

INVERSE PROBLEMS IN SPACETIME I: INVERSE PROBLEMS FOR EINSTEIN EQUATIONS – EXTENDED PREPRINT VERSION

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Abstract: *We consider inverse problems for the coupled Einstein equations and the matter field equations on a 4-dimensional globally hyperbolic Lorentzian manifold (M, g) . We give a positive answer to the question: Do the active measurements, done in a neighborhood $U \subset M$ of a freely falling observed $\mu = \mu([s_-, s_+])$, determine the conformal structure of the spacetime in the minimal causal diamond-type set $V_g = J_g^+(\mu(s_-)) \cap J_g^-(\mu(s_+)) \subset M$ containing μ ?*

More precisely, we consider the Einstein equations coupled with the scalar field equations and study the system $\text{Ein}(g) = T$, $T = T(g, \phi) + \mathcal{F}_1$, and $\square_g \phi - \mathcal{V}(\phi) = \mathcal{F}_2$, where the sources $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ correspond to perturbations of the physical fields which we control. The sources \mathcal{F} need to be such that the fields (g, ϕ, \mathcal{F}) are solutions of this system and satisfy the conservation law $\nabla_j T^{jk} = 0$. Let $(\hat{g}, \hat{\phi})$ be the background fields corresponding to the vanishing source \mathcal{F} . We prove that the observation of the solutions (g, ϕ) in the set U corresponding to sufficiently small sources \mathcal{F} supported in U determine $V_{\hat{g}}$ as a differentiable manifold and the conformal structure of the metric \hat{g} in the domain $V_{\hat{g}}$. The methods developed here have potential to be applied to a large class of inverse problems for non-linear hyperbolic equations encountered e.g. in various practical imaging problems.

Keywords: Inverse problems, active measurements, Lorentzian manifolds, non-linear hyperbolic equations, Einstein equations, scalar fields.

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1. INTRODUCTION AND MAIN RESULTS

We consider inverse problems for the non-linear Einstein equations coupled with matter field equations. In this paper, we consider for the matter fields the simplest possible model, the scalar field equations and study the perturbations of a globally hyperbolic Lorentzian manifold (M, \hat{g}) of dimension $(1 + 3)$, where the metric signature of \hat{g} is $(-, +, +, +)$.

Roughly speaking, we study the following problem: Can an observer in a space-time determine the structure of the surrounding space-time by doing measurements near its world line. More precisely, when $U_{\hat{g}}$ is a neighborhood of a time-like geodesic $\hat{\mu}$, we assume that we can control sources supported in an open neighborhood $W_{\hat{g}} \subset U_{\hat{g}}$ of $\hat{\mu}$ and measure the physical fields in the set $U_{\hat{g}}$. We ask, can the properties of the metric (the metric itself or its conformal class) be determined in a suitable larger set $J(p^-, p^+)$, $p^\pm = \hat{\mu}(s_\pm)$ that is not contained in the set $U_{\hat{g}}$, see Fig. 1(Left). This paper considers inverse problems for active measurements and the corresponding problem for passive measurements is studied in the second part of this paper, [66].

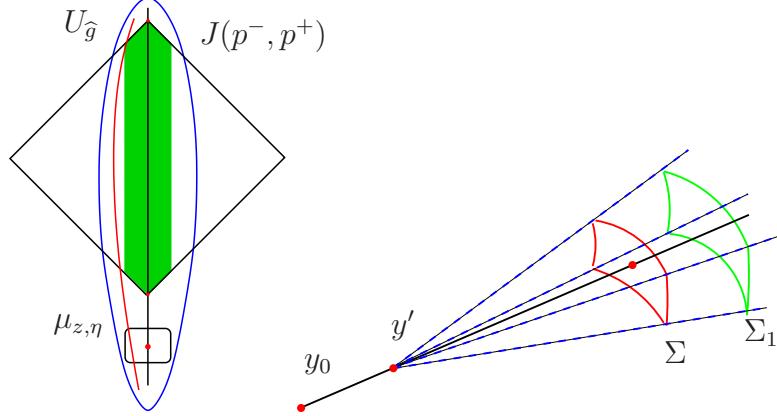


FIGURE 1. Left: This is a schematic figure in \mathbb{R}^{1+1} . The black vertical line is the freely falling observer $\hat{\mu}([-1, 1])$. The rounded black square is $\pi(\mathcal{U}_{z_0, \eta_0})$ that is a neighborhood of z_0 , and the red curve passing through $z \in \pi(\mathcal{U}_{z_0, \eta_0})$ is the time-like geodesic $\mu_{z, \eta}([-1, 1])$. The boundary of the domain $U_{\hat{g}}$ where we observe waves is shown on blue. The green area is the set $W_{\hat{g}} \subset U_{\hat{g}}$ where sources are supported, and the black “diamond” is the set $J(p^-, p^+) = J_{\hat{g}}^+(p^-) \cap J_{\hat{g}}^-(p^+)$.

Right: This is a schematic figure in the space \mathbb{R}^3 . It describes the location of a distorted plane wave (or a piece of a spherical wave) at different time moments. This wave propagates near the geodesic $\gamma_{x_0, \zeta_0}((0, \infty)) \subset \mathbb{R}^{1+3}$, $x_0 = (y_0, t_0)$ and is singular on a subset of a light cone emanated from $x' = (y', t')$. The piece of the distorted plane wave is sent from the surface $\Sigma \subset \mathbb{R}^3$, it starts to propagate, and at a later time its singular support is the surface Σ_1 .

1.0.1. Notations. Let (M, g) be a C^∞ -smooth $(1+3)$ -dimensional time-orientable Lorentzian manifold. For $x, y \in M$ we say that x is in the chronological past of y and denote $x \ll y$ if $x \neq y$ and there is a time-like path from x to y . If $x \neq y$ and there is a causal path from x to y , we say that x is in the causal past of y and denote $x < y$. If $x < y$ or $x = y$ we denote $x \leq y$. The chronological future $I^+(p)$ of $p \in M$ consist of all points $x \in M$ such that $p \ll x$, and the causal future $J^+(p)$ of p consist of all points $x \in M$ such that $p \leq x$. One defines similarly the chronological past $I^-(p)$ of p and the causal past $J^-(p)$ of p . For a set A we denote $J^\pm(A) = \cup_{p \in A} J^\pm(p)$. We also denote $J(p, q) := J^+(p) \cap J^-(q)$ and $I(p, q) := I^+(p) \cap I^-(q)$. If we need to emphasize the metric g which is used to define the causality, we denote $J^\pm(p)$ by $J_g^\pm(p)$ etc.

Let $\gamma_{x, \xi}(t) = \gamma_{x, \xi}^g(t) = \exp_x(t\xi)$ denote a geodesics in (M, g) . The projection from the tangent bundle TM to the base point of a vector is denoted by $\pi : TM \rightarrow M$. Let $L_x M$ denote the light-like directions of $T_x M$, and $L_x^+ M$ and $L_x^- M$ denote the future and past pointing light-like vectors, respectively. We also denote $\mathcal{L}_g^+(x) = \exp_x(L_x^+ M) \cup \{x\}$

the union of the image of the future light-cone in the exponential map of (M, g) and the point x .

By [10], an open time-orientable Lorenzian manifold (M, g) is globally hyperbolic if and only if there are no closed causal paths in M and for all $q^-, q^+ \in M$ such that $q^- < q^+$ the set $J(q^-, q^+) \subset M$ is compact. We assume throughout the paper that (M, g) is globally hyperbolic.

When g is a Lorentzian metric, having eigenvalues $\lambda_j(x)$ and eigenvectors $v_j(x)$ in some local coordinates, we will use also the corresponding Riemannian metric, denoted by g^+ which has the eigenvalues $|\lambda_j(x)|$ and the eigenvectors $v_j(x)$ in the same local coordinates. Let $B_{g^+}(x, r) = \{y \in M; d_{g^+}(x, y) < r\}$.

1.0.2. Perturbations of a global hyperbolic metric. Let (M, \widehat{g}) be a C^∞ -smooth globally hyperbolic Lorentzian manifold. We will call \widehat{g} the background metric on M and consider its small perturbations. A Lorentzian metric g_1 dominates the metric g_2 , if all vectors ξ that are light-like or time-like with respect to the metric g_2 are time-like with respect to the metric g_1 , and in this case we denote $g_2 < g_1$. As (M, \widehat{g}) is globally hyperbolic, it follows from [39] that there is a Lorentzian metric \widetilde{g} such that (M, \widetilde{g}) is globally hyperbolic and $\widehat{g} < \widetilde{g}$. One can assume that the metric \widetilde{g} is smooth. We use the positive definite Riemannian metric \widehat{g}^+ to define norms in the spaces $C_b^k(M)$ of functions with bounded k derivatives and the Sobolev spaces $H^s(M)$.

By [11], the globally hyperbolic manifold (M, \widetilde{g}) has an isometry Φ to the smooth product manifold $(\mathbb{R} \times N, \widetilde{h})$, where N is a 3-dimensional manifold and the metric \widetilde{h} can be written as $\widetilde{h} = -\beta(t, y)dt^2 + \kappa(t, y)$ where $\beta : \mathbb{R} \times N \rightarrow (0, \infty)$ is a smooth function and $\kappa(t, \cdot)$ is a Riemannian metric on N depending smoothly on $t \in \mathbb{R}$, and the submanifolds $\{t'\} \times N$ are C^∞ -smooth Cauchy surfaces for all $t' \in \mathbb{R}$. We define the smooth time function $\mathbf{t} : M \rightarrow \mathbb{R}$ by setting $\mathbf{t}(x) = t$ if $\Phi(x) \in \{t\} \times N$. Let us next identify these isometric manifolds, that is, we denote $M = \mathbb{R} \times N$.

For $t \in \mathbb{R}$, let $M(t) = (-\infty, t) \times N$ and, for a fixed $t_0 > 0$ and $t_1 > t_0$, let $M_j = M(t_j)$, $j = 0, 1$. Let $r_0 > 0$ be sufficiently small and $\mathcal{V}(r_0)$ be the set of metrics g on $M_1 = (-\infty, t_1) \times N$, which $C_b^s(M_1)$ -distance to \widehat{g} is less than r_0 and coincide with \widehat{g} in $M(0) = (-\infty, 0) \times N$.

1.0.3. Observation domain U . For $g \in \mathcal{V}(r_0)$, let $\mu_g : [-1, 1] \rightarrow M_1$ be a freely falling observer, that is, a time-like geodesic on (M, g) . Let $-1 < s_{-3} < s_{-2} < s_{-1} < s_{+1} < s_{+2} < s_{+3} < 1$ be such that $p^- = \mu_g(s_{-1}) \in \{0\} \times N$ and let $p^+ = \mu_g(s_{+1})$. Below, we denote $s_\pm = s_{\pm 1}$ and $\widehat{\mu} = \mu_{\widehat{g}}$.

When $z_0 = \widehat{\mu}(s_{-2}) \in M(0)$ and $\eta_0 = \partial_s \widehat{\mu}(s_{-2})$, we denote by $\mathcal{U}_{z_0, \eta_0}(h)$ the open h -neighborhood of (z_0, η_0) in the Sasaki metric of (TM, \widehat{g}^+) . We use below a small parameter $\widehat{h} > 0$. For $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}(2\widehat{h})$ we define on (M, g) a freely falling observer $\mu_{g, z, \eta} : [-1, 1] \rightarrow M$, such that

$\mu_{g,z,\eta}(s_{-2}) = z$, and $\partial_s \mu_{g,z,\eta}(s_{-2}) = \eta$. We assume that \widehat{h} is so small that $\pi(\mathcal{U}_{z_0,\eta_0}(2\widehat{h})) \subset M(0)$ and for all $g \in \mathcal{V}(r_0)$ and $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}(2\widehat{h})$ the geodesic $\mu_{g,z,\eta}([-1, 1]) \subset M$ is well defined and time-like and satisfies

$$(1) \quad \mu_{g,z,\eta}(s_{-j-1}) \in I_g^-(\mu_{g,z_0,\eta_0}(s_{-j})), \quad \mu_{g,z,\eta}(s_{j+1}) \in I_g^+(\mu_{g,z_0,\eta_0}(s_j)),$$

for $j = 1, 2$. We denote, see Fig. 1(Left), $\mathcal{U}_{z_0,\eta_0} = \mathcal{U}_{z_0,\eta_0}(\widehat{h})$ and

$$(2) \quad U_g = \bigcup_{(z,\eta) \in \mathcal{U}_{z_0,\eta_0}} \mu_{g,z,\eta}([-1, 1]), \quad \widehat{U} = U_{\widehat{g}}.$$

1.1. Formulation of the inverse problem.

1.1.1. *Inverse problems for non-linear wave equations.* The solution of inverse problems is often done by constructing the coefficients of the equations using invariant methods, e.g. using travel time coordinates. Thus in the topical studies of the subject, inverse problems are formulated invariantly, that is, on manifolds, see e.g. [2, 8, 30, 33, 42, 43, 73]. Many physical models lead to non-linear differential equations. In small perturbations, these equations can be approximated by linear equations, and most of the previous results on hyperbolic inverse problems in the multi-dimensional case concern linear models. Moreover, the existing uniqueness results are limited to the time-independent or real-analytic coefficients [2, 7, 8, 33, 61] as these results are based on Tataru's unique continuation principle [99, 100]. Such unique continuation results have been shown to fail for general metric tensors which are not analytic in the time variable [1]. Even some linear inverse problem are not uniquely solvable. In fact, the counterexamples for these problems have been used in the so-called transformation optics. This has led to models for fixed frequency invisibility cloaks, see e.g. [46] and references therein. These applications give one more motivation to study inverse problems.

Earlier studies on inverse problems for non-linear equations have concerned parabolic equations [56], elliptic equations [57, 58, 97, 98], and 1-dimensional hyperbolic equations [86]. The present paper differs from the earlier studies in that in our approach we do not consider the non-linearity as a perturbation, which effect is small with special solutions, but as a tool that helps us to solve the inverse problem. Indeed, the non-linearity makes it possible to solve a non-linear inverse problem which linearized version is not yet solved. This is the key novel feature of the paper.

1.1.2. *Einstein equations.* Below, we use the Einstein summation convention. The roman indexes i, j, k etc. run usually over indexes of spacetime variables as the greek letters are reserved to other indexes in sums. The Einstein tensor of a Lorentzian metric $g = g_{jk}(x)$ is

$$\text{Ein}_{jk}(g) = \text{Ric}_{jk}(g) - \frac{1}{2}(g^{pq} \text{Ric}_{pq}(g))g_{jk}.$$

Here, $\text{Ric}_{pq}(g)$ is the Ricci curvature of the metric g . We define the divergence of a 2-covariant tensor T_{jk} to be $(\text{div}_g T)_k = \nabla_n(g^{nj}T_{jk})$.

Let us consider the Einstein equations in the presence of matter,

$$(3) \quad \text{Ein}_{jk}(g) = T_{jk},$$

$$(4) \quad \text{div}_g T = 0,$$

for a Lorentzian metric g and a stress-energy tensor T related to the distribution of mass and energy. We recall that by Bianchi's identity $\text{div}_g(\text{Ein}(g)) = 0$ and thus the equation (4), called the conservation law for the stress-energy tensor, follows automatically from (3).

1.1.3. Reduced Einstein tensor. Let $m \geq 5$, $t_1 > t_0 > 0$ and $g' \in \mathcal{V}(r_0)$ be a C^m -smooth metric that satisfy the Einstein equations $\text{Ein}(g') = T'$ on $M(t_1)$. When r_0 above is small enough, there is a diffeomorphism $f : M(t_1) \rightarrow f(M(t_1)) \subset M$ that is a (g', \hat{g}) -wave map $f : (M(t_1), g') \rightarrow (M, \hat{g})$ and satisfies $M(t_0) \subset f(M(t_1))$. Here, $f : (M(t_1), g') \rightarrow (M, \hat{g})$ is a wave map, see [15, Sec. VI.7.2 and App. III, Thm. 4.2], if

$$(5) \quad \square_{g', \hat{g}} f = 0 \quad \text{in } M(t_1),$$

$$(6) \quad f = \text{Id}, \quad \text{in } (-\infty, 0) \times N,$$

where

$$\square_{g', \hat{g}} f = (g')^{jk} \left(\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} f^A - \Gamma'_{jk}^n(x) \frac{\partial}{\partial x^n} f^A + \widehat{\Gamma}_{BC}^A(f(x)) \frac{\partial}{\partial x^j} f^B \frac{\partial}{\partial x^k} f^C \right),$$

and $\widehat{\Gamma}_{BC}^A$ denotes the Christoffel symbols of metric \hat{g} and Γ'_{kl}^j are the Christoffel symbols of metric g' , see [15, formula (VI.7.32)]. The wave map has the property that $g = f_* g'$ satisfies $\text{Ein}(g) = \text{Ein}_{\hat{g}}(g)$, where $\text{Ein}_{\hat{g}}(g)$ is the \hat{g} -reduced Einstein tensor, see also formula (24) below,

$$(\text{Ein}_{\hat{g}} g)_{pq} = -\frac{1}{2} g^{jk} \widehat{\nabla}_j \widehat{\nabla}_k g_{pq} + \frac{1}{4} (g^{nm} g^{jk} \widehat{\nabla}_j \widehat{\nabla}_k g_{nm}) g_{pq} + P_{pq}(g, \widehat{\nabla} g),$$

where $\widehat{\nabla}_j$ is the covariant differentiation with respect to the metric \hat{g} and P_{pq} is a polynomial function of g_{nm} , g^{nm} , and $\widehat{\nabla}_j g_{nm}$ with coefficients depending on the metric \hat{g}_{nm} and its derivatives. Considering the wave map f as a transformation of coordinates, we see that $g = f_* g'$ and $T = f_* T'$ satisfy the \hat{g} -reduced Einstein equations

$$(7) \quad \text{Ein}_{\hat{g}}(g) = T \quad \text{on } M(t_0).$$

In the literature, the above is often stated by saying that the reduced Einstein equations (7) is the Einstein equations written with the wave-gauge corresponding to the metric \hat{g} . The equation (7) is a quasi-linear hyperbolic system of equations for g_{jk} . We emphasize that a solution of the reduced Einstein equations can be a solution of the original Einstein equations only if the stress energy tensor satisfies the conservation law $\nabla_j^g T^{jk} = 0$. It is usual also to assume that the energy density is non-negative. For instance, the weak energy condition requires that $T_{jk} X^j X^k \geq 0$ for all time-like vectors X . Next, we couple the Einstein

equations with matter fields and formulate the direct problem for the \hat{g} -reduced Einstein equations.

1.1.4. The initial value problem with sources. We consider metric and physical fields on a Lorentzian manifold (M, g) . This is an informal discussion. We aim to study an inverse problem with active measurements. As measurements cannot be implemented in Vacuum (as the Einstein equations is uniquely solvable with fixed initial data), we have to add matter fields in the model. We consider the coupled system of the Einstein equations and the equations for L scalar fields $\phi = (\phi_\ell)_{\ell=1}^L$ with some sources \mathcal{F}^1 and \mathcal{F}^2 .

Let \hat{g} and $\hat{\phi} = (\hat{\phi}_\ell)_{\ell=1}^L$ be C^∞ -background fields on M . Consider

$$(8) \quad \begin{aligned} \text{Ein}_{\hat{g}}(g) &= T, \quad T_{jk} = \mathbf{T}_{jk}(g, \phi) + \mathcal{F}_{jk}^1, \quad \text{in } M_0 = (-\infty, t_0) \times N, \\ \mathbf{T}_{jk}(g, \phi) &= \left(\sum_{\ell=1}^L (\partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell) \right) - V(\phi) g_{jk}, \\ \square_g \phi_\ell - V'_\ell(\phi) &= \mathcal{F}_\ell^2, \quad \ell = 1, 2, 3, \dots, L, \\ g &= \hat{g} \text{ and } \phi_\ell = \hat{\phi}_\ell \text{ in } M_0 \setminus J_g^+(p^-), \end{aligned}$$

where \mathcal{F}^1 and \mathcal{F}^2 are supported in $U_g^+ \cap J_g^+(p^-)$, $V \in C^\infty(\mathbb{R}^L)$, and $V'_\ell(s) = \frac{\partial}{\partial s_\ell} V(s)$, $s = (s_1, s_2, \dots, s_L)$. A typical model is $V(s) = \sum_{\ell=1}^L \frac{1}{2} m_\ell^2 s_\ell^2$. Above, $\square_g \phi = |\det(g)|^{-\frac{1}{2}} \partial_p (|\det(g)|^{\frac{1}{2}} g^{pq} \partial_q \phi)$. We assume that the background fields \hat{g} and $\hat{\phi}$ satisfy the equations (8) with $\mathcal{F}^1 = 0$ and $\mathcal{F}^2 = 0$. Note that above $J_g^+(p^-) \cap M_0 \subset J_{\hat{g}}^+(p^-)$ when $g \in \mathcal{V}(r_0)$ and r_0 is small enough.

To obtain a physically meaningful model, we need to assume that the physical conservation law in relativity,

$$(9) \quad \nabla_p (g^{pk} T_{kj}) = 0, \quad \text{for } j = 1, 2, 3, 4, \text{ where } T_{kj} = \mathbf{T}_{kj}(g, \phi) + \mathcal{F}_{kj}^1$$

is satisfied. Here $\nabla = \nabla^g$ is the connection corresponding to g . As will be noted in Subsection 3.1.1, the reduced Einstein tensor $\text{Ein}_{\hat{g}}(g)$ is equal to the Einstein tensor $\text{Ein}(g)$ when (g, ϕ) satisfies the system (8) and the conservation law (9). We mainly need local existence results¹ for the system (8). The global existence problem for the related systems has recently attracted much interest in the mathematical community and many important results been obtained, see e.g. [22, 26, 68, 70, 74, 75].

We encounter above the difficulty that the source $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2)$ in (8) has to satisfy the condition (9) that depends on the solution g of (8). This makes the formulation of active measurements in relativity difficult. In Appendix C we consider a model where the source term \mathcal{F}^1

¹In this paper we do not use optimal smoothness for the solutions in classical C^k spaces or Sobolev space $W^{k,p}$ but just suitable smoothness for which the non-linear wave equations can be easily analyzed using L^2 -based Sobolev spaces.

corresponds to e.g. fluid fields consisting of particles whose 4-velocity vectors are controlled and \mathcal{F}^2 contains a term corresponding to a secondary source function that adapts the changes of g , ϕ , and \mathcal{F}^1 so that the physical conservation law (9) is satisfied. This model is considered in detail in [67]. However, in this paper we replace the adaptive source functions by a general assumption of microlocal linearization stability that does not fix the physical model for the source fields \mathcal{F} .

1.1.5. *Definition of measurements.* For $r > 0$ let, see Fig. 1(Left),

$$(10) \quad W_g(r) = \bigcup_{s_- < s < s_+ - r} I_g(\mu_g(s), \mu_g(s + r)),$$

and let $r_1 > 0$ be so small that $W_g(2r_1) \subset U_g$ for all $g \in \mathcal{V}(r_0)$. We denote $W_g = W_g(r_1)$. We use Fermi-type coordinates: Let $Z_j(s)$, $j = 1, 2, 3, 4$ be a parallel frame of linearly independent time-like vectors at $\mu_g(s)$ such that $Z_1(s) = \dot{\mu}_g(s)$. Let $\Phi_g : (t_j)_{j=1}^4 \mapsto \exp_{\mu_g(t_1)}(\sum_{j=2}^4 t_j Z_j(t_1))$. We assume that $r_1 > 0$ is so small that $\Psi_g = \Phi_g^{-1}$ defines coordinates in $W_g(2r_1)$. We define the norm-like functions

$$\mathcal{N}_{\widehat{g}}^{(k)}(g) = \|(\Psi_g)_* g - (\Psi_{\widehat{g}})_* \widehat{g}\|_{C_b^k(\overline{\Psi_{\widehat{g}}(W_{\widehat{g}})})}, \quad \mathcal{N}^{(k)}(\mathcal{F}) = \|(\Psi_g)_* \mathcal{F}\|_{C_b^k(\overline{\Psi_{\widehat{g}}(W_{\widehat{g}})})},$$

where $k \in \mathbb{N}$, that measures the C^k distance of g from \widehat{g} and \mathcal{F} from zero in the Fermi-type coordinates. As we have assumed that the background metric \widehat{g} and the field $\widehat{\phi}$ are C^∞ -smooth, we can consider as smooth sources as we wish. Thus we use below smoothness assumptions on the sources that are far from the optimal ones.

Let us define the source-observation 4-tuples corresponding to measurements in U_g with sources \mathcal{F} supported in W_g . Let $\varepsilon > 0$ and $k_0 \geq 8$ and define

$$(11) \quad \begin{aligned} \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon) = & \{[(U_g, g|_{U_g}, \phi|_{U_g}, \mathcal{F}|_{U_g})] \quad ; \quad (g, \phi, \mathcal{F}) \text{ are } C^{k_0+3}\text{-smooth} \\ & \text{solutions of (8) and (9) with } \mathcal{F} \in C_0^{k_0+3}(W_g; \mathcal{B}^L), \\ & J_g^+(\text{supp } \mathcal{F}) \cap J_g^-(\text{supp } \mathcal{F}) \subset W_g, \mathcal{N}^{(k_0)}(\mathcal{F}) < \varepsilon, \mathcal{N}_{\widehat{g}}^{(k_0)}(g) < \varepsilon\}. \end{aligned}$$

Above, the sources \mathcal{F} above are considered as sections of the bundle \mathcal{B}^L , where \mathcal{B}^L is a vector bundle on M that is the product bundle of the bundle of symmetric $(0, 2)$ -tensors and the trivial vector bundle with the fiber \mathbb{R}^L . Also, $[(U_g, g, \phi, \mathcal{F})]$ denotes the equivalence class of all Lorenztian manifolds (U', g') and functions $\phi' = (\phi'_\ell)_{\ell=1}^L$ and the tensors \mathcal{F}' defined on a C^∞ -smooth manifold U' , such that there is C^∞ -smooth diffeomorphism $\Psi : U' \rightarrow U_g$ satisfying $\Psi_* g' = g$, $\Psi_* \phi'_\ell = \phi_\ell$, and $\Psi_* \mathcal{F}' = \mathcal{F}$.

In many inverse problems one considers a Dirichlet-to-Neumann map or, equivalently to that, the Cauchy data set that is the graph of the Dirichlet-to-Neumann map. Similarly, the source-observation 4-tuples $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ could be considered as graph of a “source-to-field” map but

due to the conservation law the source-to-field map could be defined only on a subset of sources supported in U_g . To avoid the difficulties related to the fact that we do not have a good characterization for this subset, nor do we know the wave map coordinates in U_g , we do not define a “source-to-field” map but use the data set $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$.

We will analyze the smoothness objects in $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ in different coordinates. Observe that when $g' \in C^{k_0+3}$ and $\mathcal{F}' \in C^{k_0+3}$, the Fermi coordinates $\Psi_{g'}$ are C^{k_0+1} -smooth and thus $(\Psi'_{g'})_* g'$ is C^{k_0+1} -smooth. Moreover, the (g', \widehat{g}) -wave map f is C^{k_0} -smooth and thus the metric g' and the source \mathcal{F}' in the wave map coordinates, that is, $f_* g'$ and $f_* \mathcal{F}'$, are C^{k_0-1} -smooth. However, to consider local existence results for the Einstein-scalar field equations, we need to consider the case when the norm of the source is small. We do this next.

Observe that when $\mathcal{N}^{(k_0)}(\mathcal{F}) < \varepsilon$, $\mathcal{N}_{\widehat{g}}^{(k_0)}(g) < \varepsilon$ we can locally solve the (g, \widehat{g}) -wave map \tilde{f} is C^{k_0-3} -smooth and thus the metric g and the source \mathcal{F} in the wave map coordinates, that is, $\tilde{f}_* g$ and $\tilde{f}_* \mathcal{F}$, are C^{k_0-4} -smooth in $W_{\tilde{f}_* g}$. Since \mathcal{F} vanishes outside the domain W_g , we see that $\tilde{f}_* \mathcal{F}$ vanishes outside $W_{\tilde{f}_* g}$ and thus $\tilde{f}_* \mathcal{F}$ is C^{k_0-4} -smooth in the set M_0 . As $k_0 \geq 8$, this implies that we can later obtain local existence of the Einstein-scalar field equations (8) when $\mathcal{N}^{(k_0)}(\mathcal{F}) < \varepsilon$, $\mathcal{N}_{\widehat{g}}^{(k_0)}(g) < \varepsilon$ and ε is small enough.

Note that $[(U_{\widehat{g}}, \widehat{g}, \widehat{\phi}, 0)]$ is the only element in $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ for which the \mathcal{F} -component is zero. Thus the collection $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ determines the isometry type of $(U_{\widehat{g}}, \widehat{g})$.

1.1.6. Linearized equations. We need also to consider the linearized version of the equations (8) that have the form (in local coordinates)

$$(12) \quad \begin{aligned} \square_{\widehat{g}} \dot{g}_{jk} + A_{jk}(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) &= f^1, \quad \text{in } M_0, \\ \square_{\widehat{g}} \dot{\phi}_\ell + B_\ell(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) &= f^2, \quad \ell = 1, 2, 3, \dots, L, \end{aligned}$$

where A_{jk} and B_ℓ are first order linear differential operators which coefficients depend on \widehat{g} and $\widehat{\phi}$. When g_ε and ϕ_ε are solutions of (8) with source \mathcal{F}_ε depending smoothly on $\varepsilon \in \mathbb{R}$ such that $(g_\varepsilon, \phi_\varepsilon, \mathcal{F}_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi}, 0)$, then $(\dot{g}, \dot{\phi}, f) = (\partial_\varepsilon g_\varepsilon, \partial_\varepsilon \phi_\varepsilon, \partial_\varepsilon \mathcal{F}_\varepsilon)|_{\varepsilon=0}$ solve (12).

Let us consider the concept of the *linearization stability (LS)* for the source problems, cf. [12, 16, 38, 40], and references therein: Let $s_0 > 4$ and consider a C^{s_0+4} -smooth source $f = (f^1, f^2)$ that is supported in $W_{\widehat{g}}$ and satisfies the linearized conservation law

$$(13) \quad \frac{1}{2} \widehat{g}^{pk} \widehat{\nabla}_p f_{kj}^1 + \sum_{\ell=1}^L f_\ell^2 \partial_j \widehat{\phi}_\ell = 0, \quad j = 1, 2, 3, 4.$$

Let $(\dot{g}, \dot{\phi})$ be the solution of the linearized Einstein equations (12) with source f . We say that f has the *LS-property* in $C^{s_0}(M_0)$ if there are

$\varepsilon_0 > 0$ and a family $\mathcal{F}_\varepsilon = (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2)$ of sources, supported in W_{g_ε} for all $\varepsilon \in [0, \varepsilon_0]$, and functions $(g_\varepsilon, \phi_\varepsilon)$ that all depend smoothly on $\varepsilon \in [0, \varepsilon_0)$ in $C^{s_0}(M_0)$ such that

- (14) $(g_\varepsilon, \phi_\varepsilon)$ satisfies the equations (8) and the conservation law (9),
 $(g_\varepsilon, \phi_\varepsilon, \mathcal{F}_\varepsilon)|_{\varepsilon=0} = (\hat{g}, \hat{\phi}, 0)$, and $(\dot{g}, \dot{\phi}, f) = (\partial_\varepsilon g_\varepsilon, \partial_\varepsilon \phi_\varepsilon, \partial_\varepsilon \mathcal{F}_\varepsilon)|_{\varepsilon=0}$.

In this case, we say that $f = (f^1, f^2)$ has the LS-property with the family \mathcal{F}_ε , $\varepsilon \in [0, \varepsilon_0)$.

Note that above (13) is obtained by linearization of the conservation law (9).

The above linearization stability, concerning the local existence of the solutions in $M_0 = (-\infty, t_0) \times N$, is valid under quite general conditions, see [13]. Below we will require that for some sources f , supported in a neighborhood V of a point y , we can find functions \mathcal{F}_ε that are also supported in the set V . The conditions when this happen are considered below.

Next, we consider sources that are conormal distributions. When $Y \subset U_{\hat{g}}$ is a 2-dimensional space-like submanifold, consider local coordinates defined in $V \subset M_0$ such that $Y \cap V \subset \{x \in \mathbb{R}^4; x^j b_j^1 = 0, x^j b_j^2 = 0\}$, where $b_j^1, b_j^2 \in \mathbb{R}$. Next we slightly abuse the notation by identifying $x \in V$ with its coordinates $X(x) \in \mathbb{R}^4$. We denote $f \in \mathcal{I}^n(Y)$, $n \in \mathbb{R}$, if in the above local coordinates, f can be written as

$$(15) \quad f(x^1, x^2, x^3, x^4) = \operatorname{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_1^1 + \theta_2 b_2^2)x^m} \sigma_f(x, \theta_1, \theta_2) d\theta_1 d\theta_2,$$

where $\sigma_f(x, \theta) \in S_{0,1}^n(V; \mathbb{R}^2)$, $\theta = (\theta_1, \theta_2)$ is a classical symbol. A function $c(x, \theta)$ that is n -positive homogeneous in θ , i.e., $c(x, s\theta) = s^n c(x, \theta)$ for $s > 0$, is the principal symbol of f if there is $\phi \in C_0^\infty(\mathbb{R}^2)$ being 1 near zero such that $\sigma_f(x, \theta) - (1 - \phi(\theta))c(x, \theta) \in S_{0,1}^{n-1}(V; \mathbb{R}^2)$. When $\eta = \theta_1 b_1 + \theta_2 b_2 \in N_x^* Y$, we say that $\tilde{c}(x, \eta) = c(x, \theta)$ is the value of the principal symbol of f at $(x, \eta) \in N^* Y$.

We need a condition that we call *microlocal linearization stability*.

Assumption μ -LS (Microlocal linearization stability): Let $n_0 \in \mathbb{Z}_-$, $Y \subset U_{\hat{g}}$ be a 2-dimensional space-like submanifold, $V \subset U_{\hat{g}}$ an open local coordinate neighborhood of $y \in Y$ with the coordinates $X : V \rightarrow \mathbb{R}^4$, $X^j(x) = x^j$ such that $X(Y \cap V) \subset \{x \in \mathbb{R}^4; x^j b_j^1 = 0, x^j b_j^2 = 0\}$. Let, in addition, $(y, \eta) \in N^* Y$ be a light-like covector, $\mathcal{W} \subset N^* Y$ be a conic neighborhood of (y, η) , $(c_{jk})_{j,k=1}^4$ be a symmetric $(0, 2)$ tensor at y that satisfies

$$(16) \quad \hat{g}^{lk}(y) \eta_l c_{kj} = 0, \quad \text{for all } j = 1, 2, 3, 4,$$

and $(d_\ell)_{\ell=1}^L \in \mathbb{R}^L$. Then, for any $n \in \mathbb{Z}_-$, $n \leq n_0$ there are $f_{jk}^1 \in \mathcal{I}^n(Y)$, $(j, k) \in \{1, 2, 3, 4\}^2$, and $f_\ell^2 \in \mathcal{I}^n(Y)$, $\ell = 1, 2, \dots, L$, supported in V with symbols that are in $S^{-\infty}$ outside the neighborhood \mathcal{W} of (y, η) . The

principal symbols of $f_{jk}^1(x)$ and $f_\ell^2(x)$ in the X -coordinates, denoted by \tilde{f}_{jk}^1 and \tilde{f}_ℓ^2 , respectively, are at (y, η) equal to $\tilde{f}_{jk}^1(y, \eta) = c_{jk}$ and $\tilde{f}_\ell^2(y, \eta) = d_\ell$. Moreover, the source $f = (f^1, f^2)$ satisfies the linearized conservation law (13) and f has the LS property (14) in $C^{s_1}(M_0)$, $s_1 \geq 13$, with a family \mathcal{F}_ε , $\varepsilon \in [0, \varepsilon_0]$ such that \mathcal{F}_ε are supported in V .

1.2. Main results. Our main result is the following uniqueness theorem for the inverse problem for the Einstein-scalar field equations:

Theorem 1.1. *Let $(M^{(1)}, \hat{g}^{(1)})$ and $(M^{(2)}, \hat{g}^{(2)})$ be globally hyperbolic manifolds and $(\hat{g}^{(j)}, \hat{\phi}^{(j)})$, $j = 1, 2$ satisfy Einstein-scalar field equations (8) with vanishing sources $\mathcal{F}^1 = 0$ and $\mathcal{F}^2 = 0$. Also, assume that there are neighborhoods $U_{\hat{g}^{(j)}}$, $j = 1, 2$ of the time-like geodesics $\hat{\mu}_j \subset M^{(j)}$ where the Assumption μ -LS is valid. Moreover, assume that, for some $\varepsilon > 0$, the sets $\mathcal{D}(\hat{g}^{(1)}, \hat{\phi}^{(1)}, \varepsilon)$ and $\mathcal{D}(\hat{g}^{(2)}, \hat{\phi}^{(2)}, \varepsilon)$ are the same, that is, the measurements done in $U_{\hat{g}^{(1)}}$ and $U_{\hat{g}^{(2)}}$ coincide. Let $p_j^- = \hat{\mu}_j(s_-)$ and $p_j^+ = \hat{\mu}_j(s_+)$. Then there is a diffeomorphism $\Psi : I_{\hat{g}^{(1)}}(p_1^-, p_1^+) \rightarrow I_{\hat{g}^{(2)}}(p_2^-, p_2^+)$ such that the metric $\Psi^*\hat{g}^{(2)}$ is conformal to $\hat{g}^{(1)}$.*

The theorem above says that the data $\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$ determine uniquely the manifold $I_{\hat{g}}(p^-, p^+) \subset M$ and the conformal type of \hat{g} in $I_{\hat{g}}(p^-, p^+)$. Reconstruction of the conformal structure of the manifold provides naturally less information than finding the whole metric structure, but the conformal structure is crucial for many questions of analysis and physics, see e.g. [17, 41]. Roughly speaking, the above result means that if the manifold (M_0, \hat{g}) is unknown, then the source-to-observation pairs corresponding to freely falling sources which are near a freely falling observer $\mu_{\hat{g}}$ and the measurements of the metric tensor and the scalar fields in a neighborhood $U_{\hat{g}}$ of $\mu_{\hat{g}}$, determine the metric tensor up to conformal transformation in the set $I_{\hat{g}}(p^-, p^+)$.

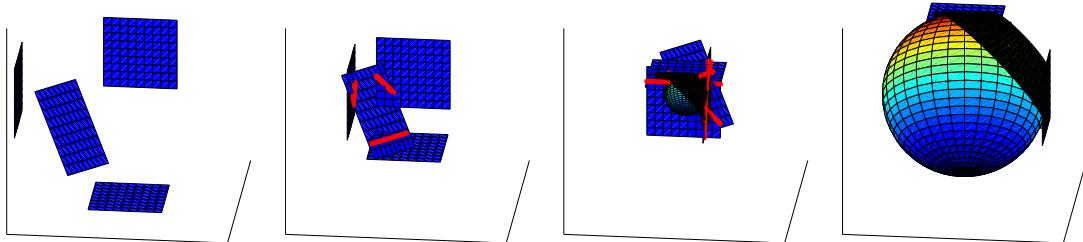


FIGURE 2. *Four plane waves propagate in space. When the planes intersect, the non-linearity of the hyperbolic system produces new waves. The four figures show the waves before the interaction of the waves start, when 2-wave interactions have started, when all 4 waves have*

just interacted, and later after the interaction. **Left:** Plane waves before interacting. **Middle left:** The 2-wave interactions (red line segments) appear but do not cause new propagating singularities. **Middle right and Right:** All plane waves have intersected and new waves have appeared. The 3-wave interactions cause new conic waves (black surface). Only one such wave is shown in the figure. The 4-wave interaction causes a point source in spacetime that sends a spherical wave to all future light-like directions. This spherical wave is essential in our considerations. For an animation on these interactions, see the supplementary video.

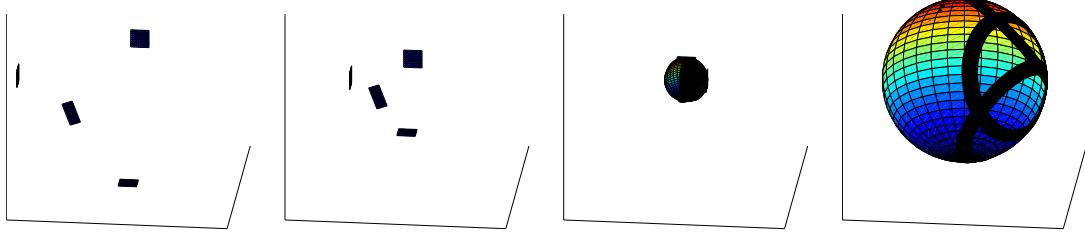


FIGURE 2B. The Figure 2 with pieces of plane waves that have smaller width. The time moments are the same as in Figure 2, but now all conic waves produced by 3-interactions (black cones) are visible. Note that there are four conic waves but one of those is not visible as it is behind the sphere.

The outline of the proof of Theorem 1.1: We consider 4 sources that send distorted plane waves from $W_{\hat{g}}$ that propagate near geodesics γ_{x_j, ξ_j} , see Fig. 1(Right). Due to the non-linearity of the Einstein equations, these waves interact and may produce a point source in the space-time, see Fig. 2 on the interaction of waves. All four waves interact if the geodesics γ_{x_j, ξ_j} intersect at a single point q of the space-time, see Fig. 3(Left). We show in Sec. 3, that if the intersection of these geodesics happens before the conjugate points of the geodesics (i.e. the caustics of the waves), then there are some sources satisfying (13) such that the produced spherical plane wave has a non-vanishing singularity on the future light cone emanating from q , see Fig. 3(Right). Thus we can observe in $U_{\hat{g}}$ the set of the earliest light observations of the point q , $\mathcal{E}_{U_{\hat{g}}}(q)$. By varying the starting directions (x_j, ξ_j) of the geodesics we can observe the collection of sets of the earliest light observations for all points $q \in I(p^-, p^+)$. This data determine uniquely the topological, differentiable and the conformal structures on $I(p^-, p^+)$, as is shown in the second part of this paper, [66]. In the proof we have to deal with several technical difficulties: First, the wave produced by the 4th order interaction consists of many terms which could cancel each other. In Sec. 3, we show that the principal symbol of this wave, considered

in the wave map coordinates, does not vanish in a generic situation. Second, we do not know the wave map coordinates in $U_{\hat{g}}$ and thus we have to consider observations in normal coordinates. This change of coordinates is a gauge transform where some of the produced singularities may vanish. However, in Sec. 4 we show that some singularities persists and can be observed. Third, caustics may produce singularities whose interactions are difficult to analyze. We avoid this by using global Lorentzian geometry (in Sec. 2) and show that no caustics affect in the earliest observations. We use this in Sec. 5 to give a step-by-step construction of the diamond set $J^+(p^-) \cap J^-(p^+)$.

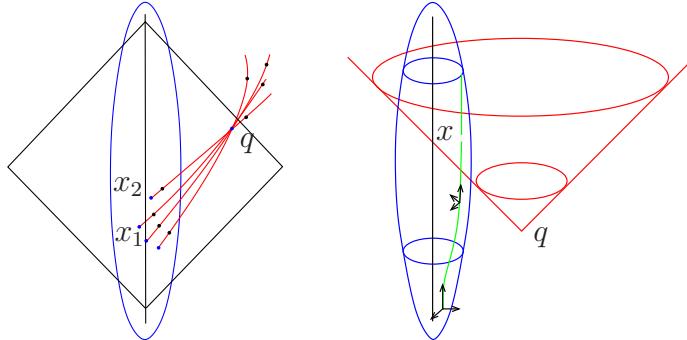


FIGURE 3. **Left:** The four light-like geodesics $\gamma_{x_j, \xi_j}([0, \infty))$, $j = 1, 2, 3, 4$ starting at the blue points x_j intersect at q before the first cut points of $\gamma_{x_j, \xi_j}([t_0, \infty))$, denoted by black points. The points $\gamma_{x_j, \xi_j}(t_0)$ are also shown as black points. **Right:** The future light cone $\mathcal{L}_g^+(q)$ emanating from the point q is shown as a red cone. The set of the light observation points $\mathcal{P}_{U_{\hat{g}}}(q)$, see Definition 2.4, is the intersection of the set $\mathcal{L}_g^+(q)$ and the set $U_{\hat{g}}$. The green curve is the geodesic $\mu = \mu_{\hat{g}, z, \eta}$. This geodesic intersects the future light cone $\mathcal{L}_g^+(q)$ at the point x . The black vectors are the frame (Z_j) that is obtained using parallel translation along the geodesic μ . Near the intersection point x we detect singularities in normal coordinates centered at x and associated to the frame (Z_j) .

We want to point out that by the main theorem, if we have two non-conformal spacetimes, a generic measurement gives different results on these manifolds. In particular, this implies that perfect space-time cloaking, in sense of light rays, see [36, 78], with a smooth metric in a globally hyperbolic universe is not possible.

The assumptions of Theorem 1.1 are valid in many cases. For instance, consider the a case when the background fields vary sufficiently:

Condition A: Assume that at any $x \in \overline{U}_{\hat{g}}$ there is a permutation $\sigma : \{1, 2, \dots, L\} \rightarrow \{1, 2, \dots, L\}$, denoted σ_x , such that the 5×5 matrix $[B_{jk}^\sigma(\hat{\phi}(x), \nabla \hat{\phi}(x))]_{j,k \leq 5}$ is invertible, where

$$\begin{aligned} B_{j\ell}^\sigma(\hat{\phi}(x), \nabla \hat{\phi}(x)) &= \frac{\partial}{\partial x^j} \hat{\phi}_{\sigma(\ell)}(x), \quad \text{for } j \leq 4, \ell = 1, 2, 3, 4, 5, \\ B_{jk}^\sigma(\hat{\phi}(x), \nabla \hat{\phi}(x)) &= \hat{\phi}_{\sigma(\ell)}(x), \quad \text{for } j = 5, \ell = 1, 2, 3, 4, 5. \end{aligned}$$

When the Condition A is valid, also the condition μ -LS is valid, see Appendix C. Very roughly speaking, Condition A means that the background fields vary so much that one could implement a measurement with some suitable sources \mathcal{F} by taking the needed “energy” from the varying ϕ_ℓ fields.

Theorem 1.1 can in some cases be improved so that also the conformal factor of the metric tensor can be reconstructed. Indeed, Theorem 1.1 and Corollary 1.3 of [66] imply that if $V \subset I_{\hat{g}}(p^-, p^+)$ is Vacuum, i.e., Ricci-flat, and all points $x \in V$ can be connected by a curve $\alpha \subset V^{int}$ to points of $U_{\hat{g}}$. Then under the assumptions of Theorem 1.1, the whole metric tensor g in V can be reconstructed.

Theorem 1.1 deals with an inverse problem for “near field” measurements. We remark that for inverse problems for linear equations the measurements of the Dirichlet-to-Neumann map or the “near field” measurements are equivalent to scattering or “far field” information [9]. Analogous considerations for non-linear equations have not yet been done but are plausible. On related inverse scattering problems, see [41, 82].

Also, one can ask if one can make an approximate image of the space-time doing only one measurement. In general, in many inverse problems several measurements can be packed together to one measurement. For instance, for the wave equation with a time-independent simple metric this is done in [49]. Similarly, Theorem 1.1 and its proof make it possible to do approximate reconstructions in a suitable class of manifolds with only one measurement, see Remark 5.1.

The techniques considered in this paper can be used also to study inverse problems for non-linear hyperbolic systems encountered in applications. For instance, in medical imaging, in the the recently developed Ultrasound Elastography imaging technique the elastic material parameters are reconstructed by sending (s-polarized) elastic waves that are imaged using (p-polarized) elastic waves, see e.g. [50, 79]. This imaging method uses interaction of waves and is based on the non-linearity of the system.

2. GEOMETRY OF THE OBSERVATION TIMES AND THE CUT POINTS

By [10], a globally hyperbolic manifold, as defined in the introduction, satisfies the strong causality condition:

- (17) For every $z \in M$ and every neighborhood $V \subset M$ of z there is a neighborhood $V' \subset M$ of z that if $x, y \in V'$ and $\alpha \subset M$ is a causal path connecting x to y then $\alpha \subset V$.

We use the following simple result:

Lemma 2.A.1 *Let $z \in M$. Then there is a neighborhood V of z so that*

- (i) *If the geodesics $\gamma_{y,\eta}([0, s]) \subset V$ and $\gamma_{y,\eta'}([0, s']) \subset V$, $s, s' > 0$ satisfy $\gamma_{y,\eta}(s) = \gamma_{y,\eta'}(s')$, then $\eta = c\eta'$ and $s' = cs$ with some $c > 0$.*
- (ii) *For any $y \in V$, $\eta \in T_y M \setminus 0$ there is $s > 0$ such that $\gamma_{x,\eta}(s) \notin V$.*

Proof. The property (i) follows from [87, Prop. 5.7]. Making $s > 0$ so small that $\overline{V} \subset B_{g^+}(z, \rho)$ with a sufficiently small ρ , the claim (ii) follows from [87, Lem. 14.13]. \square

Let us consider points $x, y \in M$. If $x < y$, we define the time separation function $\tau(x, y) \in [0, \infty)$ to be the supremum of the lengths $L(\alpha) = \int_0^1 \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))} ds$ of the piecewise smooth causal paths $\alpha : [0, 1] \rightarrow M$ from x to y . If the condition $x < y$ does not hold, we define $\tau(x, y) = 0$.

Since M is globally hyperbolic, the time separation function $(x, y) \mapsto \tau(x, y)$ is continuous in $M \times M$ by [87, Lem. 14.21] and the sets $J^\pm(q)$ are closed by [87, Lem. 14.22]. Also, any points $x, y \in M$, $x < y$ can be connected by a causal geodesic whose length is $\tau(x, y)$ by [87, Prop. 14.19].

When (x, ξ) is a light-like vector, we define $\mathcal{T}(x, \xi)$ to be the length of the maximal interval on which $\gamma_{x,\xi} : [0, \mathcal{T}(x, \xi)) \rightarrow M$ is defined. Below, to simplify notations, we sometimes use the notation $\gamma_{x,\xi}([0, \infty))$ for the geodesic $\gamma_{x,\xi}([0, \mathcal{T}(x, \xi))$.

When (x, ξ_+) is a future pointing light-like vector, and (x, ξ_-) is a past pointing light-like vector, we define the modified cut locus functions, c.f. [6, Def. 9.32],

$$(18) \quad \begin{aligned} \rho_g(x, \xi_+) &= \sup\{s \in [0, \mathcal{T}(x, \xi_+)); \tau(x, \gamma_{x,\xi_+}(s)) = 0\}, \\ \rho_g(x, \xi_-) &= \sup\{s \in [0, \mathcal{T}(x, \xi_-)); \tau(\gamma_{x,\xi_-}(s), x) = 0\}. \end{aligned}$$

The point $\gamma_{x,\xi}(s)|_{s=\rho_g(x,\xi)}$ is called the cut point on the geodesic $\gamma_{x,\xi}$.

Using [6, Thm. 9.33], we see that the function $\rho_g(x, \xi)$ is lower semi-continuous on a globally hyperbolic Lorentzian manifold (M, g) .

Below, in this section, we consider the manifold (M, \widehat{g}) and denote by $\gamma_{x,\xi}$ the geodesics of (M, \widehat{g}) and $\rho(x, \xi) = \rho_{\widehat{g}}(x, \xi)$. Also, we denote

$\mu_{\widehat{g},z_0,\eta_0} = \widehat{\mu}$, $p^\pm = \mu_{\widehat{g}}(s_\pm)$, $p_{+2} = \widehat{\mu}(s_{+2})$, and $p_{-2} = \widehat{\mu}(s_{-2})$. Recall that by (1), $\mu_{\widehat{g},z_0,\eta_0}(s_{\pm j}) \in I^\mp(\mu_{\widehat{g},z,\eta}(s_{\pm(j+1)}))$, $j = 1, 2$, for all $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$.

Definition 2.1. Let $\mu = \mu_{\widehat{g},z,\eta}$, $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$. For $x \in J^+(\mu(-1)) \cap J^-(\mu(+1))$ we define $f_\mu^\pm(x) \in [-1, 1]$ by setting

$$\begin{aligned} f_\mu^+(x) &= \inf(\{s \in (-1, 1); \tau(x, \mu(s)) > 0\} \cup \{1\}), \\ f_\mu^-(x) &= \sup(\{s \in (-1, 1); \tau(\mu(s), x) > 0\} \cup \{-1\}). \end{aligned}$$

We need the following simple properties of these functions.

Lemma 2.2. Let $\mu = \mu_{\widehat{g},z,\eta}$, $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$, and $x \in J^-(p_{+2}) \cap J^+(p_{-2})$.

- (i) The function $s \mapsto \tau(x, \mu(s))$ is non-decreasing on the interval $s \in [-1, 1]$ and strictly increasing on $s \in [f_\mu^+(x), 1]$.
- (ii) We have that $s_{-3} < f_\mu^+(x) < s_{+3}$.
- (iii) Let $y = \mu(f_\mu^+(x))$. Then $\tau(x, y) = 0$. Also, if $x \notin \mu$, there is a light-like geodesic $\gamma([0, s])$ in M from x to y with no conjugate points on $\gamma([0, s])$.
- (iv) The maps $f_\mu^+ : J^-(p_{+2}) \cap J^+(p_{-2}) \rightarrow [-1, 1]$ is continuous. The analogous results hold for $f_\mu^- : J^-(p_{+2}) \cap J^+(p_{-2}) \rightarrow [-1, 1]$.
- (v) For $q \in J^-(p^+) \setminus I^-(p^-)$ the map $F : \mathcal{U}_{z_0, \eta_0} \rightarrow \mathbb{R}$; $F(z, \eta) = f_{\mu(z, \eta)}^+(q)$ is continuous.

Proof. (i) and (ii) follows from the definition of f_μ^+ and the fact that $p_{\pm 2} \in I^\mp(\mu(s_{\pm 3}))$ by (1). Claim (iii) follows from [87, Lem. 10.51].

(iv) Assume that $x_j \rightarrow x$ in $J^-(p_{+2}) \cap J^+(p_{-2})$ as $j \rightarrow \infty$. Let $s_j = f_\mu^+(x_j)$ and $s = f_\mu^+(x)$. As τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j \rightarrow \infty} \tau(x_j, \mu(s + \varepsilon)) = \tau(x, \mu(s + \varepsilon)) > 0$ and thus for j large enough $s_j \leq s + \varepsilon$. Thus $\limsup_{j \rightarrow \infty} s_j \leq s$. Assume next that $\liminf_{j \rightarrow \infty} s_j = \tilde{s} < s$ and denote $\varepsilon = \tau(\mu(\tilde{s}), \mu(s)) > 0$. Then by the reverse triangle inequality, [87, Lem. 14.16], $\liminf_{j \rightarrow \infty} \tau(x_j, \mu(s)) \geq \liminf_{j \rightarrow \infty} \tau(\mu(s_j), \mu(s)) \geq \varepsilon$, and as τ is continuous in $M \times M$, we obtain $\tau(x, \mu(s)) \geq \varepsilon$, which is not possible since $s = f_\mu^+(x)$. Hence $s_j \rightarrow s$ as $j \rightarrow \infty$, proving (iv). The analogous results for f_μ^- follow similarly.

(v) Observe that as $J^+(q)$ is a closed set, $F(z, \eta)$ is equal to the smallest value $s \in [-1, 1]$ such that $\mu_{z,\eta}(s) \in J^+(q)$. Let $(z_j, \eta_j) \rightarrow (z, \eta)$ in (TM, g^+) as $j \rightarrow \infty$ and $s_j = F(z_j, \eta_j)$ and $\underline{s} = \liminf_{j \rightarrow \infty} s_j$. As the map $(z, \eta, s) \mapsto \mu_{z,\eta}(s)$ is continuous, we see that for a suitable subsequence $\mu_{z,\eta}(\underline{s}) = \lim_{k \rightarrow \infty} \mu_{z_{j_k}, \eta_{j_k}}(s_{j_k}) \in J^+(q)$ and hence $F(z, \eta) \leq \underline{s} = \liminf_{j \rightarrow \infty} F(z_j, \eta_j)$. This shows that F is lower-semicontinuous.

On the other hand, let $\bar{s} = F(z, \eta)$. As $\mu_{z,\eta}$ is a time-like geodesic, for any $\varepsilon \in (0, 1 - \bar{s})$ we have $\tau(q, \mu_{z,\eta}(\bar{s} + \varepsilon)) > 0$. Since τ and the map $(z, \eta, s) \mapsto \mu_{z,\eta}(s)$ are continuous, we see that there is j_0 such that if $j > j_0$ then $\tau(q, \mu_{z_j, \eta_j}(\bar{s} + \varepsilon)) > 0$. Hence, $F(z_j, \eta_j) \leq \bar{s} + \varepsilon$. Thus $\limsup_{j \rightarrow \infty} F(z_j, \eta_j) \leq \bar{s} + \varepsilon$, and as $\varepsilon > 0$ can be chosen to be

arbitrarily small, we have $\limsup_{j \rightarrow \infty} F(z_j, \eta_j) \leq \bar{s} = F(z, \eta)$. Thus F is also upper-semicontinuous. This proves (v). \square

Let $W \subset M$. We define the earliest points of set W on the curve $\mu_{z,\eta} = \mu_{z,\eta}([-1, 1])$, and in the set U , respectively, to be

$$(19) \quad \begin{aligned} \mathbf{e}_{z,\eta}(W) &= \{\mu_{z,\eta}(\inf\{s \in [-1, 1]; \mu_{z,\eta}(s) \in W\})\}, \text{ if } \mu_{z,\eta} \cap W \neq \emptyset, \\ \mathbf{e}_{z,\eta}(W) &= \emptyset, \quad \text{if } \mu_{z,\eta} \cap W = \emptyset. \end{aligned}$$

Lemma 2.A.2 *Let $K \subset M$ be a compact set. Then there is $R_1 > 0$ such that if $\gamma_{y,\theta}([0, l]) \subset K$ is a light-like geodesic with $\|\theta\|_{g^+} = 1$, then $l \leq R_1$. In the case when $K = J(p^-, p^+)$, with $q^-, q^+ \in M$ we have $\gamma_{y,\theta}(t) \notin J(q^-, q^+)$ for $t > R_1$.*

The proof of this lemma is standard, but we include it for the convenience of the reader.

Proof. Assume that there are no such R_1 . Then there are geodesics $\gamma_{y_j, \theta_j}([0, l_j]) \subset K$, $j \in \mathbb{Z}_+$ such that $\|\theta_j\|_{g^+} = 1$ and $l_j \rightarrow \infty$ as $j \rightarrow \infty$. Let us choose a subsequence (y_j, θ_j) which converges to some point (y, θ) in (TM, g^+) . As θ_j are light-like, also θ is light-like.

Then, we observe that for all $R_0 > 0$ the functions $t \mapsto \gamma_{y_j, \theta_j}(s)$, converge in $C^1([0, R_0]; M)$ to $s \mapsto \gamma_{y, \theta}(t)$, as $j \rightarrow \infty$. As $\gamma_{y_j, \theta_j}([0, l_j]) \subset K$ for all j , we see that $\gamma_{y, \theta}([0, R_0]) \subset K$ for all $R_0 > 0$. Let $z_n = \gamma_{y, \theta}(n)$, $n \in \mathbb{Z}_+$. As K is compact, we see that there is a subsequence z_{n_k} which converges to a point z as $n_k \rightarrow \infty$. Let now $V \subset M$ be a small convex neighborhood of z such that each geodesic starting from V exits the set V (cf. Lemma 2.A.1). Let $V' \subset M$ be a neighborhood of z so that the strong causality condition (17) is satisfied for V and V' . Then we see that there is k_0 such that if $k \geq k_0$ then $z_{n_k} \in V'$, implying that $\gamma_{y, \theta}([n_{k_0}, \infty)) \subset V$. This is a contradiction and thus the claimed $R_1 > 0$ exists.

