Master in Control Engineering

Process Automation
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AUTOMATICA E GESTIONALE ANTONIO RUBERTI

SAPIENZA
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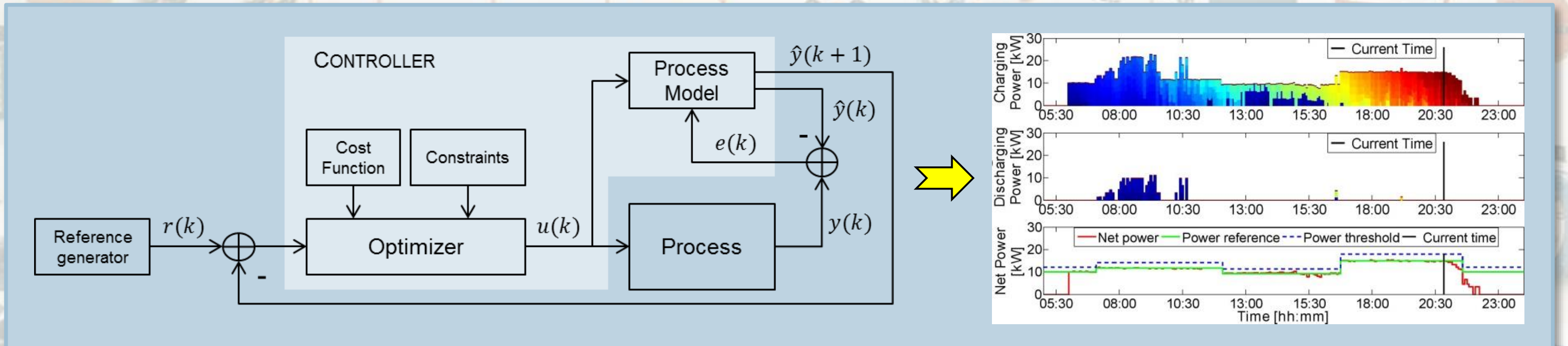
Master in Control Engineering

Process Automation

11. MPC Building Blocks

Slides based on:
E.F. Camacho, C. Bordons Alba, “Model Predictive Control”, *Advanced Textbooks in Control and Signal Processing*, Springer,-Verlag, XXII, 2nd ed., 2007, 405 p., ISBN 978-0-85729-398-5.

