

Schwarzschild Tests of the Wahlquist-Estabrook-Buchman-Bardeen Tetrad Formulation for Numerical Relativity

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A first order symmetric hyperbolic tetrad formulation of the Einstein equations developed by Estabrook and Wahlquist and put into a form suitable for numerical relativity by Buchman and Bardeen (the WEBB formulation) is adapted to explicit spherical symmetry and tested for accuracy and stability in the evolution of spherically symmetric black holes (the Schwarzschild geometry). The lapse and shift, which specify the evolution of the coordinates relative to the tetrad congruence, are reset at frequent time intervals to keep the constant-time hypersurfaces nearly orthogonal to the tetrad congruence and the spatial coordinate satisfying a kind of minimal rate of strain condition. By arranging through initial conditions that the constant-time hypersurfaces are asymptotically hyperbolic, we simplify the boundary value problem and improve stability of the evolution. Results are obtained for both tetrad gauges (“Nester” and “Lorentz”) of the WEBB formalism using finite difference numerical methods. We are able to obtain stable unconstrained evolution with the Nester gauge for certain initial conditions, but not with the Lorentz gauge.

I. INTRODUCTION

An orthonormal tetrad approach to numerical relativity has several attractive features. The metric is trivial, and most of the variables are coordinate scalars, which eliminates derivatives of the shift vector from most of the equations. There are only twenty-four connection coefficient variables in general, the Ricci rotation coefficients, as opposed to the forty connection coefficients in a metric-based formulation. While one does have to evolve the tetrad vectors in place of the metric, one does not have to deal with nonlinearities in the equations associated with the inverse metric. A number of tetrad and triad formulations for general relativity have been proposed [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], but they have not been as widely used in numerical relativity as standard 3+1 metric-based formulations. This paper describes the numerical implementation and tests of one such scheme, the WEBB formulation [15], in spherically symmetric vacuum black hole spacetimes. See Estabrook’s papers [18, 19] for an in-depth analysis of the mathematical structure of these equations (without specified gauge conditions).

The WEBB scheme takes as its primary variables the tetrad connection coefficients, and incorporates one of two alternative dynamic gauge conditions to evolve the acceleration and angular velocity of the tetrads, either the Nester gauge [20] or the Lorentz gauge [7]. The evolution equations constitute a first-order symmetrizable hyperbolic system in which all variables propagate either along the light cone or along the tetrad congruence, with the tetrad gauge information propagating along the light cone. Both the Nester and Lorentz gauges do not in general preserve hypersurface orthogonality of the tetrad congruence. The coordinate evolution is controlled by a tetrad lapse function and shift vector which we do not

evolve dynamically, but rather reset periodically to keep the constant-time hypersurfaces nearly (but not exactly) orthogonal to the congruence worldlines, and to maintain a minimal deformation condition on the spatial coordinates which is similar to the minimal strain condition of Smarr and York [21] in the 3+1 context.

Spherically symmetric spacetimes, while in a sense trivial in that they do not allow any gravitational waves to be present, provide the challenge of maintaining a stable numerical evolution in the presence of an event horizon. Additionally, they require dealing with both an excision inner boundary and an outer boundary. In our code, the numerical grid extends from just inside the event horizon to around $R = 20 M$, where R is the circumferential radius of a two-sphere. Our initial slices are constructed so that the congruence worldlines point out of the grid at both boundaries. This forces all of the eigenmodes to propagate out of the grid at the inner boundary, and the eigenmodes which travel along the congruence worldlines, as well those which travel along the outgoing light cones, to propagate out of the grid at the outer boundary. Even so, there are two eigenmodes (a “constraint” mode and a gauge mode) traveling at the speed of light into the grid at the outer boundary. Boundary conditions for the “constraint” mode are determined according to constraint-preserving boundary conditions [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], which insure that the information entering the numerical grid is consistent with the constraint equations. There is no physical constraint on the incoming gauge mode; we choose to keep its amplitude fixed as set in the initial conditions.

This paper is one of a relatively small number of numerical tests of tetrad/triad formulations in vacuum general relativity (see [16, 35, 36, 37, 38, 39]). We find that it is possible to achieve reasonable long term stability

evolving the spherically symmetric Schwarzschild geometry using the WEBB equations with the Nester gauge, but only for rather special initial conditions. Many other equation formulations and tetrad gauge conditions are possible in the context of orthonormal frames and remain to be explored. Tests in the limited context of spherical symmetry and one-dimension are far from sufficient to establish the viability of a particular formulation, but can serve to establish lack of viability.

II. VARIABLES

The application of the three-dimensional WEBB formalism presented in [15] to one-dimensional Schwarzschild black holes requires the construction of an orthonormal tetrad field well-behaved everywhere outside and in the vicinity of the event horizon. The most obvious basis vectors for a spherically symmetric spacetime consist of a timelike vector field \mathbf{e}_0 and a spacelike vector field $\mathbf{e}_{\hat{r}}$, both orthogonal to the two-spheres generated by the symmetry, and two spacelike vector fields tangent to the two-spheres. The timelike vector field defines a timelike congruence of worldlines orthogonal to the two-spheres. The problem is that orthonormal vector fields tangent to a two-sphere cannot continuously be defined everywhere on the two-sphere. If these tangent vectors are chosen to be aligned with angular polar coordinates, $\mathbf{e}_{\hat{\theta}}$ and $\mathbf{e}_{\hat{\phi}}$, they are degenerate at the poles, $\theta = 0$ and $\theta = \pi$. These “spherical” spacelike basis vectors are unsuitable for direct use as the tetrad vectors in the WEBB tetrad formalism. There would be singularities in some of the connection coefficients (Ricci rotation coefficients) at the poles. Instead, we must define a “Cartesian” triad of spacelike vectors which is rotated from the spherical triad. A simple way to do this is to invoke the same rotation as a function of the polar angles that takes spherical to Cartesian basis vectors in flat space, namely:

$$\mathbf{e}_1 = \cos \phi \sin \theta \mathbf{e}_{\hat{r}} + \cos \phi \cos \theta \mathbf{e}_{\hat{\theta}} - \sin \phi \mathbf{e}_{\hat{\phi}}, \quad (1)$$

$$\mathbf{e}_2 = \sin \phi \sin \theta \mathbf{e}_{\hat{r}} + \sin \phi \cos \theta \mathbf{e}_{\hat{\theta}} + \cos \phi \mathbf{e}_{\hat{\phi}}, \quad (2)$$

$$\mathbf{e}_3 = \cos \theta \mathbf{e}_{\hat{r}} - \sin \theta \mathbf{e}_{\hat{\theta}}. \quad (3)$$

While the Cartesian tetrad must be used to define the WEBB variables, once these are defined it is convenient to rotate back to the spherical triad to take explicit advantage of the spherical symmetry.

As in any Cauchy formulation for numerical relativity, the evolution of the spacetime is described by a sequence of spacelike hypersurfaces. Since the state of the system is specified on such a constant-time hypersurface, spatial derivatives must be evaluated at constant time.

However, the spatial triad vectors are not in general tangent to the constant-time hypersurface. As described in [15], we decompose the spatial triad vectors into a timelike component parallel to the congruence and a spacelike component tangent to the hypersurface:

$$\mathbf{e}_a = A_a \mathbf{e}_0 + B_a^k \frac{\partial}{\partial x^k}. \quad (4)$$

The three-vector \mathbf{B}_a is not a unit vector if A_a is not zero. The vectors $\mathbf{e}_{\hat{\theta}}$ and $\mathbf{e}_{\hat{\phi}}$ are tangent to the hypersurfaces, so $A_{\hat{\theta}} = A_{\hat{\phi}} = 0$. Thus, A_a only has one degree of freedom, in the $\mathbf{e}_{\hat{r}}$ direction. $A_{\hat{r}}$ is the radial 3-velocity of a tetrad observer with respect to an observer at rest in the constant-time hypersurface. We can now write the spherical triad vectors as

$$\mathbf{e}_{\hat{r}} = A_{\hat{r}} \mathbf{e}_0 + B_{\hat{r}}^r \partial_r, \quad \mathbf{e}_{\hat{\theta}} = B_{\hat{\theta}}^\theta \partial_\theta, \quad \mathbf{e}_{\hat{\phi}} = B_{\hat{\phi}}^\phi \partial_\phi, \quad (5)$$

where r is the radial coordinate in the hypersurface. Since $\mathbf{B}_{\hat{\theta}}$ and $\mathbf{B}_{\hat{\phi}}$ are unit vectors, they can be found from the metric. We find it convenient to define new symbols such that

$$B_{\hat{r}}^r \equiv B_R = e^{-\lambda}, \quad B_{\hat{\theta}}^\theta \equiv B_T = \frac{1}{R}, \quad B_{\hat{\phi}}^\phi = \frac{B_T}{\sin \theta}, \quad (6)$$

with R the circumferential radius of the two-sphere.