Finally, in the case when $K = J(q^-, q^+)$, $q^-, q^+ \in M$ we see that if $q(s) = \gamma_{y, \theta}(s) \in K$ for some $s > R_1$, then for all $\tilde{s} \in [0, s]$ we have $q(\tilde{s}) \leq q(s) \leq q^+$ and $q_- \leq q(0) \leq q(\tilde{s})$. Thus $q(\tilde{s}) \in K$ for all $\tilde{s} \in [0, s]$. As $t > R_1$, this is not possible by the above reasoning, and thus the last assertion follows. \square

Below, we use for a pair $(x, \xi) \in L^+M$ the notation

$$(20) \quad (x(h), \xi(h)) = (\gamma_{x,\xi}(h), \dot{\gamma}_{x,\xi}(h)).$$

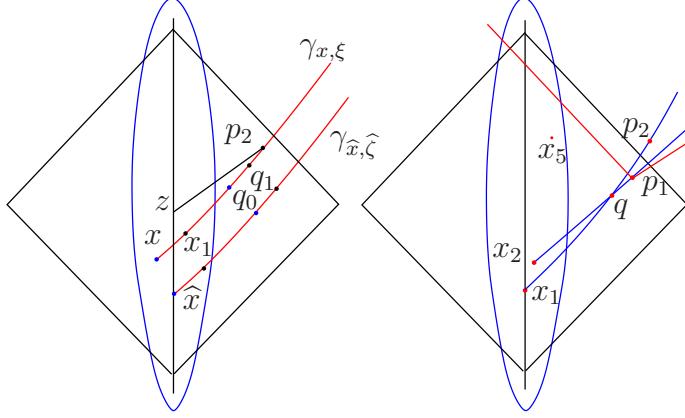


FIGURE 4. Left. Figure shows the situation in Lemma 2.3. The point $\hat{x} = \hat{\mu}(r_1)$ is on the time-like geodesic $\hat{\mu}$ shown as a black line. The black diamond is the set $J_{\hat{g}}(p^-, p^+)$, (x, ξ) is a light-like direction close to $(\hat{x}, \hat{\zeta})$, and $x_1 = \gamma_{x,\xi}(t_0) = x(t_0)$. The points $q_0 = \gamma_{x,\xi}(\rho(x, \xi))$ and $q_1 = \gamma_{x(t_0), \xi(t_0)}(\rho(x(t_0), \xi(t_0)))$ are the first cut point on $\gamma_{x,\xi}$ corresponding to the points x and x_1 , respectively. The blue and black points on $\gamma_{\hat{x}, \hat{\zeta}}$ are the corresponding cut points on $\gamma_{\hat{x}, \hat{\zeta}}$. Also, $z = \hat{\mu}(r_2)$, $r_2 = f_{\hat{\mu}}^-(p_2)$. **Right:** The figure shows the configuration in formulas (75) and (76). We send light-like geodesics $\gamma_{x_j, \xi_j}([t_0, \infty))$ from x_j , $j = 1, 2, 3, 4$. The boundary $\partial\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ is denoted by red line segments. We assume the these geodesics intersect at the point q before their first cut points p_j .

Later, we will consider wave packets sent from a point x that propagate near a geodesic $\gamma_{x,\xi}([0, \infty))$. These waves may have singularities near the conjugate points of the geodesic and due to this we analyze next how the conjugate points move along a geodesic when the initial point of the geodesic is moved from x to $\gamma_{x,\xi}(t_0)$.

Lemma 2.3. *There are $\vartheta_1, \kappa_1, \kappa_2 > 0$ such that for all $\hat{x} = \hat{\mu}(r_1)$ with $r_1 \in [s_-, s_+]$, $\hat{\zeta} \in L_{\hat{x}}^+ M$, $\|\hat{\zeta}\|_{\hat{g}^+} = 1$, $t_0 \in [\kappa_1, 4\kappa_1]$, and $(x, \xi) \in L^+ M$ satisfying $d_{\hat{g}^+}((\hat{x}, \hat{\zeta}), (x, \xi)) \leq \vartheta_1$ the following holds:*

- (i) $0 < t \leq 5\kappa_1$, then $\gamma_{x,\xi}(t) \in U_{\hat{g}}$ and $f_{\hat{\mu}}^-(\gamma_{\hat{x}, \hat{\zeta}}(t)) = r_1$.
- (ii) Assume that $t_2 \in [t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0)), T(x, \xi))$ and $p_2 = \gamma_{x,\xi}(t_2) \in J^-(\hat{\mu}(s_{+2}))$. Then $r_2 = f_{\hat{\mu}}^-(p_2)$ satisfies $r_2 - r_1 > 2\kappa_2$.

Note that above in (ii) we can choose $t_2 = t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0))$ in which case p_2 is the first cut point q_1 of $\gamma_{x,\xi}([t_0, \infty))$, see Fig. 4(Left).

Proof. Let $B = \{(\hat{x}, \hat{\zeta}) \in L^+ M; \hat{x} \in \hat{\mu}([s_-, s_+]), \|\hat{\zeta}\|_{\hat{g}^+} = 1\}$. Since B is compact, the positive and lower semi-continuous function $\rho(x, \xi)$ obtains its minimum on B . Hence we see that (i) holds when $\kappa_1 \in (0, \frac{1}{5} \inf\{\rho(\hat{x}, \hat{\zeta}); (\hat{x}, \hat{\zeta}) \in B\})$ is small enough.

(ii) Let K be the compact set $K = \{(x, \xi) \in L^+ M; d_{\hat{g}^+}((x, \xi), B) \leq \vartheta_1\}$. Also, let $T_+(x, \xi) = \sup\{t \geq 0; \gamma_{x,\xi}(t) \in J^-(p_{+2})\}$ and

$$K_0 = \{(x, \xi) \in K; \rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 \leq T_+(x, \xi)\}, \quad K_1 = K \setminus K_0.$$

Using [87, Lem. 14.13], we see that $T_+(x, \xi)$ is bounded in K . Note that for $t_0 \geq \kappa_1$ and $a > t_0$ the geodesic $\gamma_{x, \xi}([t_0, a])$ can have a cut point only if $\gamma_{x, \xi}([\kappa_1, a])$ has a cut point and thus $t_0 + \rho(x(t_0), \xi(t_0)) \geq \kappa_1 + \rho(x(\kappa_1), \xi(\kappa_1))$. If $K_0 = \emptyset$, the claim is valid as the condition $p_2 \in J^-(p_{+2})$ does not hold for any $(x, \xi) \in K_1$. Thus it is enough consider the case when $K_0 \neq \emptyset$.

We can also assume that $\vartheta_1 > 0$ is so small that for all $(x, \xi) \in K$ we have $f_{\widehat{\mu}}^-(x) > s_{-2}$. Then, by Lemma 2.2, the map $L : G_0 = \{(x, \xi, t) \in K \times \mathbb{R}_+ ; \rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 \leq t \leq T_+(x, \xi)\} \rightarrow \mathbb{R}$, defined by $L(x, \xi, t) = f_{\widehat{\mu}}^-(\gamma_{x, \xi}(t)) - f_{\widehat{\mu}}^-(x)$, is continuous. Since $\rho(x, \xi)$ is lower semi-continuous and $T_+(x, \xi)$ is upper semi-continuous and bounded we have that the sets K_0 and G_0 are compact.

For $(x, \xi, t) \in G_0$, the geodesic $\gamma_{x, \xi}([\kappa_1, t])$ has a cut point in which case we see, for $y = \gamma_{x, \xi}(t)$, that $\tau(x, y) > 0$. Thus, for $z_1 = \widehat{\mu}(f_{\widehat{\mu}}^-(x))$, we have $\tau(z_1, y) \geq \tau(z_1, x) + \tau(x, y) \geq \tau(x, y) > 0$. This shows that $L(x, \xi, t) > 0$. Since G_0 is compact and L is continuous and strictly positive, $\varepsilon_1 := \inf\{L(x, \xi, t); (x, \xi, t) \in G_0\} > 0$.

As $f_{\widehat{\mu}}^-$ is continuous and $\widehat{\mu}([-1, 1])$ is compact, we have that, by making ϑ_1 smaller if necessary, we can assume that if $\widehat{x} \in \widehat{\mu}$ and $d_{\widehat{g}^+}(x, \widehat{x}) \leq \vartheta_1$ then $|f_{\widehat{\mu}}^-(x) - f_{\widehat{\mu}}^-(\widehat{x})| < \varepsilon_1/2$. Let $\kappa_2 = \varepsilon_1/4$. Then, $\rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 < t_2 < T_+(x, \xi)$ so that $r_2 = f_{\widehat{\mu}}^-(p_2)$ and $r_3 = f_{\widehat{\mu}}^-(x)$ satisfies $r_2 - r_3 \geq \varepsilon_1$ and $r_2 - r_1 > \varepsilon_1/2$. This proves the claim. \square

Note that for proving the unique solvability of the inverse problem we need to consider two manifolds, $(M^{(1)}, \widehat{g}^{(1)})$ and $(M^{(2)}, \widehat{g}^{(2)})$ with the same data. For these manifolds, we can choose ϑ_1, κ_j so that they are the same for both manifolds.

2.0.1. Geometric results on the light observation sets. Let us first consider M with a fixed metric \widehat{g} . Denote below in this subsection $U = U_{\widehat{g}}$. See (19) for the notations we use.

Definition 2.4. We define the light-observation set of the point $q \in M$ to be $\mathcal{P}_U(q) = (\mathcal{L}_{\widehat{g}}^+(q) \cup \{q\}) \cap U = \{\gamma_{q, \eta}(r) \in M; r \geq 0, \eta \in L_q^+ M, \gamma_{q, \eta}(r) \in U\}$, see Fig. 3(Right). The set of the earliest light observations of q is $\mathcal{E}_U(q) = \bigcup_{(z, \eta) \in \mathcal{U}_{z_0, \eta_0}} \mathbf{e}_{z, \eta}(\mathcal{P}_U(q))$, that is,

$$\mathcal{E}_U(q) = \{\gamma_{q, \eta}(r) \in M; r \in [0, \rho(q, \eta)], \eta \in L_q^+ M, \gamma_{q, \eta}(r) \in U\} \subset \mathcal{P}_U(q).$$

Below, when X is a set, let $P(X) = 2^X = \{Z; Z \subset X\}$ denote the power set of X . When $\Phi : U_1 \rightarrow U_2$ is a map, we say that the power set extension of Φ is the map $\widetilde{\Phi} : 2^{U_1} \rightarrow 2^{U_2}$ given by $\widetilde{\Phi}(U') = \{\Phi(z); z \in U'\}$ for $U' \subset U$. We need the following theorem proven in [66] with $V = I_{\widehat{g}}^+(p^-) \cap I_{\widehat{g}}^-(p^+) \subset I_{\widehat{g}}^-(\mu_{\widehat{g}}(s_{+2})) \setminus I_{\widehat{g}}^-(\mu_{\widehat{g}}(s_{-2}))$.

Theorem 2.5. Let (M_j, \widehat{g}_j) , $j = 1, 2$ be two open, C^∞ -smooth, globally hyperbolic Lorentzian manifolds of dimension $(1+3)$ and let $p_j^+, p_j^- \in$

M_j be the points of a time-like geodesic $\mu_{\widehat{g}_j}([-1, 1]) \subset M_j$, $p_j^\pm = \mu_{\widehat{g}_j}(s_\pm)$. Let $U_j \subset M_j$ be a neighborhood of $\mu_{\widehat{g}_j}([s_-, s_+])$ and $V_j = I_{\widehat{g}_j}^-(p_j^+) \cap I_{\widehat{g}_j}^+(p_j^-) \subset M_j$. Then

(i) The map $\mathcal{E}_{U_1} : q \mapsto \mathcal{E}_{U_1}(q)$ is injective in V_1 .

(ii) Let us denote by $\mathcal{E}_{U_j}(V_j) = \{\mathcal{E}_{U_j}(q); q \in V_j\} \subset 2^{U_j}$ the collections of the sets of the earliest light observations on the manifold (M_j, \widehat{g}_j) of the points in the set V_j . Assume that there is a conformal diffeomorphism $\Phi : U_1 \rightarrow U_2$ such that $\Phi(\mu_1(s)) = \mu_2(s)$, $s \in [s_-, s_+]$ and the power set extension $\tilde{\Phi}$ of Φ satisfies

$$\tilde{\Phi}(\mathcal{E}_{U_1}(V_1)) = \mathcal{E}_{U_2}(V_2).$$

Then there is a diffeomorphism $\Psi : V_1 \rightarrow V_2$ such the metric $\Psi^* \widehat{g}_2$ is conformal to \widehat{g}_1 and $\Psi|_{V_1 \cap U_1} = \Phi$.

3. ANALYSIS OF THE EINSTEIN EQUATIONS IN WAVE COORDINATES

3.1. Asymptotic analysis of the reduced Einstein equations.

3.1.1. *The reduced Einstein tensor.* For the Einstein equations, we will consider a smooth background metric \widehat{g} on M and the smooth metric \widetilde{g} for which $\widehat{g} < \widetilde{g}$ and (M, \widetilde{g}) is globally hyperbolic. We also use the notations defined in Section 1.0.1. In particular, we identify $M = \mathbb{R} \times N$ and consider the metric tensor g on $M_0 = (-\infty, t_0) \times N$, $t_0 > 0$ that coincide with \widehat{g} in $(-\infty, 0) \times N$. Recall also that we consider a freely falling observer $\widehat{\mu} = \mu_{\widehat{g}} : [-1, 1] \rightarrow M_0$ for which $\widehat{\mu}(s_-) = p^- \in [0, t_0] \times N$. We denote $L_x^+ M_0 = L_x^+(M_0, \widehat{g})$ and $L^+ M_0 = L^+(M_0, \widehat{g})$, and the cut locus function on (M_0, \widehat{g}) by $\rho(x, \xi) = \rho_{\widehat{g}}(x, \xi)$. Also, recall that $\widehat{U} = U_{\widehat{g}}$ the neighborhood of the geodesic $\widehat{\mu} = \mu_{\widehat{g}}$. We denote by $\gamma_{x, \xi}(t)$ the geodesics of (M_0, \widehat{g}) .

Following [37] we recall that

$$(21) \quad \text{Ric}_{jk}(g) = \text{Ric}_{jk}^{(h)}(g) + \frac{1}{2}(g_{jq} \frac{\partial \Gamma^q}{\partial x^k} + g_{kq} \frac{\partial \Gamma^q}{\partial x^j})$$

where $\Gamma^q = g^{mn} \Gamma_{mn}^q$,

$$(22) \quad \begin{aligned} \text{Ric}_{jk}^{(h)}(g) &= -\frac{1}{2}g^{pq} \frac{\partial^2 g_{jk}}{\partial x^p \partial x^q} + P_{jk}, \\ P_{jk} &= g^{ab} g_{pq} \Gamma_{jb}^p \Gamma_{ka}^q + \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^a} \Gamma^a + g_{kl} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^j} + g_{jl} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^k} \right). \end{aligned}$$

Note that P_{jk} is a polynomial of g_{pq} and g^{pq} and first derivatives of g_{pq} .

The \widehat{g} -reduced Einstein tensor $\text{Ein}_{\widehat{g}}(g)$ and Ricci tensor $\text{Ric}_{\widehat{g}}(g)$ are

$$(23) \quad (\text{Ric}_{\widehat{g}}(g))_{jk} = \text{Ric}_{jk}g - \frac{1}{2}(g_{jn}\widehat{\nabla}_k F^n + g_{kn}\widehat{\nabla}_j F^n) \\ = \text{Ric}_{jk}^{(h)}(g) + \frac{1}{2}(g_{jq}\frac{\partial}{\partial x^k}(g^{ab}\widehat{\Gamma}_{ab}^q) + g_{kq}\frac{\partial}{\partial x^j}(g^{ab}\widehat{\Gamma}_{ab}^q)),$$

$$(24) \quad (\text{Ein}_{\widehat{g}}(g))_{jk} = (\text{Ric}_{\widehat{g}}(g))_{jk} - \frac{1}{2}(g^{ab}(\text{Ric}_{\widehat{g}}g)_{ab})g_{jk},$$

where F^n are the harmonicity functions given by

$$(25) \quad F^n = \Gamma^n - \widehat{\Gamma}^n, \quad \text{where } \Gamma^n = g^{jk}\Gamma_{jk}^n, \quad \widehat{\Gamma}^n = g^{jk}\widehat{\Gamma}_{jk}^n,$$

where Γ_{jk}^n and $\widehat{\Gamma}_{jk}^n$ are the Christoffel symbols for g and \widehat{g} , respectively. The harmonicity functions F^n of the solution (g, ϕ) of the equations (8) vanish when the conservation law (9) is valid, see [94, eq. (14.8)]. Thus by (23), the conservation law implies that the solutions of the reduced Einstein equations (7) satisfy of the Einstein equations (3).

3.1.2. Local existence of solutions. Let we consider the solutions (g, ϕ) of the equations (8) with source \mathcal{F} . To consider their local existence, let us denote $u := (g, \phi) - (\widehat{g}, \widehat{\phi})$.

It follows from by [5, Cor. A.5.4] that $\mathcal{K}_j = J_g^+(p^-) \cap \overline{M}_j$ is compact. Since $\widehat{g} < \widetilde{g}$, we see that if r_0 above is small enough, for all $g \in \mathcal{V}(r_0)$, see subsection 1.0.2, we have $g|_{\mathcal{K}_1} < \widetilde{g}|_{\mathcal{K}_1}$. In particular, we have $J_g^+(p^-) \cap M_1 \subset J_{\widetilde{g}}^+(p^-)$.

Let us assume that \mathcal{F} is small enough in the norm $C_b^4(M_0)$ and that it is supported in a compact set $\mathcal{K} = J_{\widetilde{g}}(p^-) \cap [0, t_0] \times N \subset \overline{M}_0$. Then we can write the equations (8) for u in the form

$$(26) \quad \begin{aligned} P_{g(u)}(u) &= \mathcal{F}, \quad x \in M_0, \\ u &= 0 \text{ in } (-\infty, 0) \times N, \text{ where} \\ P_{g(u)}(u) &:= g^{jk}(x; u)\partial_j\partial_k u(x) + H(x, u(x), \partial u(x)). \end{aligned}$$

Here, the notation $g^{jk}(x; u)$ is used to indicate that the metric depends on the solution u . More precisely, as the metric and the scalar field are $(g, \phi) = u + (\widehat{g}, \widehat{\phi})$, we have $(g^{jk}(x; u))_{j,k=1}^4 = (g_{jk}(x))^{-1}$. Moreover, above $(x, v, w) \mapsto H(x, v, w)$ is a smooth function which is a second order polynomial in w with coefficients being smooth functions of v , \widehat{g} , and the derivatives of \widehat{g} , [101]. Note that when the norm of \mathcal{F} in $C_b^4(M_0)$ is small enough, we have $\text{supp}(u) \cap M_0 \subset \mathcal{K}$. We note that one could also consider non-compactly supported sources or initial data, see [23]. Also, the scalar field-Einstein system can be considered with much less regularity that is done below, see [18, 19].

Let $s_0 \geq 4$ be an even integer. Below we will consider the solutions $u = (g - \widehat{g}, \phi - \widehat{\phi})$ and the sources \mathcal{F} as sections of the bundle \mathcal{B}^L on M_0 . We will consider these functions as elements of the section-valued Sobolev spaces $H^s(M_0; \mathcal{B}^L)$ etc. Below, we omit the bundle \mathcal{B}^L

in these notations and denote $H^s(M_0; \mathcal{B}^L) = H^s(M_0)$. We use the same convention for the spaces

$$E^s = \bigcap_{j=0}^s C^j([0, t_0]; H^{s-j}(N)), \quad s \in \mathbb{N}.$$

Note that $E^s \subset C^p([0, t_0] \times N)$ when $0 \leq p < s - 2$. Local existence results for (26) follow from the standard techniques for quasi-linear equations developed e.g. in [51] or [63], or [94, Section 9]. These yield that when \mathcal{F} is supported in the compact set \mathcal{K} and $\|\mathcal{F}\|_{E^{s_0}} < c_0$, where $c_0 > 0$ is small enough, there exists a unique function u satisfying equation (26) on M_0 with the source \mathcal{F} . Moreover,

$$(27) \quad \|u\|_{E^{s_0}} \leq C_1 \|\mathcal{F}\|_{E^{s_0}}.$$

(For details, see Appendix B).

3.1.3. Asymptotic expansion for the non-linear wave equation. Let us consider a small parameter $\varepsilon > 0$ and the sources $\mathcal{F} = \mathcal{F}_\varepsilon$, depending smoothly on $\varepsilon \in [0, \varepsilon_0)$, with $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$, $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = \mathbf{f}$ and $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2)$, for which the equations (26) have a solution u_ε . Denote $\vec{h} = (h_{(j)})_{j=1}^4$, where $h_{(j)} = \partial_\varepsilon^j \mathcal{F}_\varepsilon|_{\varepsilon=0}$ so that $\mathbf{f} = h_{(1)}$. Below, we always assume that \mathcal{F}_ε is supported in \mathcal{K} and $\mathcal{F}_\varepsilon \in E^s$, where $s \geq s_0 + 10$ is an odd integer. We consider the solution $u = u_\varepsilon$ of (26) with $\mathcal{F} = \mathcal{F}_\varepsilon$ and write it in the form

$$(28) \quad u_\varepsilon(x) = \sum_{j=1}^4 \varepsilon^j w^j(x) + w^{res}(x, \varepsilon).$$

To obtain the equations for w^j , we use the representation (24) for the \hat{g} -reduced Einstein tensor. Below, we use the notation $w^j = ((w^j)_{pq})_{p,q=1}^4, ((w^j)_\ell)_{\ell=1}^L$ where $((w^j)_{pq})_{p,q=1}^4$ is the g -component of w^j and $((w^j)_\ell)_{\ell=1}^L$ is the ϕ -component of w^j . Below, we use also the notation where the components of $w = ((g_{pq})_{p,q=1}^4, (\phi_\ell)_{\ell=1}^L)$ are re-enumerated so that w is represented as a $(10 + L)$ -dimensional vector, i.e., we write $w = (w_m)_{m=1}^{10+L}$ (cf. Voigt notation). Using formulas (22), (23), and (24), the solution $u_\varepsilon(x)$ of the equation (26) can be written in the form

$$\begin{aligned} u_\varepsilon &= \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \varepsilon^4 w^4 + O(\varepsilon^5), \\ w^1 &= \mathbf{Q}h_{(1)}, \\ w^2 &= \mathbf{Q}(A[w^1, w^1]) + \mathbf{Q}h_{(2)}, \\ w^3 &= 2\mathbf{Q}(A[w^1, \mathbf{Q}(A[w^1, w^1])]) + \mathbf{Q}(B[w^1, w^1, w^1]) + \mathbf{Q}h_{(3)}, \\ w^4 &= \mathbf{Q}(A[\mathbf{Q}(A[w^1, w^1]), \mathbf{Q}(A[w^1, w^1])] \\ &\quad + 4\mathbf{Q}(A[w^1, \mathbf{Q}(A[w^1, \mathbf{Q}(A[w^1, w^1])])]) + 2\mathbf{Q}(A[w^1, \mathbf{Q}(B[w^1, w^1, w^1])]) \\ &\quad + 3\mathbf{Q}(B[w^1, w^1, \mathbf{Q}(A[w^1, w^1])]) + \mathbf{Q}(C[w^1, w^1, w^1, w^1]) + \mathbf{Q}h_{(4)}, \end{aligned}$$

where $\mathbf{Q} = \mathbf{Q}_{\widehat{g}} = (\square_{\widehat{g}} + V)^{-1}$ is the causal inverse of the linearized Einstein equation, A is the notation for a generic 2nd order multilinear operator in u and derivatives of order 2 obtained via pointwise multiplication of derivatives of u , B is a 3rd order multilinear operator in u and derivatives of order 2, and C is a 4th order multilinear operator in u and derivatives of order 2, so that the sums of the orders of the derivatives appearing in the terms A , B , and C is at most two. In a more explicit way, A , B , and C are of the form

$$\begin{aligned} A[v_1, v_2] &= \sum_{|\alpha|+|\beta|\leq 2} a_{\alpha\beta pq}(x)(\partial_x^\alpha v_1^p(x)) \cdot (\partial_x^\beta v_2^q(x)), \\ B[v_1, v_2, v_3] &= \sum_{|\alpha|+|\beta|+|\alpha'|\leq 2} b_{\alpha\beta\alpha'pqr}(x)(\partial_x^\alpha v_1^p(x)) \cdot (\partial_x^\beta v_2^q(x)) \cdot (\partial_x^{\alpha'} v_3^r(x)), \\ C[v_1, v_2, v_3, v_4] &= \sum_{|\alpha|+|\beta|+|\alpha'|+|\beta'|\leq 2} c_{\alpha\beta\alpha'\beta'pqrs}(x)(\partial_x^\alpha v_1^p(x)) \cdot (\mathcal{D}^\beta v_2^q(x)) \cdot (\partial_x^{\alpha'} v_3^r(x)) \cdot (\partial_x^{\beta'} v_4^s(x)), \end{aligned}$$

where the components of $v_k = ((g_{ab}^{(k)}), \phi_{(k)}^\ell)$ are represented in form $v_k = (v_k^p)_{p=1}^{10+L}$ (cf. Voigt notation). We emphasize that above the total order of derivatives is always less or equal to two. We can write the above formula without using multilinear forms: We have that w^j , $j = 1, 2, 3, 4$ are given by

$$\begin{aligned} w^j &= (g^j, \phi^j) = \mathbf{Q}_{\widehat{g}} \mathcal{H}^j, \quad j = 1, 2, 3, 4, \text{ where} \\ \mathcal{H}^1 &= h_{(1)}, \\ \mathcal{H}^2 &= (2\widehat{g}^{jp}w_{pq}^1\widehat{g}^{qk}\partial_j\partial_kw^1, 0) + \mathcal{A}^{(2)}(w^1, \partial w^1) + h_{(2)}, \\ (29) \quad \mathcal{H}^3 &= (\mathcal{G}_3, 0) + \mathcal{A}^{(3)}(w^1, \partial w^1, w^2, \partial w^2) + h_{(3)}, \\ \mathcal{G}_3 &= -6\widehat{g}^{jl}w_{li}^1\widehat{g}^{ip}w_{pq}^1\widehat{g}^{qk}\partial_j\partial_kw^1 + \\ &\quad + 3\widehat{g}^{jp}w_{pq}^2\widehat{g}^{qk}\partial_j\partial_kw^1 + 3\widehat{g}^{jp}w_{pq}^1\widehat{g}^{qk}\partial_k\partial_jw^2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}^4 &= (\mathcal{G}_4, 0) + \mathcal{A}^{(4)}(w^1, \partial w^1, w^2, \partial w^2, w^3, \partial w^3) + h_{(4)}, \\ (30) \quad \mathcal{G}_4 &= 24\widehat{g}^{js}w_{sr}^1\widehat{g}^{rl}w_{li}^1\widehat{g}^{ip}w_{pq}^1\widehat{g}^{qk}\partial_k\partial_jw^1 + 6\widehat{g}^{jp}w_{pq}^2\widehat{g}^{qk}\partial_k\partial_jw^2 + \\ &\quad - 18\widehat{g}^{jl}w_{li}^1\widehat{g}^{ip}w_{pq}^2\widehat{g}^{qk}\partial_k\partial_jw^1 - 12\widehat{g}^{jl}w_{li}^1\widehat{g}^{ip}w_{pq}^1\widehat{g}^{qk}\partial_k\partial_jw^2 + \\ &\quad + 3\widehat{g}^{jp}w_{pq}^3\widehat{g}^{qk}\partial_k\partial_jw^1 + 3\widehat{g}^{jp}w_{pq}^1\widehat{g}^{qk}\partial_k\partial_jw^3. \end{aligned}$$

Moreover, $\mathbf{Q}_{\widehat{g}} = (\square_{\widehat{g}} + V(x, D))^{-1}$ is the causal inverse of the operator $\square_{\widehat{g}} + V(x, D)$ where $V(x, D)$ is a first order differential operator with coefficients depending on \widehat{g} and its derivatives and $\mathcal{A}^{(\alpha)}$, $\alpha = 2, 3, 4$ denotes a sum of a multilinear operators of orders m , $2 \leq m \leq \alpha$

having at a point x the representation

$$(31) \quad \begin{aligned} & (\mathcal{A}^{(\alpha)}(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3))(x) \\ &= \sum \left(a_{abcijkP_1P_2P_3pq}^{(\alpha)}(x) (v_a^1(x))^i (v_b^2(x))^j (v_c^3(x))^k \cdot \right. \\ & \quad \left. P_1(\partial v^1(x)) P_2(\partial v^2(x)) P_3(\partial v^3(x)) \right) \end{aligned}$$

where $(v_a^1(x))^i$ denotes the i -th power of a -th component of $v^1(x)$ and the sum is taken over the indexes a, b, c, p, q, n , and integers i, j, k . The homogeneous monomials $P_d(y) = y^{\beta_d}$, $\beta_d = (b_1, b_2, \dots, b_{4(10+L)}) \in \mathbb{N}^{4(10+L)}$, $d = 1, 2, 3$ having orders $|\beta_d|$, respectively, where $P_d(y) = 1$ for $d > \alpha$, and

$$(32) \quad i + 2j + 3k + |\beta_1| + 2|\beta_2| + 3|\beta_3| = \alpha,$$

$$(33) \quad |\beta_1| + |\beta_2| + |\beta_3| \leq 2.$$

Here, by (32), the term $\mathcal{A}^{(\alpha)}$ produces a term of order $O(\varepsilon^\alpha)$ when $v^j = w^j$ and condition (33) means that $\mathcal{A}^{(\alpha)}(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3)$ contain only terms where the sum of the powers of derivatives of v^1, v^2 , and v^3 is at most two.

By [15, App. III, Thm. 3.7], or alternatively, the proof of [51, Lemma 2.6] adapted for manifolds, we see that the estimate $\|\mathbf{Q}_{\hat{g}} H\|_{E^{s_1+1}} \leq C_{s_1} \|H\|_{E^{s_1}}$ holds for all $H \in E^{s_1}$, $s_1 \in \mathbb{Z}_+$ that are supported in $\mathcal{K}_0 = J_g^+(p^-) \cap \overline{M}_0$. Note that we are interested only on the local solvability of the Einstein equations.

Consider next an even integer $s \geq s_0 + 4$ and $\mathcal{F}_\varepsilon \in E^s$ that depends smoothly on ε . Then, by defining w^j via the equations (29)-(30) with $h_{(j)} \in E^s$, $j \leq 4$ and using results of [51] we obtain that in (28) we have $w^j = \partial_\varepsilon^j u_\varepsilon|_{\varepsilon=0} \in E^{s+2-j}$, $j = 1, 2, 3, 4$ and $\|w^{res}(\cdot, \varepsilon)\|_{E^{s-4}} \leq C\varepsilon^5$. Let us explain the details of this. Recall that $s \geq s_0 + 4$ is an even integer and $h_{(j)} \in E^s$. First, we have $w^1 = \mathbf{Q} h_{(1)} \in E^{s+1}$. By defining w^j via the above equations with $h_{(j)} \in E^s$, we see that in the equation for w^j we have non-linear terms where the first derivative of w^j is multiplied by the first and zeroth derivatives of w^k , $k < j$ and non-linear terms where w^j is multiplied by the second, first, and zeroth derivatives of w^k , $k < j$. Moreover, we obtain a source term where the second, first, and zeroth derivatives of w^k , $k < j$ are multiplied by each other. In other words, denoting $W^j = (w^k)_{k=1}^j$, we have

$$\begin{aligned} & (\square_{\hat{g}} + V)w^j + A^n(W^j, \partial W^j) \partial_n w^j + \\ & + B(W^j, \partial W^j, \partial^2 W^j)w^j = H(W^j, \partial W^j, \partial^2 W^j), \quad \text{in } M_0, \\ & \text{supp}(w_j) \subset \mathcal{K}, \end{aligned}$$

Using [51, Thm. I], we see that the solution exists in E^{s_0} . Let us next consider the regularity of the solution. Assuming that $W^j \in E^{s+2-j+1}$, we have that $H(W^j, \partial W^j, \partial^2 W^j) \in E^{s+2-j-1}$ and that the coefficient

$B(W^j, \partial W^j, \partial^2 W^j)$ of w^j in the equation satisfy $B(W^j, \partial W^j, \partial^2 W^j) \in E^{s+2-j-1}$. Using the fact that $\|\mathbf{Q}_\mathcal{G} H\|_{E^{s_1+1}} \leq C_{s_1} \|H\|_{E^{s_1}}$ with $s_1 \leq s+1-j$, we see that $w^j \in E^{s+2-j}$. Starting from the case $j=1$ we can repeat this argument for $j=2, 3, 4$.

We have from above that $w^j \in E^{s+2-j}$, for $j=1, 2, 3, 4$. Thus, by using Taylor expansion of the coefficients in the equation (26) we see that the approximate 4th order expansion $u_\varepsilon^{app} = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \varepsilon^4 w^4$ satisfies an equation of the form

$$(34) \quad P_{g(u_\varepsilon^{app})}(u_\varepsilon^{app}) = \mathcal{F}(\varepsilon) + H^{res}(\cdot, \varepsilon), \quad x \in M_0, \\ \text{supp } (u_\varepsilon^{app}) \subset \mathcal{K},$$

such that

$$\|H^{res}(\cdot, \varepsilon)\|_{E^{s-4}} \leq c_1 \varepsilon^5.$$

Using the Lipschitz continuity of the solution of the equation (34) with respect to the source term, see Appendix B, we have that there are $c_1, c_2 > 0$ such that for all $0 < \varepsilon < c_1$ the function $w^{res}(x, \varepsilon) = u_\varepsilon(x) - u_\varepsilon^{app}(x)$ satisfies

$$\|u_\varepsilon - u_\varepsilon^{app}\|_{E^{s-4}} \leq c_2 \varepsilon^5.$$

Thus in (28) we have $\|w^{res}(\cdot, \varepsilon)\|_{E^{s-4}} \leq C \varepsilon^5$ and $w^j = \partial_\varepsilon^j u_\varepsilon|_{\varepsilon=0} \in E^{s-4}$, $j=1, 2, 3, 4$.

3.2. Distorted plane wave solutions for the linearized equations.

3.2.1. Lagrangian distributions. Let us recall the definition of the conormal and Lagrangian distributions that we will use below. Let X be a manifold of dimension n and $\Lambda \subset T^*X \setminus \{0\}$ be a Lagrangian submanifold. Let $\phi(x, \theta)$, $\theta \in \mathbb{R}^N$ be a non-degenerate phase function that locally parametrizes Λ . We say that a distribution $u \in \mathcal{D}'(X)$ is a Lagrangian distribution associated with Λ and denote $u \in \mathcal{I}^m(X; \Lambda)$, if in local coordinates u can be represented as an oscillatory integral,

$$(35) \quad u(x) = \int_{\mathbb{R}^N} e^{i\phi(x, \theta)} a(x, \theta) d\theta,$$

where $a(x, \theta) \in S^{m+n/4-N/2}(X; \mathbb{R}^N)$, see [44, 55, 83].

In particular, when $S \subset X$ is a submanifold, its conormal bundle $N^*S = \{(x, \xi) \in T^*X \setminus \{0\}; x \in S, \xi \perp T_x S\}$ is a Lagrangian submanifold. If u is a Lagrangian distribution associated with Λ_1 where $\Lambda_1 = N^*S$, we say that u is a conormal distribution.

Let us next consider the case when $X = \mathbb{R}^n$ and let $(x^1, x^2, \dots, x^n) = (x', x'', x''')$ be the Euclidean coordinates with $x' = (x_1, \dots, x_{d_1})$, $x'' = (x_{d_1+1}, \dots, x_{d_1+d_2})$, $x''' = (x_{d_1+d_2+1}, \dots, x_n)$. If $S_1 = \{x' = 0\} \subset \mathbb{R}^n$, $\Lambda_1 = N^*S_1$ then $u \in \mathcal{I}^m(X; \Lambda_1)$ can be represented by (35) with $N = d_1$ and $\phi(x, \theta) = x' \cdot \theta$.

Next we recall the definition of $\mathcal{I}^{p,l}(X; \Lambda_1, \Lambda_2)$, the space of the distributions u in $\mathcal{D}'(X)$ associated to two cleanly intersecting Lagrangian manifolds $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$, see [25, 44, 83]. These classes have been widely used in the study of inverse problems, see [20, 34]. Let us start with the case when $X = \mathbb{R}^n$.

Let $S_1, S_2 \subset \mathbb{R}^n$ be the linear subspaces of codimensions d_1 and $d_1 + d_2$, respectively, $S_2 \subset S_1$, given by $S_1 = \{x' = 0\}$, $S_2 = \{x' = x'' = 0\}$. Let us denote $\Lambda_1 = N^*S_1$, $\Lambda_2 = N^*S_2$. Then $u \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ if and only if

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x' \cdot \theta' + x'' \cdot \theta'')} a(x, \theta', \theta'') d\theta' d\theta'',$$

where the symbol $a(x, \theta', \theta'')$ belongs in the product type symbol class $S^{\mu_1, \mu_2}(\mathbb{R}^n; (\mathbb{R}^{d_1} \setminus 0) \times \mathbb{R}^{d_2})$ that is the space of function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ that satisfy

$$(36) \quad |\partial_x^\gamma \partial_{\theta'}^\alpha \partial_{\theta''}^\beta a(x, \theta', \theta'')| \leq C_{\alpha\beta\gamma K} (1 + |\theta'| + |\theta''|)^{\mu_1 - |\alpha|} (1 + |\theta''|)^{\mu_2 - |\beta|}$$

for all $x \in K$, multi-indexes α, β, γ , and compact sets $K \subset \mathbb{R}^n$. Above, $\mu_1 = p + l - d_1/2 + n/4$ and $\mu_2 = -l - d_2/2$.

When X is a manifold of dimension n and $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$ are two cleanly intersecting Lagrangian manifolds, we define the class $\mathcal{I}^{p,l}(X; \Lambda_1, \Lambda_2) \subset \mathcal{D}'(X)$ to consist of locally finite sums of distributions of the form $u = Au_0$, where $u_0 \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ and $S_1, S_2 \subset \mathbb{R}^n$ are the linear subspace of codimensions d_1 and $d_1 + d_2$, respectively, such that $S_2 \subset S_1$, and A is a Fourier integral operator of order zero with a canonical relation Σ for which $\Sigma \circ (N^*S_1)' \subset \Lambda_1'$ and $\Sigma \circ (N^*S_2)' \subset \Lambda_2'$. Here, for $\Lambda \subset T^*X$ we denote $\Lambda' = \{(x, -\xi) \in T^*X; (x, \xi) \in \Lambda\}$, and for $\Sigma \subset T^*X \times T^*X$ we denote $\Sigma' = \{(x, \xi, y, -\eta); (x, \xi, y, \eta) \in \Sigma\}$.

In most cases, below $X = M$. We denote then $\mathcal{I}^p(M; \Lambda_1) = \mathcal{I}^p(\Lambda_1)$ and $\mathcal{I}^{p,l}(M; \Lambda_1, \Lambda_2) = \mathcal{I}^{p,l}(\Lambda_1, \Lambda_2)$. Also, $\mathcal{I}(\Lambda_1) = \cup_{p \in \mathbb{R}} \mathcal{I}^p(\Lambda_1)$.

By [44, 83], microlocally away from Λ_1 and Λ_0 ,

$$(37) \quad \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1) \subset \mathcal{I}^{p+l}(\Lambda_0 \setminus \Lambda_1) \quad \text{and} \quad \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1) \subset \mathcal{I}^p(\Lambda_1 \setminus \Lambda_0),$$

respectively. Thus the principal symbol of $u \in \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1)$ is well defined on $\Lambda_0 \setminus \Lambda_1$ and $\Lambda_1 \setminus \Lambda_0$. We denote $\mathcal{I}(\Lambda_0, \Lambda_1) = \cup_{p,q \in \mathbb{R}} \mathcal{I}^{p,q}(\Lambda_0, \Lambda_1)$.

Below, when $\Lambda_j = N^*S_j$, $j = 1, 2$ are conormal bundles of smooth cleanly intersecting submanifolds $S_j \subset M$ of codimension m_j , where $\dim(M) = n$, we use the traditional notations,

$$(38) \quad \mathcal{I}^\mu(S_1) = \mathcal{I}^{\mu+m_1/2-n/4}(N^*S_1), \quad \mathcal{I}^{\mu_1, \mu_2}(S_1, S_2) = \mathcal{I}^{p,l}(N^*S_1, N^*S_2),$$

where $p = \mu_1 + \mu_2 + m_1/2 - n/4$ and $l = -\mu_2 - m_2/2$, and call such distributions the conormal distributions associated to S_1 or product type conormal distributions associated to S_1 and S_2 , respectively. By [44], $\mathcal{I}^\mu(X; S_1) \subset L_{loc}^p(X)$ for $\mu < -m_1(p-1)/p$, $1 \leq p < \infty$.

For the wave operator \square_g on the globally hyperbolic manifold (M, g) , $\text{Char}(\square_g)$ is the set of light-like co-vectors with respect to g . For

$(x, \xi) \in \text{Char}(\square_g)$, $\Theta_{x,\xi}$ denotes the bicharacteristic of \square_g . Then, $(y, \eta) \in \Theta_{x,\xi}$ if and only if there is $t \in \mathbb{R}$ such that for $a = \eta^\sharp$ and $b = \xi^\sharp$ we have $(y, a) = (\gamma_{x,b}^g(t), \dot{\gamma}_{x,b}^g(t))$ where $\gamma_{x,b}^g$ is a light-like geodesic with respect to the metric g with the initial data $(x, b) \in LM$. Here, we use notations $(\xi^\sharp)^j = g^{jk}\xi_k$ and $(b^\sharp)_j = g_{jk}b^k$.

Let $P = \square_g + B^0 + B^j\partial_j$, where B^0 is a scalar function and B_j is a vector field. Then P is a classical pseudodifferential operator of real principal type and order $m = 2$ on M , and [83], see also [69], P has a parametrix $Q \in \mathcal{I}^{p,l}(\Delta'_{T^*M}, \Lambda_P)$, $p = \frac{1}{2} - m$, $l = -\frac{1}{2}$, where $\Delta_{T^*M} = N^*(\{(x, x); x \in M\})$ and $\Lambda_g \subset T^*M \times T^*M$ is the Lagrangian manifold associated to the canonical relation of the operator P , that is,

$$(39) \quad \Lambda_g = \{(x, \xi, y, -\eta); (x, \xi) \in \text{Char}(P), (y, \eta) \in \Theta_{x,\xi}\},$$

where $\Theta_{x,\xi} \subset T^*M$ is the bicharacteristic of P containing (x, ξ) . When (M, g) is a globally hyperbolic manifold, the operator P has a causal inverse operator, see e.g. [5, Thm. 3.2.11]. We denote it by P^{-1} and by [83], we have $P^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M}, \Lambda_g)$. We will repeatedly use the fact (see [44, Prop. 2.1]) that if $F \in \mathcal{I}^p(\Lambda_0)$ and Λ_0 intersects $\text{Char}(P)$ transversally so that all bicharacteristics of P intersect Λ_0 only finitely many times, then $(\square_g + B^0 + B^j\partial_j)^{-1}F \in \mathcal{I}^{p-3/2, -1/2}(\Lambda_0, \Lambda_1)$ where $\Lambda'_1 = \Lambda_g \circ \Lambda'_0$ is called the flowout from Λ_0 on $\text{Char}(P)$, that is,

$$\Lambda_1 = \{(x, -\xi); (x, \xi, y, -\eta) \in \Lambda_g, (y, \eta) \in \Lambda_0\}.$$

3.2.2. The linearized Einstein equations and the linearized conservation law. We will below consider sources $\mathcal{F} = \varepsilon \mathbf{f}(x)$ and solution u_ε satisfying (26), where $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)})$.

We consider the linearized Einstein equations and the linearized wave $w^1 = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0}$ in (28) that we denote by $u^{(1)} = w^1$. It satisfies the linearized Einstein equations (12) that we write as

$$(40) \quad \square_{\widehat{g}} u^{(1)} + V(x, \partial_x)u^{(1)} = \mathbf{f},$$

where $v \mapsto V(x, \partial_x)v$ is a linear first order partial differential operator with coefficients depending on \widehat{g} and its derivatives.

Assume that $Y \subset M_0$ is a 2-dimensional space-like submanifold and consider local coordinates defined in $V \subset M_0$. Moreover, assume that in these local coordinates $Y \cap V \subset \{x \in \mathbb{R}^4; x^j b_j = 0, x^j b'_j = 0\}$, where $b'_j \in \mathbb{R}$ and let $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^{n+1}(Y)$, $n \leq n_0 = -17$, be defined by

$$(41) \quad \mathbf{f}(x^1, x^2, x^3, x^4) = \text{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_m + \theta_2 b'_m)x^m} \sigma_{\mathbf{f}}(x, \theta_1, \theta_2) d\theta_1 d\theta_2.$$

Here, we assume that $\sigma_{\mathbf{f}}(x, \theta)$, $\theta = (\theta_1, \theta_2)$ is a \mathcal{B}^L -valued classical symbol and we denote the principal symbol of \mathbf{f} by $c(x, \theta)$, or componentwise, $((c_{jk}^{(1)}(x, \theta))_{j,k=1}^4, (c_\ell^{(2)}(x, \theta))_{\ell=1}^L)$. When $x \in Y$ and $\xi = (\theta_1 b_m +$

$\theta_2 b'_m) dx^m$ so that $(x, \xi) \in N^*Y$, we denote the value of the principal symbol of \mathbf{f} at (x, ξ) by $\tilde{c}(x, \xi) = c(x, \theta)$, that is component-wise, $\tilde{c}_{jk}^{(1)}(x, \xi) = c_{jk}^{(1)}(x, \theta)$ and $\tilde{c}_\ell^{(2)}(x, \xi) = c_\ell^{(2)}(x, \theta)$. We say that this is the principal symbol of \mathbf{f} at (x, ξ) , associated to the phase function $\phi(x, \theta_1, \theta_2) = (\theta_1 b_m + \theta_2 b'_m)x^m$. The above defined principal symbols can be defined invariantly, see [48].

We will below consider what happens when $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^{n+1}(Y)$ satisfies the *linearized conservation law* (13). Roughly speaking, these four linear conditions imply that the principal symbol of the source \mathbf{f} satisfies four linear conditions. Furthermore, the linearized conservation law implies that also the linearized wave $u^{(1)}$ produced by \mathbf{f} satisfies four linear conditions that we call the linearized harmonicity conditions, and finally, the principal symbol of the wave $u^{(1)}$ has to satisfy four linear conditions. Next we explain these conditions in detail.

When (13) is valid, we have

$$(42) \quad \hat{g}^{lk} \xi_l \tilde{c}_{kj}^{(1)}(x, \xi) = 0, \quad \text{for } j \leq 4 \text{ and } \xi \in N_x^*Y.$$

We say that this is the *linearized conservation law for the principal symbols*. Note that $\mathcal{I}^\mu(Y) \subset C^s(M_0)$ when $s \leq -\mu - 3$. We will later use such indexes μ so that we can use $s = 13$, cf. Assumption μ -LS.

3.2.3. The harmonicity condition for the linearized solutions. Assume that (g, ϕ) satisfy equations (8) and the conservation law (9) is valid. The conservation law (9) and the \hat{g} -reduced Einstein equations (8) imply, see e.g. [15, 94], that the harmonicity functions $\Gamma^j = g^{nm}\Gamma_{nm}^j$ satisfy

$$(43) \quad g^{nm}\Gamma_{nm}^j = g^{nm}\hat{\Gamma}_{nm}^j.$$

Next we denote $u^{(1)} = (g^1, \phi^1) = (\dot{g}, \dot{\phi})$, see (29), and discuss the implications of (43) for the metric component \dot{g} of the solution of the linearized Einstein equations.

We do next calculations in local coordinates of M_0 and denote $\partial_k = \frac{\partial}{\partial x^k}$. Direct calculations show that $h^{jk} = g^{jk}\sqrt{-\det(g)}$ satisfies $\partial_k h^{kq} = -\Gamma_{kn}^q h^{nk}$. Then (43) implies that

$$(44) \quad \partial_k h^{kq} = -\hat{\Gamma}_{kn}^q h^{nk}.$$

We call (44) the *harmonicity condition* for the metric g .

Assume now that g_ε and ϕ_ε satisfy (8) with source $\mathcal{F} = \varepsilon f$ where $\varepsilon > 0$ is a small parameter. We define $h_\varepsilon^{jk} = g_\varepsilon^{jk}\sqrt{-\det(g_\varepsilon)}$ and denote $\dot{g}_{jk} = \partial_\varepsilon(g_\varepsilon)_{jk}|_{\varepsilon=0}$, $\dot{\phi}^{jk} = \partial_\varepsilon(g_\varepsilon)^{jk}|_{\varepsilon=0}$, and $\dot{h}^{jk} = \partial_\varepsilon h_\varepsilon^{jk}|_{\varepsilon=0}$.

The equation (44) yields then²

$$(45) \quad \partial_k \dot{h}^{kq} = -\widehat{\Gamma}_{kn}^q \dot{h}^{nk}.$$

A direct computation shows that

$$\dot{h}^{ab} = (-\det(\widehat{g}))^{1/2} \kappa^{ab},$$

where $\kappa^{ab} = \dot{g}^{ab} - \frac{1}{2}\widehat{g}^{ab}\widehat{g}_{qp}\dot{g}^{pq}$. Thus (45) gives

$$(46) \quad \partial_a((-\det(\widehat{g}))^{1/2} \kappa^{ab}) = -\widehat{\Gamma}_{ac}^b (-\det(\widehat{g}))^{1/2} \kappa^{ac}$$

that implies $\partial_a \kappa^{ab} + \kappa^{nb} \widehat{\Gamma}_{an}^b + \kappa^{an} \widehat{\Gamma}_{an}^b = 0$, or equivalently,

$$(47) \quad \widehat{\nabla}_a \kappa^{ab} = 0.$$

We call (47) the *linearized harmonicity condition* for g . Writing this for \dot{g} , we obtain

$$(48) \quad -\widehat{g}^{an} \partial_a \dot{g}_{nj} + \frac{1}{2} \widehat{g}^{pq} \partial_j \dot{g}_{pq} = m_j^{pq} \dot{g}_{pq}$$

where m_j depend on \widehat{g}_{pq} and its derivatives. On similar conditions for the polarization tensor, see [89, form. (9.58) and example 9.5.a, p. 416].

3.2.4. Properties of the principal symbols of the waves. Let $K \subset M_0$ be a light-like submanifold of dimension 3 that in local coordinates $X : V \rightarrow \mathbb{R}^4$, $x^k = X^k(y)$ is given by $K \cap V \subset \{x \in \mathbb{R}^4; b_k x^k = 0\}$, where $b_k \in \mathbb{R}$ are constants. Assume that the solution $u^{(1)} = (\dot{g}, \phi)$ of the linear wave equation (40) with the right hand side vanishing in V is such that $u^{(1)} \in \mathcal{I}^\mu(K)$ with $\mu \in \mathbb{R}$. Below we use $\mu = n - \frac{1}{2}$ where $n \in \mathbb{Z}_-$, $n \leq n_0 = -17$. Let us write \dot{g}_{jk} as an oscillatory integral using a phase function $\varphi(x, \theta) = b_k x^k \theta$, and a classical symbol $\sigma_{\dot{g}_{jk}}(x, \theta) \in S_{cl}^n(\mathbb{R}^4, \mathbb{R})$,

$$(49) \quad \dot{g}_{jk}(x^1, x^2, x^3, x^4) = \operatorname{Re} \int_{\mathbb{R}} e^{i(\theta b_m x^m)} \sigma_{\dot{g}_{jk}}(x, \theta) d\theta,$$

where $n = \mu + \frac{1}{2}$. We denote the (positively homogeneous) principal symbol of \dot{g}_{jk} by $a_{jk}(x, \theta)$. When $x \in K$ and $\xi = \theta b_k dx^k$ so that $(x, \xi) \in N^*K$, we denote the value of a_{jk} at (x, θ) by $\tilde{a}_{jk}(x, \xi)$, that is, $\tilde{a}_{jk}(x, \xi) = a_{jk}(x, \theta)$.

Then, if \dot{g}_{jk} satisfies the linearized harmonicity condition (43), its principal symbol $\tilde{a}_{jk}(x, \xi)$ satisfies

$$(50) \quad -\widehat{g}^{mn}(x) \xi_m v_{nj} + \frac{1}{2} \xi_j (\widehat{g}^{pq}(x) v_{pq}) = 0, \quad v_{pq} = \tilde{a}_{pq}(x, \xi),$$

²The treatment on this de Donder-type gauge condition is known in the folklore of the field. For a similar gauge condition to (45) in harmonic coordinates, see [77, pages 6 and 250], or [72, formulas 107.5, 108.7, 108.8], or [52, p. 229-230].

where $j = 1, 2, 3, 4$ and $\xi = \theta b_k dx^k \in N_x^* K$. If (50) holds, we say that the *harmonicity condition for the symbol* is satisfied for $\tilde{a}(x, \xi)$ at $(x, \xi) \in N^* K$.

3.2.5. Distorted plane waves satisfying a linear wave equation. Next we consider a distorted plane wave whose singular support is concentrated near a geodesic. These waves, sketched in Fig. 1(Right), propagate near the geodesic $\gamma_{x_0, \zeta_0}([t_0, \infty))$ and are singular on a surface $K(x_0, \zeta_0; t_0, s_0)$, defined below in (51), that is a subset of the light cone $\mathcal{L}_{\widehat{g}}^+(x')$, $x' = \gamma_{x_0, \zeta_0}(t_0)$. The parameter s_0 gives a “width” of the wave packet and when $s_0 \rightarrow 0$, its singular support tends to the set $\gamma_{x_0, \zeta_0}([2t_0, \infty))$. Next we will define these wave packets.

We define the 3-submanifold $K(x_0, \zeta_0; t_0, s_0) \subset M_0$ associated to $(x_0, \zeta_0) \in L^+(M_0, \widehat{g})$, $x_0 \in U_{\widehat{g}}$ and parameters $t_0, s_0 \in \mathbb{R}_+$ as

$$(51) \quad K(x_0, \zeta_0; t_0, s_0) = \{\gamma_{x', \eta}(t) \in M_0; \eta \in \mathcal{W}, t \in (0, \infty)\},$$

where $(x', \zeta') = (\gamma_{x_0, \zeta_0}(t_0), \dot{\gamma}_{x_0, \zeta_0}(t_0))$ and $\mathcal{W} \subset L_{x'}^+(M_0, \widehat{g})$ is a neighborhood of ζ' consisting of vectors $\eta \in L_{x'}^+(M_0)$ satisfying $\|\eta - \zeta'\|_{\widehat{g}^+} < s_0$. Note that $K(x_0, \zeta_0; t_0, s_0) \subset \mathcal{L}_{\widehat{g}}^+(x')$ is a subset of the light cone starting at $x' = \gamma_{x_0, \zeta_0}(t_0)$ and that it is singular at the point x' . Let $S = \{x \in M_0; \mathbf{t}(x) = \mathbf{t}(\gamma_{x_0, \zeta_0}(2t_0))\}$ be a Cauchy surface which intersects $\gamma_{x_0, \zeta_0}(\mathbb{R})$ transversally at the point $\gamma_{x_0, \zeta_0}(2t_0)$. When $t_0 > 0$ is small enough, $Y(x_0, \zeta_0; t_0, s_0) = S \cap K(x_0, \zeta_0; t_0, s_0)$ is a smooth 2-dimensional space-like surface that is a subset of $U_{\widehat{g}}$.

Let $\Lambda(x_0, \zeta_0; t_0, s_0)$ be the Lagrangian manifold that is the flowout from $N^* Y(x_0, \zeta_0; t_0, s_0) \cap N^* K(x_0, \zeta_0; t_0, s_0)$ on $\text{Char}(\square_{\widehat{g}})$ in the future direction. When $K^{reg} \subset K = K(x_0, \zeta_0; t_0, s_0)$ is the set of points x that have a neighborhood W such that $K \cap W$ is a smooth 3-dimensional submanifold, we have $N^* K^{reg} \subset \Lambda(x_0, \zeta_0; t_0, s_0)$. Below, we represent locally the elements $w \in \mathcal{B}_x$ in the fiber of the bundle \mathcal{B} as a $(10 + L)$ -dimensional vector, $w = (w_m)_{m=1}^{10+L}$.

Lemma 3.1. *Let $n \leq n_0 = -17$ be an integer, $t_0, s_0 > 0$, $Y = Y(x_0, \zeta_0; t_0, s_0)$, $K = K(x_0, \zeta_0; t_0, s_0)$, $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$, and $(y, \xi) \in N^* Y \cap \Lambda_1$. Assume that $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathcal{I}^{n+1}(Y)$, is a \mathcal{B}^L -valued conormal distribution that is supported in a neighborhood $V \subset M_0$ of $\gamma_{x_0, \zeta_0} \cap Y = \{\gamma_{x_0, \zeta_0}(2t_0)\}$ and has a \mathbb{R}^{10+L} -valued classical symbol. Denote the principal symbol of \mathbf{f} by $\tilde{f}(y, \xi) = (\tilde{f}_k(y, \xi))_{k=1}^{10+L}$, and assume that the symbol of \mathbf{f} vanishes near the light-like directions in $N^* Y \setminus N^* K$.*

Let $u^{(1)} = (\dot{g}, \dot{\phi})$ be a solution of the linear wave equation (40) with the source \mathbf{f} . Then $u^{(1)}$, considered as a vector valued Lagrangian distribution on the set $M_0 \setminus Y$, satisfies $u^{(1)} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$, and its principal symbol $\tilde{a}(x, \eta) = (\tilde{a}_j(x, \eta))_{j=1}^{10+L}$ at $(x, \eta) \in \Lambda_1$, is given by

$$(52) \quad \tilde{a}_j(x, \eta) = \sum_{k=1}^{10+L} R_j^k(x, \eta, y, \xi) \tilde{f}_k(y, \xi),$$

where the pairs (y, ξ) and (x, η) are on the same bicharacteristics of $\square_{\hat{g}}$, and $y \ll x$. Observe that $((y, \xi), (x, \eta)) \in \Lambda'_{\hat{g}}$, and in addition, $(y, \xi) \in N^*Y \cap N^*K$. Moreover, the matrix $(R_j^k(x, \eta, y, \xi))_{j,k=1}^{10+L}$ is invertible.

We call the solution $u^{(1)}$ a distorted plane wave that is associated to the submanifold $K(x_0, \zeta_0; t_0, s_0)$.

Proof. It follows from [83] that the causal inverse of the scalar wave operator $\square_{\hat{g}} + V(x, D)$, where $V(x, D)$ is a 1st order differential operator, satisfies $(\square_{\hat{g}} + V(x, D))^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M_0}, \Lambda_{\hat{g}})$. Here, $\Delta_{T^*M_0}$ is the conormal bundle of the diagonal of $M_0 \times M_0$ and $\Lambda_{\hat{g}}$ is the flow-out of the canonical relation of $\square_{\hat{g}}$. A geometric representation for its kernel is given in [69]. An analogous result holds for the matrix valued wave operator, $\square_{\hat{g}}I + V(x, D)$, when $V(x, D)$ is a 1st order differential operator, that is, $(\square_{\hat{g}}I + V(x, D))^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M_0}, \Lambda_{\hat{g}})$, see [83] and [29]. By [44, Prop. 2.1], this yields $u^{(1)} \in \mathcal{I}^{n-1/2}(\Lambda_1)$ and the formula (52) where $R = (R_j^k(x, \eta, y, \xi))_{j,k=1}^{10+L}$ is obtained by solving a system of ordinary differential equation along a bicharacteristic curve. Making similar considerations for the adjoint of the $(\square_{\hat{g}}I + V(x, D))^{-1}$, i.e., considering the propagation of singularities using reversed causality, we see that the matrix R is invertible. \square

Below, let $(y, \xi) \in N^*Y \cap \Lambda_1$ and $(x, \eta) \in T^*M_0$ be a light-like co-vector such that $(x, \eta) \in \Theta_{y, \xi}$, $x \notin Y$ and, $y \ll x$.

