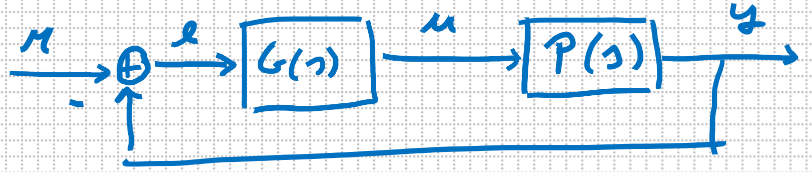
ROBUST STABILITY

PARAMETRIC UNCERTAINTY

SISO LTI



- $$P(s) = K_p \frac{N_p(s)}{D_p(s)}, \quad K_p \in [a, b]$$

  PARAMETRIC PROCESS, K_p

- Set of possible transfer functions

$$\mathcal{P}_p = \left\{ P(s) = K_p \frac{N_p(s)}{D_p(s)} \ \text{s.t.} \ K_p \in [a, b] \right\}$$

PARAMETRIC

(NOMINAL PROCESS

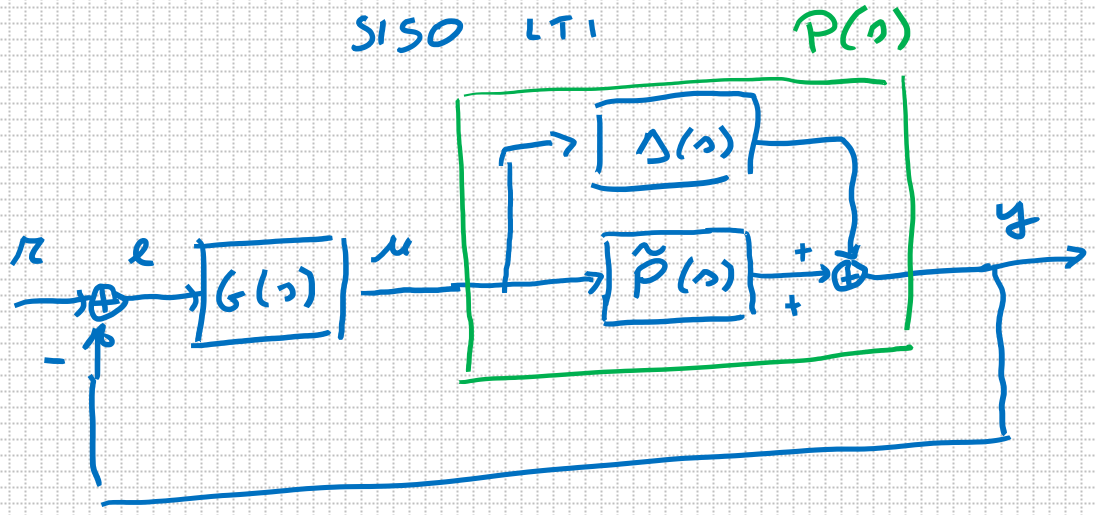
$$\tilde{P}(s) = \tilde{K}_p \frac{N_p(s)}{D_p(s)},$$

$\tilde{K}_p \in [a, b]$)

ROBUST CONTROL PROBLEM: find a controller $G(s)$ such that the closed-loop system is stabilized for all $P(s) \in \mathcal{P}_p$

ADDITIVE UNCERTAINTY

SISO LTI



$$P(s) = \tilde{P}(s) + \Delta(s)$$

$\tilde{P}(s)$: NOMINAL PROCESS MODEL

$\Delta(s)$: ADDITIVE UNCERTAINTY

- UPPERBOUND FOR $\Delta(s)$

$w(s)$ such that $|w(j\omega)| > |\Delta(j\omega)|, \ \forall \omega$

- SET OF ALL POSSIBLE TRANSFER FUNCTIONS

$$\mathcal{P}_w = \{ P(s) = \tilde{P}(s) + \Delta(s) \ | \ |\Delta(j\omega)| < |w(j\omega)|, \ \forall \omega \}$$

