Lezione 8

Esempio naturale di intervalli incapsulati:

Gli allineamenti decimali

Si dimostra che se $x \in \mathbb{R}$, allora esiste un'unica successione $(p_n)_{n \in \mathbb{N}}$ di numeri naturali tale che:

1) $p_0 \in \mathbb{Z}$ e $\forall n \in \mathbb{N}^*$: $p_n \in \{0, 1, \dots, 9\}$
2) $\forall n \in \mathbb{N}$ risulta

$$\underbrace{p_0 + \frac{p_1}{10} + \dots + \frac{p_n}{10^n}}_{a_n} \leq x < \underbrace{p_0 + \frac{p_1}{10} + \dots \frac{p_n}{10^n} + \frac{1}{10^n}}_{b_n} \qquad (1)$$

In tal caso si scrive

$$x = p_0 . p_1 p_2 p_3 p_4 \dots\dots$$

Esempio

$$\sqrt{2} = 1.41421356237309504880\dots$$

In questo caso si ha

$$a_0 = 1 < \sqrt{2} < 2 = b_0$$

$$a_1 = 1.4 < \sqrt{2} < 1.5 = b_1$$

$$a_2 = 1.41 < \sqrt{2} < 1.42 = b_2$$

$$a_3 = 1.414 < \sqrt{2} < 1.415 = b_3$$

$$a_4 = 1.4142 < \sqrt{2} < 1.4143 = b_4$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\sqrt{2} \in I_n = [a_n, b_n] \quad e$$

$$\bigcap_{n \in \mathbb{N}} I_n = \{\sqrt{2}\} \quad e \quad |I_n| = \frac{1}{10^n}.$$

Sommatorie.

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \dots + a_n$$

Se $(a_n)_{n\in\mathbb{N}}$ è una successione questa somma si può definire ricorsivamente come

$$\begin{cases} \displaystyle\sum_{k=0}^{0} a_k = 0 \\ \displaystyle\sum_{k=0}^{n+1} a_k = a_{n+1} + \sum_{k=0}^{n} a_k \end{cases}$$

Per le sommatorie valgono le seguenti proprietà

$$\left.\begin{aligned} &\sum_{k=0}^{n} (a_k + b_k) = \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k \\ &\sum_{k=0}^{n} \lambda a_k = \lambda \sum_{k=0}^{n} a_k . \end{aligned}\right\} \text{linearità}$$

Nel seguito daremo diversi esempi di sommatorie.

Proposizione

Sia $q \in \mathbb{R}$ con $q \neq 1$. Allora

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

Dim: moltiplicando la somma per q otteniamo

$$q \underbrace{\sum_{k=0}^{n} q^k} = \sum_{k=0}^{n} q^{k+1} = \sum_{k=1}^{n+1} q^k = \underbrace{\sum_{k=0}^{n} q^k} + q^{n+1} - 1$$

Perciò $1-q^{n+1} = (1-q)\sum_{k=0}^{n} q^k$ e dividendo per $1-q$ si ha la tesi.

[Gauss: $\left.\begin{array}{l} S_n = 1+q+\dots+q^n \\ qS_n = \quad q+\dots+q^n+q^{n+1} \end{array}\right\} \Rightarrow S_n(1-q) = 1-q^{n+1}$]

Proposizione

$$\sum_{k=1}^{n} k = 1+2+\dots+n$$

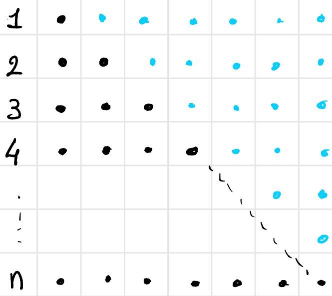
$$= \frac{n(n+1)}{2}$$

Dim (Gauss):

1	2	3	4	5	$\dots$	$n-1$	n
n	$n-1$	$n-2$	$n-3$	$n-4$		2	1
$n+1$	$n+1$	$n+1$				$n+1$	$n+1$

Quindi $2\sum_{k=0}^{n} k = n(n+1)$ da cui la tesi

Dimostrazione alternativa



$$\frac{n^2-n}{2} + n = \frac{n(n-1)+2n}{2}$$

$$= \frac{n(n+1)}{2}.$$

Si può fare anche per induzione.

I numeri del tipo $T_n = \frac{n(n+1)}{2}$ sono chiamati numeri triangololari

Es. $n=0 \longrightarrow T_0 = 0$

$n=1 \longrightarrow T_1 = 1$

$n=2 \longrightarrow T_2 = 3$

$n=3 \longrightarrow T_3 = 6$

$n=4 \longrightarrow T_4 = 10$

Corollario

La somma dei primi n numeri dispari è n^2.

