

Derivation of the Lorentz Force Law, the Magnetic Field Concept and the Faraday–Lenz Law using an Invariant Formulation of the Lorentz Transformation

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Abstract

It is demonstrated how the right hand sides of the Lorentz Transformation equations may be written, in a Lorentz invariant manner, as 4–vector scalar products. This implies the existence of invariant length intervals analogous to invariant proper time intervals. The formalism is shown to provide a short derivation of the Lorentz force law of classical electrodynamics, and the conventional definition of the magnetic field, in terms of spatial derivatives of the 4–vector potential, as well as the Faraday–Lenz Law. An important distinction between the physical meanings of the space-time and energy-momentum 4–vectors is pointed out.

Keywords; Special Relativity, Classical Electrodynamics.

PACS 03.30+p 03.50.De

1 Introduction

Numerous examples exist in the literature of the derivation of electrodynamical equations from simpler physical hypotheses. In Einstein's original paper on Special Relativity [1], the Lorentz force law was derived by performing a Lorentz transformation of the electromagnetic fields and the space-time coordinates from the rest frame of an electron (where only electrostatic forces act) to the laboratory system where the electron is in motion and so also subjected to magnetic forces. A similar demonstration was given by Schwartz [2] who also showed how the electrodynamic and magnetodynamic Maxwell equations can be derived from the Gauss laws of electrostatics and magnetostatics by exploiting the 4-vector character of the electromagnetic current and the symmetry properties of the electromagnetic field tensor. The same type of derivation of the electrodynamic and magnetodynamic Maxwell equations has recently been performed by the present author on the basis of 'space-time exchange symmetry' [3]. Frisch and Wilets [4] discussed the derivation of Maxwell's equations and the Lorentz force law by application of relativistic transforms to the electrostatic Gauss law. Dyson [5] published a proof, due originally to Feynman, of the Faraday-Lenz law of induction, based on Newton's Second Law and the quantum commutation relations of position and momentum, that excited considerable interest and a flurry of comments and publications [6, 7, 8, 9, 10, 11] about a decade ago. Landau and Lifshitz [12] presented a derivation of Ampère's Law from the electrodynamic Lagrangian, using the Principle of Least Action. By relativistic transformation of the Coulomb force from the rest frame of a charge to another inertial system in relative motion, Lorrain, Corson and Lorrain [13] derived both the Biot-Savart law, for the magnetic field generated by a moving charge, and the Lorentz force law.

In many text books on classical electrodynamics the question of what are the fundamental physical hypotheses underlying the subject, as distinct from purely mathematical developments of these hypotheses, used to derive predictions, is not discussed in any detail. Indeed, it may even be stated that it is futile to address the question at all. For example, Jackson [14] states:

At present it is popular in undergraduate texts and elsewhere to attempt to derive magnetic fields and even Maxwell equations from Coulomb's law of electrostatics and the theory of Special Relativity. It should immediately obvious that, without additional assumptions, this is impossible.'

This is, perhaps, a true statement. However, if the additional assumptions are weak ones, the derivation may still be a worthwhile exercise. In fact, in the case of Maxwell's equations, as shown in References [2, 3], the 'additional assumptions' are merely the formal definitions of the electric and magnetic fields in terms of the space-time derivatives of the 4-vector potential [15]. In the case of the derivation of the Lorentz force equation given below, not even the latter assumption is required, as the magnetic field definition appears naturally in the course of the derivation.

In the chapter on 'The Electromagnetic Field' in Misner Thorne and Wheeler's book 'Gravitation' [16] can be found the following statement:

Here and elsewhere in science, as stressed not least by Henri Poincaré, that view is

out of date which used to say, “Define your terms before you proceed”. All the laws and theories of physics, including the Lorentz force law, have this deep and subtle character, that they both define the concepts they use (here \vec{B} and \vec{E}) and make statements about these concepts. Contrariwise, the absence of some body of theory, law and principle deprives one of the means properly to define or even use concepts. Any forward step in human knowledge is truly creative in this sense: that theory concept, law, and measurement —forever inseparable—are born into the world in union.

I do not agree that the electric and magnetic fields are the fundamental concepts of electromagnetism, or that the Lorentz force law cannot be derived from simpler and more fundamental concepts, but must be ‘swallowed whole’, as this passage suggests. As demonstrated in References [2, 3] where the electrodynamic and magnetodynamic Maxwell equations are derived from those of electrostatics and magnetostatics, a more economical description of classical electromagnetism is provided by the 4–vector potential. Another example of this is provided by the derivation of the Lorentz force law presented in the present paper. The discussion of electrodynamics in Reference [16] is couched entirely in terms of the electromagnetic field tensor, $F^{\mu\nu}$, and the electric and magnetic fields which, like the Lorentz force law and Maxwell’s equations, are ‘parachuted’ into the exposition without any proof or any discussion of their interrelatedness. The 4–vector potential is introduced only in the next-but-last exercise at the end of the chapter. After the derivation of the Lorentz force law in Section 3 below, a comparison will be made with the treatment of the law in References [2, 14, 16].

The present paper introduces, in the following Section, the idea of an ‘invariant formulation’ of the Lorentz Transformation (LT) [17]. It will be shown that the RHS of the LT equations of space and time can be written as 4–vector scalar products, so that the transformed 4–vector components are themselves Lorentz invariant quantities. Consideration of particular length and time interval measurements demonstrates that this is a physically meaningful concept. It is pointed out that, whereas space and time intervals are, in general, physically independent physical quantities, this is not the case for the space and time components of the energy-momentum 4–vector. In Section 3, a derivation of the Lorentz force law, and the associated magnetic field concept, is given, based on the invariant formulation of the LT. The derivation is very short, the only initial hypothesis being the usual definition of the electric field in terms of the 4–vector potential, which, in fact, is also uniquely specified by requiring the definition to be a covariant one. In Section 4 the time component of Newton’s Second Law in electrodynamics, obtained by applying space-time exchange symmetry [3] to the Lorentz force law, is discussed.

