DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI

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Exam – 22/01/2015 – 3h

University of Rome Sapienza, DIAG, Master in Control Engineering
Process Automation 2014-2015

Problem 1 **[12/30]**

Consider a process whose step response model is given by the following coefficients:

$g_1 = 0.5,\ g_2 = 1.06,\ g_3 = 1.63,\ g_4 = 2.21,\ g_5 = 2.79,\ g_6 = 3.38, \dots, g_{20} = 12.29, \dots, g_{100} = 95.60,$

whose state-space model is:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 1.1 \end{pmatrix}, N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix},$$

and whose transfer function model is:

$$A(z^{-1})y(t) = B(z^{-1})u(t-1) + T(z^{-1})\frac{e(t)}{\Delta},$$

where $\mathrm{A}(\mathrm{z}^{-1}) = 1 - 1.1z^{-1} + 0.1z^{-2}, B(z^{-1}) = 0.5, T(z^{-1}) = 1$, and $e(t)$ is a zero-mean white noise.

Chose an appropriate MPC algorithm among DMC, MAC and GPC, chose the prediction horizon p and the control horizon m, with $p > 1$ and $m > 1$, and compute all the control actions at time $t = 4$, considering the following specifications:

- ramp reference $r(t) = 0.1 \cdot t$;
- reference trajectory: $w(t) = 0.75 \cdot w(t-1) + 0.25 \cdot r(t)$;
- known reference value (i.e., the predicted future reference samples $w(t+k|t), k = 1,2, \dots, p$ are known at time t);
- future error model: $\hat{e}(t+k|t) = \hat{e}(t|t) = y_m(t) - y(t)$, where $y_m(t)$ is the measured output at time t and $y(t)$ is the output at time t computed by the model;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory.

The data required to compute the control actions follow:

- vector of measured output at time instants $t = 0, \dots, 4$: $y_m = \begin{pmatrix} 0 & 0.025 & 0.069 & 0.128 & 0.197 \end{pmatrix}^T$;
- state variables at time $t = 3$: $x(3) = \begin{pmatrix} 0.005 \\ 0.133 \end{pmatrix}$;
- control variable at time $t = 3$: $u(3) = 0.106$.

Problem 2 **[12/30]**

Consider the process described by the following state-space model:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.9 \end{pmatrix}, N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix},$$

By using the FPC algorithm with $n_B = 2$ base functions, chose the prediction horizon p and the control horizon m, with $p > 1$ and $m > 1$, and compute the control action at time $t = 4$, considering the same specifications of Problem 1.

The data required to compute all the control actions follow:

- vector of measured output at time instants $t = 0, \dots, 4$: $y_m = \begin{pmatrix} 0 & 0.025 & 0.070 & 0.130 & 0.202 \end{pmatrix}^T$;
- state variables at time $t = 3$: $x(3) = \begin{pmatrix} 0.005 \\ 0.133 \end{pmatrix}$;
- control variable at time $t = 3$: $u(3) = 0.117$.

Questions (1/2 pg. max. for each answer) **[6/30]**

1. Which are the MPC algorithms which can be used to solve Problem 1, which are the MPC algorithms which cannot be used to solve Problem 1, and why.
2. Why MPC techniques bring safety improvements to the process automation industry?
3. Answer the following questions:
   a. Why it is difficult to extend standard control methods in the Laplace domain and why it is easier within the MPC framework?
   b. If a given chemical process control problem has a constraint on the maximum temperature of the chemical reactor, a constraint on the maximum value of a control pump which feeds a steam generator, and a constraint on the maximum pressure in a relief tank, which constraint(s) may be enforced by modifying the cost function?

Solution of Problem 1.

The state-space model indicates that the system is not stable, since the system dynamic matrix M has an eigenvalue in 1.1 (in fact, the step response samples do not show a steady state value), therefore only the GPF algorithm may be applied.

