MPC WITH NO MODEL UNCERTAINTY

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FROM LINEAR QUADRATIC OPTIMAL CONTROL TO MPC

LTI, DT systems without uncertainties and disturbances

$$\begin{cases} x(k+1) = A x(k) + B u(k) \\ y(k+1) = C x(k+1) \end{cases}$$

$x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$

$y(k) \in \mathbb{R}^{n_y}$

ASSUMPTION (A,B) STABILIZABLE

(A,C) OBSERVABLE

$x(k)$ MEASURED AT TIME k

LINEAR CONSTRAINTS

$$F x(k) + G u(k) \leq b$$

$F \in \mathbb{R}^{m_c \times m_x}$

$G \in \mathbb{R}^{m_c \times m_u}$

$b \in \mathbb{R}^{m_c}$

m_c: number of constraints

$F = 0 \Rightarrow$ input constraints $\quad G u(k) \leq b$

$G = 0 \Rightarrow$ state constraints $\quad F x(k) \leq b$

Non stationary constraints $\quad F_k, G_k$

FEASIBLE PAIRS $\quad (x(k), u(k)) \Leftrightarrow F x(k) + G u(k) \leq b$

FEASIBLE SEQUENCE $\quad \{(x(0), u(0)), (x(1), u(1)), \dots\}$

$\Leftrightarrow (x(k), u(k))$ is feasible, $k = 0, 1, \dots$

REGULATION PROBLEM

- Design of a controller that steers the system state to the origin ($x(k) \to 0$)

- QUADRATIC COST, INFINITE HORIZON, UNCONSTRAINED CASE

$$J\left(x(0), \{u(0), u(1), \dots\}\right) = \sum_{k=0}^{\infty} \left(\|x(k)\|_Q^2 + \|u(k)\|_R^2 \right)$$

- $\|v\|_S^2 = v^T S v$
- $Q \in \mathbb{R}^{n_x \times n_x}$, $R \in \mathbb{R}^{n_u \times n_u}$

ASSUMPTION

$R \succ 0$ SYMMETRIC, POSITIVE DEFINITE (eigs are real and strictly positive)

$Q \succeq 0$ " , " SEMI-DEFINITE (eigs are real and non-negative)

INFINITE HORIZON UNCONSTRAINED QUADRATIC PROBLEM

$$J^*(x(k)) = \min_{\{u(k), u(k+1), \dots\}} J\left(x(k), \{u(k), u(k+1), \dots\}\right)$$

⚠ the constrained infinite-h problem has no closed-form solution

$$\min_{\{\ \ \}} J\left(x(k), \{\ \ \}\right)$$

$$\text{s.t.} \quad F\, x(k+i) + G\, u(k+i) \leq b, \quad i = 0, 1, \dots$$

- LINEAR STATE FEEDBACK CONTROL $\quad u(k) = K\, x(k), \quad K \in \mathbb{R}^{n_u \times n_x}$

$\Rightarrow$ CLOSED-LOOP DYNAMICS

$$\begin{cases} x(k+1) = A\, x(k) + B u(k) \\ u(k) = K\, x(k) \end{cases} \rightarrow x(k+1) = (A + BK)\, x(k)$$

$$\Rightarrow \begin{cases} x(k) = (A + BK)^k\, x(0) \\ u(k) = K (A + BK)^k\, x(0) \end{cases}, \quad k = 0, 1, \dots$$

$\Rightarrow \quad J\left(x(0), \{u(0), u(1), \dots\}\right) = J(x(0))$ $\quad \leftarrow$ $u(k)$ can be expressed as function of $x(0)$

$x(k) = (A+BK)^k x(0)$

$u(k) = K(A+BK)^k x(0)$

$\Rightarrow \quad J(x(0)) = x(0)^T W x(0) = \|x(0)\|_W^2$

with $\quad W = \sum_{k=0}^{\infty} \left[\left((A+BK)^k\right)^T Q (A+BK)^k + \right.$

$\left. + \left((A+BK)^k\right)^T K^T R K (A+BK)^k \right] =$

$= \sum_{k=0}^{\infty} \left[\left((A+BK)^k\right)^T (Q + K^T R K)(A+BK)^k \right] \in \mathbb{R}^{n_x \times n_x}$

- If $(A+BK)$ is strictly stable $(|\text{eigs}| < 1) \Rightarrow W$ has a finite number of elements

- $\begin{cases} R \succ 0 \\ Q \succeq 0 \end{cases} \Rightarrow \quad J(x) \succeq 0 \;, \quad W \succeq 0$

LEMMA 1 (Lyapunov matrix equation)

W is the unique positive definite solution of

$$W = (A+BK)^T W (A+BK) + Q + K^T R K \iff (A+BK) \text{ is strictly stable}$$

THEOREM 1 (DISCRETE TIME ALGEBRAIC RICCATI EQUATION)

the optimal gain matrix K^* minimizing $J(x(0)) = \|x(0)\|_W^2$, for any $x(0) \in \mathbb{R}^{n_x}$ is

$$K^* = (B^T W B + R)^{-1} B^T W A$$

LQR $\quad u(k) = K^* x(k)$ OPTIMAL CONTROL LAW FOR THE DISCRETE TIME LTI SYSTEM UNDER QUADRATIC COST FUNCTION WITHOUT CONSTRAINTS

$$J^*(x(0)) = \|x(0)\|_W^2$$ OPTIMAL COST WHICH ONLY DEPENDS ON $x(0)$

MPC

DUAL MODE PREDICTION PARADIGM

- Presence of constraints
- Finite horizon $\Rightarrow$ RECEDING HORIZON

(the optimal solution is computed at any time k)

$u(i|k)$ control action at time $k+i$ computed at time k

- INFINITE HORIZON CONSTRAINED PROBLEM

$$\min_{\{u(0|k),\, u(1|k),\, \dots\}} J\left(x(k),\, \{u(0|k),\, u(1|k),\, \dots\}\right) =$$

$$= \min_{\{u(0|k),\, u(1|k)\}} \sum_{k=0}^{\infty} \left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2 \right)$$

s.t.

$$F\,x(i|k) + G\,u(i|k) \leq b \quad , \quad i = 0, 1, \dots$$

DUAL-MODE Divide the infinite-horizon problem in two problems

- **MODE 1** FINITE-HORIZON CONSTRAINED PROBLEM

$$\min_{\{u(0|k),\, u(1|k),\, u(N-1|k)\}} J_{RH}\left(x(k), \{u(0|k), \dots, u(N-1|k)\}\right)$$

$$\text{s.t.} \quad F x(i|k) + G u(i|k) \leq b, \quad i = 0, 1, \dots, N-1$$

- **MODE 2** INFINITE-HORIZON UNCONSTRAINED PROBLEM

$$\min_{\{u(N|k),\, u(N+1|k),\, \dots\}} J\left(x(N|k)\right) \quad \leftarrow \text{LQR}$$

$$\Rightarrow \boxed{\begin{cases} u(i|k) = K^* x(N|k), & i = N, N+1, \dots \\ J^*(x(N|k)) = \| x(N|k) \|^2_W \end{cases}} \leftarrow K^*, W \text{ are known}$$

- We use the knowledge of the inf. horiz. solution to modify J in the finite horizon problem

$$J_{RH}(x(k), \underline{u}(k)) = \underbrace{\sum_{i=0}^{N-1} \left(\|x(i|k)\|^2_Q + \|u(i|k)\|^2_R \right)}_{J \text{ over the finite horizon}} + \underbrace{\|x(N|k)\|^2_W}_{\text{final cost (penalty term)}}$$

$\underline{u}(k) : \{u(0|k), \dots, u(N-1|k)\}$

cost-to-go after N time steps under the optimal control law $u = K^* x$

- WE CAN SOLVE NUMERICALLY THE CONSTRAINED FINITE-H PROBLEM

$$\min_{\underline{u}(k)} J_{RH}(x(k), \underline{u}(k))$$

$$\text{s.t.} \quad F x(i|k) + G(i|k) \leq b, \quad i = 0, 1, \dots, N-1$$

$\Rightarrow$ RECEDING HORIZON OPTIMAL SOLUTION FOR THE FINITE HORIZON CONSTRAINED PROBLEM

$$\boxed{u(i|k) = g_{RH}(x(k)), \quad i = 0, \dots, N-1}$$

- OPTIMAL SOLUTION OF THE CONSTRAINED INFINITE-HORIZON PROBLEM DENOTED BY $\{u^{\infty}(0|k), u^{\infty}(1|k), \dots\}$

- UNDER $\{u^{\infty}(0|k), \dots\}$: $x(i|k) \longrightarrow 0$ as i grows

- $(x,u) = (0,0)$ is strictly feasible ($Fx + Gu \le b$)

$\Rightarrow$ under the optimal state feedback control law $u = K^* x$

$\Rightarrow$ The constraints are $(F + GK^*)x \le b$

$\Rightarrow$ there exists a neighborhood S of $x = 0$ such that the constraints are met : $Fx + Gu \le b, \ \forall x \in S$

$\Rightarrow$ there exists a value $\bar{N}$ under the optimal stabilizing control sequence such that
$$(F + GK^*)\, x(\bar{N}+i \mid k) \leq b, \quad i = 0, 1, \dots$$
are verified

$\Rightarrow$
$$\min_{\{u(\bar{N}|k),\, u(\bar{N}+1|k), \dots\}} J(x(\bar{N}|k)) \equiv \min_{\{u(\bar{N}|k), \dots\}} J(x(\bar{N}|k))$$

s.t.
$$F x(i|k) + G u(i|k) \leq b, \quad i = \bar{N}, \bar{N}+1, \dots$$

$\angle$ after $\bar{N}$ steps the constraints are not active

$$\Rightarrow \boxed{\begin{cases} u^{\infty}(i|k) = K^* x(i|k), & i = \bar{N}, \bar{N}+1, \dots \\ J^*(x(\bar{N}|k)) = \| x(\bar{N}|k) \|^2_{W} \end{cases}}$$

We know the optimal solution of the constrained infinite-horizon problem from $\bar{N}$ on (mode 2)

If $\underline{\underline{N \geq \bar{N}}}$ (if the prediction horizon of the MPC is suff.^{ly} long)

$$\Big\{ \underbrace{u^*(0|k), u^*(1|k), \dots, u^*(N-1|k)}_{\text{solution of the finite-horizon constrained quadratic program. with terminal cost } \|x^*(N|k)\|^2_{\infty}}, \underbrace{K^* x^*(N|k), K^* x^*(N+1|k), \dots}_{\text{optimal solution of the LQR problem starting at time } k+N} \Big\} =$$

$$= \Big\{ u^\infty(0|k), u^\infty(1|k), \dots, u^\infty(N-1|k), u^\infty(N|k), u^\infty(N+1|k), \dots \Big\}$$

SOLUTION OF THE INFINITE-HORIZON CONSTRAINED QUADRATIC PROBLEM

Theorem 2 There exists a finite horizon $\bar{N}$, which depends on $x(k)$, such that, if $N \geq \bar{N}$:

i) $\underline{u}^*(k)$ coincides with $\{u^\infty(0|k), \dots, u^\infty(N-1|k)\}$

ii) $J_{RH}(x(k), \underline{u}^*(k)) = \underbrace{J^*(k)}_{\text{OPTIMAL COST OF THE INF. HORIZON CONSTRAINED PROBLEM}}$