribution to  $(g-2)_\mu$  with large- $N_c$  Regge models, J. High Energy Phys. **03**, 101 (2020).
  - [25] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, and C. Lehner, Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD, Phys. Rev. Lett. **124**, 132002 (2020).
  - [26] A. Kurz, T. Liu, P. Marquard, and M. Steinhauser, Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order, Phys. Lett. B **734**, 144 (2014).
  - [27] F. Ambrosino *et al.* (KLOE Collaboration), Phys. Lett. B **670**, 285 (2009).
  - [28] F. Ambrosino *et al.* (KLOE Collaboration), Phys. Lett. B **700**, 102 (2011).
  - [29] F. Ambrosino *et al.* (KLOE Collaboration), Phys. Lett. B **720**, 336 (2013).
  - [30] F. Ambrosino *et al.* (KLOE Collaboration), JHEP **03**, 173 (2018).

- [31] J. P. Lees *et al.* (BABAR Collaboration), Phys. Rev. D **86** 032013, (2012).
- [32] M. Ablikim *et al.* (BABAR Collaboration), Phys. Lett. B **753**, 629 (2016), [Erratum: Phys. Lett. B **812**, 135982 (2021)].
- [33] F. V. Ignatov, arXiv.org:2302.08834 [hep-ex]. See Table 4 for  $a_\mu^{\pi\pi}$  results from CMD-3, BABAR, CMD-2, BES-III, and KLOE. The table also includes CLEO, SND, and SND2k which we have not yet analyzed.
- [34] For a pedagogical derivation of the dispersion relation between the the cross-section for hadronic production by electron-positron annihilation and the leading order hadronic vacuum polarization contribution to the muon magnetic dipole moment, see Lecture 5:  $e^+ + e^- \rightarrow$  Any Hadrons on pp. 23-30 of Ref. [54].
- [35] Axions from the local dark matter halo of the galaxy convert neutral  $\rho^0$  vector mesons into isoscalar  $\omega$  vector mesons. This creates a shift in the vector meson propagator that carries over to a shift of the resonant part of the two-pion hadronic leading order contribution  $\Delta a_\mu^{\pi\pi} = m_A E_\rho g_{A\rho\omega} \tilde{\phi}_A(q_A) a_\mu^{\pi\pi}$  [55], where,  $m_A = (0.508 \pm 0.004)$  eV [55] is the axion mass,  $\rho_A = (0.28 \pm 0.03)$  GeV/cm<sup>3</sup> [55] is the local axion energy density,  $g_{A\rho\omega} = (6.11 \pm 0.18) \times 10^{-8}$  GeV<sup>-1</sup> [55] is the axion-meson coupling,  $E_\rho$  is the neutral  $\rho^0$  vector meson energy in the axion rest-frame, and  $|\tilde{\phi}_A(q_A)|^2 = (\rho_A/m_A^2) V_e T_e$  gives the axion amplitude during the electron-positron annihilation within luminous volume  $V_e = 8\pi\sigma_x\sigma_y\sigma_z$  and bunch crossing time  $T_e = 2\sigma_z/c$  for bunches of width  $\sigma_x$ , height  $\sigma_y$ , and length  $\sigma_z$ . Due to charge conservation, the axions cannot shift determinations of the leading order hadronic contribution  $a_\mu^{\text{HLO}}(\rho^\pm)$  that are dominated by pion pair production  $\pi^\pm\pi^0(\gamma)$  at the charged  $\rho^\pm$  vector meson resonance [15, 39, 40]. Similarly, the lattice quantum chromodynamic (QCD) determination  $a_\mu^{\text{HLO}}(\text{QCD})$  [42] do not include axion effects.
- [36] The luminous volume at the VEPP-2000 collider was small during the RHO2018 campaign of measurements by the CMD-3 detector at the neutral  $\rho^0$  vector meson resonance in charged pion  $\pi^+\pi^-(\gamma)$  pair production from electron-positron annihilation (I. Logashenko, private communication, August 2023). The horizontal and vertical movements of the beam were coupled so that the effective beam profile at the interaction point was one-dimensional rather than two-dimensional. The effective cross-sectional area of the luminous volume is then much smaller than what might be inferred from the average beam radius reported to the Particle Data Group [49].
- [37] The determinations of the corrected two-pion part of the hadronic leading order contribution  $a_\mu^{\pi\pi} + \Delta a_\mu^{\pi\pi}$  from the neutral  $\rho^0$  vector meson resonance listed in Table I are independent and consistent. We perform the weighted average to obtain the Standard Model value  $a_\mu^{\pi\pi}(\text{SM})$  for the resonant part of the leading order hadronic contribution.
- [38] We take the result from Ref. [15] for the leading order hadronic vacuum polarization contributions from neutral final states aside from the dominant contribution from the  $\rho^0$  resonance for charged pion  $\pi^+\pi^-$  pair production.
