

# Refining MOND interpolating function and TeVeS Lagrangian

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## ABSTRACT

The phenomena customly called Dark Matter or Modified Newtonian Dynamics (MOND) have been argued by Bekenstein (2004) to be the consequences of a covariant scalar field, controlled by a free function (related to the MOND interpolating function  $\tilde{\mu}(g/a_0)$ ) in its Lagrangian density. In the context of this relativistic MOND theory (TeVeS), we examine critically the interpolating function in the transition zone between weak and strong gravity. Bekenstein's toy model produces too gradually varying  $\tilde{\mu}$  and fits rotation curves less well than the standard MOND interpolating function  $\tilde{\mu}(x) = x/\sqrt{1+x^2}$ . However, the latter varies too sharply and implies an implausible external field effect (EFE). These constraints on opposite sides have not yet excluded TeVeS, but made the zone of acceptable interpolating functions narrower. An acceptable "toy" Lagrangian density function with simple analytical properties is singled out for future studies of TeVeS in galaxies. We also suggest how to extend the model to solar system dynamics and cosmology, and compare with strong lensing data (see also astro-ph/0509590).

*Subject headings:* gravitation - dark matter - galaxy kinematics and dynamics

## 1. Introduction

On galaxy scales, dark matter generally dominates over baryons (stars plus gas) at large radii. At intermediate radii in a galaxy where dark matter and baryons overlap with comparable amounts, the two mass profiles are not uncorrelated (McGaugh 2005). The correlation between the Newtonian gravity of the baryons  $\mathbf{g}_N$  and the overall gravity  $\mathbf{g}$  (baryons plus dark matter) can be loosely parameterized by the Milgrom's (1983) empirical relation

$$\tilde{\mu}(g/a_0) \mathbf{g} = \mathbf{g}_N, \quad (1)$$

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where the interpolating function  $\tilde{\mu}(x)$  is a function which runs smoothly from  $\tilde{\mu}(x) = x$  at  $x \ll 1$  to  $\tilde{\mu}(x) = 1$  at  $x \gg 1$  with a dividing gravity scale  $a_0 \sim 10^{-8} \text{ cm s}^{-2} \sim cH_0/6$  at the transition. This simple correlation was taken as the basis for the MOND theory (or more precisely the quadratic Lagrangian theory) by Bekenstein & Milgrom (1984, hereafter BM84), where one modifies the Newtonian gravity of a baryonic galaxy to eliminate the need for dark matter.

Recently interests on the subject of MOND have been further stimulated since Bekenstein (2004, hereafter B04) provided a Lorentz-covariant theory (dubbed TeVeS), which passes standard tests to check General Relativity, and allows for rigorous modeling of Hubble expansion and grav-

itational lensing (e.g. Zhao et al. 2005). In TeVeS the MONDian behaviour originates from a scalar field, the dynamics of which is controlled by a Lagrangian density involving a free function that yields the expected dynamics in the low-acceleration limit (although BM84 theory is not precisely recovered). This freedom of the Lagrangian density, that echoes the freedom in the choice of the interpolating function  $\tilde{\mu}$  in MOND, means that every choice of the free function defines a distinct theory. As this class of theories do not at present derive from any basic principle and are purely phenomenological, the only constraints on the free function must come from phenomenological grounds. A refinement of the function studied by B04 as a toy model is surely needed. In this letter, we differentiate popular choices of the MOND  $\tilde{\mu}$  function by fitting a benchmark rotation curve, and argue that many of those functions are likely unphysical in the TeVeS context. We then propose a new free function for TeVeS in the domain relevant for galaxies, with a possible extension to solar system dynamics and cosmology.

### 1.1. Warming up to TeVeS

It is a tensor-vector-scalar Lorentz-covariant field theory, where the tensor is the Einstein metric  $g_{\alpha\beta}$  out of which is built the usual Einstein-Hilbert action,  $U_\alpha$  is a dynamical normalized vector field, and  $\phi$  a dynamical scalar field. The action is the sum of the Einstein-Hilbert action for the tensor  $g_{\alpha\beta}$ , the matter action, the action of the vector field  $U_\alpha$ , and the action of the scalar field  $\phi$ . Einstein-like equations are obtained for each of these fields by varying the action w.r.t. each of them. The action of the scalar field  $\phi$  involves a dimensionless parameter  $k$  (of the order of a few percents), a length scale parameter  $l$  ( $\sim \frac{\sqrt{3}kc^2}{4\pi a_0}$ ), and a free dimensionless function linking  $kl^2|\nabla\phi|^2 \propto y$  with the auxiliary nondynamical scalar field  $\mu$ .

