Exercise 1.

Consider a process whose step response model is given by the following coefficients:

$g_1 = 0,\ g_2 = 0.25,\ g_3 = 0.375,\ g_4 = 0.4375,\ g_5 = 0.475,\ g_6 = 0.475.$

a) Compute the control actions of a Dynamic Matrix Control algorithm at time $t = 2$, with:
   - control horizon $m = 2$;
   - null initial conditions;
   - constant reference $r(t) = 1, \forall t$;
   - cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory.

   In the computation, consider the following values of the measured output:
   $y_m(1) = 0$, $y_m(2) = 0.25$.

b) How might we obtain a smoother response?

Solution of exercise 1.

a)
Firstly, we note that the samples g_5 and g_6 have the same value; therefore, we select the first 4 samples as step response model, i.e., $N = 4$.
We also note that the first sample g_1 is null: this means that the process has an input-output delay $d = 1$.
Then, we need to select the prediction horizon. To avoid that the system dynamic matrix G has a null column (thus generating a singular matrix $G^T G$), we need to set $p \geq m + d = 3$. We select $p = 3$.

The dynamic matrix is then $G = \begin{pmatrix} g_1 & 0 \\ g_2 & g_1 \\ g_3 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.25 & 0 \\ 0.375 & 0.25 \end{pmatrix} \in \mathbb{R}^{p \times m}$.

The matrix G is used to compute the solution of the unconstrained optimization problem: $u = (G^T G)^{-1} G^T (w - f)$, with $(G^T G)^{-1} G^T = \begin{pmatrix} 0 & 4 & 0 \\ 0 & -6 & 4 \end{pmatrix}$. Since we are interested in the current control increment only, i.e., in the first element of the vector u, we just need the first row of the matrix $(G^T G)^{-1} G^T$. The resulting vector is the control gain $K = (0 \quad 4 \quad 0)$, and the control increment is computed as $\Delta u(t) = K(w - f)$.

Since the reference is constant, we can assume that the vector of the future reference values is always $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

By considering null initial conditions, we can compute the free response at time $t = 1,2$ over the prediction horizon, with $f(t + k) = y_m(t) + \sum_{i=1}^{N} (g_{k+i} - g_i) \Delta u(t - i)\, , k = 1, \dots, p$:

<u>$t = 1$</u> $\quad y_m(1) = 0$;
$\Delta u(1 - i) = 0, \forall i > 0$.

<u>$k = 1$</u> $\quad f(2) = y_m(1) + \sum_{i=1,\dots,4} (g_{i+1} - g_i) \Delta u(1 - i) = 0$;
<u>$k = 2$</u> $\quad f(3) = y_m(1) + \sum_{i=1,\dots,4} (g_{i+2} - g_i) \Delta u(1 - i) = 0$;
<u>$k = 3$</u> $\quad f(4) = y_m(1) + \sum_{i=1,\dots,4} (g_{i+3} - g_i) \Delta u(1 - i) = 0$.

$$\Delta u(1) = K(w - f) = (0 \quad 4 \quad 0) \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = 4.$$

$u(1) = u(0) + \Delta u(1) = 4.$

$\underline{t = 2}$ $y_m(2) = 2;$
$\Delta u(2-1) = \Delta u(1) = 4,\ \Delta u(2-i) = 0, i > 1.$

$\underline{k = 1}$ $f(3) = y_m(2) + \sum_{i=1,\dots,4}(g_{i+1} - g_i)\Delta u(2-i) = y_m(2) + (g_2 - g_1)\Delta u(1) = 0.25 + 0.25 \cdot 4 = 1.25;$
$\underline{k = 2}$ $f(4) = y_m(2) + \sum_{i=1,\dots,4}(g_{i+2} - g_i)\Delta u(2-i) = y_m(2) + (g_3 - g_1)\Delta u(1) = 0.25 + 0.375 \cdot 4 = 1.75;$
$\underline{k = 3}$ $f(5) = y_m(2) + \sum_{i=1,\dots,4}(g_{i+3} - g_i)\Delta u(2-i) = y_m(2) + (g_4 - g_1)\Delta u(1) = 0.25 + 0.4375 \cdot 4 = 2.$

$$\Delta u(2) = K(w - f) = (0 \quad 4 \quad 0)\left(\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1.25\\1.75\\2\end{pmatrix}\right) = -3.$$

