

# Onsager and Kaufman's calculation of the spontaneous magnetization of the Ising model

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## Abstract

Lars Onsager announced in 1949 that he and Bruria Kaufman had proved a simple formula for the spontaneous magnetization of the square-lattice Ising model, but did not publish their derivation. It was three years later when C. N. Yang published a derivation in Physical Review. In 1971 Onsager gave some clues to his and Kaufman's method, and there are copies of their correspondence in 1950 now available on the Web and elsewhere. Here we review how the calculation appears to have developed, and add a copy of a draft paper, almost certainly by Onsager and Kaufman, that obtains the result.

**KEY WORDS:** Statistical mechanics, lattice models, transfer matrices.

## 1 Introduction

Onsager calculated the free energy of the two-dimensional square-lattice Ising model in 1944.<sup>[1]</sup> He did this by showing that the transfer matrix is a product of two matrices, which together generate (by successive commutations) a finite-dimensional Lie algebra (the “Onsager algebra”). In 1949 Bruria Kaufman gave a simpler derivation<sup>[2]</sup> of this result, using anti-commuting spinor (free-fermion) operators, i.e. a Clifford algebra.

Onsager was the Josiah Willard Gibbs Professor of Theoretical Chemistry at Yale University. Kaufman had recently completed her PhD at Columbia University in New York, and was a research associate at the Institute for Advanced Study in Princeton.

Later that year, Kaufman and Onsager<sup>[3]</sup> went on to calculate some of the two-spin correlations. Let  $i$  label the columns of the square lattice (oriented in the usual manner, with axes horizontal and vertical), and  $j$  label the rows, as in Figure 1. Let the spin at

site  $(i, j)$  be  $\sigma_{i,j}$ , with values  $+1$  and  $-1$ . Then the total energy is

$$E = -J \sum_{ij} \sigma_{i,j} \sigma_{i,j+1} - J' \sum_{ij} \sigma_{i,j} \sigma_{i+1,j}$$

and the partition function is

$$Z = \sum_{\sigma} e^{-E/\kappa T},$$

the sum being over all values of all the spins,  $\kappa$  being Boltzmann's constant and  $T$  the temperature. Onsager defines  $H, H', H^*$  by

$$H = J/\kappa T, \quad H' = J'/\kappa T, \quad e^{-2H^*} = \tanh H. \quad (1.1)$$

The specific heat diverges logarithmically at a critical temperature  $T_c$ , where

$$\sinh(2J/\kappa T_c) \sinh(2J'/\kappa T_c) = 1.$$

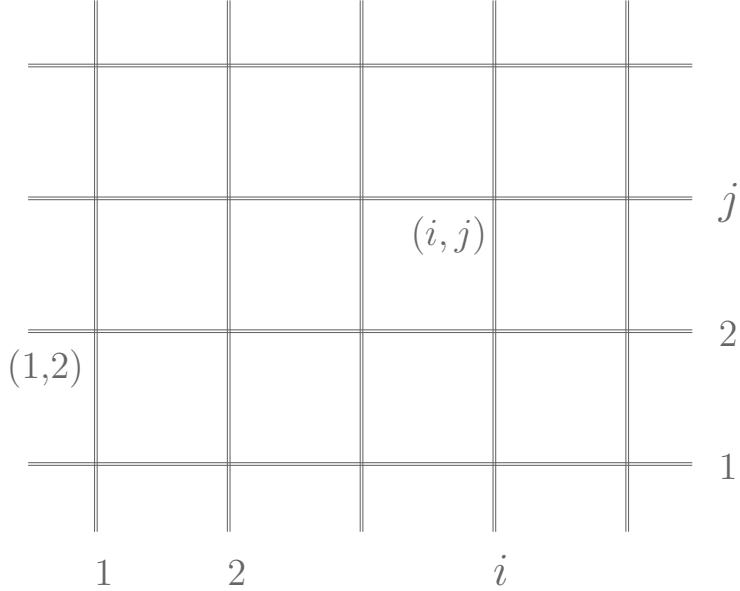


Figure 1: The square lattice.

The correlation between the two spins at sites  $(1, 1)$  and  $(i, j)$  is

$$\langle \sigma_{1,1} \sigma_{i,j} \rangle = Z^{-1} \sum_{\sigma} \sigma_{1,1} \sigma_{i,j} e^{-E/\kappa T},$$

For the isotropic case  $H' = H$ , Kaufman and Onsager[3] give in their equation 43 the formula for the correlation between two spins in the same row:<sup>1</sup>

$$\langle \sigma_{1,1} \sigma_{1,1+j} \rangle = \cosh^2 H^* \Delta_j - \sinh^2 H^* \Delta_{-j} \quad (1.2)$$

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<sup>1</sup> We have negated their  $\Sigma_r$ , which makes them the same as those in Appendix A, and corrected what appears to be a sign error. It is now the same as III.43 of the draft paper below.

Here  $\Delta_j$  and  $\Delta_{-j}$  are Toeplitz determinants:

$$\Delta_j = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \cdots & \Sigma_j \\ \Sigma_0 & \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{j-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma_{2-j} & \cdot & \cdot & \cdots & \Sigma_1 \end{vmatrix} \quad (1.3)$$

$$\Delta_{-j} = \begin{vmatrix} \Sigma_{-1} & \Sigma_{-2} & \Sigma_{-3} & \cdots & \Sigma_{-j} \\ \Sigma_0 & \Sigma_{-1} & \Sigma_{-2} & \cdots & \Sigma_{1-j} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma_{j-2} & \cdot & \cdot & \cdots & \Sigma_{-1} \end{vmatrix} \quad (1.3)$$

where

$$\Sigma_r = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\omega+i\delta'(\omega)} d\omega$$

and

$$\tan \delta'(\omega) = \frac{\sinh 2H \sin \omega}{\coth 2H' - \cosh 2H \cos \omega} . \quad (1.4)$$

Setting

$$\delta(\omega) = \delta'(\omega) + \omega , \quad (1.5)$$

this implies

$$e^{i\delta(\omega)} = \left\{ \frac{(1 - \coth H' e^{-2H+i\omega})(1 - \tanh H' e^{-2H+i\omega})}{(1 - \coth H' e^{-2H-i\omega})(1 - \tanh H' e^{-2H-i\omega})} \right\}^{1/2} . \quad (1.6)$$

(Equations (1.4), (1.6) follow from (89) of [1] and are true for the general case when  $H'$ ,  $H$  are not necessarily equal.)

Kaufman and Onsager also give the formula for the correlation between spins in adjacent rows. In particular, from their equations 17 and 20, we obtain

$$\langle \sigma_{1,1} \sigma_{2,2} \rangle = \cosh^2 H^* \Sigma_1 + \sinh^2 H^* \Sigma_{-1} . \quad (1.7)$$

The long-range order, or spontaneous magnetization, can be defined as

$$M_0 = \left( \lim_{j \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1,1+j} \rangle \right)^{1/2} . \quad (1.8)$$

It is expected to be zero for  $T$  above the critical temperature, and positive below it, as in Figure 2.

Kaufman and Onsager were obviously very close to calculating  $M_0$ : all they needed to do was to evaluate  $\Delta_j$ ,  $\Delta_{-j}$  in the limit  $j \rightarrow \infty$ . Here I shall present the evidence that they devoted their attention to doing so, and indeed succeeded. They used two methods: the first is discussed in section 2 and Appendix A, the second in sections 3, 4, 5. Many years ago (probably in the early or mid-1990's), John Stephenson (then in Edmonton, Canada) sent the author a photocopy of an eight-page typescript, bearing the names Onsager and Kaufman, that deals with the topic. Stephenson had copied it about 1965 in Adelaide,

Australia, from a copy owned by Ren Potts: both Potts and Stephenson were students of Cyril Domb.[4, 5] It's possible that Potts's copy had come from Domb when they were together in Oxford from 1949 to 1951, but more likely that he had been given it by Elliott Montroll, with whom Potts had collaborated in the early 1960's (see ref. [6]). A transcript of the author's copy is given in section 3, and a scanned copy forms Appendix B.

Both these methods start from the formulae for the pair correlation of two spins deep within an infinite lattice. There is also a third method used by the author for the superintegrable chiral Potts model (which is an  $N$ -state generalization of the Ising model)[7]: if one calculates the single-spin expectation value  $\langle \sigma_{1,M} \rangle$  in a lattice of width  $L$  and height  $2M$  with cylindrical boundary conditions and fixed-spin boundary conditions on the top and bottom rows, then one can write the result as a determinant of dimension proportional to  $L$ . (This method is similar to that of Yang.[8]) The determinant is *not* Toeplitz, but in the limit  $M \rightarrow \infty$  it is a product of Cauchy determinants, so can be evaluated directly for finite  $L$ .[9, 10]

## 2 The first method

In August 1948, Onsager silenced a conference at Cornell by writing on the blackboard an exact formula for  $M_0$ .[11, p.457] The following year, in May 1949 at a conference of the International Union of Physics on statistical mechanics in Florence, Italy,[12, p. 261] Onsager referred to the magnetization of the Ising model and announced that “B. Kaufman and I have recently solved” this problem. He gave the result as

$$M_0 = (1 - k^2)^{1/8} \quad (2.1)$$

where

$$k = 1/(\sinh 2H \sinh 2H') \quad (2.2)$$

and the result is true for  $0 < k < 1$ , when  $T < T_c$ . For  $k > 1$  the magnetization vanishes, i.e.  $M = 0$ . Figure 2 shows the resulting graph of  $M_0$  for the isotropic case  $H' = H$ .