More relevant to us, the physical metric in TeVeS near a quasi-static galaxy or the solar system is identical to that of General Relativity, with a potential

$$\Phi = \Xi\Phi_N + \phi \quad (2)$$

where  $\Xi \simeq 1$ . This means that the scalar field  $\phi$  plays the role of the dark matter gravitational potential. It is related to the Newtonian potential  $\Phi_N$  (generated by the baryonic density  $\rho$ ) through

the equation (similar to the field equation for the full  $\Phi$  in BM84)

$$\nabla \cdot [\mu_s \nabla \phi] = \nabla^2 \Phi_N = 4\pi G\rho, \quad (3)$$

where  $\mu_s$  is a function of the scalar field strength  $g_s = |\nabla\phi|$ . It is related the  $\mu$  function of B04 and the interpolating function  $\tilde{\mu}$  of MOND by

$$\mu_s \equiv \frac{\mu}{k'} = \frac{\tilde{\mu}}{1 - \tilde{\mu}}, \quad k' \equiv \frac{k}{4\pi}. \quad (4)$$

In the intermediate to deep-MOND regime, the toy free function in the scalar action of B04 gives rise to the following interpolating function:

$$\tilde{\mu}(x) = \frac{\sqrt{1+4x}-1}{\sqrt{1+4x}+1}. \quad (5)$$

### 2. Rotation Curves

Milgrom's formula (1) is a good approximation of the BM84 and TeVeS theories since for most disk galaxies the curl field is negligible or exactly zero when solving Eq. (3) (Brada & Milgrom 1995). It provides a plausible unified picture for the phenomenology of individual galaxies, which is more challenging to understand in the dark matter framework. However, while the precise functional form of the interpolating function is not necessary to make many fundamental predictions (see Sanders & McGaugh 2002), it is nevertheless important in order to fit the rotation curves of galaxies. Indeed one of the most striking successes of MOND is the ability of Milgrom's formula to fit the rotation curves of a wide range of galaxies with virtually the same value of  $a_0 = 1.2 \times 10^{-8} \text{ cm s}^{-2}$ , and the same “standard” interpolating function:

$$\tilde{\mu}(x) = \frac{x}{\sqrt{1+x^2}}. \quad (6)$$

This “standard” interpolating function was originally put in by hand, and does not derive from any physical principle. However, with the number of galaxies with good data increasing, the freedom of this function should be restrained by the observations. Famaey & Binney (2005, hereafter FB05) have e.g. found that the “simple” interpolating function,

$$\tilde{\mu}(x) = \frac{x}{1+x}, \quad (7)$$

gives a better fit to the terminal velocity curve of the Milky Way than Eq. (6), while yielding an

extremely good fit to the rotation curve of the standard external galaxy NGC3198 (cf. Fig. 1). In comparison the toy model in B04 gives rise to Eq. (5) in spherical symmetry in the intermediate to deep-MOND regime. This interpolating function triggers too slow a transition from the MONDian to the Newtonian regime in the benchmark rotation curve of NGC3198 (cf. Fig. 1); the same conclusion is reached for the terminal velocity curve of the Milky Way (FB05).

In short, from the analysis of rotation curves, Eq. (6) and Eq. (7) are preferred over Eq. (5). FB05 noted that Eq. (7) is preferred over Eq. (6) to fit the TVC of the Milky Way. They also derived the best MONDian fit (in the strong and intermediate regimes only) of the Milky Way, and found the interpolating function to transition smoothly from Eq. (7) at  $x \leq 1$  to Eq. (6) at  $x \geq 10$ . Of course, it is not obvious that all the other galaxies will give the same answer as the Milky Way and NGC3198. The observational constraints presented in this section should thus be considered as an indication more than a rigorous constraint.

### 3. The external field effect

In TeVeS the potential consists of two parts (Eq. 2): the Newtonian potential, and the scalar field which satisfies the BM84 formulation (Eq. 3). One of the key features of the BM84 theory is the existence of the dilation effect of an external field, which is why MOND does not satisfy the strong equivalence principle. Consider the perturbation generated by a low-mass  $m$  body inside a dominating uniform external scalar field strength  $g_s^e \hat{z}$ , Eq. (3) for  $\phi$  becomes

$$\nabla \cdot [\mu_s \nabla \phi] \approx \mu_s^e (\Delta_1 \partial_z^2 + \partial_x^2 + \partial_y^2) \phi = 4\pi G m \delta(r), \quad (8)$$

which differs from the Newtonian Poisson equation by the dilation factor  $\Delta_1$  given by (BM84; Zhao 2005; Zhao & Tian 2005)

$$\Delta_1 = 1 + \frac{d \ln \mu_s^e}{d \ln g_s^e}, \quad \mu_s^e \equiv \mu_s(g_s^e). \quad (9)$$