Let \mathcal{B}_x^L be the fiber of the bundle \mathcal{B}^L at x and $\mathfrak{S}_{x, \eta}$ be the space of the elements in \mathcal{B}_x^L satisfying the harmonicity condition for the symbols (50) at (x, η) . Let $(y, \xi) \in N^*Y$ and $\mathfrak{C}_{y, \xi}$ be the set of elements b in \mathcal{B}_y^L that satisfy the linearized conservation law for symbols, i.e., (42).

Let $n \leq n_0$ and $t_0, s_0 > 0$, $Y = Y(x_0, \zeta_0; t_0, s_0)$, $K = K(x_0, \zeta_0; t_0, s_0)$, $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$, and $b_0 \in \mathfrak{C}_{x, \xi}$. By Condition μ -SL, there is a conormal distribution $\mathbf{f} \in \mathcal{I}^{n+1}(Y) = \mathcal{I}^{n+1}(N^*Y)$ such that \mathbf{f} satisfies the linearized conservation law (13) and the principal symbol \tilde{f} of \mathbf{f} , defined on N^*Y , satisfies $\tilde{f}(y, \xi) = b_0$. Moreover, by Condition μ -SL there is a family of sources \mathcal{F}_ε , $\varepsilon \in [0, \varepsilon_0]$ such that $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = \mathbf{f}$ and a solution $u_\varepsilon + (\hat{g}, \hat{\phi})$ of the Einstein equations with the source \mathcal{F}_ε that depend smoothly on ε and $u_\varepsilon|_{\varepsilon=0} = 0$. Then $\dot{u} = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$.

As $\dot{u} = (\dot{g}, \dot{\phi})$ satisfies the linearized harmonicity condition (43), the principal symbol $\tilde{a}(x, \eta) = (\tilde{a}_1(x, \eta), \tilde{a}_2(x, \eta))$ of \dot{u} satisfies $\tilde{a}(x, \eta) \in \mathfrak{S}_{x, \eta}$. This shows that the map $R = R(x, \eta, y, \xi)$, given by $R : \tilde{f}(y, \xi) \mapsto \tilde{a}(x, \eta)$ that is defined in Lemma 3.1, satisfies $R : \mathfrak{C}_{y, \xi} \rightarrow \mathfrak{S}_{x, \eta}$. Since R is one-to-one and the linear spaces $\mathfrak{C}_{y, \xi}$ and $\mathfrak{S}_{x, \eta}$ have the same dimension, we see that

$$(53) \quad R : \mathfrak{C}_{y, \xi} \rightarrow \mathfrak{S}_{x, \eta}$$

is a bijection. Hence, when $\mathbf{f} \in \mathcal{I}^{n+1}(Y)$ varies so that the linearized conservation law (42) for the principal symbols is satisfied, the principal symbol $\tilde{a}(x, \eta)$ at (x, η) of the solution \dot{u} of the linearized Einstein equation achieves all values in the $(L + 6)$ dimensional space $\mathfrak{S}_{x, \eta}$.

Below, we denote $\mathbf{f} \in \mathcal{I}_C^{n+1}(Y(x_0, \zeta_0; t_0, s_0))$ when the principal symbols of \mathbf{f} satisfies the linearized conservation law for principal symbols, that is, equation (42).

3.3. Microlocal analysis of the non-linear interaction of waves. Next we consider the interaction of four C^k -smooth waves having conormal singularities, where $k \in \mathbb{Z}_+$ is sufficiently large. Interaction of such waves produces a “corner point” in the spacetime. On related microlocal tools to consider scattering by corners, see [103, 104]. Earlier considerations of interaction of three waves has been done by Melrose and Ritter [84, 85] and Rauch and Reed, [91] for non-linear hyperbolic equations in \mathbb{R}^{1+2} where the non-linearity appears in the lower order terms. Recently, the interaction of two strongly singular waves has been studied by Luc and Rodnianski [76].

3.3.1. Interaction of non-linear waves on a general manifold. Next, we introduce a vector of four ε variables denoted by $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4$. Let $s_0, t_0 > 0$ and consider $u_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}} - \hat{g}, \phi_{\vec{\varepsilon}} - \hat{\phi})$ where $v_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}}, \phi_{\vec{\varepsilon}})$ solve the equations (8) with $\mathcal{F} = \mathbf{f}_{\vec{\varepsilon}}$ where

$$(54) \quad \mathbf{f}_{\vec{\varepsilon}} := \sum_{j=1}^4 \varepsilon_j \mathbf{f}_j, \quad \mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0)),$$

and (x_j, ζ_j) are light-like vectors with $x_j \in U_{\hat{g}}$. Moreover, we assume that for some $0 < r_2 < r_1$ and $s_- + r_2 < s' < s_+$ the sources satisfy

$$(55) \quad \begin{aligned} \text{supp}(\mathbf{f}_j) \cap J_{\hat{g}}^+(\text{supp}(\mathbf{f}_k)) &= \emptyset, \quad \text{for all } j \neq k, \\ \text{supp}(\mathbf{f}_j) &\subset I_{\hat{g}}(\mu_{\hat{g}}(s' - r_2), \mu_{\hat{g}}(s')), \quad \text{for all } j = 1, 2, 3, 4, \end{aligned}$$

where r_1 is the parameter used to define $W_{\hat{g}} = W_{\hat{g}}(r_1)$, see (10). The first condition implies that the supports of the sources are causally independent.

The sources \mathbf{f}_j give raise to \mathcal{B}^L -section valued solutions of the linearized wave equations, which we denote by

$$u_j := u_j^{(1)} = \partial_{\varepsilon_j} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} = \mathbf{Q} \mathbf{f}_j \in \mathcal{I}(\Lambda(x_j, \zeta_j; t_0, s_0)),$$

where $\mathbf{Q} = \mathbf{Q}_{\hat{g}}$. In the following we use the notations $\partial_{\vec{\varepsilon}}^1 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$, $\partial_{\vec{\varepsilon}}^2 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$, $\partial_{\vec{\varepsilon}}^3 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$, and

$$\partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Next we denote the waves produced by the ℓ -th order interaction by

$$(56) \quad \mathcal{M}^{(\ell)} := \partial_{\vec{\varepsilon}}^\ell u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \ell \in \{1, 2, 3, 4\}.$$

Below, we use the notations \mathcal{B}_j^β , $j = 1, 2, 3, 4$ and S_n^β , $n = 1, 2$ to denote operators of the form

$$(57) \quad \mathcal{B}_j^\beta : (v_p)_{p=1}^{10+L} \mapsto (b_p^{r,(j,\beta)}(x) \partial_x^{k(\beta,j)} v_r(x))_{p=1}^{10+L}, \text{ and}$$

$$S_n^\beta = \mathbf{Q} \text{ or } S_n^\beta = I, \text{ or } S_n^\beta = [\mathbf{Q}, a_n^\beta(x) D^\alpha], \quad \alpha = \alpha(\beta, n), |\alpha| \leq 4$$

Here, $|k(\beta)| = \sum_{j=1}^4 k(\beta, j)$ and $|\alpha(\beta)| = |\alpha(\beta, 1)| + |\alpha(\beta, 2)|$ satisfy the bounds described below. and the coefficients $b_p^{r,(j,\beta)}(x)$ and $a_j^\beta(x)$ depend on the derivatives of \hat{g} . Here, $k(\beta, j) = k_j^\beta = \text{ord}(\mathcal{B}_j^\beta)$ are the orders of \mathcal{B}_j^β and β is just an index running over a finite set $J_4 \subset \mathbb{Z}_+$.

Computing the ε_j derivatives of the equations (29), (30), and (31) with the sources \mathbf{f}_ε , and taking into account the condition (55), we obtain:

$$\begin{aligned} (58) \quad \mathcal{M}^{(1)} &= u_1, \\ \mathcal{M}^{(2)} &= \sum_{\sigma \in \Sigma(2)} \mathbf{Q}(A[u_{\sigma(2)}, u_{\sigma(1)})], \\ \mathcal{M}^{(3)} &= \sum_{\sigma \in \Sigma(3)} \left(2\mathbf{Q}(A[u_{\sigma(3)}, \mathbf{Q}(A[u_{\sigma(2)}, u_{\sigma(1)})]) + \right. \\ &\quad \left. + \mathbf{Q}(B[u_{\sigma(3)}, u_{\sigma(2)}, u_{\sigma(1)}]) \right), \end{aligned}$$

where A is a bi-linear operator discussed below and B is a trilinear operator, or

$$\begin{aligned} (59) \quad \mathcal{M}^{(1)} &= u_1, \\ \mathcal{M}^{(2)} &= \sum_{\sigma \in \Sigma(2)} \sum_{\beta \in J_2} \mathbf{Q}(\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)}), \\ \mathcal{M}^{(3)} &= \sum_{\sigma \in \Sigma(3)} \sum_{\beta \in J_3} \mathbf{Q}(\mathcal{B}_3^\beta u_{\sigma(3)} \cdot S_1^\beta (\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)})), \\ \mathcal{M}^{(4)} &= \mathbf{Q}\mathcal{F}^{(4)}, \quad \mathcal{F}^{(4)} = \sum_{\sigma \in \Sigma(4)} \sum_{\beta \in J_4} (\mathcal{G}_\sigma^{(4),\beta} + \tilde{\mathcal{G}}_\sigma^{(4),\beta}), \end{aligned}$$

where $\Sigma(k)$ is the set of permutations, that is, bijections $\sigma : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}$, and $J_k \subset \mathbb{Z}_+$ are finite sets. Then, we have

$$(60) \quad \mathcal{G}_\sigma^{(4),\beta} = \mathcal{B}_4^\beta u_{\sigma(4)} \cdot S_2^\beta (\mathcal{B}_3^\beta u_{\sigma(3)} \cdot S_1^\beta (\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)}))$$

where the orders of the differential operators satisfy $k_4^\beta + k_3^\beta + k_2^\beta + k_1^\beta + |\alpha(\beta, 1)| + |\alpha(\beta, 2)| \leq 6$ and $k_4^\beta + k_3^\beta + |\alpha(\beta, 2)| \leq 2$ and

$$(61) \quad \tilde{\mathcal{G}}_\sigma^{(4),\beta} = \mathcal{S}_2^\beta (\mathcal{B}_4^\beta u_{\sigma(4)} \cdot \mathcal{B}_3^\beta u_{\sigma(3)}) \cdot \mathcal{S}_1^\beta (\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)}),$$

where $k_4^\beta + k_3^\beta + k_2^\beta + k_1^\beta + |\alpha(\beta, 1)| + |\alpha(\beta, 2)| \leq 6$, $k_4^\beta + k_3^\beta + |\alpha(\beta, 2)| \leq 4$, and $k_1^\beta + k_2^\beta + |\alpha(\beta, 1)| \leq 4$.

We denote below $\vec{S}_\beta = (S_1^\beta, S_2^\beta)$ and $\mathcal{M}_\sigma^{(4),\beta} = \mathbf{Q}_{\hat{g}} \mathcal{G}_\sigma^{(4),\beta}$ and $\widetilde{\mathcal{M}}_\sigma^{(4),\beta} = \mathbf{Q}_{\hat{g}} \tilde{\mathcal{G}}_\sigma^{(4),\beta}$. Let us explain how the terms above appear in the Einstein

equations: By taking $\partial_{\vec{v}}$ derivatives of the wave $u_{\vec{v}}$ we obtain terms similar to (29) and (30). In particular, we have that $\mathcal{G}_{\sigma}^{(4),\beta}$ and $\tilde{\mathcal{G}}_{\sigma}^{(4),\beta}$ can be written in the form

$$(62) \quad \mathcal{G}_{\sigma}^{(4),\beta} = A_3^{\beta}[u_{\sigma(4)}, S_2^{\beta}(A_2^{\beta}[u_{\sigma(3)}, S_1^{\beta}(A_1^{\beta}[u_{\sigma(2)}, u_{\sigma(1)}])]),$$

$$(63) \quad \tilde{\mathcal{G}}_{\sigma}^{(4),\beta} = A_3^{\beta}[S_2^{\beta}(A_2^{\beta}[u_{\sigma(4)}, u_{\sigma(3)})], S_1^{\beta}(A_1^{\beta}[u_{\sigma(2)}, u_{\sigma(1)})]],$$

where $A_j^{\beta}[V, W]$ are 2nd order multilinear operators of the form

$$(64) \quad A[V, W] = \sum_{|\alpha|+|\gamma|\leq 2} a_{\alpha\gamma}(x)(\partial_x^{\alpha} V(x)) \cdot (\partial_x^{\gamma} W(x)).$$

By commuting derivatives and the operator $\mathbf{Q}_{\widehat{g}}$ we obtain (60) and (61). Below, the terms of particular importance are the bilinear form that is given for $V = (v_{jk}, \phi)$ and $W = (w^{jk}, \phi')$ by

$$(65) \quad A_1[V, W] = -\widehat{g}^{jb} w_{ab} \widehat{g}^{ak} \partial_j \partial_k v_{pq}.$$

3.3.2. On the singular support of the non-linear interaction of three waves. Let us next consider four light-like future pointing directions (x_j, ξ_j) , $j = 1, 2, 3, 4$, and use below the notations, see (20),

$$(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4, \quad (\vec{x}(h), \vec{\xi}(h)) = ((x_j(h), \xi_j(h)))_{j=1}^4.$$

We will consider the case when we send distorted plane waves propagating on surfaces $K_j = K(x_j, \xi_j; t_0, s_0)$, $t_0, s_0 > 0$, cf. (51), and these waves interact.

Next we consider the 3-interactions of the waves. Let $\mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0)$ be set of all light-like vectors $(x, \xi) \in L^+ M_0$ that are in the normal bundles $N^*(K_{j_1} \cap K_{j_2} \cap K_{j_3})$ with some $1 \leq j_1 < j_2 < j_3 \leq 4$.

Moreover, we define $\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$ to be the set of all $y \in M_0$ such that there are $(z, \zeta) \in \mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0)$, and $t \geq 0$ such that $\gamma_{z,\zeta}(t) = y$. Finally, let

$$(66) \quad \mathcal{Y}((\vec{x}, \vec{\xi}); t_0) = \bigcap_{s_0 > 0} \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0).$$

The three wave interaction happens then on $\pi(\mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0))$ and, roughly speaking, this interaction sends singularities to $\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$.

For instance in Minkowski space, when three plane waves (whose singular supports are hyperplanes) collide, the intersections of the hyperplanes is a 1-dimensional space-like line $K_{123} = K_1 \cap K_2 \cap K_3$ in the 4-dimensional space-time. This corresponds to a point moving continuously in time. Roughly speaking, the point seems to move at a higher speed than light (i.e. it appears like a tachyonic, moving point source) and produces a (conic) shock wave type of singularity (see Fig. 2 where the interaction time is only finite). In this paper we do not analyze carefully the singularities produced by the three wave interaction near

$\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$. Our goal is to consider the singularities produced by the four wave interaction in the domain $M_0 \setminus \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$.

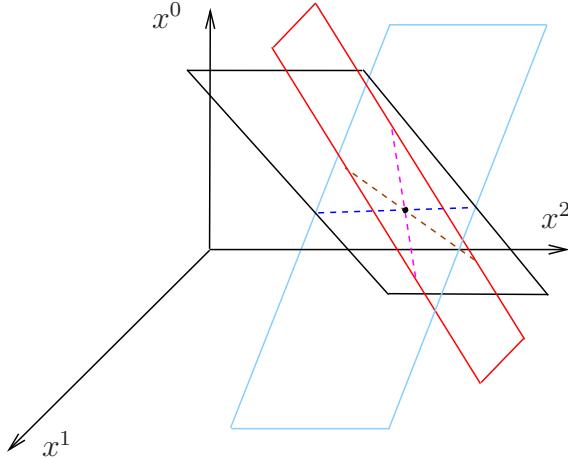


FIGURE A1: A schematic figure where the space-time is represented as the 3-dimensional set \mathbb{R}^{2+1} . In the figure 3 pieces of plane waves have singularities on strips of hyperplanes (in fact planes) K_1, K_2, K_3 , colored by light blue, red, and black. These planes have intersections, and in the figure the sets $K_{12} = K_1 \cap K_2$, $K_{23} = K_2 \cap K_3$, and $K_{13} = K_1 \cap K_3$ are shown as dashed lines with dark blue, magenta, and brown colors. These dashed lines intersect at a point $\{q\} = K_{123} = K_1 \cap K_2 \cap K_3$.

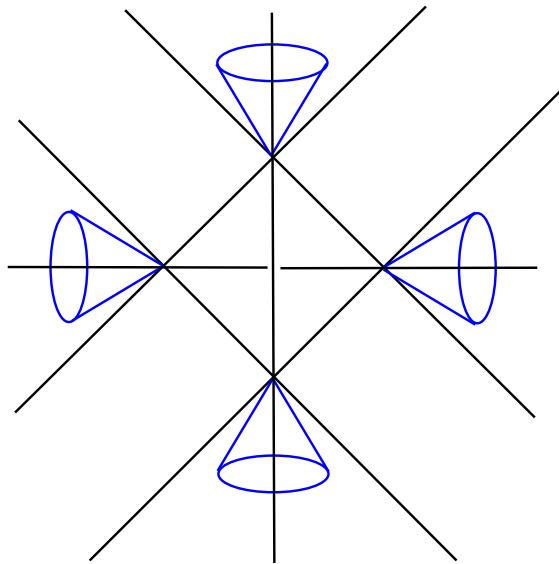


FIGURE A2: In section 3.3.2 we consider four colliding distorted plane waves. In the figure we consider the Minkowski space \mathbb{R}^{1+3} and the figure corresponds to a “time-slice” $\{T_1\} \times \mathbb{R}^3$. We assume that the distorted plane waves are sent so far away that the waves look like pieces of plane waves. The plane waves $u_j \in \mathcal{I}(K_j)$, $j = 1, 2, 3, 4$, are conormal

distributions that are solutions of the linear wave equation and their singular supports are the sets K_j , that are pieces of 3-dimensional planes in the space-time. The sets K_j are not shown in the figure. The 2-wave interaction wave $\mathcal{M}^{(2)}$ is singular on the set $\cup_{j \neq k} K_j \cap K_j$. There are 6 intersection sets $K_j \cap K_j$ that are shown as black line segments. Note that these lines have 4 intersection points, that is, the vertical and the horizontal black lines do not intersect in $\{T_1\} \times \mathbb{R}^3$. The four intersection points of the black lines are the sets $(\{T_1\} \times \mathbb{R}^3) \cap (K_j \cap K_j \cap K_n)$. These points correspond to points moving in time (i.e., they are curves in the space-time) that produce singularities of the 3-interaction wave $\mathcal{M}^{(3)}$. The points seem to move faster than the speed of the light (similarly, as a shadow of a far away object may seem to move faster than the speed of the light). Such point sources produce “shock waves”, and due to this, $\mathcal{M}^{(3)}$ is singular on the sets $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$ defined in formulas (65)-(66). The set $(\{T_1\} \times \mathbb{R}^3) \cap \mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$ is the union of the four blue cones shown in the figure.

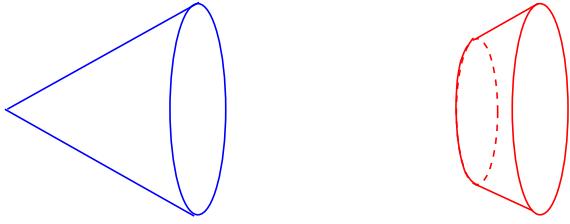


FIGURE A3: A schematic figure in 3-dimensional Euclidean space \mathbb{R}^3 where we describe the geometry of the wave produced by the interaction of three waves in the Minkowski space. The figure shows such a 3-interaction wave in a time slice $\{T_1\} \times \mathbb{R}^3$ with two different values of the parameter s_0 . On the left, s_0 is large and the 3-interaction wave has singular support on a cone. On the right, s_0 is small and the 3-interaction wave is a frustum (a truncated cone, or, a cone with its apex cut off by a plane), similar to set $\{(x^1, x^2, x^3) \in \mathbb{R}^3; (x^1)^2 + (x^2)^2 = c(x^3)^2, a < x^3 < a + h\}$, with some $a, c > 0$ and a thickness $h = c_1 s_0$.

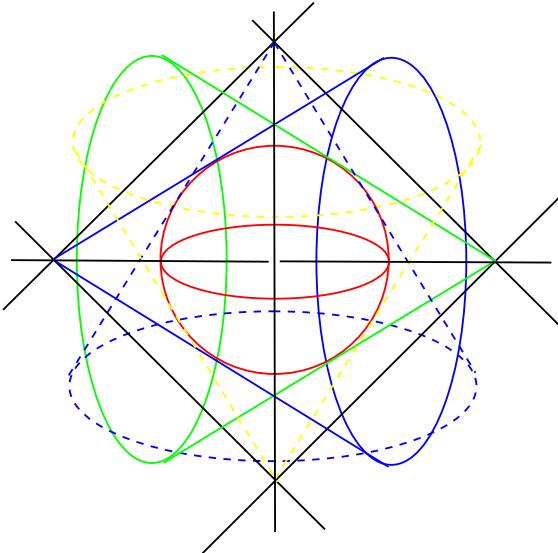


FIGURE A4: The same situation that was described in Fig. A2 is shown at a later time, that is, the figure shows the time-slice $\{T_2\} \times \mathbb{R}^3$ with $T_2 > T_1$, when the parameter s_0 is quite large. The four distorted plane waves have now collided and produced a point source in the space-time at a point $q \in K_1 \cap K_2 \cap K_3 \cap K_4$. This gives raise to the singularities of the 4-interaction wave $\mathcal{M}^{(4)}$. In the figure $T_1 < t < T_2$, where q has the time coordinate t . The four cones in the figure, shown with solid blue and green curves and dashed blue and yellow curves are the intersection of the time-slice $\{T_2\} \times \mathbb{R}^3$ and the set $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$. Inside the cones the red sphere is the set $\mathcal{L}^+(q) \cap (\{T_2\} \times \mathbb{R}^3)$ that corresponds to the spherical plane wave produced by the point source at q .

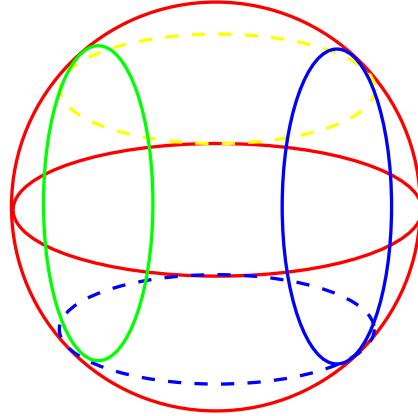


FIGURE A5: The same situation that was described in Fig. A4. That is, the figure shows in the time-slice $\{T_2\} \times \mathbb{R}^3$ the singularities produced by four colliding distorted plane waves, when the parameter s_0 is very small. In this case the truncated cones degenerate to circles. Indeed, the figure corresponds to the case when the parameter s_0 is close to zero, that is, the pieces of the distorted plane waves are concentrated near a

single geodesic in the space-time. As s_0 is small, the distorted plane waves interact only during a short time. Due to this, the sets $(K_j \cap K_n) \cap (\{T_2\} \times \mathbb{R}^3)$, that were black lines in the previous figures, are now empty, and do not appear at the figure. In the figure the red sphere is the set $\mathcal{L}^+(q) \cap (\{T_2\} \times \mathbb{R}^3)$ corresponding to the spherical wave produced by a point source at q . The solid blue and green circles on the sphere and the dashed blue and yellow circles on the sphere are the intersection of the time-slice $\{T_2\} \times \mathbb{R}^3$ and the set $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$. These circles are actually not circles but surfaces of frustums (a truncated cone, or, a cone with its apex cut off by a plane), similar to set $\{(x^1, x^2, x^3) \in \mathbb{R}^3; (x^1)^2 + (x^2)^2 = c(x^3)^2, a < x^3 < a + h\}$, with some $a, c > 0$ and a very small thickness $h > 0$. Note that the 2-Hausdorff measure of the frustums go to zero as $s_0 \rightarrow 0$ and the 3-Hausdorff measure of the set $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$ goes to zero as $s_0 \rightarrow 0$. Thus, when all the distorted plane waves intersect at a point q , the singularities of $\mathcal{M}^{(4)}$ are observed on a set whose 3-Hausdorff measure in the space-time is independent of s_0 but the singularities of $\mathcal{M}^{(3)}$ are observed on a set which 3-hHausdorff measure in the space-time is smaller than Cs_0 .

3.3.3. Gaussian beams. Our aim is to consider interactions of 4 waves that produce a new source, and to this end we use test sources that produce gaussian beams.

Let $y \in U_{\hat{g}}$ and $\eta \in T_y M$ be a future pointing light-like vector. We choose a complex function $p \in C^\infty(M_0)$ such that $\text{Im } p(x) \geq 0$ and $\text{Im } p(x)$ vanishes only at y , $p(y) = 0$, $d(\text{Re } p)|_y = \eta^\sharp$, $d(\text{Im } p)|_y = 0$ and the Hessian of $\text{Im } p$ at y is positive definite. To simplify notations, we use below complex sources and waves. The physical linearized waves can be obtained by taking the real part of these complex waves. We use a large parameter $\tau \in \mathbb{R}_+$ and define a test source

$$(67) \quad F_\tau(x) = F_\tau(x; y, \eta), \text{ where } F_\tau(x; y, \eta) = \tau^{-1} \exp(i\tau p(x))h(x)$$

where h is section on \mathcal{B}^L supported in a small neighborhood W of y . The construction of $p(x)$ and F_τ is discussed in detail below.

We consider both the usual causal solutions and the solutions of the adjoint wave equation for which time is reversed, that is, we use the anti-causal parametrix $\mathbf{Q}^* = \mathbf{Q}_{\hat{g}}^*$ instead of the usual causal parametrix $\mathbf{Q} = (\square_{\hat{g}} + V(x, D))^{-1}$.

For a function $v : M_0 \rightarrow \mathbb{R}$ the large τ asymptotics of the L^2 -inner products $\langle F_\tau, v \rangle_{L^2(M_0)}$ can be used to verify if the point $(y, \eta^\sharp) \in T^*M_0$ belongs in the wave front set $\text{WF}(v)$ of v , see e.g. [90]. Below, we will use the fact that when v is of the form $v = \mathbf{Q}_{\hat{g}} b$, we have $\langle F_\tau, v \rangle_{L^2(M_0)} = \langle \mathbf{Q}_{\hat{g}}^* F_\tau, b \rangle_{L^2(M_0)}$ where $\mathbf{Q}_{\hat{g}}^* F_\tau$ is a Gaussian beam solution to the adjoint wave equation. We will use this to analyze the singularities of $\mathcal{M}^{(4)}$ at $y \in U_{\hat{g}}$.

The wave $u_\tau = \mathbf{Q}^* F_\tau$ satisfies by [90], see also [61],

$$(68) \quad \|u_\tau - u_\tau^N\|_{C^k(J(p^-, p^+))} \leq C_N \tau^{-n_{N,k}}$$

where $n_{N,k} \rightarrow \infty$ as $N \rightarrow \infty$ and u_τ^N is a formal Gaussian beam of order N having the form

$$(69) \quad u_\tau^N(x) = e^{i\tau\varphi(x)} a(x, \tau), \quad a(x, \tau) = \sum_{n=0}^N U_n(x) \tau^{-n}.$$

Here $\varphi(x) = A(x) + iB(x)$ and $A(x)$ and $B(x)$ are real-valued functions, $B(x) \geq 0$, and $B(x)$ vanishes only on $\gamma_{y,\eta}(\mathbb{R})$, and for $z = \gamma_{y,\eta}(t)$, and $\zeta = \dot{\gamma}_{y,\eta}^\flat(t)$, $t < 0$ we have $dA|_z = \zeta$, $dB|_z = 0$. Moreover, the Hessian of B at z restricted to the orthocomplement of ζ is positive definite. The functions h and U_n defined above can be chosen to be smooth so that h is supported in any neighborhood V of y and U_N are supported in any neighborhoods of $\gamma_{y,\eta}((-\infty, 0])$. Below, $a(x, \tau)$ is the symbol and $U_0(y)$ is the principal symbol of the gaussian beam $u_\tau(x)$.

The source function F_τ can be constructed in local coordinates using the asymptotic representation of the gaussian beam, namely, by considering first

$$F_\tau^{(series)}(x) = c\tau^{-1/2} \int_{\mathbb{R}} e^{-\tau s^2} \phi(s) f_\tau(x; s) ds,$$

where $f_\tau(x; s) = \square_{\widehat{g}}(H(s - x^0)u_\tau(x))$. Here, $\phi \in C_0^\infty(\mathbb{R})$ has value 1 in some neighborhood of zero, and $H(s)$ is the Heaviside function, and we have

$$e^{-i\tau p(x)} F_\tau^{(series)}(x) = e^{-i\tau p(x)} F^{(1)}(x) \tau^{-1} + e^{-i\tau p(x)} F^{(2)}(x) \tau^{-2} + O(\tau^{-3}).$$

Then, F_τ can be defined by $F_\tau = F^{(1)}(x) \tau^{-1}$. Indeed we see that

$$\begin{aligned} \square_{\widehat{g}}^{-1} F_\tau^{(series)} &= \tau^{-1} \Phi_\tau(x) u_\tau(x), \quad \text{with} \\ \Phi_\tau(x) &= c\tau^{1/2} \int_{\mathbb{R}} e^{-\tau s^2} \phi(s) H(s - x^0) ds \\ &= \begin{cases} O(\tau^{-N}), & \text{for } x_0 < 0, \\ 1 + O(\tau^{-N}), & \text{for } x_0 > 0. \end{cases} \end{aligned}$$

Moreover, by writing $\square_{\widehat{g}} = \widehat{g}^{jk} \partial_j \partial_k + \widehat{\Gamma}^j \partial_j$, $\widehat{\Gamma}^j = \widehat{g}^{pq} \widehat{\Gamma}_{pq}^j$ we have that

$$\begin{aligned} f_\tau(x; s) &= \widehat{g}^{00} u_\tau(x) \delta'(s - x^0) \\ &\quad - 2 \sum_{j=1}^3 \widehat{g}^{j0} (\partial_j u_\tau(x)) \delta(s - x^0) - \widehat{\Gamma}^0 u_\tau(x) \delta(x^0 - s). \end{aligned}$$

Let us use normal coordinates centered at the point y so that $\Gamma^j(0) = 0$ and $\widehat{g}^{jk}(0)$ is the Minkowski metric. Then we define $p(x) = (x^0)^2 + \varphi(x)$. The leading order term of $F_\tau^{(series)}$ is given by

$$F_\tau = c\tau^{-1/2} e^{-\tau p(x)} (2x^0 \phi(x^0) U_0(x) + 2i(\partial_0 \varphi(x)) U_0(x) \phi(x^0)),$$

where $\partial_0 \varphi|_y$ does not vanish.

3.3.4. Indicator function for singularities produced by interaction of four waves. Let $y \in U_{\widehat{g}}$ and $\eta \in L_{x_0}^+(M_0, \widehat{g})$ be a future pointing light-like vector. We will next check whether $(y, \eta^\flat) \in \text{WF}(\mathcal{M}^{(4)})$.

Using the function $\mathcal{M}^{(4)}$ defined in (59) with four sources $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$, $j \leq 4$, that produce the pieces of plane waves $u_j \in \mathcal{I}^n(\Lambda(x_j, \xi_j; t_0, s_0))$ in $M_0 \setminus Y(x_j, \zeta_j; t_0, s_0)$, and the source F_τ in (67), we define the indicator functions

$$(70) \quad \Theta_\tau^{(4)} = \langle F_\tau, \mathcal{M}^{(4)} \rangle_{L^2(U)} = \sum_{\beta \in J_4} \sum_{\sigma \in \Sigma(4)} (T_{\tau, \sigma}^{(4), \beta} + \tilde{T}_{\tau, \sigma}^{(4), \beta}).$$

Here, the terms $T_{\tau, \sigma}^{(\ell), \beta}$ and $\tilde{T}_{\tau, \sigma}^{(\ell), \beta}$ correspond, for σ and β , to different types of interactions the four waves $u_{(j)}$ can have. To define the terms $T_{\tau, \sigma}^{(\ell), \beta}$ and $\tilde{T}_{\tau, \sigma}^{(\ell), \beta}$ appearing above, we use generic notations where we drop the index β , that is, we denote $\mathcal{B}_j = \mathcal{B}_j^\beta$ and $S_j = S_j^\beta$. With these notations, using the decompositions (59), (60), and (61) and the fact that $u_\tau = \mathbf{Q}^* F_\tau$, we define

$$(71) \quad \begin{aligned} T_{\tau, \sigma}^{(4), \beta} &= \langle F_\tau, \mathbf{Q}(\mathcal{B}_4 u_{\sigma(4)} \cdot S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}))) \rangle_{L^2(M_0)} \\ &= \langle \mathbf{Q}^* F_\tau, \mathcal{B}_4 u_{\sigma(4)} \cdot S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})) \rangle_{L^2(M_0)} \\ &= \langle (\mathcal{B}_4 u_{\sigma(4)}) \cdot u_\tau, S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})) \rangle_{L^2(M_0)} \\ &= \langle (\mathcal{B}_3 u_{\sigma(3)}) \cdot S_2^*((\mathcal{B}_4 u_{\sigma(4)}) \cdot u_\tau), S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(M_0)}, \end{aligned}$$

and

$$(72) \quad \begin{aligned} \tilde{T}_{\tau, \sigma}^{(4), \beta} &= \langle F_\tau, \mathbf{Q}(S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})) \rangle_{L^2(M_0)} \\ &= \langle \mathbf{Q}^* F_\tau, S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(M_0)} \\ &= \langle S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot u_\tau, S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(M_0)}. \end{aligned}$$

When σ is the identity, we will omit it in our notations and denote $T_\tau^{(4), \beta} = T_{\tau, id}^{(4), \beta}$, etc. It turns out later, in Prop. 3.5, that the term $\langle F_\tau, \mathbf{Q}(\mathcal{B}_4 u_4 \mathbf{Q}(\mathcal{B}_3 u_3, \mathbf{Q}(\mathcal{B}_2 u_2 \mathcal{B}_1 u_1))) \rangle$, where the orders k_j of \mathcal{B}_j are $k_1 = 6$, $k_2 = k_3 = k_4 = 0$ is the term that is crucial for our considerations. We enumerate this term with $\beta = \beta_1 := 1$, i.e.,

$$(73) \quad \vec{S}_{\beta_1} = (\mathbf{Q}, \mathbf{Q}) \text{ and } k_1^{\beta_1} = 6, k_2^{\beta_1} = k_3^{\beta_1} = k_2^{\beta_1} = 0.$$

This term arises from the term $\langle F_\tau, \mathcal{G}_\sigma^{(4), \beta} \rangle$, such that $\mathcal{G}_\sigma^{(4), \beta}$ is of the form (62), where $\sigma = id$ and all quadratic forms A_3^β , A_2^β , and A_1^β are of the from (65). More precisely, $\langle F_\tau, \mathcal{G}_\sigma^{(4), \beta} \rangle$ is the sum of $T_{\tau, id}^{(4), \beta_1}$ and terms analogous to that (including terms with commutators of \mathbf{Q} and differential operators), where

$$(74) \quad T_{\tau, id}^{(4), \beta_1} = -\langle \mathbf{Q}^*((F_\tau)_{nm}), u_4^{rs} \cdot \mathbf{Q}(u_3^{ab} \cdot \mathbf{Q}(u_2^{ik} \cdot \partial_r \partial_s \partial_a \partial_b \partial_i \partial_k u_1^{nm})) \rangle.$$

Here, $u_j^{ik} = \widehat{g}^{in}(x)\widehat{g}^{km}(x)(u_j(x))_{nm}$, $j = 1, 2, 3, 4$ and $(u_j(x))_{nm}$ is the metric part of the wave $u_j(x) = (((u_j(x))_{nm})_{n,m=1}^4, ((u_j(x))_\ell)_{\ell=1}^L)$.

3.3.5. Properties of the indicator functions on a general manifold. Next we analyze the indicator function for sources $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$, $j \leq 4$, and the source F_τ related to $(y, \eta) \in L^+ M_0$, $y \in U_{\widehat{g}}$. We denote in the following $(x_5, \xi_5) = (y, \eta)$.

Definition 3.2. Let $t_0 > 0$. We say that the geodesics corresponding to vectors $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ intersect and the intersection takes place at the point $q \in M_0$ if there are $t_j \in (0, \mathbf{t}_j)$, $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$ such that $q = \gamma_{x_j, \xi_j}(t_j)$ for all $j = 1, 2, 3, 4$.

Let Λ_q^+ be the Lagrangian manifold

$$\Lambda_q^+ = \{(x, \xi) \in T^* M_0; x = \gamma_{q, \zeta}(t), \xi^\sharp = \dot{\gamma}_{q, \zeta}(t), \zeta \in L_q^+ M_0, t > 0\}$$

Note that the projection of Λ_q^+ on M_0 is the light cone $\mathcal{L}_{\widehat{g}}^+(q)$.

Next we consider $x_j \in U_{\widehat{g}}$ and $\xi_j \in L_{x_j}^+ M_0$, and $\vartheta_1, t_0 > 0$ such that $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ satisfy, see Fig. 4(Right),

- (75) (i) $\gamma_{x_j, \xi_j}([0, t_0]) \subset W_{\widehat{g}}$, $x_j(t_0) \notin J_{\widehat{g}}^+(x_k(t_0))$, for $j, k \leq 4$, $j \neq k$,
- (ii) For all $j, k \leq 4$, $d_{\widehat{g}^+}((x_j, \xi_j), (x_k, \xi_k)) < \vartheta_1$,
- (iii) There is $\widehat{x} \in \widehat{\mu}$ such that for all $j \leq 4$, $d_{\widehat{g}^+}(\widehat{x}, x_j) < \vartheta_1$,

Above, $(x_j(h), \xi_j(h))$ are defined in (20). We denote

$$(76) \quad \mathcal{V}((\vec{x}, \vec{\xi}), t_0) = M_0 \setminus \bigcup_{j=1}^4 J_{\widehat{g}}^+(\gamma_{x_j(t_0), \xi_j(t_0)}(\mathbf{t}_j)),$$

where $\mathbf{t}_j := \rho(x_j(t_0), \xi_j(t_0))$. Note that two geodesics $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$ can intersect only once in $\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$. We will analyze the 4-wave interaction of waves in the set $\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ where all observed singularities are produced before the geodesics $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$ have conjugate points, that is, before the waves $u_{(j)}$ have caustics. Below we use \sim for the terms that have the same asymptotics up to an error $O(\tau^{-N})$ for all $N > 0$, as $\tau \rightarrow \infty$.

Proposition 3.3. Let $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ be future pointing light-like vectors satisfying (75) and $t_0 > 0$. Let $x_5 \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_{\widehat{g}}$ and (x_5, ξ_5) be a future pointing light-like vector such that $x_5 \notin \mathcal{V}((\vec{x}, \vec{\xi}); t_0) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$, see (66) and (76). When $n \in \mathbb{Z}_+$ is large enough and $s_0 > 0$ is small enough, the function $\Theta_\tau^{(4)}$, see (70), corresponding to $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$, $j \leq 4$, and the source F_τ , see (67), corresponding to (x_5, ξ_5) satisfy

- (i) If the geodesics corresponding to $(\vec{x}, \vec{\xi})$ either do not intersect or intersect at q and $(x_5, \xi_5) \notin \Lambda_q^+$, then $|\Theta_\tau^{(4)}| \leq C_N \tau^{-N}$ for all $N > 0$.

(ii) If the geodesics corresponding to $(\vec{x}, \vec{\xi})$ intersect at q , $q = \gamma_{x_j, \xi_j}(t_j)$ and the vectors $\dot{\gamma}_{x_j, \xi_j}(t_j)$, $j = 1, 2, 3, 4$ are linearly independent, and there are $t_5 < 0$ and $\xi_5 \in L_{x_5}^+ M_0$ such that $q = \gamma_{x_5, \xi_5}(t_5)$, then, with $m = -4n + 2$,

$$(77) \quad \Theta_\tau^{(4)} \sim \sum_{k=m}^{\infty} s_k \tau^{-k}, \quad \text{as } \tau \rightarrow \infty.$$

Moreover, let $b_j = (\dot{\gamma}_{x_j, \xi_j}(t_j))^\flat$, $j = 1, 2, 3, 4, 5$, and $\mathbf{b} = (b_j)_{j=1}^5 \in (T_q^* M_0)^5$. Let w_j be the principal symbols of the waves $u_j = \mathbf{Q}f_j$ at (q, b_j) for $j \leq 4$. Also, let w_5 be the principal symbol of $u_\tau = \mathbf{Q}F_\tau$ at (q, b_5) , and $\mathbf{w} = (w_j)_{j=1}^5$. Then there is a real-analytic function $\mathcal{G}(\mathbf{b}, \mathbf{w})$ such that the leading order term in (77) satisfies

$$(78) \quad s_m = \mathcal{G}(\mathbf{b}, \mathbf{w}).$$

(iii) Under the assumptions in (ii), the point x_5 has a neighborhood V such that \mathcal{M}^4 in V satisfies $\mathcal{M}^4|_V \in \mathcal{I}(\Lambda_q^+)$.

Proof. Below, to simplify notations, we denote $K_j = K(x_j, \xi_j; t_0, s_0)$ and $K_{123} = K_1 \cap K_2 \cap K_3$ and $K_{124} = K_1 \cap K_2 \cap K_4$, etc. We will denote $\Lambda_j = \Lambda(x_j, \xi_j; t_0, s_0)$ to consider also the singularities of K_j related to conjugate points.

We will consider below separately the case when the following linear independency condition is satisfied:

(LI) Assume if that $J \subset \{1, 2, 3, 4\}$ and $y \in J^-(x_6)$ are such that for all $j \in J$ we have $\gamma_{x_j, \xi_j}(t'_j) = y$ with some $t'_j \geq 0$, then the vectors $\dot{\gamma}_{x_j, \xi_j}(t'_j)$, $j \in J$ are linearly independent.

Let us first consider the case when (LI) is valid.

By the definition of \mathbf{t}_j , if the intersection $\gamma_{x_5, \xi_5}(\mathbb{R}_-) \cap (\cap_{j=1}^4 \gamma_{x_j, \xi_j}((0, \mathbf{t}_j)))$ is non-empty, it can contain only one point. In the case that such a point exists, we denote it by q . When q exists, the intersection of K_j at this point are transversal and we see that when s_0 is small enough, the set $\cap_{j=1}^4 K_j$ consists only of the point q . We assume below that s_0 is such that this is true.

We define two local coordinates $Z : W_0 \rightarrow \mathbb{R}^4$ and $Y : W_1 \rightarrow \mathbb{R}^4$ such that $W_0, W_1 \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$, see definition after (76). We assume that these local coordinates are such that $K_j \cap W_0 = \{x \in W_0; Z^j(x) = 0\}$ and $K_j \cap W_1 = \{x \in W_1; Y^j(x) = 0\}$ for $j = 1, 2, 3, 4$. In the Fig. 5, W_0 is a neighborhood of z and W_1 is a neighborhood of y . We note that the origin $0 = (0, 0, 0, 0) \in \mathbb{R}^4$ does not necessarily belong to the set $Z(W_0)$ or the set $Y(W_1)$, for instance in the case when the four geodesics γ_{x_j, ξ_j} do not intersect. However, this is the case when the four geodesics intersect at the point q , we need to consider the case when W_0 and W_1 are neighborhoods of 0. Note that W_0 and W_1 do not contain any cut points of the geodesic $\gamma_{x_j, \xi_j}([t_0, \infty))$.

We will denote $z^j = Z^j(x)$. We assume that $Y : W_1 \rightarrow \mathbb{R}^4$ are similar coordinates and denote $y^j = Y^j(x)$. We also denote below $dy^j = dY^j$ and $dz^j = dZ^j$. Let $\Phi_0 \in C_0^\infty(W_0)$ and $\Phi_1 \in C_0^\infty(W_1)$.

Let us next consider the map $\mathbf{Q}^* : C_0^\infty(W_1) \rightarrow C_0^\infty(W_0)$. By [83], $\mathbf{Q}^* \in I(W_1 \times W_0; \Delta'_{TM_0}, \Lambda_{\widehat{g}})$ is an operator with a classical symbol and its canonical relation $\Lambda'_{\mathbf{Q}^*}$ is associated to a union of two intersecting Lagrangian manifolds, $\Lambda'_{\mathbf{Q}^*} = \Lambda'_{\widehat{g}} \cup \Delta_{TM_0}$, intersecting cleanly [83]. Let $\varepsilon_2 > \varepsilon_1 > 0$ and $B_{\varepsilon_1, \varepsilon_2}$ be a pseudodifferential operator on M_0 which is a microlocally smoothing operator (i.e., the full symbol vanishes in local coordinates) outside of the ε_2 -neighborhood $\mathcal{V}_2 \subset T^*M_0$ of the set of the light like covectors L^*M_0 and for which $(I - B_{\varepsilon_1, \varepsilon_2})$ is microlocally smoothing operator in the ε_1 -neighborhood \mathcal{V}_2 of L^*M_0 . The neighborhoods here are defined with respect to the Sasaki metric of (T^*M_0, \widehat{g}^+) and $\varepsilon_2, \varepsilon_1$ are chosen later in the proof. Let us decompose the operator $\mathbf{Q}^* = \mathbf{Q}_1^* + \mathbf{Q}_2^*$ where $\mathbf{Q}_1^* = \mathbf{Q}^*(I - B_{\varepsilon_1, \varepsilon_2})$ and $\mathbf{Q}_2^* = \mathbf{Q}^* B_{\varepsilon_1, \varepsilon_2}$. As $\Lambda_{\mathbf{Q}^*} = \Lambda_{\widehat{g}} \cup \Delta'_{TM_0}$, we see that then there is a neighborhood $\mathcal{W}_2 = \mathcal{W}_2(\varepsilon_2)$ of $L^*M_0 \times L^*M_0 \subset (T^*M_0)^2$ such that the Schwartz kernel $\mathbf{Q}_2^*(r, y)$ of the operator \mathbf{Q}_2^* satisfies

$$(79) \quad \text{WF}(\mathbf{Q}_2^*) \subset \mathcal{W}_2.$$

Moreover, $\Lambda_{\mathbf{Q}_1^*} \subset \Delta'_{TM_0}$ implying that \mathbf{Q}_1^* is a pseudodifferential operator with a classical symbol, $\mathbf{Q}_1^* \in I(W_1 \times W_0; \Delta'_{TM_0})$, and $\mathbf{Q}_2^* \in I(W_1 \times W_0; \Delta'_{TM_0}, \Lambda_{\widehat{g}})$ is a Fourier integral operator (FIO) associated to two cleanly intersecting Lagrangian manifolds, similarly to \mathbf{Q}^* .

In the case when $p = 1$ we can write \mathbf{Q}_p^* as

$$(80) \quad (\mathbf{Q}_1^* v)(z) = \int_{\mathbb{R}^{4+4}} e^{i(y-z)\cdot\xi} q_1(z, y, \xi) v(y) dy d\xi,$$

where a classical symbol $q_1(z, y, \xi) \in S^{-2}(W_1 \times W_0; \mathbb{R}^4)$ with a real valued principal symbol

$$\widehat{q}_1(z, y, \xi) = \frac{\chi(z, \xi)}{\widehat{g}^{jk}(z) \xi_j \xi_k}$$

where $\chi(z, \xi) \in \mathbb{C}^\infty$ is a cut-off function vanishing in a neighborhood of the set where $g(\xi, \xi) = 0$. Note that then $Q_1 - Q_1^* \in \Psi^{-3}(W_1 \times W_0)$.

Furthermore, let us decompose $\mathbf{Q}_1^* = \mathbf{Q}_{1,1}^* + \mathbf{Q}_{1,2}^*$ corresponding to the decomposition $q_1(z, y, \xi) = q_{1,1}(z, y, \xi) + q_{1,2}(z, y, \xi)$ of the symbol, where

$$(81) \quad \begin{aligned} q_{1,1}(z, y, \xi) &= q_1(z, y, \xi) \psi_R(\xi), \\ q_{1,2}(z, y, \xi) &= q_1(z, y, \xi) (1 - \psi_R(\xi)), \end{aligned}$$

and $\psi_R \in C_0^\infty(\mathbb{R}^4)$ is a cut-off function that is equal to one in a ball $B(R)$ of radius R specified below.

Next we start to consider the terms $T_\tau^{(4),\beta}$ and $\widetilde{T}_\tau^{(4),\beta}$ of the type (71) and (72). In these terms, we can represent the gaussian beam $u_\tau(z)$ in

W_1 in the form

$$(82) \quad u_\tau(y) = e^{i\tau\varphi(y)} a_5(y, \tau)$$

where the function φ is a complex phase function having non-negative imaginary part such that $\text{Im } \varphi$, defined on W_1 , vanishes exactly on the geodesic $\gamma_{x_5, \xi_5} \cap W_1$. Note that $\gamma_{x_5, \xi_5} \cap W_1$ may be empty. Moreover, $a_5 \in S_{\text{clas}}^0(W_1; \mathbb{R})$ is a classical symbol.

Also, for $y = \gamma_{x_5, \xi_5}(t) \in W_1$ we have that $d\varphi(y) = c\dot{\gamma}_{x_5, \xi_5}(t)^\flat$, where $c \in \mathbb{R} \setminus \{0\}$, is light-like.

We consider first the asymptotics of terms $T_\tau^{(4),\beta}$ and $\tilde{T}_\tau^{(4),\beta}$ of the type (71) and (72) where $S_1 = S_2 = \mathbf{Q}$ and, the symbols $a_j(z, \theta_j)$, $j \leq 3$ and $a_j(y, \theta_j)$, $j \in \{4, 5\}$ are scalar valued symbols written in the Z and Y coordinates. $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are multiplication operators with Φ_0 , and \mathcal{B}_4 is a multiplication operators with Φ_1 and consider section-valued symbols and general operators later.

Let us consider functions $U_j \in \mathcal{I}^{p_j}(K_j)$, $j = 1, 2, 3$, supported in W_0 and $U_4 \in \mathcal{I}(K_4)$, supported in W_1 , having classical symbols. They have the form

$$(83) \quad U_j(x) = \int_{\mathbb{R}} e^{i\theta_j x^j} a_j(x, \theta_j) d\theta_j, \quad a_j \in S_{\text{clas}}^{p_j}(W_{\kappa(j)}; \mathbb{R}),$$

for all $j = 1, 2, 3, 4$ (Note that here the phase function is $\theta_j x^j = \theta_1 x^1$ for $j = 1$ etc, that is, there is no summing over index j). We may assume that $a_j(x, \theta_j)$ vanish near $\theta_j = 0$.

Since $x_5 \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_{\widehat{g}}$ and $W_0, W_1 \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$, we see that an example of functions (83) are $U_j(z) = \Phi_{\kappa(j)}(z) u_j(z)$, $j = 1, 2, 3, 4$, where $u_j(z)$ are the distorted plane waves. Here and below, $\kappa(j) = 0$ for $j = 1, 2, 3$ and $\kappa(4) = 1$ and we also denote $\kappa(5) = 1$. Note that $p_j = n$ correspond to the case when $U_j \in \mathcal{I}^n(K_j) = \mathcal{I}^{n-1/2}(N^* K_j)$.

Denote $\Lambda_j = N^* K_j$ and $\Lambda_{jk} = N^*(K_j \cap K_k)$. By [44, Lem. 1.2 and 1.3], the pointwise product $U_2 \cdot U_1 \in \mathcal{I}(\Lambda_1, \Lambda_{12}) + \mathcal{I}(\Lambda_2, \Lambda_{12})$ and thus by [44, Prop. 2.2], $\mathbf{Q}(U_2 \cdot U_1) \in \mathcal{I}(\Lambda_1, \Lambda_{12}) + \mathcal{I}(\Lambda_2, \Lambda_{12})$ and it can be written as

$$(84) \quad \mathbf{Q}(U_2 \cdot U_1) = \int_{\mathbb{R}^2} e^{i(\theta_1 z^1 + \theta_2 z^2)} c_1(z, \theta_1, \theta_2) d\theta_1 d\theta_2.$$

Note that here $c_1(z, \theta_1, \theta_2)$ is sum of product type symbols, see (36). As $N^*(K_1 \cap K_2) \setminus (N^* K_1 \cup N^* K_2)$ consists of vectors which are non-characteristic for the wave operator, that is, the wave operator is elliptic in a neighborhood of this subset of the cotangent bundle, the principal symbol \widehat{c}_1 of $\mathbf{Q}(U_2 \cdot U_1)$ on $N^*(K_1 \cap K_2) \setminus (N^* K_1 \cup N^* K_2)$ is given by

$$(85) \quad \widehat{c}_1(z, \theta_1, \theta_2) \sim s(z, \theta_1, \theta_2) a_1(z, \theta_1) a_2(z, \theta_2),$$

$$s(z, \theta_1, \theta_2) = 1/\widehat{g}(\theta_1 b^{(1)} + \theta_2 b^{(2)}, \theta_1 b^{(1)} + \theta_2 b^{(2)}) = 1/(2\widehat{g}(\theta_1 b^{(1)}, \theta_2 b^{(2)})).$$

Note that $s(z, \theta_1, \theta_2)$ is a smooth function on $N^*(K_1 \cap K_2) \setminus (N^* K_1 \cup N^* K_2)$ and homogeneous of order (-2) in $\theta = (\theta_1, \theta_2)$. Here, we use

\sim to denote that the symbols have the same principal symbol. Let us next make computations in the case when $a_j(z, \theta_j) \in C^\infty(\mathbb{R}^4 \times \mathbb{R})$ is positively homogeneous for $|\theta_j| > 1$, that is, we have $a_j(z, s) = a'_j(z)s^{p_j}$, where $p_j \in \mathbb{N}$ and $|s| > 1$. We consider $T_\tau^{(4),\beta} = \sum_{p=1}^2 T_{\tau,p}^{(4),\beta}$ where $T_{\tau,p}^{(4),\beta}$ is defined as $T_\tau^{(4),\beta}$ by replacing the term $\mathbf{Q}_p^*(U_4 \cdot u_\tau)$ by $\mathbf{Q}_p^*(U_4 \cdot u_\tau)$.

Let us now consider the case $p = 2$ and choose the parameters ε_1 and ε_2 that determine the decomposition $\mathbf{Q}^* = \mathbf{Q}_1^* + \mathbf{Q}_2^*$. First, we observe that for $p = 2$ we can write using Z and Y coordinates

$$(86) \quad \begin{aligned} T_{\tau,2}^{(4),\beta} &= \tau^4 \int_{\mathbb{R}^{12}} e^{i\tau\Psi_2(z,y,\theta)} c_1(z, \tau\theta_1, \tau\theta_2) \cdot \\ &\quad \cdot a_3(z, \tau\theta_3) \mathbf{Q}_2^*(z, y) a_4(y, \tau\theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz, \\ \Psi_2(z, y, \theta) &= \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + \theta_4 y^4 + \varphi(y). \end{aligned}$$

Denote $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$. Consider the case when (z, y, θ) is a critical point of Ψ_2 satisfying $\text{Im } \varphi(y) = 0$. Then we have $\theta' = (\theta_1, \theta_2, \theta_3) = 0$ and $z' = (z^1, z^2, z^3) = 0$, $y^4 = 0$, $d_y \varphi(y) = (0, 0, 0, -\theta_4)$, implying that $y \in K_4$ and $(y, d_y \varphi(y)) \in N^* K_4$. As $\text{Im } \varphi(y) = 0$, we have that $y = \gamma_{x_5, \xi_5}(s)$ with some $s \in \mathbb{R}_-$. As we have $\dot{\gamma}_{x_5, \xi_5}(s)^\flat = d_y \varphi(y) \in N_y^* K_4$, we obtain $\gamma_{x_5, \xi_5}([s, 0]) \subset K_4$. However, this is not possible by our assumption $x_5 \notin \cup_{j=1}^4 K(x_j, \xi_j; s_0)$ when s_0 is small enough. Thus the phase function $\Psi_2(z, y, \theta)$ has no critical points satisfying $\text{Im } \varphi(y) = 0$.

When the orders p_j of the symbols a_j are small enough, the integrals in the θ variable in (86) are convergent in the classical sense. We use now properties of the wave front set to compute the asymptotics of oscillatory integrals and to this end we introduce the function

$$(87) \quad \tilde{\mathbf{Q}}_2^*(z, y, \theta) = \mathbf{Q}_2^*(z, y),$$

that is, consider $\mathbf{Q}_2^*(z, y)$ as a constant function in θ . Below, denote $\psi_4(y, \theta_4) = \theta_4 y^4$ and $r = d\varphi(y)$. Note that then $d_{\theta_4} \psi_4 = y^4$ and $d_y \psi_4 = (0, 0, 0, \theta_4)$. Then in $W_1 \times W_0 \times \mathbb{R}$

$$\begin{aligned} d_{z,y,\theta} \Psi_2 &= (\theta_1, \theta_2, \theta_3, 0; r + d_y \psi_4(y, \theta_4), z^1, z^2, z^3, d_{\theta_4} \psi_4(y, \theta_4)) \\ &= (\theta_1, \theta_2, \theta_3, 0; d\varphi(y) + (0, 0, 0, \theta_4), z^1, z^2, z^3, y^4) \end{aligned}$$

and we see that if $((z, y, \theta), d_{z,y,\theta} \Psi_2) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$ and $\text{Im } \varphi(y) = 0$, we have $(z^1, z^2, z^3) = 0$, $y^4 = d_{\theta_4} \psi_4(y, \theta_4) = 0$ and $y \in \gamma_{x_5, \xi_5}$. Thus $z \in K_{123}$ and $y \in \gamma_{x_5, \xi_5} \cap K_4$.

Let us use the following notations

$$(88) \quad \begin{aligned} z \in K_{123}, \quad \omega_\theta &:= (\theta_1, \theta_2, \theta_3, 0) = \sum_{j=1}^3 \theta_j dz^j \in T_z^* M_0, \\ y \in K_4 \cap \gamma_{x_5, \xi_5}, \quad (y, w) &:= (y, d_y \psi_4(y, \theta_4)) \in N^* K_4, \\ r = d\varphi(y) &= r_j dy^j \in T_y^* M_0, \quad \kappa := r + w. \end{aligned}$$

Then, y and θ_4 satisfy $y^4 = d_{\theta_4}\psi_4(y, \theta_4) = 0$ and $w = (0, 0, 0, \theta_4)$.

Note that by definition of the Y coordinates w is a light-like covector. By definition of the Z coordinates, $\omega_\theta \in N^*K_1 + N^*K_2 + N^*K_3 = N^*K_{123}$.

Let us first consider what happens if $\kappa = r + w = d\varphi(y) + (0, 0, 0, \theta_4)$ is light-like. In this case, all vectors κ , w , and r are light-like and satisfy $\kappa = r + w$. This is possible only if $r \parallel w$, i.e., r and w are parallel, see [95, Cor. 1.1.5]. Thus $r + w$ is light-like if and only if r and w are parallel.

