TIME-DELAY SYSTEMS

PREREQUISITE : BODE & NYQUIST DIAGRAMS

$$F(j\omega) = K \frac{(1+j\omega\tau_1)(1+j\omega\tau_2)\dots}{(j\omega)^m (1+j\omega\tau_{10})(1+j\omega\tau_{11})\dots}$$

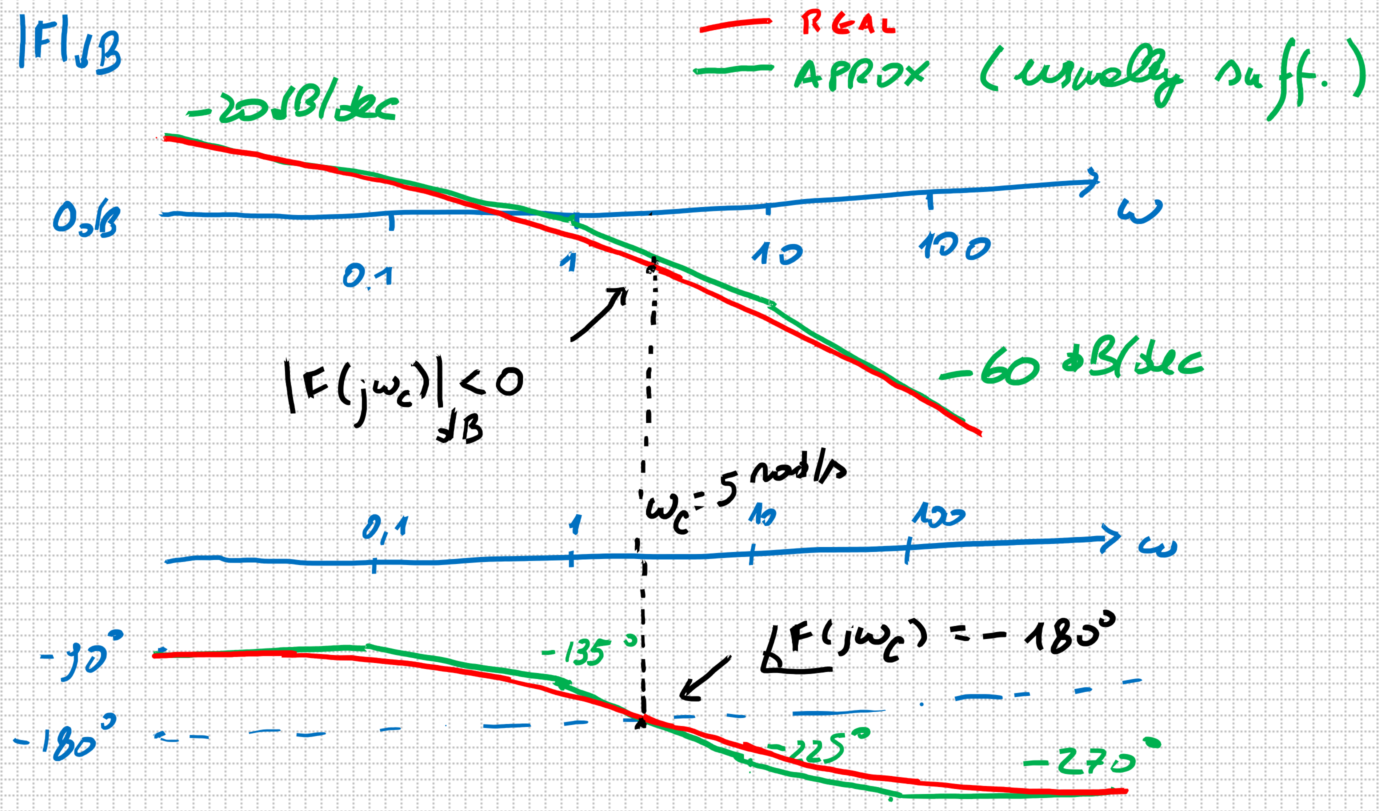
$$\to \boxed{F(s)} \to$$

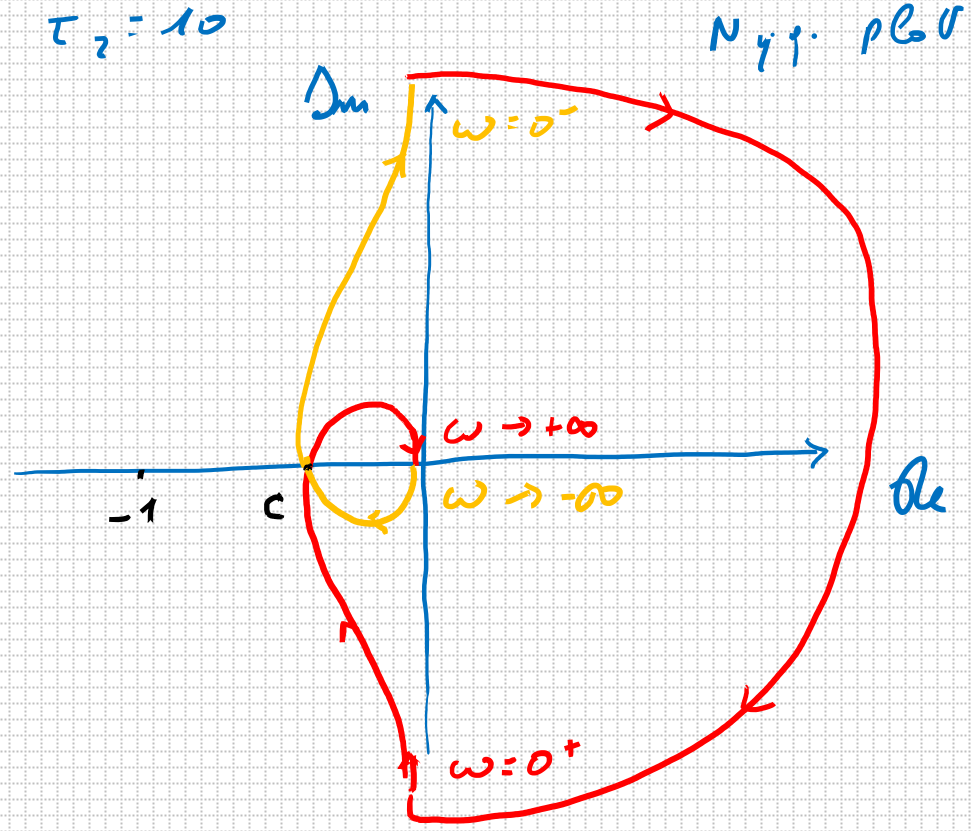
$$F(s) = K_f \frac{(s-z_1)(s-z_2)\dots}{s^m (s-p_1)\dots}$$