$u(2) = u(1) + \Delta u(2) = 1.$

Alternatively, we can compute the matrix

$$F = \begin{bmatrix}1 & g_2 - g_1 & g_3 - g_2 & g_4 - g_3 & g_5 - g_4\\ 1 & g_3 - g_1 & g_4 - g_2 & g_5 - g_3 & g_6 - g_4\\ 1 & g_4 - g_1 & g_5 - g_2 & g_6 - g_3 & g_7 - g_4\\ 1 & g_5 - g_1 & g_6 - g_2 & g_7 - g_3 & g_8 - g_4\end{bmatrix} \approx \begin{bmatrix}1 & g_2 - g_1 & g_3 - g_2 & g_4 - g_3 & g_5 - g_4\\ 1 & g_3 - g_1 & g_4 - g_2 & g_5 - g_3 & g_5 - g_4\\ 1 & g_4 - g_1 & g_5 - g_2 & g_5 - g_3 & g_5 - g_4\\ 1 & g_5 - g_1 & g_5 - g_2 & g_5 - g_3 & g_5 - g_4\end{bmatrix} =$$

$$= \begin{bmatrix}1 & 0.2500 & 0.1250 & 0.0625 & -0.0375\\ 1 & 0.3750 & 0.1875 & 0.1000 & -0.0375\\ 1 & 0.4375 & 0.2250 & 0.1000 & -0.0375\\ 1 & 0.4750 & 0.2250 & 0.1000 & -0.0375\end{bmatrix}$$

and the state vector $x(t) = [y_m(t)\ \ \Delta x(t-1)\ \ \Delta x(t-2)\ \ \Delta x(t-3)\ \ \Delta x(t-4)]^T$ to compute the free responses at time t as $\boldsymbol{f} = F\boldsymbol{x}(t)$

b)
To smoothen the system response, we can act on the reference trajectory or limit the control effort.
In the first case, we may compute the reference trajectory as:

$$w(t+k) = \begin{cases} y_m(t), k = 0 \\ \alpha \cdot w(t+k-1) + (1-\alpha) \cdot r(t+k), k = 1,2,3 \end{cases}, \text{with } \alpha \in (0,1)$$

As α increases, the predicted errors are smaller for small values of k and thus the response is smoother.
To limit the control effort, we may add constraints in the form $|\Delta u(t)| \leq \Delta u_{MAX}$, with $\Delta_{u_{MAX}} > 0$, or add an additional term to the cost function which weights the control effort: $J = e^T e + \lambda u^T u$, where λ is a positive constant which weights the control effort vs. the square error minimization. With the latter method, the optimization problem becomes unconstrained and the solution is still written in closed form as $u = (G^T G - \lambda I)^{-1} G^T (w - f)$. The former method implies that the optimization problem becomes a constrained quadratic programming one, and the solution is obtained numerically by using solvers; on the other hand, tighter control on the maximum allowed control effort is obtained.

Exercise 2

Consider a process whose impulse response model is given by the following coefficients:

$h_1 = 0.1,\ h_2 = 0.04,\ h_3 = 0.016,\ h_4 = 0.006,\ h_5 \approx 0,\ h_6 \approx 0,\ \ h_7 \approx 0,\ \ h_8 \approx 0.$

a) Compute the control action of a standard Model Algorithmic Control algorithm at time $t = 5$, with:
   - control horizon $m = 3$;
   - constant reference $r(t) = 2, \forall t$;
   - cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory;
   - $y_m(5) = 1.8$;
   - $u(1) = 5, u(2) = 4, u(3) = 3, u(4) = 2.5$.

b) Compute the control actions $u(1)$ and $u(2)$ with the same algorithm of a) but considering the cost function $J = e^T e + 0.1u^T u$, and compare the results with the control actions of point a).

