Variational Approach to Quantum Field Theory: Gaussian Approximation and the Perturbative Expansion around It[a]

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The functional Schrödinger picture formulation of quantum field theory and the variational Gaussian approximation method based on the formulation are briefly reviewed. After presenting recent attempts to improve the variational approximation, we introduce a new systematic method based on the background field method, which enables one to compute the order-by-order correction terms to the Gaussian approximation of the effective action.

1 Introduction

The variational approximation method has been proved to be a convenient non-perturbative approximation method for both the quantum mechanical and quantum field theoretic systems. The method, which was originally developed for a non-perturbative evaluation of quantum mechanical energy eigenvalues, has been extended to an action principle by Jackiw and Kerman[1], which enables one to compute the effective actions for time-dependent systems and also for quantum field theories. The usefulness of the method for quantum field theories, however, is limited by the fact that only the Gaussian functional and it's obvious generalizations can be used as trial wave functionals, which is the reason why it is often called the Gaussian approximation. Attempts to improve the Gaussian approximation by using more general non-Gaussian trial wave functionals have not been as fruitful as expected, due to the difficulties in solving the variational gap equations[2]. To overcome such limitations the authors of Ref.[3] has developed the so-called post-Gaussian expansion method, which enables one to compute the order-by-order correction terms to the Gaussian effective potentials for the time-independent systems.

It is the purpose of this note to give a brief review of the variational approximation method in quantum field theory, and to present a new systematic method of perturbative expansion around the Gaussian effective action based on the background field method, which is also applicable to the time-dependent systems.

The variational method in quantum mechanics is based on the fact that the ground state energy is the minimum of the expectation values of the Hamiltonian of the system. This can be written as a variational principle,

$$\delta_\Psi < \Psi | \hat{H}(\hat{Q}, \hat{P}) - \lambda | \Psi >= 0, \tag{1}$$

where $\hat{H}$ is the Hamiltonian of the system and $|\Psi>$ is an arbitrary state. If one starts from a trial wave function with variational parameters, then Eq.(1) provides a variational approximation of the energy eigenstates.

This simple variational method can be generalized to include the information on the time-evolution of the system, by choosing the state to satisfy the constraints,

$$
\begin{aligned}
< \Psi | \Psi > &= 1 \\
< \Psi | \hat{Q} | \Psi > &= q(t) \\
< \Psi | \hat{P} | \Psi > &= p(t).
\end{aligned} \tag{2}
$$

Minimizing the expectation value of the Hamiltonian with respect to such states,

$$\delta_\Psi < \Psi | \hat{H}(\hat{Q}, \hat{P}) | \Psi >= 0, \tag{3}$$

we obtain the effective Hamiltonian,

$$H_{eff}(q, p) =< \Psi_0 | \hat{H}(\hat{Q}, \hat{P}) | \Psi_0 >, \tag{4}$$

where $|\Psi_0>$ is the solution of Eq.(3). This effective Hamiltonian provides many useful informations on the quantum system. If one minimizes H_{eff} with respect to

the variations of $q(t)$ and $p(t)$, one obtains the ground state energy eigenvalue of the system. If one solves the Hamilton's equation, on the other hand,

$$\dot{q}(t) = \frac{\partial H_{eff}}{\partial p} \tag{5}$$
$$\dot{p}(t) = -\frac{\partial H_{eff}}{\partial q},$$

the solution q and p give the quantum mechanical informations on $< \Psi_0|\hat{Q}|\Psi_0 >$ and $< \Psi_0|\hat{P}|\Psi_0 >$, respectively.

If we set $\dot{q} = 0 = p$ in Eq.(2), we obtain the effective potential,

$$V_{eff}(q) = H_{eff}(q,p)|_{p=0}. \tag{6}$$

If one starts from a trial wave function which satisfies Eq.(2), then one obtains the variational approximation of the effective Hamiltonian or the effective potential.

This simple argument is in fact a part of the quantum action principle in the Schrödinger picture called the action principle of Dirac[1,4]. This formulation can be generalized to the case of quantum field theory, which is the functional Schrödinger picture formulation of quantum field theory[4].

2   Functional Schrödinger picture representation of quantum field theory and the variational method.

2.1   Schrödinger picture representation of scalar field theory

The scalar field theory is defined by the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\pi\dot{\phi} - \phi\dot{\pi}) - \mathcal{H}[\phi,\pi], \tag{7}$$

and the equal-time commutation relations

$$[\phi(x), \pi(x')]_{x^0=x'^0} = i\delta(\vec{x} - \vec{x}') \tag{8}$$
$$[\phi(x), \phi(x')]_{x^0=x'^0} = 0 = [\pi(x), \pi(x')]_{x^0=x'^0},$$

where $\mathcal{H}[\phi,\pi] = \frac{1}{2}\pi^2 + V(\phi)$ is the Hamiltonian density.

