

# The mass formula for a fundamental string as a BPS solution of a D-brane's worldvolume

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## Abstract

We propose a (generalized) “mass formula” for a fundamental string described as a BPS solution of a D-brane’s worldvolume. The mass formula is obtained by using the Hamiltonian density on the worldvolume, based on transformation properties required for it. Its validity is confirmed by investigating the cases of point charge solutions of D-branes in a D-8-brane (i.e. curved) background, where the mass of each of the corresponding strings is proportional to the geodesic distance from the D-brane to the point parametrized by the (regularized) value of a transverse scalar field. It is also shown that the mass of the string agrees with the energy defined on the D-brane’s worldvolume only in the flat background limit, but the agreement does not always hold when the background is curved.

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# 1 Introduction

To gain better understanding of string theories and M-theory, intersecting branes have played an important role, and worldvolume analyses have been powerful approaches to investigate the intersecting branes[1][2][3][4][5]. In ref.[1][2][3] solutions of worldvolume field theories of branes (in flat backgrounds) with nontrivial worldvolume gauge fields were obtained. In the case of D-branes, it was shown that an appropriate excitation of one of the transverse scalar fields is needed in order to obtain a *supersymmetric* (i.e. BPS) point charge solution of the worldvolume gauge field[1]. Each of the solutions was interpreted as a fundamental string ending on the D-brane on the basis of the fact that the (regularized) energy of each solution defined on the worldvolume is proportional to the (regularized) value of the scalar field, which is considered to be the length of the string.<sup>†</sup> This interpretation is also consistent with the charge conservation suggested in ref.[6] (see also ref.[7]). These analyses are very important in that they made clear “how one of intersecting two branes is described” from the viewpoint of another brane’s worldvolume.

In fact, however, *the energy defined on the brane’s worldvolume does not always agree with the target-space mass of the string*, though the agreement of the two holds for the case of ref.[1]. We can easily understand this by considering the fact that the worldvolume energy depends on the definition of the time of the worldvolume, while the mass of the string should not depend on it. That is, the derivation of the mass of the string must be considered more carefully. The main purpose of this paper is to present a generalized “mass formula” of a fundamental string described as a BPS solution of a D-brane’s worldvolume, which holds independent of definition of its worldvolume time.

Our idea to obtain the mass formula is as follows: the mass of the string does not always agree with the worldvolume energy, to be sure, but it is also true that the two are very close to each other, since the two give the same results at least in the cases discussed in ref.[1]. So, we construct the mass formula in a heuristic way, by using the Hamiltonian density defined on the worldvolume, based on the invariance of the mass under the coordinate transformations of the worldvolume. Moreover, we also construct explicitly point charge solutions (with appropriate excitations of single transverse scalar fields) of branes’ worldvolumes in a *curved* background.<sup>‡</sup> There are two advantages to consider these solutions: First, in each of the cases, the time component of the induced worldvolume metric  $\tilde{g}_{00}$  (as well as spatial ones) becomes nontrivial, So, the differences

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<sup>†</sup>Since the energies of the point charge solutions are infinite, (UV) regularization is needed to calculate them.

<sup>‡</sup>Some worldvolume solutions of branes in a curved (brane) backgrounds are discussed for other purposes in ref.[8][9].

originating from the contribution of  $\tilde{g}_{00}$  become apparent in discussing the mass formula. Second, in each case of the solutions, *the mass of the corresponding fundamental string should be proportional to the geodesic distance from the brane to the point parameterized by the (regularized) value of the scalar field*, and that the proportional coefficient should be an appropriate tension of the string, as discussed in ref.[10]. This requirement is very tight. So, once we find out the quantity which gives the mass stated above, it is expected to be the correct mass formula, even if it is constructed by hand. We will construct it based on this idea. (The discussion on the worldvolume interpretation of the string mass will be given finally in section 3.)

The worldvolume theories we discuss here are the two cases: those of a test D-4-brane and a test D-8-brane both embedded parallel to (a subspace of) the worldvolume of the D-8-brane background[11][12] (i.e. a massive IIA background). First, we present the two reasons to choose this background: One is that this background has only one transverse coordinate, leading to the fact that the harmonic function depends linearly on the coordinate. Only in this case, we can obtain *explicitly* the exact solutions of the worldvolumes without any extra assumption (as we will see later). We note that obtaining the explicit form is crucial not only in order to find out the mass formula *by hand* and but for other discussions. Another reason is that choosing this background, we can see the supersymmetry preserved in the solutions by using superalgebras in a massive IIA background via brane probes” in ref.[13]. <sup>§</sup> From the supersymmetry, we can confirm that the solutions are BPS states and that their target-space interpretation is consistent. Next, we explain why we choose the two worldvolume theories embedded in those ways: This is because at least one overall transverse space is needed *after* embedding (test) D-branes’ worldvolumes in order to obtain supersymmetric point charge solutions. In the case of a D-2-brane (a D-6-brane), only the intersection with the background D-8-brane on a string (a 5-brane) leads to the preservation of supersymmetry,<sup>¶</sup> but, there is no overall transverse space. A D-0-brane is not adequate to this worldvolume description since there is no world space. So, we consider the above two cases, which preserve at this moment 1/4 and 1/2 supersymmetry, respectively.