Solution of exercise 2.

a)
Firstly, we note that the samples $h_i, i > 4$, are null; therefore, we select the first 4 samples as step response model, i.e., $N = 4$.
The prediction horizon is equal to the control horizon $m = 3$.

The MAC prediction model is

$$y = H_1 u_+ + H_2 u_- + n,$$

where u_+ is the vector of current and future control actions,

$$u_+ = \begin{pmatrix} u(t|t) \\ u(t+1|t) \\ u(t+2|t) \end{pmatrix},$$

u_- is the vector of past control actions,

$$u_- = \begin{pmatrix} u(t-N+1) \\ u(t-N+2) \\ u(t-N+3) \end{pmatrix} = \begin{pmatrix} u(t-3) \\ u(t-2) \\ u(t-1) \end{pmatrix},$$

and n is the vector of disturbances $\hat{n}(t+k|t), k = 1,2, \dots, p$. $\hat{n}(t|t)$ is defined as the error between the measured output $y_m(t)$ and the output predicted by the impulse response model $y(t) = \sum_{j=1}^{N} h_j u(t-j)$. MAC assumes that the disturbance is constant over the prediction horizon, and that it is equal to the disturbance at time t:
$\hat{n}(t+k|t) = \hat{n}(t|t) \coloneqq y_m(t) - \sum_{j=1}^{N} h_j u(t-j)\, , k = 1, \dots, M.$
The matrixes H_1 and H_2 are computed from the impulse response samples:

$$H_1 = \begin{pmatrix} h_1 & 0 & 0 \\ h_2 & h_1 & 0 \\ h_3 & h_2 & h_1 \end{pmatrix} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0.04 & 0.1 & 0 \\ 0.016 & 0.04 & 0.1 \end{pmatrix} \in \mathbb{R}^{M \times M},$$

$$H_2 = \begin{pmatrix} h_4 & h_3 & h_2 \\ 0 & h_4 & h_3 \\ 0 & 0 & h_4 \end{pmatrix} = \begin{pmatrix} 0.006 & 0.016 & 0.04 \\ 0 & 0.006 & 0.016 \\ 0 & 0 & 0.006 \end{pmatrix} \in \mathbb{R}^{m \times (N-1)}.$$

The forced response is given by $H_1 u_+$, whereas the free response is computed as $f = H_2 u_- + n$.

The matrix H_1 is used to compute the solution of the unconstrained optimization problem: $u = \left({H_1}^T H_1 - \lambda I\right)^{-1} {H_1}^T (w - f)$; since $\lambda = 0$, the algorithm requires the calculation of $\left({H_1}^T H_1\right)^{-1} {H_1}^T = H_1^{-1}$.

We also note that the first sample h_1 is not null: this means that the process has no input-output delay and that, even if $\lambda = 0$, we can use the MAC algorithm (H_1 is singular if $\lambda = h_1 = 0$).

$$H_1^{-1} = \begin{pmatrix} 10 & 0 & 0 \\ -4 & 10 & 0 \\ 0 & -4 & 10 \end{pmatrix}$$

The reference trajectory is computed as:

$$w(t+k) = \begin{cases} y_m(t), k = 0 \\ \alpha\, w(t+k-1) + (1-\alpha)\, r(t+k), k = 1,2,3 \end{cases}, \text{ with } \alpha \in (0,1)$$

There are no indications on the choice of α; for instance, we set $\alpha = 0.5$.