- [39] M. Fujikawa *et al.* (BELLE Collaboration), Phys. Rev. D **78**, 072006 (2008), arXiv:0805.3773 [hep-ex].
- [40] S. Schael *et al.* (ALEPH Collaboration), Phys. Rept. **421**, 191 (2005), arXiv:hep-ex/0506072 [hep-ex].
- [41] M. Davier, A. Hoecker, B. Malaescu, C.-Z. Yuan, and Z. Zhang, Eur. Phys. J. C **74**, 2803 (2014). For the result  $a_\mu^{\text{HLO}}(\rho^\pm) = (703.0 \pm 4.4) \times 10^{-10}$ , see Ref. [6], pg. 32 below Eq. (2.13).
- [42] Sz. Borsanyi *et al.* (BMW Collaboration), Nature **593**, 51 (2021).
- [43] The determinations of the hadronic leading order contribution  $a_\mu^{\text{HLO}}(\rho^0)$ ,  $a_\mu^{\text{HLO}}(\rho^\pm)$ , and  $a_\mu^{\text{HLO}}(\text{QCD})$  are consistent and independent. We perform the weighted average to obtain the Standard Model value  $a_\mu^{\text{HLO}}(\text{SM})$ .
- [44] Spin precession measurements at BNL E821 [58], FNAL E989 Run 1 [57], and FNAL E989 Runs 2 and 3 [56] give muon magnetic dipole moment [56]:
- $$a_\mu(\text{Exp}) = (116\,592\,059 \pm 22) \times 10^{-11} \text{ (0.19 ppm)} \quad (11)$$
- Combining our determination of the hadronic leading order contribution  $a_\mu^{\text{HLO}}(\text{SM}) = (7033 \pm 23) \times 10^{-11}$  from Eq. (8) with the Muon  $g - 2$  Theory Initiative White Paper [6] values for the contributions from quantum electrodynamics [7, 8], the weak nuclear force [9, 10], hadronic light-by-light scattering [17–25], and higher order hadronic vacuum polarization [26], we find the Standard Model muon magnetic dipole moment:
- $$a_\mu(\text{SM}) = (116\,591\,913 \pm 23_{\text{HLO}} \pm 18_{\text{HLbL}}) \times 10^{-11}, \quad (12)$$
- where, the leading source of uncertainty remains the hadronic leading order contribution (HLO) followed by the hadronic light-by-light scattering (HLbL). Taking the difference of Eq. (11) and Eq. (12), gives the remaining tension between experiment and the Standard Model value for the muon magnetic dipole moment.
- [45] See Appendix B of Ref. [4] for the calculation of the leading order quantum electrodynamic contribution to the spin precession frequency. The axion  $\phi_A$  couples the electric field  $\mathbf{E}$  of the photon to its magnetic field  $\mathbf{B}$  through the axion electrodynamic interaction  $\mathcal{L}_{A\gamma} = g_{A\gamma\gamma} \phi_A \mathbf{E} \cdot \mathbf{B}$ , where, the coupling strength is  $g_{A\gamma\gamma} = (0.68 \pm 0.02) \times 10^{-10}$  GeV<sup>-1</sup>. The photon propagator  $D_+(x, x')$  changes at leading order in  $g_{A\gamma\gamma}$  by the amount  $\Delta D(x, x') = m_A^2 g_{A\gamma\gamma} \tilde{\phi}_A(q_A) D(x, x')$ , where,  $\tilde{\phi}_A(q_A)$  is the axion amplitude with four-momentum  $q_A$  whose components in the storage ring rest frame are all small compared to the magic momentum  $p_\mu = m_\mu c / \sqrt{a_\mu} = 3.09$  GeV/c of muons in the storage ring. The shift in the photon propagator carries over to a shift in the spin precession frequency  $\Delta a_\mu = m_A^2 g_{A\gamma\gamma} \tilde{\phi}_A(q_A) a_\mu$  of the same relative size to leading order. The axion mass is  $m_A = (0.508 \pm 0.008)$  eV, the local energy density  $\rho_A = (0.28 \pm 0.03)$  GeV/cm<sup>3</sup>, and the amplitude is  $|\tilde{\phi}_A(q_A)|^2 = (\rho_A/m_A^2) \times N_\mu V_\mu T_\mu$  [55].
- [46] The Brookhaven experiment E821 [58] had muon beam parameters  $N_\mu = 1290$ ,  $\sigma_x = 2.10$  cm, and  $\sigma_y = 1.53$  cm (J. Mott, private communication, November 2021). The Fermilab Run 1 experiment E989 [57] had  $N_\mu = 1930$ ,  $\sigma_x = 1.75$  cm, and  $\sigma_y = 1.25$  cm (J. Mott, private communication, November 2021).
