Process Automation (MCER), 2015-2016

EXAM

8 Feb. 2016

Exercise 1 (12 pts.)

Let the process be described by the transfer function: $P(s) = \gamma \frac{S-1}{(s+\gamma)(s+1)}$, with $\gamma \in (0.1,1]$.

1. Considering the process model $\tilde{P}(s) = \frac{S-1}{(s+1)^2}$ and by following the IMC design, develop a controller such that:
    a. the controlled system has 0 steady-state error for step inputs;
    b. the controlled system is robustly stable against the uncertainties of the parameter γ.
2. Compute the equivalent classic controller.

Exercise 2 (12 pts.)

Consider a process whose state-space model is:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = [1 \quad 0].$$

Compute the control actions of a Predictive Functional Control algorithm at time $t = 2$, with:

- control horizon $m = 4$;
- prediction horizon $p = 4$;
- number of coincident points $n_h \geq 2$;
- number of basis functions $n_B \geq 2$;
- constant reference $r(t) = 1, \forall t$;
- null initial conditions: $u(t) = y(t) = y_m(t) = 0, \forall t \leq 0$ and $x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \forall t \leq 0$;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory.

In the computation, consider the following transfer function to compute the measured output:
$y_m(t) = 0.5y_m(t-1) + 0.4u(t-2)$.
Moreover, consider that the plant-model error (i.e., the difference between future measured outputs $y_m(t+k)$ and the predicted model outputs $y(t+k)$ at future time instants) is equal to $\hat{e}(t+k|t) = \left(y_m(t) - y(t)\right) \cdot \frac{k+1}{2}$, $k = 1,2, \ldots$.

Questions (6 pt.)

i) Consider the IMC controller $Q(s)$ determined in Exercise 1. for the delay-free process $P(s) = \gamma \frac{S-1}{(s+\gamma)(s+1)}$, with $\gamma \in (0.1,1]$. Check whether the same controller $Q(s)$ stabilizes the process with delay $P_d(s) = \frac{S-1}{(s+1)(s+0.1)} e^{-0.5s}$ (<u>it is sufficient to write the procedure</u>). (1 pg. max, 4pt)

ii) Explain why the robust stability condition based on Nyquist theorem arguments (i.e., the condition $\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall \omega$) is conservative. (1/3 pg. max, 2pt)

Solution of exercise 1

1.

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles), therefore it is possible to design a stable IMC controller $Q(s)$ to stabilize the closed-loop nominal system.
The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)
Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:
$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$, with $\tilde{P}_+(s) = 1 - s$ and $\tilde{P}_-(s) = -\frac{1}{(1+s)^2}$,
and define the controller as follows:

$$\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = -(1+s)^2$$

Step 2)
Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$).
We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 2$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = -\frac{(1+s)^2}{(1+\lambda s)^2}$ is proper;
b) $\tilde{T}(0) = \tilde{P}(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1} f(s)\right]_{s=0} = \left[\frac{1-s}{(1+\lambda s)^2}\right]_{s=0} = 1.$

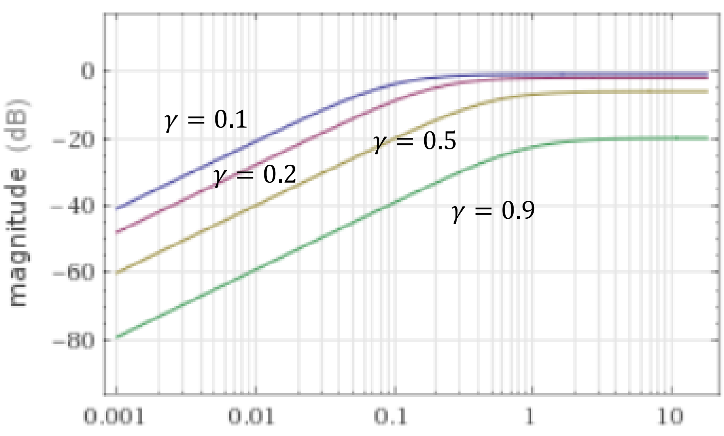
Step 3)
Determine the value of λ such that the sufficient condition for robust stability holds:

$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall\omega,$$

where $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall\omega$. By definition, the multiplicative uncertainty is defined as

$$\Delta_m(j\omega) \coloneqq \frac{P(j\omega)-\tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{-\frac{1-j\omega}{\left(1+\frac{j\omega}{\gamma}\right)(1+j\omega)}+\frac{1-j\omega}{(1+j\omega)^2}}{-\frac{1-j\omega}{(1+j\omega)^2}} = \frac{(1+j\omega)-\left(1+\frac{j\omega}{\gamma}\right)}{\left(1+\frac{j\omega}{\gamma}\right)} = \frac{\gamma-1}{\gamma}\frac{j\omega}{\left(1+\frac{j\omega}{\gamma}\right)}.$$