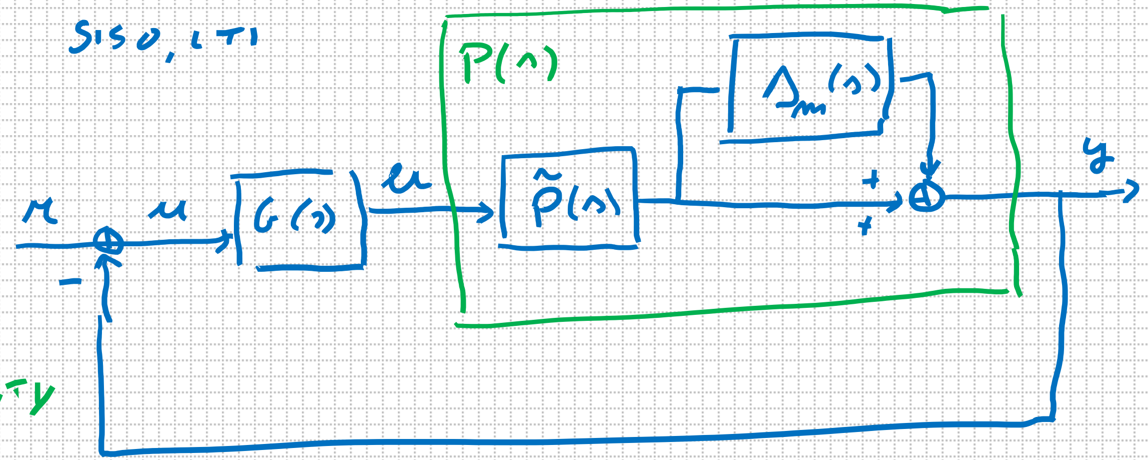
ROBUST CONTROL PROBLEM

Find $G(s)$ s.t. $P(s)$ is stabilized for all $P(s) \in \mathcal{P}_w$

MULTIPLICATIVE UNCERTAINTY

SISO, LTI

$$P(s) = \tilde{P}(s)\left(1 + \Delta_m(s)\right)$$



$\tilde{P}(s)$ NOMINAL PROCESS MODEL

$\Delta_m(s)$ MULTIPLICATIVE UNCERTAINTY

UPPERBOUND $\quad l_m(s)$ s.t. $\quad |l_m(j\omega)| > |\Delta_m(j\omega)|, \quad \forall \omega$

SET OF ALL POSSIBLE T.F.

$$\mathcal{P}_m = \left\{ P(s) = \tilde{P}(s) + (1 + \Delta_m(s)) \;\middle|\; |l_m(j\omega)| > |\Delta_m(j\omega)|, \forall \omega \right\}$$

ROBUST CONTROL PROBLEM $\quad$ Find $G(s)$ s.t. $P(s)$ is stabilized $\forall P(s) \in \mathcal{P}_m$

ROBUST STABILITY THEOREM

LEMMA. Necessary condition for the robust control problem (RPC) is that $G(s)$ stabilizes $\tilde{P}(s)$

PROOF. • $G(s)$ solves the RCP if it stabilizes all the processes $P(s) \in \mathcal{P}_w(\mathcal{P}_p, \mathcal{P}_m)$

• $\tilde{P}(s) \in \mathcal{P}_w(\mathcal{P}_p, \mathcal{P}_m)$

ROBUST STABILITY THEOREM

- SISO, LTI
- ADDITIVE UNCERT.

Under the assumptions

1) $G(s)$ stabilizes $\tilde{P}(s)$

2) $P(s)$ and $\tilde{P}(s)$ have the same number of RHP poles $\left(n_P^+ = n_{\tilde{P}}^+ \right)$

sufficient condition for robust stability is

$$\| w(j\omega)\, G(j\omega)\, \tilde{S}(j\omega) \|_\infty \leq 1$$

- $\tilde{S}(s)$ NOMINAL SENSITIVITY FUNCTION (computed for $\tilde{P}(s)$)

- $\| P(j\omega) \|_\infty = \sup_\omega |P(j\omega)|$

$$\| P(j\omega) \|_\infty \leq 1 \iff |P(j\omega)| \leq 1 \quad \forall \omega \in [0, +\infty)$$

Proof.

