

# REDUCED RELATIVE TUTTE, KAUFFMAN BRACKET AND JONES POLYNOMIALS OF VIRTUAL LINK FAMILIES

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## Abstract

This paper contains general formulae for the reduced relative Tutte, Kauffman bracket and Jones polynomials of families of virtual knots and links given in Conway notation and discussion of a counterexample to the  $Z$ -move conjecture of Fenn, Kauffman and Manturov.

*Keywords:* Conway notation; virtual knot; Tutte polynomial; Bollobás-Riordan polynomial; relative Tutte polynomial; reduced relative Tutte polynomial; Jones polynomial; Kauffman bracket polynomial; link family.

## 1. Introduction

According to Thistlethwaite's Theorem [1] and Kauffman's extension of it [2], the Jones polynomial and Kauffman bracket polynomial of any classical knot or link (shortly  $KL$ ) can be computed from the (signed) graph of the  $KL$  [2] via the Tutte polynomial for signed graphs. Without computing Tutte polynomials, the Jones polynomials of the diagrams with at most 5 twist regions in which crossings can be inserted are computed by D. Silver, A. Stoimenow, and S. Williams [3], and the general method for computing Kauffman bracket polynomial of such diagrams is outlined by A. Champanerkar and I. Kofman [4].

The general formulae for Tutte and Jones polynomials for families of classical  $KLs$  with at most 5 twisting regions, given in Conway notation, are derived in [5].

In Subsections 2.1-2.3 we compute Kauffman bracket and Jones polynomials for several families of virtual knots using the Tutte polynomials of their signed graph. In Section 4 we use Vassily Manturov's parity bracket to find a new counterexample to the  $Z$ -move conjecture, and in Section 5 plot zeros of Jones polynomials of certain families of virtual knots and links.

Knots and links (or shortly  $KLs$ ) can be given in Conway notation [6, 7, 8, 9]. For readers unfamiliar with it, we explain Conway notation, introduced in Conway's seminal paper [6] published in 1967, and effectively used since (e.g., [7, 8, 9]). Conway symbols of knots with up to 10 crossings and links with at most 9 crossings are given in the Appendix of the book [7].

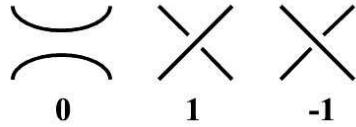


Figure 1: The elementary tangles.

The main building blocks in Conway notation are elementary tangles. We distinguish three elementary tangles, shown in Fig. 1 and denoted by 0, 1 and  $-1$ . All other tangles can be obtained by combining elementary tangles, while 0 and 1 are sufficient for generating alternating knots and links. Elementary tangles can be combined by following operations: *sum*, *product*, and *ramification* (Figs. 2-3). Given tangles  $a$  and  $b$ , image of  $a$  under reflection with mirror line NW-SE is denoted by  $-a$ , and sum is denoted by  $a+b$ . The product  $ab$  is defined as  $ab = -a+b$ , and *ramification* by  $(a,b) = -a-b$ .

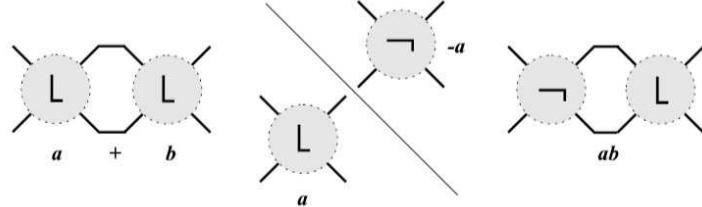


Figure 2: A sum and product of tangles.

The tangle can be closed in two ways (without introducing additional crossings): by joining in pairs NE and NW, and SE and SW ends of a tangle to obtain a *numerator closure*; or by joining in pairs NE and SE, and NW and SW ends we obtain a *denominator closure* (Fig. 4a,b).

**Definition 1.** A *rational tangle* is any finite product of elementary tangles. A *rational KL* is a numerator closure of a rational tangle.

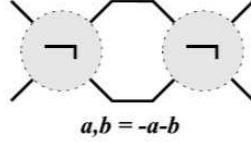


Figure 3: Ramification of tangles.

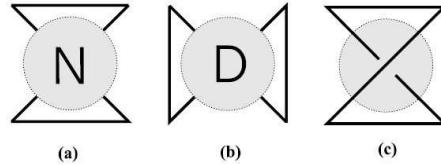


Figure 4: (a) Numerator closure; (b) denominator closure; (c) basic polyhedron 1\*.

**Definition 2.** A tangle is *algebraic* if it can be obtained from elementary tangles using the operations of sum and product. A *KL* is *algebraic* if it is a numerator closure of an algebraic tangle.

**Definition 3.** For a link or knot  $L$  given in an unreduced\* Conway notation  $C(L)$  denote by  $S$  a set of numbers in the Conway symbol excluding numbers denoting basic polyhedron and zeros (determining the position of tangles in the vertices of polyhedron) and let  $\tilde{S} = \{a_1, a_2, \dots, a_k\}$  be a non-empty subset of  $S$ . Family  $F_{\tilde{S}}(L)$  of knots or links derived from  $L$  consists of all knots or links  $L'$  whose Conway symbol is obtained by substituting all  $a_i \neq \pm 1$ , by  $sgn(a_i)|a_i + k_{a_i}|$ ,  $|a_i + k_{a_i}| > 1$ ,  $k_{a_i} \in \mathbb{Z}$ .

An infinite subset of a family is called a *subfamily*. If all  $k_{a_i}$  are even integers, the number of components is preserved within the corresponding subfamilies, i.e., adding full-twists preserves the number of components inside the subfamilies.