As in BM84, the solution is an equal potential surface given by

$$\phi(x, y, z) = -g_s^e z - \frac{Gm}{\mu_s^e \tilde{r}}, \quad \tilde{r} = \sqrt{z^2 + \Delta_1 (y^2 + x^2)}, \quad (10)$$

where  $m/\mu_s^e$  is the effective mass of the satellite and  $\tilde{r}$  is the effective distance from the centre of the satellite. Hence the perturbation scalar field is stretched in the  $z$  direction by a factor  $1/\sqrt{\Delta_1}$ . To exclude an imaginary dilation, a theory should have

$$\Delta_1 = \frac{d \ln \mu_s g_s}{d \ln g_s} > 0. \quad (11)$$

This requires models where  $g_s$  is an increasing function of  $g_N = \mu_s g_s$ . The factor  $\Delta_1$  also determines the shape of the Roche Lobe of a rotating satellite, which have a middle-to-long axis ratio  $\sqrt{\frac{2}{3\Delta_1}}$  (Zhao 2005; Zhao & Tian 2005). In fact the stretching factor  $\Delta_1^{-1/2}$  enters almost all processes involving satellites (e.g. Brada & Milgrom 2000). For these reasons we consider models with a negative  $\Delta_1$  unphysical. In the original BM84 theory, this is not a problem because  $1 < \Delta_1 < 2$  thanks to a monotonically increasing  $\tilde{\mu}(g)$  (not  $\mu_s(g_s)$  as we are concerned with), so the external field effect is a mild curiosity. This, however, is not the case if the MOND effect is created by a scalar field. From Eq. (4), we have that models with the standard  $\tilde{\mu}$  function (Eq. 6) yield a  $g_s$  which increases with  $g_N$  to some point, and then starts decreasing in the intermediate regime (see Fig. 2). At the same scalar field strength  $g_s$ , there are two different (spherical) Newtonian gravity strength, i.e., the scalar function  $\mu_s(g_s)$  becomes multi-valued for the same  $g_s$ , hence is ill-defined.<sup>1</sup> This is a general feature of any sharply increasing MOND  $\tilde{\mu}$  function (e.g., the “exponential” function  $\tilde{\mu} = 1 - \exp(-x)$ , and the best fit of FB05). This type of behaviour is undesirable in a physical model for the scalar field. On the other hand the simple function (Eq. 7) and B04 toy model (Eq. 5) give physical monotonic  $\mu_s(g_s)$ , hence positive  $\Delta_1$ .

### 4. New free function

As a toy model B04 chose for the  $\mu$ -function (see Eq. (4) above) the implicit, discontinuous formulation:

$$\frac{y(\mu)}{3k'^2} \equiv \frac{h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}}{a_0^2} = \frac{\mu^2}{k'^2} \frac{(\mu - 2)^2}{4(1 - \mu)}, \quad (12)$$

<sup>1</sup>The  $\mu_s(s)$  is given by two root branches of an essentially 4-th order polynomial equation  $s = \frac{\mu_s}{(1+\mu_s)\sqrt{1+2\mu_s}}$ , and is a very lengthy multi-valued function.

where notations are as in B04. Here cosmology ( $\mu > 2$ ) and galaxies ( $0 < \mu < 1$ ) are completely detached from each other, while the fit to galactic rotation curves is poor.

Our aim here is to propose a new *explicit, monotonic and continuous* interpolating function  $\mu_s(s) = \frac{\mu}{k'}$ . A simple choice could be

$$s \equiv \frac{g_s}{a_0} \equiv \sqrt{\frac{|y|}{3k'^2}} = \frac{\mu}{(k' + \mu)(1 - \mu)^n}, \quad n = 0, 1. \quad (13)$$

Here  $y > 0$  and  $0 < \mu < 1$  corresponds to quasi-static systems as in B04, and  $y < 0$  to cosmology. This toy function is consistent with rotation curves, since it reduces to the simple interpolating function Eq. (7) in the range for galaxies, i.e.,  $\mu = k'\mu_s \sim (0 - 10)k' \ll 1$ , where we have the explicit relations

$$\mu_s \approx \frac{g_s}{a_0 - g_s}, \quad s \approx \frac{\mu_s}{1 + \mu_s}, \quad (14)$$

which are ready to be fed into a solver for Eq. (3).

