Propositional logic: exercises

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1. The notion of theory

2. Exercises on propositional logic

Propositional theory

Definition (Propositional theory)

A theory is a set of formulas closed under the logical consequence relation, i.e. a set of formulas T is a theory iff $T \models A$ implies that $A \in T$, for every formula A.

Example (of theory)

- T_1 is the set of valid formulas $\{A \mid A \text{ is valid}\}$
- T_2 is the set of formulas which are true in the interpretation $\mathcal{I} = \{P, Q, R\}$
- T_3 is the set of formulas which are true in the set of interpretations $\{I_1, I_2, I_3\}$
- T_4 is the set of all formulas

It is easy to see that T_1, T_2, T_3, T_4 are theories.

Propositional theory

Example (of non theory)

- N_1 is the set $\{A, A \rightarrow B, C\}$
- N_2 is the set $\{A, A \rightarrow B, B, C\}$

Show that N_1 and N_2 are not theories

Axiomatization

Remark

A propositional theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas which are infinite (e.g., $A \rightarrow A$ for all formulas A).

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all formula A, $A \in T$ if and only if $\Omega \models A$.

Definition (Finitely axiomatizable theory)

A theory T is finitely axiomatizable if there exists a a finite set of axioms for it.

Important: the set of axioms of a theory is often itself called a theory. In other words, we often identify a theory with the set of its axioms.

Logical closure

Definition (Logical closure)

For any set Γ, the logical closure of Γ is defined as follows:

$$cl(\Gamma) = \{A \mid \Gamma \models A\}$$

Proposition

For any set Γ, the logical closure of Γ, $cl(\Gamma)$ is a theory.

Proposition

Γ is a set of axioms for $cl(\Gamma)$.

Axioms and theory - intuition

Compact representation of knowldge

The axiomatization of a theory is a compact way to represent a set of interpretations, in particular the set of models of the theory, and thus to represent a set of possible (acceptable) states of a domain of interest. In other words is a way to represent all the knowledge we have about the domain of interest.

Minimality and consistency

The axioms of a theory constitute the basic knowledge, and all the *generable knowledge* is obtained by logical implication. Two important features that a set of axioms may have are

- minimality: the set of axioms of a theory is minimal if no axioms can be derived from the others.
- satisfiability: the set of axioms of a theory should satisfiable, in which case the theory is said to be consistent; otherwise the theory does not have any model and is said inconsistent

Where is the mistake?

"C'è ancora chi crede che il rigore nei conti pubblici contraddica lo sviluppo dell'economia, ma anni di storia italiana con spesa facile e stagnazione economica ci insegnano esattamente il contrario."

The guy wants to refute

$$\alpha\text{: rigore} \rightarrow \neg\ \text{sviluppo}$$

and, towards this goal, he uses

$$\beta\text{: } \neg\ \text{rigore} \wedge \neg\ \text{sviluppo}$$

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This is a correct argument if and only if:

$$\beta \models \neg\alpha$$

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However,

$$\neg \text{ rigore} \wedge \neg \text{ sviluppo} \not\models \neg (\text{rigore} \rightarrow \neg \text{ sviluppo})$$

because the interpretation $\mathcal{I} = \emptyset$ is a model of the formula on the left that makes (rigore $\rightarrow \neg$ sviluppo) true, and therefore is not a model of the formula on the right.

We conclude that the argument is completely wrong.

Where is the mistake?

"C'è ancora chi crede che il rigore nei conti pubblici contraddica lo sviluppo dell'economia, ma anni di storia italiana con spesa facile e stagnazione economica ci insegnano esattamente il contrario."

At this point, it is interesting to understand which is a formula that

$$\beta\text{: } \neg \text{ rigore} \wedge \neg \text{ sviluppo}$$

refutes, i.e., a formula α such that $\beta \models \neg\alpha$.

Where is the mistake?

"C'è ancora chi crede che il rigore nei conti pubblici contraddica lo sviluppo dell'economia, ma anni di storia italiana con spesa facile e stagnazione economica ci insegnano esattamente il contrario."

