

String theories as the adiabatic limit of Yang-Mills theory

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Abstract

We consider Yang-Mills theory with a matrix gauge group G on a direct product manifold $M = \Sigma_2 \times H^2$, where Σ_2 is a two-dimensional Lorentzian manifold and H^2 is a two-dimensional open disc with the boundary $S^1 = \partial H^2$. The Euler-Lagrange equations for the metric on Σ_2 yield constraint equations for the Yang-Mills energy-momentum tensor. We show that in the adiabatic limit, when the metric on H^2 is scaled down, the Yang-Mills equations plus constraints on the energy-momentum tensor become the equations describing strings with a worldsheet Σ_2 moving in the based loop group $\Omega G = C^\infty(S^1, G)/G$, where S^1 is the boundary of H^2 . By choosing $G = \mathbb{R}^{d-1,1}$ and putting to zero all parameters in $\Omega\mathbb{R}^{d-1,1}$ besides $\mathbb{R}^{d-1,1}$, we get a string moving in $\mathbb{R}^{d-1,1}$. In arXiv:1506.02175 it was described how one can obtain the Green-Schwarz superstring action from Yang-Mills theory on $\Sigma_2 \times H^2$ while H^2 shrinks to a point. Here we also consider Yang-Mills theory on a three-dimensional manifold $\Sigma_2 \times S^1$ and show that in the limit when the radius of S^1 tends to zero, the Yang-Mills action functional supplemented by a Wess-Zumino-type term becomes the Green-Schwarz superstring action.

1. Introduction. Superstring theory has a long history [1]-[3] and pretends on description of all four known forces in Nature. In the low-energy limit superstring theories describe supergravity in ten dimensions or supergravity interacting with supersymmetric Yang-Mills (SYM) theory. On the other hand, Yang-Mills and SYM theories in four dimensions give description of three main forces in Nature not including gravity [4]-[7]. The aim of this short paper is to show that bosonic strings (both open and closed) as well as type I, IIA and IIB superstrings can be obtained as a subsector of pure Yang-Mills theory with some constraints on the Yang-Mills energy-momentum tensor. Put differently, knowing the action for superstrings with a worldsheet Σ_2 , we introduce a Yang-Mills action functional on $\Sigma_2 \times H^2$ or on $\Sigma_2 \times S^1$ such that the Yang-Mills action becomes the Green-Schwarz superstring action while H^2 or S^1 shrink to a point. We will work in Lorentzian signature, but all calculations can be repeated for the Euclidean signature of spacetime.

2. Yang-Mills equations. Consider Yang-Mills theory with a matrix gauge group G on a direct product manifold $M = \Sigma_2 \times H^2$, where Σ_2 is a two-dimensional Lorentzian manifold (flat case is included) with local coordinates $x^a, a, b, \dots = 1, 2$, and a metric tensor $g_{\Sigma_2} = (g_{ab})$, H^2 is the disc with coordinates $x^i, i, j, \dots = 3, 4$, satisfying the inequality $(x^3)^2 + (x^4)^2 < 1$, and the metric $g_{H^2} = (g_{ij})$. Then $(x^\mu) = (x^a, x^i)$ are local coordinates on M with $\mu = 1, \dots, 4$.

We start with the gauge potential $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ with values in the Lie algebra $\mathfrak{g} = \text{Lie } G$ having scalar product (\cdot, \cdot) defined either via trace Tr or, for abelian groups like $\mathbb{R}^{p,q}, T^{p,q}$ etc., via a metric on vector spaces. The gauge field $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the \mathfrak{g} -valued two-form

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \quad (1)$$

The Yang-Mills equations on M with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + g_{ij} dx^i dx^j \quad (2)$$

have the form

$$D_\mu \mathcal{F}^{\mu\nu} := \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} \mathcal{F}^{\mu\nu}) + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0, \quad (3)$$

where $g = (g_{\mu\nu})$ and $\partial_\mu = \partial/\partial x^\mu$.

The equations (3) follow from the standard Yang-Mills action on M

$$S = \frac{1}{4} \int_M d^4x \sqrt{|\det g|} (\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}), \quad (4)$$

where (\cdot, \cdot) is the scalar product on the Lie algebra \mathfrak{g} . Note that the metric g_{Σ_2} on Σ_2 is not fixed and the Euler-Lagrange equations for g_{Σ_2} yield the constraint equations

$$T_{ab} = g^{\lambda\sigma} (\mathcal{F}_{a\lambda}, \mathcal{F}_{b\sigma}) - \frac{1}{4} g_{ab} (\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 0 \quad (5)$$

for the Yang-Mills energy-momentum tensor $T_{\mu\nu}$, i.e. its components along Σ_2 are vanishing. Note that these constraints can be satisfied for many gauge configurations, e.g. for self-dual gauge fields not only $T_{ab} = 0$ but even $T_{\mu\nu} = 0$.

