A Universal Two–Bit Gate for Quantum Computation

Adriano Barenco

Clarendon Laboratory, Physics Department, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom

We prove the existence of a class of two–input, two–output gates any one of which is universal for quantum computation. This is done by explicitly constructing the three–bit gate introduced by Deutsch [Proc. R. Soc. London. A **425**, 73 (1989)] as a network consisting of replicas of a single two–bit gate.

The fact that quantum mechanical processes allow new types of computation has been known since 1985 [1]. In 1992, Deutsch and Jozsa [2] exhibited a class of problems that can be solved more rapidly on quantum computers than on classical ones and more recently Shor [3] showed that quantum computers can factor large composite integers very efficiently, a problem for which no efficient classical algorithm is known. Quantum algorithms for factoring threaten the security of public key cryptosystems such as RSA [4] which are currently considered completely reliable. This suggests that sooner or later perfect security may only be obtainable via quantum cryptography [5]. Clearly, the experimental realisation of quantum computation is a most important issue.

Computational networks built out of quantum–mechanical gates [6] provide a natural framework for constructing quantum computers. A set of gates is *adequate* if any quantum computation (*i.e.* a unitary operation on an information–carrying register) can be performed with arbitrary precision by networks consisting only of replicas of gates from that set. A gate is *universal* if by itself it forms an adequate set, *i.e.* if any quantum computation can be performed by a network containing replicas of *only* this gate.

In classical irreversible computation, there exists a universal two–input one–output gate (the NAND gate). In classical reversible computation [7] there exists a three–bit universal gate (the Toffoli gate [8]), but not two–bit universal gate; moreover there is not even an *adequate* set of two–bit gates. DiVincenzo has shown that a certain set of four gates each operating on two *qubits* (two–level quantum systems) is adequate in quantum computation. Deutsch had already shown that the operation given in the network's *computation basis* $\{|000\rangle, |001\rangle \ldots |111\rangle\}$ by the unitary matrix

$$D = \left(\begin{array}{cc} \hat{1} & \hat{0} \\ \hat{0} & \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i\cos\theta & \sin\theta \\ 0 & 0 & \sin\theta & i\cos\theta \end{array} \end{array} \right), \tag{1}$$

with θ/π irrational, defines a three–bit (i.e. three–input, three–output) gate **D** that is universal. Here $\hat{1}$ and $\hat{0}$ denote respectively the 4×4 unit matrix and the 4×4 zero matrix. Boldface symbols such as **D** denote gates, and plain symbols such as D the unitary operations performed by the corresponding gates.

In this paper, we improve on the results of both DiVincenzo and Deutsch by showing that a single *two*–bit gate is universal for quantum computation. Moreover, we present a large class of universal two–bit gates, providing evidence that such gates are very common. This result is relevant both from theoretical and experimental perspectives. The theoretical analysis of circuit complexity will be much simplified, and from the experimental point of view, it establishes that an interaction of one type between two quantum bits is sufficient to ensure universality.

Consider any two–bit gate **A** whose action is given in the computation basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ by the unitary matrix

$$A(\phi, \alpha, \theta) = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha}\cos\theta & -ie^{i(\alpha-\phi)}\sin\theta \\ 0 & 0 & -ie^{i(\alpha+\phi)}\sin\theta & e^{i\alpha}\cos\theta \end{array} \right), \tag{2}$$

where ϕ, α and θ are fixed irrational multiples of π and of each other. We shall show that any such gate is universal. The proof is by explicit construction of **D** via the three–bit gate **V** defined by

$$V(\phi, \alpha, \theta) = \left(\begin{array}{cc} \hat{1} & \hat{0} \\ \hat{0} & \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha}\cos\theta & -ie^{i(\alpha-\phi)}\sin\theta \\ 0 & 0 & -ie^{i(\alpha+\phi)}\sin\theta & e^{i\alpha}\cos\theta \end{array} \end{array} \right). \tag{3}$$

Notice first that when the first qubit (which we call the *control* qubit) is in either of the states $|0\rangle$ or $|1\rangle$, $\mathbf{A}$ induces no entanglement between the qubits. We can read off from the form of A that if the control is in the state $|0\rangle$, the second qubit (which we refer to as the *target* qubit) is unaffected by the gate. And if the control is in the state $|1\rangle$, the target undergoes the unitary operation given by the lower diagonal 2×2 block of Eq. 2. This operation is a rotation of angle 2θ about the axis $\mathbf{u} = \cos(\phi)\mathbf{x} + \sin(\phi)\mathbf{y}$ of the "spin" of the target qubit.

