Research Statement

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My Research Interests: Asymptotically hyperbolic manifolds, Normalized Ricci flow, Heat kernel estimate, Microlocal analysis, Semiclassical analysis and Asymptotically symmetric metric.

Contents

1 The Asymptotically symmetric manifolds

We will try to define the asymptotically symmetric manifolds in the light of the asymptotically hyperbolic manifolds. We will identify the boundary geometry of symmetric spaces of the non-compact type as a model parabolic geometry. By this identification, O.Biquard defined the so called asymptotically symmetric spaces at least the rank one case. For the higher rank case, the boundary geometry is not the parabolic geometry. However, we can think of it as a Cartan geometry. But the definition of the Cartan geometry is not quite suitable to define the asymptotcally symmetric manifolds. Therefore, we need to induce the above boundary cartan geometry on the tangent space of the boundary manifolds.

2 Asymptotically hyperbolic manifolds and Normalized Ricci flow

Since the seminal work of Fefferman and Graham [FG1985] there have been great interests in the study of conformally compact Einstein metrics. Lately the use of conformally compact Einstein manifolds in the so-called AdS/CFT correspondence in string theory proposed as a promising quantum theory of gravity have accelerated developments of the study of conformally compact Einstein manifolds. As it was foreseen in [FG1985], the study of conformally compact Einstein manifolds now becomes one of the most active research area in conformal geometry. But the existence of conformally compact Einstein metrics remains to be a challenging open problem in large.

In my first paper we study the normalized Ricci flows on asymptotically hyperbolic manifolds and use normalized Ricci flows to construct conformally compact Einstein metrics. We recall that Ricci flow starting from a metric g_0 on a manifold M^n is a family of metrics $g(t)$ that satisfies the following:

$$\begin{cases} \frac{d}{dt}g(t) = -2\,\mathrm{Ric}_{g(t)} \\ \quad g(0) = g_0 \end{cases}$$

We then consider the normalized Ricci flow as follows:

$$\begin{cases} \frac{d}{dt}g(t) = -2\left(\mathrm{Ric}_{g(t)} + ng(t)\right) \\ \quad g(0) = g_0 \end{cases}$$

It is easily seen that the above two equations are equivalent. In fact explicitly

$$g^N(t) = e^{-2nt} g\left(\frac{1}{2n}\left(e^{2nt} - 1\right)\right)$$

solves the second equation if and only if g(t) solves the first equation.

Naturally one initial step is to study normalized Ricci flows starting from metrics that are close to be Einstein. Such questions on compact manifolds were studied in [Ye1993], where it was observed that the normalized Ricci flow exists globally and converges exponentially to an Einstein metric if the initial metric g_0 is sufficiently Ricci pinched and is non-degenerate. There are also several works in the non-compact cases. In [LY2010], the stability of the hyperbolic space under the normalized Ricci flow was established. This stability result on the hyperbolic space in [LY2010] later is improved and extended in [Ba2015] [Ba2014] [SSS2010] [Su2009].

To be more precise we say a metric g on a manifold $\mathcal{M}^n$ is ϵ-Einstein if

$$\|h_g\| \leq \epsilon$$

on $\mathcal{M}^n$, where the Ricci pinching curvature $h_g = Ric_g + (n-1)g$. The non-degeneracy of a metric is defined to be the first L^2 eigenvalue of the linearization of the curavture tensor h as follows:

$$\lambda = \inf \frac{\int_{\mathcal{M}} \langle (\Delta_L + 2(n-1))\, u_{ij}, u_{ij} \rangle}{\int_{\mathcal{M}} \|u\|^2}$$

where the infimum is taken among symmetric 2-tensors u such that

$$\int_{\mathcal{M}} \left(|\nabla u|^2 + |u|^2\right) dv < \infty$$

and Δ_L is **Lichnerowicz Laplacian** on symmetric 2-tensors.