As in the quantum mechanical case, we choose the basis vectors for the Schrödinger representation to be the eigenstates of the field operator $\phi(x)$,

$$\phi|\varphi(\vec{x}) >= \varphi(\vec{x})|\varphi(\vec{x}) >, \tag{9}$$

which satisfy the orthogonality relation and the completeness relation,

$$< \varphi|\varphi' >= \delta[\varphi(\vec{x}) - \varphi'(\vec{x}')] \tag{10}$$
$$\int \mathcal{D}\varphi|\varphi >< \varphi| = I.$$

3

Then the state of the physical system is represented by the component, $\Psi[\varphi, t] = \; < \varphi|\Psi >$, of the expansion,

$$|\Psi >= \int \mathcal{D}\varphi|\varphi >< \varphi|\Psi >, \tag{11}$$

and the operators are represented by the matrix elements,

$$
\begin{aligned}
< \varphi|\phi|\varphi' > &= \varphi(\vec{x})\delta[\varphi - \varphi'] \\
< \varphi|\pi|\varphi' > &= -i\frac{\delta}{\delta\varphi(\vec{x})}\delta[\varphi - \varphi'],
\end{aligned}
\tag{12}
$$

which are obtained from the matrix elements of Eq.(8). The time-evolution of the system is then determined by the functional Schrödinger equation,

$$
\begin{aligned}
i\frac{d}{dt}\Psi[\varphi, t] &= < \varphi|H[\phi, \pi]|\varphi' >< \varphi'|\Psi > \\
&= H[\varphi, -i\frac{\delta}{\delta\varphi}]\Psi[\varphi, t].
\end{aligned}
\tag{13}
$$

This is a simple generalization of the Schrödinger picture representation of quantum mechanics to the case of the systems with infinite degrees of freedom.

2.2 Free Scalar field theory

To obtain physical information in the Schrödinger picture formulation one has to solve the functional Schrödinger equation (13). As an example of solving the equation of motion, we consider the free scalar field theory described by the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{14}$$

By using the representation (12), the Hamiltonian of the system can be written as

$$H = \frac{1}{2}\int d^3x[-\frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x})} - \varphi(\vec{x})(\nabla^2 - m^2)\varphi(\vec{x})]. \tag{15}$$

Since the Hamiltonian has no explicit time-dependence, there exist stationary states satisfying the energy eigenvalue equation,

$$H[\varphi, -i\frac{\delta}{\delta\varphi}]\Psi[\varphi] = E\Psi[\varphi], \tag{16}$$

where E is the energy eigenvalue. Since the Hamiltonian (15) is quadratic in $\varphi(\vec{x})$, the Gaussian functional

$$\Psi[\varphi] = e^{-\frac{1}{2}\int d^3x d^3x' \varphi(\vec{x})G(\vec{x},\vec{x}')\varphi(\vec{x}')}, \tag{17}$$

is a solution of (16). Substituting (17) into (16), we obtain.

$$\frac{1}{2}[G(\vec{x}, \vec{x}') - \varphi(\vec{x})G^2(\vec{x}, \vec{x}')\varphi(\vec{x}'') + \varphi(\vec{x})(-\nabla^2 + m^2)\varphi(\vec{x})]\Psi[\varphi] = E\Psi[\varphi], \tag{18}$$

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where the repeated arguments imply the integration over the variables. For this equation to be true for all eigenvalue $\varphi(\vec{x})$, the following relations must be satisfied,

$$E = \frac{1}{2}TrG = \frac{1}{2}\int d^3x\, G(\vec{x},\vec{x}) \tag{19}$$
$$G^2(\vec{x},\vec{x}') = (-\nabla^2 + m^2)\delta(\vec{x} - \vec{x}').$$

Solving the second equation of (19) via Fourier transformation, one finds the kernel of the Gaussian functional (17) is given by

$$G^{-1}(\vec{x},\vec{x}') = \frac{1}{(2\pi)^3}\int d^3p\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\frac{1}{\sqrt{\vec{p}^2 + m^2}}, \tag{20}$$

which is the propagator of the free scalar field theory at equal-time. From this the eigenvalue E is found to be

$$E = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x\int d^3p\sqrt{\vec{p}^2 + m^2}, \tag{21}$$

which is the well-known zero-point energy of the free scalar field theory.

We thus find that the ground state of the free scalar field theory is described by the Gaussian wave functional with the kernel given by the inverse propagator at equal-time. One can construct all the eigenstates of the Hamiltonian by considering the creation and annihilation operators. They are given by the Hermite polynomial functionals multiplied by the ground(vacuum) state[5].

For the interacting theory, however, we cannot solve the functional Schrödinger equation (13) in closed form. One thus has to use an approximation method to find solutions to the equation of motion. A natural way to evaluate an approximate solution to (13) in the Schrödinger picture is to use the variational method, which is the subject of the next subsection.

2.3 Gaussian approximation of ϕ^4 theory

To illustrate the variational approximation method in quantum field theory, we now consider the scalar ϕ^4 theory described by the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{22}$$

and compute the Gaussian approximation of the effective potential[6]. To do this we take as a trial wave functional the solution of a free field theory discussed in the last subsection, namely, the Gaussian wave functional,

$$\Psi[\varphi,t] = Ne^{-\frac{1}{4}[\varphi(\vec{x})-\hat{\phi}]G^{-1}(\vec{x},\vec{x}';t)[\varphi(\vec{x}')-\hat{\phi}]+\frac{i}{\hbar}\hat{\pi}(\vec{x},t)[\varphi(\vec{x})-\hat{\phi}]} \tag{23}$$

which satisfies the constraints,

$$\begin{aligned}
< \Psi \mid \Psi > &= 1 \\
< \Psi \mid \phi \mid \Psi > &= \hat{\phi}(\vec{x},t) \\
< \Psi \mid \pi \mid \Psi > &= \hat{\pi}(\vec{x},t).
\end{aligned} \tag{24}$$

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Since the effective potential is the effective Hamiltonian with vanishing momenta, we set $\hat{\pi}$ equal to zero and consider the space-time independent $\hat{\phi}$. Then the expectation value of the Hamiltonian with respect to the trial state(23) is given by

$$
\begin{aligned}
< H > \; = \; & \int d^3x \left(V(\hat{\phi}) + \hbar[\frac{1}{8}G^{-1}(\vec{x},\vec{x}) - \frac{1}{2}\nabla^2 G(\vec{x},\vec{x}')|_{\vec{x}=\vec{x}'} \right. \\
& \left. + V^{(2)}(\hat{\phi})G(\vec{x},\vec{x})] + \frac{\hbar^2}{8}V^{(4)}(\hat{\phi})(G(\vec{x},\vec{x}))^2 \right),
\end{aligned}
\tag{25}
$$

where $V(\hat{\phi}) = \frac{\mu^2}{2}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4$ is the classical potential and $V^{(n)}(\hat{\phi}) = (\frac{d}{d\phi})^n V(\hat{\phi})$.