Onsager and Kaufman did not publish their derivation, which has led to speculation as to why they did not do so. The first published derivation was not until 1952, by C. N. Yang,[8] who later described the calculation as “the longest in my career. Full of local, tactical tricks, the calculation proceeded by twists and turns.”[13, p.11]

Onsager did outline what happened in an article published in 1971.[14] He starts by remarking that correlations along a diagonal are particularly simple, and gives the formula

$$\langle \sigma_{1,1} \sigma_{m,m} \rangle = D_{m-1} \quad (2.3)$$

where  $D_m$ , like  $\Delta_m$ , is an  $m$  by  $m$  determinant:

$$D_m = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{m-1} \\ c_{-1} & c_0 & c_1 & \cdots & c_{m-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{1-m} & \cdot & \cdot & \cdots & c_0 \end{vmatrix} \quad (2.4)$$

the  $c_r$  being the coefficients in the Fourier expansion

$$f(\omega) = e^{i\hat{\delta}(\omega)} = \sum_{r=-\infty}^{\infty} c_r e^{ir\omega} \quad (2.5)$$

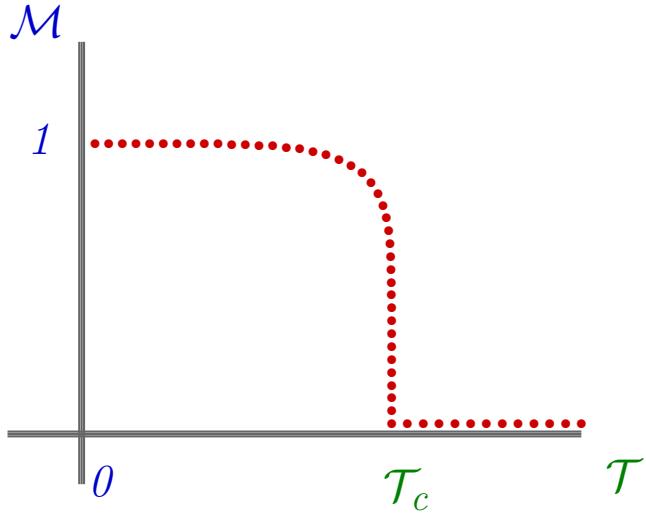


Figure 2:  $\mathcal{M}$  as a function of temperature  $\mathcal{T}$ .

of the function<sup>2</sup>

$$e^{i\delta(\omega)} = \left( \frac{1 - k e^{i\omega}}{1 - k e^{-i\omega}} \right)^{1/2}. \quad (2.6)$$

It is clear that Onsager knew this when his paper with Kaufman was submitted in May 1949, because footnote 7 of [3] states that “It can be shown that  $\delta' = \pi/2 - \omega/2$  at the critical temperature for correlations along a  $45^\circ$  diagonal of the lattice.” Indeed, this result does follow immediately from (2.6) when  $k = 1$ , provided we replace  $\delta$  in (1.4) by  $\widehat{\delta}$ . Certainly (2.3) is true, being the special case  $J_3 = v_3 = 0$  of equations (2.4), (5.13), (6.10), (6.12) of Stephenson’s pfaffian calculation[15] of the diagonal correlations of the triangular lattice Ising model. The formula (2.3) agrees with (1.7) when  $m = 2$ .

The fact that the formula depends on  $H, H'$  only via  $k$  is a consequence of the property that the diagonal transfer matrices of two models, with different values of  $H$  and  $H'$ , but the same value of  $k$ , commute.[16, §7.5]

In [14], Onsager says that he first evaluated  $D_m$  in the limit  $m \rightarrow \infty$  by using generating functions to calculate the characteristic numbers (eigenvalues) of the matrix  $D_m$  and that this leads to an integral equation with a kernel of the form

$$K(u, v) = K_1(u + v) + K_2(u - v). \quad (2.7)$$

He then obtained the determinant by taking the product of the eigenvalues and says that “This was the basis for the first announcements of the result.”. We show how this can be

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<sup>2</sup>I write the  $\delta$  of [14] as  $\widehat{\delta}$ .

done in Appendix A. One does indeed find a kernel of the form (2.7), and go on to obtain

$$\lim_{m \rightarrow \infty} D_m = (1 - k^2)^{1/4} , \quad (2.8)$$

which is the result (2.1) that Onsager announced in Cornell in 1948 and Florence in 1949.

### 3 The second method

In his 1971 article[14] Onsager goes on to say that after evaluating the particular determinant  $D_\infty$  by the integral equation method, he looked for a method for the evaluation of a general infinite-dimensional Toeplitz determinant (2.4), with arbitrary entries  $c_r$ . (The  $c_r$  must tend to zero as  $r \rightarrow \pm\infty$  sufficiently fast for the sum in (2.5) to be uniformly convergent when  $\omega$  is real.) As soon as he tried rational functions of the form

$$f(\omega) = \frac{\prod(1 - \alpha_j e^{i\omega})}{\prod(1 - \beta_k e^{-i\omega})} , \quad |\alpha_j|, |\beta_k| < 1 , \quad (3.1)$$

“the general result stared me in the face. Only, before I knew what sort of conditions to impose on the generating function, we talked to Kakutani and Kakutani talked to Szegő, and the mathematicians got there first.”

In another article in the same book [17], Onsager gives further explanation of that comment, saying that he had found “a general formula for the evaluation of Toeplitz matrices.<sup>3</sup> The only thing I did not know was how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that”. Onsager adds that six years later the mathematician Hirschman told him that he could readily have completed his proof by using a theorem of Wiener’s.

There is contemporary evidence to support these statements in the form of correspondence in 1950 between Onsager and Kaufman. There is also the photocopy of a typescript mentioned in the Introduction, which deals with the problem of calculating the  $\Delta_k$  of (1.3). Here I present a transcript of it, with approximately the original layout and pagination. A scanned copy is in Appendix B. It seems highly likely that this is Kaufman’s initial draft of paper IV in their series of papers on Crystal Statistics (see points 3 and 4 in section 5).

Hand-written additions are shown in red (for contemporary additions) and in blue (for probably later additions). Not all additions are shown. The “Fig. 1” mentioned on page 2, after equation III.45, may be Fig. 4 of [3]. Equation (17) on page 7 follows from eqn. 89b of [1] after interchanging  $H'$  with  $H^*$ ,  $\delta'$  with  $\delta^*$  and setting  $H' = H$ . There is an error in the equation between (20) and (21): the second  $e^{-2H}$  in the numerator should be  $e^{2H}$ , as should the first  $e^{-2H}$  in the denominator.

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<sup>3</sup>Refs. [14], [17] are reprinted in Onsager’s collected works, pages 232 – 241 and 37 – 45, respectively.[18]

# The unpublished draft

R.B. Potts

B. KAUFMAN

Long-Range order

Onsager

## Introduction

In a previous paper<sup>1</sup> we investigated the problem of short-range order in binary crystals of two dimensions. Short-range order in the neighbourhood of a fixed crystal-site  $(a, b)$  was described by a family of curves  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$ , each curve corresponding to a different choice of  $i$  and  $j$ . These functions give the correlation between the two crystal sites  $(a, b)$  and  $(i, j)$  as a function of temperature. In other words,  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$  is the probability that the spin at  $(i, j)$  is +1 if it is known that the spin at  $(a, b)$  is +1.

It was shown in III that for low temperatures

$$\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}} \sim 1$$

The physical meaning of this is clear: the model we used was a ferromagnet, in which, at zero temperature, all spins are aligned in the same direction; therefore all correlations (for all choices of  $(i, j)$ ) are equal to +1.

The correlation functions decrease as the temperature rises, first slowly, and then quite rapidly in the neighborhood of the critical temperature. For high temperatures the correlation functions tend to zero, as the crystal becomes more and more disordered. (Fig?)

Long-range order is the limit of the set of functions  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$  (for a fixed temperature) as the distance between  $(a, b)$  and  $(i, j)$  tends to infinity. We will evaluate this limiting function for correlations within a row, that is to say, for the case  $i = a$ . Since only the relative positions of the sites matter here, we may take  $a = b = 1$  and evaluate

$$\lim_{j \rightarrow \infty} \langle s_{1,1} s_{1,j} \rangle_{\text{Av.}}$$

It was shown in III that

$$\langle s_{1,1} s_{1,j} \rangle_{\text{Av.}} = \cosh^2 H^* \cdot \Delta_j - \sinh^2 H^* \cdot \Delta_{-j} \quad (\text{III.43})$$

Here  $H^*$  is a function of the temperature given through the relations

$$e^{-2H} = \tanh H^* , \quad H = J/kT \quad (\text{I.7.14})$$

$J$  is the energy parameter of the crystal, and  $k$  is Boltzmann's constant.

$\Delta_j$  and  $\Delta_{-j}$  are  $j$ -rowed determinants, expressed in terms of certain functions of temperature,  $\Sigma_r$  :

$$\Delta_j = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \cdots & \Sigma_j \\ \Sigma_0 & \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{j-1} \\ \Sigma_{-1} & \Sigma_0 & \Sigma_1 & \cdots & \Sigma_{j-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \Sigma_{-j+2} & \cdots & \cdots & \Sigma_1 \end{vmatrix} \quad \Delta_{-j} = \begin{vmatrix} \Sigma_{-1} & \Sigma_{-2} & \Sigma_{-3} & \cdots & \Sigma_{-j} \\ \Sigma_0 & \Sigma_{-1} & \Sigma_{-2} & \cdots & \Sigma_{-j+1} \\ \Sigma_1 & \Sigma_0 & \Sigma_{-1} & \cdots & \Sigma_{-j+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \Sigma_{j-2} & \cdots & \cdots & \Sigma_{-1} \end{vmatrix} \quad (\text{III.45})$$

(Fig. 1)

The temperature dependence of the  $\Sigma_r$  is shown in Fig.1.

Qualitatively we can see from the Figure that, since, for both high and low temperatures  $\Sigma_{-r}$  is very small,  $\Delta_{-j}$  must be small; but  $\Delta_j$ , which has  $\Sigma_1$  along its main diagonal, is small only at high  $T$ , whereas for low  $T$  its value is close to +1.

