

# Mapping Cone Connections and their Yang-Mills Functional

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## Abstract

For a given closed two-form, we introduce the cone Yang-Mills functional which is a Yang-Mills-type functional for a pair  $(A, B)$ , a connection one-form  $A$  and a scalar  $B$  taking value in the adjoint representation of a Lie group. The functional arises naturally from dimensionally reducing the Yang-Mills functional over the fiber of a circle bundle with the two-form being the Euler class. We write down the Euler-Lagrange equations of the functional and present some of the properties of its critical solutions, especially in comparison with Yang-Mills solutions. We show that a special class of three-dimensional solutions satisfy a duality condition which generalizes the Bogomolny monopole equations. Moreover, we analyze the zero solutions of the cone Yang-Mills functional and give an algebraic classification characterizing principal bundles that carry such cone-flat solutions when the two-form is non-degenerate.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Properties of cone Yang-Mills solutions</b>	<b>8</b>
2.1	Preliminaries . . . . .	8
2.2	Elliptic property of the cone Yang-Mills solutions . . . . .	10
2.3	Comparison with Yang-Mills solutions . . . . .	13
2.4	Uniqueness of cone Yang-Mills solutions for a fixed connection . . . . .	16

<b>3 Special solutions of the cone Yang-Mills functional</b>	<b>18</b>
3.1 Two-dimensional solutions with abelian gauge group . . . . .	18
3.2 Three-dimensional solutions from duality relations . . . . .	19
<b>4 Classification of cone-flat bundles with respect to a non-degenerate two-form</b>	<b>24</b>

## 1 Introduction

Let  $(M^m, g)$  be a Riemannian manifold and  $P$  a principal  $G$ -bundle over  $M$ . For simplicity, we will assume that the Lie group  $G$  is compact and a subgroup of  $SO(N)$ .

In this paper, we study a Yang-Mills type functional that involves a pair  $(A, B)$ , with  $A$  being a connection form on the associated adjoint bundle  $Ad P$ , a section  $B \in \Omega^0(M, Ad P)$ , and also a two-form  $\zeta \in \Omega^2(M)$  that is  $d$ -closed. We shall call this functional the *cone Yang-Mills functional* and it is defined by

$$\begin{aligned} S_{cYM}(A, B) &= \|F_A + \zeta B\|^2 + \|d_AB\|^2 \\ &= c_G \int_M \text{tr} [(F_A + \zeta B) \wedge * (F_A + \zeta B) + d_AB \wedge * d_AB], \end{aligned} \quad (1.1)$$

where  $F_A = dA + A \wedge A$  is the curvature,  $d_AB = dB + [A, B]$  is the covariant derivative of  $B$ , and in the second line, we have inserted the Killing form of the Lie group  $G$  which for  $SO(N)$  and  $SU(N)$  is given by the trace times a constant dependent on  $N$ , denoted by  $c_G$ . The critical points of the functional are solutions of the associated Euler-Lagrange equations

$$d_A^*(F_A + \zeta B) + [B, d_AB] = 0, \quad (1.2)$$

$$\zeta^*(F_A + \zeta B) + d_A^*d_AB = 0, \quad (1.3)$$

where  $\zeta^* : \Omega^k(M) \rightarrow \Omega^{k-2}(M)$  is the adjoint of the wedging map,  $\zeta \wedge$ , and can be explicitly expressed as

$$\zeta^* = (-1)^{(m-k)k} * \zeta * . \quad (1.4)$$

Let us give both a physical and a mathematical motivation for considering the cone Yang-Mills functional.

For the physical motivation, the cone Yang-Mills functional can be considered as the dimensional reduction of the standard Yang-Mills functional over the  $S^1$  fiber of a circle bundle,

$\pi : X \rightarrow M$ . Unlike a product space, a circle bundle generically is one where the fiber circle is non-trivially twisted over  $M$ . Such arises for instance in Kaluza-Klein monopole solutions with a non-vanishing background two-form flux  $\zeta$ . On any circle bundle  $X$ , there exists what is called a global angular one-form  $\theta$  which can be locally expressed as  $\theta = dz + a$  where  $z$  denotes the  $S^1$  fiber coordinate and  $a$  is the  $U(1)$  Kaluza-Klein gauge field. The two-form flux,  $\zeta = d\theta = d(dz + a) = da$ , is then the background field strength of the  $U(1)$  gauge field and effectively measures the twisting of the  $S^1$  fiber.

Now for the dimensional reduction of the Yang-Mills functional over the  $S^1$  fiber of the circle bundle  $\pi : X \rightarrow M$ , we take the metric to be

$$g_X = \pi^* g_M + \theta \otimes \theta, \quad (1.5)$$

and further require that the local connection one-form (Yang-Mills gauge field)  $\mathcal{A}$  on  $X$  be invariant under translation of the circle fiber. Such a connection form on  $X$  can be expressed as

$$\mathcal{A} = A + \theta B. \quad (1.6)$$

Here, both  $A$  and  $B$  takes values on  $Ad P$ , and  $B$  represents the component of the connection in the circle direction. Since  $\mathcal{A}$  is invariant under  $S^1$  translation, both  $A$  and  $B$  only have dependence on the coordinates of  $M$ . The curvature two-form of  $\mathcal{A}$  (Yang-Mills field strength) on  $X$  then has the following expression:

$$\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (F_A + \zeta B) + \theta(-d_AB), \quad (1.7)$$

where we have used  $d\theta = \zeta$ . Comparing with the cone Yang-Mills functional in (1.1), we see that  $S_{cYM} = \|\mathcal{F}_{\mathcal{A}}\|^2$ , and therefore, it is just the Yang-Mills functional on  $X$  dimensionally reduced to  $M$  over the fiber circle. And we can also interpret the cone Yang-Mills Euler-Lagrange equations (1.2)-(1.3) as the dimensionally-reduced Yang-Mills equations over the  $S^1$  fiber of  $X$ .

In physical terms, the cone Yang-Mills functional on  $M$  is a Euclidean or Wick-rotated action, situated in a curved background, with Riemannian metric  $g$ :

$$\begin{aligned} S_{cYM}(A, B) &= \int_M c_G \text{tr} (|F_A + \zeta B|^2 + |d_AB|^2) dV \\ &= \int_M c_G \text{tr} (|F_A|^2 + |d_AB|^2 + |\zeta|^2 B^2 + 2\zeta^* F_A B) dV, \end{aligned}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_C} & \text{Cone}^k(\zeta) & \xrightarrow{d_C} & \text{Cone}^{k+1}(\zeta) & \xrightarrow{d_C} & \text{Cone}^{k+2}(\zeta) & \xrightarrow{d_C} \cdots \\
\\
\cdots & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \Omega^{k+2}(M) & \xrightarrow{d} \cdots \\
& & \searrow \zeta \wedge & & \searrow \zeta \wedge & & \\
\cdots & \xrightarrow{-d} & \Omega^{k-1}(M) & \xrightarrow{-d} & \Omega^k(M) & \xrightarrow{-d} & \Omega^{k+1}(M) & \xrightarrow{-d} \cdots
\end{array}$$

Figure 1: The mapping cone complex of the  $d$ -closed two-form  $\zeta$ . The complex of elements  $\text{Cone}^*(\zeta) = \Omega^*(M) \oplus \Omega^{*-1}(M)$  arises from the  $\zeta$  map between two de Rham complexes.

where we have used a simplified notation, e.g.  $F_A \wedge *F_A = |F_A|^2 dV$ . The action thus consists of a Yang-Mills part, an adjoint-valued scalar with a “mass”  $|\zeta|$  (which may vary over  $M$ ), and an interaction term between  $A$  and  $B$ . Heuristically, the Euler-Lagrange equations of motion (1.2)-(1.3) can be thought of as a mixing of the Yang-Mills equations with a Klein-Gordon-type scalar plus an interaction term.

The cone Yang-Mills functional also has a mathematical motivation. In fact, the functional arises naturally from its relation with a mapping cone complex which we will provide an explanation here. (For a reference on the general mapping cone complex, see for example [11] and also [3].)

The  $d$ -closed two-form  $\zeta \in \Omega^2(M)$  can be thought of as an operator or a *map*, by wedge product between differential forms, i.e.  $\zeta \wedge : \Omega^*(M) \rightarrow \Omega^{*+2}(M)$ . If desired, the form  $\zeta$  can represent a geometric structure of interest on  $M$ , such as a symplectic or hermitian structure, but generally, the form  $\zeta$  can be any closed two-form on  $M$ . Letting  $\zeta$  map between two de Rham chain complexes leads to a mapping cone complex (see Figure 1). The elements of the mapping cone complex consist of pairs of differential forms:

$$\text{Cone}^k(\zeta) := \Omega^k(M) \oplus \Omega^{k-1}(M), \quad k = 0, 1, \dots, m+1. \quad (1.8)$$

The differential of the mapping cone complex  $d_C : \text{Cone}^k(\zeta) \rightarrow \text{Cone}^{k+1}(\zeta)$  is given by

$$\begin{aligned}
d_C \text{Cone}^k(\zeta) &= d_C (\Omega^k \oplus \Omega^{k-1}) \\
&= (d \Omega^k + \zeta \wedge \Omega^{k-1}) \oplus -d \Omega^{k-1}.
\end{aligned}$$

Since the grading of the two components of  $\text{Cone}^k(\zeta)$  in (1.8) are different, it is useful to introduce a formal one-form  $\theta$  to supplement the second component so that we can express a

cone form as a sum with both components having the same total degree  $k$ , i.e.

$$\text{Cone}^k(\zeta) = \Omega^k \oplus \theta \wedge \Omega^{k-1}. \quad (1.9)$$

Additionally, it is useful for us to impose that the formal one-form  $\theta$  satisfy  $d\theta = \zeta$ . For this will allow us to interpret the cone differential  $d_C$  simply as the exterior derivative:

$$\begin{aligned} d_C \text{Cone}^k(\zeta) &= d \left( \Omega^k \oplus \theta \wedge \Omega^{k-1} \right) \\ &= \left( d\Omega^k + \zeta \wedge \Omega^{k-1} \right) \oplus \theta \wedge \left( -d\Omega^{k-1} \right). \end{aligned}$$

It then becomes self-evident that  $d_C d_C = 0$ . Moreover, there is a natural product on the cone space  $\text{Cone}^*(\zeta)$  given by the usual wedge product

$$\begin{aligned} \text{Cone}^j(\zeta) \times \text{Cone}^k(\zeta) &:= (\Omega^j \oplus \theta \wedge \Omega^{j-1}) \wedge (\Omega^k \oplus \theta \wedge \Omega^{k-1}) \\ &= \left( \Omega^j \wedge \Omega^k \right) \oplus \theta \wedge \left( \Omega^{j-1} \wedge \Omega^k + (-1)^j \Omega^j \wedge \Omega^{k-1} \right). \end{aligned}$$

This product satisfies the Leibniz rule with respect to  $d_C$ . In all, we see that we have a mapping cone algebra,  $(\text{Cone}^*(\zeta), d_C, \times)$ , that satisfies the conditions of a differential graded algebra (DGA).

Now in the presence of the associated adjoint bundle, we should consider the twisted cone forms:

$$\text{Cone}^k(\zeta)(M, Ad P) = \Omega^k(M, Ad P) \oplus \theta \wedge \Omega^{k-1}(M, Ad P)$$

which take values on the associated adjoint bundle  $Ad P$ . In this context, the differential  $d_C$  must also be twisted by a cone connection form  $\mathcal{A}$

$$\dots \xrightarrow{d_C + \mathcal{A}} \text{Cone}^k(\zeta)(M, Ad P) \xrightarrow{d_C + \mathcal{A}} \text{Cone}^{k+1}(\zeta)(M, Ad P) \xrightarrow{d_C + \mathcal{A}} \text{Cone}^{k+2}(\zeta)(M, Ad P) \xrightarrow{d_C + \mathcal{A}} \dots$$

where  $\mathcal{A} = A + \theta \wedge B$ , with  $A$  being the connection form on  $M$  and  $B \in \Omega^0(M, Ad P)$ . The cone curvature then takes the form

$$\mathcal{F}_{\mathcal{A}} = (d_C + \mathcal{A})^2 = d_C \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (F_A + \zeta B) + \theta \wedge (-d_A B), \quad (1.10)$$

where  $F_A = dA + A \wedge A$ . The above twisted complex is only a differential complex if  $\mathcal{F}_{\mathcal{A}} = 0$ . This requires that the cone curvature  $\mathcal{F}_{\mathcal{A}}$  vanishes, which from (1.10) corresponds to  $(A, B)$

satisfying what we shall call the cone-flat condition with respect to  $\zeta$ , or simply just the cone-flat condition,

$$F_A + \zeta B = 0, \quad d_A B = 0. \quad (1.11)$$

The cone Yang-Mills functional of (1.1) is then just the normed square of the cone curvature,  $\|\mathcal{F}_A\|^2$ . It is worthwhile to emphasize that this mapping cone perspective only requires that the two-form  $\zeta \in \Omega^2(M)$  be  $d$ -closed and nothing more. This is in contrast with the dimensional reduction perspective, where  $\zeta$  mathematically represents the Euler class of the circle bundle  $\pi : X \rightarrow M$  and hence would need to be an element of  $H^2(M, \mathbb{Z})$ .

