How optimal inhomogeneity produces high temperature superconductivity

Steven A. Kivelson[1,2] and Eduardo Fradkin[3]

[1]*Department of Physics, Stanford University, Stanford CA 93105*

[2]*Department of Physics and Astronomy, University of California Los Angeles,*
Los Angeles, California 90095-1547, USA

[3]*Department of Physics, University of Illinois at Urbana-Champaign,*
1110 West Green Street, Urbana, Illinois 61801-3080, USA

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Before Vic Emery's untimely death, we had the privilege of working closely with him on the role of Coulomb frustrated phase separation in doped Mott insulators, and on the consequences of the resulting local electronic structures on the "mechanism" of high temperature superconductivity. In the present article, we discuss the resulting perspective on superconductivity in the cuprates, and on the more general theoretical issue of what sorts of systems can support high temperature superconductivity. We discuss some of the general, qualitative aspects of the experimental lore which we think should constrain any theory of the mechanism, and show how they are accounted for within the context of our theory.

The focus of this paper is a "dynamic inhomogeneity-induced pairing" mechanism of high temperature superconductivity (HTC) in which the pairing of electrons originates directly from strong repulsive interactions.[1] Repulsive interactions can be shown, by exact solution, to lead to a form of local superconductivity on certain mesoscale structures, but the strength of this pairing tendency decreases as the size of the structures increases above an optimal size. Moreover, the same physics responsible for pairing within a structure provides the driving force for the Coulomb frustrated phase separation that leads to the formation of mesoscale electronic structures in many highly correlated materials. From this perspective, the formation of mesoscale structures (such as "stripes") in the cuprate superconductors may not be a problem for the mechanism of superconductivity but rather a part of the mechanism itself. This mechanism is not based, as is the BCS mechanism[1], on the pairing of preexisting well defined and essentially free quasiparticles. Rather, it is based on the physics of strong correlations and low dimensionality. In this approach, coherence and quasiparticles are *emergent phenomena* at low energy, not an assumed property of the "high energy physics" from which this state derives.

The existence of strong local pairing does not guarantee a large critical temperature, since in a system of electronically isolated structures, the phase ordering (condensation) temperature is suppressed by phase fluctuations, often to $T = 0$. Thus, the highest possible superconducting transition temperature is obtained at an intermediate degree of inhomogeneity. A corollary of this

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[1] By "dynamic inhomogeneity" we mean inhomogeneity, whether static or fluctuating, which is generated dynamically by the strongly interacting degrees of freedom.

is that the optimal T_c always occurs at a point of crossover from a pairing dominated regime when the system is too homogeneous, to a phase ordering regime with a pseudo-gap when the system is too granular.

Coulomb frustrated phase separation leads to mesoscale electronic structures as a generic feature of highly correlated electronic systems. (By "mesoscale" we mean on length scales longer than but of order of the superconducting coherence length, ξ_0.) Usually this tendency leads to dominant charge density wave (CDW) and spin density wave (SDW) order, or possibly to more exotic electronic liquid crystalline phases, which can coexist with but tend to compete with superconductivity. However, we will argue that one feature that is special about the cuprate high temperature superconductors is that the intrinsic electronic inhomogeneity is strong enough to produce high temperature pairing, but strongly fluctuating enough that it does not entirely kill phase coherence.[2]

In Section I, we discuss the reasons that HTC is difficult, and hence why there are so few high temperature superconductors. In Section II we discuss the inhomogeneity induced pairing mechanism of HTC. Section III reports the latest theoretical development in this area - a solved model, the "striped Hubbard model," for which a well controlled theoretical treatment is possible, and many of the qualitative points made in the other sections can be illustrated explicitly. Then, in Section IV we briefly discuss the ways in which incipient charge order, especially due to Coulomb frustrated phase separation, can lead to the sort of local (slowly fluctuating) electronic inhomogeneities required for the proposed mechanism, as well as to a host of interesting "competing ordered" phases; a much more complete discussion of these aspects of the problem, with an extensive review of the experimental evidence in the cuprates, is contained in Ref. [3]. Sections V, which discusses the relative merits of the weak and strong coupling perspectives, and VI, which examines what is so special about the cuprates, deal explicitly with HTC in the cuprates, as opposed to the more abstract issues treated in the first part. These sections can be viewed as a set of commentaries, rather than a coherent exegesis. In Section VII, we highlight some of the salient conclusions. Finally, in the Appendix A we give a theoretical definition of HTC.

With the exception of Section III, the discussion in this paper is entirely qualitative and descriptive. For all but the most recent developments, a more detailed and technical discussion can be found in a review article, Ref. [4], which also includes extensive references to the original literature.

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[2] That the building blocks of an appropriate theory of strongly correlated systems should involve various self-organized mesoscale structures, rather than simple weakly interacting quasiparticles, is genetically related to the point of view articulate by P. W. Anderson in his famous monograph, *More is different*,[2]. He, however, may deny paternity.

I. WHY HIGH TEMPERATURE SUPERCONDUCTIVITY IS DIFFICULT

Before 1986, all but a few lonely voices proclaimed that superconductivity with transition temperatures much above 20K was impossible. Since the experimental discovery of high temperature superconductivity in the cuprates, scores of different theoretical arguments have been presented demonstrating that any number of simple model Hamiltonians are superconducting below a temperature which is "high" in the sense that it is equal to a number of order one times a microscopic electronic energy scale. These calculations, however, are typically uncontrolled, in the sense that they cannot be justified either as exact solutions of the stated model, or as asymptotic expansions in powers of a small parameter - they rely on physical intuition rather than systematic solution in any traditional sense of the word.

It seems to us that the answer cannot be so simple. The arguments (some of which are reviewed below and in Ref. [4]) made before 1986 were not ill-considered, even if they may have been accepted somewhat too uncritically - in materials that are basically good metals (Fermi liquids) there are, indeed, serious reasons to suspect that high temperature superconductivity is implausible. Moreover, even now, that we have learned to expand our horizons to include "bad metals" (*i.e.* resistively challenged materials which are not well described by Fermi liquid theory), the number of high temperature superconducting materials remains extremely small; maybe it is only the cuprates that can legitimately be called high temperature superconductors, or the class may include some subset of alkali doped C_{60}, $Ba_{1-x}K_xBiO_3$, $(TMTSF)_2ClO_4$, BEDT, MgB_2, and $Na_{0.3}CoO_{2y}H_2O$.

In Fig. 1, we show the distribution of superconducting transition temperatures among over 500 superconducting materials, as tabulated by Geballe and White[5] in 1979. The definition of what constitutes a distinct "material" is somewhat arbitrary (*e.g.* at what point, as one varies the concentration of two constituents of an alloy, does it become a new material). However, what is clear from the figure is that materials with transition temperatures above 15K are, already, extremely rare exceptions. Indeed, for reasons which, as far as we know are still not clear, all the materials known prior to 1979 with T_c in excess of 18K are alloys of Nb with the A15 crystal structure. We have added to the figure (blue hatched bars) some of the new superconductors with T_c in excess of 18K that have been discovered since this figure was made, using arbitrary definitions of our own. (See caption of Fig.1.)

The paucity of materials that exhibit high temperature superconductivity suggests that there must be a number of fairly stringent conditions on the character of the interactions that give rise to HTC. Many theories of high temperature superconductivity give no indication of why this should be the case – applying the stated (uncontrolled) analysis used in these approaches to a wide variety of Hubbard-like models on different lattices would suggest the existence of a high temperature superconducting phase in all of them.

There are several reasons why high temperature superconductivity is hard to attain,[7] and why we should be pleasantly surprised that it occurs at all, rather than being shocked that it does not
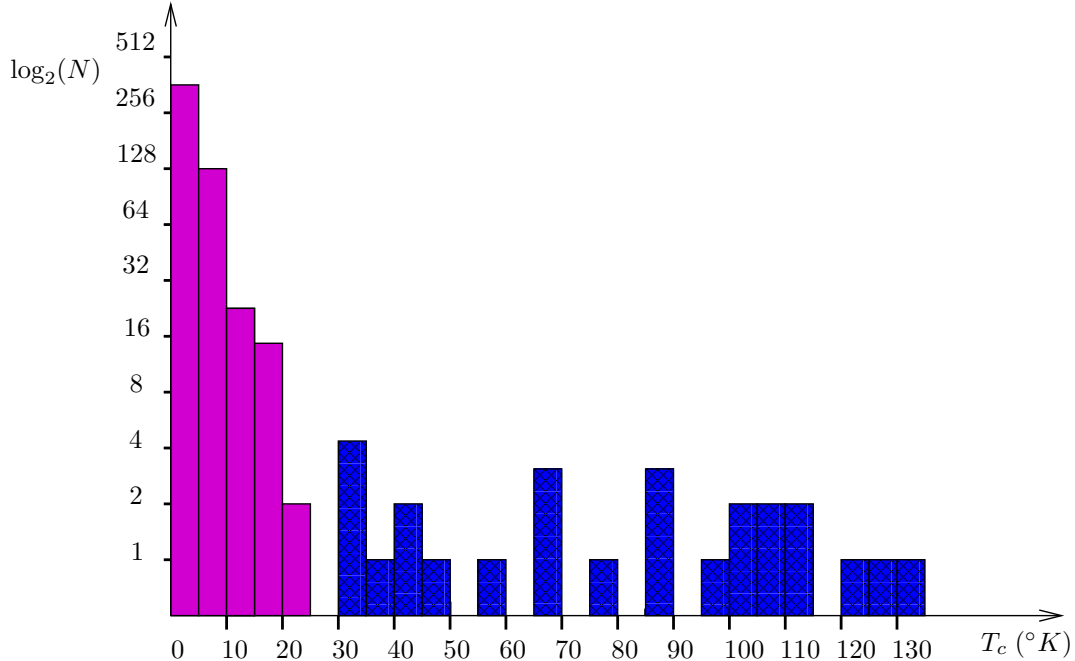
FIG. 1: Distribution of superconducting transition temperatures. The solid magenta bars represent the number of materials, N, whose transition temperatures are tabulated in Fig. VI.2 from Ref. [5], which includes over 500 superconducting materials known prior to 1979. Note that the numbers are shown on a log scale. We have added to the figure (the blue hatched bars) superconductors discovered since 1979 with transition temperatures in excess of 20K. Since all the cuprate superconductors contain nearly square Cu-O planes, which are thought to be the central structure responsible for HTC, one might think of them all as one superconducting material. However, there are also notable differences between different cuprates, including the fact that some are n-type and some p-type, they have different numbers of proximate Cu-O planes, they can have different elements making up the charge reservoir layer, etc. There were 26 distinct crystal structures for cuprate superconductors tabulated in the 1994 monograph by Shaked et $al.$[6], so we have taken this as our definition of "distinct" materials. In each case, we have reported the highest transition temperature among different materials with the same crystal structure, restricting ourselves, however, to data at atmospheric pressure in bulk materials. C_{60} can be doped with different metal ions or mixtures of metal ions, but they all have more or less the same crystal structure and charge density, so we have counted this as one material (with a maximum $T_c = 31K$ in Rb_2CsC_{60}). One point is for BaKBiO ($T_c = 31K$). We have also added one point for MgB_2 ($T_c = 39K$). All of the organic superconductors and $Na_{0.3}CoO_{2y}H_2O$ have T_c less than our arbitrary cuttoff, and so have not been included.

