IMC EXERCISES

Ex 1 · $\tilde{P}(s)$ stable

STEP 1 FACTORIZATION OF $\tilde{P}(s) = 100 \dfrac{s+1}{(s+10)^2}$

$$\tilde{P}_+(s) = 1$$

(no non-minimum phase terms)

$$\tilde{P}_-(s) = \frac{1+s}{(1+s/10)^2} \Rightarrow \tilde{Q}(s) = \tilde{P}_-^{-1}(s) = \frac{(1+s/10)^2}{1+s}$$ ← NOT PROPER

STEP 2 FILTER type 1 $\Rightarrow f(s) = \dfrac{1}{(1+\lambda s)^n}$, $n = 1$

$$Q(s) = \tilde{Q}(s) f(s) = \frac{(1+s/10)^2}{(1+s)(1+\lambda s)}$$

* $\tilde{T}(0) = \tilde{P}(0) Q(0) = \tilde{P}_+(0) \tilde{P}_-(0) \tilde{Q}(0) f(0) = \tilde{P}_+(0) f(0) = \left.\dfrac{1}{1+\lambda s}\right|_{s=0} = 1$ ✓

$\Rightarrow$ type 1 system

STEP 3 ROBUSTNESS

- MULTIPLICATIVE UNCERTAINTY $P(s) = \tilde{P}(s)\left(1 + \Delta_m(s)\right)^{*}$;

ROBUST STABILITY CONDITION

$$|\underbrace{\ell_m(j\omega)}_{\text{upper bound of } \Delta_m} \tilde{F}(j\omega)| < 1, \forall \omega$$

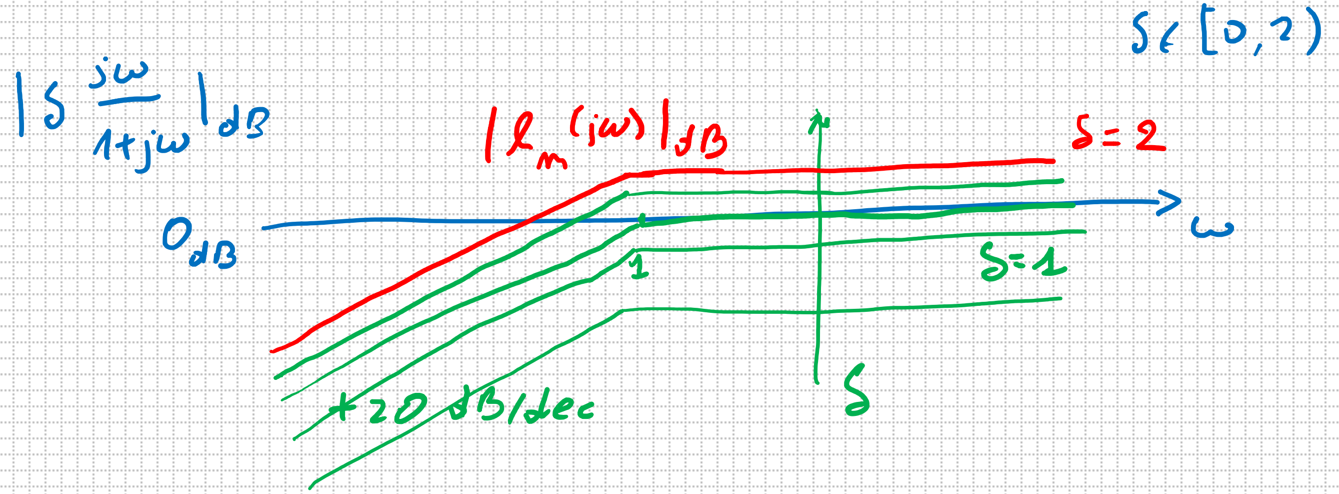
$$* \boxed{\Delta_m(j\omega)} = \frac{P(j\omega) - \tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{K\frac{j\omega - z}{(j\omega+10)^2} - 100\frac{j\omega+1}{(j\omega+10)^2}}{100\frac{j\omega+1}{(j\omega+10)^2}} =$$

$$= \frac{K(j\omega - z) - 100(j\omega+1)}{100(j\omega+1)} =$$

$$\begin{cases} K = 100(1+\delta) \\ z = -\frac{1}{1+\delta} \end{cases}$$

$$= \frac{(1+\delta)j\omega + 1 - j\omega - 1}{j\omega + 1} = \boxed{\delta\frac{j\omega}{j\omega+1}}, \quad \delta \in [0, 2)$$

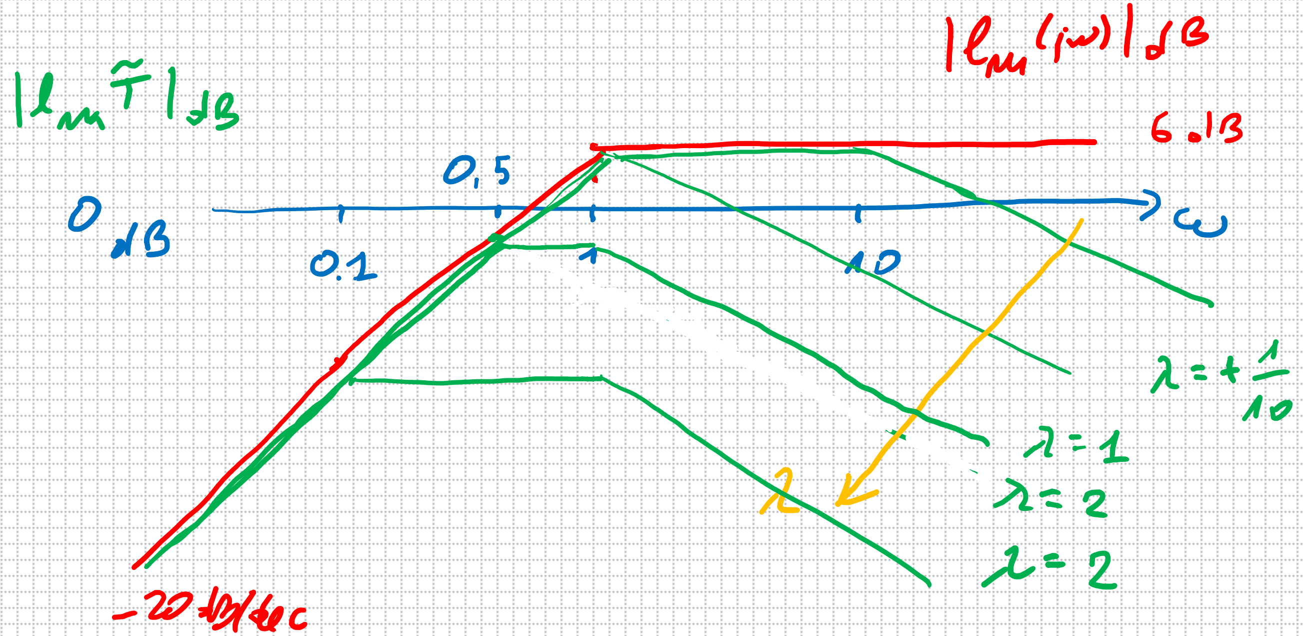
$$\boxed{\ell_m(j\omega) = 2\,\frac{j\omega}{1+j\omega}}$$