To evaluate  $\Delta_j$  exactly, we must know the analytic expressions for  $\Sigma_r$ , which have been introduced in III. It was shown there that the  $\Sigma_r$  are Fourier coefficients of a known function,  $\exp[i\delta'(\omega)]$ , and we will here make essential use of this fact.

### Evaluation of $\Delta_j$ :

What is the value of

$$\Delta_j = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{j-1} \\ c_{-1} & c_0 & c_1 & \cdots & \cdot \\ c_{-2} & c_{-1} & c_0 & \cdots & \cdot \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ c_{-j+1} & \cdots & \cdots & \cdots & c_0 \end{vmatrix} \quad (1)$$

where the  $c_n$  are the Fourier coefficients of a given function

$$f(\omega) = \sum_{-\infty}^{\infty} c_n e^{in\omega} \quad ? \quad (2)$$

At first we consider the special case where

$$f(\omega) = \frac{a_0 + a_1 e^{i\omega} + a_2 e^{2i\omega} + \cdots + a_p e^{pi\omega}}{b_0 + b_1 e^{-i\omega} + b_2 e^{-2i\omega} + \cdots + b_q e^{-qi\omega}} = \frac{g(e^{i\omega})}{h(e^{-i\omega})} . \quad (3)$$

and  $h(z) \neq 0$  for  $|z| \leq 1$

It is no restriction to assume:

$$b_0 = 1 .$$

Multiplying both sides by  $h(e^{-i\omega})$  we get

$$\begin{aligned} h(e^{-i\omega}) f(\omega) &= \left( \sum_{r=0}^q b_r e^{-ir\omega} \right) \left( \sum_{n=-\infty}^{\infty} c_n e^{in\omega} \right) = \\ &= \sum_{n=-\infty}^{\infty} (b_0 c_n + b_1 c_{n+1} + \cdots + b_q c_{n+q}) e^{in\omega} \\ &= \sum_{t=0}^p a_t e^{it\omega} = g(e^{i\omega}) \end{aligned}$$

$$\text{Hence } b_0 c_n + b_1 c_{n+1} + \cdots + b_q c_{n+q} = 0 \quad (4)$$

for all  $-\infty < n < \infty$ , except the finite set:  $n = 0, 1, 2, \dots, p$ . In particular

$$b_0 c_0 + b_1 c_1 + \cdots + b_q c_q = a_0 .$$

(4) means, when applied to the determinant  $\Delta_j$  ( $j$  larger than  $q$ ), that linear combinations of rows in  $\Delta_j$  can be chosen so that most elements in those rows vanish.

For example, suppose that  $b_0 c_n + b_1 c_{n+1} + b_2 c_{n+2} = 0$  for  $n \neq 0, 1, 2, 3$ ,

$$\text{Then, in } \Delta_4 = \begin{vmatrix} c_0 & c_1 & c_2 & c_3 \\ c_{-1} & c_0 & c_1 & c_2 \\ c_{-2} & c_{-1} & c_0 & c_1 \\ c_{-3} & c_{-2} & c_{-1} & c_0 \end{vmatrix},$$

multiply the 3rd row by  $b_0$ , and add to it the 2nd multiplied by  $b_1$  and the first multiplied by  $b_2$ . Similarly, multiply the forth row by  $b_0$  and add to it the third multiplied by  $b_1$  and the second multiplied by  $b_2$ . These operations do not affect the value of  $\Delta_3$ , and so we have:

$$\Delta_4 = \begin{vmatrix} c_0 & c_1 & c_2 & c_3 \\ c_{-1} & c_0 & c_1 & c_2 \\ 0 & 0 & a_0 & a_1 \\ 0 & 0 & 0 & a_0 \end{vmatrix} = a_0^2 \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} = a_0^2 \cdot \Delta_2$$

All elements to the left of  $a_0$  vanish, and the same will be also true for larger determinants. Therefore, in this case:

$$\Delta_j = a_0^{j-2} \cdot \Delta_2 .$$

In general case, when  $h(e^{-i\omega})$  is a polynomial of degree  $q$ , we will find:

$$\Delta_j = a_0^{j-q} \cdot \Delta_q . \quad (5)$$

$\Delta_j$  vanishes if  $a_0 = 0$ , that is, if there is no constant term in the polynomial  $g(e^{i\omega})$ . In other words  $\Delta_j = 0$ , if

$$g(e^{i\omega}) = e^{ik\omega} \cdot g_1(e^{i\omega}) \quad \text{Provided } j > q. \text{ If } j = q \text{ can't say this!}$$

where  $g_1(e^{i\omega})$  is a polynomial in  $e^{i\omega}$ , with a non-vanishing constant term.

Furthermore, if  $a_0 > 1$ , (5) shows that  $\Delta_\infty = \lim_{j \rightarrow \infty} \Delta_j$  diverges; while if  $a_0 < 1$   $\Delta_\infty = \lim_{j \rightarrow \infty} \Delta_j = 0$ . We disregard both of these uninteresting cases, and consider only the case  $a_0 = 1$ . (As mentioned above,  $b_0$  can always be chosen to equal 1). We can now replace (5) by

$$\Delta_j = \Delta_q \quad (j \geq q) . \quad (6)$$

$\Delta_q$  can always be expressed in terms of the constants in  $g(e^{i\omega})$  and  $h(e^{-i\omega})$ . Rather than use the coefficients  $a_t, b_r$ , we factorise the polynomials

$$g(e^{i\omega}) = \prod_{t=1}^p (1 - \alpha_t e^{i\omega}) \quad (7)$$

$$h(e^{-i\omega}) = \prod_{r=1}^q (1 - \beta_r e^{-i\omega})$$

and we express  $\Delta_q$  in terms of the  $\alpha$ 's and  $\beta$ 's.

We will now show that  $(1 - \alpha_t \beta_r)$  is a factor of  $\Delta_q$ . Consider  $f(\omega) = g(e^{i\omega})/h(e^{-i\omega})$  as a function of the variable  $\alpha_r$ . When  $\alpha_r = (\beta_t)^{-1}$  we get

$$\begin{aligned} \frac{g(e^{i\omega})}{h(e^{-i\omega})} &= \frac{(1 - \alpha_1 e^{i\omega}) \cdots (1 - \alpha_r e^{i\omega}) \cdots}{(1 - \beta_1 e^{-i\omega}) \cdots (1 - \beta_t e^{-i\omega}) \cdots} \\ &= \frac{\alpha_r e^{i\omega} (\beta_t e^{-i\omega} - 1)(1 - \alpha_1 e^{i\omega}) \cdots}{(1 - \beta_t e^{-i\omega})(1 - \beta_1 e^{-i\omega}) \cdots} = \frac{e^{i\omega} g_1(e^{i\omega})}{h_1(e^{-i\omega})} \end{aligned}$$

where  $g_1(e^{i\omega})$ ,  $h_1(e^{-i\omega})$  are both polynomials with constant terms. Hence by the above argument  $\Delta_j = 0$  when  $\alpha_r = (\beta_t)^{-1}$ , and we conclude that  $1 - \alpha_r \beta_t$  must be a factor of  $\Delta_q$ . Since this argument applies to all pairs  $\alpha_r, \beta_t$ , we find

$$\Delta_q = F(\beta_1, \beta_2, \dots, \beta_q) \cdot \prod_{r,t}^{q,p} (1 - \alpha_r \beta_t) \text{ Resultant} \quad (8)$$

Now to find  $F(\beta_1, \beta_2, \dots, \beta_q)$ . Since  $F$  is independent of the  $\alpha$ 's, we may choose  $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$ , and then get  $g(e^{i\omega}) = 1$

For  $f(\omega)$  we have in this case

$$f(\omega) = \frac{g(e^{i\omega})}{h(e^{-i\omega})} = \{h(e^{-i\omega})\}^{-1}, \quad \text{assumed } h(z) \neq 0 \text{ for } |z| \leq 1 !!$$

but the right-hand side will contain only terms with  $e^{-in\omega}$ ; therefore, in  $f(\omega)$  all  $c_n = 0$  for  $n > 0$ . The determinants corresponding to this  $f(\omega)$  will have the form

$$\Delta_q = \begin{vmatrix} 1 & 0 & 0 & \cdots \\ x & 1 & 0 & \cdots \\ y & x & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 1 \quad (x, y, \dots \neq 0) \quad (9)$$

On the other hand, our choice of  $\alpha_r$  gives

$$\prod_{r,t}^{q,p} (1 - \alpha_r \beta_t) = 1 \quad (10)$$

As a result

$$F(\beta_1, \beta_2, \dots, \beta_q) = 1 \quad (11)$$

and this holds for all choices of  $\alpha_r$ .

We thus have, for a general choice of  $\alpha_r, \beta_t$ :

$$\Delta_q = \prod_{r,t} (1 - \alpha_r \beta_t) \quad (12)$$

Together with (6), this gives

$$\Delta_\infty = \Delta_q = \prod_{r,t}^{q,p} (1 - \alpha_r \beta_t) \quad (13)$$

for the case where  $f(\omega)$  is the ratio of the polynomials given in (7).

(13) can be generalized. Consider the case where several roots of  $g(e^{i\omega})$  coincide, and similarly for  $h(e^{-i\omega})$ . Then we may write:

$$f(\omega) = \left\{ \prod_j (1 - \alpha_j e^{i\omega})^{m_j} \right\} \cdot \left\{ \prod_k (1 - \beta_k e^{-i\omega})^{n_k} \right\}^{-1} \quad (14)$$

and from (7) we get in this case

$$\Delta_\infty = \prod_{j,k} (1 - \alpha_j \beta_k)^{m_j n_k} \quad (15)$$

$m_j$  and  $n_k$  are integers, from the construction.

Log-range order along a row.