Since

$$A^n(\phi, \alpha, \theta) = A(\phi, n\alpha \bmod 2\pi, n\theta \bmod 2\pi), \tag{4}$$

and because of the irrationality properties that we required of α and θ, transformations of the type $A(\phi, \alpha_1, \theta_1)$, where α_1 and θ_1 are any constants in the range $[0, 2\pi]$, can be effected with arbitrary precision as a result of a sufficient but finite number n of applications of the operation $A(\phi, \alpha, \theta)$ to the same two qubits (Fig. 1). To do this with α_1 and θ_1 specified simultaneously with accuracy $\pm\epsilon$ requires $n \sim 1/\epsilon^2$. The inverse $\mathbf{A}^{-1}$ of the gate $\mathbf{A}$, defined by

$$A^{-1}(\phi, \alpha, \theta) = A(\phi, 2\pi - \alpha, 2\pi - \theta), \tag{5}$$

is clearly in this repertoire.

Let $\mathbf{A}_{ij}$ denote the three–bit gate obtained from $\mathbf{A}$ by letting qubit i be the control, qubit j the target and having the remaining qubit go through unaffected. All such gates are trivially in the repertoire. We have

$$V(\phi, \alpha, \theta) = A_{23}(\phi, \frac{\alpha}{2}, \frac{\theta}{2})A_{13}(\phi, \frac{\alpha}{2}, \frac{\theta}{2})A_{12}(\phi, \frac{\pi}{2}, \frac{\pi}{2})A_{23}^{-1}(\phi, \frac{\alpha}{2}, \frac{\theta}{2})A_{12}(\phi, \frac{\pi}{2}, \frac{\pi}{2}). \tag{6}$$

This means that a network of sequences of the gate $\mathbf{A}$, as shown in Fig 2, has the effect of $\mathbf{V}$. This construction, which greatly simplifies our proof, is similar to that proposed by Sleator and Weinfurter [10] but uses only one type of gate. $\mathbf{V}$ also has a "control-target" structure. The state of the third ("target") qubit undergoes a non-trivial unitary transformation when the first two ("control") qubits are in the state $|11\rangle$, and is unaffected if the control qubits are in any of their other three computation basis states.

If we denote by $\bar{\mathbf{V}}$ the gate obtained from $\mathbf{V}$ by permuting the second and the third qubit we easily verify that

$$P = \bar{V}(\phi, \pi/2, \pi/2) = \begin{pmatrix} \hat{1} & & \hat{0} & \\ & 1 & 0 & 0 & 0 \\ \hat{0} & 0 & 0 & 0 & e^{-i\phi} \\ & 0 & 0 & 1 & 0 \\ & 0 & e^{i\phi} & 0 & 0 \end{pmatrix} \tag{7}$$

and that

$$Q = \bar{V}(\phi, \pi/2, -\pi/2)V(\phi, \pi/2, -\pi/2)\bar{V}(\phi, \pi/2, -\pi/2) = \begin{pmatrix} \hat{1} & & \hat{0} & \\ & 1 & 0 & 0 & 0 \\ \hat{0} & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Following the construction of [6], we now note that for infinitesimal values of β the operation

$$T(\beta) = Q[V(\phi, 0, \beta)P]^2[V(\phi, 0, -\beta)P]^2Q \tag{9}$$

is a rotation of angle $2\beta^2$ along the axis $\mathbf{u}_\perp = \cos(\phi - \pi/2)\mathbf{x} + \sin(\phi - \pi/2)\mathbf{y}$ of the "spin" of the target qubit :

$$T(\beta) = 1 - i\beta^2 \begin{pmatrix} \hat{0} & & \hat{0} & \\ & 0 & 0 & 0 & 0 \\ \hat{0} & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & ie^{-i\phi} \\ & 0 & 0 & -ie^{i\phi} & 0 \end{pmatrix} + O(\beta^3). \tag{10}$$

Therefore the transformation

$$V(\phi - \pi/2, 0, \beta) = \lim_{n \to \infty} T(\sqrt{\beta/n})^n \tag{11}$$

can also be performed with arbitrary accuracy by networks of the gate $\mathbf{A}$. Similarly, so can

$$R_z(\beta) = \lim_{n\to\infty} \left[V(\phi,0,\sqrt{\beta/2n})V(\phi-\pi/2,0,\sqrt{\beta/2n})V(\phi,0,-\sqrt{\beta/2n})V(\phi-\pi/2,0,-\sqrt{\beta/2n}) \right]^n \tag{12}$$

$$= \begin{pmatrix} \hat{1} & & \hat{0} \\ & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & e^{-i\beta} \end{matrix} & \end{pmatrix}.$$

This last gate is a conditional rotation along the z axis of the "spin" of the target qubit when the two controls are in state $|11\rangle$.

