Lezione 45

Sia $f \in \mathcal{R}([-a,a])$ con $a>0$.

Allora $$\int_{-a}^{a} f = \begin{cases} 0 & \text{se } f \text{ è dispari} \\ 2\int_0^a f & \text{se } f \text{ è pari} \end{cases}$$

Dim: $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dt$$

$x = -t = \varphi(t)$ $\qquad x=-a \iff t=a$

$\qquad x=0 \iff t=0$

$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-t)(-1)\,dt = \int_0^a f(-t)\,dt$$

- se f è dispari $f(-t) = -f(t)$ e quindi
$$\int_0^a f(-t)\,dt = -\int_0^a f(t)\,dt = -\int_0^a f(x)\,dx$$
e quindi $\int_{-a}^{a} f(x) = 0$

- se f è pari $f(-t) = f(t)$ e quindi
$$\int_0^a f(-t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$$
e quindi $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

Esempi

1) $$\int_{-1}^{1} \frac{1}{\sqrt{|x|}+1}\,dx = 2\int_0^1 \frac{1}{\sqrt{x}+1}\,dx = 4\left[\sqrt{x}-\log|\sqrt{x}+1|\right]_0^1$$
$$= 4(1-\log 2)$$

2) $$\int_{-1}^{1} \frac{\operatorname{sen}(3x)}{1+x^8+\cos x}\,dx = 0$$

3) $$\int \cos(\log x)\,dx$$

$$t=\log x \iff x=\underbrace{e^t}_{\varphi(t)}$$

$$\int \cos(\log x)\,dx = \left(\int \cos t \cdot e^t\,dt\right)_{t=\log x}$$

$$\int e^t \cos t\,dt = \int e^t D\operatorname{sen} t\,dt = e^t \operatorname{sen} t - \int e^t \operatorname{sen} t\,dt$$

$$= e^t \operatorname{sen} t + \int e^t D\cos t\,dt$$

$$= e^t \operatorname{sen} t + \left[e^t \cos t - \int e^t \cos t\,dt\right]$$

$$= e^t(\operatorname{sen} t + \cos t) - \int e^t \cos t\,dt$$

$$2\int e^t \cos t\,dt = e^t(\operatorname{sen} t + \cos t) + c$$

$$\int e^t \cos t\,dt = e^t\,\frac{(\operatorname{sen} t + \cos t)}{2} + c$$

$$\int \cos(\log x)\,dx = \frac{x}{2}\left(\operatorname{sen}\log x + \cos\log x\right) + c$$

Integrazione delle funzioni razionali

$$\int f(x)\,dx \quad \text{dove} \quad f(x) = \frac{p(x)}{q(x)}$$

con $p(x)$ e $q(x)$ polinomi

La prima osservazione è che ci si può sempre ridurre al caso che grado $p(x)$ < grado $q(x)$

Infatti la divisione euclidea tra polinomi garantisce che esistono $d(x)$ e $r(x)$ polinomi tali che

$$p(x) = q(x) \cdot d(x) + r(x) \quad \text{con} \quad \text{grado } r(x) < \text{grado } q(x)$$

Allora $\displaystyle \frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}$

Quindi $$\int \frac{p(x)}{q(x)}\,dx = \underbrace{\int d(x)\,dx}_{1} + \underbrace{\int \frac{r(x)}{q(x)}\,dx}_{2}$$

e 1 è immediato perchè è un integrale

di un polinomio. Mentre in 2 il numeratore ha grado < del denominatore.