#### 4.1. Solar system

Our toy function also recovers solar system dynamics at a level similar to B04, if not better. For example, in  $n = 1$  case (the default case), the scalar field goes to infinity when  $\mu$  approaches one. This new “toy” function is easier to use than B04 because it allows for an analytic formulation for the scalar field interpolating function  $\mu_s(s) \equiv \frac{\mu}{k'} = \frac{\tilde{\mu}}{1 - \mu}$  in both strong and weak gravity, namely

$$k'\mu_s(s) = \frac{1 - k'}{2} + \frac{-1 + \sqrt{[(1 + k')s + 1]^2 - 4s}}{2s}. \quad (15)$$

The dilation factor  $\Delta_1 = 1 + \frac{d \ln \mu_s}{d \ln s} = \frac{2 + (1 - k')\mu_s}{1 + k'\mu_s^2} \sim (1 - 4)$  is in the plausible range for  $k' = 0.03 - 1$ . As we can see from Fig. 3, the new function yields a monotonically increasing scalar field strength  $g_s$ , and the correction for the solar system dynamics is less than in B04 toy model (when adopting  $k = 0.01$ , the variation from Sun to Saturn is an order smaller), which may or may not be relevant to the Pioneer Anomaly.

#### 4.2. Cosmology

Although the B04 discontinuous cosmological branch of the function  $y(\mu)$  could as well be

appended to our function in the range  $\mu > 2$ , it is more desirable to be in a universe where the Lagrangian density has a smooth transition between weakly gravitating quasi-static systems ( $y \propto (\nabla\phi)^2/a_0^2 > 0$ ) and cosmology ( $y \propto -2(\partial_t\phi)^2 < 0$ ). A possible way, as done here (Eq. 13), is to copy our function in  $y > 0$  regime into the  $y < 0$  regime by a simple mirror-imaging, (cf. Fig. 4). This means that the outer parts of galaxies would connect smoothly into the cosmological expansion as  $y$  passes from  $0+$  to  $0-$ . While a mirror-image with fine tuning is attractive, it is not a necessary condition: any positive continuous function  $\mu(y)$  going through, e.g.,  $\mu(0-) = \mu(0+) \sim 0$  is worth exploring. The next question is whether this kind of cosmology can produce realistic Hubble expansion and the Cosmic Microwave Background.

#### 5. Conclusion

Part of the amazing successes of the non-relativistic version of MOND in explaining galaxy dynamics is due to its “standard” interpolating function  $\tilde{\mu} = x/\sqrt{1 + x^2}$ . When exploring a range of other empirical functions in the context of TeVeS, we see that the fit to rotation curves becomes poorer for more gradual  $\tilde{\mu}$  function, whilst an external field effect with imaginary dilation happens for more rapid changes of the  $\tilde{\mu}$  function. These two independent constraints from opposite sides suggest a fairly narrow range of TeVeS free functions. Among these we propose a simple expression which works for both very weak and very strong gravity (Eq. 15), with a possible extension to cosmology. Unlike the one in B04, the new function has the nice feature that it links quantities of TeVeS and of MOND easily, hence facilitates future examinations using galaxy dynamics and solar system data. The explicit simple monotonic interpolating function  $\mu_s(g_s/a_0)$  could be easily fed into a numerical solver for Eq. (3) (e.g., as developed by Ciotti et al. 2005), and could allow for the modelling of realistic galaxy geometries (the curl-field of BM84, neglected here following conventional wisdom, could be put back with a realistic amount). Combined with a galaxy with sensitive kinematical data, this may confirm or falsify our “toy” function, hence further establish or squeeze the parameter space of the TeVeS theory.

As two final remarks, (i) we note that multiple-imaged gravitational systems present a challenge to all MOND/TeVeS interpolating functions (cf. Fig. 5). Zhao et al. (2005) found that elliptical galaxies of comparable luminosity and redshift show a large scatter in their Einstein ring sizes. Among the previously proposed  $\mu$  functions the B04 toy function is the most effective in lensing, but none of the functions seems to fit all lenses in the point lens case (shaded zone); the fit is poorer with realistic lens mass profile (cf. Fig. 17 of Zhao et al. 2005). (ii) We also note that the dark matter potential is fundamentally different from the scalar field; although the two are sometimes degenerate in fitting rotation curves, there is no equivalent of EFE in dark matter, hence the dark matter potential enjoys more freedom.