lurk in every third new material. At the crudest level, the dominant interaction between electrons is the strongly repulsive Coulomb interaction – for electrons to pair at all must involve subtle many-body effects which will therefore tend to be rather delicate. In BCS theory, it is the fact that the Coulomb interaction, μ, is well screened (short-ranged), and that the phonon-induced attraction, λ, is highly retarded, that combine to make a net effective attraction, $\lambda_{\mathrm{eff}} = \lambda - \mu^{\star}$, between electrons at low energy. (This important point is stressed, for instance, in the classic

treatise on the subject, Ref. [1].) Since the downward renormalization of the Coulomb repulsion, $\mu^* = \mu[1 + \mu \log(E_F/\omega_0)]^{-1}$, is only logarithmic, it is effective only when the scale of retardation, ω_0, is very small compared to the Fermi energy, E_F. Moreover, since there are all sorts of polaronic and structural instabilities which occur if λ is large compared to one, λ_{eff} can never be much larger than one. Combined, these considerations imply that superconductivity in normal metals must satisfy the hierarchy of energy scales, $E_F \gg \omega_0 \gg T_c \sim \omega_0 \exp(-1/\lambda_{\text{eff}})$.

Another important issue is that superconductivity has two distinct features: the electrons must pair and the pairs must condense. Rather than approaching the problem from the normal state, if we try to understand the physics of T_c by asking what sorts of fluctuations destroy the superconducting order as the system is heated from $T = 0$, we find that T_c is roughly determined by the lower of the two characteristic energy scales corresponding to these two features[8]. The energy scale which characterizes pair formation is the maximum gap, Δ_0. The energy scale, T_θ, of bose condensation (or more precisely, the temperature above which phase fluctuations destroy the order) is proportional to the superfluid density, $T_\theta \propto \rho_s(T = 0)/m^\star$. In good metals, T_θ is enormous. As is correctly captured by mean field theory, T_c is determined entirely by the pairing scale. However, strong interactions tend to localize electrons, either collectively (through formation of charge or spin density wave states) or through small polaron formation. Thus, as the strength of the interactions increases, Δ_0 can increase, but correspondingly T_θ will decrease. Eventually, in the strong interaction limit, T_c is set by T_θ, and so decreases as the strength of the pairing increases.

The opposing tendencies of Δ_0 and T_θ mean that there is generally an optimal T_c, *i.e.* one that does not grow without bound as the interaction strength is varied. This also suggests that, within a class of model systems, or even possibly in a class of materials, the optimal T_c will always occur at a point of crossover from a pairing dominated transition to a phase ordering transition.

II. DYNAMIC INHOMOGENEITY INDUCED PAIRING MECHANISM OF HTC

In order to obtain high temperature superconductivity, we would like to eliminate the middle man. Rather than relying on a weak induced attraction, the pairing should arise directly from the *strong short-range repulsion between electrons*. It might not be *a priori* obvious that any such mechanism exists, but we have by now demonstrated, by controlled solution of several model problems, that it does. Clearly any such pairing mechanism must be highly collective (since the pairwise interaction is repulsive), and must be "kinetic energy driven" in the sense that the energy cost of pairing two mutually repelling electrons must be more than compensated by the gain in some sort of energy of motion.[3]

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[3] This latter statement is intuitively compelling, but cannot be made completely precise since, by the time one is dealing with effective Hamiltonians, it is never completely clear how each remaining interaction is related to the microscopic kinetic energy of the constituent electrons. Note, the attractiveness of a kinetic energy driven mechanism has been emphasized by several other authors, including Refs. [9,10,11,12].

One of the main reasons we have reached the conclusion that mesoscale inhomogeneities are essential to the mechanism of high temperature superconductivity is that all the model systems in which pairing from repulsive interactions has been clearly established share this feature. This observation may reflect our limited model solving abilities rather than a characteristic of nature. However, enormous effort has been devoted to numerical searches for superconductivity in various uniform Hubbard and t-J related models, with results that are, at least, ambiguous. (For a review, see Ref. [4].) It seems to us that if superconductivity with characteristic energy and length scales of order the microscopic scales in the problem were indeed a robust feature of these models, that unambiguous evidence of it would have been found by now.

A.   Pairing in Hubbard clusters

The properties of the Hubbard model on various clusters has been studied[13,14,15,16,17,18] extensively, both numerically and analytically. A finite cluster cannot be a superconductor, but there are two local indicators of superconductivity that can be investigated: existence of a spin-gap and pair binding. If we wish to think of a Hubbard cluster as being a superconducting grain, then we certainly expect it to have a spin-gap. Even if we think of it as a grain of a d-wave superconductor, since nodal quasiparticles only occur at discrete points (sets of measure 0) in k-space, and since k is effectively quantized in a small grain, we expect there to be a true spin-gap in almost all cases. Pair-binding is less obvious - on small superconducting grains, the energy to add one quasiparticle can be less (by the charging energy) than the energy to add a pair. However, especially in models (such as the Hubbard model) in which the long-range Coulomb interaction is neglected, pair-binding is also a reasonable indicator of local superconductivity.

What is found in the cited studies is that many, but certainly not all, small Hubbard clusters exhibit spin-gaps and pair-binding in an appropriate range of strength of (repulsive) Hubbard interaction, U, and electron concentration. This effect is typically strongest at half-filling (one electron per site). It occurs most strongly for intermediate values of U/t, and the pair-binding is lost when U/t gets either very large or very small.[19] Finally, there is a general tendency for the magnitude of both the pair-binding and the spin-gap to decrease as the size of the cluster increases, suggesting that this is intrinsically an effect associated with mesoscale structure.

Among the Hubbard clusters that have been found to exhibit this locally superconducting behavior are[20] the $4n$ membered Hubbard ring, with n from 1 to 250, the cube, the truncated tetrahedron, and various pieces of the 2D square lattice on a torus. Closely related studies[21,22] have been carried out on clusters that are effectively infinite in one direction but are mesoscale transverse to it. These clusters include Hubbard ladders with up to 8 legs, and the circumference 4 Hubbard cylinder. In these "fat" 1D systems, the size of the spin-gap, and with it the magnitude

of the pair binding energy, tend to decrease exponentially with the transverse size of the clusters.[4]

The physics of spin-gap formation is at the core of this problem. It is inherited from the properties of the cluster at half-filling where, at least for large U/t, the system can better be thought of as a grain of a Mott insulator. The spin-gap is then associated with the quantum disordering of the electron spins. In the limit of infinite cluster size there is no spin gap since (except, perhaps, on special, highly frustrating lattices) the spin rotation symmetry spontaneously broken, and there are gapless spin-waves. For instance, if one considers a ladder of width L to be a finite size version of the square lattice quantum antiferromagnet, whose interacting spin-waves one treats in the continuum limit, then one can derive an expression for the spin-gap[23], $\Delta_s \sim$ 3.347 J $\exp(-0.682L/a)\left[1 + O(L/a)\right]$, which agrees quantitatively with the results of numerical simulations[24,25,26]. Again, this argument makes clear that the spin-gap is a mesoscale effect, which tends to decrease rapidly with the size of the cluster.

The remaining question is why does the spin-gap survive away from half filling, and why does the existence of a spin-gap (in many, but not all cases) lead to pair-binding? There are two distinct intuitive arguments that rationalize this observation.

The first is based[14] on the notion of a local form of spin-charge separation[27]. If we add one hole to each of two half-filled Hubbard clusters, we must make on each cluster an excitation carrying spin 1/2 and charge e. If we add two electrons to a single cluster, they can form a spin singlet, in which case we need to make excitations carrying only charge 2e. If we can approximate the excitations as holons (charge e spin 0) and spinons (charge 0 and spin 1/2), then by adding two electrons to one cluster we save twice the spinon creation energy. Even if this description is invalid (due to confinement) at long length scales, in some circumstances, it may give us a good handle on the local energetics.

The second line of argument is similar to those that lead to phase separation in doped antiferromagnets[28], or the spin-bag ideas of pairing[29]. Under some circumstances the state of the system at half-filling is anomalously stable, since the system can take maximal advantage of Umklapp scattering. A large spin-gap is a measure of this anomalous stability. When adding two electrons to two identical clusters, we have the choice of adding one electron to each cluster, in which case the particularly favorable correlations are disturbed on both clusters, or we can add both to one cluster (even if they have a direct repulsion between them), since in that case only one cluster is disturbed. Thus, paradoxically, it could be the strength of the insulating correlations in the half-filled cluster that give rise to superconductivity when the system is lightly doped.

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[4] H. Tsunetsugu, M. Troyer and T. M. Rice[22] studied arrays of two-leg $t-J$ ladders as a way to understand the physics of the translationally invariant 2D system. Although the model they studied nominally corresponds to the period 2 case we discuss below, and some of their discussion prefigures the present analysis, the questions asked by these authors were quite different. In particular they did not consider the mechanism of superconductivity in inhomogeneous 2D systems which we discuss here.