It is interesting to understand which is a formula that

$$\beta\text{: } \neg \text{ rigore} \wedge \neg \text{ sviluppo}$$

refutes. Since β is equivalent to $\neg$ (rigore $\vee$ sviluppo), we have that

$$\neg \text{ rigore} \wedge \neg \text{ sviluppo} \models \neg \text{ (sviluppo} \vee \text{rigore)}$$

and, since (sviluppo $\vee$ rigore) is equivalent to ($\neg$ sviluppo $\rightarrow$ rigore), we can conclude that

$$\beta\text{: } \neg \text{ rigore} \wedge \neg \text{ sviluppo}$$

refutes

$$\gamma\text{: } \neg \text{ sviluppo} \rightarrow \text{rigore}$$

which is exactly the inverse implication of

$$\alpha\text{: rigore} \rightarrow \neg \text{ sviluppo}$$

Where is the mistake?

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Finally, it would be interesting to understand which is a formula that correctly refutes

$$\alpha\text{: rigore} \rightarrow \neg\text{ sviluppo}$$

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Finally, it would be interesting to understand which is a formula that correctly refutes

$$\alpha\text{: rigore} \rightarrow \neg \text{ sviluppo}$$

Abduction can be used for this purpose:

Since (rigore $\wedge$ sviluppo) can be abduced from (or, is an abductive explanation of)

$$\neg\,(\text{rigore} \rightarrow \neg \text{ sviluppo})$$

we conclude that

(rigore $\wedge$ sviluppo) correctly refutes (rigore $\rightarrow \neg$ sviluppo)

A famous game

Un mazzo di carte in cui ogni carta ha una lettera in una faccia ed un numero dall'altra si dice "equilibrato" se ogni carta che ha una vocale in una faccia ha un numero pari nell'altra faccia. Sia M un mazzo di 4 carte, di ciascuna delle quali ci viene mostrata una faccia, nel modo seguente:

posizione della carta	1	2	3	4
scritta sulla faccia	A	6	K	7

Qual è il numero minimo di carte che, qualunque sia la configurazione di M, ci permette di verificare che M sia un mazzo equilibrato?

A famous game: solution 1

Un mazzo di carte in cui ogni carta ha una lettera in una faccia ed un numero dall'altra si dice "equilibrato" se ogni carta che ha una vocale in una faccia ha un numero pari nell'altra faccia. Sia M un mazzo di 4 carte, di ciascuna delle quali ci viene mostrata una faccia, nel modo seguente:

posizione della carta	1	2	3	4
scritta sulla faccia	A	6	K	7

Qual è il numero minimo di carte che dobbiamo girare affinchè sia possibile, qualunque sia la configurazione di M, verificare che M sia un mazzo equilibrato?

Soluzione 1: formalizzare le regole che stabiliscono che un mazzo è equilibrato mediante una teoria proposizionale F in modo che ogni mazzo sia una interpretazione; determinare poi il numero minimo di variabili proposizionali il cui valore di verità deve essere determinato per verificare che il mazzo M del gioco sia un modello di F.

A famous game: solution 1

Propositional variables: v_1, v_2, v_3, v_4 (vocali), and p_1, p_2, p_3, p_4 (pari)
Axioms of F: $\{v_1 \rightarrow p_1, v_2 \rightarrow p_2, v_3 \rightarrow p_3, v_4 \rightarrow p_4\}$
Partial knowledge on M: $M(v_1) =$ True, $M(p_2) =$ True, $M(v_3) =$ False, $M(p_4) =$ False
Obiettivo: determinare qual è il numero minimo di variabili proposizionali il cui valore di verità deve essere appurato per verificare che la conoscenza parziale su M possa essere estesa ad una intepretazione completa che sia un modello di F

i	card	v_i	p_i	$v_i \rightarrow p_i$
1	A	True	?	?
2	6	?	True	True
3	K	False	?	True
4	7	?	False	?

Ne segue che dobbiamo girare due carte: 1 e 4.

A famous game: solution 2

Un mazzo di carte in cui ogni carta ha una lettera in una faccia ed un numero dall'altra si dice "equilibrato" se ogni carta che ha una vocale in una faccia ha un numero pari nell'altra faccia. Sia M un mazzo di 4 carte, di ciascuna delle quali ci viene mostrata una faccia, nel modo seguente:

posizione della carta	1	2	3	4
scritta sulla faccia	A	6	K	7

Qual è il numero minimo di carte da girare per verificare che M sia un mazzo equilibrato?

