

I authorize Prof. Pietrabissa to upload the test results on the mailing list of the Process Automation group.

FAMILY NAME

GIVEN NAME

SIGNATURE

Process Automation (MCER), 2018-2019

TEST (A)

20 Dec. 2018

Exercise 1 (12 pts.)

Let the process be described by the transfer function: $P(s) = -\frac{1-s/10}{(1+s/10)(1+s/a)}$, with $a \in [10,100]$.

1. Considering the nominal value of the parameter $\tilde{a} = 10$ and by following the IMC design, develop a controller such that:
 - a. the controlled system has 0 steady-state error for step inputs,
 - b. the considered cost function for factorization is ISE;
 - c. the controlled system is robustly stable against the uncertainties of the parameter a .
2. Compute the equivalent classic controller.

Exercise 2 (12 pts.)

Consider a process whose state-space model is:

$$\begin{cases} \dot{x}(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = 0.5, N = 0.1, Q = 2.$$

Compute the control actions of a Predictive Functional Control algorithm at time $t = 5$, with:

- control horizon $m = 4$;
- prediction horizon $p = 4$;
- number of coincident points $n_h = 2$;
- number of basis functions $n_B = 2$;
- ramp reference $r(t) = t, \forall t$;
- reference trajectory $w(t+k|t) = 0.5y_m(t) + 0.5r(t+k)$; the cost function should be such that the controller reacts fast to reference variations;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory;
- plant-model error $\hat{e}(t+k|t) = k \cdot (y_m(t) - y(t)), k = 1, 2, \dots$;
- the output measure at time $t = 5$ is $y_m(5) = 5.1$; the state and the control action at time $t = 4$ are $x(4) = 1.9$ and $u(4) = 10$.

Questions (6 pt.)

- i) Consider the controller found Exercise 1 and check if it stabilizes the process $P(s) = -\frac{1-\frac{s}{10}}{(1+\frac{s}{10})^2}e^{-2s}$ (it is sufficient to write and explain the procedure). (1 pg. max, 4pt)
- ii) Briefly explain why, in the robust stability condition based on Nyquist theorem arguments (i.e., the condition $|l_m(j\omega)\tilde{T}(j\omega)| < 1, \forall \omega$), one of the assumptions is that the number of positive poles of the process model is equal to the number of positive poles of the actual process. (1/3 pg. max, 2pt).

Solution of exercise 1

1.

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles) and has the same number of positive poles of the actual the process, therefore it is possible to design a stable IMC controller $Q(s)$ to robustly stabilize the closed-loop nominal system.

The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

Factorize the nominal process $\tilde{P}(s) = -\frac{1-\frac{s}{10}}{\left(1+\frac{s}{10}\right)^2}$ in a minimum-phase term and a non-minimum-phase term, under ISE-optimal factorization:

$$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s), \text{ with } \tilde{P}_+(s) = \frac{1-\frac{s}{10}}{1+\frac{s}{10}} \text{ and } \tilde{P}_-(s) = -\frac{1}{1+\frac{s}{10}},$$

and define the controller as follows:

$$\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = -\left(1 + \frac{s}{10}\right).$$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$).

We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 1$. In fact:

$$\text{a) } Q(s) = \tilde{Q}(s)f(s) = -\frac{1+\frac{s}{10}}{1+\lambda s} \text{ is proper;}$$

$$\text{b) } \tilde{T}(0) = \tilde{P}(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1}f(s)\right]_{s=0} = \left[\frac{1-\frac{s}{10}}{1+\frac{s}{10}}\frac{1}{1+\lambda s}\right]_{s=0} = 1.$$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_m(j\omega)\tilde{T}(j\omega)| < 1, \forall \omega,$$

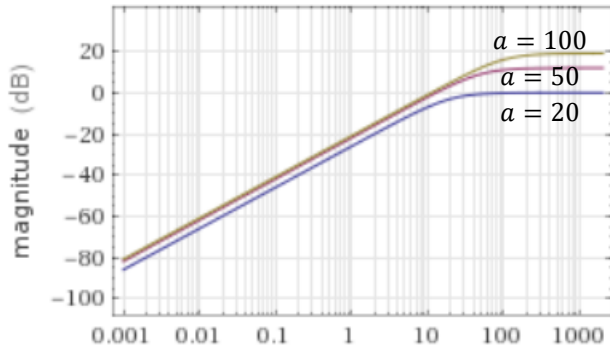
where

$$\tilde{T}(j\omega) = \tilde{P}(j\omega)Q(j\omega) = -\frac{1 - \frac{j\omega}{10}}{\left(1 + \frac{j\omega}{10}\right)(1 + \lambda j\omega)}$$

and $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall \omega$. The multiplicative uncertainty is defined as

$$\Delta_m(j\omega) := \frac{P(j\omega) - \tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{\frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{a}\right)} + \frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{10}\right)^2}}{-\frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{10}\right)^2}} = \left(\frac{1}{10} - \frac{1}{a}\right)\frac{j\omega}{1+\frac{j\omega}{a}}.$$

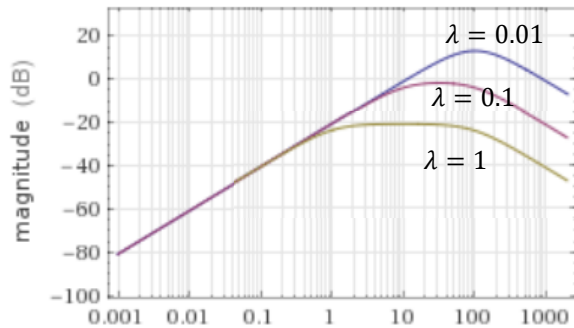
Since $|\Delta_m(j\omega)|$ grows with a (both the gain $(\frac{1}{10} - \frac{1}{a})$ and the cut-off frequency a of the pole $-1/a$ grow with a) and $a \in (10, 100]$, an upper-bound is then given by $l_m(j\omega) = 0.09 \frac{j\omega}{1 + \frac{j\omega}{100}}$.