B.   Spin-gap proximity effect

The arguments in the previous section are general and intuitive, but supported mainly by anecdotal evidence. (In a few cases, the origin of the pair-binding can be understood analytically for small U/t on the basis of perturbation theory[14,15], but here the effects are weak and the strong correlation physics, which is so central in the actual materials, is only present in ghostly form.) In the case of "fat" 1D systems, various ladders or sets of coupled ladders, we have sufficient theoretical understanding of the problem that we can analyze in some detail the conditions under which superconducting correlations emerge directly from the repulsive interactions.

In a single band one-dimensional electron gas (1DEG) with short-ranged repulsive interactions, superconductivity is suppressed relative to non-interacting electrons - there is no tendency toward a spin-gap (rather, there is quasi-long-range antiferromagnetic order) and the superconducting susceptibility is not even logarithmically divergent as $T \to 0$. Technically speaking, the low energy physics is governed by the Luttinger liquid fixed point (gapless, bosonic modes with spin-charge separation) with the charge Luttinger exponent, $K_c < 1$. However, in multiband 1D systems, under many circumstances[30,31,32], the low energy physics is governed by a strong-coupling Luther-Emery fixed point[33], with a spin-gap, Δ_s, and with a charge Luttinger exponent in the range $0 < K_c < 2$. This fixed point exhibits incipient superconductivity in the sense that the singlet superconducting susceptibility diverges for $T \ll \Delta_s$ so long as $K_c > 1/2$,

$$\chi_{SC} \sim \Delta_s / T^{2-K_c^{-1}} . \tag{2.1}$$

To complicate matters, it also exhibits incipient CDW order in the sense that the CDW susceptibility diverges at wave number $Q = 2k_F$ for $T \ll \Delta_s$ so long as $K_c < 2$,

$$\chi_{CDW}(Q) \sim \Delta_s / T^{2-K_c} . \tag{2.2}$$

Why are the multiband cases so different from the single band case? In particular, since spin-gap formation is the 1D version of singlet pairing, what is it that causes pairing to be a common feature of multiband systems and not of the single band problem? The new physics comes from interband pair scattering, and has been explained intuitively by Emery, Zachar, and one of us[11] as "the spin-gap proximity effect."

Consider coupling two distinct 1D systems. From the weak coupling perspective, one can think of these as being two bands arising from the existence of more than one atom per unit cell. From a strong coupling perspective, one could think of these as two chemically distinct chains in close physical proximity to one another. Assuming that the two systems have distinct values of the Fermi wave vector, k_F and k_F', low energy processes in which an odd number of electrons are scattered from one system to the other are forbidden by momentum conservation. Coupling of CDW fluctuations, which are singular at different values Q and Q', are negligible (*i.e.* it is an irrelevant interaction). However, scattering of electron pairs with zero center of mass

momentum from one system to the other is, under many circumstances, peturbatively relevant. It is the renormalization of these interband pair-scattering terms, and their feedback on the other interactions in the system, that can drive the system to the Luther-Emery fixed point.

The physical origin of this effect is simply understood. The electrons can gain zero-point energy by delocalizing between the two bands. In order to take advantage of this, however, the electrons need to pair, which may cost some energy. When the energy gained by delocalizing between the two bands exceeds the energy cost of pairing, the system is driven to a spin-gap phase. In this sense, the physics is very analogous to the ordinary proximity effect in superconductivity. Here, a normal metal, even one with residual repulsive interactions between electrons, is brought in contact with a superconductor. In order for the electrons to be delocalized over the combined system, the electrons in the metal must pair. In this case, even though this costs energy, the gain in zero point "kinetic energy" always makes the proximity effect favorable. In this sense this is a kinetic energy driven mechanism. As is well known, the result is that superconductivity is induced in the normal metal over a distance which diverges as $T \to 0$.

The spin-gap proximity effect is not quite so robust - it occurs only if a certain exponent inequality is satisfied. If one of the two subsystems already has a spin-gap, then the price (pairing) only needs to be paid in the other, so the exponent inequality is easier to satisfy. It is an interesting, and still largely unexplored issue, what local "chemistry" does or does not give rise to a Luther-Emery liquid with a large spin-gap in a variety of multicomponent 1D systems. We do know that the two-leg ladder in both weak and strong coupling has a robust Luther-Emery phase. We also know, as mentioned above, that the spin-gap of the half-filled 2N leg ladder in strong coupling decreases exponentially with N. Similar behavior is seen in weak coupling, where the spin-gap in the entire Luther-Emery phase can be shown[34] to decay exponentially with N. Together, these two observations reinforce our belief that pairing directly from repulsion is a mesoscopic effect, which disappears rapidly if the relevant dimensions of the system in question get too large.

III.   SUPERCONDUCTIVITY IN A STRIPED HUBBARD MODEL: A CASE STUDY

In this section, we present a theoretically well controlled solution of an explicit model in which high temperature superconductivity arises directly from the repulsive interactions and the existence of mesoscale structures.[5] In collaboration with E. Arrigoni, we discuss this model in some detail in Ref.[36].

The model has modulated interactions in one direction, so that it breaks into an array of weakly coupled two leg ladders (hence the name "striped Hubbard model"). Perhaps one can view this as a caricature of the spontaneous symmetry breaking that occurs in stripe phases in real materials, but there are troubles with this identification. Primarily, we would like this model to be viewed

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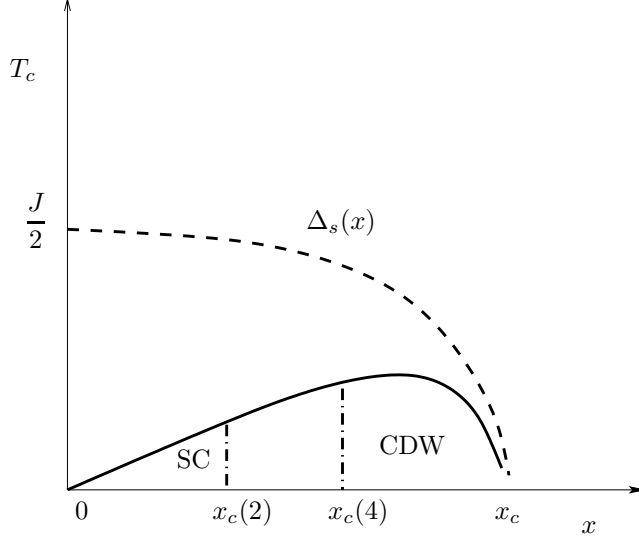[5] The same sort of physics was studied in weak coupling in Ref. [35].

K_c

2

1

$\Delta_s(2)\left(\dfrac{\delta t}{J}\right)^{\frac{4}{3}} T_c$ $\frac{1}{2}$

$\Delta_s(1)\left(\dfrac{\delta t}{J}\right)$

$\dfrac{J}{2}$

$\Delta_s(x)$

x_1

SC   CDW

0 $x_c(2)$ $x_c(4)$ x_c x

FIG. 2: Schematic phase diagram for a period 2 and a period 4 striped Hubbard model, at fixed (and small) δt. The broken line is the spin gap $\Delta_s(x)$ as a function of doping x, which labels the horizontal axis; $x_c(2)$ and $x_c(4)$ indicates the SC-CDW quantum phase transition for the period 2 and period 4 cases. These, most likely, are first order transitions. For $x \gtrsim x_c$ the isolated ladders do not have a spin gap; in this regime the physics is different involving low-energy spin fluctuations.

as a solvable model in which the basic mechanism of mesoscale inhomogeneity-induced pairing can be studied.

Because the solution of the ladder problem is so well characterized, it is possible to treat the coupled ladder problem reliably so long as the coupling between ladders is sufficiently weak. Within this model, we establish the occurrence of superconductivity directly from the repulsive interactions, document the important role of competing (CDW) order in the phase diagram, and analyze the circumstances under which the optimal T_c is obtained. A very schematic representation of the resulting phase diagram is shown in Fig.2.[6]

The the striped Hubbard model (sketched in Fig.3) is:

$$H = -\sum_{<\vec{r},\vec{r}'>,\sigma} t_{\vec{r},\vec{r}'}[c^{\dagger}_{\vec{r},\sigma}c_{\vec{r}',\sigma} + \text{h.c.}] + \sum_{\vec{r},\sigma}[\epsilon_{\vec{r}} c^{\dagger}_{\vec{r},\sigma}c_{\vec{r},\sigma} + (U/2)c^{\dagger}_{\vec{r},\sigma}c^{\dagger}_{\vec{r},-\sigma}c_{\vec{r},-\sigma}c_{\vec{r},\sigma}]$$

where $< \vec{r},\vec{r}' >$ designates nearest-neighbor sites, $c^{\dagger}_{\vec{r},\sigma}$ creates an electron on site $\vec{r}$ with spin polarization $\sigma = \pm 1$ and satisfies canonical anticommutation relations, and $U > 0$ is the repulsion

[6] In the schematic phase diagram of Fig.2, we have illustrated qualitatively several important effects discussed in the text: a) at low x, T_c grows linearly with x; b) for somewhat larger values of x, one can use the low-temperature form of the susceptibility of the spin-gap phase to estimate T_c; c) although for larger values of x non-universal effects are important, as $x \to x_c$ the spin gap vanishes and so does T_c. We have simplified the figure by taking the T_c curves for the periods 2 and 4 stripes to coincide, so as to highlight the main difference, i.e. that the critical x shifts to larger values as the period increases. In fact, however, the entire curve should be somewhat different in the two cases.
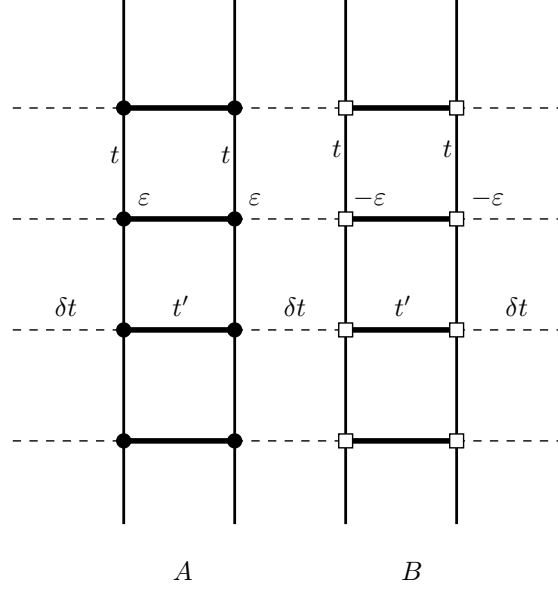
FIG. 3: Schematic representation of the striped Hubbard model analyzed in this paper. See the surrounding text for details; here A and B are the two types of ladders discussed in the text

between two electrons on the same site. In the limit of strong repulsions, $U \gg t_{\vec{r},\vec{r}'}$, this model reduces approximately to the corresponding $t - J$ model, which operates in the subspace without doubly occupied sites, but with an exchange coupling, $J_{\vec{r},\vec{r}'} = 4|t_{\vec{r},\vec{r}'}|^2/U$ between neighboring spins. Our results only depend on the low-energy physics of the ladder and, thus, apply equally to the $t - J$ and Hubbard models.

