

The Tsallis entropy and the Shannon entropy of a universal probability

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Abstract—We study the properties of Tsallis entropy and Shannon entropy from the point of view of algorithmic randomness. In algorithmic information theory, there are two equivalent ways to define the program-size complexity $K(s)$ of a given finite binary string s . In the standard way, $K(s)$ is defined as the length of the shortest input string for the universal self-delimiting Turing machine to output s . In the other way, the so-called universal probability m is introduced first, and then $K(s)$ is defined as $-\log_2 m(s)$ without reference to the concept of program-size. In this paper, we investigate the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability by means of the notion of program-size complexity. We determine the convergence or divergence of each of these three quantities, and evaluate its degree of randomness if it converges.

I. INTRODUCTION

Algorithmic information theory is a framework to apply information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of algorithmic information theory is the *program-size complexity* (or *Kolmogorov complexity*) $K(s)$ of a finite binary string s , which is defined as the length of the shortest binary program for the universal self-delimiting Turing machine U to output s . By the definition, $K(s)$ can be thought of as the information content of the individual finite binary string s . In fact, algorithmic information theory has precisely the formal properties of classical information theory (see Chaitin [2]). The concept of program-size complexity plays a crucial role in characterizing the randomness of a finite or infinite binary string.

The program-size complexity $K(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $K(s)$ without referring to such a concept, i.e., we first introduce a *universal probability* m , and then define $K(s)$ as $-\log_2 m(s)$.

In this paper, we investigate the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability, from the point of view of algorithmic randomness, by means of the notion of program-size complexity. In particular, we show the following: (i) The Shannon entropy of any universal probability diverges to infinity. (ii) If q is a computable real number with $q \geq 1$, then the power sum $\sum_s m(s)^q$ of any universal probability m has the *degree of randomness* at least $1/q$. Here the notion of degree of randomness is a stronger notion than compression rate, and is defined using the program-size complexity [9], [10]. (iii) If $0 < q < 1$, then the power sum $\sum_s m(s)^q$ diverges to infinity.

(iv) In the case where q is a computable real number with $q > 1$, the Tsallis entropy $S_q(m)$ of a universal probability m can have any computable degree of randomness. (v) If $0 < q < 1$, then the Tsallis entropy $S_q(m)$ diverges to infinity.

II. PRELIMINARIES

We start with some notation about numbers and strings which will be used in this paper.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and \mathbb{N}^+ is the set of positive integers. \mathbb{Q} is the set of rational numbers, and \mathbb{R} is the set of real numbers. $\{0, 1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$ is the set of finite binary strings where λ denotes the *empty string*, and $\{0, 1\}^*$ is ordered as indicated. We identify any string in $\{0, 1\}^*$ with a natural number in this order, i.e., we consider $\varphi: \{0, 1\}^* \rightarrow \mathbb{N}$ such that $\varphi(s) = 1s - 1$ where the concatenation $1s$ of strings 1 and s is regarded as a dyadic integer, and then we identify s with $\varphi(s)$. For any $s \in \{0, 1\}^*$, $|s|$ is the *length* of s . A subset S of $\{0, 1\}^*$ is called a *prefix-free set* if no string in S is a prefix of another string in S . $\{0, 1\}^\infty$ is the set of infinite binary strings, where an infinite binary string is infinite to the right but finite to the left. For any $\alpha \in \{0, 1\}^\infty$ and any $n \in \mathbb{N}^+$, α_n is the prefix of α of length n . For any partial function f , the domain of definition of f is denoted by $\text{dom } f$. We write “r.e.” instead of “recursively enumerable.”

Normally, $o(n)$ denotes any function $f: \mathbb{N}^+ \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any function $g: \mathbb{N}^+ \rightarrow \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \leq C$ for all $n \in \mathbb{N}^+$.

