# Epipolar Geometry

# Two-view geometry

Epipolar geometry

3D reconstruction

F-matrix comp.

Structure comp.

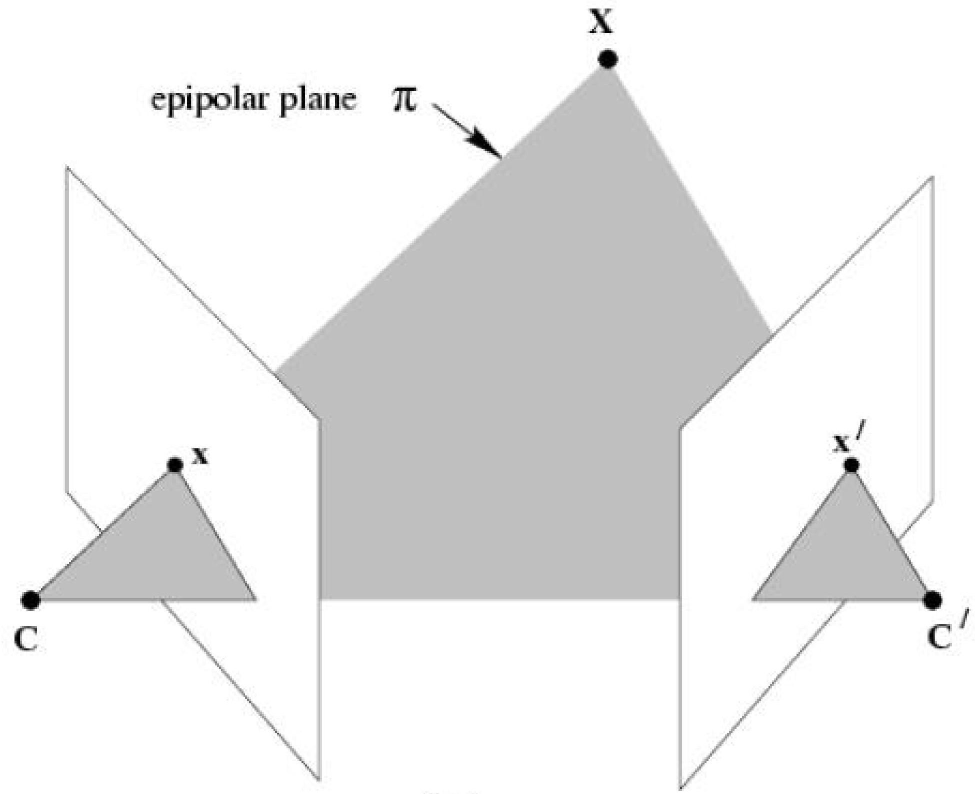
# Three questions:

(i) **Correspondence geometry:** Given an image point $x$ in the first view, how does this constrain the position of the corresponding point $x'$ in the second image?

(ii) **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, $i=1,\ldots,n$, what are the cameras P and P' for the two views?

(iii) **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

# The epipolar geometry



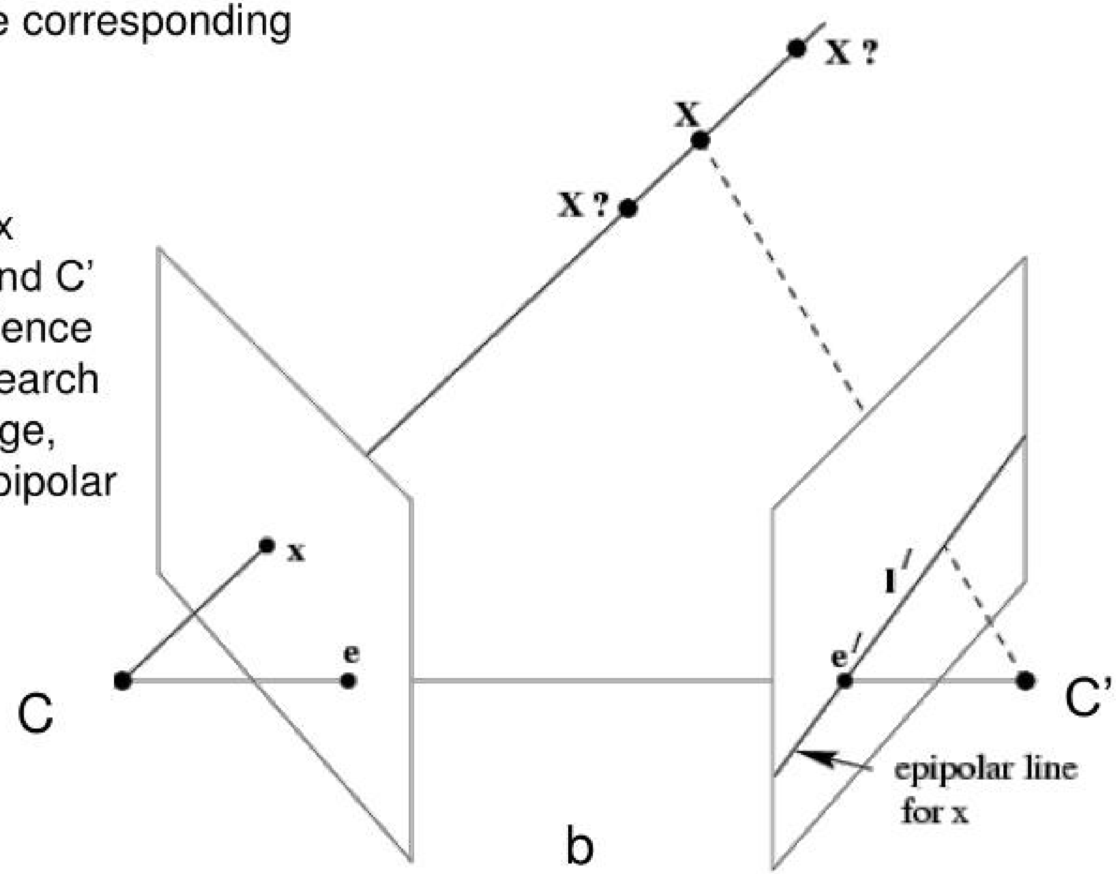

(a)

C,C',x,x' and X are coplanar

Fig. 9.1. **Point correspondence geometry.** *(a) The two cameras are indicated by their centres* $\mathbf{C}$ *and* $\mathbf{C}'$ *and image planes. The camera centres, 3-space point* $\mathbf{X}$, *and its images* $\mathbf{x}$ *and* $\mathbf{x}'$ *lie in a common plane* $\pi$. *(b) An image point* $\mathbf{x}$ *back-projects to a ray in 3-space defined by the first camera centre,* $\mathbf{C}$, *and* $\mathbf{x}$. *This ray is imaged as a line* $\mathbf{l}'$ *in the second view. The 3-space point* $\mathbf{X}$ *which projects to* $\mathbf{x}$ *must lie on this ray, so the image of* $\mathbf{X}$ *in the second view must lie on* $\mathbf{l}'$.

# The epipolar geometry

- If we know x, how is the corresponding point x' constrained?
- l' is the Epipolar line corresponding to point x
- Upshot: if we know C and C' for a stereo correspondence algorithm, no need to search all over the second image, but just only over the epipolar line.

