

Scenario for the renormalization in the 4D Yang-Mills theory

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Yang-Mills quantum field theory has unique character, allowing a self-consistent formulation in the four dimensional space-time. Two important properties — asymptotic freedom and dimensional transmutation — are characteristic features of this theory, which distinguish it from others. I think, that the typical textbook exposition of this theory, based on general paradigm of QFT, still does not underline these specifics. In my talk I propose a scheme for the description of the Yang-Mills theory which exactly does this. I do not claim finding anything new, my proposal has simply a methodological value.

As a main tool for my presentation I have chosen the object, called the effective action, defined via the background field. Following Feynman ideas I consider the functional of a background field as the generating functional for the S -matrix, whereas the Schwinger functional of external current, generating the Green functions, needs LSZ reduction formulas to define S -matrix. Moreover, the latter functional is not manifestly gauge invariant.

Ironically the standard description of the background field method [1], [2], [3] uses the external current and Legendre transformation. Alternative formulation began in [4] and entering [5] was improved in [6]. In what follows I use the latter approach.

An effective action $W(B)$ is a functional of a classical Yang-Mills field $B_\mu(x)$ given by a series in a dimensionless coupling α

$$W(B) = \frac{1}{\alpha} W_{-1}(B) + W_0(B) + \sum_{n \geq 1} \alpha^n W_n(B),$$

where $W_{-1}(B)$ is a classical action, $W_0(B)$ is a one loop correction, defined via the determinants of vector and scalar operators M_1 and M_0

$$M_0 = \nabla_\mu^2, \quad M_1 = \nabla_\sigma^2 \delta_{\mu\nu} + 2[F_{\mu\nu}, \cdot],$$

where

$$\begin{aligned}\nabla_\mu &= \partial_\mu + B_\mu, \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \\ W_0 &= -\frac{1}{2} \ln \det M_1 + \ln \det M_0\end{aligned}$$

and W_k , $k = 1, 2 \dots$ are defined as the contribution of strongly connected vacuum diagramms with $k + 1$ loops, constructed via Green functions M_0^{-1} and M_1^{-1} and vertices defined by the forms of vector and scalar fields $a_\mu(x)$, $\bar{c}(x)$, $c(x)$

$$\begin{aligned}\Gamma_3(B) &= g \int \text{tr} \nabla_\mu a_\nu [a_\mu, a_\nu] d^4x, \\ \Gamma_4(B) &= g^2 \int \text{tr} [a_\mu, a_\nu]^2 d^4x, \\ \Omega(B) &= g \int \text{tr} \nabla_\mu \bar{c} [a_\mu, c] d^4x,\end{aligned}$$

where $g = \frac{1}{2}\sqrt{\alpha}$, taking into account anticommuting properties of ghosts $\bar{c}(x)$, $c(x)$.

The divergences of the diagramms should be regularized. I believe that there exists a regularization, defined by the cut-off momentum Λ such that all infinities are powers in

$$L = \ln \frac{\Lambda}{\mu},$$

where μ is some normalization mass. Unfortunately presently I do not know a satisfactory procedure for such a regularization. That is why I call my exposition a “scenario”.

The renormalizability of the Yang-Mills theory means, that there exists a dependence of the coupling constant α on cutoff Λ such that the full action $W(B)$ is finite.

In the case of one loop everything is clear. The functional $W_0(B)$ can be defined via the proper time method of Fock [7] giving formula

$$W_0(B) = \int_0^\infty \frac{ds}{s} T(B, s),$$

where the functional $T(B, s)$ has the following behavior for small s

$$T(B, s) = T_0(B) + sT_1(B, s)$$

and

$$T_0(B) = \frac{1}{2}\beta_1 W_{-1},$$

where β_1 is a famous negative constant. So the only divergence is proportional to the classical action and can be compensated by the renormalization of the coupling constant α . In more detail, we regularize $W_0(B)$ as follows

$$W_0^{\text{reg}}(B) = \int_0^{1/\Lambda^2} ds T_1(B, s) + \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} T(B, s)$$

and choose the dependence of the coupling constant α on Λ as

$$\frac{1}{\alpha(\Lambda)} = -\beta_1 \ln \frac{\Lambda}{m},$$

where m is a new parameter with the dimension of mass. It is clear, that the regularized one loop $W(B)$ does not depend on Λ and so is finite. More explicitly, we rewrite $W_0^{\text{reg}}(B)$ as follows

$$W_{00}(B) = W_{00} + W_{01}L,$$

where

$$W_0^{\text{reg}}(B) = \int_0^{1/\mu^2} ds T_1(B, s) + \int_{1/\mu^2}^{\infty} \frac{ds}{s} T(B, s),$$

and

$$W_{01} = \beta_1 W_{-1}$$

and define the renormalized running coupling constant

$$\frac{1}{\alpha(\mu)} = -\beta_1 \ln \frac{\mu}{m},$$

so that

$$W_{1\text{loop}}^{\text{reg}}(B) = \frac{1}{\alpha(\mu)} W_{-1} + W_{00}.$$

Thus we have traded the dimensionless α for a dimensional parameter m . However m enters trivially, defining the scale. Observe that $\alpha(\Lambda)$ and $\alpha(\mu)$ are the values of the same function for two values of the argument. This function satisfies the first approximation to Gell-Mann Low equation

$$x \frac{d}{dx} \alpha(x) = \beta(\alpha) = \beta_1 \alpha^2,$$

where the RHS does not depend on x and m plays the role of the conserved integral. The shift of x can be interpreted as an abelian group action, this is the famous renormalization group. The same is true for the whole one loop functional

$$W_{\text{1loop}}^{\text{reg}}(B, \Lambda) = W_{\text{1loop}}(B, \mu)$$

and the renormalized action $W(B, \mu)$ does not depend on the running momentum μ .