**Definition 4.** A link given by Conway symbol containing only tangles  $\pm 1$  and  $\pm 2$  is called a *source link*. A link given by Conway symbol containing only tangles  $\pm 1$ ,  $\pm 2$ , or  $\pm 3$  is called a *generating link*.

For example, Hopf link 2 (link  $2_1^2$  in Rolfsen's notation) is the source link of the simplest link family  $p$  ( $p = 2, 3, \dots$ ) (Fig. 5), and Hopf link and trefoil 3 (knot  $3_1$  in the classical notation) are generating links of this family. A family of *KLs* is usually derived from its source link by substituting  $a_i \in \tilde{S}$ ,  $a_i = \pm 2$ , by  $sgn(a_i)(2 + k)$ ,  $k = 1, 2, 3, \dots$  (see Def. 1.5 and Def. 1.6).

Analogous to the Conway notation for classical *KLs* we use extended Conway notation for virtual *KLs*, by adding to the list of the elementary tangles 0, 1,  $-1$

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\*The Conway notation is called unreduced if in symbols of nonalgebraic links elementary tangles 1 in single vertices are not omitted.

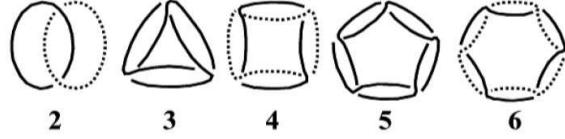


Figure 5: Hopf link  $2_1^2 = 2$  and its family  $p$  ( $p = 2, 3, \dots$ ).

the elementary tangle  $i$  denoting a virtual crossing. The extended Conway notation for virtual links differs from the standard one in the following way:

- virtual crossings are denoted by  $i$ .
- sequence of  $n$  classical crossings  $1, \dots, 1$  (positive  $n$ -twist) is denoted by  $1^n$
- sequence of  $n$  classical negative crossings  $-1, \dots, -1$  (negative  $n$ -twist) is denoted by  $(-1)^n$

The convention introduced in the Conway notation extended to virtual links is clear from the fact that every positive  $n$ -twist can be denoted as  $1, \dots, 1 = 1^n$ , and every negative  $n$ -twist by  $-1, \dots, -1$ . This convention extends to virtual knot families. For example, the simplest  $KL$  family  $2_1^2, 3_1, 4_1^2, 5_1, \dots$  of knots and links, consisting from Hopf link 2, trefoil 3, ... can be denoted by general symbol  $p$  ( $p \geq 2$ ). From them, by substituting in each of them one classical crossing by virtual one, we obtain a family of virtual  $KLs$   $(i, 1), (i, 1, 1 = i, 1^2), (i, 1, 1, 1 = i, 1^3), (i, 1, 1, 1, 1 = i, 1^4), \dots$ , and in general  $i, 1^{p-1}$  ( $p \geq 2$ ).

In comparison with other  $KL$  notations (Gauss codes, Kamada codes, PD-notation, etc.), Conway notation extended to virtual links is much shorter than any other notation for virtual  $KLs$  and the most suitable for utilizing the notion of families of knots and links and analyzing how knot and link properties change inside families. E.g, the Gauss code of the trefoil with one virtual crossing is  $U1 + O2 + O1 + U2+$ , its Kamada code is  $\{\{\{1, 3\}, \{2, 0\}\}, \{1, 1\}\}$ , its PD-code is  $PD[X[3,1,4,2], X[2,4,3,1]]$ , and its Conway symbol is  $(1, 1, i) = (1^2, i)$ , meaning that in the classical trefoil knot  $3 = (1, 1, 1) = 1^3$  the last classical crossing 1 is substituted by the virtual crossing  $i$ .

The relative Tutte polynomial of colored graphs is introduced in the paper [10] by Y. Diao and G. Hetei. An alternative approach to the computation of Bollobás-Riordan polynomial and Kauffman bracket polynomial of virtual  $KLs$ , via ribbon graphs, is given by S. Chmutov and I. Pak [11], and S. Chmutov [12].

Zeroes of the Jones polynomials of different knots and their plots are computed in [13, 14, 15, 16]. Zeroes of Jones polynomials corresponding to different families of classical (alternating and non-alternating)  $KLs$ , called “portraits of the families”, are given in [5].

First we restrict our attention to graphs corresponding only to alternating virtual  $KLs$ , hence we consider graphs corresponding to virtual  $KLs$  with all edges labeled by + or 0 (see [10], Sections 2, 5), with variables  $X_+ = X, Y_+ = Y, x_+ = x, y_+ = y$ ,

$X_0 = 1$ ,  $Y_0 = 1$ ,  $x_0 = x$ ,  $y_0 = y$ . In this setting, we have the following recursive formula for computing the relative Tutte polynomial:

$$T(G) = \begin{cases} \text{if } G \text{ is a graph with no edges} & (1) \\ X_\lambda T(G/e) \text{ if } e \text{ is a bridge} & (2) \\ Y_\lambda T(G - e) \text{ if } e \text{ is a loop} & (3) \\ y_\lambda T(G - e) + x_\lambda T(G/e) \text{ if } e \text{ is neither a loop or a bridge} & (4) \end{cases}$$

where  $G - e$  is the graph obtained from  $G$  by deleting the edge  $e$ , and  $G/e$  is the graph obtained from  $G$  by contracting  $e$ , and  $\lambda$  is the color of  $e$ .

Notice that formulae for the relative Tutte polynomial of non-alternating virtual  $KLs$ , can be obtained from general formulae for alternating virtual  $KLs$  by substituting negative values of parameters. For a graph  $G$  and its dual  $G'$  variables in their corresponding Tutte polynomials  $T(G)$  and  $T(G')$  change their places. For graphs  $G$  and  $H$ ,  $T(G * H) = T(G)T(H)$ , where  $G * H$  is the join of  $G$  and  $H$  at a vertex.