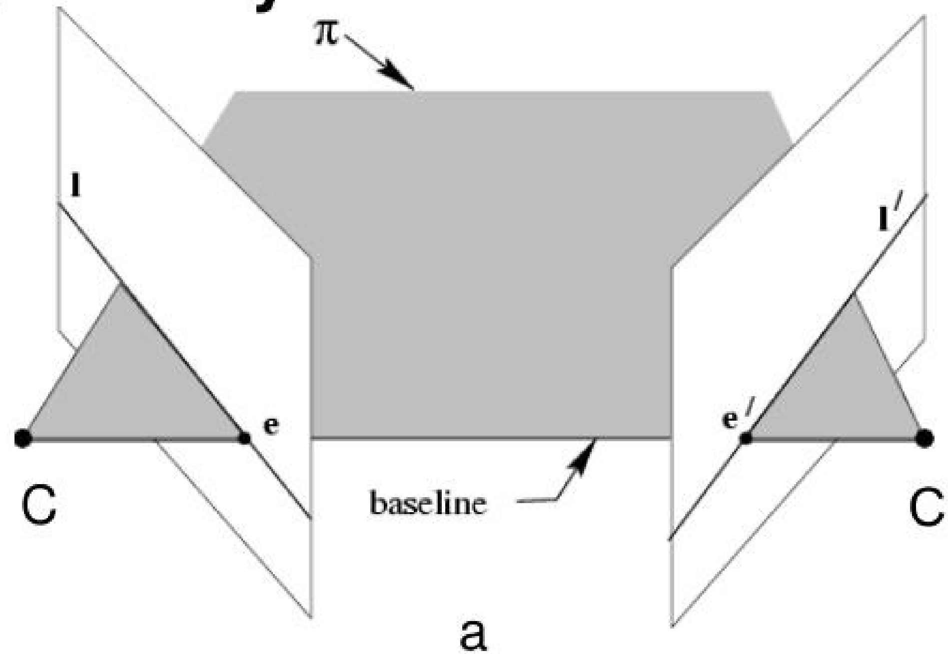



What if only C,C',x are known?

Fig. 9.1. **Point correspondence geometry.** *(a) The two cameras are indicated by their centres* **C** *and* **C'** *and image planes. The camera centres, 3-space point* **X**, *and its images* **x** *and* **x'** *lie in a common plane* $\pi$. *(b) An image point* **x** *back-projects to a ray in 3-space defined by the first camera centre,* **C**, *and* **x**. *This ray is imaged as a line* **l'** *in the second view. The 3-space point* **X** *which projects to* **x** *must lie on this ray, so the image of* **X** *in the second view must lie on* **l'**.

# The epipolar geometry

- Baseline: connects two camera centers
- Epipole: point of intersection of baseline with image plane
- Epipole: image in one view of the camera center of the other view.

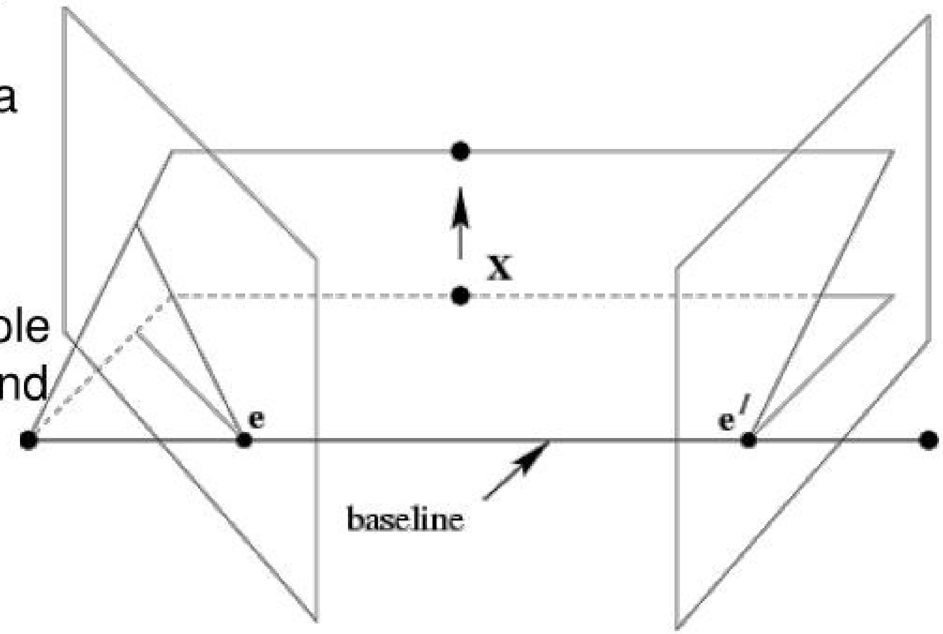



All points on π project on l and l'

Fig. 9.2. **Epipolar geometry.** *(a) The camera baseline intersects each image plane at the epipoles* **e** *and* **e**′. *Any plane π containing the baseline is an epipolar plane, and intersects the image planes in corresponding epipolar lines* **l** *and* **l**′. *(b) As the position of the 3D point* **X** *varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.*

# The epipolar geometry

- Epipolar plane: A plane containing the baseline.
- There is a one parameter family , or a pencil, of epipolar planes
- Epipolar line is the intersection of an epipolar plane with the image plane
- All epipolar lines intersect at the epipole
- An epipolar plane intersects the left and right image planes in epipolar lines, and defines the correspondence between the lines.



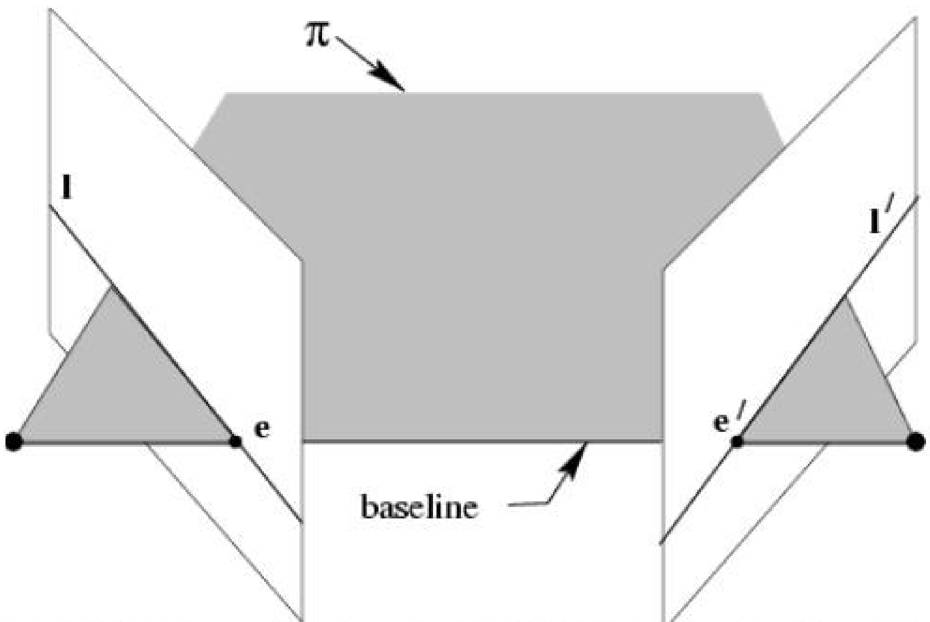

b

Family of planes $\pi$ and lines l and l'
Intersection in e and e'