Robust stability

$$|\ell_m(j\omega)\tilde{T}(j\omega)| < 1, \ \forall\omega$$

$$\Big|\underbrace{2\,\frac{j\omega}{1+j\omega}}_{\text{HP filter}}\ \underbrace{\frac{1}{1+\lambda j\omega}}_{\tilde{T}\ \text{LP filter}}\Big| < 1, \ \forall\omega$$



For $\lambda > 1$ the condition is verified

$$\boxed{\lambda = 2} \qquad \boxed{Q(s) = \frac{(1+s/10)^2}{(1+s)(1+2s)}} \qquad \text{A}$$

B.

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = \frac{\frac{(1+\frac{s}{10})^2}{(1+s)(1+2s)}}{1-\frac{1+s}{\cancel{(1+\frac{s}{10})^2}}\frac{\cancel{(1+\frac{s}{10})^2}}{(1+s)(1+2s)}} = \frac{(1+\frac{s}{10})^2}{(1+s)(1+2s)-(1+s)} =$$

$$= \frac{\frac{1}{100}s^2+\frac{1}{5}s+1}{2\,s\,(1+s)} = \frac{1}{10}\left(1+\frac{1}{0.5\,s}+\frac{1}{20}s\right)\frac{1}{1+s} = K_p\left(1+\frac{1}{T_i s}+T_D s\right)\frac{1}{1+\beta_f s}$$

PID+f (2 zeros, 1 pole in $s=0$, 1 pole)

$$\begin{cases} K_p = 0.1 \\ T_i = 0.5 \\ T_D = 0.05 \\ \beta_f = 1 \end{cases}$$

ROBUST PID+f CONTROLLER

$\boxed{2}$

RHP zero

$P(s) = K \dfrac{s-z}{(s+10)^2}$, $z = \dfrac{1}{1+\delta}$, $K = 100(1+\delta)$, $\delta \in [0, 2)$

$\tilde{P}(s) = 100 \dfrac{s-1}{(s+10)^2}$

A. $Q(s)$ s.t. $\left\{ \begin{array}{l} \text{ROBUST STABILITY} \\ \text{TYPE-1} \\ \text{IAE} \end{array} \right.$

- $\tilde{P}(s)$ is STABLE $\Rightarrow$ IMC design
- STEP 1. ~~$\tilde{P}_+(s) = s-1$ ⚠ WRONG ⚠ $\tilde{P}_+(0) = -1$~~

$\tilde{P}(s) = -\dfrac{1-s}{(1+s/10)^2}$; $\tilde{P}_+(s) = 1-s$, $\tilde{P}_- = -\dfrac{1}{(1+\frac{s}{10})^2}$ IAE

$\Rightarrow \boxed{\tilde{Q}(s) = -\left(1+\frac{s}{10}\right)^2}$

($\tilde{P}_+(s) = \dfrac{1-s}{1+s}$, $\tilde{P}_- = -\dfrac{1+s}{(1+\frac{s}{10})^2}$ ISE)

STEP 2. FILTER

type 1 $\Rightarrow$ $f(s) = \dfrac{1}{(1+\lambda s)^m}$, $m = 2$

$$Q(s) = \hat{Q}(s) f(s) = -\frac{(1+s/10)^2}{(1+\lambda s)^2}$$

$$* \quad \tilde{F}(0) = \tilde{P}_+(s) f(s) = (1-s)\frac{1}{(1+\lambda s)^2}\Bigg|_{s=0} = 1 \quad \checkmark$$

STEP 3 ROBUSTNESS

ADDITIVE UNCERTAINTY

$$\Delta(j\omega) = P(j\omega) - \tilde{P}(j\omega) \overset{*}{=}$$

$$\left(k = 100(1+\delta),\ z = \frac{1}{1+\delta}\right)$$

$$\overset{*}{=} k\frac{j\omega - z}{(j\omega+10)^2} - 100\frac{j\omega - 1}{(j\omega+10)^2} = 100\frac{(1+\delta)\left(j\omega - \frac{1}{1+\delta}\right) - j\omega + 1}{(j\omega+10)^2} = \delta\frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2}$$

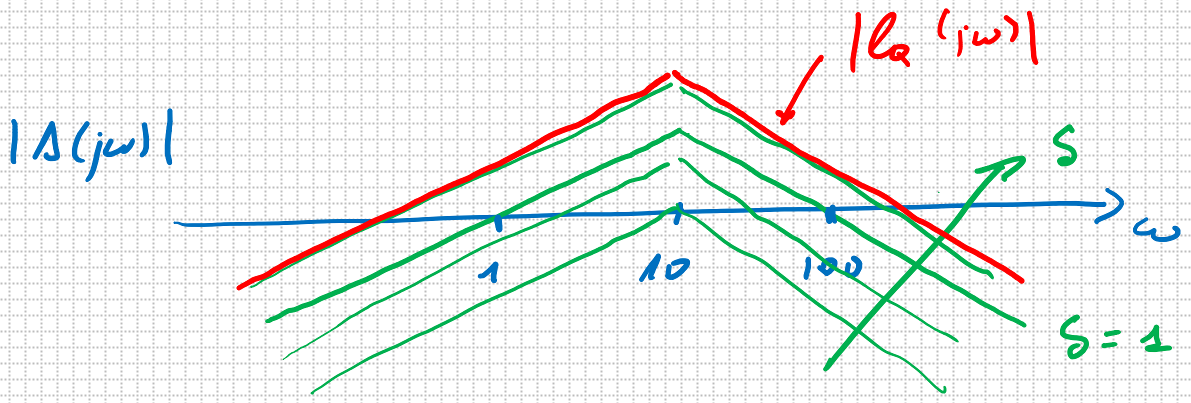
ROBUST STAB. CONDITION

$$|\underbrace{l_a(j\omega)}_{\text{UPPER-BOUND of } \Delta(j\omega)} Q(j\omega)| < 1, \ \forall \omega$$

