1 Exercise (IMC robust design)

Let the process be described by the transfer function: $P(s) = K\frac{s-z}{(s+10)^2}$, with $z = -\frac{1}{1+\delta}$, $K = 100(1+\delta)$ and $\delta \in [0,2)$, and let the process model be $\tilde{P}(s) = 100\frac{s+1}{(s+10)^2}$.

A) Design an IMC controller $Q(s)$ such that:
   i) the overall system is robustly asymptotically stable;
   ii) the overall system has 0 steady-state error for step inputs.

B) Determine the equivalent classic controller.

A)

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles), therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.
The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

$$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$$

with $\tilde{P}_+(s) = 1$ and $\tilde{P}_-(s) = 100\frac{s+1}{(s+10)^2} = \frac{1+s}{(1+s/10)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = \frac{(1+s/10)^2}{1+s}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$).
We use the standard IMC filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 1$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = \frac{(1+s/10)^2}{(1+s)(1+\lambda s)}$ is proper;

b) $\tilde{T}(0) = \tilde{P}(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1}f(s)\right]_{s=0} = \left[\frac{1}{(1+\lambda s)}\right]_{s=0} = 1.$

Step 3)

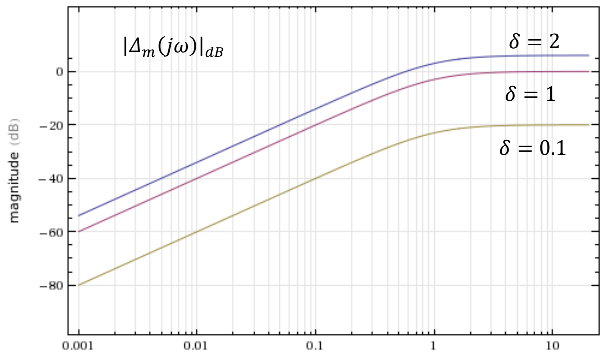
Determine the value of λ such that the sufficient condition for robust stability holds:

$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall\omega$$

where $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall\omega$ (equivalently, the additive uncertainty can be used).
By definition, the multiplicative uncertainty is defined as

$$\Delta_m(j\omega) := \frac{P(j\omega)-\tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{K\frac{j\omega-z}{(j\omega+10)^2}-100\frac{j\omega+1}{(j\omega+10)^2}}{100\frac{j\omega+1}{(j\omega+10)^2}} = \frac{(1+\delta)\left(j\omega+\frac{1}{1+\delta}\right)-(j\omega+1)}{j\omega+1} = \delta\frac{j\omega}{1+j\omega}.$$

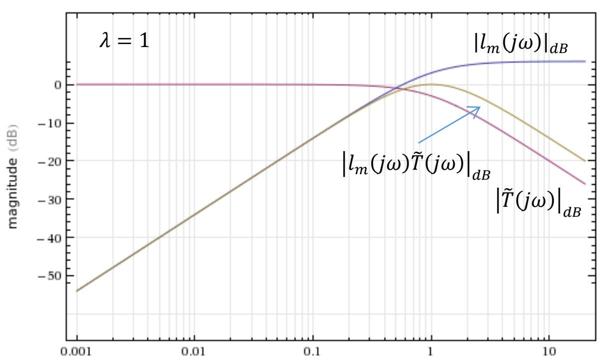
Since $\delta < 2$, an upper-bound is then simply given by $l_m(j\omega) = 2\frac{j\omega}{1+j\omega}$ (see also the Bode diagram below). The uncertainty upper-bound is a high-pass filter.

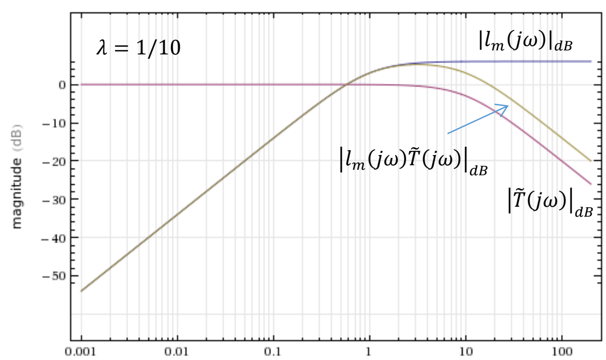


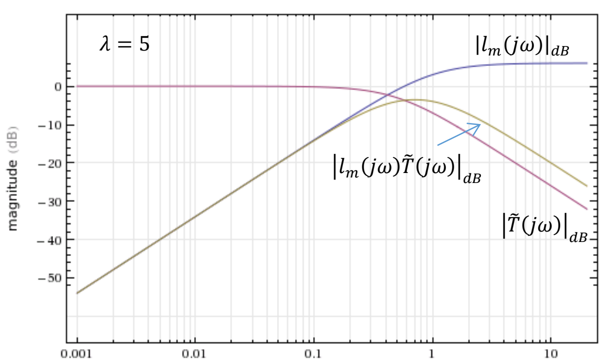
The sufficient condition for robust stability is then:

$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| = \left|2\frac{j\omega}{1+j\omega} \cdot \frac{1}{(1+\lambda s)}\right| =< 1, \forall\omega$$

Since $\tilde{T}(j\omega) = \frac{1}{(1+\lambda s)}$ is a low-pass filter, there always exist values for λ for which the condition holds. The figures below show that for $\lambda > 1$ the condition is met. For instance, we choose $\lambda = 5$.







