STABILITY OF MPC

- Discrete-time nonlinear time-invariant system

$$x(k+1) = f(x(k), u(k)), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

- ASSUMPTION $f(\cdot,\cdot)$ continuously differentiable in both arguments
  $f(0,0) = 0$
  $(0,0) \in X \times U$

- PROBLEM (MPC STABILITY): Design an MPC algorithm guaranteeing that the origin is an asymptotically stable equilibrium point

- ASSUMPTION: There exist an AUXILIARY STATE-FEEDBACK CONTROL LAW $u(k) = k_a(x(k))$ and a POSITIVELY INVARIANT SET $X^f \subseteq X$ s.t.

  i) $0 \in X^f$

  ii) $x(k+i) \in X_f, \quad i = 0, 1, \dots$
  $u(k+i) = k_a(x(k+i)) \in U, \quad i = 0, 1, \dots$

  for all $x(k) \in X^f$

MPC $\min_{\underline{u}(k)} J(x(k), \underline{u}(k), N) = \sum_{i=0}^{N-1} \left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \right) + V_f(x(N|k))$

$$x(i|k) \in X \quad , \quad i = 0, \dots, N-1$$

$$u(i|k) \in U \quad , \quad i = 0, \dots, N-1$$

$$x(N|k) \in X_f$$

with $Q \geq 0$, $R > 0$, $V_f(x(N|k))$ FINAL COST

- $V_f(x(N|k))$ plays the role of $\|x(N|k)\|_{\infty}^2$ but generally it is not the optimal final cost
- $u(k) = k_{RH}(x(k)) \leftarrow$ MPC solution for the time-steps $0, 1, \dots, N-1$

ASSUMPTION

$X^{RH}(N)$: set of initial states such that a solution $u(k) = k_{RH}(x(k))$ exists, $k = 0, \dots, N-1$

$V^f(\cdot) : X \to [0, +\infty)$:

auxiliary control law $u(k) = k_a(x(k))$

i) $\Delta V^f(x(k)) = V^f(x(k+1)) - V^f(x(k)) =$

$= V^f(f(x(k), k_a(x(k))) - V^f(x(k))$

$\Delta V^f(x(k)) + \|x(k)\|_Q^2 + \|k_a(x(k))\|_R^2 \leq 0 \quad , \forall x(k) \in X^f$

ii) $V^f(x) \leq \alpha_f(\|x(k)\|) \quad , \forall x(k) \in X^f$

with $\alpha_f : X^{RH}(N) \to [0, \infty)$, strictly increasing,

$\alpha_f(0) = 0$

Theorem If $X^{RH}(N) \neq \emptyset$, the origin is an asymptotically stable equilibrium point with region of attraction $X^{RH}(N)$

Proof (sketch)

$$V(k,N) = J\left(x(k), k_{RH}(x(k)), N\right) = \sum_{i=0}^{N-1} \left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \right) + V^f(x(N|k))$$

i) $V(k,N) \geq \omega \| x(N|k)\|^2$ for some ω ← POS. DEF.

ii) $\underline{u}^*(k,N) = \{ u^*(0|k), \dots, u^*(N-1|k) \}$ ← OPTIMAL SOLUTION AT TIME k OVER N TIME-INTERVALS

with $u^*(k) = k_{RH}(x(k))$

⇒ FEASIBLE ⇒ $x(N|k) \in X^f$

⇒ $u(N|k) = k_a(x(N|k)) \in U$

$\tilde{u}(k, N+1) = \{ \underline{u}^*(k,N), k_a(x(N|k)) \}$, ← FEASIBLE

$$\tilde{u}(k+1,N) = \{u^*(1|k),\ u^*(2|k),\ k_a(x(N|k))\}$$

feasible at time $k+1$

$$V(k+1,N) - V(k,N) \le V(k,N+1) - V(k,N) =$$

$$= \|x(N|k)\|_Q^2 + \|k_a(x(N|k))\|_R^2 + V^f\big(\underbrace{x(N+1|k)}\big) - V^f(x(N|k)) \le 0$$

by assumption

$$x(N+1|k) = f\big(x(N|k),\ k_a(x(N|k))\big)$$

Everything depends on $x(N|k)$

Example $\quad k_o(x)=0, \; X_f=\{0\}, \; V^f(x)=0$

Let $\quad u^*(k)=\{u^*(0|k),\dots,u^*(N-1|k)\}$ $\leftarrow$ OPTIMAL (FEASIBLE) CONTROL SEQUENCE

$x(N|k)=0$

$u(N|k)=k_o(x(N|k))=0$

$k+1 \quad \tilde{u}(k+1)=\{u^*(1|k),\dots,u^*(N-1|k),0\}$ $\leftarrow$ FEASIBLE

$x(N|k+1)=f(x(N-1|k+1),u(N-1|k+1))=f(0,0)$

Assumption on V^f is verified

$$V^f(f(x(k),k_o(x(k)))-V_f(x(k))+\|x(k)\|_Q^2+\|u(k)\|_R^2=0 \quad \forall x(k)\in X^f=\{0\}$$

$V^f(f(0,0))=0$ $\quad V_f(0)$ $\quad x(k)=0$ for $x(N|k)$ $\quad u(k)=0$ $\quad k_o(x(k))=0$