The variational equation,

$$
\delta_G < H >= 0,
\tag{26}
$$

yields the variational gap equation,

$$
\frac{1}{4}G^{-2}(\vec{x},\vec{x}') = [-\nabla^2 + V^{(2)}(\hat{\phi}) + \frac{\hbar}{2}V^{(4)}(\hat{\phi})G(\vec{x},\vec{x})]\delta(\vec{x}-\vec{x}').
\tag{27}
$$

Substituting(27) into (25) one obtains the effective potential,

$$
V_{eff}(\hat{\phi}) = \frac{1}{2}\mu^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\hbar}{4}G^{-1}(\vec{x},\vec{x}) - \frac{\hbar^2}{8}G(\vec{x},\vec{x})G(\vec{x},\vec{x}).
\tag{28}
$$

The first three terms Eq.(28) constitute the one-loop effective potential, and the last term contains the higher-loop contribution, which shows the non-perturbative nature of the approximation.

Using the Fourier transformation method we can solve the gap equation(27):

$$
G(\vec{x}-\vec{x}') = \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\frac{1}{\sqrt{\vec{p}^2+m^2}},
\tag{29}
$$

where the effective mass m is defined by

$$
m^2 = \mu^2 + \frac{\lambda}{2}\hat{\phi}^2 + \frac{\lambda}{2}\hbar \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\vec{p}^2+m^2}}.
\tag{30}
$$

Substituting these results into (28) yields the effective potential:

$$
\begin{aligned}
V_{eff}(\hat{\phi}) \; = \; & \frac{\mu^2}{4}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 - \frac{1}{2\lambda}\left(m^2 - \mu^2 - \frac{\lambda}{2}\hat{\phi}^2\right)^2 + \frac{m^2}{2}I_1 \\
& - \frac{m^4}{4}I_2(M) + \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{M^2} - \frac{1}{2}\right),
\end{aligned}
\tag{31}
$$

where

$$
m^2 = \mu^2 + \frac{\lambda}{2}I_1 + \frac{\lambda}{2}\hat{\phi}^2 - \frac{\lambda}{2}m^2I_2(M) + \frac{\lambda m^2}{32\pi^2}\ln\frac{m^2}{M^2},
\tag{32}
$$

and, I_1 and $I_2(M)$ are given by

$$
I_1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2|\vec{p}|} \quad , \quad I_2(M) = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2|\vec{p}|} - \frac{1}{2\sqrt{\vec{p}^2+\mu^2}}\right).
\tag{33}
$$

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The effective potential (31) has two divergences, I_1 and $I_2(M)$, and thus we need to renormalize the theory. The renormalization conditions can be written as [6],

$$\left.\frac{dV_{eff}}{dm^2}\right|_{m_0} = \frac{\mu_R^2}{\lambda_R} \quad , \quad \left.\frac{d^2V_{eff}}{d(m^2)^2}\right|_{m_0} = \frac{1}{3\lambda_R}, \tag{34}$$

which, in our case, yields

$$\frac{\mu_R^2}{\lambda_R} = \frac{\mu^2}{\lambda} + \frac{1}{2}I_1 \tag{35}$$

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{1}{2}I_2(M).$$

Substituting (35) into (31), we finally obtain the renormalized effective potential,

$$V_{eff}(\hat{\phi}) = -\frac{m^4}{2\lambda_R} + \frac{m^2}{2}\hat{\phi}^2 + \frac{\mu_R^2}{\lambda_R}m^2 + \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{M^2} - \frac{1}{2}\right), \tag{36}$$

where

$$m^2 = \mu_R^2 + \frac{\lambda_R}{2}\hat{\phi}^2 + \frac{\lambda_R}{32\pi^2}\ln\frac{m^2}{M^2}. \tag{37}$$

This result is qualitatively equivalent to the large-N approximation (where N is the number of the scalar field components)[7], and shows the non-perturbative nature of the approximation. The result (36) can be used to study the quantum nature of the theory, such as the symmetry breaking phenomena[6]. The method can be generalized to the case of the field theories in curved space[8], to the finite temperature quantum field theory[9], and also to the systems out of thermal equilibrium[10].

