Master in Control Engineering

Process Automation
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Dipartimento di Ingegneria informatica
automatica e gestionale Antonio Ruberti

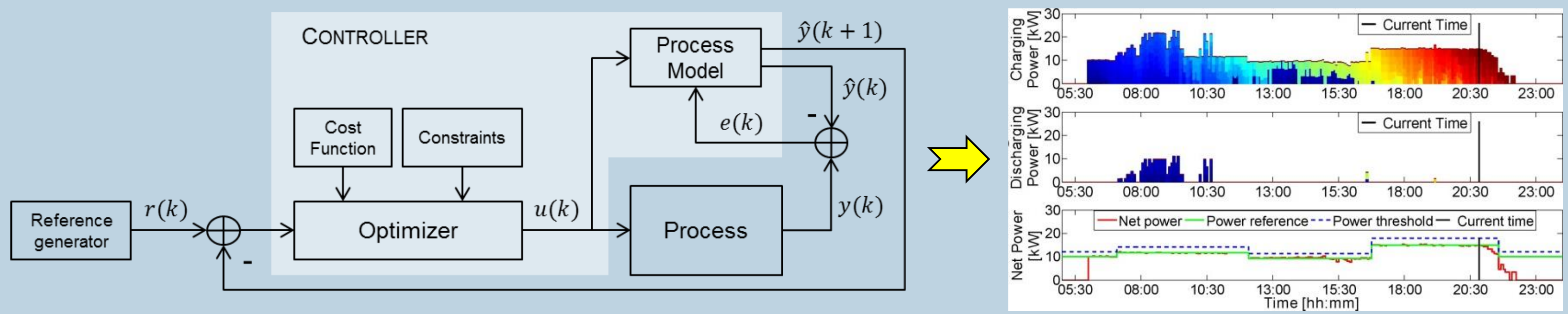
Sapienza
Università di Roma

Master in Control Engineering

Process Automation

5. Time-Delay systems

Slides based in part on:
D.E. Seborg et al., *Process Dynamics and Control (3rd ed.)*, 2009, Ch. 6.2



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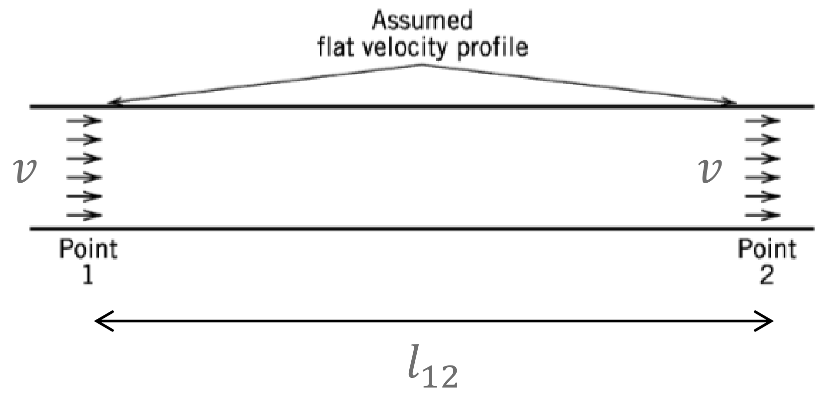
Outline

- Time-delay systems
  - Examples of delays in industrial processes
  - Reference system
    - Delay in the closed-loop transfer function
    - Delay block transfer function and effect on the reference system
  - Stability margins
    - Gain margin
    - Phase margin
    - Delay margin
  - Polynomial approximation of the delay
    - Example (cont'd)
- Summary

Examples of delays in industrial processes

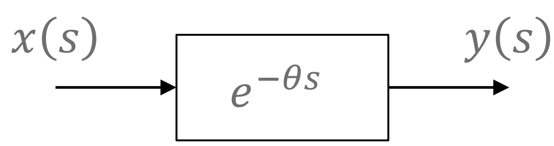
- Example 1
  - Fluid transported by a pipe
    - Assumptions
      - Flat velocity profile
        - » The fluid velocity v is constant everywhere
      - l_{12} is the length of the tube from point 1 to point 2
    - Transport delay θ:



$$\theta = \frac{l_{12}}{v}$$

  - Flow rate model:

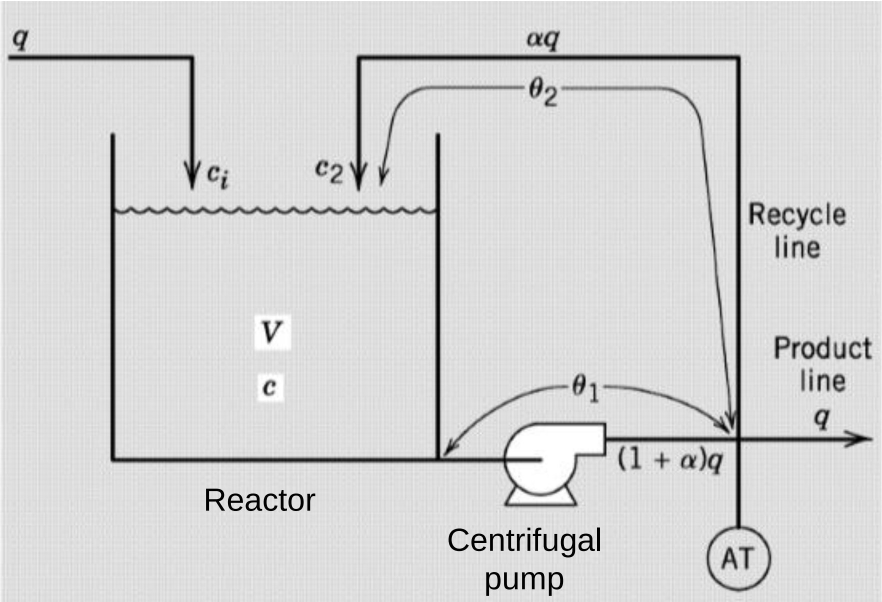
$$y(t) = x(t-\theta) \xrightarrow{\mathcal{L}} y(s) = x(s)e^{-\theta s}$$



