E.1

Studiare il seguente integrale improprio : $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{x\log x}\,dx \quad (*)$ $\qquad (*) = \lim_{\alpha\to 0} \displaystyle\int_\alpha^{\frac{1}{2}} \frac{1}{x\log x}\,dx$

$$\int_\alpha^{\frac{1}{2}} \frac{1}{x\log x}\,dx = \int_\alpha^{\frac{1}{2}} \frac{D[\log x]}{\log x}\,dx = \Big[\log|\log x|\Big]_\alpha^{\frac{1}{2}} = \log\left|\log\tfrac{1}{2}\right| - \log|\log\alpha| = \log\left|\frac{\log\frac{1}{2}}{\log\alpha}\right|$$

$$\lim_{\alpha\to 0} \log\left|\frac{\log\frac{1}{2}}{\log\alpha}\right| = -\infty \implies \text{l'integrale diverge}$$

E.2

Calcolare $\displaystyle\int_0^{+\infty} (x^2-x)e^{-x}\,dx \quad (*)$

$$(*) = \lim_{\beta\to+\infty} \int_0^{\beta} (x^2-x)e^{-x}\,dx$$

$$\int (x^2-x)e^{-x}\,dx = \int (x-x^2)\,D[e^{-x}]\,dx = e^{-x}(x-x^2) - \int (1-2x)e^{-x}\,dx = (x-x^2)e^{-x} + \int (1-2x)\,D[e^{-x}]\,dx =$$

$$= (x-x^2)e^{-x} + e^{-x}(1-2x) + 2\int e^{-x}\,dx = e^{-x}(-x^2-x-1) + c = -e^{-x}(x^2+x+1) + c$$

$$(*) = \lim_{\beta\to+\infty} \Big[-(x^2+x+1)e^{-x}\Big]_0^{\beta} = \lim_{\beta\to+\infty} -\frac{\beta^2+\beta+1}{e^\beta} + 1 = 1 \implies \text{l'integrale converge}$$

E.3

$$\int_0^{+\infty} \frac{|\operatorname{sen} x|^p}{x^2}\,dx \quad (*)$$

$$(*) = \underbrace{\int_0^1 \frac{|\operatorname{sen} x|^p}{x^2}\,dx}_{(1)} + \underbrace{\int_1^{+\infty} \frac{|\operatorname{sen} x|^p}{x^2}\,dx}_{(2)}$$

(1) per $x \in]0,1]$ $\quad f(x) = \dfrac{(\operatorname{sen} x)^p}{x^2} = \dfrac{(\operatorname{sen} x)^p}{x^p}\,\dfrac{x^p}{x^2} = \left(\dfrac{\operatorname{sen} x}{x}\right)^p x^{p-2} \sim x^{p-2} \implies \begin{cases} p-2 \geq 0 & x^{p-2} \text{ integrabile in } [0,1] \\ p-2 < 0 & x^{p-2} = \frac{1}{x^{2-p}} \text{ è integrabile su }]0,1] \iff 2-p<1,\ p>1 \end{cases}$

(2) $\displaystyle\int_1^{+\infty} \frac{|\operatorname{sen} x|^p}{x^2}\,dx \leq \int_1^{+\infty} \frac{1}{x^2}\,dx < +\infty \longleftarrow$ noto integrale che converge

E.4

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (7x-6)\cos 3x\,dx = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (7x-6)\,D[\operatorname{sen} 3x] = \frac{1}{3}\left[\operatorname{sen}(3x)(7x-6)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] - \frac{7}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sen} 3x\,dx = \frac{1}{3}\Big[\ldots\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{7}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -3\operatorname{sen} 3x\,dx$$

$$= \frac{1}{3}\Big[\ldots\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{7}{6}\Big[\cos 3x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