Esempio

$$\int \frac{x^4 - 4x^3 + 3x^2 + 5x - 4}{x^2 - 4x + 4}\, dx$$

$$\begin{array}{l|l}
x^4 - 4x^3 + 3x^2 + 5x - 4 & x^2 - 4x + 4 \\ \hline
x^4 - 4x^3 + 4x^2 & x^2 - 1 \\ \hline
= \quad = \quad -x^2 + 5x - 4 & \\
\quad\quad\quad -x^2 + 4x - 4 & \\ \hline
\quad\quad = \quad x &
\end{array}$$

$$x^4 - 4x^3 + 3x^2 + 5x - 4 = (x^2 - 4x + 4)(x^2 - 1) + x$$

$$\int \frac{x^4 - 4x^3 + 3x^2 + 5x - 4}{x^2 - 4x + 4}\, dx = \int x^2 - 1\, dx + \int \frac{x}{x^2 - 4x + 4}\, dx$$

Consideriamo il caso che grado $q(x) = 2$

Quindi necessariamente grado di $p(x) \leq 1$.

$$q(x) = ax^2 + bx + c$$

1CASO $\Delta = b^2-4ac > 0$

Siano x_1 e x_2 le due radici (distinte) di $q(x)$. Allora esistono A e $B \in \mathbb{R}$ tali che

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

Infatti $$\frac{mx+n}{ax^2+bx+c} = \frac{A}{x-x_1} + \frac{B}{x-x_2} \iff$$

$$\iff \frac{mx+n}{a\cancel{(x-x_1)(x-x_2)}} = \frac{A(x-x_2)+B(x-x_1)}{\cancel{(x-x_1)(x-x_2)}}$$

$$\iff \frac{m}{a}x+\frac{n}{a} = (A+B)x - (Ax_2+Bx_1)$$

$$\iff \begin{cases} A+B = \frac{m}{a} \\ Ax_2+Bx_1 = -\frac{n}{a} \end{cases} \iff \begin{cases} B = \frac{m}{a} - A \\ Ax_2 + \left(\frac{m}{a} - A\right)x_1 = -\frac{n}{a} \end{cases}$$

$$\iff \begin{cases} B = \frac{m}{a} - A \\ A(x_2-x_1) = -\left(\frac{m}{a}x_1 + \frac{n}{a}\right) \end{cases}$$

$$\Longleftrightarrow \begin{cases} B = \dfrac{m}{a} - A \\ A = -\dfrac{(mx_1+n)}{a(x_2-x_1)} \end{cases}$$

$$\Longleftrightarrow \begin{cases} B = \dfrac{m}{a} + \dfrac{(mx_1+n)}{a(x_2-x_1)} = \dfrac{mx_2+n}{a(x_2-x_1)} \\ A = \dfrac{mx_1+n}{a(x_1-x_2)} \end{cases}$$

<u>Es</u> $\displaystyle\int \frac{x+1}{x^2-10x+21}\,dx$

$$100-4\cdot 21 = 100-84 = 16 = 4^2.$$

$$x_{1/2} = \frac{10 \pm 4}{2} \;\begin{matrix} \nearrow 7 \\ \searrow 3 \end{matrix}$$

$$\frac{x+1}{\cancel{x^2-10x+21}} = \frac{A}{x-3} + \frac{B}{x-7} = \frac{A(x-7)+B(x-3)}{\cancel{(x-3)(x-7)}}$$

$$x+1 = (A+B)x - (7A+3B)$$

$$\begin{cases} A+B = 1 \\ 7A+3B = -1 \end{cases} \qquad \begin{cases} 3A+3B = 3 \\ 7A+3B = -1 \end{cases} \qquad \begin{cases} A+B = 1 \\ 4A = -4 \end{cases}$$

$$\Longleftrightarrow \quad A = -1 \quad e \quad B = 2$$

$$\frac{x+1}{x^2-10x+21} = -\frac{1}{x-3} + \frac{2}{x-7}$$

$$\int \frac{x+1}{x^2-10x+21}\,dx = -\int \frac{1}{x-3}\,dx + 2\int \frac{1}{x-7}\,dx$$