The directional derivative along the timelike vector of the tetrad can be related to coordinate derivatives by defining a tetrad “lapse” α and “shift” vector β^k . The shift has only a radial component, so

$$D_0 = \frac{1}{\alpha} (\partial_t - \beta^r \partial_r). \quad (7)$$

Unless the tetrad congruence is orthogonal to the constant-time hypersurfaces, the tetrad lapse and shift are different from the standard 3+1 lapse and shift.

The directional derivatives along the spatial triad directions are

$$D_a = e_a^\mu \frac{\partial}{\partial x^\mu} \quad (8)$$

for the Cartesian triad, which is related to the spherical triad by Eqs. (1), (2), and (3). The spherical directional derivatives are

$$D_{\hat{r}} = A_{\hat{r}} D_0 + e^{-\lambda} \partial_r, \quad (9)$$

$$D_{\hat{\theta}} = \frac{1}{R} \partial_\theta, \quad D_{\hat{\phi}} = \frac{1}{R \sin \theta} \partial_\phi. \quad (10)$$

The twenty-four connection coefficients are derived from the commutators of the Cartesian tetrad directional derivatives. In the WEBB formulation, a space-time split is made which groups the connection coefficients into two 3×3 dyadic matrices $N_{ab} \equiv \frac{1}{2} \varepsilon_{bcd} \Gamma_{cda}$ and $K_{ab} \equiv \Gamma_{b0a}$, plus two spatial vectors, the acceleration $a_b = \Gamma_{b00}$ and

the angular velocity (relative to Fermi-Walker transport) $\omega_b = \frac{1}{2} \varepsilon_{bcd} \Gamma_{dc0}$ of the tetrad frames.

As a consequence of the underlying spherical symmetry, only the anti-symmetric part of the N_{ab} is non-zero. The vector \mathbf{n} with Cartesian components $n_a \equiv \frac{1}{2} \varepsilon_{abc} N_{bc}$ points in the radial direction, $\mathbf{n} = n_{\hat{r}} \mathbf{e}_{\hat{r}}$, and the constraint equations arising from commutators of the spatial tetrad vectors reduce to

$$n_{\hat{r}} = \frac{1 - D_{\hat{r}} R}{R}. \quad (11)$$

The spherical symmetry also means that the K_{ab} are symmetric, which is the condition that the tetrad congruence has zero vorticity and is orthogonal to some set of spacelike hypersurfaces. The K_{ab} are the Cartesian components of the extrinsic curvature tensor of these hypersurfaces (not necessarily the same as constant-time hypersurfaces). There are only two independent degrees of freedom in the extrinsic curvature, since the only non-zero components of the extrinsic curvature with respect to the spherical triad basis are

$$K_{\hat{r}\hat{r}} \equiv K_R, \quad K_{\hat{\theta}\hat{\theta}} = K_{\hat{\phi}\hat{\phi}} \equiv K_T. \quad (12)$$

Finally, the angular velocity ω_b of a spherically symmetric congruence is identically zero, and the acceleration can only point in the radial direction perpendicular to the congruence worldlines, so the only non-zero component relative to the spherical triad basis is $a_{\hat{r}}$. The calculation from the Cartesian basis gives

$$a_1 = \cos \phi \sin \theta \ a_{\hat{r}}, \quad a_2 = \sin \phi \sin \theta \ a_{\hat{r}}, \quad a_3 = \cos \theta \ a_{\hat{r}}. \quad (13)$$

The WEBB equations are greatly simplified if K_R , K_T , $n_{\hat{r}}$, $a_{\hat{r}}$, B_R , B_T , and $A_{\hat{r}}$ as used as variables instead of the Cartesian components. This results in a reduction of the total number of variables from thirty-six to seven. The lapse function and shift vector are not evolved dynamically, but are reset periodically to optimize the evolution of the coordinates.

The spacetime metric is obtained by first calculating $g^{\mu\nu} = \eta^{\alpha\beta} e_\alpha^\mu e_\beta^\nu$. The $g^{\mu\nu}$ matrix is then inverted to give $g_{\mu\nu}$. The resulting metric is:

$$\begin{aligned} ds^2 &= [-\alpha^2 + \beta^{r^2} e^{2\lambda} (1 - A_{\hat{r}}^2) + 2 e^\lambda \alpha \beta^r A_{\hat{r}}] dt^2 \\ &\quad + 2 e^\lambda [\alpha A_{\hat{r}} + \beta^r e^\lambda (1 - A_{\hat{r}}^2)] dr dt \\ &\quad + e^{2\lambda} (1 - A_{\hat{r}}^2) dr^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2. \end{aligned} \quad (14)$$

The spatial metric of the constant-time hypersurface is

$$dl^2 = e^{2\lambda} (1 - A_{\hat{r}}^2) dr^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2. \quad (15)$$

III. TETRAD QUASIEVOLUTION AND QUASICONSTRAINT EQUATIONS

The true evolution equations must be equations for partial time derivatives of the variables in terms of partial

spatial derivatives and source terms which are functions of the variables. The natural result of the WEBB formalism is equations relating directional derivatives. Because in general there are time derivatives hidden inside the spatial directional derivatives, we call the equations expressed in terms of the tetrad directional derivatives quasievolution equations (if they contain an explicit D_0) or quasiconstraint equations if they contain only spatial directional derivatives. (See the WEBB paper [15] for a complete discussion.) Because they are simpler, we first discuss the quasievolution and quasiconstraint equations.

Since the Einstein equations are covariant under rotations of the tetrad, the quasievolution equations for K_R , K_T , and $n_{\hat{r}}$, and the quasiconstraint equations for K_T and $n_{\hat{r}}$, can be derived directly from the Einstein equations in the $\mathbf{e}_{\hat{r}}$, $\mathbf{e}_{\hat{\theta}}$, $\mathbf{e}_{\hat{\phi}}$ basis. The quasievolution equations are

$$D_0 K_R - D_{\hat{r}} a_{\hat{r}} = S_- K_R, \quad (16)$$

$$D_0 K_T - D_{\hat{r}} n_{\hat{r}} = S_- K_T, \quad (17)$$

$$D_0 n_{\hat{r}} - D_{\hat{r}} K_T = S_- n_{\hat{r}}, \quad (18)$$

with

$$\begin{aligned} S_- K_R &= a_{\hat{r}}^2 - n_{\hat{r}}^2 - K_R^2 + K_T^2 + \frac{2 n_{\hat{r}}}{R}, \\ S_- K_T &= \frac{a_{\hat{r}} + n_{\hat{r}}}{R} + K_R K_T - K_T^2 - n_{\hat{r}}^2 - a_{\hat{r}} n_{\hat{r}}, \\ S_- n_{\hat{r}} &= \frac{K_T - K_R}{R} - a_{\hat{r}} K_T + n_{\hat{r}} (K_R - 2 K_T). \end{aligned}$$

The momentum and energy quasiconstraint equations are, respectively,

$$D_{\hat{r}} K_T = \frac{(K_R - K_T) (1 - R n_{\hat{r}})}{R}, \quad (19)$$

$$D_{\hat{r}} n_{\hat{r}} = \frac{3 n_{\hat{r}}^2}{2} - \frac{2 n_{\hat{r}}}{R} - \frac{K_T}{2} (2 K_R + K_T). \quad (20)$$

The gauge quasievolution equations for $a_{\hat{r}}$ are not covariant under rotation, and must be derived from the Nester and Lorentz gauge conditions in a Cartesian basis (Eqs. (44) and (47) of [15]). The results are converted to our spherical basis variables. Both the Nester and Lorentz gauges give quasievolution equations for $a_{\hat{r}}$ of the form

$$D_0 a_{\hat{r}} - D_{\hat{r}} K_R = S_- a_{\hat{r}}. \quad (21)$$

The source, $S_- a_{\hat{r}}$ depends on the gauge. For the Nester gauge,

$$S_- a_{\hat{r}} = \frac{2 (K_R - K_T)}{R}, \quad (22)$$

and for the Lorentz gauge,

$$S \cdot a_{\hat{r}} = \frac{2(K_R - K_T)}{R} - 2(K_R n_{\hat{r}} + K_T a_{\hat{r}}). \quad (23)$$

The Nester gauge quasiconstraint equation is trivial because in spherical symmetry, the curl of a radial vector is zero.