Throughout this paper it is assumed that the electromagnetic field constitutes, together with the moving charge, a conservative system; i.e. effects of radiation, due to the acceleration of the charge, are neglected

2 Invariant Formulation of the Lorentz Transformation

The space-time LT equations between two inertial frames S and S’, written in a space-

time symmetric manner, are:

$$x' = \gamma(x - \beta x_0) \quad (2.1)$$

$$y' = y \quad (2.2)$$

$$z' = z \quad (2.3)$$

$$x'_0 = \gamma(x_0 - \beta x) \quad (2.4)$$

The frame S' moves with velocity, v , relative to S, along the common x-axis of S and S'. β and γ are the usual relativistic parameters:

$$\beta \equiv \frac{v}{c} \quad (2.5)$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \quad (2.6)$$

where c is the speed of light, and

$$x_0 \equiv ct \quad (2.7)$$

where t is the time recorded by an observer at rest in S. Clocks in S and S' are synchronised, i.e., $t = t' = 0$, when the origins of the spatial coordinates of S and S' coincide.

Eqns(2.1)-(2.4) give the relation between space and time intervals $\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$, $\Delta x_0 = c(t_2 - t_1)$, where (\vec{r}_1, t_1) and (\vec{r}_2, t_2) are space-time events, and $\vec{r} \equiv (x, y, z)$, as observed in the two frames:

$$\Delta x' = \gamma(\Delta x - \beta \Delta x_0) \quad (2.8)$$

$$\Delta y' = \Delta y \quad (2.9)$$

$$\Delta z' = \Delta z \quad (2.10)$$

$$\Delta x'_0 = \gamma(\Delta x_0 - \beta \Delta x) \quad (2.11)$$

Suppose now that a physical object, O, of Newtonian mass, m , is at rest in the frame S'; then $t' = \tau$ is the proper time of the object. The following 4-vectors are now defined [18]:

$$X \equiv (x_0; x, y, z) \quad (2.12)$$

$$V \equiv \frac{dX}{d\tau} = (\gamma c; \gamma v_x, \gamma v_y, \gamma v_z) \quad (2.13)$$

$$P \equiv mV = (p_0; p_x, p_y, p_z) \quad (2.14)$$

X , V and P are the space-time, velocity and energy-momentum 4-vectors of the object O respectively. It follows from Eqns(2.8)-(2.11) and the definition of P in (2.14) that it has the following LT between the frames S and S':

$$p'_x = \gamma(p_x - \beta p_0) \quad (2.15)$$

$$p'_y = p_y \quad (2.16)$$

$$p'_z = p_z \quad (2.17)$$

$$p'_0 = \gamma(p_0 - \beta p_x) \quad (2.18)$$

Inspection of (2.8)-(2.11) and (2.15)-(2.18) shows that the LT equations for ΔX and P are identical. However, as will be now discussed, there is an important difference in the physical interpretation of the two sets of transformation equations.

Two independent Lorentz invariant quantities may be associated with each LT (2.8)-(2.11) and (2.15)-(2.18):

$$s_{y,z}^2 \equiv (\Delta y)^2 + (\Delta z)^2 = (\Delta y')^2 + (\Delta z')^2 \quad (2.19)$$

$$s_{x,0}^2 \equiv (\Delta x)^2 - (\Delta x_0)^2 = (\Delta x')^2 - (\Delta x'_0)^2 \quad (2.20)$$

$$p_T^2 \equiv p_y^2 + p_z^2 = (p'_y)^2 + (p'_z)^2 \quad (2.21)$$

$$m_T^2 c^2 \equiv p_0^2 - p_x^2 = (p'_0)^2 - (p'_x)^2 \quad (2.22)$$

The intervals $s_{y,z}$ and $s_{x,0}$ may be combined to obtain the usual invariant interval. s :

$$\begin{aligned} s^2 &= s_{y,z}^2 + s_{x,0}^2 \\ &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta x_0)^2 \end{aligned} \quad (2.23)$$

Similarly the transverse momentum, p_T and the ‘transverse mass’ m_T may be combined to obtain the mass of the object O:

$$\begin{aligned} m^2 c^2 &= m_T^2 c^2 - p_T^2 \\ &= p_0^2 - p_x^2 - p_y^2 - p_z^2 \end{aligned} \quad (2.24)$$

The physical interpretations of the LT for ΔX (2.8)-(2.11) and P (2.15) to (2.18) will now be discussed.

The magnitude of the invariant interval $s_{x,0}$ specifies four rectangular hyperbolae on the Minkowski plot of Δx versus Δx_0 (Fig 1). The four hyperbolae result from the double sign ambiguity on taking square roots on both sides of Eqn(2.20). The equations of the hyperbolae are [19]:

$$s_x^+ = \sqrt{(\Delta x)^2 - (\Delta x_0)^2} \quad (2.25)$$

$$s_x^- = -\sqrt{(\Delta x)^2 - (\Delta x_0)^2} \quad (2.26)$$

$$s_0^+ = \sqrt{(\Delta x_0)^2 - (\Delta x)^2} \quad (2.27)$$

$$s_0^- = -\sqrt{(\Delta x_0)^2 - (\Delta x)^2} \quad (2.28)$$

where

$$s_x^+ = s_x^- = s_0^+ = s_0^- = s_{x,0}$$

The intervals Δx and Δx_0 corresponding to any pair of space-time points lie one of these hyperbolae. If the points have a space-like separation ($s_{x,0}^2 > 0$) the corresponding intervals lie on (2.25) or (2.26); if they have a time-like separation ($s_{x,0}^2 < 0$) they lie on (2.27) or (2.28). The different points on each hyperbola are the intervals of the same pair of space-time events as recorded by different inertial observers. As shown in Fig 1, the magnitude, $s_{x,0}$ of the invariant interval is equal to the distance of closest approach of each hyperbola to the origin in the Minkowski plot.

