MPC WITH INVARIANT FINAL SET

- There exists a minimum value $\bar{N}$ s.t. $F\,x(\bar{N}+i|k) + G\,u(\bar{N}+i|k) \le b$, $i = 0, 1, \dots$ under $u(k) = K^* x(k)$ ← mode 2 control law
- $\bar{N} = ?$
- <u>DEF</u> POSITIVELY INVARIANT SET

  A set $X \subseteq \mathbb{R}^{n_x}$ is POSITIVELY INVARIANT under the closed loop dynamics $x(k+1) = (A+BK)x(k)$ and under the constraints $(F+BK)x(k) \le b$

  iff
$$\begin{cases} x(k) \in X, & k = 0, 1, \dots \\ (F+BK)x(k) \le b, & k = 0, 1, \dots \end{cases}$$
  for all $x(0) \in X$

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ u(k) = K\,x(k) \end{cases}$$
$$\Downarrow$$
$$x(k+1) = (A+BK)x(k)$$

$$Fx + Gu \le b$$
$$\Downarrow$$
$$(F+GK)x \le b$$

MPC problem

$$\min_{\underline{u}(k)} J(x(k), \underline{u}(k)) = \sum_{i=0}^{N-1} \left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \right) + \|x(N|k)\|_W^2$$

$$\text{s.t.} \quad F x(i|k) + G u(i|k) \leq b, \quad i = 0, 1, \dots, N-1$$

$$x(N|k) \in X_f$$

with X_f POSITIVELY INVARIANT w.r.t. the closed-loop dynamics (mode 2) and the constraints

The MPC problem reads a

$$\min_{\underline{u}(k)} J(x(k), \underline{u}(k))$$

$$\text{s.t.} \quad \left. F x(i|k) + G u(i|k) \leq b, \quad i = 0, \dots, N-1 \right] \text{MODE 1}$$

$$\left. \begin{array}{l} (A+BK)^i x(N|k) \in X_f, \quad i = 0, 1, \dots \\ (F+GK)(A+BK)^i x(N|k) \leq b, \quad i = 0, 1, \dots \end{array} \right\} \text{MODE 2}$$

mode 2

$x(k) = (A+BK)^k x(0)$

$(F+GK) x(k) \leq b$

- Problems in solving the MPC : i) determination of X_f

ii) N must be large enough for the system to converge to X_f

- N : there is an INITIAL SET X_0 CLOSE ENOUGH TO X_f for the problem to be feasible (e.g. , $X_0 = X_f$)

- If the system is linear X_f is defined by a finite number of inequalities

time N $\quad (F + GK)\, x(N|k) \leq b$

N+1 $\quad (F + GK)(A + BK)\, x(N|k) \leq b$

N+2 $\quad (F + GK)(A + BK)^2\, x(N|k) \leq b$

...

Stabilizing control in mode 2

$\Rightarrow$ $\sigma\{A+BK\}$ is within the unit circle (stabilizing control)

$\Rightarrow$ $\underbrace{X^c(N+k)}_{\text{set defined by the constraints at time } N+k} \supseteq X^c(N+k+1)$

Example

$$\begin{cases} x(k+1) = A x(k) + B u(k), & x(k) \in \mathbb{R}^2, \ u \in \mathbb{R} \\ y(k+1) = C\, x(k) & y(k) \in \mathbb{R} \end{cases}$$