2.4 Fermionic theory

Finding the Schrödinger picture representation of fermion fields is non-trivial since the equal-time anti-commutation relation, in Hermitian representation,

$$\{\psi_\alpha(x), \psi_\beta(x')\}_{x^0=x^{0\prime}} = \delta_{\alpha\beta}\delta(\vec{x} - \vec{x}'), \tag{38}$$

implies that the field operators ψ's are their own momentum conjugates. In the non-Hermitian representation, on the other hand, the anti-commutation relation is given by

$$\{\psi_\alpha(x), \psi_\beta^\dagger(x')\}_{x^0=x^{0\prime}} = \delta_{\alpha\beta}\delta(\vec{x} - \vec{x}'), \tag{39}$$

but now the eigenstates of field operators do not form an orthonormal basis. One can show, however, that there exists a one parameter family of equivalent representations of fermion fields[4]. Floreanini and Jackiw[11], in particular, have found that the representation,

$$\psi = \frac{1}{\sqrt{2}}[\theta(\vec{x}) + \frac{\delta}{\delta\theta^\dagger(\vec{x})}] \tag{40}$$

$$\psi^\dagger = \frac{1}{\sqrt{2}}[\theta^\dagger(\vec{x}) + \frac{\delta}{\delta\theta(\vec{x})}],$$

where $\theta(\vec{x})$ is a complex Grassmann variable, satisfies the anti-commutation relation (39).

Using this representation one can follow the same procedure as in the case of the scalar field theory to evaluate the Gaussian approximation of the Fermionic field theories.

This method has been applied to the cases of (1+1)-dimensional and (2+1)-dimensional Gross-Neveu models[12], and found that the results are qualitatively equivalent to those of the large-N approximation. It has also been applied to the (2+1)-dimensional Thirring model, and in this case the Gaussian result has been found to be qualitatively better than the large-N approximation[13]. The Gaussian effective potential shows the existence of the symmetry breaking in the theory, while the large-N does not. The method has also been found to be effective in determining the existence of the bound states in the theory[14].

3   Beyond Gaussian approximation?

We have shown that the Gaussian approximation based on the functional Schrödinger picture formulation is a simple generalization of the quantum mechanical variational method, and provides a convenient non-perturbative approximation method for various quantum field theories including the systems out of thermal equilibrium. However, the method has a serious drawback in that, unlike the quantum mechanical cases, one can compute the expectation value of the quantum field theoretic Hamiltonian only for the Gaussian wave functional and its simple generalizations.

If one could develop a systematic method to compute the correction terms to this approximation, therefore, the variational method would become a very powerful technique in obtaining physical information from quantum field theories. In fact there have been some attempts in this direction recently.

One of such attempts, known as the post-Gaussian approximation method or the variational perturbation theory[3], is based on the observation that the Gaussian wave functional

$$\Psi_0 = e^{-\frac{1}{4}\tilde{\varphi}(\vec{x})G^{-1}(\vec{x}-\vec{x}')\tilde{\varphi}(\vec{x}')} \quad , \quad \tilde{\varphi} = \varphi - \hat{\phi} \tag{41}$$

is a vacuum state for a free theory with mass given by

$$m^2 = \mu^2 + \frac{\lambda}{2}\hat{\phi}^2 + \frac{\lambda}{2}\int \frac{d^3p}{(2\pi)^3}g(p), \tag{42}$$

where $g(p)$ is determined by the variational gap equation (29) and (30).

One can construct all the eigenstates of this free theory:

$$\Psi_n[\tilde{\varphi}] = H_n[\tilde{\varphi}]\Psi_0[\tilde{\varphi}] \quad , \quad n = 0, 1, 2, \cdots \tag{43}$$

where $H_n[\tilde{\varphi}]$ is the Hermite polynomial functional. One splits the Hamiltonian as

$$H = H_0 + H_I \tag{44}$$

where H_0 is the free Hamiltonian with the mass m given by Eq.(42) and H_I is the remaining part of H, and then applies the well known perturbation theory with the states (43) taken to be the basis for the perturbation.

The effective potential is the expectation value of the Hamiltonian with the constraint

$$< \Psi_0 |\phi| \Psi_0 > = \hat{\phi}, \tag{45}$$

where $|\Psi_0>$ is the ground state of the system.

Taking the eigenstates (43) to be the basis for perturbative expansion, the effective potential can be written as

$$V_{eff}(\hat{\phi}) = E_0(\hat{\phi}) + \sum_{n=0}^{\infty} < \Psi_0 |\Psi_I \left(\frac{1}{E_0 - H_0} H_I \right)^n |\Psi_0 >, \tag{46}$$

where the first term is the Gaussian effective potential and the second term represents the perturbative correction terms[3]. This method has been used to compute the first non-trivial contribution to the Gaussian effective potential for the (0+1)-dimensional ϕ^4 theory by Jeon[15] and the (1+1)- and (2+1)-dimensional ϕ^4 theories by Cea and Tedesco[3].

This method provides a conceptually simple procedure to compute the order-by-order correction terms to the Gaussian effective potential for time-independent systems. For the time-dependent systems such as those out of thermal equilibrium[10], however, we need a method that can deal with the effective action, which contains information on the time-evolution of the system. It is the subject of the next section to introduce a new systematic expansion method for effective action, which was developed by Geon Hyoung Lee[16].

4 The Background field method

We now introduce a systematic method which enables one to compute the order by order correction terms to the Gaussian approximation of the effective action, that contains the information on the time-evolution of the system. This new method is based on the background field method. So we will start to give a brief review on the background field method[17].