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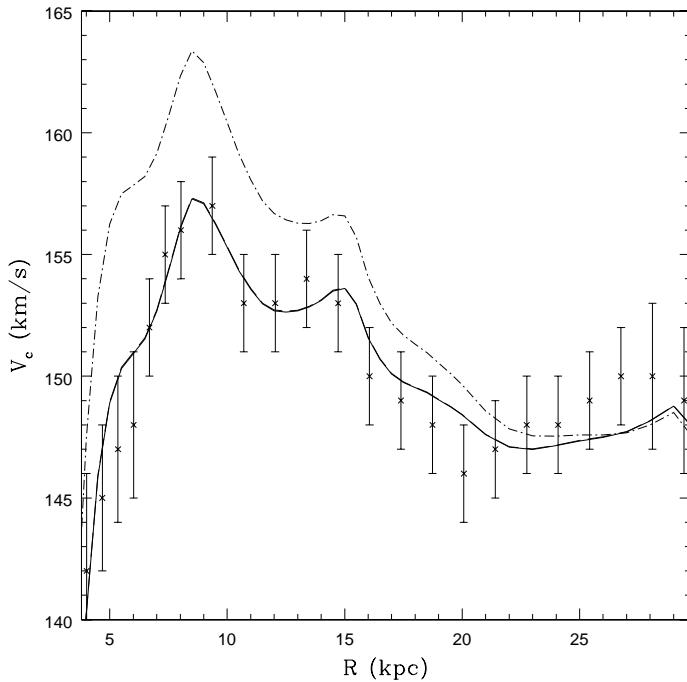
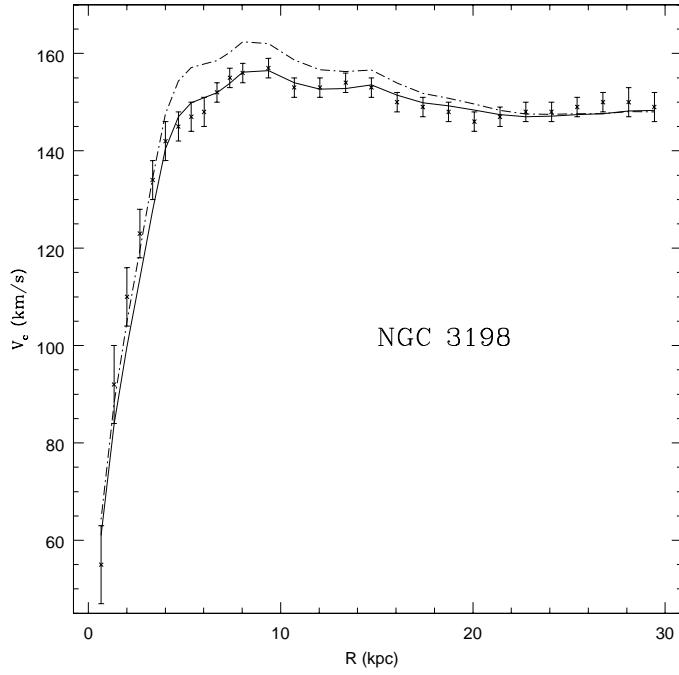


Fig. 1.— Shows the fit to the rotation curve of the “benchmark” galaxy NGC3198 using different  $\tilde{\mu}$ -functions: “simple” (Eq. (8), solid) and Bekenstein “toy” functions (Eq. (6), dashed).