In this paper, we take an important first step in understanding cone Yang-Mills solutions. We will show that a subset of the solutions involves Yang-Mills connections. For instance, in dimension two when  $M$  is compact and  $\zeta$  is taken to be the volume form, the above cone-flat condition (1.11) implies exactly the two-dimensional Yang-Mills condition

$$d_A^* F_A = - * d_A * F_A = * d_A B = 0, \quad (1.12)$$

having noted that  $*\zeta = 1$ . Conversely, in two dimensions, if  $(A, B)$  is a cone-Yang-Mills solution with  $A$  also a Yang-Mills connection, then the pair  $(A, B)$  must be cone-flat.

In higher dimensions, certain special classes of Yang-Mills connections can be paired with a scalar  $B$  to obtain cone Yang-Mills solutions. In the trivial case where we set  $\zeta = 0$ , any Yang-Mills connection  $A$  together with a covariantly constant  $B$  is trivially a cone Yang-Mills solutions. When  $\zeta \neq 0$ , Yang-Mills connections  $A$  such that the curvature two-form satisfy  $\zeta^* F_A = 0$  are cone Yang-Mills solutions with  $B = 0$ . When  $\zeta$  is a harmonic form, cone-flat solutions are always composed of a Yang-Mills connection with an appropriate scalar section  $B$ .

On the other hand, it should be evident that the space of cone Yang-Mills solutions  $(A, B)$  is generally much richer and different from that of Yang-Mills solutions. As we will show in explicit examples, not all Yang-Mills connection  $A$  can be paired with a scalar  $B$  to form a cone Yang-Mills solutions. Conversely, there are also cone Yang-Mills solutions  $(A, B)$  where  $A$  is not Yang-Mills. Furthermore, given a cone Yang-Mills solutions  $(A, B)$ , there may be other scalars  $B'$  such that  $(A, B')$  remain cone Yang-Mills.

Interestingly, when  $M$  is three-dimensional, we are able to write down a Bogomolny-type condition, that gives a sufficient condition on  $(A, B)$  to be a cone Yang-Mills solution. Such a

condition can be motivated by considering  $\mathcal{F}_A$  as the Yang-Mills curvature on a four-dimensional circle bundle  $X$  and imposing the (anti-)self-dual condition. Then, dimensionally reducing on the fiber  $S^1$  by expressing  $\mathcal{F}_A$  as in (1.10), the (anti-)self-dual condition implies the three-dimensional condition

$$F_A + \zeta B = \pm * d_A B. \quad (1.13)$$

Notice that if we set  $\zeta = 0$ , the above equation is just the Bogomolny monopole equation. Similar to the (anti-)self-dual Yang-Mills condition in four dimensions, (1.13) is a first-order condition whose solutions solve (1.2)-(1.3) in three dimensions. This duality condition will assist us in finding non-abelian, cone Yang-Mills solutions. Indeed, we shall give in Section 3.2 an explicit  $SU(2)$  solution of (1.13) that comes from dimensionally reducing the four-dimensional Taub-NUT gravitational instanton solution.

Concerning the cone-flat condition, we are able to characterize principal bundles that carry a cone-flat connection, i.e. a pair  $(A, B)$  satisfying (1.11), when  $\zeta$  is a non-degenerate two-form. This type of cone-flat connections can interestingly be classified similar to Atiyah-Bott's classification of bundles carrying Yang-Mills connections on Riemann surfaces [2]. Our classification of the cone-flat bundles however depends on the given  $\zeta$  and the second homotopy group,  $\pi_2(M)$ .

**Theorem 1.1.** *Let  $M$  be a path connected manifold,  $\zeta \in \Omega^2(M)$  be a non-zero, non-degenerate, closed two-form, and  $G$  be a Lie group. There exists a bijective correspondence between the following sets:*

$$\left\{ \begin{array}{l} \text{isomorphism classes of cone-flat connections} \\ \text{with respect to } \zeta \text{ on } G\text{-bundles over } M \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{homomorphisms } \rho : \Gamma \rightarrow G \end{array} \right\},$$

where  $\Gamma$  is an  $\mathbb{R}/\bar{H}$  extension of  $\pi_1(M)$  with  $\bar{H} \subset \mathbb{R}$  being the closure of the group

$$H := \left\{ \int_{\mathcal{S}} \zeta \mid \mathcal{S} \text{ is a representative in } \pi_2(M) \right\}.$$

This paper is organized as follows. In Section 2, after a brief description of our notations/conventions, we proceed to consider the first-order variation of the cone Yang-Mills functional to obtain its Euler-Lagrange equations. We also show that modulo gauge equivalence, the cone Yang-Mills equations are elliptic and hence has a finite-dimensional solution space on a closed manifold. We also describe properties of cone Yang-Mills solutions under certain

conditions for the two-form  $\zeta$  and structure group  $G$ , and especially emphasizing its relationship to Yang-Mills connections. In Section 3, we work out the special case of abelian cone Yang-Mills solutions in dimension two. We also discuss the three-dimensional Bogomolny-type monopole condition (1.13) and give an explicit non-trivial  $SU(2)$  cone Yang-Mills solutions on  $M = \mathbb{R}^3 - \{0\}$ , i.e. the Euclidean space with the origin removed. Finally, in Section 4, we consider bundles that can carry cone-flat solutions when  $\zeta$  is a non-degenerate, closed two-form, and prove the classification of cone-flat bundles of Theorem 1.1.

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## 2 Properties of cone Yang-Mills solutions

### 2.1 Preliminaries

Let  $M$  be a smooth manifold and  $P$  be a principal bundle over  $M$ . We consider the associated adjoint bundle  $Ad P$  equipped with an inner product that is invariant under the adjoint action. For the inner product on  $\Omega^*(M, Ad P)$ , we use the standard one induced from the Riemannian metric  $g$  and the Killing form of the Lie group  $G$ , i.e. for  $\eta_1, \eta_2 \in \Omega^*(M, Ad P)$ ,

$$\langle \eta_1, \eta_2 \rangle = c_G \int_M \text{tr} [\eta_1 \wedge * \eta_2], \quad (2.1)$$

where the constant  $c_G = -(N-2)$  if  $G = SO(N)$ . As will be useful later, we note here for  $\eta \in \Omega^0(M, Ad P)$ , we have the relation

$$\langle [\eta_1, \eta], \eta_2 \rangle = \langle \eta_1, [\eta, \eta_2] \rangle. \quad (2.2)$$

We can extend the inner product  $\langle -, - \rangle$  on  $\Omega^*(M, Ad P)$  to  $\langle -, - \rangle_C$  on  $\text{Cone}^*(\zeta)(M, Ad P) = \Omega^*(M, Ad P) \oplus \theta \Omega^{*-1}(M, Ad P)$  for a  $d$ -closed two-form,  $\zeta \in \Omega^2(M)$ . We do so by setting

$$\langle \eta_1 + \theta \xi_1, \eta_2 + \theta \xi_2 \rangle_C = \langle \eta_1, \eta_2 \rangle + \langle \xi_1, \xi_2 \rangle. \quad (2.3)$$

Note that  $\theta$  is a formal one-form with property  $d\theta = \zeta$ . When paired with another differential form, there is a wedge product,  $\theta \wedge$ , which for notational simplicity, as in (2.3), we will just assume without writing out explicitly.

With (2.1) and (2.3), we can also express  $\langle -, - \rangle_{\mathcal{C}}$  as an integral by introducing the Hodge star operator for cone forms

$$\begin{aligned} *_{\mathcal{C}} : \text{Cone}^k(\zeta)(M, Ad P) &\rightarrow \text{Cone}^{m+1-k}(\zeta)(M, Ad P) \\ \eta + \theta\xi &\mapsto * \xi + \theta (-1)^{|\eta|} * \eta \end{aligned} \quad (2.4)$$

which allows us to write

$$\langle \eta_1 + \theta\xi_1, \eta_2 + \theta\xi_2 \rangle_{\mathcal{C}} = c_G \int_M \text{tr} \frac{\partial}{\partial \theta} [(\eta_1 + \theta\xi_1) \wedge *_{\mathcal{C}} (\eta_2 + \theta\xi_2)] \quad (2.5)$$

where  $\frac{\partial}{\partial \theta}(\theta\xi) = \xi$  for any  $\xi \in \Omega^*(M, Ad P)$ .

Having defined the inner product on  $\text{Cone}^*(\zeta)(M, Ad P)$ , we write down the Yang-Mills functional associated with the cone connection form. Given a connection form  $A$  of  $Ad P$  over  $M$  and a section  $B \in \Omega^0(M, Ad P)$ , we write the cone connection form as  $\mathcal{A} = A + \theta B$ . Locally, if we write the usual covariant derivative as  $d_A = d + A$ , then the cone covariant derivative is given by  $D_{\mathcal{C}} = d + \mathcal{A} = d + (A + \theta B)$  and the cone curvature takes the form

$$\begin{aligned} \mathcal{F}_{\mathcal{A}} &= D_{\mathcal{C}} D_{\mathcal{C}} = d(A + \theta B) + (A + \theta B) \wedge (A + \theta B) \\ &= (F_A + \zeta B) - \theta d_A B, \end{aligned}$$

having used the relation  $\zeta = d\theta$ . The norm squared of the cone curvature is the cone Yang-Mills functional

$$\begin{aligned} S_{cYM}(A + \theta B) &= \|\mathcal{F}_{\mathcal{A}}\|_{\mathcal{C}}^2 = \|(F_A + \zeta B) - \theta d_A B\|_{\mathcal{C}}^2 \\ &= \|F_A + \zeta B\|^2 + \|d_A B\|^2. \end{aligned}$$

The zero points of the functional gives the cone-flat condition for the pair  $(A, B)$ :

$$F_A + \zeta B = 0, \quad d_A B = 0. \quad (2.6)$$

If satisfying (2.6), we will call the pair  $(A, B)$  cone-flat connections with respect to  $\zeta$ , or often, just simply cone-flat connections.

Now to obtain the equations for the critical points of the cone Yang-Mills functional, we consider the first order variation,  $(A, B) \rightarrow (A + t\eta, B + t\xi)$  where  $\eta \in \Omega^1(M, Ad P)$  and  $\xi \in \Omega^0(M, Ad P)$ . For the standard curvature, we find

$$F_{A+t\eta} = F_A + td_A\eta + t^2\eta \wedge \eta,$$

where locally,  $d_A\eta = d\eta + A \wedge \eta + \eta \wedge A =: d\eta + [A, \eta]$ . Furthermore, we have

$$\begin{aligned} S_{cYM}(A + t\eta, B + t\xi) &= \|F_{A+t\eta} + \zeta(B + t\xi)\|^2 + \|d_{A+t\eta}(B + t\xi)\|^2 \\ &= \|F_A + \zeta B\|^2 + \|d_AB\|^2 + 2t(\langle F_A + \zeta B, d_A\eta + \zeta\xi \rangle + \langle d_AB, [\eta, B] + d_A\xi \rangle) + o(t) \\ &= \|\mathcal{F}_A\|_C^2 + 2t(\langle d_A^*(F_A + \zeta B) + [B, d_AB], \eta \rangle + \langle \zeta^*(F_A + \zeta B) + d_A^*d_AB, \xi \rangle) + o(t) \end{aligned}$$

where for  $\dim M = m$ ,

$$d_A^* = (-1)^{mk+m+1} * d_A*, \quad (2.7)$$

and  $\zeta^*: \Omega^k(M) \rightarrow \Omega^{k-2}(M)$  is the adjoint of the  $\zeta \wedge$  map and defined to be

$$\zeta^* = (-1)^{(m-k)k} * \zeta * . \quad (2.8)$$

Hence, the stationary solutions of the cone Yang-Mills functional satisfy

$$d_A^*(F_A + \zeta B) + [B, d_AB] = 0, \quad (2.9)$$

$$\zeta^*(F_A + \zeta B) + d_A^*d_AB = 0. \quad (2.10)$$

We will call a pair  $(A, B)$  satisfying the above cone Yang-Mills equations (2.9)-(2.10) cone Yang-Mills connections.

## 2.2 Elliptic property of the cone Yang-Mills solutions

In this subsection, we shall prove the following theorem.

**Theorem 2.1.** *On a closed manifold  $M$ , the space of cone Yang-Mills connections modulo gauge equivalence is finite-dimensional.*

Our proof will follow the same line of arguments as Atiyah-Bott's proof of the analogous statement [2, Sec. 4] for the Yang-Mills functional.