Now it is interesting to note that any space or time interval Δx or Δx_0 may be identified with an invariant interval in a particular reference frame. Consider the hyperbola with Δx intercepts s_x^+ or s_x^- in Fig 1. In the inertial frame in which $\Delta x_0 = 0$ (the intersection of these hyperbolae with the Δx axis) it follows from (2.25) and (2.26) that:

$$s_x^+ = s_{x,0} = \sqrt{\Delta x^2 - (\Delta x_0)^2} = \Delta x \quad (\Delta x > 0) \quad (2.29)$$

$$s_x^- = s_{x,0} = -\sqrt{\Delta x^2 - (\Delta x_0)^2} = -\Delta x \quad (\Delta x < 0) \quad (2.30)$$

The measurement in this frame consists of taking the difference between the spatial coordinates of events at some fixed time. Such a frame may be defined for any pair of space-like separated events as a consequence of the geometry of the Minkowski plot. Notice that Δx is not *necessarily* defined in terms of such a measurement. If, following Einstein [1], the interval Δx is associated with the length, ℓ , of a measuring rod at rest in S and lying parallel to the x-axis, measurements of the ends of the rod can be made at arbitrarily different times in S. The same result $\ell = \Delta x$ will be found for the length of the rod, but the corresponding invariant intervals, s_x^+ , s_x^- as defined by Eqn(2.25),(2.26) will be different in each case. Such measurements, with $\Delta x_0 \neq 0$ are associated with all points of the hyperbolae with Δx -axis intercepts s_x^+ and s_x^- , except their intersections with the Δx -axis.

Similarly, Δx_0 may be identified with the time-like invariant interval corresponding to successive observations of a clock at a fixed position (i.e. $\Delta x = 0$) in S. In this case (2.27) and (2.28) give:

$$s_0^+ = s_{x,0} = \sqrt{(\Delta x_0)^2 - (\Delta x)^2} = \Delta x_0 \quad (\Delta x_0 > 0) \quad (2.31)$$

$$s_0^- = s_{x,0} = -\sqrt{(\Delta x_0)^2 - (\Delta x)^2} = -\Delta x_0 \quad (\Delta x_0 < 0) \quad (2.32)$$

This corresponds, in the Minkowski plot, to the inertial frame for which the hyperbolae with Δx_0 -axis intercepts s_0^+ , s_0^- intersect this axis. Such a frame exists for every pair of time-like separated events. The interval Δx_0 could also be measured by observing the difference of the times recorded by a local clock and another, synchronised, one located at a different position in S, after a suitable correction for light propagation time delay. Each such pair of clocks would yield the same value, Δx_0 , for the time difference between two events in S, but with different values of the invariant intervals defined by (2.31) or (2.32).

The invariant quantities s_0^+ , s_0^- are better known as $\pm c\Delta\tau$ where $\Delta\tau$ is the *proper time interval* in the frame S. Less well-known however is that as a consequence of the space-time symmetry manifest on the Minkowski plot, s_x^+ , s_x^- may be also identified with *invariant space intervals* $\pm\Delta\lambda$ in the frame S. This is the same as the length along the x-axis of any physical object at rest in S. Both $\Delta\tau$ and $\Delta\lambda$ may be defined by measurements corresponding to particular space-time *projections* in the frame S. As discussed above, $\Delta\tau$ corresponds to a $\Delta x = 0$ projection and $\Delta\lambda$ to a $\Delta x_0 = 0$ one. The role of such projections in the generation of the various apparent distortions of space-time in special relativity has been discussed in [20]. In every-day language these projections correspond to observations of a clock at a fixed position, or of the dimension of an object at rest, i.e. the usual way in which time and space measurements are made.

The intervals Δx and Δx_0 refer, in general, to space and time differences between different events. The latter may be, but are not necessarily, related to properties of the same physical object. As discussed above, Δx may be, for example, identified with the physical length, ℓ , of a rod, but the LT equations (2.8)-(2.11) are valid for any pair whatsoever