In the translationally invariant Hubbard model, $t_{\vec{r},\vec{r}'} = t$ and $\epsilon_{\vec{r}} = 0$. The striped version of this model is still translationally invariant along the stripe direction (which we take to be the y axis), so $t_{\vec{r},\vec{r}+\hat{y}} = t$. However, perpendicular to the stripes the hopping matrix takes on alternately large and small values: $t_{\vec{r},\vec{r}+\hat{x}} = t'$ for $r_x =$ even, and $t_{\vec{r},\vec{r}+\hat{x}} = \delta t \ll t' \sim t$ for $r_x =$ odd. This defines a "period 2 striped Hubbard model," as shown in Fig. 3. For the "period 4 striped Hubbard model," we include a modulated site energy, $\epsilon_{\vec{r}} = \pm \epsilon$ on alternate ladders with $\epsilon \gg \delta t$. Ladders with site energy ϵ will be called A ladders and ladders with site energy $-\epsilon$ will be called B ladders.

A. Zeroth order solution: Isolated 2-leg ladders

For $\delta t = 0$, the model breaks up into a series of disconnected 2-leg ladders. Considerable analytic and numerical effort has gone into studying the properties of 2-leg $t - J$ and Hubbard ladders, and much is known about them. For $x = 0$, the undoped two-leg ladder has a unique, fully gapped ground state. In the large U limit, the magnitude of the spin-gap of the undoped[24,37] ladder is approximately $\Delta_s \approx J/2$. Then, for a substantial range of x $(0 < x < x_c)$, the ladder exhibits a

Luther-Emery phase, with a spin-gap that drops smoothly[7] with increasing x, and vanishes at a critical value of the doping, $x = x_c$. (This particular Luther-Emery liquid is known[30,37,38,39,40,41] to have "d-wave-like" superconducting correlations, in the sense that the pair-field operator has opposite signs along the edge of the ladder (y direction) and on the rungs (x-direction).) For $x > x_c$, there remain uncertainties concerning the exact character of the possible gapless phases.

For the purposes of the present paper, we will confine ourselves to the range of parameters where both A and B type ladders are in the Luther-Emery phase. The low energy physics (at all energies less than Δ_s) of the two-leg ladder in the Luther-Emery phase is contained in the effective (free) bosonized Hamiltonian for the collective charge degrees of freedom,

$$H = \int dy \left\{ \frac{v_c}{2} \left[K(\partial_y \theta)^2 + \frac{1}{K}(\partial_y \phi)^2 \right] + \dots \right\} \tag{3.1}$$

where ϕ is the CDW phase and θ is the superconducting phase. These two fields are dual to each other, and so satisfy the canonical equal-time commutation relations, $[\phi(y'), \partial_y \theta(y)] = i\delta(y - y')$. Specifically, the component of the charge density operator at the wave-vector $P = 2\pi x$ of the incipient CDW order is

$$\hat{\rho}_P(y) \propto \sqrt{\Delta_s} \exp[iPy + i\sqrt{2\pi}\phi(y)] \tag{3.2}$$

while the singlet pair creation operator,

$$\hat{\Phi}(y) \propto \sqrt{\Delta_s} \exp[i\sqrt{2\pi}\theta]. \tag{3.3}$$

This effective Hamiltonian is general and physical; the precise x dependence of the spin-gap, Δ_s, the charge Luttinger exponent, K, the charge velocity, v_c, and the chemical potential, $\mu(x)$, depends on details such as the values of U/t and t'/t. For certain cases[37,38,39] Δ_s, K, v_c and v_s have been accurately computed in Monte-Carlo studies, and these studies could be straightforwardly extended to other values of the parameters.[8]

The ellipsis in Eq. (3.1) represent cosine potentials, which we will not explicitly exhibit here, that produce the Mott gap Δ_M at $x = 0$. Because of these terms, for $x \to 0$ the elementary excitations are charge $2e$ solitons that can either be viewed as spinless Fermions or hard-core bosons, with a dispersion relation $E(k) \simeq \Delta_M + \tilde{t}k^2$. One consequence of this is that[39,42] $K \to 2$ and $v_c \to 2\pi\tilde{t}x$ as $x \to 0$. A second consequence is that the renormalized harmonic theory, which retains only the explicitly exhibited terms in Eq.(3.1), is valid in a range of energies which is small in proportion to the effective Fermi energy, $\tilde{E}_F^{(1D)} = 2\pi\tilde{t}x^2$. (An estimate of $\tilde{t} \approx t/2$ can be obtained from the DMRG study of the $t - J$ ladder with $J/t = 1/3$ in Ref. [38].)

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[7] For a restricted range of x, the authors of Ref. [37] show numerical evidence indicating that the spin gap decreases smoothly with increasing x. We are not aware of any published studies that carefully trace the spin gap as a function of x, and in particular ones that accurately determine the critical doping, x_c, at which it vanishes.

[8] Note that the normalization convention on the fields used in the present paper differs from that of White and coworkers[39], so that our K is the same as their $2K_{c,+}$.

For larger x, the numerical studies[38,39,43] generally find that both K and Δ_s drop monotonically with increasing x. By the time $x = x_1 \approx 0.1$, K is close to 1, and by $x = x_c \approx 0.3$, Δ_s has dropped to values that are indistinguishable from 0, and $K \approx 0.5$. Thus, over most of the Luther-Emery phase, both the SC and the CDW susceptibilities are divergent. However, the SC susceptibility is the more divergent only at rather small values of $x < x_1$.

Before leaving the single ladder problem, it is worth mentioning a useful intuitive caricature of its electronic properties. We picture a singlet pair of electrons on neighboring sites as being a hard-core bosonic "dimer." The undoped ladder can be thought of as a Mott insulating state of these dimers, with one dimer per rung of the ladder, *i. e.* a "valence bond crystal" with lattice spacing one. To remove one electron from the system, we need to destroy one dimer and remove one electron, leaving behind a single electron with spin $1/2$ and charge e. However, when we remove a second electron from the system, we have the choice of either breaking another dimer, thus producing two quasiparticles with the quantum numbers of an electron, or of removing the unpaired electron left behind by the first removal, thus producing a new boson - a missing dimer - with charge $2e$ and spin 0. The persistence of the spin-gap upon doping the ladder can thus be interpreted as implying that the energy needed to break a dimer (of order Δ_s) is sufficiently large that one charge $2e$ boson costs less than two charge e quasiparticles. At finite x, the missing dimers can be treated as a dilute gas of hardcore bosons. That the elementary excitations of the undoped ladder can be constructed in this simple manner reflects the fact that this is a confining phase[44,45,46,47], not a spin liquid.[9]

B.   Weak Inter-ladder interactions

We now address the effect of a small, but non-zero coupling (*i. e.* single-particle hopping) between ladders, $\delta t > 0$. Because of the spin-gap, δt is an irrelevant perturbation in the renormalization group sense, and so does not directly affect the thermodynamic state of the system. However, second order processes result in various induced interactions between neighboring ladders. These consist of marginal forward scattering interactions, which are negligible for small δt, and potentially relevant Josephson tunneling and back-scattering density-density interactions.

The important (possibly relevant) low energy pieces of these latter interactions are most naturally expressed in terms of the bosonic collective variables defined above:

$$H' = -\sum_j \int dy \left\{ \mathcal{J} \cos[\sqrt{2\pi}(\theta_j - \theta_{j+1})] + \mathcal{V} \cos[(P_j - P_{j+1})y + \sqrt{2\pi}(\phi_j - \phi_{j+1})] \right\}, \qquad (3.4)$$

where $P_j = 2\pi x_j$, with x_j the concentration of doped holes on ladder j, and ϕ_j and θ_j are the charge field and its dual on each ladder. Here, again, the form of the low energy interactions between

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[9] In a confining phase, all finite energy excitations have quantum numbers equal to those of an integer number of electrons and holes; a deconfining phase supports excitations with "fractional" quantum numbers such as those of a "spinon": spin $1/2$ and charge 0.

two Luther-Emery liquids is entirely determined by symmetry considerations, but the magnitude of the Josephson coupling $\mathcal{J}$ and the induced interaction between CDW's, $\mathcal{V}$, must be computed from microscopics; they are renormalized parameters which result from "integrating" out the high energy degrees of freedom with energies between the bandwidth $W \sim 4t$ and the renormalized cutoff, Δ_s, or with wavelengths between a and $\xi_s \equiv v_s/\Delta_s$ where v_s is the spin-wave velocity.

So long as x is not too near x_c, the spin gap is large, $\Delta_s \sim J$. In this case, the spin physics really occurs on a microscopic scale, and hence the coupling constants are not qualitatively changed in this first stage or renormalization. In this case, a rough estimate of $\mathcal{J}$ and $\mathcal{V}$ can be made from second order perturbation theory:

$$\mathcal{J} \approx \mathcal{V} \sim \frac{(\delta t)^2}{J} \tag{3.5}$$

As $x \to x_c$, and hence $\Delta_s \to 0$, the problem becomes more subtle, as discussed in Ref.[36].

