The volume comparison of symmetric spaces of non-compact type of rank 1

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Outline

Introduction

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Bishop-Gromov:

Let (M^n, g) be a complete Riemannian manifold with

$$\mathrm{Ric} \geq (n-1)k \cdot g.$$

Then,

$$\mathrm{vol}\,(B_r) \leq \mathrm{vol}(B_r^k)$$

. where

- B_r: geodesic ball in (M, g)
- B_r^k: geodesic ball in the space form M^k

Introduction

Conjecture (R.Schoen):

Let (M^n, g) be a closed hyperbolic manifold. If

$$R(g_0) \geq R(g)$$

Then,

$$vol(g_0) \geq vol(g).$$

Progressions:

- **R.Hamilton and G.Perelman:** n = 3 (A corollary of geometrization);
- **L.Agol, P.A.Storm and W.P.Thurston, 2007**: hyperbolic 3-manifold with minimal surface boundary:

Relative Volume

If (M, g) is a non-compact Riemannian manifold, we need to consider the relative volume.

Definition (Relative volume(X.Hu, Y.Shi and D.Ji, 2014))

$$V_g(g_0) := \lim_{i \to +\infty} (vol(\Omega_i, g_0) - vol(\Omega_i, g))$$

where Ω_i is a compact exhaustion of M by compact sets.

On conformally compact Einstein manifold

Theorem (X.Hu, Y.Shi, D.Ji, 2014)

Let (M^n, g) be a $C^{2,\alpha}$ strictly stable conformally compact Einstein. For anther metric g_0 on M, if

$$\|\rho^\tau(g_0 - g)\|_{C^0} \leq \varepsilon$$

for some small enough $\varepsilon > 0$, $\tau > n-1$ and

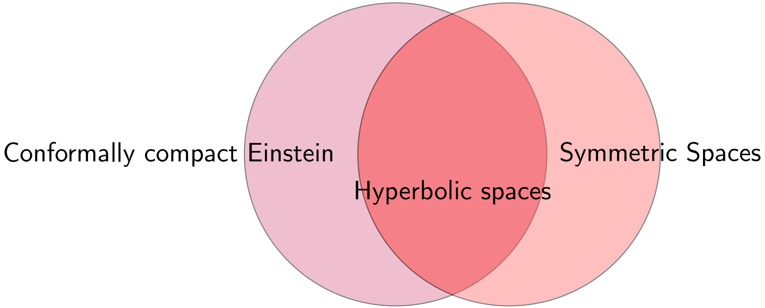
$$R_0 \geq -n(n-1)$$

where ρ is the defining function of (M, g) and R_0 is the scalar curvature of (M, g_0), then

$$Vol_g(g_0) \geq 0$$

Problem

Question: How about other kinds of non-compact manifolds?



Key Observation

Key Observation: The relative volume is non-increasing along the normalized Ricci DeTurck flow.

Normalized Ricci flows

Definitions

The normalized Ricci flow (NRF):

$$\begin{cases} \dfrac{\partial}{\partial t}\tilde{g}(t) = -2\left(\mathrm{Ric}(\tilde{g}(t)) - \dfrac{R_M}{n}\tilde{g}(t)\right) \\ \tilde{g}(0) = g_0 \end{cases}$$

The normalized Ricci DeTurck flow (NRDF):

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{g}_{ij} = -2\left(\mathrm{Ric}(\hat{g})_{ij} + \dfrac{R_M}{n}\hat{g}_{ij}\right) + \nabla_i W_j + \nabla_j W_i \\ \hat{g}(\cdot, 0) = g_0 \end{cases}$$

where $W_j = \hat{g}_{jk}\hat{g}^{pq}\left(\hat{\Gamma}^k_{pq} - \Gamma^k_{pq}\right)$ and $\hat{\Gamma}$ are the Christoffel symbols of the metric $\hat{g}$. R_M is the scalar curvature of the background Einstein metric g

The relations between NRF and NRDF

Let $\Phi_t : M \to M$ be a family of smooth diffeomorphisms satisfying the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi_t(x) = -\,W(\Phi_t(x), t) \\ \qquad \Phi_0(x) = \mathrm{id}(x) \end{cases}$$

where $W^i := \hat{g}^{ij} W_j$. It is straightforward to verifty that

$$\tilde{g}(t) = \Phi_t^* \hat{g}(t)$$

Basic definitions

Einstein operator:

$$Lu_{ij} = \Delta_g u_{ij} + 2R_{ipjq}u^{pq} \quad \text{for} \quad u \in sym^2 T^* M$$

