Robotics 1

Direct kinematics

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Kinematics of robot manipulators

- study of ...

  geometric and timing aspects of robot motion, without reference to the causes producing it

- robot seen as ...

  an (open) kinematic chain of rigid bodies interconnected by (revolute or prismatic) joints

Motivations

- functional aspects
  - definition of robot workspace
  - calibration
- operational aspects

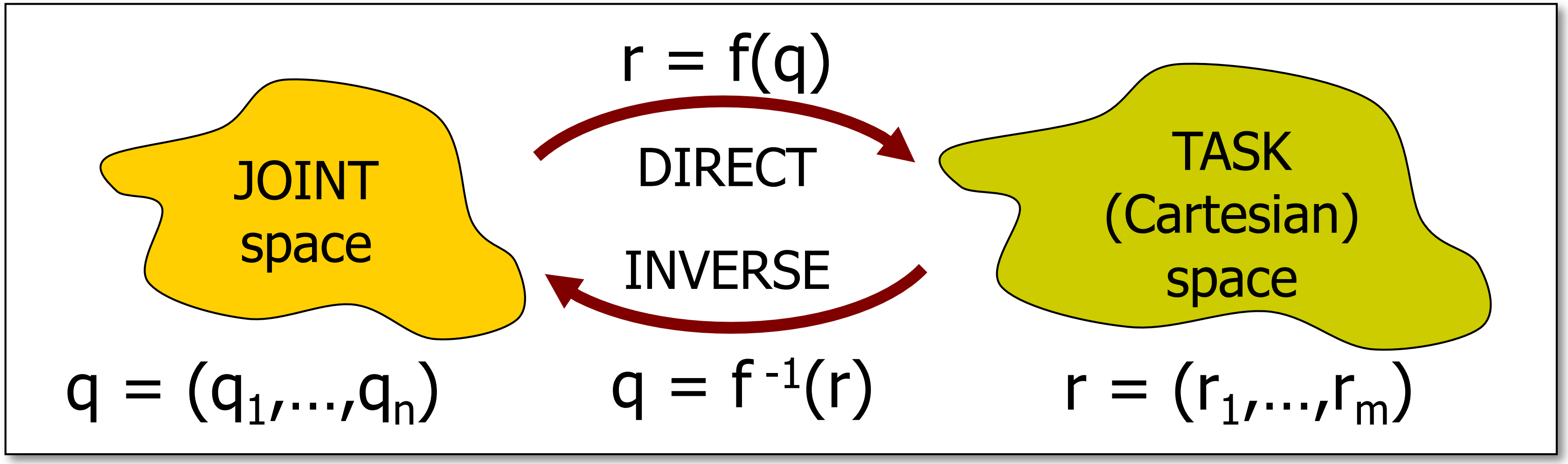
task execution (actuation by motors) ⇄ task definition and performance

two different "spaces" related by kinematic (and dynamic) maps

  - trajectory planning
  - programming
  - motion control

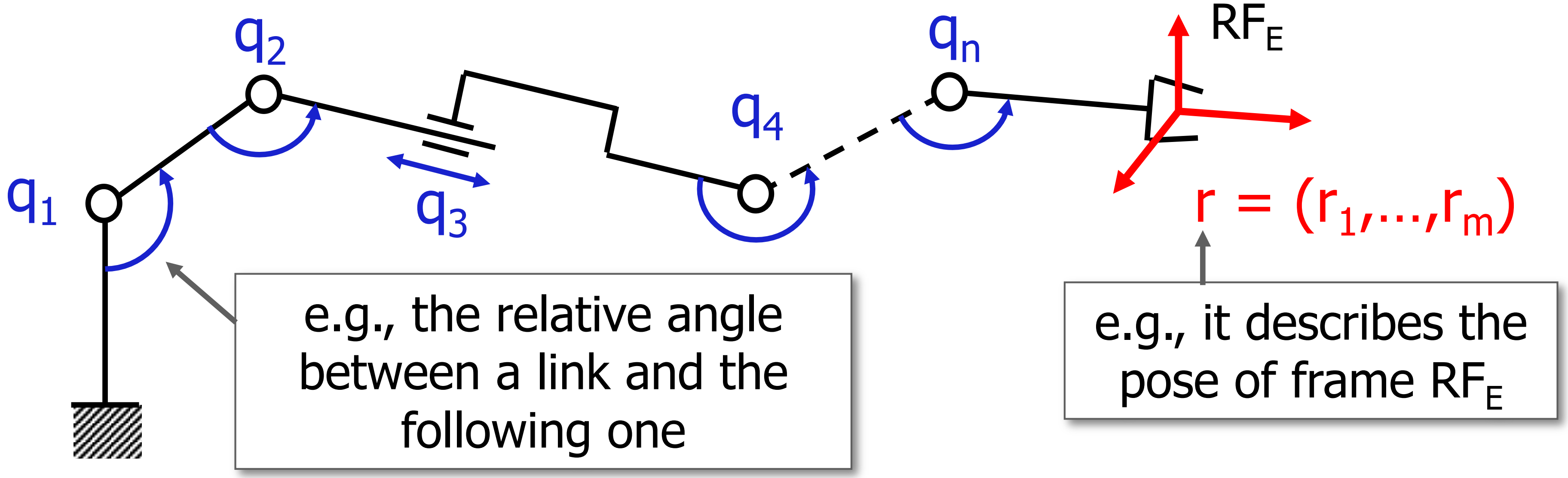
Kinematics

formulation and parameterizations



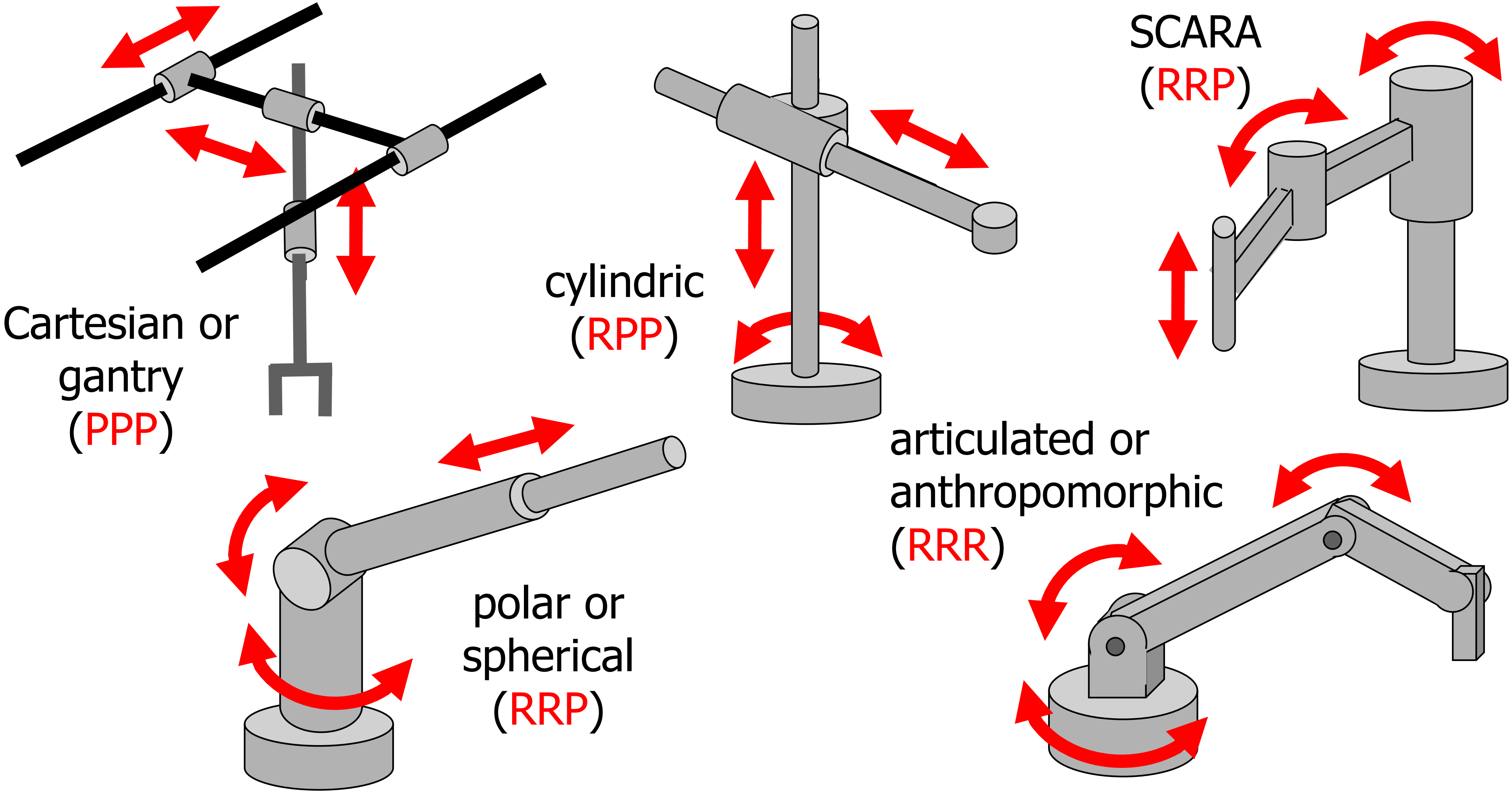
- choice of parameterization q
  - unambiguous and minimal characterization of robot configuration
  - n = # degrees of freedom (dof) = # robot joints (rotational or translational)
- choice of parameterization r
  - compact description of position and/or orientation (pose) variables of interest to the required task
  - usually, $m \leq n$ and $m \leq 6$ (but none of these is strictly necessary)

