

Drifting solutions with elliptic symmetry for the compressible Navier-Stokes equations with density-dependent viscosity

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Abstract: In this paper, we investigate the analytical solutions of the compressible Navier-Stokes equations with dependent-density viscosity. By using the characteristic method, we successfully obtain a class of drifting solutions with elliptic symmetry for the Navier-Stokes model wherein the velocity components are governed by a generalized Emden dynamical system. In particular, when the viscosity variables are taken the same as Yuen in [Yuen M.W. (2008), *Analytical Solutions to the Navier-Stokes Equations*, J. Math. Phys. **49**, 113102], our solutions constitute a generalization of that obtained by Yuen.

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1 Introduction

In this paper, we consider the following compressible Navier-Stokes equations with density-dependent viscosity coefficients

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \quad (1)$$

$$(\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \operatorname{div}(h(\rho) D(\mathbf{U})) - \nabla(g(\rho) \operatorname{div} \mathbf{U}) + \nabla P(\rho) = \mathbf{0}, \quad (2)$$

where $t \in (0, +\infty)$ is the time and $\mathbf{x} \in R^N (N \geq 2)$ is the spacial coordinate, while $\rho(x, t)$ denotes the fluid density, $\mathbf{U} = \mathbf{U}(\mathbf{x}, t) = (u_1, u_2, \dots, u_N)$ stands for the fluid velocity and $P(\rho) = \kappa \rho^\gamma$ for the pressure, respectively. And

$$D(\mathbf{U}) = \frac{\nabla \mathbf{U} + {}^t \nabla \mathbf{U}}{2} \quad (3)$$

is the strain tensor, $h(\rho)$ and $g(\rho)$ are the Lamé viscosity coefficients satisfying

$$h(\rho) > 0, \quad h(\rho) + Ng(\rho) \geq 0. \quad (4)$$

Due to the significance of the Navier-Stokes (NS) equations in various physical fields such as fluid, plasmas, astrophysics, oceanography and atmospheric dynamics, the NS equations have been studied extensively and intensively, which is manifested by a large number of related papers. For example, the mathematical derivations were derived in the simulation of flow surface in shallow region [1, 2, 3]. The existence and uniqueness of the local strong solution were analyzed by Choe and Kim [4]. While, the existence of global weak solutions was discussed by Lions [5] and other authors [6, 7, 8, 9, 10]. There are also some interesting work done on analytical solutions of the NS equations. For instance, Yuen derived a class of self-similar solutions with radial symmetry for the NS equations with $h(\rho) = \kappa_1 \rho^\theta$ and $g(\rho) = 0$ in [11]. Subsequently, Yuen constructed some self-similar solutions with elliptic symmetry for the NS equations with $h(\rho) = \mu$ and $g(\rho) = 0$ in [12]. It is noticed that these two works were based on the separation method. Recently, by using the characteristic method, An and Yuen obtained a new class of perturbational solutions with elliptic symmetry for the NS equations in [13].

What needs to point out is that most works mentioned above only hold for NS equations with special viscosity coefficients $h(\rho)$ and $g(\rho)$. A natural idea is that the analytical solutions should also exist for the NS equations with general and reasonable viscosity coefficients $h(\rho)$ and $g(\rho)$. Since the choice of viscosity coefficients is key to obtain some physically important solutions. However, up to now, except the work of Guo and Xin [14], not much related work has been done. It is remarkable that here we derive the drifting solutions with elliptic symmetry for the compressible NS equations with density-dependent viscosity via a characteristic approach. Interestingly, numerical simulation results show that such solutions can be used to explain the drifting phenomena of the propagation wave like Tsunamis in oceans.

2 Drifting solutions of the NS equations

Here, we consider the general viscosity coefficients $h(\rho)$ and $g(\rho)$, which take a form of

$$h(\rho) = \kappa_1 \rho^\theta, \quad g(\rho) = \kappa_2 \rho^\theta. \quad (5)$$

Then, the compressible Navier-Stokes system with density-dependent viscosity coefficients become

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \quad (6)$$

$$\rho [\mathbf{U}_t + \mathbf{U} \cdot \nabla \mathbf{U}] - \operatorname{div}(\kappa_1 \rho^\theta D(\mathbf{U})) - \nabla(\kappa_2 \rho^\theta \operatorname{div} \mathbf{U}) + \nabla P(\rho) = \mathbf{0}. \quad (7)$$

For simplicity, we shall take $D(\mathbf{U}) = \nabla \mathbf{U}$ as what has been chosen by Guo and Xin in [14].

