MPC with No Model Uncertainty

Remark

Section 1 summarizes chapters 2.1-2.4 of the book by K. Basil and M. Cannon, "Model predictive control", Switzerland: Springer International Publishing, 2016. Section 2 summarizes some stability results presented in the MPC course "SIDRA Summer School 2011 on Model Predictive Control" held by Prof. R. Scattolini.

1 From Linear Quadratic Optimal Control to MPC

1.1 Problem Description

Classical MPC describes to a class of control problems involving linear time invariant (LTI) systems whose dynamics are described by a discrete time model that is not subject to any uncertainty, either in the form of unknown additive disturbances or imprecise knowledge of the system parameters.

The considered system dynamics is written by means of a LTI state-space model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k+1) &= Cx(k+1) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}, u(k) \in \mathbb{R}^{n_u}, y(k) \in \mathbb{R}^{n_y}$ are the system state, the control input and the system output, and k is the discrete time index. If the system to be controlled is described by a model with continuous time dynamics (such as an ordinary differential equation), then the implicit assumption is made here that the controller can be implemented as a sampled data system and that (2.1a) defines the discrete time dynamics relating the samples of the system state to those of its control inputs.

Assumption 1. Unless otherwise stated, the state $x(k)$ of the system (1) is assumed to be measured and made available to the controller at each sampling instant $k = 0,1, \dots$.

The controlled system is also assumed to be subject to linear constraints. In general, these may involve both states and inputs and are expressed as a set of linear inequalities

$$Fx(k) + Gu(k) \leq \boldsymbol{b} \tag{2}$$

where $F \in \mathbb{R}^{n_C \times n_x}, G \in \mathbb{R}^{n_C \times n_u}$, $\boldsymbol{b} \in \mathbb{R}^{n_C}$, with n_C equal to the number of constraints, and the inequality applies elementwise. Setting F or G to zero results in constraints on inputs or states alone. A feasible pair $\big(x(k), u(k)\big)$ or feasible sequence $\big\{\big(x(0), u(0)\big), \big(x(1), u(1)\big), \dots\big\}$ for (2) is any pair or sequence satisfying (2). In general, the constraints can be time-dependent, in which case the matrixes are written as F_k and G_k.

The classical regulation problem is concerned with the design of a controller that drives the system state to some desired reference point using an acceptable amount of control effort. For the case that the state is to be steered to the origin, the controller performance is quantified conveniently for this type of problem by a quadratic cost index of the form

$$J(x(0), \{u(0), u(1), u(2), \dots\}) \coloneqq \sum_{k=0}^{\infty} \big(\|x(k)\|_Q^2 + \|u(k)\|_R^2\big) \tag{3}$$

where $\|v\|_S^2$ is the quadratic form $v^T S v$ for any $v \in \mathbb{R}^{n_v}$ and $S = S^T \in \mathbb{R}^{n_v \times n_v}$, and $Q \in \mathbb{R}^{n_x \times n_x}$and $Q \in \mathbb{R}^{n_u \times n_u}$ are weighting matrices that specify the emphasis placed on particular states and inputs in the cost. We assume that R is a symmetric positive-definite matrix (its eigenvalues are real and strictly positive), denoted with $R \succ 0$, and that Q is a symmetric positive-semidedefinite matrix (its eigenvalues are real and non-negative), denoted with $Q \succcurlyeq 0$.

At time k, the optimal value of the cost (2.3) with respect to minimization over admissible control sequences $\{u(k), u(k+1), u(k+2), \dots\}$ is denoted with

$J^*(x(k)) = \min_{u(k),u(k+1),u(k+2),\dots} J(x(k), \{u(k), u(k+1), u(k+2), \dots\})$his problem formulation leads to an optimal control problem whereby the controller is required to minimize, at time k, the performance cost (3) subject to the constraints (2). To ensure that the optimal value of the cost is well defined, we assume that the state of the model (1) is stabilizable and observable.

Assumption 2. In the system model (1) and cost (3), the pair (A, B) is stabilizable, the pair (A, C) is observable, R is positive-definite and Q is positive-semidefinite.

Given the linear nature of the controlled system, the problem of setpoint tracking (in which the output y is to be steered to a given constant setpoint) can be converted into the regulation problem considered here by redefining the state equation of (2) in terms of the deviation from a desired steady-state value. The more general case of tracking a time-varying setpoint (e.g., a ramp or sinusoidal signal) can also be tackled within the framework outlined here provided that the setpoint can itself be generated by applying a constant reference signal to a system with known LTI dynamics.

1.2 Linear Quadratic Optimal Control

The problem of minimizing the quadratic cost of (3) in the unconstrained case (i.e., when $F = \mathbf{0}$ and $G = \mathbf{0}$ in (2)) is addressed by Linear Quadratic (LQ) optimal control.