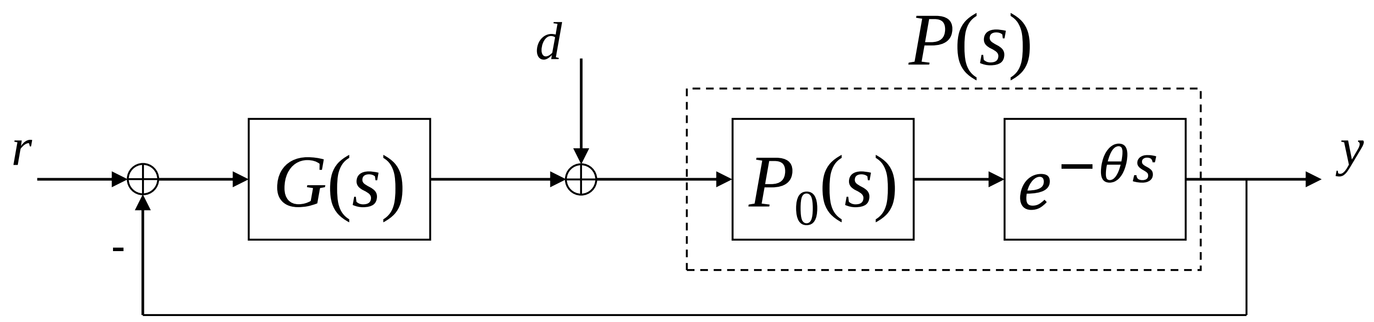
- x: flow rate at point 1
- y: flow rate at point 2

Examples of delays in industrial processes

- Trickle-bed reactor
  - Assumptions
    - Reaction rate
      $r(t) = k \cdot c(t)$
      - $-r(t)$: reactant consumed during the reaction, per volume unit
      - $c(t)$: concentration of reactant
      - k: rate constant
    - V: constant liquid volume
    - q: constant input and output rates
    - $c_1(t)$: concentration of reactant in the tank
    - $c_2(t)$: concentration of reactant in the input flow
    - q: constant input and output rates
    - θ_1: constant transport delay from the tank to the analyzer/transmitter (AT)
    - θ_2: constant transport delay from the (AT) to the tank
    - Perfect mixing hypothesis (thanks to the recycle line)

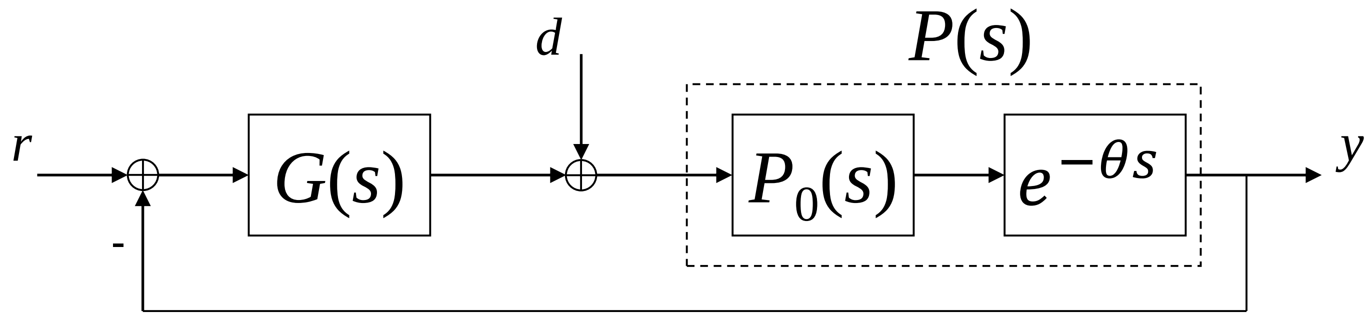


Reference system



- $P(s) = P_0(s)e^{-\theta s}$: SISO LTI process
- $P_0(s) = \frac{N_P(s)}{D_P(s)}$ Process without delay
- $\theta \geq 0$ Delay
- $G(s) = \frac{N_G(s)}{D_G(S)}$ SISO LTI controller

Delay in the closed-loop transfer function



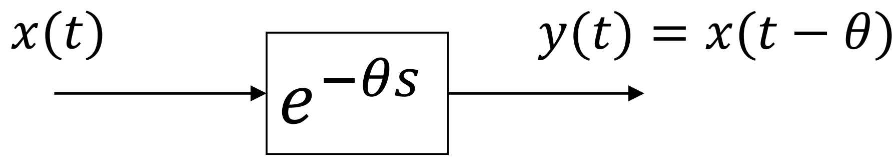
- The poles of the closed-loop system are the roots of the denominator of the closed-loop transfer function