4.1 Background field method

The generating functional for the connected Green's functions, $W[J]$, is defined by the path integral,

$$e^{iW[J]} = N \int \mathcal{D}\phi e^{iS[\phi] + iJ\phi}, \tag{47}$$

where $S[\phi]$ is the classical action, and the integral convention, $J\phi \equiv \int d^n x J(x)\phi(x)$ in n-dimensional space-time, is used in the exponent. The effective action $\Gamma[\phi]$ is then defined by the Legendre transformation,

$$\Gamma[\varphi] = W[J] - J\varphi, \tag{48}$$

where $\varphi(x)$ is the vacuum expectation value of the field operator,

$$\varphi(x) = < \phi(x) >_J = \frac{\delta}{\delta J(x)} W[J]. \tag{49}$$

The background field method is one of the most effective ways of computing the effective action in the loop expansion, and is obtained by introducing the background field $B(x)$ to the classical action:

$$S[\phi] \rightarrow S[\phi + B]. \tag{50}$$

This defines a new generating functional,

$$
\begin{aligned}
e^{i\tilde{W}[J,B]} &= N \int \mathcal{D}\phi \, e^{iS[\phi + B] + iJ\phi} \\
&= N \int \mathcal{D}\phi' \, e^{iS[\phi'] + iJ[\phi' - B]}.
\end{aligned} \tag{51}
$$

Eg.(51) implies that this new generating functional is related to the old one by

$$\tilde{W}[J,B] = W[J] - JB, \tag{52}$$

and the vacuum expectation value of the field operator in the presence of the background field is given by,

$$\tilde{\varphi} = \frac{\delta \tilde{W}}{\delta J} = \varphi - B. \tag{53}$$

Now consider the case when the vacuum expectation value of the field operator in the presense of the background field, $\tilde{\varphi}$, vanishes:

$$\frac{\delta \tilde{W}}{\delta J} = \tilde{\varphi} = 0. \tag{54}$$

This determines the configuration of J which extremizes the generating functional $\tilde{W}[J, B]$:

$$J = J[B]. \tag{55}$$

Then one can easily show from Eq.(52) that the extremum value of the generating functional $\tilde{W}$ becomes the effective action:

$$\tilde{W}[J,B]|_{\tilde{\varphi}=0} = W[J(B)] - J(B)B = \Gamma[B]. \tag{56}$$

The simpler way to compute the effective action $\Gamma[B]$ is to consider the Legendre transformation of $\tilde{W}[J, B]$:

$$\tilde{\Gamma}[\tilde{\varphi}, B] = \tilde{W}[J, B] - J\tilde{\varphi}, \tag{57}$$

which generates the one $\tilde{\varphi}$-particle irreducible diagrams in the presence of the background field. If we set $\tilde{\varphi} = 0$ in Eq.(57), we find that

$$
\begin{aligned}
\tilde{\Gamma}[\tilde{\varphi}, B]\Big|_{\tilde{\varphi}=0} &= \tilde{W}[J(B), B] - J\tilde{\varphi}\Big|_{\tilde{\varphi}=0} \\
&= \Gamma[B].
\end{aligned} \tag{58}
$$

This implies that the effective action $\Gamma[B]$ consists of one $\tilde{\varphi}$-particle irreducible bubble diagrams with no external $\tilde{\varphi}$-lines. Thus one can write the effective action as

$$e^{i\Gamma[B]} = e^{i\tilde{W}[J,B]}\Big|_{\tilde{\varphi}=0} = N \int \mathcal{D}\phi \, e^{iS[\phi + B] + iJ\phi}\Big|_{\tilde{\varphi}=0}. \tag{59}$$

To illustrate how the method facilitates the computation of the effective action, we now consider the scalar ϕ^4 theory as an example. The Lagrangian density of ϕ^4 theory can be written as

$$\mathcal{L}(\phi) = -\frac{1}{2}\phi(\partial^2 + \mu^2)\phi - \frac{\lambda}{4!}\phi^4 \equiv -\frac{1}{2}\phi(i\mathcal{D}^{-1})\phi - \frac{\lambda}{4!}\phi^4, \tag{60}$$

where D is the free propagator defined by

$$\mathcal{D} \equiv i(\partial^2 + \mu^2)^{-1}. \tag{61}$$

The shifted Lagrangian is then given by

$$\mathcal{L}(\phi + B) = -i\frac{1}{2}BD^{-1}B - \frac{\lambda}{4!}B^4 - i(BD^{-1} - i\frac{\lambda}{3!}B^3)\phi - \frac{1}{2}\phi iK^{-1}\phi - \frac{\lambda}{3!}B\phi^3 - \frac{\lambda}{4!}\phi^4, \tag{62}$$

where K is the new propagator,

$$K = i(\partial^2 + \mu^2 + \frac{\lambda}{2}B^2)^{-1}, \tag{63}$$

which includes a part of the interaction effect through the term proportional to B^2. The generating functional in the presence of the background field can then be written as,

$$e^{i\tilde{W}[J,B]} = N\det(-K)^{\frac{1}{2}}e^{\frac{1}{2}BD^{-1}B - \frac{i\lambda}{4!}B^4}e^{(BD^{-1} - \frac{i\lambda}{3!}B^3)\frac{\delta}{i\delta J} - \frac{i\lambda}{3!}B(\frac{\delta}{i\delta J})^3 - \frac{i\lambda}{4!}(\frac{\delta}{i\delta J})^4}e^{\frac{1}{2}JKJ}. \tag{64}$$

The effective action for the scalar ϕ^4 theory now becomes

$$e^{i\Gamma[B]} = e^{i\tilde{W}[J,B]}\Big|_{\tilde{\varphi}=0} = Ne^{\frac{1}{2}BD^{-1}B - \frac{i\lambda}{4!}B^4}\det(-K)^{\frac{1}{2}}I_0[B], \tag{65}$$

where

$$I_0[B] = e^{(BD^{-1} - \frac{i\lambda}{3!}B^3)\frac{\delta}{i\delta J}}\exp[-\frac{i\lambda}{3!}B(\frac{\delta}{i\delta J})^3 - \frac{i\lambda}{4!}(\frac{\delta}{i\delta J})^4]e^{\frac{1}{2}JKJ}\Big|_{\tilde{\varphi}=0}. \tag{66}$$