We first obtain an expression for the cost under linear feedback, $u = Kx$, for an arbitrary stabilizing gain matrix $K \in \mathbb{R}^{n_u \times n_x}$, using the closed-loop system dynamics

$$x(k+1) = (A + BK)x(k) \tag{4}$$

Then, we can write $x(k) = (A + BK)^k x(0)$ and $u(k) = K(A + BK)^k x(0)$, for all k. Therefore

$$J(x(0)) = J(x(0), \{Kx(0), Kx(1), \dots\}) = J(x(0), \{Kx(0), K(A + BK)x(0), K(A + BK)^2 x(0) \dots\})$$

is a quadratic function of $x(0)$ and can be written in a compact form as

$$J(x(0)) = x(0)^T W x(0)$$

$$\begin{aligned} W &= \sum_{k=0}^{\infty} \left(\left((A + BK)^k \right)^T Q (A + BK)^k + \left((A + BK)^k \right)^T K^T R K (A + BK)^k \right) \\ &= \sum_{k=0}^{\infty} \left(\left((A + BK)^k \right)^T (Q + K^T R K)(A + BK)^k \right) \in \mathbb{R}^{n_x \times n_x} \end{aligned} \tag{5}$$

If $A + BK$ is strictly stable (i.e., all its eigenvalues are strictly less than unity in absolute value), then the elements of the matrix W defined in (5) are necessarily finite. Furthermore, if $R \succ 0$ and $Q \succcurlyeq 0$, then $J(x(0))$ is a positive-definite function of ($J(x(0)) \geq 0$ for all $x(0) \neq \mathbf{0}$ and $J(x(0)) = 0$ only if $x(0) = \mathbf{0}$), which implies that W is a positive-definite matrix.

The unique matrix W satisfying (2.4) can be obtained by solving a set of linear equations.

Lemma 1 (Lyapunov matrix equation). Under Assumption 2, the matrix W in (5) is the unique positive-definite solution of the Lyapunov matrix equation

$$W = (A + BK)^T W (A + BK) + Q + K^T R K \tag{6}$$

if and only if $(A + BK)$ is strictly stable.

So far, we know that, for the LTI system (1) under a stabilizing state-feedback controller K, the cost function (3) can be expressed as a quadratic form of the initial state $x(0)$ weighted by a matrix W which can be computed starting from the matrixes K, R, Q. We are now interested in finding the optimal control gain matrix K and the corresponding matrix W such that the cost function (3) is minimized.

Theorem 1 (Discrete time algebraic Riccati equation). The feedback gain matrix K for which the control law $u = Kx$ minimizes the cost of (3) for any initial condition $x(0)$ under the dynamics of (1) is given by

$$K \ \ (B^T WB + R)^{-1} B^T WA \tag{7}$$

where $W \succ 0$ is the unique solution of

$$W = A^T WA + Q - A^T WB(B^T WB + R)^{-1} B^T WA \tag{8}$$

Under Assumption 2, $A + BK$ is strictly stable whenever there exists $W \succ 0$ satisfying (8).

1.3 The Dual-Mode Prediction Paradigm

The control law that minimizes the cost (3) is not in general a linear feedback law when constraints (2) are present. Moreover, it may not be computationally tractable to determine the optimal controller as an explicit state-feedback law.Predictive control strategies overcome this difficulty by minimizing, subject to constraints, a predicted cost that is computed for a particular initial state, namely the current plant state. This constrained minimization of the predicted cost is solved online at each time step in order to derive a feedback control law. The predicted cost corresponding to (3) can be expressed as

$$J(x(k), \{u(0|k), u(1|k), u(2|k), \dots\}) \coloneqq \sum_{i=0}^{\infty} \left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \right) \tag{9}$$

where $x(i|k)$ and $u(i|k)$ denote the predicted values of the model state and input, respectively, at time $k + i$ based on the information that is available at time k, and where $x(0|k) = x(k)$ is assumed.

The prediction horizon employed in (9) is infinite. Hence, if every element of the infinite sequence of predicted inputs $\{u(0|k), u(1|k), u(2|k), \dots\}$ were considered to be a free variable, then the constrained minimization of this cost would be an infinite-dimensional optimization problem, which is in principle intractable

$$\begin{aligned} \min_{u(0|k), u(1|k), u(2|k), \dots} \quad & J(x(k), \{u(0|k), u(1|k), u(2|k), \dots\}) \\ \text{s.t.} \quad & Fx(i|k) + Gu(i|k) \leq \boldsymbol{b}, i = 0,1, \dots \end{aligned} \qquad 10)$$

However, predictive control strategies provide effective approximations to the optimal control law that can be computed efficiently and in real time. This is possible because of a parameterization of predictions known as the *dual-mode prediction paradigm*, which enables the MPC optimization to be specified as a finite-dimensional problem.