E.5

$$\int_0^{\frac{\pi}{2}} \frac{4\cos x}{1+\sin^2 x}\,dx = 4\int^{\frac{\pi}{2}} \frac{\delta[\sin x]}{1+(\sin x)^2}\,dx = 4\,\text{arctg}\,\sin x + c$$

E.6

$$\int_0^{\pi^2} \text{sen}\sqrt{x}\,dx = \int_0^{\pi} \text{sen}t\;2t\,dt = 2\int_0^{\pi} t\,\text{sen}t\,dt = -2\int_0^{\pi} t\,\delta[\cos t]\,dt = -2\Big[t\cos t\Big]_0^{\pi} + 2\int_0^{\pi}\cos t\,dt = -2(-\pi) + 2\Big[\sin t\Big]_0^{\pi} = 2\pi$$

$t = \sqrt{x} \Rightarrow t^2 = x \Rightarrow 2t\,dt = dx$

$t(0) = 0 \quad t(\pi^2) = \pi$

E.7

$$\lim_{x\to 0} \frac{(e^{-x^3}-1)(1-\cos 2x)}{\log(1+x^5)} = \lim_{x\to 0} \frac{\frac{e^{-x^3}-1}{-x^3}(-x^3)\,\frac{1-\cos 2x}{(2x)^2}(2x)^2}{\frac{\log(1+x^5)}{x^5}\,x^5} = \lim_{x\to 0} \frac{-4x^5}{x^5} = -4$$

principio di sovrapposizione

E.8

$$\lim_{x\to 0} \frac{e^x - \text{sen}x - \cos x}{e^{x^2} - e^{x^3}} \quad (*)$$

$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + o(t^3) \qquad \sin t = t - \frac{t^3}{6} + o(t^4) \qquad \cos t = 1 - \frac{t^2}{2} + o(t^3)$

$N: \; e^t - \text{sen}t - \cos t = \cancel{1} + \cancel{t} + \frac{t^2}{2!} + \frac{t^3}{3!} + o(t^3) - \cancel{t} + \frac{t^3}{3!} + o(t^4) - \cancel{1} + \frac{t^2}{2!} + o(t^3) = t^2 + \frac{t^3}{2} + o(t^3) \Longrightarrow \quad N(x) \sim x^2$

$D: \; e^{t^2} - e^{t^3} = 1 + t^2 + \frac{t^4}{2} + o(t^6) - (1 + t^3 + o(t^9)) = t^2 - t^3 + \frac{t^4}{2} + o(t^6) \Rightarrow \quad D(x) \sim x^2$

$(*) = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$

E.9 $\lim_{n\to+\infty} (3n^2+2)\left[\sin\left(\frac{1}{n}\right) - \left(\frac{1}{n}\right)\right] \quad (*)$

Ricordando che $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{3}\cdot\lim_{x\to 0} \frac{1-\cos x}{x^2} = -\frac{1}{6} \Longrightarrow (\sin x - x) \sim -\frac{1}{6}x^3$

$(*) = \lim_{n\to+\infty} (3n^2+2)\left(-\frac{1}{6}\frac{1}{n^3}\right) = 0 \quad \left(\text{oppure se } * = \lim_{n\to+\infty} (3n^3+2)\left(-\frac{1}{6}\frac{1}{n^3}\right) = -\frac{1}{2}\right)$

E.10

$$\sum_{n=1}^{+\infty} (-1)^n \left(\frac{3n-1}{7n+4}\right)^n \quad (*)$$

Verificare convergenza assoluta

$\sum |a_n|$ converge $\Longleftrightarrow$ $(*)$ converge

$\sum_{n=1}^{+\infty} \left|(-1)^n\left(\frac{3n-1}{7n+4}\right)^n\right| = \sum_{n=1}^{+\infty} \left(\frac{3n-1}{7n+4}\right)^n \longrightarrow$ Criterio radice $\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{3n-1}{7n+4} = \frac{3}{7} < 1 \Rightarrow$ converge assolutamente $\Rightarrow$ conv. semp.

