I authorize Prof. Pietrabissa to upload the test results on the mailing list of the Process Automation group.

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Process Automation (MCER), 2018-2019

TEST (B)

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Exercise 1 (12 pts.)

Let the process be described by the transfer function: $P(s) = a\frac{s-1}{(s+1)(s+a)}$, with $a \in [1,10]$.

1. Considering the nominal value of the parameter $\tilde{a} = 1$ and by following the IMC design, develop a controller such that:
   a. the controlled system has 0 steady-state error for step inputs,
   b. the considered cost function for factorization is ISE;
   c. the controlled system is robustly stable against the uncertainties of the parameter a.
2. Compute the equivalent classic controller.

Exercise 2 (12 pts.)

Consider a process whose state-space model is:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = 0.25, N = 0.2, Q = 4.$$

Compute the control actions of a Predictive Functional Control algorithm at time $t = 8$, with:

- control horizon $m = 4$;
- prediction horizon $p = 4$;
- number of coincident points $n_h = 2$;
- number of basis functions $n_B = 2$;
- ramp reference $r(t) = t, \forall t$;
- reference trajectory $w(t+k|t) = 0.25y_m(t) + 0.75r(t+k)$;
- cost function $J = e^T e$, where e is the vector of future errors between predicted output and reference trajectory;
- plant-model error $\hat{e}(t+k|t) = \frac{k}{2} \cdot \left(y_m(t) - y(t)\right), k = 1,2,\dots$; the cost function should be such that the system response is smooth;
- the output measure at time $t = 8$ is $y_m(8) = 7.8$; the state and the control action at time $t = 7$ are $x(7) = 1.9$ and $u(7) = 5$.

Questions (6 pt.)

i) Consider the controller found Exercise 1 and check if it stabilizes the process $P(s) = 10\frac{s-1}{(s+1)(s+10)}e^{-2s}$ (*it is sufficient to write and explain the procedure*). (1 pg. max, 4pt)

ii) Briefly explain why, in the robust stability condition based on Nyquist theorem arguments (i.e., the condition $\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall\omega$), one of the assumptions is that the number of positive poles of the process model is equal to the number of positive poles of the actual process. (1/3 pg. max, 2pt).

Solution of exercise 1

1.

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles) and has the same number of positive poles of the actual the process, therefore it is possible to design a stable IMC controller $Q(s)$ to robustly stabilize the closed-loop nominal system.

The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

Factorize the nominal process $\tilde{P}(s) = \frac{s-1}{(s+1)^2} = -\frac{1-s}{(1+s)^2}$ in a minimum-phase term and a non-minimum-phase term, under ISE-optimal factorization:

$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$, with $\tilde{P}_+(s) = \frac{1-s}{1+s}$ and $\tilde{P}_-(s) = -\frac{1}{1+s}$,

and define the controller as follows:

$\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = -(1+s).$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$).

We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 1$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = -\frac{1+s}{1+\lambda s}$ is proper;

b) $\tilde{T}(0) = \tilde{P}(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1} f(s)\right]_{s=0} = \left[\frac{1-s}{1+s}\,\frac{1}{1+\lambda s}\right]_{s=0} = 1.$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall\omega,$

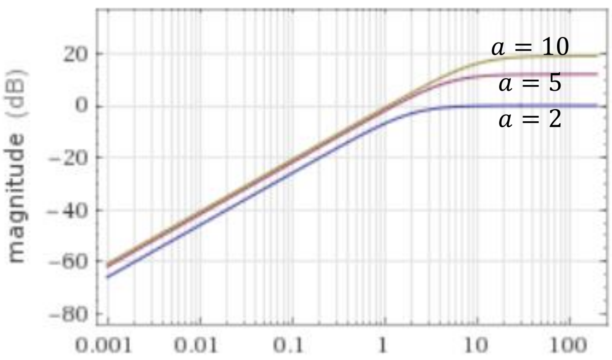
where

$$\tilde{T}(j\omega) = \tilde{P}(j\omega)Q(j\omega) = -\frac{1-j\omega}{(1+j\omega)(1+\lambda j\omega)}$$

and $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall\omega$. The multiplicative uncertainty is defined as

$\Delta_m(j\omega) \coloneqq \frac{P(j\omega)-\tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{-\frac{1-j\omega}{(1+j\omega)\left(1+\frac{j\omega}{a}\right)}+\frac{1-j\omega}{(1+j\omega)^2}}{-\frac{1-j\omega}{(1+j\omega)^2}} = \left(1-\frac{1}{a}\right)\frac{j\omega}{1+\frac{j\omega}{a}}.$

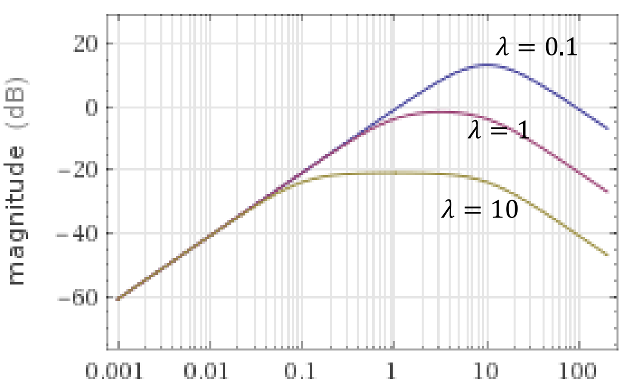
Since $|\Delta_m(j\omega)|$ grows with a (both the gain $\left(1-\frac{1}{a}\right)$ and the cut-off frequency a of the pole $-1/a$ grow with a) and $a \in [1,10]$, an upper-bound is then given by $l_m(j\omega) = 0.9\frac{j\omega}{1+\frac{j\omega}{10}}$.