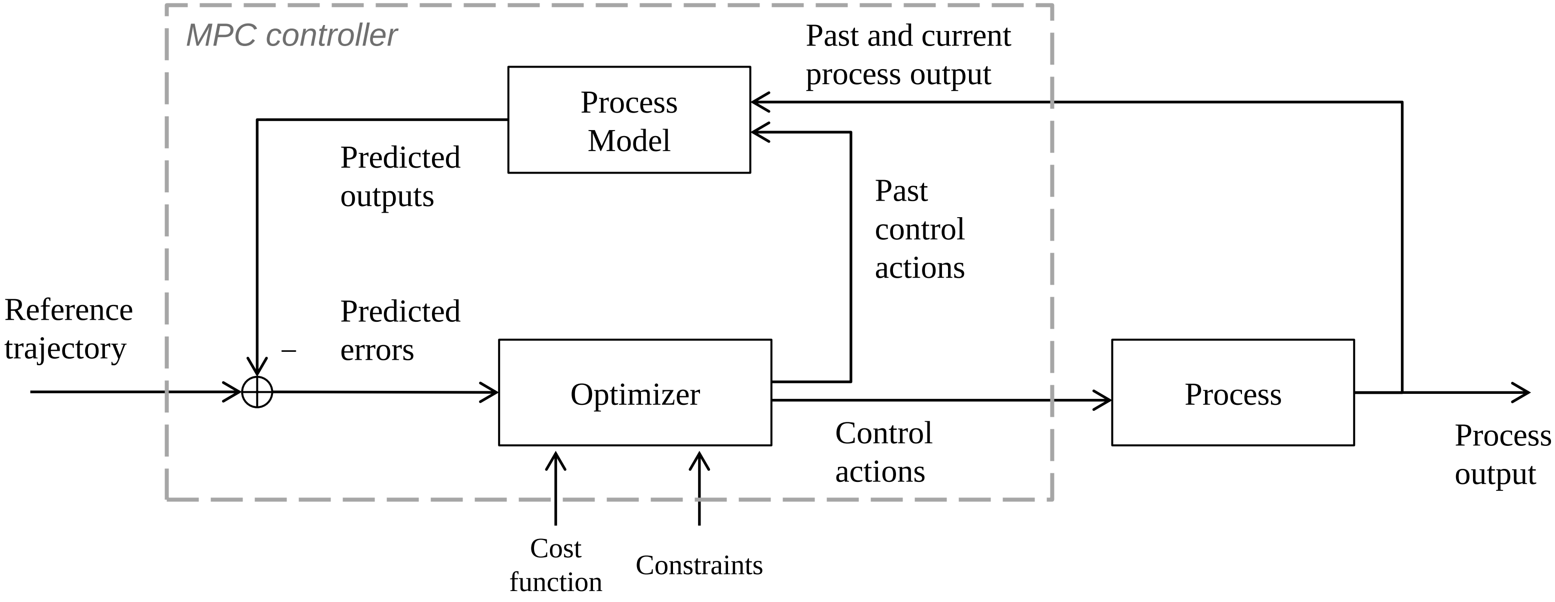


Outline

- Model Predictive Control (MPC) building blocks
  - Introduction
  - Prediction model
  - Free and forced response
  - Objective function
  - Control law
  - Summary

Introduction

- Basic elements to obtain a MPC controller
  - Definition of the process model
  - Definition of the cost function
  - Computation of the control law

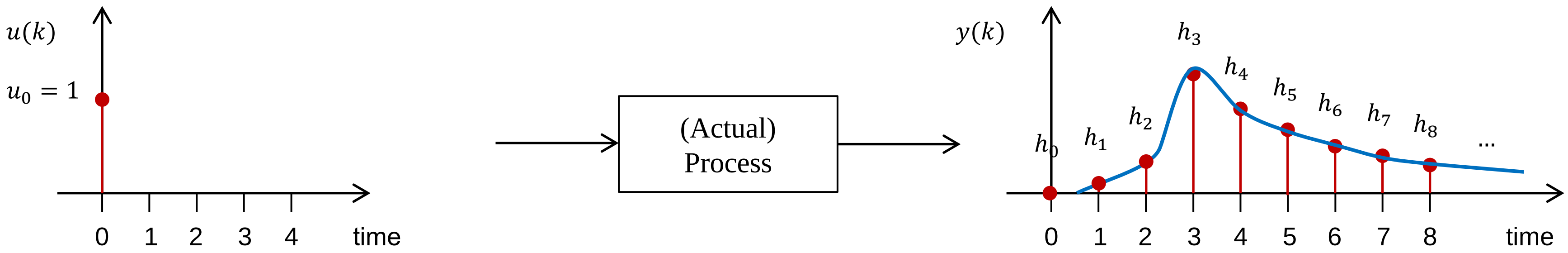


Prediction model

- Objective
  - Predict N future samples of the output sequence $\{\hat{y}(t+k|t)\}_{k=1,\dots,N}$
    - t: current time instant
      $y(t)$: measured process output at time t
      $\hat{y}(t+k)$: predicted process output at time $t+k$
      N: prediction horizon
- Models
  - Input-output (reference-output, disturbace-output) models
    - Impulse response
    - Step response
    - Transfer function
  - Sate-space model
- Trade-off
  - Complete enough to fully capture the «dominant» process dynamics
  - Simple enough to permit theoretic analysis

