Robotics 1
Material and Textbook Cross-references

http://www.diag.uniroma1.it/deluca/rob1_en.php

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This document describes the relationships between topics in the course program, content of PowerPoint slides of the lectures (available as PDF files in the course website), and associated parts (chapters/sections) in the course textbook in English.

The names of the files with lecture slides are in the format "NN_filename.pdf", with the number of pages in parentheses.

Textbook:

B. Siciliano, L. Sciavicco, G. Villani, G. Oriolo: "Robotics: Modelling, Planning and Control", Springer, 2009 (3rd Edition)

Note:

The above is the translated version of the book:

B. Siciliano, L. Sciavicco, G. Villani, G. Oriolo: "Robotica: Modellistica, pianificazione e controllo", McGraw-Hill, 2008 (3a Edizione)

Organization of chapters and sections is the same in the English and Italian versions.

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Topic in the course program	Textbook cross-references	Slides (with number of pages) and more course material
Introduction		
Course introduction and information	---	00_Introduction.pdf (35) with 22 videos
Industrial robot manipulators	Chap. 1	01_IndustrialRobots.pdf (77) with 37 videos + 2024_WorldRobotics_Presentation_Industrial&Service.pdf 2024_WorldRobotics_ExecSummary_Industrial.pdf 2024_WorldRobotics_ExecSummary_Service.pdf 2023_WorldRobotics_Presentation_Industrial&Service.pdf 2023_WorldRobotics_ExecSummary_Industrial.pdf 2023_WorldRobotics_ExecSummary_Service.pdf 2022_WorldRobotics_Presentation_Industrial&Service.pdf 2022_WorldRobotics_ExecSummary_Industrial.pdf 2022_WorldRobotics_ExecSummary_Service.pdf (and many more…)
Service applications		02_ServiceRobots.pdf (69) with 32 videos
Components		
Mechanics and Actuators	Chap. 5	03_CompsActuators.pdf (29) with 10 videos
Proprioceptive sensors		04_CompsSensorsProprio.pdf (27) with 1 video
Exteroceptive sensors		*[full version for previous programs: 05_CompsSensorsExtero_2019-20.pdf (58) with 20 videos]* 05_CompsSensorsExtero_2020-21.pdf (37) with 16 videos
Robot programming Supervision and control architectures	Chap. 6	06_ProgrammingArchitectures.pdf (47) with 5 videos
Kinematic models of manipulators		
Representations for position/orientation Homogeneous transformations	Chap. 2: Par. 2.1-2.3, 2.5-2.6	07_PositionOrientation.pdf (25)
	Chap. 2: Par. 2.4, 2.7	08_EulerRPYHomogeneous.pdf (19)
Direct kinematics	Chap. 2: Par. 2.8 (except 2.8.3), 2.10	09_DirectKinematics.pdf (33) with 2 videos 09_Exercise_DH_KukaLWR4 (8) with 3 videos + Matlab symbolic code: dirkin_SCARA.m
Further examples of direct kinematics	Chap. 2: Par. 2.9 (except 2.9.2)	Robotics1_Homework1_10-11 (6) [KUKA KR5] Robotics1_Homework1_11-12 (14) [COMAU Smart5 NJ4] Data_ABB-IRB6400.pdf Data_COMAU-SmartS2.pdf

		Data_Fanuc-2000i.pdf Product_ABB-IRB6400PE.pdf Product_Bosch-SR6SR8.pdf
Inverse Kinematics (including numerical methods)	Chap. 2: Par. 2.12 Chap. 3: Par. 3.7.1-3.7.2, only begin of 3.7.3	10_InverseKinematics.pdf (46) with 2 videos Robotics1_Homework2_10-11 (8) Article_PaulShimanoMayer_KinInvPuma600_TSMC81.pdf Article_ManseurDoty_FastInvKin6R_IJRR88.pdf
Differential kinematics (including singularities)	Chap. 3: Par. 3.1-3.4, 3.6	11_DifferentialKinematics.pdf (35) + Matlab code: subspaces_3Rplanar.m
Inverse differential kinematics	Chap. 3: Par. 3.5, 3.7.4	12a_InverseDifferentialKinematics.pdf (28) with 3 videos
Statics and force transformations	Chap. 3: Par. 3.8 (except 3.8.3)	12b_StaticsForceTransformations.pdf (17)
Manipulability	Chap. 3: Par. 3.9	
Planning of motion trajectories		
Joint space trajectories	Chap. 4: Par. 4.1-4.2	13_TrajectoryPlanningJoints.pdf (33) with 8 videos [also as 13_…._plus.pdf version (38)]
Operational space trajectories	Chap. 4: Par. 4.3	14_TrajectoryPlanningCartesian.pdf (29) with 3 videos
Motion control		
Joint- and Cartesian-level kinematic control	Chap. 8: Par. 8.1 Chap. 3: Par. 3.7.5	15_KinematicControl.pdf (30) with 2 videos [also as 15_…._plus.pdf version (32)]
Independent joint control (dynamic, single axis)	Chap. 8: Par. 8.3-8.4	16_DynamicControlSingleAxis.pdf (17)

Written exams of Robotics 1

www.diag.uniroma1.it/deluca/rob1_en/material_rob1_en.html

All materials are in English, unless indicated (very old are in Italian)

Year	Date (mm.dd)	Number of exercises	Topics	Notes
2024	11.22 (Midterm Test)	5	Use of different representations for orientation/rotation: ZXY Euler, axis-angle, RPY-type YXY angles*; Homogeneous transformation matrix for the final pose of a cylindric body after a rolling motion, followed by two rotations about a fixed and a current axis*; DH frames and table for a PPR planar robot under special requirement, with world and end-effector transformations and direct kinematics*, plus primary workspace; Analysis of a transmission with 3 toothed gears and using an incremental encoder; Assignment of DH frames from a DH table for a RPPR spatial robot plus inverse task kinematics in analytical and numeric form*	solutions; MATLAB codes
2024	11.07	4	DH frames and table for a PPR planar robot; Inverse kinematics of the PPR planar robot for end-effector position and orientation; Geometric Jacobian of the PPR planar robot with three inverse velocity problems; Planning of a circular trajectory for the end-effector of the PPR robot	---

2024	09.19	4	DH frame assignment and Table of parameters of Kawasaki RS010N 6R robot with sperical wrist; Velocity of the center of wrist and angular velocity of last DH frame for different combinations of (max) joint velocities, using the data of the Kawasaki RS010N robot; Rest-to-rest minimum time transfer for joint 2 and 3, using the data of the Kawasaki RS010N robot; End-effector resolution of Kawasaki RS010N robot, based on resolution of incremental encoders at joints 2 and 3	solutions
2024	07.08	4	DH frames and table assignment for 7R Franka Research 3 robot with all constant parameters being non-negative; RPY-type YZX angles and their time derivatives from an orientation matrix and an angular velocity; Collaborative task with two 2R planar robots, one with a sliding base on a tilded line; Rest-to-rest trajectory planning on a linear PTP Cartesian path, using a trapezoidal speed with bounds in mixed space	solutions
2024	06.12	5	PTP orientation trajectory using ZXZ angles from rotation matrices with boundary conditions on angular velocity; Sketch of a robot with frames from a given DH table, computing the end-effector position with Jacobian and its singularity analysis; Spatial resolution and field of view of a camera-lens system; Singularity-free placement of a circular path for a 3R planar robot; PTP path and trajectory planning in the joint space for a 2R planar robot with data specified in Cartesian space	solutions
2024	02.16	4	Visual localization of an object from 2 (planar) cameras and pickup via inverse kinematics of a PPR planar robot (5 items); D-H frame assignment and table for 3-dof cylindrical robot, with workspace, Jacobian, singularities, and null space directions;	---

			Rest-to-rest trajectory planning for a 2R planar robot between singular configurations, with specified tangents and uniform scaling; Kinematic control law in the Frenet space for a 3R spatial robot on helical path, traced with constant speed	
2024	01.24	6	Direct and inverse problem with ZXY angles around fixed axes (RPY-type); DH frames and table for the Yaskawa Motoman GP7 robot, with computation of e-e position; Solution of a trigonometric equation in two variables; Geometric Jacobian of a 3R robot known by its DH table, with singularity analysis and computation of the velocity of a point known in e-e frame; Inversion of a desired task trajectory for a 3R planar robot at the position, velocity, and acceleration level; Trajectory planning from rest to nonzero final velocity and maximum velocity computation	solutions
2023	11.15 (Midterm Test)	7	Euler XYZ angles to return a given Euler ZY rotation (symbolic/numeric)*; Axis-angle rotation to return two rotations about fixed axes x and v (symbolic/numeric)*; 2R workspace with limited joint ranges; 2R D-H table from unusual frame assignment*; Steady-state velocity and torque of DC motor with constant voltage, unloaded or with inertial load and reduction; D-H frame assignment and D-H table for a 5R robot, satisfying specifications on q=0 configuration and on positive direction of joint rotations*; RPR planar robot: direct kinematics, inverse kinematics for a 3-dimensional task defined with respect to a 2R robot*	solutions; MATLAB codes
2023	09.11	1 (8 parts)	4P planar robot: -DH frames and table; -homogeneous transformations with world and end-effector frames;	---

			-direct kinematics from world to end-effector; -task Jacobian; -null space of task Jacobian and range space of its transpose; -minimum norm joint velocity for a given task velocity; -rest-to-rest linear trajectory planning between two task points with max joint velocity and acceleration bounds	
2023	07.10	1 (8 parts)	RPR planar robot: -workspace with prismatic joint limits; -direct kinematics with non-DH (beta) joint variables; -inverse kinematics in closed form for the beta joint variables; -assignment of DH frames & table of parameters (with q joint variables); -direct kinematics with the DH (q) joint variables, with direct and inverse transformation between beta and q; -task Jacobian in q and its singularities; -analysis of range and null subspaces of J and J transpose in a singularity; -rest-to-rest trajectory planning between two task points (without violating limits of joint 2) --is a linear Cartesian path possible?	---
2023	06.12	3	DH frame assignment and complete table for the ABB CBR 15000 robot; Minimum time cubic rest-to-rest timing law on a circular path with bound on the Cartesian acceleration norm; Geometric Jacobian, singularity analysis and velocity pseudoinversion for a 4R spatial robot	solutions
2023	03.24	2	DH frame assignment and table for a 6-dof 3P-3R robot, geometric Jacobian with singularity check, computation of the positional direct kinematics and inversion of the velocity mapping at a given configuration; Minimum time rest-to-rest motion of a 2P Cartesian robot under bounded force inputs	---

2023	02.13	4	DH frames and table of parameters for a planar RPPR arm (with outreach computation)*; Compare the difference of ZYZ Euler angles from two rotation matrices and the ZYZ Euler angles extracted from the relative rotation matrix*; Numerical inverse kinematics solution by Newton method (for a 2R planar arm)*; Analysis of the singularities and of the relevant subspaces for the 6x4 geometric Jacobian of a 4-dof robot characterized by its DH table	solutions; MATLAB codes
2023	01.23	5	DH frames and table of parameters for the PAL TIAGo 8-dof arm*; Direct kinematics of the wrist center for the PAL TIAGo 8-dof arm with respect to a world frame*; Algebraic solution of a single kinematic equation in two unknowns; Complete analysis of the Jacobian subspaces for a planar PRPR arm*; Minimum time rest-to-rest motion along a parametrized helix, under velocity/acceleration bounds in the tangent/normal directions of the Frenet frame, with a good placement of a 3R robot to perform the task*	solutions; MATLAB codes
2022	11.18 (Midterm Test)	6	Extraction of an angle and axis from a rotation matrix (singular case)*; Inverse representation of relative rotation matrix with YXY Euler angles*; Analysis of a transmission/reduction system with HD and spur gear*; Definition of a task kinematic equation with homogeneous transformations*; DH frames, table and direct kinematics of a spatial RPR robot*; Inverse kinematics for the position of a spatial RPR robot*	solutions; MATLAB codes
2022	10.21	4	For a spatial RPR robot: - DH frames, table and direct kinematics; - Jacobian and complete singularity analysis (with subspaces); - Kinematic control for regulation without planning; Minimum time rest-to-rest motion for a 2R planar with bang-bang acceleration inputs under maximum joint velocity bounds	solutions

2022	09.09	4	DH frame assignment and table of parameter for the 6R Fanuc cr15ia robot with offsets and spherical wrist; Questionnaire (10 true/false, explain) on the inverse kinematics problem; Kinematic analysis in velocity and acceleration of a 3dof robot*; Planning minimum time motions of a single joint with velocity/acceleration bounds in the rest-to-rest and state-to-rest case*	solutions; MATLAB codes
2022	07.08	4	Relationship between derivative of XZY angles w.r.t. fixed axes and angular velocity of an end-effector, with analysis in singularity*; Closed form inverse kinematics for a 3R spatial robot with offset*; Newton iterative method for the same problem of the previous exercise*; Point-to-point path in joint space for same robot of the previous two exercises, continuous up to the acceleration and with initial velocity coming from Cartesian space*	solutions; MATLAB codes
2022	06.10	4	DH frames and table for a 3R spatial robot with offset, its primary workspace, its square Jacobian with analysis of singularities, and admissible end-effector velocity in a double singularity*; Computation of an instantaneous joint acceleration for an RP robot having a non-zero joint velocity, so as to zero the end-effector acceleration; Static equilibrium for the RP robot with a linear force applied at the tip under bounds on the joint generalized forces*; Trajectory tracking in the Cartesian space for a planar 2R robot, with error dynamics that complies with maximum joint velocity limits*	solutions; MATLAB codes
2022	04.05	2	DH frames and table for 4R spatial robot, with end-effector homogenous transformation, direct kinematics for position, angular part of the geometric Jacobian and its associated null-space joint velocity; Analysis of a double bang-bang jerk profile of motion	---
2022	02.03	5	DH frames and table for the Crane-X7 robot (7R); Axis-angle extraction from relative rotation between initial orientation expressed by YXY Euler angles and final orientation expressed by rotation matrix;	solutions

			Task nominal inversion and feedback control at the acceleration level with singularity analysis for planar 3R arm; Trajectory planning for a planar PR robot using a two-cubic spline and a via point so as to avoid an obstacle; Transmission/reduction system with gears and pulleys	
2022	01.11	7	D-H frames and table for a planar RPR robot with L-shaped forearm, draw the robot for q=0 and compute e-e position and orientation, find the rotation matrix ${}^3R_e$ and extract the XYX Euler angles; Inverse task kinematics for the above RPR robot; Task Jacobian for the above RPR robot and singularity analysis; Wrench transformation of F/T sensor measures and joint torques for static equilibrium for the above RPR robot; Cartesian trajectory planning on a parametrized elliptic path; Tracing the elliptic trajectory with joint velocity commands in the nominal case and with feedback control; Minimum number of bits for a multi-turn absolute encoder	solutions
2021	11.19 (Midterm Test)	9	Questionnaire with 9 items: sequence of axis-angle and elementary y-rotation around fixes axes; ZYZ Euler inverse formulas to match a desired relative rotation; DH frames, table and gripper frame for a 4R spatial robot with spherical shoulder; primary workspace of a 2R robot with joint limits; task kinematics of a planar 5-dof bi-manual robot; analytical inverse kinematics for the RRP planar robot, with a numerical example; motor torque computation for a desired link acceleration in a geared wheel transmission; minimum resolution of an absolute encoder in a motor-transmission-link system; three sentences to describe SCARA robots	solutions
2021	10.19	2	For a spatial PPR robot: DH frame assignment and table, gripper orientation, direct kinematics for position, Jacobian computation, singularities and Cartesian mobility;	---

			Smooth coordinated rest-to-rest joint trajectory planning for the same PPR robot	
2021	09.10	2	For a planar PRR robot: direct task kinematics, task Jacobian and singularities, null space/range space analysis in a singularity providing numerical examples, inverse task kinematics in closed form, primary and secondary workspaces*; For the same planar PRR robot, a joint trajectory planning problem for a specific position/orientation task*	solutions; MATLAB codes
2021	07.12	3	DH frame assignment and table for a 4-dof spatial RRRP robot, with direct kinematics in position, Jacobian, and singularity analysis*; Analysis of a joint axis having a DC motor with double reduction gear and encoder, with maximum torque for a bang-bang link acceleration motion*; Smooth trajectory planning in orientation, interpolating with splines the Euler ZYX angles of three given rotation matrices*	solutions; MATLAB codes
2021	06.11	3	DH frame assignment and table for a portal robot (3P) with a spherical wrist (3R); A motion task and a static balancing task for a planar RPR robot (with inverse kinematics)*; Smooth coordinated rest-to-rest trajectory planning in position and orientation, with continuity up to the acceleration*	solutions; MATLAB codes
2021	02.04 (Remote)	8	Questionnaire with 8 items: from an axis-angle representation to XYZ RPY angles and their singularity analysis; a two-jaw 5-dof gripper (DH, kinematics, and definition of task variables); small encoder errors and their effects on Cartesian accuracy for a 2R robot; primary and secondary workspace of a planar PPR robot with bounded prismatic joints; steps of Newton and Gradient methods for the inverse kinematics of a RP robot near a singularity; 3R pointing device and singularity of its angular geometric Jacobian; trajectory planning between two given orientations, with final non-zero angular velocity	solutions

			assigned; kinematic control for the self-motion of a planar 3R (using null-space projection or joint space decomposition)	
2021	01.12 (Remote)	10	Questionnaire with 10 items: inverse problem and singularity example with Euler YXZ sequence; time derivative of an orientation matrix from the angular velocity in body frame; recognize the inverse of the DH homogeneous matrix; DH frame assignment and table, with direct position Kinematics of a spatial 2R robot; analysis of a multi-turn absolute encoder; singularity analysis of a planar RRPR robot in a 3-dimensional task; joint torques for two 2R robots in equilibrium when exchanging a Cartesian force; coordinated joint trajectory planning for a 2R robot in a rendez-vous task with a moving target; minimum time for given trajectory profiles of a 2R robot under joint velocity and acceleration bounds; Cartesian kinematic control in rotated frame for the same previous rendez-vous task	solutions
2020	11.20 (Remote Midterm Test)	10	Questionnaire with 10 items: ZYX sequence around fixed axes*; axis/angle from a relative rotation*; why 4 parameters only in DH frame transformations; computational issues in products; why kinematic vs. torque commands; DH frame assignment and parameter table for a spatial PRPR robot*; inverse kinematics in analytic form for this PRPR robot*; numerical derivative of position measures by 1-step BDF (Euler) formula; link displacement and resolution in a 2R arm with a transmission belt and an incremental encoder; kinematic definition of a task*	solutions; MATLAB codes
2020	10.27 (Remote)	2	Analysis of the kinematics (direct, inverse, and differential with singularities) of a spatial RRPR robot without using DH variables; Motion computation with a jerk profile of the bang-coast-bang type	---
2020	09.11 (Remote)	5	Angular acceleration from second time derivative of a rotation matrix; DH table associated to an assigned set of frames for the UR5 manipulator;	solutions

			Cooperative task (handing over of an object) of two planar 2R and 3R robots; Singularity analysis of a 3x3 robot Jacobian matrix, with computation of linear subspaces in the rank 1 case; Minimum time in a state-to-rest task for a single mass under bounded force input	
2020	07.15 (Remote)	5	DH frames assignment and related table for a 4-dof (PRRR) planar robot; Trajectory planning on a eight-shaped path for a Cartesian planar robot under velocity and acceleration bounds; Euler YXY rotation matrix and relation between the time derivative of these angles and the angular velocity; Inverse kinematics and inverse differential kinematics for a planar 3R robot on a linear path with specified end-effector orientation; Questionnaire with 2 questions	solutions
2020	06.05 (Remote)	5	Complete a DH frame assignment and related table for a 4R spatial robot; Direct and differential kinematics for a planar RRP robot and joint torques balancing a Cartesian force in regular or singular configurations; Analysis of range and null spaces of the 3x3 Jacobian of a spatial 3R robot, with inverse differential solution in a singularity; Minimum-time smooth rest-to-rest trajectory planning for a 2R robot with joint velocity and acceleration bounds; Questionnaire with 3 questions	solutions
2020	02.12	4	DH frame assignment and table of parameters for a spatial 4-dof PRRR robot; For the same robot above: direct positional kinematics, balancing joint torque of a Cartesian force, angular Jacobian with its singularities and null space; Trajectory planning in the Cartesian space for the end-effector of a PPR planar robot moving in contact with a circle with bounds on velocity and acceleration in the joint space;	solutions

			Questionnaire with 8 questions (very mixed in nature)	
2020	01.07	4	DH frame assignment and table of parameters for the 7R Cesar arm*; Linear part of the geometric Jacobian of the 7R Cesar arm and its use for the numerical solution of an inverse kinematics problem with the Gradient method; Trajectory planning in the Cartesian space for the end-effector position and orientation of an RPR planar robot moving in contact with a linear surface*; Questionnaire with 8 questions (very mixed in nature)	solutions; MATLAB codes
2019	11.29 (Midterm Test in classroom)	5	Computation of orientation using ZYX angles w.r.t. fixed axes (RPY) and axis-angle methods, in a rotated and or in the base frame; DH table from a given assignment of frames for the 6R UR10 manipulator; Workspace analysis, DH frame assignment, and three inverse kinematics problems for a planar 2R robot with a L-shaped second link; Iterative numerical step/solution with Newton method for the inverse solution of a 3-dimensional kinematic task; Questionnaire with 7 questions (mostly on sensing and actuation)	solutions
2019	09.11	3	Joint acceleration command for zeroing the end-effector acceleration in a state (position/velocity) of a 3R planar robot (with analytic Jacobian computation); Assigned bang-(coast-)bang profiles for a RP planar robot: sketch time evolution in joint space and compute end-effector velocity and acceleration (with norms); Analysis of the resolution of a laser sensor mounted on the tip of a rotating link, driven by a DC motor with reduction and incremental encoder	solutions
2019	07.11	3	Static balancing torque for an off-centered payload in a 3R planar robot; Base placement for a RP planar robot with limited joint range in order to execute a linear path within its workspace*;	solutions; MATLAB codes

			Time-optimal motion of a joint with a prescribed structure of bang-coast-bang acceleration profile, under velocity and acceleration bounds*	
2019	06.17	2	DH assignment, joint range matching, direct kinematics, inverse kinematics of the wrist center, and linear Jacobian of the wrist center for the 6R Kawasaki S030 robot*; Specifying the end-effector velocity in different ways/representations and their relationships, with a simple numerical example*	solutions; MATLAB codes
2019	02.05	3	DH assignment and direct kinematics for a 4R spatial robot*; Geometric Jacobian and singularity analysis*; Kinematic control of a 2R planar robot along a circular trajectory in the Cartesian space, with inverse kinematics initialization*;	solutions; MATLAB codes
2019	01.11	3	DH assignment for a RRPR spatial robot*; Its geometric Jacobian and force/velocity analysis in Cartesian/joint space*; Rest-to-rest minimum time trajectory planning for a PR planar robot under joint acceleration and Cartesian acceleration norm bounds*	solutions; MATLAB codes
2018	11.16 (Midterm Test in classroom)	5	Computation on orientations using various representations*; Generating a DH homogeneous matrix; DH table from frames for a 7R anthropomorphic manipulator*; Analysis of a DC motor servo drive; Iterative step of a numerical solution of the inverse kinematics of a RP planar robot*	solutions; MATLAB codes
2018	07.11	2	Geometric analysis and direct and inverse differential mappings for a minimal representation of orientation by rotations around the sequence of fixed axes YXZ; Trajectory planning/control at the acceleration level for a RP robot executing a circular motion in the Cartesian plane	solutions
2018	06.11	2	Planar 2R robot with L-shaped second link: DH frames and table, direct kinematics; special configurations and primary workspace, inverse kinematics, analytic Jacobian, singularities and range/null spaces, inverse	solutions

			kinematics and inverse differential kinematics solutions on numerical data; Minimum time rest-to-rest trajectory planning, with joint velocity and acceleration bounds and joint coordination	
2018	03.27	2	DH frames/table for a 5-dof spatial RRPRP robot, with all non-negative constant parameters, sketch of two configurations, its geometric Jacobian, and a basis for null-space wrenches (forces/moments) in a given configuration; Questionnaire on singularity issues in 6-dof manipulators	solutions
2018	02.05	4	DH frames/table for 4R Comau e.Do robot, with all non-negative constant parameters; Questionnaire on sensors for manipulators and related measurements issues; Geometric Jacobian derivation for the 4R Comau e.Do. robot, with analysis of the singularities and computation of a null-space joint velocity; Cubic spline interpolation of four knots in time, check of velocity/acceleration limits and with uniform time scaling*	solutions, MATLAB code
2018	01.11	4	DH frames/table for 7R Franka Emika (Panda) robot, with evaluation of elementary operations in direct kinematics; Questionnaire on numerical methods for inverse kinematics; Definition of a coordinated task (position, orientation, and linear velocity of the end-effectors) for two planar 3R manipulators; Smooth rest-to-rest trajectory planning for a RP robot, with uniform time scaling to satisfy at best joint velocity and acceleration limits	solutions
2017	11.24 (Midterm Test in classroom)	4	DH frames/table for a planar RPR robot, direct kinematics with two different sets of coordinates and their relation; Analysis of a transmission/reduction assembly (incremental encoder choice to guarantee a Cartesian resolution); DH frame assignment associated to a given DH table for Stäubli robot RX 160;	solutions

			Inverse problem for an axis-angle representation of a relative rotation matrix	
2017	10.27	1	DH frame assignment and table of parameters for Stäubli robot RX 160, with comparison to joint angles and limits of the manufacturer, and computation of the position of the wrist center and of the angular part of the geometric Jacobian for the first three joints	---
2017	09.21	3	Rotations, final orientation, angular velocity, and linear velocity of the tip for a thin rod*; Rest-to-rest cubic trajectories in minimum time for a planar 2R robot under maximum joint velocity bounds; Analysis of a transmission/reduction assembly	solutions; MATLAB code
2017	07.11	3	DH assignment for a 5-dof cylindrical robot and geometric Jacobian; Cubic, quintic, and seventh-degree polynomial trajectories for a rest-to-rest motion, and their minimum time under maximum velocity or acceleration bounds; Discuss incremental vs absolute encoders, their mounting, and ways to measure the robot end-effector position	solutions
2017	06.06	3	Direct kinematics of a planar PRPR robot in different coordinates (manufacturer and DH) and their mapping; Definition of kinematic control laws for a 3R elbow-type robot in reaction to human presence in the Cartesian space sensed by laser scanning; Geometric cubic spline through four knots (for a single joint) and time properties of the associated trajectory executed with constant speed*	solutions; MATLAB code
2017	04.11	2	Inverse kinematics of a spatial 3R (elbow-type) robot in analytic form (with numerical example); Minimum-time motion of a joint with generic non-zero boundary velocities under velocity and acceleration bounds (with numerical example)	solutions
2017	02.03	3	DH frames assignment and table of parameters for the left arm of the NAO humanoid robot;	solutions

			Minimum-time rest-to-rest motion between two Cartesian points for a RP planar robot under joint velocity/acceleration bounds, followed by manipulability/singularity analysis; Kinematic control of a 3R planar robot on a linear Cartesian trajectory with continuous acceleration, imposing specified transients along the tangent and normal to the trajectory	
2017	01.11	3	DH frames assignment, table of parameters, and task Jacobian for a planar RPRP robot, with singularity analysis; Rest-to-move Cartesian planning of path and timing law for a planar 2R robot, with continuity of velocity; Completing the geometric Jacobian of a 3R spatial robot, with rank analysis, and its use for static balance of forces/torques applied to end-effector	solutions
2016	11.18 (Midterm Test in classroom)	4	Specific DH frame assignment for Universal Robot UR5; Use of homogeneous transformation matrix, with ZYX Euler angles representation; DH frames/table for a 2R robot moving in 3D, with direct kinematics computation; Inverse problem for an axis-angle representation of a (rotation?) matrix	solutions
2016	10.28	2	Interpreting a given rotation matrix parametrized by two angles in fixed and moving axes; Second-order inverse differential kinematics and control for a planar 2R robot*	solutions; MATLAB code
2016	09.12	2	Inverse kinematics for the wrist of the UR10 robot; Jacobian, singularities and inverse differential solutions for a planar RPR robot with skewed prismatic joint	solutions
2016	07.11	3	Analysis of a single cubic joint trajectory with non-zero final velocity*; Angular Jacobian from a DH table of a 3R arm and its singularities*;	solutions; MATLAB codes

			Inverse (differential) kinematics of a planar 2R arm to match a desired Cartesian velocity and design of a kinematic control law to recover initial errors*	
2016	06.06	2	DH frame assignment and table of parameters for the 6R Universal Robot UR5; (Pseudo-)code for the iterative numerical solution to the inverse kinematics of a planar 3R robot in positioning tasks	solution of Ex #1 only
2016	04.01	1	Rest-to-rest smooth and coordinated trajectory planning in minimum time for a 2R robot moving between two Cartesian positions under joint velocity, acceleration, and jerk limits	solution
2016	02.04	4	Completing the definition of a rotation matrix; Inverse kinematics of a 2-dof robot, specified by its DH table, with joint range; 4R planar robot performing a simultaneous double velocity task; Minimum time trajectory planning on a rectangular path, with bounds on the norms of the Cartesian velocity and acceleration and continuity of the velocity solution.	solutions
2016	01.11	3	Denavit-Hartenberg frame assignment for 5-dof KUKA KR60 L45; Inverse differential kinematics and static solution for a planar 3R robot*; Planning through a singularity and Cartesian kinematic control of a planar 2R robot*.	solutions; MATLAB codes
2015	10.27	2	Joint acceleration command for obtaining a desired Cartesian acceleration, at its numerical evaluation for a planar 2R robot; Analysis of a multiple-gear transmission.	solutions
2015	09.11	3	Angular velocity of a spherical wrist; Inverse kinematics in closed form for a spatial RPR robot; Singularities and null/range space analysis of the task Jacobian for a planar 3R robot.	solutions
2015	07.10	2	Analysis and displacement computation for an assigned bang-bang type profile of the snap (4^{th} time derivative);	---

			Placing of the base of a planar 2R robot for executing a straight line in its workspace and joint velocity computation at a singular configuration.	
2015	06.05	2	Path planning with an helix in 3D and minimum time rest-to-rest motion with cubic timing profile and bounded norm of Cartesian velocity; Placing of the base of an elbow-type 3R robot for executing a straight line in its workspace and joint velocity computation at a specific configuration.	---
2015	04.01	1	Minimum-time trajectory planning between two Cartesian points for a planar 2R robot under joint velocity and joint acceleration constraints.	same as 2006.07.13 (in Italian), with modified data
2015	02.06	2	Complete inverse kinematics analysis in orientation for a 3-dof robot, including singular or regular numerical cases and an inverse differential problem; Planning a Cartesian trajectory on a circular path of given radius between two points, with trapezoidal speed and bounds on the norms of the velocity, of the acceleration, and of the normal acceleration.	solutions (also longer version available)
2015	01.09	3	Effect of incremental encoder resolution on the accuracy of end-effector position measure for a planar 2R robot*; Planning of a Cartesian straight-line trajectory for a RP planar robot, to be executed in minimum time under joint range and joint velocity limits*; Kinematic control with prescribed Cartesian transient error for a 3R anthropomorphic robot*.	solutions; MATLAB codes
2014	11.21 (Test in classroom)	4	Reduction ratio and optimal inertia/acceleration of joint 2 of PUMA 560 robot; DH table of parameters from assigned frames of a PUMA 560 robot; Primary workspace of a generic planar 3R manipulator; Inverse kinematics in closed form of a 3P-3R spatial robot with spherical wrist	solutions

2014	10.27	2	Inverse representation problem and analysis of relation between angular velocity and derivative of Euler angles XYZ; Geometric Jacobian of SCARA-type robot and solution of a problem of inverse differential kinematics in a singularity	solutions
2014	09.22	1	DH frames and table for the Siemens Artis Zeego medical robot, having 7 DOFs (one prismatic and six revolute joints)	---
2014	07.15	1	7R KUKA LWR robot, with frozen last three joints: direct kinematics of the tool center point and related Jacobian, solution to the inverse kinematics when one joint angle is assigned, singularity analysis	---
2014	06.10	1	DH frames assignment and table for the COMAU RACER 7-1.4 robot, and mapping by comparison with the one used by the robot manufacturer	solution
2014	04.02	3	Draw the DH frames of a 4R robot and the direct kinematics (position only), given the DH table; For the same robot, static torques balancing a desired force; Smooth minimum time rest-to-rest motion of a single joint under velocity and acceleration bounds	---
2014	02.06	3	Definition and use of the Jacobian transpose for force transformations; A 4-3-4 trajectory planning problem: formulation and solution*; Proof of Cartesian trajectory tracking using both the Jacobian transpose (feedback) and the Jacobian inverse (feedforward)	solutions; MATLAB code
2014	01.09	3	PPR planar robot: DH frame assignment and table, primary and secondary workspace for bounded range of prismatic joints; Planning of rest-to-rest orientation trajectory using YZY Euler angles, with motion time satisfying a bound on the norm of angular velocity*; Joint velocity commands in a 6R robot with spherical wrist for planning or tracking (kinematic control) end-effector trajectories with zero desired angular velocity	solutions; MATLAB code
2013	11.29 (Test in classroom)	3	Optimal reduction ratio of a cascaded spur gear and harmonic drive transmission; K-1207 7-dof robot: DH frames and table of parameters;	solutions