Fig. 9.2. **Epipolar geometry.** *(a) The camera baseline intersects each image plane at the epipoles* **e** *and* **e**$'$. *Any plane* $\pi$ *containing the baseline is an epipolar plane, and intersects the image planes in corresponding epipolar lines* **l** *and* **l**$'$. *(b) As the position of the 3D point* **X** *varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.*

# The epipolar geometry

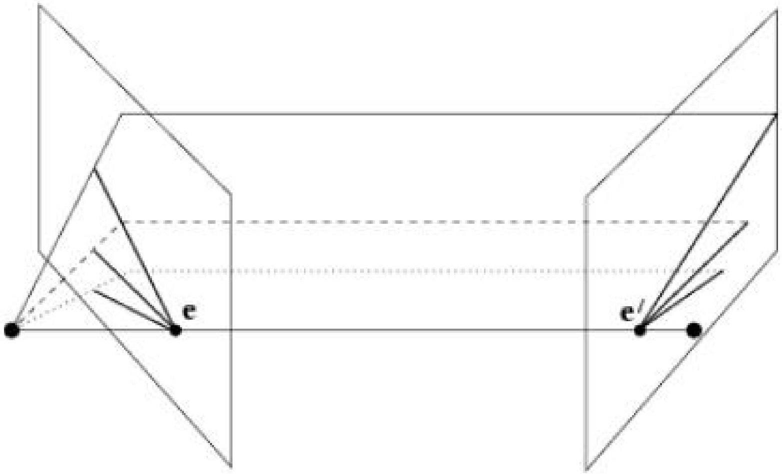
epipoles e,e'
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction




an epipolar plane = plane containing baseline (1-D family)

an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)

# Example: converging cameras



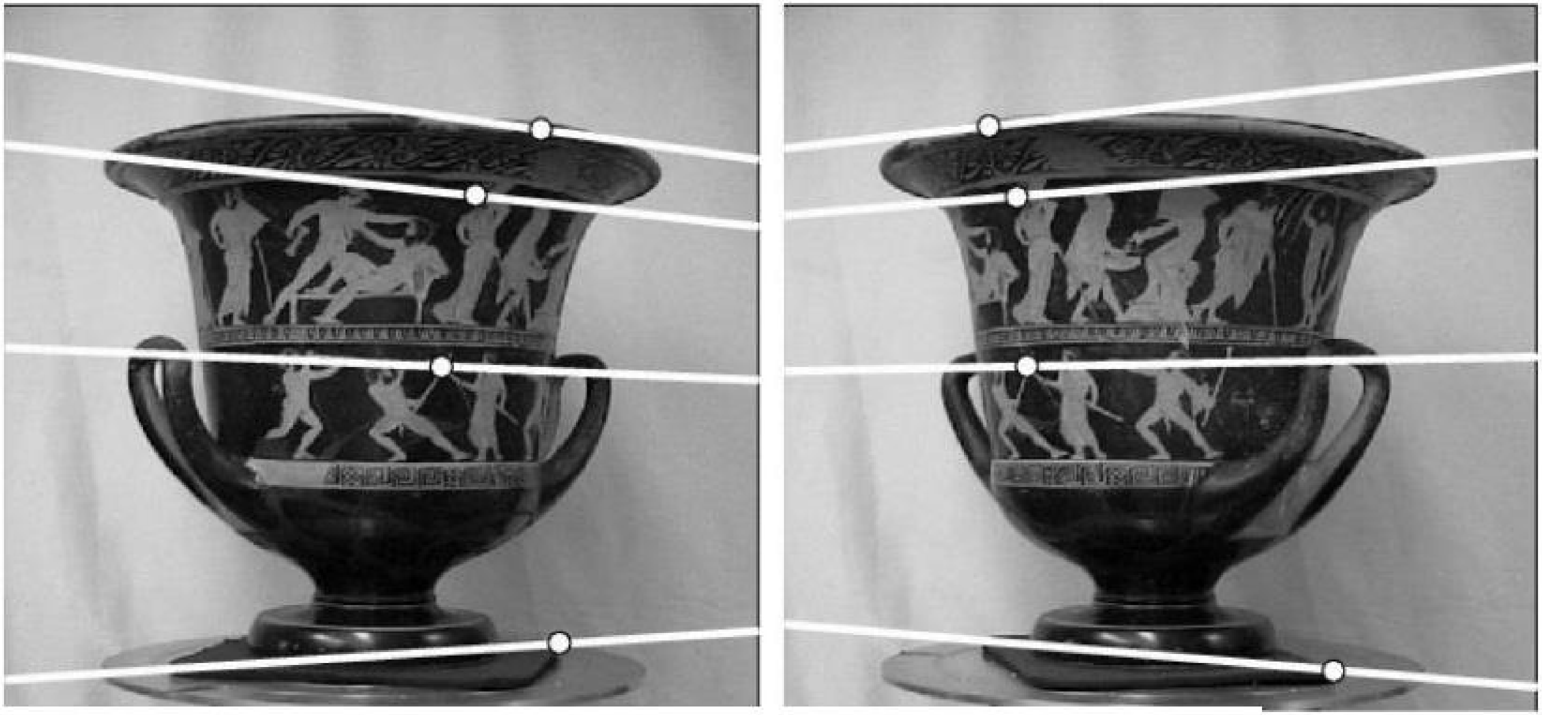



Fig. 9.3. **Converging cameras.** *(a) Epipolar geometry for converging cameras. (b) and (c) A pair of images with superimposed corresponding points and their epipolar lines (in white). The motion between the views is a translation and rotation. In each image, the direction of the other camera may be inferred from the intersection of the pencil of epipolar lines. In this case, both epipoles lie outside of the visible image.*

# Example: motion parallel with image plane

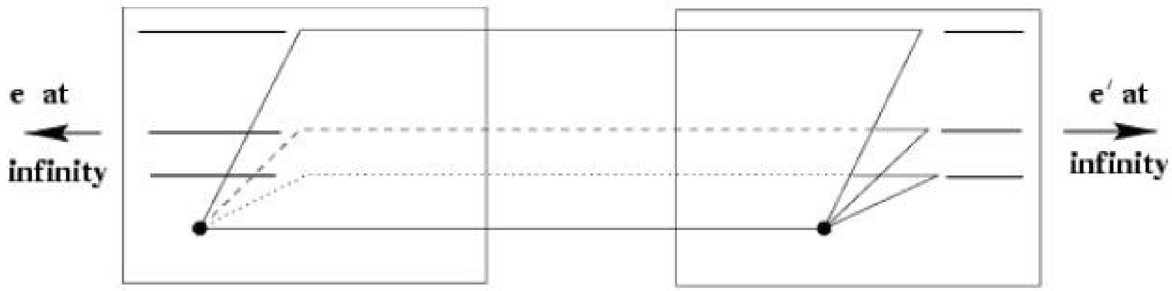



Fig. 9.4. **Motion parallel to the image plane.** *In the case of a special motion where the translation is parallel to the image plane, and the rotation axis is perpendicular to the image plane, the intersection of the baseline with the image plane is at infinity. Consequently the epipoles are at infinity, and epipolar lines are parallel. (a) Epipolar geometry for motion parallel to the image plane. (b) and (c) a pair of images for which the motion between views is (approximately) a translation parallel to the x-axis, with no rotation. Four corresponding epipolar lines are superimposed in white. Note that corresponding points lie on corresponding epipolar lines.*

