Retarded electric and magnetic fields of a moving charge: Feynman's derivation of Liénard-Wiechert potentials revisited

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Abstract

Retarded electromagnetic potentials are derived from Maxwell's equations and the Lorenz condition. The difference found between these potentials and the conventional Liénard-Wiechert ones is explained by neglect, for the latter, of the motion-dependence of the effective charge density. The corresponding retarded fields of a point-like charge in arbitary motion are compared with those given by the formulae of Heaviside, Feynman, Jefimenko and other authors. The fields of an accelerated charge given by the Feynman are the same as those derived from the Liénard-Wiechert potentials but not those given by the Jefimenko formulae. A mathematical error concerning partial space and time derivatives in the derivation of the Jefimenko equations is pointed out.

1   Introduction

The present paper is the fifth in a series written recently by the present author on relativistic classical electrodynamics (RCED). In the first of the papers [1], all of the formulae of classical electromagnetism (CEM), up to relativistic corrections of O(β^2), relating to intercharge forces, were derived from Hamilton's Principle, assuming only Coulomb's inverse-square force law of electrostatics and relativistic invariance. In the same paper it was shown that the intercharge force, mediated by the exchange of space-like virtual photons, is predicted by quantum electrodynamics (QED) to be instantaneous in the centre-of-mass frame of the interacting charges. Recently, convincing experimental evidence has been obtained [2] for the non-retarded nature of 'bound' magnetic fields with r^{-2} dependence, (associated in QED with virtual photon exchange) in a modern version, probing small r values, of the Hertz experiment [3] in which the electromagnetic waves associated with the propagation of real photons (fields with r^{-1} dependence) were originally discovered.

In two subsequent papers [4, 5] the predictions of the RCED formulae for intercharge forces derived in Ref. [1] are compared with the predictions of the CEM (Heaviside) formulae [6] for the force fields of a uniformly moving charge. Unlike the RCED formulae, the CEM ones correspond to a retarded interaction. If the latter are written in 'present time' form [7] they are found to differ from RCED formulae by terms of O(β^2). In the first paper [4], it is shown that consistent results for small-angle Rutherford scattering in different inertial frames are obtained only for RCED formulae and that stable, circular, Keplerian orbits of a system consisting of two equal and opposite charges are impossible for the case of the retarded CEM fields. The related 'Torque Paradox' of Jackson [8] is also resolved by use of the instantaneous RCED fields. The second paper [5] considers electromagnetic induction in different reference frames. Again, consistent results are obtained only in the case of RCED fields. It is demonstrated that for a particular spatial configuration of a simple two-charge 'magnet' the Heaviside formula for the electric field predicts a vanishing induction effect in the case that the 'magnet' is in motion and the test coil is at rest.

In Ref. [9], the space-time transformation properties of the RCED and CEM force fields were studied in detail and compared with those that provide the classical description of the creation, propagation, and destruction of real photons. It was shown that in the relativistic theory longitudinal (with respect the direction of motion of the source charge) electric fields contain covariance-breaking terms of O(β^2). The electric Gauss Law and Electrodynamic (Ampère Law) Maxwell Equations are also modified by the addition of covariance-breaking terms of O(β^4) and O(β^5) respectively. The retarded fields are re-derived from the Maxwell Equations and the Lorenz condition and an error in the derivation of the retarded Liénard-Wiechert (LW) [10] potentials was pointed out. The argument leading to this conclusion —which implies that retarded fields given by the Heaviside formulae are erroneous for this trivial mathematical reason, as well as being inconsistent with QED— is recalled in Sections 2 and 3 below.

The aim of the present paper is to present a more detailed discussion of retarded electromagnetic fields with a view to pointing out some of the mathematically erroneous

statements on this subject that have appeared in classical research literature, text books and modern pedagogical literature. The correct relativistic formulae for the retarded fields of an accelerated charge have previously been derived in Ref. [9]. These fields actually describe only the production and propagation of real photons whereas in text books and the pedagogical literature it is universally assumed that these fields describe both intercharge forces and radiative effects. Since the present paper is concerned only with the postulates and mathematical arguments used in different derivations of the retarded fields, the physical interpretation of the fields (in particular their relation to the quantum mechanical description of radiation), as discussed in Ref. [9], is not considered.

The structure of the paper is as follows. In the following section the retarded 4-vector potential is derived from inhomgeneous d'Alembert equations and the Lorenz condition. The reason for the difference between the potential so-obtained and the pre-relativistic LW potentials is explained. In Section 3 Feynman's derivation of the LW potentials is recalled, where the 'multiple counting' committed also in the original derivations [10] is made particularly transparent. In Section 4 some erroneous 'relativistic' derivations of the LW potentials and the Heaviside formulae that are commonly presented in text books on classical electromagnetism are discussed. In Section 5 the retarded fields of a uniformly moving charge are considered and the 'present time' formulae for the retarded RCED fields are derived for comparison with the Heaviside formulae of CEM. In Section 6 a comparison is made between different formulae for the retarded fields of an accelerated charge that have appeared in text books and the pedagogical literature including the well-known ones of Feynman and Jefimenko. Section 7 contains a brief summary.

2   Derivation of retarded electromagnetic potentials from inhomogeneous d'Alembert equations

As described in Ref. [11], retarded electromagnetic potentials may be derived from the Maxwell equations:

$$\vec{\nabla} \cdot \vec{\mathrm{E}} \;\; = \;\; 4\pi \mathrm{J}_0, \tag{2.1}$$

$$\vec{\nabla} \times \vec{\mathrm{B}} \;\; - \;\; \frac{1}{c}\frac{\partial \vec{\mathrm{E}}}{\partial t} = 4\pi \vec{\mathrm{J}} \tag{2.2}$$

and the Lorenz condition

$$\vec{\nabla} \cdot \vec{\mathrm{A}} + \frac{1}{c}\frac{\partial \mathrm{A}_0}{\partial t} = 0 \tag{2.3}$$

where the current density J is a 4-vector:

$$\mathrm{J}(\vec{x}_J(t), t) = (\mathrm{J}_0; \vec{\mathrm{J}}) \equiv (\gamma_u \rho^*; \gamma_u \vec{\beta}_u \rho^*) = \frac{u \rho^*}{c}. \tag{2.4}$$

The system of source charges is assumed to be at rest in the frame S*, where the charge density is ρ^*, and to move with velocity $\vec{u} = c\vec{\beta}_u$ relative to the frame S in which the potential is defined. The 4-vector velocity of the charge system in this last frame is:

$$u \equiv (c\gamma_u; c\gamma_u \vec{\beta}_u) \tag{2.5}$$

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where

$$\beta_u \equiv \frac{u}{c}, \quad \gamma_u \equiv \frac{1}{\sqrt{1 - \beta_u^2}}.$$

The first step of the calculation is to use the Lorenz condition (2.3) to eliminate either $\vec{J}$ or J_0 from (2.1) and (2.2) to obtain the inhomogeneous d'Alembert equations:

$$\nabla^2 A_0 - \frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} = -4\pi J_0, \tag{2.6}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi \vec{J}. \tag{2.7}$$

These equations are readily solved by introducing appropriate Green functions [11]. The solutions give the retarded 4-vector potential:

$$A_\mu^{ret}(\vec{x}_q, t) = \int dt' \int d^3x_J(t') \frac{J_\mu(\vec{x}_J(t'), t')}{|\vec{x}_q - \vec{x}_J(t')|} \delta(t' + \frac{|\vec{x}_q - \vec{x}_J(t')|}{c} - t). \tag{2.8}$$

Here $\vec{x}_q$ is the position and t the time at which the potential is defined and $\vec{x}_J(t')$ specifies the position of the volume element $d^3x_J(t')$ at the earlier time t'. The δ-function ensures that the volume element lies on the backward light cone of the field point specified by $\vec{x}_q$, as required by causality, since the potentials give the classical description of the propagation, from the source to the field point, of real (on-shell) photons at speed c. This is a consequence of the wave-equation-like structure of the terms on the left sides of the d'Alembert equations. The solutions of the corresponding homogeneous d'Alembert equations are progressive waves with phase velocity c.

In the special case of a single point-like source charge the current density in (2.8) is given by the expression:

$$J^Q(\vec{x}_J(t'), t') = \frac{Qu}{c}\delta(\vec{x}_J(t') - \vec{x}_Q(t')) \tag{2.9}$$

where $\vec{x}_Q(t')$ is the position of the charge at time t'. Inserting (2.9) in (2.8), and integrating over $\vec{x}_J$, gives

$$A_\mu^{ret}(\vec{x}_q, t) = \frac{Qu_\mu}{c} \int dt' \frac{\delta(t' - t'_Q)}{r'} \tag{2.10}$$

where

$$r' \equiv |\vec{x}_q - \vec{x}_Q(t')|, \quad t'_Q \equiv t - \frac{|\vec{x}_q - \vec{x}_Q(t'_Q)|}{c} = t - \frac{r'}{c}\bigg|_{t'=t'_Q}. \tag{2.11}$$

The retarded 4-vector potential is therefore:

$$(A_0^{ret}; \vec{A}^{ret}) = \left(\frac{Q\gamma_u}{r'}\bigg|_{t'=t'_Q} ; \frac{Q\gamma_u\vec{\beta}_u}{r'}\bigg|_{t'=t'_Q} \right). \tag{2.12}$$

A similiar result to (2.12) is obtained in the case of an extended distribution of charge in the case that its dimensions are much less than the separation between the average position of the source charge distribution, $\langle\vec{x}_J\rangle$, and the field point. In this case $\vec{x}_J$

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may be replaced in the δ-function and denominator of (2.8) by $\langle \vec{x}_J \rangle$, so that the factor $\langle r' \rangle \equiv |\vec{x}_q - \langle \vec{x}_J \rangle|$ in the denominator may be taken outside the $\vec{x}_J$ integral giving

$$
\begin{aligned}
A_\mu^{ret}(\vec{x}_q, t) &= \int \frac{dt'}{\langle r' \rangle} \int d^3x_J(t'_J) J_\mu(\vec{x}_J(t'), t') \delta(t' + \frac{|\vec{x}_q - \langle \vec{x}_J \rangle|}{c} - t) \\
&= \int \frac{dt'}{\langle r' \rangle} \frac{u_\mu}{c} \int d^3x_J(t') \rho^*(\vec{x}_J(t'), t') \delta(t' + \frac{|\vec{x}_q - \langle \vec{x}_J \rangle|}{c} - t) \\
&= \int \frac{dt'}{\langle r' \rangle} \frac{u_\mu Q}{c} \delta(t' - \langle t'_J \rangle)
\end{aligned}
\tag{2.13}
$$

where Q is the total charge of the distribution:

$$
Q = \int \rho^*(\vec{x}_J(t'), t') d^3x_J(t')
\tag{2.14}
$$

and

$$
\langle t'_J \rangle \equiv t - \frac{|\vec{x}_q - \langle \vec{x}_J \rangle|}{c} = t - \frac{\langle r' \rangle}{c}
\tag{2.15}
$$

giving the 4-vector potential:

$$
(A_0^{ret}; \vec{A}^{ret}) = \left(\left. \frac{Q\gamma_u}{\langle r' \rangle} \right|_{t'=\langle t'_J \rangle} ; \left. \frac{Q\gamma_u \vec{\beta}_u}{\langle r' \rangle} \right|_{t'=\langle t'_J \rangle} \right).
\tag{2.16}
$$

It is now of interest, in view of understanding the origin of the LW potentials, to recalculate the retarded potentials after inverting the the order of the t' and $\vec{x}_J(t')$ integrations in (2.8), so that:

$$
A_\mu^{ret}(\vec{x}_q, t) = \int d^3x_J(t') \int dt' \frac{J_\mu(\vec{x}_J(t'), t')}{|\vec{x}_q - \vec{x}_J(t')|} \delta(t' + \frac{|\vec{x}_q - \vec{x}_J(t')|}{c} - t).
\tag{2.17}
$$

Unlike in (2.10), where the insertion of the current density of a point-like charge, (2.9) simply specifies the value of t' in the δ-function to be t'_Q, as given by Eq.(2.11), on integrating over $\vec{x}_J$, the argument of the δ-function in (2.17) has a more complicated dependence on t':

$$
\delta[f(t')] = \frac{\delta(t' - t'_J)}{\left| \frac{\partial f(t')}{\partial t'} \right|_{t'=t'_J}}
\tag{2.18}
$$

where t'_J is the solution of the equation $f(t') = 0$ and

$$
f(t') \equiv t' + \frac{|\vec{x}_q - \vec{x}_J(t')|}{c} - t.
\tag{2.19}
$$

It follows from (2.19), and the definition of t'_J, that

$$
t'_J = t - \frac{|\vec{x}_q - \vec{x}_J(t'_J)|}{c}.
\tag{2.20}
$$

Differentiating (2.19) gives:

$$
\frac{\partial f(t')}{\partial t'} = 1 - \hat{r}'_J \cdot \vec{\beta}_u
\tag{2.21}
$$

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where:

$$\hat{r}'_J = \frac{\vec{x}_q - \vec{x}_J(t'_J)}{|\vec{x}_q - \vec{x}_J(t'_J)|}, \quad \vec{\beta}_u = \frac{1}{c}\frac{d\vec{x}_J(t')}{dt'} \tag{2.22}$$

so that (2.17) may be written as

$$A_\mu^{ret}(\vec{x}_q, t) = \int d^3x_J(t') \int dt' \frac{J_\mu(\vec{x}_J(t'), t')}{|\vec{x}_q - \vec{x}_J(t')|(1 - \hat{r}'_J \cdot \vec{\beta}_u)}\delta(t' - t'_J). \tag{2.23}$$

In performing the integral over t', proper account must now be taken of the appropriate current density J_μ to be inserted in (2.23). The limits of the t' integral are determined by the times at which the backward light cone of the field point coincides with the boundaries of the moving charge distribution. This is illustrated in Fig.1 for a uniform block of charge DEFG, of trapeziodal shape, moving in the plane of the figure towards a distant field point, in this plane, far to the right. The segments AA', BB' and CC' lie on the light front, LF, that coincides with the backward light cone of the field point. It is assumed that the latter is sufficiently far that LF may be approximated by a plane, with normal in the plane of the figure. The block of charge is moving with speed u in the plane of the figure at angle θ to the direction of motion of LF. The light front starts to overlap the block of charge in the position AA' and ceases to do so in the position CC'. The limits of the t' integral in (2.23) for this case then correspond to the times when the front coincides with AA' (lower limit) and with CC' (upper limit). Inspection of Fig.1 shows that, during the time interval between these limits, the average value of the charge density, $\bar{\rho}$, is less than that when the distribution it at rest, ρ^*, by the ratio:

$$\frac{\ell}{L} = \frac{\text{length of charge distribution}}{\text{length of light cone overlap region}}. \tag{2.24}$$

If $\Delta t'$ is the time during which there is overlap between LF and the block of charge, the geometry of Fig.1 gives:

$$L = u\Delta t' + \ell = \frac{c\Delta t'}{\cos\theta} \tag{2.25}$$

so that

$$\frac{\ell}{L} = 1 - \frac{u}{c}\cos\theta = 1 - \hat{r}'_J \cdot \vec{\beta}_u. \tag{2.26}$$

It can be seen from Fig.1 that the same average charge density is obtained if the uniform block of charge is replaced by a point-like charge, Q, equal to the integrated charge of the block and placed at its centre, or if the moving uniform charge distribution is replaced by the fixed one MNOP with density $\bar{\rho}$. For a single point-like charge the appropriate current density in (2.23) is then given by (2.9). (2.24) and (2.26) as:

$$J^Q(\vec{x}_J(t'), t') = (1 - \hat{r}'_J \cdot \vec{\beta}_u)\frac{Qu}{c}\delta(\vec{x}_J(t') - \vec{x}_Q(t')). \tag{2.27}$$

Inserting (2.27) in (2.23) and performing the integrals over t' and $\vec{x}_J$ recovers the result of Eq.(2.12). The increased overlap time of the light front resulting from the motion of the block of charge is exactly compensated by the reduction of the average charge density resulting from the same motion. The incorrect LW potentials are given by taking into account the time-everlap correction factor but neglecting the corresponding change in the

charge density. This gives, instead of (2.12), the potentials

$$(A_0^{ret}; \vec{A}^{ret}) \equiv (\gamma_u A_0(LW)^{ret}; \gamma_u \vec{A}(LW)^{ret}) = \left(\left. \frac{Q\gamma_u}{r'(1 - \hat{r}'_J \cdot \vec{\beta}_u)} \right|_{t'=t'_Q} ; \left. \frac{Q\gamma_u \vec{\beta}_u}{r'(1 - \hat{r}'_J \cdot \vec{\beta}_u)} \right|_{t'=t'_Q} \right)$$
(2.28)

where $A_0(LW)^{ret}$ and $\vec{A}(LW)^{ret}$ are the Liénard and Wiechert potentials. This mistake in the original Liénard and Wiechert [10] calculations has been repeated in all textbook treatments of the subject of retarded potentials. Some examples may be found in Refs. [12, 13, 14, 15, 16].