*Proof.* We compute the linearized variation of the cone Yang-Mills equations. Let  $(A, B) \rightarrow (A + t\eta, B + t\xi)$  where  $\eta \in \Omega^1(M, \text{Ad } P)$  and  $\xi \in \Omega^0(M, \text{Ad } P)$ . The linearized variation of the first cone Yang-Mills equation (2.9) is of the form

$$\begin{aligned} & d_{A+t\eta}^*(F_{A+t\eta} + \zeta(B + t\xi)) + [B + t\xi, d_{A+t\eta}(B + t\xi)] \\ &= d_A^*(F_A + \zeta B) + [B, d_A B] + t \left( d_A^* d_A \eta - [B, [B, \eta]] + d_A^*(\zeta \xi) + [B, d_A \xi] \right. \\ &\quad \left. + [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A + \zeta B)] \right) + o(t), \end{aligned} \quad (2.11)$$

and for the second equation (2.10), we find

$$\begin{aligned} & \zeta^* F_{A+t\eta} + \zeta^* (\zeta(B + t\xi)) + d_{A+t\eta}^* d_{A+t\eta}(B + t\xi) \\ &= \zeta^* F_A + \zeta^*(\zeta B) + d_A^* d_A B + t \left( d_A^* d_A \xi + \zeta^*(\zeta \xi) + \zeta^* d_A \eta - d_A^*[B, \eta] - *[\eta, *d_A B] \right) + o(t). \end{aligned} \quad (2.12)$$

With (2.11)-(2.12), we see that  $A + \theta B + t(\eta + \theta \xi) + o(t)$  describes a curve of the critical points of the cone Yang-Mills functional if and only if

$$\left\{ d_A^* d_A \eta - [B, [B, \eta]] + d_A^*(\zeta \xi) + [B, d_A \xi] \right\} + [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A + \zeta B)] = 0 \quad (2.13)$$

$$\left\{ d_A^* d_A \xi + \zeta^*(\zeta \xi) + \zeta^* d_A \eta - d_A^*[B, \eta] \right\} - *[\eta, *d_A B] = 0 \quad (2.14)$$

We can write (2.13)-(2.14) more simply in terms of  $D_C$ , the cone covariant derivative. Locally, if  $d_A = d + A$ , then  $D_C = d + (A + \theta B)$  and

$$D_C(\eta + \theta \xi) = d_A \eta + \zeta \wedge \xi + \theta([B, \eta] - d_A \xi),$$

which we can express in matrix form as

$$D_C \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} d_A & \zeta \wedge \\ [B, -] & -d_A \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad (2.15)$$

and its adjoint with respect to the metric in (2.5) by

$$D_C^* \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} d_A^* & -[B, -] \\ \zeta^* & -d_A^* \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad (2.16)$$

Here, we have noted that the adjoint  $[B, -]^* = -[B, -]$ , since for any  $\gamma, \gamma' \in \Omega^k(M, \text{Ad } P)$ , we have

$$\langle [B, -]^*(\gamma), \gamma' \rangle = \langle \gamma, [B, \gamma'] \rangle = \langle [\gamma, B], \gamma' \rangle = -\langle [B, \gamma], \gamma' \rangle,$$

having used the adjoint-invariance property of the inner product (2.2). Now, the composition

$$D_C^* D_C \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} d_A^* d_A - [B, [B, -]] & d_A^*(\zeta \wedge -) + [B, d_A -] \\ \zeta^* d_A - d_A^*[B, -] & d_A^* d_A + \zeta^*(\zeta \wedge -) \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

which reproduces exactly the terms within the curly brackets  $\{ \dots \}$  in (2.13)-(2.14). The remaining terms can be expressed as

$$\begin{aligned} (-1)^m *_C [\eta + \theta \xi, *_C \mathcal{F}_C] &= (-1)^m *_C [\eta + \theta \xi, *_C (F_A + \zeta B - \theta d_A B)] \\ &= (-1)^m *_C [\eta + \theta \xi, \theta * (F_A + \zeta B) - *d_A B] \\ &= (-1)^{m+1} *_C \{ [\eta, *d_A B] + \theta ([\xi, *d_A B] + [\eta, *(F_A + \zeta B)]) \} \\ &= [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A + \zeta B)] - \theta * [\eta, *d_A B] \end{aligned}$$

The last equation follows from the relation  $(-1)^{m+1} * [\xi, *d_A B] = [\xi, d_A B]$ . This can be seen by writing  $d_A B = \sum \mu_i \otimes \alpha_i$  and  $\xi = \sum \xi_i \otimes \alpha_i$  where  $\mu_i \in \Omega^1(M)$ ,  $\xi_i \in \Omega^0(M)$  and  $\{\alpha_i\}$  is a basis of the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ . Then

$$\begin{aligned} (-1)^{m+1} * [\xi, *d_A B] &= (-1)^{m+1} \sum *[\xi_i \otimes \alpha_i, (*\mu_j) \otimes \alpha_j] \\ &= (-1)^{m+1} \sum * * (\xi_i \mu_j) \otimes [\alpha_i, \alpha_j] \\ &= \sum \xi_i \mu_j \otimes [\alpha_i, \alpha_j] \\ &= [\xi, d_A B]. \end{aligned}$$

In all, the linearized variation condition (2.13)-(2.14) can be expressed concisely as

$$D_C^* D_C (\eta + \theta \xi) + (-1)^m *_C [\eta + \theta \xi, *_C \mathcal{F}_C] = 0. \quad (2.17)$$

Now, under a gauge transformation

$$A + \theta B \rightarrow g(A + \theta B)g^{-1} + gdg^{-1} = A + \theta B - t(D_C \alpha) + o(t)$$

having substituted on the right-hand-side  $g = e^{t\alpha}$  for  $\alpha \in \mathfrak{g}$ . To quotient out a linear variation that is a gauge transformation, we impose that the deformation satisfy the gauge-fixing condition

$$D_C^*(\eta + \theta \xi) = 0. \quad (2.18)$$

Combining (2.17) with (2.18), the linearized deformation  $(\eta + \theta\xi)$  is characterized by solving the differential system

$$(D_C^* D_C + D_C D_C^*)(\eta + \theta\xi) + (-1)^m *_C [\eta + \theta\xi, *_C \mathcal{F}_C] = 0. \quad (2.19)$$

This is an elliptic system since the cone Laplacian  $\Delta_C = D_C^* D_C + D_C D_C^*$  is elliptic. (With (2.15)-(2.16), it is easily seen that the contribution to the principal symbol of  $\Delta_C$  comes only from the standard Laplacian  $d^*d + dd^*$  on the diagonal components.) Hence, this implies that the tangent space of cone Yang-Mills connections modulo gauge equivalence is finite-dimensional.  $\square$

### 2.3 Comparison with Yang-Mills solutions

Since the cone Yang-Mills functional is closely related to the Yang-Mills functional, it is natural to ask about the relationship between their critical points (i.e. solutions). We shall study this issue starting first with some special cases.

#### (i) Cone Yang-Mills solutions with $B = 0$ .

Consider first the case of cone Yang-Mills solutions with  $B = 0$ . In this setting, the cone Yang-Mills equations simplify to

$$d_A^* F_A = 0, \quad \zeta^* F_A = 0. \quad (2.20)$$

Hence, cone Yang-Mills solutions with  $B = 0$  are a subset of Yang-Mills solutions satisfying additionally the second condition of (2.20).

In particular, when  $M$  is a Kähler manifold and  $\zeta = \omega$  is the Kähler metric, a hermitian Yang-Mills connection that satisfies the conditions

$$F_A^{2,0} = F_A^{0,2} = 0, \quad \omega^{n-1} \wedge F_A = 0,$$

is also a cone Yang-Mills solution satisfying (2.20) with  $B = 0$ . This is because  $\omega^* F_A = 0$  is equivalent to  $\omega^{n-1} \wedge F_A = 0$  for a Kähler metric.

#### (ii) Cone Yang-Mills solutions with $\zeta = 0$ .

Another special case is that of setting  $\zeta = 0$ . Notice first for the zero point of the cone Yang-Mills functional satisfying the cone-flat condition (2.6), the pair  $(A, B)$  is cone-flat if and only if  $A$  is a flat connection (i.e.  $F_A = 0$ ) and  $d_A B = 0$ . In general, we have the following:

**Lemma 2.2.** *Let  $M$  be a closed manifold and let  $\zeta = 0$ . Then  $(A, B)$  is a cone Yang-Mills solution if and only if  $A$  is a Yang-Mills connection and  $d_A B = 0$ .*

*Proof.* When  $\zeta = 0$ ,  $\zeta^*$  is a zero map. The second equation of cone Yang-Mills condition (2.10) becomes just  $d_A^* d_A B = 0$ , which implies  $d_A B = 0$  on a compact manifold. The first equation (2.9) then simplifies to  $d_A^* F_A = 0$ . Hence,  $A$  must be a Yang-Mills connection with  $B$  covariantly constant.  $\square$

### (iii) Cone Yang-Mills solutions when $\zeta$ is a harmonic form.

Instead of vanishing, suppose  $\zeta$  is a harmonic two-form, i.e.  $d\zeta = d^*\zeta = 0$ . We can obtain a similar statement to Lemma 2.2 if we require additionally that the cone Yang-Mills solution is cone-flat.

**Lemma 2.3.** *Suppose  $(A, B)$  is a cone-flat solution, i.e. a zero point of the cone Yang-Mills functional. If  $\zeta$  is a harmonic form, then  $A$  is a Yang-Mills connection.*

*Proof.* By assumption, we have  $F_A = -\zeta B$  and  $d_A B = 0$ . So

$$d_A * F_A = -d_A(*\zeta B) = -(d * \zeta)B - (-1)^{m-2}(*\zeta)d_A B = 0,$$

implying that  $A$  is a Yang-Mills connection.  $\square$

**Remark 2.4.** When  $(M^{2n}, \omega)$  is a symplectic manifold and  $\zeta = \omega$ , then a connection satisfying the curvature condition  $F_A = -\omega B$  such that  $d_A B = 0$  is called a symplectically-flat connection as introduced in [10]. Symplectically-flat connections are Yang-Mills connections with respect to a compatible metric. This agrees with Lemma 2.3 above since with respect to a compatible metric,  $*\omega = \omega^{n-1}/(n-1)!$  which implies  $d^*\omega = 0$ , i.e.  $\omega$  is a harmonic form.

### (iv) Cone Yang-Mills solutions with connection one-form not Yang-Mills.

Thus far, we have described special cone Yang-Mills solution pairs  $(A, B)$  where the connection part  $A$  is Yang-Mills. Such is not the generic case. Below, we shall give a simple cone-flat solution  $(A, B)$  where  $A$  is not Yang-Mills. In order not to contradict Lemma 2.3, the  $\zeta$  in the example below is not harmonic.

**Example 2.5.** Let  $\Sigma$  be a Riemann surface and  $A'$  a non-flat Yang-Mills connection on  $\Sigma$ . Let  $\zeta$  be the volume form of  $\Sigma$  normalized such that the total volume of  $\Sigma$  is one. Define  $M$  to be the three-dimensional circle bundle  $\pi : M \rightarrow \Sigma$  with Euler class given by  $\zeta$ . Since  $A'$  is Yang-Mills, we can write its curvature as  $F_{A'} = \zeta\Phi$  such that  $d_{A'}\Phi = 0$ . For simplicity, we will also use  $\zeta$ ,  $A'$  and  $\Phi$  to denote their pullbacks on  $M$ .

Let  $\theta \in \Omega^1(M)$  be the global angular one-form of the circle bundle  $M$ , i.e.  $d\theta = \zeta$ , and also let  $c \in \mathbb{R}$ . For  $A = A' + c\theta\Phi$  and  $B = -(1+c)\Phi$  on the pullback bundle  $\pi^*(Ad P)$ , we find

$$F_A = F_{A'} + c\zeta\Phi - c\theta d_{A'}\Phi = (1+c)\zeta\Phi = -\zeta B,$$

and

$$d_A B = -(1+c)d_{A'}\Phi - c(1+c)\theta[\Phi, \Phi] = 0,$$

that is,  $(A, B)$  is a cone-flat solution. However, the connection form  $A$  is not Yang-Mills in general with respect to the volume form on the circle bundle  $M$ ,  $dvol_M = \zeta \wedge \theta$ , since

$$d_A^* F_A = * d_A * F_A = -* d_A(\theta B) = -* \zeta B = (1+c)\theta\Phi$$

which is non-zero unless  $c = -1$ . (We have assumed  $A'$  is not a flat connection, and therefore,  $F_{A'} = \zeta\Phi \neq 0$ .) Hence, generally, for any  $c \neq -1$ ,  $A$  is not a Yang-Mills connection.

#### (v) Yang-Mills connections that can not be a part of a cone Yang-Mills solution.

For a cone Yang-Mills solution,  $(A, B)$ , we have seen that  $A$  need not be a Yang-Mills connection. In the reverse direction, we can ask if given a Yang-Mills connection  $A$ , will there always exist a  $B$  such that  $(A, B)$  is a cone Yang-Mills solution? The answer is no, as is shown in the example below.