			Planar RPR manipulator: inverse kinematics for planar pose, primary workspace for limited range of prismatic joint	
2013	09.19	1	Planar RPPR manipulator: DH frames and table of parameters, analysis of maximum reach with limits on the prismatic joints	---
2013	07.15	1	Analysis of a joint velocity motion of trapezoidal type for a planar 2R arm, with evaluation of selected Cartesian quantities (displacement, velocity, acceleration)	---
2013	06.10	1	4R spatial manipulator: assignment of DH frames, Jacobian for the linear velocity, and analysis of feasible motion at a singularity	---
2013	04.10	1	Minimum time trajectory planning for planar 3R manipulator on a three-dimensional rest-to-rest task, with joint velocity and acceleration bounds*	solution; MATLAB code
2013	02.06	2	DH assignment and geometric Jacobian of a 4-dof robotic finger; Trajectory interpolation with a class of trigonometric functions, with analysis of wandering*	solutions; MATLAB code
2013	01.09	3	Definition of a minimal representation of orientation, and singularities of the associated differential relation; Singularities and minimum norm joint velocity solution for a planar 4R arm; Effect of encoder errors on the end-effector position estimate of a 3R anthropomorphic robot	solutions
2012	09.10	1	DH frame assignment to elbow-type 3R robot, with analysis of linear and angular velocities of the end-effector in a given configuration	solution
2012	07.05	1 (4 parts)	6-dof portal robot for aeronautical industry: pointing task; inverse kinematics; positioning task and its inverse kinematics; solution for numerical data*	solutions; MATLAB code
2012	06.11	3	Derivative of a rotation matrix in fixed or rotated frame; Jacobian, singularities, and null/range spaces analysis of planar RPR arm; Resolution of incremental encoders for a Cartesian task of a 2R robot	solutions
2012	04.26	2	DH assignment and Jacobian expressed in camera frame of 3R articulated arm (symbolic MATLAB code included);	solutions; MATLAB code

			Rest-to-rest orientation planning with axis-angle method and cubic timing law*	
2012	02.09	3	Angular velocity of the COMAU NJ4 170 robot with non-spherical wrist; DH assignment, Jacobian, and singularities of RRP (polar) arm; Planning of straight Cartesian paths, singularity handling, and joint vs. Cartesian kinematic control for the RRP arm	solutions
2012	01.11	2 + bonus	Primary and secondary workspace of a planar 3R arm, singularities, and manipulability index H (bonus: write a MATLAB* program plotting H); Rest-to-rest minimum time motion between two Cartesian poses, with bounds on joint velocity and acceleration	solutions; MATLAB code
2011	09.12	1	Inverse differential kinematics for a SCARA-type robot for two 6-dimensional desired task velocities	solution
2011	07.04	1	Barrett 4-dof WAM: D-H frame check, direct kinematics, actuator transformation, linear velocity Jacobian*, singularity and joint limit check	solution; MATLAB code
2011	06.17	1	Polytopes of feasible Cartesian velocity for a 2R planar robot with joint velocity bounds in different configurations*	solution; MATLAB code
2011	02.25	1	Cyclic joint trajectory design, singularity crossing and time scaling for a 3R anthropomorphic robot*	solution; MATLAB code
2011	02.03	1	Various Jacobians with their analysis and a joint acceleration synthesis for a 3R anthropomorphic robot	solution
2010	09.15	1	Trajectory definition with double symmetric bang-coast-bang jerk profile*	solution; MATLAB code
2010	07.07	1	DH assignment for the 6R KUKA KR-30-3 robot and direct kinematics of the center of its spherical wrist	solution
2010	06.15	2	Singularities for a RP planar robot in a one-dimensional task and kinematic control at the joint acceleration level; Relation between angular velocity and derivative of Euler angles YXZ	solutions
2010	02.11	1	Path planning for a 2R planar robot among obstacles with singularity crossing*	solution; MATLAB code

2010	01.12	2 (one in common, option A or B for the other)	Cartesian trajectory planning on spiral path for position and orientation with velocity/acceleration constraints and trapezoidal speed profile*; (A) Input-output linearization control for front-wheel drive car-like; or (B) Geometric Jacobian for a cylindrical robot, singularities, and kinematic Cartesian control in acceleration	solutions (with options A and B); MATLAB code
2009	12.17 (Test in classroom)	1	Geometric Jacobian for a 4R spatial robot, feasibility of a Cartesian linear/angular velocity, minimum norm joint velocity solution, and joint torque balancing a Cartesian force/torque*	solution; MATLAB code
2009	11.10 (Test in classroom)	2	Minimal representation of orientation around fixed YXZ axes; DH assignment for a spatial 3R arm pointing a head camera, direct kinematics for the orientation, and condition for an infinite number of inverse solutions	solutions
2009	09.10	1	Jacobian of mobile manipulator, with Nomad base (unicycle) and 3R anthropomorphic manipulator (Puma, with frozen wrist)	solution
2009	07.10	2	Inverse kinematics of a RP robot, workspace with limited joint range, and number and type of inverse solutions in the workspace; Planning of a coordinated roto-translation in the Cartesian space	solutions
2009	06.10	1 (3 parts)	DH assignment for a planar PRP robot; Singularities and linear subspaces associated to the Jacobian for a planar positioning task; Kinematic control in the task space (planar position and orientation) with two case studies of feasibility with respect to joint velocity bounds	solution
2009	02.09	2	Kinematic control in the Cartesian space in acceleration; Placing the base of a planar 2R robot so as to maximize manipulability and task velocity in a given direction **(in Italian)**	solutions **(in Italian)**
2009	01.08	2	Angular velocity for an axis/angle rotation: general proof and computation of a trajectory for end-effector orientation; Direct kinematics, Jacobian and singularity analysis for a 3R supporting leg of the SmartEE parallel platform **(in Italian)**	solutions **(in Italian)**

2008	09.11	1	Second-order kinematic model of a nonholonomic mobile manipulator, a car-like mobile base with a planar 2R arm (optional: singularity analysis) **(in Italian)**	solution **(in Italian)**
2008	07.02	2	Statics of a planar 2R robot with two force applied along the links; Nonholonomic constraints of a fire-truck mobile robot (optional: kinematic model) **(in Italian)**	solutions **(in Italian)**
2008	03.20	2	Optimal planning of a trajectory composed by three velocity pieces, with initial/final sinusoidal profiles and acceleration constraint; Differential kinematics of a spatial 3R robot with eye-in-hand camera **(in Italian)**	---
2008	01.07	2	Linear Cartesian motion of a planar 3R robot and singularities; Kinematic model of a WMR with two steering wheels **(in Italian)**	solutions **(in Italian)**
2007	12.03	3	Inverse kinematics of a planar 2R robot with test on the joint range feasibility*; Angular resolution of a servo-drive with incremental encoder and sizing of the motion reduction element; Optimal trajectory planning with velocity/acceleration constraints and continuity up to acceleration* **(in Italian)**	solutions **(in Italian)**; MATLAB code
2007	09.13	1 (2 parts)	DH assignment for a spatial 3R robot and computation of the end-effector linear and angular velocity; Pseudo-code of an algorithm for numerical inverse kinematics **(in Italian)**	---
2007	06.28	2	Geometric Jacobian for the wrist frame of a KUKA KR6 Sixx robot with last three joints frozen; Planning of a piecewise polynomial trajectory through four point with boundary conditions up to the jerk and continuity in acceleration **(in Italian)**	---
2007	03.23	1	DH assignment for the KUKA KR150K robot and relationship with the "zero" configuration from the industrial robot data sheet **(in Italian)**	---
2007	01.08	3	Singularity analysis and analytical inverse kinematics for a planar RRP robot;	solutions **(in Italian)**

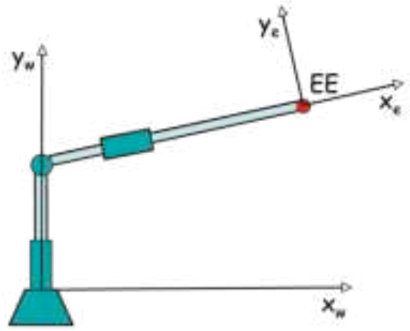
			Use of kinematic redundancy for handling joint range limits; Pros and cons of the use of vision in robot motion control **(in Italian)**	
2006	12.04	2	DH assignment, direct kinematics, and workspace of a spatial RRPR robot; Optimal planning of Cartesian straight-line trajectory for a planar RP robot with velocity/acceleration constraints and use of uniform time scaling to satisfy maximum joint velocity bounds **(in Italian)**	solutions **(in Italian)**
2006	09.11	1 (3 parts)	DH assignment for a planar 3R robot; Jacobian and its singularities; Planning of a trajectory between two Cartesian points where the robot is in a singularity, with acceleration continuity **(in Italian)**	---
2006	07.13	1	Minimum-time trajectory planning between two Cartesian points for a planar 2R robot under joint velocity and joint acceleration constraints **(in Italian)**	---
2006	06.30	1	DH assignment for the DLR LWR-III 7R robot **(in Italian)**	---
2006	04.03	1 (2 parts)	Robot-excavator: Direct kinematics; Inverse kinematics, statics, placement of robot base in the workspace (choose one) **(in Italian)**	solution **(in Italian)**
2006	01.09	1 (3 parts)	Mobile base moving in circle with a planar 2R manipulator on board: Inverse kinematics; Differential kinematics; Singularity analysis **(in Italian)**	solution **(in Italian)**
2005	12.16	1 (3 parts)	"Painting" RPPR robot: DH assignment; Direct kinematics; Minimum-time cyclic Cartesian trajectory under joint velocity constraints **(in Italian)**	solution **(in Italian)**
2005	09.22	1	DH assignment for the Comau Smart Six robot **(in Italian)**	---
2005	04.05	2	Statics and computation of joint accelerations for a constrained planar 3R robot; Computing wheel velocities so as to assign a given linear velocity to a point on the chassis of the SuperMario mobile robot **(in Italian)**	solutions **(in Italian)**
2005	01.12	2	Direct and differential kinematics, singularity analysis and control of a mobile manipulator --unicycle base with spatial 3R robot; Minimum-time trajectory planning between two Cartesian points under acceleration and, possibly, velocity constraints for a planar 2P robot (multiple solution paths) **(in Italian)**	solutions **(in Italian)**

2004	12.16	2	DH assignment, direct kinematics, singularity analysis, trajectory planning without singularities, and workspace for a spatial RPR robot; Path planning in the joint space, with given initial and final Cartesian tangents and an obstacle to be avoided* **(in Italian)**	solutions **(in Italian)**; MATLAB code
2004	04.06	1	Planning of a cyclic joint trajectory passing through three Cartesian points for a planar 2R robot **(in Italian)**	solution **(in Italian)**
2004	03.25	2	Odometry computation and minimum-time motion for the SuperMario wheeled mobile robot; Singularities, workspace, and manipulability for a planar 4R robot **(in Italian)**	solutions **(in Italian)**
2004	01.08	2	DH assignment, direct kinematics, statics, and minimum norm joint velocity computation for a (redundant) planar RRP robot; Planning of an orientation trajectory using the axis/angle method or with the YZY Euler angles* **(in Italian)**	solutions **(in Italian)**; MATLAB code
2003	12.11	2	DH assignment for a 3R pointing structure and its direct kinematics; Trajectory planning for a planar RP robot under bounds on the Cartesian acceleration norm and on the joint velocities* **(in Italian)**	solutions **(in Italian)**; MATLAB code

Note: For these* problems, MATLAB codes for computing solutions and/or for graphics are either embedded in the solution text or available to the students of the course upon request (contact deluca@diag.uniroma1.it).

Robotics I

June 10, 2009



Consider the planar PRP robot with $n = 3$ joints in the figure above. The world reference frame $RF_w = (\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$ and the end-effector frame $RF_e = (\boldsymbol{x}_e, \boldsymbol{y}_e, \boldsymbol{z}_e)$ are also shown.

- Assign the robot reference frames according to the Denavit-Hartenberg (DH) convention and write down the associated table of parameters. Moreover, specify the (constant) transformation matrices ${}^wT_0$, between the world frame and frame 0 of DH, and ${}^3T_e$, between frame 3 of DH and the end-effector frame.
- Based on the variables $\boldsymbol{q}$ defined in the Denavit-Hartenberg convention, compute the analytical Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for a task involving only the end-effector position in the plane of motion, and analyze its singular configurations. With the robot in a singular configuration $\boldsymbol{q}_0$, define a set of base vectors for each of the following four linear subspaces:

$$\mathcal{R}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) \qquad \mathcal{N}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) \qquad \mathcal{R}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) \qquad \mathcal{N}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right).$$

- For a motion task of dimension $m = 3$ specified for the robot end-effector, consider the use of a kinematic control law in the task space,

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{r}}_d + \boldsymbol{K}_P(\boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}))\right), \tag{1}$$

  where $\boldsymbol{r} \in \mathbb{R}^3$ includes the position (in the plane) as well as the orientation of the end-effector (i.e., the angle ϕ between the horizontal axis $\boldsymbol{x}_w$ and the axis $\boldsymbol{x}_e$) and $\boldsymbol{f}(\boldsymbol{q})$ is the direct kinematic function associated to these task variables. Assume that the (positive definite) gain matrix $\boldsymbol{K}_P$ is chosen as diagonal, and let the joint velocities be bounded as $|\dot{q}_i| \leq V_i$, with given values $V_i > 0$ ($i = 1, 2, 3$).

  a) With the desired task velocity being $\dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^T$, determine a joint configuration $\boldsymbol{q}^*$ which is nonsingular for the task (and 'matched' to it, i.e., $\boldsymbol{e} = \boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{0}$ for a suitable $\boldsymbol{r}_d$) and is such that the desired task *can never* be realized without violating one of the bounds on the joint velocities.

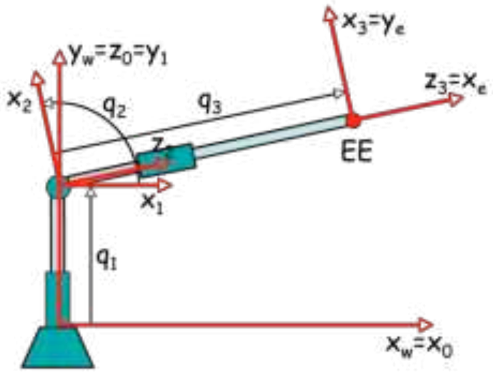
  b) Let the robot initial configuration be $\boldsymbol{q}(0) = \begin{pmatrix} 1.2 & \pi/2 & 1 \end{pmatrix}^T$, and let $\boldsymbol{r}_d(0) = \begin{pmatrix} 1.5 & 1.5 & -\pi/4 \end{pmatrix}^T$ and $\dot{\boldsymbol{r}}_d(0) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ be the specified task at time $t = 0$. Define the numerical values of the diagonal matrix $\boldsymbol{K}_P$ in the control law (1) so that the initial task error $\boldsymbol{e}(0)$ will be reduced as fast as possible *without violating* the following bounds on the joint velocities: $V_1 = 5$ [m/s], $V_2 = \pi$ [rad/s], and $V_3 = 4$ [m/s].

[180 minutes; open books]

Solution

June 10, 2009

A possible assignment of the Denavit-Hartenberg frames is shown in the figure below, together with the associated table of parameters.



i	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	q_1	0
2	$\frac{\pi}{2}$	0	0	q_2
3	0	0	q_3	0

From this, it is easy to obtain the general expression of the direct kinematics for this robot:

$$
\begin{aligned}
{}^0\boldsymbol{T}_3(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_3(\boldsymbol{q}) & {}^0\boldsymbol{p}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} &= {}^0A_1(q_1)\,{}^1A_2(q_2)\,{}^2A_3(q_3) \\
&= \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & q_3 \sin q_2 \\ 0 & -1 & 0 & 0 \\ \sin q_2 & 0 & -\cos q_2 & q_1 - q_3 \cos q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Use of the additional transformations between the frames defined in the problem text leads to

$$
\begin{aligned}
{}^w\boldsymbol{T}_e(\boldsymbol{q}) = {}^w\boldsymbol{T}_0{}^0\boldsymbol{T}_3(\boldsymbol{q}){}^3\boldsymbol{T}_e &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^0\boldsymbol{T}_3(\boldsymbol{q}) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sin q_2 & \cos q_2 & 0 & q_3 \sin q_2 \\ -\cos q_2 & \sin q_2 & 0 & q_1 - q_3 \cos q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^w\boldsymbol{R}_e(\boldsymbol{q}) & {}^w\boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
\end{aligned}
$$

from which one can obtain the kinematic functions of interest (which could be derived also by direct inspection, once the joint variables have been defined according to the Denavit-Hartenberg convention). To this end, note that the final rotation matrix ${}^w\boldsymbol{R}_e(\boldsymbol{q})$ takes the form of an elementary rotation matrix by an angle $\phi = q_2 - \pi/2$ around the world axis $\boldsymbol{z}_w$. In fact, it is

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\left(q_2 - \frac{\pi}{2}\right) & -\sin\left(q_2 - \frac{\pi}{2}\right) & 0 \\ \sin\left(q_2 - \frac{\pi}{2}\right) & \cos\left(\left(q_2 - \frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sin q_2 & \cos q_2 & 0 \\ -\cos q_2 & \sin q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^w\boldsymbol{R}_e(q_2).$$

For the (first) task involving only the end-effector position on the plane, it is

$$\boldsymbol{r}_1 = \boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} {}^w p_x \\ {}^w p_y \end{pmatrix} = \begin{pmatrix} q_3 \sin q_2 \\ q_1 - q_3 \cos q_2 \end{pmatrix},$$

and the analytical (2×3) Jacobian matrix is

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & q_3 \cos q_2 & \sin q_2 \\ 1 & q_3 \sin q_2 & -\cos q_2 \end{pmatrix}.$$

Analyzing the three minors of $\boldsymbol{J}_1(\boldsymbol{q})$, this matrix looses rank if and only if $\sin q_2 = 0$ and $q_3 = 0$, i.e., when the third robot link is oriented along the vertical direction and the third joint is completely retracted. In such a configuration, the Jacobian becomes

$$\boldsymbol{J}_1(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \mp 1 \end{pmatrix},$$

where the upper sign corresponds to the case $q_2 = 0$ and the lower sign to the case $q_2 = \pi$. The four linear subspaces indicated in the text are spanned by the following basis vectors:

$$\mathcal{N}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ \mp 1 \end{pmatrix} \right\} \qquad \mathcal{R}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) = \left\{ \begin{pmatrix} 1 \\ 0 \\ \mp 1 \end{pmatrix} \right\} \qquad \text{in } \mathbb{R}^3,$$

$$\mathcal{R}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{N}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \qquad \text{in } \mathbb{R}^2.$$

For the end-effector planar positioning and orientation task (of dimension $m = 3$), it is

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} {}^w p_x \\ {}^w p_y \\ \phi \end{pmatrix} = \begin{pmatrix} q_3 \sin q_2 \\ q_1 - q_3 \cos q_2 \\ q_2 - \pi/2 \end{pmatrix}. \tag{2}$$

The analytical (3×3) Jacobian matrix associated to this task,

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & q_3 \cos q_2 & \sin q_2 \\ 1 & q_3 \sin q_2 & -\cos q_2 \\ 0 & 1 & 0 \end{pmatrix},$$

is singular if and only if $\sin q_2 = 0$.

Under the condition of question a), namely with $\sin q_2^* \neq 0$, it is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}^*)\dot{\boldsymbol{r}}_d = \frac{1}{\sin q_2^*}\begin{pmatrix} \cos q_2^* & \sin q_2^* & -q_3^* \\ 0 & 0 & \sin q_2^* \\ 1 & 0 & -q_3^* \cos q_2^* \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sin q_2^*}\begin{pmatrix} q_3^* \\ -\sin q_2^* \\ q_3^* \cos q_2^* \end{pmatrix}.$$

It is then easy to see that, by sufficiently extending the prismatic joint 3, the robot will violate the velocity bound at the first joint for any assigned value $V_1 > 0$. More specifically, it is

$$q_3^* > V_1 \cdot |\sin q_2^*| > 0 \qquad \Rightarrow \qquad |\dot{q}_1| > V_1.$$

Under the condition of question b), the robot is not in a singularity at the initial time $t = 0$. Thus, using the problem data and eq. (2), the initial control velocity can be computed as

$$\begin{aligned} \dot{\boldsymbol{q}}(0) &= \boldsymbol{J}^{-1}(\boldsymbol{q}(0))\Big(\dot{\boldsymbol{r}}_d(0) + \boldsymbol{K}_P(\boldsymbol{r}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0))\Big) \\ &= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5\,K_{r_1} \\ 0.3\,K_{r_2} \\ -\frac{\pi}{4}K_{r_3} \end{pmatrix}\right) = \begin{pmatrix} 1 + 0.3\,K_{r_2} + \frac{\pi}{4}K_{r_3} \\ -\frac{\pi}{4}K_{r_3} \\ 0.5\,K_{r_1} \end{pmatrix}, \end{aligned}$$

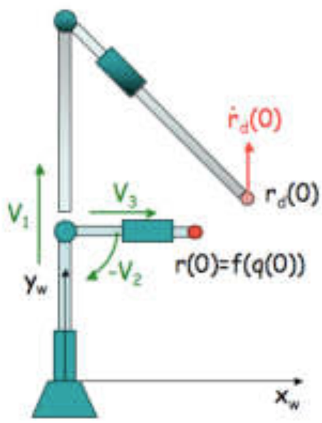
having set $\boldsymbol{K}_P = \mathrm{diag}\{K_{r_1}, K_{r_2}, K_{r_3}\}$. From these expressions, one can directly choose two out of the three control gains:

$$\begin{aligned} |\dot{q}_3(0)| \leq V_3 \quad &\Rightarrow \quad 0.5\,K_{r_1} \leq V_3 = 4 \quad \Rightarrow \quad K_{r_1} = 8 > 0, \\ |\dot{q}_2(0)| \leq V_2 \quad &\Rightarrow \quad \frac{\pi}{4}K_{r_3} \leq V_2 = \pi \quad \Rightarrow \quad K_{r_3} = 4 > 0. \end{aligned}$$

Finally, using this definition, also the remaining gain is chosen:

$$|\dot{q}_1(0)| \leq V_1 \quad \Rightarrow \quad 1 + 0.3\,K_{r_2} + \frac{\pi}{4}K_{r_3} = 1 + 0.3\,K_{r_2} + \pi \leq V_1 = 5 \quad \Rightarrow \quad K_{r_2} = \frac{10}{3}(4 - \pi) > 0.$$

With the selected gains, all joint velocities will saturate at time $t = 0$ (the second joint velocity being at its negative limit $-V_2 = -\pi$) and, as a result, the *fastest* decrease of the initial task error $\boldsymbol{e}(0) = \boldsymbol{r}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0))$ will be realized (with the task error converging anyway exponentially to zero, in a decoupled way for each task component). The situation at time $t = 0$ is depicted in the following figure, where the desired initial robot configuration is the lightly shaded one.

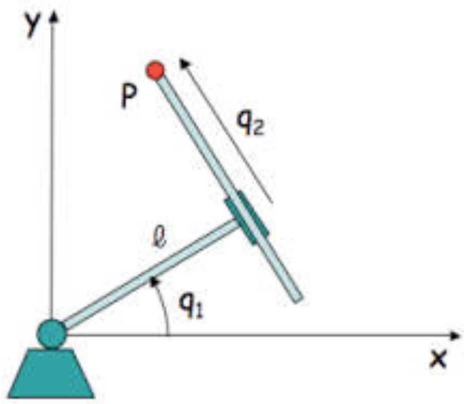


* * * * *

Robotics I

July 10, 2009

Exercise 1



Consider the planar RP robot shown in the figure, where ℓ is the length of the first link and the generalized coordinates to be used are indicated. Let $\boldsymbol{p} = (p_x \ p_y)^T$ be the position of the end-effector P.

- Solve the inverse kinematic problem for this robot, providing the number and type of the solutions for varying positions of P.
- Draw the robot primary workspace (with dimensions) in the case when the joint variables are bounded as: $q_1 \in [-\pi/2, +\pi/2]$, $q_2 \in [-L, +L]$. Discuss the presence of singularities on the boundaries of the workspace.

Exercise 2

Let an initial pose A and a final pose B be given in the robot Cartesian space, with the locations of the associated frame represented by the homogeneous transformation matrices:

$$
{}^0\boldsymbol{T}_A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^0\boldsymbol{T}_B = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

- Plan a *coordinated* motion trajectory from pose A to pose B along a straight path from O_A to O_B in a time $T = 2$ sec, and with zero initial and final linear and angular velocities.
- Provide the numerical value at time $t = T/2$ of the linear velocity (of the origin of moving frame) and of the angular velocity.

[120 minutes; open books]

Solutions

July 10, 2009

Exercise 1

The robot direct kinematics is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell \cos q_1 - q_2 \sin q_1 \\ \ell \sin q_1 + q_2 \cos q_1 \end{pmatrix}.$$

Rewriting this in the form

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \ell \\ q_2 \end{pmatrix},$$

where $\boldsymbol{R}(\theta)$ is the 2×2 planar rotation matrix by an angle θ, it immediately follows that

$$\boldsymbol{p}^T \boldsymbol{p} = p_x^2 + p_y^2 = \begin{pmatrix} \ell & q_2 \end{pmatrix}^T \boldsymbol{R}^T(q_1) \boldsymbol{R}(q_1) \begin{pmatrix} \ell \\ q_2 \end{pmatrix} = \ell^2 + q_2^2,$$

and hence

$$q_2 = \pm\sqrt{p_x^2 + p_y^2 - \ell^2}.$$

Depending on whether $\|\boldsymbol{p}\|$ is larger, equal to, or smaller than ℓ, there will be respectively two, one (singular), or no solutions. In this analysis, no joint limits are taken into account (in particular, the one for the prismatic joint).

Once q_2 is determined, in order to find the analytic expression of the solution for the first joint variable, we can rewrite the direct kinematics as

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -q_2 & \ell \\ \ell & q_2 \end{pmatrix} \begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix},$$

where the matrix that appears is always non-singular (with determinant equal to $-(q_2^2 + \ell^2) < 0$). This yields

$$\begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix} = \frac{1}{q_2^2 + \ell^2} \begin{pmatrix} \ell p_y - q_2 p_x \\ \ell p_x + q_2 p_y \end{pmatrix}.$$

Therefore,

$$q_1 = \text{ATAN2}\{\ell p_y - q_2 p_x, \ell p_x + q_2 p_y\},$$

where, in the regular case, the two solutions found for q_2 have to be replaced. In the singular case, one has only $q_2 = 0$ and thus the single associated solution $q_1 = \text{ATAN2}\{p_y, p_x\}$.

The robot workspace is shown in Figure 1. The case when only the prismatic joint variable is limited ($|q_2| \leq L$) is shown on the left, while the full requested case is given on the right. The radius of the inner and outer circumferences are $r = \ell$ and $R = \sqrt{\ell^2 + L^2}$. Note that on the external boundary of the workspace (arc of the circumference of radius R), as well as on the two straight segments belonging to the boundary, the analytic 2×2 robot Jacobian is non-singular; in fact, these limitations to the workspace are imposed by the joint limits and not by the kinematic configuration of the robot itself. Equivalently, on the parts of the workspace boundary where the Jacobian is full rank the space of admissible Cartesian velocities is still two-dimensional (though with unilaterally constrained along certain directions).

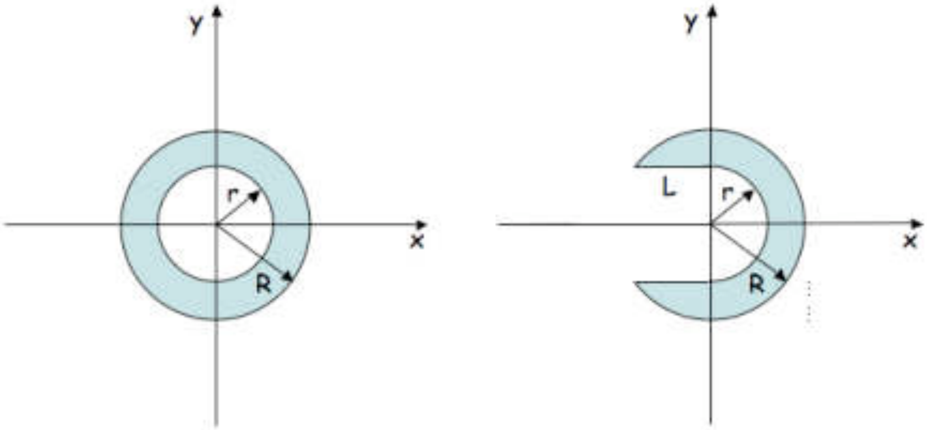


Figure 1: Robot workspace, with $r = \ell$ and $R = \sqrt{\ell^2 + L^2}$; on the left, the case when only the second joint range is limited ($|q_2| \leq L$); on the right, the full case including also $|q_1| \leq \pi/2$

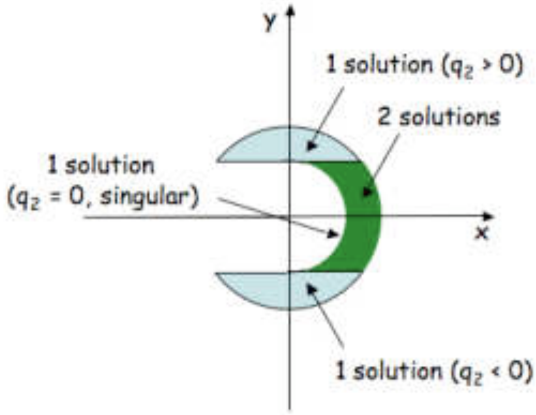


Figure 2: Number of inverse solutions in the various regions of the workspace

Finally, Figure 2 shows the partition of the workspace in terms of number of inverse kinematics solutions when joint limits are present. In particular, two solutions exist in the deep green region, including its two straight boundaries and the arc of the external circumference (on the internal one, there is only one, singular solution).

Exercise 2

The distance between the origins O_A and O_B of the frames associated to the initial and final poses is

$$L = \|{}^0\boldsymbol{p}_{0B} - {}^0\boldsymbol{p}_{0A}\| = \left\| \begin{pmatrix} 1 \\ 0 \\ -0.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix} \right\| = \sqrt{3.25}.$$

The linear path for the origin of the motion frame can be parametrized as

$$\boldsymbol{p}(s) = {}^0\boldsymbol{p}_{0A} + \frac{s}{L}\left({}^0\boldsymbol{p}_{0B} - {}^0\boldsymbol{p}_{0A}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{s}{\sqrt{3.25}} \begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}, \qquad s \in [0, L].$$

The relative rotation between pose A and pose B is given by

$${}^A\boldsymbol{R}_B = {}^0\boldsymbol{R}_A^T\,{}^0\boldsymbol{R}_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

From this, one can plan a reorientation motion using the axis/angle method. We need to compute the unit vector ${}^A\boldsymbol{r}$ (defined with respect to the initial frame) and thee angle θ_{AB} satisfying $\boldsymbol{R}({}^A\boldsymbol{r}, \theta_{AB}) = {}^A\boldsymbol{R}_B$. Denoting with r_{ij} the elements of the rotation matrix ${}^A\boldsymbol{R}_B$, from the inverse formulas of the axis/angle method we obtain

$$\begin{aligned} \theta_{AB} &= \text{ATAN2}\{+\sqrt{(r_{21}-r_{12})^2 + (r_{13}-r_{31})^2 + (r_{23}-r_{32})^2}, r_{11}+r_{22}+r_{33}-1\} \\ &= \text{ATAN2}\{\sqrt{3}, -1\} = \frac{2}{3}\pi = 2.0944 \text{ rad } (= 120°) \end{aligned}$$

and

$${}^A\boldsymbol{r} = \frac{1}{2\sin\theta_{AB}} \begin{pmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0.5774 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad (\text{being } \|{}^A\boldsymbol{r}\| = 1).$$

Note that only one of the two possible solutions is used. The orientation path can be (still linearly) parametrized as

$$\theta(s) = \frac{s}{L}\,\theta_{AB}, \qquad s \in [0, L].$$

The absolute orientation for a given value of parameter s will be

$$\boldsymbol{R}(s) = {}^0\boldsymbol{R}_A\,\boldsymbol{R}({}^A\boldsymbol{r}, \theta(s)).$$

Having use the same parameter s for the position and orientation paths, planning a single timing law $s = s(t)$, with $t \in [0, T]$, will automatically yield a coordinated motion: the translation and the rotation between the initial and final poses will be completed simultaneously.

