Process Automation (MCER), 2015-2016

# EXAM

8 Feb. 2016

## Exercise 1 (10 pts.)

Let the process be described by the transfer function: $P(s) = -K\frac{1}{(1+10s)^2}e^{-0.2s}$, with $K \in (1,2.5]$.

1. Considering the nominal value $\widetilde{K} = 2$ and following the IMC design, develop a controller such that:
   a. the controlled system has 0 steady-state error for step inputs;
   b. the controlled system is robustly stable against the uncertainties of the parameter $K$.
2. Compute the equivalent classic controller.

## Exercise 2 (10 pts.)

Consider a process whose state-space model is:

$$\begin{cases} x(t) = Mx(t-1) + Nu(t-1) \\ y(t) = Qx(t) \end{cases}, \text{ with } M = 0.2, N = -0.1, Q = 2.$$

Compute the control actions of a Predictive Functional Control algorithm at time $t = 4$, with:

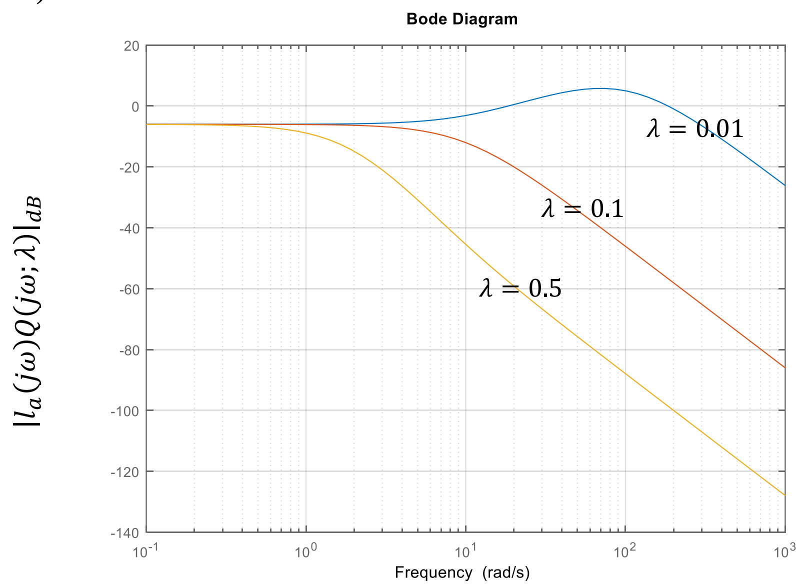
a. control horizon $m = 3$;
b. prediction horizon $p = 3$;
c. number of coincident points $n_h = 2$;
d. basis functions set: $B_1(t) = \cos\frac{\pi}{4}t$, $B_2(t) = \sin\frac{\pi}{4}t$, $n_B = 2$;
e. constant reference $w(t) = r(t) = 2.5, \forall t$;
f. cost function $J = e^T e$, where $e$ is the vector of future errors between predicted output and reference trajectory;
g. model outputs: $y(0) = 0, y(1) = 1.4, y(2) = 1.9, y(3) = 2.2$;
h. past state values: $x(0) = 0.10, x(1) = 0,.72\ x(2) = 0.94, x(3) = 1.09$;
i. past control values: $u(0) = -7, u(1) = -7, u(2) = -8, u(3) = -9$;
j. measured outputs: $y_m(0) = 0, y_m(1) = 1.6, y_m(2) = 2.1, y_m(3) = 2.4, y_m(4) = 2.7$;
k. the plant-model error (i.e., the difference between future measured outputs $y_m(t+k)$ and the predicted model outputs $y(t+k)$ at future time instants) is equal to $\hat{e}(t+k|t) = \left(y_m(t) - y(t)\right), k = 1,2,\ldots,p$.

## Questions (10 pt.)

i) Check the stability of the process $P(s) = \frac{1-\frac{s}{10}}{s(1+s)(1+\frac{s}{10})}e^{-0.5s}$ controlled by a proportional feedback controller $G(s) = 1$ (it is sufficient to use approximated Bode diagrams).
ii) List pros and cons of the prediction models of the MPC algorithms DMC, MAC, PFC and GPC (write a table with bullet lists).