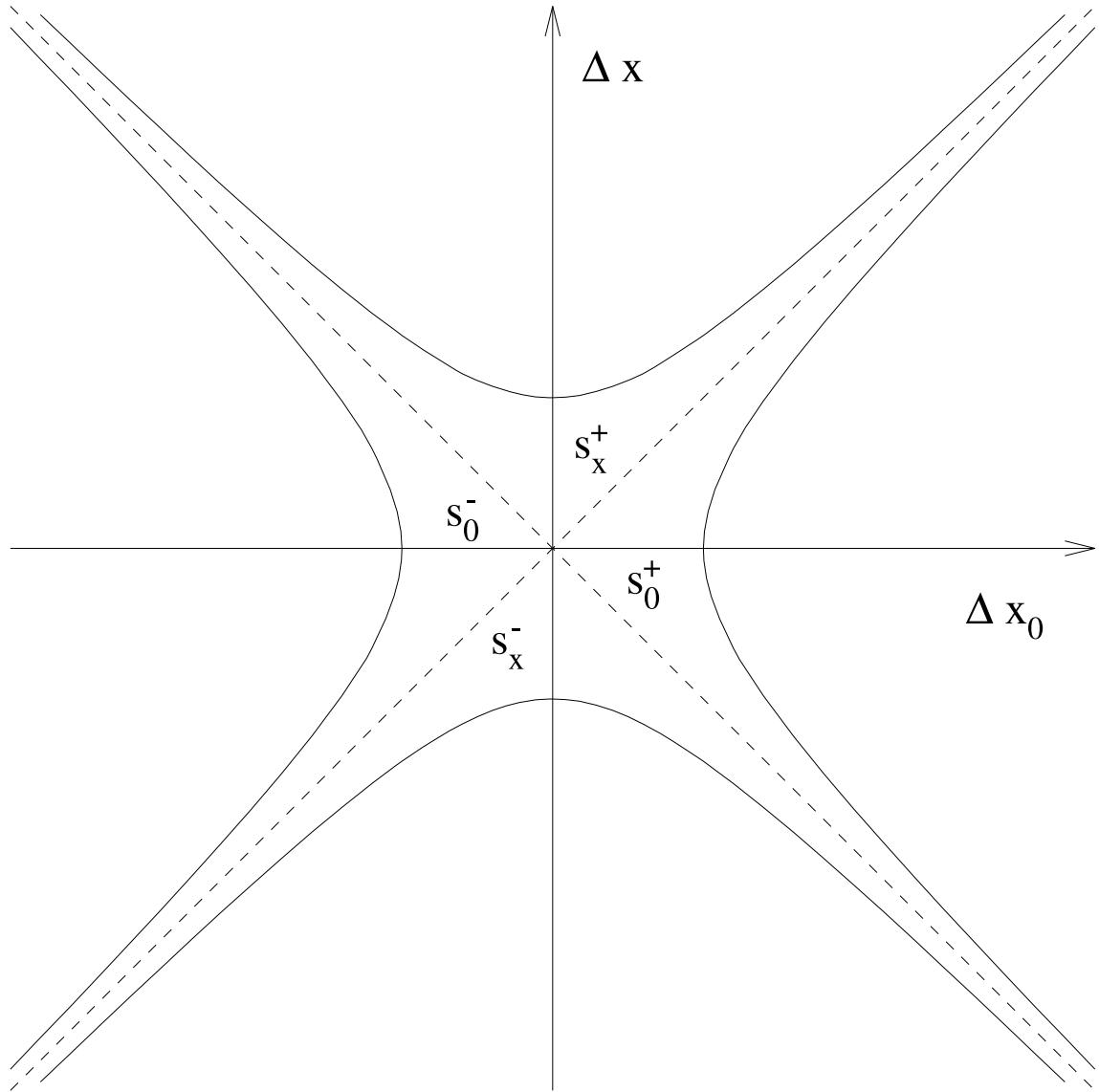


Figure 1: Space-time Minkowski Plot of Δx versus Δx_0 . The intervals corresponding to every pair of time-like separated events are seen, by different observers, to lie on the hyperbolae with Δx_0 -axis intercepts s_0^+ and s_0^- . Those for space-like separated events lie on the hyperbolae with Δx -axis intercepts s_x^+ and s_x^- . The dotted lines show the asymptotes of the hyperbolae, that are the projection of the light cone in the $\Delta x_0 - \Delta x$ plane.

of space-time events. Thus in Eqn(2.20), Δx and Δx_0 may be freely and independently chosen, each different pair describing a possible, but different event configuration corresponding to the same or different values of $s_{0,x}$.

The situation is quite different for the quantities p_0 and p_x as a consequence of the existence of the Lorentz scalar, m , the Newtonian mass, which is a fixed property of any physical object O. Because of the relation (2.24) it follows that for fixed p_T , as required by the LT equations (2.16) and (2.17), the value of p_0 is determined by that of p_x , and vice versa. Therefore only one of these quantities is independent for any physical object.

It has been shown above that *arbitrary* space and time intervals Δx and Δx_0 are equal to certain *Lorentz invariant* quantities $S_x \equiv s_x^\pm$ and $S_0 \equiv s_0^\pm$ by noting that the latter correspond to measurements of the metric in Eqn(2.20) in frames of reference where $\Delta x_0 = 0$ or $\Delta x = 0$, respectively. Another way to demonstrate this correspondence of arbitrary space and time intervals with Lorentz scalar quantities is to write the LT equations (2.8) and (2.11) in the following *invariant* form:

$$S'_x = -\bar{U}(\beta) \cdot S \quad (2.33)$$

$$S'_0 = U(\beta) \cdot S \quad (2.34)$$

where the following 4-vectors have been introduced:

$$S \equiv (S_0; S_x, 0, 0) = (\Delta x_0; \Delta x, 0, 0) \quad (2.35)$$

$$U(\beta) \equiv (\gamma; \gamma\beta, 0, 0) \quad (2.36)$$

$$\bar{U}(\beta) \equiv (\gamma\beta; \gamma, 0, 0) \quad (2.37)$$

The time-like 4-vector, U , is equal to V/c , where V is the 4-vector velocity of S' relative to S, whereas the space-like 4-vector, \bar{U} , is ‘orthogonal to U in four dimensions’[21]:

$$U(\beta) \cdot \bar{U}(\beta) = 0 \quad (2.38)$$

Since the RHS of (2.33) and (2.34) are 4-vector scalar products, S'_x and S'_0 are manifestly Lorentz invariant quantities. These 4-vector components may be defined, in terms of specific space-time measurements, by equations similar to (2.29)-(2.32) in the frame S'. Note that the 4-vectors S and S' are ‘doubly covariant’ in the sense that $S \cdot S$ and $S' \cdot S'$ are ‘doubly invariant’ quantities whose spatial and temporal terms are, individually, Lorentz invariant:

$$S \cdot S = S_0^2 - S_x^2 = S' \cdot S' = (S'_0)^2 - (S'_x)^2 \quad (2.39)$$

Every term in this equation remains invariant if the spatial and temporal intervals described above are observed from a third inertial frame S'' moving along the x-axis relative to both S and S'. This follows from the manifest Lorentz invariance of the RHS of Eqn(2.33) and (2.34) and their inverses:

$$S_x = -\bar{U}(-\beta) \cdot S' \quad (2.40)$$

$$S_0 = U(-\beta) \cdot S' \quad (2.41)$$

Since the LT Eqns(2.1) and (2.4) are valid for any 4-vector, W , it follows that:

$$W'_x = -\bar{U}(\beta) \cdot W \quad (2.42)$$

$$W'_0 = U(\beta) \cdot W \quad (2.43)$$

Again, W'_x and W'_0 are manifestly Lorentz invariant. The LT equation in invariant form (2.43), for the electromagnetic 4–vector potential, A , plays a crucial role in the derivation of the Lorentz force law presented below.

An interesting special case is the energy-momentum 4–vector P . Choosing the x-axis parallel to \vec{p} and β to correspond to the object’s velocity, so that S’ is the object’s proper frame, and since $P \equiv mcU(\beta)$, Eqns(2.33) and (2.34) yield, for this special case:

$$P'_x = -mc\bar{U}(\beta) \cdot U(\beta) = 0 \quad (2.44)$$

$$P'_0 = mcU(\beta) \cdot U(\beta) = mc \quad (2.45)$$

Since the 4–vector $U(\beta)$ is determined by the single parameter, β , then it follows from the relation $P \equiv mcU(\beta)$ that, unlike in the case of the space and time intervals in Eqn(2.20), the spatial and temporal components of the energy momentum 4–vector P are, as already discussed above, not independent. Thus, although the LT equations for the space-time and energy-momentum 4–vectors are mathematically identical, the physical interpretation of the transformed quantities is quite different in the two cases.

3 Derivation of the Lorentz force law and the Magnetic Field

In electrostatics, the electric field, \vec{E} , is customarily written in terms of the electrostatic potential, ϕ , according to the equation $\vec{E} = -\vec{\nabla}\phi$. The potential at a distance, r , from a point charge, Q , is given by Coulomb’s law $\phi(r) = Q/r$. This, together with the equation $\vec{F} = q\vec{E}$, defining the force, \vec{F} , exerted on a charge, q , by the electric field, completes the specification of the dynamical basis of classical electromagnetism.

It remains to generalise the above equation, relating the electric field to the electrostatic potential, in a manner consistent with special relativity. In relativistic notation [22], the electric field is related to the potential by the equation: $E^i = \partial^i A^0$, where ϕ is identified with the time component, A^0 , of the 4–vector electromagnetic potential $(A^0; \vec{A})$. In order to respect special relativity the electric field must be defined in a covariant manner, i.e. in the same way in all inertial frames. The electrostatic law may be generalised in two ways:

$$E^i \rightarrow E_\pm^i \equiv \partial^i A^0 \pm \partial^0 A^i \quad (3.1)$$

This equation shows the only possibilities to define the electric field in a way that respects the symmetry with respect to the exchange of space and time coordinates that is a general property of all special relativistic laws [3]. Choosing $i = 1$ in Eqn(3.1) and transforming all quantities on the RHS into the S’ frame, by use of the inverses of Eqns(2.1) and (2.4), leads to the following expressions for the 1–component of the electric field in S, in terms of quantities defined in S’:

$$E_\pm^1 = \gamma^2(1 \pm \beta^2)\partial'^1 A'^0 + \gamma^2(\beta^2 \pm 1)\partial'^0 A'^1 + \gamma^2\beta(1 \pm 1)(\partial'^0 A'^0 + \partial'^1 A'^1) \quad (3.2)$$

Only the choice $E^1 \equiv E_+^1$ yields a covariant definition of the electric field. In this case, using Eqns(2.5) and (2.6), Eqn(3.2) simplifies to:

$$E^1 = \partial'^1 A'^0 - \partial'^0 A'^1 = E'^1 \quad (3.3)$$

Which expresses the well-known invariance of the longitudinal component of the electric field under the LT.

Thus, from rotational invariance, the general covariant definition of the electric field is:

$$E^i = \partial^i A^0 - \partial^0 A^i \quad (3.4)$$

This is the ‘additional assumption’, mentioned by Jackson in the passage quoted above, that is necessary, in the present case, to derive the Lorentz force law. Note, however, that Coulomb’s law is *not* assumed; the only postulate is the existence of a 4–vector potential. The electric field is defined by Eqn(3.4) but the magnetic field concept has not yet been introduced. A further *a posteriori* justification of Eqn(3.4) will be given after derivation of the Lorentz force law. Here it is simply noted that, if the spatial part of the 4–vector potential is time-independent, Eqn(3.4) reduces to the usual electrostatic definition of the electric field.