We first, based on the ideas in [Ye1993] [Ba2015], obtain the following global existence and convergence theorem of the normalized Ricci flow on non-compact manifolds. The reason that we consider the curvature flow is that this flow is strictly elliptic flow which is easier than directly considering the Ricci DeTurck flow. In the setting of Ricci DeTurck flow, we still need to consider the long time existence and convergence of harmonic map flow.

Theorem 2.1 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifolds with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $n \geq 4$. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda, k_1) > 0$ such that if $|h|_{0,0,\delta;M} \leq \epsilon_0$, the solution of the normalized Ricci flow $g(t,x)$ has long time existence and $g(t,x)$ converges to an Einstein manifold in the sense of C^2_δ norm. Moreover, the limit metric is an Asymptotically hyperboli Einstein metric with the same conformal infinity.*

The theorem 2.1 actually is a generalization of the theorem 4.1 in [QSW2013]. In [QSW2013], they require the weight δ satisfied the following

$$\delta \in \left(\frac{n}{2} - \min\left\{\sqrt{\lambda}, \sqrt{\frac{n^2}{4} - 2}\right\}, \frac{n}{2} + \sqrt{\frac{n^2}{4} - 2}\right)$$

In this paper, we only require that

$$\delta \in \left(\frac{n}{2} - \sqrt{\frac{n^2}{4}}, \frac{n}{2} + \sqrt{\frac{n^2}{4}}\right) = (0, n)$$

The reason that we can modified the term $\sqrt{\frac{n^2}{4} - 2}$ into $\sqrt{\frac{n^2}{4}}$ is that we use a more precise isomorphism theorem of Laplacian operator on weighted space (Theorem C [Le2006]) instead of the maximal principal for L^2 norm of h.

Moreover, once we have the long time existence and convergence of normalized Ricci flow, we can derive the following stability theorem of asymptotically hyperbolic manifolds.

Theorem 2.2 *Let (M^{n+1}, g_+) be an asymptotically hyperbolic Einstein manifolds with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $n \geq 4$. Let g be another asymptotically hyperbolic metric on M^{n+1}. Then, for any $\delta \in (0, n)$, there exists $\epsilon_0(\lambda) > 0$, such that if $|g - g_+| \leq \epsilon_0 e^{-\delta d(x_0, x)}$, Then the Ricci DeTurck flow with the initial g has the long time existence and*

$$\lim_{t \to \infty} |g - g_+|_{0,0,\delta} = 0$$

For the stability result of hyperbolic space $M^{n+1} = \mathbb{H}^{n+1}$, Schulze, Schnurer and Simon ([SSS2010]) have shown stability of $n \geq 3$ for every perturbation $|g - g_{\mathbb{H}^{n+1}}|_{L^\infty}$ is bounded by a small constant depending on $\|g - g_{\mathbb{H}^{n+1}}\|_{L^2}$.

While Li and Yin ([LY2010]) have shown a stability result of $n \geq 2$ for the Riemannian curvature approaches the hyperbolic curvature like $\epsilon_1(\delta)e^{-\delta d(x_0,x)}$

Furthermore, Bamler ([Ba2015]) have shown stability of $n \geq 2$ for the perturbation $|g - g_{\mathbb{H}^{n+1}}| = h_1 + h_2$ for which

$$|h_1| \leq \frac{\epsilon_1}{d(x_0,x)+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q\right)^{\frac{1}{q}} \leq \epsilon_2$$

for every $q < \infty$.

It easy to see that the stability result of [Ba2015] just implies that the stability result of [SSS2010].

For the theorem 2.2, if we take g_+ is the standard hyperbolic metric, then this stability result is implied by the stability result of [Ba2015].

By the theorem 2.1, we can fully recover the perturbation existence results in [GL1991] [Le2006] [Bi1999]. The idea is to construct an asymptotically hyperbolic metric with prescribed boundary which satisfying the condition of theorem 2.1. Then we apply the theorem 2.1 to get the asymptotically hyperbolic Einstein metric with this boundary.