According to the Theorem 5.4 [10], the relative Tutte polynomial  $T_H(G)$  (see Section 5.1 [10]) corresponding to a virtual link diagram  $K$  gives the Kauffman bracket through the following variable substitution:

$$X \rightarrow -A^{-3}, \quad Y \rightarrow -A^3, \quad x \rightarrow A, \quad y \rightarrow A^{-1}, \quad d \rightarrow -(A^2 + A^{-2}),$$

and the Jones polynomial of  $K$  is obtained from the Kauffman bracket polynomial by substituting  $A = t^{-\frac{1}{4}}$ .

## 2. Reduction rules

In this section we consider certain reduction and transformation rules relating graphs, knots and links. For this we first recall the construction of the signed graph of a knot or link diagram. View Fig. 7. In this figure we illustrate how one first forms the two-colored checkerboard for the knot or link diagram, and then associates a signed graph  $G(K)$  to this checkerboard by assigning a node to each shaded region of the checkerboard and an edge to each crossing that is incident to two regions. We label the edges of the resulting graph with  $+1$ ,  $-1$  and  $0$  according as the crossing is classical and positive in relation to the edge, classical and negative in relation to the edge, and virtual in relation to the edge. Virtual crossings are neither over nor under and they are indicated on the knot diagram by a flat crossing with a circle around it as in Fig. 8. This figure also illustrates the  $Z$ -move (or *virtualization*), a move on virtual knot and link diagrams that does not effect the evaluation of the Kauffman bracket or the Jones polynomial. The reader who would like more information about virtual knot theory can consult [19]. In Fig. 6c we illustrate the effect of the  $Z$ -Move on the graph of a virtual knot or link. Note that we have indicated virtual crossings by corresponding graph edges that are labeled with a  $0$  and are dotted edges. We have not indicated any signs in this figure.

Since the Kauffman bracket and Jones polynomial of a virtual link  $L$  are invariant under classical and virtual Reidemeister move II and  $Z$ -move ([17], Lemma 7)

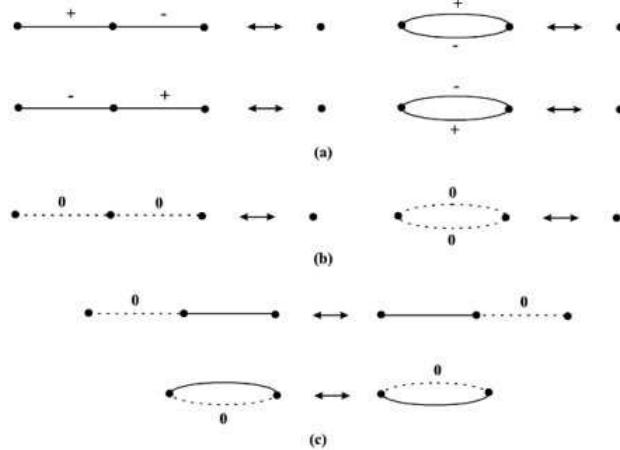


Figure 6: (a) Reidemeister move II for classical crossings; (b) Reidemeister move II for virtual crossings; (c)  $Z$ -move (virtualization) on  $KL$  graphs. Non-signed edges can have sign  $+$  or  $-$ .

on graphs, we can simplify graphs before computations. Graphs of virtual  $KL$ s can be simplified using the following reduction rules:

1. Reidemeister move II for classical crossings (Fig. 6a);
2. Reidemeister move II for virtual crossings (Fig. 6b);
3.  $Z$ -move (Fig. 6c) on  $KL$  graphs.

where 0-edges are denoted by broken lines. The relative Tutte polynomial obtained after this reduction will be called *reduced relative Tutte polynomial*.

A graph is called completely reduced if none of the reduction rules can be applied further. Twists obtained by reductions can contain crossings of the same sign, and at most one virtual crossing, so their corresponding parts of reduced graphs can contain only sequences of edges of the same sign or sets of multiple edges of the same sign, and at most one 0-edge each.

As a simple example of the application of reduction rules to graphs of virtual  $KL$ s consider virtual knots  $(1, -1, 1, 1, i)$ ,  $(1, 1, 1, i, i)$ , and  $(1, 1, i, 1, i)$  (Fig. 9). From the graph of the first knot, by the Reidemeister move II for classical crossings we conclude that its graph reduces to the graph of a trefoil with one virtual crossing (Fig. 9a). From the graph of the second knot, by the Reidemeister move II for virtual crossings, we obtain the graph of the trefoil (Fig. 9b). From the graph of the third knot, by  $Z$ -move and Reidemeister move II for virtual crossings, we conclude that its Kauffman bracket and Jones polynomial are same as those of the trefoil  $1, 1, 1$  ( $1, 1, i, 1, i \rightarrow 1, 1, 1, i, i \rightarrow 1, 1, 1 = 3$ ) (Fig. 9c). Using graph reduction,

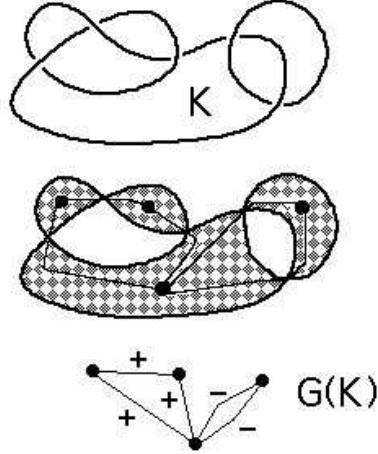


Figure 7: Signed Graph of a Knot.