Soluzione 2: formalizzare le regole che stabiliscono che un mazzo è equilibrato e la conoscenza sulle 4 carte del mazzo M mediante una teoria proposizionale F; determinare poi il numero minimo di literal (lettera proposizionale o il negato di una lettera proposizionale) da aggiungere ad F per giungere ad una teoria che rappresenti il mazzo e sia soddisfacibile.

A famous game: solution 2

Propositional variables: v_1, v_2, v_3, v_4 (vocali), and p_1, p_2, p_3, p_4 (pari)
Axioms of F: $\{v_1 \rightarrow p_1, v_2 \rightarrow p_2, v_3 \rightarrow p_3, v_4 \rightarrow p_4, v_1, p_2, \neg v_3, \neg p_4\}$
Obiettivo: determinare il minimo numero di literal da aggiungere ad F per verificare la soddisfacibilità dell'insieme risultante di assiomi.

- $F \models p_1$ and therefore $F \cup \{\neg p_1\}$ is unsatisfiable
- $F \models \neg v_4$ and therefore $F \cup \{v_4\}$ is unsatisfiable
- $F \not\models v_2 \vee p_3$ and therefore $F \cup \{\neg v_2, \neg p_3\}$ is satisfiable
- $F \not\models v_2 \vee \neg p_3$ and therefore $F \cup \{\neg v_2, p_3\}$ is satisfiable
- $F \not\models \neg v_2 \vee p_3$ and therefore $F \cup \{v_2, \neg p_3\}$ is satisfiable
- $F \not\models \neg v_2 \vee \neg p_3$ and therefore $F \cup \{v_2, p_3\}$ is satisfiable

Ne segue che dobbiamo girare due carte: 1 e 4.

Three perfect logicians at the pub

A perfect logician answers perfectly to every "boolean question" Q, with

- "yes" (if her/his knowledge implies Q true)
- "no" (if her/his knowledge implies Q false)
- "I do not know" (if her/his knowledge implies neither Q true, nor Q false)

as possible answers.

Three perfect logicians who have never met with each other enter into a pub and sit at the only table with free seats (exactly three seats). The waiter goes to the table and, since he is very busy, while cleaning the table, decides to ask collectively the following question: "Everybody at this table wants a beer?" The three logicians answer one by one, and after hearing the answers, the waiter brings exactly one beer at the table.

The question for you is: Is the waiter a perfect logician?

Three perfect logicians at the pub: solution

Three perfect logicians who have never met with each other enter into a pub and sit at the only table with free seats (exactly three seats). The waiter goes to the table and, since he is very busy, while cleaning the table, decides to ask collectively the following question: "Everybody at this table wants a beer?" The three logicians answer one by one, and after hearing the answers, the waiter brings exactly one beer at the table. Is the waiter a perfect logician?

We formalize the puzzle in terms of a theory (set of axioms) representing the knowledge of the waiter. The alphabet of the theory is simply constituted by the propositional letters b_1, b_2, b_3, b_4, where b_i means that logician i wants a beer.

- The initial theory K_0 is empty, because the waiter knows nothing.
- After the answer by logician i ($i = 1, 2, 3$) the theory changes and the new theory will be denoted by K_i. Thus, K_3 is the final theory.
- The waiter is a perfect logician if and only if, no matter what the three answers are,

$$K_3 \models (b_1 \wedge \neg b_2 \wedge \neg b_3) \vee (\neg b_1 \wedge b_2 \wedge \neg b_3) \vee (\neg b_1 \wedge \neg b_2 \wedge b_3)$$

Three perfect logicians at the pub: solution

Initially $K_0 = \{\}$; answers of the 1st logician:

- "yes" $\rightsquigarrow$ impossible (logician 1 doesn't know about the others)
- "no" $\rightsquigarrow$ all future answers will be "no", $K_1 = K_2 = K_3 = \{\neg b_1\}$ possibile outcomes are 0, 1 or 2 beers and the waiter should ask for more information
- "I do not know" $\rightsquigarrow$ $K_1 = \{b_1\}$; answers of the 2nd logician:
  - "yes" $\rightsquigarrow$ impossible (logician 2 doesn't know about logician 3)
  - "no" the third answer will be "no", $\rightsquigarrow$ $K_2 = K_3 = \{b_1, \neg b_2\}$, possibile outcomes are 1 or 2 beers and the waiter should ask for more information
  - "I do not know" $\rightsquigarrow$ $K_2 = \{b_1, b_2\}$, answers of the 3rd logician
    - "yes" $\rightsquigarrow$ $K_3 = \{b_1, b_2, b_3\}$, the waiter should bring 3 beers
    - "no" $\rightsquigarrow$ $K_3 = \{b_1, b_2, \neg b_3\}$, the waiter should bring 2 beers
    - "I do not know" $\rightsquigarrow$ impossible (logician 3 knows everything)