Theorem 2.3 *Let (M^n, g_+), be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial M, [\hat{g}])$. And suppose that the non-degeneracy of g satisfies*

$$\lambda > 0$$

Then, for any smooth metric $\hat{h}$ on ∂M, which is sufficiently $C^{2,\alpha}$ close to some $\hat{g} \in [\hat{g}]$ for any $\alpha \in (0,1)$, there is a conformally compact Einstein metric on M which is of C^2 regularity and with the conformal infinity $[\hat{h}]$

3 The stability of the asymptotically hyperbolic Einstein manifold under the Ricci flow

Let $\bar{M}$ be a $(n+1)-$dimensional compact manifold with boundary ∂M. Suppose that there is a complete Riemannian metric $\tilde{g}$ in the interior of $\bar{M}$ denoted it M, and there is a defining function ρ on $\bar{M}$, (i.e. $\rho > 0$ on M; $\rho = 0$ on ∂M; $d\rho \neq 0$ on ∂M) such that $\rho^2\tilde{g}$ can be extended into a smooth Riemannian metric on $\bar{M}$. Since for different defining functions ρ_1 and ρ_2 on $\bar{M}$, there exists a positive function f such that $\rho_1 = f\rho_2$, the interior Riemannian metric $\tilde{g}$ uniquely determine a conformal structure on boundary ∂M. We call $(M^{n+1}, \tilde{g})$ an asymptotically hyperbolic manifold with conformal boundary $\rho^2\tilde{g}|_{\partial M}$.

If $(M, \tilde{g})$ is an hyperbolic space, then we have the result of R. Bamler [**?**]

Theorem 3.1 ([?]) *Let $(M, \tilde{g})$ $\mathbb{H}^{n+1}$ for $n \geq 2$, choose a basepoint $x_0 \in M$ and let $r = d(\cdot, x_0)$ denote the radial distance function.*

There is an $\varepsilon_1 > 0$ and for every $q < \infty$ an $\varepsilon_2 = \varepsilon_2(q) > 0$ such that the following holds: If $g_0 = \tilde{g} + h$ and $h = h_1 + h_2$ satisfies

$$|h_1| < \frac{\varepsilon_1}{r+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q \, dx\right)^{1/q} < \varepsilon_2$$

then the normalized Ricci flow exists for all time and we have convergence $g_t \longrightarrow \bar{g}$ in the pointed Cheeger-Gromov sense.

We just want to generalize the above result into the case of asymptotically hyperbolic Einstein manifold in the interior of $n+1$ dimensional ball such that the conformal boundary of this asymptotically hyperbolic Einstein manifold is a perturbation of the conformal structure on the standard n dimensional sphere.

Theorem 3.2 (Our Goal) *Let $M = B^{n+1}$ be a ball wiht $n \geq 3$ and $\hat{h}$ the standard metric on the sphere S^n and $g_{\mathbb{H}}$ be the standard hyperbolic metric on B^{n+1}. For any asymptotically hyperbolic Einstein manifold (M, g) with non-positive sectional curvature and a defining function ρ such that $\hat{g} = \rho^2 g|_{\partial M}$ is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$ norm, for some $0 < \alpha < 1$ and g is sufficiently close to $g_{\mathbb{H}}$ in the sense of C^0. And choose a basepoint $x_0 \in M$ and let $r = d(., x_0)$ denote the radial distance function.*

There is an $\epsilon > 0$ such that the following holds: If $g_0 = g + h$ satisfies

$$|h| < \frac{\epsilon}{r+1}$$

then the normalized Ricci flow exists for all time and we have convergence $g_t \to g$ in the pointed Cheeger-Gromov sense.

The short time existence is a direct corollary of Shi's result. And it turns out the long time existence depends on the following heat kernel estimate.