In the following, we shall give a lemma that proves important to the constructions of drifting solutions of NS equations with dependent-density viscosity.

Lemma *For the continuity equation of the NS system, namely:*

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \quad (8)$$

there exist solutions

$$\begin{cases} \rho = \frac{f\left(\frac{x_1-d_1}{a_1}, \frac{x_2-d_2}{a_2}, \dots, \frac{x_N-d_N}{a_N}\right)}{\prod_{i=1}^N a_i} \\ u_i = \frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i, \quad \text{for } i = 1, 2, \dots, N \end{cases} \quad (9)$$

where $d_i = d_i(t)$, $a_i = a_i(t) > 0$ and an arbitrary C^1 function $f \geq 0$.

Proof. Inspired by the work of [15, 16], we perturb the velocity as this form:

$$\rho = \rho(t, \mathbf{x}), \quad u_i = \frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i. \quad (10)$$

Substitution of this ansatz into the continuity equation (8), yields

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{U}) &= \rho_t + \nabla \rho \cdot \mathbf{U} + \rho \nabla \cdot \mathbf{U} \\ &= \frac{\partial}{\partial t} \rho + \sum_{i=1}^N \frac{\partial}{\partial x_i} \rho \left[\frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i \right] + \sum_{i=1}^N \frac{\rho \dot{a}_i}{a_i} = 0. \end{aligned} \quad (11)$$

According to the classical characteristic approach [17], we have

$$\frac{dt}{1} = \frac{dx}{\frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i} = \frac{d\rho}{-\sum_{i=1}^N \frac{\rho \dot{a}_i}{a_i}} \quad (12)$$

whence, the solution is

$$F \left(\prod_{i=1}^N a_i \rho, \frac{x_1 - d_1}{a_1}, \frac{x_2 - d_2}{a_2}, \dots, \frac{x_N - d_N}{a_N} \right) = 0 \quad (13)$$

with an arbitrary C^1 function F such that $\rho \geq 0$.

For convenience, we rewrite (13) into an explicit form

$$\rho = \frac{f \left(\frac{x_1 - d_1}{a_1}, \frac{x_2 - d_2}{a_2}, \dots, \frac{x_N - d_N}{a_N} \right)}{\prod_{i=1}^N a_i}. \quad (14)$$

Therefore, the proof is completed. ■

Remark 1: It is necessary to point out that the negative symbol in the perturbational **non-constant** functions d_i for the velocity in (9) is critical to guarantee the use of the characteristic method.

On the application of the above lemma, we construct a class of drifting solutions with elliptic symmetry for the Navier-Stokes equations (6)-(7). The main result is described as follows:

Theorem 1 *For the compressible Navier-Stokes equations with dependent-density viscosity coefficients, there exists a class of drifting solutions:*

$$\begin{cases} \rho = \frac{f(s)}{\prod_{k=1}^N a_k} \\ u_i = \frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i, \quad \text{for } i = 1, 2, \dots, N \end{cases} \quad (15)$$

where

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2\theta}s} & \text{for } \theta = 1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\theta}s + \alpha\right)^{\frac{1}{\theta-1}}, 0\right) & \text{for } \theta \neq 1 \end{cases} \quad (16)$$

with

$$d_i = d_{i0} + td_{i1}, \quad s = \sum_{k=1}^N \left(\frac{x_k - d_k}{a_k} \right)^2. \quad (17)$$

In the above ξ, d_{i0}, d_{i1} and $\alpha \geq 0$ are arbitrary constants. While the auxiliary functions $a_i = a_i(t)$ are governed by the generalized Emden dynamical system:

$$\begin{cases} \ddot{a}_i(t) = \frac{-\xi \left[k_1 \sum_{k=1}^N \frac{\dot{a}_k(t)}{a_k(t)} + k_2 \frac{\dot{a}_i(t)}{a_i(t)} - \kappa \right]}{a_i(t) \left(\prod_{k=1}^N a_k(t) \right)^{\frac{1}{\theta-1}}}, & \text{for } i = 1, 2, \dots, N \\ a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1} \end{cases} \quad (18)$$

where a_{i0} and a_{i1} are initial conditions.

