I authorize Prof. Pietrabissa to upload the test results on the mailing list of the Process Automation group.

FAMYLY NAME GIVEN NAME SIGNATURE

Process Automation (MCER), 2019-2020

TEST - A

19 Dec. 2019, 2h00

Exercise 1 (12 pts.)

Let the process be described by the transfer function: $P(s) = 0.1\frac{1+\tau s}{(1+100s)^2}e^{-0.2s}$, with $\tau \in [1,11)$, and let the nominal value of the parameter be $\tilde{\tau} = 1$. Design a robust controller by following the IMC design to obtain a type 1 system under the ISE cost.

Exercise 2 (12 pts.)

Consider a process whose transfer function model is:

$\begin{cases} x(t) = Mx(t-1) + Pu(t-1) \\ y(t) = Qx(t) \end{cases}$, with $M = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}$, $P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

A) Compute the control action at time $t = 5$ by implementing a Predictive Functional Control algorithm, considering:

- prediction and control horizon $p = m = 3$;
- number of base functions $n_B = 1$;
- number of coincident points $n_H = 2$;
- smooth response preferred;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory;
- known ramp reference signal $r(t) = t$ and reference trajectory
  $w(t+k|t) = \begin{cases} y_m(t) & \text{if } k = 0 \\ 0.5\, w(t+k-1|t) + 0.5\, r(t+k) & \text{otherwise} \end{cases}$;
- predicted plant-model errors at time t equal to $\hat{n}(t+k|t) = k\big(y_m(t) - y(t)\big)$;
- available data at $t = 5$:
  - control action $u(4) = 1.5$;
  - state $x(4) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$;
  - measured output $y_m(5) = 7$.

Questions (6 pt.)

i) Analyze the stability of the controlled system with open-loop transfer function $F(s) = 0.1\frac{1-s}{s(1+s)}e^{-10s}$ (approximations are acceptable, 4pt)

ii) Explain why the stability of a system controlled by using a Smith's Predictor is not dependent on the value of the nominal delay. (1/2 pg. max, 2pt)

Remark for P1: $M^2 = \begin{bmatrix} 0.25 & 0.5 \\ 0 & 0.25 \end{bmatrix}$, $M^3 = \begin{bmatrix} 0.125 & 0.375 \\ 0 & 0.125 \end{bmatrix}$

Solution of exercise 1

Since the time-delay of the process, $0.2s$, is much smaller than the time constant – about $100s$ – of the process, we can use a Padé approximation to write the delay term as a transfer function: $P^P(s) = 0.1\frac{1+\tau s}{(1+100s)^2}\frac{1-0.1s}{1+0.1s}$.
The process : P^P is stable, therefore it is possible to design a stable controller $Q(s)$ for the closed-loop nominal system, with nominal process $\tilde{P}^P(s) = 0.1\frac{1+s}{(1+100s)^2}\frac{1-0.1s}{1+0.1s}$.
The IMC design procedure to robustly stabilize the process $P^P(s)$ consists in the following 3 steps:
Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term (ISE-optimal factorization):

$P^P(s) = P_+(s)P_-(s)$, with $P_+(s) = \frac{1-0.1s}{1+0.1s}$ and $P_-(s) = 0.1\frac{1+s}{(1+100s)^2}$.

b) Define the preliminary controller as $\tilde{Q}(s) = P_-^{-1}(s) = 10\frac{(1+100s)^2}{1+s}$.

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1. Thus, we use the IMC filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 1$.

The IMC controller is then $Q(s) = 10\frac{(1+100s)^2}{(1+s)(1+\lambda s)}$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$$\left|l_m(j\omega)\tilde{T}^P(j\omega)\right| < 1, \forall\omega$$

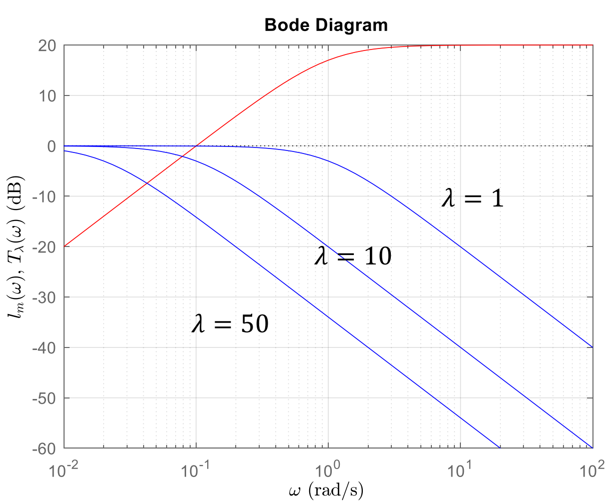
where

$$\left|\tilde{T}^P(j\omega)\right| = \left|Q(j\omega)P^P(j\omega)\right| = \left|P_-^{-1}(j\omega)f(j\omega)P_+(j\omega)P_-(j\omega)\right| = \left|\frac{1-0.1j\omega}{(1+0.1j\omega)(1+j\omega\lambda)}\right| = \left|\frac{1}{1+j\omega\lambda}\right|$$

and $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall\omega$. By definition, the multiplicative uncertainty is defined as

$$\Delta_m(s) := \frac{P^P(s)-\tilde{P}^P(s)}{\tilde{P}^P(s)} = \frac{0.1\frac{1+s\tau}{(1+100s)^2}\frac{1-0.1s}{1+0.1s} - 0.1\frac{1+s}{(1+100s)^2}\frac{1-0.1s}{1+0.1s}}{0.1\frac{1+s}{(1+100s)^2}\frac{1-0.1s}{1+0.1s}} = (\tau-1)\frac{s}{1+s}.$$