**Example 2.6.** Let  $M = T^4 = \mathbb{R}^4/2\pi\mathbb{Z}^4$  be the 4-torus described by identifications  $x_i \sim x_i + 2\pi n_i$  for  $i = 1, 2, 3, 4$  and  $n_i \in \mathbb{Z}$ . Let  $\zeta = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ . We will take the Riemannian metric to be

$$g = (dx_1)^2 + (dx_2)^2 + \frac{1}{f}(dx_3)^2 + f(dx_4)^2$$

where  $f = \frac{3 + 2 \sin 2x_2 \cos x_3}{1 - \frac{1}{2} \sin 2x_2 \cos x_3}$ .

Consider a principal  $U(1)$  bundle over  $M$  with the circle coordinate identified by

$$y \sim y + 2\pi n_5 - n_1 x_3$$

for  $n_5 \in \mathbb{Z}$ . The global connection one-form on the  $U(1)$  bundle can be taken to be

$$A = dy + \frac{1}{2\pi}x_1 dx_3 + \frac{1}{4\pi} \sin 2x_2 \sin x_3 dx_1.$$

The curvature two-form is then given by

$$F_A = dA = -\frac{1}{2\pi} \cos 2x_2 \sin x_3 dx_1 \wedge dx_2 + \frac{1}{2\pi} \left(1 - \frac{1}{2} \sin 2x_2 \cos x_3\right) dx_1 \wedge dx_3.$$

Moreover,

$$\begin{aligned} d * F_A &= \frac{1}{2\pi} d \left[ -\cos 2x_2 \sin x_3 dx_3 \wedge dx_4 - \left(1 - \frac{1}{2} \sin 2x_2 \cos x_3\right) f dx_2 \wedge dx_4 \right] \\ &= \frac{1}{2\pi} d [-\cos 2x_2 \sin x_3 dx_3 \wedge dx_4 - (3 + 2 \sin 2x_2 \cos x_3) dx_2 \wedge dx_4] = 0. \end{aligned}$$

Thus,  $F_A$  satisfies the abelian Yang-Mills equation. However, there does not exist a function  $B$  on  $T^4$  such that  $(A, B)$  satisfy the cone Yang-Mills equations (2.9)-(2.10). Equation (2.9) implies in the abelian case  $d^*(F_A + \zeta B) = 0$ . But since we know already  $d^*F_A = 0$ , this means  $B$  is required to satisfy

$$0 = d * (\zeta B) = d(\zeta B) = \zeta \wedge dB. \quad (2.21)$$

With  $\zeta$  being a non-degenerate two-form, (2.21) gives the condition  $dB = 0$ .

However, imposing the second cone Yang-Mills equation (2.10) results in a contradiction. For if  $dB = 0$ , (2.10) reduces to the condition  $\zeta^*F_A + \zeta^*\zeta B = 0$ . This implies in particular

$$B = \frac{1}{2}\zeta^*\zeta B = -\frac{1}{2}\zeta^*F_A = \frac{1}{4\pi} \cos 2x_2 \sin x_3, \quad (2.22)$$

and clearly,  $dB \neq 0$ , which contradicts the condition from (2.21). Hence, there is no  $B$  that can satisfy the cone Yang-Mills equations.

## 2.4 Uniqueness of cone Yang-Mills solutions for a fixed connection

In general, given an arbitrary connection  $A$ , there does not exist a  $B$  that would make  $(A, B)$  a cone Yang-Mills solution. To illustrate this in a simple example, let  $M$  be a closed manifold and  $\zeta = 0$ . Then by Lemma 2.2, we know that  $A$  must be a Yang-Mills connection and  $d_A B = 0$ . Hence, there exists no  $B$  that would give a cone Yang-Mills solution if  $A$  is not Yang-Mills.

However, it is interesting to ask that given a cone Yang-Mills solution  $(A, B)$ , how many different  $B$ 's with  $A$  fixed would also be a cone Yang-Mills solution? In the case of  $\zeta = 0$  and

$M$  closed, if  $(A, B)$  is a cone Yang-Mills solution, then so are all  $(A, B + \Phi)$  with  $d_A\Phi = 0$ . For  $\zeta \neq 0$ , we are able to obtain results in two special cases. First, in two dimensions and suppose  $A$  is a Yang-Mills connection, we have the following:

**Proposition 2.7.** *Suppose  $M$  is a closed Riemann surface and  $\zeta$  is its volume form. For each Yang-Mills connection  $A$ , there exists a unique  $B$  such that  $(A, B)$  is a cone Yang-Mills solution. In fact, such a pair  $(A, B)$  is always cone-flat.*

*Proof.* Since  $A$  is a Yang-Mills connection on a Riemann surface,  $F_A = \zeta\Phi$  with  $d_A\Phi = 0$ . So we can choose  $B = -\Phi$  and then  $(A, B)$  is a cone-flat solution. We will show below that this is the unique cone Yang-Mills solution for a fixed  $A$ , a Yang-Mills connection.

Generally, suppose  $(A, B)$  is a cone Yang-Mills solution. As  $\zeta$  is a volume form, the condition  $\zeta^*(F_A + \zeta B) + d_A^*d_AB = 0$  becomes

$$\Phi + B + d_A^*d_AB = 0. \quad (2.23)$$

Let  $d_A$  act on both sides. This implies  $d_AB = -d_Ad_A^*d_AB$ , and therefore,

$$\langle d_AB, d_Ad_A^*d_AB \rangle = -\|d_AB\|^2 \leq 0.$$

On the other hand,

$$\langle d_AB, d_Ad_A^*d_AB \rangle = \langle d_A^*d_AB, d_A^*d_AB \rangle = \|d_A^*d_AB\|^2 \geq 0.$$

So  $\langle d_AB, d_Ad_A^*d_AB \rangle$  has to vanish. It follows that  $\|d_AB\|^2 = 0$ , which implies,  $d_AB = 0$ , and by (2.23),  $B = -\Phi$ . So such  $B$  is unique when  $A$  is Yang-Mills.  $\square$

Another special case where we can constrain  $B$  is when  $\zeta$  is the symplectic form on a symplectic manifold. If the structure group is abelian, then  $B$  must be unique.

**Proposition 2.8.** *Suppose  $A$  is a connection on the associated adjoint bundle  $AdP$  over a symplectic manifold  $(M^{2n}, \omega)$  and the Riemannian metric is compatible with  $\omega$ . Take  $\zeta = \omega$ . Then there is at most one  $B$  that satisfies  $[B, d_AB] = 0$  and such that  $(A, B)$  is a cone Yang-Mills solution. In particular, if the structure group is abelian, there is at most one cone Yang-Mills solution pair  $(A, B)$  for any given connection  $A$ .*

*Proof.* Let  $\dim M = 2n$ . With respect to a compatible metric,  $*\zeta = *\omega = \frac{1}{(n-1)!}\omega^{n-1}$ , and  $\zeta^*\zeta = n$ .

Suppose both  $(A, B_1)$  and  $(A, B_2)$  are cone Yang-Mills solutions. Assume also  $[B_1, d_A B_1] = [B_2, d_A B_2] = 0$  which is identically true when the structure group is abelian. Then (2.9)-(2.10) imply for  $B_1$ ,

$$\begin{aligned} - * d_A (*F_A + \frac{1}{(n-1)!} \omega^{n-1} B_1) &= 0, \\ \zeta^* F_A + n B_1 + d_A^* d_A B_1 &= 0, \end{aligned}$$

and  $B_2$  satisfies identical equations. So by the first equation, we have

$$\omega^{n-1} \wedge d_A B_1 = \omega^{n-1} \wedge d_A B_2.$$

It follows that  $d_A B_1 = d_A B_2$  because  $\omega^{n-1} : \Omega^1(M) \rightarrow \Omega^{2n-1}(M)$  is an isomorphism. With this, the second equation becomes

$$n(B_1 - B_2) = d_A^* d_A (B_2 - B_1) = 0,$$

which implies,  $B_1 = B_2$ . □

The above proposition avoids considering the term  $[B, d_A B]$ . If  $[B, d_A B] = 0$  holds true for all cone Yang-Mills solutions  $(A, B)$ , then we would have obtained the uniqueness of  $B$  when  $\zeta$  is the symplectic structure. But as we shall see in Example 3.2 in the next section, a cone Yang-Mills solution in general need not satisfy  $[B, d_A B] = 0$ .

### 3 Special solutions of the cone Yang-Mills functional

#### 3.1 Two-dimensional solutions with abelian gauge group

Consider cone Yang-Mills solutions on a closed Riemann surface with abelian structure group. In this case, the Euler-Lagrange equations reduce to

$$d^*(F_A + \zeta B) = 0, \tag{3.1}$$

$$\zeta^*(F_A + \zeta B) + d^* dB = 0. \tag{3.2}$$

The first equation (3.1) implies that  $*(F_A + \zeta B)$  is a constant  $c$ . So we can assume that

$$F_A + \zeta B = c\omega, \tag{3.3}$$

where  $\omega$  is the volume form of  $M$ . By Hodge decomposition, we can write

$$\zeta = c'\omega + dd^*(f\omega) = c'\omega - d*df,$$

where  $c'$  is a constant and  $f$  is a function on  $M$ .

For any function  $\phi$  on  $M$ , we have

$$\begin{aligned} \langle \phi, \zeta^*(F_A + \zeta B) \rangle &= \langle \zeta\phi, (F_A + \zeta B) \rangle = \langle \phi(c'\omega - d*df), c\omega \rangle \\ &= \int_M c\phi(c'\omega - d*df) \\ &= \int_M cc'\phi\omega - \int_M d(c\phi*df) + \int_M c d\phi \wedge *df \\ &= \langle \phi, cc' \rangle + \langle d\phi, c df \rangle \\ &= \langle \phi, cc' + c d^*df \rangle \end{aligned}$$

and therefore,

$$\zeta^*(F_A + \zeta B) = cc' + c d^*df. \quad (3.4)$$

Plugging this into (3.2), we find that  $cc'$  is a  $d^*$ -exact constant; hence,  $cc'$  must be zero.

If  $c' \neq 0$ , then  $c$  must vanish and then  $F_A + \zeta B = 0$ . By (3.2),  $d^*dB = 0$  which implies  $dB = 0$  since  $M$  is closed. Thus, we have obtained the following statement.

**Proposition 3.1.** *Suppose  $M$  is a closed Riemann surface,  $\zeta$  is a non-exact two-form on  $M$ , and the structure group is abelian. Then all cone Yang-Mills solutions are cone-flat.*

Now we consider the case  $c' = 0$ , that is,  $\zeta = -d*df$  is a  $d$ -exact form. Together, (3.2) and (3.4) imply  $d^*d(cf + B) = 0$ , and so  $cf + B$  is a constant. We can thus write  $B = -cf + c''$  for some constant  $c''$ . Hence, by (3.3),  $F_A = c\omega - \zeta B = c\omega + (c'' - cf)d*df$ . Therefore, we find that the constants  $c$  and  $c''$  parametrize the cone Yang-Mills solutions. And finally, in the special case where  $f = 0$ , i.e.  $\zeta$  vanishes, the critical points must satisfy  $F_A = c\omega$  and  $B = c''$ .

## 3.2 Three-dimensional solutions from duality relations

Recall in four dimensions, there are special Yang-Mills solutions that satisfy the first-order self-dual/anti-self-dual conditions,  $*F_A = \pm F_A$ . The intuition that cone Yang-Mills functional can be interpreted as a dimensional reduction of the Yang-Mills functional suggests an analogous

duality condition  $*_{\mathcal{C}} \mathcal{F}_{\mathcal{A}} = \pm \mathcal{F}_{\mathcal{A}}$  over 3-manifolds, where  $\mathcal{F}_{\mathcal{A}} = (F_A + \zeta B) - \theta d_A B$ . With the  $*_{\mathcal{C}}$  acting on  $\text{Cone}^k(\zeta)$  given by (2.4), we have

$$\begin{aligned} *_{\mathcal{C}} \mathcal{F}_{\mathcal{A}} &= *_{\mathcal{C}} [(F_A + \zeta B) - \theta d_A B] \\ &= *(-d_A B) + \theta [* (F_A + \zeta B)]. \end{aligned}$$

Hence, we find that  $*_{\mathcal{C}} \mathcal{F}_{\mathcal{A}} = \mp \mathcal{F}_{\mathcal{A}}$  implies

$$F_A + \zeta B = \pm * d_A B \quad (3.5)$$

Note that when  $\zeta = 0$ , the condition becomes the Bogomolny monopole equations.