3. Adiabatic limit. On $M = \Sigma_2 \times H^2$ we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \mathcal{A}_a dx^a + \mathcal{A}_i dx^i, \quad (6)$$

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\mathcal{F}_{ab}dx^a \wedge dx^b + \mathcal{F}_{ai}dx^a \wedge dx^i + \frac{1}{2}\mathcal{F}_{ij}dx^i \wedge dx^j , \quad (7)$$

$$T = T_{\mu\nu}dx^\mu dx^\nu = T_{ab}dx^a dx^b + 2T_{ai}dx^a dx^i + T_{ij}dx^i dx^j . \quad (8)$$

By using the adiabatic approach in the form presented in [8, 9], we deform the metric (2) and introduce the metric

$$ds_\varepsilon^2 = g_{ab}dx^a dx^b + \varepsilon^2 g_{ij}dx^i dx^j , \quad (9)$$

where $\varepsilon \in [0, 1]$ is a real parameter. It is assumed that the fields \mathcal{A}_μ and $\mathcal{F}_{\mu\nu}$ smoothly depend in ε^2 , i.e. $\mathcal{A}_\mu = \mathcal{A}_\mu^{(0)} + \varepsilon^2 \mathcal{A}_\mu^{(1)} + \dots$ and $\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^{(0)} + \varepsilon^2 \mathcal{F}_{\mu\nu}^{(1)} + \dots$. Furthermore, we have $\det g_\varepsilon = \varepsilon^4 \det(g_{ab}) \det(g_{ij})$ and

$$\mathcal{F}_\varepsilon^{ab} = g_\varepsilon^{ac} g_\varepsilon^{bd} \mathcal{F}_{cd} = \mathcal{F}^{ab} , \quad \mathcal{F}_\varepsilon^{ai} = g_\varepsilon^{ac} g_\varepsilon^{ij} \mathcal{F}_{cj} = \varepsilon^{-2} \mathcal{F}^{ai} \quad \text{and} \quad \mathcal{F}_\varepsilon^{ij} = g_\varepsilon^{ik} g_\varepsilon^{jl} \mathcal{F}_{kl} = \varepsilon^{-4} \mathcal{F}^{ij} , \quad (10)$$

where indices in $\mathcal{F}^{\mu\nu}$ are raised by the non-deformed metric tensor $g^{\mu\nu}$.

For the deformed metric (9) the action functional (4) is changed to

$$S_\varepsilon = \frac{1}{4} \int_M d^4x \sqrt{|\det g_{\Sigma_2}|} \sqrt{\det g_{H_2}} \left\{ \varepsilon^2 (\mathcal{F}_{ab}, \mathcal{F}^{ab}) + 2(\mathcal{F}_{ai}, \mathcal{F}^{ai}) + \varepsilon^{-2} (\mathcal{F}_{ij}, \mathcal{F}^{ij}) \right\} . \quad (11)$$

The term $\varepsilon^{-2}(\mathcal{F}_{ij}, \mathcal{F}^{ij})$ in the Yang-Mills Lagrangian (11) diverges when $\varepsilon \rightarrow 0$. To avoid this we impose the flatness condition

$$\mathcal{F}_{ij}^{(0)} = 0 \quad \Rightarrow \quad \lim_{\varepsilon \rightarrow 0} (\varepsilon^{-1} \mathcal{F}_{ij}) = 0 \quad (12)$$

on the components of the field tensor along H^2 . Here $\mathcal{F}_{ij}^{(0)} = 0$ but $\mathcal{F}_{ij}^{(1)}$ etc. in the ε^2 -expansion must not be zero. For the deformed metric (9) the Yang-Mills equations have the form

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_i \mathcal{F}^{ib} = 0 , \quad (13)$$

$$\varepsilon D_a \mathcal{F}^{aj} + \varepsilon^{-1} D_i \mathcal{F}^{ij} = 0 . \quad (14)$$

In the deformed metric (9) the constraint equations (5) become

$$T_{ab}^\varepsilon = \varepsilon^2 \left\{ g^{cd} (\mathcal{F}_{ac}, \mathcal{F}_{bd}) - \frac{1}{4} g_{ab} (\mathcal{F}_{cd}, \mathcal{F}^{cd}) \right\} + g^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) - \frac{1}{2} g_{ab} (\mathcal{F}_{ci}, \mathcal{F}^{ci}) - \frac{1}{4} \varepsilon^{-2} g_{ab} (\mathcal{F}_{ij}, \mathcal{F}^{ij}) = 0 . \quad (15)$$

In the adiabatic limit $\varepsilon \rightarrow 0$ the Yang-Mills equations (13) and (14) become

$$D_i \mathcal{F}^{ib} = 0 , \quad (16)$$

$$D_a \mathcal{F}^{aj} = 0 , \quad (17)$$

since the ε^{-1} -term vanishes due to (12). We also keep (17) since it follows from the action (11) after taking the limit $\varepsilon \rightarrow 0$. One can see that the constraint equations (15) are nonsingular in the limit $\varepsilon \rightarrow 0$ also due to (12):

$$T_{ab}^0 = g^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) - \frac{1}{2} g_{ab} (\mathcal{F}_{ci}, \mathcal{F}^{ci}) = 0 . \quad (18)$$

Note that for the adiabatic limit of instanton equations [8, 9] the constraints (15) disappear since the energy-momentum tensor for self-dual and anti-self-dual gauge fields vanishes on any four-manifold M .