E. 11

$f(x) = (x^2+1)e^{-|x|}$

- PARITÀ - SIMMETRIE : PARI $\Rightarrow$ STUDIARE $f(x) = (x^2+1)e^{-x}$, $x>0$
- DOMINIO $A = \mathbb{R}$
- INTERSEZIONE ASSI

  $y=0 = (x^2+1)e^{-x}$ NON AMMETTE SOLUZIONE

  $x=0 \Rightarrow y=1$ $\quad P(0,1)$

- STUDIO DEL SEGNO : $f(x) = (x^2+1)e^{-x}$ SEMPRE POSITIVA

- LIMITI E ASINTOTI

$$\lim_{x\to+\infty} (x^2+1)e^{-x} = \lim_{x\to+\infty} \frac{x^2+1}{e^x} = 0 \longrightarrow y=0 \text{ ASINTOTO ORIZZONTALE}$$

- MONOTONIA, MAX E MIN

$$f(x) = (x^2+1)e^{-x} \longrightarrow f'(x) = 2xe^{-x} - (x^2+1)e^{-x} = -e^{-x}(x^2-2x+1) = -e^{-x}(x-1)^2 \longrightarrow f'(x)<0 \text{ SEMPRE} \Rightarrow f(x) \text{ STRET. DECR. IN } [0,+\infty[$$

Considerata la stretta decrescenza di $f(x)$ e la sua simmetria, $x=0$ max globale

$f'(x)=0 \iff x=1 \longrightarrow$ che punto è $x=1$? $x=1$ STAZIONARIO, MA NON ESTREMO! $\quad f(1) = \frac{2}{e} \approx 0{,}735$

- DERIVATA SECONDA, CONCAVITÀ, CONVESSITÀ E FLESSI

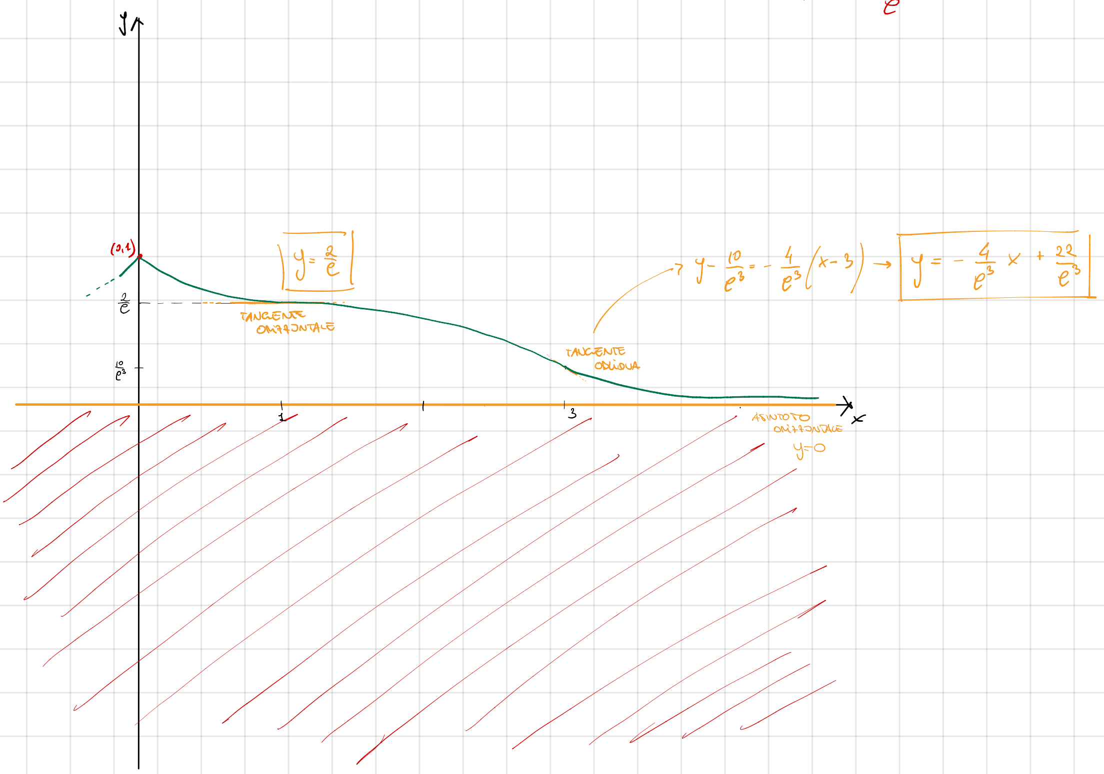
$$f'(x) = -e^{-x}(x-1)^2 \longrightarrow f''(x) = e^{-x}(x-1)^2 - e^{-x}2(x-1) = e^{-x}(x-1)\left[(x-1)-2\right] = e^{-x}(x-1)(x-3)$$

