

Optimization of Bell's Inequality Violation For Continuous Variable Systems

G. Gour*, F. C. Khanna†, A. Mann‡, M. Revzen§

*Theoretical Physics Institute, Department of Physics, University of Alberta,
Edmonton, Alberta, Canada T6G 2J1*

and

*Department of Physics, Technion-Israel Institute of Technology,
Haifa 32000, Israel*

Abstract

Two mode squeezed vacuum states allow Bell's inequality violation (BIQV) for all non-vanishing squeezing parameter (ζ). Maximal violation occurs at $\zeta \rightarrow \infty$ when the parity of either component averages to zero. For a given entangled *two spin* system BIQV is optimized via orientations of the operators entering the Bell operator (cf. S. L. Braunstein, A. Mann and M. Revzen: Phys. Rev. Lett. **68**, 3259 (1992)). We show that for finite ζ in continuous variable systems (and in general whenever the dimensionality of the subsystems is greater than 2) additional parameters are present for optimizing BIQV. Thus the expectation value of the Bell operator depends, in addition to the orientation parameters, on configuration parameters. Optimization of these configurational parameters leads to a unique maximal BIQV that depends only on ζ . The configurational parameter variation is used to show that BIQV relation to entanglement is, even for pure state, not monotonic.

Typeset using REVTEX

*E-mail: gilgour@phys.ualberta.ca

†E-mail: khanna@phys.ualberta.ca

‡E-mail: ady@physics.technion.ac.il

§E-mail: revzen@physics.technion.ac.il

Two mode squeezed vacuum states (TMSV) generated via pulsed non-degenerate parametric amplifiers are of interest following the pioneering study of Grangier et al. [1]. Such studies are, aside from their intrinsic interest, valuable for clarification of the applicability of Bell's inequalities for continuous variables in general and the original EPR state [2] in particular. The latter's relevance to the Bell inequality problem was stressed originally by Bell [3]. Such studies pertain to the question whether Quantum Mechanics can be underpinned with a theory employing local hidden variables. They gained particular attention with the demonstration, first achieved by Banaszek and Wodkiewicz [4], that negativity of the Wigner function is not a prerequisite to BIQV. In this work we demonstrate the presence of configurational parameters controlling the extent of BIQV, which are not present in the Bohm-like entangled state (since that state involves only two levels per subsystem, and therefore only orientational parameters are available for optimization of BIQV.) This is particularly relevant for *non*-maximally entangled states and should be of importance for possible applications of such states. We exhibit two exact solutions for “local maximal” BIQV for different configurational parameterizations (one was obtained earlier in [5]) and give the transformation between them. Our representative of Bells' inequalities is the CHSH inequality [6] and the observable is the parity rather than the spin. The novel configurational dependence introduced herewith should be important when such states are employed e.g. for delineating a dichotomic variable (even/odd parity in our case) as the verification involves different observables (exact photon number in one case or simply even/odd number of photons in another for the cases we consider below). BIQV for these states were observed experimentally in several laboratories [8,7] within the general study of the problem for continuous variable cases [9,10].

The state under study, $|\zeta\rangle$ (i.e. $|TMSV\rangle$), involves two beams each headed in a different direction and delineated by having the operators of the first channel designated by $\mathbf{a} = 1/\sqrt{2}(\mathbf{q} + i\mathbf{p})$ and $\mathbf{a}^+ = 1/\sqrt{2}(\mathbf{q} - i\mathbf{p})$ and the second channel by $\mathbf{b} = 1/\sqrt{2}(\mathbf{q}' + i\mathbf{p}')$ and $\mathbf{b}^+ = 1/\sqrt{2}(\mathbf{q}' - i\mathbf{p}')$. The TMSV state, $|\zeta\rangle$, is given by,

$$|\zeta\rangle = S(\zeta)|00\rangle; \quad S(\zeta) \equiv \exp[\zeta(a^+b^+ - ab)], \quad (1)$$

where ζ is real. The CHSH inequality involves the so-called Bell operator \mathcal{B} , [11], which, for our case, where parity is the dynamical variable, is given via the following observables:

(a) The parity operator (the superscript $i = 1, 2$ refers to the channel - it is omitted where clarity allows),

$$\begin{aligned} \Pi_z &= \int_0^\infty dq (|\mathcal{E}\rangle\langle\mathcal{E}| - |\mathcal{O}\rangle\langle\mathcal{O}|) \equiv \mathbf{I}_E - \mathbf{I}_O, \\ |\mathcal{E}\rangle &= \frac{1}{\sqrt{2}}[|q\rangle + |-q\rangle], \quad |\mathcal{O}\rangle = \frac{1}{\sqrt{2}}[|q\rangle - |-q\rangle], \end{aligned} \quad (2)$$

where $|q\rangle$ is the eigenstate of the position operator and \mathbf{I}_E and \mathbf{I}_O can be viewed as the identity operators in the subspaces of even and odd parity, respectively. We note that an alternative, equivalent representation for the latter operators is given in the number representation by

$$\mathbf{I}_E = \sum_{n=0}^{\infty} |2n\rangle\langle 2n|, \quad \mathbf{I}_O = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n+1|, \quad (3)$$

We refer to these two representations of $\mathbf{I}_E, \mathbf{I}_O$ given in Eq.(2) and Eq.(3) as two configurational representations of the operators. The proof of the equivalence of the two configurational representatives for the operator $\boldsymbol{\Pi}_z$ can be easily verified by considering their matrix elements, e.g., with respect to the number states.

(b) For the x and y components of $\vec{\boldsymbol{\Pi}}^{(i)}$ we choose

$$\begin{aligned}\boldsymbol{\Pi}_x &= \int_0^\infty (|\mathcal{E}\rangle\langle\mathcal{O}| + |\mathcal{O}\rangle\langle\mathcal{E}|) dq \\ \boldsymbol{\Pi}_y &= i \int_0^\infty (|\mathcal{O}\rangle\langle\mathcal{E}| - |\mathcal{E}\rangle\langle\mathcal{O}|) dq.\end{aligned}\quad (4)$$

The components of $\vec{\boldsymbol{\Pi}}^{(i)} = (\boldsymbol{\Pi}_x^{(i)}, \boldsymbol{\Pi}_y^{(i)}, \boldsymbol{\Pi}_z^{(i)})$ ($i = 1, 2$) satisfy the standard $SU(2)$ algebra, and the square of each component is equal to the identity operator.

With these definitions the Bell operator \mathcal{B} is given by

$$\begin{aligned}\mathcal{B} &= \vec{n} \cdot \vec{\boldsymbol{\Pi}}^{(1)} \otimes \vec{m} \cdot \vec{\boldsymbol{\Pi}}^{(2)} + \vec{n}' \cdot \vec{\boldsymbol{\Pi}}^{(1)} \otimes \vec{m} \cdot \vec{\boldsymbol{\Pi}}^{(2)} \\ &\quad + \vec{n} \cdot \vec{\boldsymbol{\Pi}}^{(1)} \otimes \vec{m}' \cdot \vec{\boldsymbol{\Pi}}^{(2)} - \vec{n}' \cdot \vec{\boldsymbol{\Pi}}^{(1)} \otimes \vec{m}' \cdot \vec{\boldsymbol{\Pi}}^{(2)},\end{aligned}\quad (5)$$

and the Bell inequality is,

$$|\langle \mathcal{B} \rangle| \leq 2. \quad (6)$$

Here $\vec{n}, \vec{n}', \vec{m}, \vec{m}'$ are unit vectors, specifying the orientational parameters of the first and the second channel, respectively. In the notation of Eq.(2) (i.e. \mathcal{E}, \mathcal{O}) the state $|\zeta\rangle$, Eq. (1), is given by

$$|\zeta\rangle = \int_0^\infty \int_0^\infty dq dq' [(g_+ + g_-)|\mathcal{E}\mathcal{E}'\rangle + (g_+ - g_-)|\mathcal{O}\mathcal{O}'\rangle], \quad (7)$$

with

$$g_\pm(q, q'; \zeta) = \langle qq'|S(\pm\zeta)|00\rangle,$$

giving

$$g_\pm = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2}[q^2 + q'^2 \mp 2qq' \tanh(2\zeta)] \cosh(2\zeta)\right). \quad (8)$$

We have directly $\boldsymbol{\Pi}_z^{(1)} \otimes \boldsymbol{\Pi}_z^{(2)}|\zeta\rangle = |\zeta\rangle$.