There is no input-output delay, i.e., $d = 0$, and we chose $m = p = N = 2$.
The Diophantine equation is written as $1 = E_j(z^{-1})\tilde{A}(z^{-1}) + z^{-j}F_j(z^{-1}), j = d+1, \dots, d+N$, where $\tilde{A}(z^{-1}) = \Delta A(z^{-1}) = 1 - 2.1z^{-1} + 1.2z^{-2} - 0.1z^{-3}$. Since $d = 0$ and $N = 2$, we have to solve the following two equations:

$j = d + 1 = 1$:

$1 = E_1(z^{-1})\tilde{A}(z^{-1}) + z^{-1}F_1(z^{-1})$, with $E_1(z^{-1}) = e_0$ and $F_1(z^{-1}) = f_0 + f_1 z^{-1} + f_2 z^{-2}$;

$1 = e_o(1 - 2.1z^{-1} + 1.2z^{-2} - 0.1z^{-3}) + z^{-1}(f_0 + f_1 z^{-1} + f_2 z^{-2})$;

$$\begin{cases} e_0 = 1 \\ f_0 = 2.1 \\ f_1 = -1.2 \\ f_2 = 0.1 \end{cases}.$$

It follows that

$E_1(z^{-1}) = 1$;
$G_1(z^{-1}) = E_1(z^{-1})B(z^{-1}) = 0.5$;
$F_1(z^{-1}) = 2.1 - 1.2z^{-1} + 0.1z^{-2}$.

$G_1(z^{-1})$ is written as $G_1(z^{-1}) = g_0$, with $g_0 = 0.5$.

$j = d + 2 = 2$:

$1 = E_2(z^{-1})\tilde{A}(z^{-1}) + z^{-2}F_2(z^{-1})$, with $E_2(z^{-1}) = e_0 + e_1 z^{-1}$
and $F_2(z^{-1}) = f_0 + f_1 z^{-1} + f_2 z^{-2}$;

$1 = (e_0 + e_1 z^{-1})(1 - 2.1z^{-1} + 1.2z^{-2} - 0.1z^{-3}) + z^{-2}(f_0 + f_1 z^{-1} + f_2 z^{-2})$;

$$\begin{cases} e_0 = 1 \\ e_1 = 2.1 \\ f_0 = -1.2e_0 + 2.1e_1 = 3.21 \\ f_1 = 0.1e_0 - 1.2e_1 = -2.42 \\ f_2 = 0.1e_1 = 0.21 \end{cases}.$$

It follows that

$E_2(z^{-1}) = 1 + 2.1z^{-1}$;
$G_2(z^{-1}) = E_2(z^{-1})B(z^{-1}) = 0.5 + 1.05z^{-1}$;
$F_2(z^{-1}) = 3.21 - 2.42z^{-1} + 0.21z^{-2}$.

$G_2(z^{-1})$ is written as $G_2(z^{-1}) = g_0 + g_1 z^{-1}$, with $g_0 = 0.5,\ g_1 = 1.05$.

The vector of predicted output is written as $y = Gu + G'(z^{-1})\Delta u(t-1) + F(z^{-1})y(t)$, where:

$$y = \begin{pmatrix} \hat{y}(t+1|t) \\ \hat{y}(t+2|t) \end{pmatrix};\ u = \begin{pmatrix} u(t) \\ u(t+1) \end{pmatrix};\ G = \begin{pmatrix} g_0 & 0 \\ g_1 & g_0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 1.05 & 0.5 \end{pmatrix};$$

$$G'(z^{-1}) = \begin{pmatrix} (G_1(z^{-1}) - g_0)z \\ (G_2(z^{-1}) - g_0 - g_1 z^{-1})z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \; F(z^{-1}) = \begin{pmatrix} F_1(z^{-1}) \\ F_2(z^{-1}) \end{pmatrix} = \begin{pmatrix} 2.1 - 1.2z^{-1} + 0.1z^{-2} \\ 3.21 - 2.42z^{-1} + 0.21z^{-2} \end{pmatrix}$$

At time $t = 4$, the free responses are computed as $f = G'(z^{-1})\Delta u(t-1) + F(z^{-1})y_m(t)$:

$j = d + 1 = 1$: $f_1 = 2.1y_m(4) - 1.2y_m(3) + 0.1y_m(2) = 0.168$;
$j = d + 2 = 2$: $f_2 = 3.21y_m(4) - 2.42y_m(3) + 0.21y_m(2) = 0.229$;

The control action sequence is computed as $u = G^{-1}(w - f)$, with $G^{-1} = \begin{pmatrix} 2 & 0 \\ -4.2 & 2 \end{pmatrix}$.