The universal Deutsch gate $\mathbf{D}$ may now be constructed:

$$D = R_z(\phi/2)V(\phi,\pi/2,\theta)R_z(-\phi/2). \tag{13}$$

An exceptionally simple two–bit universal quantum gate is the one that performs the operation $A(\pi/2,\pi/4,\theta)$. Note that this gate, with $\alpha = \pi/4$, does not even satisfy the irrationality constraint we imposed, but is nevertheless universal by the same proof. It is also particularly appealing from the experimental point of view. The lower diagonal 2×2 block of $A(\pi/2,\pi/2,\theta)$ is a rotation of angle $\theta/2$ about the y axis of the "spin" of the qubit. In a realistic implementation, such an operation can be realised by applying a $(\theta/2)$–pulse of resonant radiation to the spin of the qubit. A variety of experimentally realisable systems can be put forward as practical implementations of gates of this type. For instance, one possibility is to use cavity QED–type of interaction as analysed recently by Davidovich et $al.$ [11] in the context of teleportation, or use ion–ion interactions in linear traps along a similar scheme as the one recently proposed by Cirac and Zoller [12]

However, simplicity in the form of the interaction is not always to the point. It is desirable that practical gates should have as generic a form as possible. In this paper, we have presented a three–parameter family of universal gates. In another paper [13], it will be shown that starting from the result just presented, further generalisation is possible: conforming to a conjecture of Deutsch [6], it turns out that almost all two–bit quantum gates are universal.

---

[1] D. Deutsch, Proc. R. Soc. London A **400**, 97 (1985).
[2] D. Deutsch and R. Jozsa, Proc. R. Soc. London A **439**, 553 (1992).
[3] P. W. Shor, Proc. 35nd Annual Symposium on Foundations of Computer Science, Ed. Shafi Goldwasser, IEEE Press, (1994), p124-134.
[4] R. Rivest, A. Shamir and L. Adleman, *On Digital Signatures and Public-Key Cryptosystems*, MIT Laboratory for Computer Science, Technical Report, MIT/LCS/TR-212 (1979).
[5] C. H. Bennett and G. Brassard, in *Proc. IEEE Int. Conference on Computers and Signal Processing*, IEEE, New–York (1984); A. K .Ekert, Phys. Rev. Lett., **68**, 3121 (1991).
[6] D. Deutsch, Proc. R. Soc. London A **425**, 73 (1989).
[7] C. H. Bennett, IBM J. Res. Develop **32**, 16 (1988).
[8] T. Toffoli, Math. Syst. Theory **14**, 13 (1981).
[9] D. P. DiVincenzo, to appear in Phys. Rev. A, feb 1995.
[10] T. Sleator and H. Weinfurter, *Realizable Universal Quantum Logic Gates.*, preprint.
[11] L. Davidovich, N. Zagury, M. Brune, J-M. Raimond, and S. Haroche, Phys. Rev. A, **50**, R895 (1994).
[12] I. Cirac and P. Zoller, *Quantum Computations with Cold Trapped Ions*, preprint.
[13] D. Deutsch, A. Barenco and A. Ekert, *Universality in Quantum Computation*, submitted to Proc. R. Soc. London A.

FIG. 1. Any transformation of the form $A(\phi,\alpha_1,\theta_1)$ can be performed with arbitrary accuracy by a gate-sequence consisting of n replicas of a single gate $\mathbf{A}$ that effects the unitary transformation $A(\phi,\alpha,\theta)$. The control qubits of the gates and gate-sequence are indicated by black dots. An arrow points to each target qubit.

FIG. 2. A network consisting of sequences of the universal two–bit gate $\mathbf{A}$ has the effect of the three–bit gate $V(\phi,\alpha,\theta)$.
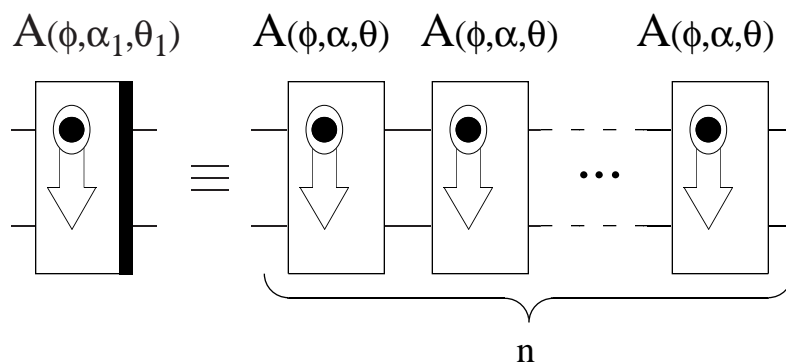
$A_{(\phi,\alpha_1,\theta_1)}$ $A_{(\phi,\alpha,\theta)}$ $A_{(\phi,\alpha,\theta)}$ $A_{(\phi,\alpha,\theta)}$

$$\equiv \quad \cdots$$

$$\underbrace{}_{n}$$

Figure 1 : A. Barenco

$A_{12}(\phi,\pi/2,\pi/2)$ $A_{12}(\phi,\pi/2,\pi/2)$ $A_{23}(\phi,\alpha/2,\theta/2)$

$$\equiv$$

$V_{(\phi,\alpha,\theta)}$ $A_{23}^{-1}(\phi,\alpha/2,\theta/2)$ $A_{13}(\phi,\alpha/2,\theta/2)$

Figure 2 : A. Barenco