The dual-mode prediction paradigm divides the prediction horizon into two intervals. Mode 1 refers to the predicted control inputs over the first N prediction time steps for some finite horizon N (chosen by the designer), while mode 2 denotes the control law over the subsequent infinite interval. The mode 2 predicted inputs are specified by a fixed state-feedback law $u = Kx$, which is usually taken to be the optimum for the problem of minimizing the cost in the absence of constraints. Therefore, the constrained QP problem is written as a finite-dimensional one:

$$\min_{\boldsymbol{u}(k)} J\big(x(k), \boldsymbol{u}(k)\big)$$

$$\text{s.t.}\ \ Fx(i|k) + Gu(i|k) \leq \boldsymbol{b}, i = 0,1, \dots, N-1 \qquad 11)$$

where

$\boldsymbol{u}(k) \coloneqq \{u(0|k), u(1|k), \dots, u(N-1|k)\} \in \mathbb{R}^N$

is the vector of the N future control inputs and the predicted cost (9) is written as

$$J\big(x(k), \boldsymbol{u}(k)\big) = \sum_{1=0}^{N-1}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big) + \|x(N|k)\|_W^2 \qquad (12)$$

where $\|x(N|k)\|_W^2$ is the final cost or terminal penalty term and accounts for the cost-to-go after N prediction time steps under the mode 2 optimal state-feedback law (i.e., the matrixes K and W are computed according to Theorem 1). A mode 1 feasible solution of the problem (10) is hereafter referred to as *receding-horizon* solution and denoted as $u(i|k) = g^{RH}\big(x(k)\big), i = 0,1, \dots, N-1$. The main question now is if it is possible to consider the problem as an unconstrained one after N time-steps.

The receding horizon implementation of MPC stipulates that, at each time-instant k, the optimal mode 1 control sequence, denoted as

$\boldsymbol{u}^*(k) \coloneqq \{u^*(0|k), u^*(1|k), \dots, u^*(N-1|k)\}$

is computed, and only the first element of this sequence is implemented, namely $u(k) = u^*(0|k)$. Thus, at each time-step, the most up-to-date measurement information (embodied in the state $x(k)$) is employed. This creates a feedback mechanism that provides some compensation for any uncertainty present in the model of (1).

The rationale behind the dual-mode prediction paradigm is as follows. Let

$$\{u^\infty(0|k), u^\infty(1|k), u^\infty(2|k), \dots\} \qquad (13)$$

denote the optimal control sequence for the constrained problem (10) over the infinite sequence $\{u(0|k), u(1|k), u(2|k), \dots\}$, for an initial condition $x(0|k) = x(k)$ such that the problem is feasible. If the weights Q and R satisfy Assumption 2, then this optimal control sequence drives the predicted state of the model (1) asymptotically to the origin, i.e., $x(i|k) \to 0$ as $i \to \infty$. Since $(x, u) = (0,0)$ is strictly feasible for the constraints $Fx + Gu \leq \boldsymbol{b}$, there exists a neighbourhood, S of $x = 0$ with the property that these constraints are satisfied at all times along trajectories of the model (1) under the *unconstrained optimal* state-feedback law, $u = Kx$, starting from any initial condition in S. Note that, if the state-feedback law, $u(i|k) = Kx(i|k)$, the constraint at time $k + i$ is written as

$$Fx(i|k) + Gu(i|k) = (F + GK)x(i|k) \leq \boldsymbol{b}$$

It is then clear that the constraint is met if $x(i|k)$ is sufficiently close to $\mathbf{0}$.

Hence, there necessarily exists a horizon $\bar{N}$ (which depends on $x(k)$) such that $x(i|k) \in S$, for all $i \geq \bar{N}$. Since, by Bellman's Principle of Optimality, the optimal trajectory for $i \geq \bar{N}$ is necessarily optimal for the problem with initial condition $x(\bar{N}|k)$, the constrained optimal sequence must therefore coincide with the unconstrained optimal feedback law, i.e.,

$$u^\infty(i|k) = \ Kx(i|k), \forall i \geq \bar{N} \qquad (14)$$

Moreover, from Theorem 1 it holds that the cost-to-go starting from time-step $\bar{N}$ can be written as a quadratic cost of the state $x(\bar{N}|k)$:

$$J(x(\bar{N}),\{u^{\infty}(0|\bar{N}),u^{\infty}(1|\bar{N}),\dots\}) = \sum_{i=0}^{\infty}\left(\|x(i|\bar{N})\|_Q^2 + \|u(i|\bar{N})\|_R^2\right) = \|x(0|\bar{N})\|_W^2 \quad (15)$$

It follows that if the mode 1 horizon is chosen to be sufficiently long, namely if $N \geq \bar{N}$, then the mode 1 control sequence $\boldsymbol{u}^*(k)$, that minimizes the cost of (11) subject to the constraints $x(i|k) + Gu(i|k) \leq \mathbf{1}, i = 0,1,\dots,N-1$ must be equal to the first N elements of the infinite optimal sequence (13) for the constrained problem (10):

$$\boldsymbol{u}^*(k) = \{u^*(0|k),u^*(1|k),\dots,u^*(N-1|k)\} = \{u^{\infty}(0|k),u^{\infty}(1|k),\dots,u^{\infty}(N-1|k)\} \quad (16)$$

From equations (14) and (15), it holds that, if $N \geq \bar{N}$, the optimal solution for the infinite-dimensional constrained QP problem (10) can be found by solving the finite-dimensional QP problem (11) to find the optimal control sequence $\boldsymbol{u}^*(k)$ and by considering the optimal state-feedback law from time-step N on.