$$D_W(s) = D_G(s)D_P(s) + e^{-\theta s}N_C(s)N_P(s)$$

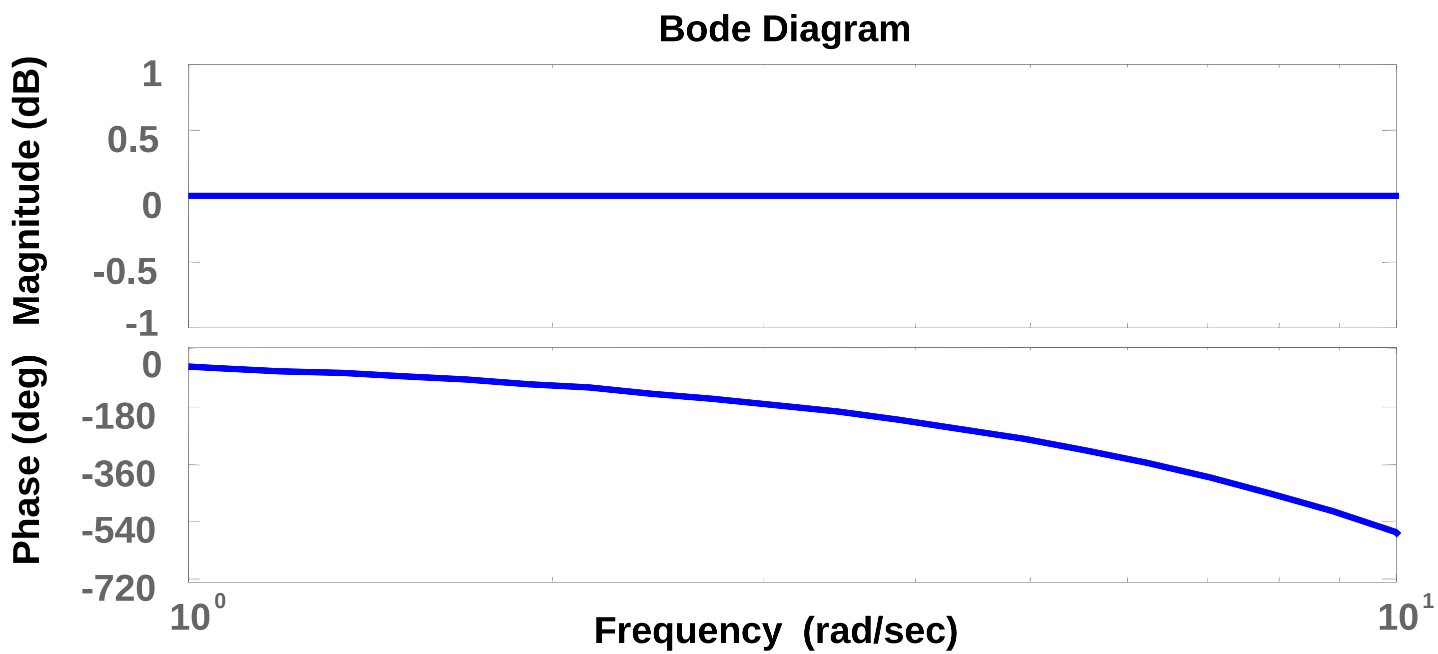
  – The system is (externally) stable iff the poles of $W(s)$ are in the LHP

- Problem
  – $D_W(s)$ is a *quasi-polinomial* because of the presence of the exponential and has infinite roots
    - The closed-loop system is infinite-dimensional (has infinite poles)

Delay block transfer function and effect on the ref. system



- Bode diagrams of the delay block ($\theta = 1$)



Delay block transfer function and effect on the ref. system

$$x(t) \longrightarrow \boxed{e^{-\theta s}} \longrightarrow y(t) = x(t-\theta)$$

- Characteristics of the delay block
  - All-pass filter

$$\left|\mathrm{e}^{-\theta s}\right|_{s=j\omega} = \left|\mathrm{e}^{-j\omega\theta}\right| = 1, \forall\omega$$

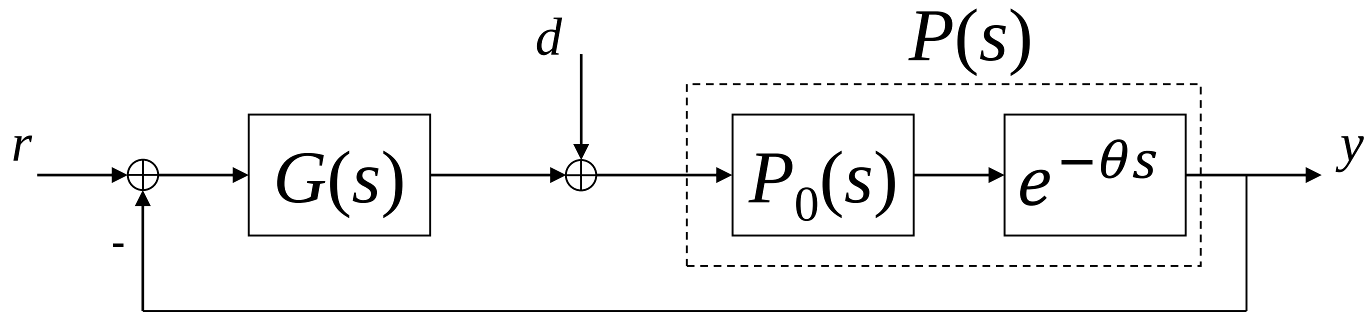
  - Phase lag proportional to the delay and to the frequency

$$\measuredangle\left\{\mathrm{e}^{-\theta s}\right\}_{s=j\omega} = \measuredangle\left\{\mathrm{e}^{-j\omega\theta}\right\} = -\omega\theta, \forall\omega$$

  - BIBO stable

$$|x(t)| < M < \infty \Rightarrow |y(t)| = |x(t-\theta)| < M < \infty$$

Delay block transfer function and effect on the ref. system



– Open-loop transfer function

$$F(s) = G(s)P(s) = G(s)P_0(s)e^{-\theta s} = F_0(s)e^{-\theta s}$$

where $F_0(s)$ is the delay-free open-loop transfer function

– The module of $F_0(j\omega)$ is the same as the module of $F(j\omega)$

$$|F(j\omega)| = \left|F_0(j\omega)e^{-\theta j\omega}\right| = |F_0(j\omega)|, \forall\omega$$

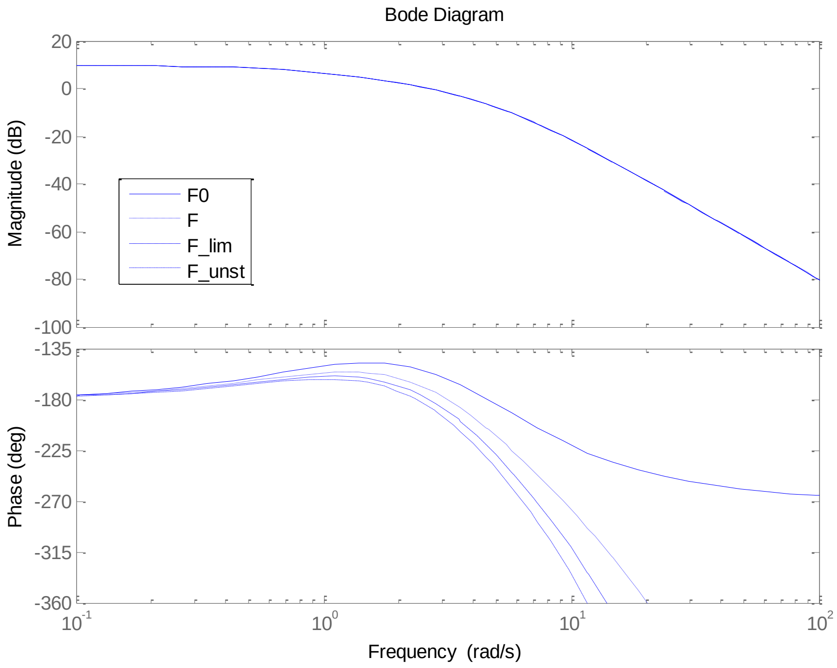
– The phase of $F_0(j\omega)$ diminishes linearly with θ and ω

$$\measuredangle\{F(j\omega)\} = \measuredangle\left\{F_0(j\omega)e^{-\theta j\omega}\right\} = \measuredangle\{F_0(j\omega)\} - \omega\theta, \forall\omega$$