C. Renormalization-group analysis and inter-ladder mean field theory

The effect of these inter-chain couplings can be deduced from an analysis of the lowest order perturbative renormalization group equations in powers of the couplings $\mathcal{V}$ and $\mathcal{J}$. However, *equivalent* results are obtained from inter-ladder mean-field theory[43,48], which is conceptually simpler. These equations are the analogue of the BCS gap equations applied to this model, and are expected to give a quantitatively accurate estimate of T_c for small $\delta t/\Delta_s$ for precisely the same reason. A discussion of the accuracy of *interchain* mean-field theory is given in the Appendix of Ref.[36]. In the present two-dimensional system, T_c should be interpreted as the onset of quasi-long range order, *i. e.* as a Kosterlitz-Thouless transition.

To implement this mean-field theory, we need to compute the expectation value $M_j(h_j) = \langle \cos[\sqrt{2\pi}\theta_j] \rangle$ of the pair creation operator on an isolated ladder, where the expectation value is taken with respect to the mean-field Hamiltonian

$$H_{MF} = H_j - h_j \int dy \, \cos[\sqrt{2\pi}\theta_j] \tag{3.6}$$

in which H_j is the effective Hamiltonian in Eq.(3.1) with parameters appropriate to ladder j, and h_j represents the mean-field due to the neighboring ladders, and so satisfies the self-consistency condition,

$$h_j = \mathcal{J}[M_{j+1} + M_{j-1}]. \tag{3.7}$$

The expression for the mean-field transition temperature can be expressed in terms of the corresponding susceptibility, $\tilde{\chi}_{SC}^{(j)} = \partial M_j(h)/\partial h|_{h=0}$, which is related to the superconducting susceptibility in Eq. (2.1) by a proportionality constant which depends on the expectation value of

the spin-fields. In the case in which all the ladders are equivalent, this yields the implicit relation $2\mathcal{J}\tilde{\chi}_{SC}(T_c) = 1$. For an alternating array of A and B type ladders, the expression for the superconducting T_c is easily seen to be

$$(2\mathcal{J})^2 \; \tilde{\chi}_{SC}^{(A)}(T_c) \; \tilde{\chi}_{SC}^{(B)}(T_c) \; = 1. \tag{3.8}$$

Notice that in the case in which the A and B type ladders are identical Eq. (3.8) reduces properly to the expression for equivalent ladders. The expression for χ_{SC} from Eq. (2.1) can be used to invert Eq. (3.8) to obtain the estimate for T_c:

$$T_c \sim \Delta_s \left(\frac{\mathcal{J}}{\widetilde{W}} \right)^\alpha; \quad \alpha = \frac{2K_A K_B}{[4K_A K_B - K_A - K_B]} \tag{3.9}$$

where $\mathcal{J}$ is the effective coupling given in Eq.(3.5), and $\widetilde{W}$ is a high energy cutoff which, so long as x is not too close to x_c, it is also of order J. Although T_c is small for small $\mathcal{J}$, it is only power law small. In fact typically $\alpha \sim 1$. A perturbative renormalization-group treatment for small $\mathcal{J}$ yields the same power law dependence as Eq. 3.9, suggesting that this expression is asymptotically exact for $\mathcal{J} << \widetilde{W}$.

The mean-field equations for the CDW order are obtained similarly. The expression for the transition temperature for CDW order with wave-vector P is

$$(2\mathcal{V})^2 \; \tilde{\chi}_{CDW}^{(A)}(P,T_c) \; \tilde{\chi}_{CDW}^{(B)}(P,T_c) \; = 1 \tag{3.10}$$

where the notation is the obvious extension of that used in the superconducting case. The best ordering vector is that which maximizes T_c. For $P = P_A$, $\chi_{CDW}^{(A)}(P_A,T)$ diverges with decreasing temperature as in Eq. (2.2), but $\chi_{CDW}^{(B)}(P_A,T)$ saturates to a finite, low temperature value when $T \sim v_c |P_A - P_B|$. Thus, even if $\chi_{CDW}^{(A)}(P_A,T)$ diverges more strongly with decreasing temperature than $\chi_{SC}^{(A)}$, there are two divergent susceptibilities in the expression for the superconducting T_c, and only one for the CDW T_c. So long as the exponent inequalities

$$2 > K_A^{-1} + K_B^{-1} - K_A; \quad 2 > K_A^{-1} + K_B^{-1} - K_B \tag{3.11}$$

are satisfied, the superconducting instability wins out.

D.   The $x \to 0$ limit

Since $K \to 2$ as $x \to 0$, there is necessarily a regime of small x in which the superconducting susceptibility on the isolated ladder is more divergent than the CDW susceptibility. Here, in the presence of weak inter-ladder coupling, even the period 2 striped Hubbard model (i. e. with $\epsilon = 0$) is superconducting. However, care must be taken in this limit, since, as mentioned above, the range of energies over which H in Eq.(3.1) is applicable vanishes in proportion to x^2. Fortunately, a complementary treatment of the problem, which takes into account the additional terms, the

ellipsis in Eq.(3.1), can be employed in this limit. The small x problem can be mapped onto a problem of dilute, hard-core charge $2e$ bosons (with concentration x per rung) with an anisotropic dispersion, $E(\vec{k}) = \tilde{t}k_y^2 - \mathcal{J}\cos[2k_x]$. (The 2 reflects the ladder periodicity.) Consequently, for small x,

$$T_c \approx 2\pi \ \sqrt{2\mathcal{J}\tilde{t}} \ xF(x) \sim |\delta t| \ x \tag{3.12}$$

where $F(x) \sim 1/\ln\ln(1/x)$ is never far from 1, and the logarithm reflects[49] the fact $d = 2$ is the marginal dimension for Bose condensation. (This result is not substantially different for the period 4 striped Hubbard model, so long as ϵ is not too large.) There is a complicated issue of order of limits when both δt and x are small; roughly, we expect that T_c will be determined by whichever expression, Eq. (3.8) or Eq. (3.12), gives the higher T_c, but with the understanding that χ_{SC} must be computed taking into account the terms represented by the ellipsis in Eq.(3.1) which cause the susceptibility to vanish as $x \to 0$.

E.   Relation to superconductivity in the cuprates

The striped Hubbard model realizes the idea that the pairing scale, in this case the spin-gap, can be inherited from a parent Mott insulating state. Moreover, like the underdoped cuprates, the gap scale is a decreasing function of increasing x, while the actual superconducting transition occurs at a T_c typically much smaller than $\Delta_s/2$, and is determined by the phase ordering temperature rather than the pairing scale. Hence, for x not too close to x_c, this model exhibits a pseudogap regime for temperatures between T_c and $T^* \sim \Delta_s/2$, reminiscent of that seen in underdoped cuprates. However, T_c is always bounded from above by Δ_s and so tends to zero as $x \to x_c$. The model also exhibits a competition between SC and CDW order, which is somewhat akin to the competition with fully developed stripe order and SC that occurs in certain cuprates.[10]

However, as mentioned above, the model cannot be thought of as a literal model of superconductivity in the cuprates. Firstly, most of the cuprates have, at most, local fluctuating charge stripe order (see Ref. [3] for an extensive discussion of the present status of this issue), and even where such order occurs, it occurs through spontaneous symmetry breaking. Moreover, the striped Hubbard model possesses a large spin-gap, and so does not contain any of the physics of low energy incommensurate spin-fluctuations which are the principle experimental signatures to date of stripe correlations in the cuprates. Thirdly, although the superconducting state is "d-wave-like" in the sense that the order parameter changes sign under rotations by $\pi/2$, since the striped Hamiltonian explicitly breaks this symmetry, there is no precise symmetry distinction between d-wave and s-wave superconductivity. Indeed, the superconducting state is not even truly adiabatically con-

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[10] For $x > x_c$ the low energy physics is dominated by spin fluctuations and by single-particle (electron) tunneling. Low T_c superconductivity can occur in this regime by conventional BCS-like mechanisms.

nected to the superconducting state observed in the cuprates, because the existence of a spin-gap implies the absence of gapless "nodal" quasiparticles in the superconducting state.[11]

There is a strong tendency in our contentious field to set up straw men which can easily be toppled by (purposely?) misinterpreting carefully caveated statements. We therefore reiterate that the striped Hubbard model is a solvable (and, we believe, fascinating) case study - not a "realistic" model of superconductivity in the striped phase of the cuprates.

IV.   WHY THERE IS MESOSCALE STRUCTURE IN DOPED MOTT INSULATORS

The cuprate high temperature superconductors are strongly correlated electronic systems, in which the short-range repulsions between the electrons are larger than the bandwidth. They are doped descendants of a strongly correlated (Mott) insulating "parent compound" which is antiferromagnetically ordered. While HTC is, seemingly, uniquely a property of the cuprates, many other aspects of the strong correlation physics are features of a much broader class of strongly correlated materials including various manganites, nickelates, cobaltates, and ruthenates. Magnetism, and various forms of charge order (to be discussed below) are among the clearest signatures of the strong correlation physics.

Of great fundamental importance is the failure of the Fermi liquid description of the "normal" state at room temperature and above. This fact was clear already at the time of the discovery of high temperature superconductivity and it has been a *leit motif* of much of the research done since then[51,52]. A directly related and associated fact is that these doped Mott insulators are "bad metals"[53]: above the superconducting T_c they exhibit a metallic T dependence of the conductivity, the famous linear resistivity, while at the same time there appears to be no evidence of well-defined quasiparticles (in the sense of Landau), and the resistivity passes the Ioffe-Regel limit without taking any notice of it. It may often be the case that well defined quasiparticles develop as emergent phenomena at low T and energy; those who treat the normal state as a Fermi liquid, despite the evidence to the contrary, are, in the immortal words of Landau[54], "Enemies of the working class."

