UNCONSTRAINED QP

$$\min_{\underline{u}} \; J(\underline{u}) \;, \qquad J(\underline{u}) = \underline{e}^T \underline{e}$$

$$\underline{e} = \underline{\hat{y}} - \underline{w} \;, \qquad \underline{\hat{y}} = G\underline{u} + \underline{f}$$

$\underline{u}$: vector of unknowns

G : constant matrix

$\underline{f}$: constant vector

$\underline{w}$: constant vector

$$J(\underline{u}) = (\underline{w} - G\underline{u} - \underline{f})^T (\underline{w} - G\underline{u} - \underline{f}) =$$

$$= \underline{w}^T\underline{w} - \underline{w}^T G\underline{u} - \underline{w}^T\underline{f} - \underline{u}^T G^T \underline{w} + \underline{u}^T G^T G \underline{u} + \underline{u}^T G^T \underline{f} +$$

$$- \underline{f}^T\underline{w} + \underline{f}^T G\underline{u} + \underline{f}^T\underline{f} = \underline{u}^T G^T G \underline{u} + 2(\underline{f}^T - \underline{w}^T) G\underline{u} + \underline{w}^T\underline{w} - 2\underline{w}^T\underline{f} + \underline{f}^T\underline{f}$$

$$\left.\frac{dJ(\underline{u})}{d\underline{u}}\right|_{\underline{u}^*} = 2G^TG\underline{u}^* - 2G^T(\underline{w} - \underline{f}) = 0$$

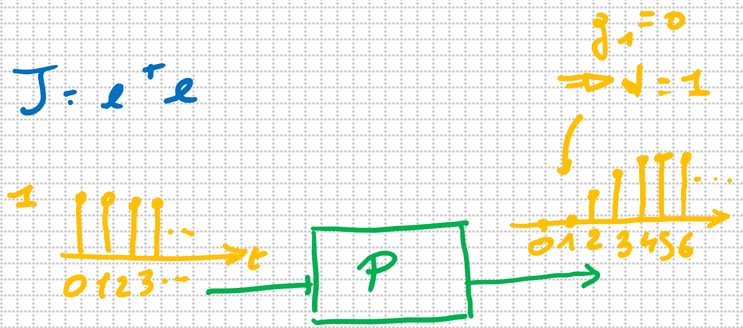
$$G^TG\underline{u}^* = G^T(\underline{w} - \underline{f}) \quad \Rightarrow \quad \boxed{\underline{u}^* = (G^TG)^{-1}G^T(\underline{w} - \underline{f})}$$

Exercise 1. STEP RESPONSE MODEL $g_1 = 0,\ g_2 = 0.25,\ g_3 = 0.375,\ g_4 = g_5 = \dots = 0.4375$

DMC $\quad u(2|2) = ?$

- $m = 2$, null init. cond., $w(t) = r(t) = 1,\ \forall t$, $\quad J = e^T e$

$y_m(1) = 0, \quad y_m(2) = 0$



- $g_{i+1} = g_i, \quad i \geq 4 \quad \Rightarrow \quad \boxed{N = 4}$
- $g_1 = 0 \quad \Rightarrow \quad \boxed{d = 0} \quad \Rightarrow \quad p \geq m + d = 3 \quad \Rightarrow \quad \boxed{p = 3}$

$$G = \begin{pmatrix} g_0 & 0 \\ g_1 & g_0 \\ g_2 & g_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.25 & 0 \\ 0.375 & 0.25 \end{pmatrix} \in \mathbb{R}^{p \times m} \quad \Rightarrow \quad (G^T G)^{-1} G^T = \begin{pmatrix} 0 & 4 & 0 \\ 0 & -6 & 4 \end{pmatrix}$$

$$u^* = (G^T G)^{-1} G^T (\underline{w} - \underline{f}) \quad ;$$

- $\underline{w}$: at time t: $\underline{w} = \begin{pmatrix} w(t+1|t) \\ w(t+2|t) \\ w(t+3|t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \forall t$ $\qquad w(t) = r(t) = 1, \forall t$

- $\underline{f}$: $f(t+k) = \underbrace{y_m(t)}_{\text{last output measure}} + \sum_{i=1,\dots,4} (g_{k+i} - g_i) \underbrace{\Delta u(t-i)}_{\text{past control actions}}, \; k = 1,2,3 \leftarrow \boxed{\begin{matrix} N=4 \\ p=3 \end{matrix}}$