– The phase lag $-\omega\theta$may drive the system to instability

Delay block transfer function and effect on the ref. system

- Example
  - F_0: delay-free open-loop transfer function
    - $m_g = 2.2 \; at \; 4.6 \frac{rad}{s}$
    - $m_\varphi = 0.42 \; at \; 2.7 \frac{rad}{s}$,
    - $m_\tau = 0.16$
  - $F = F_0 e^{-\theta s}$, with $\theta = 0.1 < m_\tau$
    - $m_g = 1.2 \; at \; 3.2 \frac{rad}{s}$
    - $m_\varphi = 0.16 \; at \; 2.7 \frac{rad}{s}$,
    - $m_\tau = 0.06$
  - $F_{lim} = F_0 e^{-\theta_{lim} s}$, with $\theta_{lim} = 0.16 = m_\tau$
    - $m_g = 1.0 \; at \; 2.7 \frac{rad}{s}$
    - $m_\varphi = 0 \; at \; 2.7 \frac{rad}{s}$,
    - $m_\tau = 0$
  - $F_{unst} = F_0 e^{-\theta_{unst} s}$, with $\theta_{unst} = 0.2 > m_\tau$



Delay block transfer function and effect on the ref. system

- Example
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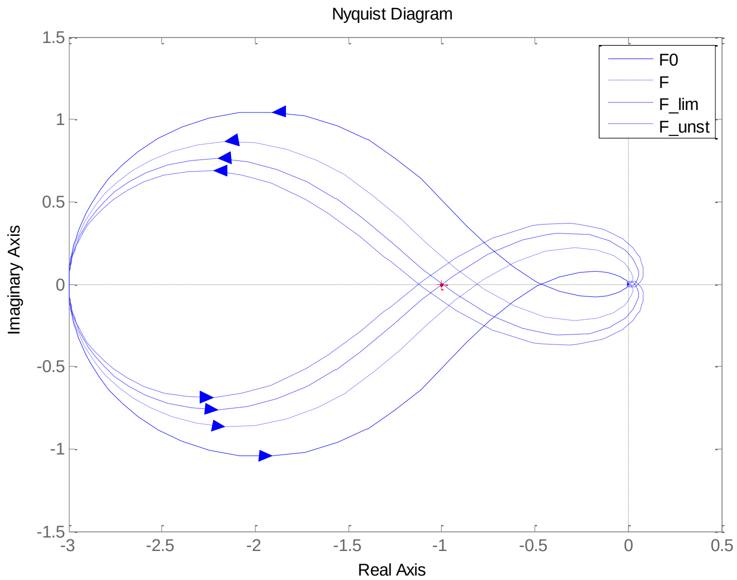
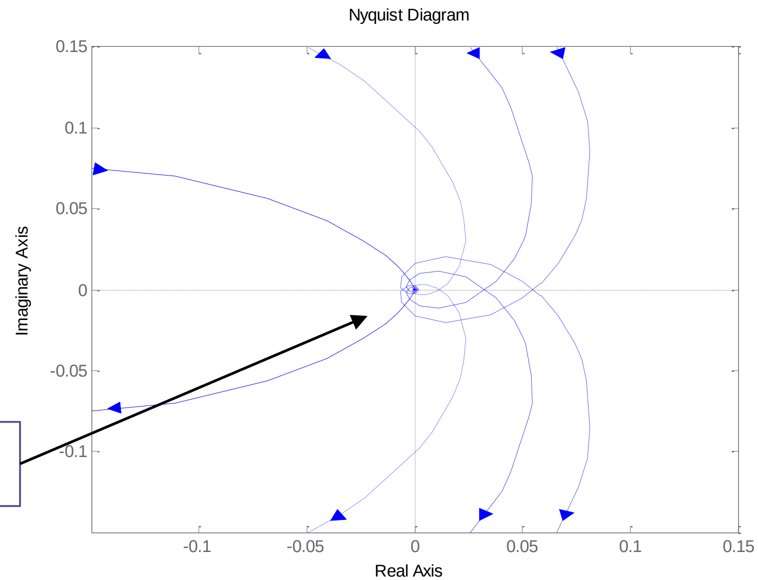




∞circles around 0 corresponding to the ∞ roots of the quasi-poliynomial $D_W(j\omega)$

Stability margins

- Gain margin
  - «Given a process $P(s)$ and a controller $G(s)$ such that the closed-loop system is stable, the gain margin is the maximum gain variation that the system can stand before becoming unstable»

- Phase margin
  - «Given a process $P(s)$ and a controller $G(s)$ such that the closed-loop system is stable, the phase margin is the maximum phase lag that the system can stand before becoming unstable»