- $F(s) = G(s)\,P(s)$, $\tilde{F}(s) = G(s)\,\tilde{P}(s)$
- $m^+_P = m^+_{\tilde{P}} \Rightarrow \boxed{m^+_F = m^+_{\tilde{F}}}$ ASSUMPTION 2
- The actual process is stabilized by $G(s)$ if $m^+_{CL} = 0$

Nyquist theorem

$$\overset{\curvearrowright}{N}^{-1}_F = m^+_{CL} - m^+_F \Rightarrow \boxed{\overset{\curvearrowleft}{N}^{-1}_F = m^+_F}$$

The nominal process is stabilized by $G(s)$ if $m^+_{CL} = 0$

$$\overset{\curvearrowright}{N}^{-1}_{\tilde{F}} = m^+_{CL} - m^+_{\tilde{F}} \Rightarrow \boxed{\overset{\curvearrowleft}{N}^{-1}_{\tilde{F}} = m^+_{\tilde{F}}}$$

← TRUE FOR ASSUMPTION 1

$\Rightarrow$ $G(s)$ stabilizes both $P(s)$ and $\tilde{P}(s)$ if $\boxed{\overset{\curvearrowleft}{N}^{-1}_F = \overset{\curvearrowleft}{N}^{-1}_{\tilde{F}}}$

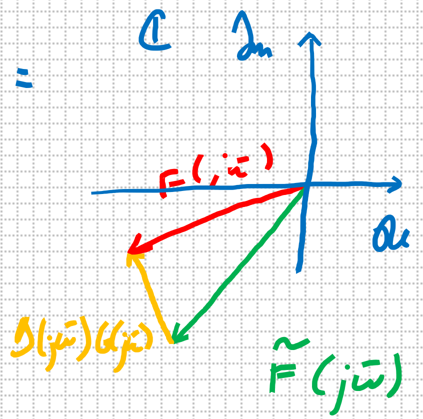
since, from ASSUMPTION 1, $G(s)$ stabilizes the nominal process

↑ STILL TO BE CHECKED

- $P(j\omega) = \tilde{P}(j\omega) + \Delta(j\omega)$ ← additive uncertainty

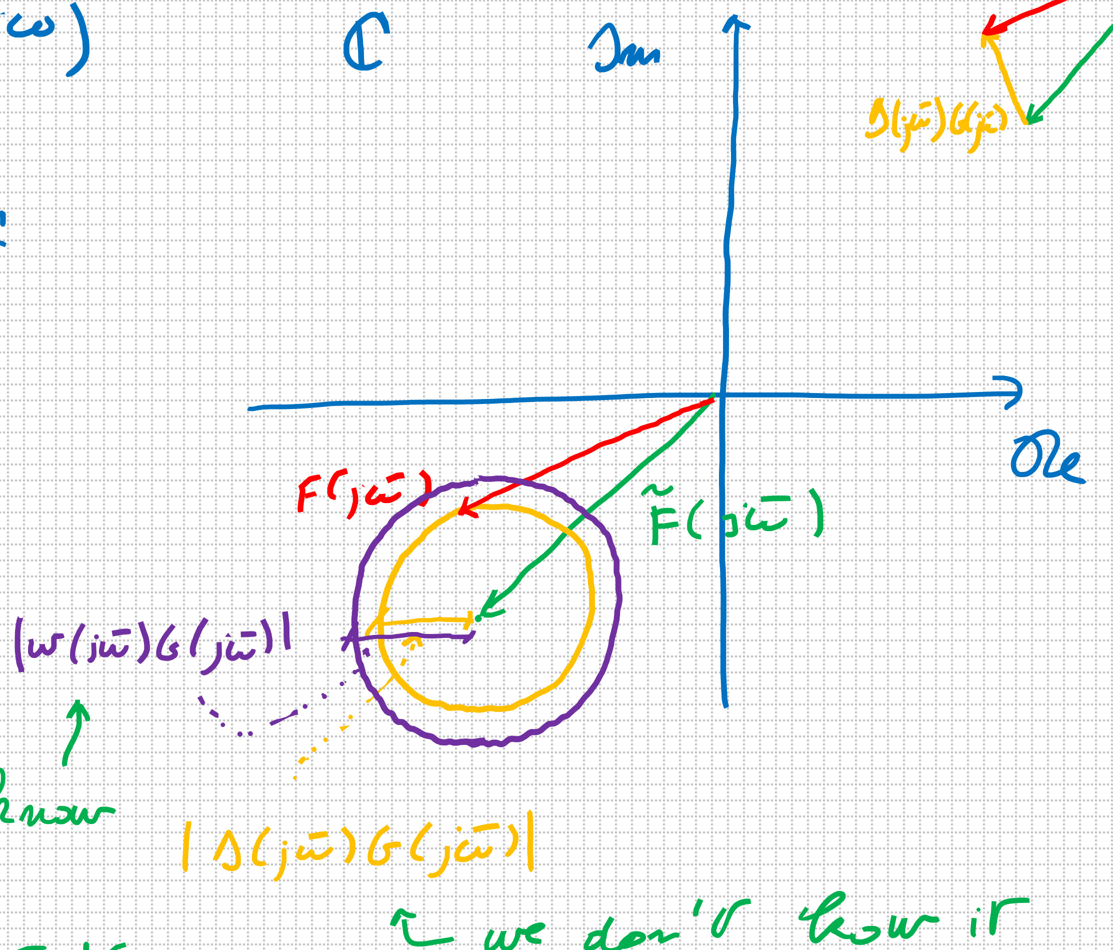
$$F(j\omega) = P(j\omega)\,G(j\omega) = \tilde{P}(j\omega)\,G(j\omega) + \Delta(j\omega)\,G(j\omega) =$$