$y_m(1) = 0$

$\Delta u(1-i) = 0$

$g_{i+1} = g_4 \quad i \geq 4$

$t = 1$

$k=1 \quad f(2) = y_m(1) + (g_2 - g_1)\Delta u(t-1) + (g_3 - g_2)\Delta u(t-2) + (g_4 - g_3)\Delta u(t-3) + (g_5 - g_4)\Delta u(t-4) = 0$

$k=2 \quad f(3) = \dots = 0$

$k=3 \quad f(4) = \dots = 0$

$m = 2$

$\underline{u}^* = \begin{bmatrix} \Delta u(t|t) \\ \Delta u(t+1|t) \end{bmatrix} \overset{t=1}{=} \begin{bmatrix} \Delta u(1|1) \\ \Delta u(2|1) \end{bmatrix} = (G^T G)^{-1} G^T (\underline{w} - \underline{f}) = \begin{bmatrix} \boxed{0 \;\; 4 \;\; 0} \\ 0 \; -6 \;\; 4 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) =$

$K = [0 \;\; 4 \;\; 0]$

$= \begin{bmatrix} 4 \\ -2 \end{bmatrix} \Rightarrow \Delta u(1|1) = 4 \Rightarrow \boxed{u(1) = u(0) + \Delta u(1|1) = 4}$

$u(0) = 0$

$\Delta u(1|1) = K(\underline{w} - \underline{f}) =$

$= [0 \;\; 4 \;\; 0] \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) =$

$= 4$

$t=2$

$y_m(2)=0$

$\Delta u(t)=0$

$t\leq 0$

$\Delta u(1)=4$

$$k=1 \quad f(3) = \cancel{y_m(2)} + (g_2-g_1)\Delta u(1) + (g_3-g_2)\cancel{\Delta u(0)} + (g_4-g_3)\cancel{\Delta u(-1)} + (g_5-g_4)\cancel{\Delta u(-2)} = (0.25-0)4 = 1$$

$$k=2 \quad f(4) = \cancel{y_m(2)} + (g_3-g_1)\Delta u(1) + (g_4-g_2)\cancel{\Delta u(0)} + (g_5-g_3)\cancel{\Delta u(-1)} + (g_6-g_4)\cancel{\Delta u(-2)} = (0.375-0)4 = 1.5$$

$$k=3 \quad f(5) = (g_4-g_1)\Delta u(1) = (0.4375-0)4 = 1.75$$

$$\Delta u(2|2) = K(\underline{w}-\underline{f}) = \begin{bmatrix}0 & 4 & 0\end{bmatrix}\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1.5\\1.75\end{bmatrix}\right) = -2$$

$$\boxed{u^*(2) = u(1) + \Delta u(2|2) = 4-2 = 2}$$

Another option: $\underline{f} = F\underline{x}$

$$F = \begin{bmatrix}1 & g_2-g_1 & g_3-g_2 & g_4-g_3\\ 1 & g_3-g_1 & g_4-g_2 & g_5-g_3\\ 1 & g_4-g_1 & g_5-g_2 & g_6-g_3\end{bmatrix} \qquad \underline{x} = \begin{bmatrix}y_m(t)\\ \Delta u(t-1)\\ \Delta u(t-2)\\ \Delta u(t-3)\end{bmatrix}$$

EXERCISE 2 (MAC)

Impulse response model $h_1 = 0.1$, $h_2 = 0.04$, $h_3 = 0.016$, $h_4 = 0.006$, $h_i \simeq 0$ $i=5,\dots$

a) $h_1 \neq 0$ $\Rightarrow$ $d = 0$

b) Process is stable without integrators

a)+b) MAC $\Leftarrow$

$u(5) = ?$

- $m = 3$ $\Rightarrow$ $p = m$ (MAC algorithm)
- $y_m(5) = 1.8$
- $N = 4$ $(h_i \simeq 0,\ i \geq 5)$
- $u(1) = 5$, $u(2) = 4$, $u(3) = 3$, $u(4) = 2.5$
- $r(t) = 2$, $t \geq 0$
- $J = \underline{e}^T \underline{e}$
- $\alpha = 0.5$ for the computation of the predicted trajectory

MAC:

$$w(t+k|t) = \begin{cases} y_m(t), & k=0 \\ \alpha\, w(t+k-1|t) + \\ \quad + (1-\alpha)\, r(t+k) \end{cases}$$

