# Logica e Metodi Probabilistici per L'Informatica
# Third session of Exercises

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**Exercise 1** (Two states Markov Chain). The most general two-stat Markov Chain has transition matrix of the following form:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Find a general formula for $p_{1,1}^{(n)}$.
**Solution:** We exploit the relation $P^{n+1} = P^n P$ to write:

$$p_{1,1}^{(n+1)} = \frac{1}{2}p_{1,2}^{(n)} + \frac{1}{3}p_{1,1}^{(n)}.$$

We also know that $p_{1,1}^{(n)} + p_{1,2}^{(n)} = 1$, so we get the recurrence:

$$p_{1,1}^{(n+1)} = -\frac{1}{6}p_{1,1}^{(n)} + \frac{1}{2},\ p_{1,1}^{1} = \frac{1}{3}.$$

With some work it is possible to disentangle the recurrence, thus obtaining:

$$p_{1,1}^{(n)} = \frac{3}{7} + \frac{4}{7}\left(-\frac{1}{6}\right)^n$$

**Exercise 2** (Virus Mutation). Suppose a virus can exist in $N$ different strains and in each generation either stays the same, or with probability $\alpha$ mutates to another strain, which is chosen at random. What is the probability that the strain in the $n^{th}$ generation is the same as that in the $0^{th}$?
**Solution:** Instead of considering the immediate Markov Chain, consider a simpler one with two states: initial state and different state. This new Markox Chain has matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \frac{\alpha}{N-1} & 1-\frac{\alpha}{N-1} \end{pmatrix}$$

With calculations similar to the ones in Exercise 1 we get that the desired probability is

$$\frac{1}{N} + \left(1 - \frac{1}{N}\right)\left(1 - \frac{\alpha N}{N-1}\right).$$

**Exercise 3** (Some linear algebra). Consider the three-state Markov Chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Find a general formula for $p_{1,1}^{(n)}$.
**Solution:** Matrix $P$ is diagonalizable, i.e. there exists an invertible matrix $U$ and three eigenvalues $1, i/2, -i/2$ such that

$$P = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{i}{2} & 0 \\ 0 & 0 & -\frac{i}{2} \end{pmatrix} U^{-1}$$

It is not hard to see that

$$P^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{i}{2}\right)^n & 0 \\ 0 & 0 & \left(\frac{-i}{2}\right)^n \end{pmatrix} U^{-1}$$

Focusing our attention to $p_{1,1}^{(n)}$, it makes sense to rewrite it in the form

$$p_{1,1}^{(n)} = a + \frac{1}{2^n}\left[b\cos\frac{n\pi}{2} + c\sin\frac{n\pi}{2}\right]$$

Using the first three value of $p_{1,1}^{(n)}$ ($n = 0, 1, 2$) and solving for $a, b$ and $c$ we obtain $a = \frac{1}{5}$, $b = \frac{4}{5}$ and $c = -\frac{2}{5}$.

**Exercise 4.** Consider the Markov Chain described in Exercise 3. Is it ergodic? Does it admit stationary distribution?
**Solution:** The Markov Chain is finite, irreducible and aperiodic, therefore ergodic. The unique stationary distribution is $(1/5, 2/5, 2/5)$.

**Exercise 5** (Gambler's ruin)**.** Alice and Bob play the following game: they repeatedly roll a fair die and observe its outcome. If the result is head, then Alice pays one euro to Bob, otherwise the converse happens. Alice and Bob have initial budgets of $a$ and $b$, respectively, with $a > b$. The game stops when one of the two players loses, i.e. has no money left.

1. Model this problem as a Markov Chain. Specify the states and the transition matrix.

2. Is the chain aperiodic? Is it irreducible? Argue about its ergodicity.

3. What is the probability that Alice wins?

**Solution.** Consider $X_t$, i.e. the amount of money Alice owns at time $t$. Clearly the relative random process is a Markov Chain represented by the random walk on the undirected graph given by the line, where the states are numbered from 0 (Alice loses) to $a+b$ (Alice wins). The chain is periodic and not irreducible, in fact there is no way to exit the absorbing states 0 or $a+b$. Clearly the chain is not ergodic since the initial state influences the probability to land in one of the two absorbing states, i.e. Alice wins or Bob wins.

To solve the last point, consider $p_i$ as the probability that Alice wins given that it starts with a budget of $i$. One can write the following recursive equations:

$$\begin{cases} p_0 & = 0 \\ p_{a+b} & = 1 \\ p_i & = \frac{1}{2}(p_{i-1} + p_{i+1}), \quad \forall\, i \in \{1, \dots, a+b-1\} \end{cases}$$

Solving the equation one gets that the required probability is $p_a = \frac{a}{a+b}$.