Stability margins

- Example
  - $F(s)$ Open-loop transfer function with no LHP poles

  - Nyquist diagram (stable process)
  - Gain margin

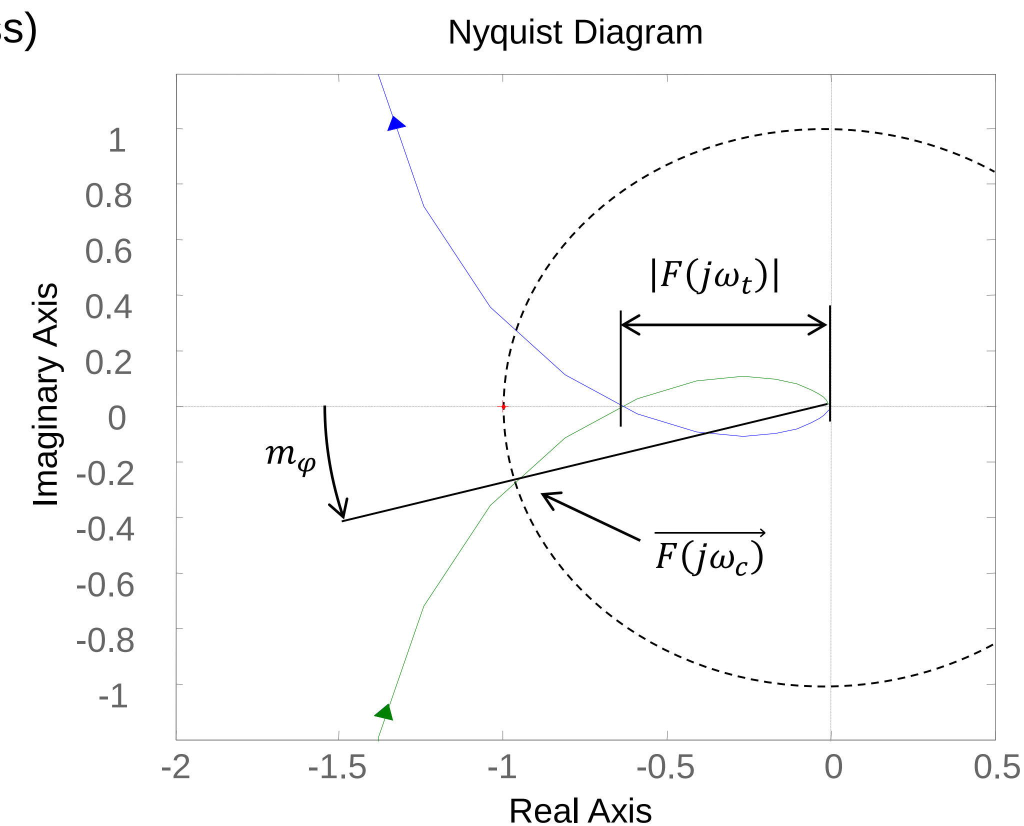
$m_g := \left|\frac{1}{F(j\omega_t)}\right|_{dB} = -20\log|F(j\omega_t)|$

with ω_t s.t. $\measuredangle\{F(j\omega_t)\} = -\pi$

  - Phase margin

$m_\varphi := \measuredangle\{F(j\omega_c)\} - (-\pi)$

with ω_c s.t. $|F(j\omega_c)| = 1$



Stability margins

- Example
  - $F(s)$ Open-loop transfer function with no LHP poles

    - Gain margin

      $m_g := \left|\frac{1}{F(j\omega_t)}\right|_{dB} = -20\log|F(j\omega_t)|$

      with ω_t s.t. $\measuredangle\{F(j\omega_t\} = -\pi$

    - Phase margin

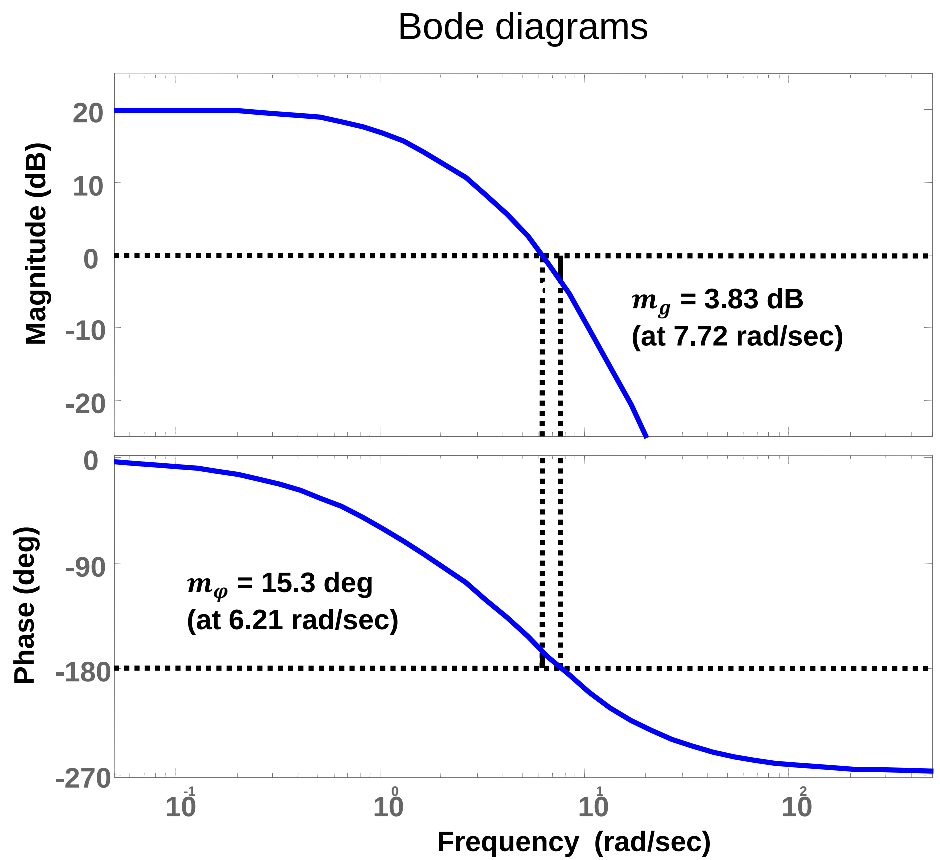
      $m_\varphi := \measuredangle\{F(j\omega_c\} - (-\pi)$

      with ω_c s.t. $|F(j\omega_c| = 1$

    - If there are multiple cut frequencies, one must consider the worst case:

      $m_\varphi := \min_{i=1,\dots,n} m_\varphi^i = \min_{i=1,\dots,n}\left(\measuredangle\{F(j\omega_c^i\} - (-\pi)\right)$

      with ω_c^i s.t. $\left|F(j\omega_c^i\right| = 1, i = 1,2,\dots,n$

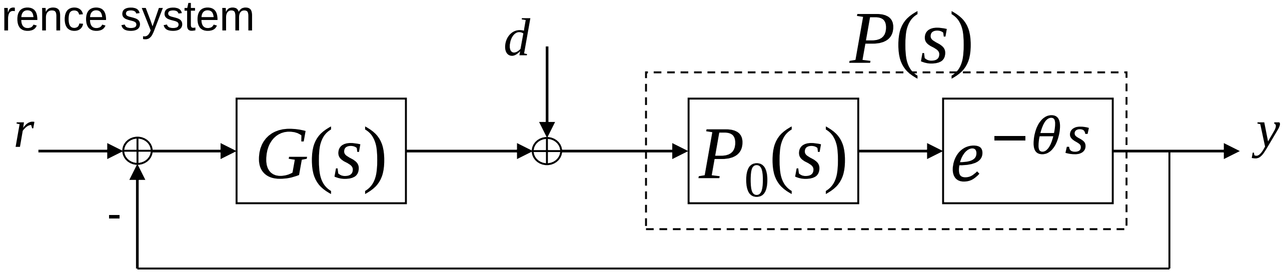


Stability margins

- Delay margin
  - «Given a process $P(s)$ and a controller $G(s)$ such that the closed-loop system is stable, the delay margin is the maximum delay variation that the system can stand before becoming unstable»

Stability margins

- Delay margin
  - Reference system



  - Let $G(s)$ be a controller such that the delay-free closed-loop transfer function is stable, i.e.:

$$G(s) \text{ such that } W_0(s) = \frac{F_0(s)}{1+F_0(s)} = \frac{P_0(s)G(s)}{1+P_0(s)G(s)} \text{ is stable}$$

$$\Rightarrow roots\{1 + P_0(s)G(s)\} \subset \mathbb{C}_{<0}$$

  - For simplicity, let the system have one cut frequency ω_c
  - Based on a continuity argument, it follows that it exists a delay θ' such that the closed-loop system $\bar{W}(s) = \frac{P_0(s)e^{-\bar{\theta}s}\,G(s)}{1+P_0(s)e^{-\bar{\theta}s}\,G(s)}$ is stable for all $\bar{\theta} \in [0, \theta')$, i.e.:

$$\Rightarrow roots\left\{1 + P_0(s)e^{-\bar{\theta}s}\,G(s)\right\} \subseteq \mathbb{C}_{<0}, \forall \bar{\theta} \in [0, \theta') \qquad (1)$$

Stability margins

- Delay margin (cont'd)
  - Let θ_{MAX} be the maximum value for θ', i.e., θ_{MAX} is the maximum value of the delay before the system becomes unstable
  - When the delay is θ_{MAX}, the closed-loop system is at the limit of stability, therefore it has at least one pole on the imaginary axis of the complex plane, i.e.:

$$\exists \bar{p} \in \left\{ roots\left\{1 + P_0(s)e^{-\theta_{MAX}s}\, G(s)\right\}\right\}\ s.t.\ \bar{p} = j\bar{\omega}$$

$$\Rightarrow\ 1 + P_0(j\bar{\omega})e^{-j\bar{\omega}\theta_{MAX}}\, G(j\bar{\omega}) = 0$$

$$\Rightarrow\ P_0(j\bar{\omega})e^{-j\bar{\omega}\theta_{MAX}}\, G(j\bar{\omega}) = -1$$

$$\Rightarrow \begin{cases} |P_0(j\bar{\omega})G(j\bar{\omega})| = 1 \\ \measuredangle\{P_0(j\bar{\omega})\, G(j\bar{\omega})\} - \bar{\omega}\theta_{MAX} = -\pi \end{cases}$$

$$\Rightarrow \begin{cases} |F_0(j\bar{\omega})| = 1 \\ \measuredangle\{F_0(j\bar{\omega})\} - \bar{\omega}\theta_{MAX} = -\pi \end{cases}$$

  - The cut frequency ω_c is, by definition, the pulsation such that $|F_0(j\omega_c)| = 1$
  - Therefore, the module condition entails that $\bar{\omega} = \omega_c$

Stability margins

- Delay margin (cont'd)
  - The phase condition becomes

$$\measuredangle\{F_0(j\omega_c)\} - \omega_c \theta_{MAX} = -\pi$$

$$\Rightarrow \omega_c \theta_{MAX} = \measuredangle\{F_0(j\omega_c)\} - (-\pi)$$

  - Recolling the definition of phase margin

$$m_\varphi := \measuredangle\{F_0(j\omega_c)\} - (-\pi)$$

  - We obtain

$$\theta_{MAX} = \frac{m_\varphi}{\omega_c} \qquad (2)$$

  - Remark
    - If the system has more cut freqiencies, one takes the worst case:

$$\theta_{MAX} = \min_{i=1,\ldots,n} \frac{m_\varphi^i}{\omega_c^i} \text{ with } \omega_c^i \text{ s.t. } \left|F(j\omega_c^i)\right| = 1, i = 1,2,\ldots,n$$

Stability margins

- Delay margin (cont'd)
  - Conclusions
    - From equation (1), from the definition of θ_{MAX} and from equation (2) it follows that the closed-loop system is stable if the delay is less than $\theta_{MAX} = \frac{m_\varphi}{\omega_c}$
    - The delay margin m_τ is then defined as

$$m_\tau := \frac{m_\varphi}{\omega_c} - \theta$$

  - Summarizing
    - If $G(s)$ stabilizes $P_0(s)$, then it stabilizes $P(s) = P_0(s)e^{-\theta s}$ iff $\theta < \frac{m_\varphi}{\omega_c}$

Polynomial approximation of the delay

- Padé approximation
  - $e^{-\theta s}$ is a non rational function
    - It cannot be expressed as the ratio between two polynomials
  - Maclaurin series expansion of $e^{-\theta s}$ (i.e., Taylor series evaluated at $s = 0$)

$$e^{-\theta s} = e^{-\theta s}|_{s=0} + (-\theta)e^{-\theta s}|_{s=0}s + \frac{(-\theta)^2}{2!}e^{-\theta s}|_{s=0}s^2 + \frac{(-\theta)^3}{3!}e^{-\theta s}|_{s=0}s^3 + \cdots =$$
$$= 1 - \theta s + \frac{\theta^2}{2!}s^2 - \frac{\theta^3}{3!}s^3 + \cdots$$

  - Padé approximations of the Maclaurin series
    - 1/1 Padé approximation
      - Numerator of order 1/denominator of order 1

$$e^{-\theta s} \cong \mathrm{G}_1(\mathrm{s}) = \frac{1-\frac{\theta}{2}s}{1+\frac{\theta}{2}s}$$

