Master in Control Engineering

Process Automation
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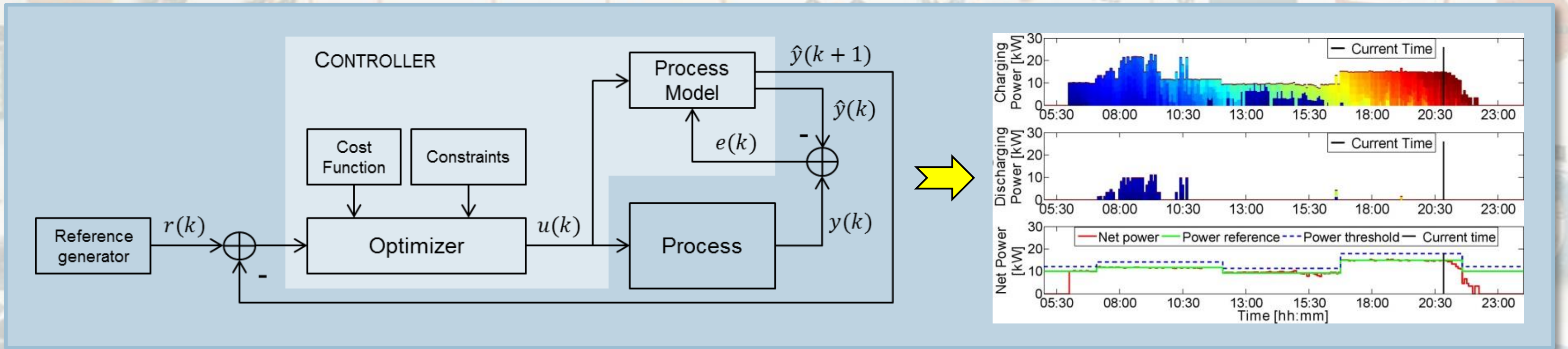
AUTOMATICA E GESTIONALE ANTONIO RUBERTI

SAPIENZA
UNIVERSITÀ DI ROMA

Master in Control Engineering

Process Automation

8. Robust stability

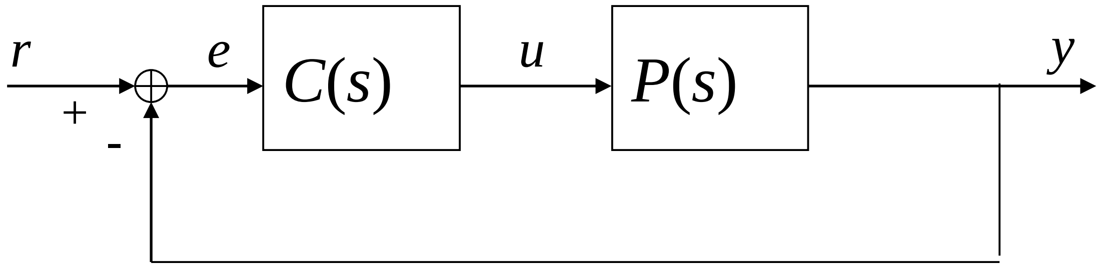


Outline

- Robust stability
  - Robust control problem and uncertainty definitions
    - Parametric uncertainty
    - Dynamic uncertainty
  - Sufficient conditions for robust stability of linear SISO systems
  - Example
  - Sufficient conditions for robust stability of linear SISO systems in the IMC approach
- Summary

Robust control problem and uncertainty definitions

- Parametric uncertainty



– Given
  - A parametric process model

$$P(s) = K\frac{s^m + a_m s^{m-1} + \cdots + a_0}{s^n + b_n s^{n-1} + \cdots + b_0} = K\frac{N_P(s;\boldsymbol{a})}{D_P(s;\boldsymbol{b})}$$

   – where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the vectors of the parameters of the numerator of $P(s)$
  - A range of admissible values for the parameters

$$K \in [K_1, K_2] = \mathcal{K}, \boldsymbol{a} \in \left[a_{m_1}, a_{m_2}\right] \times \cdots \times \left[a_{0_1}, a_{0_2}\right] = \mathcal{A}, \boldsymbol{b} \in \left[b_{n_1}, b_{n_2}\right] \times \cdots \times \left[b_{0_1}, b_{0_2}\right] = \mathcal{B}$$

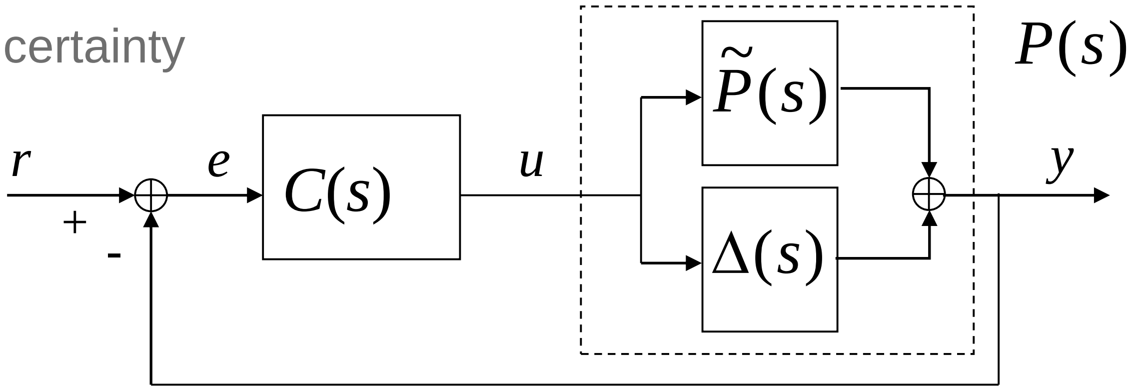
  - A set of admissible transfer functions

$$\Pi \coloneqq \left\{ P(s) = K\frac{N_P(s;\boldsymbol{a})}{D_P(s;\boldsymbol{b})} \middle| K \in \mathcal{K}, \boldsymbol{a} \in \mathcal{A}, \boldsymbol{b} \in \mathcal{B} \right\}$$

– The *robust control problem* is defined as follows:
Find a feedback controller $C(s)$ which stabilizes all the processes $P(s) \in \Pi$.