Let T be an arbitrary real number. $T \bmod 1$ denotes $T - \lfloor T \rfloor$, where $\lfloor T \rfloor$ is the greatest integer less than or equal to T . Hence, $T \bmod 1 \in [0, 1)$. We identify a real number T with the infinite binary string α such that $0.\alpha$ is the base-two expansion of $T \bmod 1$ with infinitely many zeros. Thus, T_n denotes the first n bits of the base-two expansion of the real number $T \bmod 1$ with infinitely many zeros.

We say that a real number T is *computable* if there exists a total recursive function $f: \mathbb{N}^+ \rightarrow \mathbb{Q}$ such that $|T - f(n)| < 2^{-n}$ for all $n \in \mathbb{N}^+$. We say that T is *right-computable* if there exists a total recursive function $g: \mathbb{N}^+ \rightarrow \mathbb{Q}$ such that $T \leq g(n)$ for all $n \in \mathbb{N}^+$ and $\lim_{n \rightarrow \infty} g(n) = T$. We say that T is *left-computable* if $-T$ is right-computable. It is then easy to see that, for any $T \in \mathbb{R}$, T is computable if and only if T is both right-computable and left-computable. See

e.g. Pour-El and Richards [6] and Weihrauch [14] for the detail of the treatment of the computability of real numbers and real functions on a discrete set.

A. Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [2], [3], [4]. A *computer* is a partial recursive function $C: \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that $\text{dom } C$ is a prefix-free set. For each computer C and each $s \in \{0, 1\}^*$, $K_C(s)$ is defined by $K_C(s) = \min \{ |p| \mid p \in \{0, 1\}^* \text{ & } C(p) = s \}$. A computer U is said to be *optimal* if for each computer C there exists a constant $\text{sim}(C)$ with the following property; if $C(p)$ is defined, then there is a p' for which $U(p') = C(p)$ and $|p'| \leq |p| + \text{sim}(C)$. It is easy to see that there exists an optimal computer. Note that the class of optimal computers equals to the class of functions which are computed by *universal self-delimiting Turing machines* (see [2] for the detail). We choose a particular optimal computer U as the standard one for use, and define $K(s)$ as $K_U(s)$, which is referred to as the *program-size complexity* of s , the *information content* of s , or the *Kolmogorov complexity* of s . Thus, $K(s) \leq K_C(s) + \text{sim}(C)$ for any computer C .

The program-size complexity $K(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $K(s)$ without referring to such a concept, i.e., as in the following, we first introduce a *universal probability* m , and then define $K(s)$ as $-\log_2 m(s)$. We say that r is a *semi-measure* on $\{0, 1\}^*$ if $r: \{0, 1\}^* \rightarrow [0, 1]$ such that $\sum_{s \in \{0, 1\}^*} r(s) \leq 1$. A universal probability is defined as follows [15].

Definition 1 (universal probability) We say that r is a lower-computable semi-measure if r is a semi-measure on $\{0, 1\}^*$ and there exists a total recursive function $f: \mathbb{N}^+ \times \{0, 1\}^* \rightarrow \mathbb{Q}$ such that, for each $s \in \{0, 1\}^*$, $\lim_{n \rightarrow \infty} f(n, s) = r(s)$ and $\forall n \in \mathbb{N}^+ \ 0 \leq f(n, s) \leq r(s)$. We say that a lower-computable semi-measure m is a universal probability if for any lower-computable semi-measure r , there exists a real number $c > 0$ such that, for all $s \in \{0, 1\}^*$, $cr(s) \leq m(s)$. ■

The following theorem can be then shown (see e.g. Theorem 3.4 of Chaitin [2] for its proof). Here, $P(s)$ is defined as $\sum_{U(p)=s} 2^{-|p|}$ for each $s \in \{0, 1\}^*$.

