RESEARCH STATEMENT

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My research interests lie primarily in the fields of differential geometry and geometric analysis. I am interested in studying high order geometric flows and investigating their applications to geometry.

1. Geometric Flows

1.1. **Ricci Flow.** One of the notable geometric flow is Ricci flow introduced by R. Hamilton in his famous paper [Ham82, Page 259].

Given a Riemannian manifold (M^n, g_0), the solution to Ricci flow is a one-parameter family of metric $g(t)$ defined by

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2Ric[g] \\ g(0) = g_0 \end{cases}$$

For an arbitrary smooth initial metric, the flow will always exists at least for a short time, but finite time singularities may occur which causes the flow to terminate. Ricci flow is used to prove Thurston's geometrization conjecture and the Poincaré conjecture in [Per02] [Per03b] [Per03a].

With a huge success in the Ricci flow, researchers start investigating some other higher-order geometric flows. These types of flows mostly come from the gradient of a certain energy functional.

1.2. **Calabi Flow.** E. Calabi introduces a high order geometric flow in [Cal82][Cal85], he shows that the Calabi energy is decreasing along with the Calabi flow. It is expected that the Calabi flow should converge to a constant scalar curvature metric.

In the case of Riemann surfaces, Chruściel [Chr91, Proposition 5.1] shows that the flow always converges to a constant curvature metric. After that, X.X.Chen [Che01] proves the same theorem with a different approach.

1.3. **Gradient Flow of L2 Functional of Riemann Curvature.** From 08, Jeffery Streets published a series of papers [Str08][Str12b] [Str12a][Str11] to discuss a geometric flow which deforms metric under the gradient of the following functional,

$$\mathcal{F} = \int_M |Riem|^2 d\mu$$

Since the equation is fourth-order, maximum principle techniques are not readily available, J. Streets used integral estimate to investigate properties such as long time stability of this flow. We will discuss some details about J. Streets' work in Sec 3.

1.4. **Ambient Obstruction Flow.** Another progress in this field is related to a family of tensors called obstruction tensor $\mathcal{O}_{ij}$ introduced by Fefferman and Graham in [Fef85]. In [BH11, Theorem C], Bahuaud and Helliwell studied the following flow

$$\begin{cases} \dfrac{\partial}{\partial t} g = \mathcal{O} + c_n(-1)^{\frac{n}{2}}(\Delta^{\frac{n}{2}-1}R)g_{ij} \\ g(0) = g_0 \end{cases}$$

Short time existence and uniqueness [BH15] were proved. Since that Bach tensor is the obstruction tensor in four-dimensional case, the modified Bach flow was defined by

$$\begin{cases} \dfrac{\partial}{\partial t} g = B + \dfrac{1}{2(n-1)(n-2)}\Delta R g_{ij} \\ g(0) = g_0 \end{cases}$$

2. Conformal Bach Flow

During my PhD study, I focus on the so called conformal Bach flow. This approach is inspired by the following conformal Ricci flow, which we are going to discuss.

2.1. **Conformal Ricci Flow.** In [Fis04], A. Fischer introduced a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature. These new equations are given by

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\big(Ric + \dfrac{1}{n}Rg\big) - pg \\ R[g] = -1 \end{cases}$$

for evolving metric g and a scalar function p. The conformal Ricci flow equations are analogous to the Navier–Stokes equations of fluid mechanics. Because of this analogy, the time-dependent function p is called a pressure function and it serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.

A. Fischer uses a dynamic system approach to prove the short time existence and uniqueness (Thm 4.1 in [Fis04]) to this flow with restriction to the negative Yamabe type. Shortly after, P. Lu ,J. Qing and Y. Zheng proved the short time existence for all Yamabe type in [LQZ14, Thm 3.9] with a PDE approach, based on the contractive mapping theorem in Banach spaces.

2.2. **Conformal Bach Flow.** Inspired by the conformal Ricci flow, we proposed a variant of the modified Bach flow, which named by conformal Bach flow.