Prediction model

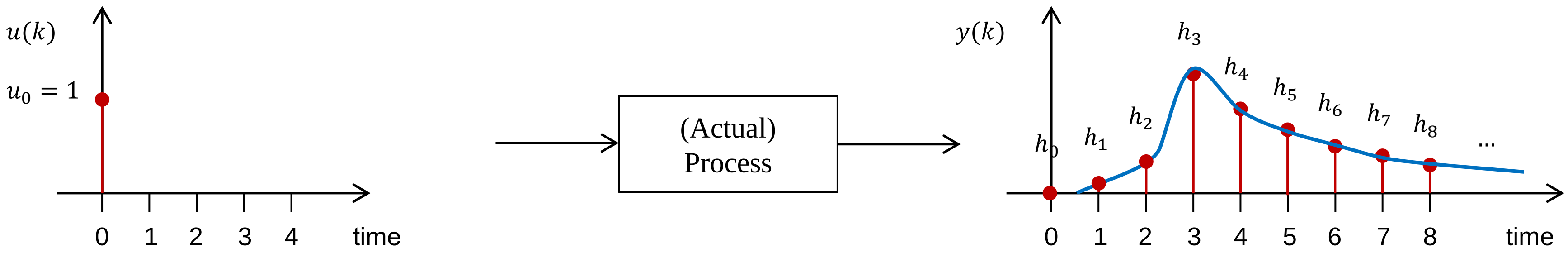
- Impulse response models



  - Impulse input: $u(t) = \delta(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0, & \text{otherwise} \end{cases}$
  - Measured sampled output: $y(t) = \sum_{i=1,\dots,\infty} h_i u(t-i)\,, t = 0,1,2,\dots$

Prediction model

- Impulse response models



– Assumption
  - Stable process without integrators
    – The response decays for $t \to \infty$
  - Approximated sampled output for sufficiently large N (usually $40 \div 50$ samples)

$$y(t) \cong \sum_{i=1,\dots,N} h_i u(t-i) \coloneqq H(z^{-1})u(t), \quad t = 0,1,2,\dots$$

with $H(z^{-1}) = h_1 z^{-1} + h_2 z^{-2} + \dots h_N z^{-N}$

z^{-1}: backward shift operator

$$z^{-1}x(t) = x(t-1)$$
$$z^{-2}x(t) = x(t-2)$$
$$\dots$$

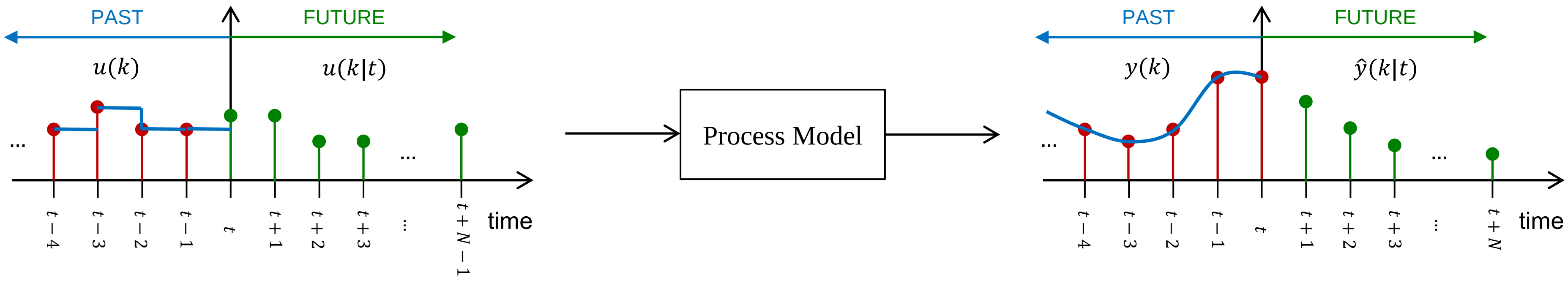
– $H(z^{-1})$ is the impulse response model of the process

– $y_\delta(k) = H(z^{-1})\delta(k) \xrightarrow{\mathcal{Z}} y_\delta(z) = H(z^{-1})\mathcal{Z}\{\delta(k)\} = H(z^{-1})$

$\mathcal{Z}\{\delta(k)\} = 1$

Prediction model

- Impulse response models



- Input
  - Past control actions at time t: $u(k),\ k = \cdots, t-3, t-2, t-1$
  - Control actions computed at time t: $u(k|t),\ k = t, t+1, \ldots, t+N-1$
- Output
  - Measured sampled output at time $k = t$: $y(k),\ \ k = \cdots, t-2, t-1, t$
  - Predicted output at time $k = t$:

$$\hat{y}(t+k|t) = \sum_{i=1,\ldots,N} h_i \underbrace{u(t+k-i|t)} \coloneqq H(z^{-1})u(t+k|t), \qquad k = 1, \ldots, N$$

$$\underbrace{u(t+k-N), \ldots, u(t-1)}_{\text{Past control actions}}, \underbrace{u(t|t), \ldots, u(t+N-1|t)}_{\text{Future control actions}}$$

Remark:
$u(k|t) = u(k)$,
$\forall k < t$

Remark
- $u(t+N-1|t)$ is needed for $k = N$
- sequance of N future control actions to be computed at time t