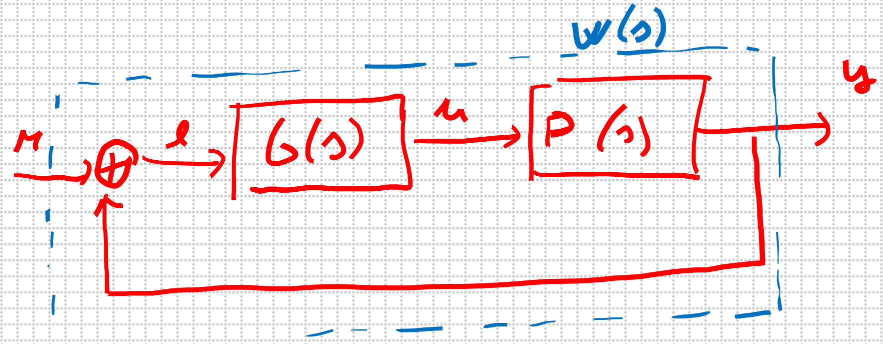
$$\begin{cases} |F(j\omega)|_{dB} := 20 \log_{10} |F(j\omega)| & \text{MAGNITUDE PLOT} \\ \\ \angle F(j\omega) := \text{phase}\,(F(j\omega)) & \text{PHASE PLOT} \end{cases}$$

Ex

$$F(j\omega) = K \frac{1}{j\omega\,(1+j\omega\tau_1)(1+j\omega\tau_2)}, \qquad K = 1, \quad \tau_1 = 1, \quad \tau_2 = 10$$







$F(s) = P(s)\,G(s)$

OPEN-LOOP T.F.