Open kinematic chains



- m = 2
  - pointing in space
  - positioning in the plane
- m = 3
  - orientation in space
  - positioning and orientation in the plane
- m = 5
  - positioning and pointing in space (like for spot welding)
- m = 6
  - positioning and orientation in space
  - positioning of two points in space (e.g., end-effector and elbow)

Classification by kinematic type
(first 3 dofs)



R = 1-dof rotational (revolute) joint
P = 1-dof translational (prismatic) joint

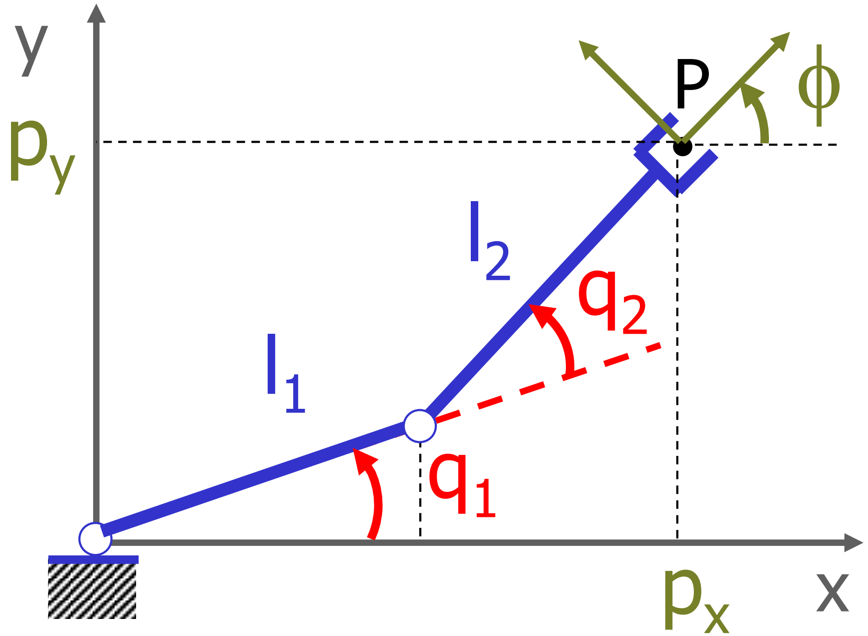
Direct kinematic map

- the structure of the direct kinematics function depends from the chosen r

$$r = f_r(q)$$

- methods for computing $f_r(q)$
  - geometric/by inspection
  - systematic: assigning frames attached to the robot links and using homogeneous transformation matrices

Example: direct kinematics of 2R arm



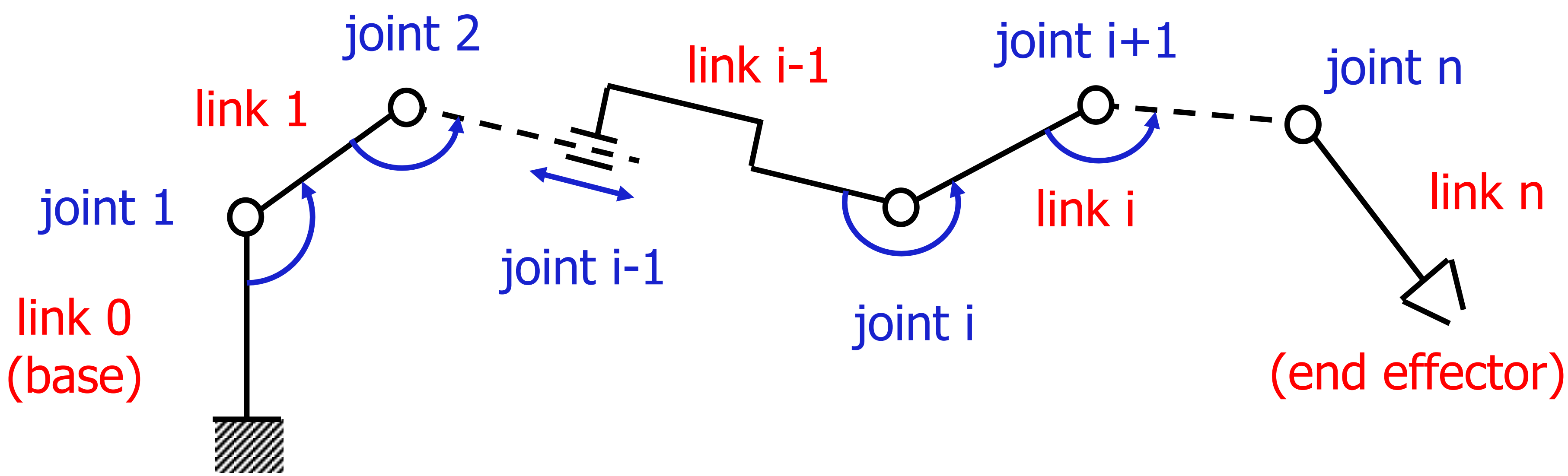
$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad n = 2$$

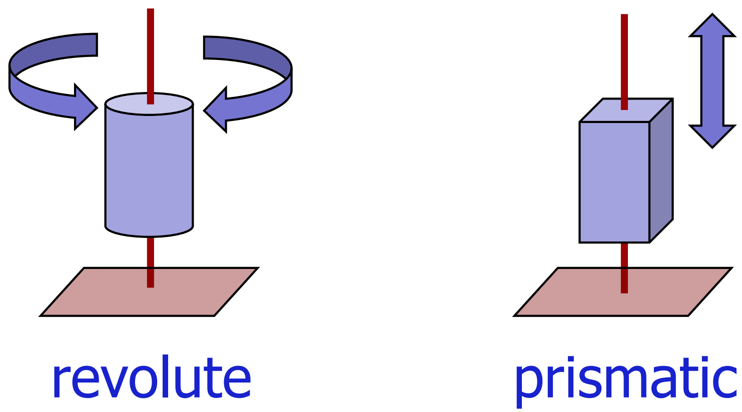
$$r = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} \qquad m = 3$$

$$p_x = l_1 \cos q_1 + l_2 \cos(q_1+q_2)$$
$$p_y = l_1 \sin q_1 + l_2 \sin(q_1+q_2)$$
$$\phi = q_1 + q_2$$

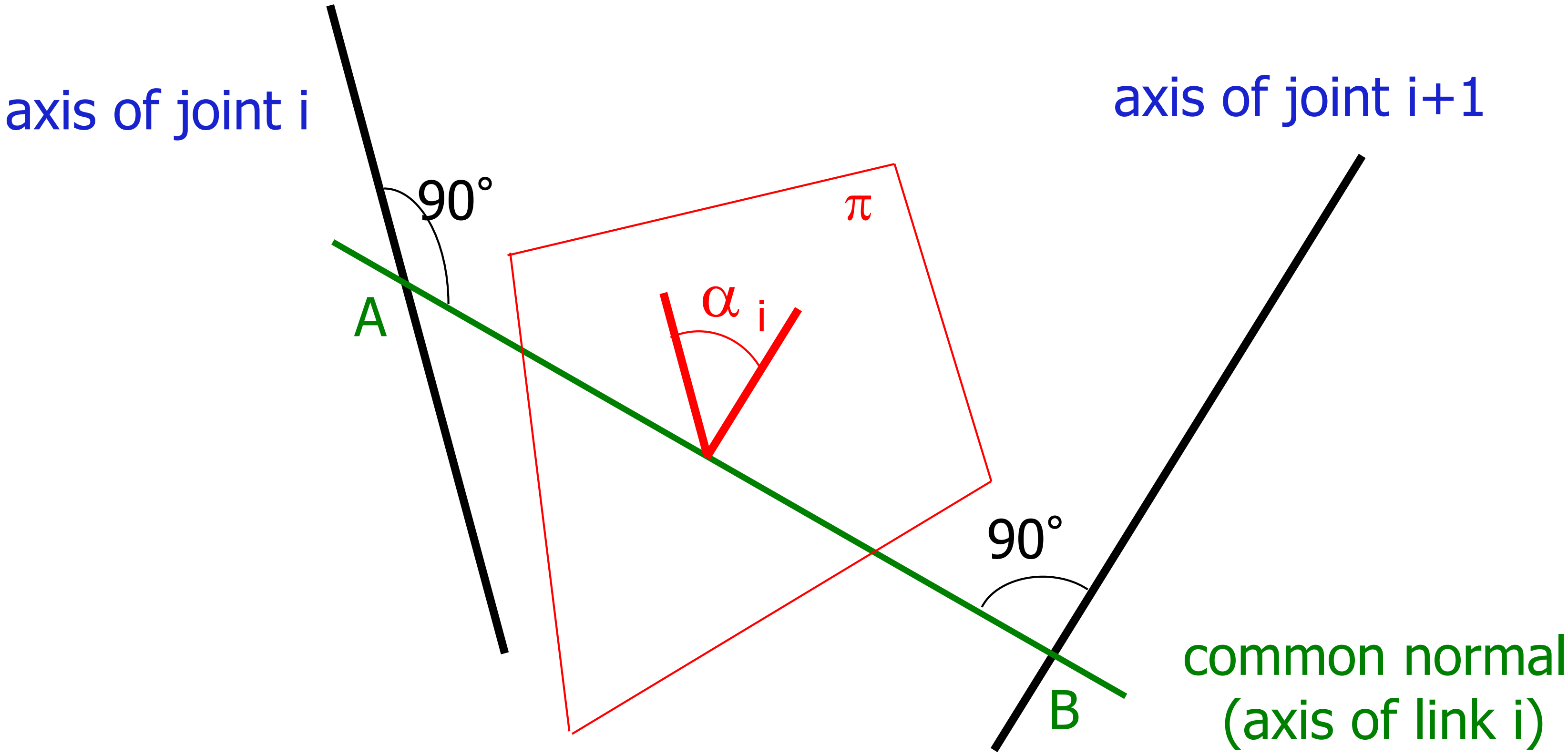
for more general cases, we need a "method"!