For the timing law, the simplest choice is a cubic (bi-normalized) polynomial with zero time derivative in $t = 0$ and $t = T$. It is

$$s(t) = L\left[-2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2\right], \qquad t \in [0, T].$$

Its time derivative is

$$\dot{s}(t) = \frac{6L}{T}\left[\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right],$$

and thus

$$\dot{s}(T/2) = \frac{3L}{2T}.$$

The linear and angular velocity during the transfer motion are

$${}^0\dot{\boldsymbol{p}}(t) = \frac{d\boldsymbol{p}}{ds}\dot{s}(t) = \frac{\dot{s}(t)}{L}\begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}$$

and

$${}^A\boldsymbol{\omega}(t) = {}^A\boldsymbol{r}\,\frac{d\theta(s)}{ds}\,\dot{s}(t) = \frac{\dot{s}(t)}{L}\,\theta_{AB}\,{}^A\boldsymbol{r}.$$

At $t = T/2 = 1$ sec, we have

$${}^0\dot{\boldsymbol{p}}(1) = \frac{3}{2}\begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}, \qquad {}^A\boldsymbol{\omega}(1) = \frac{3}{2}\cdot 2.0944\cdot 0.5774\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 1.8138\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\ \text{rad/sec},$$

and finally

$${}^0\boldsymbol{\omega}(1) = {}^0\boldsymbol{R}_A\,{}^A\boldsymbol{\omega}(1) = 1.8138\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\ \text{rad/sec}.$$

* * * * *

Robotics I

September 10, 2009

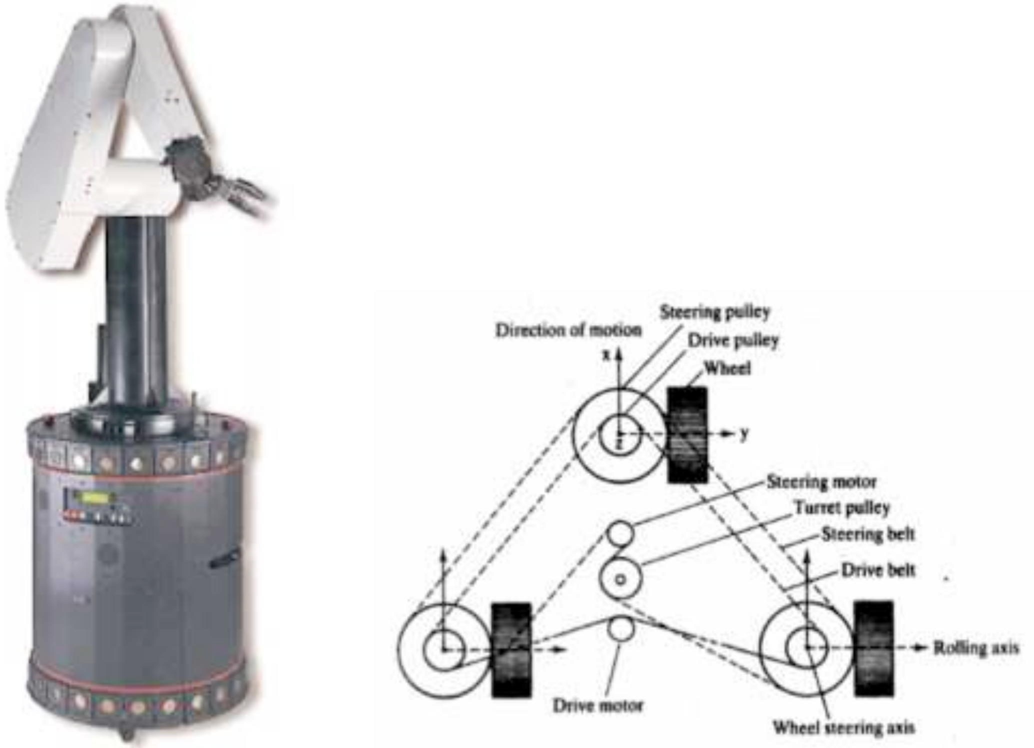


Figure 1: The mobile manipulator (left) and the *synchro-drive* of its base (right)

Consider the mobile manipulator in Figure 1, made by a nonholonomic base (Nomad) carrying a $6R$ manipulator, with shoulder and elbow off-sets (that compensate each to other) and a spherical wrist (Unimation Puma). The base has three identical steering wheels that move in coordination, driven by a *synchro-drive* actuation with one motor for driving the three wheels and one for their steering. Let v be the linear velocity of the wheels on the ground and ω the steering velocity of the wheels with respect to the base chassis. We are interested only in the position $\boldsymbol{p} \in \mathbb{R}^3$ of the center of the spherical wrist of the manipulator; the first three joints are described by the Denavit-Hartenberg coordinates $\boldsymbol{\theta} \in \mathbb{R}^3$, while the three remaining are frozen. Determine the expression of the 3×5 matrix $\boldsymbol{J}(\boldsymbol{q})$ in the relationship

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u},$$

where $\boldsymbol{q} \in \mathbb{R}^6$ is the vector of generalized coordinates for the mobile manipulator and

$$\boldsymbol{u} = \begin{pmatrix} v & \omega & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{pmatrix}^T \in \mathbb{R}^5$$

is the vector of the available input velocities.

[120 minutes; open books]

Solution

September 10, 2009

An absolute reference frame $(\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$ is chosen, with the axis $\boldsymbol{z}_w$ being normal to the motion plane. The mobile base is described by the coordinates (x, y, θ), representing the Cartesian position of its center and the absolute orientation of the three wheels with respect to the axis $\boldsymbol{x}_w$. The (differential) kinematic model of the base is that of a unicycle

$$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \omega. \end{aligned} \tag{1}$$

Note that the orientation of the base (and of its turret) *does not* change with the steering velocity ω (the orientation of the wheels changes w.r.t. the chassis body).

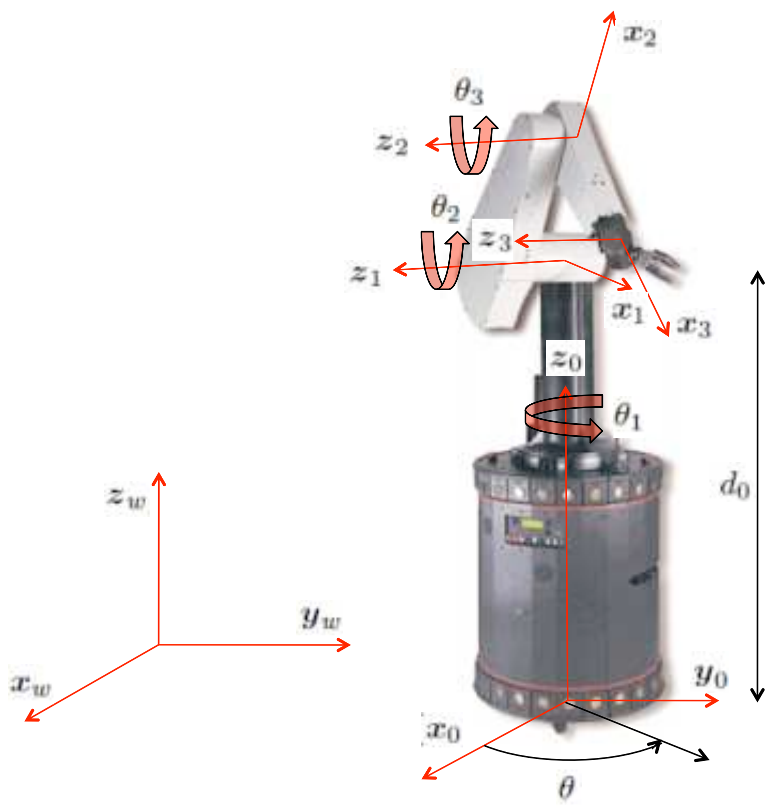


Figure 2: Reference frames for the mobile manipulator

A reference frame $(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{z}_0)$ is then chosen for the manipulator, located on the mobile base and aligned with the absolute frame (see Figure 2). The position of the wrist center relative to this frame is

$$
{}^{0}\boldsymbol{p} = \begin{pmatrix} (\ell_2 \cos\theta_2 + \ell_3 \cos(\theta_2+\theta_3)) \cos\theta_1 \\ (\ell_2 \cos\theta_2 + \ell_3 \cos(\theta_2+\theta_3)) \sin\theta_1 \\ d_0 + \ell_2 \sin\theta_2 + \ell_3 \sin(\theta_2+\theta_3) \end{pmatrix},
$$

where $(\theta_1, \theta_2, \theta_3)$ are the joint variables according to Denavit-Hartenberg and d_0 is the height from the ground of the second joint axis (which is always horizontal). The presence of shoulder and elbow offset is not relevant for the direct kinematics. It follows

$$
{}^{w}\boldsymbol{p} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + {}^{0}\boldsymbol{p}.
$$

Differentiating w.r.t. time and using (1) leads to

$$
{}^{w}\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} + {}^{0}\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u} =
$$

$$
\begin{pmatrix} c\theta & 0 & -(\ell_2 c\theta_2 + \ell_3 c(\theta_2+\theta_3))\, s\theta_1 & -(\ell_2 s\theta_2 + \ell_3 s(\theta_2+\theta_3))\, c\theta_1 & -\ell_3 s(\theta_2+\theta_3) c\theta_1 \\ s\theta & 0 & (\ell_2 c\theta_2 + \ell_3 c(\theta_2+\theta_3))\, c\theta_1 & -(\ell_2 s\theta_2 + \ell_3 s(\theta_2+\theta_3))\, s\theta_1 & -\ell_3 s(\theta_2+\theta_3) s\theta_1 \\ 0 & 0 & 0 & \ell_2 c\theta_2 + \ell_3 c(\theta_2+\theta_3) & \ell_3 c(\theta_2+\theta_3) \end{pmatrix} \begin{pmatrix} v \\ \omega \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix},
$$

where the compact notation s for sin and c for cos is used. The Jacobian $\boldsymbol{J}$ of the mobile manipulator has always a zero column, corresponding to the fact that the input ω has no effect on the wrist center velocity.

One can additionally study the singularities of $\boldsymbol{J}$ (namely, the configurations where the matrix loses its maximum rank which is 3). Note that the last three columns of this matrix are the Jacobian ${}^{0}\boldsymbol{J}_m = {}^{w}\boldsymbol{J}_m$ of the manipulator arm, when taken by itself. For the whole structure to be singular, the manipulator should necessarily be in a singularity. Since

$$
\begin{aligned}
{}^{0}\boldsymbol{J}_m(\boldsymbol{\theta}) &= {}^{0}\boldsymbol{R}_1(\theta_1)\, {}^{1}\boldsymbol{J}_m(\boldsymbol{\theta}) \\
&= {}^{0}\boldsymbol{R}_1(\theta_1) \begin{pmatrix} 0 & -(\ell_2 s\theta_2 + \ell_3 s(\theta_2+\theta_3)) & -\ell_3 s(\theta_2+\theta_3) \\ \ell_2 c\theta_2 + \ell_3 c(\theta_2+\theta_3) & 0 & 0 \\ 0 & \ell_2 c\theta_2 + \ell_3 c(\theta_2+\theta_3) & \ell_3 c(\theta_2+\theta_3) \end{pmatrix},
\end{aligned}
$$

it is easy to see that this happens if the wrist center is on the first joint axis

$$
\ell_2 \cos\theta_2 + \ell_3 \cos(\theta_2+\theta_3) = 0,
$$

or if the third link is stretched or folded

$$
\sin\theta_3 = 0,
$$

or when both situations hold together (the rank of $\boldsymbol{J}_m$ would then drop to 1). In the first kind of singularity, the manipulator wrist center cannot have a velocity along the normal to the plane of motion of the second and third link; for the base to provide mobility in this direction (through the input velocity v), the wheels must not be parallel to the plane containing the second and third link, i.e., $\theta \neq \theta_1 + k\pi$, $k = 0, 1$. In the second kind of singularity, the wrist center cannot have a velocity along the direction of alignment of the second and third link; for the base to provide mobility in this direction, the wheels must not be oriented normally to this direction, i.e., $\theta \neq \theta_1 \pm \pi/2$. When the manipulator is in a double singularity, the mobile base cannot recover the lack of mobility so as to give full row rank to $\boldsymbol{J}$.

* * * * *

Robotics I

Test — November 10, 2009

Exercise 1

Consider a minimal representation of orientation specified by the following sequence of angles, defined around fixed axes: α around Y; β around X; γ around Z.

- Compute the associated rotation matrix $\boldsymbol{R}_{YXZ}(\alpha, \beta, \gamma)$.
- Determine all sets of angles (α, β, γ) realizing the orientation specified by the matrix

$$\boldsymbol{R} = \begin{pmatrix} 0.7392 & -0.6124 & -0.2803 \\ 0.5732 & 0.3536 & 0.7392 \\ -0.3536 & -0.7071 & 0.6124 \end{pmatrix}.$$

- Characterize all rotation matrices $\boldsymbol{R}$ for which the inverse problem yields undefined angles in the sequence.

Exercise 2

Consider the kinematic structure in Figure 1, representing a camera mounted on the head of a humanoid trunk with three revolute joints.

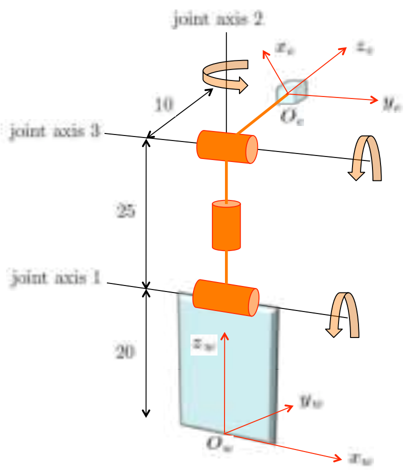


Figure 1: Kinematics of a camera head (units are in cm)

- Assign the frames according to the Denavit-Hartenberg convention in such a way that the positive (counterclockwise) joint rotations are those shown. Compute the associated table of parameters.
- Compute the expression of the rotation matrix ${}^{w}\boldsymbol{R}_e(\theta_1, \theta_2, \theta_3)$ relating the orientation of the given end-effector (camera) frame RF_e with respect to the world frame RF_w, placed as shown in Figure 1.
- Provide a rotation matrix ${}^{w}\boldsymbol{R}_e$ that can be realized by infinite pairs of values (θ_1, θ_3) and a single value of θ_2.

[120 minutes; open books]

Solutions

November 10, 2009

Exercise 1

By using the elementary rotation matrices around the coordinate axes

$$\begin{aligned}
\boldsymbol{R}_Y(\alpha) &= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \\
\boldsymbol{R}_X(\beta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}, \\
\boldsymbol{R}_Z(\gamma) &= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

and being the sequence of rotations defined around fixes axes, we obtain

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\alpha),$$

or

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\sin\alpha\cos\beta & \sin\beta & \cos\alpha\cos\beta \end{pmatrix}.$$

The inverse mapping from a given rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

to the sequence of angles (α,β,γ) is given by

$$\beta = \mathrm{ATAN2}\left\{ r_{32}, \pm\sqrt{r_{31}^2 + r_{33}^2} \right\}$$

and, provided that $r_{31}^2 + r_{33}^2 \neq 0$ (i.e., $\cos\beta \neq 0$),

$$\alpha = \mathrm{ATAN2}\left\{ \frac{-r_{31}}{\cos\beta}, \frac{r_{33}}{\cos\beta} \right\}, \qquad \gamma = \mathrm{ATAN2}\left\{ \frac{-r_{12}}{\cos\beta}, \frac{r_{22}}{\cos\beta} \right\}.$$

For the given data, we obtain the pair of solutions:

$$(\alpha,\beta,\gamma) = (0.5236, -0.7854, 1.0472)\ [\mathrm{rad}] = (30, -45, 60)\ [\mathrm{deg}]$$

and

$$(\alpha,\beta,\gamma) = (-2.6180, -2.3562, -2.0944)\ [\mathrm{rad}] = (-150, -135, -120)\ [\mathrm{deg}].$$

When $r_{31} = r_{33} = 0$, β is uniquely defined whereas the other data provide only information either on the sum $\alpha + \gamma$ or on the difference $\alpha - \gamma$. In fact, for an orientation matrix of the form

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 1 & 0 \end{pmatrix},$$

i.e., with $r_{32} = 1$, we have $\beta = \pi/2$ ($\cos\beta = 0$, $\sin\beta = 1$) and thus

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha, \pi/2, \gamma) &= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & 0 & \sin\alpha\cos\gamma + \cos\alpha\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\cos\gamma & 0 & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\gamma) & 0 & \sin(\alpha+\gamma) \\ \sin(\alpha+\gamma) & 0 & -\cos(\alpha+\gamma) \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\alpha + \gamma = \text{ATAN2}\{r_{21}, r_{11}\} = \text{ATAN2}\{r_{13}, -r_{23}\}.$$

On the other hand, for an orientation matrix of the form

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & -1 & 0 \end{pmatrix},$$

i.e., with $r_{32} = -1$, we have $\beta = -\pi/2$ ($\cos\beta = 0$, $\sin\beta = -1$) and thus

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha, -\pi/2, \gamma) &= \begin{pmatrix} \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & 0 & \sin\alpha\cos\gamma - \cos\alpha\sin\gamma \\ \cos\alpha\sin\gamma - \sin\alpha\cos\gamma & 0 & \sin\alpha\sin\gamma + \cos\alpha\cos\gamma \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha-\gamma) & 0 & \sin(\alpha-\gamma) \\ -\sin(\alpha-\gamma) & 0 & \cos(\alpha-\gamma) \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\alpha - \gamma = \text{ATAN2}\{-r_{21}, r_{11}\} = \text{ATAN2}\{r_{13}, r_{23}\}.$$

In both cases, the angles α and γ are not fully defined.

Exercise 2

Consider the assignment of Denavit-Hartenberg frames as in Figure 2, where the positive direction of the axes $\boldsymbol{z}_i$ ($i = 0, 1, 2$) has been chosen consistently with the requirement in the text. The shown configuration has $\theta_1 = 0$, $\theta_2 = 0$, and θ_3 equal to some positive angle between $\pi/2$ and $3\pi/4$.

The Denavit-Hartenberg parameters are given in Table 1, with $d_2 = 25$ cm. The associated

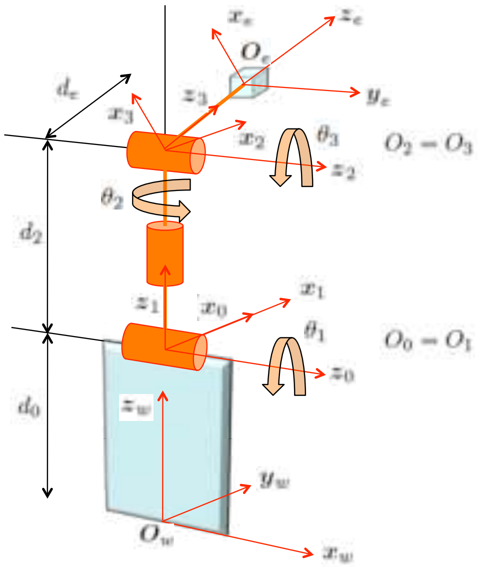


Figure 2: Denavit-Hartenberg frames

i	α_i	a_i	d_i	θ_i
1	$-\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	d_2	θ_2
3	$\frac{\pi}{2}$	0	0	θ_3

Table 1: Denavit-Hartenberg parameters

homogeneous transformation matrices are

$$
\begin{aligned}
{}^0\boldsymbol{A}_1(\theta_1) &= \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(\theta_1) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \\
{}^1\boldsymbol{A}_2(\theta_2) &= \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(\theta_2) & {}^1\boldsymbol{p}_{12} \\ \mathbf{0}^T & 1 \end{pmatrix}, \\
{}^2\boldsymbol{A}_3(\theta_3) &= \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3(\theta_3) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}.
\end{aligned}
$$

In addition, we can define the following (constant) homogenous transformation matrices

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^w\boldsymbol{R}_0 & {}^w\boldsymbol{p}_{w0} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{T}_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_e \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_e & {}^3\boldsymbol{p}_{3e} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

with $d_0 = 20$ cm and $d_e = 10$ cm. Note that ${}^3\boldsymbol{R}_e = \boldsymbol{I}$.

The orientation of frame RF_e w.r.t. the world frame RF_w is thus

$$
\begin{aligned}
{}^w\boldsymbol{R}_e(\boldsymbol{\theta}) &= {}^w\boldsymbol{R}_0\,{}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\,{}^2\boldsymbol{R}_3(\theta_3)\,{}^3\boldsymbol{R}_e \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\sin\theta_3 - \cos\theta_1\cos\theta_3 \\ -\sin\theta_2\cos\theta_3 & \cos\theta_2 & -\sin\theta_2\sin\theta_3 \end{pmatrix}.
\end{aligned}
$$

One can now proceed by solving the inverse kinematics of this three-dof robotic structure for a given orientation matrix ${}^w\boldsymbol{R}_e$. In particular, we can solve for $\boldsymbol{\theta}$ the following kinematic equation

$$
{}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\,{}^2\boldsymbol{R}_3(\theta_3) = {}^w\boldsymbol{R}_0^T\,{}^w\boldsymbol{R}_e = {}^0\boldsymbol{R}_e = \begin{pmatrix} {}^0r_{11} & {}^0r_{12} & {}^0r_{13} \\ {}^0r_{21} & {}^0r_{22} & {}^0r_{23} \\ {}^0r_{31} & r_{32} & {}^0r_{33} \end{pmatrix},
$$

where the right-hand side matrix is a constant. By similar reasoning as in Exercise 1, one can see that the inverse problem has an infinity set of values for θ_1 and θ_3 (with a prescribed sum or difference) if and only if

$$
{}^0r_{31} = {}^0r_{33} = 0 \qquad ({}^0r_{32} = \pm 1).
$$

All possible rotation matrices ${}^w\boldsymbol{R}_e$ leading to this situation are then of the form

$$
{}^w\boldsymbol{R}_e = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} {}^0r_{11} & 0 & {}^0r_{13} \\ {}^0r_{21} & 0 & {}^0r_{23} \\ 0 & \pm 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 & 0 \\ {}^0r_{11} & 0 & {}^0r_{13} \\ {}^0r_{21} & 0 & {}^0r_{23} \end{pmatrix}.
$$

For example, one candidate is

$$
{}^w\boldsymbol{R}_e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

* * * * *

Robotics I

Test 2 — December 17, 2009

Consider the robot in Figure 1, having four revolute joints. The Denavit-Hartenberg frames are already placed, with frame 0 located at the intersection of the first and second joint axis. The configuration shown corresponds (approximately) to $\boldsymbol{\theta} \simeq \begin{pmatrix} 0 & 6\pi/10 & \pi & 6\pi/10 \end{pmatrix}^T$ [rad] (or, equivalently, $\boldsymbol{\theta} \simeq \begin{pmatrix} 0 & 108 & 180 & 108 \end{pmatrix}^T$ [deg]).

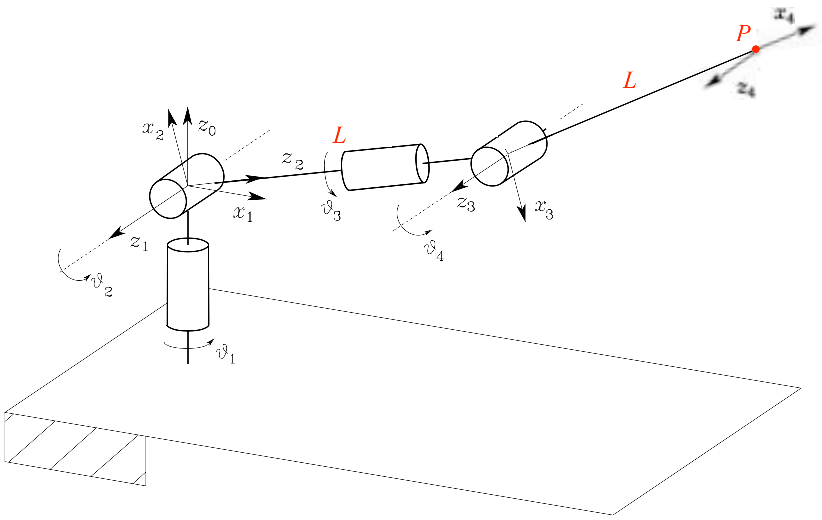


Figure 1: A 4R spatial manipulator

Let the robot be in the configuration $\boldsymbol{\theta}^* = \begin{pmatrix} 0 & 3\pi/4 & \pi & \pi \end{pmatrix}^T$ [rad], and set $L = 1$ [m] in the following if you plan to work in a numerical way.

1. Obtain the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{\theta}^*)$.
2. Show that the following Cartesian linear/angular velocity vector is feasible:
$$\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix} = \begin{pmatrix} 0 & 0 & -L & 0 & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$
3. Determine the minimum norm joint velocity vector $\dot{\boldsymbol{\theta}}$ realizing the above Cartesian velocity.
4. Compute the joint torque vector $\boldsymbol{\tau}$ that keeps the robot in static equilibrium when the following Cartesian force/torque vector is applied from the environment to the end-effector:
$$\begin{pmatrix} \boldsymbol{F}^T & \boldsymbol{M}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
5. Consider only the velocity $\boldsymbol{v}$ of point P. Verify whether the associated 3×4 Jacobian $\boldsymbol{J}_L(\boldsymbol{\theta})$ is singular or not in the configuration $\boldsymbol{\theta}^*$.

[120 minutes; open books]

Solution

December 17, 2009

The 4R spatial manipulator is made by the subset of first four joints of the DLR manipulator considered in the textbook (p. 79, Fig. 2.29)[1]. However, the fourth (and last) reference frame is different, due to the missing axes 5, 6, and 7. The Denavit-Hartenberg parameters are given in Table 1 (the first three rows are those of Table 2.7 in the textbook, with $d_3 = L$).

i	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	$\frac{\pi}{2}$	0	L	θ_3
4	0	L	0	θ_4

Table 1: Denavit-Hartenberg parameters

The associated homogeneous transformation matrices are:

$$
\begin{aligned}
{}^0\boldsymbol{A}_1(\theta_1) &= \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(\theta_1) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^1\boldsymbol{A}_2(\theta_2) &= \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(\theta_2) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^2\boldsymbol{A}_3(\theta_3) &= \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3(\theta_3) & {}^2\boldsymbol{p}_{23} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^3\boldsymbol{A}_4(\theta_4) &= \begin{pmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & L\cos\theta_4 \\ \sin\theta_4 & \cos\theta_4 & 0 & L\sin\theta_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_4(\theta_4) & {}^3\boldsymbol{p}_{34}(\theta_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
\end{aligned}
$$

The 6×4 geometric Jacobian

$$
\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{\theta}) \\ \boldsymbol{J}_A(\boldsymbol{\theta}) \end{pmatrix}
$$

can be computed symbolically or numerically for a given configuration. We present first the general symbolic derivation, and then a more direct numerical approach.

[1] Note that in Fig. 2.29 the $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ axes are drawn in a wrong way. The associated Table 2.7 of DH parameters is instead correct for the full 7R arm.

The 3×4 upper part $\boldsymbol{J}_L$ of the geometric Jacobian relates $\dot{\boldsymbol{\theta}}$ to the velocity $\boldsymbol{v}$ of point P. It can be obtained either by (analytic) differentiation of $\boldsymbol{p}_{04}$, i.e., by computing this vector as

$$\begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{\theta}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{A}_3(\theta_3)\,{}^3\boldsymbol{A}_4(\theta_4) \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

and obtaining then

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}},$$

or by the geometric formula

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{04} & \boldsymbol{z}_1 \times \boldsymbol{p}_{04} & \boldsymbol{z}_2 \times \boldsymbol{p}_{04} & \boldsymbol{z}_3 \times (\boldsymbol{p}_{04} - \boldsymbol{p}_{03}) \end{pmatrix},$$

where we used the fact that $\boldsymbol{p}_{00} = \boldsymbol{p}_{01} = \boldsymbol{p}_{02} = \mathbf{0}$ (the origins of frames 0, 1, and 2 coincide).

Thus, for deriving its explicit symbolic form we need

$$\boldsymbol{p}_{04} = L \begin{pmatrix} \cos\theta_1 \sin\theta_2 + \cos\theta_1 \sin\theta_2 \sin\theta_4 + (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ \sin\theta_1 \sin\theta_2 + \sin\theta_1 \sin\theta_2 \sin\theta_4 - (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ -\cos\theta_2 - \cos\theta_2 \sin\theta_4 + \sin\theta_2 \cos\theta_3 \cos\theta_4 \end{pmatrix},$$

and, when following the geometric construction, also

$$\boldsymbol{p}_{04} - \boldsymbol{p}_{03} = L \begin{pmatrix} \cos\theta_1 \sin\theta_2 \sin\theta_4 + (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ \sin\theta_1 \sin\theta_2 \sin\theta_4 - (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ -\cos\theta_2 \sin\theta_4 + \sin\theta_2 \cos\theta_3 \cos\theta_4 \end{pmatrix}$$

as well as

$$\begin{aligned}
\boldsymbol{z}_0 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\boldsymbol{z}_1 &= {}^0\boldsymbol{R}_1(\theta_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \\ 0 \end{pmatrix} \\
\boldsymbol{z}_2 &= {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix} \\
\boldsymbol{z}_3 &= {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\,{}^2\boldsymbol{R}_3(\theta_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta_1 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 \\ \cos\theta_1 \cos\theta_3 + \sin\theta_1 \cos\theta_2 \sin\theta_3 \\ \sin\theta_2 \sin\theta_3 \end{pmatrix}.
\end{aligned}$$

Performing symbolic computations[2], and factoring out the length L, we obtain

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = L \cdot \begin{pmatrix} \boldsymbol{J}_{L,1} & \boldsymbol{J}_{L,2} & \boldsymbol{J}_{L,3} & \boldsymbol{J}_{L,4} \end{pmatrix},$$

[2] When using the Matlab Symbolic Toolbox, take advantage of the `simplify` instruction to reduce the length/complexity of terms.

where:

$$
\boldsymbol{J}_{L,1} = \begin{pmatrix} -\sin\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2\sin\theta_4 + (\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3)\cos\theta_4 \\ \cos\theta_1\sin\theta_2 + \cos\theta_1\sin\theta_2\sin\theta_4 + (\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3)\cos\theta_4 \\ 0 \end{pmatrix}
$$

$$
\boldsymbol{J}_{L,2} = \begin{pmatrix} \cos\theta_1(\cos\theta_2 + \cos\theta_2\sin\theta_4 - \sin\theta_2\cos\theta_3\cos\theta_4) \\ \sin\theta_1(\cos\theta_2 + \cos\theta_2\sin\theta_4 - \sin\theta_2\cos\theta_3\cos\theta_4) \\ \sin\theta_2 + \sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_3\cos\theta_4 \end{pmatrix}
$$

$$
\boldsymbol{J}_{L,3} = \begin{pmatrix} (\sin\theta_1\cos\theta_3 - \cos\theta_1\cos\theta_2\sin\theta_3)\cos\theta_4 \\ -(\cos\theta_1\cos\theta_3 + \sin\theta_1\cos\theta_2\sin\theta_3)\cos\theta_4 \\ -\sin\theta_2\sin\theta_3\cos\theta_4 \end{pmatrix}
$$

$$
\boldsymbol{J}_{L,4} = \begin{pmatrix} \cos\theta_1\sin\theta_2\cos\theta_4 - (\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3)\sin\theta_4 \\ \sin\theta_1\sin\theta_2\cos\theta_4 + (\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3)\sin\theta_4 \\ -\cos\theta_2\cos\theta_4 - \sin\theta_2\cos\theta_3\sin\theta_4 \end{pmatrix}.
$$

The 3×4 lower part $\boldsymbol{J}_A$ of the geometric Jacobian, relating $\dot{\boldsymbol{\theta}}$ to the angular velocity $\boldsymbol{\omega}$ of frame 4, is given instead by

$$
\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix},
$$

where the previous symbolic expressions for $\boldsymbol{z}_i$, $i = 0, 1, 2, 3$, are used.

At this stage, the elements of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ should be evaluated at the given configuration

$$
\boldsymbol{\theta}^* = \begin{pmatrix} 0 & 3\pi/4 & \pi & \pi \end{pmatrix}^T.
$$

In this configuration, the end-effector (the origin of frame 4) is positioned along the axis of joint 1.