Consider next the case when $(x, y, \theta) \in W_1 \times W_0 \times \mathbb{R}^4$ is such that $((x, y, \theta), d_{z,y,\theta}\Psi_2) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$ and $\text{Im } \varphi(y) = 0$. Using the above notations (88), we obtain $d_{z,y,\theta}\Psi_2 = (\omega_\theta, r + w; (0, 0, 0, d_{\theta_4}\psi_4(y, \theta_4))) = (\omega_\theta, d\varphi(y) + (0, 0, 0, \theta_4); (0, 0, 0, y^4))$, where $y^4 = d_{\theta_4}\psi_4(y, \theta_4) = 0$, and thus we have

$$((z, \omega_\theta), (y, r + w)) \in \text{WF}(\mathbf{Q}_2^*) = \Lambda_{\mathbf{Q}_2^*}.$$

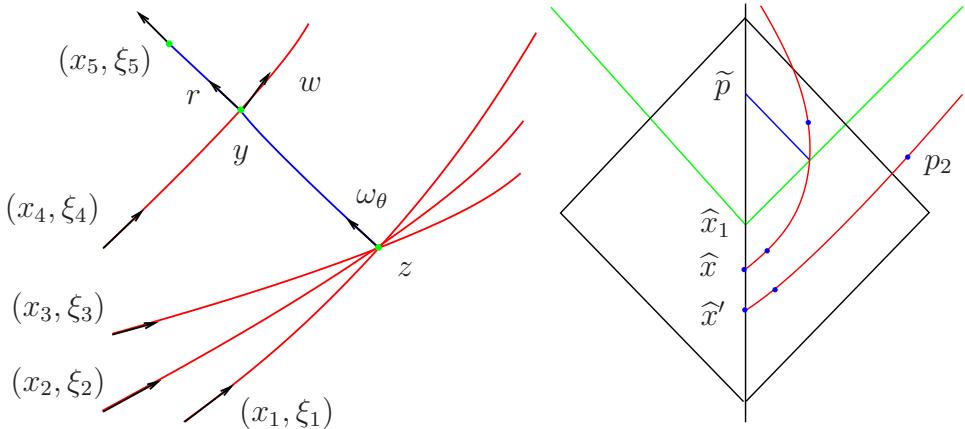


FIGURE 5. Left: In the figure we consider the case A1 where three geodesics intersect at z and the waves propagating near these geodesics interact and create a wave that hits the fourth geodesic at the point y , the produced singularities are detected by the gaussian beam source at the point x_5 . Note that z and y can be conjugate points on the geodesic connecting them. In the case A2 the points y and z are the same.
Right: Condition I is valid.

Since $\Lambda_{\mathbf{Q}_2^*} \subset \Lambda_{\tilde{g}} \cup \Delta'_{TM_0}$, this implies that one of the following conditions are valid:

$$(A1) ((z, \omega_\theta), (y, r + w)) \in \Lambda_{\tilde{g}},$$

or

$$(A2) ((z, \omega_\theta), (y, r + w)) \in \Delta'_{TM_0}.$$

Let γ_0 be the geodesic with $\gamma_0(0) = z$, $\dot{\gamma}_0(0) = \omega_\theta^\sharp$. Then (A1) and (A2) are equivalent to the following conditions:

(A1) There is $s \in \mathbb{R}$ such that $(\gamma_0(s), \dot{\gamma}_0(s)^\flat) = (y, \kappa)$ and,
the vector κ is light-like,

or

(A2) $z = y$ and $\kappa = -\omega_\theta$.

Consider next the case when (A1) is valid. As κ is light-like, r and w are parallel. Then, since $(\gamma_{x_5, \xi_5}(t_1), \dot{\gamma}_{x_5, \xi_5}(t_1)^\flat) = (y, r)$ we see that γ_0 is a continuation of the geodesic γ_{x_5, ξ_5} , that is, for some t_2 we have $(\gamma_{x_5, \xi_5}(t_2), \dot{\gamma}_{x_5, \xi_5}(t_2)) = (z, \omega_\theta) \in N^*K_{123}$. This implies that $x_5 \in \mathcal{Y}$ that is not possible by our assumptions. Hence (A1) is not possible.

Consider next the case when (A2) is valid. Then we would also have that $r \parallel w$ then r is parallel to $\kappa = -\omega_\theta \in N^*K_{123}$, and since $(\gamma_{x_5, \xi_5}(t_1), \dot{\gamma}_{x_5, \xi_5}(t_1)^\flat) = (y, r)$ we would have that $x_5 \in \mathcal{Y}$. As this is not possible by our assumptions, we see that r and w are not parallel. This implies that $\omega_\theta = -\kappa$ is not light-like.

For any given $(\vec{x}, \vec{\xi})$ and (x_5, ξ_5) there exists $\varepsilon_2 > 0$ so that $(\{(y, \omega_\theta)\} \times T^*M) \cap \mathcal{W}_2(\varepsilon_2) = \emptyset$, see (79), and thus

$$(89) \quad (\{(y, \omega_\theta)\} \times T^*M) \cap \text{WF}(\mathbf{Q}_2^*) = \emptyset.$$

Next we assume that $\varepsilon_2 > 0$ and also $\varepsilon_1 \in (0, \varepsilon_2)$ are chosen so that (89) is valid. Then there are no (z, y, θ) such that $((z, y, \theta), d\Psi_2(z, y, \theta)) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$ and $\text{Im } \varphi(y) = 0$. Thus by Corollary 1.4 in [28] or [90, Lem. 4.1] yields $T_{\tau, 2}^{(4), \beta} = O(\tau^{-N})$ for all $N > 0$. Alternatively, one can use the complex version of [32, Prop. 1.3.2], obtained using combining the proof of [32, Prop. 1.3.2] and the method of stationary phase with a complex phase, see [53, Thm. 7.7.17].

Thus to analyze the asymptotics of $T_\tau^{(4), \beta}$ we need to consider only $T_{\tau, 1}^{(4), \beta}$. Next, we analyze the case when U_4 is a conormal distribution and has the form (83).

Let us thus consider the case $p = 1$. Now

$$(90) \quad U_4(y) \cdot u_\tau(y) = \int_{\mathbb{R}^1} e^{i\theta_4 y^4 + i\tau\varphi(y)} a_4(y, \theta_4) a_5(y, \tau) d\theta_4.$$

We obtain by (80)

$$(91) \quad \begin{aligned} (\mathbf{Q}_1^*(U_4 \cdot u_\tau))(z) &= \int_{\mathbb{R}^9} e^{i((y-z) \cdot \xi + \theta_4 y^4 + \tau\varphi(y))} q_1(z, y, \xi) \cdot \\ &\quad \cdot a_4(y, \theta_4) a_5(y, \tau) d\theta_4 dy d\xi. \end{aligned}$$

Then $T_{\tau, 1}^{(4), \beta} = T_{\tau, 1, 1}^{(4), \beta} + T_{\tau, 1, 2}^{(4), \beta}$, cf. (81), where

$$(92) \quad \begin{aligned} T_{\tau, 1, k}^{(4), \beta} &= \int_{\mathbb{R}^{16}} e^{i(\theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y-z) \cdot \xi + \theta_4 y^4 + \tau\varphi(y))} c_1(z, \theta_1, \theta_2) \cdot \\ &\quad \cdot a_3(z, \theta_3) q_{1,k}(z, y, \xi) a_4(y, \theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz d\xi, \end{aligned}$$

or

$$(93) \quad T_{\tau,1,k}^{(4),\beta} = \tau^8 \int_{\mathbb{R}^{16}} e^{i\tau(\theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y-z)\cdot\xi + \theta_4 y^4 + \varphi(y))} c_1(z, \tau\theta_1, \tau\theta_2) \cdot a_3(z, \tau\theta_3) q_{1,k}(z, y, \tau\xi) a_4(y, \tau\theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz d\xi.$$

Let $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$ be a critical point of the phase function

$$(94) \quad \Psi_3(z, \theta, y, \xi) = \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y - z) \cdot \xi + \theta_4 y^4 + \varphi(y).$$

Denote $\bar{w} = (0, 0, 0, \bar{\theta}_4)$ and $\bar{r} = d\varphi(\bar{y}) = \bar{r}_j dy^j$. Then

$$(95) \quad \begin{aligned} \partial_{\theta_j} \Psi_3 &= 0, \quad j = 1, 2, 3 \quad \text{yield} \quad \bar{z} \in K_{123}, \\ \partial_{\theta_4} \Psi_3 &= 0 \quad \text{yields} \quad \bar{y} \in K_4, \\ \partial_z \Psi_3 &= 0 \quad \text{yields} \quad \bar{\xi} = \omega_{\bar{\theta}}, \\ \partial_{\xi} \Psi_3 &= 0 \quad \text{yields} \quad \bar{y} = \bar{z}, \\ \partial_y \Psi_3 &= 0 \quad \text{yields} \quad \bar{\xi} = -d\varphi(\bar{y}) - \bar{w}. \end{aligned}$$

The critical points we need to consider for the asymptotics satisfy also

(96)

$$\operatorname{Im} \varphi(\bar{y}) = 0, \quad \text{so that } \bar{y} \in \gamma_{x_5, \xi_5}, \quad \operatorname{Im} d\varphi(\bar{y}) = 0, \quad \operatorname{Re} d\varphi(\bar{y}) \in L_{\bar{y}}^{*,+} M_0.$$

Next we analyze the terms $T_{\tau,1,k}^{(4),\beta}$ starting with $k = 2$. Observe that the 3rd and 5th equations in (95) imply that at the critical points $\xi = (\partial_{y_1} \varphi(y), \partial_{y_2} \varphi(y), \partial_{y_3} \varphi(y), 0)$. Thus the critical points are bounded in the ξ variable. Let us now fix the parameter R determining $\psi_R(\xi)$ in (81) so that ξ -components of the critical points are in a ball $B(R) \subset \mathbb{R}^4$. Using the identity $e^{\Psi_3} = |\xi|^{-2} (\nabla_z - \omega_{\theta})^2 e^{\Psi_3}$ where $\omega_{\theta} = (\theta_1, \theta_1, \theta_1, 0)$ we can include the operator $|\xi|^{-2} (\nabla_z - \omega_{\theta})^2$ in the integral (93) with $k = 2$ and integrate by parts. Doing this two times we can show that this oscillatory integral (93) with $k = 2$ becomes an integral of a Lebesgue-integrable function. Then, by using method of stationary phase and the fact that $\psi_R(\xi)$ vanishes at all critical points of Ψ_3 where $\operatorname{Im} \Psi_3$ vanishes, we see that $T_{\tau,1,2}^{(4),\beta} = O(\tau^{-n})$ for all $n > 0$.

Above, we have shown that the term $T_{\tau,1,1}^{(4),\beta}$ has the same asymptotics as $T_{\tau}^{(4),\beta}$. Next we analyze this term. Let $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$ be a critical point of $\Psi_3(z, \theta, y, \xi)$ such that y satisfies (96). Let us next use the same notations (88) which we used above. Then (95) and (96) imply

$$(97) \quad \bar{z} = \bar{y} \in \gamma_{x_5, \xi_5} \cap \bigcap_{j=1}^4 K_j, \quad \bar{\xi} = \omega_{\bar{\theta}} = -\bar{r} - \bar{w}.$$

Note that in this case all the four geodesics γ_{x_j, ξ_j} intersect at the point q and by our assumptions, $\bar{r} = d\varphi(\bar{y})$ is such a co-vector that in the Y -coordinates $\bar{r} = (\bar{r}_j)_{j=1}^4$ with $\bar{r}_j \neq 0$ for all $j = 1, 2, 3, 4$. In particular, this shows that the existence of the critical point of $\Psi_3(z, \theta, y, \xi)$ implies

that there exists an intersection point of γ_{x_5, ξ_5} and $\bigcap_{j=1}^4 K_j$. Equations (97) imply also that

$$\bar{r} = \sum_{j=1}^4 \bar{r}_j dy^j = -\omega_{\bar{\theta}} - \bar{w} = -\sum_{j=1}^3 \bar{\theta}_j dz^j - \bar{\theta}_4 dy^4.$$

To consider the case when $\bar{y} = \bar{z}$, let us assume for a while that that $W_0 = W_1$ and that the Y -coordinates and Z -coordinates coincide, that is, $Y(x) = Z(x)$. Then the covectors $dz^j = dZ^j$ and $dy^j = dY^j$ coincide for $j = 1, 2, 3, 4$. Then we have

$$(98) \quad \bar{r}_j = -\bar{\theta}_j, \quad \text{i.e., } \bar{\theta} := \bar{\theta}_j dz^j = -\bar{r} = \bar{r}_j dy^j \in T_{\bar{y}}^* M_0.$$

Let us apply the method of stationary phase to $T_{\tau, 1, 1}^{(4), \beta}$ as $\tau \rightarrow \infty$. Note that as $c_1(z, \theta_1, \theta_2)$ is a product type symbol, we need to use the fact $\theta_1 \neq 0$ and $\theta_2 \neq 0$ for the critical points as we have by (98) and the fact that $\bar{r} = d\varphi(y)|_{\bar{y}} \notin N^* K_{234} \cup N^* K_{134}$ as $x_5 \notin \mathcal{Y}$ and assuming that s_0 is small enough.

In local Y and Z coordinates where $\bar{z} = \bar{y} = (0, 0, 0, 0)$ we can use the method of stationary phase, similarly to the proofs of [47, Thm. 1.11 and 3.4], to compute the asymptotics of (93) with $k = 1$. Let us explain this computation in detail. To this end, let us start with some preliminary considerations.

Let $\phi_1(z, y, \theta, \xi)$ be a smooth bounded function in $C^\infty(W_0 \times W_1 \times \mathbb{R}^4 \times \mathbb{R}^4)$ that is homogeneous of degree zero in the (θ, ξ) variables in the set $\{|\theta, \xi| > R_0\}$ with some $R_0 > 0$. Assume that $\phi_1(z, y, \theta, \xi)$ is equal to one in a conic neighborhood, with respect to (θ, ξ) , of the points where some of the θ_j or ξ_k variable is zero. Note that in this set the positively homogeneous functions $a_j(z, \theta_j/|\theta_j|)$ may be non-smooth. Let $\Sigma \subset W_0 \times W_1 \times \mathbb{R}^4 \times \mathbb{R}^4$ be a conic neighborhood of the critical points of the phase function $\Psi_3(y, z, \theta, \xi)$. Also, assume that the function $\phi_1(y, z, \theta, \xi)$ vanishes in the intersection of Σ and the set $\{|\theta, \xi| > R_0\}$. Let

$$\Psi_{(\tau)}(z, y, \theta, \xi) = \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y - z) \cdot \xi + \theta_4 y^4 + \tau \varphi(y)$$

be the phase function appearing in (92). Note that Σ contains the critical points of $\Psi_{(\tau)}$ for all $\tau > 0$. Let

$$L_\tau = \frac{\phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi)}{|d_{z,y,\theta,\xi}\Psi_{(\tau)}(z, y, \theta, \xi)|^2} (d_{z,y,\theta,\xi}\overline{\Psi_{(\tau)}(z, y, \theta, \xi)}) \cdot d_{z,y,\theta,\xi},$$

so that

$$(99) \quad L_\tau \exp(\Psi_{(\tau)}) = \phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi) \exp(\Psi_{(\tau)}).$$

Note that if $\text{Im } \varphi(y) = 0$, then $y \in \gamma_{x_5, \xi_5}$ and hence $d\varphi(y)$ does not vanish and that when τ is large enough, the function $\Psi_{(\tau)}$ has no critical points in the support of ϕ_1 . Using these we see that $\gamma_{x_5, \xi_5} \cap W_1$ has a neighborhood $V_1 \subset W_1$ where $|d\varphi(y)| > C_0 > 0$ and there are

$C_1, C_2, C_3 > 0$ so that if $\tau > C_1$, $y \in V_1$, and $(z, y, \theta, \xi) \in \text{supp}(\phi_1)$ then

$$(100) \quad |d_{z,y,\theta,\xi}\Psi_{(\tau)}(z, y, \theta, \xi)|^{-1} \leq \frac{C_2}{\tau - C_3}.$$

After these preparatory steps, we are ready to compute the asymptotics of $T_{\tau,1,1}^{(4),\beta}$. To this end, we first transform the integrals in (93) to an integral of a Lebesgue integrable function by using integration by parts of $|\xi|^{-2}(\nabla_z - \omega_\theta)^2$ as explained above. Then we decompose $T_{\tau,1,1}^{(4),\beta}$ into three terms $T_{\tau,1,1}^{(4),\beta} = I_1 + I_2 + I_3$. To obtain the first term I_1 we include the factor $(1 - \phi_1(z, y, \theta, \xi))$ in the integral (93) with $k = 1$. The integral I_1 can then be computed using the method of stationary phase similarly to the proof of [47, Thm. 3.4]. Let $\chi_1 \in C^\infty(W_1)$ be a function that is supported in V_1 and vanishes on γ_{x_5, ξ_5} . The terms I_2 and I_3 are obtained by including the factor $\phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi)\chi_1(y)$ and $\phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi)(1 - \chi_1(y))$ in the integral (92) with $k = 1$, respectively. (Equivalently, the terms I_2 and I_3 are obtained by including the factor $\phi_1(z, y, \theta, \xi)\chi_1(y)$ and $\phi_1(z, y, \theta, \xi)(1 - \chi_1(y))$ in the integral (93) with $k = 1$, respectively.) Using integration by parts in the integral (92) and inequalities (99) and (100), we see that that $I_2 = O(\tau^{-N})$ for all $N > 0$. Moreover, the fact that $\text{Im } \varphi(y) > c_1 > 0$ in $W_1 \cap V_1$ implies that $I_3 = O(\tau^{-N})$ for all $N > 0$.

Combining the above we obtain the asymptotics

$$(101) \quad T_\tau^{(4),\beta} \sim \tau^{4+4-16/2-2+\rho-2} \sum_{k=0}^{\infty} c_k \tau^{-k} = \tau^{-4+\rho} \sum_{k=0}^{\infty} c_k \tau^{-k},$$

$$c_0 = h(\bar{z}) c_1(0, -\bar{r}_1, -\bar{r}_2) \hat{a}_3(0, -\bar{r}_3) \hat{q}_1(0, 0, -(\bar{r}_1, \bar{r}_2, \bar{r}_3, 0)) \hat{a}_4(0, -\bar{r}_4) \hat{a}_5(0, 1),$$

where $\rho = \sum_{j=1}^5 p_j$ and \hat{a}_j is the principal symbol of U_j etc. The factor $h(\bar{z})$ is non-vanishing and is determined by the determinant of the Hessian of the phase function φ at q . A direct computation shows that $\det(\text{Hess}_{z,y,\theta,\xi}\Psi_3(\bar{z}, \bar{y}, \bar{\theta}, \bar{\xi})) = 1$. Above, $(\bar{z}, \bar{y}, \bar{\theta}, \bar{\xi})$ is the critical point satisfying (95) and (96), where in the local coordinates $(\bar{z}, \bar{y}) = (0, 0)$ and $h(\bar{z})$ is constant times powers of values of the cut-off functions Φ_0 and Φ_1 at zero. Recall that we considered above the case when \mathcal{B}_j are multiplication operators with these cut-off functions. The term c_k depends on the derivatives of the symbols a_j and q_1 of order less or equal to $2k$ at the critical point. If $\Psi_3(z, \theta, y, \xi)$ has no critical points, that is, q is not an intersection point of all five geodesics γ_{x_j, ξ_j} , $j = 1, 2, 3, 4, 5$ we obtain the asymptotics $T_{\tau,1,1}^{(4),\beta} = O(\tau^{-N})$ for all $N > 0$.

For future reference we note that if we use the method of stationary phase in the last integral of (93) only in the integrals with respect to z and ξ , yielding that at the critical point we have $y = z$ and

$\xi = \omega_\beta(\theta) = (\theta_1, \theta_2, \theta_3, 0)$, we see that $T_{\tau,1,1}^{(4),\beta}$ can be written as

$$(102) \quad \begin{aligned} T_{\tau,1,1}^{(4),\beta} &= c\tau^4 \int_{\mathbb{R}^8} e^{i(\theta_1 y^1 + \theta_2 y^2 + \theta_3 y^3 + \theta_4 y^4) + i\tau\varphi(y)} c_1(y, \theta_1, \theta_2) \cdot \\ &\quad \cdot a_3(y, \theta_3) q_{1,1}(y, y, \omega_\beta(\theta)) a_4(y, \theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy \\ &= c\tau^8 \int_{\mathbb{R}^8} e^{i\tau(\theta_1 y^1 + \theta_2 y^2 + \theta_3 y^3 + \theta_4 y^4 + \varphi(y))} c_1(y, \tau\theta_1, \tau\theta_2) \cdot \\ &\quad \cdot a_3(y, \tau\theta_3) q_{1,1}(y, y, \tau\omega_\beta(\theta)) a_4(y, \tau\theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy. \end{aligned}$$

Next, we consider the terms $\tilde{T}_\tau^{(4),\beta}$ of the type (72). Such term is an integral of the product of u_τ and two other factors $\mathbf{Q}(U_2 \cdot U_1)$ and $\mathbf{Q}(U_4 \cdot U_3)$. As the last two factors can be written in the form (84), one can see using the method of stationary phase that $\tilde{T}_\tau^{(4),\beta}$ has similar asymptotics to $T_\tau^{(4),\beta}$ as $\tau \rightarrow \infty$, with the leading order coefficient $\tilde{c}_0 = \tilde{h}(\bar{z}) \hat{c}_1(0, -\bar{r}_1, -\bar{r}_2) \hat{c}_2(0, -\bar{r}_3, -\bar{r}_4)$ where \hat{c}_2 is given as in (85) with symbols \hat{a}_3 and \hat{a}_4 , and moreover, $\tilde{h}(\bar{z})$ is a constant times powers of values of the cut-off functions Φ_0 and Φ_1 at zero.

This proves the claim in the special case where u_j are conormal distributions supported in the coordinate neighborhoods $W_{\kappa(j)}$, a_j are positively homogeneous scalar valued symbols, $S_j = \mathbf{Q}$, and \mathcal{B}_j are multiplication functions with smooth cut-off functions.

By using a suitable partition of unity and summing the results of the above computations, similar results to the above follows when a_j are general classical symbols that are \mathcal{B} -valued and the waves u_j are supported on $J_{\hat{g}}^+(\text{supp } (\mathbf{f}_j))$. Also, S_j can be replaced by operators of type (57) and \mathcal{B}_j can replaced by differential operator without other essential changes expect that the highest order power of τ changes. Then, in the asymptotics of terms $T_\tau^{(4),\beta}$ the function $h(\bar{z})$ in (101) is a section in dual bundle $(\mathcal{B}_L)^4$. The coefficients of $h(\bar{z})$ in local coordinates are polynomials of \hat{g}^{jk} , \hat{g}_{jk} , $\hat{\phi}_\ell$, and their derivatives at \bar{z} . Similar representation is obtained for the asymptotics of the terms $\tilde{T}_\tau^{(4),\beta}$.

As we integrated by parts two times the operator $(\nabla_z - \omega_\theta)^2$ and the total order of of \mathcal{B}_j is less or equal to 6, we see that it is enough to assume above that the symbols $a_j(z, \theta_j)$ are of order (-12) or less. The leading order asymptotics come from the term where the sum of orders of \mathcal{B}_j is 6 and $p_j = n$ for $j = 1, 2, 3, 4$, $p_5 = 0$ so that $m = -4 - \rho + 6 = 4n + 2$. We also see that the terms containing permutation $\sigma = \sigma_\beta$ of the indexes of the distorted plane waves can be analyzed analogously. This proves (77).

Note that above \bar{r} depends only on \hat{g} and \mathbf{b} and we write $\bar{r} = \bar{r}(\mathbf{b})$, omiting the \hat{g} dependency. Making the above computations explicitly, we obtain an explicit formula for the leading order coefficient s_m in (77) in terms of \mathbf{b} and \mathbf{w} . This show that s_m coincides with some real-analytic

function $G(\mathbf{b}, \mathbf{w})$. This proves the claims (i) and (ii) in the case when the linear independency condition (LI) is valid.

Next, consider the case when the linear independency condition (LI) is not valid. Again, by the definition of \mathbf{t}_j , if the intersection $\gamma_{x_5, \xi_5}(\mathbb{R}_-) \cap (\cap_{j=1}^4 \gamma_{x_j, \xi_j}((0, \mathbf{t}_j)))$ is non-empty, it can contain only one point. In the case that such a point exists, we denote it by q .

When (LI) is not valid, we have that the linear space $\text{span}(b_j; j = 1, 2, 3, 4) \subset T_q^*M_0$ has dimension 3 or less. We use the facts that for $w \in I(\Lambda_1, \Lambda_2)$ we have $\text{WF}(w) \subset \Lambda_1 \cup \Lambda_2$ and the fact, see [32, Thm. 1.3.6]

$$\begin{aligned} \text{WF}(v \cdot w) &\subset \\ \text{WF}(v) \cup \text{WF}(w) \cup \{(x, \xi + \eta); (x, \xi) \in \text{WF}(v), (x, \eta) \in \text{WF}(w)\}. \end{aligned}$$

Let us next consider the terms corresponding to the permutation $\sigma = \text{Id}$. The above facts imply that $\tilde{\mathcal{G}}^{(4),\beta}$ in (59) satisfies

$$\text{WF}(\tilde{\mathcal{G}}^{(4),\beta}) \cap T_q^*M_0 \subset \mathcal{Z}_{s_0} := \mathcal{X}_{s_0} \cup \bigcup_{1 \leq j \leq 4} N^*K_j \cup \bigcup_{1 \leq j < k \leq 4} N^*K_{jk},$$

where $\mathcal{X}_{s_0} = \mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0)$. Also, for

$$w_{123} = \mathcal{B}_3^\beta u_3 \cdot \mathcal{C}_1^\beta S_1^\beta (\mathcal{B}_2^\beta u_2 \cdot \mathcal{B}_1^\beta u_1),$$

appearing in (60), we have $\text{WF}(w_{123}) \subset \mathcal{Z}_{s_0}$ and thus using Hörmander's theorem [55, Thm. 26.1.1], we see that $\text{WF}(S_2^\beta(w_{123})) \subset \Lambda^{(3)}$, where $\Lambda^{(3)}$ is the flowout of \mathcal{Z}_{s_0} in the canonical relation of \mathbf{Q} . Then

$$\pi(\Lambda^{(3)}) \subset \mathcal{Y}_{s_0} \cup \bigcup_{1 \leq j \leq 4} K_j \cup \bigcup_{1 \leq j < k \leq 4} K_{jk},$$

where $\mathcal{Y}_{s_0} = \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$ and $\pi : T^*M_0 \rightarrow M_0$ is the projection to the base point.

Observe that $E = \text{span}(b_j; j = 1, 2, 3, 4) \subset T_q^*M_0$ has dimension 3 or less, $\Lambda^{(3)} \cap T_q^*M_0 \subset E$ and $\text{WF}(u_4) \cap T_q^*M_0 \subset E$. Thus, $\mathcal{G}^{(4),\beta} = \mathcal{B}_4^\beta u_4 \cdot \mathcal{C}_2^\beta S_2^\beta(w_{123})$ satisfies $\text{WF}(\mathcal{G}^{(4),\beta}) \cap T_q^*M_0 \subset E$. Now, $E \subset \mathcal{Z}_{s_0}$. By our assumption, $(q, b_5) \notin \mathcal{X}((\vec{x}, \vec{\xi}); t_0)$, and thus, cf. (59), we see that

$$\langle u_\tau, \mathbf{Q} \left(\sum_{\beta \leq n_1} \mathcal{G}^{(4),\beta} + \tilde{\mathcal{G}}^{(4),\beta} \right) \rangle = O(\tau^{-N})$$

for all $N > 0$ when s_0 is small enough. The terms where the permutation σ is not the identity can be analyzed similarly. This proves the claims (i) and (ii) in the case when the linear independency condition (LI) is not valid. This proves (i) and (ii).

(iii) Let us fix (x_j, ξ_j) , $j \leq 4$, s_0 , and the waves $u_j \in \mathcal{I}(K_j)$, $j \leq 4$.

First we observe that if the condition (LI) is not valid, we see similarly to the above that $\mathcal{M}^{(4)}$ is C^∞ smooth in $\mathcal{V} \setminus (\mathcal{Y} \cup \bigcup_{j=1}^4 K_j)$. Thus to prove the claim of the proposition we can assume that (LI) is valid.

Let us decompose $\mathcal{F}^{(4)}$, given by (59) and (60)-(61) as $\mathcal{F}^{(4)} = \mathcal{F}_1^{(4)} + \mathcal{F}_2^{(4)}$ where $\mathcal{F}_p^{(4)}$ is defined similarly to $\mathcal{F}^{(4)}$ in (59) and (60)-(61) by modifying these formulas so that the operator S_1^β is replaced by $S_{1,p}^\beta$, where $S_{1,p}^\beta = \mathbf{Q}_p$, when $S_1^\beta = \mathbf{Q}$, and $S_{1,p}^\beta = (2-p)I$, when $S_1^\beta = I$. Here, the operators \mathbf{Q}_p are defined as above using the parameters ε_2 and ε_1 defined below.

Using formulas (59), (83), (84), and (91) we see that near q in the Y coordinates $\mathcal{M}_1^{(4)} = \mathbf{Q}\mathcal{F}_1^{(4)}$ can be calculated using that

$$(103) \quad \mathcal{F}_1^{(4)}(y) = \int_{\mathbb{R}^4} e^{iy^j \theta_j} b(y, \theta) d\theta,$$

where K_j in local coordinates is given by $\{y^j = 0\}$ and $b(y, \theta)$ is a finite sum of terms that are products of some of the following terms: at most one product type symbol $c_l(y, \theta_j, \theta_k) \in S(W_0; \mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ (they appear in the terms (71)-(72) where the S_j^β operators are \mathbf{Q} and do not appear if these operators are the identity), and one or more term which is either the symbols $a_j(y, \theta_j) \in S^n(W_0; \mathbb{R})$, or the functions $q_1(y, y, \omega_\beta(\theta))$, cf. (102), where $\omega_\beta(\theta)$ is equal to some of the vectors $(\theta_1, \theta_2, \theta_3, 0)$, $(\theta_1, \theta_2, 0, \theta_4)$, $(\theta_1, 0, \theta_3, \theta_4)$, or $(0, \theta_1, \theta_3, \theta_4)$, depending on the permutation σ .

Let us consider next the source F_τ is determined by the functions (p, h) in (67). Then using the method of stationary phase gives the asymptotics, c.f. (102),

$$\langle u_\tau, \mathcal{F}_1^{(4)} \rangle \sim \tau^8 \int_{\mathbb{R}^8} e^{i\tau(\varphi(y) + y^j \theta_j)} (a_5(y, \tau), b(y, \tau\theta)) \widehat{G} d\theta dy \sim \sum_{k=m}^{\infty} s_k(p, h) \tau^{-k}$$

where \widehat{G} is a Riemannian metric on the fiber of \mathcal{B}^L at y , that is isomorphic to \mathbb{R}^{10+L} , and the critical point of the phase function is $y = 0$ and $\theta = -d\varphi(0)$. As we saw above, we have that when $\varepsilon_2 > 0$ is small enough then for $p = 2$ we have $\langle u_\tau, \mathcal{F}_p^{(4)} \rangle = O(\tau^{-N})$ for all $N > 0$.

Let us choose sufficiently small $\varepsilon_3 > 0$ and choose a function $\chi(\theta) \in C^\infty(\mathbb{R}^4)$ that vanishes in a ε_3 -neighborhood (in the \widehat{g}^+ metric) of \mathcal{A}_q ,

$$(104) \quad \mathcal{A}_q := N_q^* K_{123} \cup N_q^* K_{134} \cup N_q^* K_{124} \cup N_q^* K_{234}$$

and is equal to 1 outside the $(2\varepsilon_3)$ -neighborhood of this set.

Let $\phi \in C_0^\infty(W_1)$ be a function that is one near q . Also, let

$$b_0(y, \theta) = \phi(y)\chi(\theta)b(y, \theta)$$

be a classical symbol, $p = \sum_{j=1}^4 p_j$, and let $\mathcal{F}^{(4),0}(y) \in \mathcal{I}^{p-4}(q)$ be the conormal distribution that is given by the formula (103) with $b(y, \theta)$ being replaced by $b_0(y, \theta)$.

When ε_3 is small enough (depending on the point x_5), we have that F_τ is determined by functions (p, h) and the corresponding gaussian

beams u_τ propagating on the geodesic $\gamma_{x_5, \xi_5}(\mathbb{R})$ such that the geodesic passes through $x_5 \in V$, we have

$$\langle u_\tau, \mathcal{F}^{(4),0} \rangle \sim \sum_{k=m}^{\infty} s_k(p, h) \tau^{-k},$$

that is, we have $\langle u_\tau, \mathcal{F}^{(4),0} \rangle - \langle u_\tau, \mathcal{F}^{(4)} \rangle = O(\tau^{-N})$ for all N . When $\gamma_{x_5, \xi_5}(\mathbb{R})$ does not pass through q , we have that $\langle u_\tau, \mathcal{F}^{(4)} \rangle$ and $\langle u_\tau, \mathcal{F}^{(4),0} \rangle$ are both of order $O(\tau^{-N})$ for all $N > 0$.

Let $V \subset \mathcal{Y}((\vec{x}, \vec{\xi}), t_0) \setminus \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([0, \infty))$, see (76), be an open set. By varying the source F_τ , defined in (67), we see, by multiplying the solution with a smooth cut off function and using Corollary 1.4 in [28] in local coordinates, or [81], we have that the function $\mathcal{M}^4 - \mathbf{Q}\mathcal{F}^{(4),0}$ has no wave front set in $T^*(V)$ and it is thus C^∞ -smooth function in V .

Since by [44], $\mathbf{Q} : \mathcal{I}^{p-4}(\{q\}) \rightarrow \mathcal{I}^{p-4-3/2, -1/2}(N^*(\{q\}), \Lambda_q^+)$, the above implies that

$$(105) \quad \mathcal{M}^4|_{V \setminus \mathcal{Y}} \in \mathcal{I}^{p-4-3/2}(V \setminus \mathcal{Y}; \Lambda_q^+),$$

where $\mathcal{Y} = \mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$. When x_5 is fixed, choosing s_0 to be small enough, we obtain the claim (iii). \square

Next we will show that $\mathcal{G}(\mathbf{b}, \mathbf{w})$ is not vanishing identically.

3.4. Non-vanishing interaction in the Minkowski space.

3.4.1. WKB computations and the indicator functions in the Minkowski space. To show that the function $\mathcal{G}(\mathbf{b}, \mathbf{w})$, see (78), is not vanishing identically, we will next consider waves in Minkowski space.

In this section, $x = (x^0, x^1, x^3, x^4)$ are the standard coordinates in Minkowski space and $\widehat{g}_{jk} = \text{diag}(-1, 1, 1, 1)$ denote the metric in the standard coordinates of the Minkowski space \mathbb{R}^4 . Below we call the principal symbols of the linearized waves the *polarizations* to emphasize their physical meaning. We denote $\mathbf{w} = (w_{(j)})_{j=1}^5$. Then for $j \leq 4$, the polarizations $w_j = (v_{(j)}^{met}, v_{(j)}^{scal})$, represented as a pair of the metric part of the polarization $v_{(j)}^{met} \in \text{sym}(\mathbb{R}^{4 \times 4}) \equiv \mathbb{R}^{10}$ and the scalar field part of the polarization $v_{(j)}^{scal} \in \mathbb{R}^L$, are such that for the metric part $v_{(j)}^{met}$ of the polarization has to satisfy 4 linear conditions (50) with the Minkowski metric (that follow from the linearized harmonicity condition). We study the special case when all polarizations of the scalar fields ϕ_ℓ , $\ell = 1, 2, \dots, L$, vanish, that is, $v_{(j)}^{scal} = 0$ for all j . To simplify the notations, we denote below $v_{(j)}^{met} = v_{(j)}$ and $\mathbf{v} = (v_{(j)})_{j=1}^5$ so that $\mathbf{w} = (\mathbf{v}, 0)$. In this case, in Minkowski space the function $\mathcal{G}(\mathbf{v}, 0, \mathbf{b})$ can be analyzed by assuming that there are no matter fields, which we do next. Later we return to the case of general polarizations.

We assume that the waves $u_j(x)$, $j = 1, 2, 3, 4$, solving the linear wave equation in the Minkowski space, are of the form

$$(106) \quad u_j(x) = v_{(j)} \left(b_p^{(j)} x^p \right)_+^a, \quad t_+^a = |t|^a H(t),$$

where $b_p^{(j)}$, $p = 1, 2, 3, 4$ are four linearly independent light-like co-vectors of \mathbb{R}^4 , $a > 0$ and $v_{(j)}$ are constant 4×4 matrices. We also assume that $b^{(5)}$ is not in the linear span of any three vectors $b^{(j)}$, $j = 1, 2, 3, 4$. In the following, we denote $b^{(j)} \cdot x := b_p^{(j)} x^p$ and $\mathbf{b} = (b^{(j)})_{j=1}^5$.

Let us next consider the wave produced by interaction of two plane wave solutions in the Minkowski space.

Let $b^{(1)}$ and $b^{(2)}$ be light like co-vectors. We use the notations

$$u^{a_1, a_2}(x; b^{(1)}, b^{(2)}) = (b^{(1)} \cdot x)_+^{a_1} \cdot (b^{(2)} \cdot x)_+^{a_2}$$

for a product of two plane waves. We define the formal parametrix \mathbf{Q}_0 ,

$$(107) \quad \mathbf{Q}_0(u^{a_1, a_2}(x; b^{(1)}, b^{(2)})) = \frac{u^{a_1+1, a_2+1}(x; b^{(1)}, b^{(2)})}{2(a_1 + 1)(a_2 + 1) \widehat{g}(b^{(1)}, b^{(2)})}.$$

Then $\square_{\widehat{g}}(\mathbf{Q}_0(u^{a_1, a_2}(x; b^{(1)}, b^{(2)}))) = u^{a_1, a_2}(x; b^{(1)}, b^{(2)})$. Also, let

$$u_\tau^a(x; b^{(4)}, b^{(5)}) = u_4(x) u_\tau(x), \quad u_4(x) = (b^{(4)} \cdot x)_+^a, \quad u_\tau(x) = e^{i\tau b^{(5)} \cdot x},$$

so that $\square_{\widehat{g}}(u_\tau^a(x; b^{(4)}, b^{(5)})) = 2a \widehat{g}(b^{(4)}, b^{(5)}) i\tau u_\tau^{a-1, 0}(x; b^{(4)}, b^{(5)})$. Let

$$(108) \quad \mathbf{Q}_0(u_\tau^a(x; b^{(4)}, b^{(5)})) = \frac{1}{2i(a+1) \widehat{g}(b^{(4)}, b^{(5)}) \tau} u_\tau^{a+1}(x; b^{(4)}, b^{(5)}).$$

We will prove that the indicator function $\mathcal{G}(\mathbf{v}, 0, \mathbf{b})$ in (78) does not vanish identically by showing that it coincides with the *formal* indicator function $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$, defined below, which is a real-analytic function that does not vanish identically. We define the (Minkowski) indicator function (c.f. (70) and (78)) by

$$\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) = \lim_{\tau \rightarrow \infty} \tau^m \left(\sum_{\beta \leq n_1} \sum_{\sigma \in \Sigma(4)} T_{\tau, \sigma}^{(\mathbf{m}), \beta} + \widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} \right),$$

where the super-index (\mathbf{m}) refers to the word ‘‘Minkowski’’. Above, σ runs over all permutations of the set $\{1, 2, 3, 4\}$. The functions $T_{\tau, \sigma}^{(\mathbf{m}), \beta}$ and $\widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta}$ are counterparts of the functions $T_\tau^{(4), \beta}$ and $\widetilde{T}_\tau^{(4), \beta}$, see (71)-(72), obtained by replacing the distorted plane waves and the gaussian beam by the plane waves. We also replace the parametrices \mathbf{Q} and \mathbf{Q}^* by a formal parametrix \mathbf{Q}_0 . Also, we include a smooth cut off function $h \in C_0^\infty(M)$ which is one near the intersection point q of the K_j . Thus,

$$(109) \quad T_{\tau, \sigma}^{(\mathbf{m}), \beta} = \langle S_2^0(u_\tau \cdot \mathcal{B}_4 u_{\sigma(4)}), h \cdot \mathcal{B}_3 u_{\sigma(3)} \cdot S_1^0(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(\mathbb{R}^4)},$$

$$(110) \quad \widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} = \langle u_\tau, h \cdot S_2^0(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1^0(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(\mathbb{R}^4)},$$

where u_j are given by (106) with $a = -n - 1$, $j = 1, 2, 3, 4$. Here, the differential operators $\mathcal{B}_j = \mathcal{B}_j^\beta$ are in Minkowski space constant coefficient operators and finally, $S_j^0 = S_{j,\beta}^0 \in \{\mathbf{Q}_0, I\}$.

Let us now consider the orders of the differential operators appearing above. The orders $k_j = \text{ord}(\mathcal{B}_j^\beta)$ of the differential operators \mathcal{B}_j^β , defined in (57), depend on $\vec{S}_\beta^0 = (S_{1,\beta}^0, S_{2,\beta}^0)$ as follows: When β is such that $\vec{S}_\beta^0 = (\mathbf{Q}_0, \mathbf{Q}_0)$, for the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$(111) \quad k_1 + k_2 + k_3 + k_4 \leq 6, \quad k_3 + k_4 \leq 4, \quad k_4 \leq 2$$

and for the terms $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$(112) \quad k_1 + k_2 + k_3 + k_4 \leq 6, \quad k_1 + k_2 \leq 4, \quad k_3 + k_4 \leq 4.$$

When β is such that $\vec{S}_\beta^0 = (I, Q_0)$ we have for the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$

$$(113) \quad k_1 + k_2 + k_3 + k_4 \leq 4, \quad k_4 \leq 2,$$

and for terms $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$(114) \quad k_1 + k_2 + k_3 + k_4 \leq 4, \quad k_1 + k_2 \leq 2.$$

When β is such that $\vec{S}_\beta^0 = (\mathbf{Q}_0, I)$, both for the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$(115) \quad k_1 + k_2 + k_3 + k_4 \leq 4, \quad k_3 + k_4 \leq 2.$$

Finally, when β is such that $\vec{S}_\beta^0 = (I, I)$, for the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have $k_1 + k_2 + k_3 + k_4 \leq 2$.

Let us next summarize these formulas in different notations:

Recall that $k_j = \text{ord}(\mathcal{B}_j^\beta)$ are the orders of the differential operators \mathcal{B}_j^β , defined in (57). For $j = 1, 2$, we define $K_{\beta,j} = 1$ when $S_{\beta,j}^0 = \mathbf{Q}_0$ and $K_{\beta,j} = 0$ when $S_{\beta,j}^0 = I$. Then the allowed values of $\vec{k} = (k_1, k_2, k_3, k_4)$ depend on $K_{\beta,1}$ and $K_{\beta,2}$ as follows: We require that

$$(116) \quad \begin{aligned} k_1 + k_2 + k_3 + k_4 &\leq 2K_{\beta,1} + 2K_{\beta,2} + 2, & k_3 + k_4 &\leq 2K_{\beta,2} + 2, \\ k_4 &\leq 2, \text{ for all terms } T_{\tau,\sigma}^{(\mathbf{m}),\beta}, \text{ and} \\ k_1 + k_2 &\leq 2K_{\beta,1} + 2, \text{ for all terms } \tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta} \end{aligned}$$

cf. (60) and (61).

Lemma 3.A.1. *When $b^{(j)}$, $j = 1, 2, 3, 4$ are linearly independent light-like co-vectors and light-like co-vector $b^{(5)}$ is not in the linear span of any three vectors $b^{(j)}$, $j = 1, 2, 3, 4$ we have $\mathcal{G}(\mathbf{w}, \mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ when $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$.*

Proof. Let us start by considering the relation of \mathbf{Q}_0 with the causal inverse \mathbf{Q} in (108). Let

$$\begin{aligned} w_{\tau,0} &= \mathbf{Q}_0(u_{\tau}^{a,0}(\cdot; b^{(4)}, b^{(5)})) \\ &= \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P \cdot z} a_4(z, \theta_4) d\theta_4, \\ w_{\tau} &= \mathbf{Q}^*(J), \\ J &= u_4 \cdot (\chi \cdot u^{\tau}). \end{aligned}$$

Here

$$\begin{aligned} J(z) &= \chi(X^0(z)) \square w_{\tau,0} \\ &= \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P \cdot z} (\tau b_1(z, \theta_4) + b_2(z, \theta_4)) a_5(z, \tau) d\theta_4, \end{aligned}$$

where $a_5(z, \tau) = 1$. Then

$$\begin{aligned} \square(w_{\tau} - \chi w_{\tau,0}) &= J_1, \quad \text{for } x \in \mathbb{R}^4, \\ J_1 &= [\square, \chi] w_{\tau,0} \\ &= \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P \cdot z} (\tau b_3(z, \theta_4) + b_4(z, \theta_4)) d\theta_4, \end{aligned}$$

where $b_3(z, \theta_4)$ and $b_4(z, \theta_4)$ are supported in the domain $T_0 < X^0(z) < T_0 + 1$ and $w_{\tau} - \chi w_{\tau,0}$ is supported in the domain $X^0(z) < T_0 + 1$. Thus

$$w_{\tau} = \chi w_{\tau,0} + \mathbf{Q}^* J_1.$$

Here, we can write

$$\begin{aligned} (117) \quad J_1(z) &= u_4^{(1)}(z) u^{\tau,(1)}(z) + u_4^{(2)}(z) u^{\tau,(2)}(z), \quad \text{where} \\ u_4^{(1)}(z) &= \int_{\mathbb{R}} e^{i\theta_4 z^4} b_3(z, \theta_4) d\theta_4, \\ u^{\tau,(1)}(z) &= \tau u^{\tau}(z), \\ u_4^{(2)}(z) &= \int_{\mathbb{R}} e^{i\theta_4 z^4} b_4(z, \theta_4) d\theta_4, \\ u^{\tau,(2)}(z) &= u^{\tau}(z). \end{aligned}$$

Let us now substitute this in to the above microlocal computations done in the proof of Prop. 3.3.

Recall that $b^{(j)}$, $j = 1, 2, 3, 4$, are four linearly independent co-vectors. This means that a condition analogous to (LI) in the proof of Prop. 3.3 is satisfied, and that $b^{(5)}$ is not in the space spanned by any of three of the co-vectors $b^{(j)}$, $j = 1, 2, 3, 4$. Also, observe that the hyperplanes $K_j = \{x \in \mathbb{R}^4; b^{(j)} \cdot x = 0\}$ intersect at origin of \mathbb{R}^4 . Thus, we see that the arguments in the proof of Prop. 3.3 are valid mutatis mutandis if the phase function of the gaussian beam $\varphi(x)$ is replaced by the phase function of the plane wave, $b^{(5)} \cdot x$, and the geodesic γ_{x_5, ξ_5} , on which the gaussian beam propagates, is replaced by the whole space

\mathbb{R}^4 . In particular, as $b^{(5)}$ is not in the space spanned by any of three of those co-vectors, the case (A1) in the proof of Prop. 3.3 cannot occur. In particular, we have that the leading order asymptotics of the terms $T_{\tau,\sigma}^\beta$ and $\tilde{T}_{\tau,\sigma}^\beta$ do not change as these asymptotics are obtained using the method of stationary phase for the integral (102) and the other analogous integrals at the critical point $z = 0$. In other words, we can replace the gaussian beam by a plane wave in our considerations similar to those in the proof of Prop. 3.3.

Using (117) and the fact that $b_3(z, \theta_4)$ and $b_4(z, \theta_4)$ vanish near $z = 0$, we see that if u_4 and u^τ are replaced by $u_4^{(j)}$ and $u^{\tau,(j)}$, respectively, where $j \in \{1, 2\}$ and we can do similar computations based on the method of stationary phase as are done in the proof of Proposition 3.3. Then both terms $T_{\tau,\sigma}^\beta$ and $\tilde{T}_{\tau,\sigma}^\beta$ have asymptotics $O(\tau^{-N})$ for all $N > 0$ as $\tau \rightarrow \infty$. In other words, in the proof of Prop. 3.3 the term $w_\tau = \mathbf{Q}^*(u_4 u^\tau)$ can be replaced by $\chi w_{\tau,0}$ without changing the leading order asymptotics. This shows that $\mathcal{G}(\mathbf{w}, \mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$, where $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$. \square

Summarizing the above: Let $b^{(j)}$, $j = 1, 2, 3, 4$ be linearly independent light-like co-vectors and $b^{(5)}$ be a light-like co-vector that is not in the linear span of any three vectors $b^{(j)}$, $j = 1, 2, 3, 4$. Then, using the above lemma and by analyzing the microlocal computations done in the proof of Prop. 3.3, we see that $\mathcal{G}(\mathbf{w}, \mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ when $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$, $\mathbf{w} = (w_{(j)})_{j=1}^5$, and $\mathbf{v} = (v_{(j)})_{j=1}^5$.

Proposition 3.4. *Let \mathbb{X} be the set of $(\mathbf{b}, v_{(2)}, v_{(3)}, v_{(4)})$, where \mathbf{b} is a 5-tuple of light-like covectors $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$ and $v_{(j)} \in \mathbb{R}^{10}$, $j = 2, 3, 4$ are the polarizations that satisfy the equation (50) with respect to $b^{(j)}$, i.e., the harmonicity condition for the principal symbols. For $\widehat{b}^{(5)} \in \mathbb{R}^4$, let $\mathbb{X}(\widehat{b}^{(5)})$ be the set elements in \mathbb{X} where $b^{(5)} = \widehat{b}^{(5)}$. Then for any light-like $\widehat{b}^{(5)}$ there is a generic (i.e. open and dense) subset $\mathbb{X}'(\widehat{b}^{(5)})$ of $\mathbb{X}(\widehat{b}^{(5)})$ such that for all $(\mathbf{b}, v_{(2)}, v_{(3)}, v_{(4)}) \in \mathbb{X}'(\widehat{b}^{(5)})$ there exist linearly independent vectors $v_{(5)}^q$, $q = 1, 2, 3, 4, 5, 6$, with the following property:*

If $v_{(5)} \in \text{span}(\{v_{(5)}^q; q = 1, 2, 3, 4, 5, 6\})$ is non-zero, then there exists a vector $v_{(1)}$ for which the pair $(b^{(1)}, v_{(1)})$ satisfies the equation (50) and $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) \neq 0$ with $\mathbf{v} = (v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)}, v_{(5)})$.

Proof. In the proof below, let $a \in \mathbb{Z}_+$ be large enough. To show that the coefficient $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ of the leading order term in the asymptotics is non-zero, we consider a special case when the direction vectors of the intersecting plane waves in the Minkowski space are the linearly independent light-like vectors of the form

$$b^{(5)} = (1, 1, 0, 0), \quad b^{(j)} = (1, 1 - \frac{1}{2}\rho_j^2, \rho_j + O(\rho_j^3), \rho_j^3), \quad j = 1, 2, 3, 4,$$

where $\rho_j > 0$ are small parameters for which

$$(118) \quad \|b^{(5)} - b^{(j)}\|_{(\mathbb{R}^4, \hat{g}^+)} = \rho_j(1 + o(\rho_j)), \quad j = 1, 2, 3, 4.$$

With an appropriate choice of $O(\rho_k^3)$ above, the vectors $b^{(k)}$, $k \leq 5$ are light-like and

$$\begin{aligned} \hat{g}(b^{(5)}, b^{(j)}) &= -1 + (1 - \frac{1}{2}\rho_j^2) = -\frac{1}{2}\rho_j^2, \\ \hat{g}(b^{(k)}, b^{(j)}) &= -\frac{1}{2}\rho_k^2 - \frac{1}{2}\rho_j^2 + O(\rho_k\rho_j). \end{aligned}$$

Below, we denote $\omega_{kj} = \hat{g}(b^{(k)}, b^{(j)})$. We consider the case when the orders of ρ_j are

$$(119) \quad \rho_4 = \rho_2^{100}, \quad \rho_2 = \rho_3^{100}, \quad \text{and } \rho_3 = \rho_1^{100},$$

so that $\rho_4 < \rho_2 < \rho_3 < \rho_1$. When ρ_1 is small enough, b_j , $j \leq 4$ are linearly independent. Note that when ρ_1 is small enough, $b^{(5)}$ is not a linear combination of any three vectors $b^{(j)}$, $j = 1, 2, 3, 4$.

The coefficient $\mathcal{G}^{(\mathbf{m})}$ of the leading order asymptotics is computed by analyzing the leading order terms of all 4th order interaction terms, similar to those given in (109) and (110). We will start by analyzing the most important terms $T_\tau^{(\mathbf{m}), \beta}$ of the type (109) when β is such that $\vec{S}_\beta = (\mathbf{Q}_0, \mathbf{Q}_0)$. When $k_j = k_j^\beta$ is the order of \mathcal{B}_j , and we denote $\vec{k}_\beta = (k_1^\beta, k_1^\beta, k_3^\beta, k_4^\beta)$, we see that

$$\begin{aligned} (120) \quad T_\tau^{(\mathbf{m}), \beta} &= \langle \mathbf{Q}_0(\mathcal{B}_4 u_4 \cdot u^\tau), h \cdot \mathcal{B}_3 u_3 \cdot \mathbf{Q}_0(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= C \frac{\mathcal{P}_\beta}{\omega_{45}\tau \omega_{12}} \langle u_\tau^{a-k_4+1,0}(\cdot; b^{(4)}, b^{(5)}), h \cdot u_3 \cdot u^{a-k_2+1, a-k_1+1}(\cdot; b^{(2)}, b^{(1)}) \rangle \\ &= C \frac{\mathcal{P}_\beta}{\omega_{45}\tau \omega_{12}} \int_{\mathbb{R}^4} (b^{(4)} \cdot x)_+^{a-k_4+1} e^{i\tau(b^{(5)} \cdot x)} h(x) (b^{(3)} \cdot x)_+^{a-k_3} \\ &\quad \cdot (b^{(2)} \cdot x)_+^{a-k_2+1} (b^{(1)} \cdot x)_+^{a-k_1+1} dx, \end{aligned}$$

where $\mathcal{P} = \mathcal{P}_\beta$ is a polarization factor involving the coefficients of \mathcal{B}_j , the directions $b^{(j)}$, and the polarization $v_{(j)}$. Moreover, $C = C_a$ is a generic constant depending on a and β but not on $b^{(j)}$ or $v_{(j)}$.

We will analyze the polarization factors later, but as a sidetrack, let us already explain now the nature of the polarization term when $\beta = \beta_1$, see (73). Observe that this term appear only when we analyze the term $\langle F_\tau, \mathbf{Q}(A[u_4, \mathbf{Q}(A[u_3, \mathbf{Q}(A[u_2, u_1])])]) \rangle$ where all operators $A[v, w]$ are of the type $A_2[v, w] = \hat{g}^{np} \hat{g}^{mq} v_{nm} \partial_p \partial_q w_{jk}$, cf. (62) and (63). Due to this, we have the polarization factor

$$(121) \quad \mathcal{P}_{\beta_1} = (v_{(4)}^{rs} b_r^{(1)} b_s^{(1)}) (v_{(3)}^{pq} b_p^{(1)} b_q^{(1)}) (v_{(2)}^{nm} b_n^{(1)} b_m^{(1)}) \mathcal{D},$$

where $v_{(\ell)}^{nm} = \hat{g}^{nj} \hat{g}^{mk} v_{jk}^{(\ell)}$ and

$$(122) \quad \mathcal{D} = \hat{g}_{nj} \hat{g}_{mk} v_{(5)}^{nm} v_{(1)}^{jk}.$$

We will postpone the analysis of the polarization factors \mathcal{P}_β in $T_\tau^{(\mathbf{m}),\beta}$ with $\beta \neq \beta_1$ later.

Let us now return back to the computation (120). We next use in \mathbb{R}^4 the coordinates $y = (y^1, y^2, y^3, y^4)^t$ where $y^j = b_k^{(j)}x^k$, i.e., and let $A \in \mathbb{R}^{4 \times 4}$ be the matrix for which $y = A^{-1}x$. Let $\mathbf{p} = (A^{-1})^t b^{(5)}$. In the y -coordinates, $b^{(j)} = dy^j$ for $j \leq 4$ and $b^{(5)} = \sum_{j=1}^4 \mathbf{p}_j dy^j$ and

$$\mathbf{p}_j = \widehat{g}(b^{(5)}, dy^j) = \widehat{g}(b^{(5)}, b^{(j)}) = \omega_{j5} = -\frac{1}{2}\rho_j^2.$$

Then $b^{(5)} \cdot x = \mathbf{p} \cdot y$. We use the notation $\mathbf{p}_j = \omega_{j5} = -\frac{1}{2}\rho_j^2$, that is, we denote the same object with several symbols, to clarify the steps we do in the computations.

Then $\det(A) = 2\rho_1^{-3}\rho_2^{-2}\rho_3^{-1}(1 + O(\rho_1))$ and

$$T_\tau^{(\mathbf{m}),\beta} = \frac{C\mathcal{P}_\beta \det(A)}{\omega_{45}\tau \omega_{12}} \int_{(\mathbb{R}_+)^4} e^{i\tau \mathbf{p} \cdot y} h(Ay) y_4^{a-k_4+1} y_3^{a-k_3} y_2^{a-k_2+1} y_1^{a-k_1+1} dy.$$

Using repeated integration by parts we see that

(123)

$$T_\tau^{(\mathbf{m}),\beta} = C\det(A) \mathcal{P}_\beta \frac{(i\tau)^{-(12+4a-|\vec{k}_\beta|)}(1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+1+2)} \rho_3^{2(a-k_3+1)} \rho_2^{2(a-k_2+2)} \rho_1^{2(a-k_1+1+2)}}.$$

Note that here and below $O(\tau^{-1})$ may depend also on ρ_j , that is, we have $|O(\tau^{-1})| \leq C(\rho_1, \rho_2, \rho_3, \rho_4)\tau^{-1}$.

To show that $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ is non-vanishing we need to estimate \mathcal{P}_{β_1} from below. In doing this we encounter the difficulty that \mathcal{P}_{β_1} can go to zero, and moreover, simple computations show that as the pairs $(b^{(j)}, v^{(j)})$ satisfies the harmonicity condition (50) we have $v_{(r)}^{ns} b_n^{(j)} b_s^{(j)} = O(\rho_r + \rho_j)$. However, to show that $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ is non-vanishing for some \mathbf{v} we consider a particular choice of polarizations $v^{(r)}$, namely

$$(124) \quad v_{mk}^{(r)} = b_m^{(r)} b_k^{(r)}, \quad \text{for } r = 2, 3, 4, \text{ but not for } r = 1, 5$$

so that for $r = 2, 3, 4$, we have

$$\widehat{g}^{nm} b_n^{(r)} v_{mk}^{(r)} = 0, \quad \widehat{g}^{mk} v_{mk}^{(r)} = 0, \quad \widehat{g}^{nm} b_n^{(r)} v_{mk}^{(r)} - \frac{1}{2} (\widehat{g}^{mk} v_{mk}^{(r)}) b_k^{(r)} = 0.$$

Note that for this choice of $v^{(r)}$ the linearized harmonicity conditions hold. Moreover, for this choice of $v^{(r)}$ we see that for $\rho_j \leq \rho_r^{100}$

$$(125) \quad v_{(r)}^{ns} b_n^{(j)} b_s^{(j)} = \widehat{g}(b^{(r)}, b^{(j)}) \widehat{g}(b^{(r)}, b^{(j)}) = \rho_r^4 + O(\rho_r^5).$$

In particular, when $\beta = \beta_1$, so that $k_{\beta_1} = (6, 0, 0, 0)$ and the polarizations are given by (124), we have

$$\mathcal{P}_{\beta_1} = (\mathcal{D} + O(\rho_1)) \rho_1^4 \cdot \rho_1^4 \cdot \rho_1^4,$$

where \mathcal{D} is the inner product of $v^{(1)}$ and $v^{(5)}$ given in (122). Then the term $T_{\tau}^{(\mathbf{m}), \beta_1}$, which later turns out to have the strongest asymptotics in our considerations, has the asymptotics

$$(126) \quad T_{\tau}^{(\mathbf{m}), \beta_1} = \mathcal{L}_{\tau}, \quad \text{where} \\ \mathcal{L}_{\tau} = C \det(A) (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \vec{\rho}^d \rho_4^{-4} \rho_2^{-2} \rho_3^0 \rho_1^{20} \mathcal{D},$$

where $\vec{\rho} = (\rho_1, \rho_2, \rho_3, \rho_4)$, $\vec{a} = (a, a, a, a)$, and $\vec{1} = (1, 1, 1, 1)$. To compare different terms, we express ρ_j in powers of ρ_1 as explained in formula (119), that is, we write $\rho_4^{n_4} \rho_2^{n_2} \rho_3^{n_3} \rho_1^{n_1} = \rho_1^m$ with $m = 100^3 n_4 + 100^2 n_2 + 100 n_3 + n_1$. In particular, we will below write

$$\mathcal{L}_{\tau} = C_{\beta_1}(\vec{\rho}) \tau^{n_0} (1 + O(\tau^{-1})) \quad \text{as } \tau \rightarrow \infty \text{ for each fixed } \vec{\varepsilon}, \text{ and} \\ C_{\beta_1}(\vec{\varepsilon}) = c'_{\beta_1} \rho_1^{m_0} (1 + o(\rho_1)) \quad \text{as } \rho_1 \rightarrow 0.$$

Below we will show that c'_{β_1} does not vanish for generic $(\vec{x}, \vec{\xi})$ and (x_5, ξ_5) and polarizations \mathbf{v} . We will consider below $\beta \neq \beta_1$ and show that also these terms have the asymptotics

$$T_{\tau}^{(\mathbf{m}), \beta} = C_{\beta}(\vec{\rho}) \tau^n (1 + O(\tau^{-1})) \quad \text{as } \tau \rightarrow \infty \text{ for each fixed } \vec{\varepsilon}, \text{ and} \\ C_{\beta}(\vec{\varepsilon}) = c'_{\beta} \rho_1^m (1 + o(\rho_1)) \quad \text{as } \rho_1 \rightarrow 0.$$

When we have that either $n \leq n_0$ and $m < m_0$, or $n < n_0$, we say that $T_{\tau}^{(\mathbf{m}), \beta}$ has weaker asymptotics than $T_{\tau}^{(\mathbf{m}), \beta_1}$ and denote $T_{\tau}^{(\mathbf{m}), \beta} \prec \mathcal{L}_{\tau}$.