In particular, for $\xi < 0$,

(1) if all $a_{i1} < 0$, the solutions (15) blow up on or before the finite time

$$T = \min(-a_{i0}/a_{i1} : a_{i1} < 0, i = 1, 2, \dots, N); \quad (19)$$

(2) if all $a_{i1} \geq 0$ the solutions (15) exist globally.

Remark 2: We emphasize that the intrusion of the viscosity coefficients $h(\rho)$ and $g(\rho)$ not only makes the solutions are quite different from what were discussed by Guo et al [14] and Yuen [11], but also makes the occurrence of a generalized Emden dynamical system.

Remark 3: It is known that the Navier-Stokes equations can be used to describe the drifting phenomena of the propagation wave like Tsunamis in oceans. Interestingly, numerical simulations fully exhibit such drifting behaviors. Therefore, we call (15) the drifting solution and the linear time-dependent functions d_i are the drifting terms. When these functions d_i degenerate to constants, namely $d_{i1} = 0$ and $d_{i0} = \text{const}$, they coincide with the case that was discussed by Yuen in [12].

Proof of Theorem 1. According to the Lemma, it is easy to check that the function (15) satisfies the continuity equation (6). In the following, we shall validate that the function (15) also holds for the momentum equation (7).

For the i -th momentum equation of the Navier-Stokes equations (7), by defining an elliptically symmetric variable via

$$s = \sum_{k=1}^N \frac{(x_k - d_k)^2}{a_k^2(t)}. \quad (20)$$

Now we proceed with $\gamma = 2$, then we obtain

$$\begin{aligned} & \rho \left[\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k} \right] - \kappa_1 \frac{\partial}{\partial x_i} (\rho^\theta \nabla \cdot \vec{u}) - \kappa_2 \nabla \cdot (\rho^\theta \nabla u_i) + \kappa \frac{\partial}{\partial x_i} \rho^\theta \\ &= \rho \left\{ \frac{\partial}{\partial t} \left[\frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i \right] + \left[\frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i \right] \frac{\partial}{\partial x_i} \left[\frac{\dot{a}_i}{a_i} (x_i - d_i) + \dot{d}_i \right] \right\} \\ &\quad - \kappa_1 \theta \sum_{k=1}^N \frac{\dot{a}_k}{a_k} \rho^{\theta-1} \frac{\partial \rho}{\partial x_i} - \kappa_2 \theta \rho^{\theta-1} \frac{\dot{a}_i}{a_i} \frac{\partial \rho}{\partial x_i} + \kappa \theta \rho^{\theta-1} \frac{\partial \rho}{\partial x_i} \\ &= \rho \left\{ \left[\left(\frac{\ddot{a}_i}{a_i} - \frac{\dot{a}_i^2}{a_i^2} \right) (x_i - d_i) + \ddot{d}_i + \frac{\dot{a}_i^2}{a_i^2} (x_i - d_i) \right] - \theta \rho^{\theta-2} \left(\kappa_1 \sum_{k=1}^N \frac{\dot{a}_k}{a_k} + \kappa_2 \frac{\dot{a}_i}{a_i} - \kappa \right) \frac{\partial \rho}{\partial x_i} \right\} \\ &= \rho \left\{ \left[\left(\frac{\ddot{a}_i}{a_i} - \frac{\dot{a}_i^2}{a_i^2} \right) (x_i - d_i) + \ddot{d}_i + \frac{\dot{a}_i^2}{a_i^2} (x_i - d_i) \right] - \theta \left(\kappa_1 \sum_{k=1}^N \frac{\dot{a}_k}{a_k} + \kappa_2 \frac{\dot{a}_i}{a_i} - \kappa \right) \frac{f(s)^{\theta-2}}{\left(\prod_{k=1}^N a_k \right)^{\theta-1}} \frac{\partial}{\partial x_i} \frac{f(s)}{\prod_{k=1}^N a_k} \right\} \\ &= \rho \left\{ \frac{\ddot{a}_i}{a_i} (x_i - d_i) + \ddot{d}_i - 2\theta \left(\kappa_1 \sum_{k=1}^N \frac{\dot{a}_k}{a_k} + \kappa_2 \frac{\dot{a}_i}{a_i} - \kappa \right) \frac{f(s)^{\theta-2} \dot{f}(s)}{\left(\prod_{k=1}^N a_k \right)^{\theta-1}} \left(\frac{x_i - d_i}{a_i^2} \right) \right\} \\ &= \frac{\rho(x_i - d_i)}{a_i^2} \left\{ \ddot{a}_i a_i - \frac{2\theta f(s)^{\theta-2} \dot{f}(s)}{\left(\prod_{k=1}^N a_k \right)^{\theta-1}} \left(\kappa_1 \sum_{k=1}^N \frac{\dot{a}_k}{a_k} + \kappa_2 \frac{\dot{a}_i}{a_i} - \kappa \right) \right\} + \rho \ddot{d}_i \\ &= \frac{\rho(x_i - d_i)}{a_i^2} \left\{ \xi + 2\theta f(s)^{\theta-2} \dot{f}(s) \right\} + \rho \ddot{d}_i \end{aligned} \quad (22)$$

