

LEGENDRIAN SUBMANIFOLDS WITH HAMILTONIAN ISOTOPIC SYMPLECTIZATIONS

SYLVAIN COURTE

ABSTRACT. In any contact manifold of dimension $2n - 1 \geq 11$, we construct examples of closed Legendrian submanifolds which are not diffeomorphic but whose Lagrangian cylinders in the symplectization are Hamiltonian isotopic.

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1. INTRODUCTION

Let (M, ξ) be a contact manifold (ξ is cooriented) and denote by $S M$ its symplectization, i.e. the set of covectors in $T^* M$ whose kernel is equal (as cooriented hyperplane) to ξ , it comes with a natural projection $\pi : S M \rightarrow M$ which is an \mathbb{R} -principal bundle (the \mathbb{R} -action is given by multiplying covectors by e^t for $t \in \mathbb{R}$). To any Legendrian submanifold $\Lambda \subset M$, there corresponds its *symplectization* $S \Lambda = \pi^{-1}(\Lambda)$ which is a Lagrangian submanifold diffeomorphic to $\mathbb{R} \times \Lambda$. Any \mathbb{R} -equivariant Hamiltonian isotopy of $S M$ that takes $S \Lambda$ to $S \Lambda'$ induces a contact isotopy of M that takes Λ to Λ' . However, if we forget about \mathbb{R} -equivariance, we are lead to consider the following question.

Question. *If $S \Lambda$ and $S \Lambda'$ are Hamiltonian isotopic, does it follow that Λ and Λ' are Legendrian isotopic ?*

This is a relative version of the question whether contact manifolds with exact symplectomorphic symplectizations are necessarily contactomorphic. The latter question was answered negatively in [Cou14] and we explain in this paper that the same phenomenon arises in this case.

Theorem 1.1. *In any closed contact manifold (M, ξ) of dimension $2n - 1 \geq 11$, there exist closed Legendrian submanifolds which are not diffeomorphic but whose symplectizations are Hamiltonian isotopic.*

This theorem will follow from a general construction using Lagrangian h-cobordisms and a Mazur trick argument. An essential ingredient in the proof is the notion of flexible Lagrangian cobordisms recently introduced by Eliashberg, Ganatra and Lazarev in [EGL15].

2. EXACT LAGRANGIAN COBORDISMS AND THE MAZUR TRICK

Let (M, ξ) be a closed connected contact manifold, recall that its symplectization $S M$ is equipped with canonical Liouville vector field X_{can} and Liouville form λ_{can} (the restrictions of those of $T^* M$) and that a contact form for (M, ξ) is a section of the bundle $S M \rightarrow M$. We denote by $S M^{\geq \alpha}$ the subset of $S M$ above the section α and use obvious notations for similar subsets of $S M$ or subsets of a Lagrangian cylinder $S \Lambda$.

Definition 2.1. *An exact Lagrangian cobordism in $S M$ is a Lagrangian submanifold $L \subset S M$ such that there exists two sections α_- and α_+ with $\alpha_- < \alpha_+$ at each point of M with the following properties :*

- (1) *There exists two closed Legendrian submanifolds Λ_- and Λ_+ such that*

$$L \cap S M^{\geq \alpha_+} = S \Lambda_+^{\geq \alpha_+} \text{ and } L \cap S M^{\leq \alpha_-} = S \Lambda_-^{\leq \alpha_-}.$$

- (2) *The region $L \cap S M^{[\alpha_-, \alpha_+]}$ is a compact cobordism from Λ_- to Λ_+ (without any other boundary).*
- (3) *Denoting $i : L \rightarrow S M$ the inclusion, there exists a function $g : L \rightarrow \mathbb{R}$ with $i^* \lambda_{\text{can}} = dg$ which is constant on $L \cap S M^{\geq \alpha_+}$ and on $L \cap S M^{\leq \alpha_-}$.*

Remark 2.2. *The function g in definition 2.1 can be extended to $S M$ as a function (still denoted by g) constant on $S M^{\geq \alpha_+}$ and on $S M^{\leq \alpha_-}$. The Liouville vector field $X = X_{\text{can}} + X_g^{-1}$ is then tangent to L and coincides with X_{can} on $S M^{\geq \alpha_+} \cup S M^{\leq \alpha_-}$. We say that such a vector field is adapted to L .*

