Process Automation (MCER), 2015-2016

Exam

June 8, 2016

Exercise 1 (11 pt.)

Consider a process whose impulse response model is given by the following coefficients:

$$h_1 = 0,\ h_2 = 0.01,\ h_3 = 0.001,\ h_4 = 0.0001,\ h_5 = 0,\ \ h_6 = 0, \dots,$$

and whose state space model is given by the following equations:

$$\begin{cases} x_1(t) = 0.1x_1(t-1) + x_2(t-1) \\ x_2(t) = 0.1u(t-1) \\ y(t) = x_1(t) \end{cases}.$$

Develop the control action $u(4)$ of an MPC controller with the following specifications:

i) Prediction horizon $p = 4$;
ii) Ramp reference signal $r(t) = t, \forall t \geq 0$;
iii) Reference trajectory computed as $w(t) = r(t)$;
iv) Cost function $J = e^T e$, where e is the vector of predicted errors btw. predicted output and reference trajectory;
v) The plant-model error is computed as $\hat{n}(t+k|t) = k \cdot \left(y_m(t) - y(t)\right), \forall k > 0$;

In the computations, where needed, consider the following values of output, input and state variables:

$y_m(0) = 0, y_m(1) = 0.01, y_m(2) = 0.011, y_m(3) = y_m(4) = 0.111;$
$u(0) = u(1) = u(2) = u(3) = 1;$
$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, x(1) = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, x(2) = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, x(3) = \begin{pmatrix} 0.11 \\ 0.1 \end{pmatrix}.$

Exercise 2 (11 pt.) – a. y. 2015-16

Let the process be described by the transfer function: $P(s) = \frac{1+s}{1+\tau s} e^{-0.1s}$, with $\tau \in [10,20)$, and let the delay-free process model be $\tilde{P}_0(s) = \frac{1+s}{1+10s}$.

Design an IMC controller $Q(s)$ such that:

i) the overall system is robustly asymptotically stable;
ii) the overall system has 0 steady-state error for step inputs.

Exercise 2 (11 pt.) – a. y. 2014-15

Find the closed-loop characteristic polynomial of the system characterized by the transfer function model $(1 - 0.1z^{-1})y(t) = 0.1z^{-2}u(t-1) + (1 - z^{-1})e(t)/\Delta$ under a GPC controller with prediction horizon $N = 2$ and constant reference trajectory $w(t) = 1, t = 0,1, \dots$.

Questions (8 pt.)

i) Briefly discuss the characteristics of feedforward and feedback control (1/2 pg. max, 3pt).
ii) Briefly discuss pros and cons of the different system models (step and impulse response, state-space, transfer function) from the perspective of MPC (write a table with a row for each model and two columns for the pros and cons; in the table, write the pros and cons as bullet lists, 5pt).

Solution of exercise 1

Even if both a state-space model and a step response model are available, it is convenient to design a PFC controller with respect to a DMC one due to the (relatively) large prediction horizon.
To design a PFC controller, we need to select the basis functions and the coincidence points. Since the reference signal is a ramp, we chose $n_B = 2$ base functions $B_1(k) = k^0 = 1$ and $B_2(k) = k = 1, k = 0,1,2, \ldots$. Considering that the prediction horizon is $p = 4$ and that there is a time-delay $d = 1$, we chose $n_H = 2$, $h_1 = 2$ (since $d = 1$ the input at time t does not affect the output at time $t + 1$) and $h_2 = 4$ (otherwise there is no reason to set $p = 4$).

Firstly, we have to compute the model response to the base functions, denoted with $y_{B_i}, i = 1,2$, in the coincidence points, considering null initial conditions $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. By using the system model, we obtain $y_B(h_1) = (y_{B_1}(h_1) \quad y_{B_2}(h_1)) = (0.1 \quad 0)$ and $y_B(h_2) = (y_{B_1}(h_2) \quad y_{B_2}(h_2)) = (0.111 \quad 0.21)$. The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{pmatrix} 0.1 & 0 \\ 0.111 & 0.21 \end{pmatrix}$.

The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu^* = {Y_B}^{-1}(w - f)$, with ${Y_B}^{-1} = \begin{pmatrix} 10.5 & 0 \\ -5.55 & 5 \end{pmatrix}$, where μ^* is the vector of the optimal parameters at time t. The control action is the computed as $u(t) = {\mu^*}^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \ldots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 2$, we need to find a vector μ^* with two parameters.