IV. TRUE EVOLUTION EQUATIONS AND THEIR HYPERBOLIC STRUCTURE

In order to evolve K_R , K_T , $n_{\hat{r}}$, and $a_{\hat{r}}$ numerically, the evolution equations must be expressed as partial derivatives along the coordinate directions r and t . To obtain these true evolution equations, Eqs. (9) and (10) are substituted into the quasievolution equations in Sec. III. Linear combinations of the quasievolution equations are taken to isolate the time derivative of each variable. The results can be lumped together in the form

$$D_0 \mathbf{q} + \mathbf{C}^{\hat{r}} B_R \partial_r \mathbf{q} = \mathbf{S}. \quad (24)$$

In this equation,

$$\mathbf{q} = \begin{pmatrix} K_R \\ a_{\hat{r}} \\ K_T \\ n_{\hat{r}} \end{pmatrix}, \quad \mathbf{C}^{\hat{r}} = -\frac{1}{1-A_{\hat{r}}^2} \begin{pmatrix} A_{\hat{r}} & 1 & 0 & 0 \\ 1 & A_{\hat{r}} & 0 & 0 \\ 0 & 0 & A_{\hat{r}} & 1 \\ 0 & 0 & 1 & A_{\hat{r}} \end{pmatrix},$$

and

$$\mathbf{S} = \frac{1}{1-A_{\hat{r}}^2} \begin{pmatrix} S_K R + A_{\hat{r}} S_{a_{\hat{r}}} \\ S_{a_{\hat{r}}} + A_{\hat{r}} S_K R \\ S_K T + A_{\hat{r}} S_{n_{\hat{r}}} \\ S_{n_{\hat{r}}} + A_{\hat{r}} S_K T \end{pmatrix}.$$

The eigensystem of the characteristic matrix, $\mathbf{C}^{\hat{r}}$, consists of four eigenmodes: two with eigenvalue $1/(1-A_{\hat{r}})$ and amplitudes $a_{\hat{r}} + K_R$ and $n_{\hat{r}} + K_T$, and two with eigenvalue $-1/(1+A_{\hat{r}})$ and amplitudes $a_{\hat{r}} - K_R$ and $n_{\hat{r}} - K_T$. Decomposing the D_0 operator into partial derivatives gives coordinates speeds

$$s_1(r, t) = \frac{e^{-\lambda} \alpha}{1+A_{\hat{r}}} - \beta^r, \quad (25)$$

$$s_2(r, t) = -\frac{e^{-\lambda} \alpha}{1-A_{\hat{r}}} - \beta^r. \quad (26)$$

The two eigenmodes involving K_R and $a_{\hat{r}}$, are “longitudinal” modes, since they are constructed from components of the extrinsic curvature tensor and acceleration vector projected along $e_{\hat{r}}$. The two involving K_T and $n_{\hat{r}}$, are “constraint” modes, since K_T and $n_{\hat{r}}$ are the variables which appear in the principal parts of the constraint equations.

V. TRUE CONSTRAINT EQUATIONS

To eliminate the time derivatives hidden in the quasi-constraint equations, we use the true evolution equations. It is convenient to take linear combinations of the result to get decoupled equations for $B_R \partial_r K_T$ and $B_R \partial_r n_{\hat{r}}$. We call these the true momentum and energy constraint equations,

$$B_R \partial_r K_T = (K_R - K_T) \left(\frac{1}{R} - n_{\hat{r}} \right) - \frac{A_{\hat{r}}}{2} \left[\frac{2}{R} (a_{\hat{r}} - n_{\hat{r}}) - 3K_T^2 - 2a_{\hat{r}}n_{\hat{r}} + n_{\hat{r}}^2 \right], \quad (27)$$

$$B_R \partial_r n_{\hat{r}} = -\frac{2 n_{\hat{r}}}{R} - K_R K_T - \frac{K_T^2}{2} + \frac{3 n_{\hat{r}}^2}{2} + A_{\hat{r}} K_T (a_{\hat{r}} + n_{\hat{r}}). \quad (28)$$

These equations are used in calculating initial conditions, boundary conditions, and as an accuracy check for the numerical evolution.

VI. EVOLUTION AND CONSTRAINT EQUATIONS FOR $B_{\hat{r}}$, $B_{\hat{\theta}}$, AND $A_{\hat{r}}$

Evolution equations for the triad vector components are also required. Recall there are only two independent components in spherical symmetry, $B_R = e^{-\lambda}$ and $B_T = 1/R$. Using B_T rather than R as a variable in the numerics is motivated by a desire to maintain a form of the equations similar to what is necessary in a three-dimensional calculation, but also leads to a significant improvement in accuracy of the results for the Lorentz gauge. The evolution equations for $B_{\hat{r}}$ and $B_{\hat{\theta}}$ are

$$(\partial_t - \mathcal{L}_{\beta}) \mathbf{B}_{\hat{r}} = -\alpha K_R \mathbf{B}_{\hat{r}} \Rightarrow D_0 B_R = -\left(K_R + \frac{\partial_r \beta^r}{\alpha}\right) B_R, \quad (29)$$

$$(\partial_t - \mathcal{L}_{\beta}) \mathbf{B}_{\hat{\theta}} = -\alpha K_T \mathbf{B}_{\hat{\theta}} \Rightarrow D_0 B_T = -K_T B_T. \quad (30)$$

There is a constraint equation for B_T ,

$$B_R \partial_r B_T = -(B_T - n_{\hat{r}} - A_{\hat{r}} K_T) B_T, \quad (31)$$

which is equivalent to Eq. (11) for the coordinate derivative of R . This constraint is used to obtain B_T (and R) in the initial conditions, and as an additional check on the accuracy of the numerical evolution. The evolution equation for $A_{\hat{r}}$ follows from commuting D_0 with the spatial directional derivatives,

$$D_0 A_{\hat{r}} = a_{\hat{r}} - K_R A_{\hat{r}} - B_R \partial_r (\ln \alpha). \quad (32)$$

VII. THE INITIAL VALUE PROBLEM (IVP)

The initial value problem consists of finding values for $A_{\hat{r}}$, K_T , $n_{\hat{r}}$, B_T , B_R , K_R , and $a_{\hat{r}}$ with which to begin the numerical evolution. We take the congruence orthogonal to the initial hypersurface, so $A_{\hat{r}} = 0$ initially. K_T , $n_{\hat{r}}$, and R are obtained from Eqs. (27), (28), and (31) with $A_{\hat{r}} = 0$. The first integral of the constraint equations,

$$(1 - R n_{\hat{r}})^2 - K_T^2 = 1 - \frac{2M}{R}, \quad (33)$$

where the constant of integration M is the Schwarzschild mass, makes numerical integration of Eq. (28) for $n_{\hat{r}}$ unnecessary. Using Eq. (33) rather than Eq. (28) to obtain $n_{\hat{r}}$ in the initial conditions has a substantial impact on the constraint errors early in the evolution, but after a few dynamical times leads to only a modest improvement in accuracy.

The free initial data are K_R , $a_{\hat{r}}$, and B_R on the initial hypersurface. The choice of B_R is relatively trivial, since it sets the initial relationship of coordinate radius to proper radius and plays no role in the subsequent dynamics of the congruence or the geometry. With grid spacing uniform in coordinate radius, the initial choice of B_R can affect numerical accuracy simply because it affects the relative grid resolution in different parts of the domain. Two particular prescriptions for K_R and $a_{\hat{r}}$ are considered: one which, along with appropriate choices for the lapse and shift, leads analytically to a time independent solution of the evolution equations (Time Independent IVP), and one which sets a uniform value for the trace of the extrinsic curvature on the initial hypersurface (Constant Mean Curvature IVP).