Remark 2.3. *If ϕ is a diffeomorphism of $S M$ that preserve λ_{can} at infinity, then it lifts contact diffeomorphisms ϕ_- and ϕ_+ near $-\infty$ and $+\infty$ ² respectively and it is automatically exact ($\phi^* \lambda_{\text{can}} - \lambda_{\text{can}}$ is exact). These diffeomorphisms form a group denoted by \mathcal{G} , the subgroup defined by $\{\phi_- = \text{id}, \phi_+ = \text{id}\}$ will be denoted by \mathcal{G}_∂ . The image of an exact Lagrangian cobordism $(L; \Lambda_-, \Lambda_+)$ by $\phi \in \mathcal{G}$ is then an exact Lagrangian cobordism $(\phi(L); \phi_-(\Lambda_-), \phi_+(\Lambda_+))$. Exact Lagrangian cobordisms are stable in the following sense : any one-parameter family L_t , $t \in [0, 1]$, can be written $\phi_t(L_0)$ where $\phi_t \in \mathcal{G}$, $\phi_0 = \text{id}$; moreover if L_t is constant at $-\infty$ and at $+\infty$, we can require ϕ_t to lie in \mathcal{G}_∂ .*

Definition 2.4. *Two exact Lagrangian cobordisms $(L_0; \Lambda, \Lambda')$ and $(L_1; \Lambda, \Lambda')$ in $S M$ are said to be equivalent (what we write $L_0 \sim L_1$) if there exists a Hamiltonian isotopy $\phi_t : S M \rightarrow S M$, $t \in [0, 1]$, and two sections $\alpha_- < \alpha_+$ of $S M$ such that $\phi_0 = \text{id}$, $\phi_1(L_0) = L_1$ and ϕ_t equals the identity on $S M^{\geq \alpha_+} \cup S M^{\leq \alpha_-}$ (that is $\phi_t \in \mathcal{G}_\partial$ with the notations above; according to remark 2.3, this is the same as being isotopic relative boundary through exact Lagrangian cobordisms).*

¹The Hamiltonian vector field X_g is defined by $X_g \cdot \omega = -dg$.

²By that we mean, above or below some section of $S M$.

Exact Lagrangian cobordisms can be composed : given such $(L; \Lambda, \Lambda')$ and $(L'; \Lambda', \Lambda'')$ we have sections α and α' such that $L \cap S M^{\geq \alpha} = S \Lambda'^{\geq \alpha}$ and $L' \cap S M^{\leq \alpha'} = S \Lambda'^{\leq \alpha'}$. If we can find such sections with $\alpha < \alpha'$, then L and L' can naturally be glued because they both coincide with $S \Lambda'$ in $S M^{[\alpha, \alpha']}$, now observe that we can always achieve this condition by pushing up L' along the flow φ_t of X_{can} . We denote by $L \odot L'$ the resulting exact Lagrangian cobordism. This composition operation satisfies the following properties.

- (1) The equivalence class of $L \odot L'$ is independent of choices and depends only on the equivalence classes of L and L' .
- (2) $L \odot S \Lambda' \sim L$ and $S \Lambda \odot L \sim L$.
- (3) The composition is associative on equivalence classes, that is $L \odot (L' \odot L'') \sim (L \odot L') \odot L''$.
- (4) Given a sequence $(L_i; \Lambda_i, \Lambda_{i+1})$ for $i \in \mathbb{Z}$ of exact Lagrangian cobordisms, we can construct the infinite composition $\bigodot_{i \in \mathbb{Z}} L_i$ whose Hamiltonian isotopy class (not with compact support) is independent of choices and only depends on the equivalence class of each L_i .

Definition 2.5. An exact Lagrangian cobordism $(L; \Lambda, \Lambda')$ is said to be invertible if there exists another exact Lagrangian cobordism $(L'; \Lambda', \Lambda)$ such that $L \odot L' \sim S \Lambda$ and $L' \odot L \sim S \Lambda'$.

Remark 2.6. By associativity of composition, if $L \odot L' \sim S \Lambda$ and $L' \odot L'' \sim S \Lambda'$, then $L \sim L''$ and L is invertible.

Proposition 2.7. Let Λ and Λ' be closed Legendrian submanifolds of a closed contact manifold (M, ξ) . The following assertions are equivalent:

- (1) $S \Lambda$ and $S \Lambda'$ are Hamiltonian isotopic.
- (2) There exists an invertible exact Lagrangian cobordism $(L; \Lambda, \Lambda')$.