As we consider here the asymptotic of five small parameters τ^{-1} and ε_i , $i = 1, 2, 3, 4$, and compare in which order we make them tend to 0, let us explain the above ordering in detail. Above, we have chosen the order: first τ^{-1} , then $\varepsilon_4, \varepsilon_2, \varepsilon_3$ and finally, ε_1 . In correspondence with this choice we can introduce an ordering on all monomials $c\tau^{-c_{\tau}} \varepsilon_4^{n_4} \varepsilon_3^{n_3} \varepsilon_2^{n_2} \varepsilon_1^{n_1}$. Namely, we say that

$$(127) \quad C' \tau^{-n'_\tau} \varepsilon_4^{n'_4} \varepsilon_3^{n'_3} \varepsilon_2^{n'_2} \varepsilon_1^{n'_1} \prec C \tau^{-n_{\tau}} \varepsilon_4^{n_4} \varepsilon_3^{n_3} \varepsilon_2^{n_2} \varepsilon_1^{n_1}$$

if $C \neq 0$ and one of the following holds

- (i) if $n_{\tau} < n'_\tau$;
- (ii) if $n_{\tau} = n'_\tau$ but $n_4 < n'_4$;
- (iii) if $n_{\tau} = n'_\tau$ and $n_4 = n'_4$ but $n_2 < n'_2$;
- (iv) if $n_{\tau} = n'_\tau$ and $n_4 = n'_4, n_2 = n'_2$ but $n_3 < n'_3$;
- (v) if $n_{\tau} = n'_\tau$ and $n_4 = n'_4, n_3 = n'_3, n_2 = n'_2$ but $n_1 < n'_1$.

Then, we can analyze terms $T_{\tau, \sigma}^{(\mathbf{m}), \beta}$ and $\tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta}$ in the formula for

$$(128) \quad \Theta_{\tau}^{(4)} = \Theta_{\tau, \vec{\varepsilon}}^{(4)} = \sum_{\beta \in J_{\ell}} \sum_{\sigma \in \ell} \left(T_{\tau, \sigma}^{(\mathbf{m}), \beta} + \tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} \right).$$

Note that here the terms, in which the permutation σ is either the identical permutation id or the permutation $\sigma_0 = (2, 1, 3, 4)$, are the same.

Remark 3.3. We can find the leading order asymptotics of the strongest terms in the decomposition (128) using the following algorithm. First, let us multiply $\Theta_{\tau,\varepsilon}^{(4)}$ by $\tau^{\widehat{n}_\tau}$, where $\widehat{n}_\tau = \min_\beta n_\tau(\beta)$. Taking then $\tau \rightarrow \infty$ will give non-zero contribution from only those terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ where $n_\tau(\beta) = \widehat{n}_\tau$. This corresponds to step (i) above. Multiplying next by $\varepsilon_4^{-\widehat{n}_4}$, where $\widehat{n}_4 = \min_\beta n_4(\beta)$ under the condition that $n_\tau(\beta) = \widehat{n}_\tau$ and taking $\varepsilon_4 \rightarrow 0$ corresponds to selecting terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ with $n_\tau(\beta) = \widehat{n}_\tau$ and $n_4(\beta) = \widehat{n}_4$ and terms $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ with $\tilde{n}_\tau(\beta) = \widehat{n}_\tau$ and $\tilde{n}_4(\beta) = \widehat{n}_4$. This corresponds to step (ii). Continuing this process we obtain a scalar value that gives the leading order asymptotics of the strongest terms in the decomposition (128).

The next results tells what are the strongest terms in (128).

Proposition 3.5. *Assume that $v^{(r)}$, $r = 2, 3, 4$ are given by (124). In (128), the strongest term are $T_\tau^{(\mathbf{m}),\beta_1} = T_{\tau,id}^{(\mathbf{m}),\beta_1}$ and $T_{\tau,\sigma_0}^{(\mathbf{m}),\beta_1}$ in the sense that for all $(\beta, \sigma) \notin \{(\beta_1, id), (\beta_1, \sigma_0)\}$ we have $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec T_{\tau,id}^{(\mathbf{m}),\beta_1}$.*

Proof. When $\vec{S}_\beta = (S_1, S_2) = (\mathbf{Q}_0, \mathbf{Q}_0)$, similar computations to the above ones yield

$$\begin{aligned} \tilde{T}_\tau^{(\mathbf{m}),\beta} &= \langle u^\tau, h \cdot \mathbf{Q}_0(\mathcal{B}_4 u_4 \cdot \mathcal{B}_3 u_3) \cdot \mathbf{Q}_0(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= C \det(A) \mathcal{P}_\beta \frac{(i\tau)^{-(12+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+2)} \rho_3^{2(a-k_3+1+2)} \rho_2^{2(a-k_2+2)} \rho_1^{2(a-k_1+1+2)}}. \end{aligned}$$

Let us next consider the case when $\vec{S}_\beta = (S_1, S_2) = (I, \mathbf{Q}_0)$. Again, the computations similar to the above ones show that

$$\begin{aligned} T_\tau^{(\mathbf{m}),\beta} &= \langle \mathbf{Q}_0(u^\tau \cdot \mathcal{B}_4 u_4), h \cdot \mathcal{B}_3 u_3 \cdot I(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= i C \mathcal{P}_\beta \det(A) \frac{(i\tau)^{-(10+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+1+2)} \rho_3^{2(a-k_3+1)} \rho_2^{2(a-k_2+1)} \rho_1^{2(a-k_1+1)}} \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_\tau^{(\mathbf{m}),\beta} &= \langle u^\tau, h \cdot \mathbf{Q}_0(\mathcal{B}_4 u_4 \cdot \mathcal{B}_3 u_3) \cdot I(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= \mathcal{P}_\beta C \det(A) \frac{(i\tau)^{-(10+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+2)} \rho_3^{2(a-k_3+1+2)} \rho_2^{2(a-k_2+1)} \rho_1^{2(a-k_1+1)}}. \end{aligned}$$

When $\vec{S}_\beta = (S_1, S_2) = (\mathbf{Q}_0, I)$ we have $\tilde{T}_\tau^{(\mathbf{m}),\beta} = T_\tau^{(\mathbf{m}),\beta}$ and

$$\begin{aligned} T_\tau^{(\mathbf{m}),\beta} &= \langle I(u^\tau \cdot \mathcal{B}_4 u_4), h \cdot \mathcal{B}_3 u_3 \cdot \mathbf{Q}_0(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle, \\ &= i \mathcal{P}_\beta \det(A) C \frac{(i\tau)^{-(10+4a-|\vec{k}_\beta|)} (1 - O(\tau^{-1}))}{\rho_4^{2(a-k_4+1)} \rho_3^{2(a-k_3+1)} \rho_2^{2(a-k_2+2)} \rho_1^{2(a-k_1+1+2)}}, \end{aligned}$$

and finally when $\vec{S}_\beta = (S_1, S_2) = (I, I)$

$$\begin{aligned}\tilde{T}_\tau^{(\mathbf{m}),\beta} &= \langle u^\tau, h \cdot I(\mathcal{B}_4 u_4 \cdot \mathcal{B}_3 u_3) \cdot I(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= \mathcal{P}_\beta C_a \det(A) \frac{(i\tau)^{-(8+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+1)} \rho_3^{2(a-k_3+1)} \rho_2^{2(a-k_2+1)} \rho_1^{2(a-k_1+1)}}.\end{aligned}$$

We consider all β such that $\vec{S}^\beta = (\mathbf{Q}_0, \mathbf{Q}_0)$ but $\beta \neq \beta_1$. Then

$$\tilde{T}_\tau^{(\mathbf{m}),\beta} = C \det(A) (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \bar{\rho}^{\vec{d}+2\vec{k}_\beta} \rho_4^{-2} \rho_2^{-2} \rho_3^{-4} \rho_1^{-4} \cdot \mathcal{P}_\beta$$

where \vec{k}_β is as in (116). Note that for $\beta = \beta_1$ we have $\mathcal{P}_{\beta_1} = (\mathcal{D} + O(\rho_1)) \rho_1^4 \cdot \rho_1^4 \cdot \rho_1^4$ while $\beta \neq \beta_1$ we just use an estimate $\mathcal{P}_\beta = O(1)$. Then we see that $\tilde{T}_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$.

When β is such that $\vec{S}^\beta = (\mathbf{Q}_0, I)$, we see that

$$\begin{aligned}T_\tau^{(\mathbf{m}),\beta} &= C \det(A) (i\tau)^{-(10+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1})) \bar{\rho}^{-\vec{d}+2\vec{k}_\beta} \rho_4^0 \rho_2^{-2} \rho_3^0 \rho_1^{-4} \mathcal{P}_\beta, \\ \tilde{T}_\tau^{(\mathbf{m}),\beta} &= C \det(A) (i\tau)^{-(10+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1})) \bar{\rho}^{-\vec{d}+2\vec{k}_\beta} \rho_4^{-2} \rho_2^0 \rho_3^{-4} \rho_1^0 \mathcal{P}_\beta\end{aligned}$$

where $\mathcal{P}_\beta = O(1)$ and \vec{k}_β is as in (116) and hence $T_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$ and $\tilde{T}_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$.

When β is such that $\vec{S}^\beta = (I, \mathbf{Q}_0)$ or $\vec{S}^\beta = (I, I)$, using inequalities of the type (116) and (116) in appropriate cases, we see that $T_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$ and $\tilde{T}_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$.

The above shows that all terms $T_\tau^{(\mathbf{m}),\beta}$ and $\tilde{T}_\tau^{(\mathbf{m}),\beta}$ with maximal allowed k -s have asymptotics with the same power of τ but their $\bar{\rho}$ asymptotics vary, and when the asymptotic orders of ρ_j are given as explained in after (119), there is only one term, namely $\mathcal{L}_\tau = T_\tau^{(\mathbf{m}),\beta_1}$, that has the strongest order asymptotics given in (126).

Next we analyze the effect of the permutation $\sigma : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ of the indexes j of the waves u_j . We assume below that the permutation σ is not the identity map.

Recall that in the computation (120) there appears a term $\omega_{45}^{-1} \sim \rho_4^{-2}$. Since this term does not appear in the computations of the terms $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$, we see that $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$. Similarly, if σ is such that $\sigma(4) \neq 4$, the term ω_{45}^{-1} does not appear in the computation of $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and hence $T_{\tau,\sigma}^{(\mathbf{m}),\beta_1} \prec \mathcal{L}_\tau$. Next we consider the permutations for which $\sigma(4) = 4$.

Next we consider σ that is either $\sigma = (3, 2, 1, 4)$ or $\sigma = (2, 3, 1, 4)$. These terms are very similar and thus we analyze the case when $\sigma = (3, 2, 1, 4)$. First we consider the case when $\beta = \beta_2$ is such that $\vec{S}^{\beta_2} = (\mathbf{Q}_0, \mathbf{Q}_0)$, $\vec{k}_{\beta_2} = (2, 0, 4, 0)$. This term appears in the analysis of the term $A^{(1)}[u_{\sigma(4)}, \mathbf{Q}(A^{(2)}[u_{\sigma(3)}, \mathbf{Q}(A^{(3)}[u_{\sigma(2)}, u_{\sigma(1)}])])]$ when $(A^{(1)}, A^{(2)}, A^{(3)}) =$

(A_2, A_1, A_2) , see (65). By a permutation of the indexes in (120) we obtain the formula

$$(129) \quad \begin{aligned} T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} &= c'_1 \det(A) (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \vec{\rho}^{\vec{d}} \\ &\cdot (\omega_{45}\omega_{32})^{-1} \rho_4^{2(k_4-1)} \rho_1^{2k_3} \rho_2^{2(k_2-1)} \rho_3^{2(k_1-1)} \mathcal{P}_{\beta_2}, \\ \tilde{\mathcal{P}}_{\beta_2} &= (v_{(4)}^{pq} b_p^{(1)} b_q^{(1)}) (v_{(3)}^{rs} b_r^{(1)} b_s^{(1)}) (v_{(2)}^{nm} b_n^{(3)} b_m^{(3)}) \mathcal{D}. \end{aligned}$$

Hence, in the case when we use the polarizations (124), we obtain

$$T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} = c_1 (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \vec{\rho}^{\vec{d}} \rho_4^{-4} \rho_2^{-2} \rho_3^{0+4} \rho_1^{6+8} \mathcal{D}.$$

Comparing the power of ρ_3 in the above expression, we see that in this case $T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} \prec \mathcal{L}_\tau$. When $\sigma = (3, 2, 1, 4)$, we see in a straightforward way also for other β for which $\vec{S}^\beta = (\mathbf{Q}_0, \mathbf{Q}_0)$, that $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$.

When $\sigma = (1, 3, 2, 4)$, we see that for all β with $|\vec{k}_\beta| = 6$,

$$\begin{aligned} T_{\tau,\sigma}^{(\mathbf{m}),\beta} &= C \det(A) (i\tau)^{-(6+4a)} (1 + O(\frac{1}{\tau})) \vec{\rho}^{\vec{d}+2\vec{k}} \omega_{45}^{-1} \omega_{13}^{-1} \rho_4^{-2} \rho_2^0 \rho_3^{-2} \rho_1^{-2} \mathcal{P}_\beta \\ &= C \det(A) (i\tau)^{-(6+4a)} (1 + O(\frac{1}{\tau})) \vec{\rho}^{\vec{d}+2\vec{k}} \rho_4^{-4} \rho_2^0 \rho_3^{-2} \rho_1^{-4} \mathcal{P}_\beta. \end{aligned}$$

Here, $\mathcal{P}_\beta = O(1)$. Thus when $\sigma = (1, 3, 2, 4)$, by comparing the powers of ρ_2 we see that $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$. The same holds in the case when $|\vec{k}_\beta| < 6$. The case when $\sigma = (3, 1, 2, 4)$ is similar to $\sigma = (1, 3, 2, 4)$. This proves Proposition 3.5 \square

Summarizing; we have analyzed the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ corresponding to any β and all σ except $\sigma = \sigma_0 = (2, 1, 3, 4)$. Clearly, the sum $\sum_\beta T_{\tau,\sigma_0}^{(\mathbf{m}),\beta}$ is equal to the sum $\sum_\beta T_{\tau,id}^{(\mathbf{m}),\beta}$. Thus, when the asymptotic orders of ρ_j are given in (119) and the polarizations satisfy (124), we have

$$(130) \quad \begin{aligned} \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) &= \lim_{\tau \rightarrow \infty} \sum_{\beta,\sigma} \frac{T_{\tau,\sigma}^{(\mathbf{m}),\beta}}{(i\tau)^{(6+4a)}} = \lim_{\tau \rightarrow \infty} \frac{2T_{\tau,id}^{(\mathbf{m}),\beta_1} (1 + O(\rho_1))}{(i\tau)^{(6+4a)}} \\ &= 2c_1 \det(A) (1 + O(\rho_1)) \vec{\rho}^{\vec{d}} \rho_4^{-4} \rho_2^{-2} \rho_3^0 \rho_1^{20} \mathcal{D}. \end{aligned}$$

Next we consider general polarizations. Let $Y = \text{sym}(\mathbb{R}^{4 \times 4})$ and consider the non-degenerate, symmetric, bi-linear form $B : (v, w) \mapsto \hat{g}_{nj} \hat{g}_{mk} v^{nm} w^{jk}$ in Y . Then $\mathcal{D} = B(v^{(5)}, v^{(1)})$.

Let $L(b^{(j)})$ denote the subspace of dimension 6 of the symmetric matrices $v \in Y$ that satisfy equation (50) with covector $\xi = b^{(j)}$.

Let $\mathcal{W}(b^{(5)})$ be the real analytic submanifold of $(\mathbb{R}^4)^5 \times (\mathbb{R}^{4 \times 4})^3 \times (\mathbb{R}^{4 \times 4})^6 \times (\mathbb{R}^{4 \times 4})^6$ consisting of elements $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$, where $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$ is a sequence of light-like vectors with the given vector $b^{(5)}$, and $\underline{v} = (v^{(2)}, v^{(3)}, v^{(4)})$ satisfy $v^{(j)} \in L(b^{(j)})$ for all $j = 2, 3, 4$, and $V^{(1)} = (v_p^{(1)})_{p=1}^6 \in (\mathbb{R}^{4 \times 4})^6$ is a basis of $L(b^{(1)})$ and $V^{(5)} = (v_p^{(5)})_{p=1}^6 \in (\mathbb{R}^{4 \times 4})^6$ is sequence of vectors in Y such that

$B(v_p^{(5)}, v_q^{(1)}) = \delta_{pq}$ for $p \leq q$. Note that $\mathcal{W}(b^{(5)})$ has two components where the orientation of the basis $V^{(1)}$ is different.

By (109) and (109), $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ is linear in each $v^{(j)}$. For $\eta \in \mathcal{W}(b^{(5)})$, we define

$$\kappa(\eta) := \det \left(\mathcal{G}^{(\mathbf{m})}(\mathbf{v}_{(p,q)}, \mathbf{b}) \right)_{p,q=1}^6, \text{ where } \mathbf{v}_{(p,q)} = (v_p^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v_q^{(5)}).$$

Then $\kappa(\eta)$ can be written as $\kappa(\eta) = A_1(\eta)/A_2(\eta)$ where $A_1(\eta)$ and $A_2(\eta)$ are real-analytic functions on $\mathcal{W}(b^{(5)})$. In fact, $A_2(\eta)$ is a product of terms $\widehat{g}(b^{(j)}, b^{(k)})^p$ with some positive integer p , cf. (123).

Let us next consider the case when the sequence of the light-like vectors, $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$ given in (118) with $\vec{\rho}$ given in (119) with some small $\rho_1 > 0$ and let the polarizations $\underline{v} = (v^{(2)}, v^{(3)}, v^{(4)})$ be such that $v^{(j)} \in L(b^{(j)})$, $j = 2, 3, 4$, are those given by (124), and $V^{(1)} = (v_p^{(1)})_{p=1}^6$ be a basis of $L(b^{(1)})$. Let $V^{(5)} = (v_p^{(5)})_{p=1}^6$ be vectors in Y such that $B(v_p^{(5)}, v_q^{(1)}) = \delta_{pq}$ for $p \leq q$. When $\rho_1 > 0$ is small enough, formula (130) yields that $\kappa(\eta) \neq 0$ for $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$. Since $\kappa(\eta) = \kappa(\eta) = A_1(\eta)/A_2(\eta)$ where $A_1(\eta)$ and $A_2(\eta)$ are real-analytic on $\mathcal{L}(b^{(5)})$, we have that $\kappa(\eta)$ is non-vanishing and finite on an open and dense subset of the component of $\mathcal{W}(b^{(5)})$ containing η . The fact that $\kappa(\eta)$ is non-vanishing on a generic subset of the other component of $\mathcal{W}(b^{(5)})$ can be seen by changing the orientation of $V^{(1)}$. This yields that claim. \square

4. OBSERVATIONS IN NORMAL COORDINATES

We have considered above the singularities of the metric g in the wave gauge coordinates. As the wave gauge coordinates may also be non-smooth, we do not know if the observed singularities are caused by the metric or the coordinates. Because of this, we consider next the metric in normal coordinates.

Let $v^{\tilde{\varepsilon}} = (g^{\tilde{\varepsilon}}, \phi^{\tilde{\varepsilon}})$ be the solution of the \widehat{g} -reduced Einstein equations (8) with the source $\mathbf{f}_{\tilde{\varepsilon}}$ given in (54). We emphasize that $g^{\tilde{\varepsilon}}$ is the metric in the (g, \widehat{g}) -wave gauge coordinates.

Let $(z, \eta) \in \mathcal{U}_{(z_0, \eta_0)}(\widehat{h})$, $\mu_{\tilde{\varepsilon}} = \mu_{g^{\tilde{\varepsilon}}, z, \eta}$ and $(Z_{j, \tilde{\varepsilon}})_{j=1}^4$ be a frame of vectors obtained by $g^{\tilde{\varepsilon}}$ -parallel continuation of some $\tilde{\varepsilon}$ -independent frame along the geodesic $\mu_{\tilde{\varepsilon}}([-1, 1])$ from $\mu_{\tilde{\varepsilon}}(-1)$ to a point $p_{\tilde{\varepsilon}} = \mu_{\tilde{\varepsilon}}(\tilde{r})$. Recall that g and \widehat{g} coincide in the set $(-\infty, 0) \times N$ that contains the point $\mu_{\tilde{\varepsilon}}(-1)$. Let $\Psi_{\tilde{\varepsilon}} : W_{\tilde{\varepsilon}} \rightarrow \Psi_{\tilde{\varepsilon}}(W_{\tilde{\varepsilon}}) \subset \mathbb{R}^4$ denote normal coordinates of $(M_0, g^{\tilde{\varepsilon}})$ defined using the center $p_{\tilde{\varepsilon}}$ and the frame $Z_{j, \tilde{\varepsilon}}$. We say that $\mu_{\tilde{\varepsilon}}([-1, 1])$ are observation geodesics, and that $\Psi_{\tilde{\varepsilon}}$ are the normal coordinates associated to $\mu_{\tilde{\varepsilon}}$ and $p_{\tilde{\varepsilon}} = \mu_{\tilde{\varepsilon}}(\tilde{r})$. Denote $\Psi_0 = \Psi_{\tilde{\varepsilon}}|_{\tilde{\varepsilon}=0}$, and $W = W_0$. We also denote $g_{\tilde{\varepsilon}} = g^{\tilde{\varepsilon}}$ and $U_{\tilde{\varepsilon}} = U_{g^{\tilde{\varepsilon}}}$.

Lemma 4.1. *Let $v^{\vec{\varepsilon}} = (g^{\vec{\varepsilon}}, \phi^{\vec{\varepsilon}})$ and $\Psi_{\vec{\varepsilon}}$ be as above. Let $S \subset U_{\vec{\varepsilon}}$ be a smooth 3-dimensional surface such that $p_0 = p_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \in S$ and*

$$(131) \quad g^{(\alpha)} = \partial_{\vec{\varepsilon}}^\alpha g^{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \phi_\ell^{(\alpha)} = \partial_{\vec{\varepsilon}}^\alpha \phi_\ell^{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \text{for } |\alpha| \leq 4, \alpha \in \{0, 1\}^4,$$

and assume that $g^{(\alpha)}$ and $\phi_\ell^{(\alpha)}$ are in $C^\infty(W)$ for $|\alpha| \leq 3$ and $g_{pq}^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$ and $\phi_l^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$ for $\alpha_0 = (1, 1, 1, 1)$.

(i) *Assume that $S \cap W$ is empty. Then the tensors $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* g^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ and $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* \phi_\ell^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ are C^∞ -smooth in $\Psi_0(W)$.*

(ii) *Assume that $\mu_0([-1, 1])$ intersects S transversally at p_0 . Consider the conditions*

(a) *There is a 2-contravariant tensor field v that is a smooth section of $TW \otimes TW$ such that $v(x) \in T_x S \otimes T_x S$ for $x \in S$ and the principal symbol of $\langle v, g^{(\alpha_0)} \rangle|_W \in \mathcal{I}^{m_0}(W \cap S)$ is non-vanishing at p_0 .*

(b) *The principal symbol of $\phi_\ell^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$ is non-vanishing at p_0 for some $\ell = 1, 2, \dots, L$.*

If (a) or (b) holds, then either $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_ g^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ or $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* \phi_\ell^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ is not C^∞ -smooth in $\Psi_0(W)$.*

Proof. (i) is obvious.

(ii) Denote $\gamma_{\vec{\varepsilon}}(t) = \mu_{\vec{\varepsilon}}(t + \tilde{r})$. Let $X : W_0 \rightarrow V_0 \subset \mathbb{R}^4$, $X(y) = (X^j(y))_{j=1}^4$ be local coordinates in W_0 such that $X(p_0) = 0$ and $X(S \cap W_0) = \{(x^1, x^2, x^3, x^4) \in V_0; x^1 = 0\}$ and $y(t) = X(\gamma_0(t)) = (t, 0, 0, 0)$. Note that the coordinates X are independent of $\vec{\varepsilon}$. We assume that the vector fields $Z_{\vec{\varepsilon}, j}$ defining the normal coordinates are such that $Z_{0,j}(p_0) = \partial/\partial X^j$. To do computations in local coordinates, let us denote

$$\tilde{g}^{(\alpha)} = \partial_{\vec{\varepsilon}}^\alpha (X_* g^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}, \quad \tilde{\phi}_\ell^{(\alpha)} = \partial_{\vec{\varepsilon}}^\alpha (X_* \phi_\ell^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}, \quad \text{for } |\alpha| \leq 4, \alpha \in \{0, 1\}^4.$$

Let v be a tensor field given in (a) such that in the X coordinates $v(x) = v^{pq}(x) \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^q}$ so that $v^{pq}(0) = 0$ if $(p, q) \notin \{2, 3, 4\}^2$ at the point $0 = X(p_0)$ and the functions $v^{pq}(x)$ do not depend on x , that is, $v^{pq}(x) = \hat{v}^{mq} \in \mathbb{R}^{4 \times 4}$. Let $R^{\vec{\varepsilon}}$ be the curvature tensor of $g^{\vec{\varepsilon}}$ and define the functions

$$h_{mk}^{\vec{\varepsilon}}(t) = g^{\vec{\varepsilon}}(R^{\vec{\varepsilon}}(\dot{\gamma}_{\vec{\varepsilon}}(t), Z_m^{\vec{\varepsilon}}(t))\dot{\gamma}_{\vec{\varepsilon}}(t), Z_k^{\vec{\varepsilon}}(t)), \quad J_v(t) = \partial_{\vec{\varepsilon}}^{\alpha_0}(\hat{v}^{mq} h_{mq}^{\vec{\varepsilon}}(t))|_{\vec{\varepsilon}=0}.$$

The function $J_v(t)$ is an invariantly defined function on the curve $\gamma_0(t)$ and thus it can be computed in any coordinates. If $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* g^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ would be smooth near $0 \in \mathbb{R}^4$, then the function $J_v(t)$ would be smooth near $t = 0$. To show that the $\partial_{\vec{\varepsilon}}^{\alpha_0}$ -derivatives of the metric tensor in the normal coordinates are not smooth, we need to show that $J_v(t)$ is non-smooth at $t = 0$ for some values of \hat{v}^{mq} .

We will work in the X coordinates and denote $\tilde{R}(x) = \partial_{\vec{\varepsilon}}^{\alpha_0} X_*(R_{\vec{\varepsilon}})(x)|_{\vec{\varepsilon}=0}$. Moreover, $\tilde{\gamma}^j$ are the analogous 4th order ε -derivatives and we denote $\tilde{g} = \tilde{g}^{(\alpha_0)}$. For simplicity we also denote $X_* \hat{g}$ and $X_* \hat{\phi}_l$ by \hat{g} and $\hat{\phi}_l$, respectively.

We analyze the functions of $t \in I = (-t_1, t_1)$, e.g., $a(t)$, where $t_1 > 0$ is small. We say that $a(t)$ is of order n if $a(\cdot) \in \mathcal{I}^n(\{0\})$. By the assumptions of the theorem, $(\partial_x^\beta \tilde{g}_{jk})(\gamma(t)) \in \mathcal{I}^{m_0+\beta_1}(\{0\})$ when $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$. It follows from (131) and the linearized equations for the parallel transport that $(\tilde{\gamma}(t), \partial_t \tilde{\gamma}(t))$ and \tilde{Z}_k are in $\mathcal{I}^{m_0}(\{0\})$. The above analysis shows that $\tilde{R}|_{\gamma_0(I)} \in \mathcal{I}^{m_0+2}(\{0\})$. Thus in the X coordinates $\partial_{\vec{\varepsilon}}^{\alpha_0}(h_{mk}^{\vec{\varepsilon}}(t))|_{\vec{\varepsilon}=0} \in \mathcal{I}^{m_0+2}(\{0\})$ can be written as

$$\begin{aligned} \partial_{\vec{\varepsilon}}^{\alpha_0}(h_{mk}^{\vec{\varepsilon}}(t))|_{\vec{\varepsilon}=0} &= \widehat{g}(\tilde{R}(\dot{\gamma}_0(t), \tilde{Z}_m(t))\dot{\gamma}_0(t), \tilde{Z}_k(t)) + \text{s.t.} \\ &= \frac{1}{2} \left(\frac{\partial}{\partial x^1} \left(\frac{\partial \tilde{g}_{km}}{\partial x^1} + \frac{\partial \tilde{g}_{k1}}{\partial x^m} - \frac{\partial \tilde{g}_{1m}}{\partial x^k} \right) - \frac{\partial}{\partial x^m} \left(\frac{\partial \tilde{g}_{k1}}{\partial x^1} + \frac{\partial \tilde{g}_{k1}}{\partial x^1} - \frac{\partial \tilde{g}_{11}}{\partial x^k} \right) \right) + \text{s.t.}, \end{aligned}$$

where all s.t. = “smoother terms” are in $\mathcal{I}^{m_0+1}(\{0\})$.

Consider next the case (a). Assume that for given $(k, m) \in \{2, 3, 4\}^2$, the principal symbol of $\tilde{g}_{km}^{(\alpha_0)}$ is non-vanishing at $0 = X(p_0)$. Let v be such a tensor field that $v^{mk}(0) = v^{km}(0) \neq 0$ and $v^{in}(0) = 0$ when $(i, n) \notin \{(k, m), (m, k)\}$. Then the above yields (in the formula below, we do not sum over k, m)

$$J_v(t) = v^{ij} \partial_{\vec{\varepsilon}}^{\alpha_0}(h_{ij}^{\vec{\varepsilon}}(t))|_{\vec{\varepsilon}=0} = \frac{e(k, m)}{2} \widehat{v}^{km} \left(\frac{\partial}{\partial x^1} \frac{\partial \tilde{g}_{km}}{\partial x^1} \right) + \text{s.t.},$$

where $e(k, m) = 2 - \delta_{km}$. Thus the principal symbol of $J_v(t)$ in $\mathcal{I}^{m_0+2}(\{0\})$ is non-vanishing and $J_v(t)$ is not a smooth function. Thus in this case $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ is not smooth.

Next, we consider the case (b). Assume that there is ℓ such that the principal symbol of the field $\tilde{\phi}_\ell^{(\alpha_0)}$ is non-vanishing. As $\partial_t \tilde{\gamma}(t) \in \mathcal{I}^{m_0}(\{0\})$, we see that $\tilde{\gamma}(t) \in \mathcal{I}^{m_0-1}(\{0\})$. Then as $\phi_\ell^{\vec{\varepsilon}}$ are scalar fields,

$$j_\ell(t) = \partial_{\vec{\varepsilon}}^{\alpha_0} \left(\phi_\ell^{\vec{\varepsilon}}(\gamma_{\vec{\varepsilon}}(t)) \right) \Big|_{\vec{\varepsilon}=0} = \tilde{\phi}_\ell(\gamma(t)) + \text{s.t.},$$

where $j_\ell \in \mathcal{I}^{m_0}(\{0\})$ and the smoother terms (s.t.) are in $\mathcal{I}^{m_0-1}(\{0\})$. Thus in the case (b), $j_\ell(t)$ is not smooth and hence both $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ and $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* \phi_\ell^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ cannot be smooth. \square

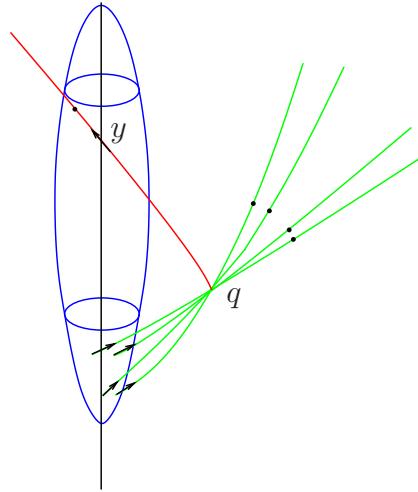


FIGURE A6: A schematic figure where the space-time is represented as the 3-dimensional set \mathbb{R}^{1+2} . The light-like geodesic emanating from the point q is shown as a red curve. The point q is the intersection of light-like geodesics corresponding to the starting points and directions $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$. A light like geodesic starting from q passes through the point y and has the direction η at y . The black points are the first conjugate points on the geodesics $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$, $j = 1, 2, 3, 4$, and $\gamma_{q, \zeta}([0, \infty))$. The figure shows the case when the interaction condition (I) is satisfied for $y \in U_{\hat{g}}$ with light-like vectors $(\vec{x}, \vec{\xi})$.

We use now the results above to detect singularities in normal coordinates. We say that the *interaction condition* (I) is satisfied for $y \in U_{\hat{g}}$ with light-like vectors $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ and $t_0 \geq 0$, with parameters (q, ζ, t) , if

- (I) There exist $q \in \bigcap_{j=1}^4 \gamma_{x_j(t_0), \xi_j(t_0)}((0, t_j))$, $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$, $\zeta \in L_q^+(M_0, \hat{g})$ and $t \geq 0$ such that $y = \gamma_{q, \zeta}(t)$.

For $\mathbf{f}_1 \in \mathcal{I}_C^{n+1}(Y((x_1, \xi_1); t_0, s_0))$ the wave $\mathbf{Q}\mathbf{f}_1 = \mathcal{M}^{(1)}$ is a solution of the linear wave equation. Thus, when \mathbf{f}_1 runs through the set $\mathcal{I}_C^{n+1}(Y((x_1, \xi_1); t_0, s_0))$ the union of the sets $\text{WF}(\mathbf{Q}\mathbf{f}_1)$ is the manifold $\Lambda((x_1, \xi_1); t_0, s_0)$. Thus, $\mathcal{D}(\hat{g}, \phi, \varepsilon)$ determines $\Lambda((x_1, \xi_1); t_0, s_0) \cap T^*U_{\hat{g}}$. In particular, using these data we can determine the geodesic segments $\gamma_{x_1, \xi_1}(\mathbb{R}_+) \cap U_{\hat{g}}$ for all $x_1 \in U_{\hat{g}}$, $\xi_1 \in L^+M_0$.

Below, in TM_0 we use the Sasaki metric corresponding to \hat{g}^+ . Moreover, let $s' \in (s_- + r_2, s_+)$, $0 < r_2 < r_1$ and $B_j \subset U_{\hat{g}}$ be open sets such that, cf. (10) and (55), for some $r'_0 \in (0, r_0)$ we have

$$(132) \quad B_j \subset \subset I_{\hat{g}}(\mu_{\hat{g}}(s' - r_2), \mu_{\hat{g}}(s')), \text{ and} \\ \text{for all } g' \in \mathcal{V}(r'_0), B_j \cap J_{g'}^+(B_k) = \emptyset \text{ for all } j \neq k.$$

We say that $y \in U_{\hat{g}}$ satisfies the singularity detection condition (D) with light-like directions $(\vec{x}, \vec{\xi})$ and $t_0, \hat{s} > 0$ if

(D) For any $s, s_0 \in (0, \hat{s})$ and $j = 1, 2, 3, 4$ there are (x'_j, ξ'_j) in the s -neighborhood of (x_j, ξ_j) , $(2s)$ -neighborhoods B_j of x_j , in $(U_{\hat{g}}, \hat{g}^+)$, satisfying (132), and sources $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$ having the LS property (14) in $C^{s_1}(M_0)$ with a family $\mathcal{F}_j(\varepsilon)$. This family is supported in B_j and satisfies $\partial_\varepsilon \mathcal{F}_j(\varepsilon)|_{\varepsilon=0} = \mathbf{f}_j$. Moreover, let $u_{\vec{\varepsilon}}$ be the solution of (8) with the source $\mathcal{F}_{\vec{\varepsilon}} = \sum_{j=1}^4 \mathcal{F}_j(\varepsilon_j)$ and $\mu_{\vec{\varepsilon}}([-1, 1])$ be observation geodesics with $y = \mu_0(\tilde{r})$, and $\Psi_{\vec{\varepsilon}}$ be the normal coordinates associated to $\mu_{\vec{\varepsilon}}$ at $\mu_{\vec{\varepsilon}}(\tilde{r})$. Then $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ or $\partial_{\vec{\varepsilon}}^{\alpha_0}((\Psi_{\vec{\varepsilon}})_* \phi_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ is not C^∞ -smooth near $0 = \Psi_0(y)$.

Lemma 4.2. *Let $(\vec{x}, \vec{\xi})$, and t_j with $j = 1, 2, 3, 4$ and $t_0 > 0$ satisfy (75)-(76). Let $t_0, \hat{s} > 0$ be sufficiently small and assume that $y \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_{\hat{g}}$ satisfies $y \notin \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, \hat{s}) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$. Then*

(i) *If y does not satisfy condition (I) with $(\vec{x}, \vec{\xi})$ and t_0 , then y does not satisfy condition (D) with $(\vec{x}, \vec{\xi})$ and $t_0, \hat{s} > 0$.*

(ii) *Assume $y \in U_{\hat{g}}$ satisfies condition (I) with $(\vec{x}, \vec{\xi})$ and t_0 and parameters q, ζ , and $0 < t < \rho(q, \zeta)$. Then y satisfies condition (D) with $(\vec{x}, \vec{\xi})$, t_0 , and any sufficiently small $\hat{s} > 0$.*

(iii) *Using the data set $\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$ we can determine whether the condition (D) is valid for the given point $y \in W_{\hat{g}}$ with the parameters $(\vec{x}, \vec{\xi})$, \hat{s} , and t_0 or not.*

Proof. (i) If $y \notin \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, \hat{s}) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$, the same condition holds also for $(\vec{x}', \vec{\xi}')$ close to $(\vec{x}, \vec{\xi})$. Thus Prop. 3.3 and Lemma 4.1 imply that (i) holds.

(ii) Let $\mu_{\vec{\varepsilon}}([-1, 1])$ be observation geodesics with $y = \mu_0(\tilde{r})$ and $\Psi_{\vec{\varepsilon}}$ be the normal coordinates associated to $\mu_{\vec{\varepsilon}}([-1, 1])$ at $\mu_{\vec{\varepsilon}}(\tilde{r})$. Our aim is to show that there is a source $\mathbf{f}_{\vec{\varepsilon}}$ described in (D) such that $\partial_{\vec{\varepsilon}}^4((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ or $\partial_{\vec{\varepsilon}}^4((\Psi_{\vec{\varepsilon}})_* \phi_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ is not C^∞ -smooth at $y = \gamma_{q, \zeta}(t)$.

Let $\eta = \partial_t \gamma_{q, \zeta}(t)$ and denote $(y, \eta) = (x_5, \xi_5)$. Let $t_j > 0$ be such that $\gamma_{x_j, \xi_j}(t_j) = q$ and denote $b_j = (\partial_t \gamma_{x_j, \xi_j}(t_j))^\flat$, $j = 1, 2, 3, 4$. Also, let us denote $b_5 = \zeta^\flat$ and $t_5 = -t$, so that $q = \gamma_{x_5, \xi_5}(t_5)$ and $b_5 = (\dot{\gamma}_{x_5, \xi_5}(t_5))^\flat$.

By assuming that $V \subset U_{\hat{g}}$ is a sufficiently small neighborhood of y , we have that $S := \mathcal{L}_{\hat{g}}^+(q) \cap V$ is a smooth 3-submanifold, as $t < \rho(q, \zeta)$.

Let $u_\tau = \mathbf{Q}_{\hat{g}}^* F_\tau$ be a gaussian beam, produced by a source $F_\tau(x) = F_\tau(x; x_5, \xi_5)$ and function $h(x)$ given in (67). Then we can use the techniques of [61, 90], see also [4, 59], to obtain a result analogous to Lemma 3.1 for the propagation of singularities along the geodesic $\gamma_{q, \zeta}([0, t])$: We have that when $h(x_5) = H$ and w is the principal symbol of u_τ at (q, b_5) , then $w = (R_{(5)})^* H$, where $R_{(5)}$ is a bijective linear map. The map $(R_{(5)})^*$ is similar to the map $R_{(1)}(q, b_5; x_5, \xi_5)$ considered in the formula (52) and it is obtained by solving a system of linear ordinary differential equations along the geodesic connecting q to x_5 .

Let $s \in (0, \hat{s})$ be sufficiently small and denote $b'_5 = b_5$. Using Propositions 3.3 and 3.4, we see that there are $(b'_j)^\sharp \in L_q^+ M_0$, $j = 1, 2, 3, 4$, in the s -neighborhoods of $b_j^\sharp \in L_q^+ M_0$, vectors $v^{(j)} \in \mathbb{R}^{10+L}$, $j \in \{2, 3, 4\}$, linearly independent vectors $v_p^{(5)} \in \mathbb{R}^{10+L}$, $p = 1, 2, 3, 4, 5, 6$, and linearly independent vectors $v_r^{(1)} \in \mathbb{R}^{10+L}$, $r = 1, 2, 3, 4, 5, 6$, that have the following properties:

(a) All $v^{(j)}$, $j = 2, 3, 4$, and $v_r^{(1)}$, $r = 1, 2, \dots, 6$ satisfy the harmonicity conditions for the symbols (50) with the covector ξ being b'_j and b'_1 , respectively.

(b) Let $X_1 = \text{span}(\{v_r^{(1)}; r = 1, 2, 3, \dots, 6\})$ and $X_5 = \text{span}(\{v_p^{(5)}; p = 1, 2, 3, \dots, 6\})$. If $v^{(5)} \in X_5 \setminus \{0\}$ then there exists a vector $v^{(1)} \in X_1$ such that for $\mathbf{v} = (v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)})$ and $\mathbf{b}' = (b'_j)_{j=1}^5$ we have $\mathcal{G}(\mathbf{v}, \mathbf{b}') \neq 0$.

Let $Y_5 := \text{sym}(T_y S \otimes T_y S) \times \mathbb{R}^L$. Since the codimension of $((R_{(5)})^*)^{-1} Y_5$ in $\text{sym}(T_q M \otimes T_q M) \times \mathbb{R}^L$ is 4 and the dimension of X_5 is 6, we see that dimension of the intersection $Z_5 = X_5 \cap ((R_{(5)})^*)^{-1} Y_5$ is at least 2. Thus there exist $v^{(j)} \neq 0$, $j = 1, 2, 3, 4, 5$ that satisfy the above conditions (a) and (b) and $v^{(5)} \in Z_5$. Let $H^{(5)} = ((R_{(5)})^*)^{-1} v^{(5)} \in ((R_{(5)})^*)^{-1} Z_5$.

Let $s_0 \in (0, \hat{s})$ and $x'_j = \gamma_{q, (b'_j)^\sharp}(-t_j)$, and $\xi'_j = \partial_t \gamma_{q, (b'_j)^\sharp}(-t_j)$, $j = 1, 2, 3, 4$. We denote $(\vec{x}', \vec{\xi}') = ((x'_j, \xi'_j))_{j=1}^4$.

Moreover, let B_j be neighborhoods of x'_j satisfying (132). Then, by condition μ -LS there are sources $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$ having the LS property (14) in $C^{s_1}(M_0)$ with some family $\mathcal{F}_j(\varepsilon)$ supported in B_j such that the principal symbols of \mathbf{f}_j at $(x'_j(t_0), (\xi'_j(t_0))^\flat)$ are equal to $w^{(j)} = R_j^{-1} v^{(j)}$, where $R_j = R_{(1)}(q, b'_j; x'_j(t_0), (\xi'_j(t_0))^\flat)$ are defined by formula (52). Then the principal symbols of $\mathbf{Q}_{\widehat{g}} \mathbf{f}_j$ at (q, b'_j) are equal to $v^{(j)}$. Let $u_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}} - \widehat{g}, \phi_{\vec{\varepsilon}} - \widehat{\phi})$ be the solution of (26) corresponding to $\mathcal{F}_{\vec{\varepsilon}} = \sum_{j=1}^4 \mathcal{F}_j(\varepsilon_j)$ with $\partial_{\varepsilon_j} \mathcal{F}_j(\varepsilon_j)|_{\varepsilon_j=0} = \mathbf{f}_j$. When s_0 is small enough, $\mathcal{M}^{(4)} = \partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ is a conormal distribution in the neighborhood V of y and $\mathcal{M}^{(4)}|_V \in \mathcal{I}(S)$. By Propositions 3.3 and 3.4, the inner product $\langle F_\tau, \mathcal{M}^{(4)} \rangle_{L^2(U_{\widehat{g}})}$ is not of order $O(\tau^{-N})$ for all $N > 0$, so that $\mathcal{M}^{(4)}$ is not smooth near y , see [47]. Let $h(x) \in C_0^\infty(U_{\widehat{g}})$ be such that $h(x_5) = H^{(5)}$. The above implies that the principal symbol of the function $x \mapsto \langle h(x), \mathcal{M}^{(4)}(x) \rangle_{BL}$ is not vanishing at (y, η^\flat) . Thus the principal symbols of the functions (131) are not vanishing and as $h(x_5) \in ((R_{(5)})^*)^{-1} Z_5$, conditions (ii) in Lemma 4.1 are satisfied. Thus either $\partial_{\vec{\varepsilon}}^4 ((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ or $\partial_{\vec{\varepsilon}}^4 ((\Psi_{\vec{\varepsilon}})_* \phi_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ is not C^∞ -smooth at $0 = \Psi_0(y)$. Thus, condition (D) is valid for y . This proves (ii).

(iii) Below, we will assume that $\mathcal{D}(\widehat{g}, \widehat{\phi}, \widehat{\varepsilon})$ is given with $\widehat{\varepsilon} > 0$.

To verify (D), we need to consider the solution $v_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}}, \phi_{\vec{\varepsilon}})$ in the wave gauge coordinates. For a general element $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in$

$\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$ we encounter the difficulty that we do not know the wave gauge coordinates in the set (U_g, g) . However, we construct the wave map (i.e. the wave gauge) coordinates for sources F of a special form. Below, we give the proof in several steps.

Step 1. Let $s_- \leq s' \leq s_0$ and $r_2 \in (0, r_1)$ be so small that $I_{\hat{g}}(\mu_{\hat{g}}(s' - r_2), \mu_{\hat{g}}(s')) \subset W_{\hat{g}}$. Assume that $\hat{\varepsilon}$ is small enough and that we are given an element $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in \mathcal{D}(\hat{g}, \hat{\phi}, \hat{\varepsilon})$ such that F is supported in $I_g(\mu_g(s' - r_2), \mu_g(s')) \subset W_g$. As we know $(U_g, g|_{U_g})$, assuming that $\hat{\varepsilon}$ is small enough, we can find the wave map $\Psi : I_g(\mu_g(s' - r_2), \mu_g(s')) \rightarrow U_{\hat{g}}$ by solving (5) in $I_g^-(\mu_g(s')) \cap U_g$. Then Ψ is the restriction of the wave map f solving (5)-(6). Then we can determine in the wave gauge coordinates Ψ the source $\Psi_* F$ and the solution $(\Psi_* g, \Psi_* \phi)$ in $\Psi(I_g^-(\mu_g(s')) \cap U_g)$. Observe that the function $\Psi_* F$ vanishes on $U_{\hat{g}}$ outside the set $\Psi(I_g^-(\mu_g(s')) \cap U_g)$. Thus the source $f_* F$ is determined in the wave gauge coordinates f in the whole set $U_{\hat{g}}$, where f solves (5)-(6). This construction can be done for all equivalence classes $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$ in $\mathcal{D}(\hat{g}, \hat{\phi}, \hat{\varepsilon})$ such that F is supported in $I_g(\mu_g(s' - r_2), \mu_g(s')) \subset W_g$. Next, we consider sources on the set $U_{\hat{g}}$ endowed with the background metric \hat{g} . Let \mathcal{F} be a source function on the set $U_{\hat{g}}$ such that $J_{\hat{g}}^-(\text{supp } (\mathcal{F})) \cap J_{\hat{g}}^+(\text{supp } (\mathcal{F})) \subset I_{\hat{g}}(\mu_{\hat{g}}(s' - r_2), \mu_{\hat{g}}(s'))$ and assume that \mathcal{F} is sufficiently small in the C^{s_1} -norm. Then, using the above considerations, we can determine the unique equivalence class $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$ in $\mathcal{D}(\hat{g}, \hat{\phi}, \hat{\varepsilon})$ for which $f_* F$ is equal to \mathcal{F} . In this case we say that the equivalence class corresponds to \mathcal{F} in the wave gauge coordinates and that \mathcal{F} is an admissible source.

Step 2. Let $k_1 \geq 8$, $s_1 \geq k_1 + 5$, and $n \in \mathbb{Z}_+$ be large enough. Let (x'_j, ξ'_j) be covectors in an s -neighborhood of (x_j, ξ_j) and B_j be neighborhoods of x'_j satisfying (132). Consider then $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$, and a family $\mathcal{F}_j(\varepsilon_j) \in C^{s_1}(M_0)$, $\varepsilon_j \in [0, \varepsilon_0]$ of functions supported in B_j that depend smoothly on ε_j . Moreover, assume that $\partial_{\varepsilon_j} \mathcal{F}_j(\varepsilon_j)|_{\varepsilon_j=0} = \mathbf{f}_j$. Then, we can use step 1 to test if all sources \mathbf{f}_j and $\mathcal{F}_j(\varepsilon_j)$, $\varepsilon_j \in [0, \varepsilon_0]$ are admissible.

Step 3. Next, assume that \mathbf{f}_j and $\mathcal{F}_j(\varepsilon_j)$, $\varepsilon_j \in [0, \varepsilon_0]$, $j = 1, 2, 3, 4$ are admissible and are compactly supported in neighborhoods B_j of x'_j satisfying (132).

Let us next consider $\vec{a} = (a_1, a_2, a_3, a_4) \in (-1, 1)^4$ and define $\mathcal{F}(\tilde{\varepsilon}, a) = \sum_{j=1}^4 \mathcal{F}_j(a_j \tilde{\varepsilon})$. Let $(g_{\tilde{\varepsilon}, \vec{a}}, \phi_{\tilde{\varepsilon}, \vec{a}})$ be the solution of (8) with the source $\mathcal{F}(\tilde{\varepsilon}, \vec{a})$. By (132), $B_j \cap J_{g_{\tilde{\varepsilon}, \vec{a}}}^+(B_k) = \emptyset$ for $j \neq k$ and $\tilde{\varepsilon}$ small enough. Hence we have that for sufficiently small $\tilde{\varepsilon}$ the conservation law (9) is satisfied for $(g_{\tilde{\varepsilon}, \vec{a}}, \phi_{\tilde{\varepsilon}, \vec{a}})$ in the set $Q = (M_0 \setminus J_{g_{\tilde{\varepsilon}, \vec{a}}}^+(p^-)) \cup \bigcup_{j=1}^4 J_{g_{\tilde{\varepsilon}, \vec{a}}}^-(B_j)$, see Fig. 6(Left). Solving the Einstein equations (8) in a neighborhood of the closure of the complement of the set Q , where the source $\mathcal{F}(\tilde{\varepsilon}, a)$ vanishes, we see that the conservation law (9) is satisfied in the whole

set M_0 , see [15, Sec. III.6.4.1]. Hence $\sum_{j=1}^4 a_j \mathbf{f}_j$ has the LS property (14) in $C^{s_1}(M_0)$ with the family $\mathcal{F}(\tilde{\varepsilon}, \vec{a})$. When $\tilde{\varepsilon} > 0$ is small enough, the sources $\mathcal{F}(\tilde{\varepsilon}, \vec{a})$ are admissible.

Step 4. Let $v_{\tilde{\varepsilon}, \vec{a}} = (g_{\tilde{\varepsilon}, \vec{a}}, \phi_{\tilde{\varepsilon}, \vec{a}})$ be the solutions of the Einstein equations (8) with the source $\mathcal{F}(\tilde{\varepsilon}, \vec{a})$.

Using Step 1, we can find for all $\vec{a} \in (-1, 1)^4$ and sufficiently small $\tilde{\varepsilon}$ the equivalence classes $[(U_{\tilde{\varepsilon}, \vec{a}}, g_{\tilde{\varepsilon}, \vec{a}}|_{U_{\tilde{\varepsilon}, \vec{a}}}, \phi_{\tilde{\varepsilon}, \vec{a}}|_{U_{\tilde{\varepsilon}, \vec{a}}}, F(\tilde{\varepsilon}, \vec{a})|_{U_{\tilde{\varepsilon}, \vec{a}}})]$ correspond to $\mathcal{F}(\tilde{\varepsilon}, \vec{a})$ in the wave guide coordinates. Then we can determine, using the normal coordinates $\Psi_{\tilde{\varepsilon}, \vec{a}}$ associated to the observation geodesics $\mu_{\tilde{\varepsilon}, \vec{a}}$ and the solution $v_{\tilde{\varepsilon}, \vec{a}}$, the function $(\Psi_{\tilde{\varepsilon}, \vec{a}})_* v_{\tilde{\varepsilon}, \vec{a}}$. Also, we can compute the derivatives of this function with respect to $\tilde{\varepsilon}$ and a_j .

Observe that $\partial_{\tilde{\varepsilon}}^4 f(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)|_{\tilde{\varepsilon}=0} = \partial_{\vec{a}}^4 (\partial_{\tilde{\varepsilon}}^4 f(a_1 \tilde{\varepsilon}, a_2 \tilde{\varepsilon}, a_3 \tilde{\varepsilon}, a_4 \tilde{\varepsilon})|_{\tilde{\varepsilon}=0})|_{\vec{a}=0}$ so that $\mathcal{M}^{(4)} = \partial_{a_1} \partial_{a_2} \partial_{a_3} \partial_{a_4} \partial_{\tilde{\varepsilon}}^4 v_{\tilde{\varepsilon}, \vec{a}}|_{\tilde{\varepsilon}=0, \vec{a}=0}$, where $\mathcal{M}^{(4)}$ given in (56). Thus, using the solutions $v_{\tilde{\varepsilon}, \vec{a}}$ we determine if the function $\partial_{\tilde{\varepsilon}}^4 ((\Psi_{\tilde{\varepsilon}})_* v_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$, corresponding to the source $f_{\tilde{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j \mathbf{f}_j$ is singular, where $\Psi_{\tilde{\varepsilon}}$ are the normal coordinates associated to any observation geodesic. Thus we can verify if the condition (D) holds. \square

5. DETERMINATION OF EARLIEST LIGHT OBSERVATION SETS

Below, we use only the metric \hat{g} and often denote $\hat{g} = g$, $U = U_{\hat{g}}$.

Our next aim is to consider the global problem of constructing the set of the earliest light observations of all points $q \in J(p^-, p^+)$. To this end, we need to handle the technical problem that in the set $\mathcal{Y}((\vec{x}, \vec{\xi}))$ we have not analyzed if we observe singularities or not. Also, we have not analyzed the waves in the set where singularities caused by caustics or their interactions may appear, see (76). To avoid these difficulties, we define next the sets $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$ of points near which we observe singularities in a 3-dimensional set.

Definition 5.1. Let $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ be a collection of light-like vectors with $x_j \in U_{\hat{g}}$ and $t_0 > 0$. We define $\mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ be the set of those $y \in U_{\hat{g}}$ that satisfies the property (D) with $(\vec{x}, \vec{\xi})$ and t_0 and some $\hat{s} > 0$. Moreover, let $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$ be the set of the points $y_0 \in \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ having a neighborhood $W \subset U_{\hat{g}}$ such that the intersection $W \cap \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ is a non-empty smooth 3-dimensional submanifold. We denote (see (19) and Def. 2.4)

$$(133) \quad \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) = cl(\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)) \cap U_{\hat{g}},$$

$$(134) \quad \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \bigcup_{(z, \eta) \in \mathcal{U}_{z_0, \eta_0}} \mathbf{e}_{z, \eta}(\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)).$$

The data $\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$ determines the set $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$. Below, we fix t_0 to be $t_0 = 4\kappa_1$, cf. Lemma 2.3.

Our goal is to show that $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ coincides with the intersection of the light cone $\mathcal{L}_g^+(q)$ and $U_{\widehat{g}}$ where q is the intersection point of the geodesics corresponding to $(\vec{x}, \vec{\xi})$, see Fig. 3(Left).

Let us next motivate the analysis we do below: We will consider how to create an artificial point source using interaction of distorted plane waves propagating along light-like geodesics $\gamma_{x_j, \xi_j}(\mathbb{R}_+)$ where $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ are perturbations of a light-like (y, ζ) , $y \in \widehat{\mu}$. We will use the fact that for all $q \in J(p^-, p^+) \setminus \widehat{\mu}$ there is a light-like geodesic $\gamma_{y, \zeta}([0, t])$ from $y = \widehat{\mu}(f_{\widehat{\mu}}^-(q))$ to q with $t \leq \rho(y, \zeta)$. We will next show that when we choose (x_j, ξ_j) to be suitable perturbations of $\partial_t \gamma_{y, \zeta}(t_0)$, $t_0 > 0$, it is possible that all geodesics $\gamma_{x_j, \xi_j}(\mathbb{R}_+)$ intersect at q before their first cut points, that is, $\gamma_{x_j, \xi_j}(r_j) = q$, $r_j < \rho(x_j, \xi_j)$. We note that we cannot analyze the interaction of the waves if the geodesics intersect after the cut points as then the distorted plane waves can have caustics. Such interactions of wave caustics can, in principle, cause propagating singularities. Thus the sets $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ contain singularities propagating along the light cone $\mathcal{L}_g^+(q)$ and in addition that they may contain singularities produced by caustics that we do not know how to analyze (that could be called “messy waves”). Fortunately, near an open and dense set of geodesics $\mu_{z, \eta}$ the nice singularities propagating along the light cone $\mathcal{L}_g^+(q)$ arrive before the “messy waves”. This is the reason why we consider below the first observed singularities on geodesics $\mu_{z, \eta}$. Let us now return to the rigorous analysis.