Robust control problem and uncertainty definitions

- Additive uncertainty



– Given a process $P(s)$ and a nominal process model $\tilde{P}(s)$, the *additive uncertainty* $\Delta(s)$ is defined as the function such that

$$P(s) = \tilde{P}(s)+\Delta(s)$$

– Given an upperbound $l_a(j\omega)$ of the additive uncertainty, i.e., a function such that

$$\|l_a(j\omega)\|_\infty > \|\Delta(j\omega)\|_\infty$$

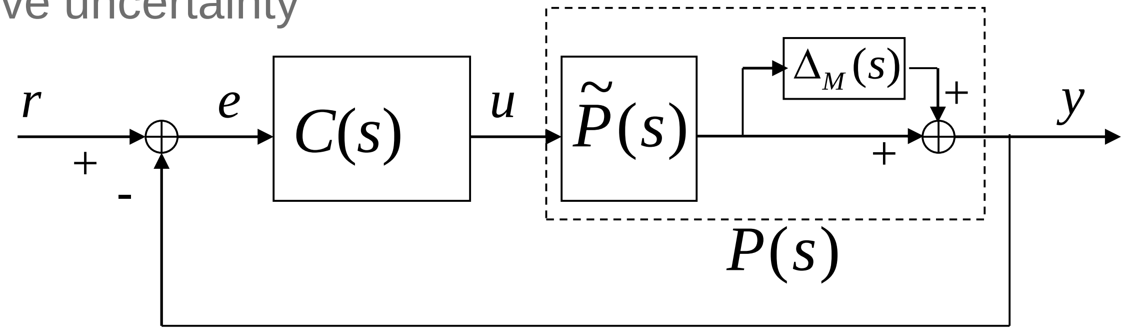
the set of admissible transfer functions is defined as

$$\Pi \coloneqq \left\{P(s) = \tilde{P}(s)+\Delta(s) \middle| |l_a(j\omega)| > |\Delta(j\omega)|, \forall\omega \right\}$$

– The *robust control problem* is defined as follows:
Find a feedback controller $C(s)$ which stabilizes all the processes $P(s) \in \Pi$.

Robust control problem and uncertainty definitions

- Multiplicative uncertainty



– Given a process $P(s)$ and a nominal process model $\tilde{P}(s)$, the *multiplicative uncertainty* $\Delta_m(s)$ is defined as the function such that

$$P(s) = \tilde{P}(s) \cdot (1 + \Delta_m(s))$$

– Given an upperbound $l_m(j\omega)$ of the multiplicative uncertainty, i.e., a function such that

$$\|l_m(j\omega)\|_\infty > \|\Delta_m(j\omega)\|_\infty$$

the set of admissible transfer functions is defined as

$$\Pi \coloneqq \left\{P(s) = \tilde{P}(s) \cdot (1 + \Delta_m(s)) \middle| |l_m(j\omega)| > |\Delta_m(j\omega)|, \forall \omega \right\}$$

– The *robust control problem* is defined as follows:
Find a feedback controller $C(s)$ which stabilizes all the processes $P(s) \in \Pi$.

Sufficient conditions for robust stability of linear SISO systems

- Lemma
  - Necessary condition for the robust stability problem is that the controller $C(s)$ stabilizes the nominal process $\tilde{P}(s)$

- Proof
  - $C(s)$ must stabilize all the process $P(s) \in \Pi$
  - $\tilde{P}(s) \in \Pi$

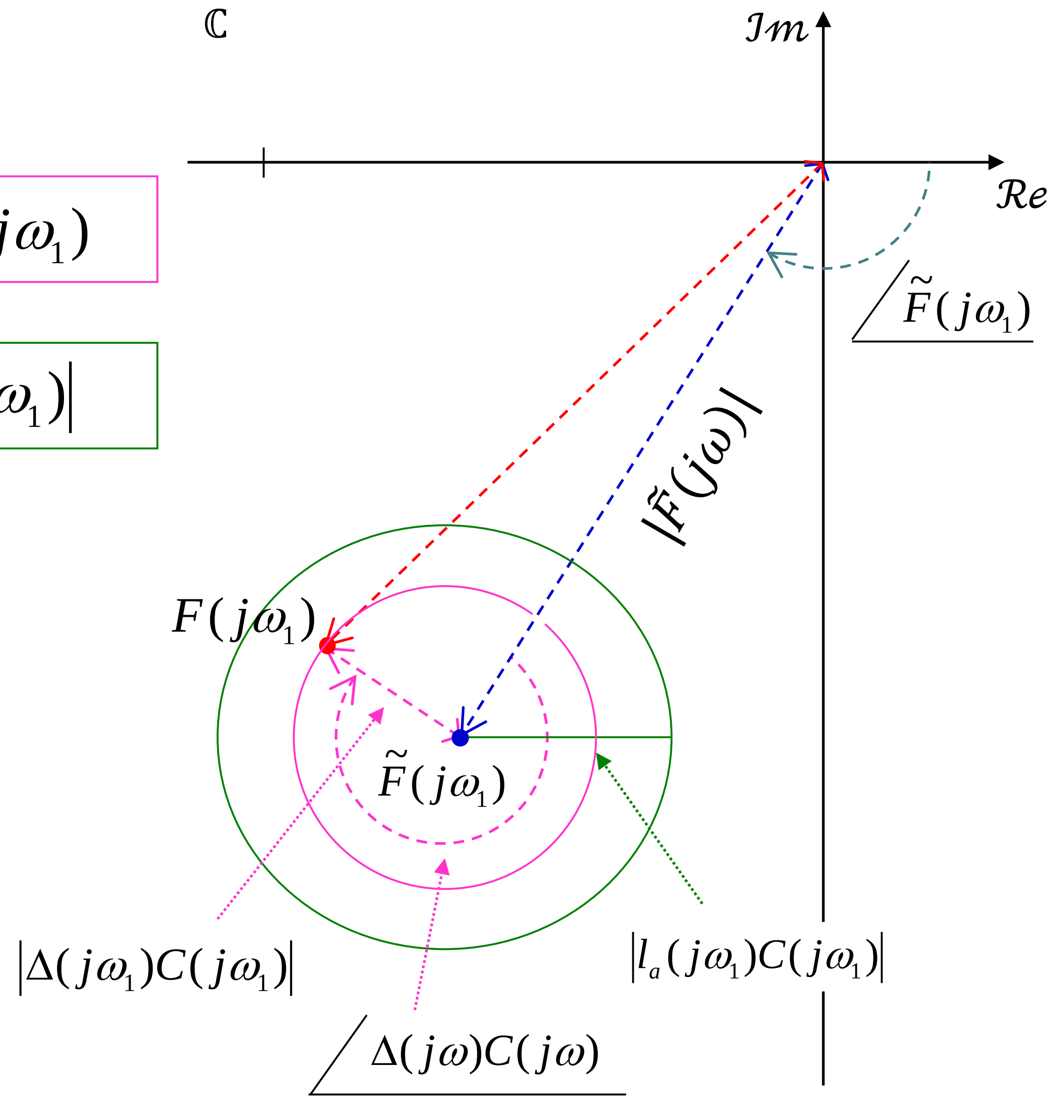
Sufficient conditions for robust stability of linear SISO systems