Proof. (1) \Rightarrow (2): Let $H_t : S M \rightarrow \mathbb{R}$ be a Hamiltonian generating an isotopy ϕ_t , $t \in [0, 1]$, of $S M$ such that $\phi_0 = \text{id}$, $\phi_1(S \Lambda) = S \Lambda'$. We pick four sections $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and two functions $\rho, \rho' : S M \rightarrow [0, 1]$ with the following properties:

- $\rho = 1$ in $S M^{\geq \alpha_2}$ and $\rho = 0$ in $S M^{\leq \alpha_1}$,
- $\rho' = 1$ in $S M^{\leq \alpha_3}$ and $\rho' = 0$ in $S M^{\geq \alpha_4}$.

Denote respectively by ψ_t , ψ'_t and θ_t the Hamiltonian isotopies generated respectively by ρH_t , $\rho' H_t$ and $\rho \rho' H_t$ (these are all well defined for $t \in [0, 1]$). Then $L = \psi_1(S \Lambda)$ and $L' = \psi'_1(S \Lambda')$ are exact Lagrangian cobordisms respectively from Λ to Λ' and from Λ' to Λ . Moreover, if we chose α_3/α_2 sufficiently big, then $L \odot L'$ sits naturally in $S M$ as $\theta_1(S \Lambda)$ and is equivalent to $S \Lambda$ (via the isotopy θ_t). We can likewise construct a right inverse for L' and we conclude using remark 2.6.

(2) \Rightarrow (1) : Let $(L'; \Lambda', \Lambda)$ be an inverse for $(L; \Lambda, \Lambda')$ and consider the infinite composition

$$L_\infty = \dots \odot L \odot L' \odot L \odot L' \odot \dots$$

By introducing parentheses in two different ways $((L \odot L') \odot (L' \odot L))$ in the above expression, we get that L_∞ is Hamiltonian isotopic to $S \Lambda$ as well as to $S \Lambda'$. \square

Remark 2.8. *It follows from proposition 2.7 together with functoriality properties of symplectic field theory that such Legendrian submanifolds have isomorphic Legendrian contact homology.*

Our goal is now to construct non-trivial invertible Lagrangian cobordisms.

3. FLEXIBLE LAGRANGIAN H-COBORDISMS

Let (M, ξ) be a contact manifold of dimension $2n - 1 \geq 5$.

Definition 3.1 ([EGL15]). *An exact Lagrangian cobordism $L \subset SM$ is called regular if there exists an adapted Liouville vector field X and a proper Morse function $f : SM \rightarrow \mathbb{R}$ for which X is a pseudo-gradient. Moreover if there exists such an adapted pair (f, X) for which f is excellent (all critical values are distinct) and the attaching spheres of critical points of index n are loose (see [Mur12]) in the complement of L , then L (as well as the pair (f, X)) is said to be flexible.*

Note that the critical points of $f|L$ are necessarily critical points of f and, in the flexible case, there cannot be any critical point of index n on L . The definition can obviously be extended to Lagrangian cobordisms into arbitrary flexible Weinstein cobordisms.

Recall that an *h-cobordism* is a cobordism which deformation retracts on its bottom boundary as well as on its top boundary. According to the s-cobordism theorem (see [Ker65]), h-cobordisms from a given closed manifold M are classified up to diffeomorphism relative to M by so-called *Whitehead torsion*, an invariant which takes values in the Whitehead group $\text{Wh}(M)$ of M (it actually depends only on $\pi_1 M$). Essentially since each element in a group has an inverse, h-cobordisms of dimension ≥ 6 are invertible for the composition of cobordisms (see [Sta65]).

Theorem 3.2. *Let (M, ξ) be a closed contact manifold of dimension ≥ 11 .*

- (1) *Let Λ a closed Legendrian submanifold in M , and $(L; \Lambda, \Lambda')$ an h-cobordism. Then L can be embedded in SM has a flexible Lagrangian cobordism starting from Λ .*
- (2) *Any flexible Lagrangian h-cobordism in SM is invertible (as an exact Lagrangian cobordism).*

We need a couple of lemmas. The first one is proved in [EGL15], proposition 2.5.