By considering the given output, input and state values, we can start computing the free response at time $t = 4$ over the coincidence points. With PFC, the output prediction is

$$\hat{y}(t + k|t) = \textstyle\sum_{i=1,\ldots,n_B} y_{B_i}(k)\mu_i(t) + QM^k x(t) + \hat{n}(t + k|t),$$

where the last two terms constitute the free response:

$$f(t + k|t) = QM^k x(t) + \hat{n}(t + k|t) = QM^k x(t) + y_m(t) - y(t).$$

$\underline{t = 4}$ $\quad y_m(4) = 0.111; x(3) = \begin{pmatrix} 0.11 \\ 0.1 \end{pmatrix}; u(3) = 1;$

Model output
$x_1(4) = 0.1x_1(3) + x_2(3) = 0.111$
$x_2(4) = 0.1u(3) = 0.1$
$y(4) = x_1(4) = 0.111.$

$\underline{h_1 = 2}$ $\qquad f(6) = QM^{h_1}x(4) + h_1 \cdot \big(y_m(4) - y(4)\big) = 0.0111;$
$\underline{h_2 = 4}$ $\qquad f(8) = QM^{h_2}x(4) + h_2 \cdot \big(y_m(4) - y(4)\big) = 0.0001.$

The reference trajectory vector is $w = \begin{pmatrix} w(t + h_1) \\ w(t + h_2) \end{pmatrix} = \begin{pmatrix} r(6) \\ r(8) \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

The optimal parameters are then:

$$\mu^* = {Y_B}^{-1}(w - f) = \begin{pmatrix} 10.5 & 0 \\ -5.55 & 5 \end{pmatrix}\left(\begin{pmatrix} 6 \\ 8 \end{pmatrix} - \begin{pmatrix} 0.0111 \\ 0.0001 \end{pmatrix}\right) = \begin{pmatrix} 62.88 \\ 6.76 \end{pmatrix};$$

$$u(4) = (B_1(0) \quad B_2(0))\mu^* = (1 \quad 0)\begin{pmatrix} 62.88 \\ 6.76 \end{pmatrix} = 62.88.$$

Solution of exercise 2 – a.y. 2015-2016

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.
The time-delay θ of the process is much smaller than the time constant τ of the process, therefore we can use a Padé approximation to write the delay term of the actual process as a transfer function.

By using the 1/1 Padé approximation $e^{-\theta s} \cong \frac{1-s\frac{\theta}{2}}{1+s\frac{\theta}{2}} = \frac{1-0.05s}{1+0.05s}$, we obtain the following approximated process:

$$P^P(s) \cong \frac{1+s}{1+\tau s} \cdot \frac{1-0.05s}{1+0.05s};$$

The IMC design procedure to robustly stabilize the approximated process $P^P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:
$\tilde{P}(s) = \frac{1+s}{1+10s}\frac{1-0.05s}{1+0.05s} = \tilde{P}_+(s)\tilde{P}_-(s)$
with $\tilde{P}_+(s) = (1-0.05s)$ and $\tilde{P}_-(s) = \frac{1+s}{(1+10s)(1+0.05s)}$

b) Define the controller as follows: $\tilde{Q}(s) = \tilde{P}^{-1}(s) = \frac{(1+10s)(1+0.05s)}{1+s}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1.
Thus, we use the well-known filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 1$.
The IMC controller is then $Q(s) = \frac{(1+10s)(1+0.05s)}{(1+s)(1+\lambda s)}$

Step 3)

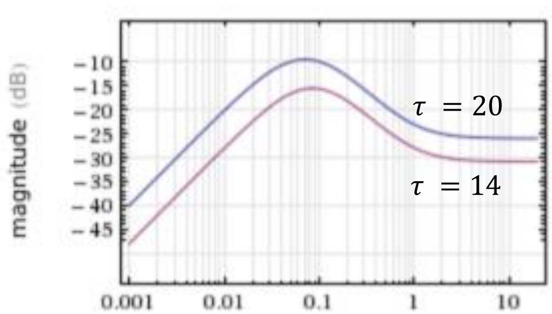
Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.
By definition, the additive uncertainty is defined as follows

$$|\Delta_a(j\omega)| = \left|P^P(j\omega) - \tilde{P}(j\omega)\right| = \left|\frac{1+j\omega}{1+\tau j\omega}\frac{1-0.05j\omega}{1+0.05j\omega} - \frac{1+j\omega}{1+10j\omega}\frac{1-0.05j\omega}{1+0.05j\omega}\right| = \left|\frac{1+j\omega}{1+\tau j\omega} - \frac{1+j\omega}{1+10j\omega}\right| = \left|(10-\tau)\frac{j\omega(1+j\omega)}{(1+\tau j\omega)(1+10j\omega)}\right|.$$

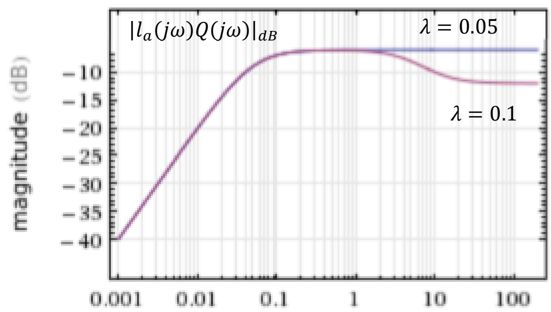
Since $\tau \in [10,20)$, an upper-bound is given by $l_a(j\omega) = 10\frac{j\omega(1+j\omega)}{(1+20j\omega)(1+10j\omega)}$ – see also the Bode plot below:



The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|10\frac{j\omega(1+j\omega)}{(1+20j\omega)(1+10j\omega)}\cdot\frac{(1+10j\omega)(1+0.05j\omega)}{(1+j\omega)(1+\lambda j\omega)}\right| = \left|10\frac{j\omega(1+0.05j\omega)}{(1+20j\omega)(1+\lambda j\omega)}\right| < 1, \forall\omega,$$

which, according to the Bode diagrams below, is true at least for $\lambda \geq 0.05$. To account for the Padé approximation we choose a conservative value $\lambda = 0.1$.