4. Flat connections. Now we start to consider the flatness equation (12), the equations (16), (17) and the constraint equations (18). From now on we will consider only zero modes in ε^2 -expansions and equations on them. For simplicity of notation we will omit the index “(0)” from all $\mathcal{A}^{(0)}$ and $\mathcal{F}^{(0)}$ tensor components. In the adiabatic approach it is assumed that all fields depend on coordinates $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha(x^a)$, $\alpha, \beta = 1, 2, \dots$, appearing in the solutions of the flatness equation (12).

Flat connection $\mathcal{A}_{H^2} := \mathcal{A}_i dx^i$ on H^2 has the form

$$\mathcal{A}_{H^2} = g^{-1} \hat{d}g \quad \text{with} \quad \hat{d} = dx^i \partial_i \quad \text{for} \quad \partial_i = \frac{\partial}{\partial x^i}, \quad (19)$$

where $g = g(\phi^\alpha(x^a), x^i)$ is a smooth map from H^2 into the gauge group G for any fixed $x^a \in \Sigma_2$.

Let us introduce on H^2 spherical coordinates: $x^3 = \rho \cos \varphi$ and $x^4 = \rho \sin \varphi$. Using these coordinates, we impose on g the condition $g(\varphi = 0, \rho^2 \rightarrow 1) = \text{Id}$ (framing) and denote by $C_0^\infty(H^2, G)$ the space of framed flat connections on H^2 given by (19). On H^2 , as on a manifold with a boundary, the group of gauge transformations for any fixed $x^a \in \Sigma_2$ is defined as (see e.g. [9, 10, 11])

$$\mathcal{G}_{H^2} = \{g : H^2 \rightarrow G \mid g \rightarrow \text{Id} \quad \text{for} \quad \rho^2 \rightarrow 1\}. \quad (20)$$

Hence the solution space of the equation (12) is the infinite-dimensional group $\mathcal{N} = C_0^\infty(H^2, G)$, and the moduli space of solutions is the based loop group [9, 10, 12]

$$\mathcal{M} = C_0^\infty(H^2, G)/\mathcal{G}_{H^2} = \Omega G. \quad (21)$$

This space can also be represented as $\Omega G = LG/G$, where $LG = C^\infty(S^1, G)$ is the loop group with the circle $S^1 = \partial H^2$ parameterized by $e^{i\varphi}$.

5. Moduli space. On the group manifold (21) we introduce local coordinates ϕ^α with $\alpha = 1, 2, \dots$ and recall that \mathcal{A}_μ 's depend on $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha = \phi^\alpha(x^a)$. Then moduli of gauge fields define a map

$$\phi : \Sigma_2 \rightarrow \mathcal{M} \quad \text{with} \quad \phi(x^a) = \{\phi^\alpha(x^a)\}. \quad (22)$$

These maps are constrained by the equations (16), (17) and (18). Since \mathcal{A}_{H^2} is a flat connection for any $x^a \in \Sigma_2$, the derivatives $\partial_a \mathcal{A}_i$ have to satisfy the linearized (around \mathcal{A}_{H^2}) flatness condition, i.e. $\partial_a \mathcal{A}_i$ belong to the tangent space $T_{\mathcal{A}} \mathcal{N}$ of the space $\mathcal{N} = C_0^\infty(H^2, G)$ of framed flat connections on H^2 . Using the projection $\pi : \mathcal{N} \rightarrow \mathcal{M}$ from \mathcal{N} to the moduli space \mathcal{M} , one can decompose $\partial_a \mathcal{A}_i$ into the two parts

$$T_{\mathcal{A}} \mathcal{N} = \pi^* T_{\mathcal{A}} \mathcal{M} \oplus T_{\mathcal{A}} \mathcal{G} \quad \Leftrightarrow \quad \partial_a \mathcal{A}_i = (\partial_a \phi^\beta) \xi_{\beta i} + D_i \epsilon_a, \quad (23)$$

where \mathcal{G} is the gauge group \mathcal{G}_{H^2} for any fixed $x^a \in \Sigma_2$, $\{\xi_\alpha = \xi_{\alpha i} dx^i\}$ is a local basis of tangent vectors at $T_{\mathcal{A}} \mathcal{M}$ (they form the loop Lie algebra $\Omega \mathfrak{g}$) and ϵ_a are \mathfrak{g} -valued gauge parameters ($D_i \epsilon_a \in T_{\mathcal{A}} \mathcal{G}$) which are determined by the gauge-fixing conditions

$$g^{ij} D_i \xi_{\alpha j} = 0 \quad \Leftrightarrow \quad g^{ij} D_i D_j \epsilon_a = g^{ij} D_i \partial_a \mathcal{A}_j. \quad (24)$$