MAC:

$$\underline{y} = \underbrace{H_1 \underline{u}_+}_{\text{FORCED RESP.}} + \underbrace{H_2 \underline{u}_- + \underline{n}}_{\text{FREE RESPONSE}}$$

$$\underline{u}_+ = \begin{bmatrix} u(t|t) \\ u(t+1|t) \\ u(t+2|t) \end{bmatrix}$$

$$H_1 = \begin{bmatrix} h_1 & 0 & 0 \\ h_2 & h_1 & 0 \\ h_3 & h_2 & h_1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.64 & 0.1 & 0 \\ 0.016 & 0.04 & 0.1 \end{bmatrix}$$

$$\underline{u}_- = \begin{bmatrix} u(t-3) \\ u(t-2) \\ u(t-1) \end{bmatrix}$$

$$H_2 = \begin{bmatrix} h_4 & h_3 & h_2 \\ 0 & h_4 & h_3 \\ 0 & 0 & h_4 \end{bmatrix} = \begin{bmatrix} 0.006 & 0.016 & 0.04 \\ 0 & 0.006 & 0.016 \\ 0 & 0 & 0.006 \end{bmatrix}$$

$$\underline{n} = \begin{bmatrix} \hat{n}(t+1|t) \\ \hat{n}(t+2|t) \\ \hat{n}(t+3|t) \end{bmatrix} = \begin{bmatrix} n(t|t) \\ n(t|t) \\ n(t|t) \end{bmatrix}$$

MAC: $\hat{n}(t+k|t) = n(t|t)$

MAC: $n(t|t) = y_m(t) - \sum_{j=1}^{N} h_j u(t-j)$

$$J = \underline{e}^T \underline{e} \Rightarrow \underline{u}^* = H_1^{-1} (\underline{w} - \underline{f}) \quad , \quad \underline{f} = H_2 \underline{u}^- + \underline{n}$$

$$\underline{w}(t+k) = \begin{cases} y_m \quad , & k = 0 \\ 0.5 \; w(t+k-1) + 0.5 \; r(t+k), & k > 0 \end{cases}$$

$$H_1^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.04 & 0.1 & 0 \\ 0.016 & 0.04 & 0.1 \end{bmatrix}^{-1} = \begin{bmatrix} \boxed{10 \quad 0 \quad 0} \\ -4 \quad 10 \quad 0 \\ 0 \quad -4 \quad 10 \end{bmatrix} \; = K$$

$r(t) = 2$

$y_m(5) = 1.8$

$\underline{t = 5}$ $\quad k = 1$

$$w(6) = 0.5 \; y_m(5) + 0.5 \; r(6) = 2.88$$
$$w(7) = 0.5 \; w(6) + 0.5 \; r(7) = 2.44$$
$$w(8) = 0.5 \; w(7) + 0.5 \; r(8) = 2.22$$

$$\underline{n} = \begin{pmatrix} n(5) \\ n(5) \\ n(5) \end{pmatrix}, \quad n(5) = y_m(5) - \sum_{j=1}^{4} h_j \cdot u(5-j) =$$

$$= 1.8 - h_1 u(4) - h_2 u(3) - h_3 u(2) - h_2 u(1) =$$

$$= 1.34$$

$$H_2 \underline{u} = \begin{bmatrix} 0.006 & 0.016 & 0.4 \\ 0 & 0.006 & 0.016 \\ 0 & 0 & 0.006 \end{bmatrix} \begin{bmatrix} u(2) \\ u(3) \\ u(4) \end{bmatrix} = \begin{bmatrix} 3.86 \\ 0.058 \\ 0.015 \end{bmatrix}$$

$$\underline{f} = H_2 \underline{u} + \underline{m} = \begin{bmatrix} 3.86 \\ 0.058 \\ 0.015 \end{bmatrix} + \begin{bmatrix} 1.34 \\ 1.34 \\ 1.34 \end{bmatrix} = \begin{bmatrix} 5 \\ 1.40 \\ 1.35 \end{bmatrix}$$

$$\underline{u} = H_1^{-1} (\underline{w} - \underline{f})$$

$$u(515) = K(\underline{w} - \underline{f}) = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 2.88 \\ 2.44 \\ 2.22 \end{bmatrix} - \begin{bmatrix} 5 \\ 1.40 \\ 1.35 \end{bmatrix} \right) =$$

$$= 10 (\underbrace{2.88}_{w(6)} - \underbrace{5}_{f(6)}) = -21.2$$

What if $\lambda = 0.1$ $\qquad J = \underline{e}^T \underline{e} + \lambda \underline{u}^T \underline{u}$

* check what happens to $u(5)$

$\left(u^* = (H_1^T H_1 + \lambda I)^{-1} H_1^T (\underline{w} - \underline{1}) \right)$

EXERCISE 3

$$\begin{cases} x(t) = M\,x(t-1) + N\,u(t-1) \\ y(t) = Q\,x(t) \end{cases} \qquad M = 0.5\ ;\ N = 0.1 \qquad Q = 1$$

PFC : $u(2) = ?$

- $m = 3$
- $p = 3$
- $h_1 = 1$, $h_2 = 3$
- $x(0) = y(0) = 0$, $y_m(0) = 0$, $u(0) = 0.1$
- $r(t) = 1$, $t \geq 0$, $w(t) = r(t)$
- $J = \underline{e}^T \underline{e}$
- $y_m(1) = 0.2$, $y_m(2) = 0.74$
- $\hat{n}(t+k|t) = \hat{n}(t|t) = y_m(t) - y(t)$, $k = 1, 2, 3$.