In order to apply the theory developed above to the problem of long-range order, we have to know the function in which the  $\Sigma_r$  are Fourier coefficients. From III we have

$$\Sigma_r = \frac{1}{\pi} \int_0^\pi \cos[rw + \delta'(w)] dw \quad (16) \quad (\text{III.})$$

About  $\delta'(w)$  we know (from I. ) that

$$\tan \delta'(w) = \frac{\sinh 2H \sin w}{\coth 2H - \cosh 2H \cdot \cos w} \quad (17)$$

$\delta'(w)$  is therefore an odd function of  $w$ , so that we can write

$$\Sigma_r = \frac{1}{\pi} \int_0^\pi e^{i\delta'(w)} \cdot e^{irw} dw \quad (18)$$

and, conversely,

$$e^{i\delta'(w)} = \sum_{-\infty}^{\infty} (\Sigma_{-r}) e^{irw} . \quad (19)$$

$e^{i\delta'(w)}$  now plays the role of  $f(w)$  from ( ), and we have to express it as a ratio of two polynomials, as in ( ). For this purpose we write

$$e^{i\delta'} = \left( \frac{1 + i \tan \delta'}{1 - i \tan \delta'} \right)^{\frac{1}{2}} = \left( \frac{\coth 2H - \cosh 2H \cdot \cos w + i \sinh 2H \cdot \sin w}{\coth 2H - \cosh 2H \cdot \cos w - i \sinh 2H \cdot \sin w} \right)^{\frac{1}{2}} \quad (20)$$

$$\begin{aligned} &= \left( \frac{\coth 2H - \frac{1}{2}e^{-2H+iw} - \frac{1}{2}e^{-2H-iw}}{\coth 2H - \frac{1}{2}e^{-2H+iw} - \frac{1}{2}e^{-2H-iw}} \right)^{\frac{1}{2}} \\ &= e^{-iw} \left( \frac{1 - 2 \coth 2H \cdot e^{-2H+iw} + e^{-4H+2iw}}{1 - 2 \coth 2H \cdot e^{-2H-iw} + e^{-4H-2iw}} \right)^{\frac{1}{2}} \end{aligned} \quad (21)$$

This is now almost in the desired form (3). If we define

$$\delta(w) = \delta'(w) + w \quad (22)$$

we have

$$e^{i\delta(w)} = \left( \frac{1 - 2 \coth 2H \cdot e^{-2H+iw} + e^{-4H+2iw}}{1 - 2 \coth 2H \cdot e^{-2H-iw} + e^{-4H-2iw}} \right)^{\frac{1}{2}} \quad (23)$$

We now factor the polynomials in the denominator and numerator, and find

$$e^{i\delta(w)} = \left\{ \frac{(1 - \coth H \cdot e^{-2H+iw})(1 - \tanh H \cdot e^{-2H+iw})}{(1 - \coth H \cdot e^{-2H-iw})(1 - \tanh H \cdot e^{-2H-iw})} \right\}^{\frac{1}{2}} \quad (24)$$

We can now find the determinants corresponding to  $e^{i\delta'(w)}$ . But we observe that

$$\begin{aligned} e^{i\delta(w)} &= e^{i\delta'(w)} \cdot e^{iw} = \sum_{r=-\infty}^{\infty} (\Sigma_{-r}) e^{irw} \cdot e^{iw} \\ &= \sum_{r=-\infty}^{\infty} (\Sigma_{-r+1}) e^{irw} = \sum_{r=-\infty}^{\infty} c_r e^{irw} \end{aligned}$$

??

Hence, the theory of section 2. gives us the values of

$$\Delta_{\infty} = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots \\ c_{-1} & c_0 & c_1 & \cdots \\ c_{-2} & c_{-1} & c_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \begin{vmatrix} \Sigma_1 & \Sigma_0 & \Sigma_{-1} & \cdots \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \cdots \\ \Sigma_3 & \Sigma_2 & \Sigma_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \cdots \\ \Sigma_0 & \Sigma_1 & \Sigma_2 & \cdots \\ \Sigma_{-1} & \Sigma_0 & \Sigma_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

$\Delta_{\infty}$  is the limit of  $\Delta_j$  in (III.45). We find from (13) and (7):<sup>10</sup>

$$\Delta_{\infty} = \{(1 - \coth^2 H \cdot e^{-4H})(1 - \tanh^2 H \cdot e^{-4H})(1 - e^{-4H})^2\}^{\frac{1}{4}}$$

But from ( ) we derive  $\tanh H^* = e^{-2H}$ , so that

$$\Delta_{\infty} = \left\{ [1 - e^{4(H^*-4H)}][1 - e^{-4(H^*+H)}](1 - e^{-4H})^2 \right\}^{\frac{1}{4}}$$

On the other hand, the limit of  $\Delta_j$  in ( ) is = 0. This is so, because the coefficients appear in  $\Delta_{-j}$  belong to the function  $e^{iw} \cdot e^{i\delta(w)}$ , and we have seen that the corresponding determinant vanishes in the limit ( ).

As a result, long-range order along a row is given by:

$$\lim_{j \rightarrow \infty} \langle s_{1,1} s_{1,j} \rangle_{\text{Av.}} = \cosh^2 H^* [1 - e^{4(H^*-H)}]^{\frac{1}{4}} [1 - e^{-4(H^*+H)}]^{\frac{1}{4}} [1 - e^{-4H}]^{\frac{1}{2}} .$$

## 4 Summary of the draft

The names at the top of the first page have been added by hand to the typescript, so it is not immediately obvious who are the authors. However, from the first sentence and their frequent references to paper III, specifically to III.45 (particularly in conjunction with their correspondence discussed below) it is clear that the paper is by Onsager and Kaufman jointly. Footnote 1 in the first line is not given, but is presumably their paper III.

The paper begins by quoting III.43, modified to (1.2) above, for the isotropic case  $H' = H$ .

It focusses on the problem of calculating a  $j$  by  $j$  Toeplitz determinant  $\Delta_j$ , of the general form (2.4), in the limit  $j \rightarrow \infty$ . The  $c_i$  are the coefficients of the Fourier expansion (2.5) of some function  $f(\omega)$ , initially allowed to be arbitrary. It first takes  $f(\omega)$

$$f(\omega) = \frac{a_0 + a_1 e^{i\omega} + \dots + a_p e^{pi\omega}}{b_0 + b_1 e^{-i\omega} + \dots + b_q e^{-qi\omega}} \quad (4.1)$$

and shows that

$$\Delta_j = 0 \text{ if } a_0 = 0 \text{ and } j > q . \quad (4.2)$$

It then takes  $f(\omega)$  to be of the form (3.1), or more specifically

$$f(\omega) = \frac{g(e^{i\omega})}{h(e^{-i\omega})} , \quad (4.3)$$

where

$$g(e^{i\omega}) = \prod_{t=1}^p (1 - \alpha_t e^{i\omega}) , \quad h(e^{-i\omega}) = \prod_{r=1}^q (1 - \beta_r e^{-i\omega}) ,$$

and goes on to show, using (4.2), that

$$\Delta_j = \prod_{t=1}^p \prod_{r=1}^q (1 - \alpha_t \beta_r)$$

provided that  $j \geq q$ . This is an algebraic identity, true for all  $\alpha_t, \beta_r$ . It is a trivial generalization then to say that if

$$f(\omega) = \frac{\prod_{j=1}^p (1 - \alpha_j e^{i\omega})^{m_j}}{\prod_{k=1}^q (1 - \beta_k e^{-i\omega})^{n_k}} ,$$

then

$$\Delta_\infty = \prod_{j=1}^p \prod_{k=1}^q (1 - \alpha_j \beta_k)^{m_j n_k} \quad (4.4)$$

for positive integers  $m_j, n_k$ .

For  $\Delta_j$ ,  $f(\omega) = e^{i\delta(\omega)}$  is given by (1.6). This is of the general form (4.4), with  $p = q = 2$  and

$$\alpha_1 = \beta_1 = \coth H e^{-2H} , \quad \alpha_2 = \beta_2 = \tanh H e^{-2H} , \quad m_1 = m_2 = n_1 = n_2 = \frac{1}{2}$$

so the  $m_j, n_j$  are no longer integers. The paper assumes that (4.4) can be generalized to this case,<sup>4</sup> so obtains (using  $\tanh H = e^{-2H^*}$ )

$$\Delta_\infty = \{[1 - e^{4(H^*-H)}][1 - e^{-4(H^*+H)}][1 - e^{-4H}]^2\}^{1/4}. \quad (4.5)$$

If we transpose the second determinant (1.3) in (1.2), then its generating function is not  $f(\omega)$  but  $e^{2i\omega}f(\omega)$ . This corresponds to the form (4.1), but with  $a_0 = 0$  (and  $a_1 = 0$ ), so  $\Delta_{-\infty}$  should vanish because of (4.2). Thus (1.2) gives

$$\lim_{j \rightarrow \infty} \langle \sigma_{1,1}\sigma_{1,1+j} \rangle = \cosh^2 H^* \{[1 - e^{4(H^*-H)}][1 - e^{-4(H^*+H)}][1 - e^{-4H}]^2\}^{1/4}.$$

For the isotropic case  $H' = H$ ,  $e^{-2H^*} = \tanh H$ , so we obtain, using (1.8),

$$M_0^2 = \lim_{j \rightarrow \infty} \langle \sigma_{1,1}\sigma_{1,1+j} \rangle = (1 - 1/\sinh^4 2H)^{1/4} = (1 - k^2)^{1/4},$$

in agreement with (2.1).