Dim: $$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= n(n+1) - n = n^2.$$

Proposizione

Sia $n \in \mathbb{N}$ $n \geq 1$. Allora

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Dim:

$$(1+k)^3 = 1 + 3k + 3k^2 + k^3$$

da cui segue che

$$\sum_{k=0}^{n} (1+k)^3 = n+1 + 3\sum_{k=0}^{n} k + 3\sum_{k=0}^{n} k^2 + \underline{\sum_{k=0}^{n} k^3}$$

$$\sum_{k=0}^{n} (1+k)^3 = \sum_{k=1}^{n+1} k^3 = \underline{\sum_{k=1}^{n} k^3} + (n+1)^3$$

$$(n+1)^3 = n+1 + 3\frac{n(n+1)}{2} + 3\sum_{k=1}^{n} k^2$$

$$3\sum_{k=1}^{n} k^2 = (n+1)\left[(n+1)^2 - 1 - \frac{3n}{2}\right]$$

$$= (n+1)\left[n^2 + 2n - \frac{3n}{2}\right]$$

$$= n(n+1)\left(n+2-\frac{3}{2}\right)$$

$$= n(n+1)\left(n+\frac{1}{2}\right)$$

$$= n(n+1)\frac{(2n+1)}{2} \qquad \square$$

Fattoriale

Si definisce per ricorrenza

$$\begin{cases} 0! = 1 \\ (n+1)! = n \cdot n! \end{cases}$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1.$$

Es. $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3! = 24$, $5! = 5 \cdot 4! = 120$

Def $n \in \mathbb{N}$, $k \in \mathbb{N}$ $k \leq n$ coefficiente binomiale di ordine n e indice k.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)\overbrace{(n-k)(n-k-1)\cdots 1}}{k!\,\underbrace{(n-k)!}}$$

$$= \frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k \text{ termini}}}{k!}$$

Es: $\binom{6}{3} = \frac{\overbrace{6\cdot 5\cdot 4}^{3 \text{ term.}}}{3!} = \frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3\cdot 2}} = 20$

$$\binom{20}{4} = \frac{\overset{5}{\cancel{20}}\cdot 19\cdot \overset{9}{\cancel{18}}\cdot 17}{\cancel{4}\cdot 3\cdot \cancel{2}} = \frac{5\cdot 19\cdot 17}{3}$$

Proprietà

(i) $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = n$

(ii) $\binom{n}{k} = \binom{n}{n-k}$

(iii) se $1 \le k$, allora $\boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$

Dim (solo (iii)):

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)\,(n-k+1)!} + \frac{n!}{k\,(k-1)!\,(n-k)!}$$

$$= \frac{k\,n! + (n-k+1)\,n!}{k!\,(n-k+1)!}$$

$$= \frac{\cancel{k\,n!} + (n+1)\,n! - \cancel{k\,n!}}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n-k+1)!}$$

$$= \binom{n+1}{k}$$

$n=0$ $\qquad \binom{0}{0}$

$n=1$ $\qquad \binom{1}{0} \quad \binom{1}{1}$

$n=2$ $\qquad \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$

$n=3$ $\qquad \binom{3}{0} \quad \boxed{\binom{3}{1} + \binom{3}{2}} \quad \binom{3}{3}$

$n=4$ $\qquad \binom{4}{0} \quad \binom{4}{1} \quad \boxed{\binom{4}{2}} \quad \binom{4}{3} \quad \binom{4}{4}$

$n=0 \quad 1$

$n=1 \quad 1 \xrightarrow{+} 1$

$n=2 \quad 1 \xrightarrow{+} 2 \rightarrow 1$

$n=3 \quad 1 \xrightarrow{+} 3 \xrightarrow{+} 3 \xrightarrow{+} 1$

$n=4 \quad 1 \xrightarrow{+} 4 \xrightarrow{+} 6 \xrightarrow{+} 4 \xrightarrow{+} 1$

$n=5 \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

Teorema (formula del binomio di Newton)
Siano $a, b \in \mathbb{R}$ e $n \in \mathbb{N}$.

Allora $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

dove $a^0 = b^0 = 1$

$$= \underline{\binom{n}{0}} a^n b^0 + \underline{\binom{n}{1}} a^{n-1} b^1 + \underline{\binom{n}{2}} a^{n-2} b^2 + \ldots$$

$$\ldots + \underline{\binom{n}{n-1}} a\, b^{n-1} + \underline{\binom{n}{n}} a^0 b^n$$

Es $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3$
$+ 5ab^4 + b^5.$

Dim: per induzione su n.

Per $n=0$. $(a+b)^0 = \sum_{k=0}^{0} \binom{n}{k} a^{n-k} b^k = \binom{0}{0} a^0 b^0$

$$1 = \underline{1}$$

Supponiamo che sia vero per $n \in \mathbb{N}$, cioè che

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

e proviamo che: $(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$

$$(a+b)^{n+1} = (a+b)(a+b)^n$$

$$= a\left[a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + b^n \right]$$

$$+ b\left[a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + b^n \right]$$

$$= a^{n+1} + \binom{n}{1} a^n b + \binom{n}{2} a^{n-1} b^2 + \dots + \binom{n}{n-1} a^2 b^{n-1} + a b^n$$

$$+ \binom{n}{0} a^n b + \binom{n}{1} a^{n-1} b^2 + \qquad + \binom{n}{n-1} a b^n + b^{n+1}$$

$$= a^{n+1} + \left[\binom{n}{1} + \binom{n}{0}\right] a^n b + \left[\binom{n}{2} + \binom{n}{1}\right] a^{n-1} b^2 + \dots + \left[\binom{n}{n} + \binom{n}{n-1}\right] a b^n + b^{n+1}$$

$$= a^{n+1} + \binom{n+1}{1} a^n b + \binom{n+2}{2} a^{n-1} b^2 + \cdots + \binom{n+1}{n} a b^n + b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k .$$