Concrete procedures are in the following: In each of the two cases we construct explicitly an point charge solution with an appropriate excitation of the only overall transverse scalar field. The consistency of the interpretation of each solution as a fundamental string is confirmed in two ways: by discussing its behavior in the flat background limit and by

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<sup>§</sup>In the previous paper[9] we have confirmed that the supersymmetry preserved in the solution of brane’s worldvolume can be derived via “superalgebras in brane backgrounds” (see also [14]).

<sup>¶</sup>We do not discuss bound states like (6|D6,D8) since the D-8-brane solution is “singular” just on the D-8-brane hyper-surface.

checking its preserved supersymmetry. Then, we propose the generalized mass formula, and examine whether it gives the geodesic distance multiplied by an appropriate tension of the string. Moreover, we also discuss the condition that the mass of the string agrees with the energy defined on the D-brane's worldvolume.

The organization of this paper is as follows: In section 2 we construct the point charge solutions and discuss their mass formula. In section 3 we give the conclusion and some discussions on the consistency and the interpretation of the mass formula

The notations in this paper is as follows: We use “mostly plus” metrics for both spacetime and worldvolumes. We denote coordinates of each p-brane's worldvolume as  $\xi^i, \xi^j, \dots$  ( $i,j = 0,1,\dots,p$ ), those of 10D spacetime as  $x^m, x^n, \dots$  ( $m,n=0,\dots,9$ ), fermionic coordinates as  $\theta^\alpha, \theta^\beta, \dots$ , and those of superspace as  $Z^M$ . We use *hatted letters* ( $\hat{M}, \hat{m}, \hat{\alpha} \dots$ ) for *all the local Lorentz frame indices* and *under-barred letters* ( $\underline{m}, \underline{i}$ ) for *spatial indices* (but not time one), respectively. We denote gamma matrices as  $\Gamma_{\hat{m}}$ , which are all real and satisfy  $\{\Gamma_{\hat{m}}, \Gamma_{\hat{n}}\} = 2\eta_{\hat{m}\hat{n}}$ .  $\Gamma_{\hat{0}}$  is antisymmetric and others symmetric. Charge Conjugation is  $\mathcal{C} = \Gamma^{\hat{0}}$ .

## 2 Point charge solutions of 10D IIA D-branes' world-volumes and their mass formula

In this section we construct the point charge solutions corresponding to fundamental strings and discuss their mass formula.

The D-p-brane action in a general 10D massive IIA background[15][12] takes the form[16][17]

$$S_{Dp} = S_{Dp}^{BI} + S_{Dp}^{WZ} = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det(\tilde{g}_{ij} + \mathcal{F}_{ij})} + T_p \int [\tilde{C} e^{\mathcal{F}}]_{(p+1)-\text{form}} + \frac{m}{(p/2 + 1)!} V(dV)^{p/2}, \quad (2.1)$$

where  $\tilde{g}_{ij} = E_i{}^{\hat{m}} E_j{}^{\hat{n}} \eta_{\hat{m}\hat{n}}$  is the induced worldvolume metric where  $E_i{}^{\hat{m}} = \partial_i Z^M E_M{}^{\hat{m}}$  where  $E_M{}^{\hat{N}}$  is the supervielbein.  $\phi$  is the dilaton field and  $\mathcal{F}_{ij}$  are the components of a modified worldvolume 2-form field strength

$$\mathcal{F} = dV - \tilde{B}_2 \quad (2.2)$$

of the worldvolume 1-form gauge field  $V$  where  $\tilde{B}_2$  is the worldvolume 2-form induced by the superspace NS-NS 2-form gauge potential  $B_2$ .  $\tilde{C}$  is a formal sum of worldvolume r-forms  $\tilde{C}^{(r)}$  ( $r=1,3,5,7,9$ ) induced by the superspace R-R r-form gauge potentials  $C^{(r)}$ .  $m$  is a mass parameter which is the dual of the 10-form field strength  $F^{(10)}$  of a R-R 9-form

$C^{(9)}$ [18][12].  $T_p$  is the “formal” (but not physical) tension of the D-p-brane which is given by[16]\*

$$T_p = \frac{1}{(2\pi)^p}. \quad (2.3)$$

We take the background of the action (2.1) to be the D-8-brane solution given by[11][12]

$$\begin{aligned} ds^2 &= H^{\epsilon/2} dx^\mu dx^\nu \eta_{\mu\nu} + H^{-5\epsilon/2-2} dy^2 \\ e^\phi &= H^{5\epsilon/4} \\ C_{01\dots 8}^{(9)} &= H^\epsilon \end{aligned} \quad (2.4)$$

where  $x^\mu$  and  $x^\nu$  ( $\mu, \nu=0,\dots,8$ ) are the spacetime coordinates parallel to the D-8-brane and  $y$  is a single transverse coordinate.  $\epsilon$  is a nonzero parameter which cannot be determined by the equations of motions of 10D massive IIA supergravity. We note that the solution (2.4) with  $\epsilon = -1$  is the standard form of the D-8-brane solution since it is obtained via T-duality from the other D-p-brane solutions[12].  $H = H(y)$  is a harmonic function on  $y$ . In this paper we set

$$H(y) = c_1 + \frac{m}{|\epsilon|} |y|, \quad (2.5)$$

which means that the D-8-brane lies at  $y = 0$ . We choose  $c_1 > 0$  and  $m > 0$  to avoid a singularity at  $y = 0$  and to get a real dilaton. We note that the solution (2.4) becomes the flat spacetime metric in the massless limit

$$\begin{cases} m \rightarrow 0 \\ c_1 \rightarrow 1 \text{ ( via diffeomorphism).} \end{cases} \quad (2.6)$$

The Killing spinor of (2.4) has the form  $\varepsilon = H^{\epsilon/8} \varepsilon_0$  where  $\varepsilon_0$  has a definite chirality, i.e.  $\Gamma_{\hat{y}} \varepsilon_0 = +\varepsilon_0$  for  $y > 0$  and  $\Gamma_{\hat{y}} \varepsilon_0 = -\varepsilon_0$  for  $y < 0$ .