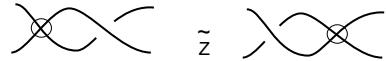


Figure 8: The  $Z$  - Move.

$Z$ -move followed by Reidemeister move II for virtual crossings and Reidemeister move II for classical crossings, we conclude that the Kauffman bracket and Jones polynomial of the knot  $1, 1, i, -1, i$  are unit  $(1, 1, i, -1, i \rightarrow 1, 1, -1, i, i \rightarrow 1, 1, -1 = 1)$  (Fig. 9d). A graph is called completely reduced if none of the reduction rules can be applied further.

### 2.1. The family $p$

Consider the family  $\mathbf{p}$  ( $p \geq 2$ ), which consists of classical knots and links of the form  $1^p$ . Graphs corresponding to links of this family are cycles of length  $p$ , satisfying the following recursion:

$$T(G(1^p)) = yX^{p-1} + xT(G(1^{p-1}))$$

with  $T(G(1^2)) = yX + xY$ , so the general formula for the reduced relative Tutte polynomial of the graph  $G((1^p))$  is

$$T(G(1^p)) = \frac{(x^p - X^p)}{x - X}y + (Y - y)x^{p-1}.$$

Virtual  $KLs$  of the form  $(i, 1^p)$  which belong to the same family, correspond to the reduced graphs shown on Fig. 10, and their Tutte polynomials satisfy the following recursion:

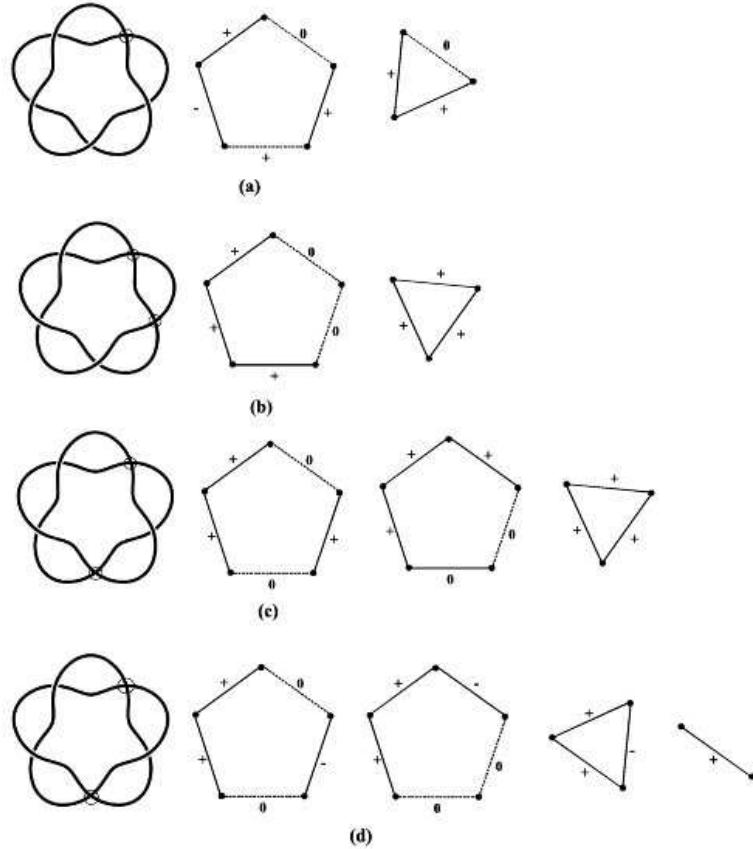


Figure 9: Reduction of the graphs of knots (a)  $1, -1, 1, 1, i$ ; (b)  $1, 1, 1, i, i$ ; (c)  $1, 1, i, 1, i$ . (d)  $1, 1, i, -1, i$ .

$$T(G(i, 1^p)) = yX^{p-1} + xT(G(i, 1^{p-1}))$$

with  $T(G(i, 1)) = x + y$ , so the general formula for the reduced relative Tutte polynomial of the graph  $G((i, 1^p))$  is

$$T(G(i, 1^p)) = x^p + \frac{(x^p - X^p)}{x - X}y.$$

As a corollary of this general formula we obtain reduced relative Tutte polynomials for positive or negative values of the parameter  $p$ , and Kauffman bracket and Jones polynomials. For example, for  $p = 2$  we obtain the reduced relative Tutte polynomial  $T(G(i, 1^2)) = x^2 + xy + Xy$  of the virtual trefoil  $(i, 1, 1)$ , and for  $p = -2$

the reduced relative Tutte polynomial  $T(G(i, (-1)^2)) = \frac{1}{x^2} - \frac{y}{xX^2} - \frac{y}{x^2X}$  of its mirror image  $(i, -1, -1)$ .

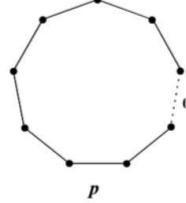


Figure 10: Graph  $G(i, 1^p)$ .

## 2.2. The family $p q$

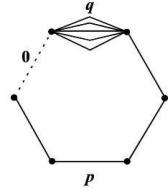


Figure 11: Graph  $G((i, 1^p) (1^q))$ .