$$= \tilde{F}(j\omega) + \Delta(j\omega)\,G(j\omega)$$



- For a given $\bar\omega$, $F(j\bar\omega)$ lies <u>on</u> a circle centered in $\tilde{F}(j\bar\omega)$ with radius $|\Delta(j\bar\omega)\,G(j\bar\omega)|$, $\forall\bar\omega$

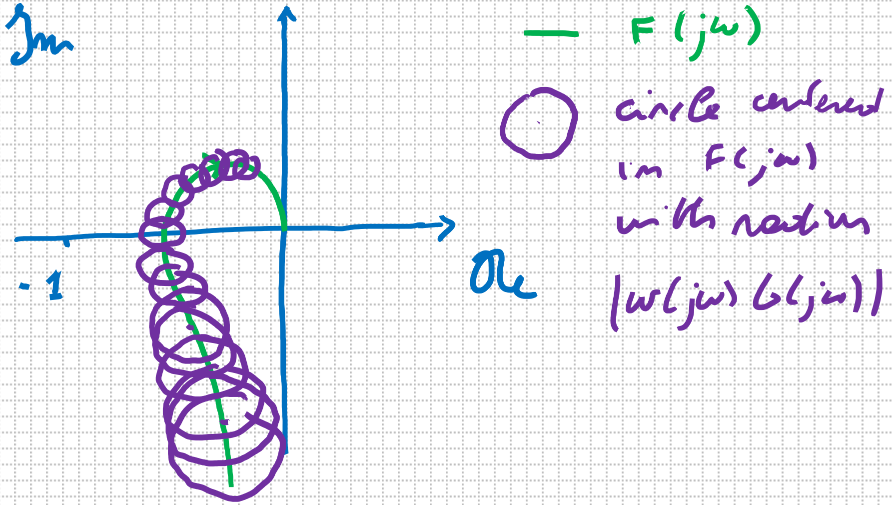
- $w(s)$ s.t. $|w(j\omega)| > |\Delta(j\omega)|, \forall\omega$



⇒ For $\bar\omega$, $F(j\bar\omega)$ lies <u>in</u> a circle centered in $\tilde{F}(j\bar\omega)$ with radius $|w(j\bar\omega)\,G(j\bar\omega)|$ ← CONSERVATIVE

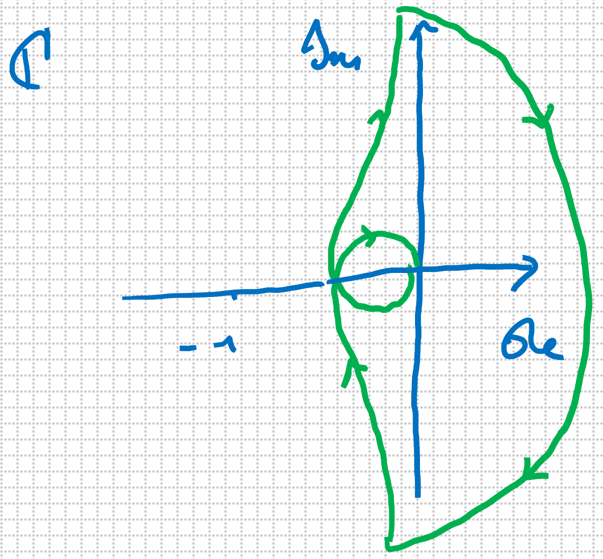
ALSO BECAUSE WE DON'T CONSIDER THE PHASE

- $F(j\omega)$ lies inside a "tube" of variable radius $|w(j\bar{\omega})G(j\bar{\omega})|$ around the Nyquist diagram of $\hat{F}(j\omega)$



