Recap of first-order logic

a logic for statements over *objects*, including quantified statements like "for all objects it holds that..." and "there exists objects such that..."

a set D called *domain* is the set of all objects

Elements

formulae are based upon:

- variables (x, y, ...), constants (c, d, ...), function symbols (f, g,...)
- predicate symbols (P, R, ...)
- propositional connectives ($\neg$, $\wedge$, $\vee$) and quantifiers $\exists$, $\forall$)

variables etc. stand for objects (elements of the domain D)

predicates, possibly combined with connectives and quantifiers, can be true or false

Terms and formulae

two kinds of values:

- elements of the domain ("objects")
- true or false

terms	build upon variables, constants and function symbols	examples: $f(x,c)$ d $h(x)$ y $g(f(y,x),c,x)$	value is an element of the domain
formulae	build upon literals, which are predicates applied to terms	examples: $P(x,f(c,d))$ $P(c) \land R(h(d))$ $\exists x\ P(f(x,c),x)$	value is true or false

Semantics

a **model** comprises:

- the domain D and
- an evaluation of everything but the variables

an **interpretation** gives values to variables
note that the value of a variable is *an element of the domain*
(not true/false like in propositional logic)

a model and an interpretation:

- assign to each term a value of D
- assign to each formula a value in *{true, false}*

Models and interpretations, formally

a model is made of a doman D and an assignments I, which assign:

an element of D to each constant
a function from D^n to D to each function symbol of arity n
a function from D^n to *{true, false}* to each predicate symbol of arity n

an interpretation μ assigns an element of D to every variable

Semantics, formally

given a model $\langle D,I\rangle$ and an interpretation μ, we can evaluate every term and formula:

constants	c	$I(c)$		
variables	x	$\mu(x)$		
terms made of functions	$f(t,\dots,s)$	$I(f)(value\ of\ t,\dots,value\ of\ s)$	(note that $I(f)$ is a function)	(*value of t*, etc. are elements of the domain obtained recursively)
literals	$P(t,\dots,s)$	$I(P)(value\ of\ t,\dots,value\ of\ s)$	(note that $I(P)$ is a function)	($t,\dots,s$ are terms)
formulae built upon propositional connectives	$P(c) \vee R(f(c,d))$	as usual: every literal is either true or false		
formulae based on an existential quantifier	$\exists x\ P(c, x)$	true if there exists μ', that differs from μ only on the value of x, such that $\langle D,I\rangle$ and μ' evaluate $P(c,x)$ to true		
formulae based on an universal quantifier	$\forall x\ P(c, x)$	true if, for all μ' that differs from μ only on the value of x, we have that $\langle D,I\rangle$ and μ' evaluate $P(c,x)$ to true		

Satisfiability

a formula is satisfiable if there exists a model and an interpretation making it true

Free variables

an occurrence of a variable in a formula is *bounded* if it falls within the scope of a quantifier, *free* otherwise

$P(c,\mathbf{x}) \land \exists x\, P(\mathbf{x},d)$
first occurrence of x is free, second is bounded

if a formula contains no free variables, then its truth value does not depend on the interpretation μ

Property of quantifiers

we can change the name of a quantified variable:

$\exists x\, P(x)$ is equivalent to $\exists y\, P(y)$

$\forall x\, P(x)$ is equivalent to $\forall y\, P(y)$

this can be done in general, if for example y is a new variable:

$\exists x\, (P(c,d) \land \forall x\, R(x) \land P(x,x))$ is equivalent to
$\exists y\, (P(c,d) \land \forall x\, R(x) \land P(y,y)$ and to
$\exists y\, (P(c,d) \land \forall y\, R(y) \land P(y,y)$

the occurrences of x relative to a quantification that is inside the formula, like $\forall x\, R(x)$ in this case, can be renamed or not

Other properties of quantifiers

relevant to automated reasoning

if x does not occur in A, then:

$$\forall x\ (A \wedge B) \equiv A \wedge \forall x\ B$$
$$\forall x\ (A \vee B) \equiv A \vee \forall x\ B$$

quantifiers can be "moved in" if the "skipped formula" does not contain the variable they quantify upon

operation can be iterated: $\exists x\ (P(c,d) \wedge (\forall y\ R(y) \vee P(d, x)) \wedge R(c))$ is equivalent to $P(c,d) \wedge (\forall y\ R(y) \vee \exists\, x\ P(d, x)) \wedge R(c)$