Note also that since $\mathcal{A}_i(\phi^\alpha, x^j)$ depends on x^a only via ϕ^α , we have

$$\partial_a \mathcal{A}_i = \frac{\partial \mathcal{A}_i}{\partial \phi^\beta} \partial_a \phi^\beta \quad \stackrel{(24)}{\Rightarrow} \quad \epsilon_a = (\partial_a \phi^\beta) \epsilon_\beta, \quad (25)$$

where the gauge parameters ϵ_β are found by solving the equations

$$g^{ij} D_i D_j \epsilon_\beta = g^{ij} D_i \frac{\partial \mathcal{A}_j}{\partial \phi^\beta}. \quad (26)$$

Recall that \mathcal{A}_i are given explicitly by (19) and \mathcal{A}_a are yet free. It is natural to choose $\mathcal{A}_a = \epsilon_a$ [5, 6] and obtain

$$\mathcal{F}_{ai} = \partial_a \mathcal{A}_i - D_i \mathcal{A}_a = (\partial_a \phi^\beta) \xi_{\beta i} = \pi_* \partial_a \mathcal{A}_i \in T_{\mathcal{M}} \mathcal{M}. \quad (27)$$

Thus if we know the dependence of ϕ^α on x^a then we can construct

$$(\mathcal{A}_\mu) = (\mathcal{A}_a, \mathcal{A}_i) = \left((\partial_a \phi^\beta) \epsilon_\beta, g^{-1}(\phi^\alpha, x^j) \partial_i g(\phi^\beta, x^k) \right), \quad (28)$$

which are in fact the components $\mathcal{A}_\mu^{(0)} = \mathcal{A}_\mu(\varepsilon=0)$.

6. Effective action. For finding equations for $\phi^\alpha(x^a)$ we substitute (27) into (16) and see that (16) are resolved due to (24). Substituting (27) into (17), we obtain the equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) g^{ij} \xi_{\beta j} + g^{ab} g^{ij} (D_a \xi_{\beta j}) \partial_b \phi^\beta = 0. \quad (29)$$

We should project (29) on the moduli space $\mathcal{M} = \Omega G$, metric $\mathbb{G} = (G_{\alpha\beta})$ on which is defined as

$$G_{\alpha\beta} = \langle \xi_\alpha, \xi_\beta \rangle = \int_{H^2} d\text{vol} g^{ij} (\xi_{\alpha i}, \xi_{\beta j}). \quad (30)$$

The projection is provided by multiplying (29) by $\langle \xi_\alpha, \cdot \rangle$ (cf. e.g. [13, 14]). We obtain

$$\begin{aligned} & \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) \langle \xi_\alpha, \xi_\beta \rangle + g^{ab} \langle \xi_\alpha, D_a \xi_\beta \rangle \partial_b \phi^\beta \\ &= \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) G_{\alpha\beta} + \langle \xi_\alpha, \nabla_\gamma \xi_\beta \rangle g^{ab} \partial_a \phi^\gamma \partial_b \phi^\beta \\ &= G_{\alpha\sigma} \left\{ \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\sigma \right) + \Gamma_{\beta\gamma}^\sigma g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma \right\} = 0, \end{aligned} \quad (31)$$

where

$$\Gamma_{\beta\gamma}^\sigma = \frac{1}{2} G^{\sigma\lambda} (\partial_\gamma G_{\beta\lambda} + \partial_\beta G_{\gamma\lambda} - \partial_\lambda G_{\beta\gamma}) \quad \text{with} \quad \partial_\gamma := \frac{\partial}{\partial \phi^\gamma}, \quad (32)$$

are the Christoffel symbols and ∇_γ are the corresponding covariant derivatives on the moduli space \mathcal{M} of flat connections on H^2 .

The equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\alpha \right) + \Gamma_{\beta\gamma}^\alpha g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma = 0 \quad (33)$$

are the Euler-Lagrange equations for the effective action

$$S_{\text{eff}} = \int_{\Sigma_2} dx^1 dx^2 \sqrt{-\det(g_{ab})} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta \quad (34)$$

obtained from the action functional (11) in the adiabatic limit $\varepsilon \rightarrow 0$; it appears from the term $(\mathcal{F}_{ai}, \mathcal{F}^{ai})$ in (11) (other terms vanish). The equations (33) are the standard sigma-model equations defining maps from Σ_2 into the based loop group ΩG .