The coefficient of $I_0[B]$ in Eq.(65) represents the one-loop effective action, and $I_0[B]$ represents the higher loop contributions to the effective action. The first factor of Eq.(66) constitutes a shift operator, which does not change the extremum value of $\tilde{W}[J,B]$. Thus this factor, which generates the tadpole diagrams, does not contribute to the effective action , and we can neglect it . One can therefore write $I_0[B]$ as

$$I_0[B] = \exp[-\frac{i\lambda}{3!}B(\frac{\delta}{i\delta J})^3 - \frac{i\lambda}{4!}(\frac{\delta}{i\delta J})^4]e^{\frac{1}{2}JKJ}\Big|_{\tilde{\varphi}=0} \tag{67}$$

which implies that the higher loop contributions to the effective action consist of one $\tilde{\varphi}$-irreducible bubble diagrams with no external $\tilde{\varphi}$-lines. One can compute the higher loop contributions by the Feynman rules:

$$\underline{\hspace{3cm}} \quad = K = \frac{i}{\partial^2 + \mu^2 + \frac{\lambda}{2}B^2} \qquad : \text{propagator}$$

$$= -i\lambda B \qquad : \text{3-vertex}$$

$$= -i\lambda \qquad : \text{4-vertex}$$

One can easily show that the two-loop contribution to the effective action consists of two bubble diagrams with no external lines[17]:

$$\Gamma^{(2)}[B] = \qquad\qquad + \qquad\qquad . \tag{68}$$

Comparing this with the conventional method of computing the effective action, one can easily convince oneself that the background field method provides a very simple and effective way to compute the effective action in the loop expansion.

4.2 Expansion around the Gaussian effective action

The reason why the procedure of computing the effective action in the background field method is simplified is that the propagator used in the Feynman rules, Eq.(63), includes a part of the interaction effect through the background field $B(x)$, and the generating functional has been rearranged in such a way that the zeroth order term in the expansion, the coefficient of $I_0[B]$ in Eq.(65), is the one-loop effective action.

We now want to proceed one step further and rearrange the effective action in such a way that the propagator used in the expansion would include the non-perturbative effect and the zeroth order contribution would be the result of the non-perturbative approximation, such as the result of the Gaussian approximation.

Note that the Gaussian approximation consists of the cactus type diagrams where an internal line comes out of a point in space-time and goes back to the same point, such as the first term of Eq.(68). So we need to extract such diagrams out of the $I_0[B]$ part of the effective action. To do this we examine the functional derivatives of the generating functional $e^{\frac{1}{2}JGJ}$ with respect to $J(x)$:

$$\frac{\delta}{\delta J_x} e^{\frac{1}{2}J_y G_{yz} J_z} = G_{xy} J_y e^{\frac{1}{2}J_y G_{yz} J_z} \equiv (GJ)_x e^{\frac{1}{2}JGJ} \tag{69}$$

$$\frac{\delta^2}{\delta J_x^2} e^{\frac{1}{2}J_y G_{yz} J_z} = ((GJ)_x^2 + G_{xx}) e^{\frac{1}{2}JGJ} \tag{70}$$

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$$\frac{\delta^3}{\delta J_x^3} e^{\frac{1}{2} J_y G_{yz} J_z} = ((GJ)_x^3 + 3G_{xx}(GJ)_x)e^{\frac{1}{2} JGJ} \tag{71}$$

$$\frac{\delta^4}{\delta J_x^4} e^{\frac{1}{2} J_y G_{yz} J_z} = ((GJ)_x^4 + 6G_{xx}(GJ)_x^2 + 3G_{xx}^2)e^{\frac{1}{2} JGJ}, \tag{72}$$

which would appear in the expansion of $I_0[B]$. Note that these derivatives, which generates 2-loop and higher order contributions to $\Gamma[B]$, generate the cactus type diagrams as can be seen in Eqs.(71) and (72).

To extract such diagrams out of $I_0[B]$, we define new functional derivatives so that the following relations satisfy:

$$\left(\frac{\delta^3}{\delta J_x^3}\right)' e^{\frac{1}{2} JGJ} = (GJ)_x^3 e^{\frac{1}{2} JGJ} \tag{73}$$

$$\left(\frac{\delta^4}{\delta J_x^4}\right)' e^{\frac{1}{2} JGJ} = (GJ)_x^4 e^{\frac{1}{2} JGJ}. \tag{74}$$

Comparing Eqs.(71)-(72) and Eqs.(73)-(74), we find that the primed derivatives must be defined by

$$\left(\frac{\delta^3}{\delta J_x^3}\right)' \equiv \frac{\delta^3}{\delta J_x^3} - 3G_{xx}\frac{\delta}{\delta J_x} \tag{75}$$

$$\left(\frac{\delta^4}{\delta J_x^4}\right)' \equiv \frac{\delta^4}{\delta J_x^3} - 6G_{xx}\frac{\delta^2}{\delta J_x^2} + 3G_{xx}^2. \tag{76}$$

Note that the primed derivatives operated on the functional $e^{\frac{1}{2} JGJ}$ do not generate the cactus type diagrams that contribute to the Gaussian effective action. This implies that by using the primed derivatives one may in fact rewrite $I_0[B]$ in such a way that all the cactus type diagrams would be included in the zeroth order term in the expansion.