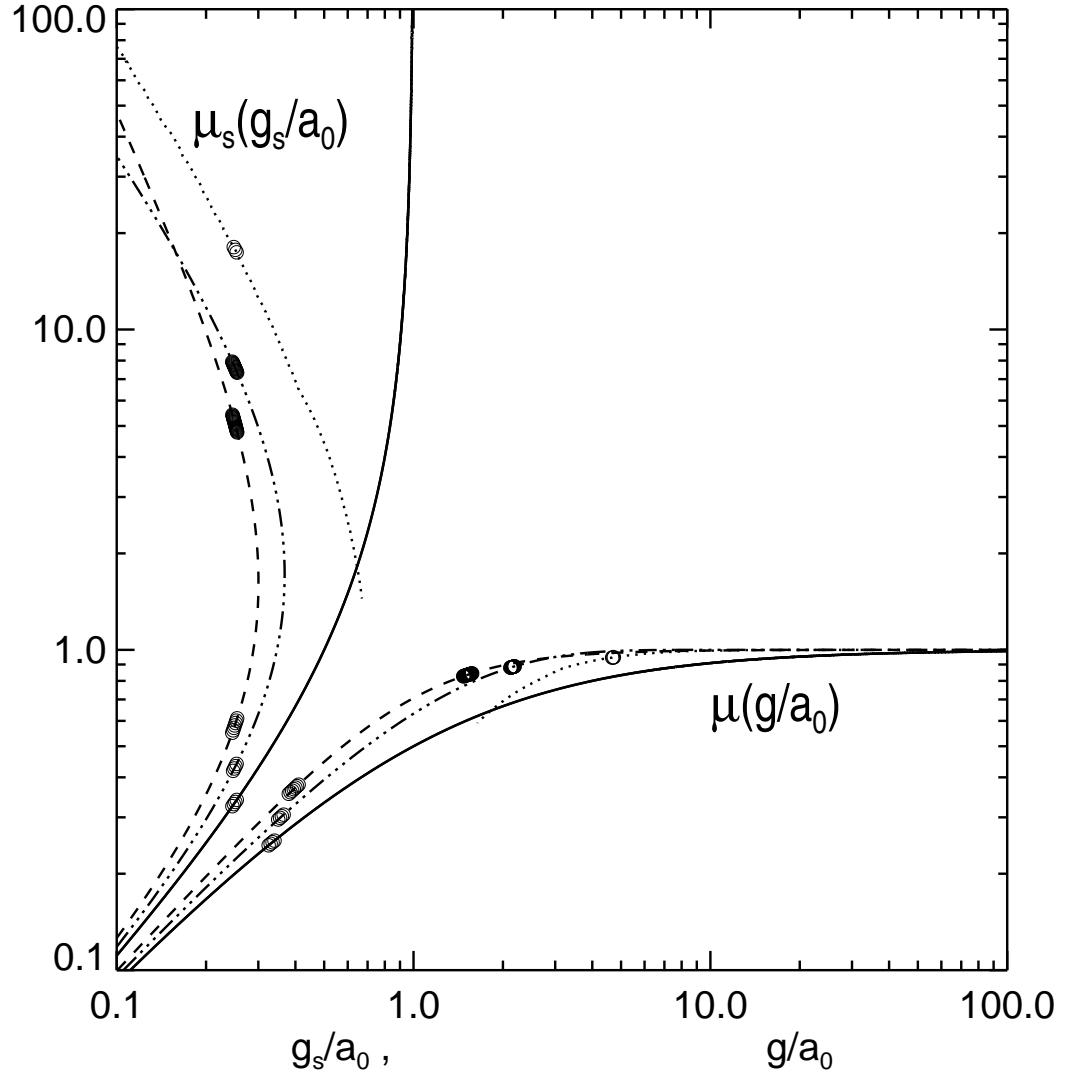


Fig. 2.— shows  $x = g/a_0$  vs.  $\tilde{\mu}(x)$  (lower right) for popular choices: “standard” (Eq. 7, thick dashed), “exponential” (dashed-dotted), “Milky Way” (fit of FB05, dotted) and “simple” (Eq. 8, solid). The corresponding scalar field  $s = g_s/a_0$  vs. its modification function  $\mu_s(s)$  is also shown (upper left). Circles mark where  $s \approx 0.25$ . Note popular  $\tilde{\mu}(x)$  functions often lead to multi-valued  $\mu_s(s)$  curves except for “simple”.