$W(s)$ CLOSED-LOOP T.F.

NYQUIST THEOREM

$$\vec{N}^{-1}_{F(j\omega)} = n^+_{CL} - n^+_{OL}$$

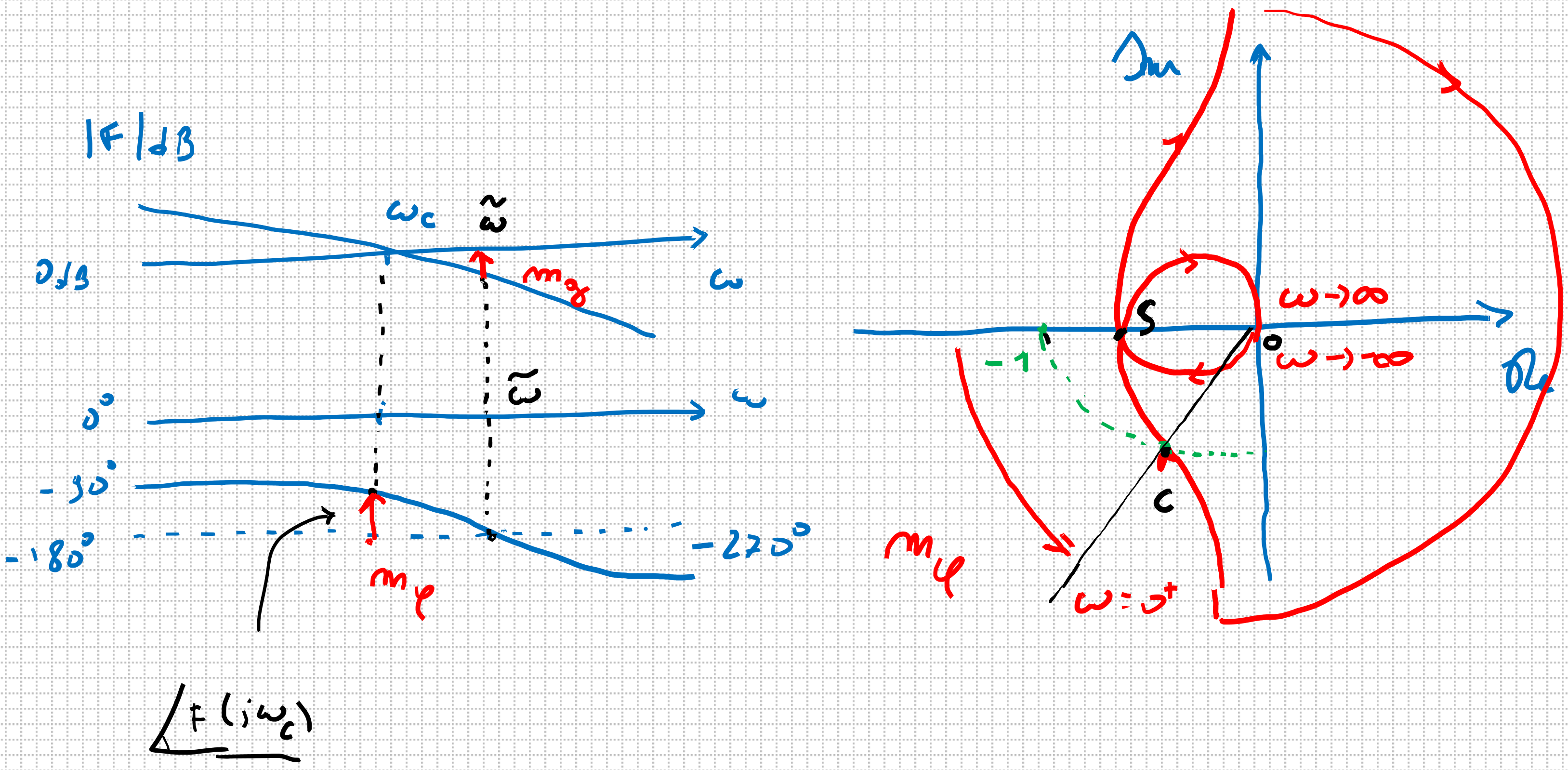
$$\Rightarrow \boxed{n^+_{CL} = \underbrace{\vec{N}^{-1}_{F}}_{0} - \underbrace{n^+_{OL}}_{0} = 0}$$