We first consider a point charge solution of a D-4-brane worldvolume parallel to the background D-8-brane. Since the solutions we construct here is a bosonic one, we set fermionic coordinates  $\theta$  to be zero. Moreover, we consider the ansatz

$$\begin{cases} x^i = \xi^i & (i = 0, 1, \dots, 4) \\ x^5, x^6, x^7, x^8 : & \text{constants} \\ y = y(r) > 0 \\ V_0 = V_0(r), & V_i = 0 \end{cases} \quad (2.7)$$

where  $r$  is defined as  $r \equiv \sqrt{\sum_{i=1}^4 (\xi^i)^2}$ . We note that the upper two columns of (2.7) mean that the D-4-brane is embedded in the 1234-hyper-plane. That is, from target-space point of view, we consider the following intersection of three branes:

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\* We choose  $\alpha' = 1$  in this paper.

background D8 (at $y = 0$ ):	0	1	2	3	4	5	6	7	8	-
worldvolume D4 (at $y = y_0$ ):	0	1	2	3	4	-	-	-	-	-
fundamental string :	0	-	-	-	-	-	-	-	-	9

where  $y_0$  is a positive constant (and  $x^9 = y$ ).

Then, since  $V_{\underline{i}} = \theta = 0$ ,  $S_{D4}^{WZ}$  does not contribute to the equations of motion, and the equations of motion to solve are given by

$$\frac{\delta \mathcal{L}^{BI}}{\delta x^m} = \partial_i \left[ \frac{\delta \mathcal{L}^{BI}}{\delta \partial_i x^m} \right], \quad \frac{\delta \mathcal{L}^{BI}}{\delta V_i} = \partial_j \left[ \frac{\delta \mathcal{L}^{BI}}{\delta \partial_j V_i} \right]. \quad (2.8)$$

So, let us examine more about  $\mathcal{L}^{BI}$ . The induced worldvolume metric  $\tilde{g}_{ij}$  is given by

$$\tilde{g}_{ij} = \begin{pmatrix} -H^{\epsilon/2} & 0 \\ 0 & H^{\epsilon/2} \cdot [\delta_{ij} + H^{-3\epsilon-2} \partial_{\underline{i}} y \partial_{\underline{j}} y] \end{pmatrix}, \quad (2.9)$$

whose determinant is  $\det \tilde{g}_{ij} = H^{5\epsilon/2} [1 + H^{-3\epsilon-2} (\partial y)^2]$ . At this moment,  $\det(\tilde{g}_{ij} + \mathcal{F}_{ij})$  arising in  $\mathcal{L}^{BI}$  is very complicated. However, setting the condition

$$\partial_{\underline{i}} y = \partial_{\underline{i}} V_0, \quad (2.10)$$

results in the simple form of the determinant:  $\det(\tilde{g}_{ij} + \mathcal{F}_{ij}) = H^{5\epsilon/2}$ .<sup>†</sup> Then, the equations of motion (2.8) become the following two simple equations

$$\sum_{\underline{i}} \partial_{\underline{i}} (H^{-3\epsilon-2} \partial_{\underline{i}} y) = 0 \quad (2.11)$$

$$\sum_{\underline{i}} \partial_{\underline{i}} (H^{-2\epsilon-1} \partial_{\underline{i}} y) = 0. \quad (2.12)$$

((2.11) arises from the 9th component of the former of (2.8), (2.12) from the time component of the latter of (2.8), and the others are solved.) So, requiring the two equations to be compatible, it needs to hold  $\epsilon = -1$ , and the two are combined into one equation

$$\sum_{\underline{i}} \partial_{\underline{i}} (H \partial_{\underline{i}} y) = 0. \quad (2.13)$$

We note that the harmonic function gives nontrivial contribution to (2.13), which means that the equations of motion of the D-4-brane worldvolume are affected by the background D-8-brane. By using (2.5) and  $y > 0$ , the equation (2.13) is written as

$$\sum_{\underline{i}} (\partial_{\underline{i}})^2 (y + \frac{c_1}{m})^2 = 0. \quad (2.14)$$

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<sup>†</sup> It is shown that no supersymmetry is preserved in this case without (2.10), by using a superalgebra via brane probe appearing later (eq.(2.20)). So, (2.10) is considered to correspond to the BPS condition.