Whether their ground states exhibit long range magnetic order or not, most models of undoped Mott insulators share an intrinsic tendency towards *electronic phase separation*[28,55], an effect which was found quite early on in analytic studies and numerical simulations of models of strongly correlated systems. The physics behind electronic phase separation is quite simple, and is related to the mechanism of pair-binding in clusters, discussed above. The addition of a single hole induces a "defect" in the correlations of the Mott insulator. The energy associated with the subsequent addition of holes is less if they clump together, since this disrupts the favorable correlations of the

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[11] However, simplified models of this type can have 2D anisotropic superconducting phases both with and without low-energy nodal quasiparticles; see, *e.g.* Ref.[50].

insulating state to a lesser extent. Thus, even though all the microscopic interactions are repulsive, there are effective attractive forces between the doped holes.

On the other hand, since the undoped systems are insulators, the long-range piece of the repulsive Coulomb interactions between the charges is poorly screened. This gives rise to *Coulomb-frustrated phase separation* – states which have as their constituents mesoscopic puddles of charges whose size and shape[56] are determined by the competition between the short-range tendency to phase separation and the Coulomb interaction. Electronic phases with self-organized mixtures of high and low density regions have been called[56,57] "electronic microemulsions." In a precise sense, the mesoscale structure defines the set of relevant degrees of freedom responsible for the low energy physics of strongly correlated systems.

At sufficiently small T, depending on how large the effective mass of a puddle, they can remain mobile (a puddle fluid), or can freeze into a variety of possible charge ordered states. (In the presence of quenched disorder, they can also be pinned.) Among the possible charge ordered states are a variety of *electronic liquid crystal phases* which exhibit a varying degree of charge inhomogeneity and spatial anisotropy[58]. As far as the mechanism of HTC is concerned, the existence of local structures on length scales greater than or of order of the superconducting coherence length, ξ_0, is what is important, not the manner in which the structures themselves order, or not. However, it is much easier experimentally to identify the states of broken spatial symmetry that arise from Coulomb frustrated phase separation. Thus, both because of their intrinsic interest, and as a way of gaining insight into the nature of the structures produced by Coulomb frustrated phase separation, there has been considerable interest in studying these phases.

Since electronic liquid crystalline phases are in some ways ordered and in some ways fluid, they are more subtle to identify in experiments than typical CDWs. Elsewhere, we have discussed the evidence in the cuprates,[3] of the existence of such ordered phases, especially smectic (stripe ordered) and Ising nematic phases. In many respects electronic liquid crystal phases are similar to the analogous phases of complex classical fluids[59]. However, while in classical liquid crystals, the rich phase diagram originates form the microscopic anisotropic structure of complex molecules (*e.g.* nematogens, chiral molecules, viruses, "molecular bananas", etc.), electronic liquid crystals are the quantum ground states of systems of point particles (holes); the role of the complex molecules is played by the self-organized structures produced by Coulomb-frustrated phase separation. It can't get more politically correct than this: complex "soft quantum matter" from self-assembling nano-structures!

V.   WEAK COUPLING VS. STRONG COUPLING PERSPECTIVES

Much of the commonly adopted theoretical analysis of the mechanism of high temperature superconductivity is, at core, the same as the BCS/Eliashberg theory, but (possibly) with a different collective excitation (spin-wave, phonon, exciton, director wave, ...) playing the role of the "glue."

However, an essential feature of BCS theory is that the normal state is a good Fermi liquid[1], with well defined quasiparticles at all energies small compared to the retardation scale (the frequency of the collective mode). It is, of course, possible to simply evaluate the same class of diagrams that are sanctified by Eliashberg theory, even when whatever peaks there are in the single particle spectral function are too broad to be classified as quasiparticles; however, in this case, there is no known justification for summing this particular class of diagrams (which sum the leading logarithms in a Fermi liquid). Whether or not one is comfortable with this sort of uncontrolled extrapolation of the (beautifully well controlled) weak coupling theory is a matter of personal taste. A distinguishing feature of these theories is that, for them, the strongly correlated nature of the cuprates is an inconvenient side issue. Indeed in all these theories, if the single particle spectral function, $A(k, \omega)$, (often taken phenomenologically from experiment) were replaced by a Fermi liquid $A(k, \omega)$, with well defined quasiparticles, the resulting calculated T_c would actually increase!

In contrast, a smaller but highly visible set of theories start from the viewpoint that the strong correlation physics is central to the physics of high temperature superconductivity. In this case, the mechanism is not based on pairing of well defined quasiparticles. Theories based on proximity to quantum critical points are of this sort. In these theories, the same physics (quantum critical fluctuations) that is supposed to be responsible for the pairing is also presumed to be responsible for the non-Fermi liquid character of the normal state, so it does not make sense to ask what would happen were the normal state replaced by a Fermi liquid. Of course, theories based on a fractionalized normal-state, with spin-charge separation, also fall in this category. The ideas we have discussed, in which mesoscale (and/or mesotime) inhomogeneity plays a crucial role in the pairing, shares some features with both of these other non-Fermi liquid based approaches. Since in the cleanest versions of our mechanism, coherence between different clusters occurs with the advent of superconducting order, these ideas provide a very concrete implementation of a mechanism of superconductivity in which the normal state has no coherently propagating quasiparticles.

It may be possible to discriminate between the strong correlation and the more BCSish approaches experimentally. In the strong correlation approaches, it would be unexpected to find a material with a high superconducting transition temperature and well defined quasiparticles in the normal state. This finds some support in the observation that, with increasing doping in the overdoped regime, as the single-particle spectral function becomes more Fermi liquid like, T_c drops rapidly. From the more BCSish viewpoint, one would be unsurprised to find some materials, even materials in which T_c is optimized, in which the normal state is well described by Fermi liquid theory, and the single-particle spectral function exhibits well defined quasiparticles.

In this context, it is important not to over-interpret ARPES evidence for or against the existence of quasiparticles. On the one hand, it is possible for quenched disorder, especially at the sample surface, to broaden what would have been a sharp peak in $A(k, \omega)$, making it too broad to be clearly identified as a quasiparticle - so long as this broadening is due to strictly elastic scattering process, a quasiparticle description remains valid despite the negative evidence from ARPES. Probably,

this can be checked with STM by looking for Friedel oscillations with random phases, but long distance power-law fall-off associated with the introduction of a known scatterer at a point in space. On the other hand, the spectral function of the one dimensional Luttinger liquid, even with moderately strong interactions, posseses a reasonably clear Fermi-liquid-like peak, although the elementary excitations of the system have no overlap with a single electron.[60] Thus, one should be cautious about concluding, without rather detailed theoretical analysis, that any particular observed spectral function is or is not exhibiting quasiparticle behavior.

VI.   WHAT IS SO SPECIAL ABOUT THE CUPRATES?

Until now, the issues we have discussed were mostly abstract, based on an analysis of the behavior of model Hamiltonians. Ultimately, however, we are interested in understanding the superconductivity in the cuprates. Moreover, since it is the one place where we all agree that a new phenomenon called high temperature superconductivity occurs, we would like to gain intuition about what is essential for high temperature superconductivity more generally, by analyzing what is essential to its occurrence in the cuprates.

A.   Is charge order, or fluctuating charge order, ubiquitous?

We have argued that some form of mesoscale spatial structure is essential to the mechanism of pairing. This structure could be static or slowly fluctuating, so long as the fluctuation frequency is less than the pairing scale. For this statement to be true, it is necessary that any material which exhibits high temperature superconductivity should also exhibit the requisite inhomogeneities. Since in the cuprates, T_c is not terribly sensitive to out of plane disorder, but, if anything, it increases as materials get cleaner, it seems implausible to us that the inhomogeneities in question can be directly linked to any sort of chemical inhomogeneity. This sort of inhomogeneity is certainly present in some materials – for instance, it is well documented[61,62,63] in STM studies on $Bi_2Sr_2CaCu_2O_{8+\delta}$, and may play a role in the superconductivity in that material.[12] However, more plausibly, in our opinion, the inhomogeneities in question are primarily associated with slow fluctuations of a proximate charge ordered state, of which the best documented example is the stripe phase[65].

Stripe order has been clearly documented in cuprates with reduced or vanishing T_c.[66] Clearly, where the stripe order is fully developed, the inhomogeneity is too strong – the superfluid density is highly suppressed and with it, T_c.[67] However, fluctuating stripe order has been clearly seen in numerous materials with moderately high T_c's, as discussed in depth in a recent review article of ours, [3]. It remains an open issue whether such fluctuating order is universal in materials with high transition temperatures. In this regard, it is most important to study the evidence[68] of stripe

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[12] For comparison, it is interesting to note that similar STM evidence of stripes has been found in the manganates.[64]

fluctuations in $YBa_2Cu_3O_{6+y}$, the material in which the greatest degree of chemical homogeneity has seemingly been achieved. While the evidence for stripe-like fluctuations in this material is not unambiguous, the magnetic structure seen with neutrons is extremely reminiscent of that seen in stripe-ordered $La_{2-x}Ba_xCuO_4$, and is in many ways suggestive of the existence of some remnant tendency to striping. (See Refs.[3], [69], [70], and [71].)

B. Does the "stuff" between the Cu-O planes matter?

One structural feature of the cuprates which has a much discussed systematic relation with T_c is the variation with the number of Cu-O planes stacked together between each "charge reservoir layer." For instance, in the sequence of materials $HgBa_2Ca_mCu_nO_y$, $T_c(n) = 98K, 128K, 135K, 125$, and $108K$ for $n = 1, 2, 3, 4, 5$, respectively. The peak in T_c at $n = 3$ is seen in all families of high temperature superconductors in which n can be systematically varied. There are many ideas concerning what this variation means. It is important to note that for $n > 2$, the different layers are not all equivalent, and so there is every reason to expect different doping levels on the different layers[72,73,74].

In the present context, three aspects of the layer number systematics seem suggestive. In the first place, this is a clear example of a situation in which there is an optimal inhomogeneity for superconductivity - apparently, $n = 3$ is in some way an optimal scale for superconductivity. Secondly, where phase fluctuations play a substantial role in determining T_c, it is clear that interplane couplings will suppress phase fluctuations and hence increase T_c. For instance[75], for the classical cubic lattice XY model on a slab n layers thick, the transition temperatures (computed by Monte Carlo) are $T_c(1) = 0.89J$, $T_c(2) = 1.38J$, $T_c(\infty) = 2.38J$. Finally, the $n = 3$ problem may reflect still more directly the way in which inhomogeneity can enhance T_c - where one has underdoped layers in good contact with overdoped layers, the combined system can inherit the high pairing scale from the underdoped layers and the large phase stiffness (superfluid density) from the overdoped layers[73].