Inspection of (2.23) shows that neglect of the charge density correction factor of (2.27) in evaluating the potentials implies that they are under-estimated when the source is receding ($\hat{r}'_J \cdot \vec{\beta}_u < 0$) and over-estimated when it is approaching ($\hat{r}'_J \cdot \vec{\beta}_u > 0$). On the other hand, the $1/r'$ dependence of the potential implies that the potentials are greater (smaller) when $\hat{r}'_J \cdot \vec{\beta}_u < 0$ ($\hat{r}'_J \cdot \vec{\beta}_u > 0$). It is shown in Section 5 below that for the 'present time' LW fields (Eqs.(4.14) and (4.15) below) the neglect of the charge density correction factor results in exact compensation of the $1/r'^2$ dependence so that the magnitudes of the fields are independent of the sign of $\hat{r}'_J \cdot \vec{\beta}_u$, as is the case for an instantaneous intercharge interaction.

Note that the potentials on the right side of (2.28), derived by neglecting the density correction factor in (2.27), differ from the retarded LW potentials by an overall factor of γ_u. This factor will be commented on at the end of Section 4 below where alternative 'relativistic' derivations of the LW potentials are discussed.

3 Feynman's derivation of the Liénard-Wiechert potentials

The erroneous nature of the retarded potentials found when the charge density correction factor of Eqs.(2.24) and (2.26) is neglected is made particularly clear by a careful examination of Feynman's derivation [17] of the LW potentials for the case of parallel motion of the source distribution and the light front, LF, corresponding to the backward light cone of the field point.

Feynman's analysis of the problem of retarded potentials is shown in Fig.2. A rectangular block of charge, of uniform density, moves towards the field point, which is sufficiently far to the right that the variation of r'_J may, as in deriving Eq.(2.16) above, be neglected in evaluating the integral that gives the potential. The light front moves across the charge distribution, sampling it. Each element of charge which is crossed by LF gives a contribution to the potential. The depth of the block of charge is ℓ and LF moves over the distance L while crossing the charge distribution. The front overlaps the charge distribution for a time interval T. The overlap distance, L, is divided into bins of width w and the contribution to the potential of each bin is considered separately. In Fig.2 the
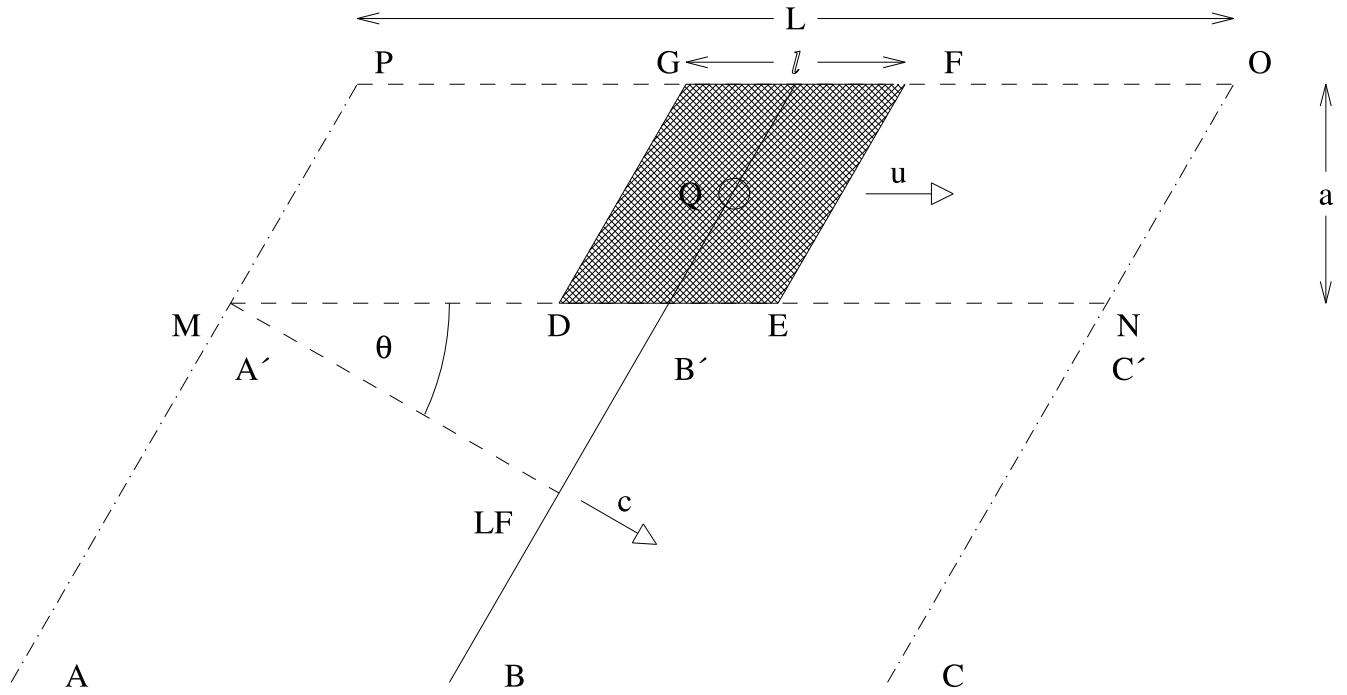
Figure 1: *The plane light front (LF) BB' crosses the block of charge DEFG with uniform charge density ρ while moving from the position AA' to CC'.The light front and the block move in the plane of the figure with speeds c and u respectively in the directions indicated. The average charge density sampled by the light front during its passage over the block is $\bar{\rho} = \rho^* \ell / L$ The retarded potential generated by the charge of the block at a distant field point to the right of the figure is the same as that that would be generated by a block of charge in the form MNOP with the same depth as DEFG,with uniform charge density $\bar{\rho}$, at rest, or by a moving point-like charge $Q = \rho^* V$, where V is the volume of the block DEFG.*
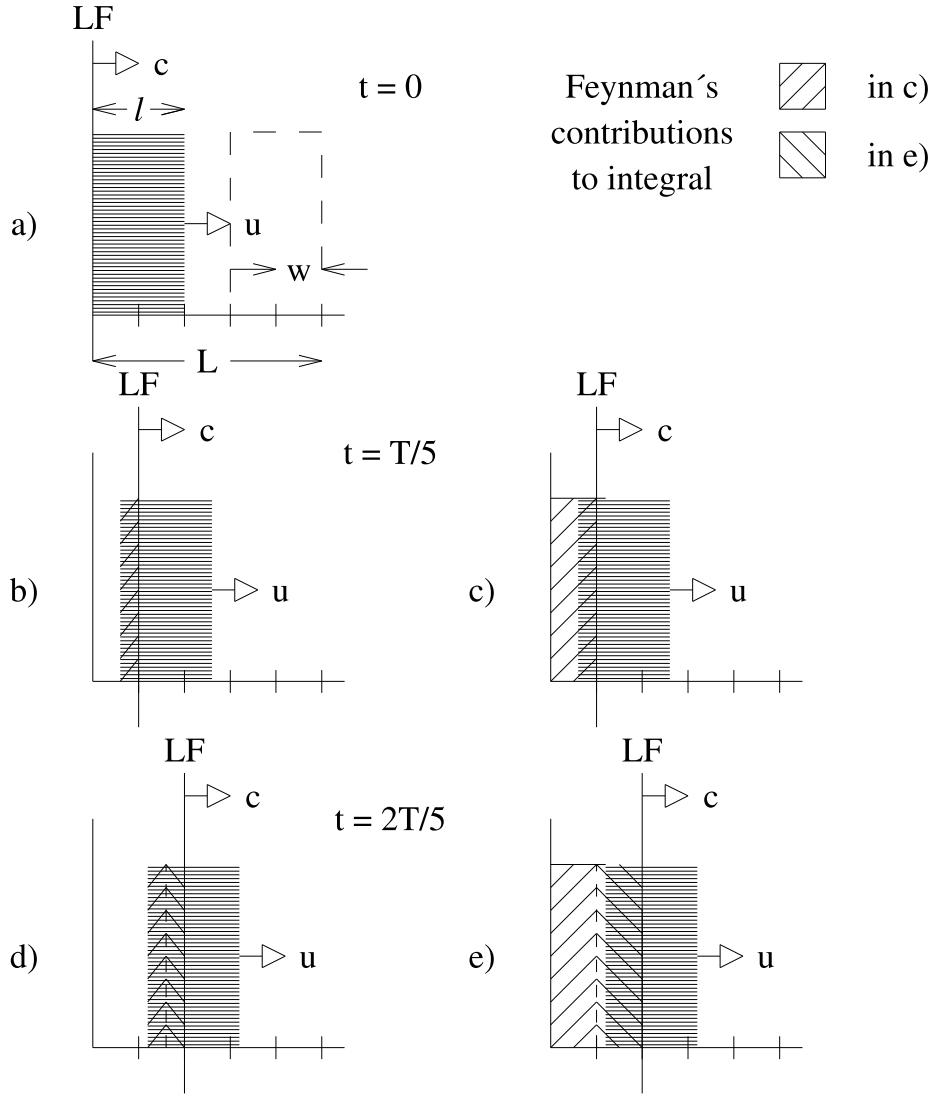
Figure 2: *Feynman's method of calculating retarded potentials [17]. A uniform rectangular block of charge of length l moves to the right with speed v towards a distant field point. The light front, LF, in causal connection with the field point, overlaps the block for a distance L and a time T. In a) LF arrives at the front of the block. The position of the block when LF overtakes it is shown shaded. b) and d) show the positions of LF at times t = T/5 and t = 2T/5 respectively. The regions of the block sampled by LF in the time intervals 0 < t < T/5 and T/5 < t < 2T/5 are shown by the SW-NE and NW-SE cross-hatched areas, respectively. The similar crossed-hatched areas in c) and e) show the charge volumes assigned to the potential integral, during the same time intervals, in Feynman's calculation. See text for discussion.*

dimensions and velocity u are chosen so that:

$$\ell = \frac{2L}{5}, \qquad w = \frac{L}{5}.$$

It then follows that $u = 3c/5$. In this figure, the positions of the charge distribution and the front LF at times 0, $T/5$, $2T/5$ respectively are shown. In Figs.2b,2d the front has crossed charge thicknesses of $0.4w$,$0.8w$ respectively. The region crossed during the time $0 < t < T/5$ is shown by SW-NE[1]diagonal cross-hatching, that crossed in $T/5 < t < 2T/5$ by NW-SE diagonal cross-hatching. Thus the average charge density in each bin is reduced, in comparison with the situation when the charges are at rest, by 60%. Integrating first over the time, as in (2.17), for each bin, then gives:

$$A_\mu = \frac{u_\mu S}{cr'_J} \sum_{bins} w\bar{\rho} = \frac{u_\mu S L \bar{\rho}}{cr'_J} \tag{3.1}$$

where S is the surface area of the charge distribution normal to its direction of motion and $\bar{\rho}$ is the average charge density. From the geometry of Fig.2a, $\bar{\rho} = 2\rho^*/5$ where ρ^* is the rest frame charge density. Since $L = 5\ell/2$ (3.1) gives:

$$A_\mu = \frac{u_\mu S \ell \rho^*}{cr'_J} = \frac{u_\mu Q}{cr'_J} \tag{3.2}$$

where Q is the total charge in the block. Allowing for the propagation time delay of the light front with respect to the time of the field point (3.2) agrees with Eq.(2.16) but not with the LW potentials in (2.28).

The contributions to the integral given by the first two bins, according to Feynman's original calculation [17] are shown by the SW-NE and NW-SE diagonal hatching in Figs.2c and 2e respectively. The movement of the charge distribution is neglected, and with it the change in the effective charge density. Feynman's result is given by replacing $\bar{\rho}$ in (3.1) by ρ^*, the density of the charge distribution at rest. This gives a result consistent with (2.28), but is evidently wrong, since charge elements are multiply counted during the passage of the light front. For example, a contribution to the integral is assigned proportional to the area of the cross-hatched region to the left of LF in Fig.2c for $t \leq T/5$. However, inspection of Fig.2b, showing the actual geometrical configuration at $t = T/5$, shows that, because of the parallel motion of the charge distribution, LF has crossed only the fraction of the region in Fig.2c that is both shaded and cross-hatched, not the entire cross-hatched region. In fact, careful inspection of Fig21-6(c) of Ref. [17] shows clearly that the contribution due to the passage of the light front over the first bin is overestimated. Only the region of the charge distribution to the left of the light front as shown in this figure has been sampled at this time, not the filled first bin of Fig21-6(b) of Ref. [17].

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[1]The points of the compass: South-West (SW), North-East (NE), North-West (NW) and South-East (SE).