The upper-bound is clearly obtained when the gain magnitude is maximized, that is for $\tau = 11$: $l_m = 10\frac{s}{1+s}$.
The figures below show that, at least, for $\lambda \geq 10$ the condition is met. For instance, we choose $\lambda = 10$, obtaining $Q(s) = 10\frac{(1+100s)^2}{(1+s)(1+10s)}$.



Solution of exercise 2

To develop the PFC controller, we need to select the $n_B = 1$ base function: we select the polynomial (step) function $B_1(k) = k^0 = 1, k = 0,1,2, \dots$.
From the process model we note that there is an input-output delay $d = 1$, as, using the model equations,

$$y(t) = Qx(t) = x_1(t) = 0.5x_1(t-1) + 0.5x_2(t-1) = 0.5x_1(t-1) + 0.5\big(0.5x_2(t-2) + u(t-2)\big).$$

Since the control horizon is 3, the only coincident points that can be selected are then $h_1 = 2$ and $h_2 = 3$.
The model responses to the base functions in the coincidence points, considering null initial conditions $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are computed as follows:

$B_1(k) = 1$
$\underline{t = h_1 = 2}$ $\qquad y_{B_1}(2) = QMPB_1(0) + QPB_1(1) = (QMP + QP)1 = [1 \;\; 0]\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 + [1 \;\; 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 = 0.5;$

$\underline{t = h_2 = 3}$ $\qquad y_{B_1}(3) = QM^2PB_1(0) + QMPB_1(1) + QPB_1(2) = (QM^2P + QMP + QP)1 = 2.$

Thus,
$y_B(h_1) = y_{B_1}(h_2) = [1.5];$
$y_B(h_2) = y_{B_1}(h_2) = [2].$

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$.
The matrix Y_B is used to compute the solution of the unconstrained optimization problem $\mu^* = (Y_B^T Y_B)^{-1} Y_B^T (w - f)$, with $(Y_B^T Y_B)^{-1} Y_B^T = \left([0.5 \;\; 2]\begin{bmatrix} 0.5 \\ 2 \end{bmatrix}\right)^{-1} [0.5 \;\; 2] = [0.12 \;\; 0.47]$,
where μ^* is the vector of the optimal parameters at time t. The control action is the computed as $u(t) = {\mu^*}^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \dots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 1$, we need to find one parameter $\mu^*(t)$.

At $t = 5$, considering that $r(t) = t$, the vector of the future reference values evaluated in the coincidence points $t + h_1 = 7$ and $t + h_2 = 8$ is computed as follows:

$w(5|5) = y_m(5) = 4;$
$w(6|5) = 0.5 \cdot w(5|5) + 0.5 \cdot r(6) = 5;$
$w(7|5) = 0.5 \cdot w(6|5) + 0.5 \cdot r(7) = 6.$
$w(8|5) = 0.5 \cdot w(7|5) + 0.5 \cdot r(8) = 7.$

By considering the given state and measured output values, we can compute the free response at time $t = 5$ over the coincidence points. With PFC, the output prediction is

$$\hat{y}(t+k|t) = y(t+k) + \hat{n}(t+k|t) = \textstyle\sum_{i=1,\dots,n_B} y_{B_i}(k)\mu_i(t) + QM^k x(t) + \hat{n}(t+k|t),$$

where the last two terms constitute the free response:

$$f(t+k|t) = QM^k x(t) + \hat{n}(t+k|t) = QM^k x(t) + k(y_m(t) - y(t)).$$

$\underline{t=5}$ $\quad y_m(5) = 7;\ x(4) = \begin{bmatrix} 4 \\ 6 \end{bmatrix};\ u(4) = 1.5;$

$$\begin{cases} x(5) = Mx(4) + Pu(4) = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ y(5) = Qx(5) = [0 \quad 1] \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 6 \end{cases};$$

$\underline{h_1 = 2}$ $\qquad f(7) = QM^{h_1}x(5) + h_1\big(y_m(5) - y(5)\big) = [0 \quad 1] \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}^2 \begin{bmatrix} 6 \\ 4 \end{bmatrix} + 2(7-6) = 3;$

$\underline{h_2 = 3}$ $\qquad f(8) = QM^{h_2}x(5) + h_1\big(y_m(5) - y(5)\big) = [0 \quad 1] \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 6 \\ 4 \end{bmatrix} + 3(7-6) = 3.5.$

$d(5) = \begin{bmatrix} \hat{w}(7) \\ \hat{w}(8) \end{bmatrix} - \begin{bmatrix} f(7) \\ f(8) \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4.5 \end{bmatrix}.$

$\mu(5) = [0.12 \quad 0.47] \begin{bmatrix} 4 \\ 4.5 \end{bmatrix} = 2.6.$

$u(5) = \mu(5)B_1(0) = 2.6.$