Alternatively (and in a much faster way for the problem at hand!), we may first evaluate numerically the homogeneous transformations at the configuration $\boldsymbol{\theta}^*$, using in this case also $L = 1$, and then perform all the required operations, including products of matrices and (vector) cross products, so as to obtain the numerical value of the geometric Jacobian. The Matlab code is:

```
% configuration data

th1=0;
th2=3*pi/4;
th3=pi;
th4=pi;
L=1;

% homogeneous transformations

A1 = [cos(th1) 0 sin(th1) 0;
      sin(th1) 0 -cos(th1) 0;
      0 1 0 0;
      0 0 0 1];
A2 = [cos(th2) 0 sin(th2) 0;
      sin(th2) 0 -cos(th2) 0;
```

```
      0 1 0 0;
      0 0 0 1];
A3 = [cos(th3) 0  sin(th3) 0;
      sin(th3) 0 -cos(th3) 0;
      0 1 0 L;
      0 0 0 1];
A4 = [cos(th4) -sin(th4) 0 L*cos(th4);
      sin(th4)  cos(th4) 0 L*sin(th4);
      0 0 1 0;
      0 0 0 1];

A12=A1*A2;
A13=A12*A3;
A14=A13*A4;

% geometric Jacobian

z0=[0 0 1]';
z1=A1(1:3,3);
z2=A12(1:3,3);
z3=A13(1:3,3);
p0=[0 0 0]';
p1=A1(1:3,4);
p2=A12(1:3,4);
p3=A13(1:3,4);
p4=A14(1:3,4);

J(1:3,1)=cross(z0,p4-p0);
J(1:3,2)=cross(z1,p4-p1);
J(1:3,3)=cross(z2,p4-p2);
J(1:3,4)=cross(z3,p4-p3);

J(4:6,1)=z0;
J(4:6,2)=z1;
J(4:6,3)=z2;
J(4:6,4)=z3;

% end
```

Whatever approach is followed, one ends up with the following matrix (where $L = 1$, if we have worked numerically):

$$
\boldsymbol{J}(\boldsymbol{\theta}^*) = \begin{pmatrix} 0 & -L\sqrt{2} & 0 & -L\dfrac{\sqrt{2}}{2} \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & -L\dfrac{\sqrt{2}}{2} \\ 0 & 0 & \dfrac{\sqrt{2}}{2} & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & \dfrac{\sqrt{2}}{2} & 0 \end{pmatrix}.
$$

It can be seen that the rank of $\boldsymbol{J}_L(\boldsymbol{\theta}^*)$ is 3, and thus the given configuration $\boldsymbol{\theta}^*$ is not singular for this sub-Jacobian. By inspection of this matrix, the desired linear/angular velocity vector $\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T$ is realized by choosing

$$
\dot{\boldsymbol{\theta}}_d = \begin{pmatrix} 0 & -\dfrac{\sqrt{2}}{2} & 0 & \sqrt{2} \end{pmatrix}^T,
$$

obtaining in fact

$$
\boldsymbol{J}(\boldsymbol{\theta}^*)\dot{\boldsymbol{\theta}}_d = \begin{pmatrix} 0 \\ 0 \\ -L \\ 0 \\ -\dfrac{\sqrt{2}}{2} \\ 0 \end{pmatrix}.
$$

Moreover, one can see that the joint velocity vector $\dot{\boldsymbol{\theta}}_d$ is the only one providing the desired linear/angular velocity. Therefore, $\dot{\boldsymbol{\theta}}_d$ is the minimum norm solution (with $\|\dot{\boldsymbol{\theta}}_d\| = 1.5811$). As a check, it can be verified that

$$
\boldsymbol{J}^{\#}(\boldsymbol{\theta}^*) \begin{pmatrix} \boldsymbol{v}_d \\ \boldsymbol{\omega}_d \end{pmatrix} = \dot{\boldsymbol{\theta}}_d,
$$

where the pseudoinverse $\boldsymbol{J}^{\#}(\boldsymbol{\theta}^*)$ can be computed either by using the Matlab function `pinv` or by its explicit expression in case of a full (column) rank matrix $\boldsymbol{J}$ with more rows than columns,

$$
\boldsymbol{J}^{\#} = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T,
$$

which applies to the present case since the rank of $\boldsymbol{J}(\boldsymbol{\theta}^*)$ is 4. Finally, the joint torque vector $\boldsymbol{\tau}$ that balances the specified Cartesian force/torque vector $\begin{pmatrix} \boldsymbol{F}^T & \boldsymbol{M}^T \end{pmatrix}^T$ is computed as

$$
\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{\theta}^*) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L\sqrt{2} \\ 0 \\ L\dfrac{\sqrt{2}}{2} \end{pmatrix},
$$

i.e., it is given by the transpose of the first row of $\boldsymbol{J}(\boldsymbol{\theta}^*)$, changed of sign (the usual convention holds also for joint torques: positive torques are counterclockwise).

* * * * *

Robotics I

B: preferred for 5 credits

January 12, 2010

Exercise 1

Consider the Cartesian path defined by

$$\boldsymbol{p} = \boldsymbol{p}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} R\cos s \\ R\sin s \\ h\,s \end{pmatrix}, \quad s \in [0, +\infty)$$

where $R > 0$ and $h > 0$. This path is a spiral around the $\boldsymbol{z}$-axis. Define a timing law $s = s(t)$ having a *trapezoidal speed* profile in $t \in [0, T]$, for a given and sufficiently large final time $T > 0$, such that the resulting planned trajectory $\boldsymbol{p}_d(t) = \boldsymbol{p}(s(t))$ satisfies the following conditions:

- $\dot{\boldsymbol{p}}_d(0) = \dot{\boldsymbol{p}}_d(T) = \boldsymbol{0}$;
- $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$, for a given $V > 0$;
- $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$, for a given and sufficiently large $A > 0$.

Provide in particular the reached height $z_d(T)$ in closed form.

Moreover, define a *coordinated motion* for the *orientation* along the above path, by specifying a moving frame that has its $\boldsymbol{x}_o$ axis always pointing and orthogonal to the central axis of the spiral (the $\boldsymbol{z}$-axis) and its $\boldsymbol{z}_o$ always parallel to it. What is the maximum value reached by the norm of the angular velocity, $\|\boldsymbol{\omega}\|$, associated to the planned trajectory?

Finally, evaluate the solution found for the following numerical data:

$$R = 0.3 \text{ [m]}, \quad h = 0.1 \text{ [m]}, \quad V = 1 \text{ [m/s]}, \quad A = 5 \text{ [m/s}^2\text{]}, \quad T = 4 \text{ [s]}.$$

Exercise 2B

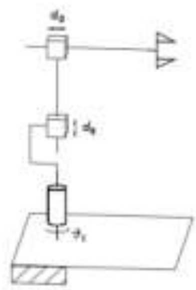


Figure 1: A cylindrical manipulator

Derive the 6×3 geometric Jacobian for the cylindrical manipulator in Fig. 1 and find the singularities of its linear velocity part. Consider a desired motion $\boldsymbol{p}_d(t)$ of the end-effector position that is twice-differentiable w.r.t. time. Taking the joint accelerations $\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{\theta}_1 & \ddot{d}_2 & \ddot{d}_3 \end{pmatrix}^T$ as control inputs and assuming that only $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are measured, define a Cartesian kinematic controller *at the acceleration level* that assigns (out of singularities) the closed-loop behavior to the system

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_D \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} = \boldsymbol{0},$$

where $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$, and $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are positive definite, diagonal matrices.

[150 minutes; open books]

Solutions

January 12, 2010

Exercise 1

The velocity vector along the path is given by

$$\dot{\boldsymbol{p}}_d = \frac{d\boldsymbol{p}_d(t)}{dt} = \frac{d\boldsymbol{p}(s)}{ds}\frac{ds(t)}{dt} = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \dot{s},$$

and thus

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{R^2+h^2}\,|\dot{s}(t)|.$$

The constraint $\|\dot{\boldsymbol{p}}_d(t)\| \le V$ on the Cartesian velocity becomes

$$|\dot{s}(t)| \le \frac{V}{\sqrt{R^2+h^2}} =: V_{\max}$$

for the speed profile $\dot{s}$.

The acceleration vector along the path is given by

$$\ddot{\boldsymbol{p}}_d = \frac{d^2\boldsymbol{p}_d(t)}{dt^2} = \frac{d\boldsymbol{p}(s)}{ds}\ddot{s}(t) + \frac{d^2\boldsymbol{p}(s)}{ds^2}\dot{s}^2(t) = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \ddot{s} + \begin{pmatrix} -R\cos s \\ -R\sin s \\ 0 \end{pmatrix} \dot{s}^2,$$

and thus

$$\|\ddot{\boldsymbol{p}}_d(t)\| = \sqrt{(R^2+h^2)\,\ddot{s}^2(t) + (R\,\dot{s}^2(t))^2}.$$

The constraint $\|\ddot{\boldsymbol{p}}_d(t)\| \le A$ on the Cartesian acceleration can be rewritten as

$$(R^2+h^2)\,\ddot{s}^2(t) \le A^2 - (R\,\dot{s}^2(t))^2$$

for the acceleration profile $\ddot{s}$. Since this constraint has to be satisfied for all $t \in [0, T]$, one should consider the worst case, i.e., $|\dot{s}| = V_{\max}$. We obtain

$$|\ddot{s}(t)| \le \sqrt{\frac{A^2 - (\frac{RV^2}{R^2+h^2})^2}{R^2+h^2}} =: A_{\max}.$$

In order to have a feasible $A_{\max} > 0$, the value of A should be sufficiently large, i.e.,

$$A > \frac{RV^2}{R^2+h^2}. \tag{1}$$

At this stage, given the total time T and the computed limits $V_{\max}$ and $A_{\max}$, the timing law with trapezoidal speed profile is fully specified. In particular, we have for the acceleration/deceleration interval time

$$T_s = \frac{V_{\max}}{A_{\max}} = \frac{V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}.$$

In order to have a complete trapezoidal profile (with at least one instant where $V_{\max}$ is reached), the total time T should be sufficiently large, i.e.,

$$T \ge 2T_s = \frac{2V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}. \tag{2}$$

The total displacement of the parameter s at time $t = T$ is then

$$s_{\max} := s(T) = (T - T_s)V_{\max} = TV_{\max} - \frac{V_{\max}^2}{A_{\max}} = \frac{TV}{\sqrt{R^2 + h^2}} - \frac{V^2}{\sqrt{(R^2 + h^2)A^2 - \frac{(RV^2)^2}{R^2+h^2}}}.$$

Therefore, the reached height at the final time $t = T$ is

$$z_d(T) = h\, s(T) = h\, s_{\max}.$$

For completeness, we compute also the curvature of the given parametric path:

$$\kappa(s) = \frac{\left\| \dfrac{d\boldsymbol{p}}{ds} \times \dfrac{d^2\boldsymbol{p}}{ds^2} \right\|}{\left\| \dfrac{d\boldsymbol{p}}{ds} \right\|^3} = \frac{R}{R^2 + h^2}.$$

Indeed, $\kappa(s)$ is constant for all s and collapses to $1/R$ for $h = 0$.

For planning the requested orientation trajectory, which has to be coordinated with the position trajectory, we define a moving frame as a function of the same parameter s. This is given by

$$\boldsymbol{R}(s) = \begin{pmatrix} \boldsymbol{x}_o(s) & \boldsymbol{y}_0(s) & \boldsymbol{z}_o(s) \end{pmatrix} = \begin{pmatrix} -\cos s & \sin s & 0 \\ -\sin s & -\cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this moving frame is *not* the Frenet frame associated to the parametrized path. Using the notations $\boldsymbol{p}'(s) = d\boldsymbol{p}(s)/ds$ and $\boldsymbol{p}''(s) = d^2\boldsymbol{p}(s)/ds^2$, the Frenet frame is specified as

$$\begin{aligned} \boldsymbol{R}_{\text{Frenet}}(s) &= \begin{pmatrix} \boldsymbol{t}(s) & \boldsymbol{n}(s) & \boldsymbol{b}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{p}'(s)}{\|\boldsymbol{p}'(s)\|} & \dfrac{\boldsymbol{p}''(s)}{\|\boldsymbol{p}''(s)\|} & \boldsymbol{t}(s) \times \boldsymbol{n}(s) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{R}{\sqrt{R^2+h^2}} \sin s & -\cos s & \frac{h}{\sqrt{R^2+h^2}} \sin s \\ \frac{R}{\sqrt{R^2+h^2}} \cos s & -\sin s & -\frac{h}{\sqrt{R^2+h^2}} \cos s \\ \frac{h}{\sqrt{R^2+h^2}} & 0 & \frac{R}{\sqrt{R^2+h^2}} \end{pmatrix}. \end{aligned}$$

In fact, the two frames coincide (modulo a rotation of $\pi/2$ around the $\boldsymbol{z}$-axis) only when $h = 0$.

Setting $\boldsymbol{R}_d(t) = \boldsymbol{R}(s(t))$, the angular velocity vector is computed from

$$\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_d \boldsymbol{R}_d^T = \dot{s}(t) \begin{pmatrix} \sin s(t) & \cos s(t) & 0 \\ -\cos s(t) & \sin s(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos s(t) & -\sin s(t) & 0 \\ \sin s(t) & -\cos s(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}(t) & 0 \\ \dot{s}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As expected (being the rotation of the moving frame only around the $\boldsymbol{z}$-axis and counterclockwise),

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{s}(t) \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{\omega}\| = |\dot{s}(t)|,$$

and the maximum value of the norm of the angular velocity vector is obviously $V_{\max}$.

With the given numerical data, which satisfy both inequalities (1) and (2), we obtain:

$$V_{\max} = \sqrt{10} = 3.1623, \quad A_{\max} = 4\sqrt{10} = 12.6491, \quad T_s = 0.25,$$

$$s_{\max} = 3.75\sqrt{10} = 11.8585, \quad z_d(T) = 0.375\sqrt{10} = 1.1859\,.$$

In the following, we show plots of the planned trajectory obtained in Matlab (code available).

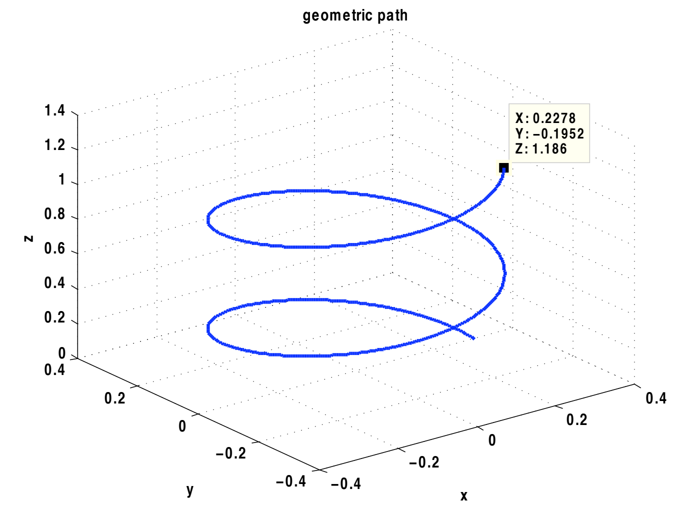


Figure 2: The spiral Cartesian trajectory (with coordinates of the final reached point at time $T = 4$ s)

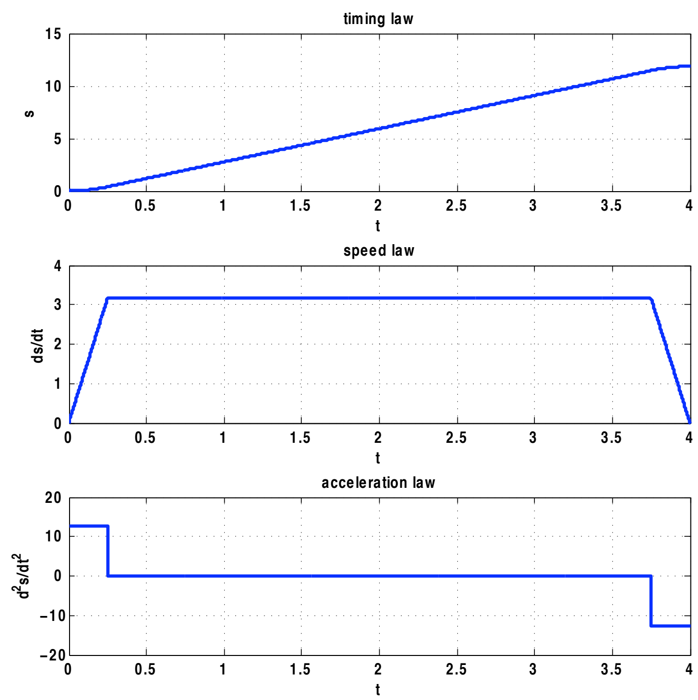


Figure 3: Timing law: Path parameter $s(t)$, speed $\dot{s}(t)$, and acceleration $\ddot{s}(t)$

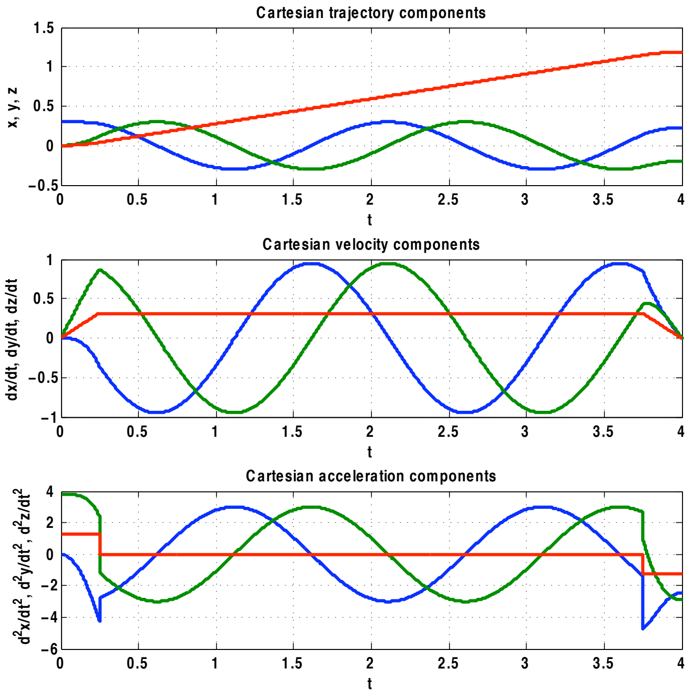


Figure 4: Components of Cartesian trajectory: Position, velocity, and acceleration (x in blue, y in green, z in red)

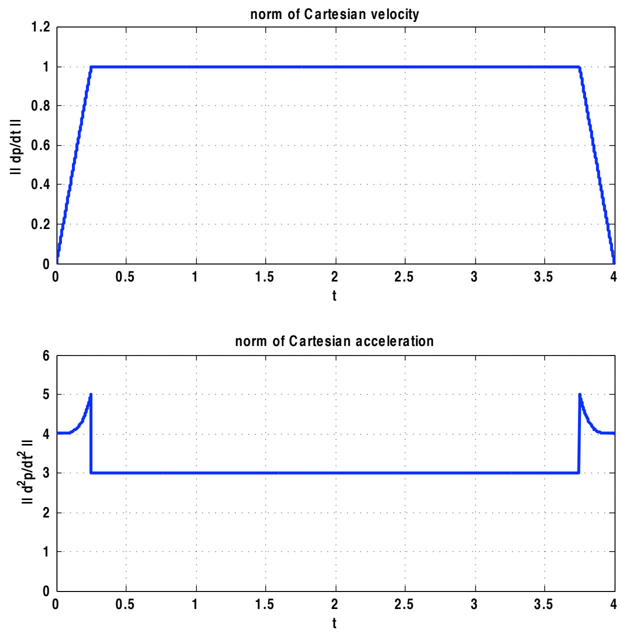


Figure 5: Norms of the Cartesian velocity and acceleration: The given bounds $\|\dot{\boldsymbol{p}}_d(t)\| \leq 1$ and $\|\ddot{\boldsymbol{p}}_d(t)\| \leq 5$ are always satisfied during motion

Exercise 2B

The Jacobian for the cylindrical (RPP) manipulator with $\boldsymbol{q} = (\theta_1, d_2, d_3)$ is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p} & \boldsymbol{z}_1 & \boldsymbol{z}_2 \\ \boldsymbol{z}_0 & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},$$

with the axes of the three joints being

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_2 = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix},$$

and the end-effector position vector given by

$$\boldsymbol{p} = \boldsymbol{k}(\boldsymbol{q}) = \begin{pmatrix} d_3 \cos\theta_1 \\ d_3 \sin\theta_1 \\ d_2 \end{pmatrix}. \tag{3}$$

Then, the expression of the geometric Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -d_3 \sin\theta_1 & 0 & \cos\theta_1 \\ d_3 \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which reveals that it is inherently impossible to rotate about the axes $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$.

The Jacobian relative to the end-effector linear velocity can be extracted by considering only the first three rows, i.e.,

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -d_3 \sin\theta_1 & 0 & \cos\theta_1 \\ d_3 \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \end{pmatrix},$$

which coincides indeed with the differentiation w.r.t. $\boldsymbol{q}$ of the direct kinematics function $\boldsymbol{k}(\boldsymbol{q})$ in (3). Its determinant is

$$\det \boldsymbol{J}_L(\boldsymbol{q}) = d_3,$$

vanishing at the singularity $d_3 = 0$. This occurs when the end-effector is located along the axis of joint 1, a situation conceptually similar to the shoulder singularity of an anthropomorphic 3R arm.

Since $\dot{\boldsymbol{p}} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}$, the differential kinematics at the acceleration level is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}_L(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}},$$

where

$$\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -\dot{d}_3 \sin\theta_1 - d_3\,\dot{\theta}_1 \cos\theta_1 & 0 & -\dot{\theta}_1 \sin\theta_1 \\ \dot{d}_3 \cos\theta_1 - d_3\,\dot{\theta}_1 \sin\theta_1 & 0 & \dot{\theta}_1 \cos\theta_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{pmatrix} = \begin{pmatrix} -2\,\dot{d}_3\,\dot{\theta}_1 \sin\theta_1 - d_3\,\dot{\theta}_1^2 \cos\theta_1 \\ 2\,\dot{d}_3\,\dot{\theta}_1 \cos\theta_1 - d_3\,\dot{\theta}_1^2 \sin\theta_1 \\ 0 \end{pmatrix}.$$

Therefore, designing the joint acceleration vector as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_L^{-1}(\boldsymbol{q})(\ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{p}}_d - \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{k}(\boldsymbol{q})) - \dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}) \tag{4}$$

yields

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}),$$

namely the desired closed-loop behavior. Note that (4) is implemented using only the measurements of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, beside the knowledge of the desired trajectory (up to its second time derivative) and of the arm direct and differential kinematics.

* * * * *

Robotics I

A: preferred for 6 credits

January 12, 2010

Exercise 1

Consider the Cartesian path defined by

$$\boldsymbol{p} = \boldsymbol{p}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} R\cos s \\ R\sin s \\ h\,s \end{pmatrix}, \quad s \in [0, +\infty)$$

where $R > 0$ and $h > 0$. This path is a spiral around the $\boldsymbol{z}$-axis. Define a timing law $s = s(t)$ having a *trapezoidal speed* profile in $t \in [0, T]$, for a given and sufficiently large final time $T > 0$, such that the resulting planned trajectory $\boldsymbol{p}_d(t) = \boldsymbol{p}(s(t))$ satisfies the following conditions:

- $\dot{\boldsymbol{p}}_d(0) = \dot{\boldsymbol{p}}_d(T) = \boldsymbol{0}$;
- $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$, for a given $V > 0$;
- $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$, for a given and sufficiently large $A > 0$.

Provide in particular the reached height $z_d(T)$ in closed form.

Moreover, define a *coordinated motion* for the *orientation* along the above path, by specifying a moving frame that has its $\boldsymbol{x}_o$ axis always pointing and orthogonal to the central axis of the spiral (the $\boldsymbol{z}$-axis) and its $\boldsymbol{z}_o$ always parallel to it. What is the maximum value reached by the norm of the angular velocity, $\|\boldsymbol{\omega}\|$, associated to the planned trajectory?

Finally, evaluate the solution found for the following numerical data:

$$R = 0.3 \text{ [m]}, \quad h = 0.1 \text{ [m]}, \quad V = 1 \text{ [m/s]}, \quad A = 5 \text{ [m/s}^2\text{]}, \quad T = 4 \text{ [s]}.$$

Exercise 2A

Extend the design of an input-output linearizing (and decoupling) trajectory controller presented in the textbook, as well as in class, for the case of a unicycle to the kinematic model of a front-wheel driven car-like vehicle. This control design should allow a suitable point B attached to the car-like vehicle to reproduce exactly (in nominal conditions) and to track in a stable way (in presence of non-persistent disturbances) *any continuous reference trajectory*, possibly having velocity discontinuities. Provide the full expression of the control law, analyzing its singularities (if any), and of the resulting closed-loop system. Discuss the pros and cons of this control approach, in particular with respect to the presence of obstacles in the vicinity of the reference trajectory.

[150 minutes; open books]

Solutions

January 12, 2010

Exercise 1

The velocity vector along the path is given by

$$\dot{\boldsymbol{p}}_d = \frac{d\boldsymbol{p}_d(t)}{dt} = \frac{d\boldsymbol{p}(s)}{ds}\frac{ds(t)}{dt} = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \dot{s},$$

and thus

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{R^2+h^2}\,|\dot{s}(t)|.$$

The constraint $\|\dot{\boldsymbol{p}}_d(t)\| \le V$ on the Cartesian velocity becomes

$$|\dot{s}(t)| \le \frac{V}{\sqrt{R^2+h^2}} =: V_{\max}$$

for the speed profile $\dot{s}$.

The acceleration vector along the path is given by

$$\ddot{\boldsymbol{p}}_d = \frac{d^2\boldsymbol{p}_d(t)}{dt^2} = \frac{d\boldsymbol{p}(s)}{ds}\ddot{s}(t) + \frac{d^2\boldsymbol{p}(s)}{ds^2}\dot{s}^2(t) = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \ddot{s} + \begin{pmatrix} -R\cos s \\ -R\sin s \\ 0 \end{pmatrix} \dot{s}^2,$$

and thus

$$\|\ddot{\boldsymbol{p}}_d(t)\| = \sqrt{(R^2+h^2)\,\ddot{s}^2(t) + (R\,\dot{s}^2(t))^2}.$$

The constraint $\|\ddot{\boldsymbol{p}}_d(t)\| \le A$ on the Cartesian acceleration can be rewritten as

$$(R^2+h^2)\,\ddot{s}^2(t) \le A^2 - (R\,\dot{s}^2(t))^2$$

for the acceleration profile $\ddot{s}$. Since this constraint has to be satisfied for all $t \in [0, T]$, one should consider the worst case, i.e., $|\dot{s}| = V_{\max}$. We obtain

$$|\ddot{s}(t)| \le \sqrt{\frac{A^2 - (\frac{RV^2}{R^2+h^2})^2}{R^2+h^2}} =: A_{\max}.$$

In order to have a feasible $A_{\max} > 0$, the value of A should be sufficiently large, i.e.,

$$A > \frac{RV^2}{R^2+h^2}. \tag{1}$$

At this stage, given the total time T and the computed limits $V_{\max}$ and $A_{\max}$, the timing law with trapezoidal speed profile is fully specified. In particular, we have for the acceleration/deceleration interval time

$$T_s = \frac{V_{\max}}{A_{\max}} = \frac{V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}.$$

In order to have a complete trapezoidal profile (with at least one instant where $V_{\max}$ is reached), the total time T should be sufficiently large, i.e.,

$$T \ge 2T_s = \frac{2V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}. \tag{2}$$

The total displacement of the parameter s at time $t = T$ is then

$$s_{\max} := s(T) = (T - T_s)V_{\max} = TV_{\max} - \frac{V_{\max}^2}{A_{\max}} = \frac{TV}{\sqrt{R^2 + h^2}} - \frac{V^2}{\sqrt{(R^2 + h^2)A^2 - \frac{(RV^2)^2}{R^2+h^2}}}.$$

Therefore, the reached height at the final time $t = T$ is

$$z_d(T) = h\, s(T) = h\, s_{\max}.$$

For completeness, we compute also the curvature of the given parametric path:

$$\kappa(s) = \frac{\left\| \dfrac{d\boldsymbol{p}}{ds} \times \dfrac{d^2\boldsymbol{p}}{ds^2} \right\|}{\left\| \dfrac{d\boldsymbol{p}}{ds} \right\|^3} = \frac{R}{R^2 + h^2}.$$

Indeed, $\kappa(s)$ is constant for all s and collapses to $1/R$ for $h = 0$.

For planning the requested orientation trajectory, which has to be coordinated with the position trajectory, we define a moving frame as a function of the same parameter s. This is given by

$$\boldsymbol{R}(s) = \begin{pmatrix} \boldsymbol{x}_o(s) & \boldsymbol{y}_0(s) & \boldsymbol{z}_o(s) \end{pmatrix} = \begin{pmatrix} -\cos s & \sin s & 0 \\ -\sin s & -\cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this moving frame is *not* the Frenet frame associated to the parametrized path. Using the notations $\boldsymbol{p}'(s) = d\boldsymbol{p}(s)/ds$ and $\boldsymbol{p}''(s) = d^2\boldsymbol{p}(s)/ds^2$, the Frenet frame is specified as

$$\begin{aligned}
\boldsymbol{R}_{\text{Frenet}}(s) &= \begin{pmatrix} \boldsymbol{t}(s) & \boldsymbol{n}(s) & \boldsymbol{b}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{p}'(s)}{\|\boldsymbol{p}'(s)\|} & \dfrac{\boldsymbol{p}''(s)}{\|\boldsymbol{p}''(s)\|} & \boldsymbol{t}(s) \times \boldsymbol{n}(s) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{R}{\sqrt{R^2+h^2}} \sin s & -\cos s & \frac{h}{\sqrt{R^2+h^2}} \sin s \\ \frac{R}{\sqrt{R^2+h^2}} \cos s & -\sin s & -\frac{h}{\sqrt{R^2+h^2}} \cos s \\ \frac{h}{\sqrt{R^2+h^2}} & 0 & \frac{R}{\sqrt{R^2+h^2}} \end{pmatrix}.
\end{aligned}$$

In fact, the two frames coincide (modulo a rotation of $\pi/2$ around the $\boldsymbol{z}$-axis) only when $h = 0$.

Setting $\boldsymbol{R}_d(t) = \boldsymbol{R}(s(t))$, the angular velocity vector is computed from

$$\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_d \boldsymbol{R}_d^T = \dot{s}(t) \begin{pmatrix} \sin s(t) & \cos s(t) & 0 \\ -\cos s(t) & \sin s(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos s(t) & -\sin s(t) & 0 \\ \sin s(t) & -\cos s(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}(t) & 0 \\ \dot{s}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As expected (being the rotation of the moving frame only around the $\boldsymbol{z}$-axis and counterclockwise),

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{s}(t) \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{\omega}\| = |\dot{s}(t)|,$$

and the maximum value of the norm of the angular velocity vector is obviously $V_{\max}$.

With the given numerical data, which satisfy both inequalities (1) and (2), we obtain:

$$V_{\max} = \sqrt{10} = 3.1623, \quad A_{\max} = 4\sqrt{10} = 12.6491, \quad T_s = 0.25,$$

$$s_{\max} = 3.75\sqrt{10} = 11.8585, \quad z_d(T) = 0.375\sqrt{10} = 1.1859\,.$$

In the following, we show plots of the planned trajectory obtained in Matlab (code available).