Theorem 2. There exists a finite horizon $\bar{N}$, which depends on $x(k)$, with the property that, whenever $N \geq \bar{N}$:

i) the sequence $\boldsymbol{u}^*(k)$ that achieves the minimum of the finite-dimensional constrained QP problem (11) is equal to the first N terms of the infinite sequence $\{u^{\infty}(0|N),u^{\infty}(1|N),\dots\}$ that minimizes the infinite-dimensional constrained QP problem (10);

ii) $J\left(x(k),\boldsymbol{u}^*(k)\right) = J(x(k),\{u^{\infty}(0|N),u^{\infty}(1|N),\dots\}) = J^*(k)$.

1.4 MPC with Invariant Final Set

The determination of the minimum prediction horizon N which ensures that the predicted state and input trajectories in mode 2 meet constraints (2) is not a trivial matter. Instead, lower bounds for this horizon were proposed but they are, in general, rather conservative, leading to the use of unnecessarily long prediction horizons. This in turn could make the online optimization of the predicted cost computationally intractable as a result of large numbers of free variables and large numbers of constraints in the minimization of predicted cost. In such cases, it becomes necessary to use a shorter horizon N while retaining the guarantee that predictions over mode 2 satisfy constraints on states and inputs.

This can be done by imposing a terminal constraint which requires that the state at the end of the mode 1 horizon should lie in a set which is positively invariant under the dynamics defined by (1) and under the constraints (2).

Definition 1 (Positively invariant set). A set $X \subseteq R^{n_x}$ is positively invariant under the closed-loop dynamics defined by (4) and the constraints (2) if and only if, for all $k = 0,1,\dots$,

i) $(F + GK)x(k) \leq \boldsymbol{b}$;

ii) $x(k) \in X$, for all $x\,(0) \in X$.

We note that the closed-loop dynamics under state-feedback and the constraints equation can be written in terms of the initial state $x(0)$:

$$x(k+i) = (A+BK)^i x(0)$$
$$(F+GK)(A+BK)^i x(0) \leq \boldsymbol{b}$$

The MPC problem at time k is then written by using the positively invariant set notion as follows.

$$\begin{gathered}\min_{\boldsymbol{u}(k)} J\big(x(k),\boldsymbol{u}(k)\big) = \textstyle\sum_{1=0}^{N-1}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big) + \|x(N|k)\|_W^2 \\ \text{s.t.} \\ Fx(i|k) + Gu(i|k) \le \boldsymbol{b}, i = 0,1,\dots,N-1 \\ x(N|k) \in X_f \end{gathered} \tag{17}$$

where X_f is the final or terminal set and is positively invariant w.r.t. the closed-loop dynamics (4) and the constraints (2). The problem (17) reads as

$$\begin{gathered}\min_{\boldsymbol{u}(k)} J\big(x(k),\boldsymbol{u}(k)\big) = \textstyle\sum_{1=0}^{N-1}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big) + \|x(N|k)\|_W^2 \\ \text{s.t.} \\ Fx(i|k) + Gu(i|k) \le \boldsymbol{b}, i = 0,1,\dots,N-1 \\ (A+BK)^i x(N|k) \in X_f, i = 0,1,\dots \\ (F+GK)(A+BK)^i x(N|k) \le \boldsymbol{b}, i = 0,1,\dots \end{gathered} \tag{18}$$

Besides the determination of the invariant set X_f, the QP problem (17) requires, again, the definition of a sufficiently large N for which a sequence of control action exists such that the system state is driven within X_f in N steps: if N is too small, the problem might become unfeasible. This problem can be seen in another way: given the number of time-steps N, there is an initial set X^0 such that there exists a sequence of N which drive the states within X^f, i.e., there is an initial set X^0 which is close enough to X^f for the problem to be feasible.

The predicted state $x(N|k) = Ax(N-1|k) + Bu(N-1|k)$ at the end of mode 1 is constrained to lie in an invariant set X^f. From time N on, in mode 1, the evolution of the state trajectory is that prescribed by the state-feedback control law $u(N+i|k) = Kx(N+i|k) = K(A+BK)^i x(N|k)$ for $i = 0,1,\dots$.

In order to increase the applicability of the MPC algorithm, and in particular to increase the size of the set of initial conditions $x(0|k) \in X^0$ for which the terminal condition $x(N|k) \in X^f$ can be met, it is important to choose the maximal positively invariant set as the terminal constraint set. This set is defined as follows.

Definition 2 (Maximal positively invariant set). The maximal positively invariant set under the closed-loop dynamics defined by (4) and the constraints (2) is the union of all sets that are positively invariant under these dynamics and constraints.

For the case of linear dynamics and linear constraints, the maximal positively invariant set is defined by a finite number of linear inequalities, which are found by accounting for the linear constraints for a limited number of time-steps v, i.e., by considering the constraints $(F+GK)(A+BK)^i x(N|k) \le \boldsymbol{b}, i = 0,1,\dots,v$.