We will show that a solution of (3.5) is automatically a solution of the cone Yang-Mills equations in two different ways. First, we can check directly that (3.5) implies the cone Yang-Mills equations (2.9)-(2.10). Applying  $d_A^*$  to (3.5), we find the following:

$$\begin{aligned} d_A^*(F_A + \zeta B) &= \pm d_A^* * d_A B \\ &= \pm * d_A d_A B = \pm * [F_A, B] \\ &= [(d_A B - * \zeta B), B] = [d_A B, B] \end{aligned}$$

where we have used the three-dimensional relations  $** = 1$ , and  $d_A^* = (-1)^k * d_A *$  acting on a  $k$ -form. The above implies the first cone Yang-Mills equation (2.9),  $d_A^*(F_A + \zeta B) + [B, d_A B] = 0$ . Furthermore, it also follows from (3.5) that

$$\begin{aligned} d_A^* d_A B &= -* d_A (* d_A B) = \mp * d_A (F_A + \zeta B) \\ &= \mp * \zeta \wedge d_A B = \mp * \zeta * (* d_A B) \\ &= -\zeta^* (F_A + \zeta B) \end{aligned}$$

which implies the second cone Yang-Mills equation (2.10),  $\zeta^* (F_A + \zeta B) + d_A^* d_A B = 0$ .

In the second method, analogous to the standard Yang-Mills instanton argument, we can express the three-dimensional cone Yang-Mills functional in the following manner:

$$\begin{aligned} \|\mathcal{F}_{\mathcal{A}}\|_{\mathcal{C}}^2 &= \int_M c_G \text{tr} [(F_A + \zeta B) \wedge * (F_A + \zeta B) + d_A B \wedge * d_A B] \\ &= \int_M c_G \text{tr} [(F_A + \zeta B \mp * d_A B) \wedge * (F_A + \zeta B \mp * d_A B)] \pm 2 \int_M c_G \text{tr} [(F_A + \zeta B) \wedge d_A B] \\ &\geq \pm 2 \int_M c_G \text{tr} [(F_A + \zeta B) \wedge d_A B]. \end{aligned} \quad (3.6)$$

The equality holds only when  $F_A + \zeta B = \pm * d_AB$ . Importantly, the bounding integral is a boundary term

$$\begin{aligned} Q &= \int_M c_G \text{tr} [(F_A + \zeta B) \wedge d_AB] = \int_M c_G \text{tr} \left[ d \left( F_A B + \frac{1}{2} \zeta B^2 \right) - (d_A F_A) B + \zeta \wedge B[A, B] \right] \\ &= \int_{\partial M} c_G \text{tr} \left[ F_A B + \frac{1}{2} \zeta B^2 \right] \end{aligned} \quad (3.7)$$

which is the action of a two-dimensional topological  $BF$ -type theory. We note that any infinitesimal local variation of  $(A, B)$  away from the boundary does not affect the bound. Hence,  $F_A + \zeta B = \pm * d_AB$  must be a critical point of the cone Yang-Mills functional.

The duality-type condition of (3.5) helps simplify the search for cone Yang-Mills solutions in three dimensions. Notably, it is first-order and hence more tractable compared with the second-order cone Yang-Mills equations (2.9)-(2.10). As mentioned, when  $\zeta = 0$ , the condition becomes the Bogomolny equations and then the known three-dimensional Bogomolny-Prasad-Sommerfeld (BPS) monopole solutions are trivially cone Yang-Mills solutions with  $Q$  of (3.7) being proportional to the magnetic monopole charge (for a review, see [9]). More generally, for  $\zeta \neq 0$ , we can look for self-dual/anti-self-dual Yang-Mills instanton solutions in four dimensions on spaces that can be described as a circle bundle over a three-manifold. If the four-dimensional Yang-Mills instanton solutions are invariant under the  $S^1$  circle action, then we can dimensionally reduce over the circle and obtain solutions that satisfy (3.5). We give such an example below coming from the Taub-NUT gravitational instanton.

**Example 3.2.** The Taub-NUT gravitational instanton solution can be thought of as a self-dual Yang-Mills solutions of a tangent bundle with structure group  $SU(2) \subset SO(4)$ . With a point removed, the four-dimensional space can be considered as a circle bundle,  $S^1 \rightarrow X \rightarrow \mathbb{R}^3 - \{0\}$  (see, for example, the description in [5]). Dimensionally reducing this solution leads to a non-abelian cone Yang-Mills solution on  $M = \mathbb{R}^3 - \{0\}$  satisfying (3.5).

We start with the Taub-NUT metric written in Gibbons-Hawking form:

$$ds_{TN}^2 = e^{2\phi} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + e^{-2\phi} \theta^2, \quad (3.8)$$

where  $e^{2\phi} = 1 + 2/r$  is a positive function on  $\mathbb{R}^3 - \{0\}$  with  $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$  and the  $d$ -closed two-form  $\zeta$  is defined to be

$$\zeta = d\theta = \pm *_0 d(e^{2\phi})$$

$$= \pm \epsilon_{ijk} \partial_k \phi e^{2\phi} dx^i \wedge dx^j \quad (3.9)$$

with  $*_0$  being the Euclidean Hodge star on  $M = \mathbb{R}^3 - \{0\}$ . The requirement that  $d\zeta = 0$  gives the following condition on  $\phi$  on  $\mathbb{R}^3 - \{0\}$ :

$$\sum_{k=1}^3 [\partial_k^2 \phi + 2(\partial_k \phi)^2] = 0. \quad (3.10)$$

For the Taub-NUT metric in (3.8), we have the following basis of moving frame (i.e. orthonormal frame) of 1-forms

$$e^i = e^\phi dx^i \quad i = 1, 2, 3, \quad \text{and} \quad e^4 = e^{-\phi} \theta.$$

These result in the following connection 1-forms (for a reference, see [8])

$$\omega^i{}_j = -\omega^j{}_i = e^{-\phi} (-\partial_i \phi e^j + \partial_j \phi e^i \mp \epsilon_{ijk} \partial_k \phi e^4) \quad (3.11)$$

$$\omega^i{}_4 = -\omega^4{}_i = e^{-\phi} (\pm \epsilon_{ijk} \partial_j \phi e^k + \partial_i \phi e^4) \quad (3.12)$$

which satisfy the torsionless condition  $de^r + \omega^r{}_s \wedge e^s = 0$ . (Regarding indices, we will let the indices  $i, j, k, l, p, q$  take values in the set  $\{1, 2, 3\}$  and  $r, s, t, u$  take values in the set  $\{1, 2, 3, 4\}$ .) Notice that these connection 1-forms are also anti-self-dual/self-dual in the sense that

$$\omega_{rs} = \mp \frac{1}{2} \epsilon_{rstu} \omega_{tu}.$$

Hence, the structure group of the bundle reduces to an  $SU(2)$  subgroup of  $SO(4)$ . It can be checked that the resulting Taub-NUT curvature two-form  $R^r{}_s = d\omega^r{}_s + \omega^r{}_t \wedge \omega^t{}_s$  is correspondingly anti-self-dual/self-dual.

Now to perform a dimensional reduction, we take the metric on  $X$  to be

$$ds_X^2 = e^{2\phi} ds_{TN}^2 = e^{4\phi} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + \theta^2, \quad (3.13)$$

conformally scaled by the  $e^{2\phi}$  factor in order to match the desired metric form of  $ds_X^2 = ds_M^2 + \theta^2$  as in (1.5). We note that the four-dimensional Yang-Mills anti-self-dual/self-dual curvature equation is invariant under an overall conformal scaling of the metric. Therefore, for the cone pair  $(A, B)$  on  $M = \mathbb{R}^3 - \{0\}$ , we can just read off from (3.11)-(3.12) by setting  $\mathcal{A}^r{}_s = (A + \theta B)^r{}_s = \omega^r{}_s$ . We find

$$A^i{}_j = -A^j{}_i = -\partial_i \phi dx^i + \partial_j \phi dx^j, \quad B^i{}_j = -B^j{}_i = \mp \epsilon_{ijk} e^{-2\phi} \partial_k \phi, \quad (3.14)$$

$$A^i{}_4 = -A^4{}_i = \pm \epsilon_{ijk} \partial_j \phi dx^k, \quad B^i{}_4 = -B^4{}_i = e^{-2\phi} \partial_i \phi. \quad (3.15)$$

These lead to the following:

$$\begin{aligned} (F_A)^i{}_j &= -\partial_i \partial_k \phi dx^k \wedge dx^j + \partial_j \partial_k \phi dx^k \wedge dx^i - \epsilon_{ikl} \epsilon_{jlpq} \partial_k \phi \partial_p \phi dx^l \wedge dx^q \\ &\quad + \partial_i \phi \partial_k \phi dx^k \wedge dx^j - \partial_j \phi \partial_k \phi dx^k \wedge dx^i - (\partial_k \phi)^2 dx^i \wedge dx^j, \\ (F_A)^i{}_4 &= \pm \left( \partial_i \phi \epsilon_{jkl} \partial_j \phi dx^k \wedge dx^l - \epsilon_{ijk} \partial_j \partial_l \phi dx^k \wedge dx^l \right), \\ (d_AB)^i{}_j &= \pm e^{-2\phi} \left[ 2 \epsilon_{ijk} \partial_k \phi \partial_l \phi dx^l - \epsilon_{ijk} \partial_k \partial_l \phi dx^l + 2 (\partial_i \phi \epsilon_{jkl} - \partial_j \phi \epsilon_{ikl}) \partial_k \phi dx^l \right], \\ (d_AB)^i{}_4 &= e^{-2\phi} (-4 \partial_i \phi \partial_j \phi dx^j + 2(\partial_j \phi)^2 dx^i + \partial_i \partial_j \phi dx^j). \end{aligned}$$

With (3.9)-(3.10), it can be straightforwardly checked that the above solution satisfies  $F_A + \zeta B = \pm * d_AB$  where the Hodge star is defined with respect to the three-dimensional metric

$$ds_M^2 = e^{4\phi} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2).$$

Hence,  $(A, B)$  as defined in (3.14)-(3.15) gives us a highly non-trivial, non-abelian solution of the cone Yang-Mills equations. Moreover, for this solution, it can be checked that  $[B, d_AB] \neq 0$ .

**Remark 3.3.** We can consider dimensionally reducing the three-dimensional duality condition  $*_{\mathcal{C}} \mathcal{F}_A = \mp \mathcal{F}_A$  over another circle down to a two-dimensional manifold  $N$ . For the metric, we will assume  $ds_M^2 = ds_N^2 + \chi^2$  where  $\chi$  is the global connection one-form (i.e. the global angular form) of the circle bundle  $M$  over  $N$ . The dimensionally reduced three-dimensional connection can then be expressed as  $A + \chi b$  where  $A$  is now a connection form over  $N$  and  $b \in \Omega^0(N, Ad P)$ . The three-dimensional cone curvature then takes the form

$$\mathcal{F}_A = F_{A+\chi b} + \zeta B - \theta d_{A+\chi b} B = F_A + \zeta B + (d\chi)b - \chi d_AB - \theta d_AB - \theta \chi [b, B],$$

with

$$*_{\mathcal{C}} \mathcal{F}_A = \theta \chi [* (F_A + \zeta B + (d\chi)b)] - \theta * d_AB + \chi * d_AB - *[b, B].$$

The condition  $*_{\mathcal{C}} \mathcal{F}_A = \mp \mathcal{F}_A$  then implies the following two equations:

$$F_A + \zeta B + (d\chi)b = \pm * [b, B], \quad (3.16)$$

$$* d_AB = \pm d_AB. \quad (3.17)$$

In the case where the circle is trivially fibered over  $N$ , then we can set  $d\chi = 0$ . Moreover, when  $\zeta = d\chi = 0$ , the above equations become equivalent to Hitchin's equations.

## 4 Classification of cone-flat bundles with respect to a non-degenerate two-form

In this section, we study the question what bundles can carry a pair  $(A, B)$  satisfying the cone-flat condition:

$$F_A + \zeta B = 0, \quad d_A B = 0. \quad (4.1)$$

Now if  $\zeta = 0$ , the cone-flat condition reduces to (1)  $F_A = 0$ , i.e.  $A$  is a flat connection, and (2)  $d_A B = 0$ , that is,  $B$  is a covariantly constant section. Since we can always set  $B = 0$ , the  $\zeta = 0$  cone-flat bundles are just the usual flat bundles, and flat bundles are well-known to be classified by the conjugacy classes of homomorphisms from  $\pi_1(M)$  to the structure group  $G$  of the fiber bundle (see, for example [6]). As the classification for the  $\zeta = 0$  case is well-understood, we will in the following always assume that  $\zeta \neq 0$ .

To simplify our consideration, we will additionally assume that  $\zeta$  is a non-degenerate two-form. Being  $d$ -closed and non-degenerate,  $\zeta$  is then a symplectic structure and  $M$  must be even-dimensional. Additionally,  $\zeta$  being non-degenerate leads to two simplifications in considering cone-flat pairs  $(A, B)$ . First, non-degeneracy ensures that  $\zeta$  nowhere vanishes, and hence, the first cone-flat equation  $F_A = -\zeta B$ , implies  $B$  is determined by the connection  $A$ . Second, the second cone-flat equation  $d_A B = 0$  is automatically satisfied and redundant when  $\dim M \geq 4$ . This is because the first cone-flat equation together with the Bianchi identity imply

$$d_A F_A = -\zeta \wedge d_A B = 0. \quad (4.2)$$

If  $\zeta$  is non-degenerate and  $\dim M \geq 4$ , then (4.2) implies  $d_A B = 0$ . In all,  $\zeta$  is non-degenerate simplifies our consideration of cone-flat pairs  $(A, B)$  to checking only that the curvature is proportional to  $\zeta$ , i.e.  $F_A = -\zeta B$ , and only if  $\dim M = 2$ , we need to also check that the resulting  $B$  is covariantly constant.