$$\Delta(j\omega) = \delta \frac{j\omega}{(1+j\omega/10)^2}$$

$\delta \in [0, 2)$

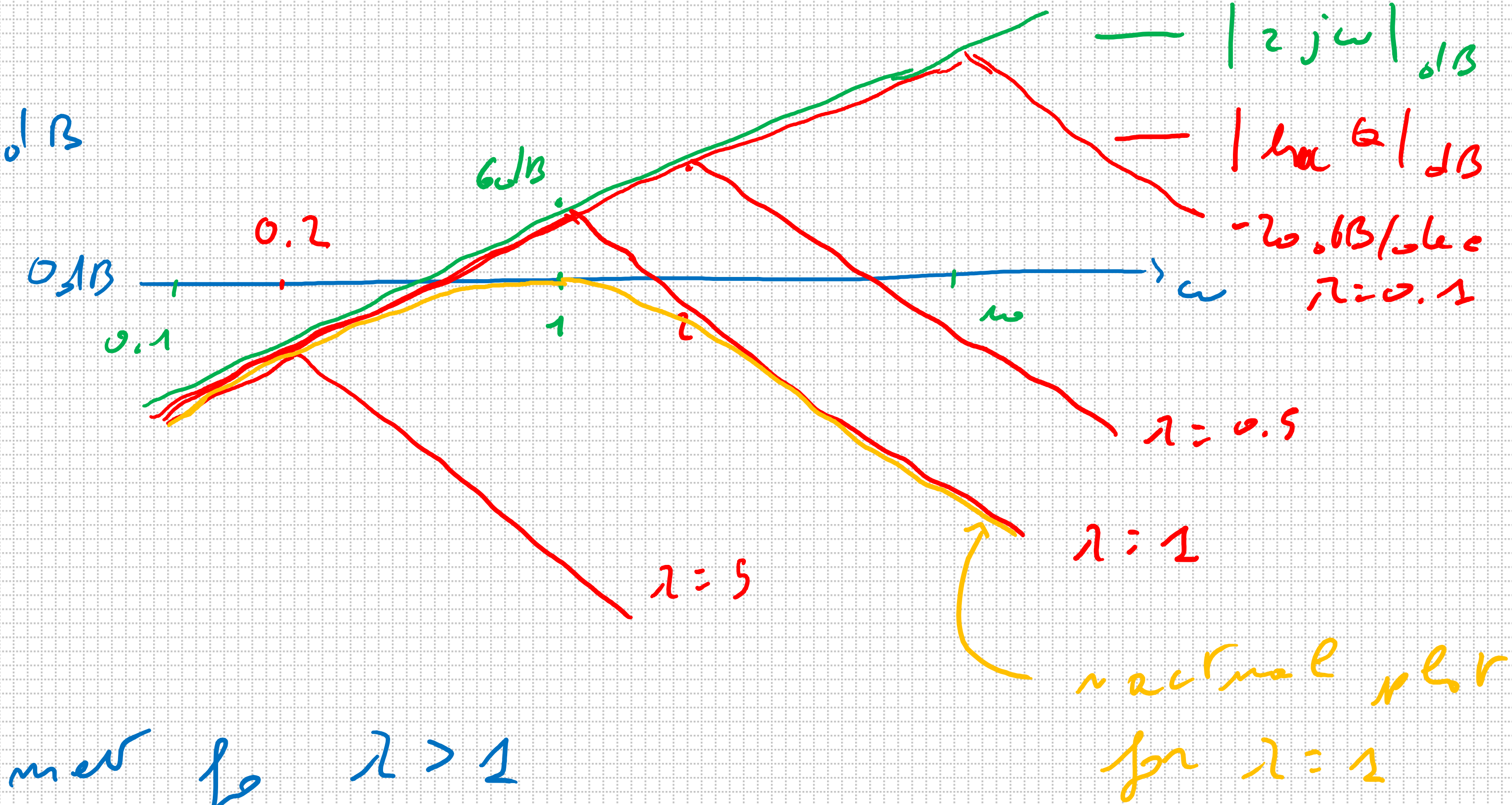
$$\boxed{\ell_a(j\omega) = 2 \frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2}}$$



$$|\ell_a(j\omega)\, Q(j\omega)| = \left| 2 \frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2} \frac{\left(1+\frac{j\omega}{10}\right)^2}{(1+\lambda j\omega)^2} \right| = \left| 2 j\omega \frac{1}{(1+\lambda j\omega)^2} \right| \not< 1, \forall \omega$$

$$\left| 2 \frac{j\omega}{(1+\lambda j\omega)^2} \right| < 1 \quad \forall \omega$$

$$\left| 2 \frac{j\omega}{(1+\lambda j\omega)^2} \right|_{dB}$$



Condition is met for $\lambda > 1$

Conservative value $\lambda = 5$

$$\boxed{Q(s) = \frac{(1+j\omega/10)^2}{(1+j\omega 5)^2}}$$

(Ex. 2b) $P(s) = k \dfrac{s-z}{(s+10)^2}$, $z = \dfrac{1}{1+\delta}$, $k = 100(1+\delta)$, $\delta \in [0, 2)$

$\tilde{P}(s) = 100 \dfrac{s-1}{(s+10)^2}$

- Design IMC controller Q : $\begin{cases} \text{robust stability} \\ \text{type 1} \\ \text{ISE} \end{cases}$

$P_+ = \prod_i \dfrac{1-\beta_i s}{1+\beta_i s} e^{-\theta s}$

$\beta_i > 0$

$\tilde{P}(s) = -\dfrac{1-s}{(1+s/10)^2}$

STEP 1. FACTORIZATION

$\tilde{P}_+(s) = \dfrac{1-s}{1+s}$, $\tilde{P}_-(s) = -\dfrac{1+s}{(1+s/10)^2}$

$(\tilde{P}_+(0) = 1)$ $\Rightarrow \tilde{Q}(s) = -\dfrac{(1+s/10)^2}{1+s}$

STEP 2. FILTER type 1 $\Rightarrow f(s) = \dfrac{1}{(1+\lambda s)^n}$, $n = 1$

$Q(s) = -\dfrac{(1+s/10)^2}{(1+s)(1+\lambda s)}$ (...)

Ex. 3 (IMC robust design, type 2 system)

$$P(s) = \frac{(1+s/10)\left((1+s\tau)+0.9\right)}{(1+s)^2(1+s\tau)}, \quad \tau \in (1, 10]$$

$$\tilde{P}(s) = \frac{1+s/10}{(1+s)^2}$$

- IMC controller Q : $\begin{cases} \text{robust stability} \\ \text{type 2 system} \\ \text{IAE} \end{cases}$

~ - ~

Check: $\tilde{P}$ stable since it has 2 poles in -1

STEP 1. $\tilde{P}_+(s) = 1$, $\tilde{P}_-(s) = \dfrac{1+s/10}{(1+s)^2} \Rightarrow \tilde{Q}(s) = \dfrac{(1+s)^2}{1+s/10}$

STEP 2. type 2 : $\tilde{T}(0) = 1$, $d\tilde{T}(s)/ds\big|_{s=0} = 0$

$$f(s) = \frac{1+\left(2\lambda - P_+'(0)\right)s}{(1+\lambda s)^2} = \frac{1+2\lambda s}{(1+\lambda s)^2} \leftarrow \frac{d\tilde{P}_+(s)}{ds} = 0$$

1) $Q(s) = \tilde{Q}(s) f(s) = \dfrac{(1+s)^2}{1+s/10} \dfrac{1+2\lambda s}{(1+\lambda s)^2}$ PROPER ✓

2) $\tilde{T}(s) = \tilde{P}(s) Q(s) = \underbrace{\tilde{P}_+(s)}_{1} \cancel{P_-(s)} \cancel{\tilde{Q}(s)} f(s) = f(s)$

$$\begin{cases} \tilde{T}(0) = f(0) = 1 \quad ✓ \\ \left.\dfrac{d\tilde{T}(s)}{ds}\right|_{s=0} = \left.\dfrac{d f(s)}{ds}\right|_{s=0} = \left[-2\lambda \dfrac{1+2\lambda s}{\underbrace{(1+\lambda s)^3}_{1}} + 2\lambda \dfrac{1}{\underbrace{(1+\lambda s)^2}_{1}} \right]_{s=0} = 0 \quad ✓ \end{cases}$$

$$\boxed{Q(s) = \frac{(1+s)^2(1+2\lambda s)}{(1+s/10)(1+\lambda s)^2}}$$

STEP 3. ROBUST STABILITY $\quad |l_a(j\omega) Q(j\omega)| < 1, \ \forall \omega$

$$\Delta_a(s) = P(s) - \tilde{P}(s) = \frac{(1+s/10)\left((1+s\tau)+0.9\right)}{(1+s)^2(1+s\tau)} - \frac{1+s/10}{(1+s)^2} =$$

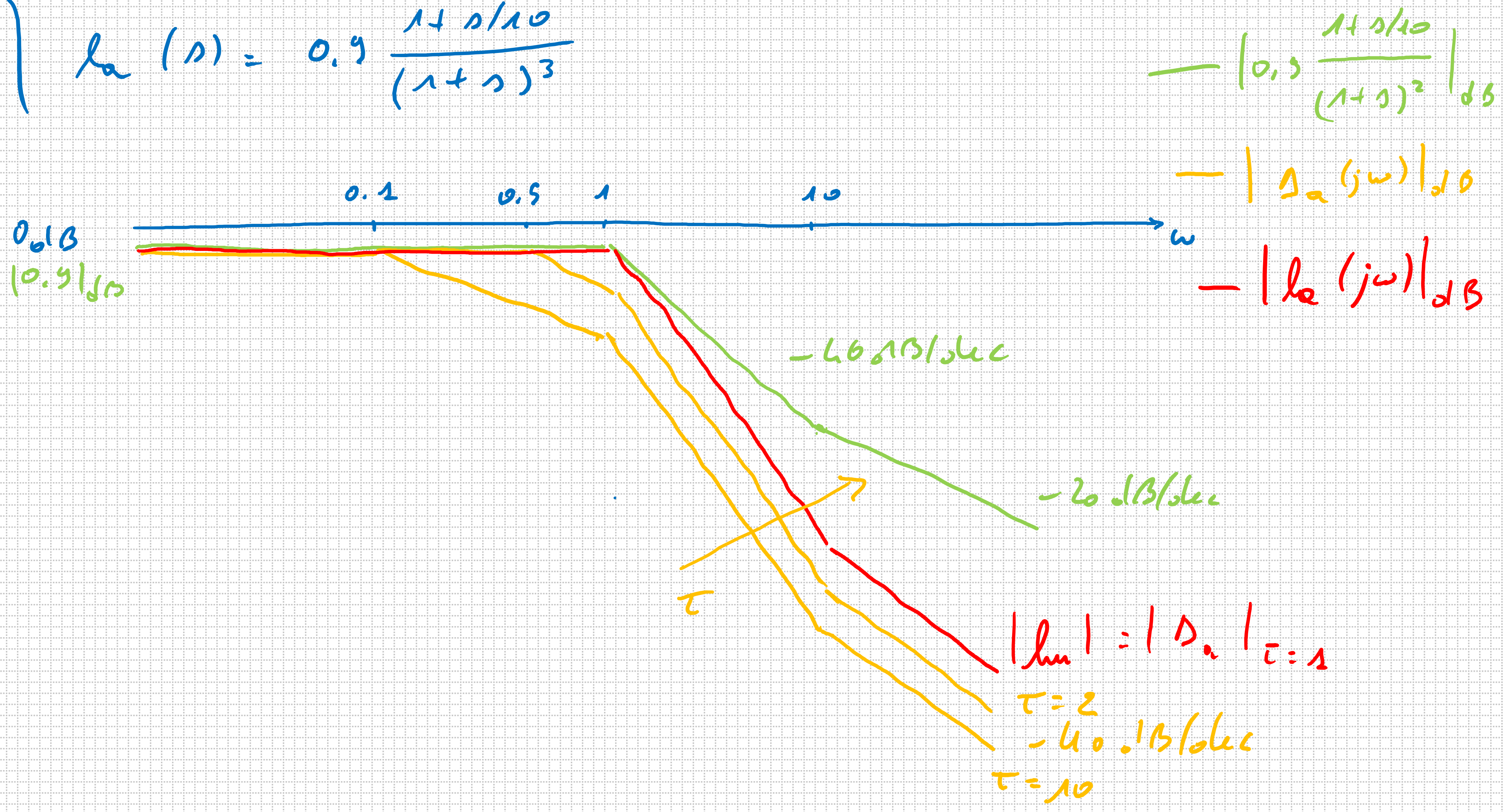
$$= \frac{\cancel{(1+s/10)(1+s\tau)} + 0.9(1+s/10) - \cancel{(1+s/10)(1+s\tau)}}{(1+s)^2(1+s\tau)} =$$

$$= 0.9\,\frac{1+s/10}{(1+s)^2(1+s\tau)}, \qquad \tau \in (1, 10]$$

$$l_a(s) : |l_a(j\omega)| > |\Delta_a(j\omega)|, \quad \forall \omega, \quad \forall \tau \in (1, 10]$$

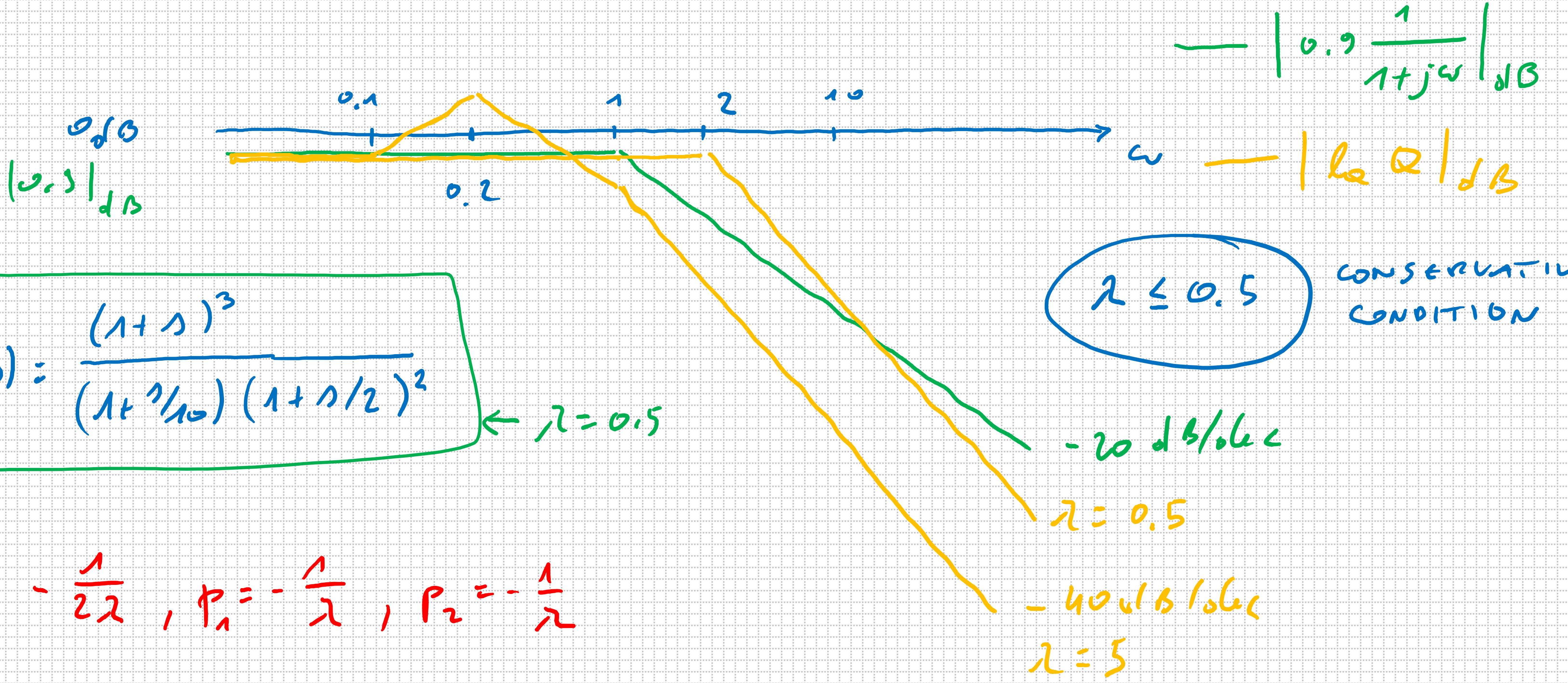
$$\begin{cases} G_a(s) = 0.9\,\dfrac{1+s/10}{(1+s)^2(1+s\tau)} \\[2ex] L_a(s) = 0.9\,\dfrac{1+s/10}{(1+s)^3} \end{cases}$$

$\tau \in (1, 10]$



Robust stability condition $\quad |\ell_a(j\omega)\, Q(j\omega)| < 1 \quad \forall \omega$

$$\Rightarrow \left| 0.9 \, \frac{\cancel{1+j\omega/10}}{(1+j\omega)^{\cancel{2}}} \; \frac{\cancel{(1+j\omega)^2}\,(1+j\omega 2\lambda)}{\cancel{(1+j\omega/10)}\,(1+j\omega\lambda)^2} \right| = \left| 0.9 \, \frac{1+j\omega 2\lambda}{(1+j\omega)(1+j\omega\lambda)^2} \right|^{*} < 1, \; \forall \omega$$



$$\boxed{Q(s) = \frac{(1+s)^3}{(1+s/10)(1+s/2)^2}} \leftarrow \lambda = 0.5$$

$\lambda \leq 0.5$ CONSERVATIVE CONDITION

$*\quad z = -\frac{1}{2\lambda}, \; p_1 = -\frac{1}{\lambda}, \; p_2 = -\frac{1}{\lambda}$

Ex. 4

$$P(s) = k \frac{1+s/10}{(1+2s)^2} e^{-\theta s}, \quad \theta = 0.2, \quad k \in [10, 20)$$

$\tilde{k} = 10$

A) IMC controller $Q(s)$: $\begin{cases} \text{robust stability} \\ \text{type 1} \\ \text{IAE} \end{cases}$

B) Verify that $\overline{Q(s)}$ stabilizes $P(s) = 10 \dfrac{1+s/10}{(1+2s)^2} e^{-0.2s}$

- $\tilde{P}(s)$ is stable (it has two poles in -0.5)

- $\theta = 0.2$, $\tau = 2$ $\Rightarrow$ Padé approximation

1/1 Padé approx.

$$e^{-\theta s} \approx \frac{1 - s\frac{\theta}{2}}{1 + s\frac{\theta}{2}} = \frac{1 - s/10}{1 + s/10}$$

$$P^r(s) = K \underbrace{\frac{1+\frac{s}{10}}{(1+2s)^2}}_{P(s)} \underbrace{\frac{1-\frac{s}{10}}{1+\frac{s}{10}}}_{\sim e^{-\theta s}} = K \frac{1-\frac{s}{10}}{(1+2s)^2}$$

$$\tilde{P}^r(s) = 10 \frac{1-s/10}{(1+2s)^2}$$

<u>STEP 1.</u>

$$\tilde{P}^r_+(s) = 1 - s/10 \quad , \quad \tilde{P}^r_-(s) = 10 \frac{1}{(1+2s)^2}$$

$$\tilde{Q}(s) = \left(\tilde{P}^r_-(s)\right)^{-1} = \frac{1}{10} (1+2s)^2$$

<u>STEP 2</u>

type 1 $\quad f(s) = \dfrac{1}{(1+\lambda s)^m}, \; m = 2$

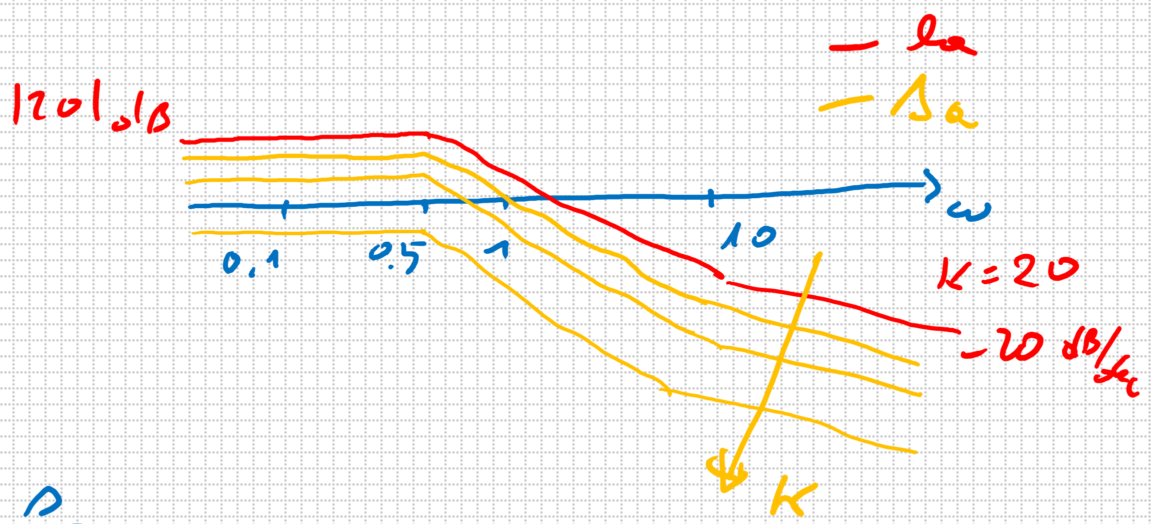
$$Q(s) = \frac{1}{10} \frac{(1+2s)^2}{(1+\lambda s)^2} \; ; \qquad \tilde{T}(0) = \tilde{P}^r_+(0) f(0) = 1 \quad \checkmark$$

STEP 3.

$$|\ell_a(j\omega)\, Q(j\omega)| < 1, \ \forall \omega$$

$$\Delta_a(s) = PP(s) - \tilde{P}P(s) = K \frac{1 - s/10}{(1+2s)^2} - 10 \frac{1-s/10}{(1+2s)^2} = (K-10) \frac{1-\frac{s}{10}}{(1+2s)^2}$$

$K \in [10, \infty)$

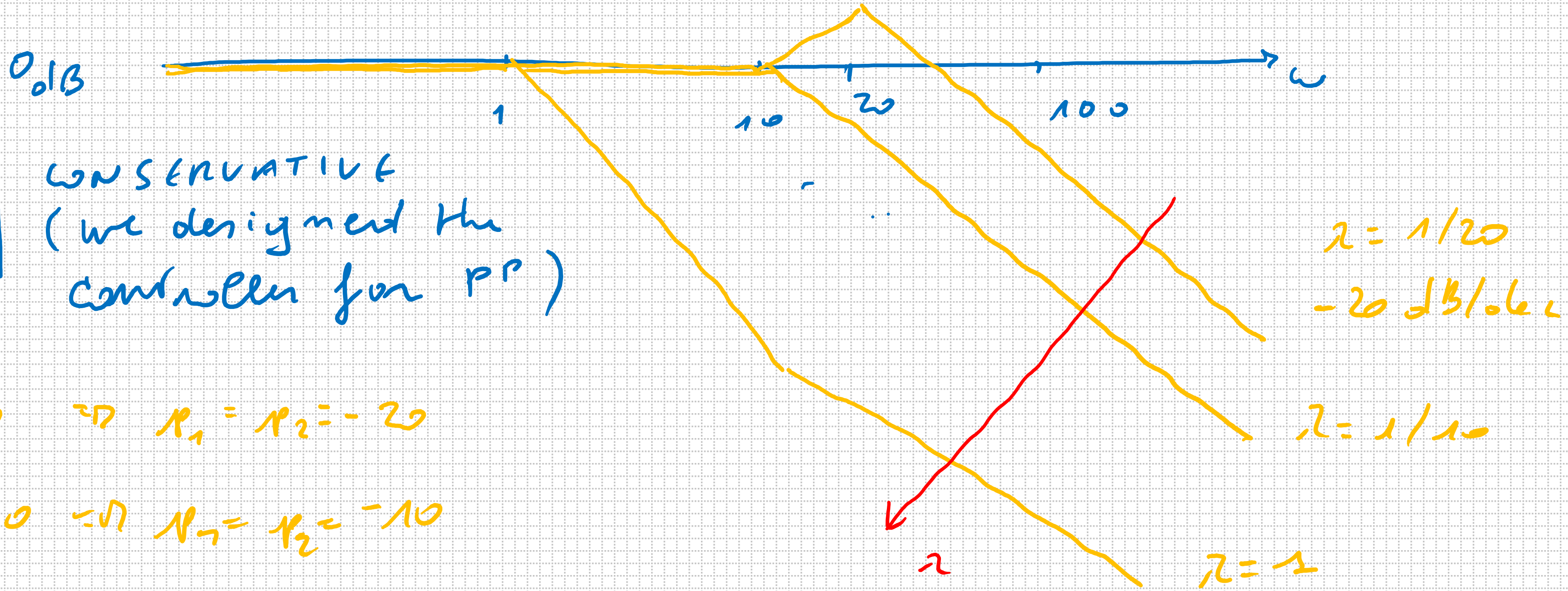


$$\ell_e(s) = 10 \frac{1 - s/10}{(1+2s)^2}$$

$$\ell_e Q = \cancel{10} \frac{1 - s/10}{\cancel{(1+2s)^2}} \frac{1}{\cancel{10}} \frac{\cancel{(1+2s)^2}}{(1+2s)^2} = \frac{1 - \frac{s}{10}}{(1+2s)^2}$$

$$\left| \frac{1 - j\omega/10}{(1 + j\omega\lambda)^2} \right|_{dB}$$

$\lambda \geq 0.1$ conservative condition



$\boxed{\lambda = 0.2}$ CONSERVATIVE (we designed the controller for PP)

$\lambda = 1/20 \Rightarrow p_1 = p_2 = -20$

$\lambda = 1/10 \Rightarrow p_1 = p_2 = -10$

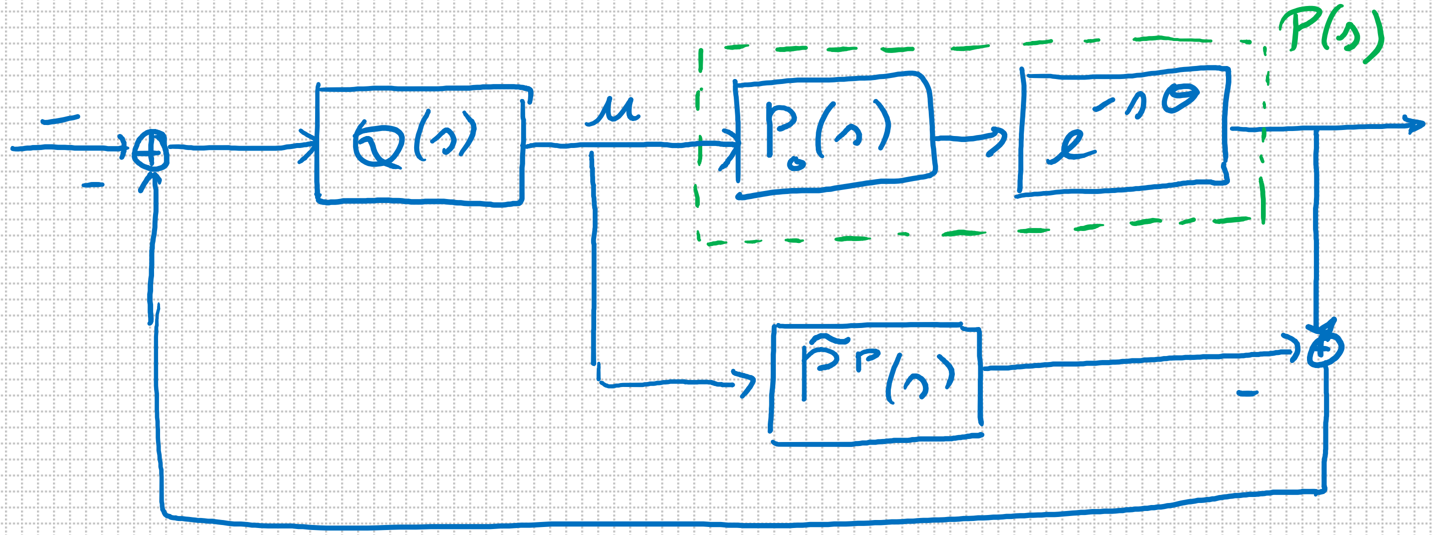
$\lambda = 1 \Rightarrow p_1 = p_2 = -1$

$$\boxed{Q(s) = \frac{1}{10}\,\frac{(1+2s)^2}{(1+0.2s)^2}}$$

controller which robustly stabilize $\tilde{P}^r(s)$

b) classic controller

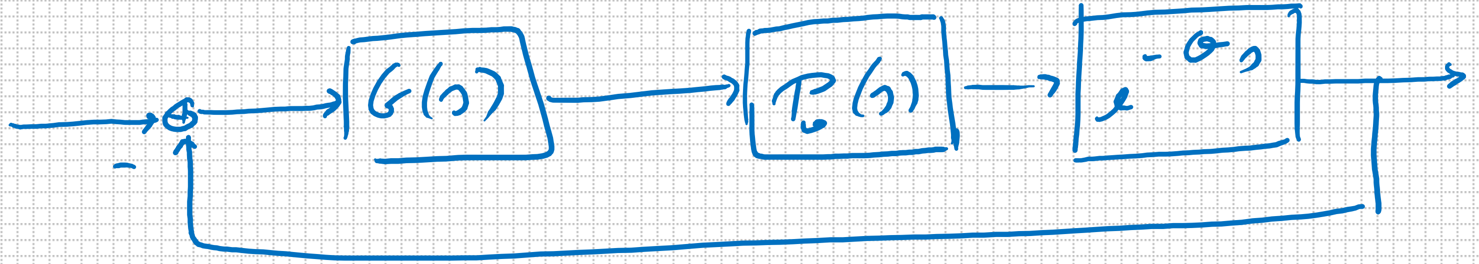
$$G(s) = \frac{Q(s)}{1 - \tilde{P}^r(s)\,Q(s)} = \frac{\frac{1}{10}\,\frac{(1+2s)^2}{(1+0.2s)^2}}{1 - \cancel{10}\,\frac{1-0.1s}{\cancel{(1+2s)^2}}\,\frac{1}{\cancel{10}}\,\frac{\cancel{(1+2s)^2}}{(1+0.2s)^2}} =^{*}$$



← We computed Q considering $\tilde{P}^r$, i.e., according this scheme

* $$G(s) = \frac{1}{5}\,\frac{(1+2s)^2}{s\left(1+\frac{s}{50}\right)} = K_c\left(1+\frac{1}{T_i s}+T_D s\right)\frac{1}{1+\beta_f s} \qquad (PID + f)$$

$$\begin{cases} K_c = 0.2 \\ T_i = 0.04 \\ T_D = 1 \\ \beta_f = 0.02 \end{cases}$$



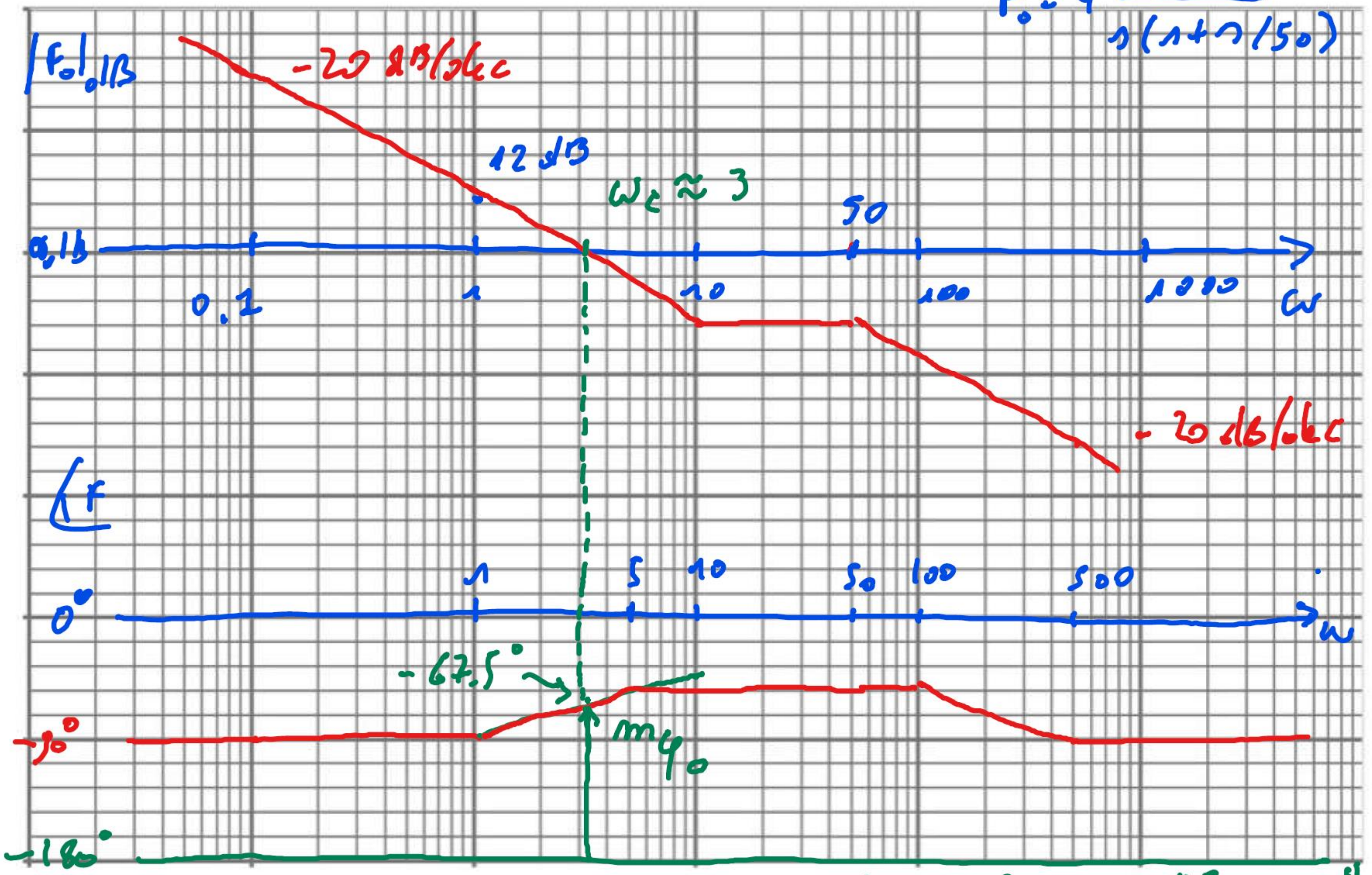
Does $G(s)$ stabilize $P(s)$??

STEP 1. Check if G stabilizes P_0 and compute its m_{τ_0}.

STEP 2. Check if $m_{\tau_0} > \theta$

(delay margin for P is $m_\tau = m_{\tau_0} - \theta$)

$$F_0 = 4\,\frac{1+s/10}{s(1+s/50)}$$



$$m_{\varphi_0} = -67.5 + 180° = 112.5° \;@\; 3\ \frac{\text{rad}}{\text{s}}$$

$$K = 20, \qquad \bar{P}_0(s) = 20\,\frac{1-0.1s}{(1+2s)^2}$$

$$F_0(s) = G(s)\,\bar{P}_0(s) = 4\,\frac{1+0.1s}{s(1+s/50)}$$

maximum delay

$$m_{\tau_0} = \frac{m\varphi_0}{\omega_c} \approx \frac{112.5^\circ \cdot \frac{\pi}{180^\circ}}{3\ \text{rad/s}} = 0.65\ s$$

$$\boxed{m_\tau} = m_{\tau_0} - \tau = 0.65 - 0.2 = \boxed{0.45\ s}$$

$\bar{P}(s)$ is stabilized by $G(s)$