Theorem 2 Both $2^{-K(s)}$ and $P(s)$ are universal probabilities. ■

By Theorem 2, we see that, for any universal probability m ,

$$K(s) = -\log_2 m(s) + O(1). \quad (1)$$

Thus it is possible to define $K(s)$ as $-\log_2 m(s)$ with a particular universal probability m instead of as $K_U(s)$. Note that the difference up to an additive constant is nonessential to algorithmic information theory. Any universal probability is not computable, as corresponds to the uncomputability of

$K(s)$. As a result, we see that $0 < \sum_{s \in \{0, 1\}^*} m(s) < 1$ for any universal probability m .

For any $\alpha \in \{0, 1\}^\infty$, we say that α is *weakly Chaitin random* if there exists $c \in \mathbb{N}$ such that, for all $n \in \mathbb{N}^+$, $n - c \leq K(\alpha_n)$ [2], [4]. As the total sum of the universal probability $2^{-K(s)}$, Chaitin [3] introduced the real number θ by

$$\theta = \sum_{s \in \{0, 1\}^*} 2^{-K(s)}. \quad (2)$$

Then [3] showed that θ is weakly Chaitin random.

In the works [9], [10], we generalized the notion of the randomness of an infinite binary string so that the degree of the randomness can be characterized by a real number D with $0 < D \leq 1$ as follows.

Definition 3 (weakly Chaitin D -random) Let $D \in \mathbb{R}$ with $D \geq 0$, and let $\alpha \in \{0, 1\}^\infty$. We say that α is weakly Chaitin D -random if there exists $c \in \mathbb{N}$ such that, for all $n \in \mathbb{N}^+$, $Dn - c \leq K(\alpha_n)$. ■

Definition 4 (D -compressible) Let $D \in \mathbb{R}$ with $D \geq 0$, and let $\alpha \in \{0, 1\}^\infty$. We say that α is D -compressible if $K(\alpha_n) \leq Dn + o(n)$, which is equivalent to $\overline{\lim}_{n \rightarrow \infty} K(\alpha_n)/n \leq D$. ■

In the case of $D = 1$, the weak Chaitin D -randomness results in the weak Chaitin randomness. For any $D \in [0, 1]$ and any $\alpha \in \{0, 1\}^\infty$, if α is weakly Chaitin D -random and D -compressible, then

$$\lim_{n \rightarrow \infty} \frac{K(\alpha_n)}{n} = D, \quad (3)$$

and therefore the compression rate of α by the program-size complexity K is equal to D . Note, however, that (3) does not necessarily implies that α is weakly Chaitin D -random.

In the work [10], we generalized θ to θ^D by

$$\theta^D = \sum_{s \in \{0, 1\}^*} 2^{-\frac{K(s)}{D}} \quad (D > 0). \quad (4)$$

Thus, $\theta = \theta^1$. If $0 < D \leq 1$, then θ^D converges and $0 < \theta^D < 1$, since $\theta^D \leq \theta < 1$. Theorem 5 below was mentioned in Remark 3.2 of Tadaki [10].

Theorem 5 (Tadaki [10]) Let $D \in \mathbb{R}$.

- (i) If $0 < D \leq 1$ and D is computable, then θ^D is weakly Chaitin D -random.
- (ii) If $0 < D \leq 1$ and D is computable, then θ^D is D -compressible.
- (iii) If $1 < D$, then θ^D diverges to ∞ . ■

III. THE SHANNON ENTROPY OF A UNIVERSAL PROBABILITY

We say that $p = (p_1, \dots, p_n)$ is a *probability distribution* if $p_i \in [0, 1]$ for all $i = 1, \dots, n$ and $p_1 + \dots + p_n = 1$. For any probability distribution $p = (p_1, \dots, p_n)$, the *Shannon entropy* $H(p)$ of p is defined by

$$H(p) = - \sum_{i=1}^n p_i \ln p_i, \quad (5)$$

where the \ln denotes the natural logarithm [7]. We say that $p = (p_1, \dots, p_n)$ is a *semi-probability distribution* if $p_i \in [0, 1]$ for all $i = 1, \dots, n$ and $p_1 + \dots + p_n \leq 1$. We define the Shannon entropy $H(p)$ also for any semi-probability distribution $p = (p_1, \dots, p_n)$ by (5). Moreover, for any semi-measure r on $\{0, 1\}^*$, we define the *Shannon entropy* $H(r)$ of r by

$$H(r) = - \sum_{s \in \{0,1\}^*} r(s) \ln r(s)$$

in a similar manner to (5).