# Fundamental Matrix

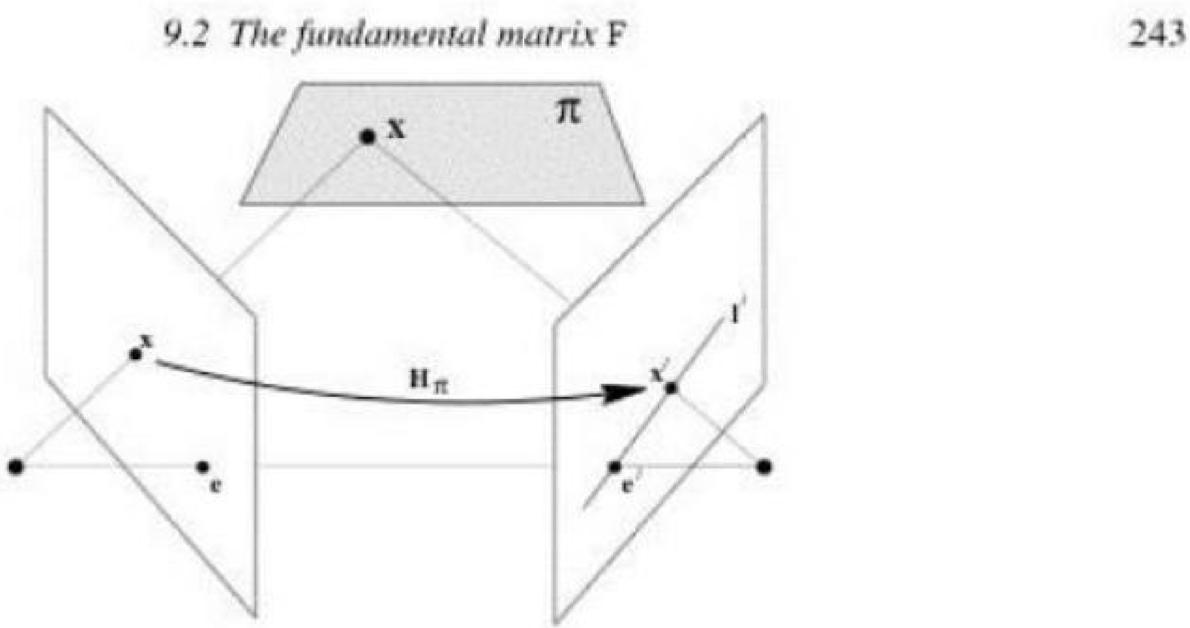



Fig. 9.5. *A point* $\mathbf{x}$ *in one image is transferred via the plane* $\pi$ *to a matching point* $\mathbf{x}'$ *in the second image. The epipolar line through* $\mathbf{x}'$ *is obtained by joining* $\mathbf{x}'$ *to the epipole* $\mathbf{e}'$*. In symbols one may write* $\mathbf{x}' = H_\pi \mathbf{x}$ *and* $\mathbf{l}' = [\mathbf{e}']_\times \mathbf{x}' = [\mathbf{e}']_\times H_\pi \mathbf{x} = F\mathbf{x}$ *where* $F = [\mathbf{e}']_\times H_\pi$ *is the fundamental matrix.*

F is a projective mapping x → l' from a point x in one image to its Corresponding epipolar line in the other image

l' = Fx

# The fundamental matrix F

algebraic representation of epipolar geometry

$$\mathrm{x} \mapsto \mathrm{l}'$$

we will see that mapping is (singular) correlation
(i.e. projective mapping from points to lines)
represented by the fundamental matrix F

# Skew Symmetric Matrix for a vector a

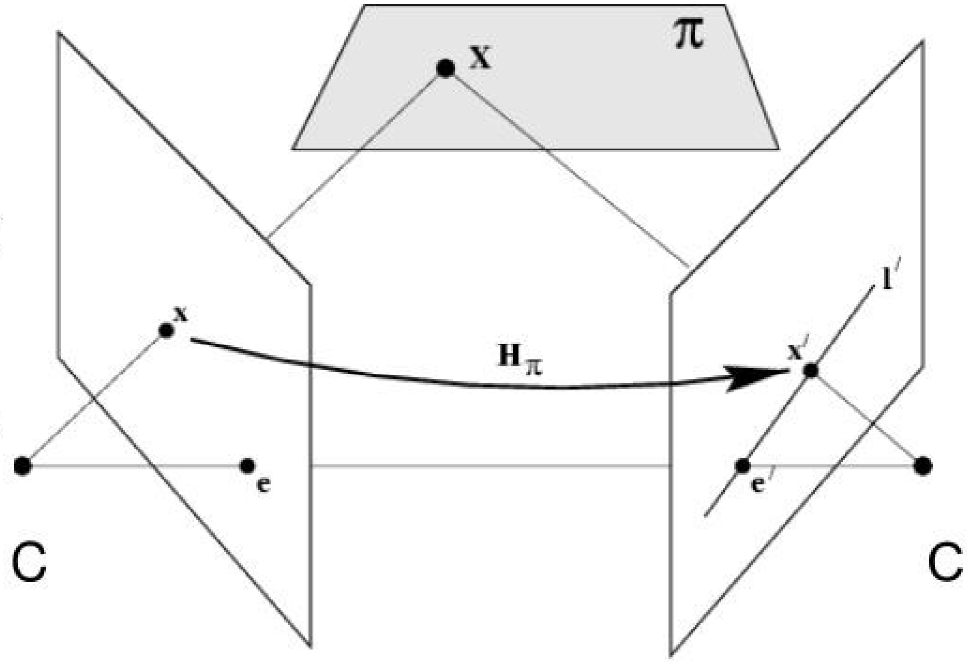
- $[a]_x$ is skew symmetric matrix for vector a
- If $a = (a_1 , a_2 , a_3 )^T$ then,

$$[a]_x = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Cross product between two vectors a and be can be written in terms of skew symmetric matrix for a:

$$axb = [a]_x\, b$$

# The fundamental matrix F
## geometric derivation




- Plane π, not passing through either of the camera centers
- Ray through C corresponding to image point x, meets plane π in a point in 3D called X.
- Project X to a point x' in the second image
- "Transfer via the plane π".
- l' is the epipolar line for x ➔ x' must like on l'

$$l' = e' \times x'$$

- x and x' are projectively equivalent to the planar point set $X_i$
- There is a 2D homography $H_\pi$ mapping each $x_i$ to $x'_i$

$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

mapping from 2-D to 1-D family (rank 2)

**Result 9.1.** *The fundamental matrix* F *may be written as* $F = [e']_\times H_\pi$, *where* $H_\pi$ *is the transfer mapping from one image to another via any plane* $\pi$. *Furthermore, since* $[e']_\times$ *has rank 2 and* $H_\pi$ *rank 3,* F *is a matrix of rank 2.*