The force \vec{F}' on an electric charge q at rest in the frame S' is given by the definition of the electric field, and Eqn(3.4) as:

$$F'^i = q(\partial'^i A'^0 - \partial'^0 A'^i) \quad (3.5)$$

Equations analogous to (2.43) above may be written relating A' and ∂' to the corresponding quantities in the frame S moving along the x' axis with velocity $-v$ relative to S' :

$$\partial'^0 = U(\beta) \cdot \partial \quad (3.6)$$

$$A'^0 = U(\beta) \cdot A \quad (3.7)$$

Substituting (3.6) and (3.7) in (3.5) gives:

$$F'^i = q \left[\partial'^i (U(\beta) \cdot A) - (U(\beta) \cdot \partial) A'^i \right] \quad (3.8)$$

This equation expresses a linear relationship between F'^i , ∂'^i and A'^i . Since the coefficients of the relation are Lorentz invariant, the same formula is valid in any inertial frame[23], in particular, in the frame S . Hence:

$$F^i = q \left[\partial^i (U(\beta) \cdot A) - (U(\beta) \cdot \partial) A^i \right] \quad (3.9)$$

This equation gives, in 4–vector notation, a spatial component of the Lorentz force on the charge q in the frame S , and so completes the derivation.

To express the Lorentz force formula in the more familiar 3–vector notation, it is convenient to introduce the relativistic generalisation of Newton’s Second Law [24]:

$$\frac{dP}{d\tau} = F \quad (3.10)$$

where F is the 4–vector force and $\tau = t'$ is the proper time (in S') that is related to the time t in S by the relativistic time dilatation formula: $dt = \gamma d\tau$. This gives, with Eqn(3.9) and (3.10):

$$\begin{aligned} \frac{dP^i}{d\tau} &= \gamma \frac{dP^i}{dt} \\ &= q(\partial^i A^0 - \partial^0 A^i)U(\beta)_\alpha \\ &= \gamma q \left[\partial^i A^0 - \partial^0 A^i - \beta_j (\partial^i A^j - \partial^j A^i) - \beta_k (\partial^i A^k - \partial^k A^i) \right] \end{aligned} \quad (3.11)$$

Introducing now the *magnetic field* according to the definition [25]:

$$B^k \equiv -\epsilon_{ijk}(\partial^i A^j - \partial^j A^i) = (\vec{\nabla} \times \vec{A})^k \quad (3.12)$$

enables Eqn(3.11) to be written in the compact form:

$$\frac{dP^i}{dt} = q \left[E^i + \beta_j B^k - \beta_k B^j \right] = q \left[E^i + (\vec{\beta} \times \vec{B})^i \right] \quad (3.13)$$

so that, in 3–vector notation, the Lorentz force law is:

$$\frac{d\vec{p}}{dt} = mc \frac{d(\gamma \vec{\beta})}{dt} = q[\vec{E} + \vec{\beta} \times \vec{B}] \quad (3.14)$$

Writing Eqn(3.4) in 3–vector notation and performing vector multiplication of both sides by the differential operator $\vec{\nabla}$ gives:

$$\vec{\nabla} \times \vec{E} = (\vec{\nabla} \times \vec{\nabla}) A^0 - \partial^0 (\vec{\nabla} \times \vec{A}) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (3.15)$$

since $\vec{W} \times \vec{W} = 0$ for any 3–vector \vec{W} , and Eqn(3.12) has been used. Eqn(3.15) is just the Faraday-Lenz induction law, i.e. the magnetodynamic Maxwell equation. This is only apparent, however, once the ‘magnetic field’ concept of Eqn(3.12) has been introduced. Thus the initial hypothesis, Eqn(3.4), is equivalent to a Maxwell equation once the magnetic field has been introduced. This is the *a posteriori* justification, mentioned above, for this covariant definition of the electric field.

Actually, since (3.12) follows from (3.4), using only the relativistic covariance of the latter, the Faraday-Lenz law has been derived above with the covariant definition of the electric field and special relativity (i.e. the LT) as the only initial postulates.

It is common in discussions of electromagnetism to introduce the second rank electromagnetic field tensor, $F^{\mu\nu}$ according to the definition:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \quad (3.16)$$

in terms of which, the electric and magnetic fields are defined as:

$$E^i \equiv F^{i0} \quad (3.17)$$

$$B^k \equiv -\epsilon_{ijk} F^{ij} \quad (3.18)$$

From the point of view adopted in the present paper both the electromagnetic field tensor and the electric and magnetic fields themselves are auxiliary quantities introduced only for mathematical convenience, in order to write the equations of electromagnetism in a compact way. Since all these quantities are completely defined by the 4–vector potential, it is the latter quantity that encodes all the relevant physical information on any electrodynamic problem [26]. This position is contrary to that commonly taken in the literature and textbooks where it is often claimed that only the electric and magnetic fields have physical significance, while the 4–vector potential is only a convenient mathematical tool. For example Röhrlich [27] makes the statement:

These functions (ϕ and \vec{A}) known as potentials have no physical meaning and are introduced solely for the purpose of mathematical simplification of the equations.

In fact, as shown above (compare Eqns(3.11) and (3.13)) it is the introduction of the electric and magnetic fields that enable the Lorentz force equation to be written in a simple manner! In other cases (e.g. Maxwell's equations) simpler expressions may be written in terms of the 4–vector potential. The quantum theory, quantum electrodynamics, that underlies classical electromagnetism, requires the introduction the 4–vector photon field A^μ in order to specify the minimal interaction that provides the dynamical basis of the theory. Similarly, the introduction of A^μ is necessary for the Lagrangian formulation of classical electromagnetism. It makes no sense, therefore, to argue that a physical concept of such fundamental importance has ‘no physical meaning’.

The initial postulate used here to derive the Lorentz force law is Eqn(3.4), which contains, explicitly, the electrostatic force law and, implicitly, the Faraday-Lenz induction law. The actual form of the electrostatic force law (Coulomb's inverse square law) is not invoked, suggesting that the Lorentz force law may be of greater generality. On the assumption of Eqn(3.4) (which has been demonstrated to be the only possible covariant definition of the electric field), the existence of the ‘magnetic field’, the ‘electromagnetic field tensor’, and finally the Lorentz force law itself have all been derived, without further assumptions, by use of the invariant formulation of the Lorentz transformation.