We may now evaluate the Bell operator, Eq. (5), and choose the angles (i.e., the unit vectors $\vec{n}, \vec{n}', \vec{m}$ and \vec{m}') to *maximize* BIQV [12,5], yielding (we refer to this as the orientational optimization):

$$\langle \zeta | \mathcal{B} | \zeta \rangle = 2\sqrt{1 + F(\zeta)^2}, \quad (9)$$

with $F(\zeta) \equiv \langle \zeta | \boldsymbol{\Pi}_x^{(1)} \otimes \boldsymbol{\Pi}_x^{(2)} | \zeta \rangle$. A straightforward calculation yields

$$F(\zeta) = 2 \int_0^\infty dq dq' (g_+^2 - g_-^2) = \frac{2}{\pi} \arctan(\sinh(2\zeta)). \quad (10)$$

Clearly at $\zeta = 0$ no violation is possible, while for $\zeta \rightarrow \infty$ the violation attains Cirel'son's [13] limit.

To demonstrate the relevance of the configuration we now compare two configurational paramerizations: the one studied above with the one studied earlier in [5] where number configurational parametrization were used. We denote by $\vec{\mathbf{S}}^{(1)}$ and $\vec{\mathbf{S}}^{(2)}$ the corresponding operators defined by [5], they are given in the number state representation by $\mathbf{\Pi}_z = \mathbf{S}_z$

$$\mathbf{S}_z = \sum_{n=0}^{\infty} (|2n\rangle\langle 2n| - |2n+1\rangle\langle 2n+1|) = \mathbf{I}_E - \mathbf{I}_O; \quad (11)$$

this is the parity operator and it is identical to the one defined in Eq. (2) (cf. Eq. (3)). The x and y components are given by

$$\begin{aligned} \mathbf{S}_x &= \sum_{n=0}^{\infty} (|2n\rangle\langle 2n+1| + |2n+1\rangle\langle 2n|) \\ \mathbf{S}_y &= -i \sum_{n=0}^{\infty} (|2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|). \end{aligned} \quad (12)$$

The difference in the value of $\langle\zeta|\mathbf{\Pi}_x^{(1)} \otimes \mathbf{\Pi}_x^{(2)}|\zeta\rangle$ stems from the different choices for $\mathbf{\Pi}_x$ and $\mathbf{\Pi}_y$. These are related via a (configurational) unitary transformation (commuting with $\mathbf{\Pi}_z$, of course). That is, $\mathbf{S}_{\pm} = \mathbf{U}\mathbf{\Pi}_{\pm}\mathbf{U}^\dagger$, where $\mathbf{S}_{\pm} = (\mathbf{S}_x \pm i\mathbf{S}_y)/2$, $\mathbf{\Pi}_{\pm} = (\mathbf{\Pi}_x \pm i\mathbf{\Pi}_y)/2$ and \mathbf{U} is a unitary operator (for example, one can take the *unitary* operator $\mathbf{U} = \mathbf{I}_E + \mathbf{S}_-\mathbf{\Pi}_+$ or $\mathbf{U} = \mathbf{I}_O + \mathbf{S}_+\mathbf{\Pi}_-$). Thus we may consider these transformations as revealing new parameters for optimization of the BIQV for the state under study. The configurational parameter dependence implies (we give an explicit example below) that BIQV is not a monotonic function of the entanglement (in our case the latter is parametrized by ζ ; $\zeta \rightarrow \infty$ gives maximal entanglement). We first show that the choice of Chen et al. [5] leads to maximal violation for all values of ζ . (Note that $\tanh 2\zeta \geq \frac{2}{\pi} \arctan(\sinh 2\zeta)$ for all values of ζ , Fig.1). Let us consider $\mathbf{\Pi}_x$ and $\mathbf{\Pi}_y$ to be arbitrary operators that satisfy the $SU(2)$ commutation rules with the parity operator $\mathbf{\Pi}_z = \mathbf{I}_E - \mathbf{I}_O$ as defined above. Thus, the requirement $\mathbf{\Pi}_x^2 = \mathbf{\Pi}_y^2 = 1$ can be written in the form

$$\mathbf{\Pi}_+ \mathbf{\Pi}_- = \mathbf{I}_E \text{ and } \mathbf{\Pi}_- \mathbf{\Pi}_+ = \mathbf{I}_O. \quad (13)$$