Next we consider virtual  $KLs$  of the form  $(i, 1^p) (1^q)$  in the family  $\mathbf{p} \mathbf{q}$ . The reduced relative Tutte polynomials of their graphs (Fig. 11) satisfy the recursion:

$$T(G(i, 1^p)) = yX^{p-1}T(\overline{G}(1^q)) + xT(G((i, 1^{p-1}) (1^q)))$$

where  $\overline{G}(1^q)$  is the dual of the graph  $G(1^q)$ , and  $T(G((i, 1)(1))) = xy + x^2y + x^3$ , so the general formula for the reduced relative Tutte polynomial of the graph  $G((i, 1^p) (1^q))$  is

$$\begin{aligned} T(G((i, 1^p) (1^q))) &= \frac{(x^{q+1} - X^{q+1})(y^{p+1} - Y^{p+1})}{(x - X)(y - Y)} \\ &- \frac{(y^{p+1} - Y^{p+1})}{y - Y} X^q - \frac{(x^{q+1} - X^{q+1})}{x - X} Y^n + \frac{(x^q - X^q)}{x - X} y^{n+1} + X^q Y^n - x^q Y^n. \end{aligned}$$

## 2.3. Family $\mathbf{p} \mathbf{1} \mathbf{q}$

Based on the link family  $\mathbf{p} \mathbf{1} \mathbf{q}$  we construct three families of virtual  $KLs$  with different reduced graphs:  $(i, 1^p) 1 (1^q)$ ,  $(i, 1^p) 1 (i, 1^q)$ , and  $(1^p) i (1^q)$ .

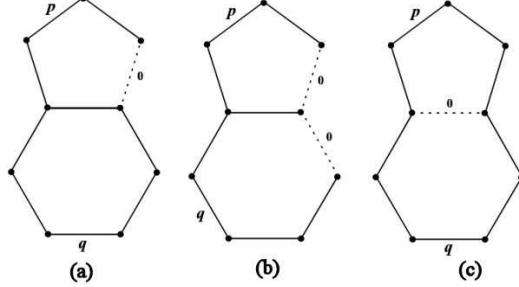


Figure 12: Graphs (a)  $G((i, 1^p) 1 (1^q))$ ; (b)  $G((i, 1^p) 1 (i, 1^q))$ , and (c)  $G((1^p) i (1^q))$ .

The reduced relative Tutte polynomials of the graphs corresponding to the family  $(i, 1^p) 1 (1^q)$  (Fig. 12a) satisfy the relation:

$$T(G((i, 1^p) 1 (1^q))) = yT(G((i, 1^{p+q}))) + xT(G((i, 1^p)))T(G((1^q))), \quad (2.1)$$

and the reduced relative Tutte polynomials of the graphs corresponding to the family  $(i, 1^p) 1 (i, 1^q)$  (Fig. 12b) satisfy the relation:

$$T(G((i, 1^p) 1 (i, 1^q))) = yT(G((1^{p+q}))) + xT(G((i, 1^p)))T(G((i, 1^q))), \quad (2.2)$$

so the general formulas for their reduced relative Tutte polynomials can be derived from the previous formulas (2.2).

The relative Tutte polynomial of the reduced graphs corresponding to the family  $(1^p) i (1^q)$  (Fig. 12c) satisfy the recursion:

$$T(G((1^p) i (1^q))) = yX^{q-1}T(G((i, 1^p))) + xT(G((1^p) i (1^{q-1}))), \quad (2.3)$$

with  $T(G((1^p) i 1)) = T(G((i, 1^p) 1))$ .

Hence, the general formula for the reduced relative Tutte polynomial of the graph  $G((1^p) i (1^q))$  is

$$\begin{aligned} T(G((1^p) i (1^q))) &= \frac{(x^p - X^p)}{x - X}(x + X)x^{q-1}y + \frac{(x^{q-p-1} - X^{q-p-1})}{x - X}x^p X^{p+1}y \\ &+ \frac{(x^{p-2} - X^{p-2})}{x - X} \frac{(x^q - X^q)}{x - X}x^2 y^2 + \frac{(x^q - X^q)}{x - X}(x + X)X^{p-2}y^2 + x^{p+q-1}Y. \end{aligned}$$

Same as before, from the general formula for reduced relative Tutte polynomials with negative values of parameters we obtain reduced relative Tutte polynomials expressed as Laurent polynomials. For example, for  $p = 4, q = -3$  from the preceding general formula we obtain reduced relative Tutte polynomial of the virtual knot  $(1, 1, 1, 1) i (-1, -1, -1)$ :

$$\frac{-x^3y}{X^3} - \frac{x^2y}{X^2} - \frac{xy}{X} + \frac{Xy}{x} + \frac{X^2y}{x^2} + \frac{X^3y}{x^3} - 3\frac{y^2}{x} - \frac{x^2y^2}{X^3} - 2\frac{xy^2}{X^2} - 3\frac{y^2}{X} - 2\frac{Xy^2}{x^2} - \frac{X^2y^2}{x^3} + Y$$

and for  $p = -4$ ,  $q = 3$  we obtain reduced relative Tutte polynomial of its mirror image  $(-1, -1, -1, -1) i (1, 1, 1)$ :

$$\frac{x^2y}{X^4} - \frac{xy}{X^3} - \frac{y}{X^2} - \frac{y}{xX} + \frac{Xy}{x^3} + \frac{X^2y}{x^4} - 2\frac{y^2}{x^3} - \frac{xy^2}{X^4} - 2\frac{y^2}{X^3} - 3\frac{y^2}{xX^2} - 3\frac{y^2}{x^2X} - \frac{Xy^2}{x^4} + \frac{Y}{x^2}.$$

Appropriate substitutions of the variables, yield their Kauffman bracket and Jones polynomials.