- Since $G(s)$ stabilizes $\tilde{P}(s)$ and $m_F^+ = m_{\tilde{F}}^+$, sufficient condition for $G(s)$ to stabilize $P(s)$ is that -1 lies outside the tube

Example

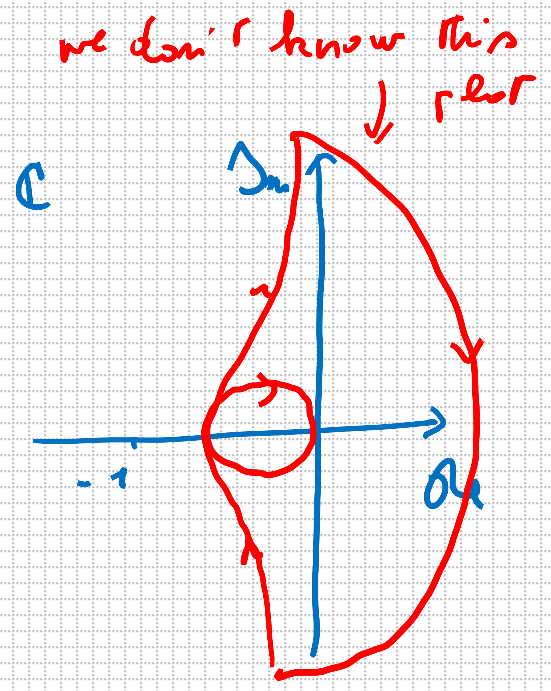


$$\tilde{F}(j\omega) = \hat{P}(j\omega)\, G(j\omega)$$

$$m^+_{\tilde{F}} = 0$$

$$\overset{\curvearrowright}{N}{}^{-1}_{\tilde{F}} = 0$$

$$\Rightarrow \quad m^+_{CL} = 0$$

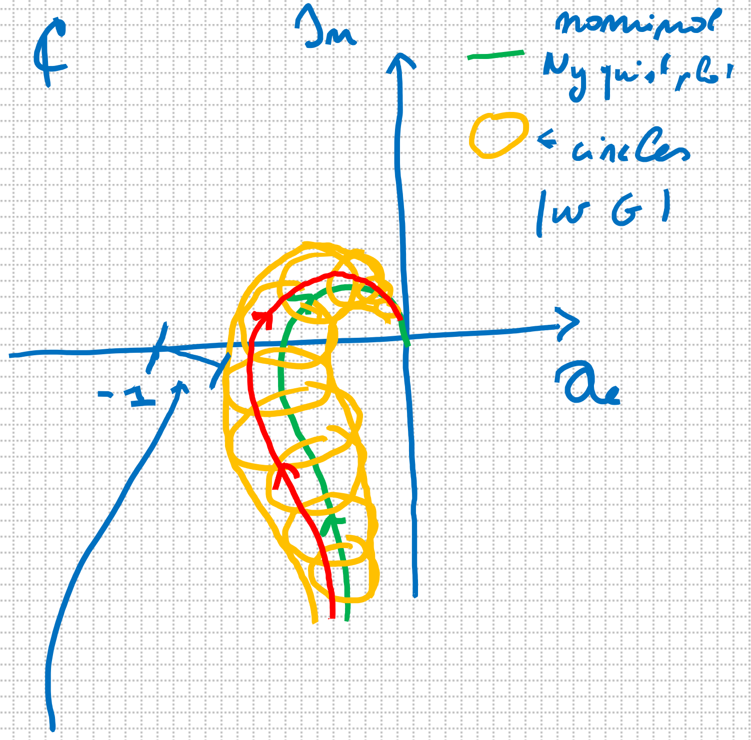


$$F(j\omega) = P(j\omega)\, G(j\omega)$$

$m^+_F = 0$ (ASSUMP. 2)

$\Rightarrow$ we need $\overset{\curvearrowright}{N}{}^{-1}_{F} = 0$

to have $m^+_{CL} = 0$



The distance between -1 and the tube > 0 for $F(j\omega)$ to be stabilized by $G(j\omega)$

WE HAVE TO CHECK THAT THE DISTANCE BETWEEN -1 AND $F(j\omega)$ IS POSITIVE $\forall\omega$

$$d(F(j\omega),-1) = |F(j\omega)-(-1)| = |\tilde{F}(j\omega)+\Delta(j\omega)G(j\omega)+1| > 0, \quad \forall\omega \qquad (1)$$