with the N -dimensional generalized Emden dynamical system given by :

$$\begin{cases} \ddot{a}_i(t) = \frac{-\xi \left[\kappa_1 \sum_{k=1}^N \frac{\dot{a}_k(t)}{a_k(t)} + \kappa_2 \frac{\dot{a}_i(t)}{a_i(t)} - \kappa \right]}{a_i(t) \left(\prod_{k=1}^N a_k(t) \right)^{\theta-1}}, & \text{for } i = 1, 2, \dots, N \\ a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1} \end{cases} \quad (23)$$

with arbitrary constants ξ , a_{i0} and a_{i1} .

If we require the function $f(s)$ satisfies the following differential equation:

$$\begin{cases} \frac{\xi}{2\theta} + f(s)^{\theta-2} \dot{f}(s) = 0 \\ f(0) = \alpha \geq 0, \end{cases} \quad \text{or} \quad \rho = 0 \quad (24)$$

then we can have

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2\theta}s} & \text{for } \theta = 1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\theta}s + \alpha\right)^{\frac{1}{\theta-1}}, 0\right) & \text{for } \theta \neq 1. \end{cases} \quad (25)$$

Therefore, the function (15) is the drifting solution with elliptic symmetry of the compressible Navier-Stokes equations with dependent-density viscosity. ■

3 Conclusion and Discussion

Due to the importance of the Naiver-Stokes equations in various branches of physics, many experts have paid great attention to them, especially to the constructions of analytical solutions. For example, when the viscosity coefficients are chosen by $h(\rho) = \kappa_1\rho^\theta, g(\rho) = 0$, Yuen obtained the self-similar solutions in [11]. Yuen also constructed some self-similar solutions with elliptic symmetry when the viscosity variables are $h(\rho) = \mu$ and $g(\rho) = 0$ in [12]. Guo and Xin derived some analytical solutions when $h(\rho) = \rho^\theta$ and $g(\rho) = (\theta - 1)\rho^\theta$ in [14]. Interestingly, here we successfully derived some drifting solutions with elliptic symmetry for the compressible Navier-Stokes equations with general forms of viscosity coefficients. Numerical simulations show that the analytical solutions obtained can be applied to explain the drifting phenomena of propagations of wave like Tsunamis in oceans. In addition, we would like to point out that the velocity components a_i are governed by the generalized Emden dynamical system, which is quite different from that the classical Emden equations obtained in [11, 12]. Does the generalized Emden system have any nice properties as the classical one? What is the relation between the generalized Emden system and the Ermakov system? These problems will be deeply considered in our future work.

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