A. Time Independent IVP

The Schwarzschild spacetime has a time Killing vector field (timelike outside the horizon, $R > 2M$), which means that coordinate systems can be found in which the metric is time independent. However, in the WEBB tetrad formulation the variables will be time independent only if the tetrad congruence is stationary as well as the (coordinate) metric. Reproducing an analytically time-independent solution is the simplest and most basic test of a numerical code. Note that the standard Schwarzschild slicing giving rise to a static metric is not satisfactory for our purposes, since the normal tetrad congruence would be singular, with infinite acceleration, on the horizon.

In a time-independent evolution, a congruence initially orthogonal to a constant-time hypersurface will be orthogonal at all times, so $A_{\hat{r}}$ will be zero at all times. This is consistent with Eq. (32) if the lapse is given by

$$B_R \partial_r (\ln \alpha) = a_{\hat{r}}. \quad (34)$$

Also, the coordinate time derivative of the geometrically defined curvature radius R must be zero, a condition

which, by Eqs. (30) and (31), requires that the shift satisfy

$$\beta^r = -\frac{\alpha B_R R K_T}{1 - R n_{\hat{r}}}. \quad (35)$$

With $\partial_t B_R = 0$, $B_R \partial_r (\beta^r / B_R) = -\alpha K_R$.

Setting the partial time derivatives of K_R and $a_{\hat{r}}$ to zero in their true evolution equations gives two simultaneous equations for the proper radial derivatives of K_R and $a_{\hat{r}}$. The equation for K_R is

$$B_R \partial_r K_R = -(1 - R n_{\hat{r}}) \times \frac{R K_T S_{-} K_R + (1 - R n_{\hat{r}}) S_{-} a_{\hat{r}}}{(1 - R n_{\hat{r}})^2 - (R K_T)^2}. \quad (36)$$

Note that $S_{-} a_{\hat{r}}$ depends on the choice of tetrad gauge, *ie.* Nester versus Lorentz.

Eq. (36) is singular when $(1 - R n_{\hat{r}})^2 - (R K_T)^2 = 0$, or $R = 2M$ (see Eq. (33)). A solution regular on the horizon is obtained by requiring that the numerator of Eq. (36) vanish at $R = 2M$, which implies a relation between the values of K_R and $R K_T$ on the horizon (which we call K_{RH} and U_0 , respectively). For the Nester gauge, this relation is

$$2 M K_{RH} = \frac{1 + 4 U_0^2 - 8 |U_0|^3}{-4 U_0 - 8 U_0 |U_0|}, \quad (37)$$

and for the Lorentz gauge, it is

$$2 M K_{RH} = \frac{-1 + 8 |U_0|^3}{4 U_0}. \quad (38)$$

U_0 is a free parameter, which must be negative for a black hole horizon. L'Hôpital's rule can be used to find the starting value for $B_R \partial_r K_R$ on the horizon.

While one can integrate the equation for $B_R \partial_r a_{\hat{r}}$ obtained along with Eq. (36), it is simpler to use the algebraic expression

$$a_{\hat{r}} = \frac{M/R^2 + R K_T K_R}{(1 - R n_{\hat{r}})}. \quad (39)$$

Eq. (39) is obtained by requiring that $\partial_t (R K_T) = 0$, eliminating radial derivatives of $R K_T$ and $n_{\hat{r}}$ using the constraint equations, eliminating the shift using Eq. (35), and finally, simplifying with the help of Eq. (33). With $a_{\hat{r}}$ given by Eq. (39), we can determine the lapse everywhere using Eq. (34).

In summary, for a particular choice of U_0 , one can obtain K_R , $a_{\hat{r}}$, α , and β^r everywhere so that the evolution of all the variables is time independent. The constant-time hypersurface generally does become singular at some point inside the horizon, so the initial hypersurface must be terminated at an excision boundary inside the horizon before $a_{\hat{r}}$ and/or K_R become too large. At large R , the generic behavior is that K_T and K_R approach the same constant value. This constant is positive if U_0 is greater (less negative) than a certain critical

value, which is exactly -0.25 for the Nester gauge and about -0.29 for the Lorentz gauge. A positive K_T at large R is highly desirable in dealing with outer boundary conditions, because then the shift at that outer boundary is negative and modes propagating along the congruence propagate out of, rather than into, the grid. Also, we find that expansion of the congruence along the radial direction ($K_R > 0$) everywhere is generally helpful in reducing growth rates of any unstable constraint-violating modes.

B. Constant Mean Curvature (CMC) Slice

An attractive slicing condition from the point of view of the conformal approach to the initial value problem [40, 41, 42] is to impose a uniform value for the trace of the extrinsic curvature of the initial hypersurface. This ‘‘Constant Mean Curvature’’ slicing, with ($\text{trace } K = K_R + 2 K_T = K_0$), allows testing of cases where the evolution of the hypersurfaces and the congruence is time dependent. In order to make the evolution of the congruence as ‘‘quiet’’ as possible, we choose the initial acceleration to satisfy the stationarity condition of Eq. (39). Then, with $K_R = K_0 - 2K_T$, the momentum and energy constraints can be integrated as differential equations for K_T and $n_{\hat{r}}$, starting at the horizon, as described in the previous section. U_0 (the value of $R K_T$ on the horizon) is a free parameter, along with the choice of K_0 . The value of $n_{\hat{r}}$ on the horizon is fixed by Eq. (33).

Together with a shift given by Eq. (35) and a lapse given by Eq. (34), this implementation of the CMC initial condition means that the time derivatives of K_T and $n_{\hat{r}}$ vanish on the initial hypersurface, though they will not stay zero. A critical test of the Nester and Lorentz tetrad gauge conditions is whether the hypersurfaces and congruence will subsequently evolve toward or away from a time-independent solution. Of course, the answer to this question also depends on how the coordinates evolve, as described in the next section.

VIII. RESETTING COORDINATE CONDITIONS

Our system of equations is symmetrizable hyperbolic for any fixed choice of lapse and shift. Accordingly, during a given time step, the lapse and shift are held fixed at the values they have at the start of the time step. While it is essential to accuracy and stability that the system be hyperbolic during the time steps [25], it is not critical that the overall evolution be hyperbolic, only well-posed [43]. In other words, the lapse and the shift can be reset at fixed time intervals according to conditions which do not necessarily preserve the hyperbolicity of the system. This gives a wide range of choices for resetting conditions. Keeping a fixed lapse and shift for many dynamical times is likely to give rise to coordinate singularities.

The lapse determines the evolution of the constant-time hypersurfaces. The hyperbolic system breaks down if $A_{\hat{r}} = 1$, which signifies that the constant-time hypersurface has become null. We reset the lapse to keep $A_{\hat{r}}$ small, which is accomplished by making the reset lapse satisfy Eq. (34). Then the time derivative of $A_{\hat{r}}$ will be small by Eq. (32) as long as $A_{\hat{r}}$ is small. Since $A_{\hat{r}} = 0$ initially, it does in fact stay very small if the lapse is reset at small time intervals. The constant of integration in solving Eq. (34) is conveniently chosen to keep the lapse constant at the outer edge of the grid, but makes no practical difference, since it is just a uniform rescaling of the time coordinate. Choosing the time step according to a Courant condition automatically compensates for any such rescaling.

The shift controls the evolution of the spatial coordinates. We work with a grid at fixed values of the radial coordinate. The evolution of the radial coordinate should be managed to keep the grid from being sucked up by the black hole, while keeping the inner edge just inside the event horizon, without any excessive stretching or compression of the grid relative to the physical curvature radius R . If made possible by the evolution of the congruence and the constant-time hypersurfaces, we want our variables to approach a stationary final state. One option would be to choose the shift to make the partial time derivative of the curvature radius R zero everywhere at each resetting. From Eqs. (30) and (31) this condition is

$$\beta^r = -\frac{\alpha B_R R K_T}{1 - R n_{\hat{r}} - A_{\hat{r}} R K_T}. \quad (40)$$

However, it seems more desirable to use a condition which is not so tied to the special circumstances of spherical symmetry.