To do this we need to find a new Green's function G such that

$$e^{\frac{\lambda B}{3!}\left(\frac{\delta}{\delta J}\right)^3 - \frac{i\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4} e^{\frac{1}{2} JKJ} = N' e^{A\frac{\delta}{\delta J}} e^{\frac{\lambda B}{3!}\left(\frac{\delta^3}{\delta J^3}\right)' - \frac{i\lambda}{4!}\left(\frac{\delta^4}{\delta J^4}\right)'} e^{\frac{1}{2} JGJ}, \tag{77}$$

where a normalization constant N' and a function A are to be determined. Using the definitions (75) and (76), one easily finds G, N' and A that satisfy Eq.(77):

$$N' = \det(-G)^{\frac{1}{2}} \det(-G^{-1} + \frac{i\lambda}{2} G_{xx})^{\frac{1}{2}} e^{\frac{i\lambda}{8} G_{xx}^2}, \tag{78}$$

$$A = \frac{\lambda}{2} B G_{xx}, \tag{79}$$

$$G_{xy}^{-1} = K_{xy}^{-1} + \frac{i\lambda}{2} G_{xx}\delta_{xy}$$

$$= \mathcal{D}_{xy}^{-1} - \frac{i\lambda}{2} B_x^2 \delta_{xy} + \frac{i\lambda}{2} G_{xx}\delta_{xy}. \tag{80}$$

Using these results we can write the generating functional in the presence of the background field as

$$e^{i\bar{W}[J,B]} = \det(-G)^{\frac{1}{2}} e^{\frac{1}{2} B\mathcal{D}^{-1}B - \frac{i\lambda}{4!} B^4 + \frac{i\lambda}{8} G_{xx}^2}$$

$$\times e^{(-iB\mathcal{D}^{-1} - \frac{\lambda}{3!} B^3 + \frac{\lambda}{2} B G_{xx})\frac{\delta}{\delta J}} e^{\frac{\lambda B}{3!}\left(\frac{\delta^3}{\delta J_x^3}\right)' - \frac{i\lambda}{4!}\left(\frac{\delta^4}{\delta J_x^4}\right)'} e^{\frac{1}{2} JGJ}. \tag{81}$$
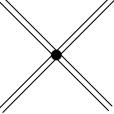
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The effective action can thus be written as

$$
\begin{aligned}
e^{i\Gamma[B]} &= \left. e^{\tilde{W}[J,B]} \right|_{\tilde{\varphi}=0} \\
&= \det(-G)^{\frac{1}{2}} e^{\frac{1}{2}B\mathcal{D}^{-1}B - \frac{i\lambda}{4!}B^4 + \frac{i\lambda}{8}G_{xx}^2} \, I[B],
\end{aligned}
\tag{82}
$$

where

$$
\begin{aligned}
I[B] &= \exp\left[\left(-iB\mathcal{D}^{-1} - \frac{\lambda}{3!}B^3 + \frac{\lambda}{2}BG_{xx} \right) \frac{\delta}{\delta J} \right] \\
&\quad \times \left. \exp[\frac{\lambda B}{3!}(\frac{\delta^3}{\delta J_x^3})' - \frac{i\lambda}{4!}(\frac{\delta^4}{\delta J_x^4})']e^{\frac{1}{2}JGJ} \right|_{\tilde{\varphi}=0} \\
&= \left. \exp[\frac{\lambda B}{3!}(\frac{\delta^3}{\delta J_x^3})' - \frac{i\lambda}{4!}(\frac{\delta^4}{\delta J_x^4})']e^{\frac{1}{2}JGJ} \right|_{\tilde{\varphi}=0}.
\end{aligned}
\tag{83}
$$

In Eq.(83) we have used the fact that the shift operator, the tadpole part of the first line, does not affect the extremum value. Note that Eq.(80), which determines the Green's function G_{xy}, is exactly the variational gap equation written in the Minkowski space notation, and therefore the coefficient of $I[B]$ in Eq.(82) gives rise to the Gaussian effective action as we have alluded to above. Since $I[B]$ can be expanded as a power series in λ as is done in the conventional background field method, we have the perturbative expansion of the effective action around the Gaussian approximation.

The structure of $I[B]$ is the same as that of $I_0[B]$ in Eq.(67), except that the propagator is now replaced by G_{xy} and the functional derivatives are replaced by the primed derivatives. Thus the $I[B]$ contribution to the effective action consists of one $\tilde{\varphi}$-particle irreducible bubble diagrams with no external $\tilde{\varphi}$-lines, and without the cactus type diagrams. The contributions from $I[B]$ can therefore be evaluated by the Feynman rules:

$$= K = \frac{i}{\partial^2 + \mu^2 + \frac{\lambda}{2}B^2 - \frac{\lambda}{2}G_{xx}} \quad \text{: propagator}$$

$$= -i\lambda B \qquad\qquad\qquad \text{: 3-vertex}$$

$$= -i\lambda \qquad\qquad\qquad\quad \text{: 4-vertex}$$

We have thus established the systematic method of computing the order-by-order correction terms to the Gaussian effective action.

One can easily show that the λ^2 order contribution of $I[B]$, for example, consists of the diagrams,

$$I^{(2)}[B] = \text{-i}\,\lambda\mathbf{B} \bigcirc\!\!\!\!\!\!\!\!\!\!\! G \text{-i}\,\lambda\mathbf{B} + \text{-i}\,\lambda \bigcirc\!\!\!\!\!\!\!\!\!\!\! G \text{-i}\,\lambda\ . \tag{84}$$

Note that the first diagram of Eq.(68) is a cactus type diagram and does not appear in the expansion of $I[B]$.