B)

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = \frac{\left(1+\frac{s}{10}\right)^2}{(1+s)(1+5s)-(1+s)} = \frac{1}{5}\frac{\left(1+\frac{s}{10}\right)^2}{s(s+1)}$$

$G(s)$ has 2 zeros, one pole in $s = 0$ and one other negative pole, therefore it can be written as a PID controller with filter:

$$G(s) = \frac{1}{5}\frac{1+\frac{s}{5}+\frac{s^2}{100}}{s}\frac{1}{s+1} = \frac{1}{25}\left(1 + \frac{1}{0.2s} + 0.05s\right)\left(\frac{1}{s+1}\right) = K_c\left(1 + \frac{1}{T_i s} + T_d s\right)\left(\frac{1}{1+\beta_f s}\right),$$

con $K_c = 0.04, T_i = 0.2, T_d = 0.05, \beta_f = 1$

2 Exercise (IMC robust design, RHP zeros)

Let the process be described by the transfer function: $P(s) = K\frac{s-z}{(s+10)^2}$, with $z = \frac{1}{1+\delta}$, $K = 100(1+\delta)$ and $\delta \in [0,2)$, and let the process model be $\tilde{P}(s) = 100\frac{s-1}{(s+10)^2}$.

A) Design an IMC controller $Q(s)$ such that:
   i) the overall system is robustly asymptotically stable;
   ii) the objective is to minimize the IAE;
   iii) the overall system has 0 steady-state error for step inputs.

B) Determine the equivalent classic controller.

A)

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles), therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.
The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

   $\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$
   with $\tilde{P}_+(s) = 1 - s$ (according to the IAE optimal factorization) and $\tilde{P}_-(s) = 100\frac{1}{(s+10)^2} = -\frac{1}{(1+s/10)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = -(1+s/10)^2$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$).
Thus, we use the well-known filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 2$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = -\frac{(1+s/10)^2}{(1+\lambda s)^2}$ is proper;

b) $\tilde{T}(0) = \tilde{P}(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1} f(s)\right]_{s=0} = \left[\frac{1-s}{(1+\lambda s)^2}\right]_{s=0} = 1.$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$ (equivalently, the multiplicative uncertainty can be used).
By definition, the additive uncertainty is defined as

$$\Delta_a(j\omega) \coloneqq P(j\omega) - \tilde{P}(j\omega) = K\frac{j\omega - z}{(j\omega+10)^2} - 100\frac{j\omega-1}{(j\omega+10)^2} = 100\frac{(1+\delta)\left(j\omega - \frac{1}{1+\delta}\right) + (1-j\omega)}{(j\omega+10)^2} = \delta\frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2}.$$

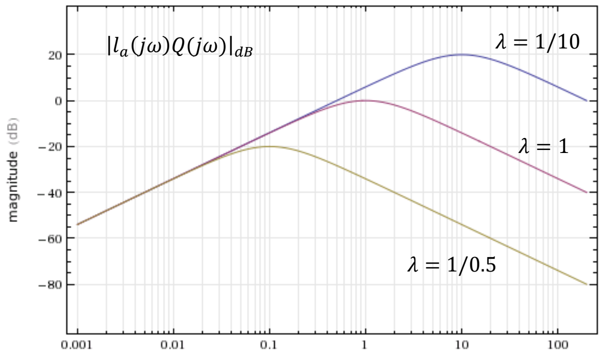
Since $\delta < 2$, an upper-bound is then simply given by $l_m(j\omega) = 2\frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2}$.

The uncertainty upper-bound is a high-pass filter.

The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|2\frac{j\omega}{\left(1+\frac{j\omega}{10}\right)^2}\frac{\left(1+\frac{j\omega}{10}\right)^2}{(1+\lambda j\omega)^2}\right| = \left|2\frac{j\omega}{(1+j\omega\lambda)^2}\right| =< 1, \forall\omega$$

The figures below show that for $\lambda > 1$ the condition is met. For instance, we choose $\lambda = 2$.



B)

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = \frac{\frac{(1+s/10)^2}{\left(1+\frac{s}{2}\right)^2}}{1-\left(-\frac{1-s}{(1+s/10)^2}\right)\cdot\frac{(1+s/10)^2}{\left(1+\frac{s}{2}\right)^2}} = \frac{\left(1+\frac{s}{10}\right)^2}{\left(1+\frac{s}{2}\right)^2-s+1} = 4\frac{\left(1+\frac{s}{10}\right)^2}{(s+j2\sqrt{2})(s-j2\sqrt{2})}$$

3 Exercise (IMC robust design, type 2 system)

Let the process be described by the transfer function: $P(s) = \frac{\left(1+\frac{s}{10}\right)((1+s\tau)+0.9)}{(1+s)^2(1+s\tau)}$, with $\tau \in (1,10]$, and let the process model be $\tilde{P}(s) = \frac{1+s/10}{(1+s)^2}$.

Design an IMC controller $Q(s)$ such that:

i) the overall system is robustly asymptotically stable (suggestion: use the additive uncertainty);

ii) the overall system has 0 steady-state error for ramp inputs (type 2 system).

The nominal process $\tilde{P}(s)$ is stable (it has 2 real negative poles), therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.

The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$

with $\tilde{P}_+(s) = 1$ and $\tilde{P}_-(s) = \frac{1+s/10}{(1+s)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = \frac{(1+s)^2}{1+\frac{s}{10}}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 2 (i.e., $\tilde{T}(0) = \tilde{P}(0)Q(0) = 1$ and $\left.\frac{d\tilde{T}(s)}{ds}\right|_{s=0} = 0$).

Thus, we use the filter $f(s) = \frac{1+\left(2\lambda - P_+'(0)\right)s}{(1+\lambda s)^2} = \frac{1+2\lambda s}{(1+\lambda s)^2}$, where $P_+'(0) = \left.\frac{dP_+(s)}{ds}\right|_{s=0} = 0$ since $\tilde{P}_+(s) = 1$.

In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = \frac{(1+s)^2}{1+\frac{s}{10}}\frac{1+2\lambda s}{(1+\lambda s)^2}$ is proper;

b) $\tilde{T}(s) = \tilde{P}(s)Q(s) = \tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1}f(s) = f(s)$;

$\tilde{T}(0) = f(0) = \left[\frac{(1+2\lambda s)}{(1+\lambda s)^2}\right]_{s=0} = 1;$

$\left.\frac{d\tilde{T}(s)}{ds}\right|_{s=0} = \left.\frac{df(s)}{ds}\right|_{s=0} = \left.\frac{d\left(\frac{1+2\lambda s}{(1+\lambda s)^2}\right)}{ds}\right|_{s=0} = \left(-2\lambda\frac{1+2\lambda s}{(1+\lambda s)^3} + 2\lambda\frac{1}{(1+\lambda s)^2}\right)_{s=0} = 0.$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

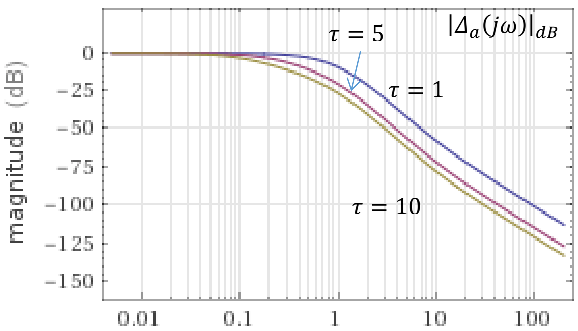
where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.

By definition, the additive uncertainty is defined as

$$\Delta_a(s) \coloneqq P(s) - \tilde{P}(s) = \frac{\left(1+\frac{s}{10}\right)((1+s\tau)+0.9)}{(1+s)^2(1+s\tau)} - \frac{1+\frac{s}{10}}{(1+s)^2} =$$

$$= \frac{\left(1+\frac{s}{10}\right)(1+s\tau)+0.9\left(1+\frac{s}{10}\right)}{(1+s)^2(1+s\tau)} - \frac{\left(1+\frac{s}{10}\right)(1+s\tau)}{(1+s)^2(1+s\tau)} = 0.9\frac{\left(1+\frac{s}{10}\right)}{(1+s)^2(1+s\tau)}.$$

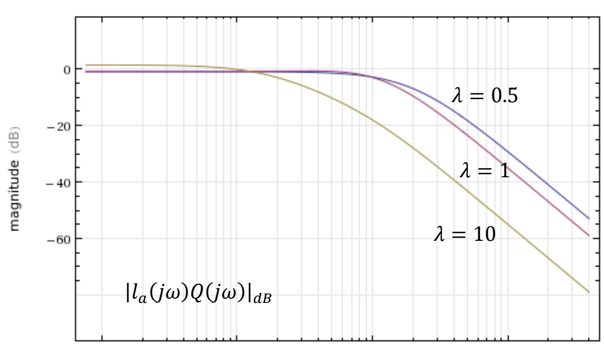
Since $1 < \tau \leq 10$, an upper-bound is given by setting $\tau = 1$, i.e., $l_a(j\omega) = 0.9\frac{\left(1+\frac{j\omega}{10}\right)}{(1+j\omega)^3}$ (see the Bode diagrams in the figure below).



The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|0.9\frac{\left(1+\frac{j\omega}{10}\right)}{(1+j\omega)^3}\cdot\frac{(1+j\omega)^2}{\left(1+\frac{j\omega}{10}\right)}\frac{1+j\omega 2\lambda}{(1+j\omega\lambda)^2}\right| = \left|0.9\frac{1+j\omega 2\lambda}{(1+j\omega)(1+j\omega\lambda)^2}\right| =< 1, \forall\omega$$

We must pay attention to the effect of the zero in $-1/2\lambda$. To counteract its effect with the pole in -1, we can set $2\lambda \leq 1$. For instance, we choose $\lambda = 0.5$, and the condition, which is now written as $\left|0.9\frac{1}{(1+j\omega/2)^2}\right| =< 1, \forall\omega$, is verified.



4 Exercise (Padé approximation)

Let the process be described by the transfer function: $P(s) = K\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-\theta s}$, with $\theta = 0.2$ and $K \in [10,20)$, and let the gain of the process model be $K = 10$.

A) Design an IMC controller $Q(s)$ such that:
   i) the overall system is robustly asymptotically stable;
   ii) the overall system has 0 steady-state error for step inputs.

B) Verify that the controller stabilizes the actual plant $\bar{P}(s) = 20\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-0.2s}$.

A)

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.

Moreover, the time-delay $\theta = 0.2s$ of the process is much smaller than the time constant $\tau = 2s$ of the process, therefore we can use a Padé approximation to write the delay term as a transfer function.

By using the 1/1 Padé approximation $e^{-\theta s} \cong \frac{1-s\frac{\theta}{2}}{1+s\frac{\theta}{2}}$, we obtain the following approximated process and nominal process:

$$P^P(s) \cong K\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2} \cdot \frac{1-\frac{s}{10}}{1+\frac{s}{10}} = K\frac{1-\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}$$

$$\tilde{P}^P(s) \cong 10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2} \cdot \frac{1-\frac{s}{10}}{1+\frac{s}{10}} = 10\frac{1-\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}$$

The IMC design procedure to robustly stabilize the approximated process $P^P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

   $\tilde{P}^P(s) = \tilde{P}_+^P(s)\tilde{P}_-^P(s)$
   with $\tilde{P}_+^P(s) = 1 - \frac{s}{10}$ and $\tilde{P}_-^P(s) = 10\frac{1}{\left(1+\frac{s}{0.5}\right)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-^P(s)\right)^{-1} = \frac{1}{10}\left(1+\frac{s}{0.5}\right)^2$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}^P(0) = \tilde{P}^P(0)Q(0) = 1$).
Thus, we use the well-known filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 2$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{(1+\lambda s)^2}$ is proper;

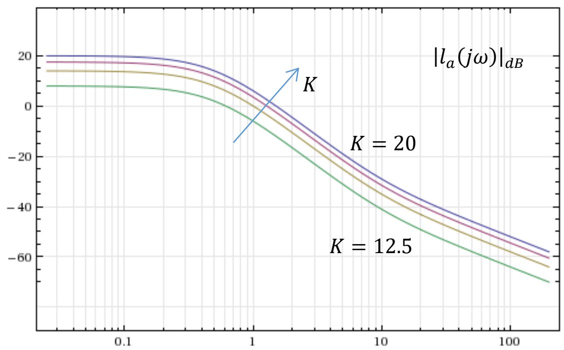
b) $\tilde{T}^P(0) = \tilde{P}^P(0)Q(0) = \left[\tilde{P}_+^P(s)\tilde{P}_-^P(s)\left(\tilde{P}_-^P(s)\right)^{-1} f(s)\right]_{s=0} = \left[\left(1-\frac{s}{10}\right)\left(\frac{\left(1+\frac{s}{0.5}\right)^2}{(1+\lambda s)^2}\right)\right]_{s=0} = 1.$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds, i.e, $|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$, where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.

By definition, the additive uncertainty is defined as $\Delta_a(j\omega) := P^P(j\omega) - \tilde{P}^P(j\omega) = (K-10)\frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{0.5}\right)^2}$.

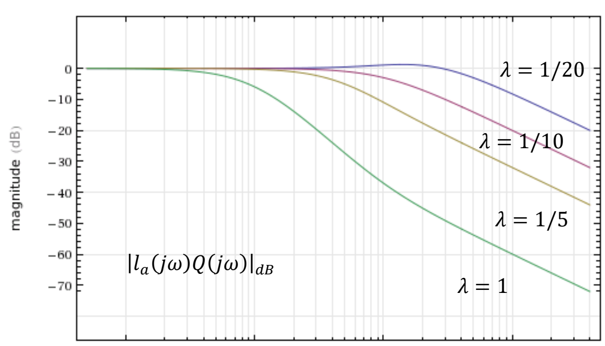
Since $K < 20$, an upper-bound is then simply given by $l_a(j\omega) = 10\frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{0.5}\right)^2}$ (see also the Bode diagram below).



The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|10\frac{1-\frac{j\omega}{10}}{\left(1+\frac{j\omega}{0.5}\right)^2}\cdot\frac{1}{10}\frac{\left(1+\frac{j\omega}{0.5}\right)^2}{(1+\lambda s)^2}\right| = \left|\frac{1-\frac{j\omega}{10}}{(1+\lambda s)^2}\right| < 1, \forall\omega$$

which is always true for $\lambda > 1/10$ (see the Bode diagram below). Recalling that we are working with an approximated process, we should pick a conservative value for λ; for instance, we choose $\lambda = 1/5$, obtaining $Q(s) = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{5}\right)^2}$.



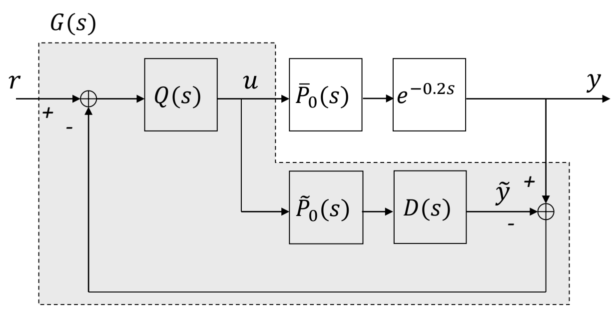
The controller $G(s)$ is calculated as:

$$G(s)=\frac{Q(s)}{1-\tilde{P}(s)Q(s)}=\frac{\frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{2}\right)^2}}{1-10\frac{1-\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}\cdot\frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{5}\right)^2}}=\frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{5}\right)^2-\left(1-\frac{s}{10}\right)}=\frac{1}{5}\frac{\left(1+\frac{s}{0.5}\right)^2}{s(1+0.08s)}=\frac{2}{25}\left(1+\frac{1}{0.4s}+10s\right)\frac{1}{1+0.08s},$$

which is a PID+filter controller with $K_P=\frac{2}{25}, T_i=0.4, T_d=10, \beta_f=0.08$.

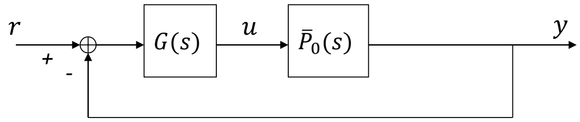
B)

Let us consider the equivalent classic controller $G(s)$ depicted in the figure below:



where $\bar{P}(s)=\bar{P}_0(s)e^{-0.2s}=20\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-0.2s}$, and $\tilde{P}(s)$ is the process model with nominal gain $K=10$ and delay model $D(s)=\frac{1-\frac{s}{10}}{1+\frac{s}{10}}$, i.e., $\tilde{P}(s)=\tilde{P}_0(s)D(s)=10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}\frac{1-\frac{s}{10}}{1+\frac{s}{10}}=10\frac{1-\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}$.