By considering the given output measures and control actions, we can compute the free response at time $t = 5$ over the prediction horizon:

<u>$t = 5$</u> $y_m(5) = 1.8$;
$u(1) = 5, u(2) = 4, u(3) = 3, u(4) = 2.5$;

$$\begin{aligned} \hat{n}(1+k|1) &= \hat{n}(1|1) \coloneqq y_m(5) - \textstyle\sum_{j=1}^{4} h_j u(5-j) = \\ &= y_m(5) - h_4 u(1) - h_3 u(2) - h_2 u(3) - h_1 u(4) = \\ &= 1.8 - 0.006 \cdot 5 - 0.016 \cdot 4 - 0.04 \cdot 3 - 0.1 \cdot 2.5 = 1.34, k = 1,2,3. \end{aligned}$$

$$f = H_2 u_- + n = \begin{pmatrix} 0.006 & 0.016 & 0.4 \\ 0 & 0.006 & 0.016 \\ 0 & 0 & 0.006 \end{pmatrix} \begin{pmatrix} u(2) \\ u(3) \\ u(4) \end{pmatrix} + \begin{pmatrix} \hat{n}(1|1) \\ \hat{n}(1|1) \\ \hat{n}(1|1) \end{pmatrix}$$
$$= \begin{pmatrix} 0.006 & 0.016 & 0.4 \\ 0 & 0.006 & 0.016 \\ 0 & 0 & 0.006 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2.5 \end{pmatrix} + \begin{pmatrix} 1.34 \\ 1.34 \\ 1.34 \end{pmatrix} = \begin{pmatrix} 5.20 \\ 0.51 \\ 0.37 \end{pmatrix}.$$

$$w = \begin{pmatrix} w(6|5) \\ w(7|5) \\ w(8|5) \end{pmatrix} = \begin{pmatrix} \alpha y_m(5) + (1-\alpha) r(6) \\ \alpha w(6|5) + (1-\alpha) r(7) \\ \alpha w(7|5) + (1-\alpha) r(8) \end{pmatrix} = \begin{pmatrix} 0.5 \cdot 1.8 + 0.5 \cdot 2 \\ 0.5 \cdot 1.9 + 0.5 \cdot 2 \\ 0.5 \cdot 1.95 + 0.5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1.9 \\ 1.95 \\ 1.975 \end{pmatrix}.$$

Since, at time $t = 5$, we are interested in the current control action only, i.e., in the first element of the vector u, we just need the first row of the matrix $H_1^{-1} = \begin{pmatrix} 10 & 0 & 0 \\ -4 & 10 & 0 \\ 0 & -4 & 10 \end{pmatrix}$. The resulting vector is the control gain $K = (10 \quad 0 \quad 0)$ [i], and the control action is computed as $u(t) = K(w - f)$

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[i] By considering that H_1 is a diagonal matrix, its inverse is computed as

$$H_1^{-1} = \begin{pmatrix} h_1 & 0 & \dots & 0 \\ h_2 & h_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_p & h_{p-1} & \dots & h_1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/h_1 & 0 & \dots & 0 \\ * & 1/h_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1/h_1 \end{pmatrix}$$

with $*$ indicating generally non-null values; it is then simple to compute $K = (1/h_1 \quad 0 \quad \dots \quad 0)$.

$$u(1) = K(w - f) = (10 \quad 0 \quad 0)\left(\begin{pmatrix} 1.9 \\ 1.95 \\ 1.975 \end{pmatrix} - \begin{pmatrix} 5.20 \\ 0.51 \\ 0.37 \end{pmatrix}\right) = -33.$$

b)

The different cost function leads to a different solution: $u = \left({H_1}^T H_1 + \lambda I\right)^{-1} {H_1}^T (w - f)$, with $\lambda = 0.1$.

$\left({H_1}^T H_1 + \lambda I\right)^{-1} {H_1}^T = \begin{pmatrix} 0.896 & 0.322 & 0.117 \\ -0.037 & 0.884 & -0.405 \\ -0.012 & -0.037 & 0.896 \end{pmatrix}$.

The control gain is then $K_\lambda = (0.896 \quad 0.322 \quad 0.117)$.