» $G_1(s)$ is an all-pass filter with a RHP zero

» $G_1(s)$ is correct up to the square term

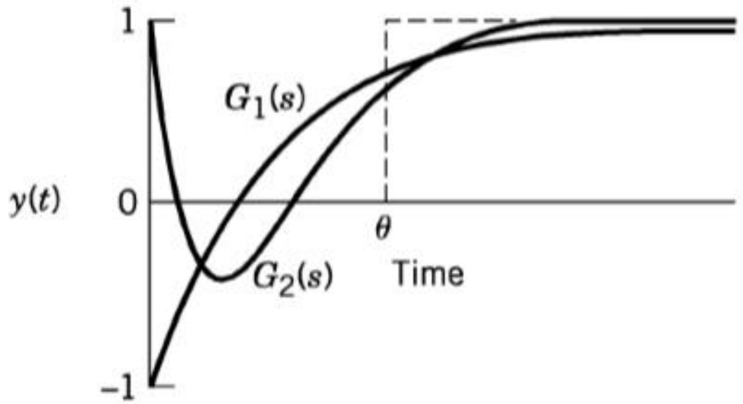
» $\mathrm{G}_1(\mathrm{s}) = \frac{1-\frac{\theta}{2}s}{1+\frac{\theta}{2}s} = 1 - \theta s + \frac{\theta^2}{2!}s^2 - \frac{\theta^3}{4}s^3 + \cdots$

Polynomial approximation of the delay

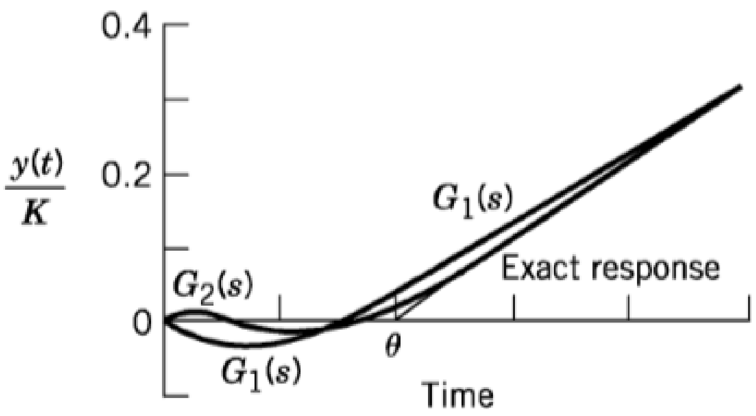
- Padé approximation
  - Padé approximations of the Maclaurin series (cont'd)
    - 2/2 Padé approximation
      - Numerator of order 2/denominator of order 2
      - $e^{-\theta s} \cong \mathrm{G}_2(\mathrm{s}) = \frac{1-\frac{\theta}{2}s+\frac{\theta^2}{12}s^2}{1+\frac{\theta}{2}s+\frac{\theta^2}{12}s^2}$
        - » $G_2(s)$ is an all-pass filter with complex conjugate poles
        - » $G_2(s)$ is correct up to the cube term

Polynomial approximation of the delay

- Padé approximation
  - Step response of a first-order system
    - $G_2(s)$ presents oscillations due to the complex poles



  - Ramp response of a first-order system
    - $P(s) = K\frac{1}{1+s\tau}e^{-\theta s}$, with $\theta \ll \tau$
    - Padé approximations works well in practice with first-order systems characterized by a slow dynamics with respect to the time-delay



Stabilization of a Time-Delay System with Padé approximation

- Actual time-delay process to control

$$P(s) = P_0(s)e^{-hs} \text{ with } h \gg \tau$$

1. Padé approximation of the delay $e^{-hs} \approx G_i(s)$

$$P(s) \approx P_0(s)G_i(s) = \tilde{P}(s)$$

2. Design a controller $C(s)$ which stabilises $\tilde{P}(s)$

3. Verify that $C(s)$ stabilises $P(s) = P_0(s)e^{-hs}$ by computing the delay margin of the delay-free controlled system with open-loop transfer function

$$F_0(s) = C(s)P_0(s)$$

i.e., by checking that

$$m_d := h_{MAX} - h = \frac{m_{\varphi_o}}{\omega_C} - h > 0$$

Example (cont'd)

- Trickle-bed reactor
  - Component balance

$$V\frac{dc}{dt} = qc_i + \alpha q c_2 - (1+\alpha)qc - Vkc$$

    - $-Vkc$ models the concentration decrease due to the reactant consume
  - Operating point

$$s = (\bar{c}, \bar{c}_i, \bar{c}_2)$$

  - Steady-state balance

$$V\left.\frac{dc}{dt}\right|_s = q\bar{c_i} + \alpha q\bar{c}_2 - (1+\alpha)q\bar{c} - Vk\bar{c} = 0$$

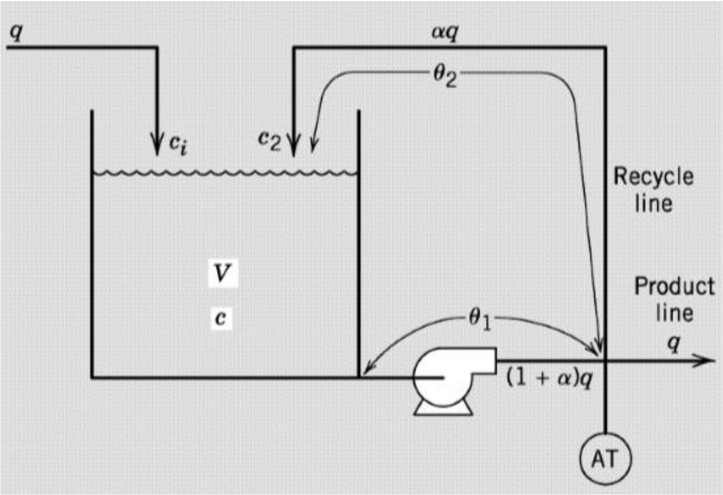
$$\Rightarrow \bar{c}_2 = \frac{\big((1+\alpha)q + Vk\big)\bar{c}}{\alpha q} - \frac{1}{\alpha}\bar{c_i}$$