In order to prove the long time existence of the Ricci flow, we need some estimates for heat kernel

$$\partial_t k_t = \Delta k_t + R(k_t) \quad \text{and} \quad k_t \to \delta_{p_0} id_{E_{p_0}} \quad \text{as } t \to 0 \tag{1}$$

where $(k_t)_{0<t<T} \in C^\infty(M; E) \otimes E_{p_0}$ and $E = Sym^2 T^* M$ and $R(h)_{il} = \tilde{g}^{jk_1} \tilde{g}^{i_1 i_2} \tilde{R}_{i i_2 l j} h_{k_1 i_1}$.

Lemma 3.1 *Let (M, g) be an asymptotically hyperbolic Einstein manifold of theorem 3.2. Choose a basepoint $x_0 \in M$ and consider the radius distance function $r = d(\cdot, x_0)$. If the heat kernel k_t defined by (1) satisfying that: For all $x_1 \in M$ and $r_1 = r(x_1)$, $t \geq 0$*

$$\int_M |k_t| (x_1, x) \, |h|(x) dx < \frac{C(\omega)}{(r_1 + 1 + a + t)^w}$$

provided that $h \in C^{\infty}\left(M; \mathrm{Sym}_2 T^*M\right)$ *and that*

$$|h|(x) < \frac{1}{(r(x)+1+a)^w}$$

for some $a \geq 0$. *The the result of theorem 3.2 holds.*

By the argument of lemma 6.3, lemma 6.4 in [**?**], the result of lemma 3.1 is determined by

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C \exp\left(\lambda_B t\right) \qquad \text{for } t > 0$$

Lemma 3.2 *Let* (M, g) *be an asymptotically hyperbolic Einstein manifold of theorem 3.2. If the heat kernel satisfies that*

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C \exp\left(\lambda_B t\right) \qquad \textit{for } t > 0$$

Then the assumption of lemma 3.1 holds.

4 Heat kernel estimate on the asymptotically hyperbolic manifolds

For the heat kernel of the scalar Laplacian operator on asymptotically hyperbolic manifolds, we have the result of Chenxi.

Theorem 4.1 (Heat kernel on asymptotically hyperbolic manifolds) *Let* (X, g) *be an* $(n+1)$ *dimensional asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Let* $H(t, z, z')$ *be the heat kernel on* (X, g). *Then* $H(t, z, z')$ *is equivalent to the Davies-Mandouvalos quantity, i.e. bounded above and below by multiples of*

$$t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} - \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)$$

uniformly over all times $t \in (0, \infty)$ *and distances* $r = d(z, z') \in (0, \infty)$.

Remark 4.1 *From the classic result of [MR], we know that the spectrum of asymptotically hyperbolic space consists of an at most finite number of eigenvalues in the interval* $(0, \frac{n^2}{4})$, *each with finite multiplicity, and continuous spectrum on the interval* $[\frac{n^2}{4}, +\infty)$. *Form [GJ], for the non-negative Yamabe type boundary of AHE, there is no resonance at* $\frac{n^2}{4}$. *Therefore the condition in Theorem 1.1 is not that strict.*

What we want to do is to generalized his result for function into the corresponding result for Lichnorwicz operator on symmetric two tensor.

Following the idea of Chenxi, first by the spectrum theorem and stone formula, we can recover the heat kernel by the corresponding resolvent.

$$e^{-t(\Delta_X)} = e^{-tn^2/4} e^{-t\left(\Delta_X - n^2/4\right)} = e^{-tn^2/4} \int_0^\infty e^{-t\sigma} dE_{(\Delta_X - n^2/4)}(\sigma) d\sigma$$

and then, via Stone's formula, in terms of the resolvent:

$$e^{-t(\Delta_X)} = \frac{\imath}{2\pi} e^{-tn^2/4} \int_{-\infty}^{\infty} e^{-t\lambda^2} R(\lambda - \imath 0) 2\lambda d\lambda, \quad \sigma = \lambda^2$$