Solution of exercise 1

1. Since the delay ($0.1s$) is at least one order of magnitude smaller than the time constant of the process ($\tau$ is between $1s$ and $10s$), we use the 1/1 Padé approximation and approximate the process as a transfer function without delays: $P^P(s) = -K\frac{1}{(1+10s)^2}\frac{1-0.1s}{1+0.1s}$. The process is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system, with nominal process $\tilde{P}^P(s) = -2\frac{1}{(1+10s)^2}\frac{1-0.1s}{1+0.1s}$.
The IMC design procedure to robustly stabilize the approximated process $P^P(s)$ consists in the following 3 steps:
**Step 1)** a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:
$\tilde{P}^P(s) = \tilde{P}^P_+(s)\tilde{P}^P_-(s)$, with $\tilde{P}^P_+(s) = (1-0.1s)$ and $\tilde{P}^P_-(s) = -2\frac{1}{(1+10s)^2(1+0.1s)}$. b) Define the controller as $\tilde{Q}(s) = \left(\tilde{P}^P_-(s)\right)^{-1} = -\frac{1}{2}(1+10s)^2(1+0.1s)$
**Step 2)** Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1. We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 3$. In fact:
a) $Q(s) = \tilde{Q}(s)f(s) = -\frac{1}{2}\frac{(1+10s)^2(1+0.1s)}{(1+\lambda s)^3}$ is proper;
b) $\tilde{T}^P(0) = \tilde{P}^P(0)Q(0) = \left[\tilde{P}^P_+(s)\tilde{P}^P_-(s)\left(\tilde{P}^P_-(s)\right)^{-1}f(s)\right]_{s=0} = \left[\frac{1-0.1s}{(1+\lambda s)^3}\right]_{s=0} = 1.$
**Step 3)** Determine the value of $\lambda$ such that the sufficient condition for robust stability holds: $|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$, where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$. By definition, the additive uncertainty is $\Delta_a(j\omega) := P^P(j\omega) - \tilde{P}^P(j\omega) = (2-K)\frac{1}{(1+10s)^2}\frac{1-0.1s}{1+0.1s}$.
Since $K \in (1,2.5]$, an upper-bound of $|\Delta_a(j\omega)|$ is simply $l_a(j\omega) = \left|\frac{1}{(1+10s)^2}\frac{1-0.1s}{1+0.1s}\right|_{s=j\omega} = \left|\frac{1}{(1+10j\omega)^2}\right|$. The robust stability condition is then $|l_a(j\omega)Q(j\omega)| = \left|\frac{1}{2}\frac{1-0.1j\omega}{(1+\lambda j\omega)^3}\right| < 1, \forall\omega$, which is satisfied at least for $\lambda > 0.1$ (see the bode diagram below).




Since we are using the Padé approximation, we set a conservative value $\lambda = 1$ and obtain the following controller $Q(s) = -\frac{1}{2}\frac{(1+10s)^2(1+0.1s)}{(1+0.5s)^3}$

2. The controller of the equivalent classic control scheme is computed from the IMC controller as follows:
$G(s) = \frac{Q(s)}{1-\tilde{P}^P(s)Q(s)} = -\frac{1}{2}\frac{(1+10s)^2(1+0.1s)}{(1+0.5s)^3-(1-0.1s)} = -\frac{1}{2}\frac{(1+10s)^2(1+0.1s)}{s(0.125s^2+0.75s+1.6)} = -4\frac{(1+10s)^2(1+0.1s)}{s(s+3+j1.95)(s+3-j1.95)}.$

Solution of exercise 2.

To develop the PFC controller, we need to select coincident points. We chose $h_1 = 1$ and, since the control horizon is $p = 3$, $h_2 = 3$.

Firstly, we have to compute the model response to the base functions in the coincidence points, considering null initial conditions:

<u>$t = h_1 = 1$</u>

$$B_1: \begin{cases} x_{B_1}(1) = 0.2x_{B_1}(0) - 0.1B_1(0) = 0 - 0.1\cos 0 = -0.1 \\ y_{B_1}(1) = 2x_{B_1}(1) = -0.2 \end{cases};$$

$$B_2: \begin{cases} x_{B_2}(1) = 0.2x_{B_2}(0) - 0.1B_2(0) = 0 - 0.1\sin 0 = 0 \\ y_{B_2}(1) = 2x_{B_2}(1) = 0 \end{cases};$$

<u>$t = 2$</u>

$$B_1: \begin{cases} x_{B_1}(2) = 0.2x_{B_1}(1) - 0.1B_1(1) = -0.02 - 0.1\cos\frac{\pi}{4} = -0.09 \\ y_{B_1}(2) = 2x_{B_1}(2) = -0.18 \end{cases};$$

$$B_2: \begin{cases} x_{B_2}(2) = 0.2x_{B_2}(1) - 0.1B_2(1) = 0 - 0.1\sin\frac{\pi}{4} = -0.07 \\ y_{B_2}(2) = 2x_{B_2}(2) = -0.14 \end{cases};$$

<u>$t = h_2 = 3$</u>

$$B_1: \begin{cases} x_{B_1}(3) = 0.2x_{B_1}(2) - 0.1B_1(2) = 0.18 - 0.1\cos\frac{\pi}{2} = -0.02 \\ y_{B_1}(3) = 2x_{B_1}(2) = -0.04 \end{cases};$$

$$B_2: \begin{cases} x_{B_2}(3) = 0.2x_{B_2}(2) - 0.1B_2(2) = -0.01 - 0.1\sin\frac{\pi}{2} = -0.11 \\ y_{B_2}(3) = 2x_{B_2}(2) = -0.22 \end{cases};$$

Thus, $y_B(h_1) = (y_{B_1}(h_1) \quad y_{B_2}(h_1)) = (-0.2 \quad 0)$, $y_B(h_2) = (y_{B_1}(h_2) \quad y_{B_2}(h_2)) = (-0.04 \quad -0.22)$.

The matrix $Y_B \in \mathbb{R}^{n_H \times n_B}$ is then $Y_B = \begin{pmatrix} y_B(h_1) \\ y_B(h_2) \end{pmatrix} = \begin{pmatrix} -0.2 & 0 \\ -0.04 & -0.22 \end{pmatrix}$.

The matrix $Y_B$ is used to compute the solution of the unconstrained optimization problem (f.): $\mu^*(t) = \left({Y_B}^T Y_B\right)^{-1} {Y_B}^T (w - f)$, with $\left({Y_B}^T Y_B\right)^{-1} {Y_B}^T = \begin{pmatrix} -5 & 0 \\ 0.9 & -4.5 \end{pmatrix}$, where $\mu^*$ is the vector of the optimal parameters at time $t$. The control action is computed as $u(t) = {\mu^*}^T B(0)$, where $B(0)$ is the column vector of base functions $B_i(k), i = 1,2$, evaluated for $k = 0$.

Since the reference is constant, the vector of the future reference values evaluated in the coincidence points $h_1 = 1$ and $h_2 = 3$ is $w(t + k) = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}, t = 1,2, \dots, k = 1,2, \dots$ .

The output predictions at time $t = 4$ are

$$\hat{y}(4 + k|4) = y(4 + k) + \hat{e}(4 + k|4) = \sum_{i=1,2} y_{B_i}(k)\mu_i(4) + QM^k x(4) + \hat{e}(4 + k|4), k = 1,2,3.$$

The last two terms constitute the free response and, by assumption k., are computed as

$$f(4+k|4) = QM^k x(4) + \hat{e}(4+k|4) = QM^k x(4) + (y_m(4) - y(4)), k = 1,2,3.$$

The measured output is given (assumption j., $, y_m(4) = 2.7$), whereas the model output is computed from assumptions h. and i.:

$$\begin{cases} x(4) = 0.2x(3) - 0.1u(3) = 0.2 \cdot 1.09 - 0.1 \cdot (-9) = 1.12 \\ \qquad\qquad\qquad y(4) = 2x(4) = 2.24 \end{cases}$$

The output predictions are then

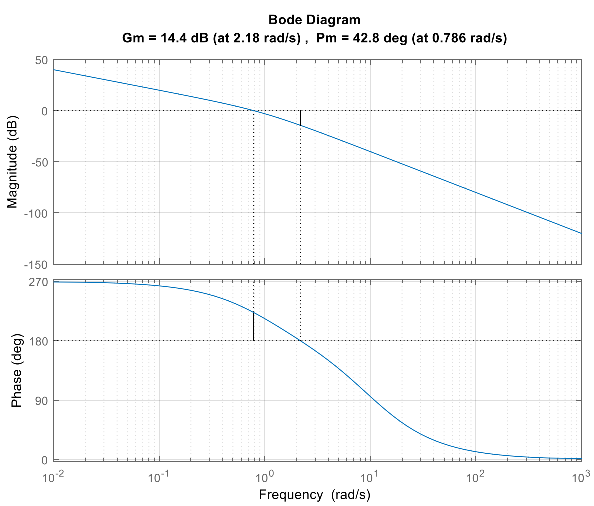
$$\underline{h_1 = 1} \qquad f(5) = QM^{h_1} x(4) + \big(y_m(4) - y(4)\big) = 0.45 + 0.46 = 0.91;$$
$$\underline{h_2 = 3} \qquad f(7) = QM^{h_2} x(4) + \big(y_m(4) - y(4)\big) = 0.07 + 0.46 = 0.53.$$

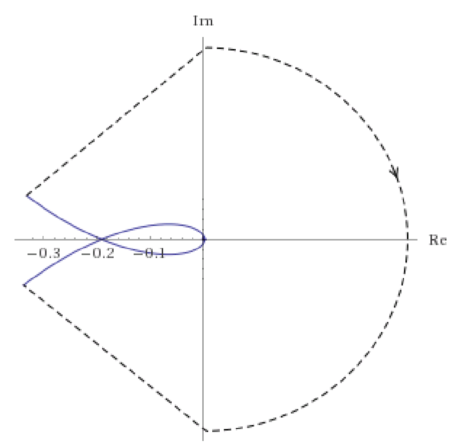
Finally we can compute the contro action as follows:

$$d = (w - f) = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 0.91 \\ 0.53 \end{pmatrix} = \begin{pmatrix} 1.59 \\ 1.97 \end{pmatrix}.$$
$$\mu^*(4) = \begin{pmatrix} \mu_1^*(4) \\ \mu_2^*(4) \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0.9 & -4.5 \end{pmatrix} \begin{pmatrix} 1.59 \\ 1.97 \end{pmatrix} = \begin{pmatrix} -7.95 \\ -7.51 \end{pmatrix};$$
$$u(4) = \mu_1^*(4) B_1(0) + \mu_2^*(4) B_2(0) = -7.95 \cdot \cos 0 - 7.51 \cdot \sin 0 = -7.95.$$

## Question i)

To check the stability of the closed-loop time delay system we check the stability of the delay-free system, with open-loop transfer function $F_0(s) = \frac{1 - \frac{s}{10}}{s(1+s)(1+\frac{s}{10})}$, and compute the delay margin. The Nyquist diagram (below, on the right) shows that the closed-loop system is stable (no poles with positive real part in $F_0(s)$ and no loops of $\vec{F}_0(j\omega)$ around -1). From the Bode diagrams (below, on the left) we evaluate the phase margin $m_\varphi = 0.747 rad$ and the crossover frequency $\omega_\varphi = 0.786 \frac{rad}{s}$.




The delay margin is then, by definition, $m_\tau := \frac{\omega_\varphi}{m_\varphi} = 1.05s$, which is larger than the delay of the system; therefore, the closed-loop time-delay system is asymptotically stable.