Solution of exercise 2 – a.y. 2014-2015

In the transfer function model we identify $A(z^{-1}) = 1 - 0.1z^{-1}$, $B(z^{-1}) = 0.1$, $d = 2$, $T(z^{-1}) = 1 - z^{-1}$.

The closed-loop characteristic polynomial is written as:
$R(z^{-1})\tilde{A}(z^{-1}) + B(z^{-1})S(z^{-1})z^{-1}$,
where the polynomials $R(z^{-1})$ and $S(z^{-1})$ are:
$R(z^{-1}) = \frac{T(z^{-1})+z^{-1}\sum_{i=3,4}k_i I_i(z^{-1})}{\sum_{i=3,4}k_i}$ and $S(z^{-1}) = \frac{\sum_{i=3,4}k_i F_i(z^{-1})}{\sum_{i=3,4}k_i}$.

The degree of the polynomials I_j's is equal to the degree of $B(z^{-1})$, denoted with d_B, minus 1. Since $d_B = 0$, the polynomials $I_j(z^{-1})$ are null, and only the polynomials F_j's are needed to compuete $R(z^{-1})$ and $S(z^{-1})$.

The polinomials F_j's and the constants k_j's are found from the Diophantine equation written as:
$T(z^{-1}) = E_j(z^{-1})\tilde{A}(z^{-1}) + z^{-j}F_j(z^{-1}), j = d+1, \dots, d+N$,
where $\tilde{A}(z^{-1}) = \Delta A(z^{-1}) = 1 - 1.1z^{-1} + 0.1z^{-2}$, the degree of $E_j(z^{-1})$ is $(j-1)$ and the degree of $F_j(z^{-1})$ is equal to the degree of $A(z^{-1})$. Since $d = 2$ and $N = 2$, we have to solve the following two equations:

$j = d + 1 = 3$:
$T(z^{-1}) = E_3(z^{-1})\tilde{A}(z^{-1}) + z^{-3}F_3(z^{-1})$, with $E_3(z^{-1}) = e_0 + e_1z^{-1} + e_2z^{-2}$ and $F_3(z^{-1}) = f_0 + f_1z^{-1}$;

$$1 - z^{-1} = (e_0 + e_1z^{-1} + e_2z^{-2})(1 - 1.1z^{-1} + 0.1z^{-2}) + z^{-3}(f_0 + f_1z^{-1});$$

$$\begin{cases} e_0 = \cdots \\ e_1 = \cdots \\ e_2 = \cdots. \\ f_0 = \cdots \\ f_1 = \cdots \end{cases}$$

It follows that:

$E_3(z^{-1}) = \cdots;$
$G_3(z^{-1}) = E_3(z^{-1})B(z^{-1}) = \cdots;$
$F_3(z^{-1}) = \cdots.$

$G_3(z^{-1})$ is written as $G_3(z^{-1}) = g_0 + (G_3(z^{-1}) - g_0)$, with $g_0 = \cdots$.

$j = d + 2 = 4$:
$T(z^{-1}) = E_4(z^{-1})\tilde{A}(z^{-1}) + z^{-4}F_4(z^{-1})$, with $E_4(z^{-1}) = e_0 + e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3}$ and $F_4(z^{-1}) = f_0 + f_1 z^{-1}$;

$$1 - z^{-1} = (e_0 + e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3})(1 - 1.1z^{-1} + 0.1z^{-2}) + z^{-4}(f_0 + f_1 z^{-1})\ ;$$

$$\begin{cases} e_0 = \cdots \\ e_1 = \cdots \\ e_2 = \cdots \\ e_3 = \cdots \\ f_0 = \cdots \\ f_1 = \cdots \end{cases}$$

It follows that

$E_4(z^{-1}) = \cdots;$
$G_4(z^{-1}) = E_4(z^{-1})B(z^{-1}) = \cdots;$
$F_4(z^{-1}) = \cdots.$

$G_4(z^{-1})$ is written as $G_4(z^{-1}) = g_0 + g_1 z^{-1} + (G_4(z^{-1}) - g_0 - g_1 z^{-1})$, with $g_0 = \cdots$, $g_1 = \cdots$.

Then we have that:

$$G'(z^{-1}) = \begin{pmatrix} (G_3(z^{-1}) - g_0)z \\ (G_4(z^{-1}) - g_0 - g_1 z^{-1})z^2 \end{pmatrix} = \cdots.$$

Let K denote the first row of G^{-1}: $K = (k_3 \quad k_4) = \cdots$.

The closed-loop characteristic polynomial is therefore

$$R(z^{-1})\tilde{A}(z^{-1}) + B(z^{-1})S(z^{-1})z^{-1} = \cdots$$

If the absolute values of the roots of the polynomial are smaller than 1, the controlled system is asymptotically stable.