Suppose that (M^n, g_0) is an n-dimensional Riemannian manifold with constant scalar curvature s_0 and $n \geq 4$, the conformal Bach flow is a family of metrics $\{g(t)\}_{t\in[0,T]}$ which satisfy

$$\begin{cases} \dfrac{\partial}{\partial t} g = 2(n-2)(B + \dfrac{1}{(n-1)(n-2)}\Delta R g + pg) \quad \text{on } M \times [0,T], \\ R[g(t)] = s_0 \quad \text{on } M \times [0,T], \end{cases} \tag{2.1}$$

where $p = p(t)$ is a family of functions on M. This is a fourth order evolution equation. The pressure function p is the "conformal change" which keeps the metric having constant scalar curvature.

One key observation is that pressure function p takes the same role as the extra term $\frac{1}{(n-1)(n-2)}\Delta R g$ introduced in [BH11, Theorem C]. Unlike the Ricci curvature, Bach curvature is not seemed to generate an appropriate geometric flow in higher dimension(unlike 2-dimensional case, see [Chr91, Prop 2.1]), i.e., DeTerck's trick is not enough to deal with the degeneracy in Bach tensor. We modified the flow in [BH11] by renormalizing the scalar curvature to a constant and it turns this conformal Bach flow to be a strongly parabolic equation coupled with an elliptic equation.

This pressure function p preserves the constant scalar curvature, from where we get more benefits than the modified Bach flow. For example, in dimension 4, combine with the solution to Yamabe problem and Gauss-Bonnet-Chern formula, under this flow, we are able to control the Sobolev constant and volume growth.

In our recent work[CLQ], we prove the short time existence, uniqueness, and improved regularity to the solution of conformal Bach flow as follows. There will be one paper coming soon. Here we will list some main theorems we proved in this paper

We first prove the local existence theorem as follows.

Theorem 2.1. *Let (M^n, g) be a closed Riemannian manifold. Suppose that $g_0 \in C^{10+\alpha}(M)$ is a Riemannian metric on M^n such that the scalar curvature $R[g_0] = s_0$ is a constant and the elliptic operator $(n-1)\Delta + s_0$ is invertible. Then there exists a small positive number T such that the conformal Bach flow $g(t)$ exists in $C^{1,4+\alpha}(M)$ for $t \in [0, T)$.*

Moreover, if the initial metric g_0 is smooth, then there exists a small positive number T such that the conformal Bach flow $g(t)$ exists in $C^{\infty,\infty}(M)$ for $t \in [0, T)$.

Once we have such existence, we are interesting in studying the singularity and long time behavior of this flow. Since this geometric partial differential equation is of order 4, instead of the maximum principle, we used an integral estimate to derive such estimates. The idea follows from [Str08, Thm 5.4]

Theorem 2.2. *Let $(M^n, g(t))$ be a solution to the conformal Bach flow, there exists positive constants $C = C(\|Rm\|_\infty, k, t, C_S)$ such that if $\|Rm\|_\infty$ and Sobolev constant remain bounded along the conformal Bach flow , the elliptic operator stays invertible,we have the following estimate*

$$\int |\nabla^k Rm|^2 d\mu \leq C \tag{2.2}$$

We obtain two additional theorems from this estimates. The first theorem gives a criterion for the development of finite singularity

Theorem 2.3. *Let $(M^n, g(t)$ be a solution to the conformal Bach flow which exists on a maximal time interval $[0, T)$ where $0 < T <= \infty$. If $T < \infty$, then at least one of the following cases is true.*

1. *C^0 norm of Riemann curvature blows up: $\lim_{t\to T} \|Rm\|_\infty \to \infty$*
2. *pressure function $p(x,t)$ blows up: $\lim_{t\to T} |p(x,t)| \to \infty$*
3. *Sobolev constant $C_S(g(t))$ blows up: $\lim_{t\to T} C_S(g(t)) \to \infty$*

Furthermore, when $n = 4$, the Sobolev constant is uniformly bounded, therefore, the last case will not happen.

This characterization allows us use previous a-prioi estimate and to extract convergent subsequences from a sequence of solutions to the conformal Bach flow with uniform C^0 curvature bound and uniform injective radius lower bound. Such compactness theorem of solutions to Ricci flow is proved by Hamilton [Ham95, Thm 2.3].