Prediction model

- Impulse response models
  - Pros
    - Linear model
      - Simple
      - Superposition principles in the MIMO LTI case

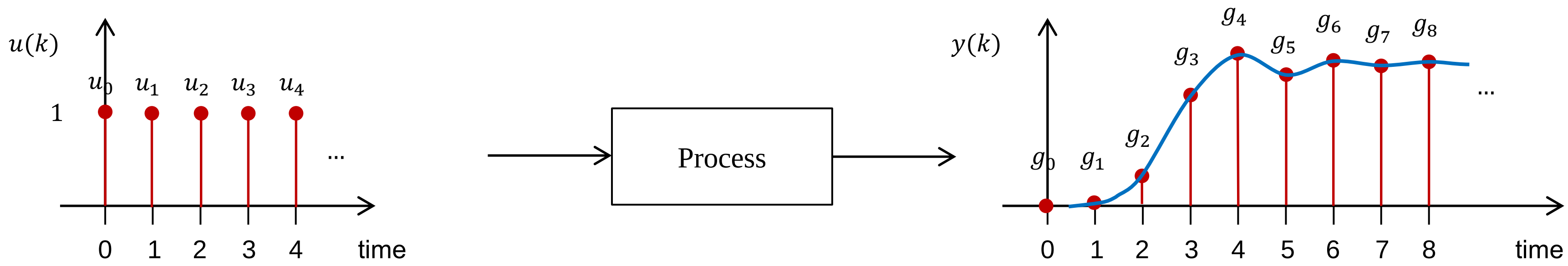
$$\hat{y}^q(t+k|t) = \sum_{p=1,\dots,P} \sum_{i=1,\dots,N} h_i^{pq} u(t+k-i|t) = \sum_{p=1,\dots,M} H^{pq}(z^{-1}) u(t+k|t)\,,$$
$$k = 1, \dots, N, q = 1, \dots, Q$$

where P and Q are the number of inputs and outputs

    - No *a priori* information on the process is needed
    - Non-minimum phase processes and processes with time-delays are dealt with straightforwardly
  - Cons
    - Applicable to stable systems without integrators
    - (Usualy) very large number N of parameters to be identified

Prediction model

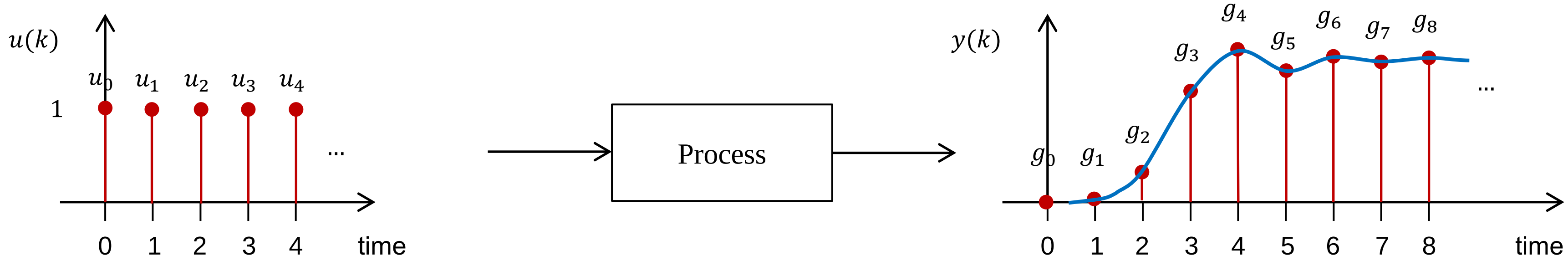
- Step response models



  - Step input: $u(k) = u_{-1}(k) = \begin{cases} 0, & \text{if } k < 0 \\ 1, & \text{if } k \geq 0 \end{cases}$
  - Measured sampled output: $y(k) = \sum_{i=1,\ldots,\infty} g_i \Delta u(k-i)\,, k = 0,1,2,\ldots$
    with $\Delta u(k) = u(k) - u(k-1)$