It is instructive to compare the derivation of the Lorentz force law given in the present paper with that of Reference [13] based on the relativistic transformation properties of the Coulomb force 3–vector. Coulomb's law is not used in the present paper. On the other hand, Reference [13] makes no use of the 4–vector potential concept, which is essential for the derivation presented here. This demonstrates an interesting redundancy among the fundamental physical concepts of classical electromagnetism.

In Reference [2], Eqns(3.4), (3.12) and (3.16) were all introduced as *a priori* initial definitions of the ‘electric field’, ‘magnetic field’ and ‘electromagnetic field tensor’ without further justification. In fact, Schwartz gave the following explanation for his introduction of Eqn(3.16) [28]:

So far everything we have done has been entirely deductive, making use only of Coulomb's law, conservation of charge under Lorentz transformation and Lorentz invariance for our physical laws. We have now come to the end of this deductive path. At this point when the laws were being written, God had to make a decision. In general there are 16 components of a second-rank tensor in four dimensions. However, in analogy to three dimensions we can make a major simplification by choosing the completely antisymmetric tensor to represent our field quantities. Then we would have only 6 independent components instead of the possible 16. Under Lorentz transformation the tensor would remain antisymmetric and we would never have need for more than six independent components. Appreciating this, and having a deep aversion to useless complication, God naturally chose the antisymmetric tensor as His medium of expression.

Actually it is possible that God may have previously invented the 4–vector potential and special relativity, which lead, as shown above, to Eqn(3.4) as the only possible covariant definition of the electric field. As also shown in the present paper, the existence of

the remaining elements of the antisymmetric field tensor, containing the magnetic field, then follow from special relativity alone. Schwartz derived the Lorentz force law, as in Einstein's original Special Relativity paper [1], by Lorentz transformation of the electric field, from the rest frame of the test charge, to one in which it is in motion. This requires that the magnetic field concept has previously been introduced as well as knowledge of the Lorentz transformation laws of the electric and magnetic fields.

In the chapter devoted to special relativity in Jackson's book [29] the Lorentz force law is simply stated, without any derivation, as are also the defining equations of the electric and magnetic fields and the electromagnetic field tensor just mentioned. No emphasis is therefore placed on the fundamental importance of the 4–vector potential in the relativistic description of electromagnetism.

In order to treat, in a similar manner, the electromagnetic and gravitational fields, the discussion in Misner Thorne and Wheeler [16] is largely centered on the properties of the tensor $F^{\mu\nu}$. Again the Lorentz force equation is introduced, in the spirit of the passage quoted above, without any derivation or discussion of its meaning. The defining equations of the electric and magnetic fields and $F^{\mu\nu}$, in terms of A^μ , appear only in the eighteenth exercise of the relevant chapter. The main contents of the chapter on the electromagnetic field are an extended discussion of purely mathematical tensor manipulations that obscure the essential simplicity of electromagnetism when formulated in terms of the 4–vector potential.

In contrast to References [2, 29, 16], in the derivation of the Lorentz force law and the magnetic field presented here, the only initial assumption, apart from the validity of special relativity, is the chosen definition, Eqn(3.4), which is the only covariant one, of the electric field in terms of the 4–vector potential A^μ . Thus, a more fundamental description of some aspects of electromagnetism, than that provided by the electric and magnetic field concepts, is indeed possible, contrary to the opinion expressed in the passage from Misner Thorne and Wheeler quoted above.

4 The time component of Newton's Second Law in Electrodynamics

Space-time exchange symmetry [3] states that physical laws in flat space are invariant with respect to the exchange of the space and time components of 4-vectors. For example, the LT of time, Eqn(2.4), is obtained from that for space, Eqn(2.1), by applying the space-time exchange (STE) operations: $x_0 \leftrightarrow x$, $x'_0 \leftrightarrow x'$. In the present case, application of the STE operation to the spatial component of the Lorentz force equation in the second line of Eqn(3.11) leads to the relation:

$$\begin{aligned} \frac{dP^0}{d\tau} &= \frac{\gamma}{c} \frac{dP^0}{dt} = q(\partial^0 A^\alpha - \partial^\alpha A^0)U(\beta)_\alpha \\ &= -qE^i U(\beta)_i = \gamma q \frac{\vec{E} \cdot \vec{v}}{c} \end{aligned} \quad (4.1)$$

where Eqns(2.5) and (3.4) and the following properties of the STE operation [3] have been

used:

$$\partial^0 \leftrightarrow -\partial^i \quad (4.2)$$

$$A^0 \leftrightarrow -A^i \quad (4.3)$$

$$C \cdot D \leftrightarrow -C \cdot D \quad (4.4)$$

Eqn(4.1) yields an expression for the time derivative of the relativistic energy, $\mathcal{E} = cP^0$:

$$\frac{d\mathcal{E}}{dt} = q\vec{E} \cdot \vec{v} = q\vec{E} \cdot \frac{d\vec{x}}{dt} \quad (4.5)$$

Integration of Eqn(4.5) gives the equation of energy conservation for a particle moving from an initial position, \vec{x}_I , to a final position, \vec{x}_F , under the influence of electromagnetic forces:

$$\int_{\mathcal{E}_I}^{\mathcal{E}_F} d\mathcal{E} = q \int_{\vec{x}_I}^{\vec{x}_F} \vec{E} \cdot d\vec{x} \quad (4.6)$$

Thus work is done on the moving charge only by the electric field. This is also evident from the Lorentz force equation, (3.14), since the magnetic force $\simeq \vec{\beta} \times \vec{B}$ is perpendicular to the velocity vector, so that no work is performed by the magnetic field. A corollary is that the relativistic energy (and hence the magnitude of the velocity) of a charged particle moving in a constant magnetic field is a constant of the motion. Of course, Eqn(4.5) may also be derived directly from the Lorentz force law, so that the time component of the relativistic generalisation of Newton's Second Law, Eqn(4.1), contains no physical information not already contained in the spatial components. This is related to the fact that, as demonstrated above, the spatial and temporal components of the energy-momentum 4-vector are not independent physical quantities.