The minimal strain condition, as introduced by Smarr and York [21] for 3+1 formulations, minimizes an integral of the square of the time derivative of the spatial metric over the hypersurface,

$$I_1 = \int \left[\frac{\partial h_{ij}}{\partial t} \frac{\partial h_{kl}}{\partial t} h^{ik} h^{jl} \right] \sqrt{h} d^3x \quad (41)$$

with respect to variations in the shift. In I_1 , h_{ij} is the spatial metric and h is the determinant of this metric. The metric time derivatives depend on the shift through

$$\frac{\partial h_{ij}}{\partial t} = -2 \alpha K_{ij} + \mathcal{L}_{\beta} h_{ij}, \quad (42)$$

where K_{ij} is the extrinsic curvature of the hypersurface.

In the context of the WEBB equations, the vectors \mathbf{B}_a carry the information corresponding to the spatial metric in a 3+1 formalism. These vectors are not unit vectors with respect to the true spatial metric if the tetrad congruence is not orthogonal to the constant-time hypersurface. As projections of the spatial triad vectors into the constant-time hypersurface, they can be thought of

as representing unit spatial displacements in the local 3-space orthogonal to the tetrad congruence, which are Lorentz-contracted when measured in the hypersurface frame. We find it convenient to pose a minimal deformation condition on the components B_a^k , which in general is *not* the same as the Smarr-York condition.

Inverting the B_a^k matrix gives the matrix B_b^a of components of one-forms dual to the \mathbf{B}_a vectors. A spatial metric with respect to which the \mathbf{B}_a are unit vectors is $h_{ij} = B_a^i B_a^j$. We use this metric, rather than the true spatial metric, in our minimal deformation condition, minimizing with respect to variations in the shift vector the action

$$I_2 = \int h_{ij} \frac{\partial B_b^i}{\partial t} \frac{\partial B_b^j}{\partial t} \sqrt{h} d^3x, \quad (43)$$

where h is the determinant of h_{ij} . In the current application, with $A_{\hat{r}}$ kept very small by resetting the lapse and a diagonal B_a^k , I_2 is very nearly equivalent to I_1 .

The dependence on the shift is through

$$\frac{\partial_t B_R}{B_R} = \beta^r \frac{\partial_r B_R}{B_R} - \alpha K_R - \partial_r \beta^r \equiv y \quad (44)$$

and

$$\frac{\partial_t B_T}{B_T} = \beta^r \frac{\partial_r B_T}{B_T} - \alpha K_T. \quad (45)$$

with

$$I_2 = \int dr 4\pi \frac{R^2}{B_R} \left[\left(\frac{\partial_t B_R}{B_R} \right)^2 + 2 \left(\frac{\partial_t B_T}{B_T} \right)^2 \right]. \quad (46)$$

The Euler-Lagrange equation is

$$\partial_r y = 2 \frac{\partial_r B_T}{B_T} \left(y + \beta^r \frac{\partial_r B_T}{B_T} - \alpha K_T \right), \quad (47)$$

which, along with Eq. (44) giving $\partial_r \beta^r$ in terms of y , can be integrated to find β^r and $\partial_r \beta^r$ everywhere. Note that $\partial_r B_T$ can be evaluated from the constraint equation (31). Strict minimization requires $y = 0$ at each boundary, but in order to keep the inner edge of the grid just inside the horizon and the outer edge at a fixed R we instead impose Eq. (40) as the shift boundary condition at both edges of the grid. After each recalculation the values of β^r and $\partial_r \beta^r$ at each grid point are stored and kept fixed until the next resetting of the lapse and shift.

IX. BOUNDARY CONDITIONS

We now discuss boundary conditions for the evolution equations. The inner boundary of the grid is an excision boundary, kept at a roughly constant $R < 2M$, inside the event horizon, by the shift condition of the last section. All characteristics of the evolution equations propagating at less than or equal to light speed (which is all of them

in the WEBB framework) are then outgoing relative to the grid at the inner boundary, since inside the horizon the entire future light cone is toward decreasing R . All information needed to update the variables at the inner boundary can be obtained from the boundary values or upwind differencing.

The outer boundary is a different story. We do not try to extend the outer boundary all the way to infinite R , though this in principle is possible. Since our asymptotically hyperbolic hypersurfaces are asymptotically null, the variables should be regular functions of $B_T = 1/R$, and the radial coordinate could be scaled to be linear in B_T approaching null infinity. We terminate our initial grid at a value of R the order of $20 M$, where variables such as $n_{\hat{r}}$, K_T , and K_R have reached nearly constant values, and keep the outer boundary at approximately the same R with our shift condition. This allows the grid to be roughly evenly spaced in proper radius, with adequate resolution near the horizon and a reasonable total number of grid points. As long as K_T is positive near the outer boundary, the shift is negative there (see Eq. (40)), and the modes propagating along the hypersurface normal are outward relative to the boundary. This is critically important, since there is no good way of specifying incoming boundary conditions for these modes. Still, the two modes with speeds s_2 are incoming at the boundary and do require boundary conditions. The amplitudes of these modes are $(a_{\hat{r}} + K_R)$ and $(n_{\hat{r}} + K_T)$. Hence, we refer to them as the incoming “longitudinal” and “constraint” modes, respectively. The “constraint” modes consist of the variables whose radial derivatives appear in the principal parts of the energy and momentum constraint equations, and have the potential of propagating constraint errors in from the boundary (see [25] for a fuller explanation). The constraints cannot be satisfied by just setting the incoming constraint mode amplitude to zero. A number of different constraint-preserving boundary conditions have been proposed [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. What we do is, first, iteratively correct the values of the constraint variables at the boundary physical points so that the energy and momentum constraints evaluated at half a step inside the grid are satisfied. Then all variables are extrapolated to the ghost point just outside the boundary, and the energy and momentum constraints are again applied iteratively halfway between the boundary physical point and the ghost point to adjust the extrapolated values of the constraint variables.

For stationary initial conditions, the longitudinal variables, K_R and $a_{\hat{r}}$, are kept equal to their initial values at the ghost point, to give boundary conditions for the incoming longitudinal mode, $K_R + a_{\hat{r}}$. For CMC initial conditions, the ghost point longitudinal variables can either be extrapolated quadratically or cubically to the ghost point, which could be dangerous for stability, or again maintained at their initial values. The incoming longitudinal mode amplitude is a purely gauge quantity, so the only real concern is not to do something which

leads to a gauge instability at the boundary.

X. NUMERICAL METHODS

The numerical methods used in the codes are designed to be second order accurate in both space and time. While we have experimented with various methods, the most accurate results, and the ones presented in this paper, were obtained with a second order Strang split method. During each full time step, the source terms alone are integrated for a half time step at each grid point using second-order Runge-Kutta, followed by a full wave propagation time step with the updated variables solving the equations with source terms omitted, and then another half time step of source term integration. In the wave propagation time step the equations are reshuffled into decoupled advection equations for the eigenmode amplitudes, of the form

$$\partial_t y + v \partial_r y = 0, \quad (48)$$

where v is the wave speed. Using characteristic tracing, the solution of the advection equation for the updated y at the i^{th} grid point is just the initial value of y at the position $r_i - \bar{v} \Delta t$, where \bar{v} is the average advection velocity along the characteristic. This average speed is approximated by advancing v to halfway through the time step by a first-order accurate Euler method at all grid points before doing the wave propagation (only necessary for the modes whose wave speed depends on B_R), and then interpolating $1/v$ to the midpoint in r along the characteristic during the wave propagation, so at the i^{th} grid point

$$\bar{v} = v_i / (1 \mp \frac{\Delta t}{2 \Delta x} (v_{i\mp1} - v_i)), \quad (49)$$

with upper and lower signs for $(v_i > 0)$ and $(v_i < 0)$, respectively. Standard second-order methods interpolate y to the characteristic location using an upwind first-order difference and a second-order difference either centered at i (Lax-Wendroff) or centered at the upwind grid point (Beam-Warming). Except at the boundaries, we use a linear combination of the two second-order differences which does the interpolation to third-order accuracy, weighting the centered second-order difference by $(2 - |v|)/3$ and the upwind second-order difference by $(1 + |v|)/3$. While the overall method is not third-order accurate, this does significantly reduce the errors compared with pure Lax-Wendroff. At the boundaries either pure Lax-Wendroff or pure Beam-Warming is used as necessary to minimize use of ghost cells, Beam-Warming for outgoing modes and Lax-Wendroff for incoming modes. Note that the upwind point changes as v changes sign, which means a small but sudden change in the interpolated upwind second-order difference and its contribution to the updated y . The effect of this is noticeable in the constraint error plots we present in Sec. XI, but has no impact on stability.