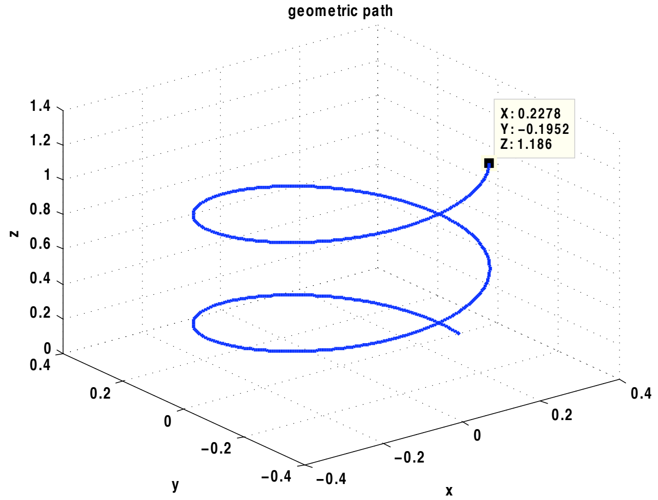


Figure 1: The spiral Cartesian trajectory (with coordinates of the final reached point at time $T = 4$ s)

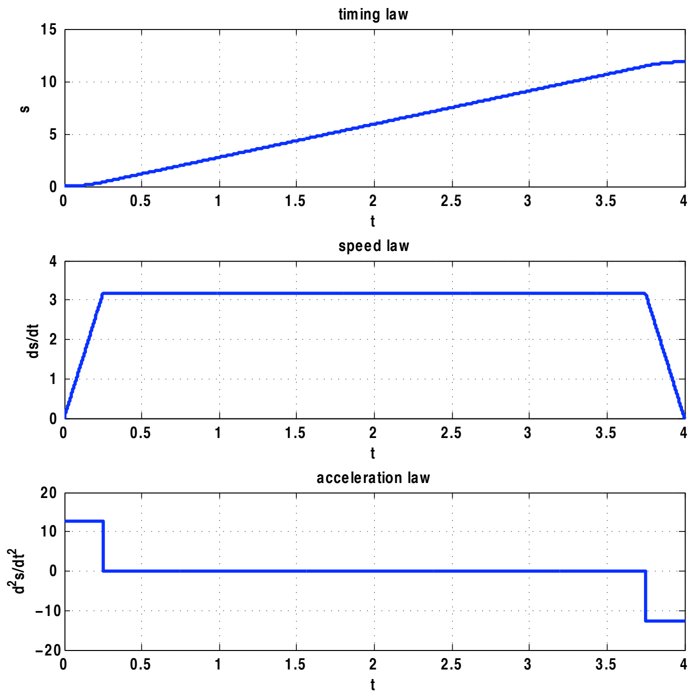


Figure 2: Timing law: Path parameter $s(t)$, speed $\dot{s}(t)$, and acceleration $\ddot{s}(t)$

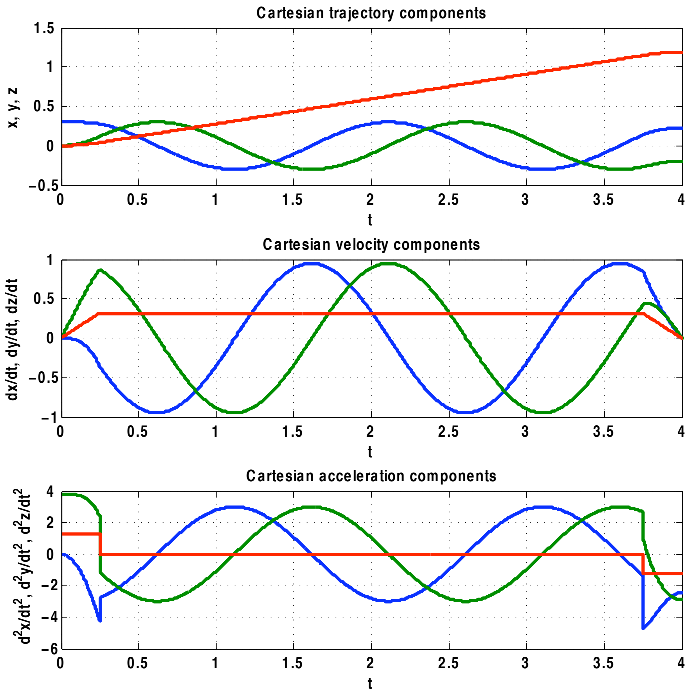


Figure 3: Components of Cartesian trajectory: Position, velocity, and acceleration (x in blue, y in green, z in red)

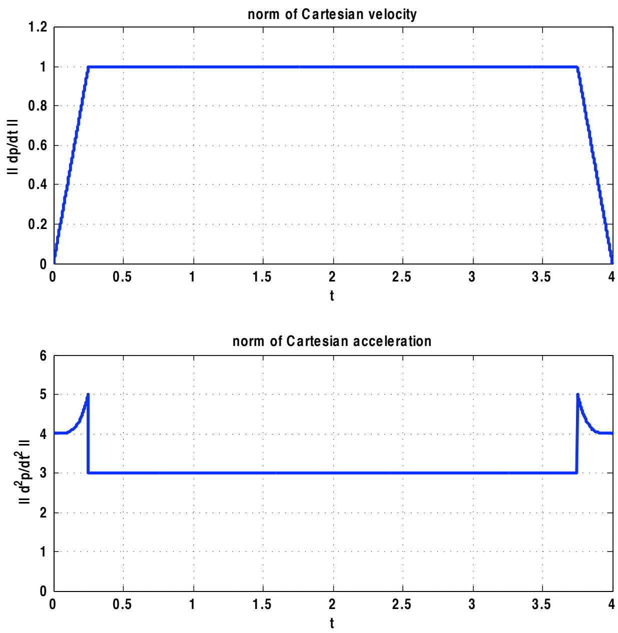


Figure 4: Norms of the Cartesian velocity and acceleration: The given bounds $\|\dot{\boldsymbol{p}}_d(t)\| \leq 1$ and $\|\ddot{\boldsymbol{p}}_d(t)\| \leq 5$ are always satisfied during motion

Exercise 2A

The kinematic model of a front-wheel driven *car-like* vehicle is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\phi \, u_1 \\ \sin\theta \cos\phi \, u_1 \\ \dfrac{\sin\phi}{\ell} \, u_1 \\ u_2 \end{pmatrix}, \tag{3}$$

where (x, y) are the coordinates of the rear wheel, θ is the absolute orientation of the vehicle (w.r.t. the $\boldsymbol{x}$ reference axis), ϕ is the steering angle of the front wheel (w.r.t. the car orientation), u_1 is the driving velocity of the front wheel and u_2 is its steering velocity.

Mimicking the trajectory controller design for the unicycle, a point B can be chosen at a distance $|b| > 0$ (b itself can be either positive or negative) from the front wheel along the direction of its absolute orientation, as given by the angle $\theta + \phi$. In this way, the velocity of point B will be affected directly by both commands u_1 and u_2. The position of B is thus given by

$$\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \ell \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + b \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix}. \tag{4}$$

Differentiating once (4) w.r.t. time and using (3), we obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_B \\ \dot{y}_B \end{pmatrix} &= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \ell\,\dot{\theta} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} + b\,(\dot{\theta} + \dot{\phi}) \begin{pmatrix} -\sin(\theta+\phi) \\ \cos(\theta+\phi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\phi) - \frac{b}{\ell}\,\sin\phi \sin(\theta+\phi) & -b\,\sin(\theta+\phi) \\ \sin(\theta+\phi) + \frac{b}{\ell}\,\sin\phi\cos(\theta+\phi) & b\,\cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \boldsymbol{T}(\theta,\phi) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

Since

$$\det \boldsymbol{T}(\theta,\phi) = b \neq 0,$$

matrix $\boldsymbol{T}$ (the so-called decoupling matrix of the system) can be inverted at any configuration. The input-output linearizing and decoupling control is then globally defined as

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \boldsymbol{T}^{-1}(\theta,\phi) \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad (5) \\ &= \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\frac{1}{b}\,\sin(\theta+\phi) - \frac{1}{\ell}\,\sin\phi\cos(\theta+\phi) & \frac{1}{b}\,\cos(\theta+\phi) - \frac{1}{\ell}\sin\phi\sin(\theta+\phi) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \end{aligned}$$

where v_x and v_y are two auxiliary inputs to be defined for asymptotically stable trajectory tracking purposes.

The closed-loop system, which is still partly nonlinear, is described by

$$\begin{pmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ \dfrac{\sin\phi}{\ell}\,(v_x\cos(\theta+\phi) + v_y \sin(\theta+\phi)) \\ \left(\dfrac{v_y}{b} - \dfrac{v_x\sin\phi}{\ell}\right)\cos(\theta+\phi) - \left(\dfrac{v_x}{b} + \dfrac{v_y\sin\phi}{\ell}\right)\sin(\theta+\phi) \end{pmatrix}, \tag{6}$$

showing that the velocity commands v_x and v_y independently drive the x and y components of the velocity of point B. For a given *continuous* reference trajectory $\boldsymbol{p}_{B,d}(t) = \begin{pmatrix} x_{B,d}(t) & y_{B,d}(t) \end{pmatrix}^T$, an asymptotically (actually, exponentially) stable tracking is obtained by choosing in (5)

$$v_x = \dot{x}_{B,d} + k_x(x_{B,d} - x_B), \qquad v_y = \dot{y}_{B,d} + k_y(y_{B,d} - y_B),$$

with $k_x > 0$ and $k_y > 0$.

The main advantage of this control design stands in its simplicity for tracking very general output trajectories. In fact, the underlying path can also have tangent discontinuities which can be executed without stopping the motion of point B. Such behavior may occur even in the presence of geometric cusps, since an instantaneous reversal of the velocity of point B is still feasible. On the other hand, the choice of a suitable value of b is critical. A small value of $|b|$ will lead to high control efforts in the presence of path tangent discontinuities to be crossed at non-negligible speed or, more in general, when sharp directional changes are required. A large value of $|b|$ will instead increase the actual area "spanned" around the nominal output trajectory by the vehicle body during motion. This should be taken into account for collision avoidance of nearby obstacles.

Moreover, as an additional issue with respect to the simpler case of a unicycle, it would be interesting to study the effect of choosing negative values for b and of varying its ratio to the car length ℓ. In summary, an investigation of the properties of boundedness of the evolution of (θ, ϕ) (the so-called *zero dynamics* of the closed-loop system (6)) should be conducted when the point B is commanded so as to exactly reproduce some specific classes of (complex) reference trajectories.

* * * * *

Robotics I

February 11, 2010

Consider a planar 2R manipulator having link lengths $\ell_1 = 0.6$, $\ell_2 = 0.5$ [m]. The joint angles θ_1, θ_2 are defined using the DH convention. The joint ranges are unlimited. The base of the manipulator is placed at the origin of the given $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ frame.

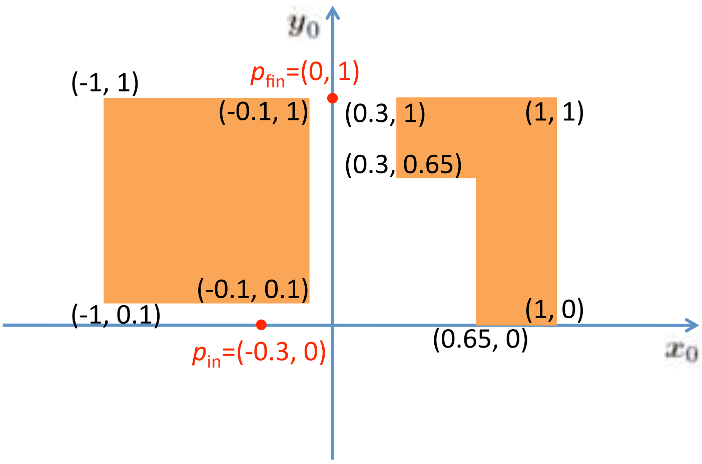


Figure 1: Workspace for the assigned task

With reference to Fig. 1, plan a continuous parametric path so that the end-effector is transferred between the Cartesian points

$$\boldsymbol{p}_{\text{in}} = \begin{pmatrix} -0.3 \\ 0 \end{pmatrix} \qquad \mapsto \qquad \boldsymbol{p}_{\text{fin}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{[m]},$$

and the following conditions are satisfied:

- the path is made of polynomial functions of the lowest possible degree;
- the path tangent is continuous with respect to the path parameter;
- the manipulator avoids collision with the two obstacles shown in orange.

Provide a solution and check graphically (e.g., using Matlab) the collision avoidance. If a singularity is encountered in the proposed solution, indicate how this situation is handled. Moreover, explain how a timing law should be assigned so that the resulting trajectory has a satisfactory behavior.

[150 minutes; open books]

Solution

February 11, 2010

The manipulator workspace is a circular ring with inner circumference of radius $|\ell_1 - \ell_2| = 0.1$ and outer circumference of radius $\ell_1 + \ell_2 = 1.1$. Therefore, the two given Cartesian points are within the manipulator workspace. Using the inverse kinematics function of the 2R robot yields the two solutions

$$\boldsymbol{\theta}_{\rm in}^{\rm left} = \begin{pmatrix} -2.1598 \\ -2.6193 \end{pmatrix} \qquad \boldsymbol{\theta}_{\rm in}^{\rm right} = \begin{pmatrix} 2.1598 \\ 2.6193 \end{pmatrix} \quad [\text{rad}]$$

for the initial Cartesian point $\boldsymbol{p}_{\rm in}$, and the two solutions

$$\boldsymbol{\theta}_{\rm fin}^{\rm left} = \begin{pmatrix} 1.9606 \\ -0.8632 \end{pmatrix} \qquad \boldsymbol{\theta}_{\rm fin}^{\rm right} = \begin{pmatrix} 1.1810 \\ 0.8632 \end{pmatrix} \quad [\text{rad}]$$

for the final Cartesian point $\boldsymbol{p}_{\rm fin}$. To avoid collision at the initial and final point, we need to choose $\boldsymbol{\theta}_{\rm in}^{\rm left}$ and $\boldsymbol{\theta}_{\rm fin}^{\rm right}$, respectively. Since these inverse solutions are of two different kinds, it is clear that the arm will need to pass through a singular configuration (stretched or folded) during motion.

Therefore, the easiest way to address the problem is to define a path in the joint space, possibly using one (or more) via points. The path can then cross singular configurations without control problems at run time during path/trajectory execution (no Jacobian inversion is needed). Indeed, the problem of avoiding collisions remains.

The straightforward interpolation of the initial and final configurations by a single linear joint path (the polynomial of lowest possible degree) is not feasible. To see this, introduce a parameter $s \in [0,1]$ for describing the path $\boldsymbol{\theta} = \boldsymbol{q}(s) = \begin{pmatrix} q_1(s) & q_2(s) \end{pmatrix}^T$. The interpolating linear path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \left(\boldsymbol{\theta}_{\rm fin}^{\rm right} - \boldsymbol{\theta}_{\rm in}^{\rm left}\right) s + \boldsymbol{\theta}_{\rm in}^{\rm left}, \qquad s \in [0,1]. \tag{1}$$

The arm will pass through the stretched singularity for $s = s^*$ such that $\theta_2 = q_2(s^*) = 0$. This happens at

$$s^* = 0.7521 \qquad \Rightarrow \qquad \theta_1 = q_1(s^*) = 0.3529 \quad [\text{rad}].$$

It is easy to see that a collision occurs with the obstacle on the right, e.g., for the manipulator configuration $\boldsymbol{\theta} = \begin{pmatrix} 0.3529 & 0 \end{pmatrix}^T$. This is confirmed graphically by a simple Matlab code, implementing joint path generation and manipulator direct kinematics, and plotting results as in Fig. 2. Initial (green) and final (red) configuration, and end-effector path (dotted) are also shown.

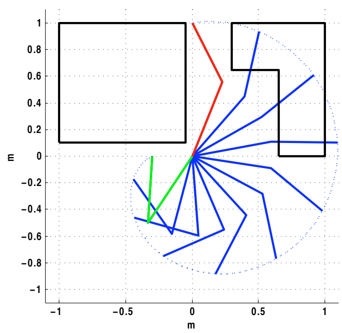


Figure 2: Stroboscopic view of manipulator motion for a linear joint path, resulting in a collision

In order to avoid this situation, we may replace in (1) the second component of $\boldsymbol{\theta}_{\text{fin}}^{\text{right}}$ by its value modulo 2π, i.e.,

$$\boldsymbol{\theta}_{\text{fin}}^{\text{right}^-} = \begin{pmatrix} 1.1810 \\ 0.8632 - 2\pi \end{pmatrix} = \begin{pmatrix} 1.1810 \\ -5.4200 \end{pmatrix} \quad \text{[rad]}.$$

The linear path of the second joint will now necessarily cross the value $\theta_2 = -\pi$, i.e., the arm will pass through the folded singularity. Unfortunately, this is not yet sufficient to avoid collision (see Fig. 3). For instance, the tip of the robot is in collision at $s^* = 0.75$, being the arm in the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0.3458 & -4.7198 \end{pmatrix}^T$ (or, $\theta_1 \approx 20°$ and $\theta_2 = -270°$).

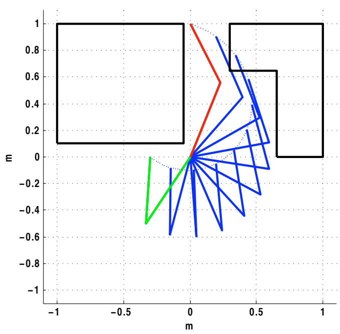


Figure 3: Stroboscopic view of manipulator motion for another linear joint path, resulting again in a collision

As a result of this analysis, we need to introduce an intermediate joint configuration, say at $s = 0.5$, which is conveniently chosen as the folded arm configuration in correspondence to the point where the passage between the obstacles begins, i.e.,

$$\boldsymbol{\theta}_{\text{mid}} = \begin{pmatrix} 0 \\ -\pi \end{pmatrix} \quad \text{[rad]} \qquad \Leftrightarrow \qquad \boldsymbol{p}_{\text{mid}} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \quad \text{[m]}.$$

The choice of a negative value $\theta_{\text{mid},2} = -\pi$ (rather than π) is based on the same previous argument: by continuity of motion, the second link will always rotate in the clockwise direction, reaching the folded singularity at the specified location $\boldsymbol{p}_{\text{mid}}$, and then unfolding itself so as to avoid collision with the obstacle on the right. The boundary conditions for the interpolating joint path are then:

$$\boldsymbol{q}(0) = \boldsymbol{\theta}_{\text{in}}^{\text{left}} = \begin{pmatrix} -2.1598 \\ -2.6193 \end{pmatrix}, \quad \boldsymbol{q}(\tfrac{1}{2}) = \boldsymbol{\theta}_{\text{mid}} = \begin{pmatrix} 0 \\ -\pi \end{pmatrix}, \quad \boldsymbol{q}(1) = \boldsymbol{\theta}_{\text{fin}}^{\text{right}^-} = \begin{pmatrix} 1.1810 \\ -5.4200 \end{pmatrix}. \tag{2}$$

Further, we need to impose now also continuity of the joint path tangent at the mid point, i.e.,

$$\left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}^-} = \left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}^+}. \tag{3}$$

Therefore, we can select (for each joint) a quadratic and a linear function of s (or viceversa) on the two tracts of the path, allowing a total of five coefficients for satisfying the five boundary conditions. Such a mixed polynomial path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \begin{cases} \boldsymbol{a}s^2 + \boldsymbol{b}s + \boldsymbol{c}, & \text{for } s \in [0, \frac{1}{2}] \\ \boldsymbol{d}s + \boldsymbol{e}, & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

where $\boldsymbol{a}, \ldots, \boldsymbol{e}$ are suitable two-dimensional vectors of coefficients. Imposing the boundary conditions (2–3), and dropping for compactness the superscripts 'left' and 'right$^-$', yields:

$$\boldsymbol{q}(s) = \begin{cases} (4\boldsymbol{\theta}_{\text{fin}} - 8\boldsymbol{\theta}_{\text{mid}} + 4\boldsymbol{\theta}_{\text{in}})\, s^2 + (6\boldsymbol{\theta}_{\text{mid}} - 4\boldsymbol{\theta}_{\text{in}} - 2\boldsymbol{\theta}_{\text{fin}})\, s + \boldsymbol{\theta}_{\text{in}}, & \text{for } s \in [0, \frac{1}{2}] \\ 2\,(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}})\, s + 2\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{fin}}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{4}$$

The planned joint path and the path tangent are given in Figs. 4 and 5, respectively. A stroboscopic view of the resulting manipulator motion is shown in Fig. 6. It can be seen that there are no collisions.

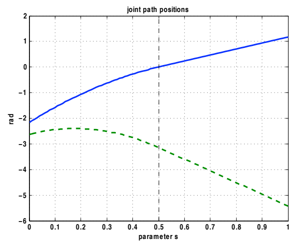


Figure 4: Quadratic/linear path in the joint space: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

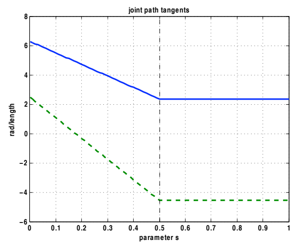


Figure 5: Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)

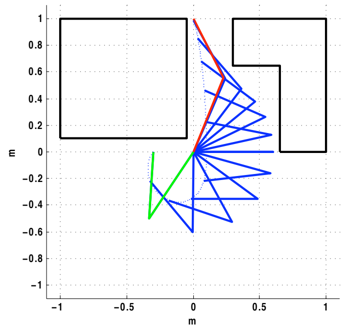


Figure 6: Stroboscopic view of manipulator motion for the quadratic/linear joint path

To convert this path into a trajectory, one should associate a timing law $s = s(t)$ for $t \in [0, T]$. Any timing law can be chosen (bang-bang in acceleration, with trapezoidal speed profile, cubic time polynomial, ...), depending on additional task requests and robot performance limits. The only care is that a *single* timing law should be chosen for both joints. Otherwise, the joint motion would be uncoordinated in time and the executed Cartesian robot path would *not* be the planned one, with a possible danger of collisions.

Additional considerations. In the following, we present complementary material to the given solution. In particular, we consider a more balanced path planning solution using two quadratic functions of s, with a total of six coefficients for each joint. It can be expected that this provides a further degree of freedom for shaping the resulting joint path. An additional constraint should be specified in this case, in order to 'square' the interpolation problem. This is obtained by imposing a specific value $\boldsymbol{\theta}'_{mid}$ for the joint path tangent at the mid point, i.e.,

$$\left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}} = \boldsymbol{\theta}'_{mid}. \tag{5}$$

Indeed, various choices can be made for this (vector) value. The fully quadratic path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \begin{cases} \boldsymbol{a}s^2 + \boldsymbol{b}s + \boldsymbol{c}, & \text{for } s \in [0, \frac{1}{2}] \\ \boldsymbol{d}s^2 + \boldsymbol{e}s + \boldsymbol{f}, & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

where $\boldsymbol{a}, \ldots, \boldsymbol{f}$ are suitable two-dimensional vectors of coefficients. Imposing the boundary conditions (2–5) yields now:

$$\boldsymbol{q}(s) = \begin{cases} \left(4(\boldsymbol{\theta}_{\text{in}} - \boldsymbol{\theta}_{\text{mid}}) + 2\boldsymbol{\theta}'_{mid}\right) s^2 + \left(4(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{in}}) - \boldsymbol{\theta}'_{mid}\right) s + \boldsymbol{\theta}_{\text{in}}, & \text{for } s \in [0, \frac{1}{2}] \\ \left(4(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}}) - 2\boldsymbol{\theta}'_{mid}\right) s^2 + \left(4(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{fin}}) + 3\boldsymbol{\theta}'_{mid}\right) s + \boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}'_{mid}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{6}$$

Accordingly, its first derivative w.r.t. s (the path tangent in the joint space) is:

$$\frac{d\boldsymbol{q}(s)}{ds} = \begin{cases} \left(8(\boldsymbol{\theta}_{\text{in}} - \boldsymbol{\theta}_{\text{mid}}) + 4\boldsymbol{\theta}'_{mid}\right) s + 4(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{in}}) - \boldsymbol{\theta}'_{mid}, & \text{for } s \in [0, \frac{1}{2}] \\ \left(8(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}}) - 4\boldsymbol{\theta}'_{mid}\right) s + 4(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{fin}}) + 3\boldsymbol{\theta}'_{mid}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

A possible choice for the vector value $\boldsymbol{\theta}'_{mid}$ is obtained by imposing at $\boldsymbol{p}_{\text{mid}}$ a tangent to the Cartesian path in the direction of $\boldsymbol{y}_0$ and of unit norm, i.e.,

$$\left.\frac{d\boldsymbol{p}(s)}{ds}\right|_{s=\frac{1}{2}} = \left.\frac{d\boldsymbol{p}}{d\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{mid}} \left.\frac{d\boldsymbol{q}}{ds}\right|_{s=\frac{1}{2}} = \boldsymbol{J}(\boldsymbol{\theta}_{mid})\,\boldsymbol{\theta}'_{mid} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7}$$

where $\boldsymbol{p} = \boldsymbol{kin}(\boldsymbol{\theta})$ is the direct kinematics of the manipulator. This choice is indeed feasible, despite the manipulator is in a (folded) singular configuration. In fact, the robot Jacobian

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -\ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1 + \theta_2) & -\ell_2 \sin(\theta_1 + \theta_2) \\ \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) & \ell_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

takes the value

$$\boldsymbol{J}(\boldsymbol{\theta}_{mid}) = \begin{pmatrix} 0 & 0 \\ \ell_1 - \ell_2 & -\ell_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.1 & -0.5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{R}(\boldsymbol{J}(\boldsymbol{\theta}_{mid})).$$

The (minimum norm) solution for $\boldsymbol{\theta}'_{mid}$ is obtained using pseudoinversion (or, equivalently, by pseudoinversion of the second row/scalar equation only) in (7), i.e.,

$$\boldsymbol{\theta}'_{mid} = \boldsymbol{J}^{\#}(\boldsymbol{\theta}_{mid}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^{\#} \cdot 1 = \frac{1}{0.26} \begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix}. \tag{8}$$

The resulting joint path, path tangent, and path curvature are given in Figs. 7–8. Note that the curvature has a discontinuity at the midpoint. A stroboscopic view of the manipulator motion is shown in Fig. 9. Also in this case, it can be seen that there are no collisions.

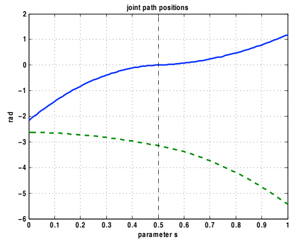


Figure 7: Fully quadratic path in the joint space: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

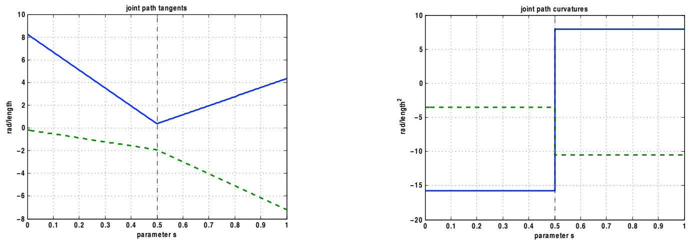


Figure 8: [Left] Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)
. [Right] Path curvature: $d^2q_1(s)/ds^2$ (solid, blue), $d^2q_2(s)/ds^2$ (dashed, green)

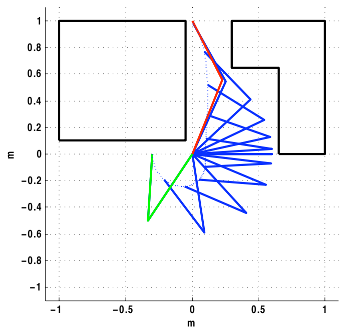


Figure 9: Stroboscopic view of manipulator motion for a fully quadratic joint path

We show also the outcome of other possible choices for $\boldsymbol{\theta}'_{mid}$. One can avoid to select a specific value, and resolve the problem by imposing also curvature continuity of the joint path at the mid point. The second derivative w.r.t. s of the interpolating path function (6) is

$$\frac{d^2\boldsymbol{q}(s)}{ds^2} = \begin{cases} 8(\boldsymbol{\theta}_{\text{in}} - \boldsymbol{\theta}_{\text{mid}}) + 4\boldsymbol{\theta}'_{mid}, & \text{for } s \in [0, \frac{1}{2}] \\ 8(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}}) - 4\boldsymbol{\theta}'_{mid} & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

i.e., piecewise constant. Imposing equality at $s = 0.5$ leads to:

$$\boldsymbol{\theta}'_{mid} = \boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{in}} = \begin{pmatrix} 3.3408 \\ -2.8007 \end{pmatrix}.$$

The resulting joint path, path tangent, and path curvature are shown in Figs. 10–11, where the curvature is now continuous (constant). However, the obtained motion is unfeasible due to the collision with the obstacle on the left, close to the reaching of the final point (Fig. 12). This is a result of the additional smoothness imposed (at least in the chosen class of interpolating functions).

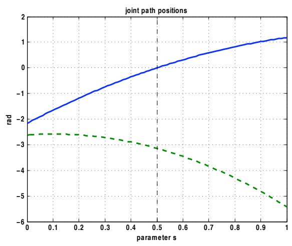


Figure 10: Quadratic path in the joint space with continuous curvature: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

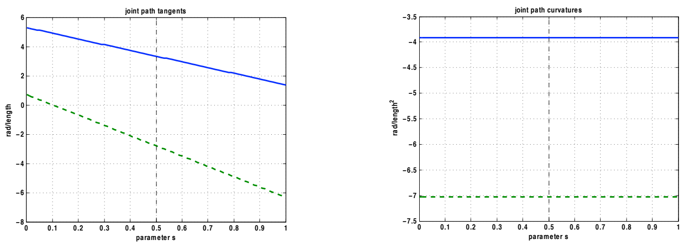


Figure 11: [Left] Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)
. [Right] Path curvature: $d^2q_1(s)/ds^2$ (solid, blue), $d^2q_2(s)/ds^2$ (dashed, green)

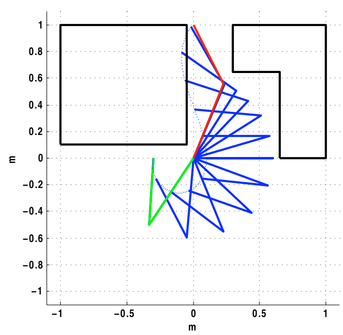


Figure 12: Stroboscopic view of manipulator motion for the quadratic path with continuous curvature, resulting in a collision

As a matter of fact, the problem is due to the large 'swing' imposed to the robot when passing through the mid point. The criticality of the choice of $\boldsymbol{\theta}'_{mid}$ in the considered class of interpolating functions becomes even more dramatically clear when setting for instance

$$\boldsymbol{\theta}'_{mid} = \frac{100}{0.26}\begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix},$$

i.e., a value that is *hundred* times larger than the minimum norm solution given in (8). The planned robot motion goes wild in this case, as shown in Fig. 13. On the other hand, the solution obtained for $\boldsymbol{\theta}'_{mid} = \mathbf{0}$ is quite similar to the one in Fig. 9.

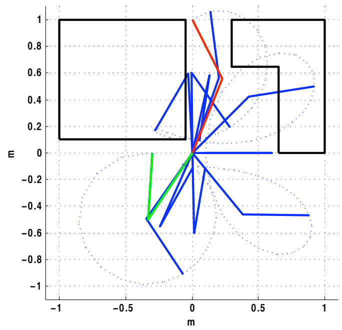


Figure 13: Stroboscopic view of manipulator motion for a quadratic path, with very large $\boldsymbol{\theta}'_{mid}$

* * * * *

Robotics I

June 15, 2010

Exercise 1

For a planar RP robot, consider a class of one-dimensional tasks defined only in terms of the y-component of the end-effector Cartesian position

$$y = p_y(q_1, q_2).$$

a) Study the singularity conditions for the robot performing this class of tasks.

b) Given a desired task trajectory $y_d(t)$, admitting second time derivative, provide the expression of a kinematic control law that is able to zero the task error $e = y_d - y$ in an exponential way starting fron any initial robot condition $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$, when the available control commands are the joint accelerations $\ddot{\boldsymbol{q}}$.

Exercise 2

For a minimal representation of the orientation of a rigid body given by Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $YX'Z''$, determine the relation

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

between the time derivatives of the Euler angles and the angular velocity $\boldsymbol{\omega}$ of the rigid body. Find the singularities of $\boldsymbol{T}(\boldsymbol{\phi})$, and provide an example of an angular velocity vector $\boldsymbol{\omega}$ that cannot be represented in a singularity.

[90 minutes; open books]

Solutions

June 15, 2010

Exercise 1

The direct kinematics associated to the end-effector position of the RP robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix},$$

where a 'natural' set of coordinates has been chosen, with q_1 being the angle between the $\boldsymbol{x}_0$ axis and the second link of the robot[1].

Being the task defined only in terms of the p_y component, it is

$$\dot{p}_y = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 & \cos q_1 \dot{q}_1 \end{pmatrix} \dot{\boldsymbol{q}}.$$

The task Jacobian $\boldsymbol{J}$ is then singular when

$$\sin q_1 = 0 \quad \text{AND} \quad q_2 = 0.$$

In this case, the rank of the $\boldsymbol{J}$ matrix is zero and the one-dimensional task cannot be correctly performed. Out of singularities, all the joint accelerations $\ddot{\boldsymbol{q}}$ that realize a desired $\ddot{y}_d$ can be written in the form

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\ddot{y}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) \right) \ddot{\boldsymbol{q}}_0,$$

being the task redundant ($M = 1$) for the RP robot ($N = 2$). Setting $\ddot{\boldsymbol{q}}_0 = \boldsymbol{0}$ one obtains the solution with minimum joint acceleration norm. Assuming full rank (equal to 1) for the task Jacobian $\boldsymbol{J}$, its pseudoinverse has the explicit expression

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \frac{1}{q_2^2 \cos^2 q_1 + \sin^2 q_1} \begin{pmatrix} q_2 \cos q_1 \\ \sin q_1 \end{pmatrix}.$$

A kinematic control law with the requested performance is defined by

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\ddot{y}_d + k_d(\dot{y}_d - \dot{p}_y) + k_p(y_d - p_y) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right),$$

where $k_d > 0$ and $k_p > 0$ and we set for simplicity $\ddot{\boldsymbol{q}}_0 = \boldsymbol{0}$. A more convenient choice would be to include an acceleration $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_D\dot{\boldsymbol{q}}$, with a diagonal, positive definite matrix $\boldsymbol{K}_D$, in the null space of the task Jacobian. As a matter of fact, such additional term allows to damp possible increases of internal joint velocity without perturbing the task.