Conclusion: the waiter is not a perfect logician, because in all cases

$$K_3 \not\models (b_1 \wedge \neg b_2 \wedge \neg b_3) \vee (\neg b_1 \wedge b_2 \wedge \neg b_3) \vee (\neg b_1 \wedge \neg b_2 \wedge b_3)$$

Formalizing English sentences

Exercise

Let $A =$"**Angelo comes to the party**", $B =$"**Bruno comes to the party**", $C =$"**Carlo comes to the party**", and $D =$"**Davide comes to the party**". Formalize the following sentences:

1. *"If Davide comes to the party, then Bruno and Carlo come too"*
2. *"Carlo comes to the party only if both Angelo and Bruno do not come"*
3. *"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"*
4. *"Carlo comes to the party provided that Davide doesn't come, but, if Davide doesn't come, then Bruno doesn't come"*
5. *"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"*
6. *"Angelo, Bruno and Carlo all come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"*

Formalizing English sentences

Exercise - Solution

1. *"If Davide comes to the party then Bruno and Carlo come too"*
   $D \rightarrow B \wedge C$
2. *"Carlo comes to the party only if Angelo and Bruno do not come"*
   $C \rightarrow \neg A \wedge \neg B$
3. *"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"*
   $D \rightarrow (\neg C \rightarrow A)$, which is equivalent to $(D \wedge \neg C) \rightarrow A$

Formalizing English sentences

Exercise - Solution

1. *"Carlo comes to the party provided that Davide doesn't come, but, if Davide doesn't come, then Bruno doesn't come"*
   $(C \rightarrow \neg D) \wedge (\neg D \rightarrow \neg B)$
2. *"A necessary condition for Angelo coming to the party, is that, if both Bruno and Carlo don't come, Davide comes"*
   $A \rightarrow (\neg B \wedge \neg C \rightarrow D)$
3. *"Angelo, Bruno and Carlo all come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"*
   $(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \rightarrow C))$

Formalizing English sentences

Exercise

Formalize the following arguments and verify whether they are correct:

- *"If you play and you study you'll pass the exams.*

  If you play and don't study you won't pass.

  Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."

Formalizing English sentences

Exercise

- *"If you play and you study you'll pass the exams.*

  If you play and don't study you won't pass.

  Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."

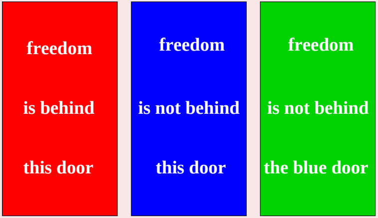
- 1. $p \wedge s \rightarrow e$
  2. $p \wedge \neg s \rightarrow \neg e$
  3. $p \rightarrow (s \wedge e) \vee (\neg s \wedge \neg e)$

  We need to prove that $(1) \wedge (2) \models (3)$

The 3 doors

Problem

Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon. After a quick search the boys find three doors, the first one red, the second one blue, and the third one green.
Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means certain death. On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

The 3 doors: solution

Language

- r: *"freedom is behind the red door"*
- b: *"freedom is behind the blue door"*
- g: *"freedom is behind the green door"*

Axioms

1. *"behind one of the door is a path to freedom, behind the other two doors is an evil dragon"*
   $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. *"at least one of the three statements is true"*
   $r \vee \neg b$
3. *"at least one of the three statements is false"*
   $\neg r \vee b$

The 3 doors: solution

Axioms

1. $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. $r \vee \neg b$
3. $\neg r \vee b$

Solution

r	b	g	1	2	3	$2 \wedge 3$	$1 \wedge 2 \wedge 3$
False	False	False	False	True	True	True	False
False	False	True	True	True	True	True	True
False	True	False	True	False	True	False	False
False	True	True	False	False	True	False	False
True	False	False	True	True	False	False	False
True	False	True	False	True	False	False	False
True	True	False	False	True	True	True	False
True	True	True	False	True	True	True	False

Since $\{1, 2, 3\} \models g$, *freedom is behind the* **green door!**

Graph coloring

Problem

Provide the formalization in propositional logic of the graph coloring problem.