The integration of the spatial ordinary differential equations during the initial condition and boundary condition subroutines is implemented with a simple second-order Predictor-Corrector scheme [44], iterated until the differences between the predicted and corrected values are all near machine precision.

XI. RESULTS

All our numerical results are obtained on a uniform grid ranging from $-0.16 \leq r \leq 9.84$, where r is the coordinate radius, with $r = 0$ at the initial location of the event horizon. The initial $B_R = 1$, so that the coordinate radius starts out equaling the proper radius. The natural logarithm of the lapse ($\ln \alpha$), the shift, (β^r), and their derivatives are reset as described in Sec. VIII. For convergence studies, these quantities are reset at a constant coordinate time interval as the grid resolution is increased and the time step is decreased in accordance with the Courant condition. Note that all values plotted are in units with $2 M = 1$.

A. Time

When presenting results, it is desirable to relate the coordinate time (t) of the numerical evolution to a physically meaningful time. The natural choice for a physical time is the proper time of an observer at infinite distance from the black hole, i.e., the Schwarzschild time coordinate t_S . Our spacelike hypersurfaces are quite different from the constant-time surfaces in the Schwarzschild metric, but we can still relate changes in our time coordinate to changes in Schwarzschild time.

The derivation is outlined assuming $A_{\hat{r}} = 0$, since $A_{\hat{r}}$ is kept close to zero with our lapse resetting condition. The proper time for a displacement perpendicular to the two-spheres as obtained from the Schwarzschild metric is

$$d\tau = \sqrt{\left(1 - \frac{2M}{R}\right) dt_S^2 - \frac{1}{1 - \frac{2M}{R}} dR^2}. \quad (50)$$

from which, with dt the change in our coordinate time,

$$\left(\frac{dt_S}{dt}\right)^2 = \frac{1}{1 - \frac{2M}{R}} \left[\left(\frac{d\tau}{dt}\right)^2 + \frac{1}{1 - \frac{2M}{R}} \left(\frac{dR}{dt}\right)^2 \right]. \quad (51)$$

For a displacement at constant coordinate radius r the metric of Eq.(14) gives $(d\tau/dt)^2 = (\alpha^2 - (\beta^r e^\lambda)^2)$. At both edges of the grid our boundary condition on the minimal deformation shift (Eq. 40) gives $\partial_t R = 0$, and simplifying with the help of Eq. (33) we arrive at

$$dt_S = \frac{\alpha}{1 - R n_{\hat{r}}} dt \quad (52)$$

for the relation between Schwarzschild time and coordinate time there. We can normalize the lapse at each resetting so that $\alpha = (1 - R n_{\hat{r}})$ at the outermost grid point, so that the change in Schwarzschild time equals the change in coordinate time at the outer edge of the grid to reasonable accuracy. For stationary initial conditions, this also makes $dt_S = dt$ hold at all interior grid points.

B. Stationary IVP

Testing the code using a stationary solution as the initial condition is the first step toward determining the viability of the code. Ideally, as the numerical evolution proceeds, all the variables stay constant. This does not occur in practice, however, because numerical errors act as perturbations to the stationary solution. Even with an accurate and stable numerical method, these perturbations may excite analytic unstable modes of the evolution equations, some of which may be constraint-violating and some of which may be purely gauge instabilities.

Figs. 1 and 2 show solutions of the stationary initial value problem for the Nester and Lorentz gauges, respectively. The solutions are one-parameter families characterized by U_0 , the value of $R K_T$ on the horizon. We only consider cases where the radial component of the extrinsic curvature K_R is positive (expansion of the normals radially) at large r , since only then do modes propagating along the hypersurface normal propagate off of, rather than onto, the grid at the outer boundary.

For the Nester gauge, only the inner portion of the spatial domain is shown, from just inside the horizon at $r = -0.16$ to $r = 4.0$, since the asymptotic behavior of most of the variables is already apparent by $r = 4.0$. The outer edge of the grid is always at $r = 9.84$. At the critical value $U_0 = -0.25$ all of the variables plotted approach zero as $R \rightarrow \infty$, and the constant-time hypersurface asymptotically approaches a Schwarzschild constant-time hypersurface. For U_0 less negative than the critical value, both K_R and K_T approach the same limiting (positive) value, with trace $K \rightarrow 0.15$ for $U_0 = -0.18$ and trace $K \rightarrow 0.24$ for $U_0 = -0.14$. The gradients in K_R and $a_{\hat{r}}$ become steeper in the vicinity of the event horizon at $r = 0$ as U_0 becomes less negative. Note that the first-integral constraint of Eq. (33) requires a change of sign of $n_{\hat{r}}$ from positive to negative going outward when K_T approaches a non-zero value at large R . This occurs outside the range of these plots, but not too far outside for $U_0 = -0.14$.

For the Lorentz gauge, we show the entire range of the grid, from just inside the horizon at $r = -0.16$ to the outer edge at $r = 9.84$. The critical value of U_0 at which the asymptotic values of K_T and K_R change sign is about -0.29 . The asymptotic behavior at large R when U_0 is less negative than the critical value is not well-behaved. Both K_R and K_T get steadily larger, eventually in a runaway indicating that the stationary condition is forcing

the hypersurface to become null. The beginning of this runaway is apparent in the plot for $U_0 = -0.25$, but is well outside the edge of the grid for $U_0 = -0.27$. This indicates a pathology in the Lorentz gauge which does not allow an approach to a stationary solution over the entire spacetime, but may not cause problems when solutions are evolved over a limited domain. The pathology is caused by the extra terms in the Lorentz gauge source for the $a_{\hat{r}}$ evolution equation (Eq. 23). Once $n_{\hat{r}}$ becomes negative the $-2 K_R n_{\hat{r}}$ term becomes positive and exerts positive feedback on K_R through Eq. (36).

To test the accuracy and stability of the time evolution with stationary initial conditions we can not only check how accurately the constraint equations of Sections V and VI are satisfied, but also take advantage of the fact that any change in any of the variables with time can only be due to some error in the calculation. The growth of some the constraint errors with time for both Nester and Lorentz gauges is shown in Fig. 3. The mass constraint error is the fractional difference between the mass as calculated at each grid point and each time from Eq. (33) and the mass used in constructing the initial conditions ($M = 1/2$ for our choice of units). The energy constraint error is the difference of the left and right-hand sides of Eq. (28), with the derivative of $n_{\hat{r}}$ evaluated numerically. The momentum constraint error (not plotted) from Eq. (27) is similar in magnitude to the energy constraint error. The energy constraint errors are weighted by a factor of R , which serves to make the errors more representative of fractional errors in individual terms of the energy constraint and also to make the amplitudes of the curves more uniform with radius. We multiply the actual errors by a factor of four because when testing convergence, we compare these curves with those from runs with double the point spacing. Assuming the quadratic convergence expected from our second-order accurate numerical method, the errors quadruple when the spacing between grid points is doubled.

The first two of the three plots show the results for the Nester gauge with $U_0 = -0.14$, up to a time of $120 M$. The errors do grow somewhat with time, but the rate of growth, after an initial oscillation in the mass constraint error, is clearly decreasing. The errors stop growing almost completely at later times, as shown in Fig. 4 for the energy constraint errors up to $t = 400 M$. The Lorentz gauge results for the energy constraint errors, the third plot in Fig. 3, clearly show exponential growth with time. The growth rate is not sensitive to numerical errors, and seems to be due to an analytically unstable constraint-violating mode of the evolution equations.

The apparent stability of the time evolution for the Nester gauge is encouraging, but it is also important to demonstrate convergence of the errors as the grid spacing is reduced. Fig. 5 does this for the Nester gauge evolution at $t = 40 M$. We compare results for $dr = 0.02$, $dr = 0.01$, and $dr = 0.005$. The errors are multiplied by the appropriate factors to make the curves lie on top of each other assuming perfect second-order convergence. While

there are some deviations close to the inner edge of the grid, the convergence is quite good everywhere else. At later times the convergence does gradually deteriorate, but that is not surprising after a very large number of dynamical times.