### 3. Virtual knots with trivial Jones polynomial and the $Z$ -move

For classical knots, the question whether non-trivial knot with unit Jones polynomial exists is still open. In the case of virtual knots, it is easy to make infinitely many non-trivial virtual knots with unit Jones polynomial [17]. Among 2171 prime virtual knots derived from classical knots with at most  $n = 8$  crossings, there are 272 knots (about 12%) with unit Jones and Kauffman bracket polynomial. The smallest virtual knot with unit Jones polynomial is  $1, 1, i, -1, i$  ([17], Fig. 17), that can be simply generalized to an infinite family of different non-trivial virtual knots of the form  $1^p, i, (-1)^q, i, p - q = 1$ , with unit Jones polynomial (where  $1^p$  denotes a sequence  $1, \dots, 1$  of the length  $p$ , and  $-1^q$  a sequence  $-1, \dots, -1$  of the length  $q$ ). In the same way, for  $p - q = 2k + 1$ , ( $k \geq 1$ ) all mutually different virtual knots  $1^p, i, (-1)^q, i$  will have the same Jones polynomial as the classical knots  $(2k + 1)_1$  ( $1_1, 3_1, 5_1, 7_1, \dots$ , or  $3, 5, 7, \dots$  in Conway notation), respectively.

For virtual  $KLs$  that can be reduced to unknot (unlink) by a series of Reidemeister moves for virtual knots and  $Z$ -moves we will say that they are  $Z$ -move equivalent to the unknot (unlink). The most of mentioned 272 knots with unit Jones polynomial are  $Z$ -move equivalent to the unknot. Hence, R. Fenn, L.H. Kauffman, and V.O. Manturov [24] proposed  $Z$ -move conjecture:

**Conjecture 3.1** *Every knot with unit Jones polynomial is  $Z$ -move equivalent to the unknot.*

The main candidates for counterexamples to this conjecture have been virtual knots  $KS = (((1, (i, 1), -1), -1), i, 1)$  (Fig. 13a,b) and  $S7 = (i, 1) 1, (i, -1) -1, (i, 1)$  (Fig. 14). For each of them, non-triviality can be proved in many different ways (e.g., for the first of them by parity arguments, or by computing their other polynomial invariants: Sawollek polynomial, Miyazawa polynomials [20], Dye-Kauffman arrow polynomial [18], or 2-cabled Jones polynomial. It is interesting to notice that the knot  $KS$  has unit Miyazawa polynomials as well.

In the next section we prove that the virtual knot  $KS$  with unit Jones polynomial is not  $Z$ -move equivalent to unknot, giving a counterexample to this conjecture (already known to be false, see [21, 19]). Methods for proving that counterexamples that we discuss below are indeed counterexamples depend on the use of parity in

virtual knot theory as introduced in [21, 25, 19]. See the next section for a detailed discussion of this method.

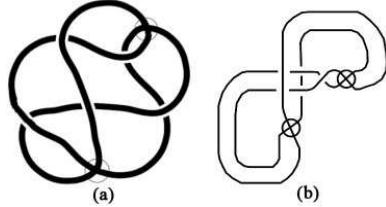


Figure 13: (a),(b) Virtual knot  $KS = (((1, (i, 1), -1), -1), i, 1)$ .

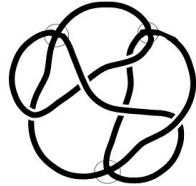


Figure 14: Virtual knot  $S7 = (i, 1) 1, (i, -1) - 1, (i, 1)$ .

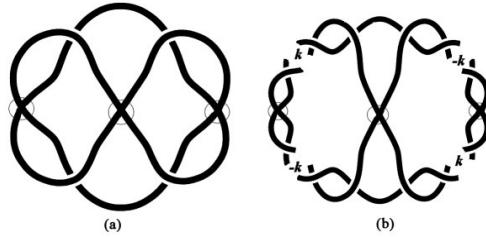


Figure 15: (a) Virtual knot  $(1, i, -1) i (1, i, -1)$ ; (b) family  $(1^k, i, (-1)^k) i (1^k, i, (-1)^k)$ .

Among virtual knots with unit Jones polynomial, probably the most interesting is the virtual knot  $(1, i, -1) i (1, i, -1)$  (Fig. 15a), which generates the family of virtual knots  $(1^k, i, (-1)^k) i (1^k, i, (-1)^k)$  ( $k \geq 1$ ) (Fig. 15b) which cannot be distinguished from unknot by any of the mentioned polynomial invariants, except 2-colored Jones polynomial. All members of this family are  $Z$ -move equivalent to the unknot.

Another interesting virtual knot is  $(i, 1)(1, i)1(-1, i)$  which has all trivial mentioned polynomial invariants, except 2-colored Jones polynomial. Moreover, it is another candidate for a counterexample to  $Z$ -move conjecture (Fig. 16).

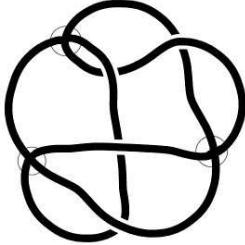


Figure 16: Virtual knot  $(i, 1)(1, i)1(-1, i)$ .

Using  $Z$ -move reduction we can prove many simple facts about Jones polynomial of virtual  $KLs$ . For example:

- if every twist of a knot or link  $L$  which contains virtual crossings has an even number of them, the Jones polynomial of  $L$  is equal to the Jones polynomial of the classical link  $L'$  obtained from  $L$  by deleting virtual crossings.
- Let be given an alternating knot  $K$  with  $n$  crossings. In order to unknot it, in any minimal diagram it is sufficient to make at most  $m = [\frac{n}{2}]$  crossing changes. If a minimal diagram  $D$  of  $K$  can be unknotted by the crossing changes in the crossings  $C_1, \dots, C_k$  ( $k \leq m$ ), the virtual knot diagram  $D'$ , obtained from  $D$  by substituting every positive crossing  $C_i$  ( $i \in \{1, \dots, k\}$ ) by  $i, -1, i$ , and every negative crossing  $C_j$  ( $j \in \{1, \dots, k\}$ ) by  $i, 1, i$ , has unit Jones polynomial. Are all virtual knots obtained in this way non-trivial? Can we obtain from two different minimal diagrams of  $K$  two different virtual knots?