Consider now two sets of such operators $\mathbf{\Pi}_{\pm}^{(i)}$ ($i = 1, 2$), both satisfying Eq. (13). Since $\mathbf{\Pi}_+^{(i)}$ transforms even parity functions into odd ones, and $\mathbf{\Pi}_-^{(i)}$ transforms odd functions into even ones, it follows that

$$F(\zeta) = \langle\zeta|\mathbf{\Pi}_x^{(1)} \otimes \mathbf{\Pi}_x^{(2)}|\zeta\rangle = \langle\zeta|\mathbf{\Pi}_+^{(1)} \otimes \mathbf{\Pi}_+^{(2)}|\zeta\rangle + \langle\zeta|\mathbf{\Pi}_-^{(1)} \otimes \mathbf{\Pi}_-^{(2)}|\zeta\rangle. \quad (14)$$

In the derivation of Eq. (9) via orientational parameters optimization [12], all the azimuthal angles of the vectors \vec{n} , \vec{n}' , \vec{m} and \vec{m}' in Eq. (5) have been set to zero, and we consider the case of

$$\langle\zeta|\mathbf{\Pi}_+^{(1)} \otimes \mathbf{\Pi}_+^{(2)}|\zeta\rangle = \langle\zeta|\mathbf{\Pi}_-^{(1)} \otimes \mathbf{\Pi}_-^{(2)}|\zeta\rangle > 0. \quad (15)$$

(Both $\mathbf{\Pi}_{\pm}$, that follow from Eq. (4), and \mathbf{S}_{\pm} , that follow from Eq. (12), satisfy this condition.) In general this requirement, Eq. (15), is relevant for the orientational parameters

optimization [12] and we assume its validity in our considerations where the orientational parameters were optimized for fixed configurational ones.

We now label $\langle 2n|\boldsymbol{\Pi}_+^{(1)}|2m+1\rangle$ by $U_{n,m}^1$ and $\langle 2n|\boldsymbol{\Pi}_+^{(2)}|2m+1\rangle$ by $U_{n,m}^2$. From Eq. (13) it follows that U^1 and U^2 are unitary matrices. Note also that $(U^1)_{n,m}^\dagger = \langle 2m|\boldsymbol{\Pi}_-^{(1)}|2n+1\rangle$, and $(U^2)_{n,m}^\dagger = \langle 2m|\boldsymbol{\Pi}_-^{(2)}|2n+1\rangle$. In the number representation our state, $|\zeta\rangle$, is given by

$$|\zeta\rangle = \frac{1}{\cosh \zeta} \sum_{n=0}^{\infty} (\tanh \zeta)^n |nn\rangle. \quad (16)$$

We thus further define the reduced density matrix

$$\rho_{n,m} \equiv \frac{(\tanh \zeta)^{2n}}{(\cosh \zeta)^2} \delta_{n,m}. \quad (17)$$

With these definitions we may write

$$\begin{aligned} F(\zeta) &= \frac{1}{2} \sinh(2\zeta) \left[\text{Tr} \left(\rho U^1 \rho U^2 \right) + \text{Tr} \left(\rho (U^1)^\dagger \rho (U^2)^\dagger \right) \right] \\ &= \sinh(2\zeta) \text{Tr} \left(\rho U^1 \rho U^2 \right), \end{aligned} \quad (18)$$

where the last equality follows from the condition (15). Hence, we have

$$F(\zeta) \leq \sinh(2\zeta) \text{Tr} \rho^2 = \tanh(2\zeta). \quad \text{QED} \quad (19)$$

We now give an explicit example of a choice of configurational paraemeters for which BIQV dependence on ζ implies $\text{BIQV} \rightarrow 0$ with $\zeta \rightarrow \infty$ while $\text{BIQV} \neq 0$ for $\zeta > 0$. i.e. BIQV is *not* a monotonic function of the entanglement even for a pure state (note: $\zeta \rightarrow \infty$ implies maximal entanglement):