We prove this compactness result by using the Cheeger-Gromov theorem to obtain subsequential convergence of solutions at a fixed time. Then, after extending estimates on the covariant derivatives of the metrics from a fixed time to the entire time interval, we obtain subsequential convergence over the entire time interval.

Theorem 2.4. *Let $\{(g_i(t), x_i(t))\}$, $t \in (\alpha, \omega)$, be a family of smooth solutions of conformal Bach flow on closed manifolds M_i^n with constant scalar curvature s_{0i}, where $-\infty \le \alpha < 0 < \omega \le \infty$. Let $\{p_i \in M_i\}$ be a sequence of points. We assume that there is a constant $K > 0$ such that the curvature of $g_i(t)$ and potential function $p_i(t)$ satisfy the following conditions:*

(1) $\sup_{x \in M_i} |Rm(x,t)|_{g_i(t)} \le K$ *for* $t \in (\alpha, \omega)$,
(2) $\sup_{x \in M_i} |p_i(x,t)| \le K$ *for* $t \in (\alpha, \omega)$
(3) *the Sobolev constant satisfies* $C_S(g_i(t)) \le K$,

Then sequence $\{(M_i, g_i(t), p_i(t), x_i)\}$ is subconverges in pointed C^∞-Cheeger-Gromov topology to a complete solution $\{(M_i, g_i(t), p_i(t), p_i)\}$, $t \in (\alpha, \omega)$ of conformal Bach flow. Furthermore, if $n = 4$, we don't need the last condition about the Sobolev constant

The following theorem is a direct consequence of this compactness theorem.

Theorem 2.5. *Let (M^4, g) be a compact solution to the conformal Bach flow which exists on a maximal time interval $[0, T)$ with a finite T. Suppose that the pressure function p is uniformly bounded on the interval. Let $\{x_i, t_i\}_{i \in \mathbb{N}} \subset M \times [0, T)$ and $t_i \to T$ be a sequence of points satisfies:*

$$|Rm(x_i, t_i)| = \sup_{x \in M} \{|Rm(x,t)|\}$$

Let $\lambda_i = |Rm(x_i, t_i)|$, then the sequence of solutions to the conformal Bach flow given by $\{(M, g_i(t), x_i)\}$, with

$$g_i(t) := \lambda_i g(t_i + \frac{t}{\lambda_i^2}), t \in (-\lambda^2 t_i, a]$$

where $a \ge 0$, subsequentially converges in the sense of families of pointed Riemannian manifolds to a nonflat, noncompact complete Bach flat steady solution $(M_\infty, g_\infty, x_\infty)$ to conformal Bach flow defined on $(-\infty, a]$ where $a \ge 0$.

This theorem states that if the pressure function is bounded along with the flow, there exists a sequence of solutions to the conformal Bach flow that converges to a nonflat, noncompact complete Bach flat steady solution.

3. Future Work

3.1. **Small Initial Data Behavior.** For most of the geometric flow, especially the gradient flow, once we have a short time existence, the next thing to ask will be under which condition, the flow exists for an arbitrarily long time? One possible condition is the low energy initial condition.

For example, in [Ye93], Ye proves long time existence and convergence of the Ricci flow to a spherical space form when the concircular curvature tensor is small and the average scalar curvature is positive.

In [KS$^+$01], Kuwert–Schazle gives convergence of the Willmore flow where the initial energy is less than a small universal constant. The proof of this fact relies on integral estimates with a blow-up argument. In later, much deeper work the same result was shown where the initial energy is only assumed less than 8π [KS04]. The overall argument combines integral estimates which exploit the uniform Sobolev constant estimate of the Michael–Simon inequality, together with a detailed blow-up analysis.