1.4.1 Example

Let us consider an example in which the state space is $\mathbb{R}^2$, the action space is $\mathbb{R}$, the constraints are defined by the following matrixes

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the optimal gain matrix for the LQ optimal control is $K = [2 \quad 0.5]$. The 5 constraints are either state- or control action-dependent:

$$F\begin{pmatrix}x_1\\x_2\end{pmatrix}+Gu\leq \boldsymbol{b}\Rightarrow\begin{cases}x_1+x_2\leq 1\\ x_1\leq 0.5\\ x_1\geq 0\\ x_2\geq 0\\ u\leq 1\\ u\geq 0\end{cases}$$

The feasible state set X is defined by the state-dependent constraints 1-4, whereas the feasible control set U is defined by the control-dependent constraints 5,6.

As the system is controlled in mode 2, i.e., from time-step N on, the closed-loop dynamics becomes

$$x(N+i|k)=(A+BK)^i x(N|k)$$

and all the constraints can be written as state-dependent ones, for $i=0,1,\ldots$: the state-feedback law translated the constraints on u to constraints on x.

For $i=0$, the constraints are written as:

$$(F+GK)\begin{pmatrix}x_1\\x_2\end{pmatrix}\leq \boldsymbol{b}\Rightarrow\left(\begin{bmatrix}1&1\\1&0\\-1&0\\0&-1\\0&0\\0&0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\1\\-1\end{bmatrix}[2\quad 0.5]\right)\begin{pmatrix}x_1\\x_2\end{pmatrix}\leq\begin{bmatrix}1\\0.5\\0\\0\\1\\0\end{bmatrix}$$

$$\Rightarrow\begin{cases}x_1+x_2\leq 1\\ x_1\leq 0.5\\ x_1\geq 0\\ x_2\geq 0\\ 2x_1+0.5x_2\leq 1\\ 2x_1+0.5x_2\geq 0\end{cases}$$

Note that the last constraint is redundant since it is always met when constraints 3 and 4 are met. Let the set defined by the constraints at time N be denoted as $X^C(N)$. It holds that $X^C(N)\subseteq X$, since, at step N, i.e., for $i=0$, additional state-constrains might be added. The added constraint is the direct result of the state-feedback control law: the control action constraint $u\leq 1$ is written as $Kx=[2\quad 0.5]\begin{pmatrix}x_1\\x_2\end{pmatrix}\leq 1$ which leads to $2x_1+0.5x_2\leq 1$; in fact, if, for instance, $x(N|k)=\begin{pmatrix}0.5\\0.5\end{pmatrix}$ (a feasible point for X), the resulting control action would exceed the control constraints: $u(N|k)=Kx(N|k)=[2\quad 0.5]\begin{pmatrix}0.5\\0.5\end{pmatrix}=1.25>1$.

For $i=1$, the constraints are written as:

$$(F+GK)(A+BK)\begin{pmatrix}x_1\\x_2\end{pmatrix}\leq \boldsymbol{b}$$

which lead to additional constraints at step $N+1$ and, thus, to a smaller set $X^C(N+1)\subseteq X^C(N)$. For LTI systems and stabilizing state-feedback control gain, by computing the constraints as i increases, after a finite number of timesteps v the constraint set remains constant, i.e.,

$$X^C(N)\supseteq X^C(N+1)\supseteq\cdots\supseteq X^C(N+1)\supseteq X^C(N+v)\equiv X^C(N+v+1)\equiv\cdots$$

The set $X^C(N+v)$ is the maximal positively invariant set.

The use of invariant sets within the dual prediction mode paradigm is illustrated in Fig. 1.