4 'Relativistic' derivations of the Liénard-Wiechert potentials and the electromagnetic fields of a uniformly moving charge

As well as the derivation of the LW potentials by consideration of retardation effects, as in the original papers of Liénard and Wiechert, text books on classical electromagnetism contain alternative derivations, where no retardation effects are considered, but instead a relativistic 'length contraction' effect is invoked. For example, in Ref. [18], the temporal component A_0 of the 4-vector electromagnetic potential is obtained by Lorentz-transformation into the frame S, where A_0 is defined, from the frame S* in which the point-like source charge Q is at rest:

$$A_0 = \gamma_u A_0^* = \gamma_u \frac{Q}{r^*} \tag{4.1}$$

where

$$r \equiv |\vec{x}_q - \vec{x}_Q|, \quad r^* \equiv |\vec{x}_q^* - \vec{x}_Q^*|. \tag{4.2}$$

The vectors $\vec{x}_q$, $\vec{x}_Q$ ($\vec{x}_q^*, \vec{x}_Q^*$) give the position of the field point and the source charge, respectively, in the frames S (S*). These coordinates are specified at a fixed time in the frame S —no retardation effects are considered. It is then assumed that the x-coordinate separations in the frames S and S* are related by the relativistic length contraction relation:

$$x_q^* - x_Q^* = \frac{x_q - x_Q}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{4.3}$$

while the y and z separations are the same in both frames. It then follows from (4.2) and (4.3) that

$$(r^*)^2 = \frac{(x_q - x_Q)^2 + (1 - \frac{u^2}{c^2})[(y_q - y_Q)^2 + (z_q - z_Q)^2]}{1 - \frac{u^2}{c^2}}. \tag{4.4}$$

Denoting by ψ the angle between the vectors $\vec{x}_q - \vec{x}_Q$ and $\vec{u}$, (4.4) may be written as:

$$\begin{aligned}(r^*)^2 &= r^2 \frac{[\cos^2 \psi + (1 - \frac{u^2}{c^2}) \sin^2 \psi]}{1 - \frac{u^2}{c^2}} \\ &= r^2 \frac{[1 - \beta_u^2 \sin^2 \psi]}{1 - \frac{u^2}{c^2}}. \end{aligned} \tag{4.5}$$

Substituting r^* from (4.5) in (4.1) than gives

$$A_0 \equiv A_0(\text{LW})^{PT} = \frac{Q}{r(1 - \beta_u^2 \sin^2 \psi)^{\frac{1}{2}}}. \tag{4.6}$$

This is the 'present time' (PT) formula [7] for the temporal component of the retarded LW potential $A_0(\text{LW})^{ret}$ given in Eq.(2.28) above. All quantities in (4.6) are defined at the instant that the potential is specified. The 'present time' form of the 3-vector potential $\vec{A}$ is calculated, in a similar manner, to obtain

$$\vec{A} \equiv \vec{A}(\text{LW})^{PT} = \frac{Q\vec{\beta}_u}{r(1 - \beta_u^2 \sin^2 \psi)^{\frac{1}{2}}} \tag{4.7}$$

It is interesting to note that that the γ_u factor in (4.1), manifesting the 4-vector character of A, is cancelled by a similar factor originating in the 'length contraction' effect of Eq.(4.3).

A similar derivation of $A_0(LW)^{PT}$ may be found in Ref. [20] where it is noted that the change of variables

$$x_q^* = \frac{x_q}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad y_q^* = y_q, \quad z_q^* = z_q \tag{4.8}$$

transforms the d'Alembert equation (2.6) into a Poisson equation, the solution of which is the Coulomb electrostatic potential Q/r^*. Expressing r^* in terms of (x_q, y_q, z_q), neglecting a multiplicative factor γ_u (which was cancelled in the derivation of Ref. [18] by the similar factor in the numerator of the right side of (4.1)) the potential $A_0(LW)^{PT}$ is obtained. It is only mentioned at the end of the calculation that the purely mathematical transformations of Eqs.(4.8) should be interpreted as physical transformations predicted by the Lorentz transformation, Unlike in Ref [18] the scalar and vector potentials are treated in non-relativistic manner, necessitating a (tacit) neglect of a factor γ_u in order to recover the LW result.

A 'relativistic' derivation of the 'present time' formulae for the electric and magnetic fields of a uniformly moving charge, by use of a similar 'length contraction' ansatz as in Refs. [18, 20] is found in Jackson's book [21][2], The conventional transformation laws of electric and magnetic fields between the frames S* and S;

$$E_x = E_x^*, \quad E_y = \gamma_u(E_y^* + \beta_u B_z^*), \quad B_z = \gamma_u(B_z^* + \beta_u E_y^*) \tag{4.9}$$

are used to transform the fields in the rest frame of the source charge:

$$E_x^* = \frac{Q(x_q^* - x_Q^*)}{(r^*)^3}, \quad E_y^* = \frac{Q(y_q^* - y_Q^*)}{(r^*)^3}, \quad E_z^* = B_x^* = B_y^* = B_z^* = 0 \tag{4.10}$$

into the frame S. Performing this transformation, and using (4.3) to express the result in terms of S frame coordinates[3] gives

$$
\begin{aligned}
E_x &= \frac{Q(x_q - x_Q)}{\gamma_u^2\{(x_q - x_Q)^2 + (1 - \frac{u^2}{c^2})[(y_q - y_Q)^2 + (z_q - z_Q)^2]\}^{\frac{3}{2}}} \\
&= \frac{Q\cos\psi}{\gamma_u^2 r^2(1 - \beta_u^2 \sin^2\psi)^{\frac{3}{2}}},
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
E_y &= \frac{Q(y_q - y_Q)}{\gamma_u^2\{(x_q - x_Q)^2 + (1 - \frac{u^2}{c^2})[(y_q - y_Q)^2 + (z_q - z_Q)^2]\}^{\frac{3}{2}}} \\
&= \frac{Q\sin\psi}{\gamma_u^2 r^2(1 - \beta_u^2 \sin^2\psi)^{\frac{3}{2}}},
\end{aligned}
\tag{4.12}
$$

$$
B_y = \beta_u E_y = \frac{Q\beta_u \sin\psi}{\gamma_u^2 r^2(1 - \beta_u^2 \sin^2\psi)^{\frac{3}{2}}}.
\tag{4.13}
$$

---

[2]A similar derivation is found in the widely-used text book on Electricity and Magnetism by Purcell [22].

[3]Actually Jackson used a relativistic time dilatation equation equivalent to Eq.(4.3).

Eqs.(4.11)-(4.13) may also be written in 3-vector notation as:

$$\vec{E} \equiv \vec{E}(H)^{PT} = \frac{Q\vec{r}}{\gamma_u^2 r^3 (1 - \beta_u^2 \sin^2 \psi)^{\frac{3}{2}}}, \tag{4.14}$$

$$\vec{B} \equiv \vec{B}(H)^{PT} = \vec{\beta_u} \times \vec{E}. \tag{4.15}$$

The label 'H' stands for 'Heaviside' who first obtained these equations [6] more than a decade before the advent of special relativity. They may also be obtained from the 'present time' potentials in (4.6) and (4.7) and the usual definitions of electric and magnetic fields in terms of derivatives of the 4-vector potential.

It is easy to show that the 'length contraction' ansatz of Eqs.(4.3) and (4.8) used to derive (4.14) and (4.15), as obtained from the retarded LW potential, but without invoking any retardation effect, is inconsistent with a fundamental reciprocity property of special relativity. This was stated in a concise way, and in a manner directly applicable to the problem considered here, by Pauli [23]:

    The contraction of lengths at rest in S* is equal to that of lengths at
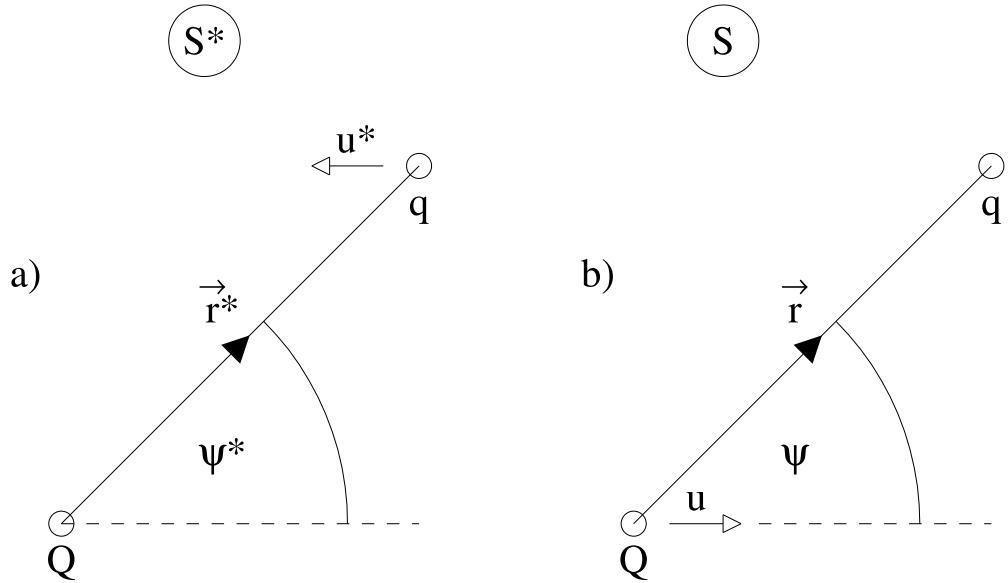   rest in S and observed in S*.

Figure 3: *Spatial configurations in the frames S* [a)] and S [b)] at corresponding instants in the two frames; for example when the origin of S*, situated at Q coincides with the origin of S.*

To make manifest the symmetry of the configurations in the frames S and S*, that is the basis of the applicability of the above reciprocity postulate in the present case, a test charge q, at rest, is placed at the field point in S. As shown in Fig.3, the 'length at rest in S*' is the separation, r^*, of the source and test charges in this frame (Fig.3a). Similarly the 'length at rest in S is equal to r (Fig.3b). However, in the case of the 'length

contraction' ansatz of Eqs.(4.3) and (4.8), r is also the contracted value of the length r^* as observed in S. i.e.

$$r = \alpha(u)r^* \tag{4.16}$$

where $\alpha(u)$ is some even function of the relative velocity u of the frames S and S*, and $\alpha(u) < 1$, $\alpha(0) = 1$. The above reciprocity postulate states that also

$$r^* = \alpha(u^*)r. \tag{4.17}$$

where $\alpha(u) = \alpha(u^*)$ Combining (4.16) and (4.17),

$$r = \alpha(u)r^* = \alpha(u)\alpha(u^*)r. \tag{4.18}$$

It follows that if $r \neq 0$, $\alpha(u)\alpha(u^*) = 1$. This requires that $u = u^* = 0$, contradicting the initial hypothesis that S and S* are in relative motion. The existence of a 'length contraction' effect respecting Pauli's reciprocity condition is therefore excluded by *reductio ad absurdum* (self-contradiction).

The length contraction ansatz of (4.3) and (4.8) is therefore incompatible with the above stated reciprocity property of special relativity. How this universally (until now) accepted length contraction effect results from a misinterpretation of the symbols in the space-time Lorentz transformation has been extensively discussed elsewhere [24, 25, 26]. In conclusion, the 'relativistic' derivation of the field equations (4.14) and (4.15) neglects retardation effects and is in fact incompatible with (correctly interpreted [24, 25, 26]) special relativity. That the same result is obtained using the incorrect LW potentials (derived by consideration of pre-relativistic retarded fields) must then be regarded, not as confirmation of the correctness of the formulae, but as purely fortuitous. The 'present time' formulae derived from the relativistically-correct retarded potentials in (2.12) are presented in the following section.

An alternative 'relativistic derivation' of $A(LW)^{PT}$, but also assuming retardation, was given by Landau and Lifshitz [27]. The retardation condition (2.11) was used to write the temporal component of A, in the rest frame of the point-like source charge as:

$$A_0^* = \frac{Q}{c(t - t'_Q)}. \tag{4.19}$$

It was the noticed that the 4-vector:

$$A \equiv \frac{Qu}{x^{ret} \cdot u} \tag{4.20}$$

where

$$x^{ret} \equiv (c(t - t'_Q); \vec{x}_q - \vec{x}_Q(t')) \tag{4.21}$$

reduces to (4.19) in the rest frame of the source charge. The right side of (4.20) is precisely the retarded LW potential in (2.28) of a point-like charge. Although it is true that (4.20) gives (4.19) in the rest frame of the source charge, the same is true of the different 4-vector potential in (2.12). The relation (4.20) is, however, nothing more than a mathematical curiosity, lacking any physical motivation, whereas the potential in (2.12), equally consistent with (4.19), is the solution of the d'Alembert equations (derived from Maxwell's equations and the Lorenz condition) for a point-like charge. The physical

meaning and method of derivation of the potential of (2.12), unlike that of (4.20), are therefore quite clear.

That the retarded LW potentials and the associated fields could be derived in a 'relativistic' calculation in which retardation effects are completely neglected, whereas in the original derivations of Liénard, Wiechert and Heaviside, performed before the advent of special relativity, the (actually spurious) length contraction effect is neglected should be serious cause for concern. This unease, however, seemed not to be shared by authors of text books, and the pedagogical literature, on classical electromgnetism, throughout the last century. There is now ample experimental verification of the predictions of correctly interpreted [24, 25, 26] special relativity, and that retardation effects do occur in processes where real photons are radiated, so that the corresponding classical fields must also be retarded. The contradiction posed by the absence of one or the other of two essential, but different, physical phenomena in the two different derivations of the Heaviside formulae was clearly stated by Jefimenko [28], but the obvious doubt shed by this on the correctness of the formulae and/or the derivations, was passed over in silence. In fact, as demonstrated in the present paper both the original 19th century and the 20th century 'relativistic' derivations' are wrong. The former because the variation of the effective charge density of the moving charge distribution was not taken into account, the latter because the 'length contraction' effect on which they are based, does not exist. It is proposed in the present paper that the correct relativistic retarded potentials of a point-like charge are those given above in Eq.(2.12). The corresponding electric and magnetic fields, for the case of a charge in uniform motion, are derived in the following section. In Section 6 the retarded fields of accelerated charges are considered, and compared with those derived from the LW potentials as well as the well-known formulae of Feynman and Jefimenko as well as some others that have appeared in text books and the pedagogical literature.

The RCED 4-vector potential and current differ from those of CEM by the multiplicative factor γ_u (see Eq.(2.28) above). This leads to to a breakdown of the Gauss law for the electric field of a moving charge [4, 9] and covariance-breaking terms in the electrodynamic Maxwell equation (Ampère's law) [9]. In text books on CEM, the validity of the electric field Gauss law for both static and moving source charges is justified by invoking the relativistic length contraction effect of Eq.(4.3) [29]. If the charge density in a moving frame transforms as the temporal component of a 4-vector $\propto \gamma_u$, and a volume element transforms $\propto (\gamma_u)^{-1}$ due to relativistic length contraction, then the charge within the volume element is Lorentz invariant. Since however, as demonstrated above, the length contraction effect is spurious, the effective charge actually varies as γ_u or as $(u/c)^2$ for small u. This effect has been experimentally observed in the vicinity of an electrically neutral superconducting magnet [30].