Given the reference $r(t) = 0.1 \cdot t$ and the trajectory equation $w(t) = 0.75 \cdot w(t-1) + 0.25 \cdot r(t)$, and given the fact that the reference is known, we can compute the vector of the future trajectory values:

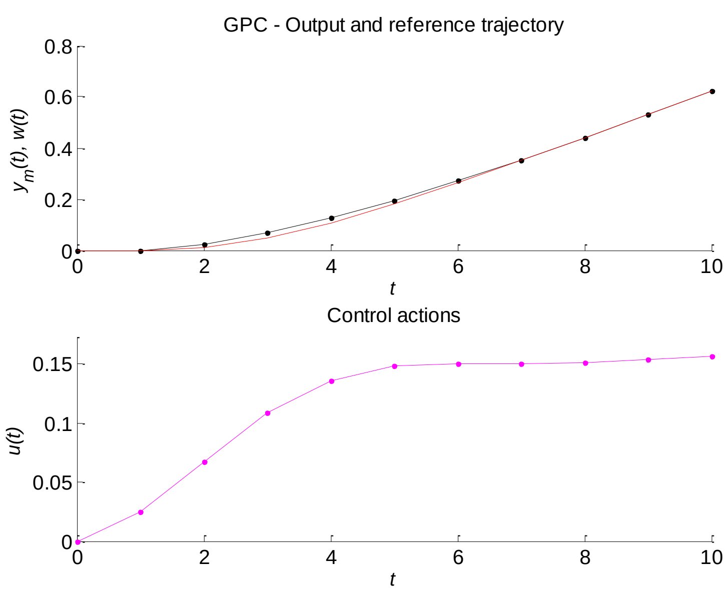
$w(1) = 0, w(2) = 0.025, w(3) = 0.069, w(4) = 0.127, w(5) = 0.195, w(6) = 0.271, \ldots$.

The control increments at time $t = 4$ are then:

$$\Delta u = \begin{pmatrix} \Delta u(4) \\ \Delta u(5) \end{pmatrix} = G^{-1}(w - f) = \begin{pmatrix} 0.5 & 0 \\ -1.05 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} w(5) - f_1 \\ w(6) - f_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4.2 & 2 \end{pmatrix} \begin{pmatrix} 0.195 - 0.168 \\ 0.271 - 0.229 \end{pmatrix} = \begin{pmatrix} 0.054 \\ -0.030 \end{pmatrix}.$$

Therefore, the control actions computed at time $t = 4$ are $u(4) = u(3) + \Delta u(4) = 0.160$, which is actually enforced, and $u(5|4) = u(4) + \Delta u(5|4) = 0.130$.

The following plot shows the output and the control actions of the process with the controller just developed:



Solution of Problem 2.

To develop the PFC controller, we need to select the control horizon $m > 1$ and prediction horizon $p > 1$, the base functions $B_i, i = 1, \dots, n_B$, with $n_B = 2$, and the coincident points $h_j, j = 1, \dots, n_H$.
Since there is no input-output delay (see, for instance, the step response samples), we chose $m = p = 2$. We also chose polynomial base functions $B_i(k) = k^{i-1},\ i = 1,2,\ k = 0,1,2, \dots$, and $n_H = 2$, with $h_1 = 1$, $h_2 = 2$ (note that at least 2 coincident points are needed since $n_B = 2$.