Prediction model

- Step response models



– Assumption
  - Stable process without integrators:
    – The response stabilizes for $k \to \infty$
  - Approximated sampled output for sufficiently large N (usually $40 \div 50$ samples)

$$y(k) \cong \sum_{i=1,\dots,N} g_i \Delta u(k-i) := G(z^{-1})\Delta u(k), \quad k = 0,1,2,\dots$$

with $G(z^{-1}) = g_1 z^{-1} + g_2 z^{-2} + \dots g_N z^{-N}$

$\mathcal{Z}\{u_{-1}(k)\} = (1 - z^{-1})^{-1}$

– $G(z^{-1})$ is the step response model of the process
  - $y_{u_{-1}}(k) = G(z^{-1})\big(u_{-1}(k) - u_{-1}(k-1)\big) \xrightarrow{\mathcal{Z}} y_{u_{-1}}(z) = G(z^{-1})(1 - z^{-1})\mathcal{Z}\{u_{-1}(k)\} = G(z^{-1})$

Prediction model

- Step response models
  - Predicted output at time $k = t$:
    - $\Delta u(k) = u(k) - u(k-1) = (1 - z^{-1})u(k), \forall k$
    - $\hat{y}(k|t) = \sum_{i=1,\dots,N} g_i \Delta u(k-i|t) = G(z^{-1})\Delta u(k|t)$

$$= G(z^{-1})(1 - z^{-1})u(k|t), \quad k = t+1, \dots, t+N$$

  - Equivalent to the impulse response model
    - Impulse input: $u(k) = \delta(k) = \begin{cases} 0, & \text{if } k < 0 \\ 1, & \text{if } k \geq 0 \end{cases} = u_{-1}(k) - u_{-1}(k-1)$
    - It follows that:

$$h_i = g_i - g_{i-1}$$

and

$$g_i = \sum_{j=0,\dots,i} h_j$$

Prediction model

- Transfer function
  - Process model:

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t-1)$$

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n_a}$$

$$B(z^{-1}) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n_b}$$

  - Predicted output:

$$\hat{y}(t+k|t) = \frac{B(z^{-1})}{A(z^{-1})} u(t+k-1|t)$$

  - Pros
    - Valid also for unstable processes
    - Smaller number of parameters
  - Cons
    - Needs *a priori* knowledge of the process
      - (at least, the order of the polynomials A and B)

Prediction model

- State space
  - Process model (LTI):
    - Implicit representation (input-state-output)

$$\begin{cases} x(t) = Ax(t-1) + Bu(t-1) \\ y(t) = Cx(t) \end{cases}$$

$x(t)$: state space variables

    - Explicit representation

$$y(t+1) = Cx(t+1) = C\big(Ax(t) + Bu(t)\big)$$

  - Predicted output:

$$\hat{y}(t+k|t) = C\hat{x}(t+k|t) = C\left(A^k x(t) + \sum_{i=1,\dots,k} A^{i-1} Bu(t+k-i|t)\right)$$

  - Pros
    - It can be extended to
      - Multivariable processes
      - Nonlinear processes
      - Hybrid systems
      - …

by means of standard control-theoretical approaches

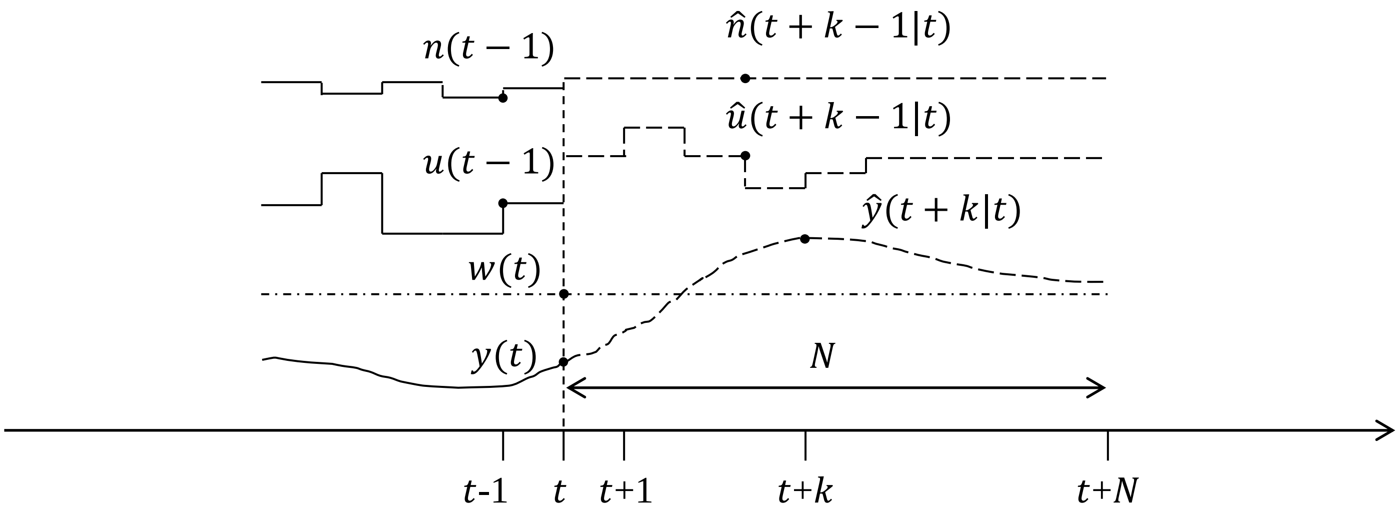
  - Cons
    - Complex
      - E.g., it often needs a state observer