Numbering links and joints





Spatial relation between joint axes



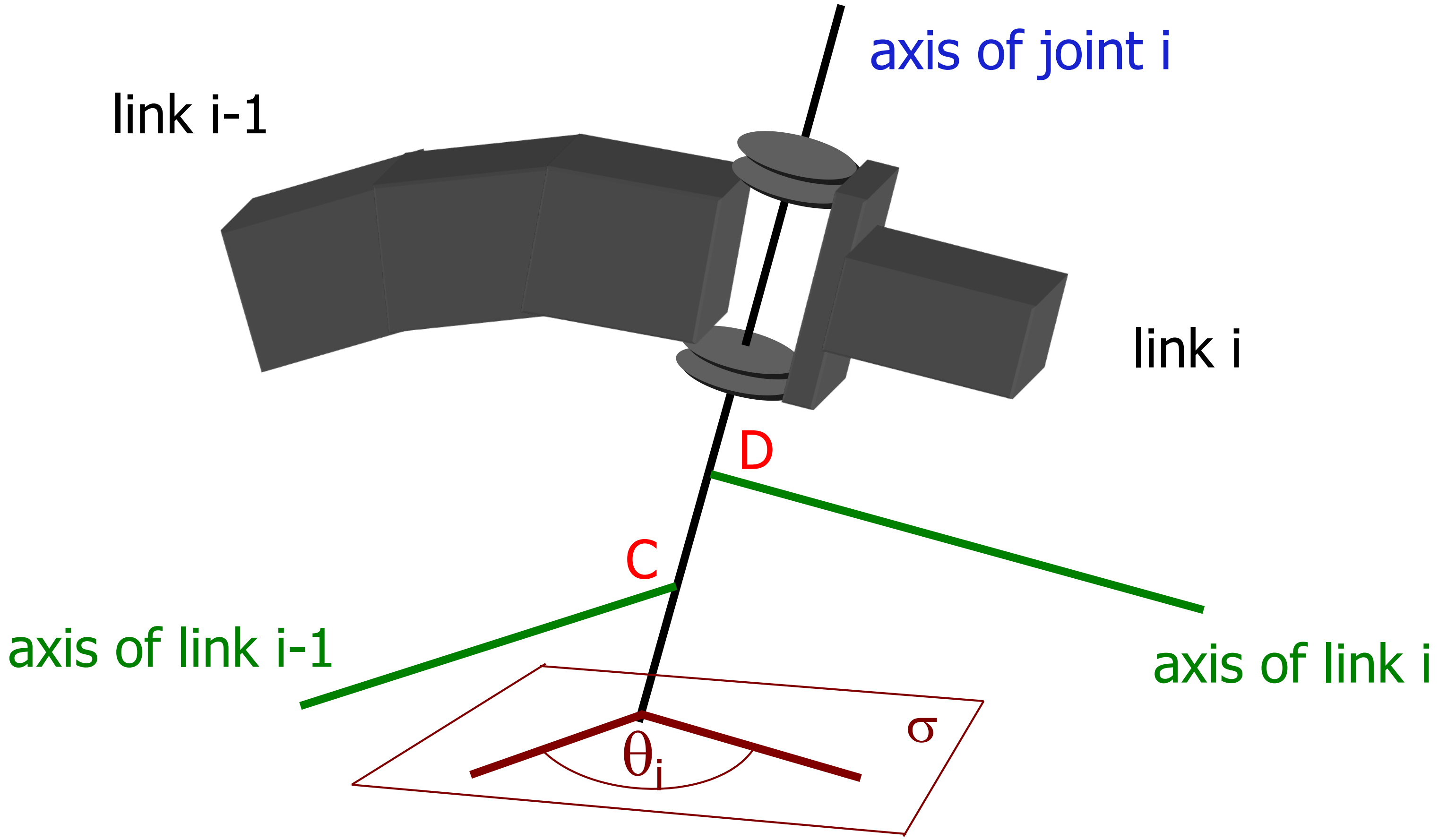
a_i = **displacement AB** between joint axes (always well defined)

α_i = **twist angle** between joint axes

— projected on a plane π orthogonal to the link axis

} with sign (pos/neg)!

Spatial relation between link axes

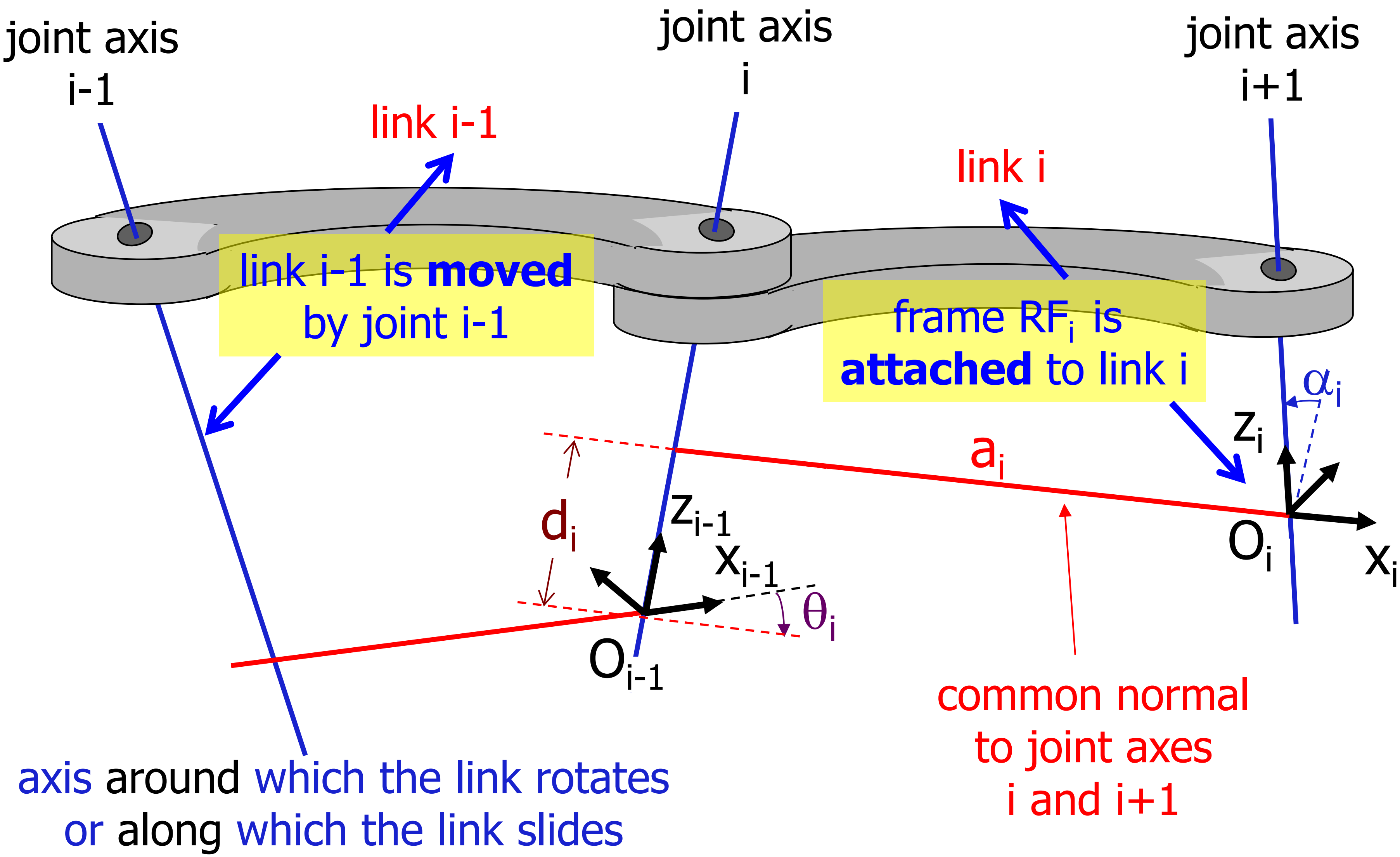


d_i = **displacement CD** (a variable if joint i is prismatic)