7. Virasoro constraints. The last undiscussed equations are the constraints (18). Substituting (27) into (18), we obtain

$$g^{ij}(\xi_{\alpha i}, \xi_{\beta j}) \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} g^{ij}(\xi_{\alpha i}, \xi_{\beta j}) \partial_c \phi^\alpha \partial_d \phi^\beta = 0 . \quad (35)$$

Integrating (35) over H^2 (projection on \mathcal{M}), we get

$$G_{\alpha\beta} \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta = 0 . \quad (36)$$

These are equations which one will obtain from (34) by varying with respect to g_{ab} . Thus

$$T_{ab}^V = G_{\alpha\beta} \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta \quad (37)$$

is the traceless stress-energy tensor and equations (36) are the Virasoro constraints accompanying the Polyakov string action (34).

8. B -field. In string theory the action (34) is often extended by adding the B -field term. This term can be obtained from the topological Yang-Mills term

$$\frac{1}{2} \int_M d^4x \sqrt{\det g_{H_2}^\varepsilon} \varepsilon_{\mu\nu\lambda\sigma} (\mathcal{F}_\varepsilon^{\mu\nu}, \mathcal{F}_\varepsilon^{\lambda\sigma}) \quad (38)$$

which in the adiabatic limit $\varepsilon \rightarrow 0$ becomes

$$\int_M d^4x \sqrt{\det g_{H_2}} \varepsilon^{ab} \varepsilon^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) = \int_{\Sigma_2} dx^1 dx^2 \varepsilon^{cd} B_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta , \quad (39)$$

where

$$B_{\alpha\beta} = \int_{H^2} d\text{vol} \varepsilon^{ij}(\xi_{\alpha i}, \xi_{\beta j}) . \quad (40)$$

are components of the two-form $\mathbb{B} = (B_{\alpha\beta})$ on the moduli space $\mathcal{M} = \Omega G$.

9. Remarks on superstrings. The adiabatic limit of supersymmetric Yang-Mills theories with a (partial) topological twisting on Euclidean manifold $\Sigma \times \tilde{\Sigma}$, where Σ and $\tilde{\Sigma}$ are Riemann surfaces, was considered in [15]. Several sigma-models with fermions on Σ (including supersymmetric ones) were obtained. Switching to Lorentzian signature and adding constraints of type (18), which were not considered in [15], one can get stringy sigma-model resembling NSR strings. However, analysis of these sigma-models demands more efforts and goes beyond the scope of our paper.

Another possibility is to consider ordinary Yang-Mills theory (11) but with Lie supergroup G as the structure group. We restrict ourselves to the $N=2$ super translation group with ten-dimensional

Minkowski space $\mathbb{R}^{9,1}$ as bosonic part. This super translation group can be represented as the coset [16, 17]

$$G = \text{SUSY}(N=2)/\text{SO}(9,1) , \quad (41)$$

with coordinates (X^α, θ^{Ap}) , where $\theta^p = (\theta^{Ap})$ are two Majorana-Weyl spinors in $d = 10, \alpha = 0, \dots, 9, A = 1, \dots, 32$ and $p = 1, 2$. The generators of G obey the Lie superalgebra $\mathfrak{g} = \text{Lie } G$,

$$\{\xi_{Ap}, \xi_{Bq}\} = (\gamma^\alpha C)_{AB} \delta_{pq} \xi_\alpha , \quad [\xi_\alpha, \xi_{Ap}] = 0 , \quad [\xi_\alpha, \xi_\beta] = 0 , \quad (42)$$

where γ^α are the γ -matrices in $\mathbb{R}^{9,1}$ and C is the charge conjugation matrix. On the superalgebra \mathfrak{g} we introduce the standard metric

$$\langle \xi_\alpha \xi_\beta \rangle = \eta_{\alpha\beta} , \quad \langle \xi_\alpha \xi_{Ap} \rangle = 0 \quad \text{and} \quad \langle \xi_{Ap} \xi_{Bq} \rangle = 0 , \quad (43)$$

where $(\eta_{\alpha\beta}) = \text{diag}(-1, 1, \dots, 1)$ is the Lorentzian metric on $\mathbb{R}^{9,1}$.