- Graph coloring problem: given a non-oriented graph with $n > 0$ nodes, check whether the graph is k-colorable, i.e., whether we can associate one of the $k > 0$ colors to each of its nodes in such a way that no pair of adjacent nodes have the same color.

Graph coloring

Problem

Provide the formalization in propositional logic of the graph coloring problem.

- Graph coloring problem: given a non-oriented graph G with $n > 0$ nodes, check whether the graph is k-colorable, i.e., whether we can associate one of the $k > 0$ colors to each of its nodes in such a way that no pair of adjacent nodes have the same color.

Soluzione:

formalizzare il problema mediante una teoria proposizionale T_G i cui assiomi siano soddisfacibili se e solo se il grafo G è colorabile.

Graph coloring: propositional formalization

Language of theory T_G

- For each $1 \leq i \leq n$ and $1 \leq c \leq k$, color_{ic} is a proposition, which intuitively means that *"the i-th node of G has the c color"*
- For each $1 \leq i \neq j \leq n$, edge_{ij} is a proposition, which intuitively means that *"the i-th node is connected in G with the j-th node"*.

Axioms of theory T_G

- for each $1 \leq i \leq n$, $\bigvee_{c=1}^{k} \text{color}_{ic}$
  "each node has at least one color"
- for each $1 \leq i \leq n$ and $1 \leq c \neq c' \leq k$, $\text{color}_{ic} \rightarrow \neg\text{color}_{ic'}$
  "every node has at most 1 color"
- for each $1 \leq i, j \leq n$ and $1 \leq c \leq k$, $\text{edge}_{ij} \rightarrow \neg(\text{color}_{ic} \wedge \text{color}_{jc})$
  "adjacent nodes do not have the same color"

It can be shown that every solution to the graph coloring problem for G corresponds to a model of T_G, and, conversely, every model of T_G corresponds to a solution of the graph coloring problem for G.

Sudoku

Problem

Provide a formalization of the Sudoku game. Sudoku is a placement puzzle. The aim of the puzzle is to enter a numeral from 1 through 9 in each cell of a grid, here a 9×9 grid made up of 3×3 subgrids (called "regions"), starting with various numerals given in some cells (the "givens"). Each row, column and region must contain only one instance of each numeral. Its grid layout is like the one shown in the following schema

		9				7		
	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

Sudoku: solution

Problem

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	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

We provide a formalization of the Sudoku game as a propositional theory, so that any truth assignment to the propositional variables that satisfy the axioms of the theory is a solution for the puzzle, and, conversely, every solution for the puzzle corresponds to a truth assignment to the propositional variables that satisfy the axioms of the theory.

Sudoku: solution

Language of theory T_S

For $1 \leq n, r, c \leq 9$,define the proposition

$$in(n, r, c)$$

which means that the number n has been inserted in the cross between row r and column c.

Sudoku: solution

Axioms of theory T_S

1. *"Every number from 1 to 9 appears in every row"*

$$\bigwedge_{n=1}^{9}\left(\bigwedge_{r=1}^{9}\left(\bigvee_{c=1}^{9}\mathbf{in}(n,r,c)\right)\right)$$

2. *"Every number from 1 to 9 appears in every column"*

$$\bigwedge_{n=1}^{9}\left(\bigwedge_{c=1}^{9}\left(\bigvee_{r=1}^{9}\mathbf{in}(n,r,c)\right)\right)$$

3. *"Every numbers from 1 to 9 appears in every region (sub-grid)"*

$$\textbf{for any} \quad 0 \leq k,h \leq 2 \qquad \bigwedge_{n=1}^{9}\left(\bigvee_{r=1}^{3}\left(\bigvee_{c=1}^{3}\mathbf{in}(n,3*k+r,3*h+c)\right)\right)$$

4. *"No cell can contain more than one number"*

$$\textbf{for any} \quad 1 \leq n,n',c,r \leq 9 \textbf{ and } n \neq n' \qquad \mathbf{in}(n,r,c) \rightarrow \neg\mathbf{in}(n',r,c)$$

5. *"the givens"* represented by appropriate $\mathbf{in}(n,r,c)$