Therefore, we need to know the schwartz kernel of $R(\lambda)$. The next thing is to generalize the result of Melrose-Sa Barreto-Vasy for the scalar Laplacian operator into the tensor Laplacian operator. The following is their result

Theorem 4.2 *Assume that (X, g) is an asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Let r denote geodesic distance on $X \times X$. Then the resolvent, $R(\lambda) := \left(\Delta_X - n^2/4 - \lambda^2\right)^{-1}$ is analytic in a neighbourhood of the closed lower half plane $\operatorname{Im} \lambda \leq 0$, and satisfies in this region of the λ-plane and for $r(1+|\lambda|) \geq 1$ (the 'off-digaonal regime') (2.7)*

$$R(\lambda)\left(z, z'\right) = e^{-i\lambda r} R_{od}(\lambda)\left(z, z'\right), \quad r = d\left(z, z'\right)$$

where

- *for $|\lambda| \leq 1, R_{od}(\lambda)$is an element of $\left(\rho_L \rho_R\right)^{n/2} \mathcal{A}^0\left(X_0^2\right)$*
- *for $|\lambda| \geq 1, R_{od}(\lambda)$ is of the form*

$$\rho_{\mathcal{L}}^{n/2} \rho_{\mathcal{R}}^{n/2} \rho_{\mathcal{A}}^{-n/2+1} \rho_{\mathcal{S}}^{-n+1} \mathcal{A}^0\left(X_0^2 \times_1 [0,1)_h\right)$$

In particular, $R_{od}(\lambda)$ is a kernel bounded pointwise by a multiple of

$$(r(1+|\lambda|))^{n/2-1} r^{-n+1} = r^{-n/2}(1+|\lambda|)^{n/2-1}$$

for $r \leq C$, and

$$e^{-nr/2}(1+|\lambda|)^{n/2-1}$$

for $r \geq C$.

This result is from the result of Melrose-Barreto-Vasy [?] [?]. CH obtain a semiclassical resolvent

$$\tilde{R}(h, \sigma) = \left(h^2 \Delta_X - h^2 n^2/4 - \sigma^2\right)^{-1} \quad \text{with } |\sigma| = 1, \operatorname{Im} \sigma \leq 0 \text{ and } h \in [0,1)$$

through the parametrix $G(h, \sigma)$ constructed by Melrose, Sà Barreto and Vasy. Then the properties of the resolvent $R(\lambda)$ in Theorem 2.2 follow from the counterparts for $\tilde{R}(h, \sigma)$. Their idea is from [?] [?]. Let (X, g) be a $n+1$ dimensional asymptotically hyperbolic manifold. They construct $0-$double space (i.e. blow up of double space X^2). Then, introduce the $0-$calculus. By this way, they construct the resolvent for Laplacian operator.

In order to generalize the function result into tensor result, we need to do some block decomposition on tensor space such that each block behaves like a function. For the standard hyperbolic space, we have the result of [OB] and [Ba] who use the spherical coordinate to decompose the homogeneous vector bundle (See section 2). Then we can use the function result for each block. So that, we can get the similar tensor results from function results. Then, by method of Melrose and Barreto and vasy, we can construct the parametrix of Lichnorwicz operator for the asymptotically hyperbolic based on hyperbolic spaces. In this way, we should be able to generalize Theorem 1.2 into tensor.

One thing is different from the function is that for function, non-negative Yamabe on the boundary is enough for no resonance at $\frac{n^2}{4}$. However, for symmetric two tensor we also require that the section curvature

$$\left(n^2-8n\right)/(8n-8)-\frac{n^2}{4}$$

to ensure that the absense of the eigenvalue in $(0,\frac{n^2}{4})$ and of the resonance at $\frac{n^2}{4}$.

Once we get the heat kernel estimate for general asymptotically hyperbolic manifolds, we can easily get the

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