Below in this section we fix t_0 to have the value $t_0 = 4\kappa_1$, cf. Lemma 2.3. Recall the notation that

$$(x(t_0), \xi(t_0)) = (\gamma_{x, \xi}(t_0), \dot{\gamma}_{x, \xi}(t_0)), \\ (\vec{x}(t_0), \vec{\xi}(t_0)) = ((x_j(t_0), \xi_j(t_0)))_{j=1}^4.$$

Lemma 5.1.A *Let $\vartheta > 0$ be arbitrary, $q \in J(p^-, p^+) \setminus \widehat{\mu}$ and let $y = \widehat{\mu}(f_{\widehat{\mu}}^-(q))$ and $\zeta \in L_y^+ M_0$, $\|\zeta\|_{g^+} = 1$ be such that $\gamma_{y, \zeta}([0, r_1])$, $r_1 > t_0 = 4\kappa_1$ is a longest causal (in fact, light-like) geodesic connecting y to q . Then there exists a set \mathcal{G} of 4-tuples of light-like vectors $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ such that the points x_j and the directions ξ_j and the points $x_j(t_0) = \gamma_{x_j, \xi_j}(t_0)$ have the following properties:*

- (i) $x_j(t_0) \in U_{\widehat{g}}$, $x_j(t_0) \notin J^+(x_k(t_0))$ for $j \neq k$,
- (ii) $d_{g^+}((x_l, \xi_l), (y, \zeta)) < \vartheta$ for $l \leq 4$,
- (iii) $q = \gamma_{x_j, \xi_j}(r_j)$ and $\rho(x_j(t_0), \xi_j(t_0)) + t_0 > r_j$,
- (iv) when $(\vec{x}, \vec{\xi})$ run through the set \mathcal{G} , the directions $(\dot{\gamma}_{x_j, \xi_j}(r_j))_{j=1}^4$ form an open set in $(L_q^+ M_0)^4$.

In addition, \mathcal{G} contains elements $(\vec{x}, \vec{\xi})$ for which $(x_1, \xi_1) = (y, \zeta)$.

Proof. Let $\eta = \dot{\gamma}_{y,\zeta}(r_0) \in L_q^+ M_0$. Let us choose light-like directions $\eta_j \in T_q M_0$, $j = 1, 2, 3, 4$, close to η so that η_j and η_k are not parallel for $j \neq k$. In particular, it is possible (but not necessary) that $\eta_1 = \eta$. Let $\mathbf{t} : M \rightarrow \mathbb{R}$ be a time-function on M that can be used to identify M and $\mathbb{R} \times N$. Moreover, let us choose $T \in ((\gamma_{y,\zeta}(r_0 - t_0)), \mathbf{t}(\gamma_{y,\zeta}(r_0 - 2t_0)))$ and for $j = 1, 2, 3, 4$, let $s_j > 0$ be such that $\mathbf{t}(\gamma_{q,\eta_j}(-s_j)) = T$. Choosing first T to be sufficiently close $\mathbf{t}(\gamma_{y,\zeta}(r_0 - t_0))$ and then all η_j , $j = 1, 2, 3, 4$ to be sufficiently close to η and defining $x_j = \gamma_{q,\eta_j}(-s_j - t_0)$ and $\xi_j = \dot{\gamma}_{q,\eta_j}(-s_j - t_0)$ we obtain the pairs (x_j, ξ_j) satisfying the properties stated in the claim. Indeed, this follows from the fact that $\rho(x, \xi)$ is lower semicontinuous function, $\rho(q, \eta) \geq r_0$ and the geodesics γ_{q,η_j} can not intersect before their first cut points. As vectors η_j can be varied in sufficiently small open sets so that the properties stated in the claim stay valid, we obtain the claim concerning the open set of light-like directions.

The last claim follows from the fact that η_1 may be equal to η and $T = \mathbf{t}(y)$. \square

Next we analyze the set $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

If the set $\cap_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))$ is non-empty we denote its earliest point by $Q((\vec{x}, \vec{\xi}), t_0)$. If such intersection point does not exists, we define $Q((\vec{x}, \vec{\xi}), t_0)$ to be the empty set. Next we consider the relation of $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$ and $\mathcal{E}_U(q)$, $q = Q((\vec{x}, \vec{\xi}), t_0)$, see Def. 2.4.

Lemma 5.2. *Let $(\vec{x}, \vec{\xi})$, $j = 1, 2, 3, 4$ and $t_0 > 0$ satisfy (75)-(76) and assume that ϑ_1 in (75) and Lemma 2.3 is so small that for all $j \leq 4$, $x_j \in I(\hat{\mu}(s_1), \hat{\mu}(s_2)) \subset W_{\hat{g}}$ with some $s_1, s_2 \in (s_-, s_+)$.*

Let $\mathcal{V} = \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ be the set defined in (76). Then

(i) Assume that $y \in \mathcal{V} \cap U_{\hat{g}}$ satisfies the condition (I) with $(\vec{x}, \vec{\xi})$ and t_0 and parameters q, ζ , and t such that $0 \leq t \leq \rho(q, \zeta)$. Then $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

(ii) Assume $y \in \mathcal{V} \cap U_{\hat{g}}$ does not satisfy condition (I) with $(\vec{x}, \vec{\xi})$ and t_0 . Then $y \notin \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

(iii) If $Q((\vec{x}, \vec{\xi}), t_0) \neq \emptyset$ and $q = Q((\vec{x}, \vec{\xi}), t_0) \in \mathcal{V}$, we have

$$\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q) \subset \mathcal{V}.$$

Otherwise, if $Q((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$, then $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$.

Proof. (i) Assume first that y is not in $\mathcal{Y}((\vec{x}, \vec{\xi}))$ and $t < \rho(q, \zeta)$. Then the assumptions in (i) and Lemma 4.2 (ii) and (iii) imply that $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. Consider next a general point y satisfying the assumptions in (i) and let $q = Q((\vec{x}, \vec{\xi}), t_0)$. Then $y \in \mathcal{E}_U(q)$. Since $\rho(x, \xi)$ is lower semi-continuous and the set $\mathcal{E}_U(q) \setminus \mathcal{Y}((\vec{x}, \vec{\xi}))$ is dense in $\mathcal{E}_U(q)$,

and y is a limit point of points $y_n \notin \mathcal{Y}((\vec{x}, \vec{\xi}))$ that satisfy the assumptions in (i). Hence $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

The claim (ii) follows from Lemma 4.2 (i).

(iii) Suppose $q = Q((\vec{x}, \vec{\xi}), t_0) \in \mathcal{V}$ and $y \in \mathcal{E}_U(q) \setminus \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))$. Let $\gamma_{q, \eta}([0, l])$ be a light-like geodesic that is the longest causal geodesic from q to y with $l \leq \rho(q, \eta)$, and let $p_j = \gamma_{x_j, \xi_j}(t_0 + \mathbf{t}_j)$, $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$, be the first cut point on the geodesic $\gamma_{x_j, \xi_j}([t_0, \infty))$. To show that y is in \mathcal{V} , we assume the opposite, $y \notin \mathcal{V}$. Then for some j there is a causal geodesic $\gamma_{p_j, \theta_j}([0, l_j])$ from p_j to y . Now we can use a short-cut argument: Let $q = \gamma_{x_j, \xi_j}(t')$. As $q \in \mathcal{V}$, we have $t' < t_0 + \mathbf{t}_j$. Moreover, as $y \notin \gamma_{x_j, \xi_j}([t_0, \infty))$, the union of the geodesic $\gamma_{x_j, \xi_j}([t', t_0 + \mathbf{t}_j])$ from q to p_j and $\gamma_{p_j, \theta_j}([0, l_j])$ from p_j to y does not form a light-like geodesic and thus $\tau(q, y) > 0$. As $y \in \mathcal{E}_U(q)$, this is not possible. Hence $y \in \mathcal{V}$. Thus by (i), $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ and hence $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))) \subset \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. Since the set $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty)))$ is dense in the closed set $\mathcal{E}_U(q)$, the above shows that $\mathcal{E}_U(q) \subset \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. Also by (ii), $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \subset \mathcal{L}^+(q)$. Using Definition 2.4 and (134), we see that $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q)$.

On the other hand, if $Q((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$, we can apply (ii) for all $y \in \mathcal{V} \cap U$ and see that $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$. This proves (iii). \square

Let

$$(135) \quad \mathcal{K}_{t_0} = \{x \in U_{\widehat{g}} \ ; \ x = \gamma_{\widehat{x}, \xi}(r), \ \widehat{x} = \widehat{\mu}(s), \ s \in [s^-, s^+], \\ \xi \in L_{\widehat{x}}^+ M_0, \ \|\xi\|_{\widehat{g}^+} = 1, \ r \in [0, 2t_0]\}.$$

Recall that $\mathcal{U}_{z_0, \eta_0} = \mathcal{U}_{z_0, \eta_0}(\widehat{h})$ was defined using the parameter \widehat{h} . We see using compactness of $\widehat{\mu}([s^-, s^+])$, the continuity of $\tau(x, y)$, and the existence of convex neighborhoods [87, Prop. 5.7], c.f. Lemma 2.A.1, that if $t'_0 > 0$ and $\widehat{h}' \in (0, \widehat{h})$ are small enough and $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}(\widehat{h}')$, then the longest geodesic from $x \in \overline{\mathcal{K}}_{t'_0}$ to the point $\mathcal{E}_{z, \eta}(x)$ is contained in \mathcal{U}_g and hence we can determine the point $\mathcal{E}_{z, \eta}(x)$ for such x and (z, η) . Let us replace the parameters \widehat{h} and t_0 by \widehat{h}' and t'_0 , correspondingly in our considerations below. Then we may assume that in addition to the data given in the original formulation of the problem, we are given also the set $\mathcal{E}_U(\mathcal{K}_{t_0})$. Next we do this. Below, we may assume that ϑ_1 is so small that $\gamma_{y, \zeta}([0, t_0]) \cap J(p^-, p^+) \subset \mathcal{K}_{t_0}$ when $y \in J(p^-, p^+)$, $d_{g^+}(y, \widehat{\mu}) < \vartheta_1$ and $\zeta \in L_y^+ M$, $\|\zeta\|_{g^+} \leq 1 + \vartheta_1$.

Let κ_1, κ_2 be constants given as in Lemma 2.3. Let $s_0 \in [s^-, s^+]$ be so close to s_+ that $J^+(\widehat{\mu}(s_0)) \cap J^-(p^+) \subset \mathcal{K}_{t_0}$. Then the given data $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ determines $\mathcal{E}_U(J^+(\widehat{\mu}(s_0)) \cap J^-(p^+))$.

Next we use a step-by-step construction: We consider $s_1 \in (s^-, s_+)$ and assume that we are given $\mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$ with $\widehat{x}_1 = \widehat{\mu}(s_1)$. Then, let $s_2 \in (s_1 - \kappa_2, s_1)$. Our next aim is to find the earliest light

observation sets $\mathcal{E}_U(J^+(\widehat{x}_2) \cap J^-(p^+))$ with $\widehat{x}_2 = \widehat{\mu}(s_2)$. To this end we need to make the following definitions (see Fig. 6).

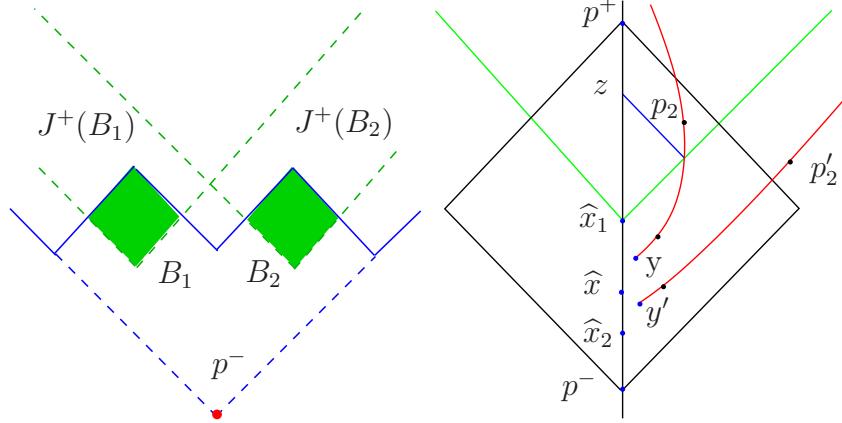


FIGURE 6. **Left:** The setting in the proof of Lemma 4.2. : The blue points on $\widehat{\mu}$ are $\widehat{x}_1 = \widehat{\mu}(s_1)$, $\widehat{x}_2 = \widehat{\mu}(s_2)$, and $\widehat{x} = \widehat{\mu}(s)$. The blue points y and y' are close to \widehat{x} . The set with the green boundary is $J^+(\widehat{x}_1)$. We consider the geodesics $\gamma_{y,\zeta}([0, \infty))$ and $\gamma_{y',\zeta'}([0, \infty))$. These geodesics corresponding to the cases when the geodesic $\gamma_{y,\zeta}([0, \infty))$ enters in $J^-(p^+) \cap J^+(\widehat{x}_1)$, and the case when the geodesic $\gamma_{y',\zeta'}([0, \infty))$ does not enter this set. The point p_2 is the cut point of $\gamma_{y,\zeta}([t_0, \infty))$ and p'_2 is the cut point of $\gamma_{y',\zeta'}([t_0, \infty))$. At the point $z = \widehat{\mu}(\mathbb{S}(y, \zeta, s_1))$ we observe for the first time on the geodesic $\widehat{\mu}$ that the geodesic $\gamma_{y,\zeta}([0, \infty))$ has entered $J^+(\widehat{x}_1)$.

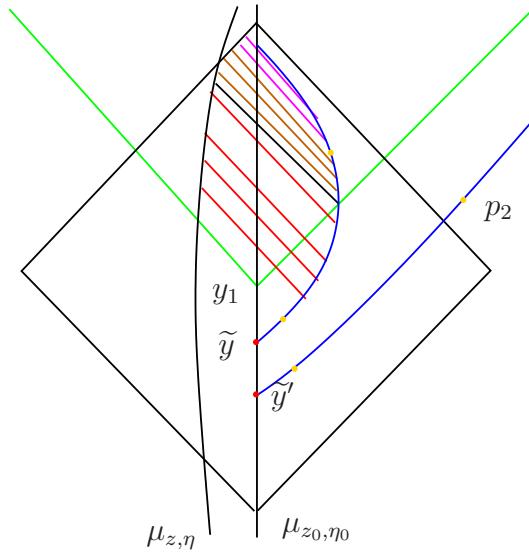


FIGURE A8: A schematic figure where the space-time is represented as the 2-dimensional set \mathbb{R}^{1+1} . We consider the geodesics emanating from a point $x(r) = \gamma_{\tilde{y}, \tilde{\zeta}}(r)$. When r is smallest value for which $x(r) \in J^+(y_1)$, a light-like geodesic (black line segment) emanating from $x(r)$ is observed at the point $\tilde{p} \in \hat{\mu}$. Then $\tilde{p} = \hat{\mu}(\mathbb{S}(\tilde{y}, \tilde{\zeta}, s_1))$. When r is small enough so that $x(r) \notin J^+(y_1)$, the light-geodesics (red line segments) can be observed on $\hat{\mu}$ in the set $J^-(\tilde{p})$. Moreover, when r is such that $x(r) \in J^+(y_1)$, the light-geodesics can be observed at $\hat{\mu}$ in the set $J^+(\tilde{p})$. The golden point is the cut point on $\gamma_{\tilde{y}, \tilde{\zeta}}([t_0, \infty))$ and the singularities on the light-like geodesics starting before this point (brown line segments) can be analyzed, but after the cut point the singularities on the light-like geodesics (magenta line segments) are not analyzed in this paper.

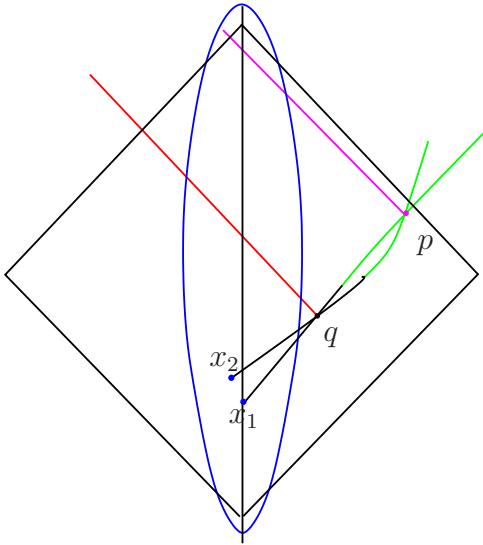


FIGURE A9: A schematic figure where the space-time is represented as the 2-dimensional set \mathbb{R}^{1+1} . In section 5 we consider geodesics $\gamma_{x_j, \xi_j}([0, \infty))$, $j = 1, 2, 3, 4$, that all intersect for the first time at a point q . When we consider geodesics $\gamma_{x_j, \xi_j}([t_0, \infty))$, $j = 1, 2$ with $t_0 > 0$, they may have cut points at $x_j^{cut} = \gamma_{x_j, \xi_j}(t_j)$. In the figure $\gamma_{x_j, \xi_j}([0, t_j))$ are colored by black and the geodesics $\gamma_{x_j, \xi_j}([t_j, \infty))$ are colored by green. In the figure the geodesics intersect at the point q before the cut point and for the second time at p after the cut point. We can analyze the singularities caused by the distorted plane waves that interact at q but not the interaction of waves after the cut points of the geodesics. It may be that e.g. the intersection at p causes new singularities to appear and we observe those in $U_{\hat{g}}$. As we cannot analyze these singularities, we consider these singularities as “messy waves”. However, the “nice” singularities caused by the interaction at q propagate along the future light cone of the point q and in $U_{\hat{g}} \setminus \cup_{j=1}^4 \gamma_{x_j, \xi_j}$ these “nice” singularities are observed before the “messy waves”. Due to this the first singularities we observe near $\hat{\mu}$ come from the point q .

Definition 5.3. Let $s_- \leq s_2 \leq s < s_1 \leq s_+ \leq s_2 + \kappa_2$, $\hat{x}_j = \hat{\mu}(s_j)$, $j = 1, 2$, and $\hat{x} = \hat{\mu}(s)$, $\hat{\zeta} \in L_{\hat{x}}^+ U$, $\|\hat{\zeta}\|_{g^+} = 1$. Let $(y, \zeta) \in L^+ U$ be in ϑ_1 -neighborhood of $(\hat{x}, \hat{\zeta})$ such that $y \in J^+(\hat{x}_2)$ and the geodesic $\gamma_{y, \zeta}(\mathbb{R}_+)$ does not intersect $\hat{\mu}$.

Let $r_1(y, \zeta, s_1) = \inf\{r > 0; \gamma_{y, \zeta}(r) \in J^+(\hat{\mu}(s_1))\}$. Also, define $r_2(y, \zeta) = \inf\{r > 0; \gamma_{y, \zeta}(r) \in M \setminus I^-(\hat{\mu}(s_2))\}$ and $r_0(y, \zeta, s_1) = \min(r_2(y, \zeta), r_1(y, \zeta, s_1))$.

When $\gamma_{y, \zeta}(\mathbb{R}_+)$ intersects $J^+(\hat{\mu}(s_1)) \cap J^-(p^+)$ we define

$$(136) \quad \mathbb{S}(y, \zeta, s_1) = f_{\hat{\mu}}^+(q_0),$$

where $q_0 = \gamma_{y, \zeta}(r_0)$ and $r_0 = r_0(y, \zeta, s_1)$. In the case when $\gamma_{y, \zeta}(\mathbb{R}_+)$ does not intersect $J^+(\hat{\mu}(s_1)) \cap J^-(p^+)$, we define $\mathbb{S}(y, \zeta, s_1) = s^+$.

We note that above $r_2(y, \zeta)$ is finite by [87, Lem. 14.13]. Below we use the notations used in Def. 5.3. We saw in Sec. 4 that using solutions of the linearized Einstein equations we can find $\gamma_{x,\xi} \cap U_{\widehat{g}}$ for any $(x, \xi) \in L^+U$. Thus we can check for given (x, ξ) if $\gamma_{x,\xi} \cap \widehat{\mu} = \emptyset$.

Definition 5.4. Let $0 < \vartheta < \vartheta_1$ and $\mathcal{R}_\vartheta(y, \zeta)$ be the set of $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ that satisfy (i) and (ii) in formula (75) with ϑ_1 replaced by ϑ and $(x_1, \xi_1) = (y, \zeta)$. We say that the set $S \subset U_{\widehat{g}}$ is a genuine observation associated to the geodesic $\gamma_{y,\zeta}$ if there is $\widehat{\vartheta} > 0$ such that for all $\vartheta \in (0, \widehat{\vartheta})$ there are $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$ such that $S = \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$.

Lemma 5.5. Suppose $\max(s_-, s_1 - \kappa_2) \leq s < s_1 < s^+$ and let $\widehat{x} = \widehat{\mu}(s)$, $\widehat{x}_1 = \widehat{\mu}(s_1)$, and $\widehat{\zeta} \in L_{\widehat{x}}^+ M$, $\|\widehat{\zeta}\|_{g^+} = 1$. Moreover, let (y, ζ) be in a ϑ_1 -neighborhood of $(\widehat{x}, \widehat{\zeta})$. Assume that the geodesic $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$.

(A) Then the cut point $p_0 = \gamma_{y(t_0), \zeta(t_0)}(\mathbf{t}_*)$, $\mathbf{t}_* = \rho(y(t_0), \zeta(t_0))$ of the geodesics $\gamma_{y(t_0), \zeta(t_0)}([0, \infty))$, if it exists, satisfies either

(i) $p_0 \notin J^-(\widehat{\mu}(s_{+2}))$,

or

(ii) $r_0 = r_0(y, \zeta, s_1) < r_2(y, \zeta)$ and $p_0 \in I^+(\widehat{x}_1)$.

(B) There is $\vartheta_2(y, \zeta, s_1) \in (0, \vartheta_1)$ such that if $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$, $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$, and the geodesics $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$, $j \in \{1, 2, 3, 4\}$, has a cut point $p_j = \gamma_{x_j(t_0), \xi_j(t_0)}(\mathbf{t}_j)$, then the following holds:

If either the point p_0 does not exist or it exists and (i) holds then $p_j \notin J^-(p^+)$. On the other hand, if p_0 exists and (ii) holds, then $f_{\widehat{\mu}}^+(p_j) > f_{\widehat{\mu}}^+(q_0)$, where $q_0 = \gamma_{y,\zeta}(r_0(y, \zeta, s_1))$.

Note that $f_{\widehat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$.

Proof. (A) Assume that (i) does not hold, that is, $p_0 = \gamma_{y,\zeta}(t_0 + \mathbf{t}_*) \in J^-(\widehat{\mu}(s_{+2}))$. By Lemma 2.3 (ii) we have $f_{\widehat{\mu}}^-(p_0) > s + 2\kappa_2 \geq s_1$ that yields $p_0 \in I^+(\widehat{x}_1)$. Thus, the geodesic $\gamma_{y(t_0), \zeta(t_0)}([0, \rho(y(t_0), \zeta(t_0))])$ intersects $J^+(\widehat{x}_1) \cap J^-(\widehat{\mu}(s_{+2}))$. Hence the alternative (ii) holds with $0 < r_0 < r_2(y, \zeta)$ and moreover, $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$.

(B) If (i) holds, the claim follows since the function $(x, \xi) \mapsto \rho(x, \xi)$ is lower semi-continuous and $(x, \xi, t) \mapsto \gamma_{x,\xi}(t)$ is continuous.

In the case (ii), we saw above that $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$. Let $q_0 = \gamma_{y,\zeta}(r_0)$. Then by using a short cut argument and the fact that $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$ we see similarly to the above that $f_{\widehat{\mu}}^+(p_0) > f_{\widehat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$. Since the function $(x, \xi, t) \mapsto f_{\widehat{\mu}}^+(\gamma_{x,\xi}(t))$ is continuous and $t \mapsto f_{\widehat{\mu}}^+(\gamma_{x,\xi}(t))$ is non-decreasing, and the function $(x, \xi) \mapsto \rho(x, \xi)$ is lower semi-continuous, we have that the function $(x, \xi) \mapsto f_{\widehat{\mu}}^+(\gamma_{x(t_0), \xi(t_0)}(\rho(x(t_0), \xi(t_0))))$ is lower semi-continuous, and the claim follows. \square

Definition 5.6. Let $s_- \leq s_2 \leq s < s_1 \leq s_+$ satisfy $s_1 < s_2 + \kappa_2$, $\widehat{x}_j = \widehat{\mu}(s_j)$, $j = 1, 2$, and $\widehat{x} = \widehat{\mu}(s)$, $\widehat{\zeta} \in L_{\widehat{x}}^+ U$, $\|\widehat{\zeta}\|_{g^+} = 1$. Also, let $(y, \zeta) \in L^+ U$ be in ϑ_1 -neighborhood of $(\widehat{x}, \widehat{\zeta})$ and $\mathcal{G}(y, \zeta, s_1)$ be the set of the genuine observations $S \subset U_{\widehat{g}}$ associated to the geodesic $\gamma_{y, \zeta}$ such that $S \in \mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$. Moreover, define $T(y, \zeta, s_1)$ to be the infimum of $s' \in [-1, s_+]$ such that $\widehat{\mu}(s') \in S \cap \widehat{\mu}$ with some $S \in \mathcal{G}(y, \zeta, s_1)$. If no such s' exists, we define $T(y, \zeta, s_1) = s^+$.

Let us next consider $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$ where $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$. Here, $\vartheta_2(y, \zeta, s_1)$ is defined in Lemma 5.5. Assume that for some $j = 1, 2, 3, 4$ we have that $\rho(x_j(t_0), \xi_j(t_0)) < \mathcal{T}(x_j(t_0), \xi_j(t_0))$ and consider the cut point $p_j = \gamma_{x_j(t_0), \xi_j(t_0)}(\rho(x_j(t_0), \xi_j(t_0)))$. Then either the case (i) or (ii) of Lemma 5.5, (B) holds. If (i) holds, p_j satisfies $p_j \notin J^-(p^+)$ and thus $f_{\widehat{\mu}}^+(p_j) > s_+ \geq \mathbb{S}(y, \zeta, s_1)$. If (ii) holds, there exists $r_0 = r_0(y, \zeta, s_1) < r_1(y, \zeta)$ such that $q_0 = \gamma_{y, \zeta}(r_0) \in J^+(\widehat{x}_1)$ and $f_{\widehat{\mu}}^+(p_j) > f_{\widehat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$. Thus both in case (i) and (ii) we have

$$(137) \quad f_{\widehat{\mu}}^+(p_j) > \mathbb{S}(y, \zeta, s_1).$$

Next, consider a point $q = \gamma_{y, \zeta}(r) \in J^-(p^+)$, where $t_0 < r \leq r_0 = r_0(y, \zeta, s_1)$. Let $\theta = -\dot{\gamma}_{y, \zeta}(r) \in T_q M$. By Lemma 2.3 (ii), the geodesic $\gamma_{y, \zeta}([t_0, r]) = \gamma_{q, \theta}([0, r - t_0])$ has no cut points. Denote $\widetilde{x} = \widehat{\mu}(\mathbb{S}(y, \zeta, s_1))$. Note that then $q \in J^-(\widetilde{x})$. Then, consider four geodesics that emanate from q to the past, in the light-like direction $\eta_1 = \theta$ and in the light-like directions $\eta_j \in T_q M$, $j = 2, 3, 4$ that are sufficiently close to the direction θ . Let $\gamma_{q, \eta_j}(r_j)$ be the intersection points of γ_{q, η_j} with the surface $\{x \in M; \mathbf{t}(x) = c_0\}$, on which the time function $\mathbf{t}(x)$ has the constant value $c_0 := \mathbf{t}(\gamma_{y, \zeta}(t_0))$. Note that such $r_j = r_j(\eta_j)$ exists by the inverse function theorem when η_j is sufficiently close to η_1 . Choosing $x_j = \gamma_{q, \eta_j}(r_j + t_0)$ and $\xi_j = -\dot{\gamma}_{q, \eta_j}(r_j + t_0)$ we see that when $\vartheta_3 \in (0, \vartheta_2(y, \zeta, s_1))$ is small enough, for all $\vartheta \in (0, \vartheta_3)$, there is $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{R}_\vartheta(y, \zeta)$ such that the geodesics corresponding to $(\vec{x}, \vec{\xi})$ intersect at q . As the set $(U_{\widehat{g}}, \widehat{g})$ is known, that for sufficiently small ϑ one can verify if given vectors $(\vec{x}, \vec{\xi})$ satisfy $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{R}_\vartheta(y, \zeta)$, cf. [87, Prop. 5.7]. Also, note that as then $\vartheta < \vartheta_2(y, \zeta, s_1)$, the inequality (137) yields $\widetilde{x} \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ and thus $q \in J^-(\widetilde{x}) \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$. Then Lemma 5.2 (iii) yields $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q)$. As $\vartheta \in (0, \vartheta_3)$ above can be arbitrarily small, we have that for any $q = \gamma_{y, \zeta}(r) \in J^-(p^+)$ where $t_0 < r \leq r_0 = r_0(y, \zeta, s_1)$, we obtain

$$(138) \quad \begin{aligned} S &= \mathcal{E}_U(q) \text{ is a genuine observation associated to } \gamma_{y, \zeta} \text{ and} \\ S \cap \widehat{\mu} &= \{\widehat{\mu}(\widehat{s})\}, \quad \widehat{s} := f_{\widehat{\mu}}^+(q) \leq \mathbb{S}(y, \zeta, s_1). \end{aligned}$$

Lemma 5.7. Assume that $\gamma_{y, \zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$. Then we have $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$.

Proof. Let us first prove that $T(y, \zeta, s_1) \geq \mathbb{S}(y, \zeta, s_1)$. To this end, let $s' < \mathbb{S}(y, \zeta, s_1)$ and $x' = \widehat{\mu}(s')$. Assume $S \in \mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$ is a genuine observation associated to the geodesic $\gamma_{y,\zeta}$ and $S \cap \widehat{\mu} = \{\widehat{\mu}(s')\}$. Let $q \in J^+(\widehat{x}_1) \cap J^-(p^+)$ be such that $S = \mathcal{E}_U(q)$.

Then for arbitrarily small $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$ there is $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$ satisfying $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = S$. Let $\mathcal{V} = \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$. Then by (137), we have $x' \in \mathcal{V}$.

If the geodesics corresponding to $(\vec{x}, \vec{\xi})$ intersect at some point $q' \in J^-(x')$, then by Lemma 5.2 (i), (ii) we have $S = \mathcal{E}_U(q')$. Then $\widehat{\mu}(s') = \widehat{\mu}(f_\mu^+(q'))$ implying $f_\mu^+(q') = s'$. Moreover, we have then that $\mathcal{E}_U(q) = \mathcal{E}_U(q')$ and Theorem 2.5 (i) yields $q' = q$. Since $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$ implies $(x_1, \xi_1) = (y, \zeta)$, we see that $q' \in \gamma_{y,\zeta}([t_0, \infty))$. As $q \in J^+(\widehat{x}_1)$, we see that $q = q' \in \gamma_{y,\zeta}([t_0, \infty)) \cap J^+(\widehat{x}_1) = \gamma_{y,\zeta}([r_0(y, \zeta, s_1), \infty))$. However, then $f_\mu^+(q') \geq \mathbb{S}(y, \zeta, s_1) > s'$ and thus $S \cap \widehat{\mu} = \mathcal{E}_U(q) \cap \widehat{\mu}$ can not be equal to $\{\widehat{\mu}(s')\}$.

On the other hand, if the geodesics corresponding to $(\vec{x}, \vec{\xi})$ do not intersect at any point in $J^-(x') \subset \mathcal{V}$, then either they intersect in some $q'_1 \in (M \setminus J^-(x')) \cap \mathcal{V}$, do not intersect at all, or intersect at $q'_2 \in M \setminus \mathcal{V}$. In the first case, $S = \mathcal{E}_U(q'_1)$ do not satisfy $S \cap \widehat{\mu} \in \widehat{\mu}((-1, s'))$. In the other cases, Lemma 5.2 (iii) yields $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$. As $x' = \widehat{\mu}(s') \in \mathcal{V}$, we see that $S \cap \widehat{\mu}$ can not be equal to $\{\widehat{\mu}(s')\}$. Since above $s' < \mathbb{S}(y, \zeta, s_1)$ is arbitrary, this shows that $T(y, \zeta, s_1) \geq \mathbb{S}(y, \zeta, s_1)$.

Let us next show that $T(y, \zeta, s_1) \leq \mathbb{S}(y, \zeta, s_1)$. Assume the opposite. Then, if $\mathbb{S}(y, \zeta, s_1) = s_+$, we see by Def. 5.6 that $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$ which leads to a contradiction. However, if $\mathbb{S}(y, \zeta, s_1) < s_+$, by Def. 5.3, we have (137). This implies the existence of $q_0 = \gamma_{y,\zeta}(r_0)$, $r_0 = r_0(y, \zeta, s_1)$ such that $q_0 \in J^+(\widehat{x}_1) \cap J^-(p^+)$ and by (138), $S = \mathcal{E}_U(q_0)$ is a genuine observation associated to the geodesic $\gamma_{y,\zeta}$. By Lemma 5.5 (ii), $\mathbb{S}(y, \zeta, s_1) = f_\mu^-(q_0)$ which implies, by Def. 5.6 that $T(y, \zeta, s_1) \leq \mathbb{S}(y, \zeta, s_1)$. Thus, $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$. \square

Next we reconstruct $\mathcal{E}_U(q)$ when q runs over a geodesic segment.

Lemma 5.8. *Let $s_- \leq s_2 \leq s < s_1 \leq s_+$ with $s_1 < s_2 + \kappa_2$, let $\widehat{x}_j = \widehat{\mu}(s_j)$, $j = 1, 2$, and $\widehat{x} = \widehat{\mu}(s)$, $\widehat{\zeta} \in L_{\widehat{x}}^+ U$, $\|\widehat{\zeta}\|_{g^+} = 1$. Let $(y, \zeta) \in L^+ U$ be in the ϑ_1 -neighborhood of $(\widehat{x}, \widehat{\zeta})$ such that $y \in J^+(\widehat{x}_2)$. Assume that $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$. Then, if we are given the data set $\mathcal{D}(\widehat{g}, \phi, \varepsilon)$, we can determine the collection $\{\mathcal{E}_U(q); q \in G_0(y, \zeta, s_1)\}$, where $G_0(y, \zeta, s_1) = \{q \in \gamma_{y,\zeta}([t_0, \infty)) \cap (I^-(p^+) \setminus J^+(\widehat{x}_1))\}$.*

Proof. Let $s' = \mathbb{S}(y, \zeta, s_1)$, $x' = \widehat{\mu}(s')$, and Σ be the set of all genuine observations S associated to the geodesic $\gamma_{y,\zeta}$ such that S intersects $\widehat{\mu}([-1, s'])$.

Let $q = \gamma_{y,\zeta}(r) \in G_0(y, \zeta, s_1)$. Since $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$, using a short cut argument for the geodesics from q to $q_0 = \gamma_{y,\zeta}(r_0(y, \zeta, s_1))$

and from q_0 to x' , we see that $f_{\widehat{\mu}}^-(q) < s'$. Then, $q \in I^-(p^+) \setminus J^+(\widehat{x}_1)$ and $r < r_0(y, \zeta, s_1)$, and we have using (138) that $S = \mathcal{E}_U(q)$ is a genuine observation associated to the geodesic $\gamma_{y, \zeta}$ and $S \cap \widehat{\mu} = \{\widehat{\mu}(f_{\widehat{\mu}}^-(q))\}$ with $f_{\widehat{\mu}}^-(q) < s'$. Thus $\mathcal{E}_U(q) \in \Sigma$ and we conclude that $\mathcal{E}_U(G_0(y, \zeta, s_1)) \subset \Sigma$.

Next, suppose $S \in \Sigma$. Then there is $\widehat{\vartheta} \in (0, \vartheta_2(y, \zeta, s_1))$ such that for all $\vartheta \in (0, \widehat{\vartheta})$ there is $(\vec{x}, \vec{\xi}) \in \mathcal{R}_{\vartheta}(y, \zeta)$ so that $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = S$. Observe that by (137) we have $\widehat{\mu}([-1, s')) \subset J^-(x') \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$.

First, consider the case when the geodesics corresponding to $(\vec{x}, \vec{\xi})$ do not intersect at any point in $I^-(x')$. Then Lemma 5.2 (iii) yields that $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$ is either empty or does not intersect $I^-(x')$. Thus $S \cap I^-(x') = \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap I^-(x')$ is empty and S does not intersect $\widehat{\mu}([-1, s'))$. Hence S cannot be in Σ .

Second, consider the case when the geodesics corresponding to $(\vec{x}, \vec{\xi})$ intersect at some point $q \in I^-(x')$. Then, Lemma 5.2 (iii) yields $S = \mathcal{E}_U(q)$. Since $(\vec{x}, \vec{\xi}) \in \mathcal{R}_{\vartheta}(y, \zeta)$ implies $(x_1, \xi_1) = (y, \zeta)$, the intersection point q has a representation $q = \gamma_{x_1, \xi_1}(r)$. As $q \in I^-(x')$, this yields $q \in G_0(y, \zeta, s_1)$ and $S \in \mathcal{E}_U(G_0(y, \zeta, s_1))$. Hence $\Sigma \subset \mathcal{E}_U(G_0(y, \zeta, s_1))$.

Combining the above arguments, we see that $\Sigma = \mathcal{E}_U(G_0(y, \zeta, s_1))$. As Σ is determined by the data set, the claim follows. \square

Let $B(s_2, s_1)$ be the set of all (y, ζ, t) such that there are $\widehat{x} = \widehat{\mu}(s)$, $s \in [s_2, s_1]$ and $\widehat{\zeta} \in L_{\widehat{x}}^+ U$, $\|\widehat{\zeta}\|_{g^+} = 1$ so that $(y, \zeta) \in L^+ U$ in ϑ_1 -neighborhood of $(\widehat{x}, \widehat{\zeta})$, $y \in J^+(\widehat{x}_2)$, and $t \in [t_0, r_0(y, \zeta, s_1)]$. Moreover, let $B_0(s_2, s_1)$ be the set of all $(y, \zeta, t) \in B(s_2, s_1)$ such that $t < r_0(y, \zeta, s_1)$ and $\gamma_{y, \zeta}(\mathbb{R}_+) \cap \widehat{\mu} = \emptyset$. Lemma 5.8 and the fact that $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ determines $\gamma_{y, \zeta} \cap U_{\widehat{g}}$ show that, when we are given the data set $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, we can determine the collection $\Sigma_0(s_2, s_1) := \{\mathcal{E}_U(q); q = \gamma_{y, \zeta}(t), (y, \zeta, t) \in B_0(s_2, s_1)\}$. We denote also $\Sigma(s_2, s_1) := \{\mathcal{E}_U(q); q = \gamma_{y, \zeta}(t), (y, \zeta, t) \in B(s_2, s_1)\}$.

Note that the sets $\mathcal{E}_U(q) \subset U$, where $q \in J := J^-(p^+) \cap J^+(p^-)$, can be identified with the function, $F_q : \mathcal{U}_{z_0, \eta_0} \rightarrow \mathbb{R}$, $F_q(z, \eta) = f_{\mu(z, \eta)}^+(q)$, c.f. (1). Let $\mathcal{U} = \mathcal{U}_{z_0, \eta_0}$. When we endow the set $\mathbb{R}^{\mathcal{U}}$ of maps $\mathcal{U} \rightarrow \mathbb{R}$ with the topology of pointwise convergence, Lemma 2.2 yields that $F : q \mapsto F_q$ is continuous map $F : J \rightarrow \mathbb{R}^{\mathcal{U}}$. By Theorem 2.5, F is one-to-one, and since J is compact and $\mathbb{R}^{\mathcal{U}}$ is Hausdorff, we have that $F : J \rightarrow F(J)$ is homeomorphism. Next, we identify $\mathcal{E}_U(q)$ and F_q .

Using standard results of differential topology, we see that any neighborhood of $(y, \zeta) \in L^+ U$ contains $(y', \zeta') \in L^+ U$ such that the geodesic $\gamma_{y', \zeta'}([0, \infty))$ does not intersect $\widehat{\mu}$. Since $(y, \zeta) \mapsto r_0(y, \zeta, s_1)$ is lower semicontinuous, this implies that $\Sigma_0(s_2, s_1)$ is dense in $\Sigma(s_2, s_1)$. Hence we obtain the closure $\overline{\Sigma}(s_2, s_1)$ of $\Sigma(s_2, s_1)$ as the limits points of $\Sigma_0(s_2, s_1)$.

Then, we obtain the set we $\mathcal{E}_U(J^+(\widehat{\mu}(s_2)) \cap J^-(p^+))$ as the union $\overline{\Sigma}(s_2, s_1) \cup \mathcal{E}_U(J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)) \cup \mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s_2)))$, see (135).

Let $s_0, \dots, s_K \in [s_-, s_+]$ be such that $s_j > s_{j+1} > s_j - \kappa_2$ and $s_K = s_-$. Then, by iterating the above construction so that the values of the parameters s_1 and s_2 are replaced by s_j and s_{j+1} , respectively, we can construct the set $\mathcal{E}_U(J^+(\widehat{\mu}(s_-)) \cap J^-(\widehat{\mu}(s_+)))$.

Moreover, similarly to the above construction, we can find the sets $\mathcal{E}_U(J^+(\widehat{\mu}(s')) \cap J^-(\widehat{\mu}(s'')))$ for all $s_- < s' < s'' < s_+$, and taking their union, we find the set $\mathcal{E}_U(I(\widehat{\mu}(s_-), \widehat{\mu}(s_+))$. By Theorem 2.5 we can reconstruct the manifold $I(\widehat{\mu}(s_-), \widehat{\mu}(s_+))$ and the conformal structure on it. This proves Theorem 1.1. \square

Remark 5.1. The proof of Theorem 1.1 can be used to analyze approximative reconstruction of the set $I_{\widehat{g}}(p^-, p^+) = I_{\widehat{g}}^+(p^-) \cap I_{\widehat{g}}^-(p^+)$ and its conformal structure with only one measurement: We choose one suitably constructed source \mathcal{F} , supported in $W_{\widehat{g}}$, measure the fields (f, ϕ) produced by this source in $U_{\widehat{g}}$ and aim to construct an approximation of the conformal class of the metric g in $I_{\widehat{g}}(p^-, p^+)$. In the proof above we showed that it is possible to use the non-linearity to create an artificial point source at a point $q \in J_{\widehat{g}}(p^-, p^+)$. Using the same method we see that it is possible to create with a single source \mathcal{F} an arbitrary number of artificial point sources. To see this, let $\delta > 0$ and $P, Q \in \mathbb{Z}_+$ and consider points $x_p \in U_{\widehat{g}} \cap J_{\widehat{g}}(p^-, p^+)$, $p = 1, 2, \dots, P$. Let $\xi_{p,k} \in \Sigma_p = \{\xi \in T_{x_p}^* M; \|\xi\|_{\widehat{g}} = 1\}$, $k = 1, 2, \dots, n_Q$ be a maximal $1/Q$ net on the set Σ_p and $\Sigma_{p,k}(R)$ be an R -neighborhood of $\xi_{p,k}$ in Σ_p . Let $R_1 = 1/Q$ and $R_2 = 2/Q$, and consider real numbers $a(p, k) \in (-31, -30)$, chosen to be $a(p, k) = -31 + 1/j(p, k)$, where $j : \mathbb{Z}_+^2 \rightarrow \mathcal{P}$ is a bijection from \mathbb{Z}_+^2 to the set \mathcal{P} of the prime numbers. Let $F_{p,k} \in \mathcal{I}^{a(p,k)}(\Sigma_{p,k}(R_2))$ be Lagrangian distributions whose principal symbols are non-vanishing on $\Sigma_{p,k}(R_1)$.

Assume next that (M, g) is in generic manifold (i.e., it is in the intersection of countably many open and dense sets in a suitable space of smooth manifolds), points x_p have generic positions, and let $\delta > 0$. Note that the sets $\Sigma_{p,k}(R_1)$ are a covering of the unit sphere Σ_p and the linearized waves $u_{p,k} = \mathbf{Q}_{\widehat{g}}(F_{p,k})$ are singular on a subset of the light-cones $\mathcal{L}_{\widehat{g}}^+(x_p)$. Let $\varepsilon > 0$ be small enough and consider a suitable source F_ε , with $\partial_\varepsilon F_\varepsilon|_{\varepsilon=0} = \sum_{p,k} F_{p,k}$, that produces the perturbation $u_\varepsilon(x) = \sum_{n=1}^4 \varepsilon^n u_n(x) + O(\varepsilon^5)$ for $(\widehat{g}, \widehat{\phi})$. Assume that we measure the singular supports of the waves u_n , $n = 1, 2, 3, 4$ produced by n -th order interaction of the waves. When ε is small enough, this could be done e.g. by using thresholding of the curvelet coefficients, of a suitable order, of the solution [14, 27]. Then in $I_{\widehat{g}}(p^-, p^+)$, outside the singular support of the wave u_3 , the wave u_4 is a sum of a smooth wave and the waves produced by artificial point sources \mathcal{F}_ℓ located at q_ℓ , $\ell = 1, 2, \dots, L$. Here $q_\ell \in J_{\widehat{g}}(p^-, p^+)$ are the intersection points

of any four light cones $\mathcal{L}_{\widehat{g}}^+(x_{p_1(\ell)})$, $p_1(\ell), p_2(\ell), p_3(\ell), p_4(\ell) \leq N$. When Q and P are large enough, so that N is large, the points q_ℓ are a δ -dense subset of $J_{\widehat{g}}(p^-, p^+)$. Moreover, for the chosen orders $a(p, q)$, the orders of the sources \mathcal{F}_ℓ at points q_ℓ are all different and thus the waves $\mathbf{Q}_{\widehat{g}}(\mathcal{F}_\ell)$ have different orders. As a very rough analogy, we can produce in $J_{\widehat{g}}(p^-, p^+)$ an arbitrarily dense collection of artificial points sources having all different colors. Using this observation one can show, using similar methods to those in [2], that in a suitable compact class of Lorentzian manifolds having no conjugate points the measurement with the source F_ε , defined using sufficiently large P and Q and generic values of principal symbols on $\Sigma_{p,q}$, determines a δ -approximation (in a suitable sense) of the conformal class of the manifold $(J_{\widehat{g}}(p^-, p^+), \widehat{g})$. The details of this construction will be considered elsewhere.

APPENDIX A: REDUCED EINSTEIN EQUATION

In this section we review known results on the Einstein equations and wave maps.

A.1. Summary of the used notations. Let us recall some definitions given in Introduction. Let (M, \widehat{g}) be a C^∞ -smooth globally hyperbolic Lorentzian manifold and \widetilde{g} be a C^∞ -smooth globally hyperbolic metric on M such that $\widehat{g} < \widetilde{g}$. Let us start by explaining how one can construct a C^∞ -smooth metric \widetilde{g}' such that $\widehat{g} < \widetilde{g}'$ and (M, \widetilde{g}') is globally hyperbolic: When $v(x)$ is an eigenvector corresponding to the negative eigenvalue of $\widehat{g}(x)$, we can choose a smooth, strictly positive function $\eta : M \rightarrow \mathbb{R}_+$ such that

$$\widetilde{g}' := \widehat{g} - \eta v \otimes v < \widetilde{g}.$$

Then (M, \widetilde{g}') is globally hyperbolic, \widetilde{g}' is smooth and $\widehat{g} < \widetilde{g}'$. Thus we can replace \widetilde{g} by the smooth metric \widetilde{g}' having the same properties that are required for \widetilde{g} .

Recall that there is an isometry $\Phi : (M, \widetilde{g}) \rightarrow (\mathbb{R} \times N, \widetilde{h})$, where N is a 3-dimensional manifold and the metric \widetilde{h} can be written as $\widetilde{h} = -\beta(t, y)dt^2 + \overline{h}(t, y)$ where $\beta : \mathbb{R} \times N \rightarrow (0, \infty)$ is a smooth function and $\overline{h}(t, \cdot)$ is a Riemannian metric on N depending smoothly on $t \in \mathbb{R}$. As in the main text we identify these isometric manifolds and denote $M = \mathbb{R} \times N$. Also, for $t \in \mathbb{R}$, recall that $M(t) = (-\infty, t) \times N$. We use parameters $t_1 > t_0 > 0$ and denote $M_j = M(t_j)$, $j \in \{0, 1\}$. We use the time-like geodesic $\widehat{\mu} = \mu_{\widehat{g}}$, $\mu_{\widehat{g}} : [-1, 1] \rightarrow M_0$ on (M_0, \widehat{g}) and the set $\mathcal{K}_j := J_{\widehat{g}}^+(\widehat{\mu}(-1)) \cap M_j$ with $\widehat{\mu}(-1) \in (-\infty, t_0) \times N$. Then $J_{\widehat{g}}^+(\widehat{\mu}(-1)) \cap M_j$ is compact. Also, there exists $\varepsilon_0 > 0$ such that if g is a Lorentz metric in M_1 such that $\|g - \widehat{g}\|_{C_b^0(M_1; \widehat{g}^+)} < \varepsilon_0$, then $g|_{\mathcal{K}_1} < \widetilde{g}|_{\mathcal{K}_1}$. In particular, this implies that we have $J_g^+(p) \cap M_1 \subset \mathcal{K}_1$ for all $p \in \mathcal{K}_1$. Later, we use this property to deduce that when g satisfies the \widehat{g} -reduced Einstein equations in M_1 , with a source that is

supported in \mathcal{K}_1 and has small enough norm is a suitable space, then g coincides with \widehat{g} in $M_1 \setminus \mathcal{K}_1$ and satisfies $g < \widehat{g}$.

Let us use local coordinates on M_1 and denote by $\nabla_k = \nabla_{X_k}$ the covariant derivative with respect to the metric g in the direction $X_k = \frac{\partial}{\partial x^k}$ and by $\widehat{\nabla}_k = \widehat{\nabla}_{X_k}$ the covariant derivative with respect to the metric \widehat{g} to the direction X_k .

A.2. Reduced Ricci and Einstein tensors. Following [37] we recall that

$$(139) \quad \text{Ric}_{\mu\nu}(g) = \text{Ric}_{\mu\nu}^{(h)}(g) + \frac{1}{2}(g_{\mu q} \frac{\partial \Gamma^q}{\partial x^\nu} + g_{\nu q} \frac{\partial \Gamma^q}{\partial x^\mu})$$

where $\Gamma^q = g^{mn} \Gamma_{mn}^q$,

$$(140) \quad \begin{aligned} \text{Ric}_{\mu\nu}^{(h)}(g) &= -\frac{1}{2}g^{pq} \frac{\partial^2 g_{\mu\nu}}{\partial x^p \partial x^q} + P_{\mu\nu}, \\ P_{\mu\nu} &= g^{ab} g_{ps} \Gamma_{\mu b}^p \Gamma_{\nu a}^s + \frac{1}{2} \left(\frac{\partial g_{\mu\nu}}{\partial x^a} \Gamma^a + g_{\nu l} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^\mu} + g_{\mu l} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^\nu} \right). \end{aligned}$$

Note that $P_{\mu\nu}$ is a polynomial of g_{jk} and g^{jk} and first derivatives of g_{jk} . The harmonic Einstein tensor is

$$(141) \quad \text{Ein}_{jk}^{(h)}(g) = \text{Ric}_{jk}^{(h)}(g) - \frac{1}{2}g^{pq} \text{Ric}_{pq}^{(h)}(g) g_{jk}.$$

The harmonic Einstein tensor is extensively used to study the Einstein equations in local coordinates where one can use the Minkowski space \mathbb{R}^4 as the background space. To do global constructions with a background space (M, \widehat{g}) one uses the reduced Einstein tensor. The \widehat{g} -reduced Einstein tensor $\text{Ein}_{\widehat{g}}(g)$ and the \widehat{g} -reduced Ricci tensor $\text{Ric}_{\widehat{g}}(g)$ are given by

$$(142) \quad (\text{Ein}_{\widehat{g}}(g))_{pq} = (\text{Ric}_{\widehat{g}}(g))_{pq} - \frac{1}{2}(g^{jk}(\text{Ric}_{\widehat{g}}g)_{jk})g_{pq},$$

$$(143) \quad (\text{Ric}_{\widehat{g}}(g))_{pq} = \text{Ric}_{pq}(g) - \frac{1}{2}(g_{pn} \widehat{\nabla}_q \widehat{F}^n + g_{qn} \widehat{\nabla}_p \widehat{F}^n)$$

where \widehat{F}^n are the harmonicity functions given by

$$(144) \quad \widehat{F}^n = \Gamma^n - \widehat{\Gamma}^n, \quad \text{where } \Gamma^n = g^{jk} \Gamma_{jk}^n, \quad \widehat{\Gamma}^n = g^{jk} \widehat{\Gamma}_{jk}^n,$$

where Γ_{jk}^n and $\widehat{\Gamma}_{jk}^n$ are the Christoffel symbols for g and \widehat{g} , correspondingly. Note that $\widehat{\Gamma}^n$ depends also on g^{jk} . As $\Gamma_{jk}^n - \widehat{\Gamma}_{jk}^n$ is the difference of two connection coefficients, it is a tensor. Thus \widehat{F}^n is tensor (actually, a vector field), implying that both $(\text{Ric}_{\widehat{g}}(g))_{jk}$ and $(\text{Ein}_{\widehat{g}}(g))_{jk}$ are 2-covariant tensors. A direct calculation shows that the \widehat{g} -reduced Einstein tensor is the sum of the harmonic Einstein tensor and a term that is a zeroth order in g ,

$$(145) \quad (\text{Ein}_{\widehat{g}}(g))_{\mu\nu} = \text{Ein}_{\mu\nu}^{(h)}(g) + \frac{1}{2}(g_{\mu q} \frac{\partial \widehat{\Gamma}^q}{\partial x^\nu} + g_{\nu q} \frac{\partial \widehat{\Gamma}^q}{\partial x^\mu}).$$

We also use the wave operator

$$(146) \quad \square_g \phi = \sum_{p,q=1}^4 (-\det(g(x)))^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left((-\det(g(x)))^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} \phi(x) \right),$$

can be written for as

$$(147) \quad \square_g \phi = g^{jk} \partial_j \partial_k \phi - g^{pq} \Gamma_{pq}^n \partial_n \phi = g^{jk} \partial_j \partial_k \phi - \Gamma^n \partial_n \phi.$$

A.3. Wave maps and reduced Einstein equations. Let us consider the manifold $M_1 = (-\infty, t_1) \times N$ with a C^m -smooth metric g' , $m \geq 8$, which is a perturbation of the metric \widehat{g} and satisfies the Einstein equation

$$(148) \quad \text{Ein}(g') = T' \quad \text{on } M_1,$$

or equivalently,

$$\text{Ric}(g') = \rho', \quad \rho'_{jk} = T'_{jk} - \frac{1}{2}((g')^{nm} T'_{nm}) g'_{jk} \quad \text{on } M_1.$$

Assume also that $g' = \widehat{g}$ in the domain A , where $A = M_1 \setminus \mathcal{K}_1$ and $\|g' - \widehat{g}\|_{C_b^2(M_1, \widehat{g}^+)} < \varepsilon_0$, so that (M_1, g') is globally hyperbolic. Note that then $T' = \widehat{T}$ in the set A and that the metric g' coincides with \widehat{g} in particular in the set $M^- = \mathbb{R}_- \times N$

We recall next the considerations of [15]. Let us consider the Cauchy problem for the wave map $f : (M_1, g') \rightarrow (M, \widehat{g})$, namely

$$(149) \quad \square_{g', \widehat{g}} f = 0 \quad \text{in } M_1,$$

$$(150) \quad f = \text{Id}, \quad \text{in } \mathbb{R}_- \times N,$$

where $M_1 = (-\infty, t_1) \times N \subset M$. In (149), $\square_{g', \widehat{g}} f = g' \cdot \widehat{\nabla}^2 f$ is the wave map operator, where $\widehat{\nabla}$ is the covariant derivative of a map $(M_1, g') \rightarrow (M, \widehat{g})$, see [15, Ch. VI, formula (7.32)]. In local coordinates $X : V \rightarrow \mathbb{R}^4$ of $V \subset M_1$, denoted by $X(z) = (x^j(z))_{j=1}^4$ and $Y : W \rightarrow \mathbb{R}^4$ of $W \subset M$, denoted by $Y(z) = (y^A(z))_{A=1}^4$, the wave map $f : M_1 \rightarrow M$ has the representation $Y(f(X^{-1}(x))) = (f^A(x))_{A=1}^4$ and the wave map operator in equation (149) is given by

$$(151) \quad (\square_{g', \widehat{g}} f)^A(x) = (g')^{jk}(x) \left(\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} f^A(x) - \Gamma'_{jk}^n(x) \frac{\partial}{\partial x^n} f^A(x) + \widehat{\Gamma}_{BC}^A(f(x)) \frac{\partial}{\partial x^j} f^B(x) \frac{\partial}{\partial x^k} f^C(x) \right)$$

where $\widehat{\Gamma}_{BC}^A$ denotes the Christoffel symbols of metric \widehat{g} and Γ'_{kl}^j are the Christoffel symbols of metric g' . When (149) is satisfied, we say that f is a wave map with respect to the pair (g', \widehat{g}) .

It follows from [15, App. III, Thm. 4.2 and sec. 4.2.2], that if $g' \in C^m(M_0)$, $m \geq 5$ is sufficiently close to \widehat{g} in $C^m(M_0)$, then (5)-(6) has a unique solution $f \in C^0([0, t_1]; H^{m-1}(N)) \cap C^1([0, t_1]; H^{m-2}(N))$. This

comes from the fact that the Christoffel symbols of g' are in $C^{m-1}(M_0)$. Moreover, when m is even, using [63, Thm. 7] for f and $\partial_t^p f$, we see that the solution f is in $f \in \cap_{p=0}^{m-1} C^p([0, t_1]; H^{m-1-p}(N)) \subset C^{m-3}(M_0)$ and f depends in $C^{m-3}(M_0)$ continuously on $g' \in C^m(M_0)$. We note that these smoothness results for f are not optimal.

The wave map operator $\square_{g', \hat{g}}$ is a coordinate invariant operator. The important property of the wave maps is that, if f is wave map with respect to the pair (g', \hat{g}) and $g = f_* g'$ then, as follows from (151), the identity map $Id : x \mapsto x$ is a wave map with respect to the pair (g, \hat{g}) and, the wave map equation for the identity map is equivalent to (cf. [15, p. 162])

$$(152) \quad \Gamma^n = \hat{\Gamma}^n, \quad \text{where } \Gamma^n = g^{jk} \Gamma_{jk}^n, \quad \hat{\Gamma}^n = g^{jk} \hat{\Gamma}_{jk}^n$$

where the Christoffel symbols $\hat{\Gamma}_{jk}^n$ of the metric \hat{g} are smooth functions.

Since $g = g'$ outside a compact set $\mathcal{K}_1 \subset (0, t_1) \times N$, we see that this Cauchy problem is equivalent to the same equation restricted to the set $(-\infty, t_1) \times B_0$, where $B_0 \subset N$ is an open relatively compact set such that $\mathcal{K}_1 \subset (0, t_1] \times B_0$ with the boundary condition $f = Id$ on $(0, t_1] \times \partial B_0$. Moreover, by the uniqueness of the wave map, we have $f|_{M_1 \setminus \mathcal{K}_1} = id$ so that $f(\mathcal{K}_1) \cap M_0 \subset \mathcal{K}_0$.

As the inverse function of the wave map f depends continuously, in $C_b^{m-3}([0, t_1] \times N, g^+)$, on the metric $g' \in C^m(M_0)$ we can also assume that ε_1 is so small that $M_0 \subset f(M_1)$.

Denote next $g := f_* g'$, $T := f_* T'$, and $\rho := f_* \rho'$ and define $\hat{\rho} = \hat{T} - \frac{1}{2}(\text{Tr } \hat{T})\hat{g}$. Then g is C^{m-6} -smooth and the equation (148) implies

$$(153) \quad \text{Ein}(g) = T \quad \text{on } M_0.$$

Since f is a wave map and $g = f_* g'$, we have that the identity map is a (g, \hat{g}) -wave map and thus g satisfies (152) and thus by the definition of the reduced Einstein tensor, (23)-(24), we have

$$\text{Ein}_{pq}(g) = (\text{Ein}_{\hat{g}}(g))_{pq} \quad \text{on } M_0.$$

This and (153) yield the \hat{g} -reduced Einstein equation

$$(154) \quad (\text{Ein}_{\hat{g}}(g))_{pq} = T_{pq} \quad \text{on } M_0.$$

This equation is useful for our considerations as it is a quasilinear, hyperbolic equation on M_0 . Recall that g coincides with \hat{g} in $M_0 \setminus \mathcal{K}_0$. The unique solvability of this Cauchy problem is studied in e.g. [15, Thm. 4.6 and 4.13], [51] and Appendix B below.