In [Str12a, Thm 1] , J. Streets proves that if we have Sobolev constant and small initial L^2 norm of Riemann curvature, then the gradient flow will exist for all time on round sphere in dimension 4. Base on this result, a similar result to the gap theorem in [CGY03] was proved, this gradient flow converges to standard four sphere given the small initial data and positive Yamabe constant. And in [Str12b, Thm 1.3], he prove a more general case, by assuming addition initial condition, this flow exists for all time and converges exponentially to a flat metric.

One of the potential future work will be this low energy behavior of conformal Bach flow.

3.2. **Curvature Pinch.** In Riemannian geometry, sphere theorem states that under what conditions on the curvature can we conclude that a smooth, closed Riemannian manifold is diffeomorphic or conformal to the sphere?

First result is from the work of Margerin, a sharp pointwise geometric characterisation of the smooth structure of $\mathbb{S}^4$. The powerful tool in this approach is the Ricci flow.

Theorem 3.1. [Mar98, Thm 1] *Given a closed manifold with positive Yamabe constant and the following curvature pinching condition:*

$$|W|^2 + 2|E|^2 < \frac{1}{6}R^2$$

then M^4 is diffeomorphic to standard $\mathbb{S}^4$ or $\mathbb{RP}^4$.

Remark 3.2. This theorem is sharp, both $(\mathbb{CP}^2, g_{FS})$ and $(\mathbb{S}^3 \times \mathbb{S}^1, g_{prod})$ satisfies the equality.

This theorem is improved by A.Chang, P.Yang and M.Gursky, a global curvature condition was proposed to get the same result.

Theorem 3.3. [CGY03, Thm A'] *Given a closed manifold with positive Yamabe constant and the following curvature pinching condition:*

$$\int_M |W|^2 + 2|E|^2 d\mu < \frac{1}{6}\int_M R^2 d\mu$$

then M^4 is diffeomorphic to standard $\mathbb{S}^4$ or $\mathbb{RP}^4$.

Later on, Q.Jie, A.Chang, P.Yang improved this result by directly comparing the curvature quantity to $16\pi^2$

Theorem 3.4. [CQY07, Thm A] *There is an $\epsilon > 0$, for any Bach flat manifold (M^4, g) with positive Yamabe type, if*

$$\int_M -\frac{1}{2}|E|^2 + \frac{1}{6}R^2 d\mu > (1-\epsilon)16\pi^2$$

then $(M^4, [g]) \overset{conf}{\cong} (\mathbb{S}^4, g_{\mathbb{S}^4})$.

In this theorem, Bach flat condition is assumed, the proof of the theorem builds upon some estimates in the work of Tian–Viaclovsky [TV05] on the compactness of Bach-flat metrics on 4-manifolds.

Will Bach flow helps us here? On key fact is that both conformal Bach flow and modified Bach flow are gradient flows of L^2-norm of Weyl curvature in dimension 4. If we can prove such flow exists for a long time with small initial data, i.e. L^2-norm of Weyl is small, we might have hope to improve the gap theorem in [CQY07, Thm A].

3.3. **Singular Model.** From my previous work, and some results from [Lop18], we want to investigate the singular model when we construct a sequence via blow up argument. In Lopez's case, the singular model for modified Bach flow will be a solution to

$$\frac{\partial}{\partial t}g = \frac{1}{12}\Delta R g$$

In our case, some extra assumption needs to be proposed, since we also need to consider either pressure function or the elliptic operator for the pressure function. If such a thing can be overcome, we will have a better singular model, which is a steady manifold.

In both cases, the singular model will be a Bach flat manifold. One of the classical arguments is to investigate such singular models to derive some contradiction by the gap theorem for critical metrics. For example, the rigidity of non-compact complete Bach-flat manifolds is proved by S.Kim .

Theorem 3.5. [Kim15, Thm 1] *Let (M^4, g) be a non-compact complete Bach-flat Riemannian 4-manifold with zero scalar curvature and positive Yamabe type. Then there exists a small number ϵ such that if $\|Rm\|_{L^2} \leq \epsilon$, then (M^4, g) is flat.*

We might hope such rigidity theorem can derive some contradiction on the singular model.

References

[BH11] Eric Bahuaud and Dylan Helliwell. Short-time existence for some higher-order geometric flows. *Communications in Partial Differential Equations*, 36(12):2189–2207, 2011.