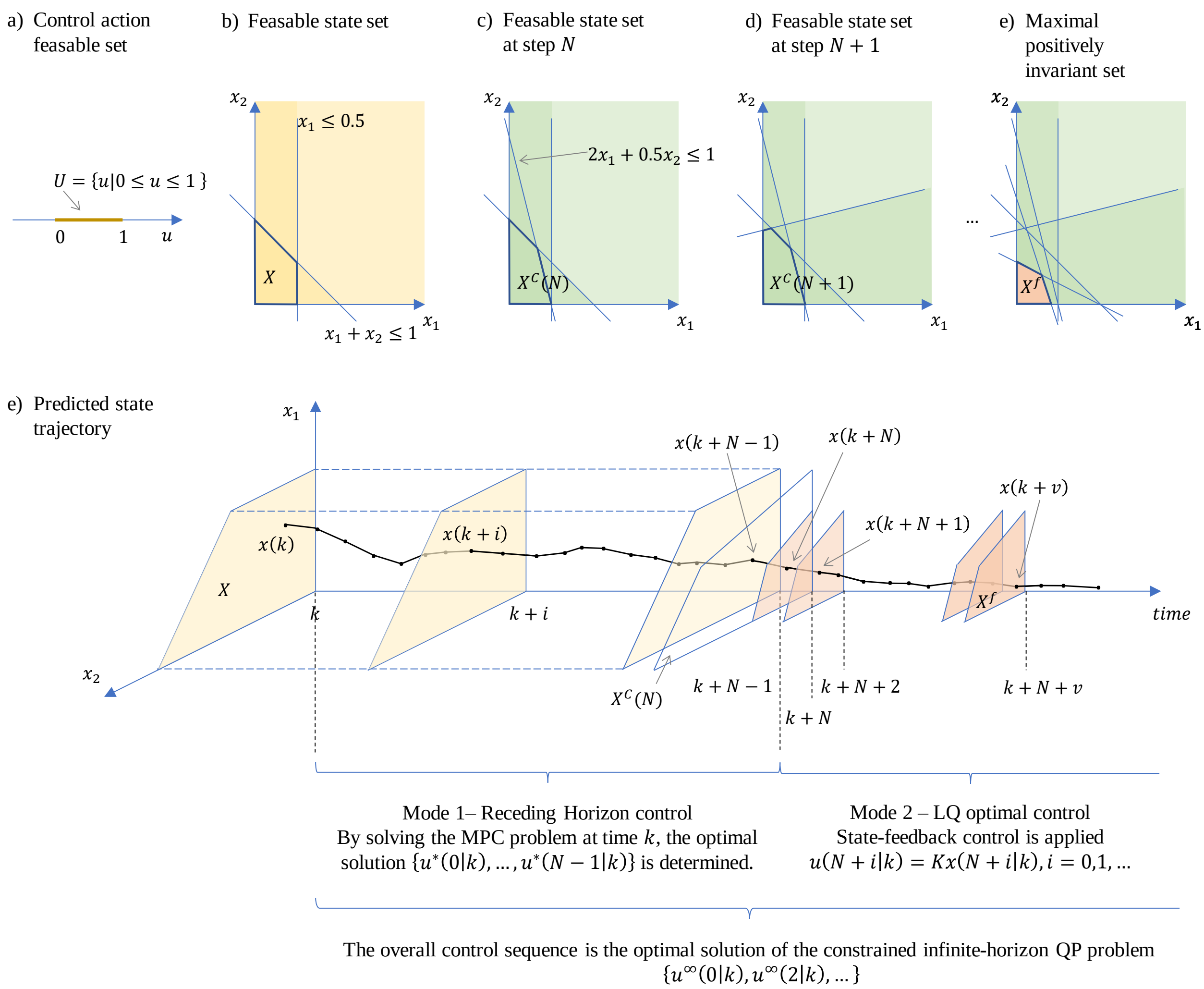

Figure 1: example of dual-mode control.

2 Stability of MPC with no uncertainties

In this section we examine the stability of a discrete-time nonlinear time-invariant system

$$x(k+1) = f\big(x(k), u(k)\big) \tag{19}$$

with $x \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$, under the constraints

$$\begin{aligned} &x(k) \in X \subseteq \mathbb{R}^n, k = 0,1, \dots \\ &u(k) \in U \subseteq \mathbb{R}^m, k = 0,1, \dots \end{aligned} \tag{20}$$

and under the following assumption.

Assumption 3. In the system model (19), $f(\cdot,\cdot)$ is continuously differentiable in both arguments and $f(0,0) = 0$. Moreover, the state-constraints set X and the control action-constraints set U contain the origin, i.e., $\mathbf{0} \in X \times U$.

2.1 MPC stability theorem.

Problem 1 (MPC stability problem). Considering the system (19) under constraints (20) and assumptions 3, design an MPC algorithm guaranteeing that the origin of the closed-loop system is an asymptotically stable equilibrium point.

Motivated by the qualitative analysis of the linear case of section 1, we formulate the following further assumptions.

Assumption 4. There exists an *auxiliary state-feedback control law*

$$u(k) = k_a\big(x(k)\big) \tag{21}$$

and a positively invariant set $X^f \subseteq X$ such that

i) $\mathbf{0} \in X^f$;
ii) for any $x(k) \in X^f$, one has $u(k+i) = k_a\big(x(k+i)\big) \in U$ and $x(k+i) \in X^f$, for $i = 0,1, \dots$.

We can now formulate the MPC problem as a constrained finite-horizon QP problem.

$$\begin{gathered} \min_{\boldsymbol{u}(k)} J(x(k), \boldsymbol{u}(k), N) = \sum_{1=0}^{N-1} \big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \big) + V_f\big(x(N|k)\big) \\ \text{s.t.} \\ x(i|k) \in X, i = 0,1, \dots, N-1 \\ u(i|k) \in U, i = 0,1, \dots, N-1 \\ x(N|k) \in X^f \end{gathered} \tag{22}$$

where $Q \succcurlyeq 0$, $R \succ 0$ and $V_f\big(x(N|k)\big) \geq 0$ is the final cost or terminal cost.

Clearly, in (22), $V_f\big(x(N|k)\big)$ plays the role of $\|x(N|k)\|_W^2$ of the linear case (17). The main difference is that the auxiliary control law (21)is not required to be an optimal one and, therefore, $V_f\big(x(N|k)\big)$ is not, generally, the optimal final cost.

Hereafter, we will denote with $u(k) = k_{RH}\big(x(k)\big)$ the control law underlying a feasible solution of the MPC problem (22).

The following technical assumptions are needed for the stability theorem.