The sufficient condition for robust stability is then:

$$|l_m(j\omega)\tilde{T}(j\omega)| = \left| 0.09 \frac{j\omega}{1 + \frac{j\omega}{100}} \frac{1 - j\omega/10}{(1 + j\omega/10)(1 + \lambda j\omega)} \right| = \left| 0.09 \frac{j\omega}{(1 + \frac{j\omega}{100})(1 + \lambda j\omega)} \right| < 1, \forall \omega.$$

which holds, approximatively, for $\lambda \geq 1/10$.



Thus, we set $\lambda = 1/10$ and obtain $Q(s) = -1$.

2.

In the equivalent classic control scheme, the controller is a PID+filter controller computed from the IMC controller as follows:

$$G(s) = \frac{Q(s)}{1 - \bar{P}(s)Q(s)} = \frac{1}{1 - \frac{1-s/10}{(1+s/10)^2}} = 10 \frac{1 + \frac{s}{10} + \frac{s^2}{100}}{s(1 + \frac{s}{10})} = 2 \left(1 + \frac{0.2}{s} + 0.05s \right) \frac{1}{s(1 + \frac{s}{10})}.$$

Solution of exercise 2.

To develop the PFC controller, we need to select the base functions and the coincident points. We chose $n_B = 2$ base functions $B_i(k) = k^{i-1} = 1, k = 0, 1, \dots$. Since $n_h = 2$, the prediction horizon is $p = 4$ and we want a fast response we chose $h_1 = 1$ and $h_2 = 4$.

Firstly, we have to compute the model response to the base functions in the coincidence points, considering null initial conditions $x(0) = 0$:

$$t = h_1 = 1$$

$$B_1: y_{B_1}(1) = QNB_1(0) = 0.2 \cdot 1 = 0.2;$$

$$B_2: y_{B_2}(1) = QNB_2(0) = 0.2 \cdot 0 = 0.$$

$$t = h_2 = 4$$

$$B_1: y_{B_1}(4) = QM^3NB_1(0) + QM^2NB_1(1) + QMNB_1(2) + QNB_1(3) = 0.025 \cdot 1 + 0.05 \cdot 1 + 0.1 \cdot 1 + 0.2 \cdot 1 = 0.375.$$

$$B_2: y_{B_2}(4) = QM^3NB_2(0) + QM^2NB_2(1) + QMNB_2(2) + QNB_2(3) = 0.025 \cdot 0 + 0.05 \cdot 1 + 0.1 \cdot 2 + 0.2 \cdot 3 = 0.85.$$

$$\text{Thus, } y_B(h_1) = (y_{B_1}(h_1) \quad y_{B_2}(h_1)) = (0.2 \quad 0), y_B(h_2) = (y_{B_1}(h_2) \quad y_{B_2}(h_2)) = (0.375 \quad 0.85).$$

$$\text{The matrix } Y_B \in \mathbb{R}^{n_H \times n_B} \text{ is then } Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{pmatrix} 0.2 & 0 \\ 0.375 & 0.85 \end{pmatrix}.$$

The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu^* = Y_B^{-1}(w - f)$, where μ^* is the vector of the optimal parameters at time t and $Y_B^{-1} = \begin{pmatrix} 5 & 0 \\ -2.21 & 1.18 \end{pmatrix}$. The control action is the computed as $u(t) = \mu^{*T} B(0)$, where $B(0)$ is the column vector of base functions $B_i(k)$, $i = 1, 2, \dots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 2$, we need to find a two-column vector $\mu^*(t)$.

At time $t = 5$, the future reference trajectory values in the coincidence points are

$$k = h_1 = 1, \quad w(5 + 1|5) = 0.25y_m(5) + 0.75r(5 + 1) = 0.5 \cdot 5.1 + 0.5 \cdot 6 = 5.55;$$

$$k = h_2 = 4, \quad w(5 + 4|5) = 0.25y_m(5) + 0.75r(5 + 4) = 0.5 \cdot 5.1 + 0.5 \cdot 9 = 7.05.$$

With PFC, the free response at time $t = 5$ in the coincidence points is given by

$$d(t + k|t) = f(t + k|t) + \hat{e}(t + k|t) = QM^k x(t) + k \cdot (y_m(t) - y(t)),$$

where $y(t)$ is the model output at time $t = 5$:

$$x(5) = Mx(4) + Nu(4) = 1.95;$$

$$y(5) = Qx(5) = 3.9.$$

$$t = 5$$

$$h_1 = 1 \quad f(8 + 1|8) = QMx(5) + 1 \cdot (y_m(5) - y(5)) = 3.15;$$

$$h_2 = 4 \quad f(8 + 4|8) = QM^4x(8) + 4 \cdot (y_m(8) - y(8)) = 5.04.$$

$$d = (w - f) = \begin{pmatrix} 5.55 \\ 7.05 \end{pmatrix} - \begin{pmatrix} 3.15 \\ 5.04 \end{pmatrix} = \begin{pmatrix} 2.40 \\ 2.01 \end{pmatrix}.$$

$$\mu(1) = \begin{pmatrix} 5 & 0 \\ -2.21 & 1.18 \end{pmatrix} \begin{pmatrix} 2.40 \\ 2.01 \end{pmatrix} = \begin{pmatrix} 12 \\ -2.93 \end{pmatrix};$$

$$u(1) = \mu_1(1)B_1(0) + \mu_2(1)B_2(0) = 12 \cdot 1 - 2.93 \cdot 0 = 12.$$