In this example we take $\boldsymbol{\Pi}_z^{(1)} = \boldsymbol{\Pi}_z^{(2)} \equiv \boldsymbol{\Pi}_z = \mathbf{S}_z$ to be the parity operator as represented in Eq.(2) and $\boldsymbol{\Pi}_x^{(1)} = \boldsymbol{\Pi}_x^{(2)} \equiv \boldsymbol{\Pi}_x$ and $\boldsymbol{\Pi}_y^{(1)} = \boldsymbol{\Pi}_y^{(2)} \equiv \boldsymbol{\Pi}_y$ are defined as follows:

$$\boldsymbol{\Pi}_+^\dagger = \boldsymbol{\Pi}_- = \frac{\boldsymbol{\Pi}_x - i\boldsymbol{\Pi}_y}{2} = \sum_{n=0}^{\infty} i^n |2n+1\rangle \langle 2n|. \quad (20)$$

It is simple to check that $\boldsymbol{\Pi}_x$, $\boldsymbol{\Pi}_y$ and $\boldsymbol{\Pi}_z$ satisfy the conventional $SU(2)$ commutation relations. Note that these definitions corresponds to $U_{n,m}^1 = U_{n,m}^2 = (-i)^n \delta_{n,m}$. Thus,

$$F(\zeta) = \sinh(2\zeta) \text{Tr} \left(\rho U^1 \rho U^2 \right) = \frac{\sinh(2\zeta)}{\cosh^4 \zeta} \sum_{n=0}^{\infty} (-1)^n (\tanh \zeta)^{4n} = \sinh(2\zeta) \frac{(1 - \tanh^2 \zeta)^2}{1 + \tanh^4 \zeta}. \quad (21)$$

Note that $F(\zeta = 0) = F(\zeta \rightarrow \infty) = 0$ while $F(\zeta) > 0$ for finite (non-zero) values of ζ .

The dichotomic (± 1) parity operator is studied for the continuous variable two channel system of two mode squeezed vacuum state. The maximal BIQV for the Bell operator is demonstrated explicitly for two cases. The violation of the inequality is shown to depend on the choice of these components for non-maximal entanglement thus revealing configurational parameters whose choice allows optimization of the violation. We show that the particular choice, selected in [5] attains this optimization and that the dependence on these new (configurational) parameters implies that Bell's inequality violation is *not* a monotonic function of the entanglement.

ACKNOWLEDGMENT

We thank R. Teshima for his help. G.G. research is supported by the Killam Trust; F.K. research is supported by NSERC.

FIGURES

FIG. 1. Variation of $F(\zeta)$ (see Eq. (9)) with ζ for two configurational schemes.

REFERENCES

- [1] P. Grangier, M. J. Potasek, and B. Yurke, Phys. Rev A **38**, 6, 3132 (1988).
- [2] A. Peres *Quantum Theory: Concepts and Methods*, (Kluwer, Dordrecht, 1993).
- [3] J. S. Bell *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1987).
- [4] K. Banaszek, and K. Wodkiewicz, Phys. Rev. Lett. **76**, 23, 4344 (1996); **82**, 10, 2009 (1999).
- [5] Zeng-Bing Chen, Jian-Wei Pan, Guang Hou, and Yong-De Zhang, Phys. Rev. Lett. **88**, 040406 (2002).
- [6] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969); Clauser J. F. and A. Shimony , Rep. Prog. Phys. **41**, 1881 (1978).
- [7] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. **68**, 25, 3663 (1992).
- [8] A. Kuzmich, I. A. Walmsley, and L. Mandel, Phys. Rev. Lett. **85**, 7, 1349 (2000).
- [9] P. G. Kwiat, A. M. Steinberg, and R. Y. Chiao, Phys. Rev. A **47**, R2472 (1993).
- [10] J. D. Franson, Phys. Rev. Lett., **62**, 2205 (1989).
- [11] S. L. Braunstein, A. Mann and M. Revzen , Phys. Rev. Lett. **68**, 3259 (1992).
- [12] N. Gisin, Phys. Lett. A **154**, 201 (1991).
- [13] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).