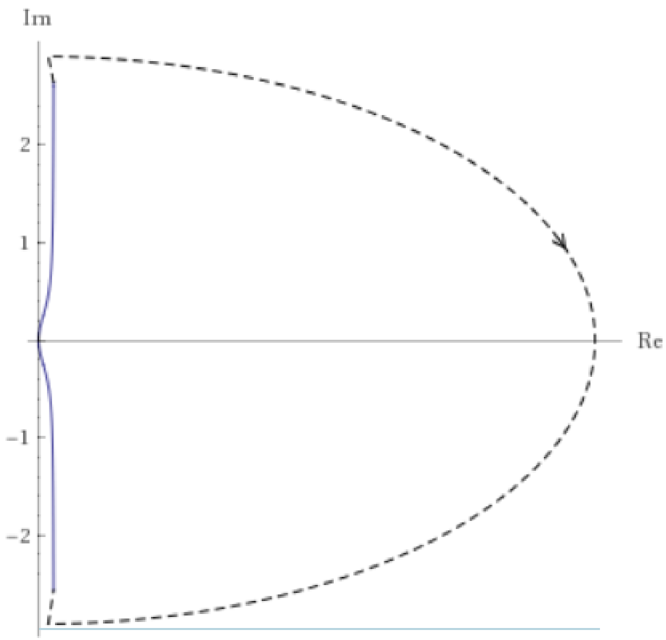
Now we check that $G(s)$ stabilizes $\bar{P}_0(s)$ (in the classic control scheme depicted below) and compute the delay margin m_τ.

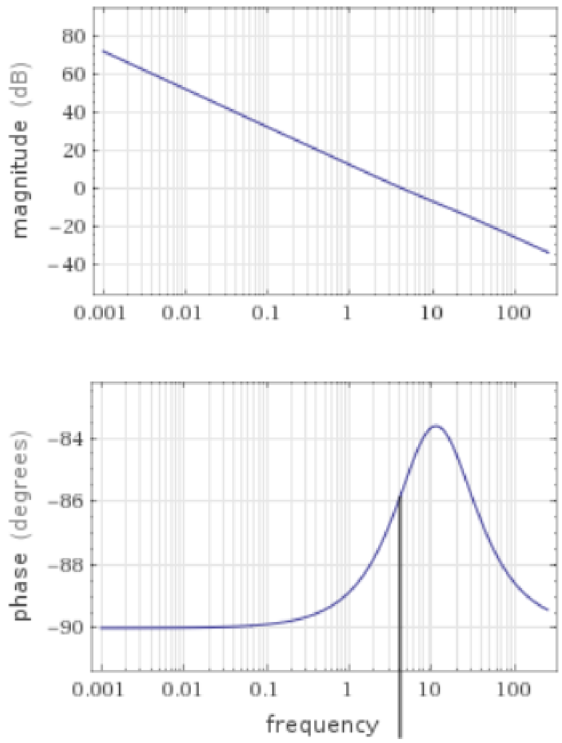


The delay-free open-loop transfer function is:

$$\bar{F}_0(s)=\bar{P}_0(s)G(s)=20\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}\cdot\frac{1}{5}\frac{\left(1+\frac{s}{0.5}\right)^2}{s(1+0.08s)}=4\frac{1+\frac{s}{10}}{s(1+0.08s)}.$$

By the Nyquist theorem, the closed-loop system is asymptotically stable (see the Nyquist diagram below) with phase margin $m_\varphi=94.1°$ at $\omega_c=4.11\,\frac{rad}{s}$ (see the Bode diagrams below).





The delay margin is then $m_\tau := \frac{m_\varphi}{\omega_c} = \frac{94.1° \cdot \frac{\pi\, rad}{180°}}{4.11\, \frac{rad}{s}} = 0.4s$.

Since $m_\tau > \theta = 0.2$, the controller $G(s)$ and, therefore, the IMC controller $Q(s)$ stabilizes $\bar{P}(s)$.

5 Exercise (with time-delay)

Let the process be described by the transfer function: $P(s) = K\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-\theta s}$, with $\theta = 2$ and $K \in [10,20)$, and let the nominal gain of the process model be $\widetilde{K} = 10$.

A) Design an IMC controller $Q(s)$ under the IAE cost such that:
   iii) the overall system is robustly asymptotically stable;
   iv) the overall system has 0 steady-state error for step inputs.

B) Compute the classical controller as a Smith Predictor controller.

A)

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.

Moreover, the time-delay $\theta = 2s$ of the process is in the same magnitude order of the time constant $\tau = 2s$ of the process. The IMC design procedure to robustly stabilize the process consists in the following 3 steps:

Step 1)

a) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

$$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$$

with $\tilde{P}_+(s) = e^{-2s}$ and $\tilde{P}_-(s) = 10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{1+\frac{s}{10}}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}^P(0) = \tilde{P}(0)Q(0) = 1$).

We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 12$. In fact:

c) $Q(s) = \tilde{Q}(s)f(s) = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{10}\right)(1+\lambda s)}$ is proper;

d) $\tilde{T}(0) = P(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1}f(s)\right]_{s=0} = \left[e^{-2s}\frac{1}{1+\lambda s}\right]_{s=0} = 1.$

Step 3)

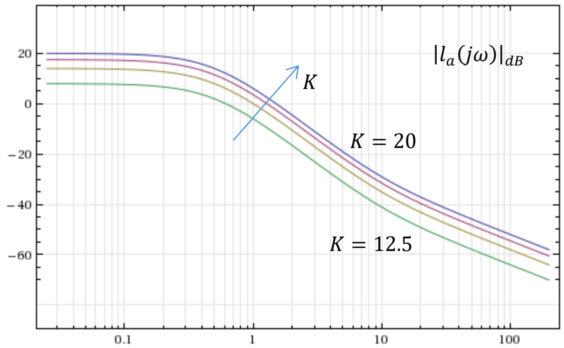
Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.

By definition, the additive uncertainty is defined as $\Delta_a(j\omega) \coloneqq P(j\omega) - \tilde{P}(s)(j\omega) = (K - 10)\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-2s}$.

Since $10 \le K < 20$, an upper-bound is then simply given, for $K = 20$, by $l_a(j\omega) = 10\frac{1+\frac{j\omega}{10}}{\left(1+\frac{j\omega}{0.5}\right)^2}$ (see also the Bode diagram below).



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The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|10\frac{1+\frac{j\omega}{10}}{\left(1+\frac{sj\omega}{0.5}\right)^2}e^{-2j\omega} \cdot \frac{1}{10}\frac{\left(1+\frac{j\omega}{0.5}\right)^2}{\left(1+\frac{j\omega}{10}\right)(1+\lambda j\omega)}\right| = \left|\frac{1}{1+\lambda j\omega}\right| < 1, \forall\omega$$

which is always true. For instance, we choose $\lambda = 1/10$.

We obtain $Q(s) = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{10}\right)^2}$.

C) The classic controller $G(s)$ is computed as

$$G(s) = \frac{Q(s)}{1-\tilde{P}(s)Q(s)} = \frac{1}{Q^{-1}(s)-\tilde{P}(s)}$$
$$= \frac{1}{10\frac{\left(1+\frac{s}{10}\right)^2}{\left(1+\frac{s}{0.5}\right)^2}-10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-2s}} = \frac{1}{10\frac{\frac{s^2}{100}+\frac{s}{10}+1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}-10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-2s}} = \frac{1}{10\frac{\frac{s^2}{100}-\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}+10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}-10\frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}e^{-2s}} ==$$
$$\frac{1}{G_0^{-1}(s)+\tilde{P}_0(s)(1-e^{-2s})} = \frac{G_0(s)}{1+G_0(s)\tilde{P}_0(s)(1-e^{-2s})},$$

with $G_0(s) = \frac{\left(1+\frac{s}{0.5}\right)^2}{s(\frac{1}{10}s-1)} = K_C\left(1+\frac{1}{T_i s}+T_d s\right)\frac{1}{1+\beta_f s}$, $K_C = 1, T_i = 1, T_d = 4, \beta_f = -0.1$.

6 Exercise (with time-delay)

Let the process be described by the transfer function: $P(s) = K \frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2} e^{-\theta s}$, with $\theta = 2$ and $K \in [10,20)$, and let the nominal gain of the process model be $\widetilde{K} = 10$.

D) Design an IMC controller $Q(s)$ under the IAE cost such that:
   v) the overall system is robustly asymptotically stable;
   vi) the overall system has 0 steady-state error for step inputs.

E) Compute the classical controller as a Smith Predictor controller.

A)

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.

Moreover, the time-delay $\theta = 2s$ of the process is in the same magnitude order of the time constant $\tau = 2s$ of the process. The IMC design procedure to robustly stabilize the process consists in the following 3 steps:

Step 1)

c) Factorize the nominal process in a minimum-phase term and a non-minimum-phase term:

$\tilde{P}(s) = \tilde{P}_+(s)\tilde{P}_-(s)$

with $\tilde{P}_+(s) = e^{-2s}$ and $\tilde{P}_-(s) = 10 \frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2}$.

d) Define the controller as follows: $\tilde{Q}(s) = \left(\tilde{P}_-(s)\right)^{-1} = \frac{1}{10} \frac{\left(1+\frac{s}{0.5}\right)^2}{1+\frac{s}{10}}$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $\tilde{T}^P(0) = \tilde{P}(0)Q(0) = 1$).

We use the filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 12$. In fact:

e) $Q(s) = \tilde{Q}(s)f(s) = \frac{1}{10} \frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{10}\right)(1+\lambda s)}$ is proper;

f) $\tilde{T}(0) = P(0)Q(0) = \left[\tilde{P}_+(s)\tilde{P}_-(s)\left(\tilde{P}_-(s)\right)^{-1} f(s)\right]_{s=0} = \left[e^{-2s}\frac{1}{1+\lambda s}\right]_{s=0} = 1.$

Step 3)

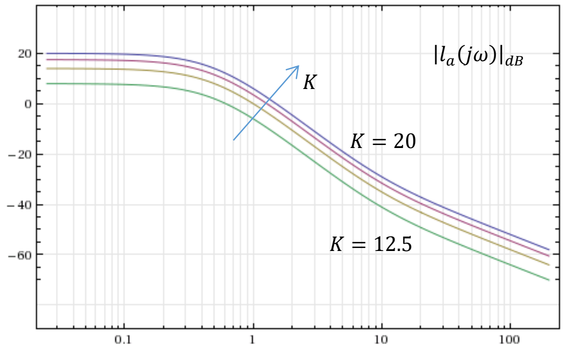
Determine the value of λ such that the sufficient condition for robust stability holds:

$$|l_a(j\omega)Q(j\omega)| < 1, \forall\omega$$

where $l_a(j\omega)$ is an upper-bound of the additive uncertainty $\Delta_a(j\omega)$, i.e., a function such that $|l_a(j\omega)| > |\Delta_a(j\omega)|, \forall\omega$.

By definition, the additive uncertainty is defined as $\Delta_a(j\omega) := P(j\omega) - \tilde{P}(s)(j\omega) = (K - 10) \frac{1+\frac{s}{10}}{\left(1+\frac{s}{0.5}\right)^2} e^{-2s}$.

Since $10 \leq K < 20$, an upper-bound is then simply given, for $K = 20$, by $l_a(j\omega) = 10\frac{1+\frac{j\omega}{10}}{\left(1+\frac{j\omega}{0.5}\right)^2}$ (see also the Bode diagram below).



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The sufficient condition for robust stability is then:

$$|l_a(j\omega)Q(j\omega)| = \left|10\frac{1+\frac{j\omega}{10}}{\left(1+\frac{sj\omega}{0.5}\right)^2}e^{-2j\omega}\cdot\frac{1}{10}\frac{\left(1+\frac{j\omega}{0.5}\right)^2}{\left(1+\frac{j\omega}{10}\right)(1+\lambda j\omega)}\right| = \left|\frac{1}{1+\lambda j\omega}\right| < 1, \forall\omega$$

which is always true. For instance, we choose $\lambda = 1/10$.

We obtain $Q(s) = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{10}\right)^2}$.

F) The Smith's Predictor primary controller $G_0(s)$ is computed as

$$G_0(s) = \frac{Q(s)}{1-\tilde{P}_0(s)Q(s)} = \frac{N_Q(s)D_{\tilde{P}_0}(s)}{D_Q(s)D_{\tilde{P}_0}(s)-N_Q(s)N_{\tilde{P}_0}(s)} = \frac{1}{10}\frac{\left(1+\frac{s}{0.5}\right)^2}{\left(1+\frac{s}{10}\right)^2-\left(1+\frac{s}{10}\right)} = \frac{\left(1+\frac{s}{0.5}\right)^2}{s\left(1+\frac{s}{10}\right)} = K_C\left(1+\frac{1}{T_i s}+T_d s\right)\frac{1}{1+\beta_f s}$$

with $K_C = 4, T_i = 4, T_d = 1, \beta_f = 0.1$.

7 Exercise (robust SP)

Let the process be described by the transfer function: $P(s) = 10\frac{1-s}{(1+s)^2}e^{-\theta s}$, with $\theta = 2+\delta$ and $\delta \in [0,2)$, and let the process model be $\tilde{P}(s) = 10\frac{1-s}{(1+s)^2}e^{-2s}$.

A) Design an IMC controller $Q(s)$, under the IAE cost, such that:
   i) the overall system is robustly asymptotically stable;
   ii) the overall system has 0 steady-state error for step inputs.

B) Determine the Smith Predictor controller.

A)

The nominal process $\tilde{P}(s)$ is stable, therefore it is possible to design a stable controller $Q(s)$ to stabilize the closed-loop nominal system.
Moreover, since the time-delay $\tilde{\theta} = 2s$ of the process is comparable to the time constant $\tau = 1s$ of the process, we cannot use a Padé approximation to write the delay term as a transfer function.
Then, we use a Smith Predictor controller, depicted in the figure below:

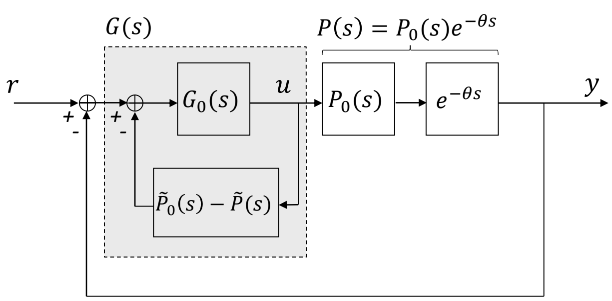


Figure 1)

The IMC form of the SP controller of Figure 1) is shown in the figure below:

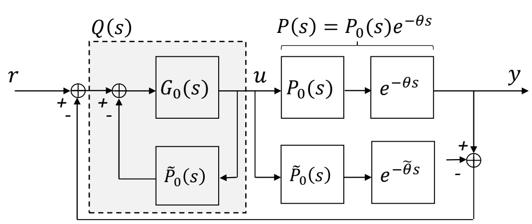


Figure 2)

Thanks to the Smith's principle, we can design an IMC controller for the delay-free model and then use it to develop the primary controller of the SP.
The IMC design procedure to robustly stabilize the process $P(s)$ consists in the following 3 steps:

Step 1)

a) Factorize the nominal delay-free process $\tilde{P}_0(s) = P_0(s)$ in a minimum-phase term and a non-minimum-phase term:

$$P_0(s) = P_+(s)P_-(s)$$

with $P_+(s) = (1-s)$ and $P_-(s) = 10\frac{1}{(1+s)^2}$.

b) Define the controller as follows: $\tilde{Q}(s) = \left(P_{-}(s)\right)^{-1} = \frac{1}{10}(1+s)^2$

Step 2)

Design the controller $Q(s) = \tilde{Q}(s)f(s)$, where the IMC filter $f(s)$ must be such that a) the controller $Q(s)$ is proper and b) the overall system is of type 1 (i.e., $T_0(0) = P_0(0)Q(0) = 1$).
Thus, we use the well-known filter $f(s) = \frac{1}{(1+\lambda s)^n}$ with $n = 2$. In fact:

a) $Q(s) = \tilde{Q}(s)f(s) = \frac{1}{10}\frac{(1+s)^2}{(1+\lambda s)^2}$ is proper;

b) $T_0(0) = P_0(0)Q(0) = \left[P_{+}(s)P_{-}(s)\left(P_{-}(s)\right)^{-1}f(s)\right]_{s=0} = \left[\frac{1-s}{(1+\lambda s)^2}\right]_{s=0} = 1.$

Step 3)

Determine the value of λ such that the sufficient condition for robust stability holds:

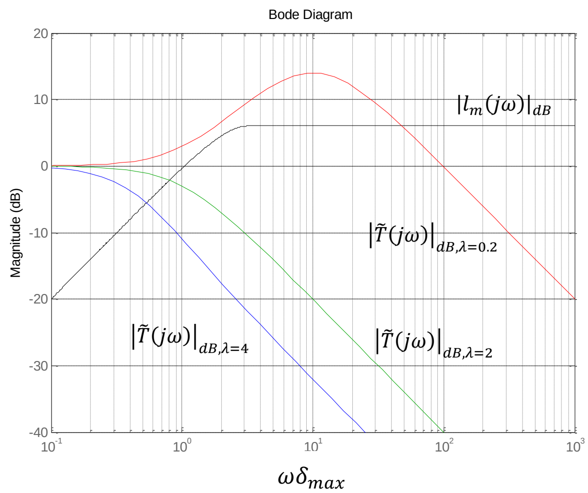
$$\left|l_m(j\omega)\tilde{T}(j\omega)\right| < 1, \forall\omega$$

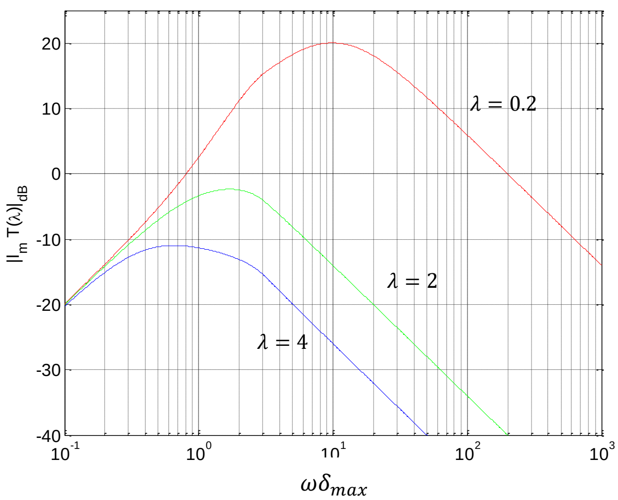
where $\left|\tilde{T}(j\omega)\right| = \left|\tilde{P}(s)Q(s)\right| = \left|10\frac{1-j\omega}{(1+j\omega)^2}e^{-2s}\frac{1}{10}\frac{(1+j\omega)^2}{(1+j\omega\lambda)^2}\right| = \left|\frac{1-j\omega}{(1+j\omega\lambda)^2}\right|$ and $l_m(j\omega)$ is an upper-bound of the multiplicative uncertainty $\Delta_m(j\omega)$, i.e., a function such that $|l_m(j\omega)| > |\Delta_m(j\omega)|, \forall\omega$.
By definition, the multiplicative uncertainty is defined as

$$\Delta_m(j\omega) := \frac{P(j\omega)-\tilde{P}(j\omega)}{\tilde{P}(j\omega)} = \frac{10\frac{1-j\omega}{(1+j\omega)^2}e^{-\theta j\omega} - 10\frac{1-j\omega}{(1+j\omega)^2}e^{-2j\omega}}{10\frac{1-j\omega}{(1+j\omega)^2}e^{-2j\omega}} = e^{-j\omega\delta} - 1.$$

From theory, we know that an upper-bound is defined as $l_m(j\omega) = \begin{cases} e^{-s\delta_{max}} - 1, & if\ \omega \le \frac{\pi}{\delta_{max}} \\ 2, & if\ \omega > \frac{\pi}{\delta_{max}} \end{cases}$, where, in our example, $\delta_{max} = 2$.

The figures below show that for $\lambda \geq 2$ the condition is met. For instance, we choose $\lambda = 2$.





The resulting IMC controller is then $Q(s) = \frac{1}{10}\frac{(1+s)^2}{(1+2s)^2}$.

B)

From the scheme of Figure 2), it follows that the controller G_0 is computed as

$$G_0(s) = \frac{Q(s)}{1-Q(s)\tilde{P}_0(s)} = \frac{\frac{1}{10}\frac{(1+s)^2}{(1+2s)^2}}{1-\frac{1}{10}\frac{(1+s)^2}{(1+2s)^2}10\frac{1-s}{(1+s)^2}} = \frac{1}{10}\frac{(1+s)^2}{(1+2s)^2-(1-s)} = \frac{1}{50}\frac{(1+s)^2}{s\left(1+s\frac{4}{5}\right)}.$$

The primary controller has 2 zeros, one pole in s=0 and one negative pole, therefore it is a PID + filter controller: $G_0(s) = \frac{1}{50}\frac{1+2s+s^2}{s}\frac{1}{1+0.8s} = K_c\left(1+\frac{1}{T_i s}+T_d s\right)\frac{1}{1+\beta_f s}$, with $K_c = \frac{1}{50}, T_i = 2, T_d = 0.5, \beta_f = 0.8$.

The overall controller (in Figure 1) is

$$G(s) = \frac{G_0(s)}{1+G_0(s)\left(\tilde{P}_0(s)-\tilde{P}(s)\right)} = \frac{K_c\left(1+\frac{1}{T_i s}+T_d s\right)\frac{1}{1+\beta_f s}}{1+K_c\left(1+\frac{1}{T_i s}+T_d s\right)\cdot 10\frac{1-s}{(1+s)^2}(1-e^{-2s})}$$