- Theorem 1 (additive uncertainty)
  - By assuming that
    - the controller $C(s)$ stabilizes the nominal process $\tilde{P}(s)$
    - the process $P(s)$ and the nominal process $\tilde{P}(s)$ have the same number of RHP poles:
      $$n_P^+ = n_{\tilde{P}}^+$$
  - then a sufficient condition for the solution of the robust stability problem is that the controller $C(s)$ is such that
    $$\left\| l_a(j\omega)C(j\omega)\tilde{S}(j\omega) \right\|_\infty < 1$$
- Proof
  - Since $n_P^+ = n_{\tilde{P}}^+$, to satisfy the Nyquist theorem, the number of counter-clockwise encirclements about $-1 + j0$ of the vectors $\overrightarrow{F(j\omega)} = \overrightarrow{C(j\omega)P(j\omega)}$ and $\overrightarrow{\tilde{F}(j\omega)} = \overrightarrow{C(j\omega)\tilde{P}(j\omega)}$, as the frequency ω ranges from $-\infty$ to ∞, must be the shame
  - By the additive uncertainty definition, it holds that:
    $$F(j\omega) = C(j\omega)P(j\omega) = C(j\omega)\left(\tilde{P}(j\omega) + \Delta(j\omega)\right) = \tilde{F}(j\omega) + C(j\omega)\Delta(j\omega)$$
  - Since $|l_a(j\omega)| > |\Delta(j\omega)|, \forall\omega$ by definition, it follows that, for all ω, the complex vector $\overrightarrow{F(j\omega)}$ lies in a circle centered in $\tilde{F}(j\omega)$ and with radius $|C(j\omega)l_a(j\omega)|$

Teorema: condizioni sufficienti per la stabilità robusta – dimostrazione (cont.)

$$F(j\omega_1) = \tilde{F}(j\omega_1) + \Delta(j\omega_1)C(j\omega_1)$$

$$\left|l_a(j\omega_1)C(j\omega_1)\right| > \left|\Delta(j\omega_1)C(j\omega_1)\right|$$



Sufficient conditions for robust stability of linear SISO systems

- Proof (cont'd)
  - Therefore, the Nyquist diagram of $F(j\omega)$ lies in a 'tube' with variable radius $|C(j\omega)l_a(j\omega)|$ around the Nyquist diagram of $\tilde{F}(j\omega)$
  - Since, by assumption, $C(s)$ stabilizes $\tilde{P}(s)$ and $n_P^+ = n_{\tilde{P}}^+$, sufficient condition for $C(s)$ to stabilize all the processes $P(s) \in \Pi$ is that the point -1 lies outside the tube
  - Therefore, the distance between all P's and -1 must be always positive:

$$|F(j\omega) - (-1)| > 0, \forall\omega, \forall\Delta(j\omega)$$

$$\Rightarrow \left|\tilde{F}(j\omega) + C(j\omega)\Delta(j\omega) + 1\right| > 0, \forall\omega, \forall\Delta(j\omega) \qquad (1)$$

  - Since $|a + b| \geq |a| - |b|, \forall a, b \in \mathbb{C}$, it follows that

$$\left|1 + \tilde{F}(j\omega) + C(j\omega)\Delta(j\omega)\right| > \left|1 + \tilde{F}(j\omega)\right| - |C(j\omega)\Delta(j\omega)|, \forall\omega, \forall\Delta(j\omega) \qquad (2)$$

  - From equation (2), it follows that a sufficient condition for equation (1) to be verified is:

$$\left|1 + \tilde{F}(j\omega)\right| - |C(j\omega)\Delta(j\omega)| > 0, \forall\omega, \forall\Delta(j\omega) \qquad (3)$$

Sufficient conditions for robust stability of linear SISO systems

- Proof (cont'd)
  - From the definition of upperbound of the additive uncertainty, it follows from equation (3) that:

$$\left|1+\tilde{F}(j\omega)\right| - |C(j\omega)\Delta(j\omega)| > \left|1+\tilde{F}(j\omega)\right| - |C(j\omega)l_a(j\omega)|, \forall\omega, \forall\Delta(j\omega) \quad (4)$$

  - From equation3 (3) and (4), it follows that a sufficient condition for equation (1) to be verified is:

$$\left|1+\tilde{F}(j\omega)\right| - |C(j\omega)l_a(j\omega)| > 0, \forall\omega \quad (5)$$

$$\Rightarrow |C(j\omega)l_a(j\omega)| < \left|1+\tilde{F}(j\omega)\right|, \forall\omega \quad (6)$$

$$\Rightarrow |C(j\omega)l_a(j\omega)| \cdot \left|1+\tilde{F}(j\omega)\right|^{-1} < 1, \forall\omega \quad (7)$$

  - Recalling the definition of sensitivity function:

$$\tilde{S}(s) := \frac{1}{1+\tilde{P}(s)C(s)} = \frac{1}{1+\tilde{F}(s)} \quad (8)$$

from equations (7) and (8) it follows that:

$$\left|l_a(j\omega)C(j\omega)\tilde{S}(s)\right| < 1, \forall\omega \quad (9)$$

Sufficient conditions for robust stability of linear SISO systems

- Theorem 2 (multiplicative uncertainty)
  - By assuming that
    - the controller $C(s)$ stabilizes the nominal process $\tilde{P}(s)$
    - the process $P(s)$ and the nominal process $\tilde{P}(s)$ have the same number of RHP poles:

$$n_P^+ = n_{\tilde{P}}^+$$

  - then a sufficient condition for the solution of the robust stability problem is that the controller $C(s)$ is such that

$$\left\| l_m(j\omega)\tilde{T}(j\omega) \right\|_\infty < 1$$

- Proof
  - From the definitions of additive and multiplicative uncertainties, it follows that:

$$\Delta_m(\mathrm{s}) = \Delta(\mathrm{s})\, \tilde{P}^{-1}(s)$$

  - From the definitions of sensitivities functions, it follows that:

$$\tilde{T}(s) = \frac{\tilde{F}(s)}{1 + \tilde{F}(s)} = C(s)\tilde{P}(s)\tilde{S}(s)$$

  - Therefore, the sufficient condition (9) becomes

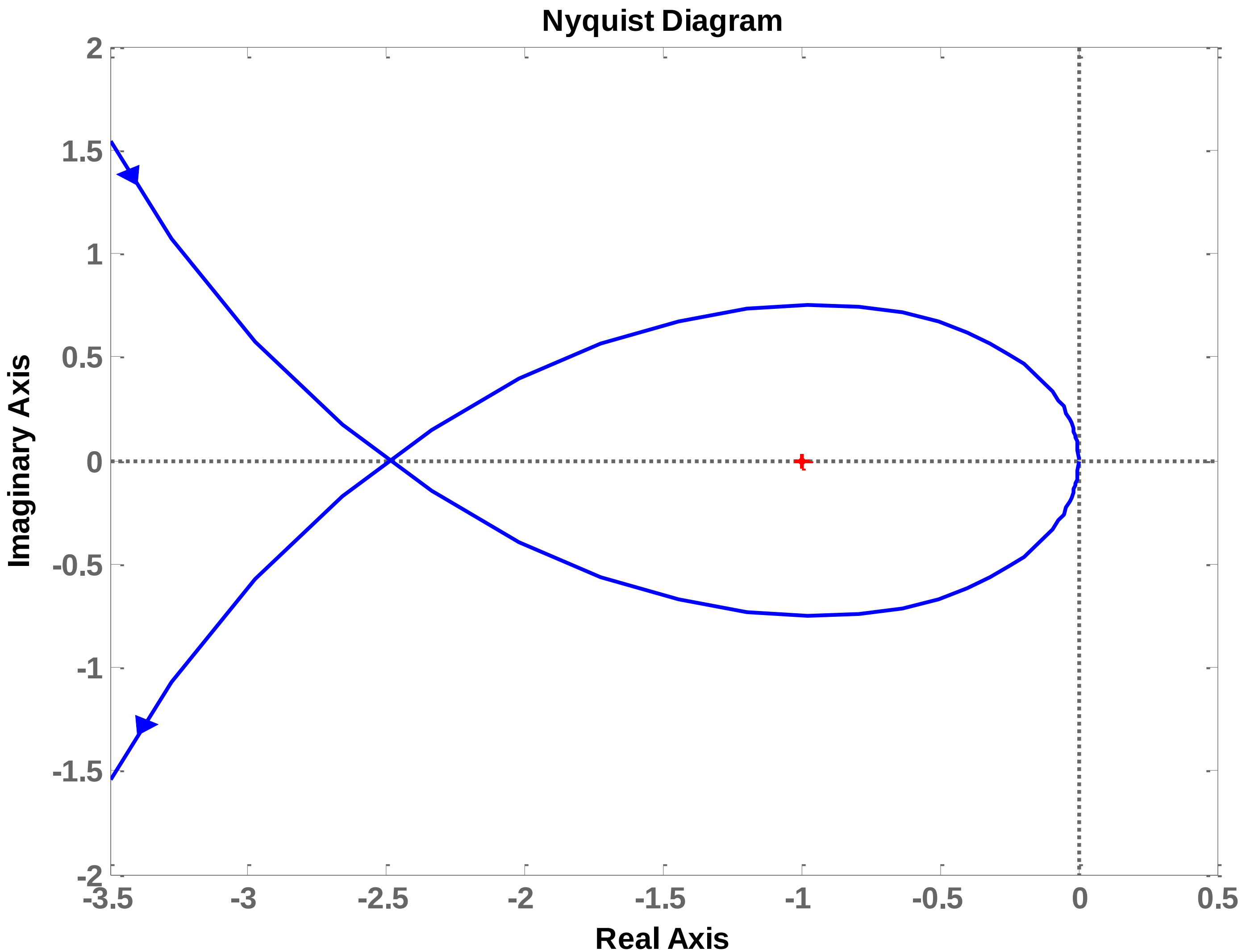
$$\left| l_a(j\omega)C(j\omega)\tilde{S}(s) \right| = \left| l_m(j\omega)\tilde{P}^{-1}(s)\tilde{P}(s)\tilde{T}(s) \right| = \left| l_m(j\omega)\tilde{T}(s) \right| < 1, \forall\omega$$

Example

Process with parametric uncertainty	$P(s) = K_P \frac{(s+1)}{s(s-1)};$ $K_P \in [1.5\,, 2.5]$
Nominal process	$\tilde{P}(s) = K_{\tilde{P}} \frac{(s+1)}{s(s-1)} = 2.5 \frac{(s+1)}{s(s-1)};$ $K_{\tilde{P}} = 2.5$
Controller	$C(s) = 1$
Nominal open-loop transfer function	$\tilde{F}(s) = 2.5 \frac{(s+1)}{s(s-1)}$
Stable nominal closed-loop transfer function	$\tilde{W}(s) = \frac{\tilde{F}(s)}{1+\tilde{F}(s)} = 2.5 \frac{s+1}{s^2 + 1.5s + 2.5}$

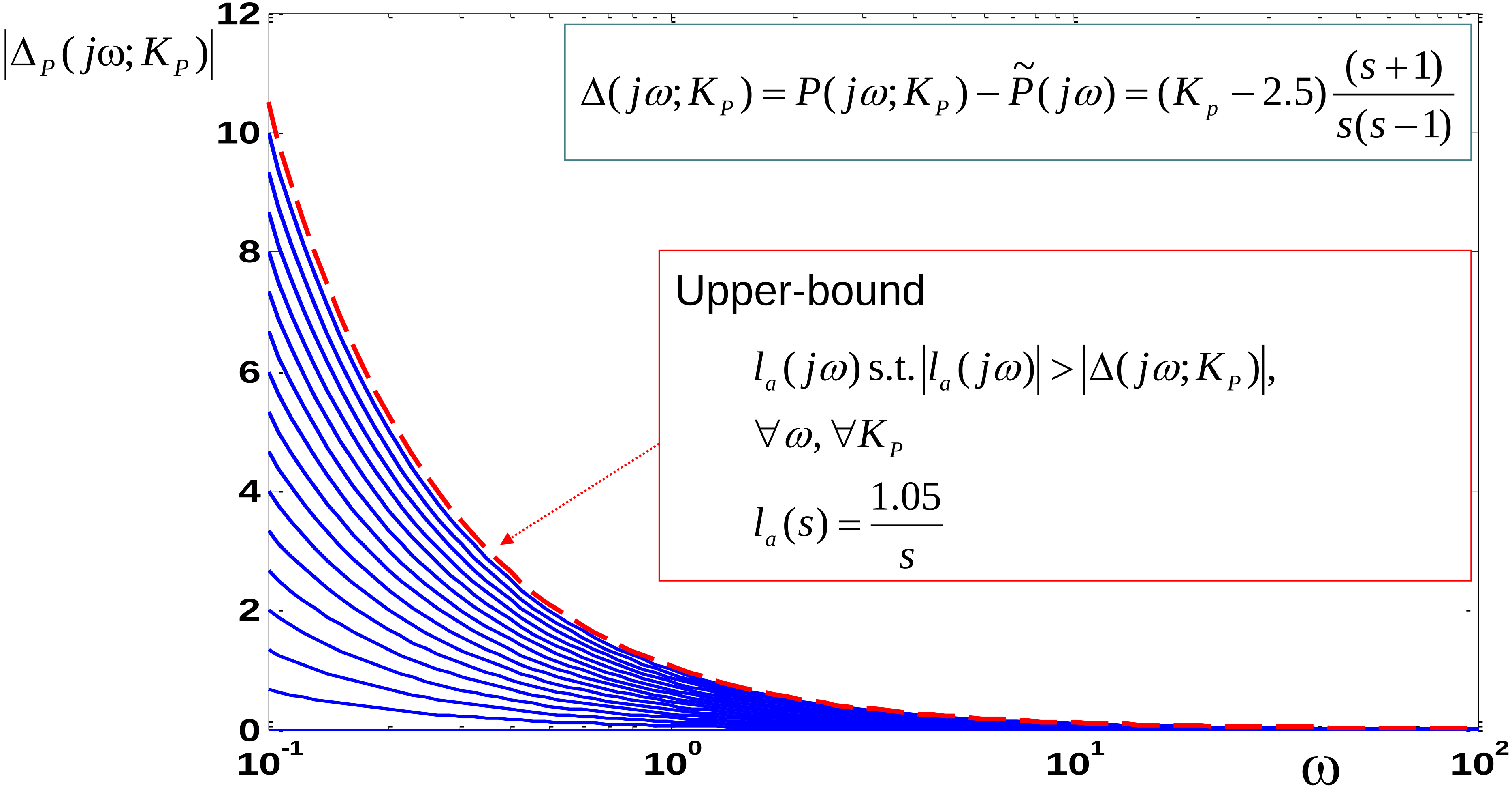
Example

– Nyquist diagram of the nominal open-loop system



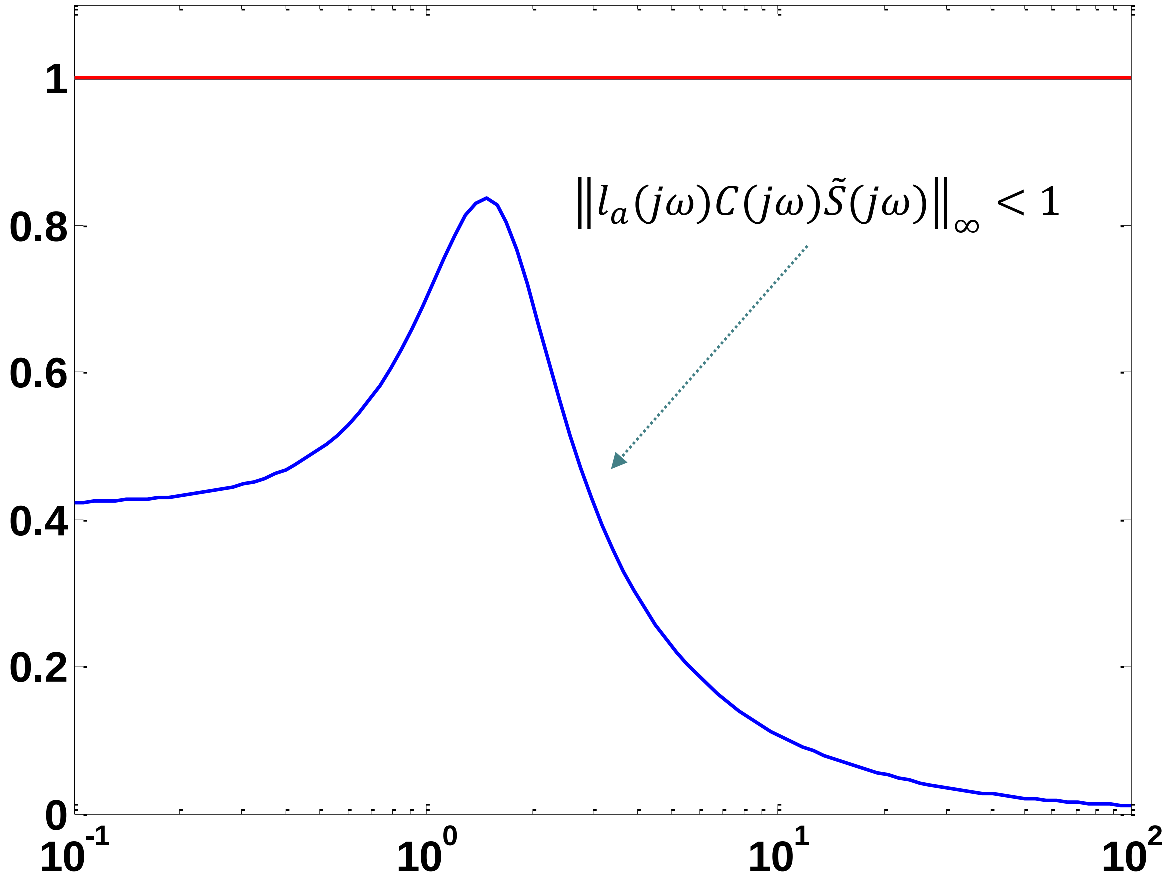
Example

– From parametric uncertainty to additive uncertainty



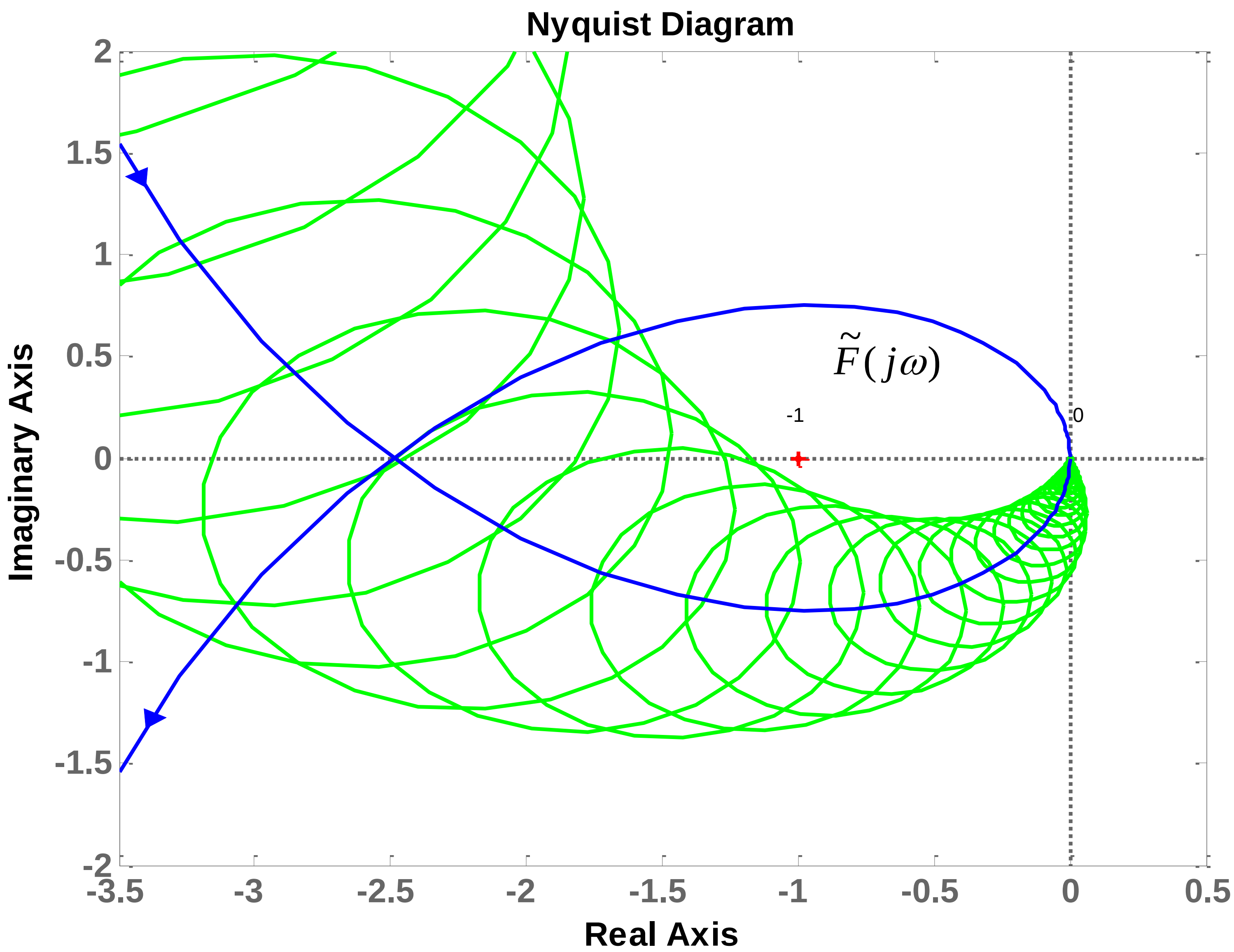
Example

– Suff. Condition for robust stability $\left\| l_a(j\omega) C(j\omega) \tilde{S}(j\omega) \right\|_\infty < 1$



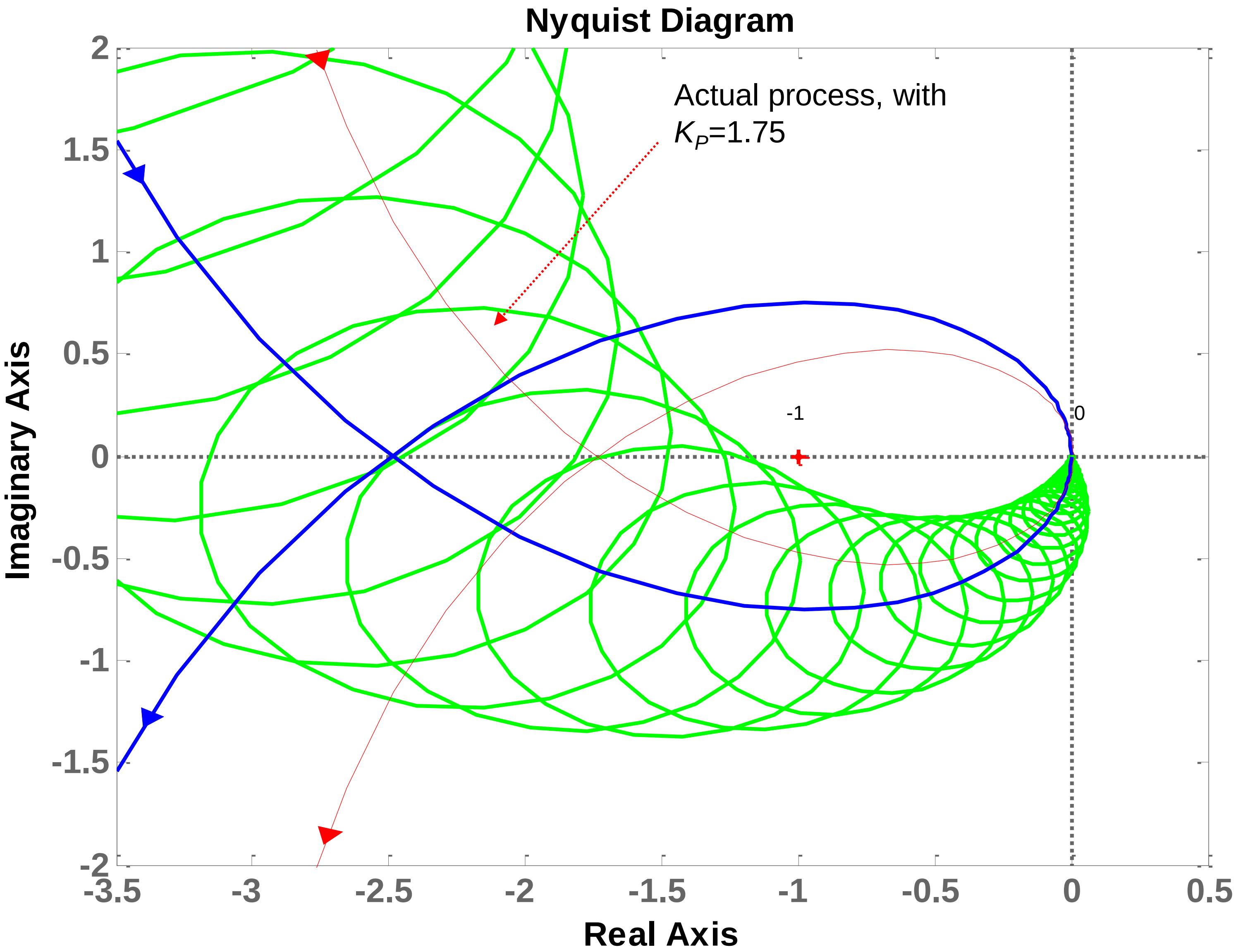
– Remark: condition verified conservatively

Example

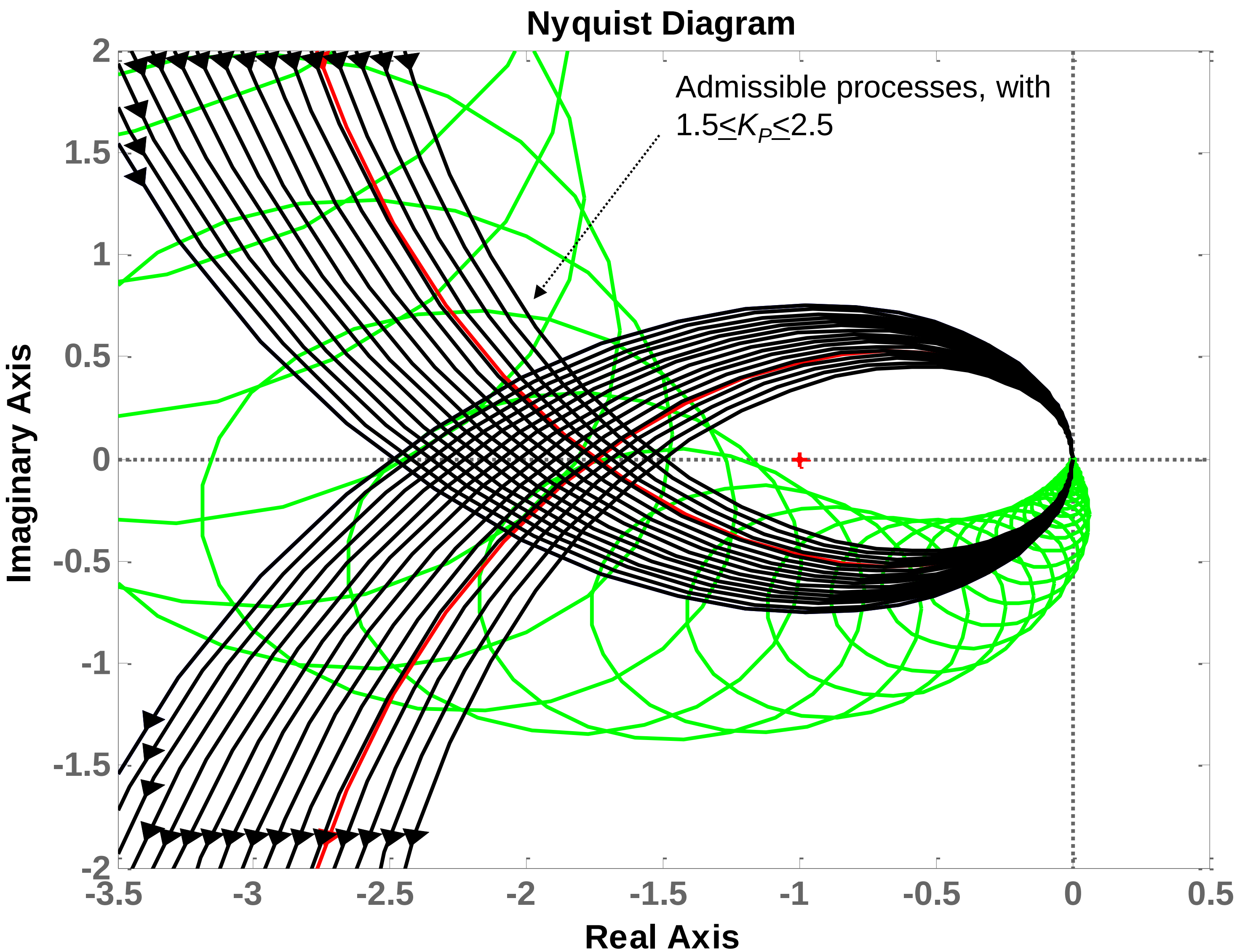


Nyquist Diagram
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
$\tilde{F}(j\omega)$
-1
0

Example



Nyquist Diagram
Actual process, with
K_P=1.75
-1
0
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5


Nyquist Diagram
Admissible processes, with
1.5≤K_P≤2.5
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Sufficient conditions for robust stability of linear SISO systems

- Theorem 3 (additive uncertainty, IMC approach)
  - By assuming that
    - the controller $Q(s)$ is stable and the nominal process $\tilde{P}(s)$ is stable
    - the process $P(s)$ and the nominal process $\tilde{P}(s)$ have the same number of RHP poles:
    $$n_P^+ = n_{\tilde{P}}^+$$
  - then a sufficient condition for the solution of the robust stability problem is that the controller $Q(s)$ is such that
  $$\|l_a(j\omega)\, Q(j\omega)\|_\infty < 1$$
- Proof
  - In the IMC design approach, the sensitivity function is $\tilde{S}(s) = 1 - Q(s)\tilde{P}(s)$, and the controller $C(s)$ is written in terms of the controller $Q(s)$ as $C(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)}$
  - It follows that

$$\left\|l_a(j\omega)C(j\omega)\tilde{S}(j\omega)\right\|_\infty = \left\|l_a(j\omega)\frac{Q(s)}{1-\tilde{P}(s)Q(s)}\left(1-\tilde{P}(s)Q(s)\right)\right\|_\infty = \|l_a(j\omega)\, Q(s)\|_\infty < 1$$

Sufficient conditions for robust stability of linear SISO systems

- Theorem 4 (multiplicative uncertainty, IMC approach)
  - By assuming that
    - the controller $Q(s)$ is stable and the nominal process $\tilde{P}(s)$ is stable
    - the process $P(s)$ and the nominal process $\tilde{P}(s)$ have the same number of RHP poles:
    $$n_P^+ = n_{\tilde{P}}^+$$
  - then a sufficient condition for the solution of the robust stability problem is that the controller $Q(s)$ is such that
  $$\left\| l_m(j\omega)\tilde{P}(s)Q(s) \right\|_\infty < 1$$
- Proof
  - In the IMC design approach, the sensitivity function is $\tilde{T}(j\omega) = Q(s)\tilde{P}(s)$
  - It follows that

$$\left\| l_m(j\omega)\tilde{T}(j\omega) \right\|_\infty = \left\| l_m(j\omega)\tilde{P}(s)Q(s) \right\|_\infty < 1$$

Summary

- Robust control
  - Definitions of parametric and dynamic (additive, multiplicative) uncertainty
  - Definitions of the robust control problem
  - Sufficient conditions for robust stability of linear SISO systems
    - Additive uncertainty
    - Multiplicative uncertainty
    - IMC design
  - Example with parametric uncertainty