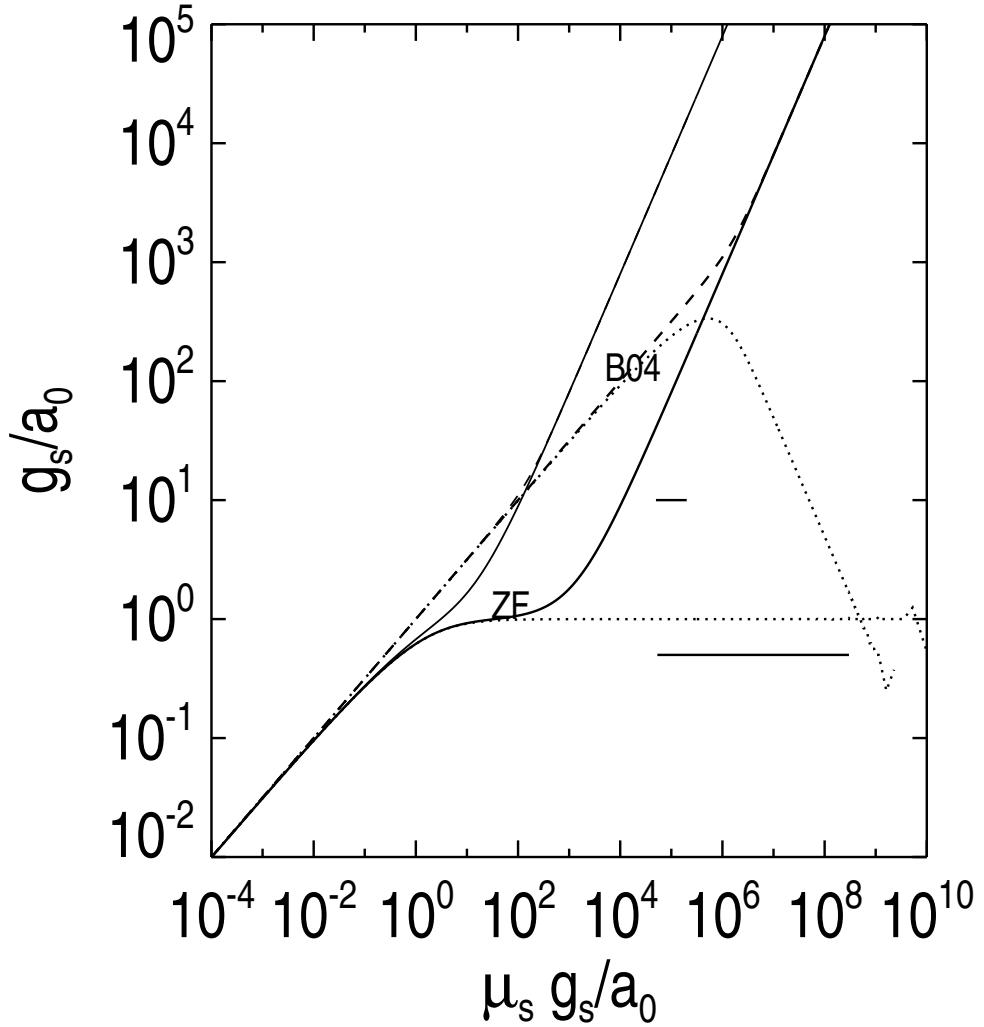


Fig. 3.— Compares our proposed (solid, marked ZF) model with the B04 model (dashed) in the parameter space of the rescaled scalar field strength  $s = g_s/a_0$  vs. Newtonian field strength  $g_N/a_0 \equiv s\mu_s(s)$  in cases of  $k = 4\pi k' = 1$  (thick) and  $k = 0.01$  (thin). Also overplotted is the deviation from the  $\propto r^{-2}$  force in the solar system (flatter dotted line for our model, other dotted line for B04). This should be compared with constraints from planets and the measurement of the Pioneer Anomaly (long and short horizontal thick lines).

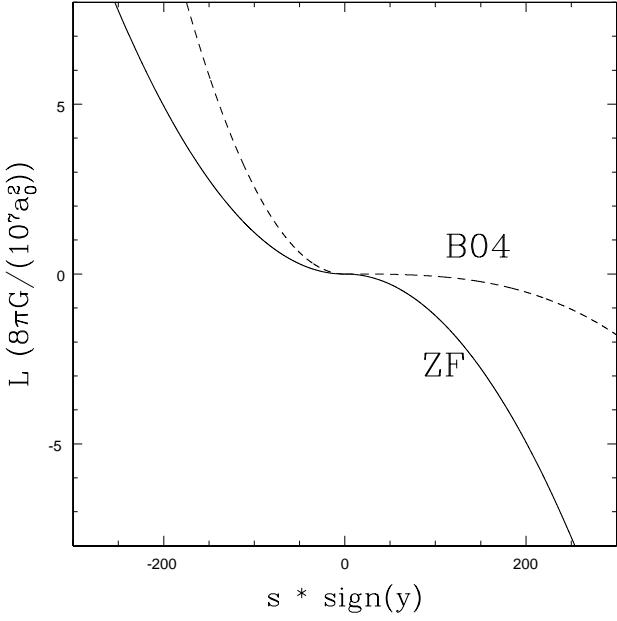
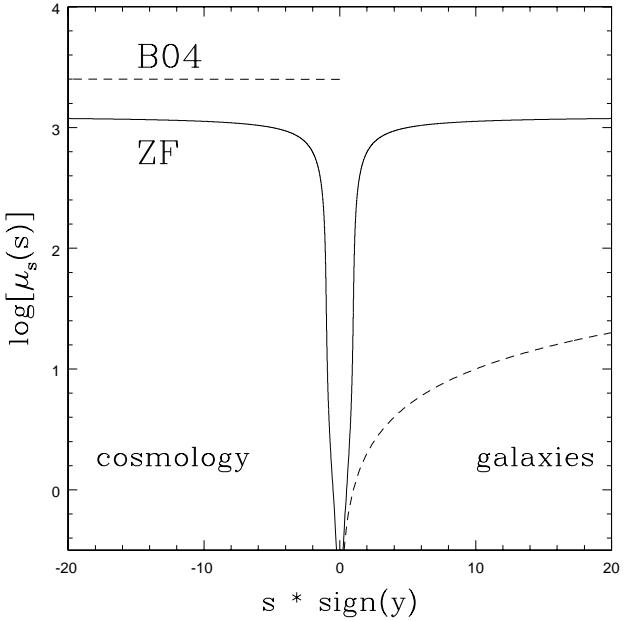