The sufficient condition for robust stability is then:

$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| = \left|0.9\frac{j\omega}{1+\frac{j\omega}{10}}\frac{1-j\omega}{(1+j\omega)(1+\lambda j\omega)}\right| = \left|0.9\frac{j\omega}{\left(1+\frac{j\omega}{10}\right)(1+\lambda j\omega)}\right| < 1, \forall\omega.$$

which holds, approximatively, for $\lambda \geq 1$.



Thus, we set that $Q(s) = -1$.

2.
In the equivalent classic control scheme, the controller is a PID+filter controller computed from the IMC controller as follows:

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = \frac{1}{1-\frac{1-s}{(1+s)^2}} = \frac{1}{3}\frac{1+2s+s^2}{s\left(1+\frac{s}{3}\right)} = \frac{2}{3}\left(1+\frac{1}{2s}+0.5s\right)\frac{1}{1+s/3}.$$

Solution of exercise 2.

To develop the PFC controller, we need to select the base functions and the coincident points. We chose $n_B = 2$ base functions $B_i(k) = k^{i-1} = 1, k = 0,1, \ldots$. Since $n_h = 2$, the prediction horizon is $p = 4$ and we want a smooth response we chose $h_1 = 3$ and $h_2 = 4$.

Firstly, we have to compute the model response to the base functions in the coincidence points, considering null initial conditions $x(0) = 0$:

$\underline{t = h_1 = 3}$

B_1: $y_{B_1}(3) = QM^2NB_1(0) + QMNB_1(1) + QNB_1(2) = 0.05 \cdot 1 + 0.2 \cdot 1 + 0.8 \cdot 1 = 1.05$;
B_2: $y_{B_2}(3) = QM^2NB_2(0) + QMNB_2(1) + QNB_2(1) = 0.05 \cdot 0 + 0.2 \cdot 1 + 0.8 \cdot 2 = 1.8$.

$\underline{t = h_2 = 4}$
B_1: $y_{B_1}(4) = QM^3NB_1(0) + QM^2NB_1(1) + QMNB_1(2) + QNB_1(3) = 0.0125 \cdot 1 + 0.05 \cdot 1 + 0.2 \cdot 1 + 0.8 \cdot 1 = 1.0625$.
B_2: $y_{B_2}(4) = QM^3NB_2(0) + QM^2NB_2(1) + QMNB_2(2) + QNB_2(3) = 0.0125 \cdot 0 + 0.05 \cdot 1 + 0.2 \cdot 2 + 0.8 \cdot 3 = 2.85$.

Thus, $y_B(h1) = (y_{B_1}(h_1) \quad y_{B_2}(h_1)) = (1.05 \quad 1.0625)$, $y_B(h2) = (y_{B_1}(h_2) \quad y_{B_2}(h_2)) = (1.8 \quad 2.85)$.

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{pmatrix} 1.05 & 1.0625 \\ 1.8 & 2.85 \end{pmatrix}$.
The matrix Y_B is used to compute the solution of the unconstrained optimization problem: $\mu^* = Y_B^{-1}(w - f)$, where μ^* is the vector of the optimal parameters at time t and $Y_B^{-1} = \begin{pmatrix} 2.65 & -0.98 \\ -1.67 & 0.97 \end{pmatrix}$. The control action is the computed as $u(t) = {\mu^*}^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2, \dots, n_B$, evaluated for $k = 0$. In our problem, since $n_B = 2$, we need to find a two-column vector $\mu^*(t)$.

At time $t = 8$, the future reference trajectory values in the coincidence points are
$k = h_1 = 3,\ w(8 + 3|8) = 0.25y_m(8) + 0.75r(8 + 3) = 0.25 \cdot 7.8 + 0.75 \cdot 11 = 10.2$;
$k = h_2 = 4,\ w(8 + 4|8) = 0.25y_m(8) + 0.75r(8 + 4) = 0.25 \cdot 7.8 + 0.75 \cdot 12 = 10.95$.

With PFC, the free response at time $t = 8$ in the coincidence points is given by
$d(t + k|t) = f(t + k|t) + \hat{e}(t + k|t) = QM^k x(t) + \frac{k}{2} \cdot \left(y_m(t) - y(t)\right)$,
where $y(t)$ is the model output at time $t = 8$:
$x(8) = Mx(7) + Nu(7) = 1.475$;
$y(8) = Qx(8) =$5.9.

$\underline{t = 8}$
$\underline{h_1 = 3}$ $\quad f(8 + 3|8) = QM^3 x(8) + \frac{3}{2}\left(y_m(8) - y(8)\right) = 2.94$;
$\underline{h_2 = 4}$ $\quad f(8 + 4|8) = QM^4 x(8) + 2\left(y_m(8) - y(8)\right) = 3.82$.

$d = (w - f) = \begin{pmatrix} 10.2 \\ 10.95 \end{pmatrix} - \begin{pmatrix} 2.94 \\ 3.82 \end{pmatrix} = \begin{pmatrix} 7.26 \\ 7.13 \end{pmatrix}$.
$\mu(1) = \begin{pmatrix} 2.65 & -0.98 \\ -1.67 & 0.97 \end{pmatrix} \begin{pmatrix} 7.26 \\ 7.13 \end{pmatrix} = \begin{pmatrix} 12.14 \\ -5.17 \end{pmatrix}$;
$u(1) = \mu_1(1)B_1(0) + \mu_2(1)B_2(0) = 12.14 \cdot 1 - 5.15 \cdot 0 = 12.14$.