Disturbance model

- As important as the process model
  - Process output depends on both input and disturbance

$$y(t+k|t) = f\big(u(t+k-1|t)\big) + g\big(n(t+k-1|t)\big)$$

$n(t)$: disturbance



  - Output prediction requires both input and disturbance predictions

$$\hat{y}(t+k|t) = f\big(\hat{u}(t+k-1|t)\big) + g\big(\hat{n}(t+k-1|t)\big)$$

Disturbance model

- Controlled AutoRegressive Integrated Moving Average (CARIMA)

$$n(t) = \frac{C(z^{-1})}{D(z^{-1})} e(t)$$

  - $e(t)$: white noise
  - $C(z^{-1})$: used to model coloured noises
    - $C(z^{-1})e(t) = \begin{cases} \text{coloured noise if } C(z^{-1}) \neq 1 \\ \text{white noise if } C(z^{-1}) = 1 \end{cases}$
  - The disturbance models both the actual disturbances and model uncertainties
    - Often the model imperfections are attributed to (modelled as) a disturbance

$$e(t) \coloneqq y(t) - \hat{y}(t)$$

- Appropriate for
  - Random changes occurring at random instants
    - E.g., changes in the quality of a material
  - Brownian motion

Free and forced response

- Definition of free and forced input
  - $u(t) \coloneqq u_f(t) + u_c(t)$,

    Free input: $u_f(t+k) \coloneqq \begin{cases} u(t+k), \forall k < 0 \\ u(t-1), \forall k \geq 0 \end{cases}$

    Forced input: $u_c(t+k) \coloneqq \begin{cases} 0, \forall k < 0 \\ u(t+k|t) - u(t-1) \coloneqq \Delta u(t+k|t), \forall k \geq 0 \end{cases}$