A.4. Relation of the reduced Einstein equations and the original Einstein equation. The metric g which solves the \hat{g} -reduced Einstein equation $\text{Ein}_{\hat{g}}(g) = T$ is a solution of the original Einstein equations $\text{Ein}(g) = T$ if the harmonicity functions \hat{F}^n vanish identically. Next we recall the result that the harmonicity functions vanish

on M_0 when

$$(155) \quad \begin{aligned} (\text{Ein}_{\widehat{g}}(g))_{jk} &= T_{jk}, \quad \text{on } M_0, \\ \nabla_p T^{pq} &= 0, \quad \text{on } M_0, \\ g &= \widehat{g}, \quad \text{on } M_0 \setminus \mathcal{K}_0. \end{aligned}$$

To see this, let us denote $\text{Ein}_{jk}(g) = S_{jk}$, $S^{jk} = g^{jn}g^{km}S_{nm}$, and $T^{jk} = g^{jn}g^{km}T_{nm}$. Following standard arguments, see [15], we see from (24) that in local coordinates

$$S_{jk} - (\text{Ein}_{\widehat{g}}(g))_{jk} = \frac{1}{2}(g_{jn}\widehat{\nabla}_k\widehat{F}^n + g_{kn}\widehat{\nabla}_j\widehat{F}^n - g_{jk}\widehat{\nabla}_n\widehat{F}^n).$$

Using equations (155), the Bianchi identity $\nabla_p S^{pq} = 0$, and the basic property of Lorentzian connection, $\nabla_k g^{nm} = 0$, we obtain

$$\begin{aligned} 0 &= 2\nabla_p(S^{pq} - T^{pq}) \\ &= \nabla_p(g^{qk}\widehat{\nabla}_k F^p + g^{pm}\widehat{\nabla}_m\widehat{F}^q - g^{pq}\widehat{\nabla}_n\widehat{F}^n) \\ &= g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + (g^{qp}\nabla_n\widehat{\nabla}_p\widehat{F}^n - g^{qp}\nabla_p\widehat{\nabla}_n\widehat{F}^n) \\ &= g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + W^q(\widehat{F}) \end{aligned}$$

where $\widehat{F} = (\widehat{F}^q)_{q=1}^4$ and the operator

$$W : (\widehat{F}^q)_{q=1}^4 \mapsto (g^{qk}(\nabla_p\widehat{\nabla}_k\widehat{F}^p - \nabla_k\widehat{\nabla}_p\widehat{F}^p))_{q=1}^4$$

is a linear first order differential operator which coefficients are polynomial functions of \widehat{g}_{jk} , \widehat{g}^{jk} , g_{jk} , g^{jk} and their first derivatives.

Thus the harmonicity functions \widehat{F}^q satisfy on M_0 the hyperbolic initial value problem

$$\begin{aligned} g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + W^q(\widehat{F}) &= 0, \quad \text{on } M_0, \\ \widehat{F}^q &= 0, \quad \text{on } M_0 \setminus \mathcal{K}_0, \end{aligned}$$

and as this initial Cauchy problem is uniquely solvable by [15, Thm. 4.6 and 4.13] or [51], we see that $\widehat{F}^q = 0$ on M_0 . Thus equations (155) yield that the Einstein equations $\text{Ein}(g) = T$ hold on M_0 .

We note that in the (g, \widehat{g}) -wave map coordinates, where $\widehat{F}^q = 0$, the wave operator (147) has the form

$$(156) \quad \square_g \phi = g^{jk}\partial_j\partial_k\phi - g^{pq}\widehat{\Gamma}_{pq}^n\partial_n\phi.$$

Thus, the scalar field equation $\square_g \phi - m^2\phi = 0$ does not involve derivatives of g .

A.5. Linearized Einstein-scalar field equations. Next we consider the linearized equation are obtained as the derivatives of the solutions of the non-linear, generalized Einstein-matter field equations (8).

Observe that if a family $\mathcal{F}_\varepsilon = (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2)$ of sources and a family $(g_\varepsilon, \phi_\varepsilon)$ of functions are solutions of the non-linear reduced Einstein-scalar field equations (8) that depend smoothly on $\varepsilon \in [0, \varepsilon_0]$ in $C^{15}(M_0)$ and

satisfy $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$, $(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi})$, then $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = f = (f^1, f^2)$, and $\partial_\varepsilon(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\dot{g}, \dot{\phi})$, satisfy the linearized version of the equation (8) that has the form (in the local coordinates),

$$(157) \quad \begin{aligned} \square_{\widehat{g}} \dot{g}_{jk} + A_{jk}(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) &= f_{jk}^1, \quad \text{in } M_0, \\ \square_{\widehat{g}} \dot{\phi}_\ell + B_\ell(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) &= f_\ell^2, \quad \ell = 1, 2, 3, \dots, L. \end{aligned}$$

Here A_{jk} and B_ℓ are first order linear differential operators whose coefficients depend on \widehat{g} and $\widehat{\phi}$. Let us write these equations in more explicit form. We see that the linearized reduced Einstein tensor is in local coordinates of the form

$$\begin{aligned} e_{pq}(\dot{g}) &:= \partial_\varepsilon(\text{Ein}_{\widehat{g}} g_\varepsilon)_{pq}|_{\varepsilon=0} \\ &= -\frac{1}{2}\widehat{g}^{jk}\widehat{\nabla}_j\widehat{\nabla}_k\dot{g}_{pq} + \frac{1}{4}(\widehat{g}^{nm}\widehat{g}^{jk}\widehat{\nabla}_j\widehat{\nabla}_k\dot{g}_{nm})\widehat{g}_{pq} \\ &\quad + A_{pq}^{abn}\widehat{\nabla}_n\dot{g}_{ab} + B_{pq}^{ab}\dot{g}_{ab}, \end{aligned}$$

where $A_{pq}^{abn}(x)$ and $B_{pq}^{ab}(x)$ depend on \widehat{g}_{jk} and its derivatives (these terms can be computed explicitly using (140)). The linearized scalar field stress-energy tensor is the linear first order differential operator

$$\begin{aligned} t_{pq}^{(1)}(\dot{g}) + t_{pq}^{(2)}(\dot{\phi}) &:= \partial_\varepsilon(\mathbf{T}_{jk}(g_\varepsilon, \phi_\varepsilon))|_{\varepsilon=0} \\ &= \sum_{\ell=1}^L \left(\partial_j \widehat{\phi}_\ell \partial_k \dot{\phi}_\ell + \partial_j \dot{\phi}_\ell \partial_k \widehat{\phi}_\ell \right. \\ &\quad - \frac{1}{2}\dot{g}_{jk}\widehat{g}^{pq}\partial_p \widehat{\phi}_\ell \partial_q \widehat{\phi}_\ell - \frac{1}{2}\widehat{g}_{jk}\dot{g}^{pq}\partial_p \widehat{\phi}_\ell \partial_q \widehat{\phi}_\ell \\ &\quad - \frac{1}{2}\widehat{g}_{jk}\widehat{g}^{pq}\partial_p \dot{\phi}_\ell \partial_q \widehat{\phi}_\ell - \frac{1}{2}\widehat{g}_{jk}\widehat{g}^{pq}\partial_p \widehat{\phi}_\ell \partial_q \dot{\phi}_\ell \\ &\quad \left. - m^2 \widehat{\phi}_\ell \dot{\phi}_\ell \widehat{g}_{jk} - \frac{1}{2}m^2 \widehat{\phi}_\ell^2 \dot{g}_{jk} \right). \end{aligned}$$

Thus when $\mathcal{F}_\varepsilon = (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2)$ is a family of sources and $(g_\varepsilon, \phi_\varepsilon)$ a family of functions that satisfy the non-linear reduced Einstein-scalar field equations (8), the ε -derivatives $\dot{u} = (\dot{g}, \dot{\phi})$ and $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = f = (f^1, f^2)$ satisfy in local (g, \widehat{g}) -wave coordinates

$$(158) \quad \begin{aligned} e_{pq}(\dot{g}) - t_{pq}^{(1)}(\dot{g}) - t_{pq}^{(2)}(\dot{\phi}) &= f_{pq}^1, \\ \square_{\widehat{g}} \dot{\phi}_\ell - \widehat{g}^{nm}\widehat{g}^{kj}(\partial_n \partial_j \widehat{\phi}_\ell + \widehat{\Gamma}_{nj}^p \partial_p \widehat{\phi}_\ell) \dot{g}_{mk} - m^2 \phi_\ell &= f_\ell^2. \end{aligned}$$

We call this the linearized Einstein-scalar field equation.

A.6. Linearization of the conservation law. Assume that $u = (g, \phi)$ and $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2)$ satisfy equation (8). Then the conservation

law (9) gives for all $j = 1, 2, 3, 4$ equations (see [15, Sect. 6.4.1])

$$\begin{aligned} 0 &= \frac{1}{2} \nabla_p^g (g^{pk} T_{jk}) \\ &= \frac{1}{2} \nabla_p^g (g^{pk} (\mathbf{T}_{jk}(g, \phi) + \mathcal{F}_{jk}^1)) \\ &= \sum_{\ell=1}^L (g^{pk} \nabla_p^g \partial_k \phi_\ell) \partial_j \phi_\ell - (m_\ell^2 \phi_\ell \partial_p \phi_\ell) \delta_j^p + \frac{1}{2} \nabla_p^g (g^{pk} \mathcal{F}_{jk}^1) \\ &= \sum_{\ell=1}^L (g^{pk} \nabla_p^g \partial_k \phi_\ell - m_\ell^2 \phi_\ell) \partial_j \phi_\ell + \frac{1}{2} \nabla_p^g (g^{pk} \mathcal{F}_{jk}^1). \end{aligned}$$

This yields by (8)

$$(159) \quad \frac{1}{2} g^{pk} \nabla_p^g \mathcal{F}_{jk}^1 + \sum_{\ell=1}^L \mathcal{F}_\ell^2 \partial_j \phi_\ell = 0, \quad j = 1, 2, 3, 4.$$

Next assume that $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ and \mathcal{F}_ε satisfy equation (8) and C^1 -smooth functions of $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ taking values in $H^1(N)$ -tensor fields, and $(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\hat{g}, \hat{\phi})$ and $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$. Denote $(f^1, f^2) = \partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0}$. Then by taking ε -derivative of (159) at $\varepsilon = 0$ we get

$$(160) \quad \frac{1}{2} \hat{g}^{pk} \hat{\nabla}_p f_{jk}^1 + \sum_{\ell=1}^L f_\ell^2 \partial_j \hat{\phi}_\ell = 0, \quad j = 1, 2, 3, 4.$$

We call this the linearized conservation law.

APPENDIX B: STABILITY AND EXISTENCE OF THE DIRECT PROBLEM

Let us next consider existence and stability of the solutions of the Einstein-scalar field equations. Let $t = \mathbf{t}(x)$ be local time so that there is a diffeomorphism $\Psi : M \rightarrow \mathbb{R} \times N$, $\Psi(x) = (\mathbf{t}(x), Y(x))$, and

$$S(T) = \{x \in M_0; \mathbf{t}(x) = T\}, \quad T \in \mathbb{R}$$

are Cauchy surfaces. Let $t_0 > 0$. Next we identify M and $\mathbb{R} \times N$ via the map Ψ and just denote $M = \mathbb{R} \times N$. Let us denote

$$M(t) = (-\infty, t) \times N, \quad M_0 = M(t_0).$$

By [5, Cor. A.5.4] the set $\mathcal{K} = J_g^+(\widehat{p}^-) \cap M_0$, where $\widehat{p}^0 \in M_0$, is compact. Let $N_1, N_2 \subset N$ be such open relatively compact sets with smooth boundary so that $N_1 \subset N_2$ and $Y(J_g^+(\widehat{p}^0) \cap M_0) \subset N_1$.

To simplify citations to the existing literature, let us define \tilde{N} to be a compact manifold without boundary such that N_2 can be considered as a subset of \tilde{N} . Using a construction based on a suitable partition of unity, the Hopf double of the manifold N_2 , and the Seeley extension of the metric tensor, we can endow $\widetilde{M} = (-\infty, t_0) \times \tilde{N}$ with a smooth Lorentzian metric \widehat{g}^e (index e is for "extended") so that

$\{t\} \times \tilde{N}$ are Cauchy surfaces of \tilde{N} and that \hat{g} and \hat{g}^e coincide in the set $\Psi^{-1}((-\infty, t_0) \times N_1)$ that contains the set $J_{\hat{g}}^+(\hat{p}^0) \cap M_0$. We extend the metric \tilde{g} to a (possibly non-smooth) globally hyperbolic metric \tilde{g}^e on $\tilde{M}_0 = (-\infty, t_0) \times \tilde{N}$ such that $\hat{g}^e < \tilde{g}^e$.

To simplify notations below we denote $\hat{g}^e = \hat{g}$ and $\tilde{g}^e = \tilde{g}$ on the whole \tilde{M}_0 . Our aim is to prove the estimate (27).

Let us denote by $t = t(x)$ the local time. Recall that when (g, ϕ) is a solution of the scalar field-Einstein equation, we denote $u = (g - \hat{g}, \phi - \hat{\phi})$. We will consider the equation for u , and to emphasize that the metric depends on u , we denote $g = g(u)$ and assume below that both the metric g is dominated by \tilde{g} , that is, $g < \tilde{g}$. We use the pairs

$$\mathbf{u}(t) = (u(t, \cdot), \partial_t u(t, \cdot)) \in H^1(\tilde{N}) \times L^2(\tilde{N})$$

the notations $\mathbf{v}(t) = (v(t), \partial_t v(t))$ etc. We consider sources \mathcal{F} and F , cf. Appendix C. Let us consider a generalization of the system (26) of the form

$$(161) \quad \square_{g(u)} u + V(x, D)u + H(u, \partial u) = R(x, u, \partial u, F, \partial F) + \mathcal{F}, \quad x \in \tilde{M}_0, \\ \text{supp } (u) \subset \mathcal{K},$$

where $\square_{g(u)}$ is the Lorentzian Laplace operator operating on the sections of the bundle \mathcal{B}^L on M_0 and $\text{supp } (F) \cup \text{supp } (\mathcal{F}) \subset \mathcal{K}$. Note that above $u = (g - \hat{g}, \phi - \hat{\phi})$ and $g(u) = g$. Also, $F \mapsto R(x, u, \partial u, F, \partial F)$ is a non-linear first order differential operator where R is a smooth function that depends on all variables x , $u(x)$, $\partial_j u(x)$, $F(x)$, and $\partial_j F(x)$ smoothly and also on the derivatives of $(\hat{g}, \hat{\phi})$ at x . Let

$$V(x, D) = V^j(x) \partial_j + V(x)$$

be a linear first order differential operator whose coefficients at x are depending smoothly on the derivatives of $(\hat{g}, \hat{\phi})$ at x , and finally, $H(u, \partial u)$ is a polynomial of $u(x)$ and $\partial_j u(x)$ which coefficients at x are depending smoothly on the derivatives of $(\hat{g}, \hat{\phi})$ such that

$$\partial_v^\alpha \partial_w^\beta H(v, w)|_{v=0, w=0} = 0 \text{ for } |\alpha| + |\beta| \leq 1.$$

By [94, Lemma 9.7], the equation (161) has at most one solution with given C^2 -smooth source functions F and \mathcal{F} . Next we consider the existence of u and its dependency on F and \mathcal{F} .

Below we use notations, c.f. (26) and (34) and Appendix C,

$$\mathcal{R}(\mathbf{u}, F) = R(x, u(x), \partial u(x), F(x), \partial F(x)), \quad \mathcal{H}(\mathbf{u}) = H(u(x), \partial u(x)).$$

Note that $u = 0$, i.e., $g = \hat{g}$ and $\phi = \hat{\phi}$ satisfies (161) with $F = 0$ and $\mathcal{F} = 0$. Let us use the same notations as in [51] cf. also [63, section 16], to consider quasilinear wave equation on $[0, t_0] \times \tilde{N}$. Let $\mathbb{H}^{(s)}(\tilde{N}) = H^s(\tilde{N}) \times H^{s-1}(\tilde{N})$ and

$$Z = \mathbb{H}^{(1)}(\tilde{N}), \quad Y = \mathbb{H}^{(k+1)}(\tilde{N}), \quad X = \mathbb{H}^{(k)}(\tilde{N}).$$

The norms on these space are defined invariantly using the smooth Riemannian metric $h = \widehat{g}|_{\{0\} \times \tilde{N}}$ on \tilde{N} . Note that $\mathbb{H}^{(s)}(\tilde{N})$ are in fact the Sobolev spaces of sections on the bundle $\pi : \mathcal{B}_K \rightarrow \tilde{N}$, where \mathcal{B}_K denotes also the pull back bundle of \mathcal{B}_K on \tilde{M} in the map $id : \{0\} \times \tilde{N} \rightarrow \tilde{M}_0$, or on the bundle $\pi : \mathcal{B}_L \rightarrow \tilde{N}$. Below, ∇_h denotes the standard connection of the bundle \mathcal{B}_K or \mathcal{B}_L associated to the metric h .

Let $k \geq 4$ be an even integer. By definition of H and R we see that there are $0 < r_0 < 1$ and $L_1, L_2 > 0$, all depending on $\widehat{g}, \widehat{\phi}, \mathcal{K}$, and t_0 , such that if $0 < r \leq r_0$ and

$$(162) \quad \begin{aligned} \|\mathbf{v}\|_{C([0,t_0];\mathbb{H}^{(k+1)}(\tilde{N}))} &\leq r, & \|\mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(k+1)}(\tilde{N}))} &\leq r, \\ \|F\|_{C([0,t_0];H^{(k+1)}(\tilde{N}))} &\leq r^2, & \|\mathcal{F}\|_{C([0,t_0];H^{(k)}(\tilde{N}))} &\leq r^2 \\ \|F'\|_{C([0,t_0];H^{(k+1)}(\tilde{N}))} &\leq r^2, & \|\mathcal{F}'\|_{C([0,t_0];H^{(k)}(\tilde{N}))} &\leq r^2 \end{aligned}$$

then

$$(163) \quad \begin{aligned} \|g(\cdot; v)^{-1}\|_{C([0,t_0];H^s(\tilde{N}))} &\leq L_1, \\ \|\mathcal{H}(\mathbf{v})\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\leq L_2 r^2, & \|\mathcal{H}(\mathbf{v}')\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\leq L_2 r^2, \\ \|\mathcal{H}(\mathbf{v}) - \mathcal{H}(\mathbf{v}')\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\leq L_2 r \|\mathbf{v} - \mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(s)}(\tilde{N}))}, \\ \|\mathcal{R}(\mathbf{v}', F')\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\leq L_2 r^2, & \|\mathcal{R}(\mathbf{v}, F)\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\leq L_2 r^2, \\ \|\mathcal{R}(\mathbf{v}, F) - \mathcal{R}(\mathbf{v}', F')\|_{C([0,t_0];H^{s-1}(\tilde{N}))} &\\ &\leq L_2 r \|\mathbf{v} - \mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(s)}(\tilde{N}))} + L_2 \|F - F'\|_{C([0,t_0];H^{s+1}(\tilde{N})) \cap C^1([0,t_0];H^s(\tilde{N}))}, \end{aligned}$$

for all $s \in [1, k+1]$.

Next we write (161) as a first order system. To this end, let $\mathcal{A}(t, \mathbf{v}) : \mathbb{H}^{(s)}(\tilde{N}) \rightarrow \mathbb{H}^{(s-1)}(\tilde{N})$ be the operator

$$\mathcal{A}(t, \mathbf{v}) = \mathcal{A}_0(t, \mathbf{v}) + \mathcal{A}_1(t, \mathbf{v})$$

where in local coordinates and in the local trivialization of the bundle \mathcal{B}^L

$$\mathcal{A}_0(t, \mathbf{v}) = - \left(\frac{1}{g^{00}(v)} \sum_{j,k=1}^3 g^{jk}(v) \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \quad \frac{I}{\frac{1}{g^{00}(v)} \sum_{m=1}^3 g^{0m}(v) \frac{\partial}{\partial x^m}} \right)$$

with $g^{jk}(v) = g^{jk}(t, \cdot; v)$ is a function on \tilde{N} and

$$\mathcal{A}_1(t, \mathbf{v}) = \frac{-1}{g^{00}(v)} \left(\begin{array}{cc} 0 & 0 \\ \sum_{j=1}^3 B^j(v) \frac{\partial}{\partial x^j} & B^0(v) \end{array} \right)$$

where $B^j(v)$ depend on $v(t, x)$ and its first derivatives, and the connection coefficients (the Christoffel symbols) corresponding to $g(v)$. We denote $\mathcal{S} = (F, \mathcal{F})$ and

$$\begin{aligned} f_{\mathcal{S}}(t, \mathbf{v}) &= (f_{\mathcal{S}}^1(t, \mathbf{v}), f_{\mathcal{S}}^2(t, \mathbf{v})) \in \mathbb{H}^{(k)}(\tilde{N}), \quad \text{where} \\ f_{\mathcal{S}}^1(t, \mathbf{v}) &= 0, \quad f_{\mathcal{S}}^2(t, \mathbf{v}) = \mathcal{R}(\mathbf{v}, F)(t, \cdot) - \mathcal{H}(\mathbf{v})(t, \cdot) + \mathcal{F}(t, \cdot). \end{aligned}$$

Note that when (162) are satisfied with $r < r_0$, inequalities (163) imply that there exists $C_2 > 0$ so that

$$(164) \quad \begin{aligned} \|f_S(t, \mathbf{v})\|_Y + \|f_{S'}(t, \mathbf{v}')\|_Y &\leq C_2 r^2, \\ \|f_S(t, \mathbf{v}) - f_S(t, \mathbf{v}')\|_Y &\leq C_2 r \|\mathbf{v} - \mathbf{v}'\|_{C([0, t_0]; Y)}. \end{aligned}$$

Let $U^\mathbf{v}(t, s)$ be the wave propagator corresponding to metric $g(\mathbf{v})$, that is, $U^\mathbf{v}(t, s) : \mathbf{h} \mapsto \mathbf{w}$, where $\mathbf{w}(t) = (w(t), \partial_t w(t))$ solves

$$(\square_{g(\mathbf{v})} + V(x, D))w = 0 \quad \text{for } (t, y) \in [s, t_0] \times \tilde{N}, \quad \text{with } \mathbf{w}(s, y) = \mathbf{h}.$$

Let

$$S = (\nabla_h^* \nabla_h + 1)^{k/2} : Y \rightarrow Z$$

be an isomorphism. As k is an even integer, we see using multiplication estimates for Sobolev spaces, see e.g. [51, Sec. 3.2, point (2)], that there exists $c_1 > 0$ (depending on r_0, L_1 , and L^2) so that

$$\mathcal{A}(t, \mathbf{v})S - S\mathcal{A}(t, \mathbf{v}) = C(t, \mathbf{v}),$$

where $\|C(t, \mathbf{v})\|_{Y \rightarrow Z} \leq c_1$ for all \mathbf{v} satisfying (162). This yields that the property (A2) in [51] holds, namely that

$$S\mathcal{A}(t, \mathbf{v})S^{-1} = \mathcal{A}(t, \mathbf{v}) + B(t, \mathbf{v})$$

where $B(t, \mathbf{v})$ extends to a bounded operator in Z for which

$$\|B(t, \mathbf{v})\|_{Z \rightarrow Z} \leq c_1$$

for all \mathbf{v} satisfying (162). Alternatively, to see the mapping properties of $B(t, \mathbf{v})$ we could use the fact that $B(t, \mathbf{v})$ is a zeroth order pseudo-differential operator with H^k -symbol.

Thus the proof of [51, Lemma 2.6] shows that there is a constant $C_3 > 0$ so that

$$(165) \quad \|U^\mathbf{v}(t, s)\|_{Z \rightarrow Z} \leq C_3 \quad \text{and} \quad \|U^\mathbf{v}(t, s)\|_{Y \rightarrow Y} \leq C_3$$

for $0 \leq s < t \leq t_0$. By interpolation of estimates (165), we see also that

$$(166) \quad \|U^\mathbf{v}(t, s)\|_{X \rightarrow X} \leq C_3,$$

for $0 \leq s < t \leq t_0$.

Let us next modify the reasoning given in [63]: let $r_1 \in (0, r_0)$ be a parameter which value will be chosen later, $C_1 > 0$ and E be the space of functions $\mathbf{u} \in C([0, t_0]; X)$ for which

$$(167) \quad \|\mathbf{u}(t)\|_Y \leq r_1 \quad \text{and}$$

$$(168) \quad \|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_X \leq C_1 |t_1 - t_2|$$

for all $t, t_1, t_2 \in [0, t_0]$. The set E is endowed by the metric of $C([0, t_0]; X)$. We note that by [63, Lemma 7.3], a convex Y -bounded, Y -closed set is closed also in X . Similarly, functions $G : [0, t_0] \rightarrow X$ satisfying (168) form a closed subspace of $C([0, t_0]; X)$. Thus $E \subset X$ is a closed set implying that E is a complete metric space.

Let

$$W = \{(F, \mathcal{F}) \in C([0, t_0]; H^{k+1}(\tilde{N})) \times C([0, t_0]; H^k(\tilde{N})); \\ \sup_{t \in [0, t_0]} \|F(t)\|_{H^{k+1}(\tilde{N})} + \|\mathcal{F}(t)\|_{H^k(\tilde{N})} < r_1\}.$$

Following [63, p. 44], we see that the solution of equation (161) with the source $\mathcal{S} \in W$ is found as a fixed point, if it exists, of the map $\Phi_{\mathcal{S}} : E \rightarrow C([0, t_0]; Y)$ where $\Phi_{\mathcal{S}}(\mathbf{v}) = \mathbf{u}$ is given by

$$\mathbf{u}(t) = \int_0^t U^{\mathbf{v}}(t, \tilde{t}) f_{\mathcal{S}}(\tilde{t}, \mathbf{v}) d\tilde{t}, \quad 0 \leq t \leq t_0.$$

Below, we denote $\mathbf{u}^{\mathbf{v}} = \Phi_{\mathcal{S}}(\mathbf{v})$.

As $\Phi_{\mathcal{S}_0}(0) = 0$ where $\mathcal{S}_0 = (0, 0)$, we see using the above and the inequality $\|\cdot\|_X \leq \|\cdot\|_Y$ that the function $\mathbf{u}^{\mathbf{v}}$ satisfies

$$\|\mathbf{u}^{\mathbf{v}}\|_{C([0, t_0]; Y)} \leq C_3 C_2 t_0 r_1^2, \\ \|\mathbf{u}^{\mathbf{v}}(t_2) - \mathbf{u}^{\mathbf{v}}(t_1)\|_X \leq C_3 C_2 r_1^2 |t_2 - t_1|, \quad t_1, t_2 \in [0, t_0].$$

When $r_1 > 0$ is so small that $C_3 C_2 (1 + t_0) < r_1^{-1}$ and $C_3 C_2 r_1^2 < C_1$ we see that

$$\|\Phi_{\mathcal{S}}(\mathbf{v})\|_{C([0, t_0]; Y)} < r_1, \\ \|\Phi_{\mathcal{S}}(\mathbf{v})\|_{C^{0,1}([0, t_0]; X)} < C_1.$$

Hence $\Phi_{\mathcal{S}}(E) \subset E$ and we can consider $\Phi_{\mathcal{S}}$ as a map $\Phi_{\mathcal{S}} : E \rightarrow E$.

As $k > 1 + \frac{3}{2}$, it follows from Sobolev embedding theorem that $X = \mathbb{H}^{(k)}(\tilde{N}) \subset C^1(\tilde{N})^2$. This yields that by [63, Thm. 3], for the original reference, see Theorems III-IV in [62],

$$(169) \quad \begin{aligned} & \| (U^{\mathbf{v}}(t, s) - U^{\mathbf{v}'}(t, s)) \mathbf{h} \|_X \\ & \leq C_3 \left(\sup_{t' \in [0, t]} \| \mathcal{A}(t', \mathbf{v}) - \mathcal{A}(t', \mathbf{v}') \|_{Y \rightarrow X} \| U^{\mathbf{v}}(t', 0) \mathbf{h} \|_Y \right) \\ & \leq C_3^2 \| \mathbf{v} - \mathbf{v}' \|_{C([0, t_0]; X)} \| \mathbf{h} \|_Y. \end{aligned}$$

Thus,

$$\begin{aligned} & \| U^{\mathbf{v}}(t, s) f_{\mathcal{S}}(s, \mathbf{v}) - U^{\mathbf{v}'}(t, s) f_{\mathcal{S}}(s, \mathbf{v}') \|_X \\ & \leq \| (U^{\mathbf{v}}(t, s) - U^{\mathbf{v}'}(t, s)) f_{\mathcal{S}}(s, \mathbf{v}) \|_X + \| U^{\mathbf{v}'}(t, s) (f_{\mathcal{S}}(s, \mathbf{v}) - f_{\mathcal{S}}(s, \mathbf{v}')) \|_X \\ & \leq (1 + C_3)^2 C_2 r_1^2 \| \mathbf{v} - \mathbf{v}' \|_{C([0, t_0]; X)}. \end{aligned}$$

This implies that

$$\| \Phi_{\mathcal{S}}(\mathbf{v}) - \Phi_{\mathcal{S}}(\mathbf{v}') \|_{C([0, t_0]; X)} \leq t_0 (1 + C_3)^2 C_2 r_1^2 \| \mathbf{v} - \mathbf{v}' \|_{C([0, t_0]; X)}.$$

Assume next that $r_1 > 0$ is so small that we have also

$$t_0 (1 + C_3)^2 C_2 r_1^2 < \frac{1}{2}.$$

cf. Thm. I in [51] (or (9.15) and (10.3)-(10.5) in [63]). For $\mathcal{S} \in W$ this implies that

$$\Phi_{\mathcal{S}} : E \rightarrow E$$

is a contraction with a contraction constant $C_L \leq \frac{1}{2}$, and thus $\Phi_{\mathcal{S}}$ has a unique fixed point \mathbf{u} in the space $E \subset C^{0,1}([0, t_0]; X)$.

Moreover, elementary considerations related to fixed points of the map $\Phi_{\mathcal{S}}$ show that \mathbf{u} in $C([0, t_0]; X)$ depends in $E \subset C([0, t_0]; X)$ Lipschitz-continuously on $\mathcal{S} \in W \subset C([0, t_0]; H^{k+1}(\tilde{N}) \times H^k(\tilde{N}))$. Indeed, if

$$\|\mathcal{S} - \mathcal{S}'\|_{C([0, t_0]; H^{k+1}(\tilde{N}) \times H^k(\tilde{N}))} < \varepsilon,$$

we see that

$$(170) \quad \|f_{\mathcal{S}}(t, \mathbf{v}) - f_{\mathcal{S}'}(t, \mathbf{v})\|_Y \leq C_2 \varepsilon, \quad t \in [0, t_0],$$

and when (164) and (165) are satisfied with $r = r_1$, we have

$$\|\Phi_{\mathcal{S}}(\mathbf{v}) - \Phi_{\mathcal{S}'}(\mathbf{v}')\|_{C([0, t_0]; Y)} \leq C_3 C_2 t_0 r_1^2.$$

Hence

$$\|\Phi_{\mathcal{S}}(\mathbf{v}) - \Phi_{\mathcal{S}'}(\mathbf{v})\|_{C([0, t_0]; Y)} \leq t_0 C_3 C_2 \varepsilon.$$

This and standard estimates for fixed points, yield that when ε is small enough the fixed point \mathbf{u}' of the map $\Phi_{\mathcal{S}'} : E \rightarrow E$ corresponding to the source \mathcal{S}' and the fixed point \mathbf{u} of the map $\Phi_{\mathcal{S}} : E \rightarrow E$ corresponding to the source \mathcal{S} satisfy

$$(171) \quad \|\mathbf{u} - \mathbf{u}'\|_{C([0, t_0]; X)} \leq \frac{1}{1 - C_L} t_0 C_3 C_2 \varepsilon.$$

Thus the solution \mathbf{u} depends in $C([0, t_0]; X)$ Lipschitz continuously on $\mathcal{S} \in C([0, t_0]; H^{k+1}(\tilde{N}) \times H^k(\tilde{N}))$ (see also [63, Sect. 16], and [94]). In fact, for analogous systems it is possible to show that u is in $C([0, t_0]; Y)$, but one can not obtain Lipschitz or Hölder stability for u in the Y -norm, see [63], Remark 7.2.

Finally, we note that the fixed point \mathbf{u} of $\Phi_{\mathcal{S}}$ can be found as a limit $\mathbf{u} = \lim_{n \rightarrow \infty} \mathbf{u}_n$ in $C([0, t_0]; X)$, where $\mathbf{u}_0 = 0$ and $\mathbf{u}_n = \Phi_{\mathcal{S}}(\mathbf{u}_{n-1})$. Denote $\mathbf{u}_n = (g_n - \hat{g}, \phi_n - \hat{\phi})$. We see that if

$$\text{supp}(\mathbf{u}_{n-1}) \subset J_{\hat{g}}(\text{supp}(\mathcal{S}))$$

then also

$$\text{supp}(g_{n-1} - \hat{g}) \subset J_{\hat{g}}(\text{supp}(\mathcal{S})).$$

Hence for all $x \in M_0 \setminus J_{\hat{g}}(\text{supp}(\mathcal{S}))$ we see that

$$J_{g_{n-1}}^-(x) \cap J_{\hat{g}}(\text{supp}(\mathcal{S})) = \emptyset.$$

Then, using the definition of the map $\Phi_{\mathcal{S}}$ we see that $\text{supp}(\mathbf{u}_n) \subset J_{\hat{g}}(\text{supp}(\mathcal{S}))$. Using induction we see that this holds for all n and hence we have that the solution \mathbf{u} satisfies

$$(172) \quad \text{supp}(\mathbf{u}) \subset J_{\hat{g}}(\text{supp}(\mathcal{S})).$$

APPENDIX C: AN EXAMPLE SATISFYING MICROLOCAL LINEARIZATION STABILITY

C.1. Formulation of the direct problem. Let us define some physical fields and introduce a model as a system of partial differential equations. Later we will motivate this system by discussion of the corresponding Lagrangians, but we postpone this discussion to Appendix C.3 as it is not completely rigorous.

We assume that there are C^∞ -background fields \hat{g} , $\hat{\phi}$, on M .

We consider a Lorentzian metric g on M_0 and $\phi = (\phi_\ell)_{\ell=1}^L$ where ϕ_ℓ are scalar fields on $M_0 = (-\infty, t_0) \times N$.

Let $P = P_{jk}(x)dx^j dx^k$ be a symmetric tensor on M_0 , corresponding below to a direct perturbation to the stress energy tensor, and $Q = (Q_\ell(x))_{\ell=1}^K$ where $Q_\ell(x)$ are real-valued functions on M_0 , where $K \geq L + 1$. We denote by $\mathcal{V}(\phi_\ell; S_\ell)$ the potential functions of the fields ϕ_ℓ ,

$$(173) \quad \mathcal{V}(\phi_\ell; S_\ell) = \frac{1}{2}m^2 \left(\phi_\ell + \frac{1}{m^2} S_\ell \right)^2.$$

These potentials depend on the source variables S_ℓ . The way how S_ℓ , called below the adaptive source functions, depend on other fields is explained later. We assume that there are smooth background fields \hat{P} and \hat{Q} . For a while we consider the case when $\hat{P} = 0$ and $\hat{Q} = 0$, and discuss later the generalization to non-vanishing background fields.

Using the ϕ and P fields, we define the stress-energy tensor

$$(174) T_{jk} = \sum_{\ell=1}^L (\partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell - \mathcal{V}(\phi_\ell; S_\ell) g_{jk}) + P_{jk}.$$

We assume that P and Q are supported on $\mathcal{K} = J_{\tilde{g}}^+(\hat{p}^-) \cap M_0$. Let us represent the stress energy tensor (174) in the form

$$\begin{aligned} T_{jk} &= P_{jk} + Z g_{jk} + \mathbf{T}_{jk}(g, \phi), \quad Z = -\left(\sum_{\ell=1}^L S_\ell \phi_\ell + \frac{1}{2m^2} S_\ell^2\right), \\ \mathbf{T}_{jk}(g, \phi) &= \sum_{\ell=1}^L (\partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}), \end{aligned}$$

where we call Z the stress energy density caused by the sources S_ℓ .

Now we are ready to formulate the direct problem for the adaptive Einstein-scalar field equations. Let g and ϕ satisfy

$$(175) \text{Ein}_{\hat{g}}(g) = P_{jk} + Z g_{jk} + \mathbf{T}_{jk}(g, \phi), \quad Z = -\left(\sum_{\ell=1}^L S_\ell \phi_\ell + \frac{1}{2m^2} S_\ell^2\right),$$

$$\square_g \phi_\ell - \mathcal{V}'(\phi_\ell; S_\ell) = 0 \quad \text{in } M_0, \quad \ell = 1, 2, 3, \dots, L,$$

$$S_\ell = \mathcal{S}_\ell(g, \phi, \nabla \phi, Q, \nabla Q, P, \nabla^g P), \quad \text{in } M_0,$$

$$g = \hat{g}, \quad \phi_\ell = \hat{\phi}_\ell, \quad \text{in } M_0 \setminus \mathcal{K}.$$

Above, $\mathcal{V}'(\phi; s) = \partial_\phi \mathcal{V}(\phi; s)$ so that $\mathcal{V}'(\phi_\ell; S_\ell) = m^2 \phi_\ell + S_\ell$. We assume that the background fields $\hat{g}, \hat{\phi}$, satisfy these equations with $\hat{Q} = 0$ and $\hat{P} = 0$.

We consider here $P = (P_{jk})_{j,k=1}^4$ and $Q = (Q_\ell)_{\ell=1}^K$ as fields that we can control and call those the controlled source fields. Local existence of the solution for small sources P and Q is considered in Appendix B.

To obtain a physically meaningful model, we need to consider how the adaptive source functions \mathcal{S}_ℓ should be chosen so that the physical conservation law in relativity

$$(176) \quad \nabla_k (g^{kp} T_{pq}) = 0$$

is satisfied. Here $\nabla = \nabla^g$ is the connection corresponding to the metric g .

We note that the conservation law is a necessary condition for the equation (175) to have solutions for which $\text{Ein}_{\hat{g}}(g) = \text{Ein}(g)$, i.e., that the solutions of (175) are solutions of the Einstein field equations.

The functions $\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$ model the devices that we use to perform active measurements. Thus, even though the Assumption S below may appear quite technical, it can be viewed as the instructions on how to build a device that can be used to measure the structure of the spacetime far away. Outside the support of the measurement device (that contain the union of the supports of Q and P) we have just assumed that the standard coupled Einstein-scalar field equations hold, c.f. (177). We can consider them in the form

$$\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P) = Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$$

for $\ell = 1, 2, \dots, L$ where Q_ℓ are the primary sources and \mathcal{S}_ℓ^{2nd} , that depend also on Q_ℓ with $\ell = L+1, L+2, \dots, K$, corresponds to the response of the measurement device that forces the conservation law to be valid.

The solution (g, ϕ) of (175) is a solution of the equations (8) when we denote

$$\begin{aligned} \mathcal{F}_{jk}^1 &= P_{jk} + Z g_{jk}, \\ \mathcal{F}_\ell^2 &= \mathcal{V}'(\phi_\ell; S_\ell) - \mathcal{V}'(\phi_\ell; 0) = S_\ell. \end{aligned}$$

Our next goal is to construct suitable adaptive source functions \mathcal{S}_ℓ and consider what kind of sources \mathcal{F}^1 and \mathcal{F}^2 of the above form can be obtained by varying P and Q .

We will consider adaptive source functions \mathcal{S}_ℓ satisfying the following conditions:

Assumption S:

The adaptive source functions $\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$ have the following properties:

(i) Denoting $c = \nabla\phi$, $C = \nabla^g P$, and $H = \nabla Q$ we assume that $\mathcal{S}_\ell(g, \phi, c, Q, H, P, C)$ are smooth non-linear functions, of the pointwise values $g_{jk}(x), \phi(x), \nabla\phi(x), Q(x), \nabla Q(x), P(x)$, and $\nabla^g P(x)$, defined near $(g, \phi, c, Q, H, P, C) = (\widehat{g}, \widehat{\phi}, \nabla\widehat{\phi}, 0, 0, 0, 0)$, that satisfy

$$(177) \quad \mathcal{S}_\ell(g, \phi, c, 0, 0, 0, 0) = 0.$$

(ii) We assume that \mathcal{S}_ℓ is independent of $P(x)$ and the dependency of \mathcal{S} on $\nabla^g P$ and ∇Q is only due to the dependency in the term $g^{pk}\nabla_p^g(P_{jk} + Zg_{jk}) = g^{pk}\nabla_p^g P_{jk} + \nabla_j^g Q_K$, associated to the divergence of the perturbation of T , that is, there exist functions $\widetilde{\mathcal{S}}_\ell$ so that

$$\mathcal{S}_\ell(g, \phi, c, Q, H, P, C) = \widetilde{\mathcal{S}}_\ell(g, \phi, c, Q, R), \quad R = (g^{pk}\nabla_p^g(P_{jk} + Q_K g_{jk}))_{j=1}^4.$$

Below, denote $\widehat{R} = \widehat{g}^{pk}\widehat{\nabla}_p\widehat{P}_{jk} + \widehat{\nabla}_j\widehat{Q}_K$. Note that we still are considering the case when $\widehat{Q} = 0$ and $\widehat{P} = 0$ so that $\widehat{R} = 0$, too. This implies that for the background fields that adaptive source functions \mathcal{S}_ℓ vanish.

To simplify notations, we also denote below $\widetilde{\mathcal{S}}_\ell$ just by \mathcal{S}_ℓ and indicate the function which we use by the used variables in these functions.

Below we will denote $Q = (Q', Q_K)$, $Q' = (Q_\ell)_{\ell=1}^{K-1}$. There are examples when the background fields $(\widehat{g}, \widehat{\phi})$ and the adaptive source functions \mathcal{S}_ℓ exists and satisfy the Assumption S. This is shown later in the case the following condition is valid for the background fields:

Condition A: Assume that at any $x \in \overline{U}_{\widehat{g}}$ there is a permutation $\sigma : \{1, 2, \dots, L\} \rightarrow \{1, 2, \dots, L\}$, denoted σ_x , such that the 5×5 matrix $[B_{jk}^\sigma(\widehat{\phi}(x), \nabla\widehat{\phi}(x))]_{j,k \leq 5}$ is invertible, where

$$[B_{jk}^\sigma(\phi(x), \nabla\phi(x))]_{k,j \leq 5} = \begin{bmatrix} (\partial_j \phi_{\sigma(\ell)}(x))_{\ell \leq 5, j \leq 4} \\ (\phi_{\sigma(\ell)}(x))_{\ell \leq 5} \end{bmatrix}.$$

Below, for a permutation $\sigma : \{1, 2, \dots, L\} \rightarrow \{1, 2, \dots, L\}$ we denote by $U_{\widehat{g}, \sigma}$ the open set of points $x \in U_{\widehat{g}}$ for which $[B_{jk}^\sigma(\widehat{\phi}(x), \nabla\widehat{\phi}(x))]_{j,k \leq 5}$ is invertible. So, we assume that the sets $U_{\widehat{g}, \sigma}$, $\sigma \in \Sigma(L)$ is an open covering of $U_{\widehat{g}}$.

Our next aim is to prove the following:

Theorem 5.9. *Let $L \geq 5$ and assume that $\widehat{Q} = 0$ and $\widehat{P} = 0$ so that $\widehat{R} = 0$. Moreover, assume that Condition A is valid. Then for all permutations $\sigma : \{1, 2, \dots, L\} \rightarrow \{1, 2, \dots, L\}$ there exists functions $\mathcal{S}_{\ell, \sigma}$ satisfying Assumption S such that*

(i) *For all $x \in U_{\widehat{g}, \sigma}$ the differential of*

$$\mathcal{S}_\sigma(\widehat{g}, \widehat{\phi}, \nabla\widehat{\phi}, Q, R) = (\mathcal{S}_{\ell, \sigma}(\widehat{g}, \widehat{\phi}, \nabla\widehat{\phi}, Q, R))_{\ell=1}^L$$

with respect to Q and R , that is, the map

$$(178) \quad D_{Q,R} \mathcal{S}_\sigma(\widehat{g}(x), \widehat{\phi}(x), \nabla \widehat{\phi}(x), Q, R)|_{Q=\widehat{Q}(x), R=\widehat{R}(x)} : \mathbb{R}^{K+4} \rightarrow \mathbb{R}^L$$

is surjective.

(ii) The adaptive source functions \mathcal{S}_σ are such that for $(Q_\ell)_{\ell=1}^K$ and (P_{jk}) that are sufficiently close to $\widehat{Q} = 0$ and $\widehat{P} = 0$ in the $C_b^3(M_0)$ -topology and supported in $U_{\widehat{g},\sigma}$ the equations (175) with source functions \mathcal{S}_σ have a unique solution (g, ϕ) and the conservation law (176) is valid.

(iii) Under the same assumptions as in (ii), when (g, ϕ) is a solution of (175) with the controlled source functions P and Q , we have $Q_K = Z$. This means that the physical field Z can be directly controlled.

Proof. As one can enumerate the ℓ -indexes of the fields ϕ_ℓ as one wishes, it is enough to prove the claim with one σ . We consider below the case when $\sigma = Id$.

Consider a symmetric (0,2)-tensor P and a scalar functions Q_ℓ that are C^3 -smooth and compactly supported in $U_{\widehat{g},\sigma}$. Let $[P_{jk}(x)]_{j,k=1}^4$ be the coefficients of P in local coordinates and $Q(x) = (Q_\ell(x))_{\ell=1}^L$.

To obtain adaptive required adaptive source functions, let us start implications of the conservation law (176). To this end, consider C^2 -smooth functions $S_\ell(x)$ on $U_{\widehat{g},\sigma}$.

Note that since $[\nabla_p, \nabla_n] = [\partial_p, \partial_n] = 0$ (see [15, Sect. III.6.4.1]),

$$\begin{aligned} & \nabla_p(g^{pj}\mathbf{T}_{jk}(g, \phi)) = \\ & \sum_{\ell=1}^L \nabla_p g^{pj} (\partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{nm} \partial_n \phi_\ell \partial_m \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}) \\ &= \sum_{\ell=1}^L (g^{pj} \nabla_p \partial_j \phi_\ell) \partial_k \phi_\ell + \sum_{\ell=1}^L (g^{pj} \partial_j \phi_\ell \nabla_p \partial_k \phi_\ell) \\ & \quad - \frac{1}{2} \sum_{\ell=1}^L \delta_k^p (g^{nm} (\nabla_p \partial_n \phi_\ell) \partial_m \phi_\ell + g^{nm} \partial_n \phi_\ell (\nabla_p \partial_m \phi_\ell)) - \sum_{\ell=1}^L m^2 \delta_k^p \phi_\ell \partial_p \phi_\ell \\ &= \sum_{\ell=1}^L (g^{pj} \nabla_p \partial_j \phi_\ell) \partial_k \phi_\ell + \sum_{\ell=1}^L (g^{pj} \partial_j \phi_\ell (\nabla_p \partial_k \phi_\ell)) \\ & \quad - \frac{1}{2} \sum_{\ell=1}^L (g^{nm} \partial_m \phi_\ell (\nabla_n \partial_k \phi_\ell) + g^{nm} \partial_n \phi_\ell (\nabla_m \partial_k \phi_\ell)) - \sum_{\ell=1}^L m^2 \phi_\ell \partial_k \phi_\ell \\ &= \sum_{\ell=1}^L (g^{pj} \nabla_p \partial_j \phi_\ell - m^2 \phi_\ell) \partial_k \phi_\ell. \end{aligned}$$

Thus conservation law (176) gives for all $j = 1, 2, 3, 4$ equations

$$\begin{aligned}
0 &= \nabla_p^g(g^{pk}T_{jk}) \\
&= \nabla_p^g(g^{pk}(\mathbf{T}_{jk}(g, \phi) + P_{jk} + Zg_{jk})) \\
&= \sum_{\ell=1}^L \left((g^{pk}\nabla_p^g \partial_k \phi_\ell) \partial_j \phi_\ell - (m_\ell^2 \phi_\ell \partial_p \phi_\ell) \delta_j^p \right. \\
&\quad \left. - \nabla_p^g(g^{pk}g_{jk}(S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2) + g^{pk}P_{jk}) \right) \\
&= \sum_{\ell=1}^L \left((g^{pk}\nabla_p^g \partial_k \phi_\ell - m_\ell^2 \phi_\ell) \partial_j \phi_\ell \right. \\
&\quad \left. - \nabla_p^g(g^{pk}g_{jk}(S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2) + g^{pk}P_{jk}) \right) \\
&= \sum_{\ell=1}^L S_\ell \partial_j \phi_\ell - \nabla_p^g(g^{pk}g_{jk}(S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2)) + g^{pk}\nabla_p^g P_{jk} \\
&= \left(\sum_{\ell=1}^L S_\ell \partial_j \phi_\ell \right) - \partial_j \left(\sum_{\ell=1}^L S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2 \right) + g^{pk}\nabla_p^g P_{jk}.
\end{aligned}$$

Summarizing, the conservation law yields

$$(179) \quad \left(\sum_{\ell=1}^L S_\ell \partial_j \phi_\ell \right) - \partial_j \left(\sum_{\ell=1}^L S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2 \right) + g^{pk}\nabla_p^g P_{jk} = 0,$$

for $j = 1, 2, 3, 4$.

Recall that the field Z has the definition

$$(180) \quad \sum_{\ell=1}^L S_\ell \phi_\ell + \frac{1}{2m^2}S_\ell^2 = -Z.$$

Then, the conservation law (176) holds if we have

$$(181) \quad \sum_{\ell=1}^L S_\ell \partial_j \phi_\ell = -g^{pk}\nabla_p^g V_{jk}, \quad V_{jk} = (P_{jk} + g_{jk}Z),$$

for $j = 1, 2, 3, 4$.

Equations (180) and (181) give together five point-wise equations for the functions S_1, \dots, S_L .

Recall that we consider here the case when $\sigma = Id$. By Condition A, at any $x \in U_{\hat{g}, \sigma}$ that the 5×5 matrix $(B_{jk}^\sigma(\hat{\phi}(x), \nabla \hat{\phi}(x)))_{j,k \leq 5}$ is invertible, where

$$(B_{jk}^\sigma(\phi(x), \nabla \phi(x)))_{j,k \leq 5} = \begin{pmatrix} (\partial_j \phi_\ell(x))_{j \leq 4, \ell \leq 5} \\ (\phi_\ell(x))_{\ell \leq 5} \end{pmatrix}.$$

We consider a \mathbb{R}^K valued function $Q(x) = (Q'(x), Q_K(x))$, where

$$Q' = (Q_\ell)_{\ell=1}^{K-1}.$$

Also, below $R_j = g^{pk} \nabla_p^g V_{jk}$, $V_{jk} = P_{jk} + g_{jk} Z$ and we require that identity

$$(182) \quad Q_K = Z$$

holds.

Motivated by equations (180), (181), and (182), our next aim is to consider a point $x \in U_{\hat{g},\sigma}$, and construct functions $\mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)$, $\ell = 1, 2, \dots, L$ that satisfy

$$(183) \quad \sum_{\ell=1}^5 \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g) \partial_j \phi_\ell = -R_j - \sum_{\ell=6}^L Q_{\sigma,\ell} \partial_j \phi_\ell,$$

$$(184) \quad \sum_{\ell=1}^5 \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g) \phi_\ell = -\left(Q_K + \sum_{\ell=6}^L Q_{\sigma,\ell} \phi_\ell + \right.$$

$$(185) \quad \left. + \sum_{\ell=1}^L \frac{1}{2m^2} \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)^2 \right).$$

Let

$$\begin{aligned} (Y_\sigma(\phi, \nabla\phi))(x) &= \psi(x)(B^\sigma(\phi, \nabla\phi))^{-1}, \quad \text{for } x \in U_{\hat{g},\sigma}, \\ (Y_\sigma(\phi, \nabla\phi))(x) &= 0, \quad \text{for } x \notin U_{\hat{g},\sigma}, \end{aligned}$$

where $\psi \in C_0^\infty(U_{\hat{g},\sigma})$ has value 1 in $\text{supp}(Q) \cup \text{supp}(P)$.

Then we define $\mathcal{S}_{\sigma,\ell} = \mathcal{S}_{\sigma,\ell}(g, \phi, \nabla\phi, Q', Q_K, R)$, $\ell = 1, 2, \dots, L$, to be the solution of the system

$$(186) \quad (S_{\sigma,\ell})_{\ell \leq 5} = Y_\sigma(\phi, \nabla\phi) \begin{pmatrix} (-R_j - \sum_{\ell=6}^L Q_{\sigma,\ell} \partial_j \phi_\ell)_{j \leq 4} \\ -Q_K - \sum_{\ell=6}^L Q_{\sigma,\ell} \phi_\ell - \sum_{\ell=1}^L \frac{1}{2m^2} S_{\sigma,\ell}^2 \end{pmatrix}$$

$$(S_{\sigma,\ell})_{\ell \geq 6} = (Q_\ell)_{\ell \geq 6}.$$

When Q and R are sufficiently small, this equation can be solved pointwisely, at each point $x \in U_{\hat{g},\sigma}$, using iteration by the Banach fixed point theorem.

Let

$$(K_{jk}^\sigma(\phi(x), \nabla\phi(x)))_{j,k \leq 5} = \begin{pmatrix} (\partial_j \phi_\ell(x))_{j \leq 4, 6 \leq \ell \leq L} \\ (\phi_\ell(x))_{6 \leq \ell \leq L} \end{pmatrix}.$$

Then we see that the differential of $\mathcal{S}_\sigma = (\mathcal{S}_{\sigma,\ell})_{\ell=1}^L$ with respect to (Q', Q_K, R) at $(Q, R) = (0, 0)$, that is,

$$(187)$$

$$D_{Q', Q_K, R} \mathcal{S}_\sigma(\hat{g}, \hat{\phi}, \nabla\hat{\phi}, Q', Q_K, R)|_{Q=0, R=0} : \mathbb{R}^{K+4} \rightarrow \mathbb{R}^L,$$

$$(Q', Q_K, R) \mapsto - \begin{pmatrix} Y_\sigma(\hat{\phi}, \nabla\hat{\phi}) & Y_\sigma(\hat{\phi}, \nabla\hat{\phi}) K(\hat{\phi}, \nabla\hat{\phi}) \\ 0 & I_\sigma \end{pmatrix} \begin{pmatrix} \begin{pmatrix} R \\ Q_K \\ Q' \end{pmatrix} \end{pmatrix},$$

is surjective, where $I_\sigma = [\delta_{k,j+5}]_{k \leq K-1, j \leq L-5} \in \mathbb{R}^{(K-1) \times (L-5)}$. Hence (i) is valid.

By their construction, the functions $\mathcal{S}_\sigma = (\mathcal{S}_{\sigma,\ell})_{\ell=1}^L$ satisfy the equations (180) and (181) for all $x \in U_{\hat{g},\sigma}$ and also equation (182) holds.

Hence (iii) is valid.

Above, the equation (181) is valid by construction of the functions $(\mathcal{S}_\ell)_{\ell=1}^L$. Thus the conservation law is valid. This proves (ii). \square

Note that as the adaptive source functions \mathcal{S}_ℓ were constructed in Theorem 5.9 using the inverse function theorem, the results of Theorem 5.9 are valid also if \hat{Q} and \hat{P} are sufficiently small non-vanishing fields and \hat{g} and $\hat{\phi}$ satisfy the Einstein scalar field equations (175) with these background fields. Next we return to the case when $\hat{P} = 0$ and $\hat{Q} = 0$.

C.2. Microlocal linearization stability. Below we consider the case when $\hat{P} = 0$ and $\hat{Q} = 0$ and use the adaptive source functions \mathcal{S}_ℓ constructed in Theorem 5.9 and its proof.

Assume that $Y \subset M_0$ is a 2-dimensional space-like submanifold and consider local coordinates defined in $V \subset M_0$. Moreover, assume that in these local coordinates $Y \cap V \subset \{x \in \mathbb{R}^4; x^j b_j = 0, x^j b'_j = 0\}$, where $b'_j \in \mathbb{R}$ and let $\mathbf{p} \in \mathcal{I}^n(Y)$, $n \leq n_0 = -17$, be defined by

$$(188) \quad \mathbf{p}_{jk}(x^1, x^2, x^3, x^4) = \operatorname{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_m + \theta_2 b'_m)x^m} v_{jk}(x, \theta_1, \theta_2) d\theta_1 d\theta_2.$$

Here, we assume that $v_{jk}(x, \theta)$, $\theta = (\theta_1, \theta_2)$ are classical symbols and we denote their principal symbols by $\sigma_p(\mathbf{p}_{jk})(x, \theta)$. When $x \in Y$ and $\xi = (\theta_1 b_m + \theta_2 b'_m)dx^m$ so that $(x, \xi) \in N^*Y$, we denote the value of the principal symbol $\sigma_p(\mathbf{p})$ at (x, θ_1, θ_2) by $\tilde{v}_{jk}^{(a)}(x, \xi)$, that is, $\tilde{v}_{jk}^{(a)}(x, \xi) = \sigma_p(\mathbf{p}_{jk})(x, \theta_1, \theta_2)$, and say that it is the principal symbol of \mathbf{p}_{jk} at (x, ξ) , associated to the phase function $\phi(x, \theta_1, \theta_2) = (\theta_1 b_m + \theta_2 b'_m)x^m$. The above defined principal symbols can be defined invariantly, see [48].

We assume that also $\mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ have representations (188) with classical symbols. Below we consider symbols in local coordinates. Let us denote the principal symbols of $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ by $\tilde{v}^{(a)}(x, \xi)$, $\tilde{w}_1^{(a)}(x, \xi)$, $\tilde{w}_2^{(a)}(x, \xi)$, respectively and let $\tilde{v}^{(b)}(x, \xi)$ and $\tilde{w}_2^{(b)}(x, \xi)$ denote the sub-principal symbols of \mathbf{p} and \mathbf{z} , correspondingly, at $(x, \xi) \in N^*Y$.

We will below consider what happens when $(\mathbf{p}_{jk} + \mathbf{z}\hat{g}_{jk}) \in \mathcal{I}^n(Y)$ satisfies

$$(189) \quad \hat{g}^{lk} \nabla_l^\hat{g} (\mathbf{p}_{jk} + \mathbf{z}\hat{g}_{jk}) \in \mathcal{I}^n(Y), \quad j = 1, 2, 3, 4.$$

Note that a priori this function is only in $\mathcal{I}^{n+1}(Y)$, so the assumption (189) means that $\hat{g}^{lk} \nabla_l^\hat{g} (\mathbf{p}_{jk} + \mathbf{z}\hat{g}_{jk})$ is one degree smoother than it should be a priori.

When (189) is valid, we say that *the leading order of singularity of the wave satisfies the linearized conservation law*. This corresponds to the assumption that the principal symbol of the sum of divergence of

the first two terms appearing in the stress energy tensor on the right hand side of (175) vanishes.

By [48], the identity (189) is equivalent to the vanishing of the principal symbol on N^*Y , that is,

$$(190) \quad \widehat{g}^{lk}\xi_l(\widetilde{v}_{kj}^{(a)}(x, \xi) + \widehat{g}_{kj}(x)\widetilde{w}_2^{(a)}(x, \xi)) = 0, \text{ for } j \leq 4 \text{ and } \xi \in N_x^*Y.$$

We say that this is the *linearized conservation law for principal symbol of R* .

Let us consider source fields that have the form $Q'_\varepsilon = ((Q_\varepsilon)_\ell)_{\ell=1}^{K-1} = \varepsilon \mathbf{q}'$, $(Q_\varepsilon)_K = \varepsilon \mathbf{z}$ and $P_\varepsilon = \varepsilon \mathbf{p}$. We denote $\mathbf{q} = (\mathbf{q}', \mathbf{z})$. We assume that \mathbf{q}' , \mathbf{z} , and \mathbf{p} are supported in $\widehat{V} \subset\subset U_{\widehat{g}}$.