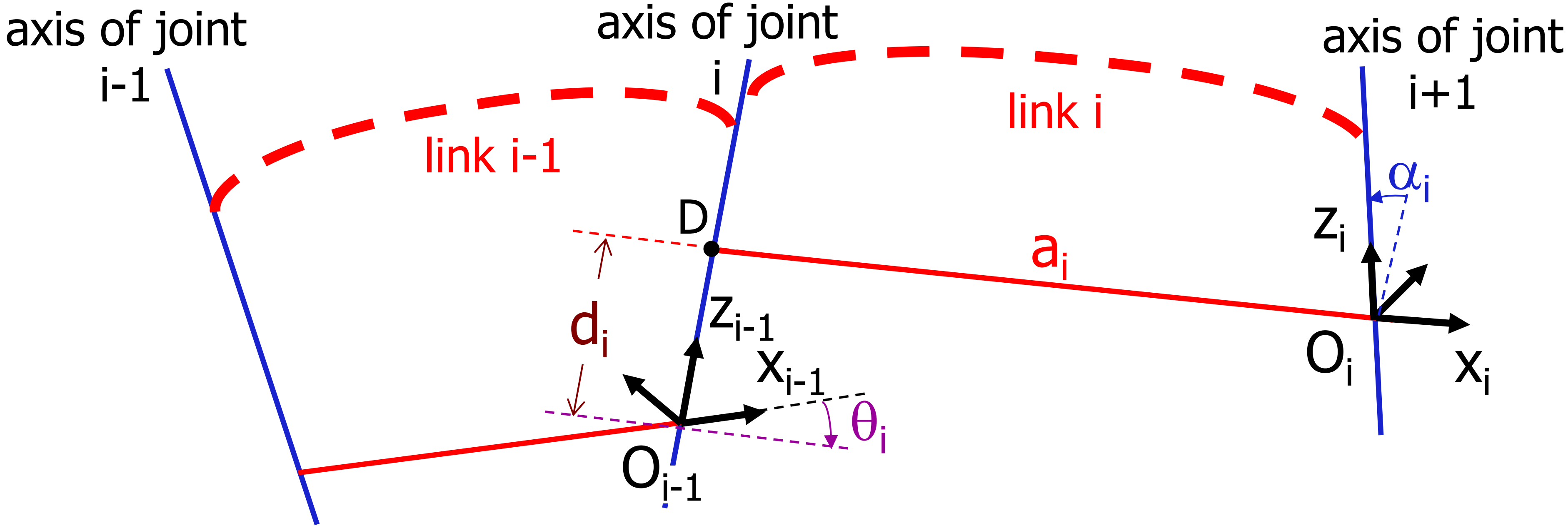
θ_i = **angle between link axes** (a variable if joint i is revolute)
— projected on a plane σ orthogonal to the joint axis

with sign (pos/neg)!

Denavit-Hartenberg (DH) frames

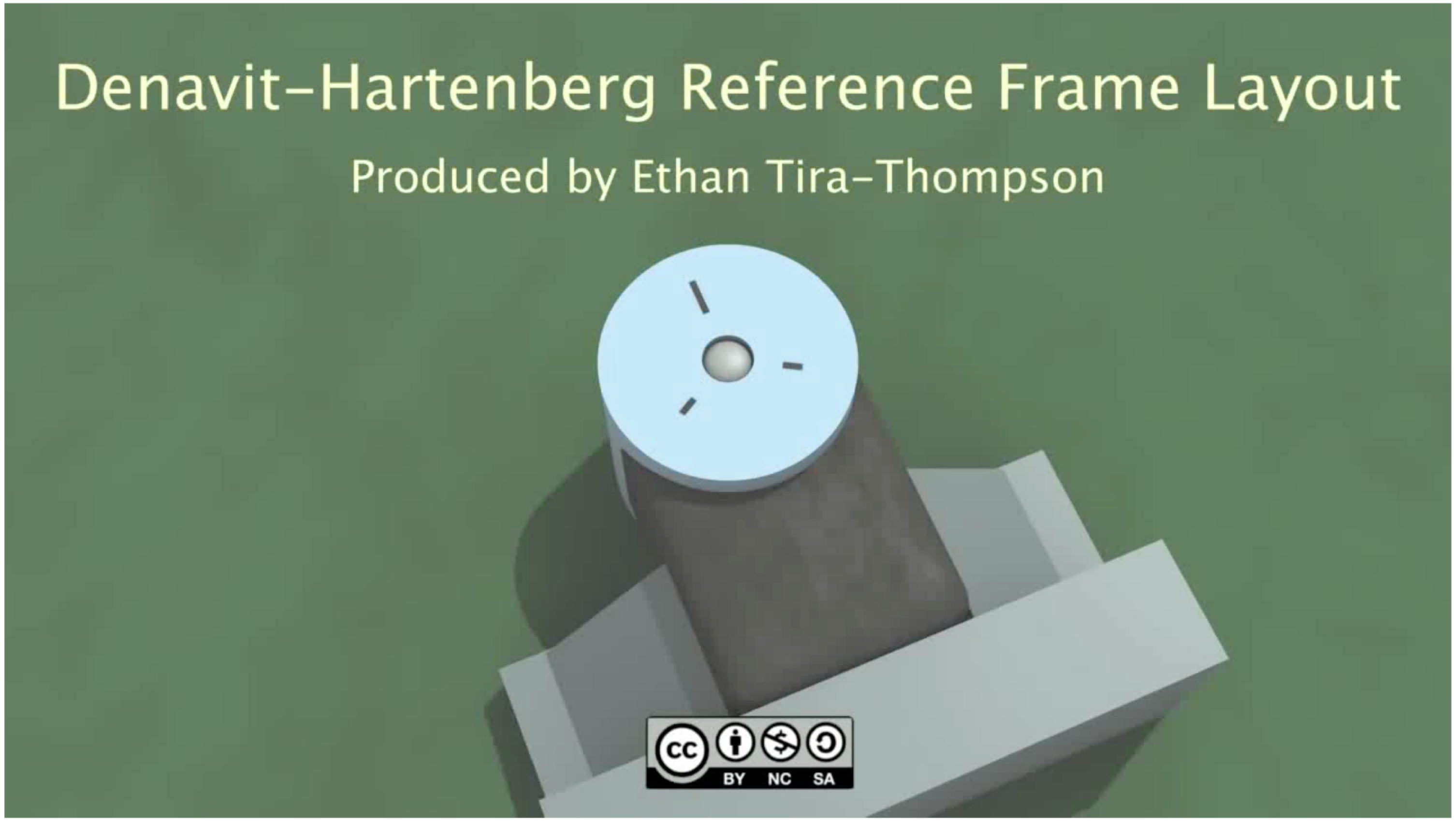


Denavit-Hartenberg parameters



- unit vector z_i along axis of joint i+1
- unit vector x_i along the common normal to joint i and i+1 axes (i $\rightarrow$ i+1)
- a_i = distance DO_i — positive if oriented as x_i (constant = "length" of link i)
- d_i = distance $O_{i-1}D$ — positive if oriented as z_{i-1} (variable if joint i is PRISMATIC)
- α_i = twist angle between z_{i-1} and z_i around x_i (constant)
- θ_i = angle between x_{i-1} and x_i around z_{i-i} (variable if joint i is REVOLUTE)

Denavit-Hartenberg layout made simple
(a popular 3-minute illustration...)



video

https://www.youtube.com/watch?v=rA9tm0gTln8

- **note**: the authors of this video use r in place of a, and do not add subscripts!

Ambiguities in defining DH frames

- *$frame_0$:* origin and x_0 axis are arbitrary
- *$frame_n$:* z_n axis is not specified (but x_n must be orthogonal to and intersect z_{n-1})
- *positive* direction of z_{i-1} (up/down on joint i) is arbitrary
  - choose one, and try to avoid "flipping over" to the next one
- *positive* direction of x_i (on axis of link i) is arbitrary
  - we often take $x_i = z_{i-1} \times z_i$ when successive joint axes are incident
  - when natural, we follow the direction "from base to tip"
- if z_{i-1} and z_i are *parallel:* common normal not uniquely defined
  - O_i is chosen arbitrarily along z_i , but try to "zero out" parameters
- if z_{i-1} and z_i are *coincident:* normal x_i axis may be chosen at will
  - again, we try to use "simple" constant angles (0, $\pi/2$)
  - this case may occur only if the two joints are of different kind (P & R)

Homogeneous transformation

between successive DH frames (from frame i-1 to frame i)

- roto-translation around and along z_{i-1}

$$
{}^{i-1}A_{i'}(q_i) = \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

rotational joint $\Rightarrow$ $q_i = \theta_i$

prismatic joint $\Rightarrow$ $q_i = d_i$

- roto-translation around and along x_i

$$
{}^{i'}A_i = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

always a constant matrix

Denavit-Hartenberg matrix

J. Denavit and R.S. Hartenberg, "A kinematic notation for lower-pair mechanisms based on matrices," *Trans. ASME J. Applied Mechanics,* **23**: 215–221, 1955

$$
{}^{i-1}A_i\,(q_i) = {}^{i-1}A_{i'}\,(q_i)\,{}^{i'}A_i = \left[\begin{array}{ccc|c} c\theta_i & -c\alpha_i\, s\theta_i & s\alpha_i\, s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i\, c\theta_i & -s\alpha_i\, c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