It was shown in [18] that the action functional for Yang-Mills theory on $\Sigma_2 \times H^2$ with the gauge group G , defined by (42),

$$S_\varepsilon = \frac{1}{2\pi} \int_{\Sigma_2 \times H^2} d^4x \sqrt{|\det g_{\Sigma_2}|} \sqrt{\det g_{H^2}} \left\{ \varepsilon^2 \langle \mathcal{F}_{ab} \mathcal{F}^{ab} \rangle + 2 \langle \mathcal{F}_{ai} \mathcal{F}^{ai} \rangle + \varepsilon^{-2} \langle \mathcal{F}_{ij} \mathcal{F}^{ij} \rangle \right\} \quad (44)$$

plus the Wess-Zumino-type term

$$S_{WZ} = \frac{1}{\pi} \int_{\Sigma_3 \times H^2} dx^{\hat{a}} \wedge dx^{\hat{b}} \wedge dx^{\hat{c}} \wedge dx^3 \wedge dx^4 f_{\Gamma\Delta\Lambda} \mathcal{F}_{\hat{a}i}^\Gamma \xi^i \mathcal{F}_{\hat{b}j}^\Delta \xi^j \mathcal{F}_{\hat{c}k}^\Lambda \xi^k \quad (45)$$

yield the Green-Schwarz superstring action [17] in the adiabatic limit $\varepsilon \rightarrow 0$. Here Σ_3 is a Lorentzian manifold with the boundary $\Sigma_2 = \partial\Sigma_3$ and local coordinates $x^{\hat{a}}, \hat{a} = 0, 1, 2$; the structure constants $f_{\Gamma\Delta\Lambda}$ are given in [16] and $(\xi_i) = (\sin \varphi, -\cos \varphi)$ is the unit vector on H^2 running the boundary $S^1 = \partial H^2$.

10. Superstrings from $d = 3$ Yang-Mills. Here we will show that the Green-Schwarz superstrings with a worldsheet Σ_2 can also be associated with a Yang-Mills model on $\Sigma_2 \times S^1$. When the radius of S^1 tends to zero, the action of this Yang-Mills model becomes the Green-Schwarz superstring action. So, we consider Yang-Mills theory on a direct product manifold $M^3 = \Sigma_2 \times S^1$, where Σ_2 is a two-dimensional Lorentzian manifold discussed before and S^1 is the unit circle parameterized by $x^3 \in [0, 2\pi]$ with the metric tensor $g_{S^1} = (g_{33})$ and $g_{33} = 1$. As the structure group G of Yang-Mills theory we consider the super translation group in $d = 10$ auxiliary dimensions (41) with the generators (42) and the metric (43) on the Lie superalgebra $\mathfrak{g} = \text{Lie } G$. As in (20), we impose framing over S^1 , i.e. consider the group of gauge transformations equal to the identity over S^1 . Coordinates on G are X^α and θ^{Ap} introduced in the previous section. The one-forms

$$\Pi^\Delta = \{\Pi^\alpha, \Pi^{Ap}\} = \{dX^\alpha - i\delta_{pq}\bar{\theta}^p\gamma^\alpha d\theta^q, d\theta^{Ap}\} \quad (46)$$

form a basis of one-forms on G [16].

By using the adiabatic approach, we deform the metric on $\Sigma_2 \times S^1$ and introduce

$$ds_\varepsilon^2 = g_{\mu\nu}^\varepsilon dx^\mu dx^\nu = g_{ab} dx^a dx^b + \varepsilon^2 (dx^3)^2 , \quad (47)$$

where $\varepsilon \in [0, 1]$ is a real parameter, $a, b = 1, 2$, $\mu, \nu = 1, 2, 3$. This is equivalent to the consideration of the circle S_ε^1 of radius ε . It is assumed that for the fields \mathcal{A}_μ and $\mathcal{F}_{\mu\nu}$ there exist limits $\lim_{\varepsilon \rightarrow 0} \mathcal{A}_\mu$ and $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_{\mu\nu}$. Indices are raised by $g_\varepsilon^{\mu\nu}$ and we have

$$\mathcal{F}_\varepsilon^{ab} = g_\varepsilon^{ac} g_\varepsilon^{bd} \mathcal{F}_{cd} = \mathcal{F}^{ab} , \quad \mathcal{F}_\varepsilon^{a3} = g_\varepsilon^{ac} g_\varepsilon^{33} \mathcal{F}_{c3} = \varepsilon^{-2} \mathcal{F}^{a3} , \quad (48)$$

where indices in $\mathcal{F}^{\mu\nu}$ are raised by the non-deformed metric tensor.

We consider the Yang-Mills action of the form

$$S_\varepsilon = \int_{M^3} d^3x \sqrt{|\det g_{\Sigma_2}|} \left\{ \frac{\varepsilon^2}{2} \langle \mathcal{F}_{ab} \mathcal{F}^{ab} \rangle + \langle \mathcal{F}_{a3} \mathcal{F}^{a3} \rangle \right\} , \quad (49)$$

which for $\varepsilon = 1$ coincides with the standard Yang-Mills action. Variations with respect to \mathcal{A}_μ and g_{ab} yield the equations

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_3 \mathcal{F}^{3b} = 0 , \quad D_a \mathcal{F}^{a3} = 0 , \quad (50)$$

$$T_{ab}^\varepsilon = \varepsilon^2 \left(g^{cd} \langle \mathcal{F}_{ac} \mathcal{F}^{bd} \rangle - \frac{1}{4} g_{ab} \langle \mathcal{F}_{cd} \mathcal{F}^{cd} \rangle \right) + \langle \mathcal{F}_{a3} \mathcal{F}_{b3} \rangle - \frac{1}{2} g_{ab} \langle \mathcal{F}_{c3} \mathcal{F}^{c3} \rangle . \quad (51)$$