In particular, for g is Einstein, the NRDF is equivalent to

$$\frac{\partial}{\partial t} u = Lu + Q(u) \quad \text{for} \quad u \in sym^2 T^* M$$

where

- $u = \hat{g} - g$
- $Q(u) = \hat{g}^{-1} * \hat{g}^{-1} * \nabla u * \nabla u + (\hat{g}^{-1} - g^{-1}) * \nabla^2 u$

Basic definitions

Strictly stable:

A complete Riemannian manifold (M^n, g) is called strictly stable if

$$\lambda = \inf_{u} \frac{\int_M < Lu, u > d\mu_g}{\int_M |u|^2 d\mu_g} > 0$$

where the infimum is taken among all nonzero symmetric 2-tensor u such that

$$\int_M (|u|^2 + |\nabla u|^2) d\mu_g < +\infty$$

Short time existences

- **R.Hamilton, 1982:** Closed manifolds
- **W.Shi, 1989:** Completed manifolds with $|Rm(g_0)| < k_0$
- **M.Simon, 2002:** C^0 metrics: $|g_0 - g| < \varepsilon$ and (M, g) is completed with $|Rm(g)| < k_0$
- **J.Qing, Y.Shi, J.Wu, 2011:** Conformally compact manifolds (The flow preserves the conformal infinity)

Long time existences and convergences

Compact manifolds

- **R.Ye, 1993:** Closed strictly stable manifolds with $|Ric(g_0) - \frac{R(g_0)}{n} g_0|$ small enough.

Long time existences and convergences

Non-compact manifolds

- **O.Schnürer, F.Schulze, M.Simon, 2011:** g_0 is a L^2 perturbation of a hyperbolic space
- **H.Li, H.Yin, 2011:** g_0 is a C^0 perturbation of a hyperbolic space
- **J.Qing, Y.Shi, J.Wu, 2011** g_0 is a C^0 or L^2 perturbation of a strictly stable conformally compact Einstein manifold
- **R.Bamler, 2015:** g_0 is a C^0 or L^p perturbation of symmetric spaces of non-compact type

Theorem

Theorem

Let (M^n, g) be a strictly stable Einstein manifold satisfying $|Rm| < k_0$, $vol(B(x,1)) > v_0 > 0$ and

$$\int_M e^{-\alpha d(x,x_0)} dvol_g < +\infty$$

for some positive number k_0, v_0, α. Then, if

$$\|g_0 - g\|_{C^0} \leq \varepsilon_1 \quad \text{and} \quad \int_M |g_0 - g|^2 dvol_g \leq \varepsilon_0,$$

then, $\hat{g}(t)$ exists for all the time and satisfies that

$$\|\hat{g}(t) - g\|_{C^i} \leq C(i, k_0, v_0, t_0)\varepsilon_0 e^{-\lambda_{\varepsilon_1} t} \quad \text{for any} \quad t \in [t_0, +\infty)$$

where $t_0 > 0$ is an arbitrary given number.

The non-increasing of the relative volume along the NRDF

The evolution equation of the relative volume

Taking a compact exaustion of M, $\Omega_i \subset M$. Then we have that

$$\frac{d}{dt}\int_{\Omega_i}\sqrt{\det(\hat{g}(t))}-\sqrt{\det(g)}dvol_g$$
$$=-\int_{\Omega_i}(R(\hat{g})-R_M)dvol_g+\int_{\partial\Omega_i}<W,v>d\sigma_g$$

where

- $W_i=\hat{g}_{ik}\hat{g}^{ab}(\Gamma^k_{ab}(\hat{g}(t))-\Gamma^k_{ab})$
- v: the inward norm vector of $\partial\Omega_i$
- $d\sigma_g$: the induced volume form of g on $\partial\Omega_i$

Goals

Goal1:

- $\lim_{i\to\infty} \frac{d}{dt} \int_{\Omega_i} \sqrt{\det(\hat{g}(t))} - \sqrt{\det(g)} dvol_g = \frac{d}{dt} \lim_{i\to\infty} \int_{\Omega_i} \sqrt{\det(\hat{g}(t))} - \sqrt{\det(g)} dvol_g$
- $\lim_{i\to 0} \int_{\partial\Omega_i} < W, v > d\sigma_g = 0$

which would imply that

$$\frac{d}{dt} \int_M \sqrt{\det(\hat{g}(t))} - \sqrt{\det(g)} dvol_g = -\int_M (R(\hat{g}) - R_M) dovl_g$$