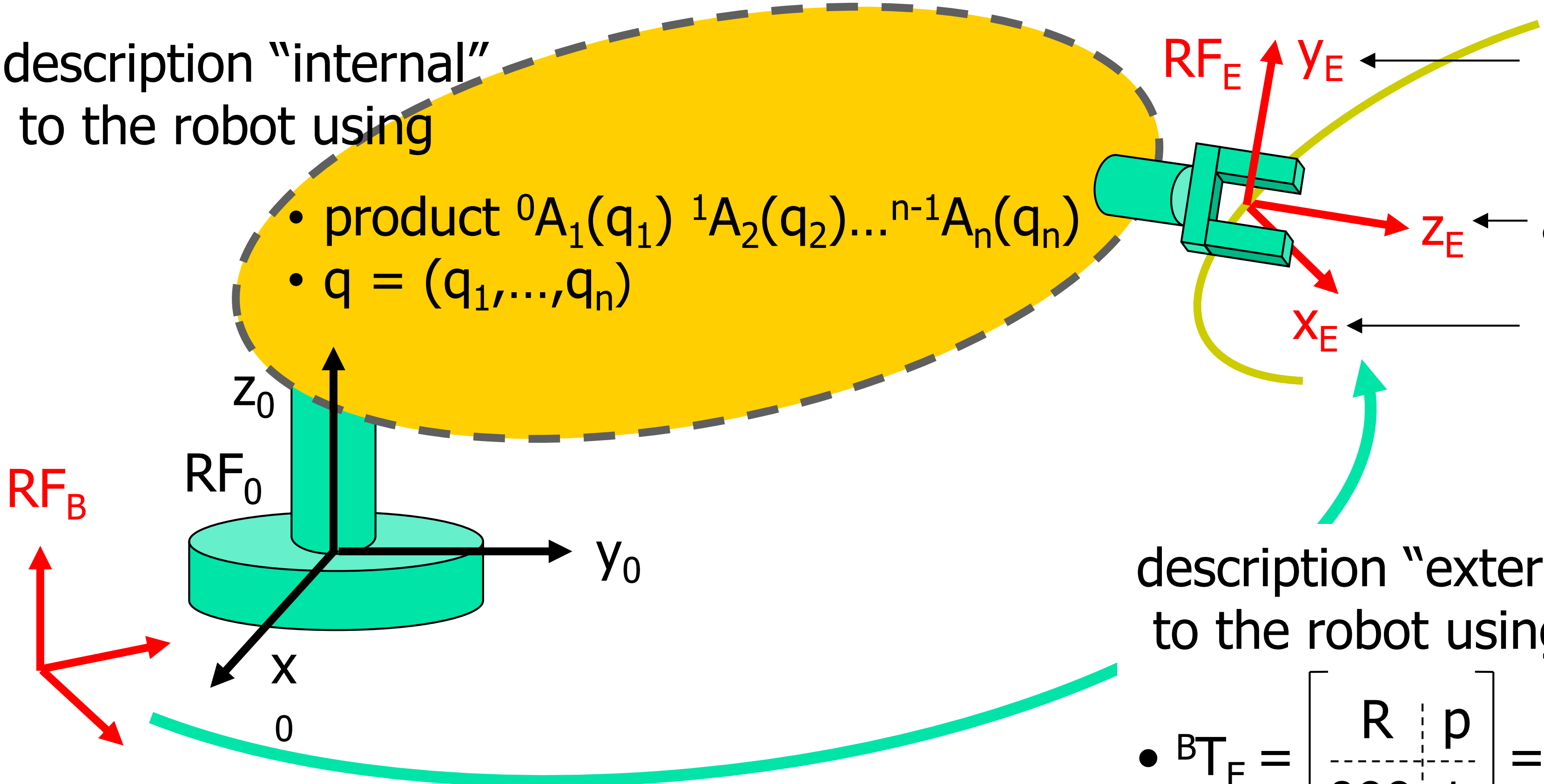
compact notation: c = cos, s = sin

super-compact notation: $c_i = \cos q_i$, $s_i = \sin q_i$

Direct kinematics of manipulators

description "internal" to the robot using

- product ${}^0A_1(q_1)\ {}^1A_2(q_2)\ldots{}^{n-1}A_n(q_n)$
- $q = (q_1,\ldots,q_n)$



description "external" to the robot using

- ${}^BT_E = \begin{bmatrix} R & p \\ 000 & 1 \end{bmatrix} = \begin{bmatrix} n\ s\ a & p \\ 0\ 0\ 0 & 1 \end{bmatrix}$
- $r = (r_1,\ldots,r_m)$

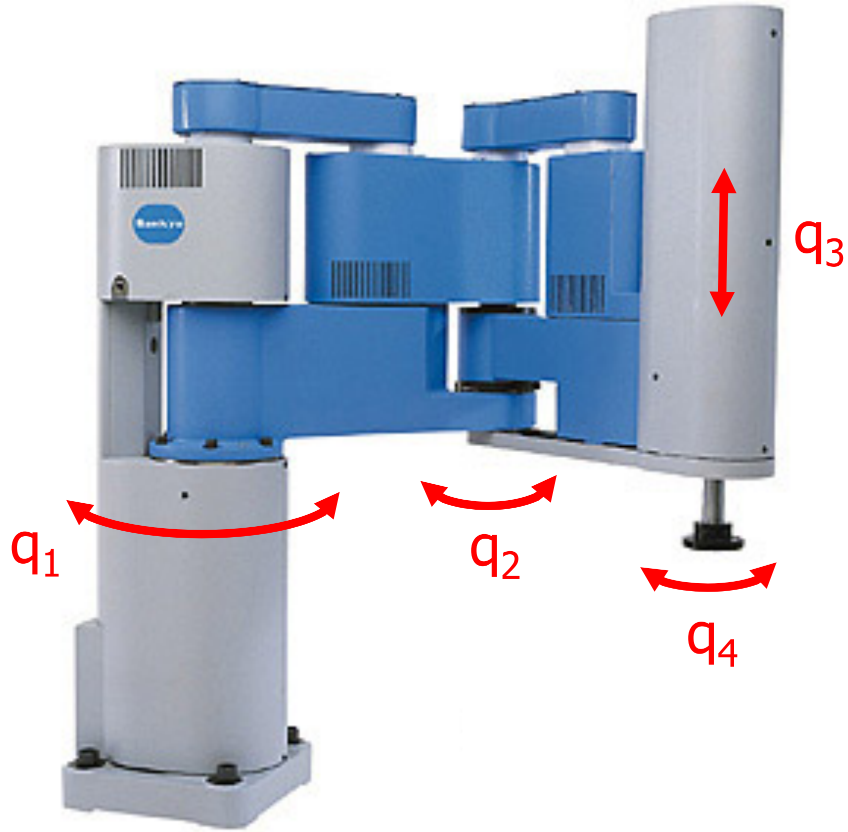
$$\boxed{\begin{aligned} {}^BT_E &= {}^BT_0\ {}^0A_1(q_1)\ {}^1A_2(q_2)\ldots{}^{n-1}A_n(q_n)\ {}^nT_E \\ r &= f_r(q) \end{aligned}}$$

alternative descriptions of the **direct kinematics** of the robot

Example: SCARA robot

video

Sankyo SCARA 8438

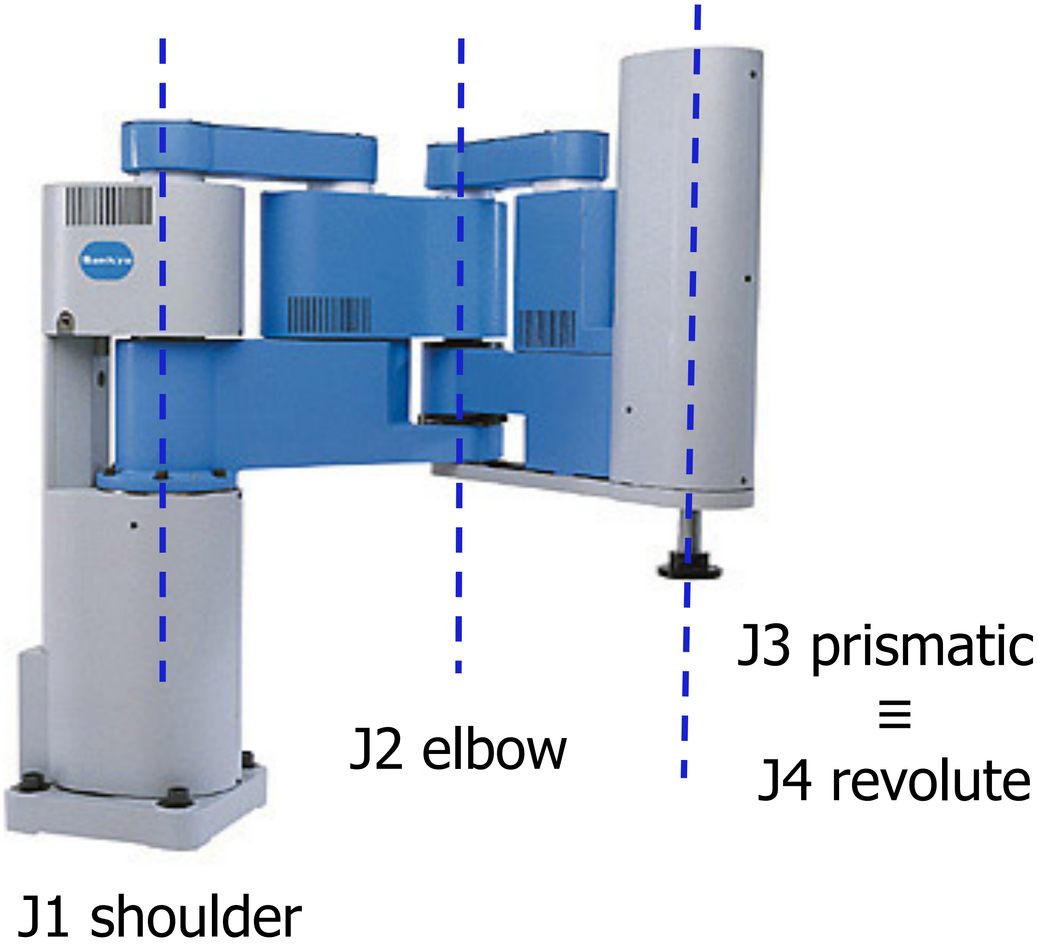


Sankyo SCARA SR 8447

Step 1: joint axes

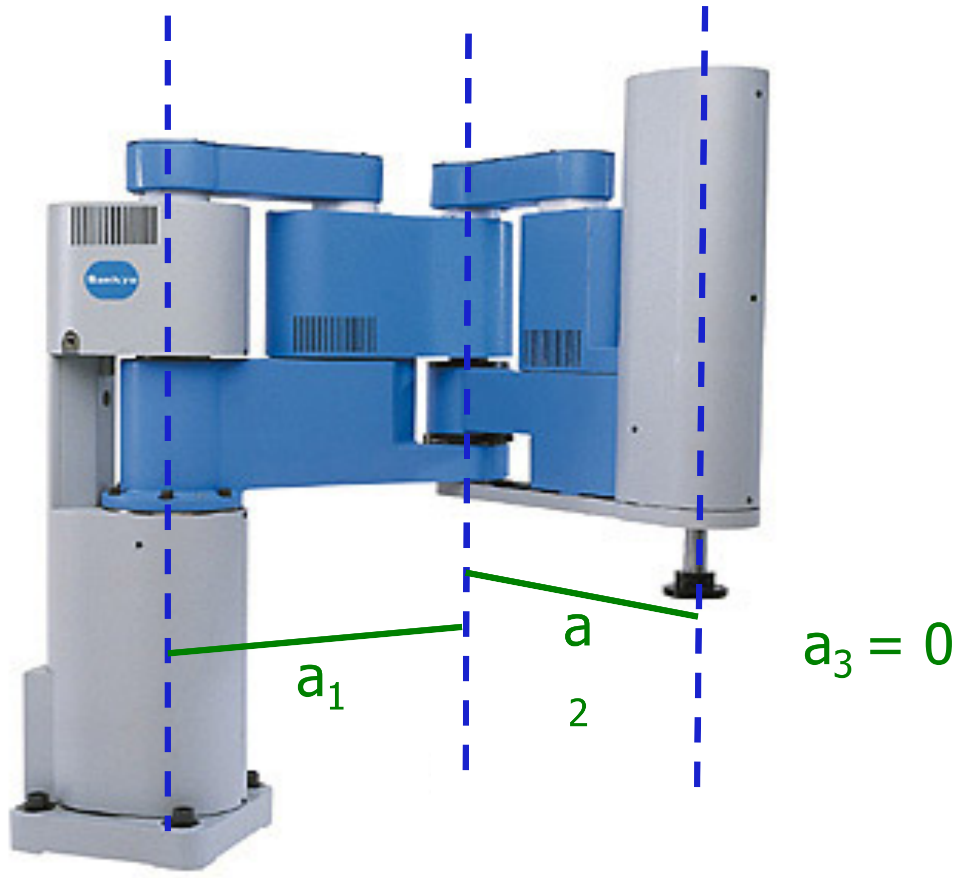
all parallel
(or coincident)

twists $\alpha_i = 0$
or π



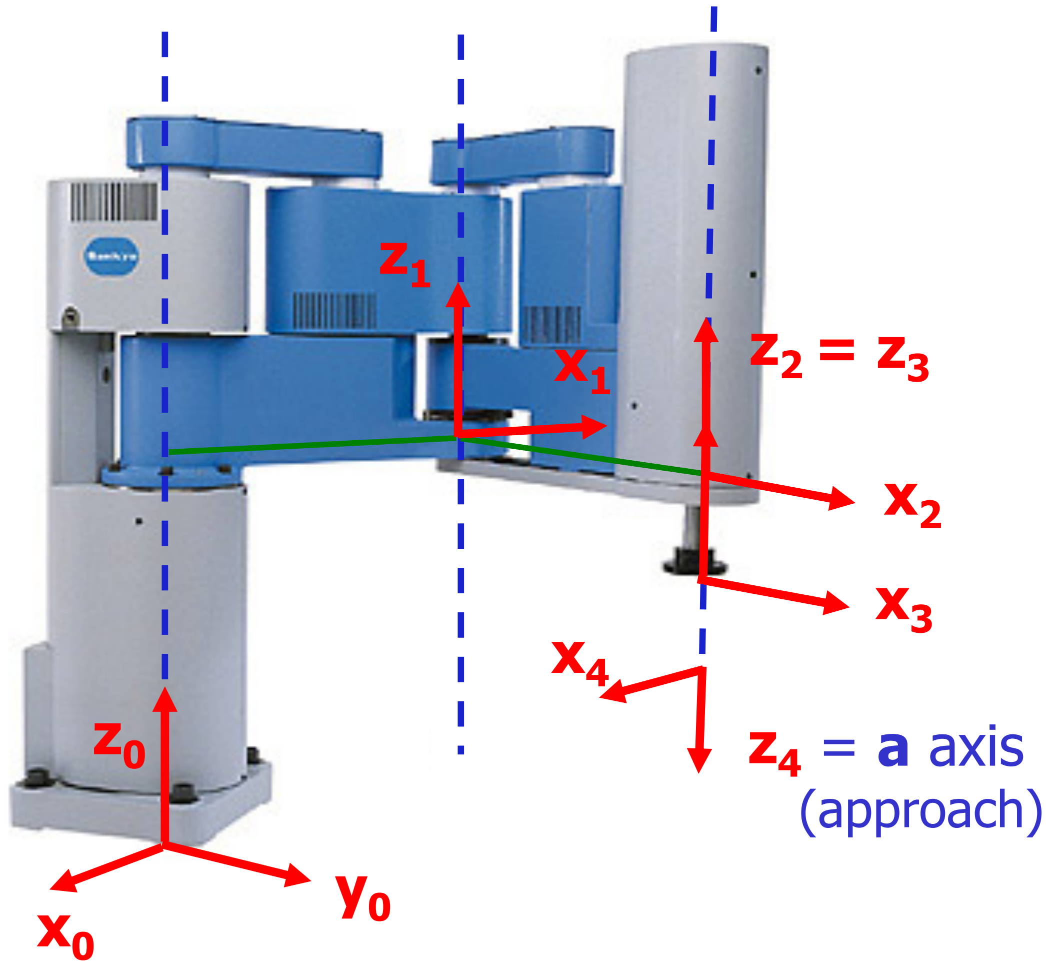
Step 2: link axes

the vertical "heights"
of the link axes
are arbitrary
(for the time being)

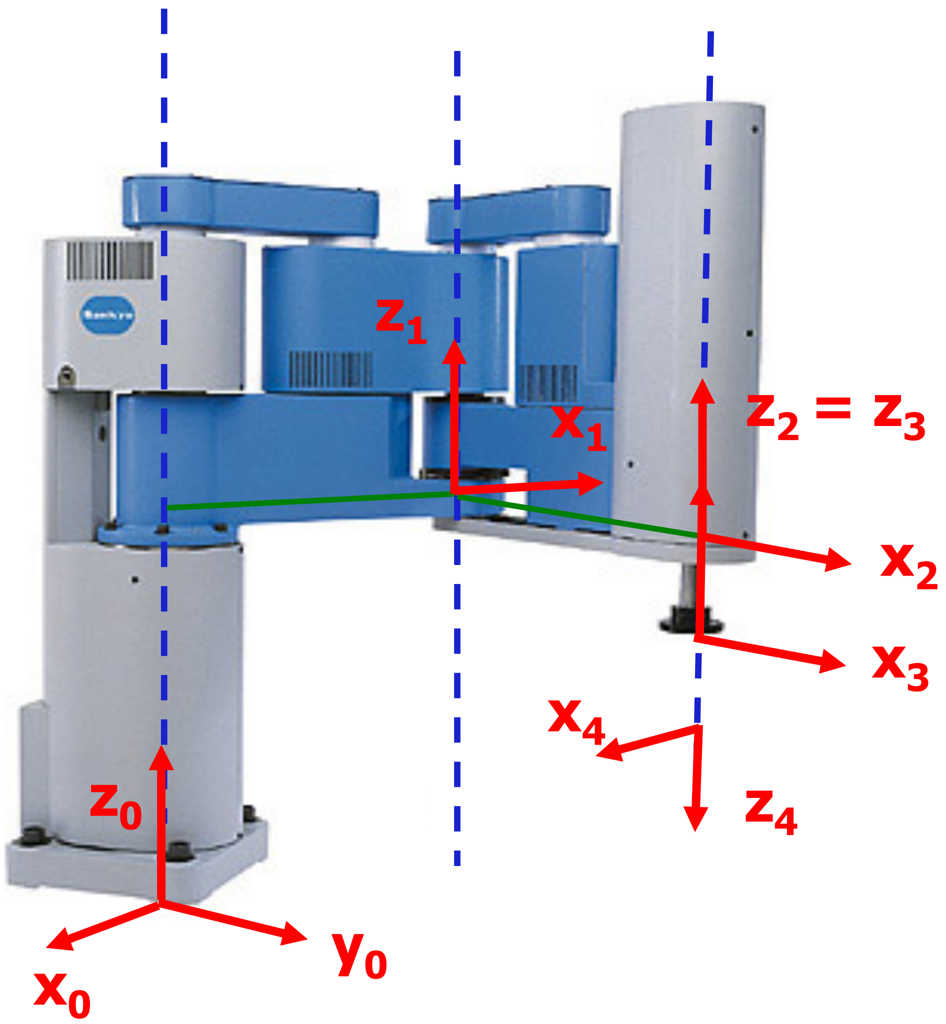


Step 3: frames

axes $\mathbf{y_i}$ for i > 0
are not shown
(nor needed; they form
right-handed frames)



Step 4: DH table of parameters



i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	q_1
2	0	a_2	0	q_2
3	0	0	q_3	0
4	π	0	d_4	q_4

note that:

- d_1 and d_4 could be set = 0
- here, it is $d_4 < 0$

Step 5: transformation matrices

$$ {}^0A_1(q_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1A_2(q_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^2A_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ q = (q_1, q_2, q_3, q_4) = (\theta_1, \theta_2, d_3, \theta_4) $$

$$ {}^3A_4(q_4) = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 & 0 \\ s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Step 6a: direct kinematics

as homogeneous matrix ${}^{B}T_E$ (products of ${}^{i}A_{i+1}$)

$$
{}^{0}A_3(q_1,q_2,q_3) = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & d_1 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}A_4(q_4) = \begin{bmatrix} c_4 & s_4 & 0 & 0 \\ s_4 & -c_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$R(q_1,q_2,q_4)=[\ n\ s\ a\ ]$

$p = p(q_1,q_2,q_3)$

$$
{}^{B}T_E = {}^{0}A_4(q_1,q_2,q_3,q_4) = \begin{bmatrix} c_{124} & s_{124} & 0 & a_1c_1 + a_2c_{12} \\ s_{124} & -c_{124} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 + q_3 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$({}^{B}T_0 = {}^{4}T_E = I)$

Step 6b: direct kinematics

as task vector $r \in \mathbb{R}^m$

$$
{}^0A_4(q_1,q_2,q_3,q_4) = \begin{bmatrix} \boxed{\begin{matrix} c_{124} & s_{124} & 0 \\ s_{124} & -c_{124} & 0 \\ 0 & 0 & -1 \end{matrix}} & \boxed{\begin{matrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ d_1 + q_3 + d_4 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}
$$

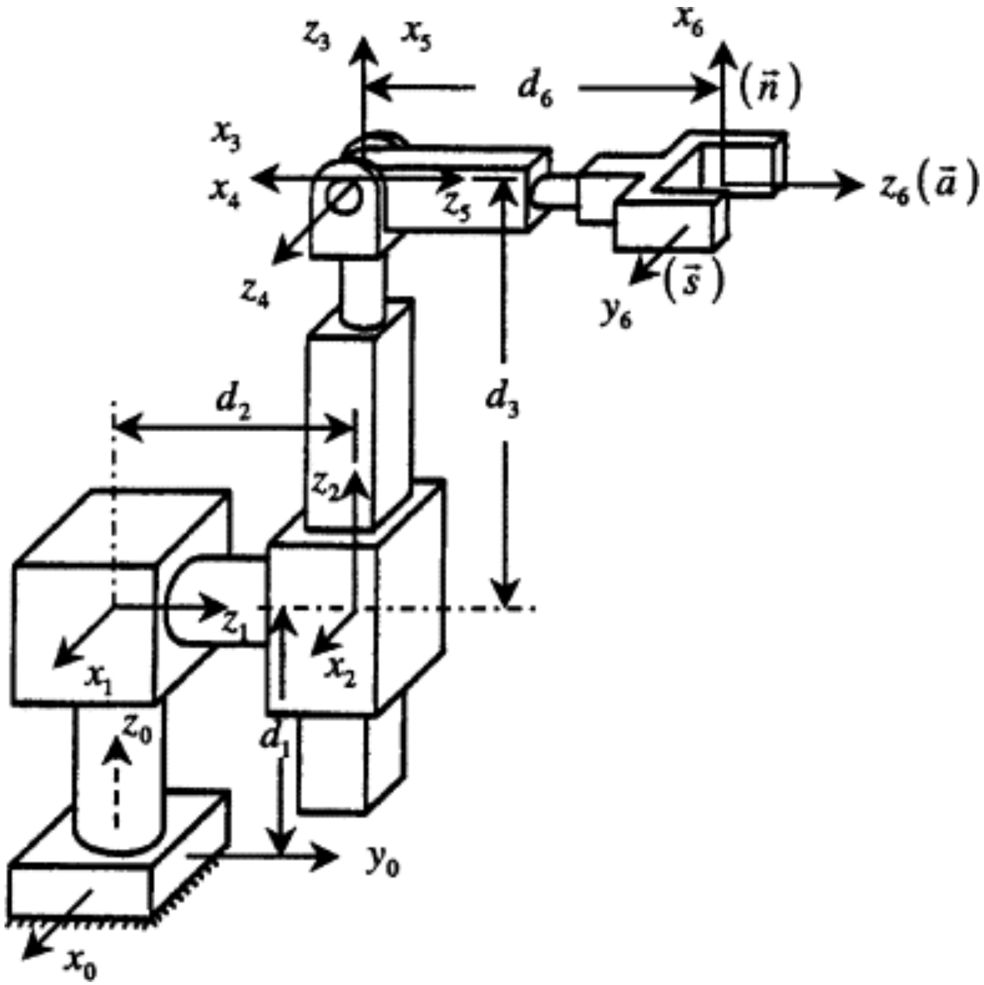
extract α_z from $R(q_1,q_2,q_4)$

take $p(q_1,q_2,q_3)$ as such

$$
r = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha_z \end{bmatrix} = f_r(q) = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ d_1 + q_3 + d_4 \\ q_1 + q_2 + q_4 \end{bmatrix} \in \mathbb{R}^4
$$

Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)



shoulder offset

"one possible" DH assignment of frames is shown

determine the associated

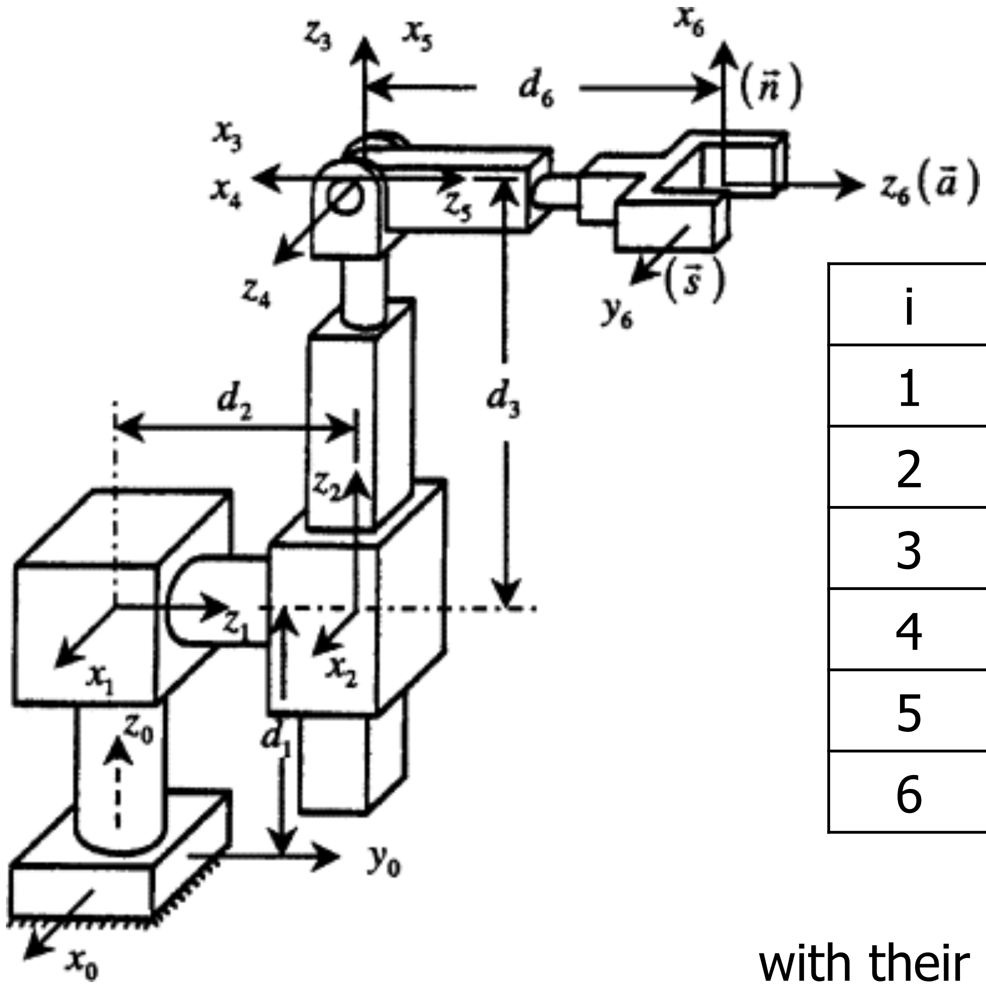
- DH parameters table
- homogeneous transformation matrices
- direct kinematics

write a program for computing the direct kinematics

- numerically (Matlab)
- symbolically (Mathematica, Maple, Symbolic Manipulation Toolbox of Matlab, …)