$f''(x) > 0$ (0, 1, 3: $+$, $-$, $+$)

$$f''(x) = \begin{cases} e^{-x}(x-1)(x-3) > 0 & x \in]0,1[\,\cup\,]3,+\infty[\\ e^{-x}(x-1)(x-3) < 0 & x \in]1,3[\end{cases}, \quad f''(x) \text{ è } \begin{cases} \text{CONVESSA in } [0,1] \text{ e in } [3,+\infty[\\ \text{CONCAVA IN } [1,3] \end{cases}$$

$f(3) = \frac{10}{e^3}$ $\quad$ ATTN.: $x=1$ PUNTO DI FLESSO MA ANCHE STAZIONARIO $\Rightarrow$ $x=1$ FLESSO TANGENTE ORIZZONTALE

$x=3$ PUNTO DI FLESSO A TANGENTE OBLIQUA : $f'(3) = -\frac{4}{e^3}$



E.12 Verificare se $f(x)=\begin{cases} \left(\frac{2}{2+x}\right)^{\frac{1}{x}} & -2<x<0 \\ \frac{1}{\sqrt{x+e}} & x\geq 0 \end{cases}$ è continua in 0

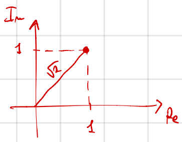
$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{1}{\sqrt{x+e}}=\frac{1}{e}$$

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}\left(\frac{2}{2+x}\right)^{\frac{1}{x}}=\lim_{x\to0^-}\left(\frac{1}{\frac{2+x}{2}}\right)^{\frac{1}{x}}=\lim_{x\to0^-}\left(\frac{1}{1+\frac{x}{2}}\right)^{\frac{1}{x}}=\lim_{x\to0^-}\left[\left(\frac{1}{1+\frac{1}{\frac{2}{x}}}\right)^{\frac{2}{x}}\right]^{\frac{1}{2}}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}$$

$$f(0)=\frac{1}{\sqrt{e}}$$

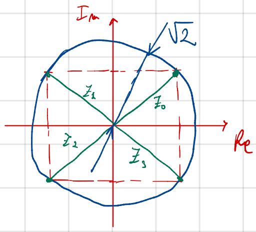
Quindi $\lim_{x\to0^+} f(x)=\lim_{x\to0^-} f(x)=f(0)$

E.13 Determinare le radici quarte di $w=1+i$



$z^4=1+i$, cioè $z_k=\sqrt[4]{1+i}=\sqrt[4]{w}$

$$|w|=\sqrt{2} \Rightarrow w=\sqrt{2}\left(\cos\frac{\pi}{4}+i\,\mathrm{sen}\frac{\pi}{4}\right)=$$



$$z_k=\sqrt[8]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)\right],\quad k=0,1,2,3$$

$z_0=1+i \qquad z_1=-1+i \qquad z_2=-1-i \qquad z_3=1-i$