5 Retarded electric and magnetic fields of a point-like charge in uniform motion

The electric and magnetic fields corresponding to the 4-vector potential (2.12) are obtained by straightforward application of the definitions of electric and magnetic fields in terms of derivatives of the potential:

$$\vec{E} \equiv -\vec{\nabla} A_0 - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \tag{5.1}$$

$$\vec{B} \equiv \vec{\nabla} \times \vec{A} \tag{5.2}$$

where, without loss of generality, it may be assumed that the electric field is confined to the x-y plane,

$$\vec{\nabla} \equiv \hat{\imath} \frac{\partial}{\partial x_q} + \hat{\jmath} \frac{\partial}{\partial y_q}.$$

Unit vectors along the x- y- and z-axes are denoted as $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. To perform the calculation, the retardation condition

$$t' = t - \frac{|\vec{x}_q - \vec{x}_Q(t')|}{c} = t - \frac{r'}{c} \tag{5.3}$$

must be used to express the derivatives with respect to t in (5.1) in terms of t', since the retarded position of the source charge is a function of t', not of t. Assuming that u is constant, (2.12) gives:

$$\left. \frac{\partial A_0}{\partial x_q} \right|_t = -\frac{Q \gamma_u}{(r')^2} \left. \frac{\partial r'}{\partial x_q} \right|_t. \tag{5.4}$$

Differentiating the geometrical relation:

$$(r')^2 = [x_q - x_Q(t')]^2 + y_q^2 \tag{5.5}$$

with respect to x_q gives

$$r' \left. \frac{\partial r'}{\partial x_q} \right|_t = (x_q - x_Q(t')) \left(1 - \frac{dx_Q(t')}{dt'} \left. \frac{\partial t'}{\partial x_q} \right|_t \right). \tag{5.6}$$

Differentiating (5.3) with respect to x_q,

$$\left. \frac{\partial t'}{\partial x_q} \right|_t = -\frac{1}{c} \left. \frac{\partial r'}{\partial x_q} \right|_t. \tag{5.7}$$

Combining (5.6) and (5.7), rearranging, and noting that $dx_Q(t')/dt' = c\beta_u$ gives

$$\left. \frac{\partial r'}{\partial x_q} \right|_t = \frac{x_q - x_Q(t')}{r'(1 - \hat{r}' \cdot \vec{\beta}_u)} \tag{5.8}$$

where $\hat{r}' \equiv \vec{r}'/r'$. Combining (5.4) and (5.8)

$$\left. \frac{\partial A_0}{\partial x_q} \right|_t = -\frac{Q \gamma_u [x_q - x_Q(t')]}{(1 - \hat{r}' \cdot \vec{\beta}_u)(r')^3}. \tag{5.9}$$

An analogous relation is obtained for $\partial \mathcal{A}_0 / \partial y_q$ so that

$$-\vec{\nabla} A_0 = \frac{Q \gamma_u \vec{r'}}{(1 - \hat{r}' \cdot \vec{\beta}_u)(r')^3}. \tag{5.10}$$

Considering now the second term on the right side of (5.1), (2.12) gives

$$-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\frac{\hat{i}}{c} \frac{\partial A_x}{\partial t}\bigg|_{x_q, y_q} = -\frac{Q \gamma_u \vec{\beta}_u}{(r')^2} \frac{\partial r'}{\partial t'}\bigg|_{x_q, y_q} \frac{\partial t'}{\partial t}\bigg|_{x_q, y_q}. \tag{5.11}$$

Differentiating (5.5) with respect to t':

$$r' \frac{\partial r'}{\partial t'}\bigg|_{x_q, y_q} = -[x_q - x_Q(t')] \frac{dx_Q(t')}{dt'} \tag{5.12}$$

or

$$\frac{\partial r'}{\partial t'}\bigg|_{x_q, y_q} = -c \hat{r}' \cdot \vec{\beta}_u. \tag{5.13}$$

Differentiating (5.3) with respect to t:

$$\frac{\partial t'}{\partial t}\bigg|_{x_q, y_q} = 1 - \frac{1}{c} \frac{\partial r'}{\partial t'}\bigg|_{x_q, y_q} \times \frac{\partial t'}{\partial t}\bigg|_{x_q, y_q} \tag{5.14}$$

Combining (5.13) and (5.14) and rearranging

$$\frac{\partial t'}{\partial t}\bigg|_{x_q, y_q} = \frac{1}{1 - \hat{r}' \cdot \vec{\beta}_u}. \tag{5.15}$$

Combining (5.1),(5.10),(5.11),(5.13) and (5.15) gives, for the retarded RCED electric field:

$$\vec{E}(RCED)^{ret} = \frac{Q \gamma_u}{(1 - \hat{r}' \cdot \vec{\beta}_u)} \left[\frac{[\vec{r'} - \vec{\beta}_u(\vec{r'} \cdot \vec{\beta}_u)]}{(r')^3} \right]\bigg|_{t'=t'_Q}. \tag{5.16}$$

Since $\vec{A} = \hat{i} A$,

$$\vec{\nabla} \times \vec{A} = -\hat{k} \frac{\partial A_x}{\partial y_q}\bigg|_t = -\hat{k} \frac{Q \gamma_u \beta_u}{(r')^2} \frac{\partial r'}{\partial y_q}\bigg|_t. \tag{5.17}$$

Similarly to (5.8)

$$\frac{\partial r'}{\partial y_q}\bigg|_t = \frac{y_q}{r'(1 - \hat{r}' \cdot \vec{\beta}_u)} \tag{5.18}$$

So that

$$\vec{B}(RCED)^{ret} = \vec{\nabla} \times \vec{A} = \frac{Q \gamma_u \vec{\beta}_u \times \vec{r'}}{(r')^3(1 - \hat{r}' \cdot \vec{\beta}_u)}\bigg|_{t'=t'_Q} = \vec{\beta}_u \times \vec{E}(RCED)^{ret}. \tag{5.19}$$

Apart from the retarded time argument and an overall factor $1/(1 - \hat{r}' \cdot \vec{\beta}_u)$ (the Jacobian of the transformation from t to t', see Eq.(5.15)) Eqs.(5.16) and (5.19) are the same as the formulae for the instantaneous force fields of RCED [1].

For comparison with the Heaviside formulae (4.14) and (4.15), that may be derived from the LW potentials it is of interest to write $\vec{E}(RCED)^{ret}$ and $\vec{B}(RCED)^{ret}$ in the 'present time' form.
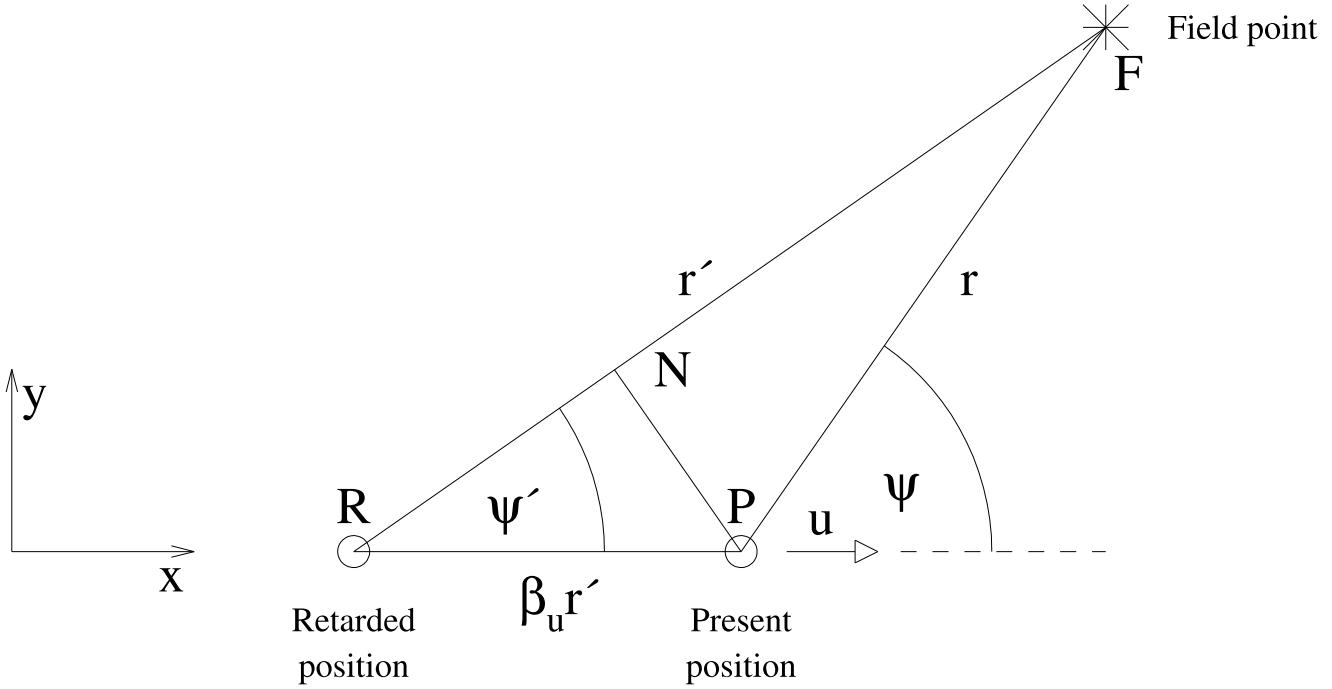
16

Figure 4: *Geometry for the calculation of the retarded potential of a charge moving with uniform velocity u along the x-axis in terms of the 'present time' coordinates r, ψ. R is the position of the charge from which a light signal was emitted so as to arrive at the field point F at the instant shown. P is the position of the charge at this instant. The line segment PN is perpendicular to RF.*

Consider a point-like charge, Q, moving with speed u along the x-axis (Fig.4). The field point at which the fields are to be specified is denoted by F, the present position of the charge by P and the retarded position, lying on the backward light cone of F, by R. If N is the foot of the normal to RF passing through P, the geometry of Fig.4 gives:

$$
\begin{aligned}
\text{NF} &= r' - \beta_u r' \cos \psi' = r'(1 - \hat{r}' \cdot \vec{\beta}_u) \\
&= \sqrt{r^2 - \beta_u^2 (r' \sin \psi')^2} = \sqrt{r^2 - \beta_u^2 (r \sin \psi)^2} \\
&= r(1 - \beta_u^2 \sin^2 \psi)^{\frac{1}{2}} \equiv r f_u.
\end{aligned}
\tag{5.20}
$$

Solving the quadratic equation obtained by applying the cosine rule to the triangle RFP:

$$
(r')^2 = r^2 + \beta_u^2 (r')^2 + 2\beta_u r r' \cos \psi
\tag{5.21}
$$

gives the retarded separation between the source charge and field point, r' in terms of the 'present time' parameters r and ψ

$$
r' = r \frac{(\beta_u \cos \psi + f_u)}{1 - \beta_u^2}.
\tag{5.22}
$$

Also

$$
\sin \psi' = \frac{r \sin \psi}{r'} = \frac{(1 - \beta_u^2) \sin \psi}{\beta_u \cos \psi + f_u}
\tag{5.23}
$$

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and

$$\cos\psi' = = \frac{\beta_u f_u + \cos\psi}{\beta_u \cos\psi + f_u} \tag{5.24}$$

$$\hat{r}' = \hat{\imath}\cos\psi' + \hat{\jmath}\sin\psi' \tag{5.25}$$

$$\hat{r}' \cdot \vec{\beta}_u = \beta_u \cos\psi'. \tag{5.26}$$

Eqs.(5.20)-(5.26) may now be used to express the retarded fields in terms of 'present time' coordinates:

$$\vec{E}(RCED)^{PT} = \frac{Q(1-\beta_u)[\hat{\imath}(\beta_u f_u + \cos\psi) + \hat{\jmath}(1+\beta_u)\sin\psi]}{r^2\gamma_u(\beta_u\cos\psi + f_u)^2 f_u}, \tag{5.27}$$

$$\vec{B}(RCED)^{PT} = \frac{Q\hat{k}\sin\psi}{r^2\gamma_u^3(\beta_u\cos\psi + f_u)^2 f_u} = \vec{\beta}_u \times \vec{E}(RCED)^{PT}. \tag{5.28}$$

These expressions replace, in relativistic classical electrodynamics, the pre-relativistic Heaviside formulae (4.14) and (4.15). Important differences are that (5.27), unlike (4.14), is not radial, and in consequence does not respect Gauss' Law, and does not revert to a radial Coulomb field on neglecting terms of $O(\beta_u^2)$.

The manifestly incorrect physical behaviour of the retarded electric field given by the Heaviside formula (4.14) is evident on comparing it with that given by (5.27). This is done in Figs. 5 and 6 which show curves of $E_L^{PT} r^2/Q$ and $E_T^{PT} r^2/Q$, respectively, as a function of ψ, where $\vec{E}^{PT} = \hat{\imath}E_L^{PT} + \hat{\jmath}E_T^{PT}$, for different values of β_u, as given by (4.14) and (5.27). Elementary physical considerations require that if $\psi < 90°$ (i.e. the source charge is approaching the field point) the magnitude of the retarded field must be less than when $\psi > 90°$ and the charge is receding from the field point. This is because in the former case the source charge was further from the field point than its present-time position when the retarded field was emitted, and closer to it in the latter case. Because the strength of the field is inversely proportional to the square of the source-field point separation, at the time of emission of the retarded field, the magnitude of the field must be greater at an angle $\psi_+ = 90° + \alpha$ than at an angle $\psi_- = 90° - \alpha$ where $\alpha > 0$. The fields given by (5.27) demonstrate this property, whereas $\vec{E}(H)^{PT}$ given by (4.14) gives symmetric behaviour for E_T:

$$E_T(H)^{PT}(\psi_+) = E_T(H)^{PT}(\psi_-) \tag{5.29}$$

and antisymmetric behaviour for E_L:

$$E_L(H)^{PT}(\psi_+) = -E_L(H)^{PT}(\psi_-). \tag{5.30}$$

These symmetry properties are those of instantaneous [1, 4], not retarded, force fields[4]. As explained in Section 2 above, the antisymmetry of the $E_L(H)^{PT}$ curves in Fig. 5, about $\psi = 90°$ and the symmetry of the $E_T(H)^{PT}$ curves in Fig. 6, about $\psi = 90°$ in Fig. 6 is a consequence of the neglect of the velocity dependence of the source charge density in deriving the LW potentials of Eq.(2.28).

