This question paper consists of 4 printed pages, each of which is identified by the Code Number COMP394001.

This is a closed book examination. No material is permitted.

School of Computing

January 2018

COMP394001

Graph Algorithms and Complexity Theory

Answer all four questions

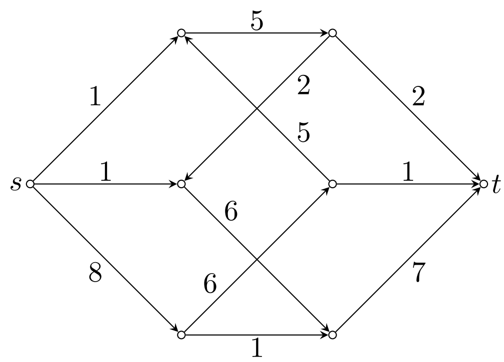
Time allowed: two hours

Question 1

(a) Suppose directed graphs are already defined. Define

- network **[2 marks]**
- capacity function **[1 mark]**
- flow **[3 marks]**
- s, t-cut **[2 marks]**

(b) Determine a maximum flow and a minimum cut (state the partition of the vertex set and edges in the cut) in the following network. Numbers next to edges denote capacities. Show your work. **[12 marks]**



[question 1 total: 20 marks]

Question 2

This is a question about matchings in undirected graphs

(a) Define

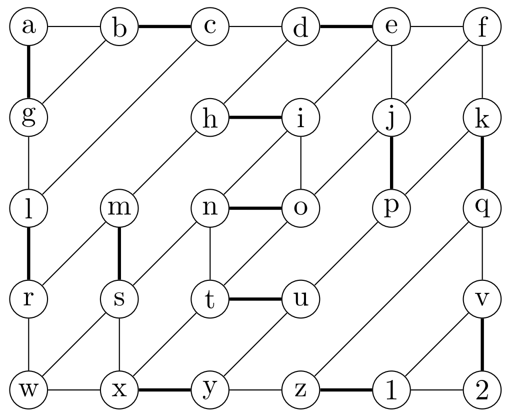
(i) a matching, **[1 mark]**

(ii) a maximum matching, and **[1 mark]**

(iii) an augmenting path. **[2 marks]**

(b) Berge's lemma says that M is a maximum matching of G if and only if there is no M-augmenting path in G. Prove this lemma. You may assume that the symmetric difference between any two matchings consists of alternating paths and alternating cycles. **[6 marks]**

(c) Illustrate Edmonds' blossom algorithm at the graph depicted below. An initial matching is indicated by bold lines. **[10 marks]**



[question 2 total: 20 marks]

Question 3

This question deals with polynomial-time many-one reducibility.

(a) Define the relation $\leq_{\mathrm{m}}^{\mathrm{p}}$. **[5 marks]**

(b) Prove that $\leq_{\mathrm{m}}^{\mathrm{p}}$ is reflexive, that is, show for all decision problems Π that $\Pi \leq_{\mathrm{m}}^{\mathrm{p}} \Pi$ holds. **[5 marks]**

(c) Prove that $\leq_{\mathrm{m}}^{\mathrm{p}}$ is transitive, that is, show for all decision problems Π_1, Π_2 and Π_3 that if $\Pi_1 \leq_{\mathrm{m}}^{\mathrm{p}} \Pi_2$ and $\Pi_2 \leq_{\mathrm{m}}^{\mathrm{p}} \Pi_3$ hold then $\Pi_1 \leq_{\mathrm{m}}^{\mathrm{p}} \Pi_3$ holds as well. **[10 marks]**

[question 3 total: 20 marks]

Question 4

Let $\alpha(G)$ denote the maximum size of an independent set of the graph G. We consider the following decision problem related to this parameter:

MIS (maximum independent set) is defined by
instance: An undirected graph G on $n > 0$ vertices and a natural number k.
question: Is $\alpha(G) \geq k$?

It is known that MIS is $\mathbb{NP}$-complete. For integers i with $1 \leq i \leq 6$ we consider functions f_i that map instances (G, k) of MIS to other such instances. Which of these functions are polynomial transformations from MIS to itself? Your answer can be an unconditional "yes" or "no", or it can depend on whether $\mathbb{P} = \mathbb{NP}$ or $\mathbb{P} \neq \mathbb{NP}$. A justification is not required.

For two graphs $G = (V, E)$ and $H = (U, F)$ with $V \cap U = \varnothing$ let $G + H = (V \cup U, E \cup F)$. If $V \cap U \neq \varnothing$ we replace one of the graphs by an isomorphic copy such that their vertex sets become disjoint. Especially, $G + G$ is the union of two disjoint copies of G. By K_n we denote the complete graph on n vertices.

(a) $f_1(G, k) = (G + G, 2k)$ **[3 marks]**

(b) $f_2(G, k) = (G + K_n, k + 1)$ **[3 marks]**

(c) $f_3(G, k) = (G + K_{2^n}, k + 1)$ **[3 marks]**

(d) $f_4(G, k) = (K_1, 1)$ if G has no matching of size $n - k + 1$ and
$f_4(G, k) = (K_1, 2)$ if this is not the case. **[4 marks]**

(e) $f_5(G, k) = f_4(G, k)$ if G is bipartite and $f_5(G, k) = (G, k)$ otherwise. **[4 marks]**

(f) $f_6(G, k) = (K_1, 1)$ if $\alpha(G) \geq k$ and $f_6(G, k) = (K_1, 2)$ if $\alpha(G) < k$. **[3 marks]**

[question 4 total: 20 marks]

[grand total: 80 marks]