Lemma 3.3. *For any regular Lagrangian cobordism L together with an adapted pair (f, X) , we can find a homotopy (f_t, X_t) of adapted pairs such that $(f_0, X_0) = (f, X)$ and for all critical point of f_1 on L the index is the same for f_1 and $f_1|L$. Moreover if (f_0, X_0) is flexible, we can require (f_t, X_t) to be flexible for all t .*

Lemma 3.4. *Let (M, ξ) be a contact manifold of dimension ≥ 5 . Let $(L; \Lambda, \Lambda')$ be a flexible Lagrangian cobordism of SM which is diffeomorphic to $\Lambda \times [0, 1]$, then there exists an adapted pair without critical points.*

Proof. We start with a flexible adapted pair (f, X) . By lemma 3.3, we can assume that the critical points on L have same index for $f|L$ and f . Since there are no X -trajectories going from critical points outside of L to critical points on L , we can reorder the critical values so that the critical points on L lie below all the others. Since L is diffeomorphic to $\Lambda \times [0, 1]$, the function $g = f|L$ can be deformed via a homotopy g_t , $t \in [0, 1]$, to

a function without critical points and moreover this can be done without introducing any maximum along the deformation. We then extend the homotopy g_t to a homotopy (f_t, X_t) of flexible adapted pairs supported into an arbitrary small neighbourhood of the support of the homotopy g_t (see [CE12] lemma 12.8). We then proceed to the cancellation of the remaining critical points which are all outside of L , following the proof of the h-cobordism theorem :

- Cancel index 0 critical points with some index 1 critical points.
- Trade critical points of index i for critical points of index $i + 2$, until there only remains critical points of index $n - 1$ and n .
- Cancel together critical points of index $n - 1$ and n .

We have to go through these steps keeping (f, X) fixed near L . We claim this is possible because every X -trajectory between critical points are disjoint from L . The main point to notice is that the isotopies of the attaching spheres needed to arrange cancellation positions can be done in the complement of L because they can be localized near Whitney 2-disks which are generically disjoint from L . Subcriticality or looseness in the complement of L then allows to realize this isotopies as isotropic isotopies as in [CE12] chapter 14 (see lemma 14.10 for example). \square

Proof of theorem 3.2. (1) Recall that any h-cobordism of dimension at least 6 can be presented with a Morse function having only critical points of index 2 and 3 (see [Ker65]). We first construct a flexible Weinstein cobordism $(W; M, M')$ containing a flexible Weinstein Lagrangian cobordism $(L; \Lambda, \Lambda')$ by attaching Weinstein handles of index 2 and 3 on Λ . Denoting by $\tau \in \text{Wh}(L)$ the Whitehead torsion of L , we note that the ambient cobordism W is also an h-cobordism and its torsion is $i(\tau)$ where $i : \text{Wh}(\Lambda) \rightarrow \text{Wh}(M)$ is the map induced by inclusion. We now attach handles of index 2 and 3 on top of M' away from Λ' to produce a flexible Weinstein h-cobordism W' with torsion $-i(\tau) \in \text{Wh}(M')$ (we identify $\text{Wh}(M) \simeq \text{Wh}(M')$ via the homotopy equivalence induced by W). The Lagrangian L can be continued inside of W' by composing with the Lagrangian cylinder $S\Lambda'$. The composition $W \odot W'$ is a flexible Weinstein cobordism and it is diffeomorphic to $M \times [0, 1]$ since its Whitehead torsion vanishes. We can therefore cancel all the handles and show that $W \odot W'$ is equivalent to $S M$ relative to the negative boundary (see [CE12] corollary 14.2). Thus L now sits as a flexible Lagrangian cobordism in $S M$.

(2) Let $(L'_1; \Lambda', \Lambda)$ be an inverse cobordism for $(L; \Lambda, \Lambda')$. Using the first point, we can embed L' as a flexible Lagrangian cobordism in $S M$. Denote by Λ_1 the positive Legendrian boundary of L'_1 , note that it is a priori different from Λ . Now lemma 3.4 allows to find an adapted pair (f, X) without critical points for the composition $L \odot L'_1$. By sending the trajectories of X_{can} to that of X we find a symplectic pseudo-isotopy ψ of $S M$ (that is $\psi \in \mathcal{G}$ with $\psi_- = \text{id}$) that takes $S\Lambda$ to $L \odot L'_1$. We undo this pseudo-isotopy by composing L'_1 further with $L'_2 = \psi^{-1}(S\Lambda_1)$, we then get a flexible Lagrangian h-cobordism $L' = L'_1 \odot L'_2$ from Λ' to Λ such that $L \odot L'$ is equivalent to $S\Lambda$. We can repeat the same argument to produce a right inverse for L' and the result now follows from remark 2.6. \square

Remark 3.5. Starting from an exact Lagrangian filling F of a Legendrian Λ , the same method shows that F is Hamiltonian isotopic to the composition of F with any flexible Lagrangian h-cobordism starting from Λ .