### Generalization to the anisotropic case

In their paper III,[3] Kaufman and Onsager focus on the isotropic case  $H' = H$ , as does the above draft. It does in fact appear that their results generalize immediately to the anisotropic case when  $H, H'$  are independent, and  $H^*, \delta'$  are defined as in [1] (i.e. by eqns. (1.1), (1.5), (1.6) above), and that (1.2), (1.7) are then correct as written.<sup>5</sup> Replacing equation (24) of the draft by (1.6), and repeating the last few steps, one obtains

$$\begin{aligned} \lim_{j \rightarrow \infty} \langle \sigma_{1,1}\sigma_{1,1+j} \rangle &= \cosh^2 H^* \{[1 - \coth^2 H'e^{-4H}][1 - \tanh^2 H'e^{-4H}][1 - e^{-4H}]^2\}^{1/4} \\ &= [1 - 1/(\sinh 2H \sinh 2H')^2]^{1/4} = (1 - k^2)^{1/4}, \end{aligned} \quad (4.6)$$

which again agrees with (2.1).

## 5 Further comments

There are several letters on the Onsager archive in Norway between Onsager and Kaufman relating to this second method, at

[http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009\\_0097.html](http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009_0097.html)  
and

[http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009\\_0096.html](http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009_0096.html)  
We shall refer to these two files as 0097 and 0096, respectively.

1. On pages 21 – 24 of 0097, (also in [19]) is a letter dated April 12, 1950 from Onsager to Kaufman giving the argument of the above draft up to equation 15 therein. Onsager states that the result admits considerable generalization. He defines

$$\eta_+(\omega) = \log g(e^{i\omega}), \quad \eta_-(\omega) = -\log h(e^{-i\omega}) \quad (5.1)$$

---

<sup>4</sup>This vital point is considered by Onsager in the first letter quoted in section 5.

<sup>5</sup>I have verified that (1.7) agrees with (2.3) - (2.6). I have not verified (1.2) directly, but have checked algebraically, with the aid of Mathematica, that it agrees with eqn. 56 of [6] for  $j \leq 8$ .

so  $f(\omega) = e^{\eta_+(\omega) + \eta_-(\omega)}$ . He remarks that if  $\log f(\omega)$  is analytic in a strip which contains the real axis, then these functions may be approximated by polynomials which have “no wrong zeros” in such a manner that the corresponding determinants converge. This implies that the  $\alpha_j, \beta_j$  all have modulus less than one. He goes on to say that (15) is equivalent to

$$\log \Delta_\infty = \frac{i}{2\pi} \int_{\omega=0}^{2\pi} \eta_+ d\eta_-(\omega) , \quad (5.2)$$

If

$$\log f(\omega) = \sum_{n=-\infty}^{\infty} b_n e^{in\omega} \quad (5.3)$$

and  $b_0 = 0$ , then (5.2) is equivalent to

$$\log \Delta_\infty = \sum_{n=1}^{\infty} n b_n b_{-n} , \quad (5.4)$$

which is the result now known as Szegő’s theorem.[20, 21, 6]

Onsager then says “we get the degree of order from C.S. III without much trouble. It equals  $(1 - k^2)^{1/8}$  as before.”

**2.** On pages 32, 33 of 0096 is a letter from Kaufman to Onsager. It is undated, but was presumably written after April 12th and before April 18th 1950 (the date of the next letter we discuss). She thanks him for his letter and comments that his method is elegant and simple, far superior to the integral-equations method for this purpose.

She also mentions the formula (2.6), i.e.

$$f(\omega) = \left( \frac{1 - ke^{i\omega}}{1 - ke^{-i\omega}} \right)^{1/2}$$

for the generating function for long-range order along a diagonal.

She goes on to say that the mathematician Kakutani had written to her saying that he had spoken to Onsager about this. He had been very interested in Onsager’s letter, which he saw in her house and immediately copied down.

**3.** On pages 30, 31 of 0096 is a letter from Kaufman to Onsager dated April 18, 1950, she says she’s glad to hear that Onsager is coming to Princeton the following Friday and refers to Onsager’s 1944 paper I, to her 1949 paper II, to their joint paper III, and then, significantly to a paper IV “on long-range order”. She says she would like Onsager to see her m.s.

**4.** Then on page 34 of 0096 is a letter from Kaufman to Onsager dated May 10, saying “Here is a draft of Crystal Statistics IV”.

All this fits with Onsager’s recollections of 1971[14, 17]. He and Kaufman had evaluated the determinants in III.43[3], or alternatively in (2.4) - (2.6), by the integral equation method (Appendix A) and by the general Toeplitz determinant formulae (15) of the draft, i.e (5.2). He preferred the second method, but apparently was content to let Kakutani and Szegő take it over and put in the rigorous mathematics. Kaufman’s draft of paper IV was never published, but it seems highly likely that the draft given here is indeed that.

Szegő did publish his resulting general theorem[20, [21, p.76] on the large-size limit of a Toeplitz determinant, but not until 1952. He also restricted his attention to Hermitian

forms, where  $f^*(\omega) = f(\omega)$  and  $b_n^* = b_{-n}$ ,[\[6, footnote 17\]](#) so his result needed further generalization before it could be applied to the Ising model magnetization.

The first derivation of  $M_0$  published was in 1952 by C.N. Yang.[\[8\]](#) He used the spinor operator algebra to write  $M_0$  as the determinant of an  $L$  by  $L$  matrix and evaluated the determinant by calculating the eigenvalues of the matrix in the limit  $L \rightarrow \infty$ . Intriguingly, he mentions Onsager and Kaufman's papers I, II and III, and in footnote 10 of his paper, Yang thanks Kaufman for showing him her notes on Onsager's work. However, his method is quite different from theirs.

Later, combinatorial ways were found of writing the partition function of the Ising model on a finite lattice directly as a determinant or a pfaffian (the square root of an anti-symmetric determinant).[\[22, 23\]](#). Then it was realised that the problem could be solved by first expressing it as one of filling a planar lattice with dimers.[\[24, 25, 26, 27\]](#) In 1963 Montroll, Potts and Ward[\[6\]](#) used these pfaffian methods to show that  $\langle \sigma_{1,1} \sigma_{1,j+1} \rangle$  could be written as a single Toeplitz determinant. They evaluated its value in the limit  $j \rightarrow \infty$  limit by using Szegő's theorem.

So there is no reason to doubt that Onsager and Kaufman had derived the formula (2.1) by May 1949, and ample evidence that they had obtained the result by what is now known as Szegő's theorem by May 1950. They did not publish the calculation, perhaps because the mathematicians beat them to the remaining problem of "how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that".

## 6 Acknowledgements

The author is most grateful to John Stephenson for giving him a copy of the draft paper reproduced here, and thanks Jacques Perk for telling him of ref.[\[19\]](#) and the material on the Lars Onsager Online archive under "Selected research material and writings" at <http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/arkiv5.php>

in particular items 9.96 and 9.97, and for pointing out a number of typographical errors in the original form of this paper. He also thanks Harold Widom for sending him a copy of the letter [\[19\]](#), and Richard Askey for alerting him to page 41 of Onsager's collected works. He is grateful to the reviewers for a number of helpful comments.

## Appendix A: An integral equation method

We regard  $D = D_m$  as a matrix and write  $z$  for  $e^{i\omega}$  and  $C(z)$  for  $e^{i\widehat{\delta}(\omega)}$ , so (2.5), (2.6) become

$$C(z) = \left( \frac{1 - kz}{1 - k/z} \right)^{1/2} = \sum_{m=-\infty}^{\infty} c_r z^r . \quad (\text{A1})$$

For  $0 < k < 1$ , the determinant of  $D_m$  tends to a non-zero limit as  $m \rightarrow \infty$ . The eigenvalues of  $D$  itself lie on an arc in the complex plane, and appear to tend to a continuous distribution as  $m \rightarrow \infty$ . By contrast, in this limit the individual eigenvalues  $\lambda_r$  of  $D^T D$  occur in discrete pairs, lying on the real axis, between 0 and 1. If one orders them so that  $\lambda_r \leq \lambda_{r+1}$ , then for given  $r$  the eigenvalue  $\lambda_r$  appears to tend to a limit as  $m \rightarrow \infty$ , and these limiting values tend to one as  $r \rightarrow \infty$ .

We assume these properties and seek to calculate the eigenvalues of  $D^T D$  in the limit  $m \rightarrow \infty$ , and hence the determinant of  $D^T D$ , which is  $(\det D)^2$ . Writing them as  $\lambda^2$ , we

can write the eigenvalue equation as

$$\lambda x = Dy , \quad \lambda y = D^T x . \quad (\text{A2})$$

For finite  $m$ ,  $x = \{x_0, x_1, \dots, x_{m-1}\}$ ,  $y = \{y_0, y_1, \dots, y_{m-1}\}$ . Let  $P$  be the  $m$  by  $m$  matrix with entries

$$P_{ij} = \delta_{i,m-1-j}$$

so  $Px = \{x_{m-1}, x_{m-2}, \dots, x_0\}$ . Then

$$D^T = PDP , \quad D^T D = (PD)^2 .$$

For finite  $m$ , the eigenvalues of  $D^T D$  are distinct, so the eigenvectors are those of  $PD$ . For large  $m$  and a given  $\lambda$ , this means that the elements  $x_i, y_i$  are of order one if  $i$  is close to zero or to  $m - 1$ , but tend to zero in between, when both  $i$  and  $m - i$  become large.

However, in the limit  $m \rightarrow \infty$ , the eigenvalues occur in equal pairs. One can then choose the two corresponding eigenvectors so that one has the property that

$$x_i, y_i \rightarrow 0 \quad \text{as } i \rightarrow \infty , \quad (\text{A3})$$

while for the other eigenvector  $x_{m-i}, y_{m-i} \rightarrow 0$ . The eigenvectors are transformed one to another by replacing  $x_i$  and  $y_i$  by  $y_{m-1-i}$  and  $x_{m-1-i}$ , respectively.

Since the eigenvalues are equal, we can and do restrict our attention to eigenvectors with the property (A3).