[1] When using the Denavit-Hartenberg formalism, one would define $q_2^{\mathrm{DH}} = q_2 \pm \frac{\pi}{2}$. The rest of the developments follows accordingly in a similar way.

Exercise 2

The orientation of a rigid body is represented, using the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $YX'Z''$, by the product of elementary rotation matrices

$$\boldsymbol{R} = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta)\boldsymbol{R}_{Z''}(\gamma).$$

The angular velocity $\boldsymbol{\omega}$ due to $\dot{\boldsymbol{\phi}}$ can be obtained as the sum of the three angular velocities contributed by, respectively, $\dot{\alpha}$ (along the unit vector $\boldsymbol{Y}$), $\dot{\beta}$ (along $\boldsymbol{X}'$), and $\dot{\gamma}$ (along $\boldsymbol{Z}''$)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{Y}\dot{\alpha} + \boldsymbol{X}'\dot{\beta} + \boldsymbol{Z}''\dot{\gamma}$$

where the unit vectors $\boldsymbol{Y}$, $\boldsymbol{X}$ e $\boldsymbol{Z}''$ are expressed with respect to the initial reference frame. It is

$$\boldsymbol{Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{X}' = \boldsymbol{R}_Y(\alpha)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{Z}'' = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, it is sufficient to compute

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \quad \boldsymbol{R}_{X'}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix},$$

$$\boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta) = \begin{pmatrix} * & * & \sin\alpha\cos\beta \\ * & * & -\sin\beta \\ * & * & \cos\alpha\cos\beta \end{pmatrix}$$

in order to obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\alpha} + \begin{pmatrix} \cos\alpha \\ 0 \\ -\sin\alpha \end{pmatrix}\dot{\beta} + \begin{pmatrix} \sin\alpha\cos\beta \\ -\sin\beta \\ \cos\alpha\cos\beta \end{pmatrix}\dot{\gamma} = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \\ 0 & -\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

Note also, as a general property, that matrix $\boldsymbol{T}$ depends only on the first two Euler angles. Matrix $\boldsymbol{T}$ is singular when

$$\det \boldsymbol{T} = -\cos\beta = 0 \quad \iff \quad \beta = \pm\frac{\pi}{2}.$$

In this condition, an angular velocity vector (with norm k) of the form

$$\boldsymbol{\omega} = k\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{T}(\alpha, \pm\frac{\pi}{2})\right\}$$

cannot be represented by any choice of $\dot{\boldsymbol{\phi}}$.

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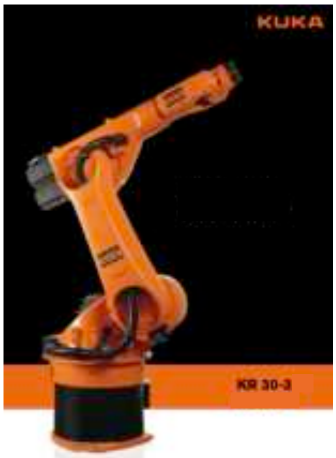
Robotics I

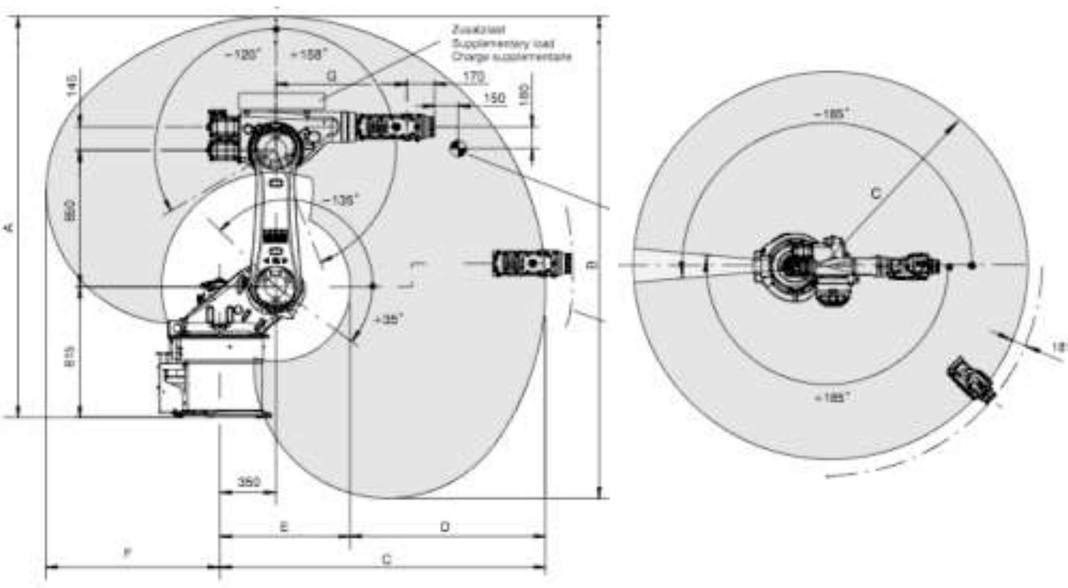
July 7, 2010

Based on the data sheet of the KUKA KR 30-3 robot with six revolute joints and spherical wrist:

i) assign the reference frames according to the Denavit-Hartenberg formalism, putting the origin of the reference frame RF_0 on the floor, and derive the associated table of parameters;

ii) using the numerical data (in mm), determine the position of the spherical wrist center with respect to RF_0 in the configuration $\boldsymbol{\theta} = (0, \pi/2, 0, \theta_4, \theta_5, \theta_6)$, for arbitrary θ_i, $i = 4, 5, 6$.





	A	B	C	D	E	F	G
KR 30-3	2498	3003	2033	1218	815	1084	820

[120 minutes; open books]

Solution

July 7, 2010

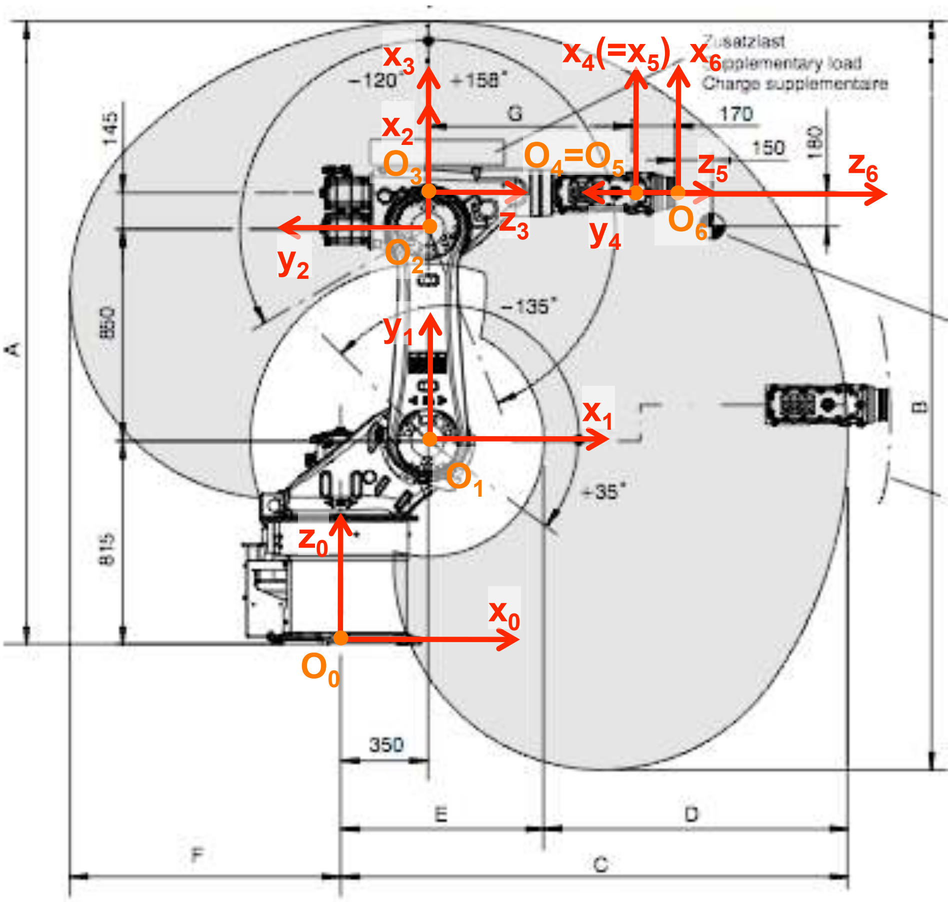


Figure 1: D-H frame assignment for the KUKA KR 30-3 robot

With reference to Fig. 1, the table of Denavit-Hartenberg parameters is the following:

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	$d_1 = 815$	$a_1 = 350$	θ_1
2	0	0	$a_2 = 850$	θ_2
3	$\pi/2$	0	$a_3 = 145$	θ_3
4	$-\pi/2$	$d_4 = 820$	0	θ_4
5	$\pi/2$	0	0	θ_5
6	0	$d_6 = 170$	0	θ_6

The numerical values (in mm) assigned to the constant parameters in the table are taken from the data sheet. The robot configuration shown corresponds to $\boldsymbol{\theta} = (0\ \frac{\pi}{2}\ 0\ 0\ 0\ 0)^T$. Only the first three homogeneous transformation matrices are needed for determining the position of the center $W = 0_4 = 0_5$ of the spherical wrist:

$$
{}^0\boldsymbol{A}_1(\theta_1) = \begin{pmatrix} c_1 & 0 & s_1 & a_1 c_1 \\ s_1 & 0 & -c_1 & a_1 s_1 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^1\boldsymbol{A}_2(\theta_2) = \begin{pmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^2\boldsymbol{A}_3(\theta_3) = \begin{pmatrix} c_3 & 0 & s_3 & a_3 c_3 \\ s_3 & 0 & -c_3 & a_3 s_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

In fact, we may compute the position ${}^0\boldsymbol{p}_{04}$ of O_4 as

$$
{}^0\boldsymbol{p}_{04,hom} = \begin{pmatrix} {}^0\boldsymbol{p}_{04} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{A}_3(\theta_3)\,{}^3\boldsymbol{A}_4(\theta_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}
$$

but also as

$$
{}^0\boldsymbol{p}_{04,hom} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{A}_3(\theta_3) \begin{pmatrix} {}^3\boldsymbol{p}_{34} \\ 1 \end{pmatrix} \qquad \text{with} \quad {}^3\boldsymbol{p}_{34} = \begin{pmatrix} 0 \\ 0 \\ d_4 \end{pmatrix}.
$$

The most efficient way is to perform computations as matrix/vector products:

$$
{}^2\boldsymbol{p}_{24,hom} = {}^2\boldsymbol{A}_3(\theta_3)\,{}^3\boldsymbol{p}_{34,hom} = \begin{pmatrix} a_3 c_3 + d_4 s_3 \\ a_3 s_3 - d_4 c_3 \\ 0 \\ 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{p}_{14,hom} = {}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{p}_{24,hom} = \begin{pmatrix} (a_2 + a_3 c_3 + d_4 s_3) c_2 - (a_3 s_3 - d_4 c_3) s_2 \\ (a_2 + a_3 c_3 + d_4 s_3) s_2 + (a_3 s_3 - d_4 c_3) c_2 \\ 0 \\ 1 \end{pmatrix},
$$

$$
{}^0\boldsymbol{p}_{04,hom} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{p}_{14,hom} = \begin{pmatrix} (a_1 + (a_2 + a_3 c_3 + d_4 s_3) c_2 - (a_3 s_3 - d_4 c_3) s_2) c_1 \\ (a_1 + (a_2 + a_3 c_3 + d_4 s_3) c_2 - (a_3 s_3 - d_4 c_3) s_2) s_1 \\ d_1 + (a_2 + a_3 c_3 + d_4 s_3) s_2 + (a_3 s_3 - d_4 c_3) c_2 \\ 1 \end{pmatrix}.
$$

At the desired configuration (for the first three joints), substituting the numerical values, yields

$$
{}^0\boldsymbol{p}_{04}(0, \frac{\pi}{2}, 0) = \begin{pmatrix} a_1 + d_4 \\ 0 \\ d_1 + a_2 + a_3 \end{pmatrix} = \begin{pmatrix} 1170 \\ 0 \\ 1810 \end{pmatrix} \text{ [mm]}.
$$

* * * * *

Robotics I

September 15, 2010

For a revolute robot joint, consider the rest-to-rest motion $q = q(t)$ defined by the jerk profile $\dddot{q}(t)$ shown in Fig. 1, with given $j_{max} > 0$. The motion starts from $q(0) = q_0$ at time $t = 0$, with zero initial velocity ($\dot{q}(0) = 0$) and zero initial acceleration ($\ddot{q}(0) = 0$).

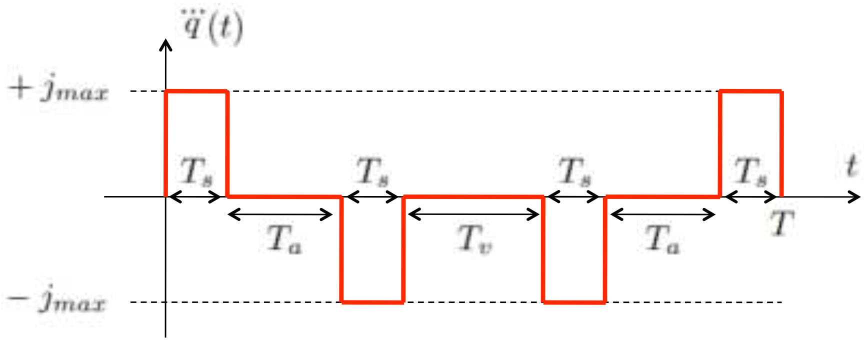


Figure 1: Jerk profile

i) Let the bounds $|\dot{q}(t)| \leq v_{max}$, $|\ddot{q}(t)| \leq a_{max}$ (with $v_{max} > 0$ and $a_{max} > 0$) be assigned, as well as the time interval $T_v \geq 0$. Under the assumption

$$\frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \geq 0,$$

determine the analytic expression of the *maximum feasible displacement* $\Delta q = q(T) - q_0$ that can be realized. Provide the numerical solution for

$$j_{max} = 12 \text{ [rad/s}^3] \qquad a_{max} = 5 \text{ [rad/s}^2] \qquad v_{max} = 3 \text{ [rad/s}^3] \qquad T_v = 2 \text{ [s]}.$$

ii) Let the bounds $|\dot{q}(t)| \leq v_{max}$, $|\ddot{q}(t)| \leq a_{max}$ (with $v_{max} > 0$ and $a_{max} > 0$) be assigned, as well as the total displacement $\Delta q > 0$. Under the assumptions

$$\frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \geq 0 \qquad \Delta q \geq v_{max}\left(\frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}}\right),$$

determine the analytic expression of the *minimum feasible motion time* T that can be realized. Provide the numerical solution for

$$j_{max} = 10 \text{ [rad/s}^3] \qquad a_{max} = 4 \text{ [rad/s}^2] \qquad v_{max} = 2 \text{ [rad/s}^3] \qquad \Delta q = 3 \text{ [rad]}.$$

[90 minutes; open books]

Solution

September 15, 2010

The solution is obtained by integration of the jerk profile, using the given initial conditions at time $t = 0$ and then the suitable boundary conditions at the instants of jerk switching. In addition, due to the symmetry of the trajectory derivatives with respect to $T/2$, it is sufficient to analyze only the first half of the motion. Without loss of generality, we set $q_0 = 0$ (only the displacement w.r.t. the initial position matters). We will also see that the assumptions made on the velocity, acceleration, and jerk bounds, as well as on the total displacement assigned in problem *ii)*, guarantee that none of the motion segments will vanish.

- *First segment:* $\dddot{q}(t) = j_{max}$, for $t \in [0, T_s)$

$$\begin{aligned} \ddot{q}(t) &= j_{max}\, t \\ \dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, t^2 \\ q(t) &= \tfrac{1}{6}\, j_{max}\, t^3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \ddot{q}(T_s) &= j_{max}\, T_s = a_{max} \quad \Rightarrow \quad T_s = \frac{a_{max}}{j_{max}} \\ \dot{q}(T_s) &= \tfrac{1}{2}\, j_{max}\, T_s^2 \\ q(T_s) &= \tfrac{1}{6}\, j_{max}\, T_s^3 \end{aligned}$$

- *Second segment:* $\dddot{q}(t) = 0$, for $t \in [T_s, T_s + T_a)$

$$\begin{aligned} \ddot{q}(t) &= a_{max} \\ \dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\,(t - T_s) \\ q(t) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 (t - T_s) + \tfrac{1}{2}\, a_{max}\,(t - T_s)^2 \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} \ddot{q}(T_s + T_a) &= a_{max} \\ \dot{q}(T_s + T_a) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a \\ q(T_s + T_a) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2 \end{aligned}$$

- *Third segment:* $\dddot{q}(t) = -j_{max}$, for $t \in [T_s + T_a, 2T_s + T_a)$

$$\begin{aligned} \ddot{q}(t) &= a_{max} - j_{max}\,(t - (T_s + T_a)) \\ \dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a + a_{max}\,(t - (T_s + T_a)) - \tfrac{1}{2}\, j_{max} (t - (T_s + T_a))^2 \\ q(t) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2 + (\tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a)(t - (T_s + T_a)) \\ &\quad + \tfrac{1}{2}\, a_{max}\,(t - (T_s + T_a))^2 - \tfrac{1}{6}\, j_{max} (t - (T_s + T_a))^3 \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} \ddot{q}(2T_s + T_a) &= a_{max} - j_{max}\, T_s = 0 \\ \dot{q}(2T_s + T_a) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a + a_{max}\, T_s - \tfrac{1}{2}\, j_{max}\, T_s^2 = v_{max} \quad \Rightarrow \quad T_a = \frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \\ q(2T_s + T_a) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2 + (\tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a)\, T_s \\ &\quad + \tfrac{1}{2}\, a_{max}\, T_s^2 - \tfrac{1}{6}\, j_{max}\, T_s^3 \end{aligned}$$

- *First half of fourth segment:* $\dddot{q}(t) = 0$, for $t \in [2T_s + T_a, 2T_s + T_a + T_v/2)$.

$$\begin{aligned} \ddot{q}(t) &= 0 \\ \dot{q}(t) &= v_{max} \\ q(t) &= q(2T_s + T_a) + v_{max}(t - (2T_s + T_a)) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \ddot{q}\left(2T_s + T_a + \frac{T_v}{2}\right) &= 0 \\ \dot{q}\left(2T_s + T_a + \frac{T_v}{2}\right) &= v_{max} \\ q\left(2T_s + T_a + \frac{T_v}{2}\right) &= q(2T_s + T_a) + v_{max}\left(\frac{T_v}{2}\right) \end{aligned}$$

Since we have that $\frac{T}{2} = 2T_s + T_a + \frac{T_v}{2}$, due to the symmetry of the trajectory, we have

$$q\left(\frac{T}{2}\right) = \frac{\Delta q}{2},$$

or

$$\frac{1}{2} j_{max} T_s^2 T_a + \frac{1}{2} a_{max} T_a^2 + (\frac{1}{2} j_{max} T_s^2 + a_{max} T_a) T_s + \frac{1}{2} a_{max} T_s^2 + v_{max}\left(\frac{T_v}{2}\right) = \frac{\Delta q}{2}.$$

Substituting the expressions of T_s and T_a and simplifying, we obtain finally

$$T_v = \frac{\Delta q}{v_{max}} - \frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}}. \tag{1}$$

While

$$T_s = \frac{a_{max}}{j_{max}} > 0$$

always hold, we note that the assumptions made on the relative amplitudes of the bounds v_{max}, a_{max}, j_{max} and on Δq simultaneously guarantee that

$$T_a \geq 0, \qquad T_v \geq 0.$$

As a result, the total motion time is given by

$$T = T_v + 2T_a + 4T_s = \frac{\Delta q}{v_{max}} + \frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}}, \tag{2}$$

which is the minimum feasible time under the made assumptions. For specific choices of data, some of the motion segments may collapse, and the actual duration of each of them (and thus the total motion time) should be computed accordingly.

If T_v is assigned, the maximum feasible displacement is obtained from (1) as

$$\Delta q = v_{max}\left(T_v + \frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}}\right). \tag{3}$$

Plugging the data of problem *i)* in eq. (3) yields $\Delta q = 9.05$ [rad] (with a total time $T = 4.033$ [s]). The obtained profiles of position, velocity, acceleration, and jerk are shown in Fig. 2. With the data of problem *ii)*, from eq. (2) we have $T = 2.4$ [s]. The associated profiles of position, velocity, acceleration, and jerk are shown in Fig. 3. Matlab sources are available.

$* * * * *$

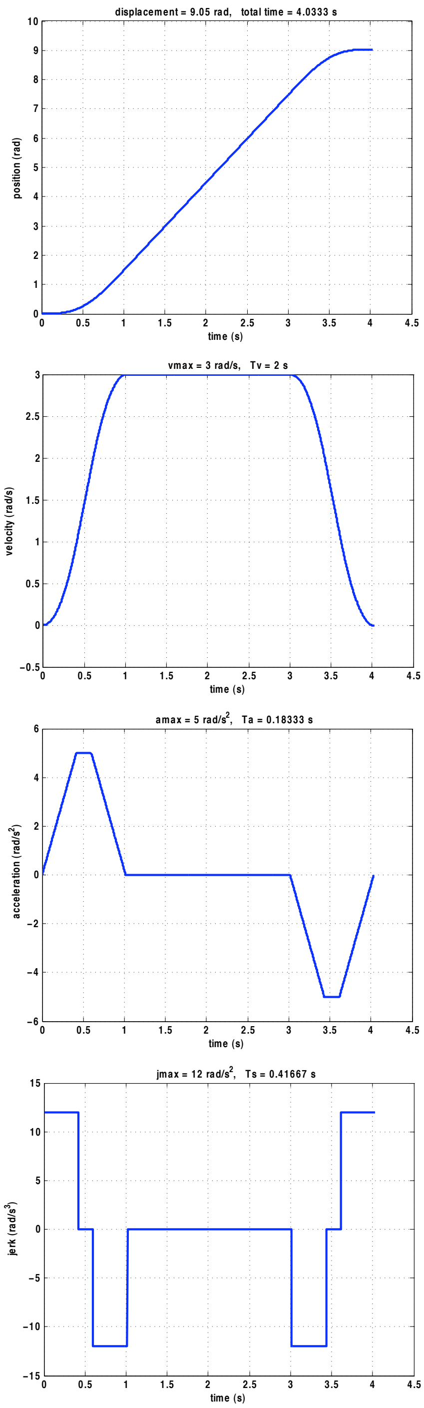


Figure 2: Position, velocity, acceleration, and jerk profiles for the solution to problem *i)*

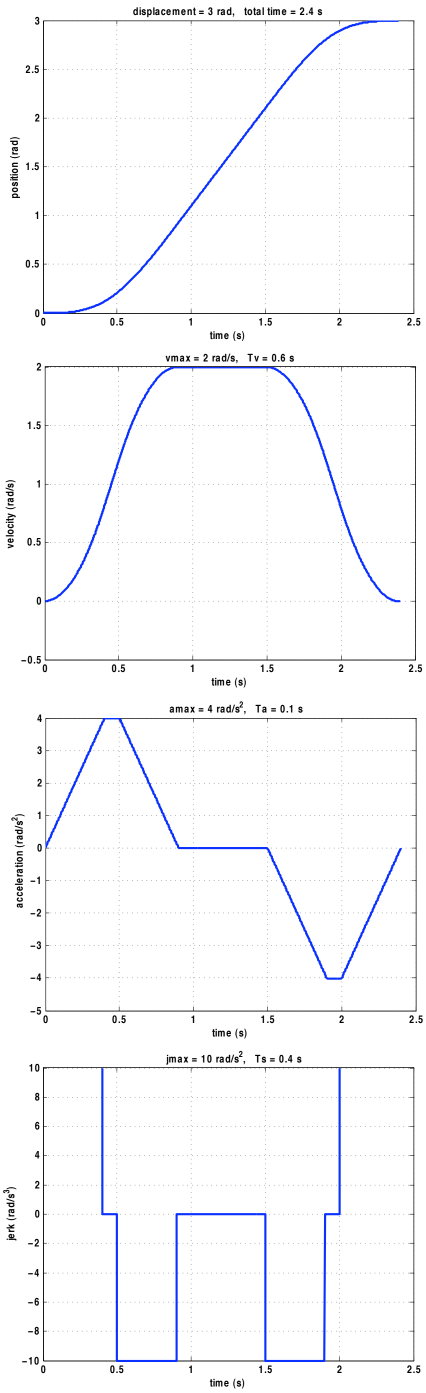


Figure 3: Position, velocity, acceleration, and jerk profiles for the solution to problem *ii)*

Robotics I

February 3, 2011

Consider a 3R anthropomorphic robot mounted on the floor and characterized by the Denavit-Hartenberg parameters in Table 1, where D, L_1, L_2, and L_3 are all strictly positive values.

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	D	L_1	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Table 1: Table of DH parameters

1. Obtain the 3×3 Jacobian matrix ${}^0\boldsymbol{J}_L(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity ${}^0\boldsymbol{v}$ of the origin O_3 of frame 3 expressed in frame 0.

2. Characterize the singular configurations $\boldsymbol{q}$ of the Jacobian ${}^3\boldsymbol{J}_L(\boldsymbol{q})$ relating $\dot{\boldsymbol{q}}$ to the linear velocity ${}^3\boldsymbol{v}$ of the origin O_3 of frame 3 expressed in frame 3.

3. Obtain the 3×3 Jacobian matrix ${}^0\boldsymbol{J}_A(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the angular velocity ${}^0\boldsymbol{\omega}$ of frame 3 expressed in frame 0. Show that this matrix is always singular and provide an explanation of this result.

4. Assume that the robot is in the configuration

$$\boldsymbol{q}^* = \begin{pmatrix} 0 & \frac{\pi}{4} & -\frac{\pi}{4} \end{pmatrix}^T \quad \text{[rad]}$$

   with a joint velocity

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{q}_1^* & 0 & 0 \end{pmatrix}^T \quad \text{[rad/s]}, \qquad \text{with } \dot{q}_1^* \neq 0.$$

   Determine the joint acceleration $\ddot{\boldsymbol{q}}$ that should be imposed so that the resulting linear Cartesian acceleration of the origin O_3 is directed along $\boldsymbol{y}_3$ and has an intensity $A \neq 0$. Provide some comment on the structure of the obtained solution. In particular, is there a value A such that only one joint needs to accelerate?

[150 minutes; open books]

Solution

February 3, 2011

For item 1, we are interested in the velocity of point O_3, whose position $\boldsymbol{p} = {}^0\boldsymbol{p}$ is given by the direct kinematics map

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1\,(L_1 + L_2\cos q_2 + L_3\cos(q_2+q_3)) \\ \sin q_1\,(L_1 + L_2\cos q_2 + L_3\cos(q_2+q_3)) \\ D + L_2\sin q_2 + L_3\sin(q_2+q_3) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

The Jacobian ${}^0\boldsymbol{J}_L(\boldsymbol{q})$ can be obtained either by analytical differentiation of $\boldsymbol{f}(\boldsymbol{q})$ in (1) w.r.t. $\boldsymbol{q}$ or by using the expression of the first three rows of the geometric Jacobian. Using the usual short notation for trigonometric functions, the result is in both cases

$${}^0\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(L_1 + L_2c_2 + L_3c_{23}) & -c_1(L_2s_2 + L_3s_{23}) & -L_3c_1s_{23} \\ c_1(L_1 + L_2c_2 + L_3c_{23}) & -s_1(L_2s_2 + L_3s_{23}) & -L_3s_1s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \end{pmatrix}. \tag{2}$$

For item 2, we have that

$$\det {}^3\boldsymbol{J}_L(\boldsymbol{q}) = \det\left({}^2\boldsymbol{R}_3^T(q_3)\,{}^1\boldsymbol{R}_2^T(q_2)\,{}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{J}_L(\boldsymbol{q})\right) = \det {}^0\boldsymbol{J}_L(\boldsymbol{q}).$$

Nonetheless, it is useful to rewrite the Jacobian in the successive frames 1, 2, and 3, because the resulting expressions will be simplified. From Table 1, we have

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix},\ {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ {}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From these we obtain

$${}^1\boldsymbol{J}_L(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -(L_2s_2 + L_3s_{23}) & -L_3s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},$$

$${}^2\boldsymbol{J}_L(\boldsymbol{q}) = {}^1\boldsymbol{R}_2^T(q_1)\,{}^1\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -L_3s_3 & -L_3s_3 \\ 0 & L_2 + L_3c_3 & L_3c_3 \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},$$

and

$${}^3\boldsymbol{J}_L(\boldsymbol{q}) = {}^2\boldsymbol{R}_3^T(q_1)\,{}^2\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & L_2s_3 & 0 \\ 0 & L_3 + L_2c_3 & L_3 \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix}.$$

In particular from the last expression, it is immediate to see that for any $i \in \{1, 2, 3\}$

$$\det {}^i\boldsymbol{J}_L(\boldsymbol{q}) = -L_2L_3(L_1 + L_2c_2 + L_3c_{23})s_3. \tag{3}$$

Therefore, the singular configurations of $\boldsymbol{J}_L(\boldsymbol{q})$ are:

$$\begin{array}{rcll} s_3 = 0 & \Longleftrightarrow & q_3 = \{0, \pm\pi\} & \text{(third link is stretched or folded)} \\ L_1 + L_2 c_2 + L_3 c_{23} = 0 & \Longleftrightarrow & p_x = p_y = 0 & \text{(}O_3\text{ is on the axis }\boldsymbol{z}_0\text{ of joint 1)} \end{array}$$

For item 3, we compute the expression of the lower three rows of the geometric Jacobian. It is

$$\begin{aligned} {}^0\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

Matrix ${}^0\boldsymbol{J}_A(\boldsymbol{q})$ is always singular, having constant rank equal to 2. This can be easily explained as follows. The three degrees of freedom of the considered manipulator allow placing the end-effector in any point of the robot primary workspace, and imposing a linear velocity in any direction when the arm is out of singularities. However, the orientation of the end-effector frame can never be changed around the unitary axis $\boldsymbol{n}(q_1) = \begin{pmatrix} c_1 & s_1 & 0 \end{pmatrix}^T$. In fact, $\boldsymbol{\omega} = \alpha\,\boldsymbol{n}(q_1) \notin \mathcal{R}\{{}^0\boldsymbol{J}_A(\boldsymbol{q})\}$, for every $\boldsymbol{q}$ and for any scalar α.