In this section, we prove that the Shannon entropy $H(m)$ of an arbitrary universal probability m diverges to ∞ . For convenience, however, we first prove the following more general theorem, Theorem 6, from which the result follows. For example, Theorem 6 itself can be used to determine the properties of the notions of *thermodynamic quantities* introduced by Tadaki [12] into algorithmic information theory.

Theorem 6 *Let A be an infinite r.e. subset of $\{0, 1\}^*$ and let $f: \mathbb{N}^+ \rightarrow \mathbb{N}$ be a total recursive function such that $\lim_{n \rightarrow \infty} f(n) = \infty$. Then the following hold.*

- (i) $\sum_{U(p) \in A} f(|p|)2^{-|p|}$ diverges to ∞ .
- (ii) If there exists $l_0 \in \mathbb{N}^+$ such that $f(l)2^{-l}$ is a nonincreasing function of l for all $l \geq l_0$, then $\sum_{s \in A} f(K(s))2^{-K(s)}$ diverges to ∞ .

Proof: (i) Contrarily, assume that $\sum_{U(p) \in A} f(|p|)2^{-|p|}$ converges. Then, there exists $d \in \mathbb{N}^+$ such that $\sum_{U(p) \in A} f(|p|)2^{-|p|} \leq d$. We define the function $r: \{0, 1\}^* \rightarrow [0, \infty)$ by

$$r(s) = \frac{1}{d} \sum_{U(p)=s} f(|p|)2^{-|p|}$$

if $s \in A$; $r(s) = 0$ otherwise. We then see that $\sum_{s \in \{0,1\}^*} r(s) \leq 1$ and therefore r is a lower-computable semi-measure. Since $P(s)$ is a universal probability by Theorem 2, there exists $c \in \mathbb{N}^+$ such that $r(s) \leq cP(s)$ for all $s \in \{0, 1\}^*$. Hence we have

$$\sum_{U(p)=s} (cd - f(|p|))2^{-|p|} \geq 0 \quad (6)$$

for all $s \in A$. On the other hand, since A is an infinite set and $\lim_{n \rightarrow \infty} f(n) = \infty$, there is $s_0 \in A$ such that $f(|p|) > cd$ for all p with $U(p) = s_0$. Therefore we have $\sum_{U(p)=s_0} (cd - f(|p|))2^{-|p|} < 0$. However, this contradicts (6), and the proof of (i) is completed.

(ii) We first note that there is $n_0 \in \mathbb{N}$ such that $K(s) \geq l_0$ for all s with $|s| \geq n_0$. Now, let us assume contrarily that $\sum_{s \in A} f(K(s))2^{-K(s)}$ converges. Then, there exists $d \in \mathbb{N}^+$ such that $\sum_{s \in A} f(K(s))2^{-K(s)} \leq d$. We define the function $r: \{0, 1\}^* \rightarrow [0, \infty)$ by

$$r(s) = \frac{1}{d} f(K(s))2^{-K(s)}$$

if $s \in A$ and $|s| \geq n_0$; $r(s) = 0$ otherwise. We then see that $\sum_{s \in \{0,1\}^*} r(s) \leq 1$ and therefore r is a lower-computable

semi-measure. Since $2^{-K(s)}$ is a universal probability by Theorem 2, there exists $c \in \mathbb{N}^+$ such that $r(s) \leq c2^{-K(s)}$ for all $s \in \{0, 1\}^*$. Hence, if $s \in A$ and $|s| \geq n_0$, then $cd \geq f(K(s))$. On the other hand, since A is an infinite set and $\lim_{n \rightarrow \infty} f(n) = \infty$, there is $s_0 \in A$ such that $|s_0| \geq n_0$ and $f(K(s_0)) > cd$. Thus, we have a contradiction, and the proof of (ii) is completed. ■

From Theorem 6 (ii), we obtain the following result, as desired.