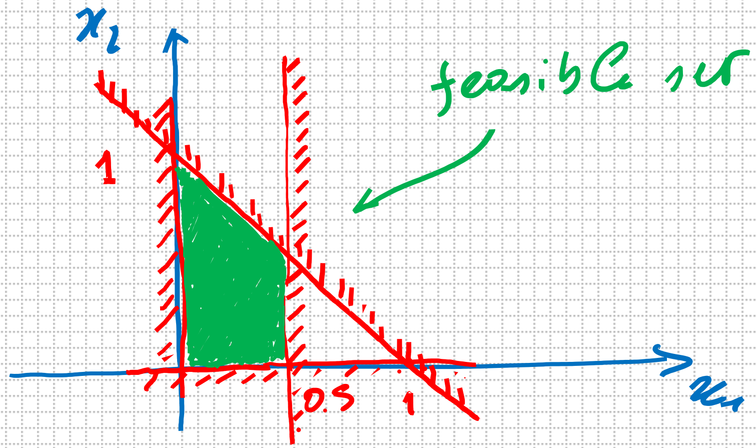
$$F x(k) + G u(k) \leq b$$

mode 2. $K = [2 \quad 0.5]$ for same R, Q

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

STATE CONSTR.

CONTROL ACTION CONST.



feasible set X

$n_C = 6 \leftarrow$ number of constraints

MODE 1 CONSTRAINTS

$$F \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + G u \leq b \Rightarrow \begin{cases} x_1 + x_2 \leq 1 \\ x_1 \leq 0.5 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ u \leq 1 \\ u \geq 0 \end{cases}$$

STATE CONST.

CONTROL ACTION CONST.

feasible set U

MODE 2

$$\begin{cases} x(N+i|k) = (A+BK)^i x(N|k) \\ u(N+i|k) = K\, x(N+i|k) \end{cases}$$

← closed-loop dynamics

$Fx + Gu \le b$

$Fx + GKx \le b \Rightarrow (F+GK)x \le b$ ← ALL STATE-DEPENDENT CONSTRAINTS

$i=0$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 0.5 \end{bmatrix} \right) x(N|k) \le \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

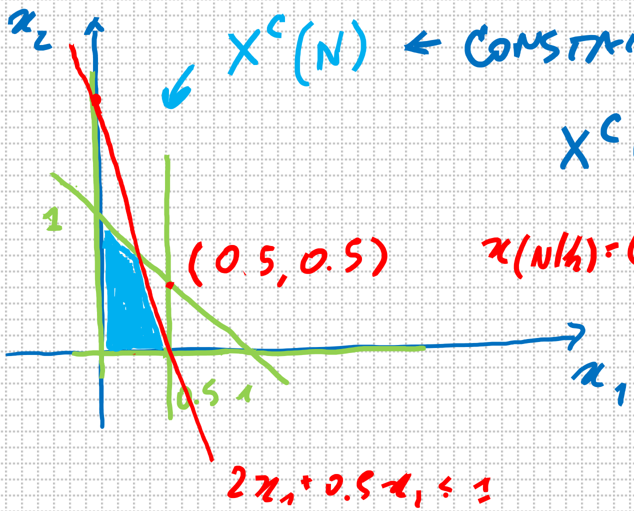
← constraints at time N with $x(N|k) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\left\{ \begin{array}{ll} x_1 + x_2 & \le 1 \\ x_1 & \le 0.5 \\ x_1 & \ge 0 \\ x_2 & \ge 0 \\ 2x_1 + 0.5x_2 & \le 1 \\ 2x_1 + 0.5x_2 & \ge 0 \end{array} \right.$$

SAME STATE CONSTR.

NEW STATE CONSTR.

LAST CONST. ALWAYS MET



$X^c(N)$ ← CONSTRAINED SET AT TIME N

$X^c(N) \subseteq X$

$x(N|k) = (0.5, 0.5) \in X$

$(0.5, 0.5) \notin X^c(N)$

$u(N|k) = \begin{bmatrix} 2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 1.5 \notin U$

- After a finite number of time-steps ν

$$X \supseteq X^c(N) \supseteq X^c(N+1) \supseteq \dots \supseteq X^c(N+\nu) = X^c(N+\nu+1) = X^c(N+\nu+2) = \dots$$

$\Rightarrow$ $X^c(N+\nu)$ IS THE SOUGHT POSITIVELY INVARIANT SET

$$X_f = X^c(N+\nu)$$