Fig. 4.— The upper panel presents our newly proposed function  $\mu_s(s)$  (Eq.(16), solid line), compared to B04 toy function (dashed). We also present a possible extension to the cosmological regime  $y < 0$  by mirror-imaging (this kind of extension is far from being unique). The lower panel displays the above extention of our new Lagrangian density  $L$  from the weak gravity  $y = 0+$  into cosmology  $y < 0$ . B04 Lagrangian is shown as dashed line for comparison. Note  $y \propto (\nabla\phi)^2/a_0^2$  is large and positive for the solar system, and  $y \propto -(\partial_t\phi)^2$  is negative for cosmology.

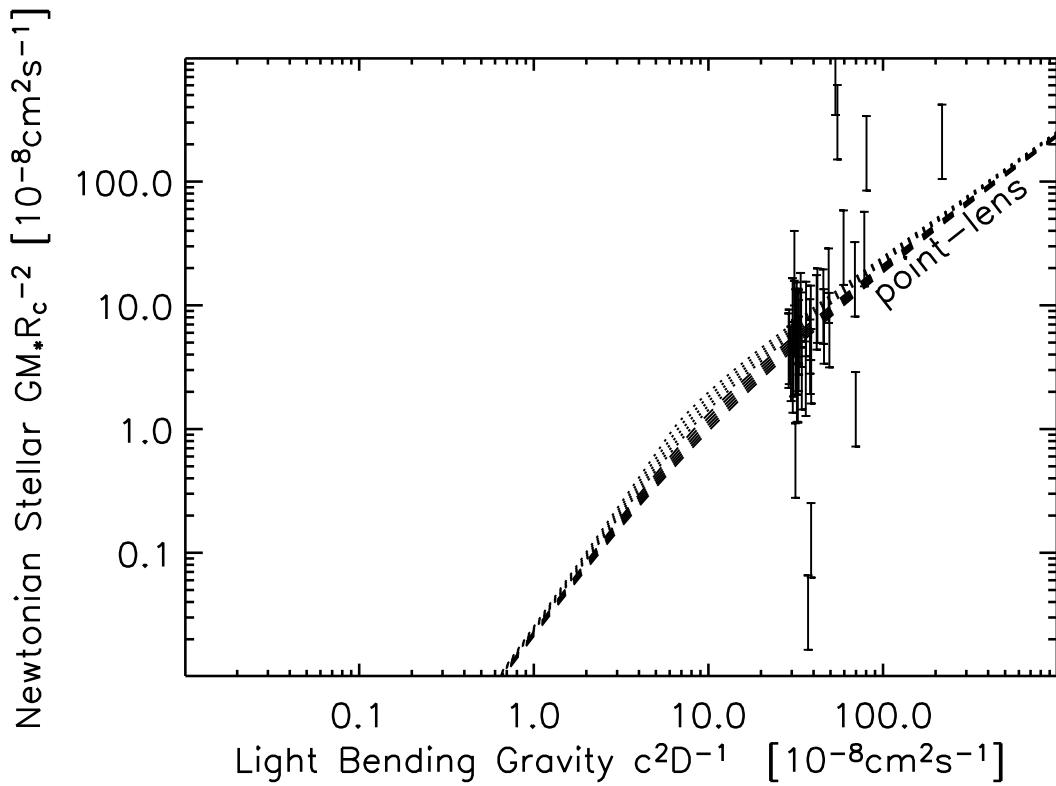


Fig. 5.— Error bars show the stellar Newtonian gravity vs. the light-bending gravity for CASTELS sample of galaxy-quasar lenses (Zhao et al. 2005), where we allow for a factor of two uncertainty with the stellar mass-to-light ratio. Predictions are made for point lenses with a  $\mu$  function in between the B04 toy model (lower boundary of the hatched zone) and a  $\mu$  with a sudden transition (upper boundary); models in the upper hatched zone imply unphysical scalar field. Galaxies would require more mass to bend the light than the point-lens predictions here.