Corollary 7 *Let m be a universal probability. Then the Shannon entropy $H(m)$ of m diverges to ∞ .*

Proof: We first note that there is a real number $x_0 > 0$ such that the function $x2^{-x}$ of a real number x is decreasing for $x \geq x_0$. For this x_0 , there is $n_0 \in \mathbb{N}$ such that $-\log_2 m(s) \geq x_0$ for all s with $|s| \geq n_0$. On the other hand, by (1), there is $c \in \mathbb{N}$ such that $-\log_2 m(s) \leq K(s) + c$ for all $s \in \{0, 1\}^*$. Thus, we see that

$$\begin{aligned} & - \sum_{s \in \{0,1\}^* \text{ & } |s| \geq n_0} m(s) \log_2 m(s) \\ & \geq \sum_{s \in \{0,1\}^* \text{ & } |s| \geq n_0} (K(s) + c)2^{-K(s)-c} \\ & = 2^{-c} \sum_{s \in \{0,1\}^* \text{ & } |s| \geq n_0} K(s)2^{-K(s)} \\ & \quad + c2^{-c} \sum_{s \in \{0,1\}^* \text{ & } |s| \geq n_0} 2^{-K(s)}. \end{aligned} \quad (7)$$

Using Theorem 6 (ii) with $A = \{0, 1\}^*$ and $f(n) = n$, we see that $\sum_{s \in \{0,1\}^*} K(s)2^{-K(s)}$ diverges to ∞ . It follows from (7) that $-\sum_{s \in \{0,1\}^*} m(s) \log_2 m(s)$ also diverges to ∞ . This completes the proof. ■

IV. THE POWER SUM OF A UNIVERSAL PROBABILITY

In this section, we investigate the convergence or divergence of the power sum $\sum_{s \in \{0,1\}^*} m(s)^q$ of a universal probability m , and evaluate its degree of randomness if it converges, by means of the notions of the weak Chaitin D -randomness and the D -compressibility. We first consider the notion of the weak Chaitin D -randomness of the power sum of a universal probability. We can generalize Theorem 5 (i) and (iii) on the specific universal probability $2^{-K(s)}$ over an arbitrary universal probability as follows.

Theorem 8 *Let m be a universal probability, and let $q \in \mathbb{R}$.*

- (i) If $q \geq 1$ and q is a right-computable real number, then $\sum_{s \in \{0,1\}^*} m(s)^q$ converges to a left-computable real number which is weakly Chaitin $1/q$ -random.
- (ii) If $0 < q < 1$, then $\sum_{s \in \{0,1\}^*} m(s)^q$ diverges to ∞ . ■

Theorem 8 (i) shows that, for any $q \in \mathbb{R}$ with $q \geq 1$, the right-computability of q results in the weak Chaitin $1/q$ -randomness of the power sum $\sum_{s \in \{0,1\}^*} m(s)^q$ of a universal probability m . On the other hand, Theorem 9 below shows that

the converse in a certain sense holds. Theorem 9 can be proved based on the techniques used in the proof of *the fixed point theorem on compression rate* [12].