4.3 Effective potential

If we consider the space-time independent background field in the above formulation, we obtain the effective potential defined by

$$V_{eff}(B) \equiv -\frac{\Gamma[B]}{\int d^n x}. \tag{85}$$

Since G_{xy} is a function of $(x-y)$ for a constant B, one can solve Eq.(80) by Fourier transformation,

$$G(x-y) \equiv \int \frac{d^n p}{(2\pi)^n} g(p) e^{ip(x-y)}. \tag{86}$$

The Green's function G_{xy} in the momentum space can then be written as

$$g(p) = \frac{1}{i(p^2 - m_G^2)}, \tag{87}$$

where

$$m_G^2 \equiv \mu^2 + \frac{\lambda}{2} B^2 - \frac{\lambda}{2} \int \frac{d^n p}{(2\pi)^n} g(p). \tag{88}$$

Note that Eq.(87) is the variational gap equation of Gaussian approximation written in the Minkowski notation.

Up to the λ^2 order contribution of $I[B]$, the effective potential is therefore given by

$$V_{eff}(B) = V_G + V_P, \tag{89}$$

where

$$V_G \equiv \frac{\mu^2}{2} B^2 + \frac{\lambda}{4!} B^4 - \frac{1}{2\lambda}(m^2 - \mu^2 - \frac{\lambda}{2} B^2)^2 + i\text{Tr} \log \sqrt{-G}, \tag{90}$$

and

$$V_P \equiv i\frac{\lambda^2}{12} B^2 G^3 - i\frac{\lambda^2}{48} G^4. \tag{91}$$

V_G of (90) is the Gaussian effective potential and V_P of (91) is the first non-trivial contribution from $I[B]$, which is represented by the Feynman diagrams shown in

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(84). This result is the same as that of Cea and Tedesco[3] computed from the variational perturbation theory mentioned earlier.

To illustrate the effectiveness of the perturbative correction method, we now consider the $(0+1)$-dimensional case, which is the quantum mechanical anharmonic oscillator. It is convenient to write the Green's function in Euclidean space which is defined by

$$p^2 = -p_E^2, \quad d^n p = i d^n p_E. \tag{92}$$

Then $G(x,x)$ in Eq.(88) can be written as

$$\int \frac{dp}{2\pi} \frac{1}{i(p^2 - m^2)} = -\int \frac{dp_E}{2\pi} \frac{1}{p_E^2 + m^2} = -\frac{1}{\sqrt{2}} \frac{1}{\sqrt{m^2}}, \tag{93}$$

and the gap equation becomes

$$m^2 = \mu^2 + \frac{\lambda}{2}B^2 + \frac{\lambda}{4\sqrt{m^2}}. \tag{94}$$

The effective potential up to the λ^2 order of $I[B]$ becomes

$$V_{eff}(B) = V_G + V_P, \tag{95}$$

where

$$V_G = \frac{\mu^2}{2}B^2 + \frac{\lambda}{4!}B^4 - \frac{1}{2\lambda}(m^2 - \mu^2 - \frac{\lambda}{2}B^2)^2 + \frac{\sqrt{m^2}}{2}, \tag{96}$$

and

$$V_P = -\frac{\lambda^2}{144m^4}B^2 - \frac{\lambda^2}{1536(\sqrt{m^2})^5}. \tag{97}$$

We can compare this result with other approximation methods. For the comparison of the effectiveness of various approximation methods it is convenient to compare the ground state energy eigenvalue of the anharmonic oscillator, which is defined by the minimum value of V_{eff}. The table below shows the errors of various approximation methods compared to the numerical results[18]. This shows that our

Table 1: Errors of $E_0(\mu^2 = 1)$ compared to the numerical result

λ	1-loop	2-loop	Gaussian Approx.	Improved Gaussian
0.24	1.4%	0.05%	0.006%	0.0003%
24	38%	56%	1.1%	0.37%

method greatly improves the Gaussian approximation even at the first non-trivial correction level, for all values of the coupling constant. Note that, for strong coupling, the Gaussian and the improved Gaussian approximation display much better results than the loop expansion methods, which shows the non-perturbative nature of the approximation methods.

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5 Conclusion

We have shown that the functional Schrödinger picture formulation of quantum field theory provides a conceptually simple non-perturbative approximation method, namely the Gaussian approximation. It has been proved to be a reliable method to obtain non-perturbative informations on the symmetry breaking phenomena, existence of bound states and time-evolution of the time-dependent systems with relatively little computational difficulties. The method, however, has a serious drawback in that it does not provide a systematic procedure to improve the approximation.

We have shown that the background field method provides a solution to this limitation of the variational method. We were able to rearrange the effective action in such a way that the zeroth order contribution in the expansion of the effective action consists of the Gaussian effective action. This provides a new systematic expansion method which enables one to compute the order-by-order correction terms to the Gaussian effective action.

The effective potentials for the $(0 + 1)$, $(1 + 1)$ and $(2 + 1)$ dimensional ϕ^4 theory computed by the new method agree with those of the variational perturbation theory. It has been shown that the perturbative correction greatly improves the Gaussian approximation even at the first non-trivial correction level. Our method can also be applied to the time-dependent systems and can easily be extended to the cases of fermionic and gauge field theories.

Although the effective action expanded around the Gaussian approximation for the lower-dimensional ϕ^4 theories can easily be shown to be renormalizable, one has to study the renormalization procedure systematically for the $(3 + 1)$ dimensional case.

Acknowledgements

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