We choose the boundary condition

$$\begin{cases} y & \rightarrow y_0 (> 0) \\ V_0 & \rightarrow 0 \end{cases} \quad (2.15)$$

for  $r \rightarrow \infty$ , which means that the D-4-brane lies at  $y = y_0$ . Then, the solution is obtained, with the following unusual form, as

$$\begin{aligned} y(r) &= [(\frac{c_1}{m} + y_0)^2 + \frac{c_2}{r^2}]^{1/2} - \frac{c_1}{m} \\ V_0 &= [(\frac{c_1}{m} + y_0)^2 + \frac{c_2}{r^2}]^{1/2} - (\frac{c_1}{m} + y_0) \end{aligned} \quad (2.16)$$

where  $c_2$  is a constant proportional to the electric charge of the gauge field. The electric charge  $Q_1$  is defined as [3]

$$Q_1 = \int_{S_3} \star D \quad (2.17)$$

where  $S_n$  is the n-sphere,  $\star$  is the worldvolume Hodge dual and  $D$  is the 2-form defined by  $D^{ij} = -\frac{1}{T_4} \frac{\delta \mathcal{L}_{D4}}{\delta F_{ij}}$ . Then, we have

$$Q_1 = \int_{S_3} \star (H dV) = m c_2 \Omega_3 \quad (2.18)$$

where  $\Omega_n$  is the volume of the unit n-sphere. We note that in this definition the string coupling  $g_s = e^\phi$  is included in  $Q_1$  since  $\mathcal{L}_{D4}^{BI}$  is proportional to the inverse of  $g_s$ .

Since the solution has been obtained explicitly, we next give some pieces of evidence that the solution corresponds to the fundamental string ending on the D-4-brane. First, we discuss the massless limit of the solution (2.16). We assume here that the charge  $Q_1$  is independent of  $m$ . (The validity of the assumption is discussed later.) Then, in the massless limit (2.6), the solution (2.16) with (2.18) behaves as

$$\begin{aligned} y &\rightarrow y_0 + \frac{Q_1}{2\Omega_3 r^2} \\ V_0 &\rightarrow \frac{Q_1}{2\Omega_3 r^2}. \end{aligned} \quad (2.19)$$

The right hand side of (2.19) is exactly the solution of a D-4-brane's worldvolume in the 10D *flat* spacetime, which corresponds to the fundamental string[1]. So, it is expected to correspond to a fundamental string. Next, we check the preserved supersymmetry of the solution by using “superalgebras in brane backgrounds via brane probes”[14][13][9]. The superalgebra in a D-8 brane background via a D-4-brane probe is given in ref. [13] as

$$\begin{aligned} \{Q_\alpha^+, Q_\beta^+\} &= 2 \int_{M_4} d^4\xi \Pi_\mu (\mathcal{C}\Gamma^\mu)_{\alpha\beta} + 2 \int_{M_4} d^4\xi \mathcal{P}^{(0)i} \partial_i y (\mathcal{C}\Gamma_y \Gamma_{11})_{\alpha\beta} \\ &\quad + \frac{2T_4}{4!} \int_{M_4} H^{5/4} dx^{\mu_1}..dx^{\mu_4} (\mathcal{C}\Gamma_{\mu_1..\mu_4} \Gamma_{11})_{\alpha\beta} + 2T_4 \int_{M_4} H^{5/4} dx^\mu dy dV (\mathcal{C}\Gamma_{\mu y})_{\alpha\beta} \\ &\quad + 2T_4 \int_{M_4} H^{5/4} (dV)^2 (\mathcal{C}\Gamma_{11})_{\alpha\beta} + \mathcal{O}(\theta^2) \end{aligned} \quad (2.20)$$

where  $Q_\alpha^+ = \frac{1+\Gamma_{\hat{y}}}{2} Q_\alpha$  is the supercharge preserved in (2.4) and  $\mathcal{M}_p$  is the worldspace of the D-p-brane.  $\mathcal{P}^{(0)\underline{i}}$  is almost equivalent to the conjugate momentum of  $V_{\underline{i}}$ . (The contributions of the Chern-Simons term are subtracted.) Substituting the solution for the right hand side of (2.20), the superalgebra can be written as

$$\{Q_\alpha^+, Q_\beta^+\} = 4T_4 \int_{\mathcal{M}_4} d^4\xi [H^{1/4}(\frac{1+\mathcal{C}\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}}\Gamma_{11}}{2})_{\alpha\beta} + H^{5/4}(\partial_{\hat{y}})^2(\frac{1+\mathcal{C}\Gamma_{\hat{y}}\Gamma_{11}}{2})_{\alpha\beta}] \quad (2.21)$$

Since the three gamma matrix products  $\Gamma_{\hat{y}}, \mathcal{C}\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}}\Gamma_{11}$  and  $\mathcal{C}\Gamma_{\hat{y}}\Gamma_{11}$  (arising in (2.21)) commute with each other, all of them can be simultaneously diagonalized. Since the square of each matrix product is equal to the identity and each is traceless, both of the matrices  $\frac{1+\mathcal{C}\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}}\Gamma_{11}}{2}$  and  $\frac{1+\mathcal{C}\Gamma_{\hat{y}}\Gamma_{11}}{2}$  are projection operators. So, we conclude that the solution has  $1/8$  supersymmetry, hence is consistent with the target-space interpretation. This also shows that the solution is a BPS state.