---

[4]See, for example, the comparision of the 'present time' retarded LW fields with the instantaneous RCED fields in Figs. 2 and 3 of Ref. [4].
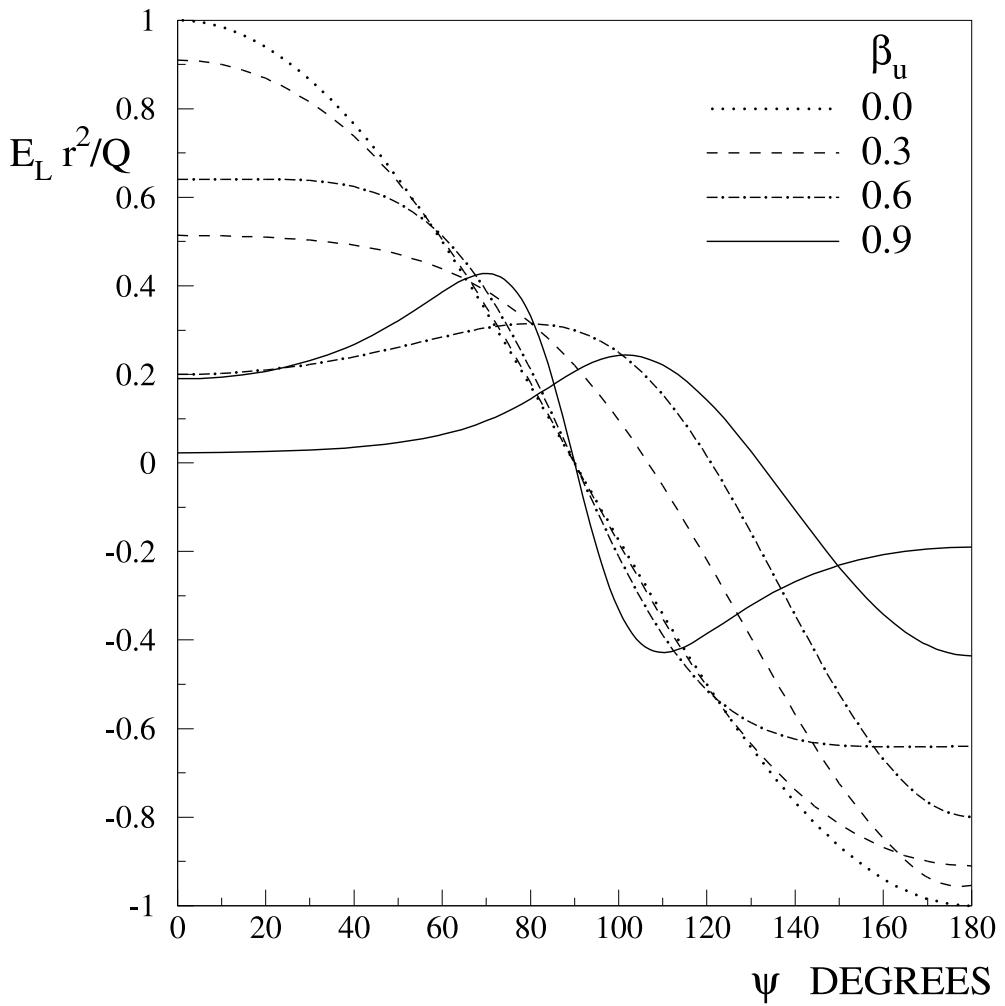
Figure 5: *Scaled retarded present time longitudinal electric field: $E_L r^2/Q$ as a function of ψ for various values of the source charge velocity $\beta_u = u/c$. The curves of $E_L(H)r^2/Q$ are antisymmetric relative to $\psi = 90°$ and so do not display the expected ψ dependence of retarded fields as seen in the $E_L(RCED)r^2/Q$ curves where $|E_L(RCED)|$ for $\psi < 90°$ (source charge approaching the field point) is less than $|E_L(RCED)|$ for $\psi > 90°$ (source charge receding from the field point).*
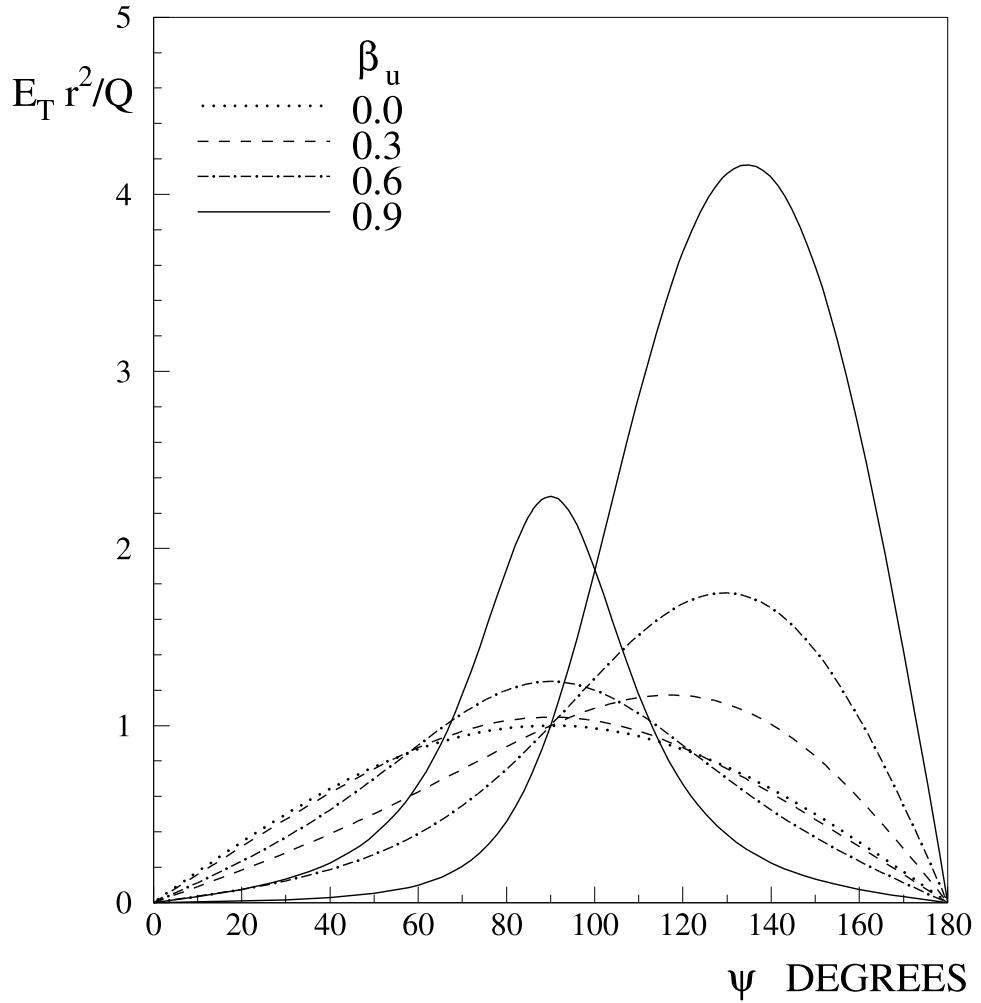
Figure 6: *Scaled retarded present time transverse electric field:* $E_T r^2/Q$ *as a function of* ψ *for various values of the source charge velocity* $\beta_u = u/c$. *The curves of* $E_T(H)r^2/Q$ *are symmetric relative to* $\psi = 90°$ *and so do not display the expected* ψ *dependence of retarded fields as seen in the* $E_T(RCED)r^2/Q$ *curves where* $|E_T(RCED)|$ *for* $\psi < 90°$ *(source charge approaching the field point) is less than* $|E_T(RCED)|$ *for* $\psi > 90°$ *(source charge receding from the field point).*

6 Retarded electric and magnetic fields of an accelerated point-like charge: the RCED, Liénard-Wiechert, Feynman and Jefimenko equations

The generalisation of the RCED formulae (5.16) and (5.19) to the case of a non-constant value of the source charge velocity $\vec{u}$ is straightforward. The details of the calculation may be found in Ref. [9]. Including the terms generated by the acceleration of the source charge gives:

$$\vec{E}(\text{RCED})^{ret} = \left\{ \frac{Q\gamma_u}{K} \left[\frac{[\hat{r}' - \vec{\beta}_u(\hat{r}' \cdot \vec{\beta}_u)]}{(r')^2} + \frac{[\gamma_u^2 \beta_u \dot{\beta}_u(\hat{r}' - \vec{\beta}_u) - \dot{\vec{\beta}}_u]}{cr'} \right] \right\} \Bigg|_{t'=t'_Q}, \quad (6.1)$$

$$\vec{B}(\text{RCED})^{ret} = \left\{ \frac{Q\gamma_u(\vec{\beta}_u \times \hat{r}')}{K} \left[\frac{1}{(r')^2} + \frac{\gamma_u^2 \dot{\beta}_u}{c\beta_u r'} \right] \right\} \Bigg|_{t'=t'_Q} \quad (6.2)$$

where

$$K \equiv (1 - \hat{r}' \cdot \vec{\beta}_u), \quad \dot{\beta}_u \equiv |\dot{\vec{\beta}}_u|. \quad (6.3)$$

It follows from (6.1) that

$$\vec{\beta}_u \times \vec{E}(\text{RCED})^{ret} = \left\{ \frac{Q\gamma_u(\vec{\beta}_u \times \hat{r}')}{K} \left[\frac{1}{(r')^2} + \frac{\gamma_u^2 \beta_u \dot{\beta}_u}{cr'} \right] - \frac{Q\gamma_u(\vec{\beta}_u \times \dot{\vec{\beta}}_u)}{Kcr'} \right\} \Bigg|_{t'=t'_Q} \quad (6.4)$$

and

$$\hat{r}' \times \vec{E}(\text{RCED})^{ret} = \left\{ \frac{Q\gamma_u(\vec{\beta}_u \times \hat{r}')}{K} \left[\frac{\hat{r}' \cdot \vec{\beta}_u}{(r')^2} + \frac{\gamma_u^2 \beta_u \dot{\beta}_u}{cr'} \right] - \frac{Q\gamma_u(\hat{r}' \times \dot{\vec{\beta}}_u)}{Kcr'} \right\} \Bigg|_{t'=t'_Q}. \quad (6.5)$$

The relation $\vec{B}(\text{RCED})^{ret} = \vec{\beta}_u \times \vec{E}(\text{RCED})^{ret}$ than holds only if $\dot{\beta}_u = 0$, as in Eq.(5.19) above, while, in all cases, $\vec{B}(\text{RCED})^{ret} \neq \hat{r}' \times \vec{E}(\text{RCED})^{ret}$

These formulae may be compared with those derived by inserting the LW potentials of Eq.(2.28) into the defining equations (5.1) and (5.2) of the electric and magnetic fields, making use of the Jacobian of (5.15) to relate derivatives with respect to t and t'. This calculation is given in Appendix A. It is found that:

$$\vec{E}(\text{LW})^{ret} = \left\{ \frac{Q}{K^3} \left[\frac{\hat{r}' - \vec{\beta}_u}{\gamma_u^2(r')^2} + \frac{\hat{r}' \times [(\hat{r}' - \vec{\beta}_u) \times \dot{\vec{\beta}}_u]}{cr'} \right] \right\} \Bigg|_{t'=t'_Q}, \quad (6.6)$$

$$\vec{B}(\text{LW})^{ret} = \left\{ \frac{Q(\vec{\beta}_u \times \hat{r}')}{K^3} \left[\frac{1}{\gamma_u^2(r')^2} + \frac{(\dot{\beta}_u(1 - \hat{r}' \cdot \vec{\beta}_u) + \beta_u(\hat{r}' \cdot \dot{\vec{\beta}}_u)}{cr'\beta_u} \right] \right\} \Bigg|_{t'=t'_Q}. \quad (6.7)$$

The markedly different angular dependence of these fields to that of the RCED fields of (6.1) and (6.2) may be noticed. Eq.(6.6) gives

$$\vec{\beta}_u \times \vec{E}(\text{LW})^{ret} = \left\{ \frac{Q}{K^3} \left[(\vec{\beta}_u \times \hat{r}') \left(\frac{1}{\gamma_u^2(r')^2} + \frac{\hat{r}' \cdot \dot{\vec{\beta}}_u}{cr'} \right) \right] \right\} \Bigg|_{t'=t'_Q}, \quad (6.8)$$

$$\hat{r}' \times \vec{E}(LW)^{ret} = \vec{B}(LW)^{ret} \tag{6.9}$$

The relation $\vec{B}(LW)^{ret} = \vec{\beta}_u \times \vec{E}(LW)^{ret}$ then hold only if $\dot{\vec{\beta}}_u = 0$.

The RCED retarded fields (6.1) and (6.2) are now compared with those obtained from formulae given by Feynman, Jefimenko and other authors. The consistency of the latter fields with the LW ones of (6.6) and (6.7) will also be considered. In the 'Feynman Lectures in Physics' compact formulae for the retarded fields of an accelerated point-like charge are given, but not derived [33, 34]. In the notation of the present paper they are

$$\vec{E}(Feyn)^{ret} = Q \left[\frac{\hat{r}'}{(r')^2} + \frac{r'}{c} \frac{d}{dt} \left(\frac{\hat{r}'}{(r')^2} \right) + \frac{1}{c^2} \frac{d^2 \hat{r}'}{dt^2} \right] \Bigg|_{t'=t'_Q}, \tag{6.10}$$

$$\vec{B}(Feyn)^{ret} = \hat{r}'|_{t'=t'_Q} \times \vec{E}(Feyn)^{ret}. \tag{6.11}$$

Since (see Eq.(5.3)) r' is a function of the retarded time t', not of the present time t it is necessary to introduce the Jacobian of Eq.(5.15) in order to evaluate the derivatives in Eq.(6.10). Although Feynman uses the symbol for a total time derivative rather than a partial one in Eq.(6.10) it is clear from the definitions of the fields in terms of potentials in (5.1) and (5.2) that the time derivatives should be understood as partial ones for a fixed position of the field point $\vec{x}_q$ as in (5.15). The straightforward but somewhat lengthy calculation, which is analogous to that shown in the previous section, leading to Eqs.(5.16) and (5.19), is presented in Appendix B. The following formula for the electric field is obtained:

$$\vec{E}(Feyn)^{ret} = Q \left\{ \frac{\hat{r}'}{(r')^2} + \frac{1}{K} \frac{[3\hat{r}'(\hat{r}' \cdot \vec{\beta}_u) - \vec{\beta}_u]}{(r')^2} \right.$$
$$+ \frac{1}{K^2} \left[\frac{\hat{r}'[3(\hat{r}' \cdot \vec{\beta}_u)^2 - \beta_u^2] - 2\vec{\beta}_u(\hat{r}' \cdot \vec{\beta}_u)}{(r')^2} + \frac{[\hat{r}'(\hat{r}' \cdot \dot{\vec{\beta}}_u) - \dot{\vec{\beta}}_u]}{cr'} \right]$$
$$\left. + \frac{[\hat{r}'(\hat{r}' \cdot \vec{\beta}_u) - \vec{\beta}_u]}{K^3} \left[\frac{(\hat{r}' \cdot \vec{\beta}_u)^2 - \beta_u^2}{(r')^2} + \frac{\hat{r}' \cdot \dot{\vec{\beta}}_u}{cr'} \right] \right\} \Bigg|_{t'=t'_Q}. \tag{6.12}$$

Collecting terms on the right side of (6.12) proportional to $\hat{r}'$, $\vec{\beta}_u$ and $\dot{\vec{\beta}}_u$ recovers, together with (6.11), the LW formulas (6.6) and (6.7).

A formula for the retarded potentials, similar to Eq.(2.8) above, is obtained in in Ref. [35] by using Green functions to solve the inhomogeneous d'Alembert equations (2.6) and (2.7). However, subsequently, the usual mistake is made of neglecting the motion-dependence of the charge density, as explained in Section 2. After performing the spatial integration, instead of replacing $r'(t')$ in the argument of the δ-function by $r'(t'_Q)$, (see Eq.(2.10) above) the appropriate retarded position of the point-like source charge at time t, the same functional dependence on t' is assumed as before the spatial integration and the formula (2.18) is then used to transform the argument of the δ-function, leading to the spurious retardation factor $1/K$ of the LW potentials. In this way the formula (19) of Ref. [35] was obtained which was then shown to give the Feynman's formula Eq. (6.10) above. It was also correctly stated (but not demonstrated) that the same formula was equivalent to the LW field of Eq. (6.6) above.

The Jefimenko formulae for the fields of an accelerated charge distribution are [36]:

$$\vec{E}(\text{Jefi})^{ret} = \int \left\{ \hat{r}' \left[\frac{[\rho]}{(r')^2} + \frac{1}{cr'} \frac{\partial[\rho]}{\partial t} \right] - \frac{1}{c^2 r'} \frac{\partial[\vec{J}]}{\partial t} \right\} d^3 x_J, \tag{6.13}$$

$$\vec{B}(\text{Jefi})^{ret} = \frac{1}{c} \int \left[\frac{[\vec{J}]}{(r')^2} + \frac{1}{cr'} \frac{\partial[\vec{J}]}{\partial t} \right] \times \hat{r}' d^3 x_J. \tag{6.14}$$

The square brackets around the charge density ρ and the non-relativistic current density $\vec{J}$ indicate that they are evaluated at the retarded time $t' = t - r'/c$, as is also the spatial separation r' of the current element from the field point. The volume element, $d^3 x_J$, is also specified at the retarded time t'. Important differences from the RCED, LW and Feynman formulae are that the time derivatives act only on the charge and current densities, not on r', and that, as compared to the Feynman formula, only first order time derivatives appear. However a time derivative of $\vec{r}'$ is implicit in the definition of the current $\vec{J}$.