Goal2:

$$\lim_{t\to+\infty} \int_M \sqrt{\det(\hat{g}(t))} - \sqrt{\det(g)} dvol_g = 0$$

Goal3:

$$R(\hat{g}(t)) \geq R_M$$

The strategy for Goal 1 & Goal 2

The background metric:

(M^n, g): a strictly stable Cartan-Hadamard Einstein manifold satisfying that

(1) $|Rm| < k_0, vol_g(B(x,1)) > v_0$ for some $k_0, v_0 > 0$

(2) $\tau_0 := \lim_{d \to \infty} \Delta_g d$ exists, where $d(x) = dist(x, p_0)$;

(3) $\mathrm{R}_{ijkl} h^{ik} h^{jl} < \gamma |h|^2$ for any $h \in sym^2 T^* M$ and some $\gamma > 0$

Remark:

In particular, all the irreducible symmetric spaces of non-compact type satisfy the above condtions.

Lemma

For any

$$\delta \in (0, \frac{\tau_0}{2} + \sqrt{\frac{\tau_0^2}{4} - 2\frac{R_M}{n}}) \cap (\tau_0, \tau_0 + 2\sqrt{\frac{\tau_0^2}{4} - 2\gamma}),$$

there exists $\varepsilon_0 > 0$ and $\lambda_R > 0$, such that for any other Riemannian metric g_0, if

$$\|e^{\delta d}(g_0 - g)\|_{C^1} < \varepsilon_0,$$

there exits a solution to NRDF $\hat{g}(t)$ with initial $\hat{g}(0) = g_0$ for $t \in [0, \infty)$, satisfying that for $t \in [0, +\infty)$,

$$|\hat{R} - R_M| \leq C(n, \delta, \tau_0, k_0, v_0, t_0, \varepsilon_0)(1 + \frac{1}{\sqrt{t}})e^{-\lambda_R t}e^{-\delta d(x)} \quad (1)$$

$$|W(t, x)| \leq C(n, \delta, \tau_0, k_0, v_0, t_0, \varepsilon_0)e^{-\lambda_R t}e^{-\delta d(x)} \quad (2)$$

where $\hat{R}$ is the scalar curvature of $\hat{g}(t)$.

The strategy for Goal3

The evolution equation of $R(\tilde{g}) - R_M$ is

$$\frac{\partial}{\partial t}(\tilde{R} - R_M) = \Delta_{\tilde{g}}(\tilde{R} - R_M) + 2|\tilde{Ric} - \frac{R_M}{n}\tilde{g}|^2_{\tilde{g}} + 2\frac{R_M}{n}(\tilde{R} - R_M)$$

which implies that

$$\frac{\partial}{\partial t}(\tilde{R} - R_M) \geq \Delta_{\tilde{g}}(\tilde{R} - R_M) + 2\frac{R_M}{n}(\tilde{R} - R_M)$$

The Goal2 follows from the maximal principle.

The key issue

Question:

Whether

$$(0, \frac{\tau_0}{2} + \sqrt{\frac{\tau_0^2}{4} - 2\frac{R_M}{n}}) \cap (\tau_0, \tau_0 + 2\sqrt{\frac{\tau_0^2}{4} - 2\gamma})$$

is not empty?

Remark: the hyperbolic space $(M, g) = \mathbb{H}^n$: $\tau_0 = n-1$, $R_M = -n(n-1)$, $\gamma = 1$. The above intersection is nonempty.

Symmetric spaces of noncompact type

Basic concepts

Definition (Symmetric space)

A Riemmaninan manifold (M, g) is called **symmetric space** if for arbitrary point $p \in M$ there exist a a reflection Φ_p at p.

Definition (Symmetric space of noncompact type)

A symmetric space is of noncompact type, if its sectional curvature is strictly negative.

Theorem (Parallel curvature)

A Riemannian manifold is a symmetric space if and only if it is simply connected and $\nabla R \equiv 0$.

Basic properties

Definition (Irreducible symmetric space)

A symmetric space is called irreducible if it can not be decomposed into a product of two symmetric space.

Theorem (De Rham decomposition)

Let M be symmetric space. Then M is a product

$$M = M_1 \times \ldots \times M_r$$

where the factors M_i are irreducible.

Theorem

Irreducible symmetric spaces are Einstein manifolds.