Let $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ be the solution of (175) with source P_ε and Q_ε . Then u_ε depends C^4 -smoothly on ε and $(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi})$. Denote $\partial_\varepsilon(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\dot{g}, \dot{\phi})$. When ε_0 is small enough, P_ε and Q_ε are supported in U_{g_ε} for all $\varepsilon \in (0, \varepsilon_0)$.

Let

$$R_\varepsilon = g_\varepsilon^{pk} \nabla_p^{g_\varepsilon} ((P_\varepsilon)_{jk} + g_{jk}^\varepsilon (Q_\varepsilon)_K)$$

and

$$(S_\varepsilon)_\ell = \mathcal{S}_\ell(g_\varepsilon, \phi_\varepsilon, \nabla \phi_\varepsilon, Q'_\varepsilon, (Q_\varepsilon)_K, R_\varepsilon),$$

where \mathcal{S}_ℓ are the adaptive source functions constructed in Theorem 5.9 and its proof.

Then $S_\varepsilon|_{\varepsilon=0} = 0$ and $\partial_\varepsilon S_\varepsilon|_{\varepsilon=0} = \dot{S}$ satisfy

(191)

$$\dot{S}_\ell = D_{Q', Q_K, R} \mathcal{S}_\ell(\widehat{g}, \widehat{\phi}, \nabla \widehat{\phi}, Q', Q_K, R) \Big|_{Q'=0, Q_K=0, R=0} \begin{pmatrix} \mathbf{q}' \\ \mathbf{z} \\ \mathbf{r} \end{pmatrix},$$

where $\mathbf{r} = \widehat{g}^{pk} \widehat{\nabla}_p (\mathbf{p}_{jk} + \widehat{g}_{jk} \mathbf{z})$.

Functions $\dot{u} = (\dot{g}, \dot{\phi})$ satisfy the linearized Einstein-scalar field equation (158). The linearized Einstein-scalar field equation (158) is

$$\begin{aligned} e_{pq}(\dot{g}) - t_{pq}^{(1)}(\dot{g}) - t_{pq}^{(2)}(\dot{\phi}) &= \mathbf{f}_{pq}^1 \\ \square_{\widehat{g}} \dot{\phi}^\ell - \widehat{g}^{nm} \widehat{g}^{kj} (\partial_n \partial_j \widehat{\phi}_\ell + \widehat{\Gamma}_{nj}^p \partial_p \widehat{\phi}_\ell) \dot{g}_{mk} - m^2 \dot{\phi}^\ell &= \mathbf{f}_\ell^2, \end{aligned}$$

where

$$(192) \quad \begin{aligned} \mathbf{f}_{pq}^1 &= \mathbf{p}_{pq} - \left(\sum_{\ell=1}^L \dot{S}_\ell \widehat{\phi}_\ell \right) \widehat{g}_{pq}, \quad \text{and} \quad - \left(\sum_{\ell=1}^L \dot{S}_\ell \widehat{\phi}_\ell \right) = \mathbf{z}, \\ \mathbf{f}_\ell^2 &= \dot{S}_\ell. \end{aligned}$$

By Theorem 5.9 (ii), $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ satisfy the conservation law (176). This yields that $\dot{u} = (\dot{g}, \dot{\phi})$ satisfies the linearized Einstein-scalar field equation (158) and the linearized conservation law (13) is valid, too.

The linearized conservation law (13) gives, by the considerations before (192),

$$(193) \quad \left(\sum_{\ell=1}^L \mathbf{f}_\ell^2 \partial_j \hat{\phi}_\ell \right) + \hat{g}^{pk} \hat{\nabla}_p \mathbf{f}_{kj}^1 = 0.$$

Below, we use the adaptive source functions \mathcal{S}_ℓ constructed in Theorem 5.9 and its proof.

We see that

$$(194) \quad \mathbf{f} = F(x; \mathbf{p}, \mathbf{q}) = (F^{(1)}(x; \mathbf{p}, \mathbf{q}), F^{(2)}(x; \mathbf{p}, \mathbf{q}))$$

has by formulas (191) and (192) and Assumption S the form

$$(195) \quad F_{jk}^{(1)}(x; \mathbf{p}, \mathbf{q}) = \mathbf{p}_{jk} + \mathbf{z}(x) \hat{g}_{jk}(x)$$

and

$$(196) \quad F^{(2)}(x; \mathbf{p}, \mathbf{q}) = M_{(2)} \mathbf{q}' + L_{(2)} \mathbf{z} + N_{(2)}^j \hat{g}^{lk} \hat{\nabla}_l (\mathbf{p}_{jk} + \mathbf{z} \hat{g}_{jk}),$$

where

$$\begin{aligned} M_{(2)} &= M_{(2)}(\hat{\phi}(x), \hat{\nabla} \hat{\phi}(x), \hat{g}(x)), \\ L_{(2)} &= L_{(2)}(\hat{\phi}(x), \hat{\nabla} \hat{\phi}(x), \hat{g}(x)), \\ N_{(2)}^j &= N_{(2)}^j(\hat{\phi}(x), \hat{\nabla} \hat{\phi}(x), \hat{g}(x)) \end{aligned}$$

are, in local coordinates, matrices whose elements are smooth functions of $\hat{\phi}(x)$, $\hat{\nabla} \hat{\phi}(x)$, and $\hat{g}(x)$. By Thm. 5.9 (i), the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N_{(2)}^j(x)$, $j = 1, 2, 3, 4$, span the space \mathbb{R}^L for all $x \in U_{\hat{g}, \sigma}$.

Consider $n \in \mathbb{Z}$, $t_0, s_0 > 0$, $Y = Y(x_0, \zeta_0; t_0, s_0)$, $K = K(x_0, \zeta_0; t_0, s_0)$ and $(x, \xi) \in N^*Y$ (to recall the definitions of these notations, see formula (51) and definitions below it). Let $\mathcal{Z} = \mathcal{Z}(x, \xi)$ be the set of the values of the principal symbol $\tilde{f}(x, \xi) = (\tilde{f}_1(x, \xi), \tilde{f}_2(x, \xi))$, at (x, ξ) , of the source $\mathbf{f} = (f_1(x), f_2(x)) \in \mathcal{I}^n(Y)$ that satisfy the linearized conservation law for principal symbols (16).

We use the following auxiliary result:

Lemma 5.10. *Assume the Condition A is satisfied and $\widehat{Q} = 0$ and $\widehat{P} = 0$. Let $k_0 \geq 8$, $s_1 \geq k_0 + 5$, and $Y \subset W_{\hat{g}}$ be a 2-dimensional space-like submanifold and $y \in Y$, $\xi \in N_y^*Y$, and let \mathcal{W} be a conic neighborhood of (y, ξ) in T^*M . Also, let $y \in U_{\hat{g}, \sigma}$ with some permutation σ . Let us consider an open, relatively compact local coordinate neighborhood $V \subset U_{\hat{g}, \sigma}$ of y such that in the coordinates $X : V \rightarrow \mathbb{R}^4$, $X^j(x) = x^j$, we have $X(Y \cap V) \subset \{x \in \mathbb{R}^4; x^j b_j^1 = 0, x^j b_j^2 = 0\}$. Let $n_1 \in \mathbb{Z}_+$ be sufficiently large and $n \leq -n_1$. Let us consider $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$, supported in V , that have classical symbols with principal symbols $\tilde{v}^{(a)}(x, \xi)$,*

$\tilde{w}_1^{(a)}(x, \xi)$, $\tilde{w}_2^{(a)}(x, \xi)$, correspondingly, at $(x, \xi) \in N^*Y$. Moreover, assume that the principal symbols of \mathbf{p} and \mathbf{z} satisfy the linearized conservation law for the principal symbols, that is, (190), at all $N^*Y \cap N^*K$ and assume that they vanish outside the conic neighborhood \mathcal{W} of (y, ξ) in T^*M . Let $\mathbf{f} = (f^1, f^2) \in \mathcal{I}^n(Y)$ be given by (191) and (192).

Then the principal symbol $\tilde{f}(y, \xi) = (\tilde{f}_1(y, \xi), \tilde{f}_2(y, \xi))$ of the source \mathbf{f} at (y, ξ) is the set $\mathcal{Z} = \mathcal{Z}(y, \xi)$. Moreover, by varying $\mathbf{p}, \mathbf{q}', \mathbf{z}$ so that the linearized conservation law (190) for principal symbols is satisfied, the principal symbol $\tilde{f}(y, \xi)$ at (y, ξ) achieves all values in the $(L + 6)$ -dimensional space \mathcal{Z} .

Proof. Let us use local coordinates $X : V \rightarrow \mathbb{R}^4$ where $V \subset M_0$ is a neighborhood of x . In these coordinates, let $\tilde{v}^{(b)}(x, \xi)$ and $\tilde{w}_2^{(b)}(x, \xi)$ denote the sub-principal symbols of \mathbf{p} and \mathbf{z} , respectively, at $(x, \xi) \in N^*Y$. Moreover, let $\tilde{v}_j^{(c)}(x, \xi) = \frac{\partial}{\partial x^j} \tilde{v}^{(a)}(x, \xi)$ and $\tilde{d}_j^{(c)}(x, \xi) = \frac{\partial}{\partial x^j} \tilde{d}_2^{(a)}(x, \xi)$, $j = 1, 2, 3, 4$ be the x -derivatives of the principal symbols and let us denote

$$\tilde{v}^{(c)}(x, \xi) = (\tilde{v}_j^{(c)}(x, \xi))_{j=1}^4, \quad \tilde{d}^{(c)}(x, \xi) = (\tilde{d}_j^{(c)}(x, \xi))_{j=1}^4.$$

Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) = F(x; \mathbf{p}, \mathbf{q})$ be defined by (195) and (196). When the principal symbols of $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ are such that the linearized conservation law (190) for principal symbols is satisfied, we see that $\mathbf{f} \in \mathcal{I}^n(Y)$ has the principal symbol $\tilde{f}(x, \xi) = (\tilde{f}_1(x, \xi), \tilde{f}_2(x, \xi))$ at (x, ξ) , given by

$$\begin{aligned} \tilde{f}_1(x, \xi) &= s_1(x, \xi), \\ \tilde{f}_2(x, \xi) &= s_2(x, \xi), \end{aligned}$$

where

$$\begin{aligned} (197) \quad s_1(x, \xi) &= (\tilde{v}^{(a)} + \hat{g}\tilde{w}_2^{(a)})(x, \xi), \\ s_2(x, \xi) &= \left(M_{(2)}(x)\tilde{w}_1^{(a)} + J_{(2)}(\tilde{v}^{(c)} + \hat{g}\tilde{d}^{(c)}) + \right. \\ &\quad \left. + L_{(2)}\tilde{w}_2^{(a)} + N_{(2)}^j \hat{g}^{lk} \xi_l (\tilde{v}_1^{(b)} + \hat{g}\tilde{w}_2^{(b)})_{jk} \right)(x, \xi). \end{aligned}$$

Here, roughly speaking, the $J_{(2)}$ term appears when the ∇ -derivatives in R hit to the symbols of the conormal distributions having the form (188). We emphasize that here the symbols $s_1(x, \xi)$ and $s_2(x, \xi)$ are well defined objects (in fixed local coordinates) also when the linearized conservation law (190) for principal symbols is not valid. When (190) is valid, $\mathbf{f} \in \mathcal{I}^n(Y)$ and $s_1(x, \xi)$ and $s_2(x, \xi)$ coincide with the principal symbols of \mathbf{f}_1 and \mathbf{f}_2 .

Observe that the map $(c_{jk}^{(b)}) \mapsto (\widehat{g}^{lk}\xi_l c_{jk}^{(b)})_{j=1}^4$, defined as $\text{Symm}(\mathbb{R}^{4 \times 4}) \rightarrow \mathbb{R}^4$, is surjective. Denote

$$\begin{aligned}\widetilde{m}^{(a)} &= (\widetilde{v}^{(a)} + \widehat{g}\widetilde{w}_2^{(a)})(x, \xi), \\ \widetilde{m}^{(b)} &= (\widetilde{v}^{(b)} + \widehat{g}\widetilde{w}_2^{(b)})(x, \xi), \\ \widetilde{m}^{(c)} &= (\widetilde{v}^{(c)} + \widehat{g}\widetilde{d}^{(c)})(x, \xi).\end{aligned}$$

As noted above, by (187), the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N_{(2)}^j(x)$, $j = 1, 2, 3, 4$, span the space \mathbb{R}^L for all $x \in U_{\widehat{g}}$. Hence the map

$$\mathbf{A} : (\widetilde{m}^{(a)}, \widetilde{m}^{(b)}, \widetilde{m}^{(c)}, \widetilde{w}_1, \widetilde{w}_2^{(a)})|_{(x, \xi)} \mapsto (s_1(x, \xi), s_2(x, \xi)),$$

given by (197), considered as a map $\mathbf{A} : \mathcal{Y} = (\text{Symm}(\mathbb{R}^{4 \times 4}))^{1+1+4} \times \mathbb{R}^K \times \mathbb{R} \rightarrow \text{Symm}(\mathbb{R}^{4 \times 4}) \times \mathbb{R}^L$, is surjective. Let \mathcal{X} be the set of elements $(\widetilde{m}^{(a)}(x, \xi), \widetilde{m}^{(b)}(x, \xi), \widetilde{m}^{(c)}(x, \xi), \widetilde{w}_1^{(a)}(x, \xi), \widetilde{w}_2^{(a)}(x, \xi)) \in \mathcal{Y}$ where $\widetilde{m}^{(a)}(x, \xi) = (\widetilde{v}^{(a)} + \widehat{g}\widetilde{w}_2^{(a)})(x, \xi)$ is such that the pair $(\widetilde{v}^{(a)}(x, \xi), \widetilde{w}_2^{(a)}(x, \xi))$ satisfies the linearized conservation law for principal symbols, see (190). Then \mathcal{X} has codimension 4 in \mathcal{Y} and we see that the image $\mathbf{A}(\mathcal{X})$ has in $\text{Symm}(\mathbb{R}^{4 \times 4}) \times \mathbb{R}^L$ co-dimension less or equal to 4.

By (193) and the considerations above it, we have that \mathbf{f} satisfies the linearized conservation law (13). This implies that its principal symbol $\mathbf{A}((\widetilde{m}^{(a)}(x, \xi), \widetilde{m}^{(b)}(x, \xi), \widetilde{m}^{(c)}(x, \xi), \widetilde{w}_1^{(a)}(x, \xi), \widetilde{w}_2^{(a)}(x, \xi)))$ has to satisfy the linearized conservation law for principal symbols (16) and hence $\mathbf{A}(\mathcal{X}) \subset \mathcal{Z}$. As \mathcal{Z} has codimension 4, this and the above prove that $\mathbf{A}(\mathcal{X}) = \mathcal{Z}$. \square

Now we are ready to prove the microlocal stability result for the Einstein-scalar field equation (8). Note that the claim of the following theorem does not involve the adaptive source functions constructed in Theorem 5.9 as these functions are needed only as an auxiliary tool in the proof.

Theorem 5.11. (μ -LS i.e., Microlocal linearization stability) *Let $k_0 \geq 8$, $s_1 \geq k_0 + 5$, and $Y \subset W_{\widehat{g}}$ be a 2-dimensional space-like submanifold, and $y \in Y$ and let \mathcal{W} be a conic neighborhood of (y, η) in T^*M . Also, consider in an open local coordinate neighborhood $V \subset W_{\widehat{g}}$ of $y \in Y$ such that in the coordinates $X : V \rightarrow \mathbb{R}^4$, $X^j(x) = x^j$, we have $X(Y \cap V) \subset \{x \in \mathbb{R}^4; x^j b_l^1 = 0, x^j b_l^2 = 0\}$. Then there there is $n_1 \in \mathbb{Z}_+$ such that if $(y, \eta) \in N^*Y$ is light-like, $n \leq -n_1$ and $\widetilde{c}_{jk}(x, \theta)$ and $\widetilde{d}_\ell(x, \theta)$ are positively n -homogeneous in the θ -variable in the domain $\{(x, \theta) \in V \times \mathbb{R}^2; |\theta| > 1\}$ and satisfy*

$$(198) \quad \widehat{g}^{lk}(\theta_1 b_l^1 + \theta_2 b_l^2) \widetilde{c}_{kj}(x, \theta_1, \theta_2) = 0, \quad |\theta| > 1, \quad j = 1, 2, 3, 4,$$

then there are $f^1 \in \mathcal{I}^n(Y)$ and $f^2 \in \mathcal{I}^n(Y)$ supported in V such that the principal symbols of these distributions vanish outside \mathcal{W} and are equal to $\widetilde{c}_{jk}(y, \eta)$ and $\widetilde{d}_\ell(y, \eta)$ at (y, η) , respectively. Moreover, $f = (f^1, f^2)$

satisfies the linearized conservation law (13) and f has the LS-property (14) in $C^{s_1}(M_0)$ with a family \mathcal{F}_ε , $\varepsilon \in [0, \varepsilon_0)$ such that all functions $\mathcal{F}_\varepsilon, \varepsilon \in [0, \varepsilon_0)$ are supported in V .

Proof. Let $\sigma \in \Sigma(K)$ be such that $y \in U_{\hat{g}, \sigma}$. Let \mathbf{p} and \mathbf{q} be the functions constructed in Lemma 5.10 such that they are supported in V and their principal symbols vanish outside \mathcal{W} . We can assume that these functions are supported in $W_0 = V \cap U_{\hat{g}, \sigma}$. Let $P_\varepsilon = \varepsilon \mathbf{p}$ and $Q_\varepsilon = \varepsilon \mathbf{q}$ be sources depending on $\varepsilon \in \mathbb{R}$ and $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ be the solution of (175) with the sources P_ε and Q_ε . Also, let

$$\begin{aligned} \mathcal{F}_\varepsilon^1 &= P_\varepsilon + Z_\varepsilon g_\varepsilon, \quad Z_\varepsilon = -\left(\sum_{\ell=1}^L S_\ell^\varepsilon \phi_\ell^\varepsilon + \frac{1}{2m^2} (S_\ell^\varepsilon)^2\right), \\ (\mathcal{F}_\varepsilon^2)_\ell &= S_\ell^\varepsilon, \end{aligned}$$

where

$$S_\ell^\varepsilon = \mathcal{S}_\ell(g_\varepsilon, \phi_\varepsilon, \nabla \phi_\varepsilon, Q_\varepsilon, \nabla Q_\varepsilon, P_\varepsilon, \nabla^{g_\varepsilon} P_\varepsilon),$$

where \mathcal{S}_ℓ are the adaptive source functions constructed in Theorem 5.9 and its proof.

By (177) also S_ℓ^ε and the family \mathcal{F}_ε , $\varepsilon \in [0, \varepsilon_0]$ of non-linear sources are supported in V_0 and we have shown that $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ and \mathcal{F}_ε satisfy the reduced Einstein-scalar field equation (8) and the conservation law (176). This proves Theorem 5.11. \square

C.3. A special case when the whole metric is determined. Finally, we consider the case when \widehat{Q} and \widehat{P} are non-zero, and we define

$$\begin{aligned} \mathcal{D}^{mod}(\widehat{g}, \widehat{\phi}, \varepsilon) &= \{[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \quad ; \quad (g, \phi, F) \text{ are smooth solutions,} \\ &\quad \text{of (175) with } F = (P - \widehat{P}, Q - \widehat{Q}), F \in C_0^{19}(W_g; \mathcal{B}^K), \\ &\quad J_g^+(\text{supp}(F)) \cap J_g^-(\text{supp}(F)) \subset W_g, \mathcal{N}^{(17)}(F) < \varepsilon, \mathcal{N}_{\widehat{g}}^{(17)}(g) < \varepsilon\}. \end{aligned}$$

Next we consider the case when $\widehat{Q}^{(j)}$ and $\widehat{P}^{(j)}$ are not assumed to be zero, see Fig. A11.

Corollary 5.12. Assume that $(M^{(1)}, \widehat{g}^{(1)})$ and $(M^{(2)}, \widehat{g}^{(2)})$ are globally hyperbolic manifolds and $\widehat{\phi}^{(j)}$, $\widehat{Q}^{(j)}$, and $\widehat{P}^{(j)}$ are background fields satisfying (175). Also, assume that there are neighborhoods $U_{\widehat{g}^{(j)}}$, $j = 1, 2$ of time-like geodesics $\mu_j \subset M^{(j)}$ where the Condition μ -LS is valid, and points $p_j^- = \mu_j(s_-)$ and $p_j^+ = \mu_j(s_+)$. Moreover, assume also that for $j = 1, 2$ there are sets $W_j \subset M_j$ such that $\widehat{\phi}^{(j)}$, $\widehat{Q}^{(j)}$ and $\widehat{P}^{(j)}$ are zero (and thus the metric tensors \widehat{g}_j have vanishing Ricci curvature) in W_j and that $I_{\widehat{g}_j}(\widehat{p}_j^-, \widehat{p}_j^+) \subset W_j \cup U_{\widehat{g}}$. If there is $\varepsilon > 0$ such that

$$\mathcal{D}^{mod}(\widehat{g}^{(1)}, \widehat{\phi}^{(1)}, \varepsilon) = \mathcal{D}^{mod}(\widehat{g}^{(2)}, \widehat{\phi}^{(2)}, \varepsilon)$$

then the metric $\Psi^* \widehat{g}_2$ is isometric to \widehat{g}_1 in $I_{\widehat{g}_1}(\widehat{p}_1^-, \widehat{p}_1^+)$.

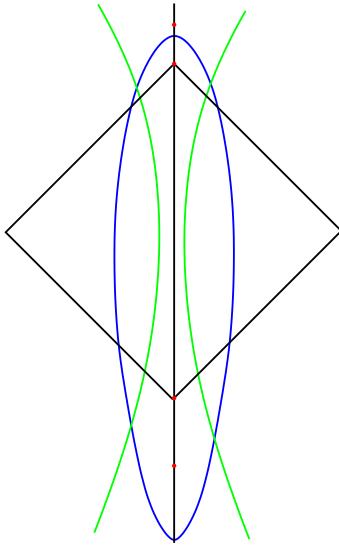


FIGURE A11: A schematic figure where the space-time is represented as the 2-dimensional set \mathbb{R}^{1+1} on the setting in Corollary 5.12. The set $J_{\widehat{g}}(\widehat{p}^-, \widehat{p}^+)$, i.e., the diamond with the black boundary, is contained in the union of the blue set $U_{\widehat{g}}$ and the set W . The set W is in the figure the area outside of the green curves. The sources are controlled in the set $U_{\widehat{g}} \setminus W$ and the set W consists of vacuum.

Proof. (of Corollary 5.12) In the above proof of Theorem 1.1, we used the assumption that $\widehat{Q} = 0$ and $\widehat{P} = 0$ to obtain equations (59). In the setting of Cor. 5.12 where the background source fields \widehat{Q} and \widehat{P} are not zero, we need to assume in the computations related to sources (54) that there are neighborhoods V_j of the geodesics γ_j that satisfy $\text{supp}(\mathbf{f}_j) \subset V_j$, the linearized waves $u_j = \mathbf{Q}_{\widehat{g}} \mathbf{f}_j$ satisfy $\text{singsupp}(u_j) \subset V_j$, and $V_i \cap V_j \cap (\text{supp}(\widehat{Q}) \cup \text{supp}(\widehat{P})) = \emptyset$ for $i \neq j$. To this end, we have first consider measurements for the linearized waves and check for given $(\vec{x}, \vec{\xi})$ that no two geodesic $\gamma_{x_j, \xi_j}(\mathbb{R}_+)$ do intersect at $U_{\widehat{g}}$ and restrict all considerations for such $(\vec{x}, \vec{\xi})$. Notice that such $(\vec{x}, \vec{\xi})$ form an open and dense set in $(TU_{\widehat{g}})^4$. If then the sources are supported in balls so small that each ball is in some set $U_{\widehat{g}, \sigma}$ and the width \widehat{s} of the used distorted plane waves is chosen to be small enough, we see that condition $V_i \cap V_j \cap (\text{supp}(\widehat{Q}) \cup \text{supp}(\widehat{P})) = \emptyset$ is satisfied.

The above restriction causes only minor modifications in the above proof and thus, mutatis mutandis, we see that we can determine the conformal type of the metric in relatively compact sets $I_{\widehat{g}}(\widehat{\mu}(s'), \widehat{\mu}(s'')) \setminus (\text{supp}(\widehat{Q}) \cup \text{supp}(\widehat{P}))$ for all $s_- < s' < s'' < s_+$. By glueing these manifolds and $U_{\widehat{g}}$ together, we find the conformal type of the metric in $I_{\widehat{g}}(\widehat{p}^-, \widehat{p}^+)$. After this the claim follows similarly to the proof of Theorem 1.1 using Corollary 1.3 of [66]. \square

In the setting of Corollary 5.12 the set W is such $I(\widehat{p}^-, \widehat{p}^+) \cap (M \setminus W) \subset U$. This means that if we restrict to the domain $I(\widehat{p}^-, \widehat{p}^+)$ then we have the Vacuum Einstein equations in the unknown domain $I(\widehat{p}^-, \widehat{p}^+) \setminus U$ and have matter only in the domain U where we implement our measurement (c.f. a space ship going around in a system of black holes, see [24]). This could be considered as an “Inverse problem for the vacuum Einstein equations”.

C.4. Motivation using Lagrangian formulation. To motivate the system (175) of partial differential equations, we give in this subsection a non-rigorous discussion.

Following [15, Ch. III, Sect. 6.4, 7.1, 7.2, 7.3] and [3, p. 36] we start by considering the Lagrangians, associated to gravity, scalar fields $\phi = (\phi_\ell)_{\ell=1}^L$ and non-interacting fluid fields, that is, the number density four-currents $\mathbf{n} = (\mathbf{n}_\kappa(x))_{\kappa=1}^J$ (where each \mathbf{n}_κ is a vector field, see [3, p. 33]). We consider also products of vector fields $\mathbf{n}_\kappa(x)$ and $\frac{1}{2}$ -density $|\det(g)|^{1/2}$ that denote \mathbf{p}_κ ,

$$(199) \quad \mathbf{p}_\kappa^j(x) \frac{\partial}{\partial x^j} = \mathbf{n}_\kappa^j(x) |\det(g)|^{1/2} \frac{\partial}{\partial x^j}.$$

see [31, p. 53]. Also, $\rho = (-g_{jk} \mathbf{n}_\kappa^j \mathbf{n}_\kappa^k)^{1/2}$ corresponds to the energy density of the fluid. Below, we use the variation of density with respect to the metric,

$$(200) \quad \begin{aligned} \frac{\delta}{\delta g_{jk}} \left(\sum_{\kappa=1}^J (-g_{nm} \mathbf{p}_\kappa^n \mathbf{p}_\kappa^m)^{1/2} \right) &= - \sum_{\kappa=1}^J \frac{1}{2} (-g_{nm} \mathbf{p}_\kappa^n \mathbf{p}_\kappa^m)^{-1/2} \mathbf{p}_\kappa^j \mathbf{p}_\kappa^k \\ &= - \sum_{\kappa=1}^J \frac{1}{2} \rho \mathbf{n}_\kappa^j \mathbf{n}_\kappa^k |\det(g)|^{1/2}. \end{aligned}$$

Due to this, we denote

$$(201) \quad P = \sum_{\kappa=1}^J \frac{1}{2} \rho \mathbf{n}_\kappa^\kappa \mathbf{n}_\kappa^\kappa dx^j \otimes dx^k, \quad \text{where } \mathbf{n}_\kappa^\kappa = g_{ki} \mathbf{n}_\kappa^i = g_{ki} \mathbf{p}_\kappa^i |\det(g)|^{-1/2}.$$

Below, we consider a model for g , ϕ , and \mathbf{p} . We also add in to the model a Lagrangian associated with some scalar valued source fields $S = (S_\ell)_{\ell=1}^L$ and $Q = (Q_k)_{k=1}^K$. We consider the action corresponding

to the coupled Lagrangian

$$\begin{aligned}\mathcal{A} &= \int_M \left(L_{grav}(x) + L_{fields}(x) + L_{source}(x) \right) dV_g(x), \\ L_{grav} &= \frac{1}{2} R(g), \\ L_{fields} &= \sum_{\ell=1}^L \left(-\frac{1}{2} g^{jk} \partial_j \phi_\ell \partial_k \phi_\ell - \mathcal{V}(\phi_\ell; S_\ell) \right) + \\ &\quad + \sum_{\kappa=1}^J \left(-\frac{1}{2} (-g_{jk} \mathbf{p}_\kappa^j \mathbf{p}_\kappa^k)^{\frac{1}{2}} \right) |\det(g)|^{-\frac{1}{2}}, \\ L_{source} &= \varepsilon \mathcal{H}_\varepsilon(g, S, Q, \mathbf{p}, \phi),\end{aligned}$$

where $R(g)$ is the scalar curvature, $dV_g = (-\det(g))^{1/2} dx$ is the volume form on (M, g) ,

$$(202) \quad \mathcal{V}(\phi_\ell; S_\ell) = \frac{1}{2} m^2 \left(\phi_\ell + \frac{1}{m^2} S_\ell \right)^2$$

are energy potentials of the scalar fields ϕ_ℓ that depend on S_ℓ , and $\mathcal{H}_\varepsilon(g, S, Q, \mathbf{p}, \phi)$ is a function modeling the measurement device we use. We assume that \mathcal{H}_ε is bounded and its derivatives with respect to S, Q, \mathbf{p} are very large (like of order $O((\varepsilon)^{-2})$) and its derivatives with respect of g and ϕ are bounded when $\varepsilon > 0$ is small. We note that the above Lagrangian for the fluid fields is the sum of the single fluid Lagrangians. where for all fluids the master function $\Lambda(s) = s^{1/2}$, that is, the energy density of each fluid is given by $\rho = \Lambda(-g_{jk} \mathbf{n}^j \mathbf{n}^k)$. On fluid Lagrangians, see the discussions in [3, p. 33-37], [15, Ch. III, Sect. 8], [31, p. 53], and [102] and [35, p. 196].

When we compute the critical points of the Lagrangian L and neglect the $O(\varepsilon)$ -terms, the equation $\frac{\delta \mathcal{A}}{\delta g} = 0$, together with formulas (200) and (201), give the Einstein equations with a stress-energy tensor T_{jk} defined in (174). The equation $\frac{\delta \mathcal{A}}{\delta \phi} = 0$ gives the wave equations with sources S_ℓ . We assume that $O(\varepsilon^{-1})$ order equations obtained from the equation $(\frac{\delta \mathcal{A}}{\delta S}, \frac{\delta \mathcal{A}}{\delta Q}, \frac{\delta \mathcal{A}}{\delta \mathbf{p}}) = 0$ fix the values of the scalar functions Q and the fields \mathbf{p}^κ , $\kappa = 1, 2, \dots, J$, and moreover, yield for the sources $S = (S_\ell)_{\ell=1}^L$ equations of the form

$$(203) \quad S_\ell = \mathcal{S}_\ell(g, \phi, \nabla \phi, Q, \nabla Q, P, \nabla^g P)$$

where P is given by (201). Let us aslo write (203) using different notations, as

$$S_\ell = Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla \phi, Q, \nabla Q, P, \nabla^g P).$$

Summarizing, we have obtained, up to the above used approximations, the model (175). However, note that above the field P is not directly controlled but instead, we control \mathbf{p} and the value of the field P is

determined by the solution n and formula (201). In this sense P is not controlled, but an observed field.

Above, the function \mathcal{H}_ε models the way the measurement device works. Due to this we will assume that \mathcal{H}_ε and thus functions \mathcal{S}_ℓ may be quite complicated. The interpretation of the above is that in each measurement event we use a device that fixes the values of the scalar functions Q , \mathbf{p} , and gives the equations for S^{2nd} that tell how the sources of the ϕ -fields adapt to these changes so that the physical conservation laws are satisfied.

APPENDIX D: AN INVERSE PROBLEM FOR A NON-LINEAR WAVE EQUATION

In this appendix we explain how a problem for a scalar wave equation can be solved with the same techniques that we used for the Einstein equations.

Let (M_j, g_j) , $j = 1, 2$ be two globally hyperbolic $(1+3)$ dimensional Lorentzian manifolds represented using global smooth time functions as $M_j = \mathbb{R} \times N_j$, $\mu_j = \mu_j([-1, 1]) \subset M_j$ be a time-like geodesic and $U_j \subset M_j$ be open, relatively compact neighborhood of $\mu_j([s_-, s_+])$, $-1 < s_- < s_+ < 1$. Let $M_j^0 = (-\infty, T_0) \times N_j$ where $T_0 > 0$ is such that $U_j \subset M_j^0$. Consider the non-linear wave equation

$$(204) \quad \begin{aligned} \square_{g_j} u(x) + a_j(x) u(x)^2 &= f(x) \quad \text{on } M_j^0, \\ \text{supp}(u) &\subset J_{g_j}^+(\text{supp}(f)), \end{aligned}$$

where $\text{supp}(f) \subset U_j$,

$$\square_g u = \sum_{p,q=1}^4 (-\det(g))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g))^{1/2} g^{pq} \frac{\partial}{\partial x^q} u(x) \right),$$

$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(U_j)$ is a controllable source, and a_j is a non-vanishing C^∞ -smooth function. Our goal is to prove the following result:

Theorem 5.13. *Let (M_j, g_j) , $j = 1, 2$ be two open, smooth, globally hyperbolic Lorentzian manifolds of dimension $(1+3)$. Let $p_j^+ = \mu_j(s_+)$, $p_j^- = \mu_j(s_-) \in M_j$ the points of a time-like geodesic $\mu_j = \mu_j([-1, 1]) \subset M_j$, $-1 < s_- < s_+ < 1$, and let $U_j \subset M_j$ be an open relatively compact neighborhood of $\mu_j([s_-, s_+])$ given in (2). Let $a_j : M_j \rightarrow \mathbb{R}$, $j = 1, 2$ be C^∞ -smooth functions that are non-zero on M_j .*

Let L_{U_j} , $j = 1, 2$ be measurement operators defined in an open set $\mathcal{W}_j \subset C_0^6(U_j)$ containing the zero function by setting

$$(205) \quad L_{U_j} : f \mapsto u|_{U_j}, \quad f \in C_0^6(U_j),$$

where u satisfies the wave equation (204) on (M_j^0, g_j) .

Assume that there is a diffeomorphic isometry $\Phi : U_1 \rightarrow U_2$ so that $\Phi(p_1^-) = p_2^-$ and $\Phi(p_1^+) = p_2^+$ and the measurement maps satisfy

$$((\Phi^{-1})^* \circ L_{U_1} \circ \Phi^*)f = L_{U_2}f$$

for all $f \in \mathcal{W}$ where \mathcal{W} is some neighborhood of the zero function in $C_0^6(U_2)$.

Then there is a diffeomorphism $\Psi : I(p_1^-, p_1^+) \rightarrow I(p_2^-, p_2^+)$, and the metric Ψ^*g_2 is conformal to g_1 in $I(p_1^-, p_1^+) \subset M_1$, that is, there is $\beta(x)$ such that $g_1(x) = \beta(x)(\Psi^*g_2)(x)$ in $I(p_1^-, p_1^+)$.

We note that the smoothness assumptions assumed above on the functions a and the source f are not optimal. The proof, presented below, is based on using the interaction of singular waves. The techniques used can be modified to study different non-linearities, such as the equations $\square_{\tilde{g}}u + a(x)u^3 = f$, $\square_{\tilde{g}}u + a(x)u_t^2 = f$, or $\square_{g(x,u(x))}u = f$, but these considerations are outside the scope of this paper.

Theorem 5.13 can be applicable for example in the mathematical analysis of non-destructive testing or imaging in non-linear medium e.g, in imaging the non-linearity of the acoustic material parameter inside a given body when it is under large, time-varying, possibly periodic, changes of the external pressure and at the same time the body is probed with small-amplitude fields. Such acoustic measurements are analogous to the recently developed Ultrasound Elastography imaging technique where the interaction of the elastic shear and pressure waves is used for medical imaging, see e.g. [50, 79, 80, 88]. There, the slowly progressing shear wave is imaged using a pressure wave and the image of the shear wave inside the body is used to determine approximately the material parameters. In other words, the changes which the elastic wave causes in the medium are imaged using the interaction of the s-wave and p-wave components of the elastic wave.

Let us also consider some implications of theorem 5.13 for inverse problems for a non-linear equation involving a time-independent metric

$$(206) \quad g(t, y) = -dt^2 + \sum_{\alpha, \beta=1}^3 h_{\alpha\beta}(y)dy^\alpha dy^\beta, \quad (t, y) \in \mathbb{R} \times N.$$

The metric (206) corresponds to the hyperbolic operator $\partial_t^2 - \Delta_h$, with a time-independent Riemannian metric $h = (h_{\alpha\beta}(y))_{\alpha, \beta=1}^3$, $y \in N$, where N is a 3-dimensional manifold and

$$\Delta_h u(y, t) = \sum_{\alpha, \beta=1}^3 \frac{\partial}{\partial y^\alpha} \left(h^{\alpha\beta}(y) \frac{\partial}{\partial y^\beta} u(t, y) \right).$$

Corollary 5.14. *Let (M_j, g_j) , $M_j = \mathbb{R} \times N_j$, $j = 1, 2$ be two open, smooth, globally hyperbolic Lorentzian manifolds of dimension $(1+3)$. Assume that g_j is the product metric of the type (206), $g_j = -dt^2 +$*

$h_j(y)$, $j = 1, 2$. Assume that μ_j is a time-like geodesic $\mu_j([-1, 1]) \subset \mathbb{R} \times \{p_j\}$, where $p_j \in M_j$.

Let $p_j^+ = \mu_j(s_+)$, $p_j^- = \mu_j(s_-) \in (0, T_0) \times N_j$, $-1 < s_- < s_+ < 1$ and assume that $\mu_j(s_-) = (1, p_j)$. Moreover, Let $U_j \subset (0, T_0) \times N_j$ be an open relatively compact neighborhood of $\mu_j([s_-, s_+])$ given in (2). Let $a_j : M_j \rightarrow \mathbb{R}$, $j = 1, 2$ be C^∞ -smooth functions that are non-zero on M_j and $x = (t, y) \in \mathbb{R} \times N$.

For $j = 1, 2$, consider the non-linear wave equations

$$(207) \quad \begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \Delta_{h_j} \right) u(t, y) + a_j(y, t)(u(t, y))^2 &= f(t, y) \quad \text{on } (0, T_0) \times N_j, \\ \text{supp}(u) &\subset J_{g_j}^+(\text{supp}(f)), \end{aligned}$$

where $f \in C_0^6(U_j)$, $j = 1, 2$. Let $L_{U_j} : f \mapsto u|_{U_j}$ be the measurement operator (205) for the wave equation (207) with the Riemannian metric $h_j(x)$ and the coefficient $a_j(x, t)$ for $j = 1, 2$, defined in some $C_0^6(U_j)$ neighborhood of the zero function.

Assume that there is a diffeomorphism $\Phi : U_1 \rightarrow U_2$ of the form $\Phi(t, y) = (t, \phi(y))$ so that

$$((\Phi^{-1})^* \circ L_{U_1} \circ \Phi^*)f = L_{U_2}f$$

for all $f \in \mathcal{W}$ where \mathcal{W} is some neighborhood of the zero function in $C_0^6(U_2)$.

Then there is a diffeomorphism $\Psi : I^+(p_1^-) \cap I^-(p_1^+) \rightarrow I^+(p_2^-) \cap I^-(p_2^+)$ of the form $\Psi(t, y) = (\psi(y), t)$, the metric $\Psi^* g_2$ is isometric to g_1 in $I^+(p_1^-) \cap I^-(p_1^+)$, and $a_1(t, y) = a_2(t, \psi(y))$ in $I^+(p_1^-) \cap I^-(p_1^+)$.

Next we consider the proofs.

Proof. (of Theorem 5.13). We will explain how the proof of Theorem 1.1 for the Einstein equations needs to be modified to obtain the similar result for the non-linear wave equation.

Let (M, \hat{g}) be a smooth globally hyperbolic Lorentzian manifold that we represent using a global smooth time function as $M = (-\infty, \infty) \times N$, and consider $M^0 = (-\infty, T) \times N \subset M$. Assume that the set U , where the sources are supported and where we observe the waves, satisfies $U \subset [0, T] \times N$.

The results of section 3.1.2 concerning the direct problem for Einstein equations can be modified for the wave equation

$$(208) \quad \begin{aligned} \square_{\hat{g}} u + au^2 &= f, \quad \text{in } M^0 = (-\infty, T) \times N, \\ u|_{(-\infty, 0) \times N} &= 0, \end{aligned}$$

where $a = a(x)$ is a smooth, non-vanishing function. Here we denote the metric by \hat{g} to emphasize the fact that it is independent on the solution u . Below, let Q be the causal inverse operator of $\square_{\hat{g}}$.

When f in $C_0([0, t_0]; H_0^6(B)) \cap C_0^1([0, t_0]; H_0^5(B))$ is small enough, we see by using [94, Prop. 9.17] and [51, Thm. III], see also (171)

in Appendix B, that the equation (208) has a unique solution $u \in C([0, t_0]; H^5(N)) \cap C^1([0, t_0]; H^4(N))$. Moreover, we can consider the case when $f = \varepsilon f_0$ where $\varepsilon > 0$ is small. Then, we can write

$$u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$$

where w_j and the reminder term E_ε satisfy

$$\begin{aligned} w_1 &= Qf, \\ w_2 &= -Q(a w_1 w_1), \\ w_3 &= -2Q(a w_1 w_2) \\ &\quad = 2Q(a w_1 Q(a w_1 w_1)), \\ w_4 &= -Q(a w_2 w_2) - 2Q(a w_1 w_3) \\ &\quad = -Q(a Q(a w_1 w_1) Q(a w_1 w_1)) \\ &\quad \quad + 4Q(a w_1 Q(a w_1 w_2)) \\ &\quad = -Q(a Q(a w_1 w_1) Q(a w_1 w_1)) \\ &\quad \quad - 4Q(a w_1 Q(a w_1 Q(a w_1 w_1))), \\ \|E_\varepsilon\|_{C([0, t_0]; H_0^4(N)) \cap C^1([0, t_0]; H_0^3(N))} &\leq C\varepsilon^5. \end{aligned}$$

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^4 \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec{\varepsilon}}$ of (208), we see that

$$\begin{aligned} \mathcal{M}^{(4)} &= \partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \\ &= \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \\ (209) \quad &= - \sum_{\sigma \in \Sigma(4)} \left(Q(a Q(a u_{(\sigma(1))} u_{(\sigma(2))}) Q(a u_{(\sigma(3))} u_{(\sigma(4))})) \right. \\ &\quad \left. + 4Q(a u_{(\sigma(1))} Q(a u_{(\sigma(2))} Q(a u_{(\sigma(3))} u_{(\sigma(4))}))) \right), \end{aligned}$$

where $u_{(j)} = Qf_{(j)}$ and ℓ is the set of permutations of the set $\{1, 2, 3, \dots, \ell\}$.

The results of Lemma 3.1 can be replaced by the results of [44, Prop. 2.1] as follows. Using the same notations as in Lemma 3.1, let $Y = Y(x_0, \zeta_0; t_0, s_0)$, $K = K(x_0, \zeta_0; t_0, s_0)$, and $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$, and consider a source $f \in \mathcal{I}^{n+1}(Y)$. Then $u = Qf$ satisfies $u|_{M_0 \setminus Y} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$. Assume that $(x, \xi), (y, \eta) \in L^+ M$ are on the same bicharacteristics of $\square_{\tilde{g}}$, and $x < y$, that is, $((x, \xi), (y, \eta)) \in \Lambda'_{\tilde{g}}$. Moreover, assume that $(x, \xi) \in N^* Y$. Let $\tilde{b}(x, \xi)$ be the principal symbol of f at (x, ξ) and $\tilde{a}(y, \eta)$ be the principal symbol of u at (y, η) . Then $\tilde{a}(y, \eta)$ depends linearly on $\tilde{f}(x, \xi)$ and $\tilde{a}(y, \eta)$ vanishes if and only if $\tilde{f}(x, \xi)$ vanishes.

Analogously to the Einstein equations, we consider the indicator function

$$(210) \quad \Theta_\tau^{(4)} = \langle F_\tau, \mathcal{M}^{(4)} \rangle_{L^2(U)},$$

where $\mathcal{M}^{(4)}$ is given by (209) with $u_{(j)} = Qf_{(j)}$, $j = 1, 2, 3, 4$, where $f_{(j)} \in \mathcal{I}^{n+1}(Y(x_j, \xi_j; t_0, s_0))$, $n \leq -n_1$, and F_τ is the source producing a gaussian beam Q^*F_τ that propagates to the past along the geodesic $\gamma_{x_5, \xi_5}(\mathbb{R}_-)$, see (67).

Similar results to the ones given in Proposition 3.3 are valid. Let us consider next the case when (x_5, ξ_5) comes from the 4-intersection of rays corresponding to $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ and q is the corresponding intersection point, that is, $q = \gamma_{x_j, \xi_j}(t_j)$ for all $j = 1, 2, 3, 4, 5$. Then

$$(211) \quad \Theta_\tau^{(4)} \sim \sum_{k=m}^{\infty} s_k \tau^{-k}$$

as $\tau \rightarrow \infty$ where $m = -4n + 4$. Moreover, let $b_j = (\dot{\gamma}_{x_j, \xi_j}(t_j))^\flat$ and $\mathbf{b} = (b_j)_{j=1}^5 \in (T_q^* M_0)^5$, w_j be the principal symbols of the waves $u_{(j)}$ at (q, b_j) , and $\mathbf{w} = (w_j)_{j=1}^5$. Then we see as in Proposition 3.3 that there is a real-analytic function $\mathcal{G}(\mathbf{b}, \mathbf{w})$ such that the leading order term in (77) satisfies

$$(212) \quad s_m = \mathcal{G}(\mathbf{b}, \mathbf{w}).$$

The proof of Prop. 3.4 dealing with the Einstein equations needs significant changes and we need to prove the following:

Proposition 5.15. *The function $\mathcal{G}(\mathbf{b}, \mathbf{w})$ given in (212) for the nonlinear wave equation is a non-identically vanishing real-analytic function.*

Proof. Let us use the notations introduced in Prop. 3.4.

As for the Einstein equations, we consider light-like vectors

$$b_5 = (1, 1, 0, 0), \quad b_j = (1, 1 - \frac{1}{2}\rho_j^2, \rho_j + O(\rho_j^3), \rho_j^3), \quad j = 1, 2, 3, 4,$$

in the Minkowski space \mathbb{R}^{1+3} , endowed with the standard metric $g = \text{diag } (-1, 1, 1, 1)$, where the terms $O(\rho_k^3)$ are such that the vectors b_j , $j \leq 5$, are light-like. Then

$$g(b_5, b_j) = -\frac{1}{2}\rho_j^2, \quad g(b_k, b_j) = -\frac{1}{2}\rho_k^2 - \frac{1}{2}\rho_j^2 + O(\rho_k \rho_j).$$

Below, we denote $\omega_{kj} = g(b_k, b_j)$. Note that if $\rho_j < \rho_k^4$, we have $\omega_{kj} = -\frac{1}{2}\rho_k^2 + O(\rho_k^3)$.

For the wave equation, we use different parameters ρ_j than for the Einstein equations, and define (so, we use here the "unordered" numbering 4-2-1-3)

$$(213) \quad \rho_4 = \rho_2^{100}, \quad \rho_2 = \rho_1^{100}, \quad \text{and} \quad \rho_1 = \rho_3^{100}.$$

Below in this proof, we denote $\vec{\rho} \rightarrow 0$ when $\rho_3 \rightarrow 0$ and ρ_4, ρ_2 , and ρ_1 are defined using ρ_3 as in (213).

Let us next consider in Minkowski space the coordinates $(x^j)_{j=1}^4$ such that $K_j = \{x^j = 0\}$ are light-like hyperplanes and the waves $u_j = u_{(j)}$ that satisfy in the Minkowski space $\square u_j = 0$ and can be written as

$$u_j(x) = \int_{\mathbb{R}} e^{ix^j \theta} a_j(x, \theta) d\theta, \quad a_j(x, \theta') \in S^n(\mathbb{R}^4; \mathbb{R} \setminus 0), \quad j \leq 4,$$

and

$$u^\tau(x) = \chi(x^0) w_{(5)} \exp(i\tau b^{(5)} \cdot x).$$

Note that the singular supports of the waves u_j , $j = 1, 2, 3, 4$, intersect then at the point $\cap_{j=1}^4 K_j = \{0\}$. Analogously to the definition (78) we considered for the Einstein equations, we define the (Minkowski) indicator function

$$\mathcal{G}^{(\mathbf{m})}(v, \mathbf{b}) = \lim_{\tau \rightarrow \infty} \tau^m \left(\sum_{\beta \leq n_1} \sum_{\sigma \in \Sigma(4)} T_{\tau, \sigma}^{(\mathbf{m}), \beta} + \tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} \right),$$

where

$$\begin{aligned} T_{\tau, \sigma}^{(\mathbf{m}), \beta} &= \langle Q_0(u^\tau \cdot \alpha u_{\sigma(4)}), h \cdot \alpha u_{\sigma(3)} \cdot Q_0(\alpha u_{\sigma(2)} \cdot u_{\sigma(1)}) \rangle, \\ \tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} &= \langle u^\tau, h \alpha Q_0(\alpha u_{\sigma(4)} \cdot u_{\sigma(3)}) \cdot Q_0(\alpha u_{\sigma(2)} \cdot u_{\sigma(1)}) \rangle. \end{aligned}$$

As for the Einstein equations, we see that when α is equal to the value of the function $a(t, y)$ at the intersection point $q = 0$ of the waves, we have $\mathcal{G}^{(\mathbf{m})}(v, \mathbf{b}) = \mathcal{G}(v, \mathbf{b})$.

Similarly to the Lemma 3.3 we analyze next the functions

$$\Theta_\tau^{(\mathbf{m})} = \sum_{\beta \in J_\ell} \sum_{\sigma \in \Sigma(4)} (T_{\tau, \sigma}^{(\mathbf{m}), \beta} + \tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta}).$$

Here (\mathbf{m}) refers to ‘‘Minkowski’’. We denote $T_\tau^{(\mathbf{m}), \beta} = T_{\tau, id}^{(\mathbf{m}), \beta}$ and $\tilde{T}_\tau^{(\mathbf{m}), \beta} = \tilde{T}_{\tau, id}^{(\mathbf{m}), \beta}$.

Let us first consider the case when the permutation $\sigma = id$. Then, as in the proof of Prop. 3.4, in the case when $\vec{S}^\beta = (Q_0, Q_0)$, we have

$$\begin{aligned} T_\tau^{(\mathbf{m}), \beta} &= C_1 \det(A) \cdot (i\tau)^m \left(1 + O\left(\frac{1}{\tau}\right)\right) \bar{\rho}^{2\vec{n}} (\omega_{45}\omega_{12})^{-1} \rho_4^{-4} \rho_2^{-4} \rho_1^{-4} \rho_3^2 \cdot \mathcal{P} \\ &= C_2 \det(A) \cdot (i\tau)^m \left(1 + O\left(\frac{1}{\tau}\right)\right) \bar{\rho}^{2\vec{n}} \rho_4^{-4-2} \rho_2^{-4} \rho_1^{-4-2} \rho_3^{-2} \cdot \mathcal{P} \end{aligned}$$

where \mathcal{P} is the product of the principal symbols of the waves u_j at zero, $\bar{\rho}^{2\vec{n}} = \rho_1^{2n} \rho_2^{2n} \rho_3^{2n} \rho_4^{2n}$, and C_1 and C_2 are non-vanishing. Similarly, a direct computation yields

$$\begin{aligned} \tilde{T}_\tau^{(\mathbf{m}), \beta} &= C_1 \det(A) \cdot (i\tau)^n \left(1 + O\left(\frac{1}{\tau}\right)\right) \bar{\rho}^{2\vec{n}} (\omega_{43}\omega_{21})^{-1} \rho_4^{-4} \rho_2^{-4} \rho_1^{-4} \rho_3^{-4} \cdot \mathcal{P} \\ &= C_2 \det(A) \cdot (i\tau)^m \left(1 + O\left(\frac{1}{\tau}\right)\right) \bar{\rho}^{2\vec{n}} \rho_4^{-4} \rho_2^{-4} \rho_1^{-4-2} \rho_3^{-4-2} \cdot \mathcal{P}, \end{aligned}$$

where again, \mathcal{P} is the product of the principal symbols of the waves u_j at zero and C_1 and C_2 are non-vanishing.

Considering formula (209), we see that for the wave equation we do not need to consider the terms that for the Einstein equations correspond to the cases when $\vec{S}^\beta = (Q_0, I)$, $\vec{S}^\beta = (I, Q_0)$, or $\vec{S}^\beta = (I, I)$ as the corresponding terms do not appear in formula (209).

Let us now consider permutations σ of the indexes $(1, 2, 3, 4)$ and compare the terms

$$\begin{aligned} L_\sigma^{(\mathbf{m}), \beta} &= \lim_{\tau \rightarrow \infty} \tau^m T_{\tau, \sigma}^{(\mathbf{m}), \beta}, \\ \tilde{L}_\sigma^{(\mathbf{m}), \beta} &= \lim_{\tau \rightarrow \infty} \tau^m \tilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta}. \end{aligned}$$

Due to the presence of $\omega_{45}\omega_{12}$ in the above computations, we observe that all the terms $\tilde{L}_{\tau, \sigma}^{(\mathbf{m}), \beta}/L_{\tau, id}^{(\mathbf{m}), \beta} \rightarrow 0$ as $\vec{\rho} \rightarrow 0$, see (213). Also, if $\sigma \neq (1, 2, 3, 4)$ and $\sigma \neq \sigma_0 1 = (2, 1, 3, 4)$, we see that $\tilde{L}_{\tau, \sigma}^{(\mathbf{m}), \beta}/L_{\tau, id}^{(\mathbf{m}), \beta} \rightarrow 0$ as $\vec{\rho} \rightarrow 0$. Also, we observe that $L_{\tau, \sigma_1}^{(\mathbf{m}), \beta} = L_{\tau, id}^{(\mathbf{m}), \beta}$. Thus we see that the equal terms $L_{\tau, \sigma_1}^{(\mathbf{m}), \beta} = L_{\tau, id}^{(\mathbf{m}), \beta}$ that give the largest contributions as $\vec{\rho} \rightarrow 0$ and that when $\mathcal{P} \neq 0$ the sum

$$S(\vec{\rho}, \mathcal{P}) = \sum_{\sigma \in \Sigma(4)} (L_{\tau, \sigma}^{(\mathbf{m}), \beta} + \tilde{L}_{\tau, \sigma}^{(\mathbf{m}), \beta})$$

is non-zero when $\rho_3 > 0$ is small enough and ρ_4, ρ_2 , and ρ_1 are defined using ρ_3 as in (213). Since the indicator function is real-analytic, this shows that the indicator function is non-vanishing in a generic set. \square

We need also to change the singularity *detection condition* (D) with light-like directions $(\vec{x}, \vec{\xi})$ as follows: We define that point $y \in U_{\widehat{g}}$, satisfies the singularity *detection condition* (D') with light-like directions $(\vec{x}, \vec{\xi})$ and $t_0, \widehat{s} > 0$ if

(D') For any $s, s_0 \in (0, \widehat{s})$ there are $(x'_j, \xi'_j) \in \mathcal{W}_j(s; x_j, \xi_j)$, $j = 1, 2, 3, 4$, and $f_{(j)} \in \mathcal{I}_S^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$, and such that if $u_{\vec{\varepsilon}}$ is the solution of (208) with the source $f_{\vec{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j f_{(j)}$, then the function $\partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ is not C^∞ -smooth in any neighborhood of y .

When condition (D) is replaced by (D'), the considerations in the Sections 4 and 5 show that we can recover the conformal class of the metric. This proves Theorem 5.13. \square

Proof. (of Corollary 5.14). Let us denote $W_j = I^+(p_j^-) \cap I^-(p_j^+) \subset M_j$. By Theorem 5.13, there is a map $\Psi : W_1 \rightarrow W_2$ such that the product type metrics $g_1 = -dt^2 + h_1(y)$ and $g_2 = -dt^2 + h_2(y)$ are conformal. Note that the above methods to determine the conformal class of the metric but not the metric itself.

To construct the metric, let us consider the linearized waves. Let $B_j(y, r)$ denote the Riemannian ball of (N_j, h_j) with center y and radius

r. Consider a set $K_j = (s'_j, s''_j) \times B_j(p_j, r) \subset W_j$. On $(s'_j, s''_j) \times (N_j \setminus B_j(p_j, r)) \subset M_j$ we can consider the linear wave equation

$$(214) \quad \begin{aligned} \square_{g_j} w^{(j)} &= 0, \quad \text{on } (s'_j, s''_j) \times (N_j \setminus B_j(p_j, r)), \\ w^{(j)}|_{(s'_j, s''_j) \times \partial B_j(p_j, r)} &= h \\ w^{(j)}|_{t=s'_j} &= 0, \quad \partial_t w^{(j)}|_{t=s'_j} = 0. \end{aligned}$$

For this wave equation we define the Dirichlet-to-Neumann operator $\Lambda^{(j)} : h \mapsto \partial_\nu w^{(j)}|_{(s'_j, s''_j) \times \partial B_j(p_j, r)}$. We observe that any solution $w^{(j)}$ of (214) can be continued to a solution of

$$(215) \quad \begin{aligned} \square_{g_j} u^{(j)} &= F_j, \quad \text{on } (s'_j, s''_j) \times N_j, \\ u^{(j)}|_{t=s'_j} &= 0, \quad \partial_t u^{(j)}|_{t=s'_j} = 0. \end{aligned}$$

where F_j is some source supported in K_j , that is, for all $w^{(j)}$ and h there exists $u^{(j)}$ and F_j such that $u^{(j)} = w^{(j)}$ on $(s'_j, s''_j) \times (N_j \setminus B_j(p_j, r))$.

Consider next a given h . We see that if F_j is such that it satisfies

$$(216) \quad h = \partial_\varepsilon \left(L_{U_j}(\varepsilon F_j)|_{(s'_j, s''_j) \times \partial B_j(p_j, r)} \right) \Big|_{\varepsilon=0},$$

then

$$\begin{aligned} \Lambda_j h &= \partial_\nu w^{(j)}|_{(s'_j, s''_j) \times \partial B_j(p_j, r)} \\ &= \partial_\varepsilon \left(\partial_\nu (L_{U_j}(\varepsilon F_j))|_{(s'_j, s''_j) \times \partial B_j(p_j, r)} \right) \Big|_{\varepsilon=0}, \end{aligned}$$

and by the above considerations, for any h there exists some F_j for which (216) is valid.

This means that by linearizing the map L_{U_j} we can determine the map Λ_j . Now, using [60, 61], see also [8, 64], we see that if $L_{U_1} = L_{U_2}$ then (W_1, g_1) and (W_2, g_2) are isometric.

As g_1 and g_2 are independent of t , we see that there is a diffeomorphism $\Psi : W_1 \rightarrow W_2$ of the form $\Psi(t, y) = (t, \psi(y))$ such that $g_1 = \Psi^* g_2$. Note also that if $\pi_2 : (t, y) \mapsto y$, then $h_1 = \psi^* h_2$ on $\pi_2(W_1)$. Thus, the metric tensors h_1 and h_2 are isometric.

As the linearized waves $u^{(j)} = Qf^{(j)}$ depend only on W_j and the metric g_j , using the proof of Theorem 5.13 we see that the indicator functions $\mathcal{G}(\mathbf{b}, \mathbf{w})$ for (U_1, g_1, a_1) and (U_1, g_1, a_1) coincide for all \mathbf{b} and \mathbf{w} . This implies that $a_1(t, y)^3 = a_2(t, y)^3$ for all $(t, y) \in W$. Hence $a_1(t, y) = a_2(t, y)$ for all $(t, y) \in W$. \square

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