Now, we discuss the mass of the string. Let us first consider the energy of the solution. For this purpose, we pass to the Hamiltonian formalism as done in ref.[1]. If we assume that  $V$  is purely electric and that only the scalar  $y$  is excited,  $S_{D4}$  reduces to

$$S_{D4} = -T_4 \int d^5\xi \sqrt{\{1 - H(F_{0\underline{i}})^2\}\{1 + H(\partial_{\underline{i}}y)^2\} + H^2(F_{0\underline{i}}\partial_{\underline{i}}y)^2 - H\dot{y}^2} \quad (2.22)$$

(where  $H$  is the harmonic function). The canonical momenta of  $y$  and  $V_{\underline{i}}$ , are defined respectively as

$$\begin{aligned} P &= \frac{T_4 H \dot{y}}{\sqrt{\{1 - H(F_{0\underline{i}})^2\}\{1 + H(\partial_{\underline{i}}y)^2\} + H^2(F_{0\underline{i}}\partial_{\underline{i}}y)^2 - H\dot{y}^2}} \\ \Pi_{\underline{i}} &= \frac{T_4 H [F_{0\underline{i}}\{1 + H(\partial_{\underline{i}}y)^2\} - H\partial_{\underline{i}}y(F_{0\underline{j}}\partial_{\underline{j}}y)]}{\sqrt{\{1 - H(F_{0\underline{i}})^2\}\{1 + H(\partial_{\underline{i}}y)^2\} + H^2(F_{0\underline{i}}\partial_{\underline{i}}y)^2 - H\dot{y}^2}}. \end{aligned} \quad (2.23)$$

The Hamiltonian  $\bar{H}$  is constructed as  $\bar{H} \equiv \int_{\mathcal{M}_4} d^4\xi \mathcal{H}$  where  $\mathcal{H}$  is the Hamiltonian density given by

$$\mathcal{H} = T_4 \sqrt{\{1 + H(\partial_{\underline{i}}y)^2\}(1 + T_4^{-2}H^{-1}P^2) + T_4^{-2}H^{-1}(\Pi_{\underline{i}})^2 + T_4^{-2}(\Pi_{\underline{i}}\partial_{\underline{i}}y)^2}. \quad (2.24)$$

We note that  $\Pi_{\underline{i}}$  is subject to the constraint  $\partial_{\underline{i}}\Pi_{\underline{i}} = 0$ . Substituting the solution (2.16) for  $\bar{H}$ , we can obtain the energy of the solution  $E$  defined on the worldvolume. We note that for a BPS solution like this case, it generically happens that the square root of  $\bar{H}$  becomes a perfect square and that the energy becomes a sum of the two parts: the part originating from the D-p-brane and that from the string. so, we denote the first part of  $\mathcal{H}$  as  $\mathcal{H}_1$  and the second part as  $\mathcal{H}_2$ . Concretely, the energy in this case takes the form

$$E = T_4 \int_{\mathcal{M}_4} d^4\xi [1 + H(\partial_{\underline{i}}y)^2] \equiv E_1 + E_2. \quad (2.25)$$

The first term  $E_1$  is the “energy” of the D-4-brane itself, and the second term  $E_2$  is the energy of the excitation (i.e. the string), both defined on the worldvolume. Since we are interested in the second part, we compute only  $E_2$  here.  $E_2$  is infinite in this case, but if we regularize it by introducing a small parameter  $\delta$ , we can get the energy for  $r \geq \delta$  as

$$\begin{aligned} E_2 &= T_4 \int d\Omega_3 \int_{\delta}^{\infty} r^3 dr (c_1 + my(r)) (\partial_{\underline{i}} y(r))^2 \\ &= T_4 Q_1 (y(\delta) - y_0). \end{aligned} \quad (2.26)$$

That is, the energy is (again) proportional to the difference of the coordinate. Thus, we conclude that *the energy defined on the brane’s worldvolume does not agree with the mass of the string in the case of D-branes in curved backgrounds*. We note that this result is rather reasonable, in a sense, in the case of  $\tilde{g}_{00} \neq 1$ , because the energy has the same transformation property as  $\partial_0$  under the reparametrization of  $\xi^0$ .

Now, we construct the mass formula. Since it should be invariant under the reparametrization of  $\xi^0$ , we propose the generalized mass formula  $M$  for a string described as a solution of a D-p-brane’s worldvolume, as

$$M = \int_{\mathcal{M}_p} d^p \xi \sqrt{-\tilde{g}^{00}} \mathcal{H}_2 \quad (2.27)$$

where  $\mathcal{H}_2$  is the second part of the Hamiltonian density defined on the D-p-brane (originating from the excitation corresponding to a string).

Let us calculate the mass of the string in this case, based on the formula. Substituting the solution for (2.27), we find

$$\begin{aligned} M(\delta) &= T_4 \int_{\mathcal{M}_4} d^4 \xi H^{5/4} (\partial_{\underline{i}} y)^2 = T_4 \int d\Omega_3 \int_{\delta}^{\infty} r^3 dr (c_1 + my(r))^{5/4} (\partial_{\underline{i}} y)^2 \\ &= \frac{4T_4 Q_1}{5m} [ \{(c_1 + my_0)^2 + \frac{c_2}{\delta^2}\}^{5/8} - (c_1 + my_0)^{5/4} ]. \end{aligned} \quad (2.28)$$