By following the same step of the point a), we obtain the control input

$$u(1) = K(w - f) = (0.896 \quad 0.322 \quad 0.117)\left(\begin{pmatrix} 1.9 \\ 1.95 \\ 1.975 \end{pmatrix} - \begin{pmatrix} 5.20 \\ 0.51 \\ 0.37 \end{pmatrix}\right) = -2.31,$$

which clearly shows how the control effort is reduced if compared with the previous solution, obtained without weighting the control action in the cost function.

Exercise 3.

Consider a process whose state-space model is:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = 0.5, N = 0.1, Q = 1.$$

Compute the control actions of a Predictive Functional Control algorithm at time $t = 2$, with:

- control horizon $m = 3$;
- prediction horizon $p = 3$;
- coincident points $h_1 = 1, h_2 = 3$;
- initial conditions:
  $x(0) = y(0) = 0 = y_m(0) = 0$;
  $u(0) = 0.1$;
- constant reference $r(t) = 1, \forall t$;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory.

In the computation, consider the following measured output values: $y_m(1) = 0.02,\ \ y_m(2) = 1.35$.
Moreover, consider the plant-model error (i.e., the difference between future measured outputs $y_m(t+k)$ and the predicted model outputs $y(t+k)$ at future time instants) equal to $\hat{e}(t+k|t) = k\big(y_m(t) - y(t)\big)$.

Solution of exercise 3.

To develop the PFC controller, we need to select the base functions. For the sake of simplicity, and since the reference signal is constant, we chose $n_B = 1$ base function $B_1(k) = k^0 = 1, k = 0,1,2, \dots$ (step function).

Firstly, we have to compute the model response to the base function in the coincidence points, considering null initial conditions $x(0) = 0$:

$\underline{t = h_1 = 1}$ $$\begin{cases} x(1) = 0.5x(0) + 0.1B_1(0) = 0.1 \\ y_{B_1}(1) = x(1) = 0.1 \end{cases};$$

$\underline{t = 2}$ $$\begin{cases} x(2) = 0.5x(1) + 0.1B_1(1) = 0.15 \\ y_{B_1}(2) = x(2) = 0.15 \end{cases};$$

$\underline{t = h_2 = 3}$ $$\begin{cases} x(3) = 0.5x(2) + 0.1B_1(2) = 0.175 \\ y_{B_1}(3) = x(3) = 0.175 \end{cases}.$$

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_{B_1}(h_1) \\ y_{B_1}(h_2) \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.175 \end{pmatrix}$.
The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu^* = \left({Y_B}^T Y_B\right)^{-1} {Y_B}^T (w - f)$, with $\left({Y_B}^T Y_B\right)^{-1} {Y_B}^T = (2.46 \quad 4.31)$, where μ^* is the vector of the optimal parameters at time t. The control action is computed as $u^*(t) = {\mu^*}^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \dots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 1$, we need to find a single parameter $\mu^*(t)$.

For the sake of simplicity we define the reference trajectory as $w(t+k|t) = r(t)$. Since the reference is constant, the vector of the future reference values evaluated in the coincidence points $h_1 = 1$ and $h_2 = 3$ is $w(t+k|t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \forall t, k \geq 0$.

By considering null initial conditions, we can start computing the free response at time $t=1$ over the coincidence points. With the PFC, the output prediction is

$$\hat{y}(t+k|t) = y(t+k) + \hat{e}(t+k|t) = \sum_{i=1,\dots,n_B} y_{B_i}(k)\mu_i(t) + QM^k x(t) + \hat{n}(t+k|t),$$

where the disturbance model is $\hat{n}(t+k|t) = k(y_m(t) - y(t))$.