STABILITY MARGINS

$F(s) = P(s)\,G(s)$

- ASSUMPTIONS: $F(s)$ has no poles with positive real part

$m^+_{OL} = 0$, $\vec{N}^{-1}_F = 0 \Rightarrow m^+_{CL} = 0$



m_g: $|F(j\tilde{\omega})|$

$\tilde{\omega}$ s.t. $\angle F(j\tilde{\omega}) = -180°$

$F = K P \quad (G(s) = K)$

K_{MAX} : $K\,|F(j\tilde{\omega})| = 1$

$K_{MAX} = |F(j\tilde{\omega})|^{-1}$

$m_g = -20 \log_{10} |F(j\tilde{\omega})|$

m_φ: $\angle F(\omega_c)$: ω_c s.t. $|F(j\omega_c)| = 1$

$m_\varphi = \angle F(j\omega_c) - \angle(-1) =$

$= \angle F(j\omega_c) - (-180°)$

(Example : trickle bed reactor)

COMPONENT BALANCE

$$V \frac{dc(t)}{dt} = q_1 c_i(t) + \alpha q c_2(t) - (1+\alpha) q c(t) - V k c(t) \qquad (1)$$

STATIC ANALYSIS

OP. POINT $\bar{s} = (\bar{c}, \bar{c}_i, \bar{c}_2)$

$$V \left.\frac{dc(t)}{dt}\right|_{\bar{s}} = q_1 \bar{c}_i + \alpha q \bar{c}_2 - (1+\alpha) q \bar{c} - V k \bar{c} = 0$$

$$\Rightarrow \quad \bar{c}_2 = \frac{1}{\alpha q}\left[(1+\alpha) q \bar{c} + V k \bar{c} - q \bar{c}_i\right] = \frac{(1+\alpha) q + V k}{\alpha q} \bar{c} - \frac{1}{\alpha} \bar{c}_i$$

DYNAMIC ANALYSIS

Deviation variables
$$\begin{cases} c'(t) = c(t) - \bar{c}(t) & \text{STATE VAR.} \\ c_i'(t) = c_i(t) - \bar{c}_i(t) & \text{INPUT VAR.} \\ c_1'(t) = c_1(t) - \bar{c}_1(t) & \text{OUTPUT VAR} \end{cases}$$

SYSTEM MODEL

$$\begin{cases} V \dot{c}'(t) = q_1 c_i'(t) + \alpha q c_2'(t) - (1+\alpha) q c'(t) - V k c'(t) & (2) \leftarrow \text{eq}(1) \\ c_1'(t) = c'(t - \theta_1) & \text{EXIT LINE} \\ c_2'(t) = c_1'(t - \theta_2) = c'(t - \theta_1 - \theta_2) = c'(t - \theta_3) & \text{RECYCLE LINE} \end{cases}$$

$$\theta_3 := \theta_1 + \theta_2$$

(2) :

$$V \dot{c}'(t) = q\, c_i'(t) + \alpha q\, c'(t - \theta_3) - (1+\alpha) q\, c'(t) - V k\, c'(t) \qquad (3)$$