  - Deviation variables

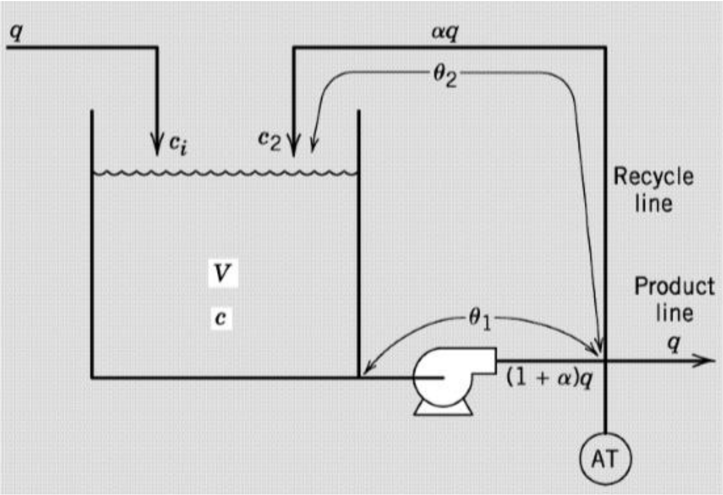
$$\begin{cases} c' = c - \bar{c} \\ c_i' = c_i - \bar{c_i} \\ c_2' = c_2 - \bar{c_2} \end{cases}$$

  - Dynamic balance

$$V\dot{c}'(t) = qc_i'(t) + \alpha q c_2'(t) - (1+\alpha)qc'(t) - Vkc'(t) \qquad (1)$$



Example (cont'd)



- Trickle-bed reactor
  - Exit and recycle lines

$$\begin{cases} c_1'(t) = c'(t-\theta_1) \\ c_2'(t) = c_1'(t-\theta_2) \end{cases} \Rightarrow \begin{cases} c_1'(t) = c'(t-\theta_1) \\ c_2'(t) = c'(t-\theta_3) \end{cases} \qquad (2)$$

with $\theta_3 = \theta_1 + \theta_2$

  - Overall dynamic equation (1)+(2)

$$V\dot{c}'(t) = qc_i'(t) + \alpha q c'(t-\theta_3) - (1+\alpha)qc'(t) - Vkc'(t) \qquad (3)$$

  - Transfer function
    - Select c_i' as input andc_1' as output
    - Laplace-transform of (3)

$$sVc'(s) = qc_i'(s) + \alpha q c'(s)e^{-s\theta_3} - (1+\alpha)qc'(s) - Vkc'(s)$$

$$\Rightarrow c'(s) = \frac{q}{sV - \alpha q e^{-s\theta_3} + (1+\alpha)q + Vk} c_i'(s)$$

$$\Rightarrow c'(s) = \frac{q}{sV + q + Vk + \alpha q\left(1 - e^{-s\theta_3}\right)} c_i'(s)$$

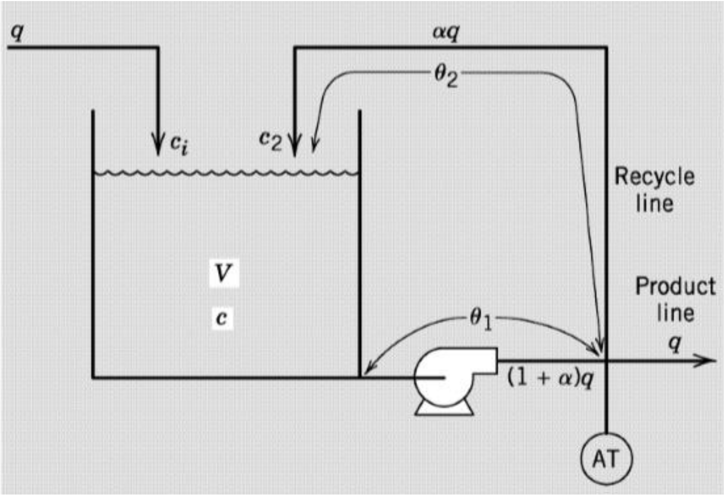
By defining $\tau \coloneqq \frac{V}{q+Vk}$ and $K \coloneqq \frac{q}{q+Vk}$, we obtain:

$$c'(s) = \frac{K}{1 + s\tau + \alpha K\left(1 - e^{-s\theta_3}\right)} c_i'(s) \qquad (4)$$

Example (cont'd)

- Trickle-bed reactor
  - Transfer function (cont'd)
    - From (2) and (4):

$$P(s) \coloneqq \frac{c_1'(s)}{c_i'(s)} = \frac{Ke^{-s\theta_1}}{1+s\tau+\alpha K\left(1-e^{-s\theta_3}\right)} \qquad (5)$$



  - Padé approximation
    - Assumption: $\theta_3 \ll \tau = \frac{V}{q+Vk}$
    - 1/1 Padé approximation of the feedback delay:

$$e^{-\theta_3 s} \cong \frac{1-\frac{\theta_3}{2}s}{1+\frac{\theta_3}{2}s} \qquad (6)$$

    - From (5)+(6) we obtain a second-order system cascaded by a delay:

$$P(s) \cong \frac{K}{1+s\tau+\alpha K\left(1-\frac{1-\frac{\theta_3}{2}s}{1+\frac{\theta_3}{2}s}\right)}e^{-s\theta_1} = \frac{K}{1+s\tau+\alpha K\left(\frac{\theta_3 s}{1+\frac{\theta_3}{2}s}\right)}e^{-s\theta_1} = \frac{K\left(1+\frac{\theta_3}{2}s\right)}{\left(1+\frac{\theta_3}{2}s\right)(1+s\tau)+\alpha K\theta_3 s}e^{-s\theta_1} =$$

$$= \frac{K(1+\tau_3 s)}{(1+\tau_1 s)(1+\tau_2 s)}e^{-s\theta_1} \qquad (7)$$

Summary

- Industrial process control must deal with delays
  - Transport delays
  - Reaction delays
  - …
- Delays affect the stability of the closed-loop system
  - The delay margin expresses the maximum delay tolerated by a system
- If the delay is ‘small’ with respect to the time constants of the system, it can be approximated by a transfer function (Padé polynomial approximation)