<u>$t=1$</u> $y_m(0) = 0$; $x(0) = 0$; $u(0) = 0.1$;

Measured output
$y_m(1) = 0.02$

Model output
$x(1) = 0.5x(0) + 0.1u(0) = 0.01$
$y(1) = x(1) = 0.01.$

<u>$h_1 = 1$</u> $f(2) = QM^{h_1}x(1) + \big(y_m(1) - y(1)\big)\cdot h_1 = 0.015;$
<u>$h_2 = 3$</u> $f(4) = QM^{h_2}x(1) + \big(y_m(1) - y(1)\big)\cdot h_3 = 0.011.$

$d(1) = (w - f) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0.015\\0.011\end{pmatrix} = \begin{pmatrix}0.985\\0.989\end{pmatrix}.$
$\mu(1) = (2.46 \quad 4.31)\begin{pmatrix}0.985\\0.989\end{pmatrix} = 6.68;$
$u(1) = \mu(1)B_1(0) = 6.68.$

<u>$t=2$</u>

Measured output
$y_m(2) = 1.35$

Model output
$x(2) = 0.5x(1) + 0.1u(1) = 0.67$
$y(2) = x(2) = 0.67.$

<u>$h_1 = 1$</u> $f(3) = QM^{h_1}x(2) + \big(y_m(2) - y(2)\big)\cdot h_1 = 0.5^1x(2) + \big(y_m(2) - y(2)\big)\cdot 1 = 1.01;$
<u>$h_2 = 3$</u> $f(5) = QM^{h_2}x(2) + \big(y_m(2) - y(2)\big)\cdot h_2 = 0.5^3x(2) + \big(y_m(2) - y(2)\big)\cdot 3 = 2.1.$

$d(2) = (w - f) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1.01\\2.1\end{pmatrix} = \begin{pmatrix}-0.01\\-1.1\end{pmatrix}.$
$\mu(2) = (2.47 \quad 4.31)\begin{pmatrix}-0.01\\-1.1\end{pmatrix} = -4.67;$
$u(2) = \mu(2)B_1(0) = -4.67.$

Exercise 3.

Solve the problem of Exercise 2 by using the Dynamic Matrix Control algorithm and compare the results with the ones of Exercise 2.

Exercise 4.

Consider a process with model described by the equations: $x(t) = 0.2x(t-1) + 0.1u(t-2)$, $y(t) = 2x(t)$.

a) Compute the control actions of a Predictive Functional Control algorithm at time $t = 2$, with:
   - control horizon $m = 2$;
   - prediction horizon $p = 2$;
   - number of coincidence points $n_H = 1$;
   - $n_B = 1$ base function: $B_1(k) = 1$;
   - conditions at time $t \leq 0$: $x(t) = y(t) = 0, u(t) = 0, y_m(t) = 0$;
   - constant reference $r(t) = 1, \forall t$;
   - cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory.

   In the computation, consider the following transfer function to compute the measured output:
   $y(t) = 0.4\, y(t-1) \; + \; 0.2\, u(t-2)$,
   and consider the difference between future measured outputs $y_m(t+k)$ and the predicted model outputs $y(t+k)$ at future time instants constant and equal to $\left(y_m(t) - y(t)\right)$.

b) Which coincidence point was to be chosen in Exercise 6 a) and why.

c) Would the control action sequence change if n_H is chosen equal to 2 (i.e., $n_H = p = 2$)?

NOTES: Due to the input-state delay, the state-space model is written as

$$\begin{cases} x_1(t) = 0.2x_1(t-1) + 0.1x_2(t-1) \\ x_2(t) = u(t-1) \\ y(t) = x_1(t) \end{cases}.$$

Also, due to the delay, the response of the system to the control action at time t is null until time $t+2$, therefore no solution is obtained if we chose $n_H = 1$ coincident point at time $h_1 = 1$: the coincident point to be chosen is $h_1 = 2$. For the same reason, no advantage is gained if we chose $n_H = 2$ (and, therefore, $h_1 = 1$ and $h_2 = 2$): the result would be the same as the results obtained with $n_H = 1$ and $h_1 = 2$.