This question paper consists of 5 printed pages, each of which is identified by the Code Number COMP391001

The template for Question 3 (a) is provided on a separate sheet of paper. It can be used to illustrate the operation of the network simplex method. Please attach the template sheet to the exam script book.

A non-programmable calculator can be used

School of Computing

May/June 2018

COMP3910

Combinatorial Optimisation

Answer ALL THREE questions.

Time allowed: 2 hours

Formula Sheet

The rules for updating the simplex tableau are:

(1) Interchange the variables at the head of the pivot column and the left of the pivot row.

(2) Replace the pivot by: $\frac{1}{\texttt{pivot}}$.

(3) Replace other entries in the pivot row by: $\frac{\texttt{old entry}}{\texttt{pivot}}$.

(3) Replace other entries in the pivot column by: $-\frac{\texttt{old entry}}{\texttt{pivot}}$.

(4) Replace all remaining entries by:

$$\text{old entry} - \frac{\texttt{old entry in same row, pivot column} \times \texttt{old entry in same column, pivot row}}{\texttt{pivot}}.$$

Question 1

(a) A company is considering seven capital investments $i = 1, \dots, 7$. Each investment can be done only once. Investments differ in expected profit that they will generate, and in the amount of capital required. For each investment i, the estimated profit p_i and the capital required c_i are known.

The total capital available for these investments is K. The objective is to select the combination of capital investments that will maximise the total estimated long-run profit.

(i) Formulate an ILP for the problem. **[2 marks]**

(ii) Model the following additional constraints:

- Investment opportunities 3 and 4 are mutually exclusive; **[1 marks]**
- Neither 5 nor 6 can be undertaken unless at least one of investments 1 and 2 is undertaken; **[2 marks]**
- At least three and at most four investment opportunities have to be undertaken from the set {1,3,5,6,7}. **[2 marks]**

(b) Suppose an ILP model with $0-1$ variables δ_1, δ_2, δ_3 and δ_4 contains the constraint $\delta_1+\delta_2+\delta_3-3\delta_4 \leq 0$.

(i) Explain which logical implication(s) this constraint models. **[1 mark]**

(ii) Suggest a stronger formulation of the above constraint. Explain why the new formulation is stronger. **[3 marks]**

(c) In the *travelling salesman problem*, a salesman must visit each of n cities and return to his start city. The order in which the cities are visited can be freely chosen, and a particular order constitutes a *tour*. The distances d_{ij} from city i to every other city j are given, and these are symmetric, i.e. $d_{ij} = d_{ji}$ for all $i, j = 1, 2, \dots, n$. The aim is to find the tour having the smallest possible length.

The following example gives a distance table for $n = 5$:

	d_{ij} (i \ j)	1	2	3	4	5
i	1	0	3	1	2	4
	2	3	0	4	3	2
	3	1	4	0	1	2
	4	2	3	1	0	2
	5	4	2	2	2	0

For example, tour $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ has the total length $3+4+1+2+4 = 14$.

(i) Devise one *construction heuristic* for this problem. Apply it to the above data. **[4 marks]**

(ii) Devise also a *local search* heuristic (e.g., iterative improvement) for the travelling salesman problem. Explain how neighbours are generated and how a transition from a current solution to a new one is performed. Illustrate the described approach by considering the tour $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ as a current solution. Perform one iteration; it is sufficient to generate three neighbours. **[5 marks]**

[20 marks total]

Question 2

(a) The simplex tableau below is the optimal solution to a linear program in which z is to be maximised. It corresponds to the root of the branch-and-bound tree for an integer linear program in which x_1, x_2 and x_3 are required to have non-negative integer values. Using the branching strategy based on penalties, find the optimal integer solution, and prove its optimality. Show all your working. Draw the search tree.

LP_0	x_2	x_4	x_5	
x_1	$\frac{13}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
x_3	$-\frac{5}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{40}{3}$
z	$\frac{140}{3}$	$\frac{40}{3}$	$\frac{10}{3}$	$\frac{1100}{3}$

[14 marks]

(b) The performance of the branch-and-bound algorithm heavily depends on the procedure for calculating the bounds. Often such a procedure is based on relaxing some of the problem constraints. Explain what kind of relaxation can be done when solving the following problems and how the appropriate bounds can be calculated:

(i) an ILP with $0 - 1$ variables, **[1 mark]**

(ii) the knapsack problem with integer variables representing how many items of each type should be placed in the knapsack, **[1 mark]**

(iii) the travelling salesman problem. **[4 marks]**

[20 marks total]

Question 3

(a) Consider the uncapacitated minimum cost flow problem shown in the figure below. The numbers displayed next to the nodes are supplies/demands. The numbers on the arcs are costs. Bold arcs represent the initial spanning tree.

(i) Specify the initial feasible flow and its cost. **[2 marks]**

(ii) Apply one iteration of the network simplex method to find an improved solution. **[6 marks]**

(iii) State whether the found solution is optimal and justify your answer. **[4 marks]**

(iv) Specify the found flow and its cost. **[1 mark]**

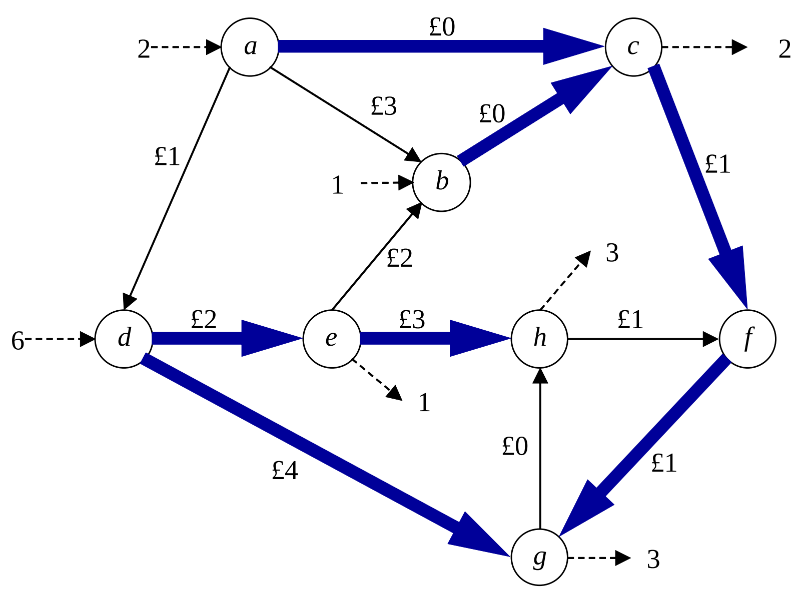

Figure 1: Input graph for Question 3 (a)

(b) The next two questions are on the justification of the network simplex method.

(i) Why is it justified to use the simplex method for finding an integer-valued flow of minimum cost, assuming that supply/demand values are integers? **[2 marks]**

(ii) Why does the notion of a spanning tree play an important role in the network simplex method? **[2 marks]**

(c) Describe how the problems stated below can be modelled as a minimum-cost flow problem. For each problem, explain the structure of the network, specify the supply/demand values and arc costs. Explain why a flow solution of minimum cost corresponds to an optimal solution of the original problem.

(i) *Assignment problem*: n jobs should be assigned to n persons, one job per person and one person per job. For each combination of a person i and a job j, the cost c_{ij} of the assignment is given. The task is to find an optimal assignment of jobs to persons minimising the total cost. **[1 mark]**

(ii) *Shortest path problem*: given a directed acyclic graph, find a path of minimum total length from a specified source node s to another specified sink node t, assuming that each arc (i, j) has an associated length c_{ij}. **[2 marks]**

[20 marks total]