Firstly, we have to compute the <u>model</u> response to the base functions in the coincidence point, considering null initial conditions $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$\underline{B_1(k) = k^0 = 1}$

$\underline{k = 1}$ $\quad \begin{cases} x(1) = M \cdot x(0) + N \cdot B_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \\ y(1) = Q \cdot x(1) = 0.5 \end{cases}$

$\underline{k = h_1 = 2}$ $\quad \begin{cases} x(2) = M \cdot x(1) + N \cdot B_1(1) = \begin{pmatrix} 0.1 \\ 1.9 \end{pmatrix}. \\ y(2) = Q \cdot x(1) = 1 \end{cases}$

Thus, $y_{B_1} = (y(h_1) \quad y(h_2)) = (0.5 \quad 1)$.

$\underline{B_2(k) = k^1}$

$\underline{k = 1}$ $\quad \begin{cases} x(1) = M \cdot x(0) + N \cdot B_2(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \\ y(1) = Q \cdot x(1) = 0 \end{cases}$

$\underline{k = h_1 = 2}$ $\quad \begin{cases} x(2) = M \cdot x(1) + N \cdot B_2(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \\ y(2) = Q \cdot x(1) = 0.5 \end{cases}$

Thus, $y_{B_2}(y(h_1) \quad y(h_2)) = (0 \quad 0.5)$.

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_{B_1}^T & y_{B_2}^T \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix}$.
The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu = \left(Y_B{}^T Y_B\right)^{-1} Y_B{}^T (w - f) =$, with $\left(Y_B{}^T Y_B\right)^{-1} Y_B{}^T = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$, where μ is the vector of the optimal parameters at time t. The control action is the computed as $u(t) = \mu^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \dots, n_B$, evaluated for $k = 0$: in our case, $B(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Given the reference $r(t) = 0.1 \cdot t$ and the trajectory equation $w(t) = 0.75 \cdot w(t-1) + 0.25 \cdot r(t)$, and given the fact that the reference is known, we can compute the vector of the future trajectory values:

$w(1) = 0, w(2) = 0.025, w(3) = 0.069, w(4) = 0.127, w(5) = 0.195, w(6) = 0.271, \dots$.

We can now compute the free response at time $t = 4$ over the coincidence points. With PFC, the output prediction is

$\hat{y}(t+k|t) = y(t+k) + \hat{e}(t+k|t) = \sum_{i=1,\dots,n_B} y_{B_i}(k)\mu_i(t) + QM^k x(t) + \hat{e}(t+k|t),$

where the last two terms constitute the free response: $f(t+k|t) = QM^k x(t) + \hat{e}(t+k|t)$ (we recall that the error model is $\hat{e}(t+k|t) = \hat{e}(t|t) = y_m(t) - y(t)$).

$\underline{t=4}$ $\quad y_m(4) = 0.128;\ x(3) = \begin{pmatrix} 0.005 \\ 0.133 \end{pmatrix};\ u(3) = 0.117;$

Model output

$x(4) = Mx(3) + Nu(3) = \begin{pmatrix} 0.014 \\ 0.237 \end{pmatrix}$

$y(4) = Qx(4) = 0.125.$

$e(4) = y_m(4) - y(4) = 0.005$

$\underline{h_1 = 1}$ $\qquad f(5) = QM^1 x(4) + e(4) = 0.124;$

$\underline{h_2 = 2}$ $\qquad f(6) = QM^2 x(4) + e(4) = 0.113.$

$d(4) = \begin{pmatrix} w(5) - f(5) \\ w(6) - f(6) \end{pmatrix} = \begin{pmatrix} 0.071 \\ 0.158 \end{pmatrix}.$

$\mu(4) = \left(Y_B{}^T Y_B\right)^{-1} Y_B{}^T d(4) = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0.071 \\ 0.158 \end{pmatrix} = \begin{pmatrix} 0.142 \\ 0.033 \end{pmatrix};$

$u(4) = \mu(4)^T B(0) = 0.142.$

The following plot shows the output and the control actions of the process with the controller just developed: