Process Automation (MCER), 2017-2018

Exam - January 23, 2018 (3h00)

Exercise 1 (11 pt.)

Let the process be described by the transfer function: $P(s) = K\frac{s+0.2}{(s+1)^2}e^{-\theta s}$, with $K \in [1,1.9)$ and $\theta = 0.1s$, and let the nominal gain value be $\widetilde{K} = 1$.

A) By using the Padé approximation, under the IAE cost function, design an IMC controller $Q(s)$ such that:
   - i) the overall system is robustly asymptotically stable;
   - ii) the overall system has 0 steady-state error for step inputs.

B) Describe (without calculations) how to check that the found controller stabilizes the real process $P^R(s) = 1.9\frac{s+0.2}{(s+1)^2}e^{-0.1}$ – suggestion: consider the equivalent controller $G(s)$ and the classic feedback control scheme.

Exercise 2 (11 pt.)

Consider a process whose state space model is given by the following equations:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = \begin{bmatrix} 0.2 & 0.4 \\ 0 & -0.2 \end{bmatrix}, N = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, Q = [0.25 \;\; 0.75].$$

Compute the control action $u(5)$ of a PFC controller with the following specifications:

- i) Prediction horizon $p = 4$;
- ii) Base functions: $B_1(k) = 1, B_2(k) = k^2, k \geq 0$;
- iii) Number of coincidence points $n_H = 2$, chosen to prefer robustness to performance;
- iv) Reference signal $r(t) = \begin{cases} t/2, & t = 0, \dots, 6 \\ 3, & t > 6 \end{cases}$;
- v) Reference trajectory computed as $w(t+k|t) = r(t)$;
- vi) Cost function $J = e^T e$.
- vii) The plant-model error is computed as $\hat{n}(t+k|t) = y_m(t) - y(t), \forall k > 0$, with $y_m(0) = 0$, $y_m(1) = 0.2$, $y_m(2) = 0.8$, $y_m(3) = 1.75$, $y_m(4) = 2.5$, $y_m(5) = 3$.
- viii) State and control values $x(4) = \begin{pmatrix} 15 \\ 2 \end{pmatrix}, u(4) = 5$.

Questions (8 pt.)

- i) Discuss why MPC may improve the safety of plants (1/2 pg. max, 4pt).
- ii) Discuss why the length of the prediction horizon depends on the accuracy of the plant model (consider the two cases: step response model and state-space model)? (1/2 pg. max, 4pt).

Solution of exercise 1

A)

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a (stable) IMC controller $Q(s)$ to stabilize the closed-loop nominal system.

The time-delay $\theta = 0.1s$ of the process is much smaller than the time constant $\tau = 1s$ of the process, therefore we can use a Padé approximation to write the delay term of the actual process as a transfer function. By using the 1/1 Padé approximation $e^{-\theta s} \cong \frac{1-s\frac{\theta}{2}}{1+s\frac{\theta}{2}} = \frac{1-0.05s}{1+0.05s}$, we obtain the following approximated and nominal processes:

$$P^P(s) = 0.2K\frac{1+5s}{(1+s)^2} \cdot \frac{1-0.05s}{1+0.05s};\ \tilde{P}(s) = \tilde{K}P^P(s) = 0.2\frac{1+5s}{(1+s)^2} \cdot \frac{1-0.05s}{1+0.05s}.$$

The IMC design procedure to robustly stabilize the approximated process $P^P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term (IAE-optimal factorization):

$$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s),$$

with $\tilde{P}_+(s) = (1-0.05s)$ and $\tilde{P}_-(s) = 0.2\frac{1+5s}{(1+s)^2(1+0.05s)}$

b) Define the controller as follows: $\tilde{Q}(s) = \tilde{P}^{-1}(s) = 5\frac{(1+s)^2(1+0.05s)}{1+5s}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1.

Thus, we use the IMC filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 2$.

The IMC controller is then $Q(s) = 5\frac{(1+s)^2(1+0.05s)}{(1+5s)(1+\lambda s)^3}$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.

By definition, the additive uncertainty is defined as follows

$$|\Delta_a(j\omega)| = |P^P(j\omega) - \tilde{P}(j\omega)| = \left|0.2K\frac{1+5j\omega}{(1+j\omega)^2} \cdot \frac{1-0.05j\omega}{1+0.05j\omega} - 0.2\frac{1+5j\omega}{(1+j\omega)^2} \cdot \frac{1-0.05j\omega}{1+0.05j\omega}\right|$$

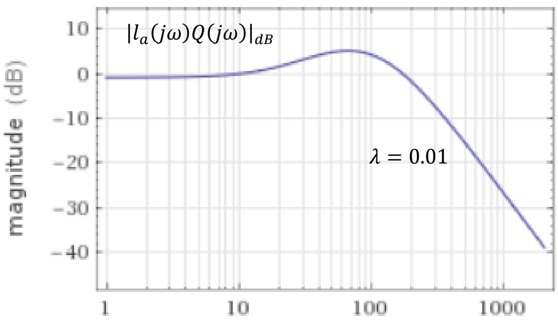
$$= \left|0.2(K-1)\frac{1+5j\omega}{(1+j\omega)^2}\right|.$$

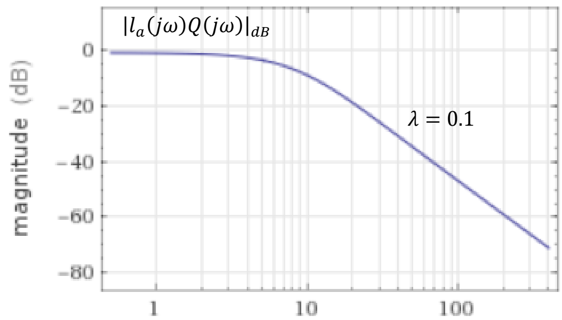
Since $K \in [1,1.9)$, an upper-bound is given by $l_a(j\omega) = 0.18\frac{1+5j\omega}{(1+j\omega)^2}$.

The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|0.18\frac{1+5j\omega}{(1+j\omega)^2}5\frac{(1+j\omega)^2(1+0.05j\omega)}{(1+5j\omega)(1+\lambda j\omega)^2}\right| = \left|0.9\frac{1+0.05j\omega}{(1+\lambda j\omega)^2}\right| < 1, \forall\omega,$$

which, according to the Bode diagrams below, is true at least for $\lambda \geq 0.05$ (the poles must be placed 'before' the zero). To account for the Padé approximation we choose a conservative value $\lambda = 0.1$.





B)

It suffices to compute the equivalent controller $G(s) = \frac{Q(s)}{1-G(s)Q(s)}$ and the phase margin m_ϕ of the transfer function $F(s) = G(s)P^R(s)$, and to check that the time margin $m_\tau \coloneqq \frac{m_\phi}{\omega_c}$ (where ω_c is the cross-over pulsation) is positive.

Solution of exercise 2

To develop the PFC controller, we need to select the coincidence points. Considering that we want to achieve a robust system behaviour, we should aim at smooth output trajectories by not weighting the predicted errors in the first samples; then, we select the coincidence points $h_1 = 3$ and $h_2 = 4$. According to the PFC procedure, as a first step we have to compute the model response to the base functions, denoted with y_{B_1} and y_{B_2}, at the coincidence points, considering the system output $y_{B_i}(k) = QM^{k-1}NB_i(0) + QM^{k-2}NB_i(1) + \cdots + QNB_i(k-1)$, with $u(k) = B_1(k)$ and $u(k) = B_2(k)$, respectively, and null initial conditions $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$B_1(k) = 1$

$k = h_1 = 3$

$y_{B_1}(3) = QM^2NB_1(0) + QMNB_1(1) + QNB_1(2) = [0.25 \;\; 0.75]\begin{bmatrix} 0.2 & 0.4 \\ 0 & -0.2 \end{bmatrix}^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot 1 + [0.25 \;\; 0.75]\begin{bmatrix} 0.2 & 0.4 \\ 0 & -0.2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot 1 + [0.25 \;\; 0.75]\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot 1 = 0.02 - 0.1 + 0.5 = 0.402$

$k = h_2 = 4$

$y_{B_1}(4) = QM^3NB_1(0) + QM^2NB_1(1) + QMNB_1(2) + QNB_1(3) = [0.25 \;\; 0.75]\begin{bmatrix} 0.2 & 0.4 \\ 0 & -0.2 \end{bmatrix}^3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot 1 + 0.02 \cdot 1 - 0.1 \cdot 1 + 0.5 \cdot 1 = -0.004 + 0.402 = 0.398$

$B_2(k) = k^2$

$k = h_1 = 3$

$y_{B_1}(3) = QM^2NB_2(0) + QMNB_2(1) + QNB_2(2) = 0.02 \cdot 0 - 0.1 \cdot 1 + 0.5 \cdot 4 = 1.98$

$k = h_2 = 4$

$y_{B_1}(4) = QM^3NB_2(0) + QM^2NB_2(1) + QMNB_2(2) + QNB_2(3) = -0.004 \cdot 0 + 0.02 \cdot 1 - 0.1 \cdot 4 + 0.5 \cdot 9 = 4.12.$

The matrix $Y_B = \begin{pmatrix} y_{B_1}(h_1) & y_{B_2}(h_1) \\ y_{B_1}(h_2) & y_{B_2}(h_2) \end{pmatrix} = \begin{pmatrix} 0.402 & 1.98 \\ 0.398 & 4.12 \end{pmatrix}$ is used to compute the solution of the unconstrained optimization problem: $\mu^* = Y_B^{-1}(w - f)$, with $Y_B{}^{-1} = \begin{pmatrix} 4.75 & -2.28 \\ -0.46 & 0.46 \end{pmatrix}$. The control action is the computed as $u(t) = B(0)\mu^*$, where $B(0) = \big(B_1(0) \;\; B_2(0)\big) = (1 \;\; 0)$.

By considering the given state and control values, we can start computing the free response at time $t = 5$ over the coincidence points, computed as $f(5 + k|5) = QM^k x(5) + \hat{n}(5 + k|5) = QM^k x(5) + y_m(5) - y(5)$.
Since $y_m(5)$, $x(4)$ and $u(4) = 5$ are given, we can directly compute the state and the model output at time $t = 5$:

$$\begin{cases} x_1(5) = 0.2x_1(4) + 0.4x_2(4) - u(4) = -1.2 \\ x_2(5) = -0.2x_2(4) + u(4) = 4.6 \\ y(5) = 0.25x_1(5) + 0.75x_2(5) = 3.15 \end{cases}$$

$\underline{h_1 = 3}$ $\quad f(3) = QM^4x(5) + y_m(5) - y(5) = [0.25 \;\; 0.75]\begin{bmatrix} 0.2 & 0.4 \\ 0 & -0.2 \end{bmatrix}^3 \begin{bmatrix} -1.2 \\ 4.6 \end{bmatrix} + 3.15 - 3 = 0.138;$

$\underline{h_2 = 4}$ $\quad f(4) = QM^4x(5) + y_m(5) - y(5) = 0.155.$

The reference trajectory vector is $w = \begin{pmatrix} w(t + h_1|t) \\ w(t + h_2|t) \end{pmatrix} = \begin{pmatrix} r(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} r(5) \\ r(5) \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$.

$\mu(5) = Y_B{}^{-1}(w - f) = \begin{pmatrix} 4.75 & -2.28 \\ -0.46 & 0.46 \end{pmatrix}\left(\begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 0.138 \\ 0.155 \end{pmatrix}\right) = \begin{pmatrix} 5.87 \\ -0.01 \end{pmatrix};$

$u(5) = (B_1(0) \;\; B_2(0))\mu(5) = (1 \; 0)\begin{pmatrix} 5.87 \\ -0.01 \end{pmatrix} = 5.87.$