On the other hand, the geodesic distance  $l$  from the D-4-brane (lying at  $y_0$ ) to the point parametrized by  $y(\delta)$  is given by

$$l(y(\delta); y_0) \equiv \int_{y_0}^{y(\delta)} \sqrt{g_{yy}} dy = \frac{4}{5} m^{1/4} [(y(\delta) + \frac{c_1}{m})^{5/4} - (y_0 + \frac{c_1}{m})^{5/4}]. \quad (2.29)$$

So, we obtain the proportional relation:

$$M = T_4 Q_1 \cdot l(y(\delta); y_0). \quad (2.30)$$

Furthermore, we can show that the coefficient  $T_4 Q_1$  reproduces the tension of the fundamental string correctly. To derive this, we discuss the unit electric charge for a (1,0) (i.e. a fundamental) string. First, we review the discussion about the case of point charge

solutions of the D-p-brane in the flat spacetime[1] (and ref.[19]). Let us consider a triple junction of strings: a (0,1) string, a (n,0) string and a (n,1) string. If the string coupling  $g_s$  is small, it holds

$$\frac{\Delta T}{T_{(0,1)}} = \frac{(g_s)^2 n^2}{2} \quad (2.31)$$

where  $T_{(p,q)}$  is the tension of a (p,q) string and  $\Delta T$  is the additional tension  $\Delta T \equiv T_{(n,1)} - T_{(0,1)}$ . On the other hand, the solution of a D-1-brane worldvolume corresponding to the above string junction is given in ref.[19]. In the flat background with the ansatz that  $x^0 = \xi^0, x^1 = \xi^1, x^9 = y(\xi^1), x^m = \text{constant}$  for  $m=2,\dots,8$ , and  $\theta = 0$ , the D-1-brane action is written as

$$S_{D1} = -\frac{T_1}{g_s} \int d^2\xi \sqrt{1 + (\partial_1 y)^2 - (F_{01})^2 - (\partial_0 y)^2}. \quad (2.32)$$

The solution of the D-1-brane's worldvolume as the triple string junction with an electric charge  $q_1$  is[19]

$$y(\xi_1) = V_0(\xi_1) = \begin{cases} -q_1 \xi_1 & \text{for } \xi_1 > 0 \\ 0 & \text{for } \xi_1 < 0. \end{cases} \quad (2.33)$$

The energy of the solution can be computed by using the Hamiltonian, and the additional tension  $\Delta T$  is also derived from this correctly. By taking into account the bending of the (n,1) string[19], it is obtained as  $\Delta T/T_{(0,1)} = (1/2)(F_{01})^2$ . Comparing this with (2.31), the “charge quantization condition”  $F_{01} = q_1 = g_s n$  is deduced for a point charge  $q_1$  (for an integer  $n$ ). By T-dualizing with respect to the directions of  $x^m$  for  $m=2,\dots,8$ , the charge quantization condition for a electric point charge  $q_1$  of the D-p-brane's worldvolume is shown to be

$$q_1 \equiv \frac{1}{(2\pi)^{p-1}} \int_{S_{p-1}} F_{0r} = g_s n. \quad (2.34)$$

Next, we discuss the case which is related by T-duality to the case of (2.16). Let us suppose a D-1-brane parallel to a subspace of the worldvolume of a D-5-brane background, and that some number  $n$  of fundamental strings are absorbed in the D-1-brane. (This is also a BPS configuration since 1/8 spacetime supersymmetry is preserved[20].) If we consider a D-5-brane background solution, the string coupling  $g_s$  becomes a *local* function on the transverse coordinates  $y^a$  ( $g_s = e^\phi = H^{-1/2}$  where  $H$  is the harmonic function on  $y^a$ ), and so is the tension  $T_{(n,1)}$ . If the “test” D-1-brane is put near the D-5-brane, the string coupling  $g_s$  is considered to be sufficiently small around the D-1-brane. So, the equation (2.31) with  $g_s = H^{-1/2}$  holds on the basis of the same discussion. On the other

hand, suppose that a electric point charge  $q'_1$  is added to the D-1-brane's worldvolume with an excitation of a scalar field  $y^9$ . Then, the D-1-brane action is

$$S_{D1} = -T_1 \int d^2\xi \sqrt{1 + H(\partial_1 y^9)^2 - H(F_{01})^2 - H(\partial_0 y^9)^2}. \quad (2.35)$$

If we assume the existence of the correponding point charge solution, the additional tension due to the field strength is derived in the same way, as

$$\frac{\Delta T}{T_{(0,1)}} = \frac{1}{2(T_1)^2} \cdot H^{-1}(\Pi_1)^2 = \frac{1}{2} \cdot H(F_{01})^2. \quad (2.36)$$

So, comparing this with (2.31) we have the charge quantization condition  $H^{1/2}F_{01} = n g_s$  for an integer  $n$ . By using the the “electric induction”  $D_{01} = -\frac{1}{T_1} \cdot \frac{\delta \mathcal{L}_{D1}}{\delta F_{01}}$  ( $= HF_{01}$  in this case) it can be rewritten in a more generic form as

$$D_{01} = n. \quad (2.37)$$

The higher dimensional D-brane cases are related to (2.37) by T-dualities.<sup>‡</sup> So, we have

$$\frac{1}{(2\pi)^{p-1}} \int_{S_{p-1}} D_{0r} = n. \quad (2.38)$$

This is the “generalized” charge quantization condition for the point charge of the D-p-brane parallel to the worldvolume of a D-(p+4)-brane background.