DH table for Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

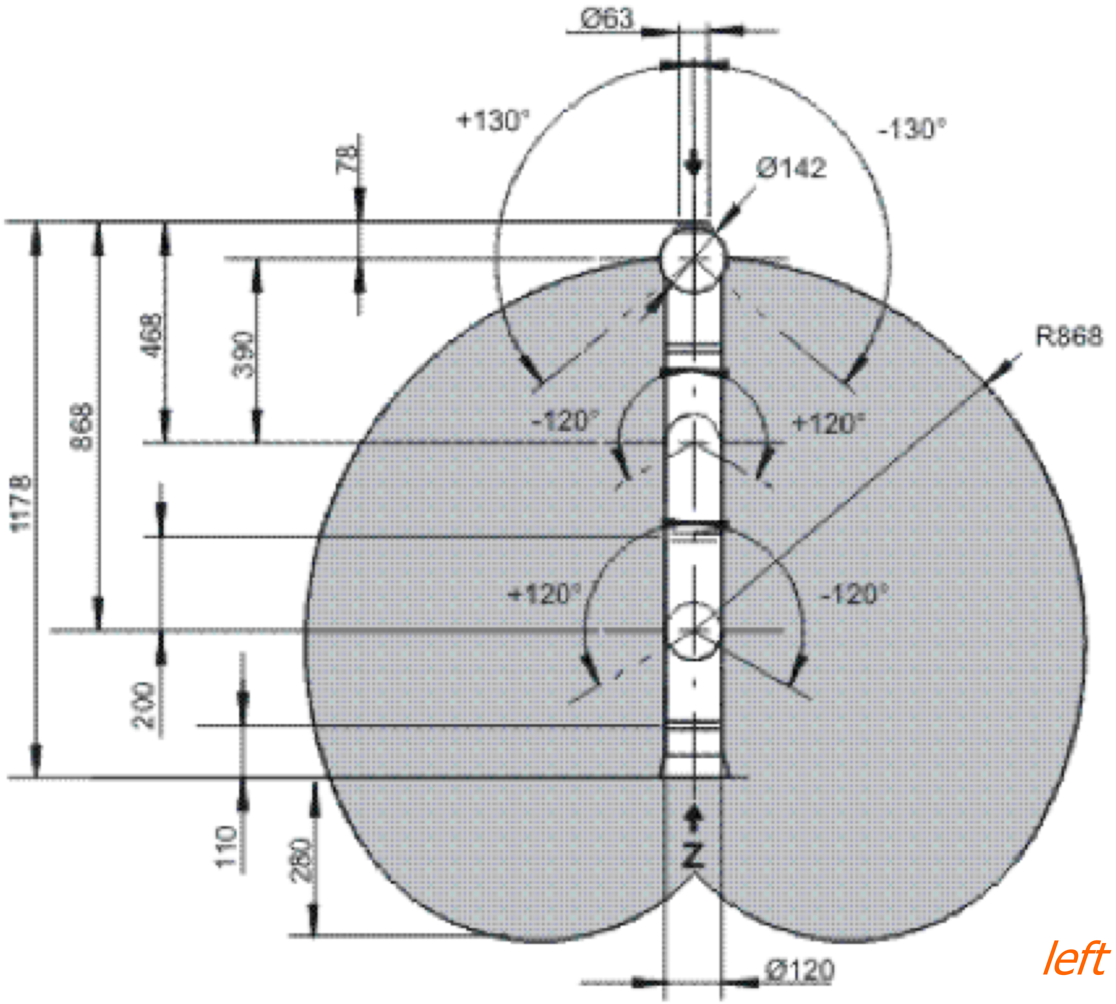


i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1>0$	$q_1=0$
2	$\pi/2$	0	$d_2>0$	$q_2=0$
3	0	0	$q_3>0$	$-\pi/2$
4	$-\pi/2$	0	0	$q_4=0$
5	$\pi/2$	0	0	$q_5=-\pi/2$
6	0	0	$d_6>0$	$q_6=0$

joint variables are in red,
with their current value in the shown configuration

KUKA LWR 4+

- 7R (no offsets, spherical shoulder and spherical wrist)



side view (from the left)

left  right

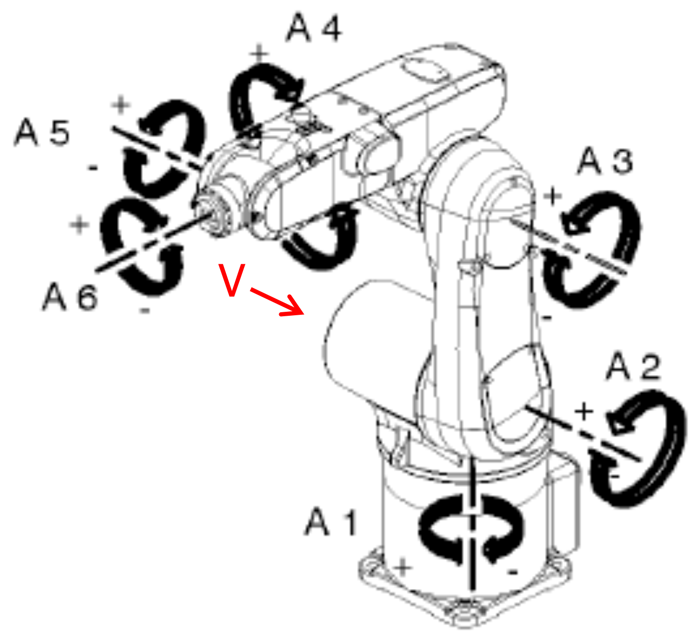
frontal view

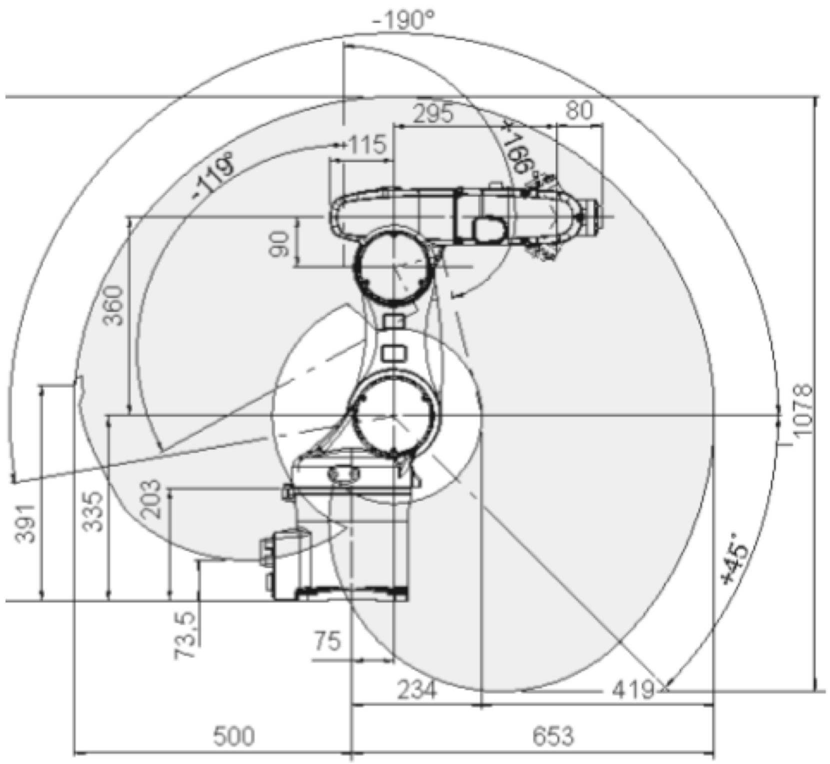
available at
DIAG Robotics Lab

- determine
  - frames and table of DH parameters
  - homogeneous transformation matrices
  - direct kinematics
  - d_1 and d_7 can be set = 0 or not (as needed)

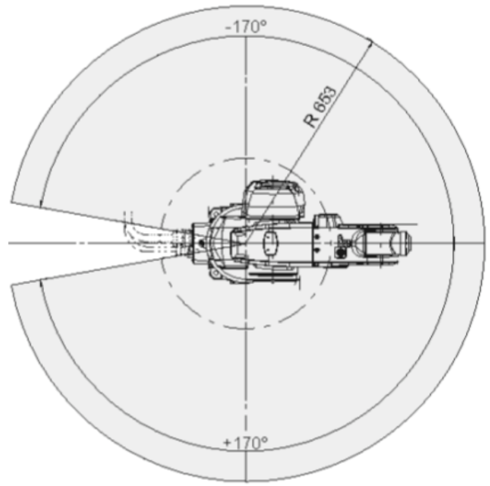
KUKA KR5 Sixx R650

- 6R (offsets at shoulder and elbow, spherical wrist)





side view (from observer in V)



top view

- determine
  - frames and table of DH parameters
  - homogeneous transformation matrices
  - direct kinematics

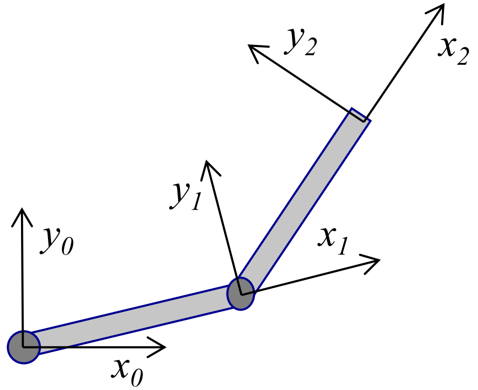
available at
DIAG Robotics Lab

Appendix: Modified DH convention

- a modified version used in J. Craig's book "Introduction to Robotics", 1986
  - has z_i axis on joint i
  - a_{i-1} & α_{i-1} = distance & twist angle from z_{i-1} to z_i, measured along & about x_{i-1}
  - d_i & θ_i = distance & angle from x_{i-1} to x_i, measured along & about z_i
  - source of much confusion... if you are not aware of it (or don't mention it!)
  - convenient with link flexibility: a rigid frame at the base, another at the tip...

classical (or distal)

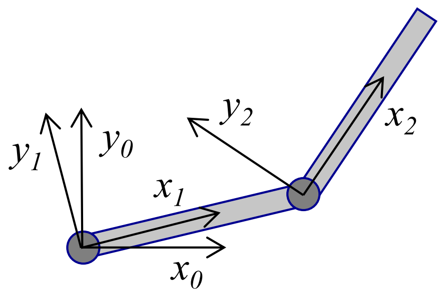
$$ {}^{i-1}A_i = \begin{pmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



planar 2R example

modified (or proximal)

$$ {}^{i-1}A_i^{\text{mod}} = \begin{pmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1} s\theta_i & c\alpha_{i-1} c\theta_i & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\alpha_{i-1} s\theta_i & s\alpha_{i-1} c\theta_i & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



modified DH tends to place frames at the base of each link