Assumption 5. Let $X^{RH}(N)$ be the set of states in X where at least one solution $\boldsymbol{u}(k)$ of the MPC problem (22) exists. The function $V_f(\cdot): X \to [0, \infty)$ describing the final cost of the MPC problem (22) is such that

i) $V_f\left(f\left(x(k), k_a(x(k))\right)\right) - V_f(x(k)) + \|x(k)\|_Q^2 + \left\|k_a(x(k))\right\|_R^2 \leq 0$, for all $x(k) \in X^f$;

ii) $V_f(x) \leq \alpha_f(\|x\|)$, for all $x(k) \in X^f$, where $\alpha_f: X^{RH} \to [0, \infty)$ is a class $\mathcal{K}$ function[1].

The MPC stability theorem is finally formulated below.

Theorem 3 (MPC stability). Considering the MPC problem (22) under Assumptions 3-5, if $X^{RH}(N) \neq \emptyset$, the origin of the closed-loop system is an asymptotically stable equilibrium point with region of attraction $X^{RH}(N)$. Moreover, if $\alpha_f(\|x\|) = b\|x\|^2$, with $b > 0$, and $X^f = X^{RH}(N)$, the origin is exponentially stable in $X^{RH}(N)$.

Proof.

Preliminarily, let us define

$$V(x, N) \coloneqq J(x, k_{RH}(x), N) \tag{23}$$

It holds that, for all $x(k) \in X^{RH}(N)$ and $k = 0,1, \ldots$,

$$V(x(k), N) = \sum_{1=0}^{N-1} \left(\|x(i|k)\|_Q^2 + \left\|k_{RH}(x(i|k))\right\|_R^2\right) + V_f(x(N|k)) \geq \|x(k)\|_Q^2$$

which implies that, for some $\omega > 0$,

$$V(x, N) \geq \omega\|x\|^2, \text{ for all } x \in X^{RH}(N) \tag{24}$$

Let $x(k) \in X^{RH}(N)$, let

$$\boldsymbol{u}^*(k) = \{u^*(0|k), u^*(1|k), \ldots, u^*(N-1|k)\} \tag{25}$$

be the optimal control sequence for the MPC problem (22) at time k and let

$$\tilde{\boldsymbol{u}}(k) = \{\tilde{u}(0|k+1), \ldots, \tilde{u}(N|k+1)\} = \left\{u^*(1|k), u^*(2|k), \ldots, u^*(N-2|k), k_a(x(N|k))\right\} \tag{26}$$

be a candidate solution at step $k+1$, where the last control action is computed based on the auxiliary control law $\tilde{u}(N|k+1) = k_a(x(N|k))$. Since $\boldsymbol{u}^*(k)$ is a feasible solution, it holds that $x(N|k) \in X^f$ and, by Assumption 4.ii, $x(N+1|k+1)$ is feasible under the auxiliary control law, i.e.,

$$x(N|k+1) = f\left(x(N-1|k+1), u(N-1|k+1)\right) = f\left(x(N|k), k_a(x(N|k))\right) \in X^f.$$

Therefore, $\tilde{\boldsymbol{u}}(k)$ is a feasible solution for the MPC problem at time $k+1$ and $x(k+1) = x(k+1|k) \in X^{RH}(N)$.

Now, let

$$\tilde{\boldsymbol{u}}_{N+1}(k) = \left\{\boldsymbol{u}^*(k), k_a(x(N|k))\right\} \tag{27}$$

[1] That is, $\alpha_f(x)$ is strictly increasing and such that $\alpha_f(0) = 0$.

be a candidate solution for the MPC problem (22) at time k, with prediction horizon $N+1$. The candidate solution (27) is feasible for the MCP problem with horizon $N+1$, as results from the analysis of the candidate solution (26). The corresponding cost is computed as

$$J(x(k),\tilde{\boldsymbol{u}}_{N+1}(k),N+1)=\sum_{1=0}^{N}\left(\|x(i|k)\|_Q^2+\left\|\tilde{u}_{N+1}\big(x(i|k)\big)\right\|_R^2\right)+V_f\big(x(N+1|k)\big)=$$
$$=\sum_{1=0}^{N-1}\left(\|x(i|k)\|_Q^2+\left\|u^*\big(x(i|k)\big)\right\|_R^2\right)+\|x(N|k)\|_Q^2+\left\|k_a\big(x(N|k)\big)\right\|_R^2+V_f\big(x(N+1|k)\big) \tag{28}$$

Recalling that

$$J(x(k),\boldsymbol{u}^*(k),N)=\sum_{i=0}^{N-1}\left(\|x(i|k)\|_Q^2+\left\|u^*\big(x(i|k)\big)\right\|_R^2\right)+V_f\big(x(N|k)\big)$$

equation (28) can be written as

$$J(x(k),\tilde{\boldsymbol{u}}_{N+1}(k),N+1)=$$
$$=J(x(k),\boldsymbol{u}^*(k),N)-V_f\big(x(N|k)\big)+\|x(N|k)\|_Q^2+\left\|k_a\big(x(N|k)\big)\right\|_R^2+V_f\big(x(N+1|k)\big) \tag{29}$$

(The cost for the MPC problem with horizon $N+1$ under control $\tilde{\boldsymbol{u}}_{N+1}(k)$ is equal to the optimal cost for the MPC problem with horizon N at time k plus the cost of the $(N+1)$-th step plus the difference between the final costs.)