4. EXAMPLES

An example where Λ and Λ' are not diffeomorphic. For $n \geq 6$, consider the manifold $\Lambda = L(4, 1) \times T^{n-4}$. It was proved in [FH67], that there exists an h-cobordism $(L; \Lambda, \Lambda')$ such that Λ' is not diffeomorphic to Λ . We claim that Λ admits a Legendrian embedding into \mathbb{R}^{2n-1} endowed with its standard contact structure. Indeed, Λ is parallelizable so we can find a Legendrian bundle monomorphism $T\Lambda \rightarrow \mathbb{R}^{2n-1}$ and then turn it into a Legendrian embedding via Gromov's h-principle (see [EM02] theorem 16.1.3, and note that a generic Legendrian immersion is an embedding). This Legendrian embedding of Λ can be implanted in any contact manifold via a Darboux chart. Theorem 1.1 now follows from theorem 3.2 and proposition 2.7.

An example where Λ and Λ' are smoothly isotopic but not Legendrian isotopic. The following construction is very similar to that in [Cou] section 3, but we repeat some of the arguments there for the convenience of the reader.

Consider the closed 7-dimensional manifold $\Lambda = L(5, 1) \times S^4$. Note that Λ is parallelizable and that $\pi_3\Lambda = \pi_3 L(5, 1) = \mathbb{Z}$ (a generator is given by the universal covering map $S^3 \rightarrow L(5, 1)$).

Lemma 4.1. (1) There exists an h-cobordism $(L; \Lambda, \Lambda)$ such that the induced map $f : \Lambda \rightarrow \Lambda$ acts by multiplication by -1 on $\pi_3\Lambda$.
(2) No diffeomorphism of Λ may act by multiplication by -1 on $\pi_3\Lambda$.

Proof. (1): There are exactly two homotopy classes of maps $L(5, 1) \rightarrow L(5, 1)$ of degree -1 (these are automatically homotopy equivalences) and they respectively induce multiplication by 2 and -2 on $\pi_1 L(5, 1) = \mathbb{Z}/5\mathbb{Z}$ (see [Coh73], 29.5). We pick such a map and perturb it to an embedding $j : L(5, 1) \rightarrow L(5, 1) \times \text{int } D^5$ using Whitney's embedding theorem. The normal bundle of j is trivial because it is stably trivial and has rank greater than the dimension of the base. We can therefore extend j to an embedding $L(5, 1) \times D^5 \rightarrow L(5, 1) \times \text{int } D^5$ that we still denote by j . The region $L = L(5, 1) \times D^5 \setminus j(L(5, 1) \times \text{int } D^5)$ is an h-cobordism from Λ to itself (see [Mil61] lemma 2 p.579). The map $f : \Lambda \rightarrow \Lambda$ induced by the cobordism L can be defined as $f = r \circ i$ where $i : \Lambda \rightarrow L$ is the inclusion of the negative boundary and $r : L \rightarrow \Lambda$ is a deformation retraction on the positive boundary (the homotopy class of f is independent of choices). Since we started with a map of degree -1 on $L(5, 1)$, we see that j induces multiplication by -1 on $H_3(L(5, 1) \times D^5; \mathbb{Z}) \simeq \mathbb{Z}$ as well as on $\pi_3(L(5, 1) \times D^5) \simeq \mathbb{Z}$ because the Hurewicz homomorphism $\pi_3 L(5, 1) \rightarrow H_3(L(5, 1); \mathbb{Z})$ is non zero. It follows from the commutativity up to homotopy of the following diagram (the vertical arrows are obvious inclusions)

$$\begin{array}{ccc} L(5, 1) \times D^5 & \xrightarrow{j} & L(5, 1) \times D^5 \\ \uparrow & & \uparrow \\ \Lambda & \xrightarrow{f} & \Lambda \end{array}$$

that the map f also induces multiplication by -1 on $\pi_3\Lambda$.