Acknowledgements

I should like to thank O.L. de Lange for asking the question whose answer, presented in Section 4, was the original motivation for the writing of this paper, and an anonymous referee of an earlier version, submitted to another journal, for pointing out to me related material, in the books of Jackson and Misner, Thorne and Wheeler, which is discussed in some detail in this version. I also thank a second anonymous referee of the other journal for critical comments that have enabled me to much improve the presentation of the material in Section 2.

References

- [1] A.Einstein, **17** 891 (1905).
- [2] M.Schwartz, ‘Principles of Electrodynamics’, (McGraw-Hill, New York, 1972) Ch 3.
- [3] J.H.Field, Am. J. Phys. **69** 569 (2001).
- [4] D.H.Frisch and L.Wilets, Am. J. Phys. **24** 574 (1956).
- [5] F.J.Dyson, Am. J. Phys. **58** 209 (1990).
- [6] N.Dombey, Am. J. Phys. **59** 85 (1991).
- [7] R.W.Breheme, Am. J. Phys. **59** 85 (1991).
- [8] J.L.Anderson, Am. J. Phys. **59** 86 (1991).
- [9] I.E.Farquhar, Am. J. Phys. **59** 87 (1991).
- [10] S.Tanimura, Ann. Phys. (N.Y.) **220** 229 (1992).
- [11] A.Vaidya and C.Farina, Phys. Lett. **153 A** 265 (1991).
- [12] L.D.Landau and E.M.Lifshitz, ‘The Classical Theory of Fields’, (Pergamon Press, Oxford, 1975) Section 30, P93.
- [13] P.Lorrain, D.R.Corson and F.Lorrain, ‘Electromagnetic Fields and Waves’, (W.H.Freeman, New York, Third Edition, 1988) Section 16.5, P291.
- [14] J.D.Jackson, ‘Classical Electrodynamics’, (John Wiley and Sons, New York, 1975) Section 12.2, P578.
- [15] Actually, a careful examination of the derivation of Ampère’s from the Gauss law of electrostatics in Reference[3] shows that, although Eqn(3.4) of the present paper is a necessary initial assumption, the definition of the magnetic field in terms of the spatial derivatives of the 4–vector potential occurs naturally in the course of the derivation (see Eqns(5.16) and (5.17) of Reference[3]) so it is not necessary to assume, at the outset, the expression for the spatial components of the electromagnetic field tensor as given by Eqn(5.1) of Reference[3].
- [16] C.W.Misner, K.S.Thorne and J.A.Wheeler, ‘Gravitation’, (W.H.Freeman, San Francisco, 1973) Ch 3, P71.
- [17] This should not be confused with a manifestly *covariant* expression for the LT, where it is written as a linear 4-vector relation with Lorentz-invariant coefficients, as in: D.E.Fahnline, Am. J. Phys. **50** 818 (1982).
- [18] Capital letters are used consistently to denote 4-vectors.
- [19] Positive square roots are taken in Eqns(2.25)-(2.28).
- [20] J.H.Field, Am. J. Phys. **68** (2000), 367-374.

[21] The 4-vector \bar{U} has previously been used in the covariant formulation of advanced and retarded potentials in classical electrodynamics. See, for example, F.Röhrlich, ‘Classical Charged Particles’, (Addison-Wesley, Reading, MA, 1990) Section 4.7, Eqn(4.89).

[22] A time-like metric is used for 4-vector products with the components of a 4–vector, W , defined as:

$$W_t = W^0 = W_0, \quad W_{x,y,z} = W^{1,2,3} = -W_{1,2,3}$$

and an implied summation over repeated contravariant (upper) and covariant (lower) indices. Repeated Greek indices are summed from 0 to 3, repeated Roman ones from 1 to 3. Also

$$\partial^\mu \equiv \left(\frac{\partial}{\partial x^0}; -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3} \right) = (\partial^0; -\vec{\nabla})$$

[23] That is to say that the relation is a covariant one.

[24] H.Goldstein, ‘Classical Mechanics’, (Addison-Wesley, Reading Massachusetts, 1959) P200, Eqn(6-30).

[25] The alternating tensor, ϵ_{ijk} , equals 1 (−1) for even (odd) permutations of ijk .

[26] The explicit form of A^μ , as derived from Coulomb’s law, is given in standard textbooks on classical electrodynamics. For example, in Reference[13], it is to be found in Eqns(17-51) and (17-52). A^μ is actually proportional, in relativistic theory, to the 4-vector velocity, V , of the charged particle that is the source of the electromagnetic field. See J.H.Field, ‘Classical Electromagnetism as a Consequence of Coulomb’s Law, Special Relativity and Hamilton’s Principle and its Relationship to Quantum Electrodynamics’, arXiv pre-print: physics/0501130.

[27] Reference in [21] above, P65.

[28] Reference [2] above, Ch 3, P127.

[29] Reference [14] above, Section 11.9, P547.