$$= -\log|x-3| + 2\log|x-7| + c$$

<u>2 CASO</u> $\Delta = b^2 - 4ac = 0$

Sia x_0 la radice doppia. Allora

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_0} + \frac{B}{(x-x_0)^2}$$

$$\Longleftrightarrow \frac{mx+n}{ax^2+bx+c} = \frac{A(x-x_0)+B}{(x-x_0)^2}$$

$$\Longleftrightarrow \frac{mx+n}{a\cancel{(x-x_0)^2}} = \frac{A(x-x_0)+B}{\cancel{(x-x_0)^2}}$$

$$\Longleftrightarrow \frac{m}{a}x + \frac{n}{a} = Ax + B - Ax_0$$

$$\Longleftrightarrow \frac{m}{a} = A \quad e \quad \frac{n}{a} = B - Ax_0 \Longleftrightarrow A = \frac{m}{a} \quad e \quad B = \frac{mx_0+n}{a}$$

Es. $\dfrac{x}{x^2-4x+4} = \dfrac{x}{(x-2)^2}$

$\Delta = 16 - 4\cdot 4 = 0$

$x_0 = \dfrac{4}{2} = 2$

$$\frac{x}{\cancel{x^2-4x+4}} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{A(x-2)+B}{\cancel{(x-2)^2}}$$

$$x = Ax + B - 2A \iff A = 1 \text{ e } B-2=0$$

$$\iff A = 1 \text{ e } B = 2.$$

$$\int \frac{x}{x^2-4x+4}\,dx = \int \frac{1}{x-2}\,dx + 2\int \frac{1}{(x-2)^2}\,dx$$

$$= \log|x-2| + 2\int (x-2)^{-2}\,dx$$

$$= \log|x-2| + 2\,\frac{(x-2)^{-1}}{-1} + c$$

$$= \log|x-2| - \frac{2}{x-2} + c$$

3 CASO $\quad \Delta = b^2 - 4ac < 0$

$$\frac{p(x)}{q(x)} = \frac{A\,q'(x)}{q(x)} + \frac{B}{q(x)}$$

$$\iff mx + n = A(2ax + b) + B$$

$$\iff mx + n = 2aAx + (Ab + B)$$

$$\iff \begin{cases} m = 2aA \\ n = Ab + B \end{cases} \iff \begin{cases} A = \dfrac{m}{2a} \\ B = n - Ab = n - \dfrac{mb}{2a} \end{cases}$$

$$\int \frac{p(x)}{q(x)}\,dx = A \int \frac{q'(x)}{q(x)}\,dx + B \int \frac{1}{q(x)}\,dx$$

$$= A \log|q(x)| + B \int \frac{1}{ax^2 + bx + c}\,dx.$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 + 2\frac{b}{2a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right]$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{-\Delta}{4a^2}}\,dx$$

$$= \frac{1}{a}\cdot\frac{4a^2}{-\Delta}\int \frac{1}{\Big(\underbrace{\frac{2ax+b}{\sqrt{-\Delta}}}_{\varphi(x)}\Big)^2+1}\,dx$$

$$= \frac{2}{\sqrt{-\Delta}}\int \frac{\varphi'(x)}{1+\varphi(x)^2}\,dx$$

$$= \frac{2}{\sqrt{-\Delta}}\,\mathrm{arctg}\left(\frac{2ax+b}{\sqrt{-\Delta}}\right)+c\,.$$

Es. $$\int \frac{1}{x^2+x+2}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+2-\frac{1}{4}}\,dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\,dx$$

$$= 4\int \frac{1}{(2x+1)^2+7}\,dx$$

$$= \frac{4}{7}\int \frac{1}{\Big(\underbrace{\frac{2x+1}{\sqrt{7}}}_{\varphi(x)}\Big)^2+1}\,dx$$

$$= \frac{2}{\sqrt{7}}\int \frac{\varphi'(x)}{1+\varphi'(x)}\,dx$$

$$= \frac{2}{\sqrt{7}}\ \operatorname{arctg}\left(\frac{2x+1}{\sqrt{7}}\right)$$