The good results for the Nester gauge with $U_0 = -0.14$ unfortunately deteriorate rather rapidly as U_0 is varied away from this value. For $U_0 = -0.19$, for instance, the errors grow much more rapidly initially, and then start oscillating with slowly increasing amplitude for $t > 100 M$ or so. Convergence starts breaking down around $t = 60M$. The errors at late times are at least a couple of order of magnitudes greater than they are for $U_0 = -0.14$. There is no real indication of exponential growth, but accuracy is not very good even for 1000 points in the grid.

The apparent need to fine-tune the initial conditions to get errors under good control with the Nester gauge suggests that the present equation formulation does not give the kind of robust accuracy and stability that is necessary for dealing with physically more interesting problems of black hole dynamics and mergers. The Lorentz gauge, with its exponentially growing instabilities, seems clearly unsuitable for dealing with black hole problems.

C. Constant Mean Curvature (CMC) IVP

A more demanding test of the code is to start from initial conditions which do not correspond to a stationary solution, and see if the gauge conditions allow relaxation toward a stationary solution. Certainly one only expects to be able to evolve black hole spacetimes accurately for many dynamical times if this is the case. The obvious first choice, consistent with the conformal approach to the initial value problem pioneered by York [40, 41, 42], is to require that the trace of the extrinsic curvature be uniform on the initial hypersurface, which is what we mean by constant mean curvature. This is the condition for separability of the momentum and energy constraint equations in the conformal approach, and has been assumed in much of the work on initial value problem over the last few decades. The most common assumption has been that the initial hypersurface is maximal, trace $K = 0$, but hyperbolic hypersurfaces, with trace $K > 0$, are more desirable from our point of view, since they simplify boundary conditions at the outer edge of the grid and seem to be necessary for the stability of our evolution equations.

The value of the constant mean curvature, K_0 , is a free parameter, as is the value of U_0 ($R K_T$ on the horizon). While the initial acceleration of the tetrad congruence is in principle a free function, we fix it according to the stationary condition of Eq. (39). This gives the time evolution a quiet start, in that the initial partial time derivative of $R K_T$ is zero, but nothing more than that.

Fig. 6 shows the Nester gauge time evolution of a_T for three different choices of K_0 and U_0 . With $K_0 = 0.2$

and $U_0 = -0.14$, the solution approaches a stationary solution, with the value of $R K_T$ on the event horizon equal to -0.15 at $t = 200 M$. With $K_0 = 0.2$ and $U_0 = -0.18$, the solution also approaches a stationary solution, with the value of $R K_T$ on the horizon equal to -0.16 at $t = 200 M$. However, with $K_0 = 0.4$ and $U_0 = -0.14$, the solution becomes pathological due to an increasingly sharp gradient of the acceleration somewhat outside the horizon, and the code crashes.

Similar initial conditions with the Lorentz gauge do not lead to any pathologies in the acceleration, but in all cases roughly exponential growth of constraint errors prevents meaningful continuation of the solutions much beyond a time of $40 M$.

XII. DISCUSSION

The WEBB tetrad formulation for numerical relativity has been implemented and tested in spherically symmetric black hole spacetimes, with two types of tetrad gauge conditions, which we call the Nester and Lorentz gauge conditions. While there is freedom in the choice of the initial velocity and acceleration of the tetrad congruence, subsequent evolution of the tetrad frames follows uniquely from the tetrad gauge conditions. The coordinate evolution is determined by elliptic equations for a tetrad lapse and shift in such a way that hypersurfaces of constant time stay perpendicular to the tetrad congruence, and the spatial coordinates evolve according to a minimal deformation condition, with boundary conditions which keep both the inner and outer edges of the grid at roughly constant Schwarzschild curvature radius R . While we pay particular attention to initial conditions consistent with a stationary evolution, we also consider non-stationary evolution from constant mean curvature initial hypersurfaces. Some representative results have been presented, but we have experimented with a variety of other parameter choices, numerical methods, etc. We have done sufficient testing to convince ourselves that the instabilities we find are not due to problems with numerical methods, but represent genuine analytic instabilities of the evolution equations, with numerical errors only playing a role of seeding the instabilities.

There are several potential factors which can influence the existence and severity of instabilities, such as (1) the choice of tetrad gauge, (2) the equation formulation, including choices of variables and the possible addition of constraint equations to the evolution equations in various combinations, (3) initial conditions and how the coordinates are evolved, and (4) boundary conditions. Of course, all of these factors interact in various ways and it is not always clear just what is behind a particular instability.

The equation formulation, gauge conditions, and the choice of variables determine the hyperbolic structure of the evolution system. The WEBB framework makes specific choices for all three of these, which give a par-

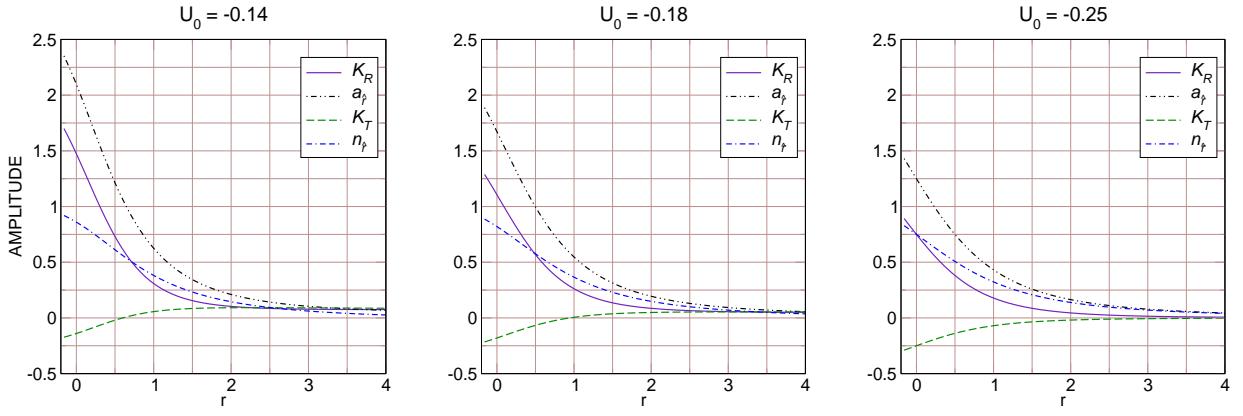


FIG. 1: Stationary Schwarzschild solution, Nester gauge. Shown is the region $-0.16 < r < 4.0$. The outer boundary is located at $r = 9.84$.

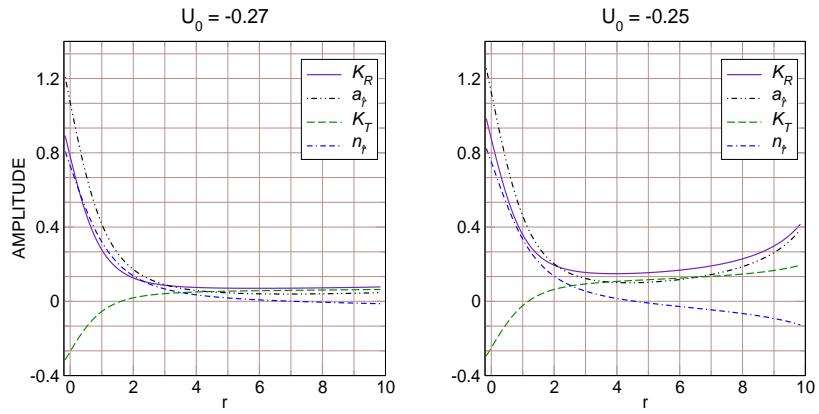


FIG. 2: Stationary Schwarzschild solution, Lorentz gauge.

ticularly simple symmetrizable hyperbolic system with all propagation either along the light cone or along the tetrad congruence. All we do in this paper is adapt the variables to explicit spherical symmetry, and compare two alternative tetrad gauge conditions, the Nester and Lorentz gauges. The fairly trivial choices of variables that we have explored ($B_T \equiv 1/R$ versus R and $B_R \equiv e^{-\lambda}$ versus λ) have some effect on accuracy, but no real effect on stability.

We have discovered potential problems with both Nester and Lorentz gauges which cannot be fixed by playing with the equation formulation. The Lorentz gauge is not consistent with stationary solutions of the evolution equations which have asymptotically hyperbolic spacelike hypersurfaces. With the Nester gauge the tetrad congruence can develop a kind of gauge shock, with steep gradients in the acceleration, for certain initial conditions. These issues would remain even if other problems could be fixed with a change in equation formulation.