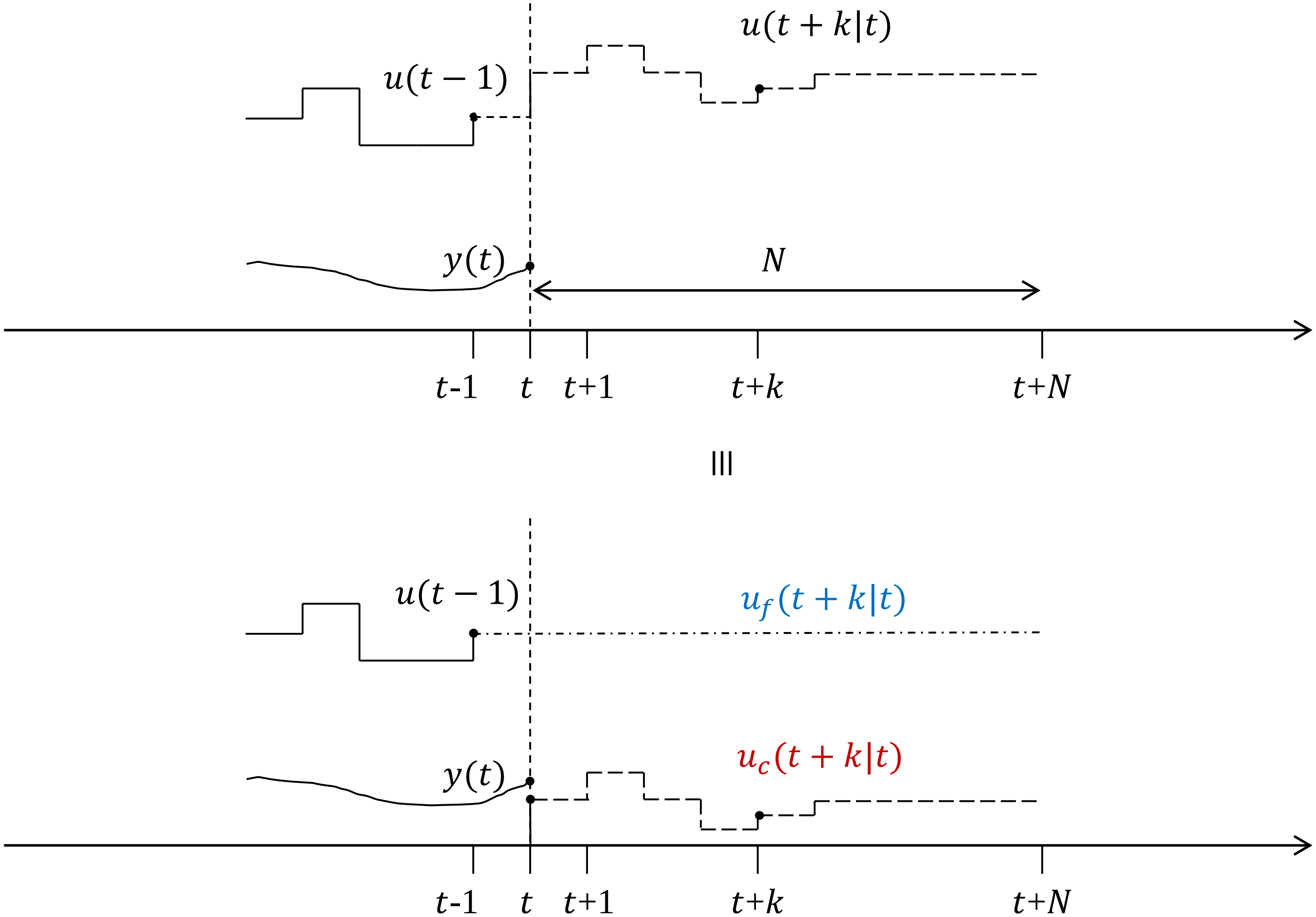
- Free and forced output prediction

$$\hat{y}(t+k|t) \coloneqq \hat{y}_f(t+k|t) + \hat{y}_c(t+k|t), \forall k \geq 1$$

  - Free response
    - Output prediction considering $\Delta u(t+k) = 0, \forall k \geq 0$
    - Evolution of the process from current state
  - Forced response
    - Output prediction considering the forced control action

      $u_C(t+k) \coloneqq \Delta u(t+k) - u(t-1), \forall k \geq 0$
    - Evolution of the process due to the control action variations

Free and forced responce



Objective function

- Generally, it penalizes
  - The difference between the future output and the reference during the prediction horizon
  - The control effort during the control horizon

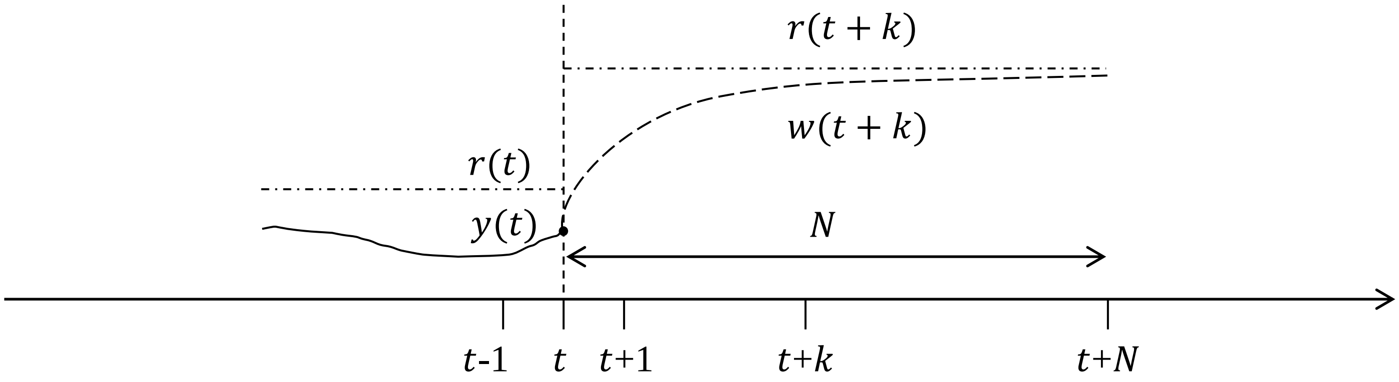
$$J(N_1, N_2, N_u) = \sum_{k=N_1}^{N_2} \delta(k)\big(\hat{y}(t+k|t) - \widehat{w}(t+k|t)\big)^2 + \sum_{k=1}^{N_u} \lambda(k)\big(\Delta u(t+k-1)\big)^2$$