Different "families" of high temperature superconductors are defined by subtle differences in the crystal structure and in the chemical character of the "charge reservoir layers" that lie between the Cu-O planes. There are substantial differences between the optimal T_c's in different families. For instance, double layer YBCO has an optimal $T_c \approx 92K$, while double layer Tl 2212 has $T_c = 118K$ and double layer Hg 2212 has $T_c = 128K$. The differences are still more extreme if we compare the single layer cuprates, where the optimal T_c in the 214 family is $T_c = 42K$ for Stage IV O doped LCO, while it is $T_c = 94K$ in Hg 1221. Thus, the variation of T_c with family is stronger still than its variation with n, as has been stressed by Leggett[72], by Chakravarty, Kee and Voelker[74], and by Geballe and Moyzhes[76]. Relatively little thought has been given to this striking observation, possibly because it makes one reflect uncomfortably about the importance of the solid state chemistry. One exception is the appealing idea of Geballe and Moyzhes[76], which is discussed
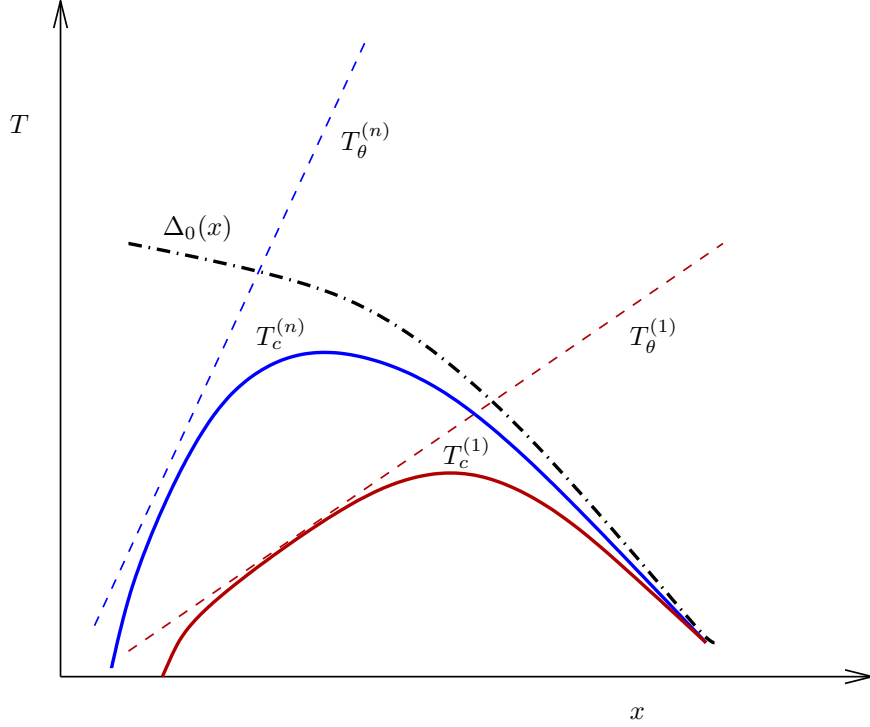
FIG. 4: Schematic phase diagram for a high temperature superconductors for a single layer and a multilayer cuprate (n layers) as a function of doping x. The rationale for this figure is discussed in the text. The dashed lines are the putative classical phase-ordering temperature (were all other fluctuations suppressed), and the dashed-dotted curve is the pairing scale or mean-field transition temperature. The solid lines are the transition temperatures.

in the article by Geballe elsewhere in this volume[77]. It is clear to us that this is an issue worth considerably more attention than it has so far received.

While it may well be true that interlayer tunneling[74] and/or electronic interactions in the charge reservoir layers in some way enhances the pairing, there is another possible explanation for the strong dependence of T_c on the three dimensional structure of the materials. This is illustrated in the schematic phase diagram in Fig. 4. We suppose, as indicated by the dashed-dotted line, that the pairing scale, *i.e.* the superconducting gap magnitude $\Delta_0(x)$, is a monotonically falling function of doping, x. Were fluctuations negligible, the material would order at a mean-field transition temperature $\sim \Delta_0/2$. However, in the underdoped regime, the small superfluid density implies[8,78,79] a large, fluctuation induced reduction of T_c to a phase ordering temperature, $T_\theta \sim Ax$, as shown by the dashed lines in Fig.4.

Since pairing involves short-distance physics (on the scale of ξ_0), we take as a working hypothesis that it is largely a single plane property, so $\Delta_0(x)$ is largely insensitive to structures outside of the

Cu-O plane.[13] However, since the phase ordering involves long-wave-length fluctuations (at length scales large compared to ξ_0), it is reasonable to expect the proportionality constant, A, to depend on the number of layers, n, and the electronic structure of the charge reservoir layer. Specifically, from the Monte-Carlo calculations on the classical XY model mentioned above[75], we know that it is reasonable for A to vary by 50% or so with n. Since the pair tunneling amplitude through the charge reservoir layer can clearly depend on its electronic structure[80], it is likewise possible that A depends on "family."

The two different T_θ lines in the figure are thus supposed to represent materials with different three dimensional structures.[14] The actual superconducting transition, T_c, is bounded above by T_θ and $\Delta_0/2$, as shown schematically by the solid curves in the figure. (In drawing the figure, we have assumed that quantum fluctuations will drive $T_c \to 0$ at a critical $x_c > 0$.) A consequence of this scheme is that in comparing the properties of "optimally doped" materials, those with a higher $T_c(x_{\mathrm{opt}})$ should (unsurprisingly) have a larger gap, $\Delta_0(x_{\mathrm{opt}})$, and a *smaller* value of the optimal doping, x_{opt}. (This latter correlation, which as far as we know has never been tested, is a slightly non-trivial prediction.)

C. What about phonons?

There are phonons in the cuprates – they are seen in neutron scattering and thermal conductivity. They show up clearly in the optical absorption spectrum, so they must involve charge motion. There is evidence in support of the obvious fact that they affect the electron dynamics obtained from an analysis of the ARPES spectra, and the Raman spectra.[81] Despite the moral injunction against mentioning the "P word" in certain company, it is respectable - even desirable - to think about the relevance of phonons for high temperature superconductivity.

Two obvious facts argue against the usual role for phonons in the mechanism. Firstly, there is the d-wave character of the superconductivity: most phonons are pair-breaking in the d-wave channel[82]. Secondly, the isotope effect is nearly zero at optimal doping; it is, of course, possible to have zero isotope effect even in the context of a conventional phonon-mediated BCS mechanism from a competition between the isotope dependence of the prefactor and $\mu^\star$. However, were this to occur precisely where T_c is maximum would smack of a joke by a malicious deity.

In underdoped cuprates, there is often an appreciable isotope effect, one that can be larger than those observed in simple metallic superconductors and which can apparently diverge as $x \to 1/8$ in some cases[83,84]. However, the fact that this isotope effect occurs where T_c is suppressed, and in

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[13] This is certainly an oversimplification. For instance, in $La_{2-x}Sr_xCuO_4$ the gap at all doping levels is much smaller than in $YBa_2Cu_3O_{6+y}$, at optimal doping.

[14] For graphical simplicity, we have assumed that in all cases, T_θ, which is proportional to the low frequency Drude weight, is linear in x, but the same qualitative physics is obtained if a more complex x dependence is assumed; what is important is that T_θ vanishes as x gets small (approaching the Mott insulator) and increases monotonically with increasing x.

particular its singular doping dependence near $x = 1/8$, suggests that the isotope effect is indirect as far as superconductivity is concerned, and is probably better thought of as an isotope dependent enhancement of the tendency to stripe order. In the underdoped regime, where the inhomogeneity is more than optimal, if replacing O^{17} with O^{18} tends to further stabilize the charge order, it will consequently tend to suppress the superconducting T_c.

D.   What about magnetism?

The empirical evidence suggests that antiferromagnetic correlations are an important feature of the electronic correlations in the cuprates, even when doped. Exactly what role this plays in the mechanism of HTC is much debated. It seems clear, by now, that whatever antiferromagnetism survives in the optimally doped superconductor is very short-ranged, so exchange of well defined magnon like elementary excitations cannot be the mechanism of HTC. In addition, as shown by Schrieffer[85], excitations that too closely resemble Goldstone modes decouple from the electrons, and so are particularly ineffective for inducing pairing. However, short range magnetic correlations can[86], and in our opinion are likely do play a role in the mechanism of HTC. These are the principle correlations responsible for the pair binding on Hubbard clusters.

In other strongly correlated systems, such as the manganites and nickelates, there is ample magnetism, but no superconductivity. Any mechanism that involves magnetism must rationalize why these other materials are not superconducting. In our view, there are several features that are responsible for this. The higher spin (spin 1/2 in the cuprates, spin 1 in the nickelates and spin 3/2 in the manganates) means that the magnetism is less quantum mechanical, and less easily quantum disordered in the presence of weak inhomogeneity. In addition, the presence of stronger electron phonon coupling and of other orbital degrees of freedom increases the tendency of these other materials to condense into other (non-superconducting) ordered ground states. In particular, the strong electron-phonon coupling in many standard perovskites, much enhances their tendency to form insulating "classical" charge-ordered states relative to the cuprates.