When  $M$  is a two-dimensional Riemann surface,  $\zeta$  being non-degenerate implies that it is the product of a nowhere vanishing function times the volume form. If we take  $\zeta$  to be the volume form, then we saw in (1.12) that the cone-flat condition (4.1) becomes equivalent to the Yang-Mills equation for  $A$  with  $B = -* F_A$ . Concerning Yang-Mills connections in two dimensions, Atiyah-Bott [2, Theorem 6.7] gave a classification of principal bundles over Riemann surfaces that carry a Yang-Mills connection. So we ask whether Atiyah-Bott's classification of Yang-Mills bundles in two dimensions can be extended to cone-flat bundles in  $\dim M = 2n \geq 2$  with

respect to a closed, non-degenerate two-form  $\zeta$ . Indeed, we obtain the following for principal bundles which generalizes Atiyah-Bott's result.

**Theorem 4.1.** *Let  $M$  be a path connected manifold,  $\zeta \in \Omega^2(M)$  be a non-zero, non-degenerate, closed two-form, and  $G$  be a Lie group. There exists a bijective correspondence between the following sets:*

$$\left\{ \begin{array}{l} \text{isomorphism classes of cone-flat connections} \\ \text{with respect to } \zeta \text{ on } G\text{-bundles over } M \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{homomorphisms } \rho : \Gamma \rightarrow G \end{array} \right\},$$

where  $\Gamma$  is an  $\mathbb{R}/\bar{H}$  extension of  $\pi_1(M)$  with  $\bar{H} \subset \mathbb{R}$  being the closure of the group

$$H := \left\{ \int_{\mathcal{S}} \zeta \mid \mathcal{S} \text{ is a representative in } \pi_2(M) \right\}. \quad (4.3)$$

As in the theorem, we will assume in the remainder of this section that  $M$  is path-connected. We can give a more explicit description of  $\Gamma$  in terms of path spaces. To do so, we first introduce some notations.

Denote by  $\Psi(M)$  the path space consisting of equivalence classes of closed, piecewise smooth paths on  $M$ . Two paths  $\alpha_1, \alpha_2 : [0, 1] \rightarrow M$  are considered to be equivalent in  $\Psi(M)$  if they have the same image and orientation. Specifically,  $\alpha_1$  and  $\alpha_2$  are equivalent if there exists a piecewise smooth increasing function  $\phi : [0, 1] \rightarrow [0, 1]$  with  $\phi(0) = 0$  and  $\phi(1) = 1$  such that  $\alpha_2 = \alpha_1 \circ \phi$ .

For two paths  $\alpha_1, \alpha_2 : [0, 1] \rightarrow M$  such that  $\alpha_1(1) = \alpha_2(0)$ , we define their multiplication by concatenation:

$$(\alpha_2 \alpha_1)(s) = \begin{cases} \alpha_1(2s), & s \in [0, \frac{1}{2}], \\ \alpha_2(2s - 1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

A connected subpath of a path is also a path. We define

$$\alpha_{(t)}(s) = \alpha(st), \quad t \in [0, 1].$$

which is a one-parameter family of subpaths, parametrized by  $t$ , all with the same starting point  $\alpha(0)$ , and ending point being at  $\alpha(t)$ .

The following subsets of  $\Psi(M)$  will be of interest:

- $\Psi_a^b(M)$  be the space of paths from a base point  $a \in M$  to another point  $b \in M$ .
- $\Psi_a^a(M)$  be the semigroup in  $\Psi(M)$  that consists of loops with base point  $a$ .
- $\Psi_{a,0}^a(M)$  be the subsemigroup of  $\Psi_a^a(M)$  that consists of contractible loops.
- $\Psi_{a,\zeta}^a(M)$  be the subsemigroup of  $\Psi_{a,0}^a(M)$  that consists of loops which are the boundary of some disk (contractible 2-chain with proper orientation)  $D$  in  $M$  satisfying  $\int_D \zeta = 0$ .

Quotienting them leads to the following spaces:

1.  $\Psi_a^a(M)/\Psi_{a,0}^a(M) = \pi_1(M)$  is the fundamental group where the identity element is the equivalence class of the constant path at  $a$ , and the inverse reverses the path orientation, i.e.  $\alpha^{-1}(s) = \alpha(1-s)$ .
2.  $\Psi_{a,0}^a(M)/\Psi_{a,\zeta}^a(M) \cong \mathbb{R}/H$ . As  $\zeta \neq 0$ , for any real number  $c$  there exists a contractible loop  $\gamma$  that is the boundary of some disk  $D$  such that  $\int_D \zeta = c$ . So  $\Psi_{a,0}^a(M)/\Psi_{a,\zeta}^a(M)$  is equivalent to a quotient group  $\mathbb{R}/H$  by the identification of a loop  $\gamma \mapsto [\int_D \zeta]$ , where

$$H = \left\{ \int_S \zeta \mid S \text{ is a representative in } \pi_2(M) \right\}.$$

The group  $H$  comes from noting that it is possible that the sum of two contractible curves  $\partial D_1 + \partial D_2$  can come from the vanishing boundary of a two-sphere formed by two hemisphere disks,  $D_1$  and  $D_2$ , glued together at the equator such that  $\partial D_1 = -\partial D_2$ .

3.  $\Psi_a^a(M)/\Psi_{a,\zeta}^a(M) =: \Gamma$ . This is the extension of  $\pi_1(M)$  by  $\mathbb{R}/H$  that appears in the statement of the theorem. In the case where  $H$  is dense and thus  $\mathbb{R}/H$  is not a Lie group,  $\Gamma = \pi_1(M)$ .

Explicitly, there are three possibilities for  $H$ . Let  $H^+$  be the subset of  $H$  with positive numbers.

**Case 1.** When  $H^+$  is empty,  $H = 0$  and  $\mathbb{R}/H = \mathbb{R}$ . In this case,  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  is an  $\mathbb{R}$ -extension of  $\pi_1(M)$ . (For instance, this occurs for any Riemann surface  $\Sigma_g$  with genus  $g \geq 1$ , since then  $\pi_2(\Sigma_g) = 0$ .)

**Case 2.** When  $H^+$  has a minimal number,  $H \simeq \mathbb{Z}$  and  $\mathbb{R}/H \simeq S^1$ . In this case  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  is an  $S^1$ -extension of  $\pi_1(M)$ . (This occurs for Riemann surface of genus  $g = 0$  and  $\zeta$  is not an exact form.)

**Case 3.** When  $H^+$  is non-empty and has no minimal number,  $H$  is dense in  $\mathbb{R}$  and  $\mathbb{R}/H$  is not a Lie group. In this case,  $\Gamma = \pi_1(M)$ . This case is impossible when  $c\zeta$  is an integral cohomology class for some number  $c \neq 0$ .

We now proceed to prove the classification theorem Theorem 4.1. Our proof will be similar to that given by Morrison [7] for Yang-Mills bundles in the two-dimensional case.

We will first consider the case that  $\mathbb{R}/H$  is a Lie group. By the discussion above,  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  is either an  $\mathbb{R}$  or an  $S^1$  extension of  $\pi_1(M)$ .

**Proof for Case 1 and 2:  $\mathbb{R}/H = \mathbb{R}$  or  $S^1$ .**

**Step 1.** Given a homomorphism  $\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \rightarrow G$ , construct a corresponding principal  $G$ -bundle  $P_\rho$  with a connection  $A_\rho$ .

Let  $\Psi_a(M)$  be the space of classes of paths starting at  $a \in M$ , where any two paths  $\delta_1, \delta_2$  are identified if  $\delta_1(1) = \delta_2(1)$  and  $(\delta_2)^{-1}\delta_1 \in \Psi_{a,\zeta}^a(M)$ . Notice that  $\Psi_a(M)$  is a principal bundle over  $M$ . The projection to the base space  $M$ ,  $\tau : \Psi_a(M) \rightarrow M$ , is given by  $\delta \mapsto \delta(1)$ , and its structure group is  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ , since any element  $\gamma \in \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  can act on  $\delta \in \Psi_a(M)$  on the right by  $\delta \mapsto \delta\gamma$ .

Given a homomorphism  $\rho : \Gamma \rightarrow G$ , we define the principal  $G$ -bundle  $P_\rho$  as an associated  $G$ -bundle to  $\Psi_a(M)$ :

$$P_\rho := \Psi_a(M) \times_\rho G = (\Psi_a(M) \times G) / (\delta\gamma, g) \sim (\delta, \rho(\gamma)g)$$

where  $\gamma \in \Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ . To denote the equivalence class, we use the bracket to denote a point  $u = [\delta, g] \in P_\rho$ . Note that  $G$  acts on  $P_\rho$  by  $[\delta, g]h = [\delta, gh]$  for  $h \in G$ .

To define a connection on a principal bundle, recall that there are two ways to do so. We can define a connection on  $P_\rho$  either as a horizontal subspace,  $H_u \subset (TP_\rho)_u$  at all  $u \in P_\rho$ , or as a one-form  $A_\rho \in \Omega^1(P_\rho, \mathfrak{g})$ . They are related by  $H_u = \ker A_\rho|_u$ . On  $P_\rho$ , we define a horizontal distribution by defining the horizontal lifts for any path  $\alpha$  on  $M$  as follows. Given an arbitrary point  $[\delta, g] \in (P_\rho)_{\alpha(0)}$  on the fiber of base point  $\alpha(0) = \delta(1) \in M$ , we define the horizontal lift of  $\alpha$  starting from  $[\delta, g] \in P_\rho$  by  $\tilde{\alpha}(t) = [\alpha_{(t)}\delta, g]$  where  $\alpha_{(t)}$  denotes the one-parameter family of paths within  $\alpha$  starting at  $\alpha(0)$  and ending at  $\alpha(t)$ . It is straightforward to check that this construction is well-defined and satisfies the  $G$  invariance condition for a connection on  $P_\rho$ . We will not need to explicitly write down  $A_\rho$  as a one-form in order to check that  $A_\rho$  satisfies the cone-flat condition. We will instead express the curvature in terms of the holonomy group.

**Step 2.** Verify that the connection  $A_\rho$  is cone-flat, that is,  $F_{A_\rho} = -\zeta B_\rho$  such that  $B_\rho$  is covariantly constant.

**Lemma 4.2.** *The connection  $A_\rho$  constructed above is cone-flat.*

*Proof.* Let  $p \in M$  be an arbitrary point and  $v_1, v_2 \in T_p M$  be arbitrary linearly independent vectors. Suppose  $\zeta(v_1, v_2) = c \neq 0$ . By Darboux's theorem, we can find a local coordinate system  $\{x_1, \dots, x_m\}$  such that  $v_1 = \frac{\partial}{\partial x_1}$ ,  $v_2 = \frac{\partial}{\partial x_2}$ , and  $\zeta = c dx_1 \wedge dx_2 + \bar{\zeta}$ , where  $\bar{\zeta} = dx_3 \wedge dx_4 + \dots$ , if  $\dim M \geq 4$ . Let  $D_t$  be an infinitesimal parallelogram spanned by  $\sqrt{t}v_1$  and  $\sqrt{t}v_2$  in the local coordinate system, and  $\gamma_t = \partial D_t$  be its boundary. Then we have  $\int_{D_t} \zeta = ct$ .

At arbitrary  $[\delta, g] \in P_\rho$  on the fiber of  $p$ , let  $v_1^H$  and  $v_2^H$  be the horizontal lift of  $v_1$  and  $v_2$ , respectively. Then the curvature

$$F_{A_\rho}(v_1^H, v_2^H) = \frac{\partial}{\partial t} \text{hol}(\gamma_t) \Big|_{t=0}.$$

Here,  $\text{hol}(\gamma_t) \in G$  denotes the holonomy along  $\gamma_t$  at  $[\delta, g]$ .