Base function response

$$u(t+k|t) = \sum_{i=1}^{n_B} \mu_i(t) B_i(k)$$

$$n_B = 1$$

$$B_1(k) = 1, \quad k \geq 0$$

$$y_{B_i}(k) = Q M^{k-1} N B_i(0) + Q M^{k-2} N B_i(1) + \dots + Q N B_i(k-1)$$

$$x_{B_1}(0) = 0$$

$Q = 1, \; M = 0.5$
$N = 0.1$

$B_i(k) = 1$

$k = h_1 = 1 \qquad y_{B_1}(1) = Q N B_i(0) = 0.1$

$k = h_2 = 3 \qquad y_{B_1}(3) = Q M^2 N B_i(2) + Q M N B_1(1) + Q N B_i(0) = 0.175$

$$\underline{y}_B(h_1) = y_{B_1}(1) = 0.1$$

$$\underline{y}_B(h_2) = y_{B_1}(3) = 0.175$$

$$Y_B = \begin{bmatrix} \underline{y}_B(h_1) \\ \underline{y}_B(h_2) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.175 \end{bmatrix}$$

- $(Y_B{}^T Y_B)^{-1} Y_B{}^T = [\,2.46 \quad 4.31\,] \Rightarrow \underline{\mu}^* = (Y_B{}^T Y_B)^{-1} Y_B{}^T \underline{d}$

$w(t) = 1(t) = 1$

- $\underline{d} = \underline{w} - \underline{f}$

- $\underline{w} = \begin{bmatrix} \hat{w}(t+h_1|t) \\ \hat{w}(t+h_2|t) \end{bmatrix} = \begin{bmatrix} \hat{w}(t+1|t) \\ \hat{w}(t+3|t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \geq 0$

- $f(t+k|t) = QM^k x(t) + \hat{n}(t+k|t)$

$t = 1, \quad k = h_1 = 1 \qquad f(2|1) = QM\,x(1) + [\,y_m(1) - y(1)\,] = 0.01$

$y_m(1) = 0.2$

$x(1) = M\,\cancel{x(0)}^{\,0} + N\,u(0) = 0.01$

$u(0) = 0.1$

$y(1) = Q\,x(1) = 0.01$

$\underline{t=1}$ $\quad k = h_2 = 3 \quad f(4|1) = QM^3 x(1) + (y_m(1) - y(1)) = 0.03$

$$\underline{d}(1) = (\underline{w} - \underline{f}) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.01 \\ 0.03 \end{bmatrix} \right) = \begin{bmatrix} 0.99 \\ 0.97 \end{bmatrix}$$

$$\mu(1) = (Y_B^T Y_B)^{-1} Y_B^T \underline{d} = [2.46 \quad 4.31] \begin{bmatrix} 0.99 \\ 0.97 \end{bmatrix} = 6.62 \qquad \leftarrow n_B = 1$$

$$u(1) = \mu_1(1) B_1(0) = 6.62$$

$y_m(2) = 0.74$

$\underline{t=2}$ $\quad k = h_1 = 1 \quad f(3|2) = QM\, x(2) + (y_m(2) - y(2)) = 0.07$

$$x(2) = M x(1) + N u(1) = 0.67$$

$$y(2) = Q x(2) = 0.67$$

$k = h_2 = 3 \quad f(5|2) = QM^3 x(2) + (y_m(2) - y(2)) = 0.21$

$$\underline{d}(2) = \underline{w} - \underline{f} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.07 \\ 0.21 \end{bmatrix} \right) = \begin{bmatrix} 0.93 \\ 0.79 \end{bmatrix}$$

$$\mu^*(2) = (Y_B^T Y_B)^{-1} Y_B^T \underline{d} = [2.47 \quad 4.31] \begin{bmatrix} 0.93 \\ 0.79 \end{bmatrix} = 5.9$$

$$\boxed{\mu(2) = \mu^*(2) B_1(0) = 5.9}$$