Examples

- **Real hyperbolic case** the half plane model

$$\mathbb{R}H^{n+1} = \{(\mathbf{x}, y) | \mathbf{x} \in \mathbb{R}^n, \textit{and } y > 0\}$$

with metric

$$\bar{g} = \frac{1}{y^2}(|dx|^2 + |dy|^2)$$

- **Complex hyperbolic case** the siegel half plane model

$$\mathbb{C}H^{n+1} = \{(\mathbf{z}, \rho, v) | \mathbf{z} \in \mathbb{C}^n, \rho > 0, v \in \mathbb{R}\}$$

with metric

$$\bar{g} = \frac{4|d\mathbf{z}|^2}{\rho^2} + \frac{1}{\rho^4}(4\rho^2(d\rho)^2 + 4(dv + \mathrm{Im}(\mathbf{z}d\mathbf{z}))^2$$

Lie algebra structures

The symmetric space (M, g) can be identified as homogeneous space $M \cong G/K$, where

- G: isometric group of M. (G is semisimple and transitive)
- $K \triangleq \{g \in G | g(p_0) = p_0\}$: isotropic group of $p_0 \in M$.

Lie algebra structures

Let $\mathfrak{g}$ and $\mathfrak{l}$ be the Lie algebra of G and K respectively.

- **The involution on $\mathfrak{g}$** There is the induced involution $\sigma^2 = Id$ from the reflection at p_0.
- **The Cartan decomposition $\mathfrak{g}$** The Lie algebra of $\mathfrak{g}$ can have the following decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$$

  where
  - $\mathfrak{p}$: the eigenspace of -1 of σ ($\mathfrak{p} \cong T_{p_0}M$)
  - $\mathfrak{l}$: the eigenspace of 1 of σ
- **The inner product**

$$(.,.) \triangleq -< ., \sigma(.) >$$

  where $< .,. >$ is the Killing form on $\mathfrak{g}$. In particular, $(.,.)|_{\mathfrak{p}} = c \cdot g(.,.)|_{T_{p_0}M}$ for irreducible symmetric spaces, where $c > 0$ is a constant.

Lie algebra structures

- **The rank of symmetric space**

$$\dim(\mathfrak{a})$$

where $\mathfrak{a} \subset \mathfrak{p}$ is the maximal Abelian subalgebra

- **Root system** $\alpha \in \Delta \subset \mathfrak{a}^*$ such that

$$[v, X] = \alpha(v)X$$

for any $v \in \mathfrak{a}$ and eigenvector X.

- **Positive root**

$$\Delta_+ = \{\alpha \in \Delta : \alpha(v_0) > 0\}$$

- **Root system properties** $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\sigma(\Delta_+) = -\Delta_+ \subset \Delta$.
- **Root decomposition** $\mathfrak{g} = \mathfrak{a} \oplus_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \oplus \mathfrak{l}_0$

Lie algebra structures

- **Frame**

$$\alpha_1, \cdots, \alpha_{n-r} \in \Delta_+$$
$$x_1, \cdots, x_{n-r} \in \mathfrak{g}$$

Let $y_i = \sigma(x_i)$. Then,

$$\mathfrak{p}_i = \frac{1}{\sqrt{2}}(x_i - y_i), \quad k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$$

forms an orthonormal basis of $\mathfrak{a}^\perp \oplus \mathfrak{l}_0^\perp$

- **Curvature** For $X, Y, Z \in \mathfrak{p} \cong T_{p_0}M$, the Riemannian curvature of the symmetric space, M, is

$$R(X, Y)Z|_{p_0} = [Z, [X, Y]]|_{p_0}$$

The computation of τ_0

Lemma

Let (M^n, g) be an irreducible Riemannian symmetric space of non-compact with rank r. Then,

$$\mathrm{Ric} = -\sum_{i=1}^{n-r} (\alpha_i^2(v_0)) g$$

and

$$\Delta_g d|_{\exp(v_0 t)(p_0)} = -\sum_{i=1}^{n-r} \frac{ch(-\alpha_i(v_0)d)}{sh(-\alpha_i(v_0)d)} \alpha_i(v_0)$$

The computation of γ

For the operator

$$
\begin{aligned}
Rm : sym^2 T^* M &\to sym^2 T^* M \\
h &\mapsto R_{ijkl} h^{jl}
\end{aligned}
$$

And the above operator can be extended into $T^*M \otimes T^*M$ as following

$$
\begin{aligned}
Rm : T^* M \otimes T^* M &\to T^* M \otimes T^* M \\
u \otimes v &\mapsto R_{ijkl} u^j u^l
\end{aligned}
$$