## Generating functions

Taking the limit  $m \rightarrow \infty$ , we can write (A2) explicitly as

$$\lambda x_i = \sum_{j=0}^{\infty} c_{j-i} y_j , \quad \lambda y_i = \sum_{j=0}^{\infty} c_{i-j} x_j \quad (\text{A4})$$

for  $i = 0, 1, 2, \dots$ .

We can extend these equations to negative  $i$ , defining  $x_i, y_i$  to then be given by the left-hand sides of the equations. Let  $X(z), Y(z)$  be the generating functions

$$X(z) = \sum_{i=-\infty}^{\infty} x_i z^i , \quad Y(z) = \sum_{i=-\infty}^{\infty} y_i z^i .$$

Then (A4) gives

$$\lambda X(z) = C(1/z)\tilde{Y}(z) , \quad \lambda Y(z) = C(z)\tilde{X}(z) , \quad (\text{A5})$$

where

$$\tilde{X}(z) = \sum_{i=0}^{\infty} x_i z^i , \quad \tilde{Y}(z) = \sum_{i=0}^{\infty} y_i z^i .$$

A Wiener–Hopf argument (or simple contour integration on each term) gives, for  $|z| < 1$ ,

$$\tilde{X}(z) = \frac{1}{2\pi i} \oint \frac{X(w)}{w-z} dw , \quad \tilde{Y}(z) = \frac{1}{2\pi i} \oint \frac{Y(w)}{w-z} dw , \quad (\text{A6})$$

the integrations being round the unit circle in the complex  $w$ -plane.

Thus (A5) is a pair of coupled integral equations for  $\lambda, X(z), Y(z)$ .

## The integral equations in terms of elliptic functions

To get rid of the square root in (A1) we introduce Jacobi elliptic functions of modulus  $k$  and set

$$z = k \operatorname{sn}^2 u .$$

If  $K, K'$  are the complete elliptic integrals and  $u = \alpha - K - iK'/2$ , then from (8.181), (8.191) of [28], or (15.1.5), (15.1.6) of [16],

$$z = s \prod_{n=1}^{\infty} \frac{(1 + p^{4n-3}/s)^2(1 + p^{4n-1}s)^2}{(1 + p^{4n-3}s)^2(1 + p^{4n-1}/s)^2}$$

where

$$p = q^{1/2} = e^{-\pi K'/2K} , \quad s = e^{i\pi\alpha/K} , \quad 0 < q < 1 .$$

If  $0 < k < 1$ , then  $0 < p, q < 1$ , so we see that  $z$  goes round the unit circle as  $\alpha$  goes from  $-K$  to  $K$  along the real axis, i.e.  $u$  goes along a horizontal line in the complex plane from  $-2K - iK'/2$  to  $-iK'/2$ .

From the elliptic function relations  $\operatorname{cn}^2 u = 1 - \operatorname{sn}^2 u$ ,  $\operatorname{dn}^2 u = 1 - k^2 \operatorname{sn}^2 u$ ,

$$\begin{aligned} C(z) &= i \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} , \\ &= \prod_{n=1}^{\infty} \frac{(1 + (-1)^n p^{2n-1}s)(1 - (-1)^n p^{2n-1}/s)}{(1 - (-1)^n p^{2n-1}s)(1 + (-1)^n p^{2n-1}/s)} \end{aligned} \quad (\text{A7})$$

Replace  $w$  in (A6) by

$$w = k \operatorname{sn}^2 v ,$$

where

$$\operatorname{Im}(v) = -K'/2 , \quad -K'/2 < \operatorname{Im}(u) < K'/2 . \quad (\text{A8})$$

Then

$$dw = 2k \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v dv . \quad (\text{A9})$$

It is helpful to set

$$X(u) = \frac{-i\widehat{X}(u)}{\operatorname{sn} u \operatorname{dn} u} , \quad Y(u) = \frac{i\widehat{Y}(u)}{\operatorname{cn} u} . \quad (\text{A10})$$

Then the integral equations (A5) become

$$\lambda \widehat{X}(u) = \frac{1}{2\pi} \int M(u, v) \widehat{Y}(v) dv , \quad \lambda \widehat{Y}(u) = \frac{1}{2\pi} \int M(v, u) \widehat{X}(v) dv , \quad (\text{A11})$$

where the integrations are from  $-2K - iK'/2$  to  $-iK'/2$  and

$$M(u, v) = 2 \frac{\operatorname{cn} u \operatorname{sn} v \operatorname{dn} v}{\operatorname{sn}^2 v - \operatorname{sn}^2 u} .$$

Using the Liouville theorem-type arguments of sections 15.3, 15.4 of [16], one can establish that

$$M(u, v) = \frac{\operatorname{dn}(v+u)}{\operatorname{sn}(v+u)} + \frac{\operatorname{dn}(v-u)}{\operatorname{sn}(v-u)} .$$

We see that this kernel is indeed of the form (2.7).

We require the functions  $\tilde{X}(z), \tilde{Y}(z)$  to be analytic for  $|z| < 1$ , i.e. for  $|\text{Im}(u)| < K'/2$ , and in particular for real  $u$ . From (A5), (A7) and (A10),  $\tilde{X}(z) = \lambda \widehat{Y}(u)/\text{snu dnu}$ ,  $\tilde{Y}(z) = \lambda \widehat{X}(u)/\text{cnu}$ . Since  $\text{sn} 0 = \text{cn} K = 0$ , it follows that

$$\widehat{X}(K) = \widehat{Y}(0) = 0 . \quad (\text{A12})$$

Also,  $M(u, v)$  is an analytic function of  $u$  within the domain (A8), and we can negate  $u$  within the domain. Since  $M(u, v)$  is an even function of  $u$ , it follows from (A11) (provided  $\lambda$  is not zero) that  $\widehat{X}(u)$  is also an even function. Similarly,  $\widehat{Y}(u)$  is an odd function.

### Solution by Fourier transforms

The sum and difference form of the kernel  $M(\alpha, \beta)$  suggests solving (A11) by Fourier transforms. The functions  $\widehat{X}, \widehat{Y}, M$  are anti-periodic of period  $2K$ , while  $\widehat{X}(u), \widehat{Y}(u)$  are even and odd, respectively. We therefore try

$$\widehat{X}(u) = A \cos \pi(2r - 1)u/2K , \quad \widehat{Y}(u) = B \sin \pi(2r - 1)u/2K , \quad (\text{A13})$$

$r$  being a positive integer. We note that this ansatz immediately satisfies (A12).

Integrating round a period rectangle and using Cauchy's theorem, or using (8.146) of [28], for all integers  $r$  we obtain

$$\frac{1}{2\pi} \int \frac{\text{dn}(v-u)}{\text{sn}(v-u)} e^{i\pi(2r-1)v/2K} dv = i \frac{e^{i\pi(2r-1)u/2K}}{1 + q^{2r-1}} . \quad (\text{A14})$$

Negating  $2r - 1$  and/or  $u$  (but not  $v$ ) and taking appropriate sums and differences, we obtain the identities, true when both  $v_0 + u$  and  $v_0 - u$  have imaginary parts between  $-2K'$  and zero and the integration is from  $v_0 - 2K$  to  $v_0$ :

$$\frac{1}{2\pi} \int M(u, v) \sin \frac{\pi(2r-1)v}{2K} dv = \frac{1 - q^{2r-1}}{1 + q^{2r-1}} \cos \frac{\pi(2r-1)u}{2K} \quad (\text{A15})$$

$$\frac{1}{2\pi} \int M(v, u) \cos \frac{\pi(2r-1)v}{2K} dv = \frac{1 - q^{2r-1}}{1 + q^{2r-1}} \sin \frac{\pi(2r-1)u}{2K} . \quad (\text{A16})$$

We see that the ansatz (A13) does indeed satisfy (A11), with  $A = B = 1$  and

$$\lambda = \frac{1 - q^{2r-1}}{1 + q^{2r-1}} . \quad (\text{A17})$$

More generally, we can allow  $\widehat{X}(u), \widehat{Y}(u)$  each to be an arbitrary linear combination of  $\exp[i\pi(2r-1)u/2K]$  and  $\exp[-i\pi(2r-1)u/2K]$ . We obtain the above solution, plus another where  $\lambda = 0$ . However, this second solution does not satisfy the necessary condition (A12).

### Determinant of $D$

For every positive integer  $r$  there is one and only one eigenvector of  $D^T D$  of the form (A3), with eigenvalue  $\lambda^2$ ,  $\lambda$  being given by (A17). However, from the discussion before

(A3), there is also an equal eigenvalue whose eigenvector has elements  $x_i, y_i$  of order 1 only when  $i$  is close to  $m$ . Hence, using (8.197.4) of [28] or (15.1.4b) of [16],

$$\det D^T D = (\det D)^2 = \prod_{r=1}^{\infty} \left( \frac{1 - q^{2r-1}}{1 + q^{2r-1}} \right)^4 = k' = (1 - k^2)^{1/2} . \quad (\text{A18})$$

so

$$\det D = (1 - k^2)^{1/4} \quad (\text{A19})$$

as in (2.8).

## Appendix B: The typewritten draft

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Long-Range Order

### Introduction

Onsager

In a previous paper<sup>1</sup> we investigated the problem of short-range order in binary crystals of two dimensions. Short-range order in the neighborhood of a fixed crystal-site  $(a, b)$  was described by a family of curves  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$ , each curve corresponding to a different choice of  $i$  and  $j$ . These functions give the correlation between the two crystal sites  $(a, b)$  and  $(i, j)$  as a function of temperature. In other words,  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$  is the probability that the spin at  $(i, j)$  is +1 if it is known that the spin at  $(a, b)$  is +1.

It was shown in III that for low temperatures

$$\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}} \sim 1$$

The physical meaning of this is clear: the model we used was a ferromagnet, in which, at zero temperature, all spins are aligned in the same direction; therefore all correlations (for all choices of  $(i, j)$ ) are equal to +1. The correlation functions decrease as the temperature rises, first slowly, and then quite rapidly in the neighborhood of the critical temperature. For high temperatures the correlation functions tend to zero, as the crystal becomes more and more disordered. (Fig?)