Theorem 9 *Let m be a universal probability, and let $q \in \mathbb{R}$ with $q \geq 1$. If $\sum_{s \in \{0,1\}^*} m(s)^q$ is a right-computable real number, then q is weakly Chaitin $1/q$ -random.* ■

Next, we consider the notion of the D -compressibility of the power sum of a universal probability. Theorem 5 (ii) shows that, for the specific universal probability $m(s) = 2^{-K(s)}$, if q is a computable real number with $q > 1$, then the power sum $\sum_{s \in \{0,1\}^*} m(s)^q$ is $1/q$ -compressible. Thus, the following question naturally arises: Is $\sum_{s \in \{0,1\}^*} m(s)^q$ a $1/q$ -compressible real number for any universal probability m and any computable real number $q > 1$? As shown in Theorem 10, however, we can answer this question negatively.

Theorem 10 *There exists a universal probability m such that, for every computable real number $q > 1$, $\sum_{s \in \{0,1\}^*} m(s)^q$ is weakly Chaitin random and therefore not $1/q$ -compressible.*

Proof: We choose any one universal probability r , and then choose any one $c \in \mathbb{N}$ with $2^{-c}\theta \leq r(\lambda)$, where θ is defined by (2). We define the function $m: \{0,1\}^* \rightarrow [0, \infty)$ by $m(s) = 2^{-c}\theta$ if $s = \lambda$; $m(s) = r(s)$ otherwise. Since $\sum_{s \in \{0,1\}^*} r(s) \leq 1$, it follows that $\sum_{s \in \{0,1\}^*} m(s) \leq 1$. Therefore, since θ is left-computable and r is a lower-computable semi-measure, we see that m is a lower-computable semi-measure. Note that $dr(s) \leq m(s)$ for all $s \in \{0,1\}^*$, where $d = 2^{-c}\theta/r(\lambda) > 0$. Thus, since r is a universal probability, m is also a universal probability.

On the other hand, since θ is weakly Chaitin random, $m(\lambda)$ is also weakly Chaitin random. Let q be an arbitrary computable real number with $q > 1$. Then, since q is a computable real number with $q \neq 0$, it follows that $K((a^q)_n) = K(a_n) + O(1)$ for any real number $a > 0$. Thus, $K((m(\lambda)^q)_n) = K((m(\lambda))_n) + O(1)$ and therefore $m(\lambda)^q$ is weakly Chaitin random. Note that $K(a_n) \leq K((a+b)_n) + O(1)$ for any left-computable real numbers a, b . This can be proved using the condition 2 of Lemma 4.4 and Theorem 4.9 of [1]. Thus, since $m(\lambda)^q$ and $\sum_{s \neq \lambda} m(s)^q$ are left-computable, we see that $\sum_{s \in \{0,1\}^*} m(s)^q$ is weakly Chaitin random. It follows from $q > 1$ that $\sum_{s \in \{0,1\}^*} m(s)^q$ is not $1/q$ -compressible. ■

V. THE TSALLIS ENTROPY OF A UNIVERSAL PROBABILITY

The notion of Tsallis entropy has been introduced by Tsallis [13]. Let q be a positive real number with $q \neq 1$. For any probability distribution $p = (p_1, \dots, p_n)$, the *Tsallis entropy* $S_q(p)$ of p is defined by

$$S_q(p) = \frac{1 - \sum_{i=1}^n p_i^q}{q - 1}. \quad (8)$$

When $q \rightarrow 1$, the Tsallis entropy recovers the Shannon entropy for any probability distribution. See [13], [5] for the detail of the theory and applications of Tsallis entropy.

We generalize the definition (8) for any semi-probability distribution $p = (p_1, \dots, p_n)$ by

$$S_q(p) = \frac{\sum_{i=1}^n \{p_i - p_i^q\}}{q - 1}. \quad (9)$$

In fact, we see that, for any semi-probability distribution p , $\lim_{q \rightarrow 1} S_q(p) = H(p)$, and therefore this generalization (9) is consistent with the Shannon entropy for a semi-probability distribution, defined in Section III. Thus, we define the *Tsallis entropy* $S_q(r)$ of any semi-measure r on $\{0,1\}^*$ by

$$S_q(r) = \frac{1}{q - 1} \sum_{s \in \{0,1\}^*} \{r(s) - r(s)^q\}$$

in a similar manner to (9).