- [47] Using the muon beam parameters [46], we find FNAL E989 Run 1 had shift  $\Delta a_\mu(\text{FNAL, Run 1}) = (165 \pm 10) \times 10^{-11}$  while the experimental precision was  $\sigma_{a_\mu}(\text{FNAL, Run 1}) = a_\mu \times 460 \text{ ppm} = 53 \times 10^{-11}$  [57]. BNL E821 had shift  $\Delta a_\mu(\text{BNL}) = (161 \pm 10) \times 10^{-11}$  and precision  $\sigma_{a_\mu}(\text{BNL}) = a_\mu \times 540 \text{ ppb} = 63 \times 10^{-11}$  [58]. The relevant data needed to compute the shift are not

yet publicly available for FNAL Runs 2 and 3 but the precision was  $\sigma_{a_\mu}(\text{FNAL, Runs 2 + 3}) = a_\mu \times 215 \text{ ppm} = 25 \times 10^{-11}$  [56] and the beam parameters are roughly the same as Run 1 (J. Mott, private communication, 14 August 2023). For example, the average field index for Runs 2 and 3 was around  $n = 0.108$  (D. Kawall, private communication, September 2023) while the pitch correction was  $C_p = 170 \pm 10 \text{ ppb}$ . Together these give a rough estimate of the beam-width  $\sigma_y(\text{Runs 2 + 3}) = R\sqrt{2C_p/n} = 1.27 \pm 0.04$  which is consistent at this level of precision with the value  $\sigma_y(\text{Run1}) = 1.25 \text{ cm}$  from Run 1. Meanwhile, the Brookhaven and Fermilab Run 1 experiments give consistent values for the shift. We perform the weighted average to obtain the shift  $\Delta a_\mu$ . The uncertainty of the shift is fully correlated between the two experiments so we simply take the uncertainty to be that of the shift of a single experiment.

- [48] F. Ambrosino *et al.* (KLOE Collaboration), Beam parameters from first year at DAΦNE, *Eur. J. Phys. C* **47**, 589 (2006).
- [49] V. Shiltsev and F. Zimmerman, 32. High Energy Collider Parameters in Ref. [52].
- [50] R.R. M. Barnett *et al.* (Particle Data Group), *Phys. Rev. D* **54**, 1 (1996). See Section 21. High Energy Colliders: e+e- colliders (II) on pg. 129.
- [51] A.J. Bevan, B. Golob, Th. Mannel, S. Prell, and B.D. Yabsley, *Eur. Phys. J. C* **74**, 3026 (2014). Equation 2.1.2 on pg. 18 gives  $\gamma = (E_+ + E_-)/(2E) = (3.1 + 9.0)/(2 \times 5.282) = 1.15$  and  $\beta = (E_- - E_+)/(E_- + E_+) = (9.0 - 3.1)/(9.0 + 3.1) = 0.59$ .
- [52] P. A. Zyla *et al.* (Particle Data Group Collaboration), *Prog. Theor. Exp. Phys.* **2020**, 083C01 (2020).
- [53] By momentum conservation, the neutral  $\rho^0$  vector meson momentum  $p_\rho = E/c - m_\rho^2 c^3/(4E)$  in the center-of-mass frame is simply that of the initial state radiation photon for colliders such as DAFNE, BEPC-2, and PEP-II with beam energy off the  $\rho^0$  resonance at beam energy  $E = m_\rho c^2/2$ . For asymmetric electron-positron colliders, such as PEP-II, we assume the initial state radiation photon makes a large lab-frame angle with the high energy beam direction in accord with the data selection by the BABAR experiment [31, 32]. In the center-of-mass frame, this photon must then come out approximately along the low energy beam direction. We then boost the center-of-mass energy  $\sqrt{p_\rho^2 c^2 + m_\rho^2 c^4}$  of the  $\rho^0$  to the lab frame using the time-dilation factor  $\gamma$  and the speed  $\beta c$  to find its lab-frame energy  $E_\rho = \gamma\sqrt{p_\rho^2 c^2 + m_\rho^2 c^4} - \gamma\beta p_\rho c$ .
- [54] R. P. Feynman, *Photon-Hadron Interaction*, Addison-Wesley, Redwood City, CA (1989).
- [55] N. Bray-Ali, “Weighing the Axion with Muon Haloscopy,” arXiv.org:2108.12243 [hep-ph].
- [56] D. P. Aguillard *et al.* (Muon  $g - 2$  Collaboration), arXiv.org:2308.06230 [hep-ex] (2023).
- [57] B. Abi *et al.* (Muon  $g - 2$  Collaboration), *Phys. Rev. Lett.* **126**, 141801 (2021).
- [58] G.W. Bennett *et al.* (Muon  $g - 2$  Collaboration), *Phys. Rev. D* **73**, 072003 (2006).