Finally, for item 4 we use the second-order differential map

$${}^0\ddot{\boldsymbol{p}} = {}^0\boldsymbol{J}_L(\boldsymbol{q})\ddot{\boldsymbol{q}} + {}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{5}$$

evaluated at $\boldsymbol{q} = \boldsymbol{q}^*$, $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^*$. The Cartesian acceleration is specified as

$${}^0\ddot{\boldsymbol{p}} = {}^0\boldsymbol{R}_3(\boldsymbol{q})\,{}^3\ddot{\boldsymbol{p}} = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3) \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} = \begin{pmatrix} -A\,c_1 s_{23} \\ -A\,s_1 c_{23} \\ A\,c_{23} \end{pmatrix},$$

which, when evaluated at $\boldsymbol{q} = \boldsymbol{q}^*$, yields the desired value

$${}^0\ddot{\boldsymbol{p}}_d = {}^0\ddot{\boldsymbol{p}}|_{\boldsymbol{q}=\boldsymbol{q}^*} = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix}, \tag{6}$$

i.e., the acceleration of the end-effector should be directed along $\boldsymbol{z}_0$, the vertical direction. Since the determinant (3) of the associated Jacobian is nonzero at the given configuration, the solution for the joint acceleration is obtained from (5) as

$$\ddot{\boldsymbol{q}} = {}^0\boldsymbol{J}_L^{-1}(\boldsymbol{q}^*) \left({}^0\ddot{\boldsymbol{p}}_d - {}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q}^*)\dot{\boldsymbol{q}}^* \right),$$

where

$${}^0\boldsymbol{J}_L^{-1}(\boldsymbol{q}^*) = \begin{pmatrix} 0 & -L_2\dfrac{\sqrt{2}}{2} & 0 \\ L_1 + L_2\dfrac{\sqrt{2}}{2} + L_3 & 0 & 0 \\ 0 & L_2\dfrac{\sqrt{2}}{2} + L_3 & L_3 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \dfrac{1}{L_1 + L_2\frac{\sqrt{2}}{2} + L_3} & 0 \\ -\dfrac{\sqrt{2}}{L_2} & 0 & 0 \\ \dfrac{1}{L_3} + \dfrac{\sqrt{2}}{L_2} & 0 & \dfrac{1}{L_3} \end{pmatrix}. \tag{7}$$

Let ${}^0\boldsymbol{J}_1$ be the first column of the Jacobian ${}^0\boldsymbol{J}_L$. Thanks to the simple structure of $\dot{\boldsymbol{q}}^*$, for the term involving the time derivative of the Jacobian we need only to compute

$$\begin{aligned}\left({}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*,\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}^*} &= \left({}^0\dot{\boldsymbol{J}}_1(\boldsymbol{q})\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*,\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}^*}\dot{q}_1^* = \left(\frac{\partial\,{}^0\boldsymbol{J}_1(\boldsymbol{q})}{\partial q_1}\dot{q}_1^*\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*}\dot{q}_1^* \\ &= \begin{pmatrix} -c_1(L_1+L_2c_2+L_3c_{23}) \\ -s_1(L_1+L_2c_2+L_3c_{23}) \\ 0 \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}^*}(\dot{q}_1^*)^2 = \begin{pmatrix} -(L_1+L_2\frac{\sqrt{2}}{2}+L_3) \\ 0 \\ 0 \end{pmatrix}(\dot{q}_1^*)^2 .\end{aligned} \tag{8}$$

As a result, from (6–8) the final solution is

$$\ddot{\boldsymbol{q}} = A\begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_3} \end{pmatrix} + \left(L_1+L_2\frac{\sqrt{2}}{2}+L_3\right)(\dot{q}_1^*)^2\begin{pmatrix} 0 \\ -\dfrac{\sqrt{2}}{L_2} \\ \dfrac{1}{L_3}+\dfrac{\sqrt{2}}{L_2} \end{pmatrix}.$$

We note that no acceleration should be applied to the first joint ($\ddot{q}_1 = 0$), as could be argued already from (6). In fact, any angular acceleration imposed to joint 1 (along the vertical joint axis $\boldsymbol{z}_0$) would produce a centrifugal acceleration on the end-effector, which is in contrast with the requested zero acceleration along the $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ axes in (6). Moreover, if

$$A = -\left(1+\frac{L_3}{L_2}\sqrt{2}\right)\left(L_1+L_2\frac{\sqrt{2}}{2}+L_3\right)(\dot{q}_1^*)^2$$

then $\ddot{q}_1 = \ddot{q}_3 = 0$ in the solution.

* * * * *

Robotics I

February 25, 2011

Consider a 3R anthropomorphic robot whose direct kinematics is given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right) \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2+q_3) \end{pmatrix}, \tag{1}$$

where

$$d_1 = 0.5, \qquad a_2 = 1.5, \quad a_3 = 1. \tag{2}$$

Let the following two Cartesian positions be assigned for the robot end-effector:

$$\boldsymbol{p}_1 = \begin{pmatrix} 0 \\ 1.0607 \\ 2.5607 \end{pmatrix}, \qquad \boldsymbol{p}_2 = \begin{pmatrix} 1.5309 \\ 1.5309 \\ 0.25 \end{pmatrix}. \tag{3}$$

a) Design a infinite cyclic trajectory of period $T = 2$ s in the *joint space* such that:

- the robot end-effector is in $\boldsymbol{p}_1$ at $t = 0, T, 2T, \ldots$ with zero velocity, and in $\boldsymbol{p}_2$ at $t = T/2, 3T/2, 5T/2, \ldots$ with zero velocity;
- the trajectory is guaranteed to be *smooth* (i.e., continuously differentiable up to any order) at all times.

  Hint: It is sufficient to define the trajectory only in the first period $[0, T]$.

b) How do we check that the designed trajectory remains always in the robot workspace? Are there problems associated with the possible crossing of kinematic singularities?

c) Design a cyclic trajectory with duration $T = 2$ s that traces the *same joint path* of the trajectory determined in a) and is such that:

- the robot end-effector starts in $\boldsymbol{p}_1$ at $t = 0$ with initial zero velocity and acceleration, passes in $\boldsymbol{p}_2$ at $t = T/2$ with zero velocity, and returns in $\boldsymbol{p}_1$ at $t = T$ with final zero velocity and acceleration;
- the trajectory is guaranteed to be at least continuous up to the acceleration for $t \in [0, T]$.

[150 minutes; open books]

Solution

February 25, 2011

As a preliminary step, since a joint trajectory solution is sought, we need to associate joint configurations to the end-effector positions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in (3). For each position, there are four different solutions to the inverse kinematics (see Homework 2, 2010/11, on the Robotics 1 course web site: the 3R robot considered here is a special case, with offset $a_1 = 0$). Using the robot kinematic parameters given in (2), a configuration corresponding to $\boldsymbol{p}_1$ is

$$\boldsymbol{q}_1 = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \end{pmatrix}^T = \begin{pmatrix} \frac{\pi}{2} & \frac{\pi}{4} & \frac{\pi}{4} \end{pmatrix}^T \approx \begin{pmatrix} 1.5708 & 0.7854 & 0.7854 \end{pmatrix}^T \quad \text{[rad]}, \tag{4}$$

and one corresponding to $\boldsymbol{p}_2$ is

$$\boldsymbol{q}_2 = \begin{pmatrix} q_{2,1} & q_{2,2} & q_{2,3} \end{pmatrix}^T = \begin{pmatrix} \frac{\pi}{4} & -\frac{\pi}{6} & \frac{\pi}{3} \end{pmatrix}^T \approx \begin{pmatrix} 0.7854 & -0.5236 & 1.0472 \end{pmatrix}^T \quad \text{[rad]}. \tag{5}$$

Note that these two inverse kinematic solutions belong to the same class, namely having the 'elbow down' ($q_3 > 0$). Any other pair of feasible inverse kinematic solutions could have been chosen. The following developments remain the same.

Problem a) requires the use of a smooth and periodic joint trajectory, with period T. Thus, it is convenient to adopt a class of trigonometric paths parametrized by a scalar λ. Assigning a timing law $\lambda = \lambda(t)$ will define the trajectory completely. As suggested in the text, it is sufficient to design a suitable trajectory for the first period $[0, T]$ (and then to repeat it indefinitely). The following derivation is useful also for problem c).

Let $\lambda(t)$ be a monotonically increasing function of time, satisfying the boundary conditions

$$\lambda(0) = 0, \qquad \lambda(T) = 1. \tag{6}$$

In view of the zero velocity conditions at $t = 0$ and $t = T$, we choose for each joint i ($i = 1, 2, 3$)

$$q_i(t) = q_{1,i} + A_i\big(1 - \cos(2\pi\lambda(t))\big), \qquad \text{with } A_i \neq 0 \text{ yet to be defined.} \tag{7}$$

The trajectory (7) guarantees automatically the passage for $\boldsymbol{p}_1$, i.e.,

$$q_i(0) = q_i(T) = q_{1,i}, \qquad \text{for } i = 1, 2, 3.$$

Its velocity is

$$\dot{q}_i(t) = 2\pi A_i \sin(2\pi\lambda(t))\,\dot{\lambda}(t), \tag{8}$$

implying, for any form of $\lambda(t)$,

$$\dot{q}_i(0) = \dot{q}_i(T) = 0.$$

For later use, the acceleration is

$$\ddot{q}_i(t) = 2\pi A_i \sin(2\pi\lambda(t))\,\ddot{\lambda}(t) - 4\pi^2 A_i \cos(2\pi\lambda(t))\,\dot{\lambda}^2(t). \tag{9}$$

For problem a), it is sufficient to use $\lambda(t) = t/T$ (a normalization in time). From (7) and (8)

$$q_i(t) = q_{1,i} + A_i\left(1 - \cos\frac{2\pi t}{T}\right), \qquad \dot{q}_i(t) = \frac{2\pi}{T}\,A_i \sin\frac{2\pi t}{T},$$

which imply at $t = T/2$

$$q_i\left(\frac{T}{2}\right) = q_{1,i} + 2A_i, \qquad \dot{q}_i\left(\frac{T}{2}\right) = 0.$$

Choosing then

$$A_i = \frac{q_{2,i} - q_{1,i}}{2}, \qquad \text{for } i = 1, 2, 3,$$

guarantees automatically the passage for $\boldsymbol{p}_2$ at $t = T/2$ with zero velocity.

However, note that the associated acceleration is from (9)

$$\ddot{q}_i(t) = \left(\frac{2\pi}{T}\right)^2 A_i \cos\frac{2\pi t}{T},$$

and thus

$$\ddot{q}_i(0) = \ddot{q}_i(T) = \left(\frac{2\pi}{T}\right)^2 A_i \neq 0, \qquad \text{for } i = 1, 2, 3.$$

The acceleration (and, similarly, all time derivatives of higher but even order) will not be zero at the boundaries of each period. This makes the present solution unsuitable for problem c). On the other hand, it is easy to see that continuity of all time derivatives up to any order (i.e., smoothness) will hold at any time, even in the passage from one period to the other (i.e., at $t = kT$, for $k = 1, 2, \dots$). We note that this would not have been the case if using two cubic polynomials (in time) to address the same problem, one from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ in time $T/2$ and another for the symmetric reverse motion. Such an approach would be sufficient for satisfying the boundary conditions on the specified motion, but would fail to guarantee continuity of jerk when reversing the motion in $\boldsymbol{q}_2$. Similarly, using a single higher-order polynomial (from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ and back to $\boldsymbol{q}_1$) would lead to a discontinuity in some derivative at $t = kT$.

For $T = 2$ s, Figures 1–3 show the position, velocity, acceleration, jerk (third derivative), and snap (fourth derivative) of the designed joint trajectory. Figure 4 shows the resulting robot end-effector path $\boldsymbol{p}(\lambda)$, obtained from eq. (1) evaluated along the designed trajectory $\boldsymbol{q}(t)$ for $t \in [0, T]$. Two different 3D-views are shown. Note that the path is traced twice (forth and back) from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ (in $T/2 = 1$ s) and vice versa (in $T/2 = 1$ s).

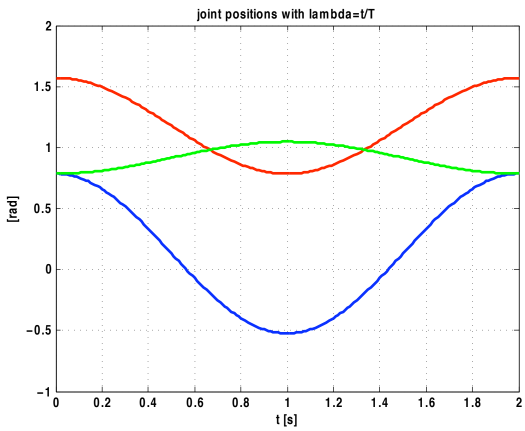


Figure 1: Solution trajectory for problem a) — joint 1 (red), joint 2 (blue), joint 3 (green)

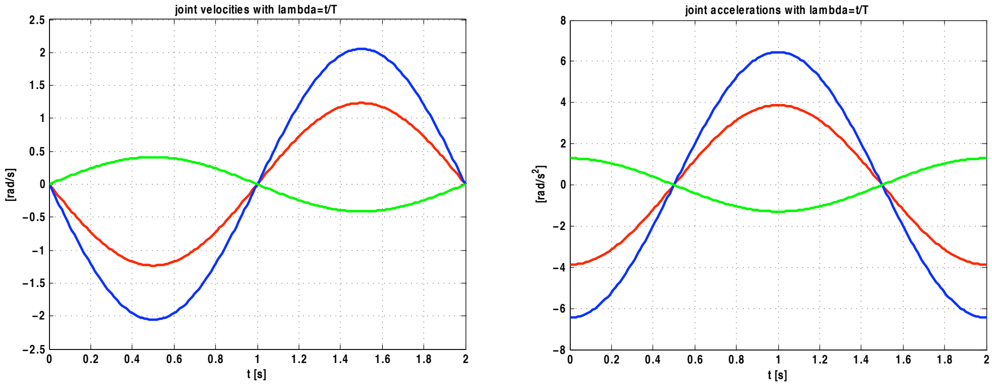


Figure 2: Velocity and acceleration of the trajectory for problem a)

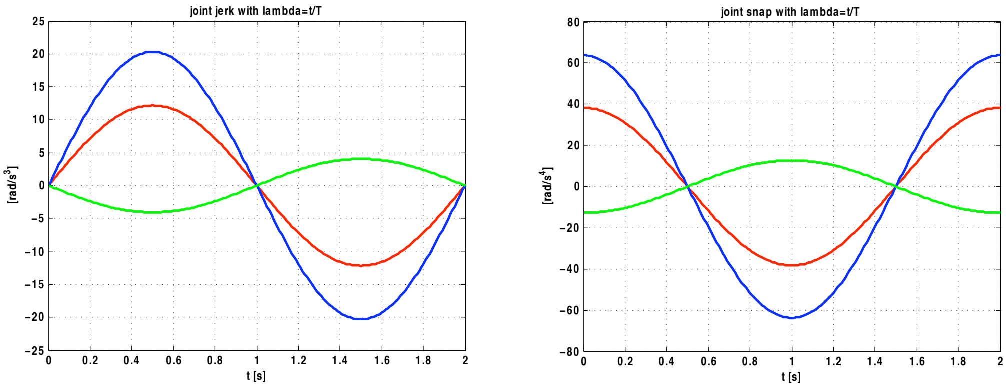


Figure 3: Jerk and snap of the trajectory for problem a)

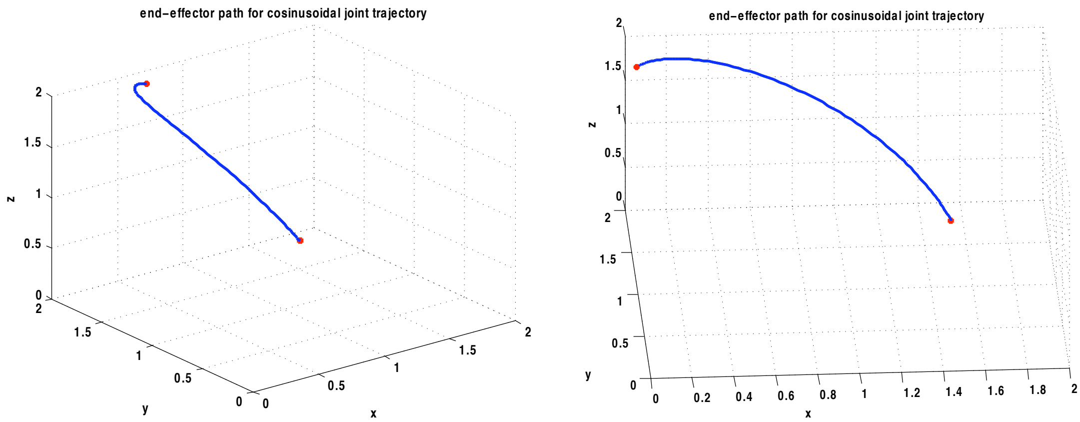


Figure 4: Two 3D-views of the end-effector path with the joint trajectory of problem a) — the red point on the left/top is $\boldsymbol{p}_1$, the other is $\boldsymbol{p}_2$

The two questions of problem b) have both a trivial answer. Since trajectories are being defined at the joint level, they will always be feasible and the end-effector cannot leave the workspace at any time. Moreover, singularity crossing is not an issue: apart from statically transforming the two Cartesian positions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in the joint space, inversion of a Cartesian trajectory, and thus of the robot Jacobian, is not needed neither for planning nor for control purposes. As a matter of fact, the designed joint trajectory does not cross any singularity of the considered manipulator, thanks to the choice (4–5) for the inverse kinematics. A different choice, e.g., between the elbow down configuration in $\boldsymbol{p}_1$ as in (4) and the elbow up configuration

$$\boldsymbol{q}_2^{\mathrm{up}} = \begin{pmatrix} \frac{\pi}{4} & 0.2937 & -\frac{\pi}{3} \end{pmatrix}^T \quad \text{[rad]}. \tag{10}$$

locating the end-effector in $\boldsymbol{p}_2$, would have led to crossing twice the singularity $q_3 = 0$, with the third link fully stretched. Still, this would have been irrelevant.

To address problem c), a slightly more general form for the time parameterization $\lambda(t)$ is needed. Taking into account the expression (9) of the joint acceleration, we note that is sufficient to impose for the time derivative of $\lambda(t)$

$$\dot{\lambda}(0) = 0, \qquad \dot{\lambda}(T) = 0 \tag{11}$$

for zeroing also the acceleration in $t = 0$ and $t = T$. In fact, being the sine function equal to zero in these two instants, the value of $\ddot{\lambda}$ will not affect the acceleration. Therefore, the boundary conditions (6) and (11) can be satisfied by the following cubic polynomial (normalized in time)

$$\lambda(t) = -2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2, \qquad \text{for } t \in [0, T], \tag{12}$$

having first two derivatives

$$\dot{\lambda}(t) = \frac{6}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right), \qquad \ddot{\lambda}(t) = \frac{6}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right). \tag{13}$$

The joint trajectory is then defined by (7) with (12). Accordingly, its velocity and acceleration will be defined by (8) and (9), with (13).

The new joint trajectory is shown in Figs. 5–7, where the time derivatives are reported up to the fourth order (snap). As expected, the starting and ending phases are somewhat slower at the expense of larger velocity peaks with respect to the previous case. For instance, comparing the left sides of Figs. 2 and 6, we see that the velocity of joint 2 (in blue) goes from a peak (in absolute value) slightly larger than 2 rad/s to more than 2.7 rad/s. It is easy to realize that, when this trajectory is repeated for any multiple of the same period T, one obtains again a periodic solution that is smooth everywhere. Therefore, this trajectory is another solution to problem a), with the additional property of having also zero acceleration when the end-effector is in $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Finally, as a check, Figure 8 confirms that the resulting robot end-effector path $\boldsymbol{p}(\lambda)$ has not changed.

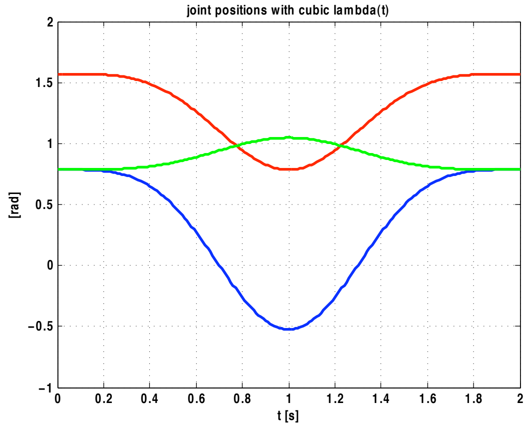
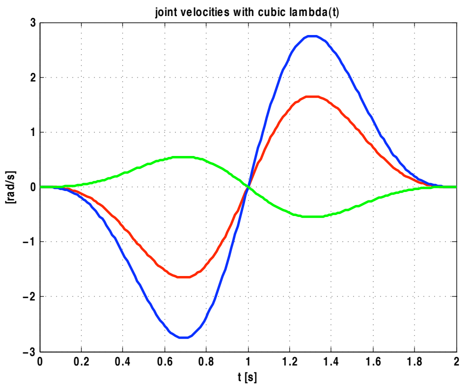


Figure 5: Solution trajectory for problem c) — joint 1 (red), joint 2 (blue), joint 3 (green)



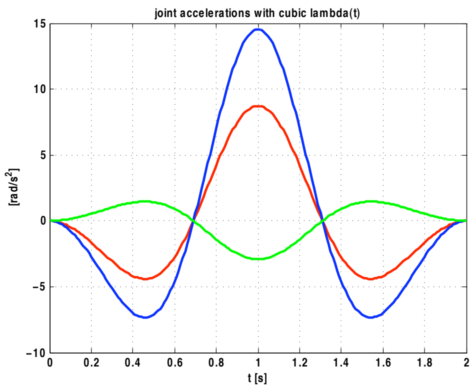
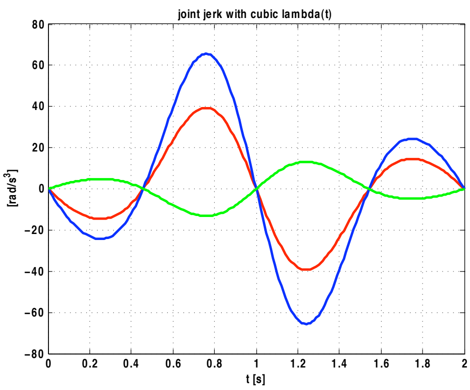


Figure 6: Velocity and acceleration of the trajectory for problem c)



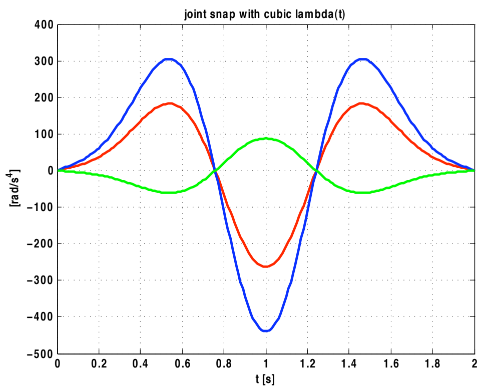


Figure 7: Jerk and snap of the trajectory for problem c)

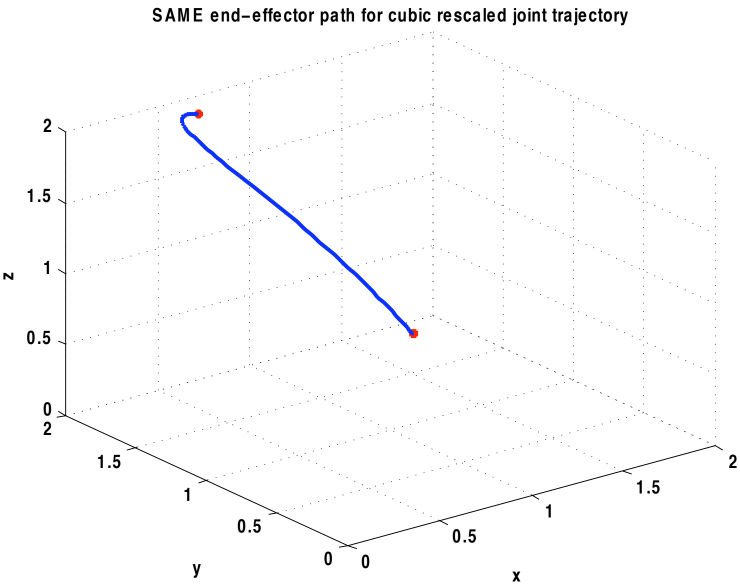


Figure 8: The same end-effector path is obtained with the trajectory of problem c) — the red point on the left/top is $\boldsymbol{p}_1$, the other is $\boldsymbol{p}_2$

* * * * *

Robotics I

June 17, 2011

Consider a 2R planar robot having link lengths $\ell_1 = 3$ and $\ell_2 = 2$ [m]. The joint velocities are limited by

$$|\dot{q}_1| \leq 1\,\text{rad/s}, \qquad |\dot{q}_2| \leq 1.5\,\text{rad/s}.$$

Determine the feasible Cartesian velocity $\boldsymbol{v} = (v_x, v_y)$ of the end-effector which is the largest in norm at the configuration $\boldsymbol{q}_a = (\pi/6, \pi/3)$ [rad], providing also the joint velocities $\dot{\boldsymbol{q}}$ realizing it and the resulting norm $\|\boldsymbol{v}\|$. Repeat this analysis for a second configuration $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$ [rad]. Draw a figure illustrating all feasible Cartesian velocities of the end-effector at least for the first case. Also, illustrate what happens to this figure when the robot is in a singular configuration $\boldsymbol{q}_s$.

[90 minutes; open books]

Solution

June 17, 2011

The solution is a straightforward application of linear algebra. Figure 1 shows a rectangle R_j in the $(\dot{q}_1, \dot{q}_2)$ space representing the region of feasible joint velocities, having the four vertices $\dot{\boldsymbol{q}}_A$ to $\dot{\boldsymbol{q}}_D$ with both joints at their maximum (positive) or minimum (negative, symmetric) bounds.

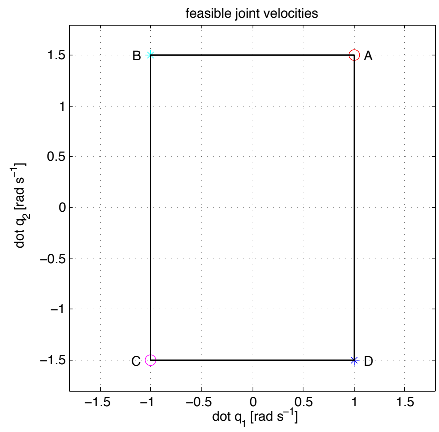


Figure 1: Joint velocity limits — the vertices of the rectangle R_j are labeled as A to D with reference to the following Cartesian plots

The Jacobian of the 2R planar robot is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\left(\ell_1 \sin q_1 + \ell_2 \sin(q_1+q_2)\right) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1+q_2) & \ell_2 \cos(q_1+q_2) \end{pmatrix}.$$

When evaluated at the configuration $\boldsymbol{q}_a = (\pi/6, \pi/3)$, and with the given link lengths, we have the constant matrix

$$\boldsymbol{J}_a = \boldsymbol{J}(\boldsymbol{q}_a) = \begin{pmatrix} -3.5 & -2 \\ 2.5981 & 0 \end{pmatrix}$$

that will generate all feasible Cartesian velocities $\boldsymbol{v} \in P_c$ as

$$\boldsymbol{v} = \boldsymbol{J}_a \dot{\boldsymbol{q}}, \qquad \forall \boldsymbol{q} \in R_j.$$

In particular, the vertices A to D of R_j will map respectively into the homonymous vertices of the region P_c (see Fig. 2):

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{J}_a \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -6.5 \\ 2.5981 \end{pmatrix} & \qquad \boldsymbol{v}_B &= \boldsymbol{J}_a \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -2.5981 \end{pmatrix} \\ \boldsymbol{v}_C &= \boldsymbol{J}_a \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6.5 \\ -2.5981 \end{pmatrix} & \qquad \boldsymbol{v}_D &= \boldsymbol{J}_a \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 2.5981 \end{pmatrix}. \end{aligned} \tag{1}$$

The four boundaries of P_c are characterized by points that are linear combinations of the above four vertices, taken two by two in alphabetic sequence. Therefore, P_c will be a convex polytope.

The largest Cartesian velocity in norm occurs at one (or more) vertex of P_c which is the farthest away from the origin (in the following figures, the origin of the (v_x, v_y) space is located at the robot end-effector for a more intuitive visualization). Hence,

$$V_{\mathrm{max,a}} = \max\{\|\boldsymbol{v}\|, \text{for } \boldsymbol{v} \in P_c\} = \max\{\|\boldsymbol{v}_A\|, \|\boldsymbol{v}_B\|, \|\boldsymbol{v}_C\|, \|\boldsymbol{v}_D\|\} = \|\boldsymbol{v}_A\| = \|\boldsymbol{v}_C\| = 7 \text{ [m/s]}.$$

There are indeed two opposite and saturated joint velocities, $\dot{\boldsymbol{q}}_A$ and $\dot{\boldsymbol{q}}_C$, that provide the maximum norm of the Cartesian velocity.

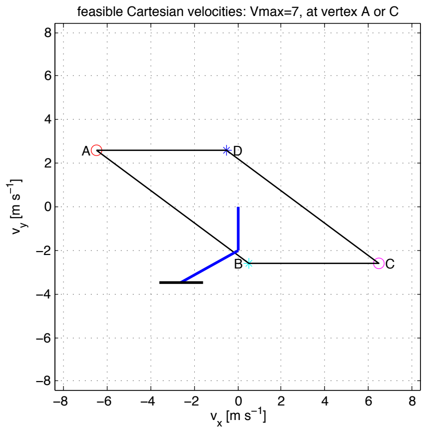


Figure 2: Polytope of feasible Cartesian velocities at $\boldsymbol{q}_a = (\pi/6, \pi/3)$, centered at the end-effector of the robot (shown in blue on its fixed base) — the vertices A to D are the images of those in Fig. 1

The analysis is identical at the configuration $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$, see Fig. 3 (note the different scale). There, the Jacobian takes the numerical values

$$\boldsymbol{J}_b = \boldsymbol{J}(\boldsymbol{q}_b) = \begin{pmatrix} -1.2389 & 0.2611 \\ 0.6152 & -1.9829 \end{pmatrix}$$

and we obtain, in place of (1),

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{J}_b \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -0.8474 \\ -2.3591 \end{pmatrix} & \qquad \boldsymbol{v}_B &= \boldsymbol{J}_b \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1.6305 \\ -3.5895 \end{pmatrix} \\ \boldsymbol{v}_C &= \boldsymbol{J}_b \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 0.8474 \\ 2.3591 \end{pmatrix} & \qquad \boldsymbol{v}_D &= \boldsymbol{J}_b \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} -1.6305 \\ 3.5895 \end{pmatrix}, \end{aligned} \tag{2}$$

from which

$$V_{\mathrm{max,b}} = \|\boldsymbol{v}_B\| = \|\boldsymbol{v}_D\| = 3.9425 \text{ [m/s]}.$$

The two opposite and saturated joint velocities that provide the maximum norm of the Cartesian velocity are now $\dot{\boldsymbol{q}}_B$ and $\dot{\boldsymbol{q}}_D$. This change is due to the different configuration, in much the same way as in the analysis of manipulability (the associated velocity ellipsoid is not related to the presence of hard bounds on the joint velocities, since in that case we map all possible $\dot{\boldsymbol{q}}$, normalized with $\|\dot{\boldsymbol{q}}\| = 1$). However, the maximum norm of the Cartesian velocity is now almost halved w.r.t.

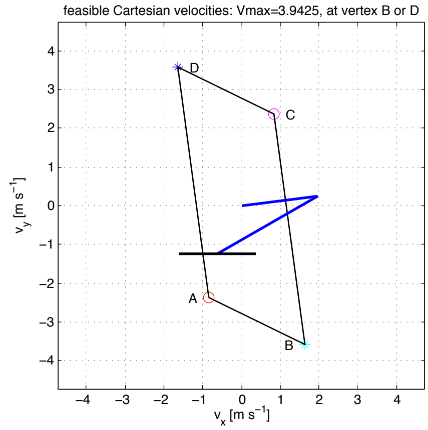


Figure 3: Polytope of feasible Cartesian velocities at $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$, centered at the robot end-effector

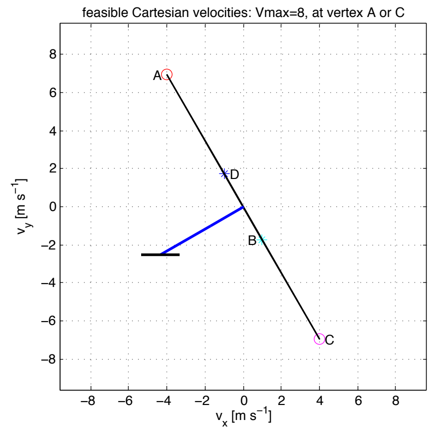


Figure 4: Polytope of feasible Cartesian velocities at the singular configuration $\boldsymbol{q}_s = (\pi/6, 0)$ — the area of the polytope vanishes and only one possible direction for the end-effector velocity is left, with limited amplitude

the previous case and also the area of the polytope is reduced. This is because the robot is close to a singular configuration, the folded one.

To complete the analysis, we consider the singular configuration $\boldsymbol{q}_s = (\pi/6, 0)$, i.e., with the arm stretched (see Fig. 4). In this case, the Jacobian

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -2.5 & -1 \\ 4.3301 & 1.7321 \end{pmatrix}$$

is singular and the polytope collapses. It is

$$
\begin{aligned}
\boldsymbol{v}_A &= \boldsymbol{J}_s \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -4 \\ 6.9282 \end{pmatrix} = -\boldsymbol{v}_C \\
\boldsymbol{v}_B &= \boldsymbol{J}_s \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.7321 \end{pmatrix} = -\boldsymbol{v}_D,
\end{aligned} \tag{3}
$$

and

$$V_{\mathrm{max,s}} = \|\boldsymbol{v}_A\| = \|\boldsymbol{v}_C\| = 8 \text{ [m/s]}.$$

This value is the largest of all cases. Indeed, the saturated joint velocity $\dot{\boldsymbol{q}}_A$ (or $\dot{\boldsymbol{q}}_C$) is maximally contributing to the Cartesian velocity in just one single direction when the arm is fully stretched, similarly to when a human is throwing of a ball.

The Matlab source code generating the solution and the plots is available upon request.