#### 4. The Parity Bracket Polynomial

In this section we introduce the Parity Bracket Polynomial of Vassily Manturov [21]. This is a generalization of the bracket polynomial to virtual knots and links that uses the parity of the crossings. A crossing is *odd* (of odd parity) if it flanks an odd number of symbols in the Gauss code of that diagram. A crossing that flanks an even number of symbols is said to be *even*. For example, if we have a Gauss code of the form 1212 (here we have specified only the crossings 1 and 2, and nothing about their signs or whether they occur over or under) then both crossings are odd. On the other hand in the code 123123, all crossings are even and in the code 123213, the crossings 1 and 2 are odd, while the crossing 3 is even. In a classical knot diagram, it is easy to see (by the Jordan curve theorem) that all crossings are even. But in a virtual knot diagram it is possible to have odd crossings.

We define a *Parity State* of a virtual diagram  $K$  to be a labeled virtual graph obtained from  $K$  as follows: For each odd crossing in  $K$  replace the crossing by a

graphical node. For each even crossing in  $K$  replace the crossing by one of its two possible smoothings, and label the smoothing site by  $A$  or  $A^{-1}$  in the usual way. Then we define the parity bracket by the state expansion formula

$$\langle K \rangle_P = \sum_S A^{n(S)} [S]$$

where  $n(S)$  denotes the number of  $A$ -smoothings minus the number of  $A^{-1}$  smoothings and  $[S]$  denotes a combinatorial evaluation of the state defined as follows: *First reduce the state by Reidemeister two moves on nodes as shown in Fig.17.* The reader should note that we have labeled even crossings by  $e$  and odd crossings by  $o$ . There should be no confusion between this notation and the notation we used previously for a virtual edge in a knot graph.

Here the graphs are taken up to virtual equivalence (planar isotopy plus detour moves on the virtual crossings. See [19].). We regard the reduced state as a disjoint union of standard state loops (without nodes) and graphs that irreducibly contain nodes. With this we write

$$[S] = (-A^2 - A^{-2})^{l(S)} [G(S)]$$

where  $l(S)$  is the number of standard loops in the reduction of the state  $S$  and  $[G(S)]$  is the disjoint union of reduced graphs that contain nodes. In this way, we obtain a sum of Laurent polynomials in  $A$  multiplying reduced graphs as the Manturov Parity Bracket. It is not hard to see that this bracket is invariant under regular isotopy and detour moves and that it behaves just like the usual bracket under the first Reidemeister move. However, the use of parity to make this bracket expand to graphical states gives it considerable extra power in some situations. For example, consider the Kishino diagram in Fig.18. We see that all the classical crossings in this knot are odd. Thus the parity bracket is just the graph obtained by putting nodes at each of these crossings. The resulting graph does not reduce under the graphical Reidemeister two moves, and so we conclude that the Kishino knot is non-trivial and non-classical. Since we can apply the parity bracket to a flat knot by taking  $A = -1$ , we see that this method shows that the Kishino flat is non-trivial.

**Remark.** It should be mentioned that the graph-link theory and free knot theory of Manturov-Ilyutko is the best setting for the Manturov bracket since it is abstract, graphical and does not feel the  $Z$ -move since the free knot theory is already invariant under the  $Z$ -move. The interested reader should see [21] for more details about this theory. Here, we do not use the free knot theory but rather, we use standard virtual knot theory and we formulate a version of the parity bracket in this context. This formulation is given in [19]. The basic idea of the parity bracket is due to Manturov.

In Fig. 19 we illustrate the  $Z$ -move and the *graphical Z-move*. Two virtual knots or links that are related by a  $Z$ -move have the same standard bracket polynomial. This follows directly from our discussion in the previous section. We would like to analyze the structure of  $Z$ -moves using the parity bracket. In order to do this we need a version of the parity bracket that is invariant under the  $Z$ -move. In order to accomplish this, we need to add a corresponding  $Z$ -move in the graphical

$$\begin{aligned}
 \left\langle \overbrace{\diagup \diagdown}^e \right\rangle_P &= A \left\langle \overbrace{\diagup \diagdown}^o \right\rangle_P + A^{-1} \left\langle \overbrace{\quad} \right\rangle_P \\
 \left\langle \overbrace{\diagup \diagdown}^o \right\rangle_P &= \left\langle \overbrace{\diagup \diagdown}^* \right\rangle_P \\
 \left\langle \overbrace{\diagup \diagdown}^* \right\rangle_P &= \left\langle \overbrace{\quad} \right\rangle_P
 \end{aligned}$$

Figure 17: Parity bracket expansion.

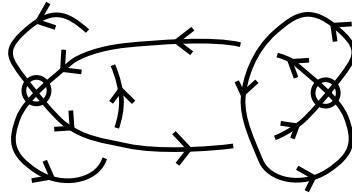


Figure 18: Kishino diagram.

reduction process for the parity bracket. This extra graphical reduction is indicated in Fig. 19 where we show a graphical  $Z$ -move. The reader will note that graphs that are irreducible without the graphical  $Z$ -move can become reducible if we allow graphical  $Z$ -moves in the reduction process. See Fig.19 for an example of this process as well as an illustration of the graphical  $Z$ -move. For example, the graph associated with the Kishino knot is reducible under graphical  $Z$ -moves. However, there are examples of graphs that are not reducible under graphical  $Z$ -moves and Reidemeister two moves. An example of such a graph occurs in the parity bracket of the knot  $KS$  shown in Fig. 20. This knot has one even classical crossing and four odd crossings. One smoothing of the even crossing yields a state that reduces to a loop with no graphical nodes, while the other smoothing yields a state that is irreducible even when the  $Z$ -move is allowed (see Fig. 21). The upshot is that this knot  $KS$  is not  $Z$ -equivalent to any classical knot. Since one can verify that  $KS$  has unit Jones polynomial, this example is a counterexample to a conjecture of Fenn, Kauffman and Manturov [24] that suggested that a knot with unit Jones polynomial should be  $Z$ -equivalent to a classical knot. The existence of such counterexamples via parity was first pointed out by Vassily Manturov in 2009.