Long-range order is the limit of the set of functions  $\langle s_{a,b} s_{i,j} \rangle_{\text{Av.}}$  (for a fixed temperature) as the distance between  $(a, b)$  and  $(i, j)$  tends to infinity. We will evaluate this limiting function for correlations within a row, that is to say, for the case  $i = a$ . Since only the relative positions of the sites matter here, we may take  $a = b = 1$  and evaluate

$$\lim_{j \rightarrow \infty} \langle s_{1,1} s_{1,j} \rangle_{\text{Av.}}$$

It was shown in III that

$$\langle \epsilon_{j,j} \rangle_{A\gamma} = \cosh^2 H^2 \cdot \Delta_j + \sinh^2 H^2 \cdot \Delta_{-j} \quad (\text{III. 43})$$

Here  $H^2$  is a function of the temperature given through the relations

$$e^{-2H} = \tanh H^2, \quad H = J/kT \quad (\text{I. 7}, 14)$$

$J$  is the energy parameter of the crystal, and  $k$  is Boltzmann's constant.

$\Delta_j$  and  $\Delta_{-j}$  are  $j$ -rowed determinants, expressed in terms of certain functions of temperature,  $\Sigma_r$ :

$$\Delta_j = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \dots & \dots & \dots & \Sigma_j \\ \Sigma_{-1} & \Sigma_1 & \Sigma_2 & \dots & \dots & \dots & \Sigma_{j-1} \\ \Sigma_0 & \Sigma_{-1} & \Sigma_{-2} & \dots & \dots & \dots & \Sigma_{j-1}+1 \\ \Sigma_{-1} & \Sigma_0 & \Sigma_1 & \dots & \dots & \dots & \Sigma_{j-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma_{-j+2} & \dots & \dots & \dots & \dots & \dots & \Sigma_1 \end{vmatrix} \quad \Delta_{-j} = \begin{vmatrix} \Sigma_{-1} & \Sigma_{-2} & \Sigma_{-3} & \dots & \dots & \dots & \Sigma_{-j} \\ \Sigma_0 & \Sigma_{-1} & \Sigma_{-2} & \dots & \dots & \dots & \Sigma_{-j+1} \\ \Sigma_1 & \Sigma_0 & \Sigma_{-1} & \dots & \dots & \dots & \Sigma_{-j+2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma_{j-2} & \dots & \dots & \dots & \dots & \dots & \Sigma_{-1} \end{vmatrix} \quad (\text{III. 15})$$

(Fig. 1)

The temperature dependence of the  $\Sigma_r$  is shown in Fig. 1.

Qualitatively we can see from the Figure that, since, for both high and low temperatures  $\Sigma_r$  is very small,  $\Delta_{-j}$  must be small; but  $\Delta_j$ , which has  $\Sigma_1$  along its main diagonal, is small only at high  $T$ , whereas for low  $T$  its value is close to +1.

To evaluate  $\Delta_j$  exactly, we must know the analytic expressions for  $\Sigma_r$ , which have been introduced in III. It was shown there that the  $\Sigma_r$  are Fourier coefficients of a known function,  $\exp[i\delta(\omega)]$ , and we will here make essential use of this fact.

#### Evaluation of $\Delta_j$ :

What is the value of

$$\Delta_j = \begin{vmatrix} c_0 & c_1 & c_2 & \dots & \dots & c_{j-1} \\ c_{-1} & c_0 & c_1 & \dots & \dots & \dots \\ c_{-2} & c_{-1} & c_0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{-j+1} & \dots & \dots & \dots & \dots & c_0 \end{vmatrix} \quad (1)$$

where the  $c_n$  are Fourier coefficients of a given function

$$z(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega} \quad ? \quad (2)$$

At first we consider the special case where

$$f(\omega) = \frac{a_0 + a_1 e^{i\omega} + a_2 e^{2i\omega} + \dots + a_p e^{pi\omega}}{b_0 + b_1 e^{i\omega} + b_2 e^{2i\omega} + \dots + b_q e^{qi\omega}} = \frac{f(e^{i\omega})}{h(e^{-i\omega})}. \quad (3)$$

It is no restriction to assume:

$$\text{and } h(z) \neq 0 \text{ for } |z| \leq 1$$

$$b_0 = 1. \quad c_{pn} = 0 \text{ for } n > p$$

Multiplying both sides by  $h(e^{-i\omega})$  we get

$$\begin{aligned} h(e^{-i\omega}) f(\omega) &= (\sum_{n=0}^q b_n e^{-in\omega}) (\sum_{n=-\infty}^{\infty} c_n e^{in\omega}) = \\ &= \sum_{n=-\infty}^{\infty} (b_0 c_n + b_1 c_{n+1} + \dots + b_q c_{n+q}) e^{in\omega} \\ &= \sum_{t=0}^p a_t e^{it\omega} = g(e^{i\omega}) \end{aligned}$$

$$\text{Hence } b_0 c_n + b_1 c_{n+1} + \dots + b_q c_{n+q} = 0 \quad (4)$$

for all  $-\infty < n < \infty$ , except the finite set:  $n = 0, 1, 2, \dots, p$ . In particular

$$b_0 c_0 + b_1 c_1 + \dots + b_q c_q = a_0.$$

(4) means, when applied to the determinant  $\Delta_j$  ( $j$  larger than  $q$ ), that linear combinations of rows in  $\Delta_j$  can be chosen so that most elements in those rows vanish.

For example, suppose that  $b_0 c_n + b_1 c_{n+1} + b_2 c_{n+2} = 0$  for  $n \neq 0, 1, 2, 3$ ,

Then, in  $\Delta_4 = \frac{1}{b_0^2} \begin{vmatrix} c_0 & c_1 & c_2 & c_3 \\ c_{-1} & c_0 & c_1 & c_2 \\ c_{-2} & c_{-1} & c_0 & c_1 \\ c_{-3} & c_{-2} & c_{-1} & c_0 \end{vmatrix}$

multiply the 3rd row by  $b_0$ , and add to it the 2nd multiplied by  $b_1$  and the first multiplied by  $b_2$ . Similarly, multiply the forth row by  $b_0$  and add to it the third multiplied by  $b_1$  and the second multiplied by  $b_2$ . These operations do not affect the value of  $\Delta_4$ , and so we have :

$$\Delta_4 = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \\ 0 & 0 & 0 & a_0 \end{vmatrix} = \frac{a_0^2}{b_0^2} \begin{vmatrix} a_0 & a_1 \\ a_{-1} & a_0 \end{vmatrix} = \frac{a_0^2}{b_0^2} \cdot \Delta_2$$

All elements to the left of  $a_0$  vanish, and the same will be also true for larger determinants. Therefore, in this case:

$$\Delta_j = a_0^{j-2} \cdot \Delta_2$$

In general case, when  $h(e^{-i\omega})$  is a polynomial of degree  $q$ , we will find:

$$\Delta_j = a_0^{j-q} \cdot \Delta_q. \quad \text{unless } j \leq q \quad (5)$$

$\Delta_j$  vanishes if  $a_0 = 0$ , that is, if there is no constant term in the polynomial  $g(e^{i\omega})$ . In other words  $\Delta_j = 0$ , if

$$g(e^{i\omega}) = e^{ik\omega} g_s(e^{i\omega}) \quad \text{Provided } j > q. \quad \text{If } j = 1 \text{ can't say this!}$$

where  $g_s(e^{i\omega})$  is a polynomial in  $e^{i\omega}$ , with a non-vanishing constant term.

Furthermore, if  $a_0 > 1$ , (5) shows that  $\Delta_\infty = \lim_{j \rightarrow \infty} \Delta_j$  diverges;

while if  $a_0 < 1$   $\Delta_\infty = \lim_{j \rightarrow \infty} \Delta_j = 0$ . We disregard both of these uninteresting cases, and consider only the case  $a_0 = 1$ . (As mentioned above,  $b_0$  can always be chosen to equal 1). We can now replace (5) by

$$\boxed{\Delta_j = \Delta_q} \quad (j \geq q). \quad (6)$$

$\Delta_q$  can be expressed in terms of the constants in  $g(e^{i\omega})$  and  $h(e^{-i\omega})$ . Rather than use the coefficients  $a_t$ ,  $b_r$ , we factorize the polynomials

$$g(e^{i\omega}) = \prod_{t=1}^T (1 - \alpha_t e^{i\omega}) \quad (7)$$

$$h(e^{-i\omega}) = \prod_{r=1}^R (1 - \beta_r e^{-i\omega})$$

and we express  $\Delta_q$  in terms of the  $\alpha$ 's and  $\beta$ 's.