On the other hand,

$$[\delta, g]\text{hol}(\gamma_t) = [\gamma_t \delta, g] = [\delta(\delta^{-1}\gamma_t \delta), g] = [\delta, g(g^{-1}\rho(\delta^{-1}\gamma_t \delta)g)].$$

This implies

$$\text{hol}(\gamma_t) = g^{-1}\rho(\delta^{-1}\gamma_t \delta)g.$$

Note that  $\delta^{-1}\gamma_t \delta$  is contractible and its base point is  $a \in M$ , i.e.  $\delta^{-1}\gamma_t \delta \in \Psi_{a,0}^a(M)$ . We can express  $\delta^{-1}\gamma_t \delta = \exp(ct\xi)$ , where  $\xi$  is an element of the Lie algebra of  $\Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  that generates the subgroup  $\Psi_{a,0}^a(M)/\Psi_{a,\zeta}^a(M) \cong \mathbb{R}/H$  and  $\int_{D_t} \zeta = \zeta(v_1, v_2)t = ct \in \mathbb{R}/H$ . Note that  $\xi$  is independent of the choice of  $v_1$  and  $v_2$ . We thus find at  $p \in M$

$$F_{A_\rho}(v_1^H, v_2^H) = \frac{\partial}{\partial t} \text{hol}(\gamma_t) \Big|_{t=0} = c \text{Ad}_{g^{-1}} d\rho(\xi) = \zeta(v_1, v_2) \text{Ad}_{g^{-1}} d\rho(\xi) = -\zeta(v_1, v_2) B_\rho,$$

where  $d\rho$  maps the Lie algebra of  $\Gamma$  into  $\mathfrak{g} = \text{Lie}(G)$ . It is clear that  $B_\rho$  as obtained above is covariantly constant as it is a constant when evaluated along horizontally lifted curves.

Thus far, we have assumed  $\zeta(v_1, v_2) \neq 0$ . If however  $\zeta(v_1, v_2) = 0$  which may occur in  $\dim M \geq 4$ , then we can find a local coordinate system  $\{x_1, \dots, x_m\}$  such that  $v_1 = \frac{\partial}{\partial x_1}$ ,  $v_2 = \frac{\partial}{\partial x_3}$ , and  $\zeta = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \bar{\zeta}$ , where  $\bar{\zeta}$ , if non-zero, is generated by other  $dx_i \wedge dx_j$  locally. The proof goes through as above but with  $c$  set to zero.  $\square$

**Step 3.** Show that the morphism  $\rho \mapsto (P_\rho, A_\rho)$  is invariant under conjugation.

**Lemma 4.3.** Suppose  $\rho$  and  $\bar{\rho}$  are conjugate homomorphisms from  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  to  $G$ , and also,  $(P_\rho, A_\rho)$  and  $(P_{\bar{\rho}}, A_{\bar{\rho}})$  are principal bundles with cone-flat connections constructed as above. Then  $(P_\rho, A_\rho)$  and  $(P_{\bar{\rho}}, A_{\bar{\rho}})$  are equivalent.

*Proof.* Suppose  $\rho$  and  $\bar{\rho}$  are conjugate, i.e.  $\bar{\rho} = g_0 \rho g_0^{-1}$  for some  $g_0 \in G$ . We consider the automorphism on  $\Psi_a(M) \times G$  given by  $(\delta, g) \mapsto (\delta, g_0 g)$ . This automorphism induces the desired bundle isomorphism

$$\begin{aligned} f : \quad \Psi_a(M) \times_\rho G &\longrightarrow \quad \Psi_a(M) \times_{\bar{\rho}} G \\ (\delta\gamma, g) \sim (\delta, \rho(\gamma)g) &\mapsto (\delta\gamma, g_0g) \sim (\delta, \bar{\rho}(\gamma)g_0g) = (\delta, g_0\rho(\gamma)g) \end{aligned}$$

where the second line shows the map on the equivalence class for all  $\gamma \in \Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ .

Now to show  $f^*A_{\bar{\rho}} = A_\rho$ , let  $[\delta, g]_\rho \in \Psi_a(M) \times_\rho G$  and  $\alpha$  on  $M$  be an arbitrary path starting at  $\alpha(0) = \delta(1)$ . As described in Step 1 in defining  $A_\rho$ , the horizontal lift of  $\alpha$  starting at  $[\delta, g]_\rho$  is defined as  $\tilde{\alpha}_\rho(t) = [\alpha_{(t)}\delta, g]_\rho$ . Likewise, the horizontal lift of  $\alpha$  starting at  $f([\delta, g]_\rho) = [\delta, g_0g]_{\bar{\rho}}$  is defined as  $\tilde{\alpha}_{\bar{\rho}}(t) = [\alpha_{(t)}\delta, g_0g]_{\bar{\rho}}$ . Clearly, we have  $\tilde{\alpha}_{\bar{\rho}}(t) = f \circ \tilde{\alpha}_\rho(t)$ , which implies  $f_*$  sends horizontal vectors to horizontal vectors as desired.  $\square$

By this lemma, given any conjugacy classes  $[\rho]$  of the homomorphisms from  $\Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  to  $G$ , there is a corresponding  $G$ -bundle  $P_\rho$  with a cone-flat connection  $A_\rho$ . It remains to show that this correspondence is bijective.

**Step 4.** The morphism  $[\rho] \mapsto (P_\rho, A_\rho)$  is injective.

**Lemma 4.4.** Let  $\rho, \bar{\rho} : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \rightarrow G$ . If  $(P_\rho, A_\rho)$  and  $(P_{\bar{\rho}}, A_{\bar{\rho}})$  are equivalent, then  $\rho$  and  $\bar{\rho}$  are conjugate.

*Proof.* Suppose  $f : P_\rho \rightarrow P_{\bar{\rho}}$  is a  $G$ -bundle isomorphism and  $f^*A_{\bar{\rho}} = A_\rho$ . Let us define  $h : \Psi_a(M) \rightarrow G$  such that

$$f([\delta, e]_\rho) = [\delta, e]_{\bar{\rho}}h(\delta)$$

where  $\delta \in \Psi_a(M)$  and  $e$  is the identity of  $G$ . Then, for each  $\gamma \in \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ , we have

$$\begin{aligned} [\delta, e]_{\bar{\rho}}h(\delta)\rho(\gamma) &= f([\delta, e]_\rho)\rho(\gamma) = f([\delta, e]_\rho\rho(\gamma)) \\ &= f([\delta\gamma, e]_\rho) = [\delta\gamma, e]_{\bar{\rho}}h(\delta\gamma) = [\delta, e]_{\bar{\rho}}\bar{\rho}(\gamma)h(\delta\gamma). \end{aligned}$$

This implies  $\rho(\gamma) = h(\delta)^{-1}\bar{\rho}(\gamma)h(\delta\gamma)$ . We will prove that  $h$  is a constant.

For arbitrary  $\delta \in \Psi_a(M)$ , let  $\delta_{(t)}$  denote the  $t$ -parametrized subpaths of  $\delta$  starting at  $\delta(0) = a$  and ending at  $\delta(t)$ . By the construction of the connections,  $[\delta_{(t)}, e]_\rho$  and  $[\delta_{(t)}, e]_{\bar{\rho}}$  are horizontal paths on  $P_\rho$  and  $P_{\bar{\rho}}$ , respectively. The projection of these two paths on  $M$  are exactly  $\delta_{(t)}$ . By the definition of  $h$ , we have

$$f([\delta_{(t)}, e]_\rho) = [\delta_{(t)}, e]_{\bar{\rho}}h(\delta_{(t)}). \quad (4.4)$$

On the other hand, since  $f^*A_{\bar{\rho}} = A_\rho$ ,  $f([\delta_{(t)}, e]_\rho)$  is a horizontal lift of  $\delta_{(t)}$  on  $P_{\bar{\rho}}$  passing through  $f([\delta_{(0)}, e]_\rho)$ . As horizontal subspaces are invariant under right  $G$ -action,  $[\delta_{(t)}, e]_{\bar{\rho}}h(\delta_{(0)})$  is also a horizontal lift of  $\delta_{(t)}$  on  $P_{\bar{\rho}}$  passing through  $[\delta_{(0)}, e]_{\bar{\rho}}h(\delta_{(0)}) = f([\delta_{(0)}, e]_\rho)$ . Therefore, we have

$$f([\delta_{(t)}, e]_\rho) = [\delta_{(t)}, e]_{\bar{\rho}}h(\delta_{(0)}). \quad (4.5)$$

Comparing (4.4)-(4.5), we have  $h(\delta_{(t)}) = h(\delta_{(0)})$  for all  $t \in [0, 1]$ . This implies  $h(\delta_{(t)})$  is equal to  $h$  acting on the constant path at  $a$ . So  $h$  is a constant, and therefore,  $\rho$  and  $\bar{\rho}$  are conjugate.  $\square$

**Step 5.** The morphism  $[\rho] \mapsto (P_\rho, A_\rho)$  is surjective.

The following lemma leads to surjectivity. It also holds when  $\mathbb{R}/H$  is not a Lie group, which we will discuss later.

**Lemma 4.5.** *Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle, and let  $A$  be a cone-flat connection on  $P$  with curvature  $F_A = -\zeta B$ . Then there exists  $\xi \in \mathfrak{g}$  such that for any contractible loop  $\gamma$  starting at  $a \in M$  and any oriented disk  $D$  in  $M$  with  $\partial D = \gamma$  ( $\partial D$  and  $\gamma$  also have the same orientation), the holonomy along  $\gamma$  is*

$$\text{hol}(\gamma) = \exp\left[\left(\int_D \zeta\right)\xi\right].$$

*Proof.* Take a point  $u_0 \in P$  on the fiber of  $a \in M$ . Consider the holonomy bundle

$$\hat{P} = \{u \in P \mid \text{there exists a horizontal curve } \tilde{\delta} \text{ such that } \tilde{\delta}(0) = u_0, \tilde{\delta}(1) = u\}.$$

$\hat{P}$  is a principal bundle over  $M$ , and its structure group  $\hat{G}$  is the holonomy group of  $P$  at  $u_0$ . Let  $\hat{A}$  be the restriction of  $A$  on  $\hat{P}$ . By the holonomy theorem of Ambrose-Singer [1], the Lie algebra of  $\hat{G}$  is

$$\hat{\mathfrak{g}} = \text{span} \left\{ F_{\hat{A}}(v_1^H, v_2^H) \mid v_1^H, v_2^H \text{ are horizontal vectors at } u \text{ for some } u \in \hat{P} \right\}.$$

By assumption,  $F_A = -\zeta B$ , or more precisely  $F_A = -(\pi^*\zeta)B$ . Since  $B$  is covariantly constant, it is equal to some  $\xi \in \hat{\mathfrak{g}}$  at any point in  $\hat{P}$ . Hence,  $F_{\hat{A}}(v_1^H, v_2^H) \in \mathbb{R}\xi$ . So  $\hat{\mathfrak{g}}$  is 1-dimensional and abelian. Then  $\hat{A} = -\theta \otimes \xi$  for some  $\theta \in \Omega^1(\hat{P})$  and  $d\theta = \hat{\pi}^*\zeta$ , where  $\hat{\pi} : \hat{P} \rightarrow M$  is the projection.

For an arbitrary contractible loop  $\gamma$  and a disk  $D$  such that  $\partial D = \gamma$ , there exists a contractible neighborhood  $U \subset M$  of  $D$ . Then  $\hat{P}|_U = U \times \hat{G}$  is trivial. Let  $\sigma : U \rightarrow \hat{P}|_U$  be a local section and  $\psi : \hat{P}|_U \rightarrow U \times \hat{G}$  be a trivialization such that  $\psi \circ \sigma(p) = (p, e)$  for  $p \in M$ . Then  $(\psi^{-1})^*\hat{A} = (-\sigma^*\theta \otimes \xi, 0) + (0, MC_{\hat{G}})$ , where  $MC_{\hat{G}} : T\hat{G} \rightarrow \hat{\mathfrak{g}}$  is the Maurer-Cartan form of  $\hat{G}$  sending a vector to the corresponding invariant vector field. Observe that  $d(\sigma^*\theta) = \zeta$ . The horizontal lift  $\tilde{\gamma}$  of  $\gamma$  with  $\tilde{\gamma}(0) = \sigma(a)$  satisfies  $\psi \circ \tilde{\gamma}(t) = (\gamma(t), g(t))$  with  $g(t) \in \hat{G}$  and  $g(0) = e$ . With the horizontal vectors in the kernel space of the connection one-form, we have

$$\begin{aligned} 0 &= \hat{A}(\tilde{\gamma}'(t)) \Big|_{\gamma(t_0)} = (\psi^{-1*}\hat{A})(\gamma'(t), 0) \Big|_{(\gamma(t_0), g(t_0))} + (\psi^{-1*}\hat{A})(0, g'(t)) \Big|_{(\gamma(t_0), g(t_0))} \\ &= -(\sigma^*\theta)(\gamma'(t)) \cdot \xi \Big|_{\gamma(t_0)} + MC_{\hat{G}}(g'(t)) \Big|_{g(t_0)}. \end{aligned}$$

Hence, we find

$$g(t_0) = \exp \left[ \left( \int_0^{t_0} (\sigma^*\theta)(\gamma'(t)) dt \right) \xi \right],$$

and the holonomy along  $\gamma$  is

$$hol(\gamma) = g(1) = \exp \left[ \left( \int_\gamma \sigma^*\theta \right) \xi \right] = \exp \left[ \left( \int_D \zeta \right) \xi \right].$$

□

Now we proceed to prove surjectivity.