In the adiabatic limit $\varepsilon \rightarrow 0$ equations (50), (51) become

$$D_3 \mathcal{F}^{3b} = 0 , \quad D_a \mathcal{F}^{a3} = 0 , \quad (52)$$

$$T_{ab}^0 = \langle \mathcal{F}_{a3} \mathcal{F}_{b3} \rangle - \frac{1}{2} g_{ab} \langle \mathcal{F}_{c3} \mathcal{F}^{c3} \rangle . \quad (53)$$

Notice that as a function of $x^3 \in S^1$, the field \mathcal{A}_3 belongs to the loop algebra $L\mathfrak{g} = \mathfrak{g} \oplus \Omega\mathfrak{g}$, where $\Omega\mathfrak{g}$ is the Lie superalgebra of the based loop group ΩG . Let us denote by \mathcal{A}_3^0 the zero-mode in the expansion of \mathcal{A}_3 in $\exp(i x^3) \in S^1$ (Wilson line). The generic \mathcal{A}_3 can be represented in the form

$$\mathcal{A}_3 = h^{-1} \mathcal{A}_3^0 h + h^{-1} \partial_3 h , \quad (54)$$

where G -valued function h depends on x^a and x^3 . For fixed $x^a \in \Sigma_2$ one can choose $h \in \Omega G = \text{Map}(S^1, G)/G$. We denote by \mathcal{N} the space of all \mathcal{A}_3 given by (54) and define the projection $\pi : \mathcal{N} \rightarrow G$ on the space G parametrizing \mathcal{A}_3^0 since we want to keep only \mathcal{A}_3^0 in the limit $\varepsilon \rightarrow 0$. We denote by Q the fibres of the projection π .

In the adiabatic approach it is assumed that \mathcal{A}_3^0 depends on $x^a \in \Sigma_2$ only via the moduli parameters $(X^\alpha, \theta^{Ap}) \in G$. Therefore, the moduli define the maps

$$(X, \theta^p) : \Sigma_3 \rightarrow G \quad (55)$$

which are not arbitrary, they are constrained by the equations (52), (53). The derivatives $\partial_a \mathcal{A}_3$ of $\mathcal{A}_3 \in \mathcal{N}$ belong to the tangent space $T_{\mathcal{A}_3} \mathcal{N}$ of the space \mathcal{N} . Using the projection $\pi : \mathcal{N} \rightarrow G$, one can decompose $\partial_a \mathcal{A}_3$ into two parts

$$T_{\mathcal{A}_3} \mathcal{N} = \pi^* T_{\mathcal{A}_3^0} G \oplus T_{\mathcal{A}_3} Q \quad \Leftrightarrow \quad \partial_a \mathcal{A}_3 = \Pi_a^\Delta \xi_{\Delta 3} + D_3 \epsilon_a , \quad (56)$$

where $\Delta = (\alpha, Ap)$ and

$$\Pi_a^\alpha := \partial_a X^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^\alpha \partial_a \theta^q , \quad \Pi_a^{Ap} := \partial_a \theta^{Ap} . \quad (57)$$

In (56), ϵ_a are \mathfrak{g} -valued parameters ($D_3\epsilon_a \in T_{\mathcal{A}_3}Q$) and the vector fields $\xi_{\Delta 3}$ on G can be identified with the generators $\xi_{\Delta} = (\xi_{\alpha}, \xi_{Ap})$ of G .

On $\xi_{\Delta 3}$ we impose the gauge fixing condition

$$D_3\xi_{\Delta 3} = 0 \quad \xrightarrow{(56)} \quad D_3D_3\epsilon_a = D_3\partial_a\mathcal{A}_3 . \quad (58)$$

Recall that \mathcal{A}_3 is fixed by (54) and \mathcal{A}_a are yet free. In the adiabatic approach one chooses $\mathcal{A}_a = \epsilon_a$ (cf. [5, 6]) and obtains

$$\mathcal{F}_{a3} = \partial_a\mathcal{A}_3 - D_3\mathcal{A}_a = \Pi_a^{\Delta}\xi_{\Delta 3} \in T_{\mathcal{A}_3}G . \quad (59)$$

Substituting (59) into the first equation in (52), we see that they are resolved due to (58). Substituting (59) into the action $S_0 = \lim_{\varepsilon \rightarrow 0} S_{\varepsilon}$ given by (49) and integrating over x^3 , we obtain the effective action

$$S_0 = 2\pi \int_{\Sigma_2} d^2x \sqrt{|\det g_{\Sigma_2}|} g^{ab} \Pi_a^{\alpha} \Pi_b^{\beta} \eta_{\alpha\beta} , \quad (60)$$

which coincides with the kinetic part of the Green-Schwarz superstring action [17]. One can show (cf. [14]) that the second equations in (52) are equivalent to the Euler-Lagrange equations for $(X^{\alpha}, \theta^{Ap})$ following from (60). Finally, substituting (59) into (53), we obtain the equations

$$\Pi_a^{\alpha} \Pi_b^{\beta} \eta_{\alpha\beta} - \frac{1}{2} g_{ab} g^{cd} \Pi_c^{\alpha} \Pi_d^{\beta} \eta_{\alpha\beta} = 0 \quad (61)$$

which can also be obtained from (60) by variation of g^{ab} .