Let us return to to the case of (2.16). the left hand side of (2.38) with  $p = 4$  is equivalent to the charge  $Q_1$  of (2.16) multiplied by  $1/(2\pi)^3$ . So, the unit charge  $Q_1$  for a (1,0) string in this case is  $Q_1 = (2\pi)^3$ . That is, the unit charge is the same as the one in the case of the flat background, (This result is consistent from physical point of view, since the unit charge  $Q_1$  is considered to be independent of the background.) Thus, the proportional coefficient of (2.30) for a (1,0) string is obtained as  $T_4 Q_1 = 1/2\pi$ , which is exactly the tension of the string. So, the quantity  $M$  defined in (2.27) certainly gives the mass of the string correctly !

As another case, we consider the case of a test D-8-brane parallel to the background D-8-brane. The worldvolume action is given in (2.1) for p=8. In this case, we consider the ansatz

$$\begin{cases} x^i = \xi^i & (i = 0, 1, \dots, 8) \\ y = y(r) (> 0) \\ V_0 = V_0(r), \quad V_i = 0. \end{cases}$$

where  $r$  is defined as  $r \equiv \sqrt{\sum_{i=1}^8 (\xi^i)^2}$ . Then, combined with  $\theta = 0$ , only the term including  $\tilde{C}^{(9)}$  in  $S_{D8}^{WZ}$  does contribute to the equations of motion ( $\tilde{C}_{01..8}^{(9)} = H^\epsilon$ ). The expression of

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<sup>‡</sup> We note that the background fields should also be transformed by T-dualities simultaneously.

the induced worldvolume metric  $\tilde{g}_{ij}$  is the same as (2.9) except for the range of the indices (in this case  $i, j = 1, 2, \dots, 8$ ). Setting the same condition as (2.10) makes the determinant of  $(\tilde{g}_{ij} + \mathcal{F}_{ij})$  simple, such as  $\det(\tilde{g}_{ij} + \mathcal{F}_{ij}) = H^{9\epsilon/2}$ . Then, we find the equations of motion

$$\sum_{\underline{i}} \partial_{\underline{i}}(H^{-2\epsilon-2}\partial_{\underline{i}}y) = 0 \quad (2.39)$$

$$\sum_{\underline{i}} \partial_{\underline{i}}(H^{-\epsilon-1}\partial_{\underline{i}}y) = 0. \quad (2.40)$$

So, these two equations are again compatible only if  $\epsilon = -1$ , and the equations to solve become a single equation

$$\sum_{\underline{i}} (\partial_{\underline{i}})^2 y = 0. \quad (2.41)$$

We note that unlike the D-4-brane case, the harmonic function  $H$  does not appear in (2.41). So, choosing the same boundary condition as (2.15), the solution is obtained easily as

$$\begin{aligned} y &= y_0 + \frac{c'_2}{r^6} \\ V_0 &= \frac{c'_2}{r^6} \end{aligned} \quad (2.42)$$

where  $c'_2$  is a constant proportional to the electric charge of the solution. By using the definition of the charge similar to (2.17), we have the electric charge  $Q'_1$ :

$$Q'_1 \equiv -\frac{1}{T_8} \int_{S_7} \star \left( \frac{\delta \mathcal{L}_{D4}}{\delta F_{ij}} \right) = \int_{S_7} \star(dV) = 6c'_2 \Omega_7. \quad (2.43)$$

We note that the form of this point charge solution (2.42) is completely the same as that of the D-8-brane worldvolume in the *flat* background.

Here, We derive the preserved supersymmetry of the solution (2.42) in the same way. 1/4 supersymmetry is expected to be preserved. The superalgebra in a D-8 brane background via a D-8-brane probe is[13]

$$\begin{aligned} \{Q_\alpha^+, Q_\beta^+\} &= 2 \int_{\mathcal{M}_8} d^8\xi \Pi_\mu (\mathcal{C}\Gamma^\mu)_{\alpha\beta} + 2 \int_{\mathcal{M}_8} d^8\xi \mathcal{P}^{(0)\underline{i}} \partial_{\underline{i}} y (\mathcal{C}\Gamma_y \Gamma_{11})_{\alpha\beta} \\ &+ \frac{2T_8}{5!} \int_{\mathcal{M}_8} H^{5/4} dx^{\mu_1}..dx^{\mu_5} dy dV (\mathcal{C}\Gamma_{\mu_1..\mu_5 y})_{\alpha\beta} + \frac{2T_8}{4!} \int_{\mathcal{M}_8} H^{5/4} dx^{\mu_1}..dx^{\mu_4} (dV)^2 (\mathcal{C}\Gamma_{\mu_1..\mu_4} \Gamma_{11})_{\alpha\beta} \\ &+ 2T_8 \int_{\mathcal{M}_8} H^{5/4} dx^\mu dy (dV)^3 (\mathcal{C}\Gamma_{\mu y})_{\alpha\beta} + 2T_8 \int_{\mathcal{M}_8} H^{5/4} (dV)^4 (\mathcal{C}\Gamma_{11})_{\alpha\beta} + \mathcal{O}(\theta^2). \end{aligned} \quad (2.44)$$