(2): If $\psi : \Lambda \rightarrow \Lambda$ was such a diffeomorphism, then the map $L(5, 1) \rightarrow L(5, 1)$, obtained by composing the inclusion of a factor with ψ and then projection, would have degree -1 . But then ψ necessarily acts by multiplication by ± 2 on π_1 , in which case the Whitehead torsion of ψ must be non zero (see [Cou] lemma 3.2) contradicting the fact that ψ is a diffeomorphism. \square

Let $(L; \Lambda, \Lambda)$ be an h-cobordism given by the lemma above. We fix a framing of Λ and extend it to a framing of L by using an isomorphism $T L \rightarrow \mathbb{R} \times T \Lambda$ lifting the retraction map $r : L \rightarrow \Lambda$ on the positive boundary. Note that the induced framing of $T \Lambda \times \mathbb{R}$ on the negative boundary a priori differs from the given one : it is the image of the given framing by a map $A : \Lambda \rightarrow O(8) \subseteq U(8)$. Recall that any Legendrian immersion $\Lambda \rightarrow \mathbb{R}^{15}$ gives rise to a map $\Lambda \rightarrow U(7)$ well-defined up to homotopy and Gromov's h-principle (see [EM02] theorem 16.1.3) implies that this classifies Legendrian regular homotopy classes. Given an embedding of L as a Lagrangian cobordism in $S\mathbb{R}^{15}$, we get maps $g : \Lambda \rightarrow U(7)$, $g' : \Lambda \rightarrow U(7)$ and $G : L \rightarrow U(8)$ associated respectively to $\partial_- L$, $\partial_+ L$ and L . These maps are related by the following formulas:

$$A.s \circ g \sim G \circ i, \quad s \circ g' \circ r \sim G$$

where \sim here means homotopic, $s : U(7) \rightarrow U(8)$ is the stabilization map (note that this is an isomorphism on π_3), r, i are defined as in the proof of the lemma 4.1 and the dot denotes multiplication in $U(8)$. In particular, we get $s \circ g'$ out of $s \circ g$:

$$s \circ g' \sim A.s \circ g \circ f^{-1}.$$

Recall from Bott periodicity that $\pi_3 U(8) \simeq \mathbb{Z}$. Identifying $\pi_3 \Lambda$ and $\pi_3 U(8)$ with \mathbb{Z} , the map induced on π_3 by $s \circ g$, $s \circ g'$ and A are respectively multiplication by integers b, b' and a and the equation above reads:

$$b' = a - b$$

(note that multiplication on $U(8)$ induces addition on $\pi_3 U(8)$).

We now observe that, whatever a is, we can choose g such that $b' \neq b$ and therefore g' is not homotopic to g . Indeed

- if $a \neq 0$, we take g to be constant so that $b = 0$ and $b' \neq 0$,
- if $a = 0$, we take $g = \alpha \circ h \circ p_1$ where $p_1 : \Lambda \rightarrow L(5, 1)$ is the projection on the first factor, $h : L(5, 1) \rightarrow S^3$ is a map of degree 1 and $\alpha : S^3 \rightarrow U(7)$ corresponds to $1 \in \mathbb{Z} = \pi_3 U(7) = \pi_3 U(8)$, so that $b = 5$ and $b' = -5$.

The rest of the construction is the same as in the first example above: we take a Legendrian embedding $\phi : \Lambda \rightarrow \mathbb{R}^{15}$ that induces the map g and use theorem 3.2 to obtain an embedding of L as a flexible Lagrangian cobordism in $S\mathbb{R}^{15}$ with negative boundary ϕ and a new Legendrian embedding $\phi' : \Lambda \rightarrow \mathbb{R}^{15}$ on the positive boundary which induces the map g' . The Legendrian embeddings ϕ and ϕ' are not homotopic through Legendrian immersions and moreover using the second point of lemma 4.1, we see that this cannot be arranged by composing ϕ' by a diffeomorphism of Λ . Hence the Legendrian submanifolds $\phi(\Lambda)$ and $\phi'(\Lambda)$ are not Legendrian isotopic though they have Hamiltonian isotopic symplectizations and by Haefliger's embedding theorem (see [Hae61]) they are smoothly isotopic.

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UPPSALA UNIVERSITET, SWEDEN
E-mail address: sylvain.courte@math.uu.se
URL: <http://www2.math.uu.se/~sylco859/>