Considering Assumption 5.i and that, by definition (23), $V(x,N)\coloneqq J(x,k_{RH}(x),N)$, the following inequality holds

$$V(x(k),N+1)=V(x(k),N)-V_f\big(x(N|k)\big)+\|x(N|k)\|_Q^2+\left\|k_a\big(x(N|k)\big)\right\|_R^2+V_f\big(x(N+1|k)\big) \tag{30}$$

Thus, the monotonicity property with respect to N holds and is written as

$$V(x,N+1)\leq V(x,N),\text{for all } x\in X^{RH}(N) \tag{31}$$

with $V(x,0)=V_f(x)$, for all $x\in X^f$.

Equations (24) and (31) yelds

$$V(x,N+1)\leq V(x,N)\leq V_f(x)\leq \alpha_f(\|x\|),\text{for all } x\in X^f \tag{32}$$

and the condition

$$V(x,N)\leq \alpha_f(\|x\|),\text{for all } x\in X^f \tag{33}$$

is satisfied.

Finally, considering that

$$J(x(k+1),\boldsymbol{u}_{N-1}^*(k+1),N-1)=\sum_{i=1}^{N-1}\left(\|x(i|k)\|_Q^2+\left\|u^*\big(x(i|k)\big)\right\|_R^2\right)+V_f\big(x(N|k)\big)=$$
$$=J(x(k),\boldsymbol{u}_N^*(k),N)-\|x(k)\|_Q^2-\left\|u\big(x(k|k)\big)\right\|_R^2 \tag{34}$$

and that $x(k+1|k)=f\left(x(k|k),k_{RH}\big(x(k|k)\big)\right)$, we can write $V(x,N)$ as

$$V(x,N)=J\big(f\big(x,k_{RH}(x)\big),\boldsymbol{u}_{N-1}^*(k+1),N-1\big)+\|x\|_Q^2+\|k_{RH}(x)\|_R^2=$$
$$=V\big(f\big(x,k_{RH}(x)\big),N-1\big)+\|x\|_Q^2+\|k_{RH}(x)\|_R^2=$$
$$\geq V\big(f\big(x,k_{RH}(x)\big),N\big)+\|x\|_Q^2+\|k_{RH}(x)\|_R^2=$$
$$\geq V\big(f\big(x,k_{RH}(x)\big),N\big)+\|x\|_Q^2,\text{ for all } x\in X^{RH}(N) \tag{35}$$

and the condition

$$\Delta V(x,N) \leq -r(\|x\|), \text{for all } x \in X^{RH}(N) \tag{36}$$

is satisfied.

From equations (33) and (36), we can conclude that $V(x,N)$ is a Lyapunov function and, thus, that the origin is an asymptotically stable equilibrium point. Moreover, if $\alpha_f(\|x\|) = b\|x\|^2$ for some $b > 0$ and $X_f = X^{RH}(N)$, the origin is an exponentiaally stable equilibrium point. (q.e.d.)

Remarks.

1) Longer horizons extend the feasible state set, since $X^{RH}(N+1) \supseteq X^{RH}(N)$.
2) $X^{RH}(N) \supseteq X_f$, since the auxiliary control law can be used by the optimization algorithm, i.e., one can set $k_{RH}(x) = k_a(x)$ (which, in general, is not the optimal solution).

2.2 Example

In the simplest case, we select $k_a(x) = 0, X_f = \{0\}, V_f = 0$, i.e., at each time-step k, we aim at reaching the origin at time-step $N + k$.

Let, at time k, the optimal sequence be

$$\boldsymbol{u}^*(k) = \{u^*(0|k), u^*(1|k), \ldots, u^*(N-1|k)\}$$

The optimal sequence is such that $x^*(N|k) = 0$ (and, therefore, $u(N|k) = k_a\big(x^*(N|k)\big) = 0$).

At time $k+1$, the sequence

$\boldsymbol{u}(k+1) = \{u^*(1|k), u^*(2|k), \ldots, u^*(N-1|k), 0\}$

is feasible, since, recalling that, by Assumption 1, $f(0{,}0) = 0$, it is such that

$$x(N|k+1) = f\big(x(N-1|k+1), u(N-1|k+1)\big) = f(x^*(N|k), 0) = 0$$

Assumprion 5.i which we rewrite below for the readers' convenience

$$V_f\left(f\left(x(k), k_a\big(x(k)\big)\right)\right) - V_f\big(x(k)\big) + \|x(k)\|_Q^2 + \left\|k_a\big(x(k)\big)\right\|_R^2 \leq 0, \text{ for all } x(k) \in X^f = \{0\}$$

is then verified since $X^f = \{0\}$ and, therefore, $x(k) = 0$ and all the terms are null.

We note that the terminal constraint $x(N|k) = 0$ might be conservative and hard to verify for nonlinear systems.