E.   Must we consider Cu-O chemistry and the three-band model?

It is a standard assumption in this field that the 2D Hubbard model, *i.e.* without any additional interactions or other embellishments, is "the Standard Model of Strongly Correlated Systems".[15] There are other 'simple models', such as the Emery or three-band model[87,88], which are more complicated (and hence "uglier") but which may be, in some ways, more "realistic." It is unclear to us whether the microscopic differences between the Emery and Hubbard models are essential to

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[15] The enshrinement of this simple model as a sort of "Theory of Everything" is peculiar in a field that stresses the fundamental importance of "emergent" and the misleading assertions of the "fundamental".

the mechanism of HTC, or unimportant. However, one thing that we have realized recently is that the Emery model, by virtue of its greater complexity, can be studied in various limits where certain aspects of the physics can be seen more simply and with better mathematical control than in the Hubbard model. For instance, the Emery model has an even stronger tendency to electronic phase separation that its simpler cousin, the Hubbard model. In addition, we have recently shown[89] the Emery model supports charge (Ising) nematic long range order, and probably other electron liquid crystal phases. (See also Refs.[90,91].) Hence, competing interactions over microscopic length scales can (and do) give rise to relevant mesoscale structures.

F. Is d-wave crucial?

The answer to this question depends on what one means by 'd-wave'. If by d-wave one means a precise symmetry under rotations by $\pi/2$ this is clearly not essential as many materials, notably $YBa_2Cu_3O_{6+y}$, are orthorhombic. In the particular case of $YBa_2Cu_4O_8$ the anisotropy is so large that the ratio of the superfluid densities in the a and b directions is as large as $\rho_s^a/\rho_s^b \sim 7$; this material is essentially quasi one-dimensional.[92] (At the very least, this means that there must be order one s-d mixing.) On the other hand, even in this case, the *sign* of the order parameter alternates as seen clearly in corner junction[93] and tri-crystal[94] experiments. So far, all the existing experimental evidence in the cuprates is consistent with "d-wave like" superconductivity, in this sense.

What is less evident is how essential are the nodal quasiparticles. The experimental evidence in most cuprates[81,95,96,97] is consistent with the existence of nodal excitations in the superconducting state[16], while they are either manifestly absent or poorly defined above T_c, in the pseudogap regime[81,97,98]. One of the puzzles of this problem, and one that makes it interesting, is why there are nodal quasiparticles below T_c even though they do not exist in the 'normal'. In the BCS mechanism, or in any other weak coupling approach, the quasiparticles of the superconducting state are a 'left-over' of the states of the parent normal (Fermi liquid) state. While it is clear that as the interactions become stronger the *symmetry* of the superconducting state may be 'protected', it is not obvious that the quasiparticles themselves should be. From the perspective of a strong coupling approach, such as the one advocated here which does not assume a state with well defined quasiparticles in the parent state, the nodal quasiparticles are an emergent phenomenon, and one can perfectly conceive a d-wave state with or without nodal quasiparticles. In fact, the transition between a node-less and nodal d-wave-like state was studied in Refs. [50,99], where it was found to be a mean field (Lifshitz) transition with relatively little effect on T_c.

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[16] In fact, even in the superconducting state the nodal quasiparticles in high temperature superconductors are never as well defined as in conventional metals, *e.g.* even at temperatures as low as 5 K, the energy width of a nodal quasiparticle is at least comparable to its energy.

G.   Is electron fractionalization relevant?

The discovery of high temperature superconductivity and the realization that the underlying physics of these systems is inconsistent with the venerable Landau Theory of the Fermi Liquid, launched an all-out effort to develop a "new" theory of strongly correlated systems. Many interesting and novel phases of matter were (and are) proposed, some of which were hoped to contain the fundamental (pardon our language) correlations responsible for high temperature superconductivity}, and in particular for the high values of T_c. Thus, in addition to the conventional Néel antiferromagnetic state, other non-magnetic ground states have been proposed, such as spin liquids with and without time-reversal symmetry breaking, as well as valence bond crystals which break translation and rotation invariance to various degrees.[46,47,100,101,102] However, perhaps following the "Bell Labs Rule" (a New Jersey version of Occam's Razor) that of all possible theories the most boring one (the one with the standard answer) is the one most likely to be correct, it has turned out that the ground states of simple models of undoped strongly correlated systems are typically antiferromagnets with long range Néel order.[103,104]

A number of interesting theories of spin liquid states, with[105,106] and without[27,107,108,109,110,111] time-reversal symmetry breaking, have been proposed over the years. Electron fractionalization and deconfinement are a defining feature of all these spin liquid phases. However, while recent advances in this subject[112] have put some of these proposals on firmer theoretical footing (by proving that they are the ground states of reasonably local Hamiltonians), most simple models of strongly correlated systems do not seem to naturally have these phases[4,45,101]. Moreover, in apparent accordance with the Bell Labs Rule, there is no compelling experimental evidence (yet) in support of their relevance, at least in the cuprates. Typically, the simple spin models thus far explored, even those with significant ring exchange interactions, have either spin ordered phases or valence bond ordered phases, and confinement on relatively short length scales, although there are known counterexamples[113]. We should note, however, that it is also possible to have phases with extremely long confinement length scales, *e.g.* the Cantor Deconfinement phases of Ref.[114], which for all practical purposes can do the job just as well.

As noted in Section II.A, both the spin liquid scenario and the mechanism explored here have in common the existence of a high energy pairing scale associated with spin-gap formation.

VII.   CODA: HIGH TEMPERATURE SUPERCONDUCTIVITY IS DELICATE BUT ROBUST

By whatever measure one might devise, the set of materials which exhibit high temperature superconductivity is a very small subset of electronically active materials. However, within the cuprates, materials that share the basic motif of Cu-O planes, high temperature superconductivity is robust in the sense that the transition temperature is not wildly sensitive to many sorts of chem-

ical substitutions, structural differences, and degrees of quenched disorder.[17] It seems reasonable to us to expect that any theory of high temperature superconductivity should be able to answer the question: why is high temperature superconductivity so rare?

Part of the answer is clearly the role of competing order. At weak coupling, the only instability of a Fermi liquid is the Cooper instability, so low temperature superconductivity should be (and is) reasonably generic. At strong coupling, many sorts of ordered states can be stabilized, including spin and charge density wave states, and more exotic states such as orbital antiferromagnetism[115] (dDW), which, in general, compete with superconductivity. Thus, precisely in those materials in which the couplings are strong enough that they could produce high T_c, other ordered phases occur which can quench the superconductivity substantially.

In our view, another feature is the necessity of an optimal degree (and character) of inhomogeneity - self-organized or otherwise. If the system is too homogeneous, then a high pairing scale is unattainable. If the system is too inhomogeneous, the coherence scale is strongly suppressed, and with it T_c. Obtaining a high T_c requires a rather delicate balance between these two extremes.

There are several other special features of the cuprates which likely also are essential. It seems to us that the fact that the cuprates are doped Mott insulators (with local moments), and that the insulating state in question is highly quantum mechanical (spin 1/2) are likely to be essential features of the physics, although the fact that the undoped system has a Néel ordered ground-state is probably not crucial. It is clear to us that overly strong electron-phonon coupling would produce too strong a tendency toward charge ordering[116], and hence would be destructive of high temperature superconductivity. From this point of view, the relatively *weakness* of the electron-phonon coupling in the cuprates in comparison with other perovskites (*e.g.* the nickelates and the manganates) is one of the important features of the cuprates that makes them high temperature superconductors. On the other hand, it seems to us likely that the tendency toward self-organized inhomogeneity found in theoretical studies of the Hubbard and related models is too weak to provide the necessary mesoscale inhomogeneity. In this sense, the electron-phonon coupling in the cuprates likely plays an important role in producing high temperature superconductivity - not that phonons serve as the glue but that they help with the self-organization of the necessary inhomogeneities.

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[17] T_c is sensitive to some changes, such as Cu substitution in the Cu-O planes and some features of the interplane arrangements and chemistry, so this statement has some exceptions.

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APPENDIX A: WHAT DEFINES "HIGH TEMPERATURE SUPERCONDUCTIVITY"

The term "high temperature superconductivity" is rather vague, since of course the question arises, high compared to what? From Fig. 1, it is clear that, from a material science viewpoint, high temperature superconductivity means T_c larger than 20 K. However, as an abstract issue in theory, it is less clear what is meant.

What we would like to find are models that are "physical," although not necessarily "realistic," and which have superconducting transition temperatures that are the of order of a microscopic energy scale. By "physical," we mean that the model must satisfy certain sets of constraints, such as having electrons with spin-1/2 which are fermions with dominantly repulsive bare interactions. Of course, in some sense, the closer a model is to reflecting the essential solid state chemistry of a particular material of interest, the more clearly physical it is, but for the purposes of understanding the mechanism, we would prefer to study as simple a model as possible, rather than one that has extraneous bells and whistles that happen to be part of the electronic structure of one material or another.

Alas, upon reflection, this rough definition of what constitutes high temperature superconductivity ceases to make any sense. Presumably, in any model in which the strength of the various interactions are all comparable to each other, if the model is superconducting at all, T_c must be equal to a number of order 1 times a microscopic scale. It then becomes a question of how big the number of order 1 must be to be considered high. (For the negative U Hubbard model with $U = -4t$ the superconducting transition temperature has been estimated[117] from quantum Monte-Carlo to be $T_c = 0.14t$. Putting aside the "unphysical" nature of the microscopically attractive interactions in this model, it is not clear whether one should or should not classify this as "high temperature superconductivity.")

We[36] have therefore proposed a different purely theoretical definition of HTC. In all cases we know of in which T_c can be computed reliably (other than by Monte-Carlo or related numerical methods), there is a small parameter, $\lambda \ll 1$, which is exploited in the calculation. In BCS theory, λ is the dimensionless electron-phonon coupling, and T_c depends exponentially on $1/\lambda$. If we agree that we can trust BCS theory when $\lambda < 1/5$ (to choose a number arbitrarily), this means that on the basis of this theory, we can claim to have a good understanding of the mechanism

of superconductivity only so long as T_c is at least two orders of magnitude smaller than the typical microscopic scale. In contrast, mechanisms we wish to associate with high temperature superconductivity should have a much weaker dependence on the small parameter, $T_c \propto \lambda^\alpha$, where the smaller α the better. For such a mechanism, say with $\alpha \sim 1$, if we accept the same criterion for the range of λ for which the theory is trustworthy, we have a valid theoretical understanding of the superconductivity even when T_c is fully 1/5 of a microscopic scale.

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