My scenario is based on the assumption that the whole $W(B)$ has the same properties. I believe that all infinities in $W(B)$ can be combined into the form

$$\begin{aligned} W(B, \Lambda) = & \frac{1}{\alpha} W_{-1} + W_{00} + W_{01} L + \\ & + \alpha (W_{10} + W_{11} L) + \dots + \\ & + \alpha^n (W_{n0} + W_{n1} L + \dots W_{nn} L^n) + \dots, \end{aligned}$$

where the functionals $W_{k0}(B)$, $k = 0, \dots$ are finite and depend on μ . The coupling constant $\alpha(\Lambda)$ should satisfy the full Gell-Mann Low equation

$$\Lambda \frac{d\alpha}{d\Lambda} = \beta(\alpha) = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \dots \beta_n \alpha^n + \dots$$

and the full $W(B)$ should be independent of L and so it is finite.

The equation

$$\frac{dW}{dL} = 0$$

immediately gives

$$W_{11} = \beta_2 W_{-1}$$

and leads to an equation, expressed via double series in powers of α and L . The condition that the corresponding coefficients vanish leads to the recurrent relations expressing W_{nm} via $W_{n-1,m}$, $n = 2, 3, \dots$, $n \geq m$.

Here are examples of such equations at the lowest orders

$$\begin{aligned} \beta_1 W_{10} + W_{21} &= \beta_3 W_{-1} \\ \beta_2 W_{10} + 2\beta_1 W_{20} + W_{31} &= \beta_4 W_{-1} \\ \beta_1 W_{11} + 2W_{22} &= 0 \\ \beta_2 W_{11} + 2\beta_1 W_{21} + 2W_{32} &= 0. \end{aligned}$$

One can solve some of these equations exactly. For instance for the highest coefficients W_{nn} we get relation

$$(n-1)\beta_1 W_{n-1,n-1} + nW_{nn} = 0$$

and as W_{11} is proportional to W_{-1} the same is true for all W_{nn} . Thus their contribution $\sum \alpha^n L^n W_{nn}$ is summed up to

$$\frac{\beta_2}{\beta_1} \ln(1 + \beta_1 \alpha L) W_{-1}$$

and this gives the next correction to the coefficient in front of W_{-1} in $W(B)$, namely

$$\frac{1}{\alpha} + \beta_1 L + \frac{\beta_2}{\beta_1} [\ln \alpha + \ln(\frac{1}{\alpha} + \beta_1 L)]$$

leading to the next approximation for the renormalized coupling constant

$$\frac{1}{\alpha_r} = \frac{1}{\alpha} + \beta_1 L + \frac{\beta_2}{\beta_1} \ln L,$$

consistent with the Gell-Mann Low equation.

We see that all coefficients W_{nm} , $n \geq m$, $n \geq 1$ are expressed via the finite ones W_{n0} . More detailed investigation of these equations [8] shows, that the full expression for $W(B)$ can be rewritten in terms of W_{n0} and powers of renormalized coupling constant

$$W(B, \mu) = \frac{1}{\alpha_r} W_{-1} + W_{00} + \sum \alpha_r^n W_{n0}.$$

This is consistent with the equation

$$W_{\text{reg}}(B, \Lambda) = W(B, \mu)$$

and

$$L|_{\Lambda=\mu} = 0.$$

Let us comment, that the recurrence relations make sense when the only nonzero coefficients in β -function are β_1 and β_2 . This makes possible to speculate, that there exist the regularizations in which $\beta_3 = \beta_4 = \dots = 0$.

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References

- [1] B. S. DeWitt, “Quantum Theory of Gravity. 2. The Manifestly Covariant Theory,” *Phys. Rev.* **162** (1967) 1195.
B. S. DeWitt, “Quantum Theory of Gravity. 3. Applications of the Covariant Theory,” *Phys. Rev.* **162** (1967) 1239.
- [2] G. ’t Hooft, “The Background Field Method in Gauge Field Theories,” In *Karpacz 1975, Proceedings, Acta Universitatis Wratislaviensis No.368, Vol.1*, Wroclaw 1976, 345-369.
- [3] L. F. Abbott, “Introduction to the Background Field Method,” *Acta Phys. Polon. B* **13** (1982) 33.
- [4] I. Y. Arefeva, L. D. Faddeev and A. A. Slavnov, “Generating Functional for the s Matrix in Gauge Theories,” *Theor. Math. Phys.* **21** (1975) 1165 [*Teor. Mat. Fiz.* **21** (1974) 311].
- [5] L. D. Faddeev and A. A. Slavnov, “Gauge Fields. Introduction To Quantum Theory,” *Front. Phys.* **50** (1980) 1, [*Front. Phys.* **83** (1990) 1].
- [6] L. D. Faddeev, “Mass in Quantum Yang-Mills Theory: Comment on a Clay Millenium problem,” arXiv:0911.1013 [math-ph].
- [7] V. Fock, “Proper time in classical and quantum mechanics,” *Phys. Z. Sowjetunion* **12** (1937) 404.
- [8] S. Derkachov, L. Faddeev, A. Ivanov, to be published.