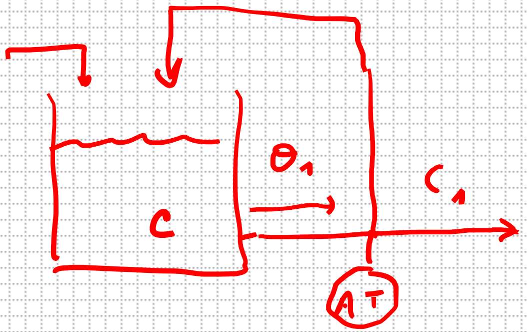
$$\xrightarrow{\mathcal{L}} \quad sV c'(s) = q\, c_i'(s) + \alpha q\, c'(s)\, e^{-\theta_3 s} - (1+\alpha) q\, c'(s) - V k\, c'(s) \qquad (4)$$

$$c'(s) = \frac{q}{sV - \alpha q\, e^{-\theta_3 s} + (1+\alpha) q + V k}\, c_i'(s)$$

multiply num and den by $\frac{1}{q+Vk}$

$$c'(s) = \frac{\frac{q}{q+Vk}}{s\frac{V}{q+Vk} + 1 + \frac{\alpha q}{q+Vk}\left(1 - e^{-\theta_3 s}\right)}\, c_i'(s)$$

$$\begin{cases} K := \dfrac{q}{q+Vk} \\[2ex] \tau := \dfrac{V}{q+Vk} \end{cases}$$



$$c'(s) = \frac{K}{1 + s\tau + \alpha K\left(1 - e^{-\theta_3 s}\right)}\, c_i'(s) \qquad 5)$$

$$c_1'(s) = c'(s)\, e^{-\theta_1 s} \qquad 6)$$

5) + 6).

$$P(s) = \frac{K}{1 + s\tau + \alpha K\left(1 - e^{-\theta_3 s}\right)} e^{-\theta_1 s} = \frac{c'_1(s)}{c'(s)}$$

- $\theta_3 = \theta_1 + \theta_2$ is the feedback delay appearing in the DEN of $P(s)$
- θ_1 also appears in NUM of $P(s)$
- If $\theta_1 \simeq \theta_2 \simeq 0$ $\quad P(s) = \dfrac{K}{1 + s\tau}$

K, τ : gain and time constant of a recycle reactor without delays (STIRRED TANK REACTOR)

- Assume that θ is "small" w.r.t. τ $\quad (\theta < 0.1\tau)$ but it is not negligible

Padé approximation $\quad e^{-\theta_3 s} \simeq \dfrac{1-\frac{\theta_3}{2}s}{1+\frac{\theta_3}{2}s}$

$$P(s)=\frac{K}{1+s\tau+\alpha K\left(1-\dfrac{1-\frac{\theta_3}{2}s}{1+\frac{\theta_3}{2}s}\right)}e^{-\theta_1 s}=\frac{K}{1+s\tau+\alpha K\left(\dfrac{\theta_3 s}{1+\frac{\theta_3}{2}s}\right)}e^{-\theta_1 s}=$$

$$=\frac{K\left(1+\frac{\theta_3}{2}s\right)}{\underbrace{(1+s\tau)\left(1+\frac{\theta_3}{2}s\right)+\alpha K\theta_3 s}_{2^{nd}\text{ order polynomial}}}e^{-\theta_1 s}=\underbrace{K\frac{1+\tau_3 s}{(1+\tau_1 s)(1+\tau_2 s)}}_{\substack{2^{ND}\text{ ORDER T.F.}\\ \text{(DELAY - FREE)}}}\underbrace{e^{-\theta_1 s}}_{\substack{\text{CASCADED}\\ \text{DELAY}}}$$

CONCLUSIONS

phase lag due to the delay θ

If $\theta\omega_c \ll 1$ $\Rightarrow$ Padé approx if $\theta \ll \frac{1}{\omega_c}$

(Rule of thumb: $\theta \leq 0.1\,\tau$, τ is the time-constant of the transfer function)

CONTROL DESIGN

$\theta \ll \frac{1}{\omega_c}$

Transfer function of the Padé approx.

- Step 1. $e^{-\theta s} \longrightarrow G_i(s)$
- Step 2. "Normal" control design for the delay-free t.f.: $P_0(s)G_i(s)$
  $\Rightarrow C(s) \leftarrow$ controller
- Step 3. Verify that $C(s)$ stabilizes $P_0(s)$ with $m_{\tau_0} > \theta$

m_{τ_0}: delay margin of $C(s)P_0(s)$