We will now show that  $(1 - \alpha_t \beta_r)$  is a factor of  $\Delta_q$ . Consider  $f(\omega) = g(e^{i\omega})/h(e^{-i\omega})$  as a function of the variable  $\alpha_r$ . When  $\alpha_r = (\beta_t)^{-1}$  we get

$$\frac{f(e^{i\omega})}{h(e^{-i\omega})} = \frac{(1 - \alpha_1 e^{i\omega}) \dots (1 - \alpha_r e^{i\omega}) \dots}{(1 - \beta_1 e^{-i\omega}) \dots (1 - \beta_t e^{-i\omega}) \dots} =$$

$$= \cancel{\alpha_r e^{i\omega} \cdot (\alpha_r e^{-i\omega} - 1)} \cdot \frac{(1 - \alpha_1 e^{i\omega}) \dots}{(1 - \beta_1 e^{-i\omega}) (1 - \beta_t e^{-i\omega}) \dots} = \frac{e^{i\omega} \cancel{\alpha_r} (e^{-i\omega})}{h(e^{-i\omega})}$$

where  $g_r(e^{i\omega})$ ,  $h(e^{-i\omega})$  are both polynomials with constant terms. Hence by the above argument  $\Delta_q = 0$  when  $\alpha_r = (\beta_t)^{-1}$ , and we conclude that  $1 - \alpha_r \beta_t$  must be a factor of  $\Delta_q$ . Since this argument applies to all pairs  $\alpha_r, \beta_t$  we find

$$\Delta_q = F(\beta_1, \beta_2, \dots, \beta_q) \cdot \frac{q!}{r_1 t} (1 - \cancel{\alpha_r \beta_t}) \quad \text{Resultant} \quad (8)$$

Now to find  $F(\beta_1, \beta_2, \dots, \beta_q)$ . Since  $F$  is independent of the  $\alpha$ 's, we may choose  $\alpha_1 = \alpha_2 = \dots = \alpha_p = 0$ , and then  $g(e^{i\omega}) = 1$

For  $f(\omega)$  we have, in this case

$$f(\omega) = \frac{f(e^{i\omega})}{h(e^{-i\omega})} = \left\{ h(e^{-i\omega}) \right\}^{-1}, \quad \Rightarrow \text{assumed } h(z) \neq 0 \text{ for } |z| \leq 1!!$$

but the right-hand side will contain only terms with  $e^{-in\omega}$ ; therefore, in  $f(\omega)$  all  $c_n = 0$  for  $n > 0$ . The determinants corresponding to this  $f(\omega)$  will have the form

$$\Delta_q = \begin{vmatrix} 1 & 0 & 0 & \dots & 1 \\ x & 1 & 0 & \dots & \\ y & x & 1 & \dots & \\ \vdots & \ddots & \ddots & \ddots & \end{vmatrix} = 1 \quad (x, y, \dots, y' \neq 0) \quad (9)$$

On the other hand, our choice of  $\alpha_r$  gives

$$\frac{1}{r! t!} (1 - \alpha_r \beta_t)^{-1} = 1 \quad (10)$$

Since  $r! t! \neq 0$

$$F(\beta_1, \beta_2, \dots, \beta_q) = 1 \quad (11)$$

and this holds now for all choices of  $\alpha_r$

We thus have, for a general choice of  $\alpha_r, \beta_t$ ,

$$\Delta_q = \frac{\prod}{r_i t} (1 - \alpha_r \beta_t) = \text{Resultant of } f \text{ and } g \quad (12)$$

Together with (6), this gives

$$\Delta_{\infty} = \Delta_q = \frac{\prod}{r_i t} (1 - \alpha_r \beta_t) \quad (13)$$

for the case where  $f(\omega)$  is the ratio of the polynomials given in (7).

(#3) can be generalized. Consider the case where several roots of  $e^{i\omega}$  coincide, and similarly for  $h(e^{-i\omega})$ . Then we may write:

$$f(\omega) = \left\{ \frac{1}{j} (1 - \alpha_j e^{i\omega})^{n_j} \right\} \cdot \left\{ \frac{1}{k} (1 - \beta_k e^{-i\omega})^{n_k} \right\} \quad (14)$$

and from (7) we get in this case

$$\Delta_{\infty} = \prod_{j,k} (1 - \alpha_j \beta_k)^{n_j n_k} \quad (15)$$

$n_j$  and  $n_k$  are integers, from the construction.  $\alpha_j = \frac{a_j}{b_j}$ ,  $\beta_k = \frac{c_k}{d_k}$ .

$$\Delta_q = 1 \quad \text{if } \alpha_j = \infty \quad \text{i.e. } j=0. \quad f(\omega) = \frac{a_0}{h(e^{-i\omega})}$$

$$\Rightarrow \frac{1}{1 + b_1 e^{-i\omega} + b_2 e^{-i\omega} + \dots + b_N e^{-i\omega}} = \prod_{j=1}^N e^{a_j e^{-i\omega}} \begin{vmatrix} a_0 & a_1 & & \\ a_1 & a_2 & & \\ \vdots & \vdots & \ddots & \\ a_N & & & a_N \end{vmatrix} =$$

Q. Does  $h(e^{-i\omega})$  have any zeros inside  $|z| \leq N$ ?

Satisfy if  $a_j \approx \sqrt{2}$ .

Long = range order along a row.

In order to apply the theory developed above to the problem of long-range order, we have to know the function in which the  $\Sigma_{\mathbf{F}}$  are Fourier coefficients. From III we have

$$\Sigma_{\mathbf{F}} = \frac{1}{\pi} \int_0^{\pi} \cos[nw + f^*(w)] dw \quad (16) \quad (\text{III.})$$

About  $f^*(w)$  we know (from I. ) that

$$\tan f^*(w) = \frac{\sinh 2H \sin w}{\coth 2H - \cosh 2H \cos w} \quad (\text{I.}) \quad \tan f^* = \frac{\sin w \sinh 2H}{\coth 2H - \cosh 2H}$$

$f^*(w)$  is therefore an odd function of  $w$ , so that we can write

$$\Sigma_{\mathbf{F}} = \frac{1}{\pi/2} \int_0^{\pi} e^{i f^*(w)} \cdot e^{i nw} dw \quad \begin{matrix} \text{sh } 2H \text{ is } 1 \\ \text{ch } 2H = 1 \\ \text{ch } 2H - \cosh 2H \end{matrix}$$

and, conversely,

$$e^{i f^*(w)} = \sum_{-\infty}^{\infty} (\Sigma_{\mathbf{F}}) e^{i nw}. \quad (17)$$

$e^{i f^*(w)}$  now plays the role of  $f(w)$  from ( ), and we have to express it as a ratio of two polynomials, as in ( ). For this purpose we write

$$\begin{aligned} e^{i f^*} &= \frac{(1 + i \tan f^*)}{(1 - i \tan f^*)}^{\frac{1}{2}} = \left( \frac{\coth 2H - \cosh 2H \cos w + i \sinh 2H \sin w}{\coth 2H - \cosh 2H \cos w - i \sinh 2H \sin w} \right)^{\frac{1}{2}} \quad (20) \\ &= \left( \frac{\coth 2H - \frac{1}{2} e^{-2H} + iw - \frac{1}{2} e^{-2H} - iw}{\coth 2H - \frac{1}{2} e^{-2H} + iw - \frac{1}{2} e^{-2H} - iw} \right)^{\frac{1}{2}} \\ &= e^{-iw} \left( \frac{1 - 2 \coth 2H \cdot e^{-2H} + iw + e^{-4H} + 2 iw}{1 - 2 \coth 2H \cdot e^{-2H} - iw + e^{-4H} - 2 iw} \right)^{\frac{1}{2}} \quad (21) \end{aligned}$$

This is now almost in the desired form (3). If we define

$$f(w) = f^*(w) + w$$

we have

$$e^{i f^*(w)} = \left( \frac{1 - 2 \coth 2H \cdot e^{-2H} + iw + e^{-4H} + 2 iw}{1 - 2 \coth 2H \cdot e^{-2H} - iw + e^{-4H} - 2 iw} \right)^{\frac{1}{2}} \quad (22)$$

We now factor the polynomials in the denominator and numerator, and find

$$e^{i f(w)} = \frac{(1 - \coth H \cdot e^{-2H} + iw)(1 - \tanh H \cdot e^{-2H} + iw)}{(1 - \coth H \cdot e^{-2H} - iw)(1 - \tanh H \cdot e^{-2H} - iw)}^{\frac{1}{2}}$$

- 8 -

We can now find the determinants corresponding to  $e^{i\delta(w)}$ . But we observe that

$$\begin{aligned} e^{i\delta(w)} &= e^{i\delta(w)}, e^{iw} = \sum_{n=-\infty}^{\infty} (\Sigma_{-n}) e^{iwn} \cdot e^{iw} \\ 27 &= \sum_{n=-\infty}^{\infty} (\Sigma_{-n+1}) e^{iwn} = \sum_{n=-\infty}^{\infty} c_n e^{iwn} \end{aligned}$$

Hence, the theory of section 2. gives us the values of

$$\Delta = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-1} & \cdots \\ c_{-2} & c_{-1} & c_0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \Sigma_1 & \Sigma_0 & \Sigma_{-1} & \cdots & \cdots & \cdots \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \cdots & \cdots & \cdots \\ \Sigma_3 & \Sigma_2 & \Sigma_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \cdots & \cdots & \cdots \\ \Sigma_0 & \Sigma_1 & \Sigma_2 & \cdots & \cdots & \cdots \\ \Sigma_{-1} & \Sigma_0 & \Sigma_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$\Delta_{\infty}$  is the limit of  $\Delta_n$  in (27). We find from (13) and (14),<sup>10</sup>

$$\Delta_{\infty} = \left\{ (1 - \coth^2 H \cdot e^{-4H}) (1 - \tanh^2 H \cdot e^{-4H}) (1 - e^{-4H})^2 \right\} \frac{1}{4}$$

But from (14) we derive  $\tanh H^* = e^{-2H}$ , so that

$$\Delta_{\infty} = \left\{ [1 - e^{-4(H^* - H)}] [1 - e^{-4(H^* + H)}] [1 - e^{-4H}]^2 \right\} \frac{1}{4}$$

On the other hand, the limit of  $\Delta_n$  in (27) is 0. This is so, because the coefficients appear in  $\Delta_n$  belong to the function  $e^{iw} \cdot e^{i\delta(w)}$ , and we have seen that the corresponding determinant vanishes in the limit. (1).

As a result, long-range order along a row is given by:

$$\lim_{j \rightarrow \infty} \langle s_{i,j}, s_{i,j} \rangle_{Av.} = \cosh^2 H^* [1 - e^{-4(H^* - H)}]^{\frac{1}{4}} [1 - e^{-4(H^* + H)}]^{\frac{1}{4}} [1 - e^{-4H}]^{\frac{1}{2}}.$$

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