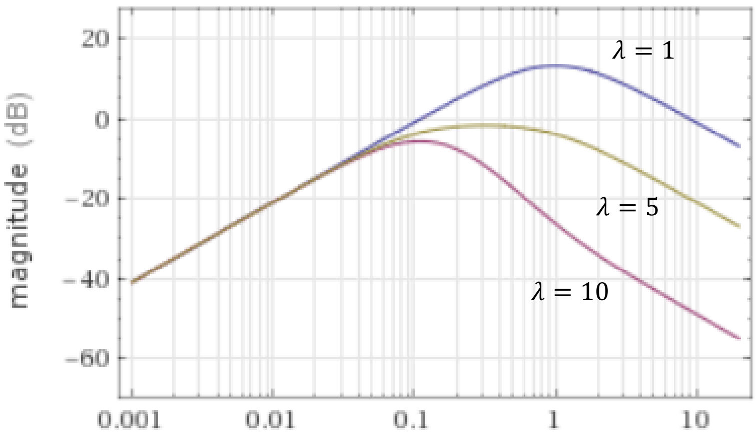
Since $\gamma \in (0.1,1]$, an upper-bound is then simply given by $l_m(j\omega) = -9\frac{j\omega}{\left(1+\frac{j\omega}{0.1}\right)}$, as shown in the figure below:



The sufficient condition for robust stability is then:

$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| = \left|-9\frac{j\omega}{\left(1+\frac{j\omega}{0.1}\right)}\cdot\frac{1-j\omega}{(1+\lambda j\omega)^2}\right| < 1, \forall\omega.$$

The figures below show that the condition is met for example for different values of λ. Already for $\lambda = 5$ the condition is met. We peeka conservative value $\lambda = 10$.



Thus, we have that $Q(s) = -\frac{(1+s)^2}{(1+10s)^2}$.

2.
In the equivalent classic control scheme, the controller is a PID+filter and is computed from the IMC controller as follows:

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = G(s) = \frac{-\frac{(1+s)^2}{(1+10s)^2}}{1-\frac{(1+s)^2}{(1+10s)^2}\frac{1-s}{(1+s)^2}} = 21\frac{-s^2-2s-1}{s\left(s+\frac{100}{21}\right)} = -42\left(1+\frac{1}{2s}+\frac{1}{2}s\right)\frac{1}{s+\frac{100}{21}}.$$

Solution of exercise 2.

To develop the PFC controller, we need to select the base functions and the coincident points. We chose $n_B = 2$ base functions $B_i(k) = k^{i-1} = 1, k = 0,1, ...$. Since there is an input-output delay (the output at time t does not depend directly on the input but it depends on the first component of the state vector $x_1(t-1)$; in turn, $x_1(t-1)$ does not depend directly on the input but it depends on the second component of the state vector $x_2(t-1)$, which, finally, depends on the input $u(t-2)$), and since the prediction horizon is $p = 4$, we chose $h_1 = 2$ and $h_2 = 4$ (another suitable choice would have been $h_1 = 3$ and $h_2 = 4$).

Firstly, we have to compute the model response to the base function in the coincidence points, considering null initial conditions $x(0) = 0$:

$\underline{t = 1}$

$$B_1: \begin{cases} x_1(1) = 0.5x_1(0) + 0.5x_2(0) = 0 \\ x_2(1) = B_1(0) = 1 \\ y(1) = x_1(1) = 0 \end{cases}; \qquad B_2: \begin{cases} x_1(1) = 0.5x_1(0) + 0.5x_2(0) = 0 \\ x_2(1) = B_2(0) = 0 \\ y(1) = x_1(1) = 0 \end{cases};$$

$\underline{t = h_1 = 2}$

$$B_1: \begin{cases} x_1(2) = 0.5x(1) + 0.5x_2(1) = 0.5 \\ x_2(2) = B_1(1) = 1 \\ y(2) = x_1(2) = 0.5 \end{cases}; \qquad B_2: \begin{cases} x_1(2) = 0.5x_1(1) + 0.5x_2(1) = 0 \\ x_2(2) = B_2(1) = 1 \\ y(2) = x_1(2) = 0 \end{cases};$$