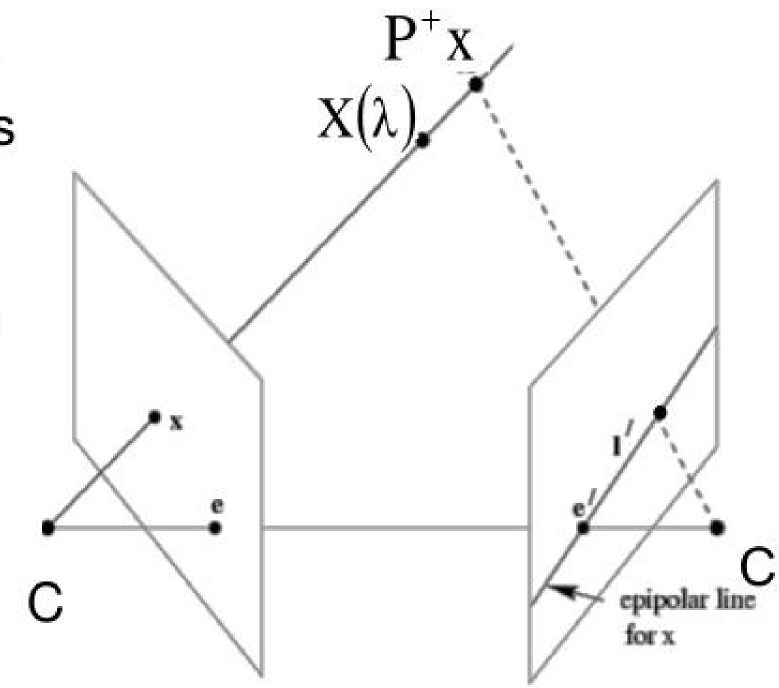
# The fundamental matrix F

## algebraic derivation

$P^+$ is pseudo inverse of P

$$\left(P^+P = I\right)$$

$$l' = P'C \times P'P^+x$$




- Line l' joints two points: can be written as cross product of those two points:
- First point is P'C which is e'
- Second point is projection P' of $P^+x$ onto second image plane

l' = e' cross product with ( P' $P^+$ x )

$$F = [e']_\times P'P^+$$

(note: doesn't work for C=C' $\Rightarrow$ F=0)

# The fundamental matrix F

## correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

Combine these two: $\left(x'^T l' = 0\right)$ $l' = Fx$

$$x'^T Fx = 0$$

**Result 9.3.** *The fundamental matrix satisfies the condition that for any pair of corresponding points* $\mathbf{x} \leftrightarrow \mathbf{x}'$ *in the two images*

$$\mathbf{x}'^T F \mathbf{x} = 0.$$

- Upshot: A way of characterizing fundamental matrix without reference to camera matrices, i.e. only in terms of corresponding image points
- How many correspondences are needed find F? at least 7.

# The fundamental matrix F

F is the unique 3x3 rank 2 matrix that satisfies $x'^T F x = 0$ for all $x \leftrightarrow x'$

(i) **Transpose:** if F is fundamental matrix for (P,P'), then $F^T$ is fundamental matrix for (P',P)

(ii) **Epipolar lines:** for any point x in the first image, the corresponding epipolar line is $l' = Fx$ ; same with converse: $l = F^T x'$ represents the epipolar line corresponding to x' in the second image

(i) **Epipoles:** for any point x, the epipolar line $l' = Fx$ contains the epipole e'. Thus $e'^T F x = 0, \forall x \Rightarrow e'^T F = 0$; similarly $Fe = 0$
e' is the left null vector of F; e is the right null vector of F

(i) **F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)

(ii) **F** is a correlation, projective mapping from a point x to a line $l' = Fx$ (not a proper correlation, i.e. not invertible)
If l and l' are corresponding epipolar lines, then any point x on l is mapped to the same line l' ➔ no inverse mapping ➔ F not proper correlation

# Epipolar Line Homography

9.3 Fundamental matrices arising from special motions 247

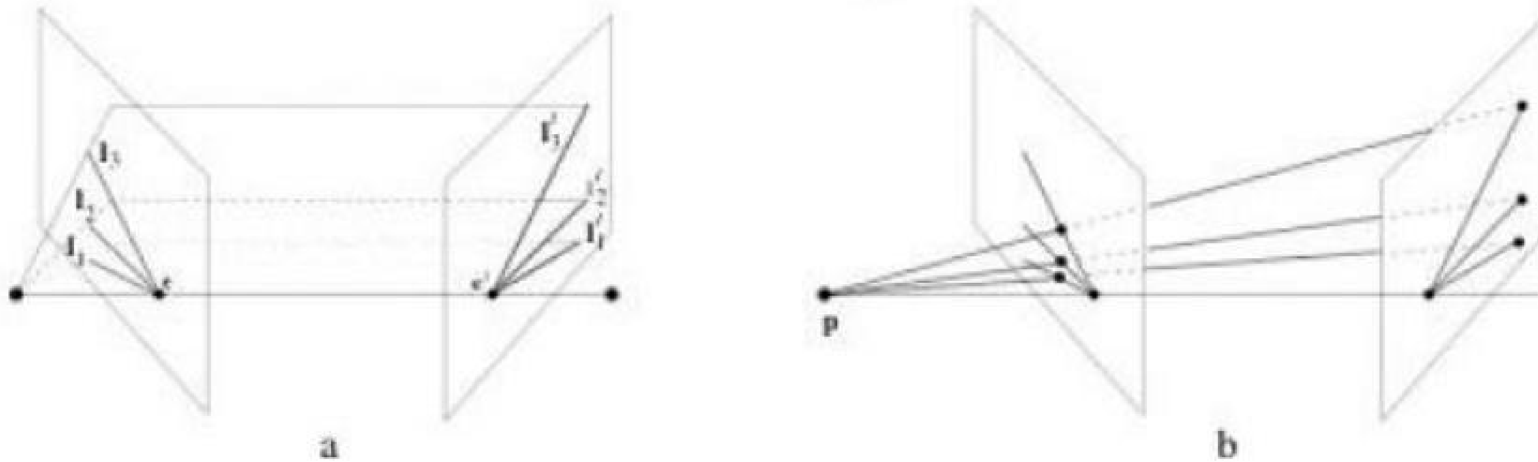
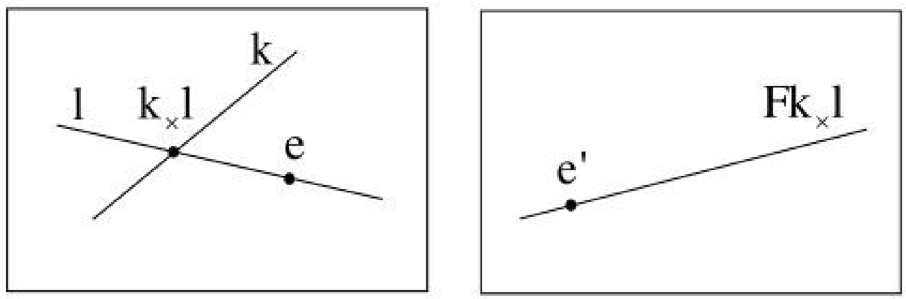