In particular, if (M, g) is a symmetric spaces, then

$$
\begin{aligned}
Rm : \mathfrak{p} \otimes \mathfrak{p} &\to \mathfrak{p} \otimes \mathfrak{p} \\
u \otimes v &\mapsto \sum_{(k_i) \text{ basis of } \mathfrak{l}} [k_i, u] \otimes [k_i, v]
\end{aligned}
$$

The computation of γ

$$Rm = \frac{1}{2}(\mathfrak{C}(\mathfrak{l}, sym^2\mathfrak{p}) - \mathfrak{C}(\mathfrak{l}, \mathfrak{p}))$$

where

- $\mathfrak{C}(\mathfrak{l}, sym^2\mathfrak{p})$: The Casimir operator of $\mathfrak{l}$ on $sym^2\mathfrak{p}$
- $\mathfrak{C}(\mathfrak{l}, \mathfrak{p})$: The Casimir operator of $\mathfrak{l}$ on $\mathfrak{p}$
- The γ is the largest eigenvalue of the operator Rm.

Rank 1 cases

- $\mathbb{H}^n = SO(1,n)/SO(n)$
- $\mathbb{CH}^n = SU(1,n)/U(n)$
- $\mathbb{HH}^n = Sp(1,n)/Sp(1)Sp(n)$
- $\mathbb{OH}^2 = F_4^{-20}/Spin_9$

The dimensions are n, 2n, 4n, 16, respectively.

- **Real hyperbolic cases:** $\lim_{d\to+\infty} \Delta_g d = n-1$
- **Complex hyperbolic cases:** $\lim_{d\to+\infty} \Delta_g d = 2n$
- **Quaternionic hyperbolic cases:** $\lim_{d\to+\infty} \Delta_g d = 4n+2$
- **Octonionic hyperbolic cases:** $\lim_{d\to+\infty} \Delta_g d = 22$

Except the real hyperbolic case, $\gamma = 4$ for all the other cases. (See W.Fulton and J.Harris, Representation theory)

Strictly stable

A Bochner formula

$$(Lu)_{ij} = (div^* div + d^* d)u_{ij} - Rm(u)_{ij} - \frac{1}{2}(R_{ik}u_j^k + R_{jk}u_i^k)$$

where

- $div : C^\infty(M, sym^2 T^*) \to C^\infty(M, T^*),\ u_{ij} \mapsto \nabla^i u_i j$
- $div^* : C^\infty(M, T^*) \to C^\infty(M, sym^2 T^*),\ \gamma_i \mapsto \frac{1}{2}(\nabla_i \gamma_j + \nabla_j \gamma_i)$
- $d : C^\infty(M, sym^2 T^*) \to C^\infty(M, \Lambda^2 T^* \otimes T^*),\ u_{ij} \mapsto \nabla_i u_{jk} - \nabla_j u_{ik}$
- $d^* : C^\infty(M, \Lambda^2 T^* \otimes T^*) \to C^\infty(M, sym^2 T^*),$
  $\gamma_{ijk} \mapsto -\frac{1}{2}(\nabla^k \gamma_{kij} + \nabla^k \gamma_{kji})$

Strictly stable

$$
\begin{aligned}
A : sym^2 T^* &\to sym^2 T^* \\
u_{ij} &\mapsto -Rm(u)_{ij} - \frac{1}{2}(R_{ik}u_j^k + R_{jk}u_i^k)
\end{aligned}
$$

is self-adjoint and nonnegative definite for symmetric spaces of noncompact type.

Main theorem

Theorem

Suppose that (M^n, g) is an irreducible Riemannian symmetric space of noncompact type of rank 1 with $n \geq N$, where $N = 3, 10, 8, 4$ for real, complex, quaternionic and octonionic hyperbolic spaces respectively. And g_0 is another complete noncompact Riemannian metric on M. Then for $\tau > |\lim_{d\to\infty} \Delta_g d(x)|$, there exists some $\varepsilon > 0$ such that if

$$\|e^{\tau d}(g_0 - g)\|_{C^1} \leq \varepsilon \quad \text{and} \quad R \geq R_M$$

where R_M is the scalar curvature of (M, g), then

$$V_g(g_0) \geq 0.$$

In particular, $V_g(g_0) = 0$, there exists a C^∞ diffeomorphism $\Phi : M \to M$, such that $g_0 = \Phi^* g$.

Thanks for your time!