- $$|\tilde{F}(j\omega)+1+\Delta(j\omega)G(j\omega)| > |1+\tilde{F}(j\omega)| - |\Delta(j\omega)G(j\omega)|, \quad \forall\omega$$

- CONSERVATIVE
$$\boxed{w(j\omega) > \Delta(j\omega), \forall\omega} \Rightarrow |1+\tilde{F}(j\omega)|-|\Delta(j\omega)G(j\omega)| > |1+\tilde{F}(j\omega)|-|w(j\omega)G(j\omega)|, \quad \forall\omega \qquad (2)$$

$$(1)+(2) \quad d(F(j\omega),-1) > |1+\tilde{F}(j\omega)| - |w(j\omega)G(j\omega)| > 0$$

we want that

$$\Leftrightarrow |w(j\omega)G(j\omega)| < |1+\tilde{F}(j\omega)|, \quad \forall\omega$$

$$|w(j\omega)G(j\omega)|\,|1+\tilde{F}(j\omega)|^{-1} < 1 \quad \forall\omega$$

$$\boxed{|w(j\omega)G(j\omega)\hat{S}(j\omega)| < 1, \quad \forall\omega}$$

$$\hat{S}(j\omega) = \frac{1}{1+\tilde{F}(j\omega)} = \frac{1}{1+\check{P}(j\omega)G(j\omega)}$$

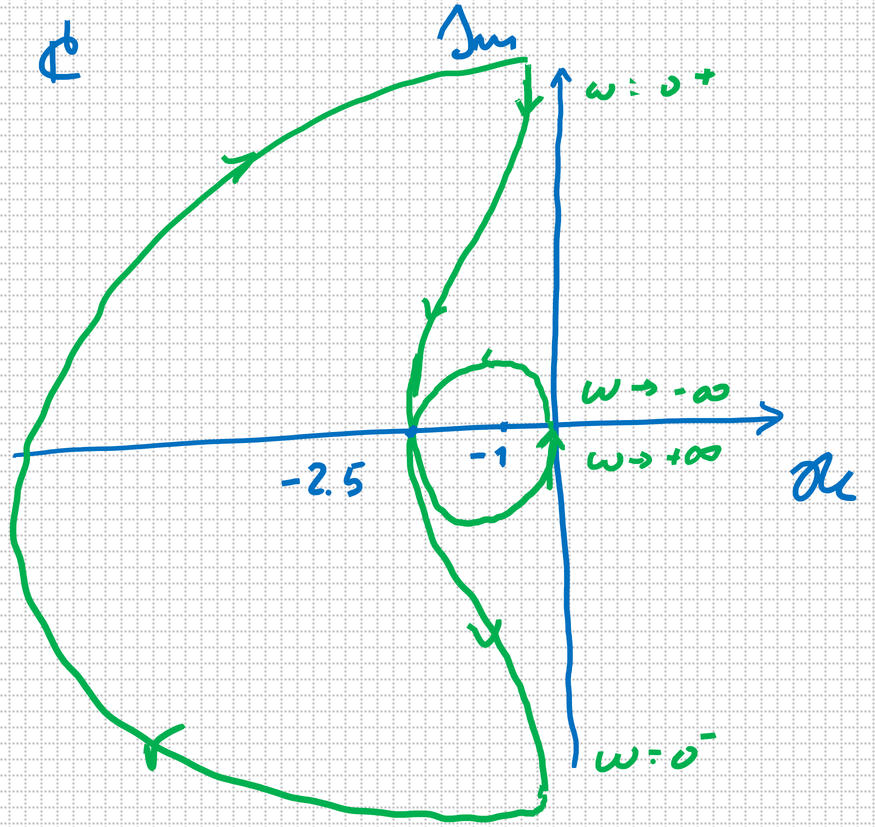
EXAMPLE

$$\begin{cases} P(s) = K_p \dfrac{s+1}{s(s-1)} \\ K_p \in [1.5, 2.5] \\ \tilde{K}_p = 2.5 \end{cases}$$

$$\Rightarrow \tilde{P}(s) = 2.5 \frac{s+1}{s(s-1)}$$

$$G(s) = 1$$

$$\tilde{F}(s) = 2.5 \frac{s+1}{s(s-1)}$$



Nyquist plot of

$$\tilde{F}(j\omega) = -2.5 \frac{1+j\omega}{j\omega(1-j\omega)}$$

ASSUMPTION 1

NECESSARY CONDITION

$G(s)$ stabilizes $\tilde{P}(s)$

$$D_{\tilde{W}}(s) = N_{\tilde{F}}(s) + D_{\tilde{F}}(s) = s^2 + 1.5s + 2.5 \quad ✓$$