In experiments with various boundary conditions on the evolution equations, we found that it is critically important to use some form of constraint-preserving boundary conditions. Otherwise constraint errors are generated at the boundary and propagate into the grid at the speed of light. The particular scheme we implement is not as elegant as some that have been proposed in the literature, but seems to work well, at least in this simple one-dimensional context. The other key to boundary conditions is to arrange to have the "zero-velocity" modes along the tetrad congruence propagate out of the grid at the boundary. This is no problem at the inner excision boundary, but requires expansion of the tetrad congruence at the outer boundary. Since we keep our constant-time hypersurfaces close to being orthogonal to the tetrad congruence, this is equivalent to having asymptotically hyperbolic hypersurfaces bending toward the future. With our choice of sign for extrinsic curvature, this means positive extrinsic curvature for the hypersurface.

For the Nester gauge, there is only a rather narrow range of initial conditions which give rise to stable and reasonably accurate solutions of the evolution equations. For stationary initial conditions, if the parameter $U_0 = R K_T$ is too close to zero, the acceleration and the radial extrinsic curvature K_R have very large gradients near the horizon, which is bad for stability. However, U_0 needs to be much less negative than -0.25 in order that

the hyperbolic character of the hypersurface be strong enough to give stability. There is a rather narrow window around $U_0 = -0.14$ where constraint and other errors do not grow substantially. We have not found any initial conditions for which the evolution in the Lorentz gauge is stable.

The importance of positive extrinsic curvature for stability has also been noted in a much more rigorous analytic treatment of the stability of the Weyl tensor evolution equations by Frauendiener and Vogel [45].

Finally, we note that in the context of black hole evolution it is critical to control the relation of the inner edge of the coordinate grid to the apparent event horizon, and highly desirable to keep the outer edge of the grid at a constant physical radius. We do this through the boundary conditions on our minimal deformation equation for the shift vector. Failure to do this leads to severe numerical problems.

We conclude that the WEBB equation formulation as presented in [15], with either the Nester or Lorentz gauge conditions, does not seem particularly promising for black hole evolutions. While the Nester gauge can give good results for some initial conditions, these results are not very robust. The Lorentz gauge does not seem to be viable at all. However, it may be possible, by adding constraint damping terms to the evolution equations, to improve accuracy and stability for a wider range of initial conditions, while preserving a reasonable hyperbolic structure. We have begun to explore other equation formulations and gauge conditions in the context of the tetrad framework, some of which seem considerably more promising.

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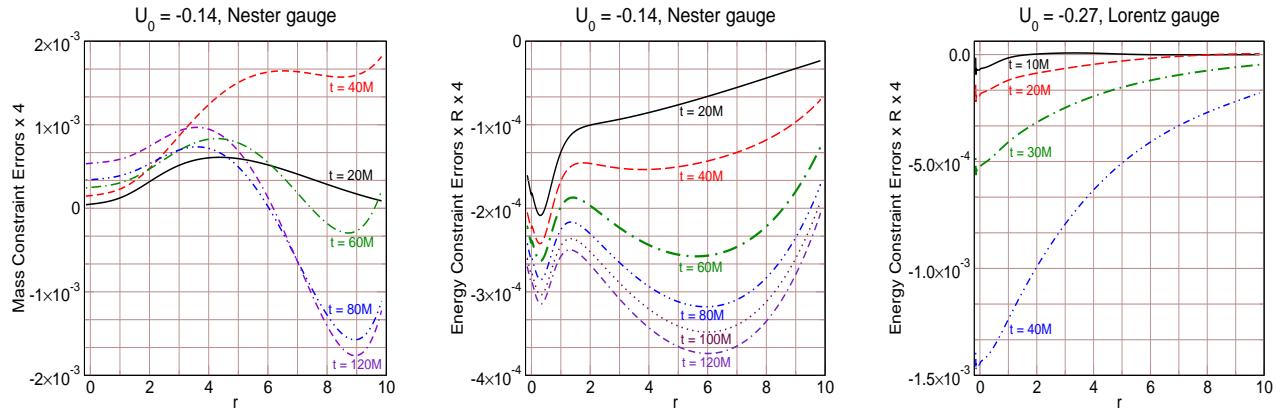


FIG. 3: Constraint errors vs. time for stationary IVP. $dr = 0.01$, $dt = 0.001$. The lapse and shift are reset every $dt = 0.002$.

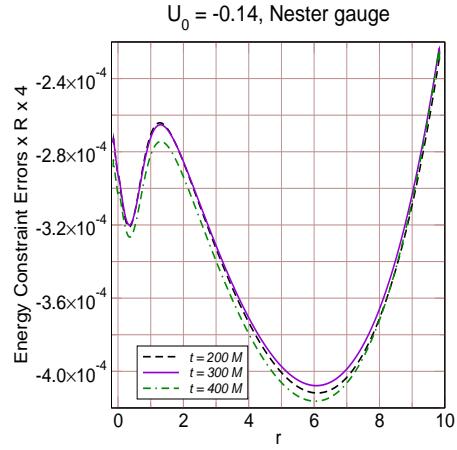


FIG. 4: Energy constraint error at late times for stationary IVP and Nester gauge. $dr = 0.01$, $dt = 0.001$. The lapse and shift are reset every $dt = 0.002$.

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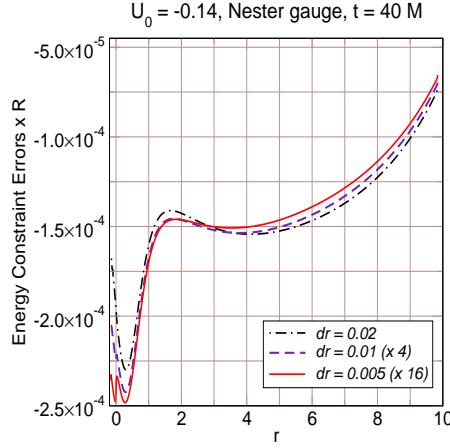


FIG. 5: Convergence of energy constraint errors at $t = 40 M$ for stationary IVP and Nester gauge. The results at resolution $dr = 0.01$ have been multiplied by 4, and those at $dr = 0.005$ have been multiplied by 16. $dr/dt = 10.0$ for all resolutions. The lapse and shift are reset every $dt = 0.002$.

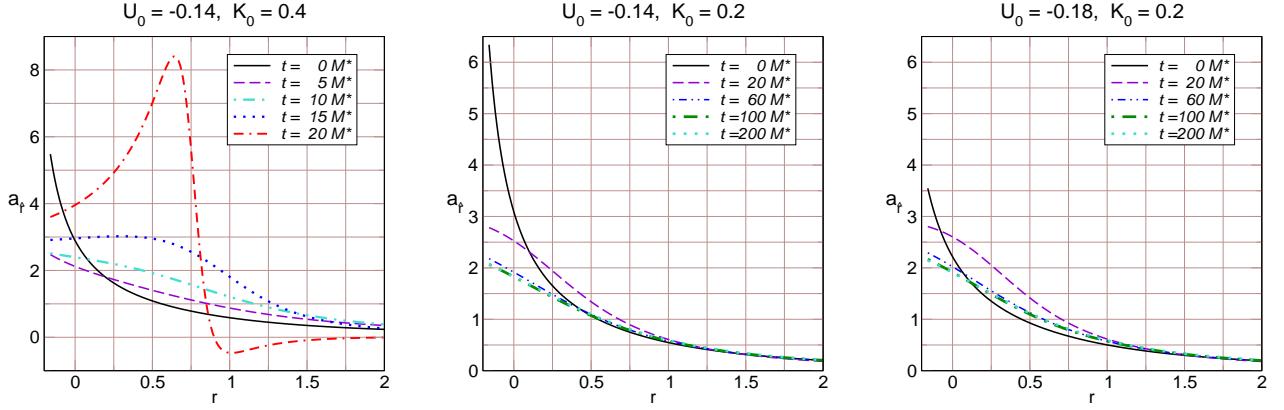


FIG. 6: CMC initial conditions and Nester gauge: With $K_0 = 0.4$, a_r evolves pathologically, but with $K_0 = 0.2$, a_r approaches a stationary solution (shown is region near event horizon). *These times are for $r = 9.84$.