For getting the full Green-Schwarz superstring action one should add to (60) a Wess-Zumino-type term which is described as follows [16, 17]. One should consider a Lorentzian 3-manifold Σ_3 with the boundary $\Sigma_2 = \partial\Sigma_3$ and coordinates $x^{\hat{a}}$, $\hat{a} = 0, 1, 2$. On Σ_3 one introduces the 3-form [16]

$$\Omega_3 = i dx^{\hat{a}} \Pi_{\hat{a}}^{\alpha} \wedge (\check{d}\bar{\theta}^1 \gamma^{\beta} \wedge \check{d}\theta^1 - \check{d}\bar{\theta}^2 \gamma^{\beta} \wedge \check{d}\theta^2) \eta_{\alpha\beta} = \check{d}\Omega_2 , \quad (62)$$

where

$$\Omega_2 = -i \check{d}X^{\alpha} \wedge (\bar{\theta}^1 \gamma^{\beta} \check{d}\theta^1 - \bar{\theta}^2 \gamma^{\beta} \check{d}\theta^2) \quad \text{with} \quad \check{d} = dx^{\hat{a}} \frac{\partial}{\partial x^{\hat{a}}} .$$

Then the term

$$S_{WZ} = \int_{\Sigma_3} \Omega_3 = \int_{\Sigma_2} \Omega_2 \quad (63)$$

is added to (60) with a proper coefficient \varkappa and $S_{GS} = S_0 + \varkappa S_{WZ}$ is the Green-Schwarz action for the superstrings of type I, IIA and IIB.

To get (63) from Yang-Mills theory we consider the manifold $\Sigma_3 \times S^1$ and notice that in addition to (59) we now have the components

$$\mathcal{F}_{03} = \Pi_0^{\Delta} \xi_{\Delta 3} = (\partial_0 X^{\alpha} - i \delta_{pq} \bar{\theta}^p \gamma^{\alpha} \partial_0 \theta^q) \xi_{\alpha 3} + (\partial_0 \theta^{Ap}) \xi_{Ap 3} . \quad (64)$$

Introduce one-forms $F_3 := \mathcal{F}_{\hat{a}3} dx^{\hat{a}}$ on Σ_3 , where $\mathcal{F}_{\hat{a}3}(\varepsilon)$ are general Yang-Mills fields on $\Sigma_3 \times S^1$ which take the form (59),(64) only in the limit $\varepsilon \rightarrow 0$, and consider the functional

$$S_{WZ}^{YM} = \int_{\Sigma_3 \times S^1} f_{\Delta\Lambda\Gamma} F_3^{\Delta} \wedge F_3^{\Lambda} \wedge F_3^{\Gamma} \wedge dx^3 , \quad (65)$$

where the explicit form of the constant $f_{\Delta\Lambda\Gamma}$ can be found in [16]. Therefore, the Yang-Mills action (49) plus (65) in the adiabatic limit $\varepsilon \rightarrow 0$ becomes the Green-Schwarz action. This result can be considered as a generalization of the Green result [19] who derived the superstring theory in a fixed gauge from Chern-Simons theory on $\Sigma_2 \times \mathbb{R}$.

11. Concluding remarks. We have shown that bosonic strings and Green-Schwarz superstrings can be obtained via the adiabatic limit of Yang-Mills theory on manifolds $\Sigma_2 \times H^2$ with a Wess-Zumino-type term. Notice that the constraint equations (15) on the Yang-Mills energy momentum tensor with $\varepsilon > 0$ are important for restoring the unitarity of Yang-Mills theory on $\Sigma_2 \times H^2$. More interestingly, the same result is also obtained by considering Yang-Mills theory on three-dimensional manifolds $\Sigma_2 \times S_\varepsilon^1$ with the radius of the circle S_ε^1 given by $\varepsilon \in [0, 1]$. For $\varepsilon \neq 0$ we have well-defined quantum Yang-Mills theory on $\Sigma_2 \times S_\varepsilon^1$. For $\varepsilon \rightarrow 0$ we get superstring theories. This raises hopes that various results for superstring theories can be obtained from results of the associated Yang-Mills theory on $\Sigma_2 \times S_\varepsilon^1$.

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