  - N_1, N_2 define the horizon during which it is desirable that $\hat{y} \rightarrow w$
    - E.g., if N_1 is too small, the smoothness of the output may be affected
    - E.g., if the process has a dead time τ, it must hold that $N_1 \geq \tau$
    - E.g., in case of a non-minimum phase process, with inversion period τ, it must hold that $N_1 \geq \tau$
  - N_u defines the control horizon
  - $\delta(k), \lambda(k)$ are parameters which weight the future behaviour
    - E.g., $\delta(k) = \alpha_2^{N-k}, \text{with } \alpha \in (0,1)$
      - First errors are more penalized for faster control

Objective function

- Reference
  - $\hat{w}$ is the prediction of the reference signal
  - The reference trajectory may be known a priori
    - E.g., robotics, servos, …
  - The reference trajectory w may not coincide with the real reference r
    - E.g., at a set-point change at time t, the reference may be computed as a moving average starting from $y(t)$

$$w(t) = y(t)$$
$$w(t+k) = \alpha w(t+k-1) + (1-\alpha) r(t+k), k = 1,2,\dots, \alpha \in (0,1)$$



- E.g.,
$$w(t+k) = r(t+k) - \alpha^k \big(y(t) - r(t)\big), k = 1,2,\dots, \alpha \in (0,1)$$

Constraints

- E.g., quadratic optimization problem

$$\min_u J(u)$$
$$s.t.$$
$$u_{min} \leq u(t) \leq u_{max}, \forall t$$
$$\Delta u_{min} \leq u(t) - u(t-1) \leq \Delta u_{max}, \forall t$$
$$y_{min} \leq y(t) \leq y_{max}, \forall t$$

  – Generally, no closed-form solution

Control law

- Control horizon N_u

$$J(N_1, N_2, N_u) = \sum_{k=N_1}^{N_2} \delta(k)\big(\hat{y}(t+k|t) - \hat{w}(t+k|t)\big)^2 + \sum_{k=1}^{N_u} \lambda(k)\big(\Delta u(t+k-1)\big)^2$$

  - N_u is usually less than N_2 for scalability issue
  - Free evolution of $u(t+k)$ for $k > N_u$ may produce undesirable high-frequency control signals, up to instability
  - Control horizon concept:

$$u(t+k) = u(t+k-1), \qquad k = N_u+1, N_u+2, \ldots$$
$$\Delta u(t+k) = 0, \qquad k = N_u+1, N_u+2, \ldots$$

  - Equivalently, in the control function, we may set $\lambda(k) = \infty, k = N_u+1, N_u+2, \ldots$

Control law

- Structured control laws
  - The control signal can be expressed as a linear combination of base functions

$$u(t+k) = \sum_{i=1}^{n_B} \mu_i(t) B_i(k)$$

- n_B: number of base functions
- $B_i(k)$: k-th sample of the i-th base function
  - Selected based on the nature of the process
  - E.g., polynomial base functions

$$B_0 = 1, B_1 = k, B_2 = k^2, \ldots$$

- $\mu_i(t)$: i-th coefficient computed at time t

- At time t, the optimization problem has n_B unknowns (vs. N_u of the standard objective f.)

Summary

- MPC building blocks introduced
  - Introduced the main prediction models
    - Input-output models
      - Impulse response model
      - Step response model
      - Transfer function model
    - State-space models
    - Disturbance model
      - CARIMA
  - Introduced the free and forced response concept
  - Defined the optimization problem
    - Objective function
    - Control law