The momentum  $\Pi_\mu$  includes the following two terms:

$$\Pi_\mu = \Pi_\mu^{(0)} + \frac{T_8}{8!} H^{-2} m y e^{0i_1..i_8} \partial_{i_1} x^{\nu_1}..\partial_{i_8} x^{\nu_8} \epsilon_{\mu\nu_1..\nu_8 y} \quad (2.45)$$

where  $\Pi_\mu^{(0)}$  is the contribution of  $S_{D8}^{BI}$  and the second term is that of  $S_{D8}^{WZ}$  (where  $\epsilon^{01..8} = 1$ ). Substituting the solution for the right hand side of (2.44), the superalgebra can be written as

$$\{Q_\alpha^+, Q_\beta^+\} = 4T_8 \int_{\mathcal{M}_8} d^8\xi H^{1/4} (\partial_{\underline{i}} y)^2 \left( \frac{1 + \mathcal{C}\Gamma_{\hat{y}}\Gamma_{11}}{2} \right)_{\alpha\beta}. \quad (2.46)$$

By the same discussion as done in the D-4-brane case, it is shown that 1/4 supersymmetry is preserved in this configuration, which is consistent with the spacetime interpretation. So, we interpret the solution (2.42) as a fundamental string again.

Applying the mass formula (2.27) to this case, we obtain the result:

$$\begin{aligned} M &\equiv \int_{\mathcal{M}_8} \sqrt{-\tilde{g}^{00}} \mathcal{H}_2 = T_8 \int_{\mathcal{M}_8} d^8\xi H^{1/4} (\partial_{\underline{i}} y)^2 \\ &= T_8 \int d\Omega_7 \int_{\delta}^{\infty} r^7 dr (c_1 + my(r))^{1/4} (\partial_{\underline{i}} y)^2 \\ &= T_8 Q'_1 \cdot l(y(\delta); y_0). \end{aligned} \quad (2.47)$$

Based on the same discussion as that done in the D-4-brane case, the unit electric charge  $Q'_1$  for a (1,0) string ending on a D-8-brane can also be derived as  $Q'_1 = (2\pi)^7$ . So, the tension of the string is reproduced correctly again, and (2.27) also gives the mass of the string for the solution (2.42) correctly. Therefore, we conclude that (2.42) is the correctly generalized mass formula, which holds when the background of the D-brane is curved. We note that the energy defined on the worldvolume again gives the difference of the coordinate  $y$  multiplied by  $T_8 Q'_1$ . That is, this case is another example that the worldvolume energy does not agree with the mass of the string.

### 3 Summary and discussions

In summary, we have proposed (2.42) as the generalized mass formula which holds when the background of the D-brane is curved, and have shown explicitly using the two examples that the formula certainly gives the mass of a fundamental string described as a BPS solution of a D-brane's worldvolume. In addition, based on the obtained formula, we can see that the mass of the string agrees with the worldvolume energy only in the cases  $\tilde{g}_{00} = -1$  (where  $\tilde{g}_{ij}$  is the induced worldvolume metric). which include the case discussed in ref.[1].

Here, we discuss the consistency of the mass formula from another point of view, especially focusing on the factor  $\sqrt{-\tilde{g}^{00}}$ . Suppose we consider the D-4-brane embedded parallel to the D-8-brane background (2.4) *with no excitation of worldvolume fields* (i.e.  $x^i = \xi^i$  for  $i=0,..,4$  and  $x^5,..,x^8, y : \text{constants}$ ). Then, on the analogy of the mass formula

of a (p,q) string given by Sen in ref.[10], the target-space mass  $m_{D4}$  of the D-4-brane should be proportional to its spatial volume element *measured by the geodesic distances* in the spacetime. So, it should be given by

$$m_{D4} = \int_{\mathcal{M}_4} d^4\xi T_4 e^{-\phi} \sqrt{\det \tilde{g}'_{ij}} \quad (3.1)$$

where  $\tilde{g}'_{ij}$  is the induced *world-space* metric of the D-4-brane. In fact, the mass  $m_{D4}$  obtained in this way can be shown to agree with the quantity  $M_{D4}$  defined as

$$M_{D4} = \int_{\mathcal{M}_4} d^4\xi \sqrt{-\tilde{g}^{00}} \mathcal{H}_1 (= \int_{\mathcal{M}_4} d^4\xi H^{1/4} \cdot 1) \quad (3.2)$$

where  $\mathcal{H}_1$  is the first part (i.e. originating from the D-4-brane) of the Hamiltonian density of the solution (2.16) defined on the D-4-brane. This means that the information of the D-4-brane mass can also be extracted from the solution (2.16) by integrating the Hamiltonian density multiplied by the factor  $\sqrt{-\tilde{g}^{00}}$  with respect to world-space coordinates. Thus, the factor  $\sqrt{-\tilde{g}^{00}}$  arising in the formula is consistent in this sense, too.

Finally, let us discuss the worldvolume interpretation of the mass formula (2.27). If we define a worldvolume proper time  $\tau$  as  $d\tau \equiv \sqrt{-\tilde{g}_{00}} d\xi^0$ , the Hamiltonian density multiplied by  $\sqrt{-\tilde{g}^{00}}$  might be regarded as the energy density defined with respect to  $\tau$ . So, we might say that from worldvolume point of view, the mass of the string is interpreted as “the energy defined with respect to the worldvolume proper time”.

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