Fig. 9.6. **Epipolar line homography.** *(a) There is a pencil of epipolar lines in each image centred on the epipole. The correspondence between epipolar lines,* $\mathbf{l}_i \leftrightarrow \mathbf{l}'_i$, *is defined by the pencil of planes with axis the baseline. (b) The corresponding lines are related by a perspectivity with centre any point* $\mathbf{p}$ *on the baseline. It follows that the correspondence between epipolar lines in the pencils is a 1D homography.*

- Set of epipolar lines in each of the images forms a pencil of lines passing through the epipole
- Such a pencil of lines is a 1D projective space
- Epipolar lines are perspectively related
- There is a homography between epipolar lines centered at e in the first view and the pencil centered at e' in the second .
- A homography between such 1D projective spaces as 3 degrees of freedom
- Count degrees of freedom of fundamental matrix: 2 for e, 2 for e', 3 for 1D homography ➔ total of 7

# The epipolar line homography

$l, l'$ epipolar lines, $k$ line not through $e$
$\Rightarrow l' = F[k]_\times l$ and symmetrically $l = F^T[k']_\times l'$




(pick $k=e$, since $e^T e \neq 0$)

$$\mathbf{l}' = F[\mathbf{e}]_\times \mathbf{l} \qquad \mathbf{l} = F^T[\mathbf{e}']_\times \mathbf{l}'$$

Epipolar Line Homography

**Result 9.5.** *Suppose* $\mathbf{l}$ *and* $\mathbf{l}'$ *are corresponding epipolar lines, and* $\mathbf{k}$ *is any line not passing through the epipole* $\mathbf{e}$*, then* $\mathbf{l}$ *and* $\mathbf{l}'$ *are related by* $\mathbf{l}' = \mathtt{F}[\mathbf{k}]_\times \mathbf{l}$*. Symmetrically,* $\mathbf{l} = \mathtt{F}^\mathsf{T}[\mathbf{k}']_\times \mathbf{l}'$.

# Pure Translation camera motion

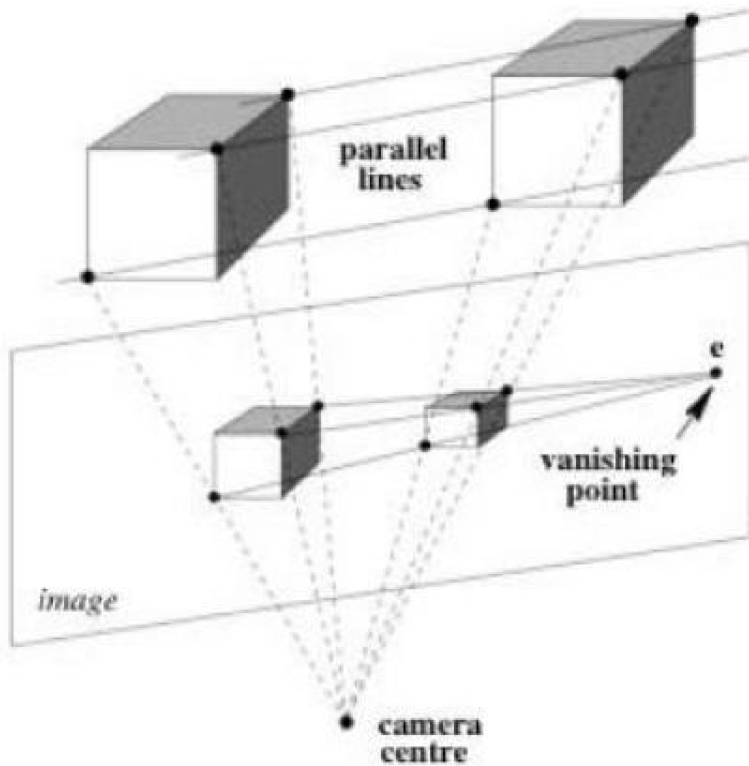



Fig. 9.7. *Under a pure translational camera motion, 3D points appear to slide along parallel rails. The images of these parallel lines intersect in a vanishing point corresponding to the translation direction. The epipole* **e** *is the vanishing point.*

- Pure translation = no rotation; no change in internal parameters
- Pure translation of camera is equivalent to camera is stationary and the world undergoes a translation –t;
- Points in 3 space move on straight lines parallet to t

# Fundamental matrix for pure translation

Forward motion

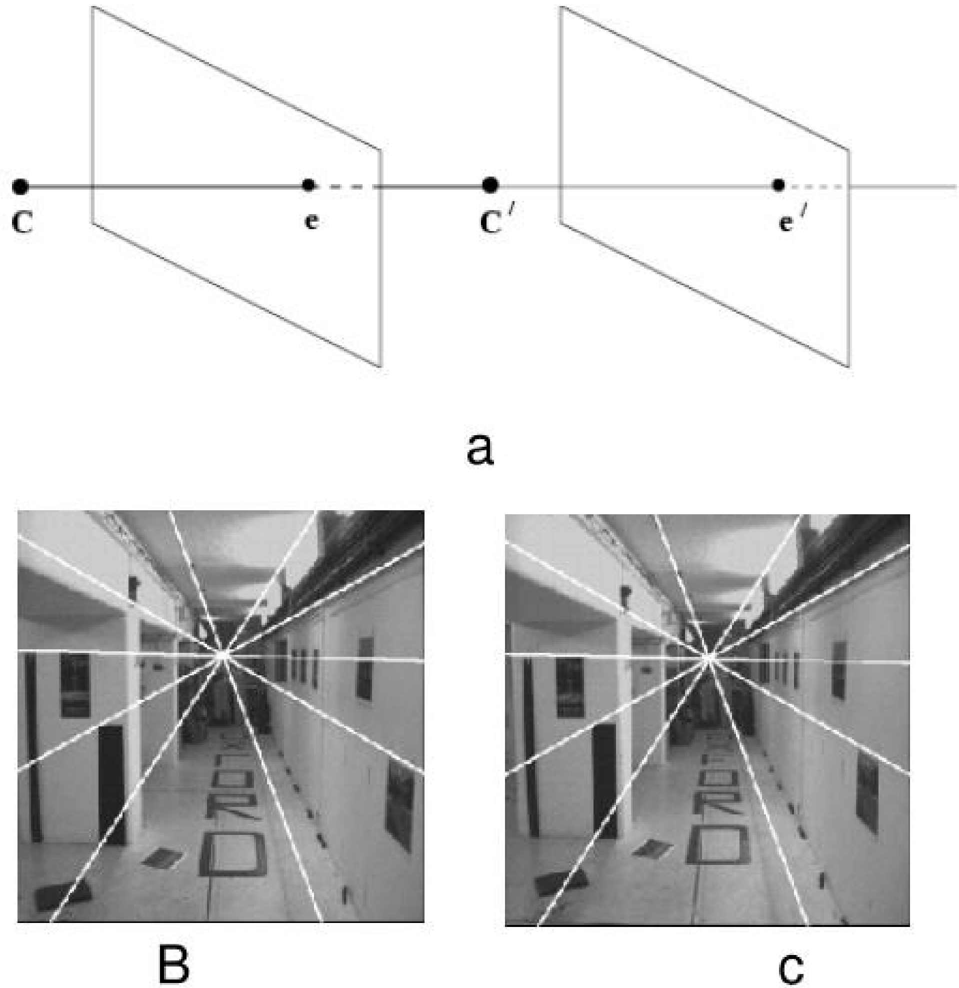



Fig. 9.8. **Pure translational motion.** *(a) under the motion the epipole is a fixed point, i.e. has the same coordinates in both images, and points appear to move along lines radiating from the epipole. The epipole in this case is termed the* Focus of Expansion *(FOE). (b) and (c) the same epipolar lines are overlaid in both cases. Note the motion of the posters on the wall which slide along the epipolar line.*

# Fundamental matrix for pure translation

P = K [ I | 0] ; P' = K [ I | t]

$$F = [e']_\times H_\infty = [e']_\times \qquad (H_\infty = K^{-1}RK)$$

example: Camera translation parallel to x axis

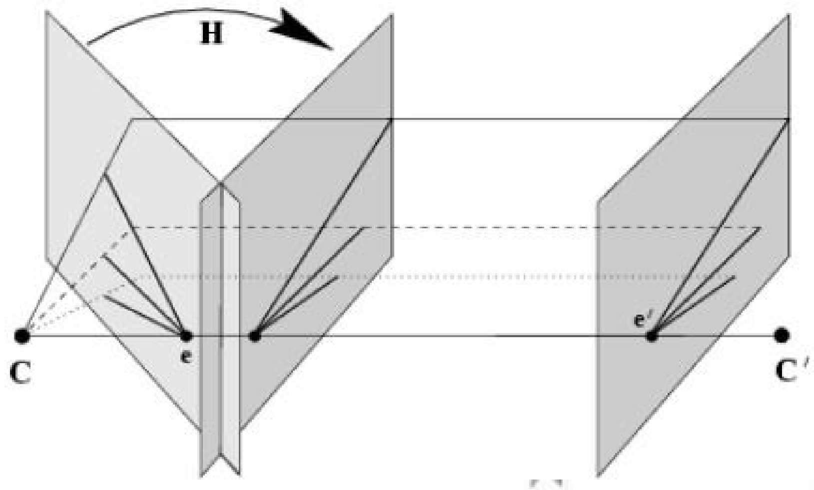
$$e' = (1,0,0)^T \qquad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times \quad x'^T F x = 0 \Leftrightarrow y = y'$$

$$x = PX = K[I \mid 0]X \qquad (X,Y,Z)^T = K^{-1}x/Z$$

$$x' = P'X = K[I \mid t]\begin{bmatrix} K^{-1}x \\ Z \end{bmatrix} \qquad \boxed{x' = x + Kt/Z}$$

- Z = depth of point X = distance of X from the camera center measured along the principal axis of the first camera
- Motion starts at x and moves towards e; extent of motion depends on magnitude of t and Inversely proportional to Z; points closer to camera appear to move faster than those further away faster depending on Z➔ *looking out of the train window*
- pure translation: F only 2 d.o.f., $x^T[e]_x x=0 \Rightarrow$ auto-epipolar

$P = K [I \mid 0]$ ; $P' = K' [R \mid t]$;




# General motion

- Rotate the first camera so that it is aligned with the 2nd camera ➔ projective transformation
- Further correction to the first image to account for differences in calibration matrices
- Result of these two corrections is projective transformation H of the first image: $H = K' R K^{-1}$
- Then the effective relation between 2 images is pure translation

Fig. 9.9. **General camera motion.** *The first camera (on the left) may be rotated and corrected to simulate a pure translational motion. The fundamental matrix for the original pair is the product* $F = [e']_\times H$*, where* $[e']_\times$ *is the fundamental matrix of the translation, and* $H$ *is the projective transformation corresponding to the correction of the first camera.*

$$x'^T [e']_\times Hx = 0$$

$$x'^T [e']_\times \hat{x} = 0$$

$$x' = K' R K^{-1} x + K' t/Z$$

First term depends on image position, x, but not point's depth Z, and takes into account camera rotation R and change of internal parameters

Second term depends on depth, but not image position x, and takes account of translation

# Projective transformation and invariance

F invariant to transformations of projective 3-space

**Result 9.8.** *If* H *is a* $4 \times 4$ *matrix representing a projective transformation of 3-space, then the fundamental matrices corresponding to the pairs of camera matrices* (P, P′) *and* (PH, P′H) *are the same.*

Consequence; F does not depend on the choice of world frame

$$(\mathrm{P},\mathrm{P}') \mapsto \mathrm{F}$$ unique

$$\mathrm{F} \mapsto (\mathrm{P},\mathrm{P}')$$ not unique

canonical form for a pair of camera matrices for a given F: Choose first Camera as P = [I | 0 ]

$$\begin{aligned}\mathrm{P} &= [\mathrm{I} \mid 0] \\ \mathrm{P}' &= [\mathrm{M} \mid \mathrm{m}]\end{aligned} \qquad \mathrm{F} = [\mathrm{m}]_\times \mathrm{M}$$

**Result 9.9.** *The fundamental matrix corresponding to a pair of camera matrices* $\mathrm{P} = [\mathrm{I} \mid \mathbf{0}]$ *and* $\mathrm{P}' = [\mathrm{M} \mid \mathbf{m}]$ *is equal to* $[\mathbf{m}]_\times \mathrm{M}$.

# Projective ambiguity of cameras given F

previous slide: at least projective ambiguity
this slide: no more than projective ambiguity!

Show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'),
there exists a projective transformation H so that
$\tilde{P}$=HP and $\tilde{P}$'=HP'

**Theorem 9.10.** *Let* $\mathrm{F}$ *be a fundamental matrix and let* $(\mathrm{P}, \mathrm{P}')$ *and* $(\tilde{\mathrm{P}}, \tilde{\mathrm{P}}')$ *be two pairs of camera matrices such that* $\mathrm{F}$ *is the fundamental matrix corresponding to each of these pairs. Then there exists a non-singular* $4 \times 4$ *matrix* $\mathrm{H}$ *such that* $\tilde{\mathrm{P}} = \mathrm{PH}$ *and* $\tilde{\mathrm{P}}' = \mathrm{P}'\mathrm{H}$.

Upshot: a given fundamental matrix determines the pair of camera matrices up to a right multiplication by a projective transformation.

**Lemma 9.11.** *Suppose the rank 2 matrix* $\mathrm{F}$ *can be decomposed in two different ways as* $\mathrm{F} = [\mathbf{a}]_\times \mathrm{A}$ *and* $\mathrm{F} = [\tilde{\mathbf{a}}]_\times \tilde{\mathrm{A}}$*; then* $\tilde{\mathbf{a}} = k\mathbf{a}$ *and* $\tilde{\mathrm{A}} = k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^\mathsf{T})$ *for some non-zero constant* $k$ *and 3-vector* $\mathbf{v}$.

P and P' each have 11 dof ( 3 x 4 – 1 = 11). Total of 22 dof for both;
Projective world frame H has ( 4 x 4 – 1= 15 ) dof;
Remove dof of world frame from the two cameras: 22 – 15 = 7 = dof of Fundamental matrix.

# Canonical cameras given F

**Result 9.12.** *A non-zero matrix* $\mathtt{F}$ *is the fundamental matrix corresponding to a pair of camera matrices* $\mathtt{P}$ *and* $\mathtt{P}'$ *if and only if* $\mathtt{P}'^{\mathsf{T}}\mathtt{F}\mathtt{P}$ *is skew-symmetric.*

F matrix corresponds to P,P' iff $P'^{T}FP$ is skew-symmetric

$$\left(X^{T}P'^{T}FPX = 0, \forall X\right)$$

**Result 9.13.** *Let* $\mathtt{F}$ *be a fundamental matrix and* $\mathtt{S}$ *any skew-symmetric matrix. Define the pair of camera matrices*

$$\mathtt{P} = [\mathtt{I} \mid \mathbf{0}] \quad \textit{and} \quad \mathtt{P}' = [\mathtt{SF} \mid \mathbf{e}'],$$

*where* $\mathbf{e}'$ *is the epipole such that* $\mathbf{e}'^{\mathsf{T}}\mathtt{F} = \mathbf{0}$*, and assume that* $\mathtt{P}'$ *so defined is a valid camera matrix (has rank 3). Then* $\mathtt{F}$ *is the fundamental matrix corresponding to the pair* $(\mathtt{P}, \mathtt{P}')$.

Possible choice: $S = [e']_{x}$

**Result 9.14.** *The camera matrices corresponding to a fundamental matrix* $\mathtt{F}$ *may be chosen as* $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$ *and* $\mathtt{P}' = [[\mathbf{e}']_{\times}\mathtt{F} \mid \mathbf{e}']$.

**Result 9.15.** *The general formula for a pair of canonic camera matrices corresponding to a fundamental matrix* $\mathtt{F}$ *is given by*

$$\mathtt{P} = [\mathtt{I} \mid \mathbf{0}] \quad \mathtt{P}' = [[\mathbf{e}']_{\times}\mathtt{F} + \mathbf{e}'\mathbf{v}^{\mathsf{T}} \mid \lambda\mathbf{e}'] \tag{9.10}$$

*where* $\mathbf{v}$ *is any 3-vector, and* $\lambda$ *a non-zero scalar.*

# The essential matrix

~fundamental matrix for calibrated cameras; remove K to get Essential matrix

- P = K [ R | t]; x = PX; known K $\rightarrow \hat{x} = K^{-1} x = [R \mid t] X$
- $\hat{x}$ is the image point expressed in normalized coordinates; image of point X w.r.t. camera [R | t] having identity I as calibration matrix

$$E = [t]_\times R = R[R^T t]_\times$$

Definition: $$\hat{x}'^T E \hat{x} = 0 \qquad \left(\hat{x} = K^{-1}x;\ \hat{x}' = K^{-1}x'\right)$$

Relation between E and F $$E = K'^T F K$$

5 d.o.f. (3 for R; 2 for t up to scale)

**Result 9.17.** *A $3 \times 3$ matrix is an essential matrix if and only if two of its singular values are equal, and the third is zero.*

**Result 9.18.** *Suppose that the SVD of* E *is* $U \operatorname{diag}(1,1,0) V^T$. *Using the notation of (9.13), there are (ignoring signs) two possible factorizations* $E = SR$ *as follows:*

$$S = UZU^T \quad R = UWV^T \quad or \quad UW^TV^T. \tag{9.14}$$

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

# Four possible reconstructions from E

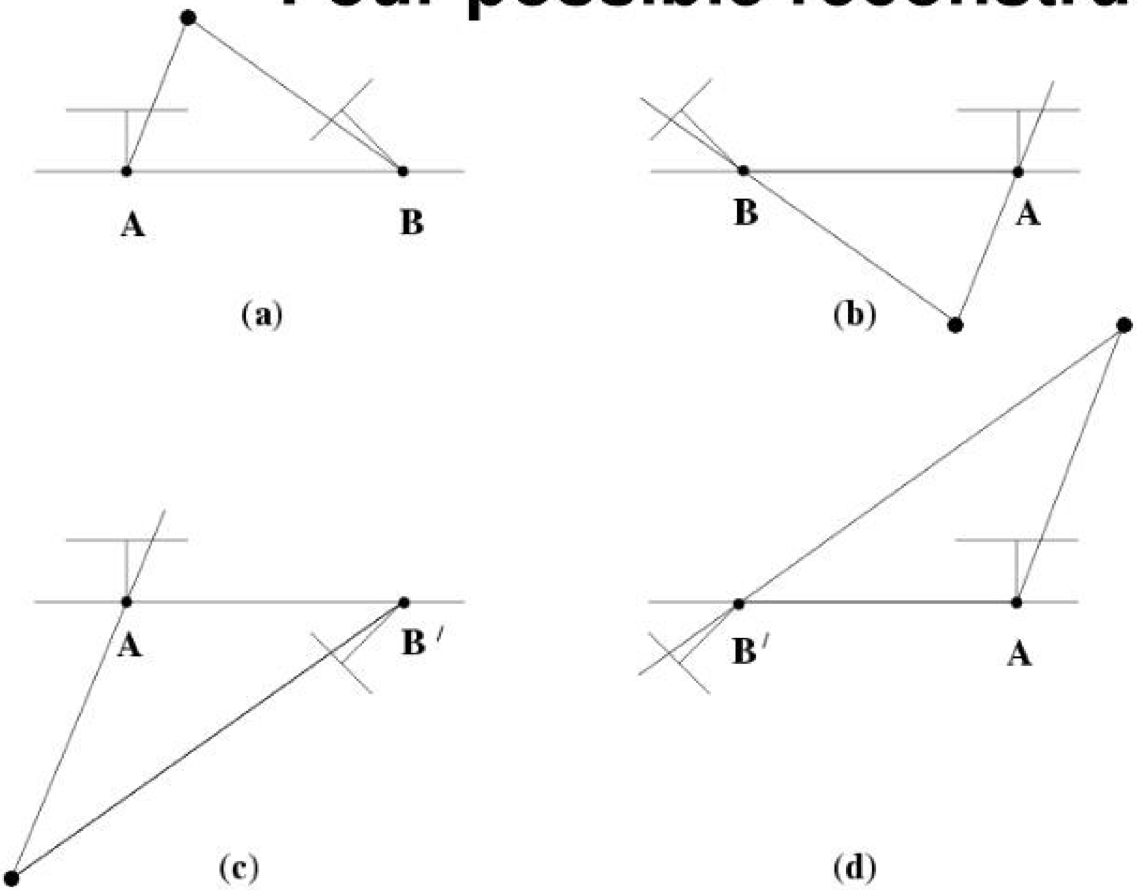



Fig. 9.12. **The four possible solutions for calibrated reconstruction from E.** *Between the left and right sides there is a baseline reversal. Between the top and bottom rows camera B rotates 180° about the baseline. Note, only in (a) is the reconstructed point in front of both cameras.*

**Result 9.19.** *For a given essential matrix* $\mathtt{E} = \mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{V}^\mathsf{T}$, *and first camera matrix* $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$, *there are four possible choices for the second camera matrix* $\mathtt{P}'$, *namely*

$$\mathtt{P}' = [\mathtt{UWV}^\mathsf{T} \mid +\mathbf{u}_3] \text{ or } [\mathtt{UWV}^\mathsf{T} \mid -\mathbf{u}_3] \text{ or } [\mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T} \mid +\mathbf{u}_3] \text{ or } [\mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T} \mid -\mathbf{u}_3].$$

(only one solution where points is in front of both cameras)