In what follows, we investigate the convergence or divergence of the Tsallis entropy $S_q(m)$ of a universal probability m , and evaluate its degree of randomness if it converges, in the same manner as the previous section. We first investigate the convergence and divergence of $S_q(m)$ as follows.

Theorem 11 *Let m be a universal probability, and let $q \in \mathbb{R}$.*

- (i) *If $q > 1$, then $S_q(m)$ converges.*
- (ii) *If $0 < q < 1$, then $S_q(m)$ diverges to ∞ .*

Proof: Theorem 11 follows immediately from Theorem 8. ■

Theorem 12 below shows that, if the total sum of a universal probability m is small, then the Tsallis entropy of m has to be maximally random with respect to the degree of randomness.

Theorem 12 *Let m be a universal probability, and let q be a computable real number with $q > 1$. If $m(s) \leq q^{\frac{1}{1-q}}$ for all $s \in \{0,1\}^*$, then $S_q(m)$ is left-computable and weakly Chaitin random.*

Proof: By Theorem 11 (i), there is $d \in \mathbb{N}^+$ such that $S_q(m) \leq d$. We define $r: \{0,1\}^* \rightarrow (0, \infty)$ by $r(s) = F(m(s))/d$, where $F: (0,1] \rightarrow [0, \infty)$ with $F(x) = (x - x^q)/(q - 1)$. We show that r is a universal probability.

Obviously, $\sum_{s \in \{0,1\}^*} r(s) \leq 1$. Since m is a lower-computable semi-measure, there exists a total recursive function $f: \mathbb{N}^+ \times \{0,1\}^* \rightarrow \mathbb{Q}$ such that, for each $s \in \{0,1\}^*$, $\lim_{n \rightarrow \infty} f(n, s) = m(s)$ and $\forall n \in \mathbb{N}^+ \ 0 < f(n, s) \leq m(s)$. Since $F(x)$ is continuous and increasing for all $x \in (0, q^{\frac{1}{1-q}}]$, it follows that, for each $s \in \{0,1\}^*$, $\lim_{n \rightarrow \infty} F(f(n, s)) = F(m(s))$ and $\forall n \in \mathbb{N}^+ \ 0 \leq F(f(n, s)) \leq F(m(s))$. On the other hand, since q is computable, there exists a total recursive function $g: \mathbb{N}^+ \times \{0,1\}^* \rightarrow \mathbb{Q} \cap [0, \infty)$ such that, for each $s \in \{0,1\}^*$ and each $n \in \mathbb{N}^+$,

$$F(f(n, s)) - 2^{-n} \leq g(n, s) \leq F(f(n, s)).$$

Hence, r is a lower-computable semi-measure. Note that $x/q \leq F(x)$ for all $x \in (0, q^{\frac{1}{1-q}}]$. It follows that $m(s)/(qd) \leq r(s)$ for all $s \in \{0,1\}^*$. Thus, since m is a universal probability, r is also a universal probability.

It follows from Theorem 8 (i) that $\sum_{s \in \{0,1\}^*} r(s) = S_q(m)/d$ is weakly Chaitin random. Note that $K(a_n) \leq K((ab)_n) + O(1)$ for any left-computable real numbers $a, b > 0$. This can be proved using the condition 4 of Lemma 4.4 and Theorem 4.9 of [1]. Thus, since $\sum_{s \in \{0,1\}^*} r(s)$ and d are left-computable positive real numbers, we see that $S_q(m)$ is weakly Chaitin random and, obviously, left-computable. ■

Based on Theorem 12, we can show a stronger result than Theorem 12 with respect to the range of the degree of randomness of the Tsallis entropy $S_q(m)$. Theorem 13 and Corollary 14 below show that the Tsallis entropy of a universal probability can have any computable degree of randomness D . Note, however, that Theorem 13 is not a generalization of Theorem 12. The reason is as follows: The Tsallis entropy $S_q(m)$ is right-computable in Theorem 13 whereas it is not right-computable in Theorem 12.