Parity is clearly an important theme in virtual knot theory and will figure in many future investigations of this subject. The type of construction that we have indicated for the bracket polynomial in this section can be varied and applied to other invariants. Furthermore the notion of describing a parity for crossings in a diagram is also susceptible to generalization. For more on this theme the reader

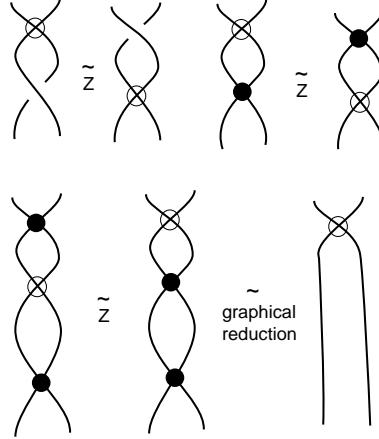


Figure 19:  $Z$ -move and graphical  $Z$ -move.

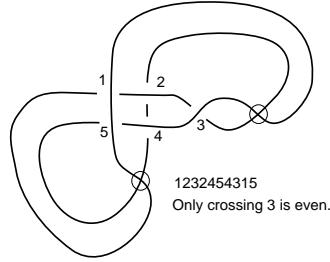


Figure 20: A knot  $KS$  with unit Jones polynomial.

should consult [22, 23] and [25] for our original use of parity for another variant of the bracket polynomial.

## 5. Portraits of families of virtual $KL$ s

Recursive and general formulas for the reduced relative Tutte polynomials can be computed for different families of virtual  $KL$ s, and from them we obtain Jones polynomials and Kauffman bracket polynomials of the considered families of virtual  $KL$ s.

Obtained results can be used to study properties of the reduced relative Tutte polynomials of virtual  $KL$ s and zeros of Kauffman bracket polynomials and Jones polynomials. The plot of zeros of Jones polynomials of virtual  $KL$  family is specific to the family and will be referred to as the “portrait of a virtual link family”.

The portrait of the virtual link family  $(i, 1^p)(1^q)$  ( $1 \leq p \leq 20$ ,  $2 \leq q \leq 20$ ) is shown in Fig. 22.

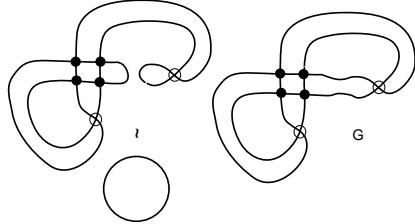


Figure 21: Parity bracket states for the knot  $KS$ .

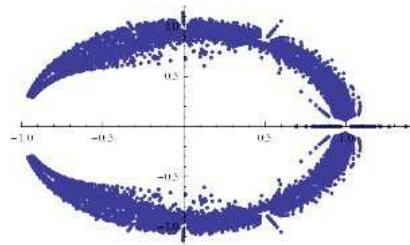


Figure 22: Plot of the zeroes of Jones polynomial for virtual  $KL$  family  $(i, 1^p) (1^q)$  ( $1 \leq p \leq 20, 2 \leq q \leq 20$ ).

Portraits of families of virtual  $KLs$  obtained for different choices of signs of parameters are compared in Figs. 15 and 16. Fig. 23 is the portrait of the family  $(1^p) i (1^q)$  for  $2 \leq p \leq 20, 2 \leq p \leq 20$ , and the Fig. 24 corresponds to the family  $(1^p) i ((-1)^q)$  for  $p, q \in Z^+$ . More detailed results of this kind will be given in the forthcoming paper.

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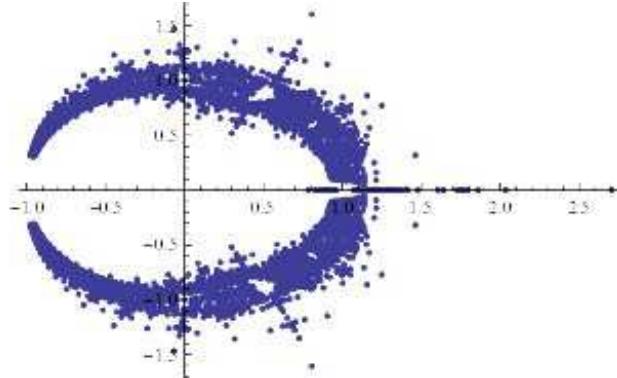


Figure 23: Plot of the zeroes of Jones polynomial for virtual  $KL$  family  $(1^p)i(1^q)$  ( $2 \leq p \leq 20$ ,  $2 \leq q \leq 20$ ).

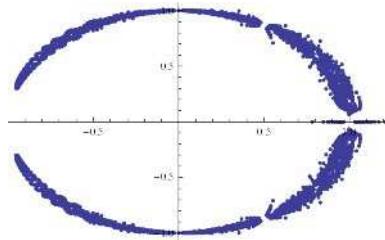


Figure 24: Plot of the zeroes of Jones polynomial for virtual  $KL$  family  $(1^p)i((-1)^q)$  ( $2 \leq p \leq 20$ ,  $2 \leq q \leq 20$ ).

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