In order to discuss the Jefimenko equations for the case of a point-like charge, it will be found convenient to explicitly impose the retardation condition by including an integration over the retarded time t' together with the corresponding δ-function as in Eq.(2.8). Indeed, much confusion about, and misinterpretation of, formulae for retarded fields result from not properly taking into account integrations over both space and time. This must be done to correctly describe the essential physical properties of the problem under consideration —a spatially extended distribution of charge[5] in motion that is probed, in time, by the backward light cone of the field point. As will be seen, attempts to simplify formulae by omitting time integrals, and the associated δ-functions, as in (6.13) and (6.14), and in many text-book treatments of retarded potentials, often lead to erroneous results. Specifying explicitly the retardation condition, (6.13) and (6.14) are written as:

$$\vec{E}(\text{Jefi})^{ret} = \int dt' \int \left\{ \left[\frac{[\rho(\vec{x}_J(t'), t')]}{(r')^2} + \frac{1}{cr'} \frac{\partial[\rho(\vec{x}_J(t'), t')]}{\partial t} \right] \hat{r}' \right.$$
$$\left. - \frac{1}{c^2 r'} \frac{\partial[\vec{J}(\vec{x}_J(t'), t')]}{\partial t} \right\} \delta(t' + \frac{r'(t')}{c} - t) d^3 x_J(t'), \tag{6.15}$$

$$\vec{B}(\text{Jefi})^{ret} = \int dt' \int \frac{1}{c} \left[\frac{[\vec{J}(\vec{x}_J(t'), t')]}{(r')^2} \right.$$
$$\left. + \frac{1}{cr'} \frac{\partial[\vec{J}(\vec{x}_J(t'), t')]}{\partial t} \right] \times \hat{r}' \delta(t' + \frac{r'(t')}{c} - t) d^3 x_J(t'). \tag{6.16}$$

A point-like charge Q has, in non-relativistic approximation, the following charge and current densities:

$$\rho_Q(\vec{x}_J(t'), t') = Q\delta(\vec{x}_J(t') - \vec{x}_Q(t')), \tag{6.17}$$
$$\vec{J}_Q(\vec{x}_J(t'), t') = Q\vec{u}(t')\delta(\vec{x}_J(t') - \vec{x}_Q(t')). \tag{6.18}$$

Substituting (6.17) and (6.18) into (6.15) and (6.16) and performing the spatial integrations gives:

$$\vec{E}(\text{Jefi})^{ret} = Q \int dt' \left[\frac{\hat{r}'}{(r')^2} - \frac{1}{c^2 r'} \frac{\partial \vec{u}(t')}{\partial t} \right] \delta(t' - t'_Q)$$

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[5]In the real world, consisting of an ensemble of identical point-like charged particles.

$$= Q\left[\frac{\hat{r}'}{(r')^2} - \frac{1}{c^2 r'}\frac{\partial t'}{\partial t}\bigg|_{\vec{x}_q}\frac{d\vec{u}(t')}{dt'}\right]\bigg|_{t'=t'_Q}$$

$$= Q\left[\frac{\hat{r}'}{(r')^2} - \frac{1}{Kc^2 r'}\frac{d\vec{u}(t')}{dt'}\right]\bigg|_{t'=t'_Q}, \tag{6.19}$$

$$\vec{B}(\text{Jefi})^{ret} = Q\int dt'\left[\frac{\vec{u}(t')}{c(r')^2} + \frac{1}{c^2 r'}\frac{\partial\vec{u}(t')}{\partial t}\right]\times\hat{r}'\delta(t'-t'_Q)$$

$$= Q\left\{\left[\frac{\vec{u}(t')}{c(r')^2} + \frac{1}{c^2 r'}\frac{\partial t'}{\partial t}\bigg|_{\vec{x}_q}\frac{d\vec{u}(t')}{dt'}\right]\times\hat{r}'\right\}\bigg|_{t'=t'_Q}$$

$$= Q\left\{\left[\frac{\vec{u}(t')}{c(r')^2} + \frac{1}{Kc^2 r'}\frac{d\vec{u}(t')}{dt'}\right]\times\hat{r}'\right\}\bigg|_{t'=t'_Q} \tag{6.20}$$

where

$$t'_Q \equiv t - \frac{r'(t'_Q)}{c}. \tag{6.21}$$

For a uniformly moving charge, in contrast to the RCED, LW and Feynman equations the Jefimenko equations therefore predict the same (Coulombic) electric field as in the electrostatic case.

In a paper on time-dependent generalisations of of the Biot and Savart and Coulomb laws where the Jefimenko equations were extensively discussed [37], it was claimed that the Jefimenko, Liénard-Wiechert and Feynman formulae for the retarded fields are all consistent with each other. The arguments given in support of this conclusion are critically examined below, but first the claim of the authors of Ref. [37] to *derive* the Jefimenko equations from the defining formulae (5.1) and (5.2) of electric and magnetic fields and retarded potentials is considered. The assumed form of the potentials in Ref. [37] in the notation and choice of units of the present paper, is:

$$A_0 = \int\frac{[\rho]}{r'}d\tau, \tag{6.22}$$

$$\vec{A} = \int\frac{[\vec{J}]}{r'}d\tau. \tag{6.23}$$

The volume element $d\tau \equiv d^3 x_J(t')$ and the quantities in square brackets, as well as the distance r' between the volume element and the field point are all specified at the retarded time $t' = t - r'/c$. Note that unlike the solutions of the D'Alembert equations in (2.8) there is no integral over the retarded time in these equations and also they do not contain the $1/K$ factor of the LW potentials. Substitution of (6.22) and (6.23) into (5.1) gives[6]:

$$\vec{E}(\text{GH1})^{ret} = -\int\left[\frac{1}{r'}\vec{\nabla}[\rho] + [\rho]\vec{\nabla}\left(\frac{1}{r'}\right) + \frac{1}{c^2 r'}\frac{\partial[\vec{J}]}{\partial t} + \frac{[\vec{J}]}{c^2}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right)\right]d\tau. \tag{6.24}$$

This aleady disagrees with the corresponding equation (21) of Ref. [37], where the last term on the right side of (6.30) is omitted. Indeed, this term does not vanish but the last

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[6]The field is labelled according to the initials of the authors, Griffiths and Heald, of Ref. [37]

factor in it is:

$$\frac{\partial}{\partial t}\left(\frac{1}{r'}\right)\bigg|_{\vec{x}_q} = \frac{\partial t'}{\partial t}\bigg|_{\vec{x}_q}\frac{\partial}{\partial t'}\left(\frac{1}{r'}\right)\bigg|_{\vec{x}_q} = -\frac{1}{(r')^2}\frac{\partial t'}{\partial t}\bigg|_{\vec{x}_q}\frac{\partial r'}{\partial t'}\bigg|_{\vec{x}_q} = \frac{c(\hat{r}'\cdot\vec{\beta}_u)}{K(r')^2} \tag{6.25}$$

where (5.13) and (5.15) have been used. This result is implicit in the later Eq.(44) of Ref. [37], so the omission of the term in Eq.(21) of this reference in hard to understand. The retarded density $[\rho]$ may be a function of the retarded time t' and the position $\vec{x}_J(t')$ of the volume element $d\tau$, but does not depend of the position $\vec{x}_q$ of the field point. Therefore $\vec{\nabla}[\rho]$ vanishes. More formally[7]

$$\begin{aligned}\vec{\nabla}[\rho] &= \hat{i}\frac{\partial[\rho]}{\partial x_q}\bigg|_t + \ldots \\ &= \hat{i}\frac{\partial t'}{\partial x_q}\bigg|_t\frac{d[\rho]}{dt'} + \ldots \\ &= \hat{i}\frac{\partial t'}{\partial x_q}\bigg|_t\frac{\partial t}{\partial t'}\bigg|_t\frac{\partial[\rho]}{\partial t}\bigg|_t + \ldots \\ &= 0 \end{aligned} \tag{6.26}$$

to be compared with the relation

$$\vec{\nabla}[\rho] = -\frac{1}{c}\frac{\partial[\rho]}{\partial t}\hat{r}' \tag{6.27}$$

given in Ref. [37]. The mathematical error leading to the incorrect equation (6.27) is a subtle one concerning the precise definitions of partial derivatives. Combining Eqs.(5.7) and (5.8) gives:

$$\frac{\partial t'}{\partial x_q}\bigg|_t = -\frac{1}{c}\frac{(x_q - x_Q)}{Kr'} \tag{6.28}$$

so that the second line of (6.26) may be written as:

$$\vec{\nabla}[\rho] = -\frac{\hat{i}}{c}\frac{(x_q - x_Q)}{Kr'}\frac{d[\rho]}{dt'} + \ldots . \tag{6.29}$$

Now it seems plausible, in view of Eq.(5.15) above to make the substitution

$$\frac{1}{K}\frac{d}{dt'} \to \frac{\partial}{\partial t} \tag{6.30}$$

in (6.29) thus yielding (6.27). But all spatial partial derivatives in (5.1) and hence in (6.26) and (6.29) are evaluated *at constant t* whereas the operator relation of (6.30) is (see Eq.(5.15)) valid only *at constant $\vec{x}_q$*, and so is inapplicable in relation to derivatives with respect to x_q, y_q or z_q.

In fact the spurious relation (6.27) was also used by Jefimenko in the original derivation of Eq.(6.13) [36]. Maxwell's equations and Eq.(6.23), called the 'Vector Identity' V-33 [38], are used to obtain (6.19) from an integral vector identity: the 'Vector wave field theorem' V-31 [38].

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[7]The ellipsis in (6.26) and subsequent equations indicates the contribution of the x- and y-components.

The term containing $\vec{\nabla}(1/r')$ in Eq.(6.24) is the same, up to a constant multiplicative factor, as one which has been previously calculated in Section 5 (Eq.(5.10)) so that:

$$-\vec{\nabla}\left(\frac{1}{r'}\right) = \frac{\hat{r}'}{K(r')^2}.$$

(6.31)

Combining (6.24), (6.25), (6.26) and (6.27) gives:

$$\vec{E}(GH1)^{ret} = \int \left[\frac{[\rho]\hat{r}' - (\hat{r}' \cdot \beta_u)[\vec{J}]}{K(r')^2} - \frac{1}{c^2 r'}\frac{\partial[\vec{J}]}{\partial t}\right] d\tau.$$

(6.32)

Note that the first term on the right side of (6.32) differs from the corresponding one in Jefimenko's formula by a factor $1/K$. This factor was missed in the calculation of Ref. [37]. In summary, the claimed-to-be-derived Jefimenko formula, Eq.(19) of Ref. [37] is missing a factor $1/K$ on the first term; the second term vanishes, and the fourth term in (6.24) (the second in (6.32)) is omitted. Indeed, only the last term of Eq.(19) of Ref. [37] is correct.

Combining (5.2) and (6.23) gives:

$$\vec{B}(GH1)^{ret} = \frac{1}{c}\int\left[\vec{\nabla}\times\frac{[\vec{J}]}{r'} - [\vec{J}]\times\vec{\nabla}\left(\frac{1}{r'}\right)\right]d\tau.$$

(6.33)

Choosing $[\vec{J}]$ parallel to the x-axis

$$\vec{\nabla}\times[\vec{J}] = -\hat{k}\left.\frac{\partial|\vec{J}|}{\partial y_q}\right|_t = -\hat{k}\left.\frac{\partial t'}{\partial y_q}\right|_t\left.\frac{\partial t}{\partial t'}\right|_t\left.\frac{\partial|\vec{J}|}{\partial t}\right|_t = 0$$

(6.34)

whereas the authors of Ref. [37] state that

$$\vec{\nabla}\times[\vec{J}] = \frac{1}{c}\frac{\partial[\vec{J}]}{\partial t}\times\hat{r}'.$$

(6.35)

This results from a similar misuse of partial derivatives to that described above in connection with Eq.(6.27). Eqs.(6.33),(6.31) and (6.34) give

$$\vec{B}(GH1)^{ret} = \int\frac{[\vec{J}]\times\hat{r}'}{cK(r')^2}d\tau$$

(6.36)

which differs from the Jefimenko equation (6.15) by an overall factor $1/K$ and the absence of the $\partial[\vec{J}]/\partial t$ term. Again the 'derivation' of the Jefimenko equation in Ref. [37] , based now on the incorrect formula (6.35), is fallacious. Jefimenko [36] also assumed this formula in order to derive the second term on the right side of (6.14).

In Section IV of Ref. [37] it is claimed to derive the LW fields of Eqs.(6.4) and (6.5) from the Jefimenko formulae. However this derivation starts not from the Jefimenko formulae (6.13) and (6.14) but instead from the different equations [8] (given here the label

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[8]Eq. (6.37)is Eq. (38) of Ref. [37]

'GH2'):

$$\vec{E}(GH2)^{ret} = \int \left[\frac{[\rho]\hat{r}'}{(r')^2} + \frac{\partial}{\partial t}\left(\frac{[\rho]\hat{r}'}{cr'} \right) - \frac{\partial}{\partial t}\left(\frac{[\vec{J}]}{c^2 r'} \right) \right] d^3 x_J, \tag{6.37}$$

$$\vec{B}(GH2)^{ret} = \int \left[\frac{[\vec{J}] \times \hat{r}'}{c(r')^2} + \frac{\partial}{\partial t}\left(\frac{[\vec{J}] \times \hat{r}'}{c^2 r'} \right) \right] d^3 x_J \tag{6.38}$$

which differ from the Jefimenko equations in that the time derivatives operate not only on the charge and current densities but also on the reciprocal of the retarded source-field point separation r' and the unit vector $\hat{r}'$.

The authors of Ref. [37] introduce into Eqs.(6.37) and (6.38) point-like non-relativistic charge and current densities according to Eq.(6.17) and (6.18). Since the integration over t' is omitted, it is then implicit in these equations that $t' = t'_Q$, where the fixed time, t'_Q, is the solution of Eq.(6.21), corresponding to a fixed position of the source charge for given values of $\vec{x}_q$ and t. It then follows that for point-like charges (6.37) and (6.38) are written as:

$$\vec{E}(GH2)^{ret} = Q \int \left[\frac{\hat{r}'}{(r')^2} + \frac{\partial}{\partial t}\left(\frac{\hat{r}'}{cr'} \right) - \frac{\partial}{\partial t}\left(\frac{\vec{\beta}_u}{cr'} \right) \right] \delta(\vec{x}_J(t'_Q) - \vec{x}_Q(t'_Q)) d^3 x_J$$

$$= Q \left[\frac{\hat{r}'}{(r')^2} + \frac{\partial}{\partial t}\left(\frac{\hat{r}'}{cr'} \right) - \frac{\partial}{\partial t}\left(\frac{[\vec{\beta}_u]}{cr'} \right) \right] \Bigg|_{t'=t'_Q}, \tag{6.39}$$

$$\vec{B}(GH2)^{ret} = Q \int \left[\frac{\vec{\beta}_u \times \hat{r}'}{(r')^2} + \frac{\partial}{\partial t}\left(\frac{\vec{\beta}_u \times \hat{r}'}{cr'} \right) \right] \delta(\vec{x}_J(t'_Q) - \vec{x}_Q(t'_Q)) d^3 x_J$$

$$= Q \left\{ \left[\frac{\vec{\beta}_u \times \hat{r}'}{(r')^2} + \frac{\partial}{\partial t}\left(\frac{\vec{\beta}_u \times \hat{r}'}{cr'} \right) \right] \right\} \Bigg|_{t'=t'_Q}. \tag{6.40}$$

However, these formulae are not the ones obtained from (6.37) and (6.38) in Ref. [37]. Instead, a change of variable is introduced into the δ-functions in the first lines of (6.39) and (6.40):

$$\vec{z}(t'_Q) \equiv \vec{x}_J(t'_Q) - \vec{x}_Q(t'_Q). \tag{6.41}$$

The Jacobian, J, relating the volume elements $d^3\vec{z}$ and $d^3\vec{x}_J$ according to

$$d^3\vec{z} = Jd^3\vec{x}_J \tag{6.42}$$

is introduced. It is then stated, without derivation, that $J = K$ where K is Jacobian relating dt to dt', as given by Eqs.(5.15) and (6.3) above. The x-component of (6.41) is

$$z_x(t'_Q) = x_J(t'_Q) - x_Q(t'_Q). \tag{6.43}$$

Since $x_Q(t'_Q)$ is constant it follows from (6.49) that $dz_x = dx_J$. Similarly. $dz_y = dy_J$ and $dz_z = dz_J$ so that the Jacobian J in (6.42) is unity, not K, as claimed in Ref. [37].