all coeff. are positive $\Rightarrow$ roots$\{D_{\tilde{W}}(s)\} \subset \mathbb{C}_{<0}$

$$\vec{N}_{\tilde{F}}^{-1} = n_{CL}^+ - n_{\tilde{F}}^+ \quad ✓$$

$$-1 = n_{CL}^+ - 1$$

ASSUMPTION 2 $\quad m^+_F = m^+_{\tilde{F}} = 1$ ✓

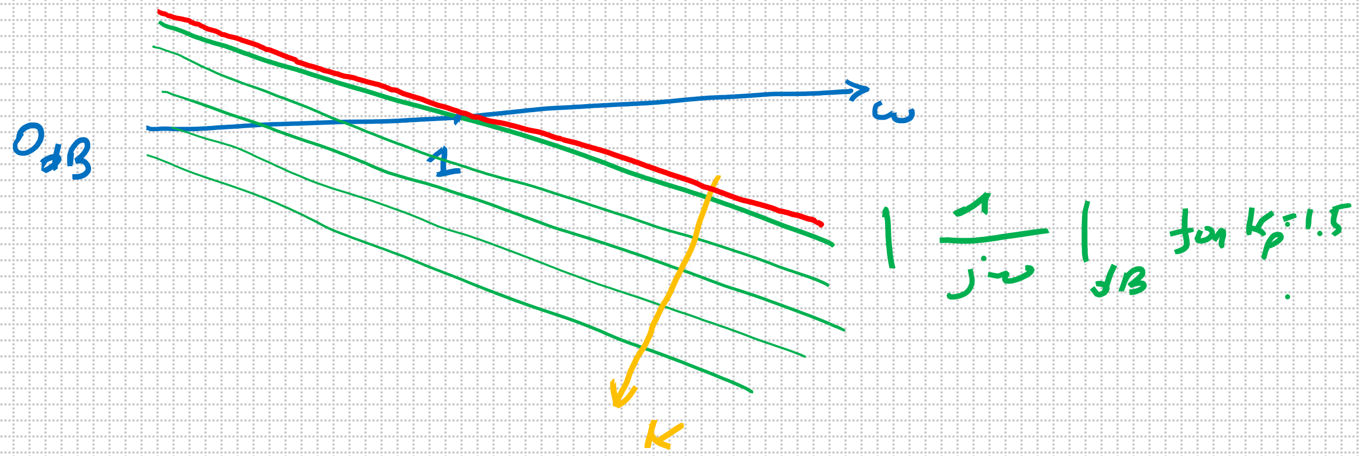
• ADDITIVE UNCERTAINTY

$$\Delta(j\omega) := P(j\omega) - \tilde{P}(j\omega) = -(K_p - 2.5)\,\frac{1+j\omega}{j\omega(1-j\omega)}$$

$$|\Delta(j\omega)| = |K_p - 2.5|\,\frac{|1+j\omega|}{|j\omega|\,|1-j\omega|} = \left|\frac{K_p-2.5}{j\omega}\right|, \quad K_p \in [1.5, 2.5]$$

$$|W(j\omega)| > \left|\frac{K_p}{j\omega}\right|, \quad \forall K_p \in [1.5, 2.5], \ \forall \omega$$

$$|W(j\omega)| > \left|\frac{1}{j\omega}\right|, \quad K_p = 1.5, \ \forall \omega$$



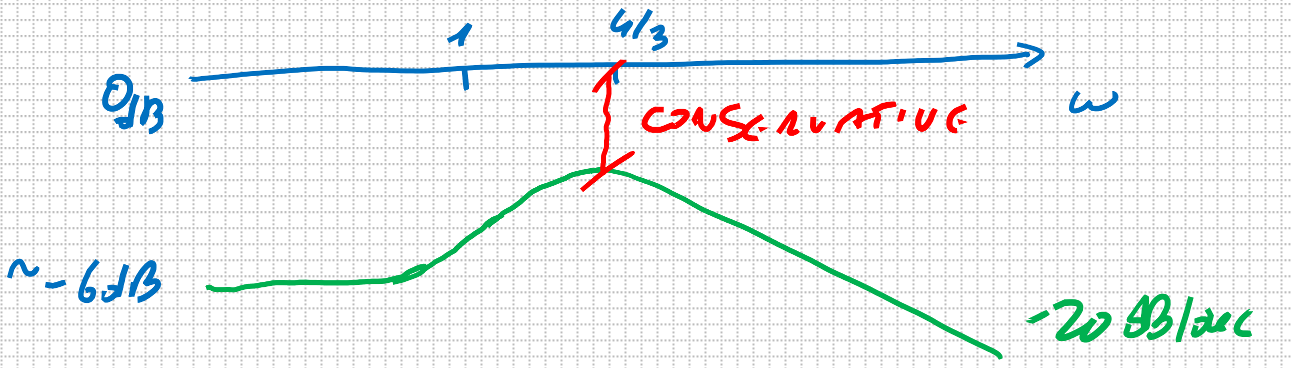
$$\boxed{W(j\omega) = \frac{1.05}{j\omega}}$$

SUFF. CONDITION $\quad \left| W(j\omega)\, G(j\omega)\, \tilde{S}(j\omega) \right| < 1 \quad \forall \omega$

$$\tilde{S}(j\omega) = \frac{1}{1+\hat{\tilde{F}}(j\omega)} = \frac{1}{1+G(j\omega)\tilde{P}(j\omega)} = \frac{1}{1-2.5\,\dfrac{1+j\omega}{j\omega(1-j\omega)}}$$

$$\Rightarrow \left| W(j\omega)\, G(j\omega)\, \tilde{S}(j\omega) \right| = \left| 1.05\, \frac{1-j\omega}{(j\omega+0.75-j1.4)(j\omega+0.75+j1.4)} \right|$$

$|W G \tilde{S}|$



SUFFICIENT CONDITION VERIFIED

ROBUST STABILITY THEOREM with multiplicative uncertainty

$$P(s) = \tilde{P}(s) + \Delta(s)$$

$$P(s) = \tilde{P}(s)\left(1 + \Delta_m(s)\right) = \hat{P}(s) + \Delta_m(s)\,\hat{P}(s) \quad \Rightarrow \quad \begin{cases} \Delta(s) = \Delta_m(s)\,\tilde{P}(s) \\ \Delta_m(s) = \Delta(s)\,\tilde{P}(s)^{-1} \end{cases}$$

$$\left|\Delta(j\omega)\, G(j\omega)\, \tilde{S}(j\omega)\right| < 1 \quad \forall\omega$$

$$\tilde{S}(j\omega) = \left(1 + \hat{P}(j\omega)\, G(j\omega)\right)^{-1}$$

$$\left|\Delta(j\omega)\, G(j\omega)\left(1 + \hat{P}(j\omega)\, G(j\omega)\right)^{-1}\right| < 1 \quad \forall\omega$$

$$\Delta(j\omega) = \Delta_m(j\omega)\, \tilde{P}(j\omega)$$

$$\left|\Delta_m(j\omega)\, \hat{P}(j\omega)\, G(j\omega)\left(1 + \tilde{P}(j\omega)\, \tilde{G}(j\omega)\right)^{-1}\right| < 1 \quad \forall\omega$$

$$\begin{cases} \tilde{T}(j\omega) = \dfrac{\tilde{P}(j\omega)\, G(j\omega)}{1 + \tilde{P}(j\omega)\, G(j\omega)} \\ \left|l_m(j\omega)\right| > \left|\Delta_m(j\omega)\right|, \ \forall\omega \end{cases}$$

$$\boxed{\left|l_m(j\omega)\, \tilde{T}(j\omega)\right| < 1, \quad \forall\omega}$$

ROBUST STABILITY THEOREM (IMC)

Additive uncertainty
$$|w(j\omega)\,G(j\omega)\,\tilde{S}(j\omega)| = |w(j\omega)\,Q(j\omega)| < 1, \ \forall\omega$$

$$\begin{cases} G(j\omega) = \dfrac{Q(j\omega)}{1-\tilde{p}(j\omega)Q(j\omega)} \\ \tilde{S}(j\omega) = 1-\tilde{p}(j\omega)Q(j\omega) \end{cases}$$

Multiplicative uncertainty
$$|l_m(j\omega)\,\tilde{T}(j\omega)| = |l_m(j\omega)\,\tilde{p}(j\omega)\,Q(j\omega)| < 1, \ \forall\omega$$

$$\tilde{T}(j\omega) = \tilde{p}(j\omega)\,Q(j\omega)$$