$\underline{t = 3}$

$$B_1: \begin{cases} x_1(3) = 0.5x(2) + 0.5x_2(2) = 0.75 \\ x_2(3) = B_1(2) = 1 \\ y(3) = x_1(3) = 0.75 \end{cases}; \qquad B_2: \begin{cases} x_1(3) = 0.5x(2) + 0.5x_2(2) = 0.5 \\ x_2(3) = B_2(2) = 2 \\ y(3) = x_1(3) = 0.5 \end{cases};$$

$\underline{t = h_2 = 4}$

$$B_1: \begin{cases} x_1(4) = 0.5x(3) + 0.5x_2(3) = 0.875 \\ x_2(4) = B_1(3) = 1 \\ y(4) = x_1(4) = 0.875 \end{cases}; \qquad B_2: \begin{cases} x_1(4) = 0.5x(3) + 0.5x_2(3) = 1.5 \\ x_2(4) = B_2(3) = 3 \\ y(4) = x_1(4) = 1.25 \end{cases};$$

Thus, $y_B(h1) = (y_{B_1}(h_1) \quad y_{B_2}(h_1)) = (0.5 \quad 0)$, $y_B(h_2) = (y_{B_1}(h_2) \quad y_{B_2}(h_2)) = (0.875 \quad 1.25)$.

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0.875 & 1.25 \end{pmatrix}$.

The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu^* = \left(Y_B{}^T Y_B\right)^{-1} Y_B{}^T (w - f)$, with $\left(Y_B{}^T Y_B\right)^{-1} Y_B{}^T = \begin{pmatrix} 2 & 0 \\ -1.4 & 0.8 \end{pmatrix}$, where μ^* is the vector of the optimal parameters at time t. The control action is the computed as $u(t) = \mu^{*T} B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \dots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 2$, we need to find a two-column vector $\mu^*(t)$.

Since the reference is constant, the vector of the future reference values evaluated in the coincidence points $h_1 = 2$ and $h_2 = 4$ is always $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

By considering null initial conditions, we can start computing the free response at time $t = 1$ over the coincidence points. With PFC, the output prediction is

$$\hat{y}(t + k|t) = y(t + k) + \hat{e}(t + k|t) = \sum_{i=1,\dots,n_B} y_{B_i}(k)\mu_i(t) + QM^k x(t) + \hat{e}(t + k|t),$$

where the last two terms constitute the free response:
$f(t+k|t) = QM^k x(t) + \hat{e}(t+k|t) = QM^k x(t) + (y_m(t) - y(t)) \cdot \frac{k+1}{2}$.

$\underline{t=1}$

Measured output
$y_m(1) = 0.5y_m(0) + 0.4u(-1) = 0$

Model output
$$\begin{cases} x_1(1) = 0.5x_1(0) + 0.5x_2(0) = 0 \\ x_2(1) = u(0) = 0 \\ y(1) = x_1(0) = 0 \end{cases}$$

$\underline{h_1 = 2}$ $\quad f(3) = QM^{h_1}x(1) + \big(y_m(1) - y(1)\big) \cdot \frac{h_1+1}{2} = 0;$
$\underline{h_2 = 4}$ $\quad f(5) = QM^{h_2}x(1) + \big(y_m(1) - y(1)\big) \cdot \frac{h_2+1}{2} = 0.$

$d(1) = (w - f) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$
$\mu(1) = \begin{pmatrix}2 & -1.4\\0 & 0.8\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0.6\\0.8\end{pmatrix};$
$u(1) = \mu_1(1)B_1(0) + \mu_2(1)B_2(0) = 0.6.$

$\underline{t=2}$

Measured output
$y_m(2) = 0.5y_m(1) + 0.4u(0) = 0$

Model output
$$\begin{cases} x_1(2) = 0.5x_1(2) + 0.5x_2(2) = 0 \\ x_2(2) = u(1) = 0.6 \\ y(2) = x_1(2) = 0 \end{cases}$$

$\underline{h_1 = 2}$ $\quad f(4) = QM^{h_1}x(2) + \big(y_m(2) - y(2)\big) \cdot \frac{h_1+1}{2} = 0;$
$\underline{h_2 = 4}$ $\quad f(5) = QM^{h_2}x(2) + \big(y_m(2) - y(2)\big) \cdot \frac{h_2+1}{2} = 0.$

$d(1) = (w - f) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$
$\mu(1) = \begin{pmatrix}2 & -1.4\\0 & 0.8\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0.6\\0.8\end{pmatrix};$
$u(1) = \mu_1(1)B_1(0) + \mu_2(1)B_2(0) = 0.6.$