Since the authors of Ref. [37] nevertheless *did* insert a factor $1/K$ multiplying the δ-functions in the first lines of (6.39) and (6.40), before performing the spatial integrations,

the equations obtained were not those in the last lines of (6.39) and (6.40) but instead:

$$\vec{E}(\text{GH2})^{ret}_{J=K} = Q\left[\frac{\hat{r}'}{K(r')^2} + \frac{\partial}{\partial t}\left(\frac{\hat{r}'}{cKr'}\right) - \frac{\partial}{\partial t}\left(\frac{[\vec{\beta}_u]}{cKr'}\right)\right]\Bigg|_{t'=t'_Q}, \qquad (6.44)$$

$$\vec{B}(\text{GH2})^{ret}_{J=K} = Q\left\{\left[\frac{\vec{\beta}_u \times \hat{r}'}{K(r')^2} + \frac{\partial}{\partial t}\left(\frac{\vec{\beta}_u \times \hat{r}'}{cKr'}\right)\right]\right\}\Bigg|_{t'=t'_Q} \qquad (6.45)$$

where the subscript '$J = K$' distinguishes these equations from the formally correct ones (6.39) and (6.40), i.e. the correctly calculated point-like charge versions of the general equations (6.37) and (6.38), claimed to be the Jefimenko equations but actually given, without derivation, in Ref. [37].

After writing (6.44) and (6.45) (the equivalents, in gaussian units, of the MKS Eqs.(41) and (42) of Ref [37]) it is stated that they are 'essentially in the form made famous by Feynman'. In fact Eq. (6.44) is equivalent to Eq. (19) of Ref. [35] on transforming the t' derivatives in the latter into the t derivatives of the former. It is correctly stated, but not demonstrated, in Ref. [35] that their Eq. (19) is the same as the LW retarded electric field.

The introduction of the factor $(1/K)$ inside the time derivatives in (6.44) and (6.45) is equivalent to replacing the retarded potentials in (6.22) and (6.23) by the LW potentials. It is shown in Ref [37] that (6.44) is equivalent to the LW electric field of Eq. (6.6) and stated (without proof) that the magnetic field is given by the relation $\vec{B}(\text{LW})^{ret} = \hat{r}' \times \vec{E}(\text{LW})^{ret}$.

Summarising the results obtained so far in this section: Ref. [35] does demonstrate the consistency of the Feynman and LW formulae for retarded electric fields. The 'derivation' of the Jefimenko formulae from the defining equations (5.1) and (5.2) of the electric and magnetic fields and the retarded potentials (6.22) and (6.23) given in Ref. [37] is erroneous due to mathematical misinterpretation of spatial partial derivatives. The same remark applies to Jefimenko's original derivation [36] of these equations. The Eqs.(6.44) and (6.45) given in Ref. [37] are not the Jefimenko equations but are obtained from them by introducing an overall multiplicative factor $1/K$ in each term and allowing the time derivatives to act on all factors in the terms of the equation instead of uniquely on the charge and current densities as in the Jefimenko formulae. This is tantamount to replacing the potentials of (6.22) and (6.23) by the retarded LW potentials of Eq. (2.28). Eq.(6.44) does give the LW field of (6.6), as claimed in Ref. [37].

It was pointed out in Ref. [39] that an equation for the magnetic field identical to the Jefimenko equation (6.14) and a formula equivalent to the Jefimenko electric field, (6.13), had been given earlier in the second edition of the book 'Classical Electricity and Magnetism' by Panofsky and Phillips [40]. The equivalent electric field formula is:

$$\vec{E}(\text{PP})^{ret} = \int\left\{\frac{\hat{r}'[\rho]}{(r')^2} + \frac{([\vec{J}]\cdot\hat{r}')\hat{n} + ([\vec{J}]\times\hat{r}')\times\hat{r}'}{c(r')^2} + \frac{[\dot{\vec{J}}]\times\hat{r}')\times\hat{r}'}{c^2r'}\right\}d^3x_{\text{J}}. \qquad (6.46)$$

A calculation claiming to show the equivalence of (6.13) and (6.46) was given in Section II of Ref. [39] The first step was to repeat the erroneous derivation of (6.13) from the

defining equation (5.1) of the electric field and the non-relativistic potentials (6.22) and (6.23), previously given in Ref. [37] and discussed above. The term proportional to $\partial[\rho]/\partial t$ is manipulated to obtain Eq.(6.46). As shown above, this term actually vanishes.

7   Summary

Retarded potentials are derived from homogeneous d'Alembert equations for electromagnetic potentials and the Lorenz condition. The potentials so-obtained in Eq.(2.12) differ from the LW potentials of CEM. It is shown that the incorrect LW potentials result from neglect of the dependence of the effective density of a moving charge distribution on its speed. This point is made particularly clear by the careful re-examination of Feynman's derivation of the LW potentials presented in Section 3. In Section 4, several 'relativistic' derivations of the LW potentials or the corresponding retarded fields given in text books are reviewed. It is shown that they all contain misapplications of special relativity –in particular by invoking a spurious 'length contraction' effect. In all of the relativistic derivations, retardation effects are neglected, whereas in the original 19th Century derivations of the LW potentials or the corresponding retarded fields, no relativistic effects are considered. There are therefore two independent, logically incompatible, and incorrect, derivations of retarded potentials and their associated fields. In Section 5 the retarded RCED fields of a uniformly moving charge are derived and expressed in 'present time' form. Except for an overall multiplicative factor $1/(1 - \hat{r}' \cdot \vec{\beta}_u)$ and the retarded time argument, they are the same as the instantaneous force fields of RCED [1]. In Section 6 the consistency claimed in the pedagogical literature between various different formulae for the fields of an accelerated charge (LW, Feynman, Jefimenko) is considered. The Feynman formula for the retarded electric field of a charge in arbitary motion is (as previously shown in Ref. [35]) consistent with the LW field. The electric field of a uniformly moving charge given by the Jefimenko formula is found to be, unlike CEM and RCED fields, Coulombic.

The considerations of the present paper are of a primarily mathematical nature. The physical interpretation of retarded (radiation) and instantaneous (force) fields in RCED has been discussed in some detail previously [1, 9].

Appendix A

In this appendix, retarded electric and magnetic fields are derived from the LW potentials as well as from the equations (6.44) and (6.45) equivalent to those given in Ref [37] and claimed there to be the same as the LW fields. To derive the LW fields the potentials

$$A_0(\text{LW})^{ret} = \left. \frac{Q}{Kr'} \right|_{t'=t'_Q}, \tag{A.1}$$

$$\vec{A}(\text{LW})^{ret} = \left. \frac{\vec{\beta}_u}{Kr'} \right|_{t'=t'_Q} \tag{A.2}$$

where $K \equiv (1 - \hat{r}' \cdot \vec{\beta}_u)$, are substituted into the defining equations (5.1) and (5.2) of electric and magnetic fields to give

$$\vec{E}(\text{LW})^{ret} = -\vec{\nabla}A_0(\text{LW})^{ret} - \frac{1}{c}\frac{\partial \vec{A}(\text{LW})^{ret}}{\partial t}, \tag{A.3}$$

$$\vec{B}(\text{LW})^{ret} = \vec{\nabla} \times \vec{A}(\text{LW})^{ret}. \tag{A.4}$$

For simplicity, all labels, superscripts and subscripts on the fields and potentials are omitted in the following.

Taking into account, by the chain rule, the contribution to the fields of each factor in the potentials, (A.3) and (A.4) give:

$$\vec{E} = -\frac{Q}{K}\vec{\nabla}\left(\frac{1}{r'}\right) - \frac{Q}{r'}\vec{\nabla}\left(\frac{1}{K}\right) - \frac{Q\vec{\beta}_u}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) - \frac{Q\vec{\beta}_u}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) - \frac{Q}{cKr'}\frac{\partial \vec{\beta}_u}{\partial t}, \tag{A.5}$$

$$\vec{B} = -\frac{Q}{K}\vec{\beta}_u \times \vec{\nabla}\left(\frac{1}{r'}\right) - \frac{Q}{r'}\vec{\beta}_u \times \vec{\nabla}\left(\frac{1}{K}\right) + \frac{Q}{Kr'}(\vec{\nabla} \times \vec{\beta}_u). \tag{A.6}$$

In these and the following equations it is understood that all spatial partial derivatives hold t constant and all temporal partial deivatives hold $\vec{x}_q$, the field position, constant. The derivatives in the successive terms on the right sides of these equations are now evaluated.

The first term on the right side of (A.5) gives:

$$-\frac{Q}{K}\vec{\nabla}\left(\frac{1}{r'}\right) = -\frac{\hat{\imath}Q}{K}\frac{\partial}{\partial x_q}\left(\frac{1}{r'}\right) + \dots.$$

$$= \frac{\hat{\imath}Q}{K(r')^2}\frac{\partial r'}{\partial x_q} + \dots.$$

$$= \frac{\hat{\imath}Q(x_q - x_Q)}{K^2(r')^3} + \dots.$$

$$= \frac{\hat{r}'}{K^2(r')^2} \tag{A.7}$$

where Eq.(5.8) has been used.

Considering the second term on the right side of (A.5),

$$-\frac{Q}{r'}\vec{\nabla}\left(\frac{1}{K}\right) = -\frac{\hat{\imath}Q}{r'K^2}\frac{\partial}{\partial x_q}(\hat{r}' \cdot \vec{\beta}_u) + \dots$$

$$= -\frac{\hat{\imath}Q}{r'K^2}\left[\vec{\beta}_u \cdot \frac{\partial \hat{r}'}{\partial x_q} + \hat{r}' \cdot \frac{\partial \vec{\beta}_u}{\partial x_q}\right] + \ldots$$

$$= -\frac{\hat{\imath}Q}{r'K^2}\left[\vec{\beta}_u \cdot \frac{\partial \hat{r}'}{\partial x_q} + \frac{\partial t'}{\partial x_q}(\hat{r}'\cdot\dot{\vec{\beta}}_u)\right] + \ldots . \qquad (A.8)$$

Now,

$$\frac{\partial \hat{r}'}{\partial x_q} = \frac{\partial}{\partial x_q}\left(\frac{\vec{r}'}{r'}\right) = \frac{1}{r'}\frac{\partial \vec{r}'}{\partial x_q} - \frac{\vec{r}'}{(r')^2}\frac{\partial r'}{\partial x_q}$$

$$= \frac{\hat{\imath}}{r'}\left(1 - \frac{dx_Q}{dt'}\frac{\partial t'}{\partial x_q}\right) - \frac{\hat{r}'}{r'}\frac{\partial r'}{\partial x_q}$$

$$= \frac{\hat{\imath}}{r'} + \frac{\hat{\imath}\beta_u - \hat{r}'}{r'}\frac{\partial r'}{\partial x_q}$$

$$= \frac{\hat{\imath}}{r'} + \frac{(\hat{\imath}\beta_u - \hat{r}')(x_q - x_Q)}{K(r')^2} \qquad (A.9)$$

where Eqs.(5.7) and (5.8) have been used, as well as the assumption that $\vec{u}$ is parallel to the x-axis. Substituting (A.9) in (A.8) and again using Eqs.(5.7) and (5.8) gives:

$$-\frac{Q}{r'}\vec{\nabla}\left(\frac{1}{K}\right) = -\frac{Q\vec{\beta}_u}{K^2(r')^2} - Q\hat{\imath}\frac{[\beta_u^2 - (\hat{r}' \cdot \vec{\beta}_u)]}{K^3(r')^2}\frac{(x_q - x_Q)}{r'} + Q\hat{\imath}\frac{(\hat{r}' \cdot \dot{\vec{\beta}}_u)]}{cK^3 r'}\frac{(x_q - x_Q)}{r'} + \ldots$$

$$= -\frac{Q\vec{\beta}_u}{K^2(r')^2} - Q\hat{r}'\frac{[\beta_u^2 - (\hat{r}' \cdot \vec{\beta}_u)]}{K^3(r')^2} + Q\hat{r}'\frac{(\hat{r}' \cdot \dot{\vec{\beta}}_u)]}{cK^3 r'} + \ldots . \qquad (A.10)$$

The third term on the right side of (A.5) gives

$$-\frac{Q\vec{\beta}_u}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) = -\frac{Q\vec{\beta}_u}{cK}\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left(\frac{1}{r'}\right)$$

$$= \frac{Q\vec{\beta}_u}{cK^2(r')^2}\frac{\partial r'}{\partial t'}$$

$$= -\frac{Q\vec{\beta}_u(\hat{r}' \cdot \vec{\beta}_u)}{K^2(r')^2} \qquad (A.11)$$

where (5.15) and (5.13) have been used. The fourth term on the right side of (A.5) gives

$$-\frac{Q\vec{\beta}_u}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) = -\frac{Q\vec{\beta}_u}{cr'}\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left(\frac{1}{K}\right) \qquad (A.12)$$

But

$$\frac{\partial}{\partial t'}\left(\frac{1}{K}\right) = \frac{\partial}{\partial t'}\left(\frac{1}{1-(\hat{r}' \cdot \vec{\beta}_u)}\right) = \frac{1}{K^2}\frac{\partial(\hat{r}' \cdot \vec{\beta}_u)}{\partial t'} = \frac{1}{K^2}\left[\vec{\beta}_u \cdot \frac{\partial \hat{r}'}{\partial t'} + \hat{r}' \cdot \dot{\vec{\beta}}_u\right] \qquad (A.13)$$

and also

$$\frac{\partial \hat{r}'}{\partial t'} = \frac{\partial}{\partial t'}\left(\frac{\vec{r}'}{r'}\right) = -c\frac{\vec{\beta}_u}{r'} - \frac{\vec{r}'}{(r')^2}\frac{\partial r'}{\partial t'} = c\frac{[\hat{r}'(\hat{r}' \cdot \vec{\beta}_u) - \vec{\beta}_u]}{r'} \qquad (A.14)$$

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where (5.13) has been used. (A.12)-(A.14) together with (5.15) then give

$$-\frac{Q\vec{\beta}_u}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) = -\frac{Q\vec{\beta}_u}{K^3}\left\{\frac{(\hat{r}'\cdot\vec{\beta}_u)^2-\beta_u^2)}{(r')^2}+\frac{\hat{r}'\cdot\dot{\vec{\beta}}_u}{cr'}\right\}. \tag{A.15}$$

Collecting together (A.7), (A.10), (A.11) and (A.15) gives, for the electric field derived from the LW potentials:

$$\begin{aligned}
\vec{E} &= \frac{Q}{K^2(r')^2}\left\{\hat{r}'-\vec{\beta}_u[1+(\hat{r}'\cdot\vec{\beta}_u)]-\frac{\vec{\beta}_u[(\hat{r}'\cdot\vec{\beta}_u)^2-\beta_u^2]+\hat{r}'[\hat{r}'\cdot\vec{\beta}_u-\beta_u^2]}{K}\right\} \\
&\quad -\frac{Q}{K^2cr'}\left[\dot{\vec{\beta}}_u+\frac{(\vec{\beta}_u-\hat{r}')\hat{r}'\cdot\dot{\vec{\beta}}_u}{K}\right] \\
&= \frac{Q}{K^3}\left[\frac{\hat{r}'-\vec{\beta}_u}{\gamma_u^2(r')^2}+\frac{\hat{r}'\times[(\hat{r}'-\beta_u)\times\dot{\vec{\beta}}_u]}{cr'}\right]. \tag{A.16}
\end{aligned}$$

The retarded LW magnetic field given by (A.6) is now calculated.