[BH15] Eric Bahuaud and Dylan Helliwell. Uniqueness for some higher-order geometric flows. *Bulletin of the London Mathematical Society*, 47(6):980–995, 2015.

[Cal82] Eugenio Calabi. Extremal kähler metrics. In *Seminar on differential geometry*, volume 102, pages 259–290, 1982.

[Cal85] Eugenio Calabi. Extremal kähler metrics ii. In *Differential geometry and complex analysis*, pages 95–114. Springer, 1985.

[CGY03] Sun-Yung A Chang, Matthew J Gursky, and Paul C Yang. A conformally invariant sphere theorem in four dimensions. *Publications Mathématiques de l'IHÉS*, 98:105–143, 2003.

[Che01] XX Chen. Calabi flow in riemann surfaces revisited: a new point of view. *International Mathematics Research Notices*, 2001(6):275–297, 2001.

[Chr91] Piotr T Chruściel. Semi-global existence and convergence of solutions of the robinson-trautman (2-dimensional calabi) equation. *Communications in mathematical physics*, 137(2):289–313, 1991.

[CLQ] Jiaqi Chen, Peng Lu, and Jie Qing. Conformal bach flow(in progress).

[CQY07] Sun-Yung A Chang, Jie Qing, and Paul Yang. On a conformal gap and finiteness theorem for a class of four-manifolds. *GAFA Geometric And Functional Analysis*, 17(2):404–434, 2007.

[Fef85] Charles Fefferman. Conformal invariants. *" Elie Cartan et les Mathematiques d Aujourd hui, Asterisque, hors serie*, pages 95–116, 1985.

[Fis04] Arthur E Fischer. An introduction to conformal ricci flow. *Classical and Quantum Gravity*, 21(3):S171, 2004.

[Ham82] Richard S. Hamilton. Three-manifolds with positive ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[Ham95] Richard S Hamilton. A compactness property for solutions of the ricci flow. *American journal of mathematics*, 117(3):545–572, 1995.

[Kim15] Seongtag Kim. Rigidity of bach-flat manifolds. In *Extended Abstracts Fall 2013*, pages 35–39. Springer, 2015.

[KS+01] Ernst Kuwert, Reiner Schätzle, et al. The willmore flow with small initial energy. *Journal of Differential Geometry*, 57(3):409–441, 2001.

[KS04] Ernst Kuwert and Reiner Schätzle. Removability of point singularities of willmore surfaces. *Annals of Mathematics*, pages 315–357, 2004.

[Lop18] Christopher Lopez. Ambient obstruction flow. *Transactions of the American Mathematical Society*, 370(6):4111–4145, 2018.

[LQZ14] Peng Lu, Jie Qing, and Yu Zheng. A note on conformal ricci flow. *Pacific Journal of Mathematics*, 268(2):413–434, 2014.

[Mar98] Christophe Margerin. A sharp characterization of the smooth 4-sphere in curvature terms. *Communications in Analysis and Geometry*, 6(1):21–65, 1998.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds, 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds, 2003.

[Str08] Jeffrey D Streets. The gradient flow of l2 of rm. *Journal of Geometric Analysis*, 18(1):249–271, 2008.

[Str11] Jeffrey Streets. The gradient flow of the l2 curvature energy on surfaces. *International Mathematics Research Notices*, 2011(23):5398–5411, 2011.

[Str12a] Jeffrey Streets. The gradient flow of the l2 curvature energy near the round sphere. *Advances in Mathematics*, 231(1):328–356, 2012.

[Str12b] Jeffrey Streets. The gradient flow of the l2 curvature functional with small initial energy. *Journal of Geometric Analysis*, 22(3):691–725, 2012.

[TV05] Gang Tian and Jeff Viaclovsky. Bach-flat asymptotically locally euclidean metrics. *Inventiones mathematicae*, 160(2):357–415, 2005.

[Ye93] Rugang Ye. Ricci flow, einstein metrics and space forms. *Transactions of the american mathematical society*, 338(2):871–896, 1993.