**Lemma 4.6.** *Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle with a cone-flat connection  $A$  with respect to  $\zeta$ . Then there exists a homomorphism  $\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \rightarrow G$  such that  $(P_\rho, A_\rho)$  and  $(P, A)$  are equivalent.*

*Proof.* For any contractible loop  $\gamma \in \Psi_{a,\zeta}^a(M)$  and a disk  $D$  such that  $\partial D = \gamma$ , it follows from the definition of  $\Psi_{a,\zeta}^a(M)$  that  $\int_D \zeta = 0$ . By Lemma 4.5, the holonomy along  $\gamma$  in  $(P, A)$  is given by  $hol(\gamma) = \exp \left[ \left( \int_D \zeta \right) \xi \right] = e$ . Hence, we define the following homomorphism:

$$\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \rightarrow G, \quad \gamma \mapsto \rho(\gamma) = hol(\gamma). \quad (4.6)$$

We will show that the resulting  $(P_\rho, A_\rho)$  is equivalent to  $(P, A)$ . We start by defining a  $G$ -bundle isomorphism between the two principal bundles.

Given  $[\delta, g] \in P_\rho$ , let  $\tilde{\delta}$  be the horizontal lift of  $\delta$  in  $P$  with  $\tilde{\delta}(0) = u_0$ . We define

$$f : P_\rho \rightarrow P, \quad [\delta, g] \mapsto \tilde{\delta}(1) g. \quad (4.7)$$

Note that the definition of  $f$  utilizes the horizontal lift determined by the connection  $A$  in  $P$ . Let us show that this map is well-defined. Suppose  $[\delta_1, g_1]$  and  $[\delta_2, g_2]$  represent the same class in  $P_\rho$ . Then there must exist a  $\gamma_0 \in \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  such that

$$(\delta_1, g_1) = (\delta_2 \gamma_0, \rho(\gamma_0^{-1}) g_2) \sim (\delta_2, g_2). \quad (4.8)$$

Thus,  $\gamma_0^{-1} \delta_2^{-1} \delta_1 \in \Psi_{a,\zeta}^a(M)$ , and its holonomy in  $(P, A)$  is trivial according to Lemma 4.5. Moreover, we have

$$\text{hol}(\delta_2^{-1} \delta_1) = \text{hol}(\gamma_0) = \rho(\gamma_0), \quad (4.9)$$

where the last equality follows from our definition of  $\rho$  in (4.6). Now let  $\tilde{\delta}_1$  and  $\tilde{\delta}_2$  denote the horizontal lift in  $(P, A)$  of  $\delta_1$  and  $\delta_2$ , respectively, starting at  $u_0$ . Since  $\tilde{\delta}_1$  and the horizontal lift of  $\delta_2 \gamma_0 = \delta_2(\delta_2^{-1} \delta_1)$  starting at  $u_0$  have the same ending point, so does  $\tilde{\delta}_1$  and the horizontal lift of  $\delta_2$  starting at  $u_0 \cdot \text{hol}(\gamma_0)$ . Hence, the ending points of  $\tilde{\delta}_1$  and  $\tilde{\delta}_2$  satisfy  $\tilde{\delta}_1(1) = \tilde{\delta}_2(1) \cdot \text{hol}(\gamma_0)$ . Together with (4.7)-(4.9), we find

$$f([\delta_1, g_1]) = \tilde{\delta}_1(1) g_1 = \tilde{\delta}_2(1) \rho(\gamma_0) g_1 = \tilde{\delta}_2(1) g_2 = f([\delta_2, g_2]).$$

Hence,  $f$  is well-defined. It is also straightforward to check that  $f$  is a  $G$ -bundle isomorphism.

Finally, we check that the definition of  $A_\rho$  described earlier in Step 1 is consistent with  $f^*A = A_\rho$ . For  $[\delta, g] \in P_\rho$  and an arbitrary path  $\alpha$  on  $M$  such that  $\alpha(0) = \delta(1)$ , the horizontal lift of  $\alpha$  at  $[\delta, g]$  is  $\tilde{\alpha}(t) = [\alpha_{(t)} \delta, g]$ . Here again,  $\alpha_{(t)}$  is the subpath of  $\alpha$  starting at  $\alpha(0)$  and ending at  $\alpha(t)$ . Notice that  $f([\alpha_{(t)} \delta, g]) = (\widetilde{\alpha_{(t)} \delta})(1) g$  is the ending point of the path on  $P$  that is the horizontal lift of  $\alpha_{(t)}$  in  $(P, A)$  starting at  $f([\delta, g]) = \tilde{\delta}(1) g$ . Clearly then,  $f \circ \tilde{\alpha}$  is a horizontal path on  $(P, A)$ . Hence,  $f_*$  sends horizontal vectors to horizontal vectors, and therefore,  $f^*(A) = A_\rho$ .  $\square$

Combining the lemmas above, we have proved that  $[\rho] \mapsto (P_\rho, A_\rho)$  is an isomorphism.

### Proof for Case 3: $\mathbb{R}/H$ is not a Lie group

We now turn to the case when  $\mathbb{R}/H$  is not a Lie group. In this case,  $H^+$  is non-empty and has no minimal number.

**Lemma 4.7.** *When  $H^+$  is non-empty and has no minimal number, a cone-flat connection is a flat connection.*

*Proof.* By Lemma 4.5, there exists some  $\xi \in \mathfrak{g}$  such that for any contractible loop  $\gamma$  and disk  $D$  with  $\partial D = \gamma$ , we have  $hol(\gamma) = \exp\left[\left(\int_D \zeta\right)\xi\right]$ .

Let us show that  $\xi$  must be zero. For if  $\xi \neq 0$ , then there exists some small enough  $t_0$  such that for any  $0 < t < t_0$ ,  $\exp(t\xi) \neq e$  the identity element of  $G$ . Now since  $H^+$  has no minimal number, there exists a closed sphere  $S \subset M$  such that  $\int_S \zeta = t$  for some  $0 < t < t_0$ . Let  $\gamma$  be a constant loop at some point  $p \in S$ , and  $D = S \setminus \{p\}$ . Then  $\partial D = \gamma$  so that  $hol(\gamma) = \exp\left[\left(\int_D \zeta\right)\xi\right]$ . But  $hol(\gamma) = e$  as  $\gamma$  is the identity loop. On the other hand,  $\int_D \zeta = \int_S \zeta = t$ , and therefore,  $\exp(t\xi) = e$ , which gives a contradiction. Thus, we conclude that  $\xi$  is zero.

With  $\xi = 0$ , the holonomy of any contractible loop is trivial. Hence, the curvature vanishes and the connection is flat.  $\square$

The classification of  $G$ -bundles with flat connections can be represented by the conjugacy classes of homomorphisms from  $\pi_1(M) \rightarrow G$  (c.f. [6, Theorem 2.9]). So we have proved Theorem 4.1 in this case. This completes the proof of the theorem.

We point out that the classification of cone-flat bundles generally depends on the choice of  $\zeta$ . In Theorem 4.1, the  $\zeta$  dependence explicitly appears in the definition of  $H$  in (4.3). Below, we will work out the classification and demonstrate its dependence on  $\zeta$  in the simple example of  $U(1)$  bundles over the four-dimensional torus  $M = T^4$ .

**Example 4.8.** We describe  $T^4$  as  $\mathbb{R}^4 / \sim$ , with the identification  $(x_1, x_2, x_3, x_4) \sim (x_1 + a, x_2 + b, x_3 + c, x_4 + d)$  where  $a, b, c, d \in \mathbb{Z}$ , and  $U(1)$  as  $\{z \in \mathbb{C} \mid |z| = 1\}$ . Let

$$\zeta = c_1 dx_1 \wedge dx_2 + c_2 dx_3 \wedge dx_4$$

be a closed 2-form with  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ . By Theorem 4.1, the equivalent classes of  $U(1)$  bundles with a cone-flat connection are in 1-1 correspondence with homomorphisms  $\rho : \Gamma \rightarrow U(1)$ , where  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ .

Since  $\pi_2(T^4)$  is trivial,  $\Gamma$  is an  $\mathbb{R}$ -extension of  $\pi_1(T^4)$ . To describe its group structure explicitly, let  $a_i$  for  $i = 1, 2, 3, 4$  be the straight line path in  $\mathbb{R}^4$  starting at the origin and ending at the point where the  $i$ -th coordinate,  $x_i = 1$ , and  $x_j = 0$  for  $j \neq i$ . When projected to  $T^4$ ,  $\{a_1, a_2, a_3, a_4\}$  become the generators of  $\pi_1(T^4)$ . Although  $\pi_1(T^4)$  is abelian, the elements of  $\{a_1, a_2, a_3, a_4\}$  may no longer commute in  $\Gamma$ .

For a contractible loop  $b$ , there is an oriented disk  $D$  such that  $\partial D = b$  and  $\partial D$  has the same orientation as  $b$ . Let  $|b| = \int_D \zeta$ , and we note that  $|b|$  is independent of the choice of  $D$ . Now recall that  $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$  is a quotient of loops by contractible ones whose integral,  $|b| = \int_D \zeta = 0$ . Moreover, contractible loops  $b$  and  $b'$  would represent different classes in  $\Gamma$  if and only if  $|b| \neq |b'|$ . So, a class of contractible loops  $b \in \Gamma$  can be represented by the real number  $|b|$ .

On  $T^4$  then, we can think of  $\Gamma$  as being generated by  $\{a_1, a_2, a_3, a_4, |b|\}$ , with multiplication defined by

$$a_i|b| = |b|a_i \quad |b'b| = |b| + |b'|.$$

However,  $a_j^{-1}a_i^{-1}a_ja_i$  can represent some non-trivial class of a contractible loop. Specifically, we have

$$\begin{aligned} |a_2^{-1}a_1^{-1}a_2a_1| &= c_1, \\ |a_4^{-1}a_3^{-1}a_4a_3| &= c_2, \\ |a_j^{-1}a_i^{-1}a_ja_i| &= 0, \text{ for other } i, j, \end{aligned}$$

Now, the possible homomorphisms of  $\rho : \Gamma \rightarrow U(1)$  is dependent on whether  $\frac{c_1}{c_2} \in \mathbb{Q}$ . Since  $U(1)$  is abelian,

$$\rho(c_1) = \rho(a_2^{-1})\rho(a_1^{-1})\rho(a_2)\rho(a_1) = \rho(a_2)^{-1}\rho(a_2)\rho(a_1)^{-1}\rho(a_1) = 1.$$

Similarly  $\rho(c_2) = 1$ , so  $\rho(pc_1 + qc_2) = 1$  for any  $p, q \in \mathbb{Z}$ .

When  $\frac{c_1}{c_2} \in \mathbb{Q}$ ,  $\{(pc_1 + qc_2) \mid p, q \in \mathbb{Z}\}$  has a minimal positive number  $c_0$ . Then  $\rho(b)$  must have the form  $e^{2\pi i \frac{n|b|}{c_0}}$  for some  $n \in \mathbb{Z}$ . In this case, the Euler class of the circle bundle is  $\frac{n}{c_0}\zeta$ , and the choice of  $\rho(a_i)$  for  $i = 1, \dots, 4$  determines the connection.

When  $\frac{c_1}{c_2} \notin \mathbb{Q}$ ,  $\{(pc_1 + qc_2) \mid p, q \in \mathbb{Z}\}$  is dense in  $\mathbb{R}$ . So  $\rho(b)$  must be 1 for any  $|b| \in \mathbb{R}$ . Thus, the classification is only dependent on  $\rho(a_i)$ , and becomes equivalent to the classification of flat

connections. Actually, the Euler class in this case is  $c\zeta$  for some  $c \in \mathbb{R}$ . But the Euler class is an integral class, and hence, the only possible  $c$  is 0, i.e. every cone-flat connection is flat.

**Remark 4.9.** When  $M$  is simply-connected, the Hurewicz homomorphism  $\pi_2(M) \rightarrow H_2(M, \mathbb{Z})$  is surjective. Assuming  $\zeta$  is not  $d$ -exact,  $H$  is discrete if and only if there exists some non-zero constant  $c$  such that  $c[\zeta] \in H^2(M, \mathbb{Z})$ . Therefore, if such  $c$  exists, then  $\Gamma = \mathbb{R}/\mathbb{Z}$  and the classification of isomorphism classes of cone-flat connections on  $G$ -bundles is given by the conjugacy classes of  $\text{Hom}(S^1, G)$ . If such  $c$  does not exist, then  $\Gamma$  is trivial and then there does not exist non-trivial cone-flat connections.

As an application of this observation, let  $M$  be simply-connected and closed. If  $M$  is also a projective manifold, then we can choose  $\zeta$  to be the Kähler form which is an integral class, and so,  $\zeta \in H^2(M, \mathbb{Z})$ . The isomorphism classes of cone-flat connections on  $G$ -bundles is then given by the conjugacy classes of  $\text{Hom}(S^1, G)$ . If however  $M$  is a non-projective Kähler manifold and we still let  $\zeta$  be the Kähler form, then there is no non-zero constant  $c$  such that  $c[\zeta] \in H^2(M, \mathbb{Z})$  (see, for a reference, [4, Corollary 5.3.3]) and the isomorphism classes of cone-flat connections would be trivial.

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