Since $\vec{u}$ is assumed to be parallel to the x-axis, it follows that

$$\begin{aligned}
-\frac{Q}{K}\vec{\beta}_u\times\vec{\nabla}\left(\frac{1}{r'}\right) &= -\frac{Q\hat{k}\beta_u}{K}\frac{\partial}{\partial y_q}\left(\frac{1}{r'}\right) \\
&= \frac{Q\hat{k}\beta_u}{K(r')^2}\frac{\partial r'}{\partial y_q} = \frac{Q\hat{k}\beta_u y_q}{K^2(r')^3} \\
&= \frac{Q(\vec{\beta}_u\times\hat{r}')}{K^2(r')^2}. \tag{A.17}
\end{aligned}$$

Similarly

$$\begin{aligned}
-\frac{Q}{K}\vec{\beta}_u\times\vec{\nabla}\left(\frac{1}{K}\right) &= -\frac{Q\hat{k}\beta_u}{K}\frac{\partial}{\partial y_q}\left(\frac{1}{K}\right) \\
&= -\frac{Q\hat{k}\beta_u}{K^2r'}\frac{\partial(\hat{r}'\cdot\vec{\beta}_u)}{\partial y_q} \\
&= -\frac{Q\hat{k}\beta_u}{K^2r'}\left(\hat{r}'\cdot\frac{\partial\vec{\beta}_u}{\partial y_q}+\vec{\beta}_u\cdot\frac{\partial\hat{r}'}{\partial y_q}\right). \tag{A.18}
\end{aligned}$$

Evaluating the first term in brackets on the right side of Eq.(A.18),

$$\hat{r}'\cdot\frac{\partial\vec{\beta}_u}{\partial y_q} = \frac{\partial t'}{\partial y_q}(\hat{r}'\cdot\dot{\vec{\beta}}_u) = -\frac{y_q(\hat{r}'\cdot\dot{\vec{\beta}}_u)}{cKr'} \tag{A.19}$$

where the relation
$$\frac{\partial t'}{\partial y_q} = -\frac{1}{c}\frac{\partial r'}{\partial y_q} \tag{A.20}$$

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given by differentiating the retardation condition $t' = t - r'/c$ as well as Eq.(5.18) have been used. The second tern in brackets on the right side of(A.18) is

$$\vec{\beta}_u \cdot \frac{\partial \hat{r}'}{\partial y_q} = \vec{\beta}_u \cdot \frac{\partial}{\partial y_q}\left(\frac{\vec{r}'}{r'}\right) = \vec{\beta}_u \cdot \left(\frac{1}{r'}\frac{\partial \vec{r}'}{\partial y_q} - \frac{\vec{r}'}{(r')^2}\frac{\partial r'}{\partial y_q}\right). \tag{A.21}$$

Assuming, without loss of generality, that the vector $\vec{r}'$ is confined to the $x - y$ plane,

$$\frac{\partial \vec{r}'}{\partial y_q} = -\hat{\imath}\frac{dx_Q}{dt'}\frac{\partial t'}{\partial y_q} + \hat{\jmath}. \tag{A.22}$$

Combining (A.21) and (A.22), again using (A.20) and (5.18), gives

$$\vec{\beta}_u \cdot \frac{\partial \hat{r}'}{\partial y_q} = \frac{[\beta_u^2 - \hat{r}' \cdot \vec{\beta}_u]y_q}{K(r')^2}. \tag{A.23}$$

Combining A(18), (A.19) and (A.23),

$$-\frac{Q}{K}\vec{\beta}_u \times \vec{\nabla}\left(\frac{1}{K}\right) = \frac{Q\vec{\beta}_u \times \hat{r}'}{K^3}\left[\frac{[\hat{r}' \cdot \vec{\beta}_u - \beta_u^2]}{(r')^2} + \frac{\hat{r}' \cdot \dot{\vec{\beta}}_u}{cr'}\right] \tag{A.24}$$

The third term on the right side of (A.6) is

$$\begin{aligned}\frac{Q}{Kr'}(\vec{\nabla} \times \vec{\beta}_u) &= -\frac{Q\hat{k}}{Kr'}\frac{\partial \beta_u}{\partial y_q} = -\frac{Q\hat{k}}{Kr'}\frac{\partial t'}{\partial y_q}\dot{\beta}_u \\ &= \frac{Q\hat{k}}{cKr'}\frac{\partial r'}{\partial y_q}\dot{\beta}_u = \frac{Q(\vec{\beta}_u \times \hat{r}')}{cK^2r'\beta_u}\dot{\beta}_u\end{aligned} \tag{A.25}$$

where (A.20) and (5.18) have been used.

Collecting together (A.17), (A.24) and (A.25), the magnetic field generated by the LW potentials is:

$$\begin{aligned}\vec{B} &= \left\{\frac{Q(\vec{\beta}_u \times \hat{r}')}{K^3}\left[\frac{K + \hat{r}' \cdot \vec{\beta}_u - \beta_u^2}{(r')^2} + \frac{K\dot{\beta}_u + \beta_u(\hat{r}' \cdot \dot{\vec{\beta}}_u)}{cr'\beta_u}\right]\right\}\Bigg|_{t'=t'_Q}, \\ &= \left\{\frac{Q(\vec{\beta}_u \times \hat{r}')}{K^3}\left[\frac{1}{\gamma_u^2(r')^2} + \frac{(\dot{\beta}_u(1 - \hat{r}' \cdot \vec{\beta}_u) + \beta_u(\hat{r}' \cdot \dot{\vec{\beta}}_u)}{cr'\beta_u}\right]\right\}\Bigg|_{t'=t'_Q}. \end{aligned} \tag{A.26}$$

(A.16) and (A.26) are the formulae (6.6) and (6.7) of the text.

The consistency of the fields of Eqs.(6.44) and (6.45) with the LW fields of (6.6) and (6.7) claimed in Ref. [37] is now investigated. The equations analogous to (A.5) and (A.6) given by using the chain rule to expand the derivatives in (6.44) and (6.45) are:

$$\begin{aligned}\vec{E} &= \frac{Q\hat{r}'}{K(r')^2} + \frac{Q\hat{r}'}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) + \frac{Q\hat{r}'}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) + \frac{Q}{cKr'}\frac{\partial \hat{r}'}{\partial t} \\ &\quad -\frac{Q\vec{\beta}_u}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) - \frac{Q\vec{\beta}_u}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) - \frac{Q}{cKr'}\frac{\partial \vec{\beta}_u}{\partial t}, \end{aligned} \tag{A.27}$$

$$\begin{aligned}\vec{B} &= \frac{Q(\vec{\beta}_u \times \hat{r}')}{K(r')^2} + \frac{Q(\vec{\beta}_u \times \hat{r}')}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) + \frac{Q(\vec{\beta}_u \times \hat{r}')}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) \\ &\quad + \frac{Q}{cKr'}\left(\vec{\beta}_u \times \frac{\partial \hat{r}'}{\partial t}\right) - \frac{Q}{cKr'}\left(\hat{r}' \times \frac{\partial \vec{\beta}_u}{\partial t}\right). \end{aligned} \tag{A.28}$$

. Comparison with (A.5) and (A.6) shows that the last three terms in (A.5) and (A.27) (originating from the time derivative in (A.3)) are the same but all other terms in (A.27) and (A.28) differ from those in (A.5) and (A.6). Thus to compare (A.5) and (A.27) the derivatives in the second. third and fourth terms on the right of (A.27) must be evaluated, while to compare (A.6) and (A.28) all derivatives on the right side of (A.28) must be evaluated. This is readily done using, *mutatis mutandis*, the formulae obtained above.

The second term in (A.27) is

$$\frac{Q\hat{r}'}{cr'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) = \frac{Q\hat{r}'}{cKr'}\frac{\partial}{\partial t'}\left(\frac{1}{K}\right) = \frac{Q\hat{r}'}{K^3}\left\{\frac{(\hat{r}'\cdot\vec{\beta}_u)^2 - \beta_u^2)}{(r')^2} + \frac{\hat{r}'\cdot\dot{\vec{\beta}}_u}{cr'}\right\} \tag{A.29}$$

by analogy with Eq.(A.15).

The third term is

$$\frac{Q\hat{r}'}{cK}\frac{\partial}{\partial t}\left(\frac{1}{r'}\right) = \frac{Q\hat{r}'}{cK^2}\frac{\partial}{\partial t'}\left(\frac{1}{r'}\right) = -\frac{Q\hat{r}'}{cK^2(r')^2}\frac{\partial r'}{\partial t'} = \frac{Q\hat{r}'(\hat{r}'\cdot\vec{\beta}_u)}{K^2(r')^2} \tag{A.30}$$

where (5.15) and (5.13) have been used

The fourth term is

$$\frac{Q}{cKr'}\frac{\partial\hat{r}'}{\partial t} = \frac{Q}{cK^2r'}\frac{\partial\hat{r}'}{\partial t'} = \frac{Q[\hat{r}'(\hat{r}'\cdot\vec{\beta}_u) - \vec{\beta}_u]}{K^2(r')^2}. \tag{A.31}$$

Substituting (A.29)-(A.31) into (A.22) as well as the previously obtained terms, and performing some algebraic simplification, gives

$$\vec{E} = \frac{Q}{K^3}\left[\frac{\hat{r}' - \vec{\beta}_u}{\gamma_u^2(r')^2} + \frac{\hat{r}' \times [(\hat{r}' - \beta_u) \times \dot{\vec{\beta}}_u]}{cr'}\right]. \tag{A.32}$$

Which is the LW field of Eq.(A.16)

Noting that the first three terms of (A.28) differ from those of (A.27) by the replacement $\hat{r}' \rightarrow \vec{\beta}_u \times \hat{r}'$ and using the above results for the latter terms, gives, after algebraic simplification, the LW magnetic field of Eq. (6.7).

Appendix B

For clarity the total time derivatives in (6.6) are replaced the corresponding partial derivatives with respect to the present time, t, for a fixed value of the field point position $\vec{x}_q$.

The second term on the right side of (6.10) contains the derivative:

$$\frac{\partial}{\partial t}\left(\frac{\hat{r}'}{(r')^2}\right) = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left(\frac{\vec{r}'}{(r')^3}\right) = \frac{\partial t'}{\partial t}\left[\frac{1}{(r')^3}\frac{\partial \vec{r}'}{\partial t'} - \frac{3\vec{r}'}{(r')^4}\frac{\partial r'}{\partial t'}\right]. \tag{B.1}$$

Since the vector $\vec{r}'$ is confined to the x-y plane,

$$\frac{\partial \vec{r}'}{\partial t'} = \hat{\imath}\frac{\partial(x_q - x_Q)}{\partial t'} + \hat{\jmath}\frac{\partial y_q}{\partial t'} = -c\vec{\beta}_u \tag{B.2}$$

since $\partial y_q/\partial t' = 0$ and $c\beta_u = dx_Q/dt$. (5.13), (5.15), (B.1) and (B.2) give:

$$\frac{r'}{c}\frac{\partial}{\partial t}\left(\frac{\hat{r}'}{(r')^2}\right) = \frac{1}{1 - \hat{r}'\cdot\vec{\beta}_u}\left[\frac{3\hat{r}'(\hat{r}'\cdot\vec{\beta}_u) - \vec{\beta}_u}{(r')^2}\right]. \tag{B.3}$$

Considering now the last term on the right side of (6.10):

$$\frac{\partial^2 \hat{r}'}{\partial t^2} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left[\frac{\partial t'}{\partial t}\frac{\partial \hat{r}'}{\partial t'}\right] = \frac{\partial t'}{\partial t}\left[\frac{\partial \hat{r}'}{\partial t'}\frac{\partial}{\partial t'}\left(\frac{\partial t'}{\partial t}\right) + \frac{\partial t'}{\partial t}\frac{\partial^2 \hat{r}'}{\partial t'^2}\right] \tag{B.4}$$

where

$$
\begin{aligned}
\frac{\partial}{\partial t'}\left(\frac{\partial t'}{\partial t}\right) &= \frac{\partial}{\partial t'}\left(\frac{1}{1 - \hat{r}'\cdot\vec{\beta}_u}\right) = \frac{1}{(1 - \hat{r}'\cdot\vec{\beta}_u)^2}\frac{\partial(\hat{r}'\cdot\vec{\beta}_u)}{\partial t'} \\
&= \frac{1}{(1 - \hat{r}'\cdot\vec{\beta}_u)^2}\left[c\frac{[(\hat{r}'\cdot\vec{\beta}_u)^2 - \beta_u^2]}{r'} + (\hat{r}'\cdot\dot{\vec{\beta}}_u)\right]
\end{aligned} \tag{B.5}
$$

where (A.14) has been used. It also follows from (A.14) that

$$
\begin{aligned}
\frac{\partial^2 \hat{r}'}{\partial t'^2} &= -\frac{c[\hat{r}'(\hat{r}'\cdot\vec{\beta}_u) - \vec{\beta}_u]}{(r')^2}\frac{\partial r'}{\partial t'} + \frac{c}{r'}\left[\left(\frac{\partial \hat{r}'}{\partial t'}\right)(\hat{r}'\cdot\vec{\beta}_u) + \hat{r}'\left(\frac{\partial \hat{r}'}{\partial t'}\right)\cdot\vec{\beta}_u + \hat{r}'(\hat{r}'\cdot\dot{\vec{\beta}}_u) - \dot{\vec{\beta}}_u\right] \\
&= c^2\left[\frac{\hat{r}'[3(\hat{r}'\cdot\vec{\beta}_u)^2 - \beta_u^2] - 2\vec{\beta}_u(\hat{r}'\cdot\vec{\beta}_u)}{(r')^2}\right] + \frac{c[\hat{r}'(\hat{r}'\cdot\dot{\vec{\beta}}_u) - \dot{\vec{\beta}}_u]}{r'}.
\end{aligned} \tag{B.6}
$$

Combining (5.15) and (A.14),(B.5) and (B.6) then gives

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 \hat{r}'}{\partial t^2} &= \frac{1}{(1 - \hat{r}'\cdot\vec{\beta}_u)}\left\{\frac{[\hat{r}'(\hat{r}'\cdot\vec{\beta}_u) - \vec{\beta}_u]}{(1 - \hat{r}'\cdot\vec{\beta}_u)^2}\left[\frac{\hat{r}'(\hat{r}'\cdot\vec{\beta}_u)^2 - \beta_u^2}{(r')^2} + \frac{\hat{r}'\cdot\dot{\vec{\beta}}_u}{cr'}\right]\right. \\
&\left. + \frac{1}{(1 - \hat{r}'\cdot\vec{\beta}_u)}\left[\frac{\hat{r}'[3(\hat{r}'\cdot\vec{\beta}_u)^2 - \beta_u^2] - 2\vec{\beta}_u(\hat{r}'\cdot\vec{\beta}_u)}{(r')^2} + \frac{\hat{r}'(\hat{r}'\cdot\dot{\vec{\beta}}_u) - \dot{\vec{\beta}}_u}{cr'}\right]\right\} \tag{B.7}
\end{aligned}
$$

Inserting (B.3) and (B.7) into (6.10) yields Eq.(6.12) of the text.

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