Theorem 13 *Let q be a computable real number with $q > 1$. Then, for any right-computable real number $y \in (0, q^{\frac{1}{1-q}}]$, there exists a universal probability m such that $S_q(m) = y$.*

Proof: Let $F: (0, 1] \rightarrow [0, \infty)$ with $F(x) = (x - x^q)/(q - 1)$, and let x_0 be the unique real number such that $q^{\frac{1}{1-q}} < x_0 < 1$ and $F(x_0) = y/2$. We choose any one rational number c such that $0 < c \leq \min\{q^{\frac{1}{1-q}}, 1 - x_0, (q-1)y/2\}$. We also choose any one universal probability r . We then define a universal probability $r_1: \{0, 1\}^* \rightarrow (0, 1)$ by $r_1(s) = cr(s)$. Since $r_1(s) \leq q^{\frac{1}{1-q}}$ for all $s \in \{0, 1\}^*$, it follows from Theorem 12 that $S_q(r_1)$ is left-computable.

Let $\Theta = S_q(r_1)$. From $\sum_{s \in \{0,1\}^*} r(s) \leq 1$ we have $\sum_{s \in \{0,1\}^*} r_1(s) \leq c$. Therefore,

$$\Theta = \sum_{s \in \{0,1\}^*} F(r_1(s)) < \frac{1}{q-1} \sum_{s \in \{0,1\}^*} r_1(s) \leq \frac{c}{q-1}.$$

Since $c/(q-1) \leq y/2$, it follows that $y/2 < y - \Theta < y$.

Note that $F(x)$ is continuous and decreasing for all $x \in [q^{\frac{1}{1-q}}, 1]$. Thus, since $F(q^{\frac{1}{1-q}}) = q^{\frac{q}{1-q}} \geq y$ and $y/2 > F(1) = 0$, there exists the unique real number a such that $q^{\frac{1}{1-q}} < a < x_0$ and $F(a) = y - \Theta$. We see that a is left-computable. This is because $y - \Theta$ is right-computable, q is computable, and $F(x)$ is decreasing for all $x \in (q^{\frac{1}{1-q}}, x_0)$.

We define the function $m: \{0, 1\}^* \rightarrow (0, \infty)$ by $m(s) = a$ if $s = \lambda$; $m(s) = r_1(s-1)$ otherwise. Note here that $\{0, 1\}^*$ is identified with \mathbb{N} . Then, it follows from $c \leq 1 - x_0$ and $a < x_0$ that $\sum_{s \in \{0,1\}^*} m(s) < 1$. Thus, since r_1 is a lower-computable semi-measure and a is left-computable, we see that m is a lower-computable semi-measure. Since r_1 is a universal probability and $a > 0$, we further see that m is a universal probability. On the other hand, $S_q(m) = F(a) + S_q(r_1) = F(a) + \Theta = y$. This completes the proof. ■

Corollary 14 *Let q be a computable real number with $q > 1$. Then, for any computable real number $D \in [0, 1]$, there exists a universal probability m such that $S_q(m)$ is weakly Chaitin D -random and D -compressible.*

Proof: In the case of $D = 0$, consider a rational number $y \in (0, q^{\frac{1}{1-q}}]$ in Theorem 13. In the case of $D > 0$, consider $y = a(1 - \theta^D)$ in Theorem 13, where a is any one rational number with $a \in (0, q^{\frac{1}{1-q}}]$ and θ^D is defined by (4). In this case, the result follows from Theorem 5 (i) and (ii). ■

VI. CONCLUSION

In this paper, we have investigated the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability, from the point of view of algorithmic randomness. Future work may aim at generalizing Rényi entropy over a universal probability properly and investigating its randomness properties.

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