* * * * *

Robotics I

July 4, 2011

Several views of the Barrett 4-dof WAM arm are shown in Fig. 1, together with the frame assignment used by the manufacturer. Six reference frames are considered, including one attached to the Base (fixed platform) and one attached at the Tool (end-effector). All data are in mm. The only information missing in the drawings is that the displacement at the elbow joint is equal to 45 mm. Also, the origin of the Tool frame is placed at the end of link 4 (the drawing may be misleading).

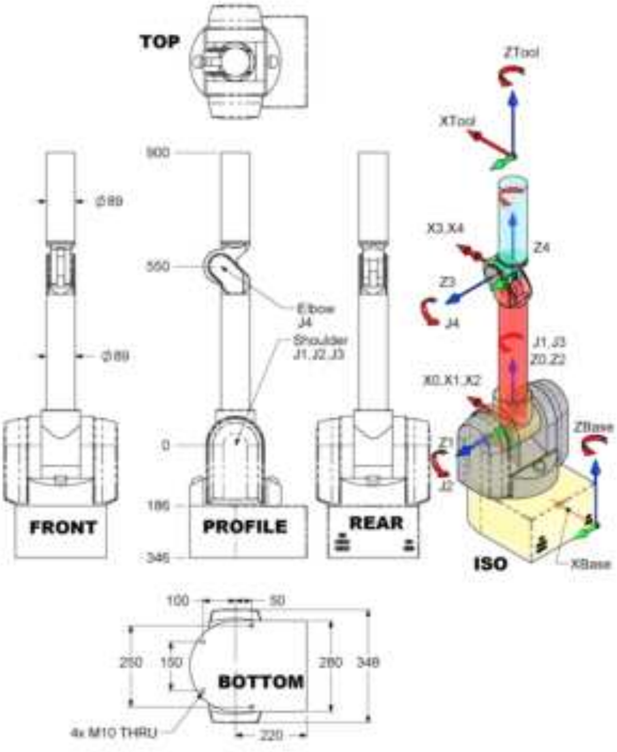


Figure 1: The Barrett 4-dof WAM arm, with the frame assignment used by the manufacturer

1. Check whether the frame assignment used by the manufacturer is fully consistent with the Denavit-Hartenberg (D-H) convention, and modify it if needed. Derive the correct table of D-H parameters.

2. Determine the direct kinematic map for the position ${}^{B}\boldsymbol{p}_T$ of the origin of the Tool frame with respect to the Base frame (use symbols for geometric quantities, specifying separately their numerical values):

$$\begin{pmatrix} {}^{B}\boldsymbol{p}_T \\ \\ 1 \end{pmatrix} = {}^{B}T_0\,{}^{0}T_1(\theta_1)\,{}^{1}T_2(\theta_2)\,{}^{2}T_3(\theta_3)\,{}^{3}T_4(\theta_4)\,{}^{4}T_T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

3. Four motors are used to drive the four joints through suitable transmission elements, with two motors (located at the shoulder) cooperating for the motion of joints 2 and 3. The mapping between the motor velocity vector $\dot{\boldsymbol{\theta}}_m$ and the joint velocity vector $\dot{\boldsymbol{\theta}}$ is

$$\dot{\boldsymbol{\theta}} = \begin{pmatrix} -\frac{1}{N_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2N_2} & -\frac{1}{2N_2} & 0 \\ 0 & -\frac{n_3}{2N_2} & -\frac{n_3}{2N_2} & 0 \\ 0 & 0 & 0 & -\frac{1}{N_4} \end{pmatrix} \dot{\boldsymbol{\theta}}_m, \tag{2}$$

with $N_1 = 42$, $N_2 = 28.25$, $n_3 = 1.68$, $N_4 = 18$. Determine the mapping *from* the torque vector $\boldsymbol{\tau}_m$ at the output shaft of each motor *to* the torque vector $\boldsymbol{\tau}$ producing work on the joint coordinates $\boldsymbol{\theta}$.

4. Find the 3×4 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ relating the joint velocity vector $\dot{\boldsymbol{\theta}}$ to the velocity $\boldsymbol{v}_T$ of the origin of the Tool frame

$$\boldsymbol{v}_T = \dot{\boldsymbol{p}}_T = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}.$$

Choose your preferred reference frame for expressing $\boldsymbol{v}_T$. *Note: In order to obtain the full expression of the associated Jacobian, the use of a symbolic manipulation program is appropriate.*

5. Determine one singularity of $\boldsymbol{J}(\boldsymbol{\theta})$. Verify whether this singularity is within the following joint limits of the robot or not:

$$-150^\circ \leq \theta_1 \leq +150^\circ, \quad -113^\circ \leq \theta_2 \leq +113^\circ, \quad -157^\circ \leq \theta_3 \leq +157^\circ, \quad -50^\circ \leq \theta_4 \leq +180^\circ.$$

[150 minutes; open books]

Solution

July 4, 2011

For item 1., the frame assignment used by the manufacturer is fully consistent with the Denavit-Hartenberg convention and there is no need of modifications. The D-H parameters are given in Tab. 1, with $a_3 = 45$ and $d_3 = 550$ [mm]. In Fig. 1, the arm is shown in its "zero configuration" ($\boldsymbol{\theta} = \mathbf{0}$).

i	α_i	a_i	d_i	θ_i
1	$-\dfrac{\pi}{2}$	0	0	θ_1
2	$\dfrac{\pi}{2}$	0	0	θ_2
3	$-\dfrac{\pi}{2}$	a_3	d_3	θ_3
4	$\dfrac{\pi}{2}$	$-a_3$	0	θ_4

Table 1: D-H parameters for the Barrett 4-dof WAM arm

For item 2., Table 1 is used to compute the four homogeneous transformations ${}^{i-1}T_i(\theta_i)$, for $i = 1, \ldots, 4$. The remaining two constant transformations are

$$
{}^{B}T_0 = \begin{pmatrix} & {}^{B}p_{0x} \\ \boldsymbol{I} & {}^{B}p_{0y} \\ & {}^{B}p_{0z} \\ \mathbf{0}^T & 1 \end{pmatrix},
$$

with ${}^{B}p_{0x} = 220$, ${}^{B}p_{0y} = 140$, ${}^{B}p_{0z} = 346$ [mm], and

$$
{}^{4}T_T = \begin{pmatrix} & 0 \\ \boldsymbol{I} & 0 \\ & L \\ \mathbf{0}^T & 1 \end{pmatrix},
$$

with $L = 350$ [mm]. Note that these constant transformations may not be expressed in general as D-H homogeneous matrices (i.e., in terms of the usual four, now all constant, D-H parameters). In particular, ${}^{4}T_T$ has the structure of a D-H matrix (with $\alpha_T = \theta_T = 0$, $a_T = 0$, and $d_T = L$), whereas ${}^{B}T_0$ cannot be associated to D-H parameters.

In order to find the expression of ${}^{B}\boldsymbol{p}_T$ in (1), there is no need to compute the full product of all the transformation matrices, i.e., ${}^{B}T_T$ (or even just ${}^{0}T_4$). In fact, expressing all vectors in homogenous coordinates, we can compute recursively:

$$\begin{aligned}
{}^4\boldsymbol{p}_T &= {}^4T_T{}^T\boldsymbol{p}_T = {}^4T_T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ L \\ 1 \end{pmatrix} \\
{}^3\boldsymbol{p}_T &= {}^3T_4(\theta_4){}^4\boldsymbol{p}_T \\
&\vdots \\
{}^0\boldsymbol{p}_T &= {}^0T_1(\theta_1){}^1\boldsymbol{p}_T \\
{}^B\boldsymbol{p}_T &= {}^BT_0{}^4\boldsymbol{p}_T.
\end{aligned}$$

It is then tedious but straightforward to obtain

$${}^B\boldsymbol{p}_T = \begin{pmatrix} {}^Bp_{0x} + (c_1c_2c_3 - s_1s_3)(a_3 + Ls_4 - a_3c_4) + c_1s_2(d_3 - a_3s_4 - Lc_4) \\ {}^Bp_{0y} + (s_1c_2c_3 + c_1s_3)(a_3 + Ls_4 - a_3c_4) + s_1s_2(d_3 - a_3s_4 - Lc_4) \\ {}^Bp_{0z} + s_2c_3(a_3 + Ls_4 - a_3c_4) + c_2(d_3 + a_3s_4 + Lc_4) \end{pmatrix}, \tag{3}$$

where the short notations $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ have been used. At this point, we can substitute the numerical values previously defined.

For item 3., writing eq. (2) as $\dot{\boldsymbol{\theta}} = \boldsymbol{A}\dot{\boldsymbol{\theta}}_m$, we have from the principle of virtual work

$$\boldsymbol{\tau}^T\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}^T\boldsymbol{A}\dot{\boldsymbol{\theta}}_m = \boldsymbol{\tau}_m^T\dot{\boldsymbol{\theta}}_m, \qquad \forall\dot{\boldsymbol{\theta}}_m,$$

and thus

$$\boldsymbol{\tau}_m = \boldsymbol{A}^T\boldsymbol{\tau} \qquad \Longleftrightarrow \qquad \boldsymbol{\tau} = \boldsymbol{A}^{-T}\boldsymbol{\tau}_m.$$

Therefore, the mapping is

$$\boldsymbol{\tau} = \begin{pmatrix} -N_1 & 0 & 0 & 0 \\ 0 & N_2 & -N_2 & 0 \\ 0 & -\dfrac{N_2}{n_3} & -\dfrac{N_2}{n_3} & 0 \\ 0 & 0 & 0 & -N_4 \end{pmatrix} \boldsymbol{\tau}_m.$$

We can now replace in this expression the numerical data provided in the text.

For item 4., the Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$ of interest is computed either geometrically or by analytic differentiation of eq. (3) w.r.t. $\boldsymbol{\theta}$. The result can be obtained by hand or by using symbolic/algebraic manipulation tools (e.g., the *Matlab Symbolic Toolbox*). However, very complex expressions are typically obtained which are difficult to be simplified automatically. A possible relatively simpler form for $\boldsymbol{J}(\boldsymbol{\theta})$ is obtained when looking at the expression of the task velocity $\boldsymbol{v}_T$ (naturally defined in the zero-th reference frame) in a suitable reference frame. In fact, we have

$${}^i\boldsymbol{v}_T = {}^0\boldsymbol{R}_i^T(\theta_1, \ldots, \theta_i)\,{}^0\boldsymbol{v}_T = {}^0\boldsymbol{R}_i^T(\theta_1, \ldots, \theta_i)\,{}^0\boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \Rightarrow \qquad {}^i\boldsymbol{J}(\boldsymbol{\theta}) = {}^0\boldsymbol{R}_i^T(\theta_1, \ldots, \theta_i)\,{}^0\boldsymbol{J}(\boldsymbol{\theta})$$

where the Jacobian ${}^i\boldsymbol{J}(\boldsymbol{\theta})$ is expressed in the i-th frame of the robot, and

$${}^0\boldsymbol{R}_i(\theta_1, \ldots, \theta_i) = {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\ldots{}^{i-1}\boldsymbol{R}_i(\theta_i)$$

is the composition of the rotation matrices defined by the D-H parameters of the robot. For the Barrett 4-dof WAM arm, a convenient choice is to refer the Jacobian to frame 2, i.e.,

$$^2\boldsymbol{J}(\boldsymbol{\theta}) = {}^1\boldsymbol{R}_2^T(\theta_2)\,{}^0\boldsymbol{R}_1^T(\theta_1)\,{}^0\boldsymbol{J}(\boldsymbol{\theta}).$$

For illustration, we report in the Appendix the $3 \times 4 = 12$ elements j_{ik} of $^2\boldsymbol{J}(\boldsymbol{\theta})$, i.e.,

$$^2\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} j_{11} & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ j_{31} & j_{32} & j_{33} & j_{34} \end{pmatrix}, \tag{4}$$

provided as output of a *Matlab Symbolic Toolbox* program (available upon request).

For item 5., it is rather immediate to see from Fig. 1 that the robot configuration having $\theta_2 = \theta_3 = \theta_4 = 0$ (and arbitrary θ_1) is certainly singular, since the arm is fully stretched. Plugging in these values of joint angles, the symbolic Jacobian matrix becomes

$$^2\boldsymbol{J}(\boldsymbol{\theta})|_{\theta_2=\theta_3=\theta_4=0} = \begin{pmatrix} 0 & (L+d_3)\,c_1 & 0 & L\,c_1 \\ 0 & (L+d_3)\,s_1 & 0 & L\,s_1 \\ 0 & 0 & 0 & a_3 \end{pmatrix},$$

having, as expected, rank 2 for any θ_1. In particular, this is true at the zero configuration (i.e., with $\theta_1 = 0$, in addition to the previously selected values for the other joints) illustrated in Fig. 1. This singularity is clearly within the assigned joint limits.

* * * * *

Appendix

With reference to (4), using the short notations for$s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ $(i = 1, \ldots, 4)$, we have:

$$
\begin{aligned}
j_{11} &= -c_2\left(c_1\left(c_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) + s_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right)\right. \\
&\quad \left. + s_1\left(s_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) - c_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right)\right) \\
j_{12} &= s_2\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right) - c_2\left(c_1\left(c_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right.\right. \\
&\quad \left.\left. - c_1 c_2\left(d_3 + L c_4 + a_3 s_4\right)\right) - s_1\left(c_2 s_1\left(d_3 + L c_4 + a_3 s_4\right) - s_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right) \\
j_{13} &= -s_3\left(2 a_3 \sin\left(\frac{\theta_4}{2}\right)^2 + L s_4\right) {s_2}^2 - c_2\left(c_1\left(c_3 s_1 + c_1 c_2 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right)\right. \\
&\quad \left. - s_1\left(c_1 c_3 - c_2 s_1 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right)\right) \\
j_{14} &= s_2\left(c_2\left(L s_4 - a_3 c_4\right) + c_3 s_2\left(L c_4 + a_3 s_4\right)\right) - c_2\left(c_1\left(c_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right)\right.\right. \\
&\quad \left.\left. + s_1 s_3\left(L c_4 + a_3 s_4\right)\right) + s_1\left(s_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right) - c_1 s_3\left(L c_4 + a_3 s_4\right)\right)\right) \\
j_{21} &= s_1\left(c_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) + s_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right) \\
&\quad - c_1\left(s_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) - c_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right) \\
j_{22} &= s_1\left(c_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right) - c_1 c_2\left(d_3 + L c_4 + a_3 s_4\right)\right) + c_1\left(c_2 s_1\left(d_3 + L c_4 + a_3 s_4\right)\right. \\
&\quad \left. - s_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right) \\
j_{23} &= c_1\left(c_1 c_3 - c_2 s_1 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right) + s_1\left(c_3 s_1 + c_1 c_2 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right) \\
j_{24} &= s_1\left(c_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right) + s_1 s_3\left(L c_4 + a_3 s_4\right)\right) \\
&\quad - c_1\left(s_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right) - c_1 s_3\left(L c_4 + a_3 s_4\right)\right) \\
j_{31} &= -s_2\left(c_1\left(c_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) + s_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right)\right. \\
&\quad \left. + s_1\left(s_1\left(a_3 s_3 + s_3\left(L s_4 - a_3 c_4\right)\right) - c_1\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right)\right) \\
j_{32} &= -s_2\left(c_1\left(c_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right) - c_1 c_2\left(d_3 + L c_4 + a_3 s_4\right)\right) - s_1\left(c_2 s_1\left(d_3 + L c_4 + a_3 s_4\right)\right.\right. \\
&\quad \left.\left. - s_1 s_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right)\right) - c_2\left(s_2\left(d_3 + L c_4 + a_3 s_4\right) + c_2\left(a_3 c_3 + c_3\left(L s_4 - a_3 c_4\right)\right)\right) \\
j_{33} &= c_2 s_2 s_3\left(2 a_3 \sin\left(\frac{\theta_4}{2}\right)^2 + L s_4\right) - s_2\left(c_1\left(c_3 s_1 + c_1 c_2 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right)\right. \\
&\quad \left. - s_1\left(c_1 c_3 - c_2 s_1 s_3\right)\left(a_3 + L s_4 - a_3 c_4\right)\right) \\
j_{34} &= -c_2\left(c_2\left(L s_4 - a_3 c_4\right) + c_3 s_2\left(L c_4 + a_3 s_4\right)\right) - s_2\left(c_1\left(c_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right)\right.\right. \\
&\quad \left.\left. + s_1 s_3\left(L c_4 + a_3 s_4\right)\right) + s_1\left(s_1\left(s_2\left(L s_4 - a_3 c_4\right) - c_2 c_3\left(L c_4 + a_3 s_4\right)\right) - c_1 s_3\left(L \cos\theta_4 + a_3 \sin\theta_4\right)\right)\right).
\end{aligned}
$$

* * * * *

Robotics I

September 12, 2011

The direct kinematics of a 4-dof robot of the SCARA type (2R-1P-1R) is

$$
\begin{aligned}
x &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\
y &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\
z &= q_3 \\
\alpha &= 0 \\
\beta &= 0 \\
\gamma &= q_1 + q_2 + q_4,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{p} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ is the position of the end-effector and $\boldsymbol{\phi} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}^T$ is its orientation, expressed in XYZ angles (roll-pitch-yaw) with respect to the fixed base frame. The length of links 1 and 2 is $l_1 = l_2 = 1$ [m].

In the configuration $\boldsymbol{q}_A = \begin{pmatrix} 0 & \pi/2 & 1 & -\pi/2 \end{pmatrix}^T$, check if the following end-effector generalized (linear/angular) velocity vector

$$
\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}
\tag{2}
$$

is feasible or not. Provide then the expression and the numerical value of the joint velocity vector $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that guarantees the exact execution of the desired end-effector velocity (if the task (2) is feasible), or at least the minimization of the Cartesian/task velocity error norm in a least squares sense (if (2) is unfeasible).

Repeat the above analysis and the synthesis of a joint velocity vector with the same properties when the robot is in the configuration $\boldsymbol{q}_B = \begin{pmatrix} \pi/4 & 0 & 1 & 0 \end{pmatrix}^T$.

[120 minutes; open books]

Solution

September 12, 2011

Denote the 6-dimensional task vector associated to the end-effector pose of the robot as

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

where $\boldsymbol{f}(\boldsymbol{q})$ is the direct kinematics expressed by the right-hand side of (1). The 6×4 analytic Jacobian matrix $\boldsymbol{J}_a$ relating $\dot{\boldsymbol{q}}$ to $\dot{\boldsymbol{r}}$

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_a(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and the 6×4 geometric Jacobian matrix $\boldsymbol{J}_g$ relating $\dot{\boldsymbol{q}}$ to the linear and angular velocity of the end-effector

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \boldsymbol{J}_g(\boldsymbol{q})\dot{\boldsymbol{q}}$$

will coincide in this case ($\boldsymbol{J}_a = \boldsymbol{J}_g = \boldsymbol{J}$), since the only admissible rotations of the end-effector frame of this robot occur around a single constant axis (the Z axis). In particular, it will always be $\omega_x = \dot{\alpha} = 0$, $\omega_y = \dot{\beta} = 0$, and $\omega_z = \dot{\gamma}$.

Therefore, we compute (for convenience, analytically from (1))

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\left(l_1 \sin q_1 + l_2 \sin(q_1+q_2)\right) & -l_2 \sin(q_1+q_2) & 0 & 0 \\ l_1 \cos q_1 + l_2 \cos(q_1+q_2) & l_2 \cos(q_1+q_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Evaluating this matrix at the configuration $\boldsymbol{q}_A = \begin{pmatrix} 0 & \pi/2 & 1 & -\pi/2 \end{pmatrix}^T$, and with the given link lengths, provides

$$\boldsymbol{J}_A = \boldsymbol{J}(\boldsymbol{q}_A) = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

which has full row rank equal to 4 ($\det \boldsymbol{J}_A^T \boldsymbol{J}_A = 1 \neq 0$). The desired linear and angular velocity (2), due to its structure with zeros in the fourth and fifth components, will certainly be feasible and a joint velocity solution is computed as

$$
\begin{aligned}
\dot{\boldsymbol{q}}_A &= \boldsymbol{J}_A^{\#} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \left(\boldsymbol{J}_A^T \boldsymbol{J}_A\right)^{-1} \boldsymbol{J}_A^T \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ [rad/s]}.
\end{aligned} \tag{3}
$$

We note that in this case $\dot{\boldsymbol{q}}_A$ is also the unique joint velocity realizing the desired task velocity.

At the configuration $\boldsymbol{q}_B = \begin{pmatrix} \pi/4 & 0 & 1 & 0 \end{pmatrix}^T$, we have

$$
\boldsymbol{J}_B = \boldsymbol{J}(\boldsymbol{q}_B) = \begin{pmatrix} -\sqrt{2} & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2} & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},
$$

which has clearly rank equal to 3. Still the desired linear and angular velocity lies in the range of matrix $\boldsymbol{J}_B$ and thus it is feasible. The only difference with respect to the previous case is that a joint velocity solution should be computed using the numerical algorithm for the pseudoinverse of $\boldsymbol{J}_B$ (i.e., using the SVD of $\boldsymbol{J}_B$, or by simply calling the `pinv` function in Matlab). We obtain, with some numerical rounding,

$$
\begin{aligned}
\dot{\boldsymbol{q}}_B &= \boldsymbol{J}_B^{\#} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} \\
&= \begin{pmatrix} -0.3536 & 0.3536 & 0 & 0 & 0 & -0.1667 \\ 0 & 0 & 0 & 0 & 0 & 0.3333 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3536 & -0.3536 & 0 & 0 & 0 & 0.8333 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.5404 \\ -0.3333 \\ 1 \\ -0.1262 \end{pmatrix} \text{ [rad/s]}.
\end{aligned} \tag{4}
$$

The solution $\dot{\boldsymbol{q}}_B$ is not unique in this case, but it has automatically the minimum norm property among all joint velocities realizing the task. For illustration, another joint velocity satisfying the task is

$$
\dot{\boldsymbol{q}}_B' = \begin{pmatrix} -0.1322 \\ -1.1498 \\ 1 \\ 0.2820 \end{pmatrix} \text{ [rad/s]},
$$

which is obtained by adding to $\dot{\boldsymbol{q}}_B$ a vector $\boldsymbol{n}_B$ in the one-dimensional null space of $\boldsymbol{J}_B$:

$$
\dot{\boldsymbol{q}}_B' = \dot{\boldsymbol{q}}_B + \boldsymbol{n}_B, \qquad \boldsymbol{n}_B = \begin{pmatrix} 0.4082 \\ -0.8165 \\ 0 \\ 0.4082 \end{pmatrix}, \qquad \text{being } \boldsymbol{J}_B \boldsymbol{n}_B = \boldsymbol{0}.
$$

Checking the norm of these two joint velocity solutions gives

$$\|\dot{\boldsymbol{q}}_B\| = 1.1913 < 1.554 = \|\dot{\boldsymbol{q}}'_B\|,$$

as expected.

In both cases A and B (as well as in B'), the task velocity error is zero, since the desired linear and angular velocity (2) is exactly realized. Indeed, one can verify that

$$\boldsymbol{J}_A\,\dot{\boldsymbol{q}}_A = \boldsymbol{J}_B\,\dot{\boldsymbol{q}}_B = \boldsymbol{J}_B\,\dot{\boldsymbol{q}}'_B = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

In case B, there exist other task velocity vectors (still with structural zeros in the fourth and fifth components) that are not feasible. The joint velocity obtained by using the pseudoinverse of $\boldsymbol{J}_B$ will still provide a task velocity error of minimum norm, although this error would then be different from zero. The reader is invited to provide an example.

We finally note that, due to the presence of structural zeros in the fourth and fifth rows of $\boldsymbol{J}$, all previous results could have been obtained by reducing the problem to the 4-dimensional task space $\bar{\boldsymbol{r}} = \begin{pmatrix} x & y & z & \gamma \end{pmatrix}^T$ and working with the square 4×4 matrix $\bar{\boldsymbol{J}}$ obtained by deleting rows 4 and 5 from $\boldsymbol{J}$. For case A, we would obtain the same solution $\dot{\boldsymbol{q}}_A$ given in (3) by computing $\dot{\boldsymbol{q}}_A = \bar{\boldsymbol{J}}^{-1}\dot{\bar{\boldsymbol{r}}}$, while for case B the same solution $\dot{\boldsymbol{q}}_B$ given in (4) would be obtained as $\dot{\boldsymbol{q}}_B = \bar{\boldsymbol{J}}^{\#}\dot{\bar{\boldsymbol{r}}}$.

In fact we can state in general that, given a $m \times n$ matrix $\boldsymbol{J}$ (with $m > n$) of the form

$$\boldsymbol{J} = \boldsymbol{T}\begin{pmatrix} \bar{\boldsymbol{J}} \\ \boldsymbol{O} \end{pmatrix},$$

where $\boldsymbol{T}$ is a row permutation (and thus, an orthonormal) matrix and $\bar{\boldsymbol{J}}$ is a square $m \times m$ matrix, the following expression holds for the pseudoinverse:

$$\boldsymbol{J}^{\#} = \bar{\boldsymbol{J}}^{\#}\begin{pmatrix} \boldsymbol{I}_{m\times m} & \boldsymbol{O} \end{pmatrix}\boldsymbol{T}^T.$$

Morever, when $\det \bar{\boldsymbol{J}} \neq 0$, it is $\bar{\boldsymbol{J}}^{\#} = \bar{\boldsymbol{J}}^{-1}$. In our application, this relation means that we can simply work with the reduced task space and the associated reduced Jacobian, discarding the zero rows.

* * * * *

Robotics I

January 11, 2012

Exercise 1

Consider a *planar* 3R robot having link lengths $\ell_1 = 1$, $\ell_2 = 0.5$, and $\ell_3 = 0.25$ [m].

- Draw the primary workspace in the plane (x_0, y_0).
- Draw the secondary workspace, i.e., the set of all points that can be reached by the end-effector with any admissible approach angle in the plane (x_0, y_0).
- Following the Denavit-Hartenberg convention for the definition of the joint variables $\boldsymbol{q}$, find all singular configurations of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity vector $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the end-effector velocity vector $\boldsymbol{v} = \begin{pmatrix} v_x & v_y \end{pmatrix}^T \in \mathbb{R}^2$.
- Give the explicit expression of the manipulability index $H = \sqrt{\det(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}))}$ in a form that shows its independence from the variable q_1.

Exercise 2

For the same robot of Exercise 1, consider a *rest-to-rest* motion task for the end-effector from point $\boldsymbol{p}_A = \begin{pmatrix} 1.75 & 0 \end{pmatrix}^T$ [m], with orientation (with respect to the axis x_0) $\phi_A = 0$, to point $\boldsymbol{p}_B = \begin{pmatrix} 1.25 & 0.5 \end{pmatrix}^T$ [m], with orientation $\phi_B = 0$.

- For this motion task, define a minimum time trajectory under the joint velocity and acceleration bounds
$$|\dot{q}_i| \le V_i, \qquad |\ddot{q}_i| \le A_i, \qquad i = 1, 2, 3,$$
with the following numerical data:
$$V_1 = 90°/\text{s}, \qquad V_2 = 120°/\text{s}, \qquad V_3 = 60°/\text{s},$$
$$A_1 = 150°/\text{s}^2, \qquad A_2 = 160°/\text{s}^2, \qquad A_3 = 240°/\text{s}^2.$$
Provide also the minimum motion time T.
- Sketch the velocity and acceleration profiles of each joint for the minimum time trajectory that has been found.
- Is the obtained minimum time trajectory unique for this problem? Motivate the answer.

[150 minutes; open books]

Bonus Exercise (30 additional minutes available)

Write a Matlab program that plots in 3D the manipulability index H of the considered planar 3R robot as a function of q_2 and q_3. Using such a program, determine (at least approximately) a configuration that globally maximizes this index and provide the associated value of H.

Solutions

January 11, 2012

Exercise 1

The primary workspace is the set of positions that can be reached by the robot end-effector, independently from its orientation. For a planar 3R robot, the primary workspace is an annulus (*it: corona circolare*). The outer radius is simply $R_1 = \ell_1 + \ell_2 + \ell_3$. For the inner radius, the following general formula holds

$$r_1 = \max\left\{|\ell_1 - \ell_2| - \ell_3, |\ell_2 - \ell_3| - \ell_1, |\ell_1 - \ell_3| - \ell_2, 0\right\},$$

which takes into account all possible cases of (non-negative) lengths for the three links. Note that the inner 'hole' would vanish in particular (but not only) for equal $\ell_1 = \ell_2 = \ell_3$. With the given numerical data, it is

$$r_1 = 0.25\,[\mathrm{m}], \qquad R_1 = 1.75\,[\mathrm{m}].$$

Figure 1 shows in gray the primary workspace and the robot arm in a stretched (on the right) and in a folded configuration (on the left), in which the end-effector reaches respectively the outer and inner boundaries of the primary workspace.

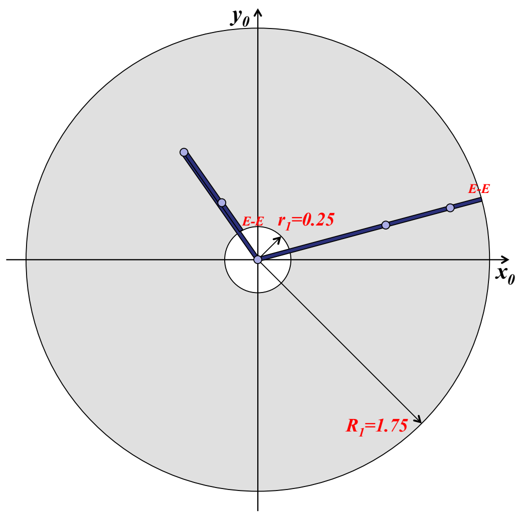


Figure 1: Primary workspace of the planar 3R robot

The secondary workspace is the set of positions that can be reached by the robot end-effector with any of its admissible orientations. For a planar robot, all admissible orientations are defined as rotations by an angle ϕ around an axis orthogonal to the plane of motion. The angle ϕ can take any value in $(-\pi, \pi]$. The secondary workspace for the considered planar 3R robot is again an annulus with inner and outer radii given by

$$r_2 = \ell_1 - \ell_2 + \ell_3 = 0.75\,[\mathrm{m}], \qquad R_2 = \ell_1 + \ell_2 - \ell_3 = 1.25\,[\mathrm{m}],$$

as shown by the dark gray area in Fig. 2. The geometric construction of this secondary workspace is rather straightforward. Take any point A in the primary workspace (by the central symmetry with respect to q_1, it is sufficient to consider only points lying on a ray starting from the origin) and draw a circle of radius ℓ_3 centered in A. In order for the third link to assume any orientation in the plane when the end-effector is placed in A, all points on this circle should be reachable by the tip of the sub-arm made by the first two links. Therefore, we are interested in the (primary) workspace of this auxiliary two-link robot (with link lengths ℓ_1 and ℓ_2). This auxiliary workspace is an annulus of inner and outer radii $r_a = |\ell_1 - \ell_2| = 0.5$ [m] and $R_a = \ell_1 + \ell_2 = 1.5$ [m], as represented by the dashed circles in Fig. 2. The two boundaries of the secondary workspace for the 3R robot are obtained when folding the third link (of length $\ell_3 = 0.25$ [m]) by $\pm\pi$ while the tip of the two-link sub-arm is on the boundary of the auxiliary workspace, as exemplified by the two robot configurations reported in Fig. 2.

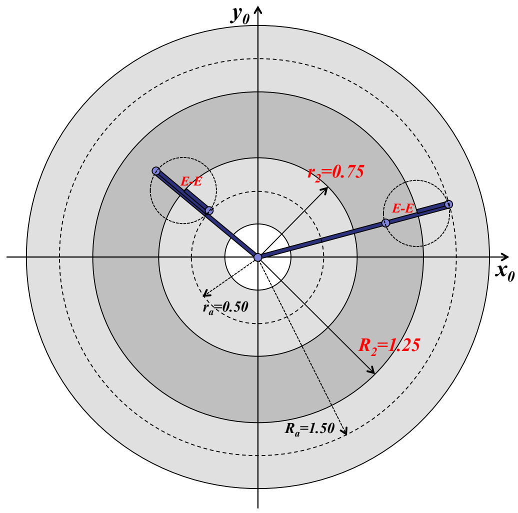


Figure 2: Secondary workspace of the planar 3R robot

Let $\boldsymbol{q} = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}^T$ be the joint configuration vector of this robot, with θ_i $(i = 1, 2, 3)$ defined according to the Denavit-Hartenberg convention. The direct kinematics is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} \\ \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} \end{pmatrix},$$

with the usual shorthand notations (e.g., $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$). The Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of interest in the velocity relation

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

is the (2×3) matrix obtained by differentiation of the direct kinematics:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123}) & -(\ell_2 s_{12} + \ell_3 s_{123}) & -\ell_3 s_{123} \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} & \ell_2 c_{12} + \ell_3 c_{123} & \ell_3 c_{123} \end{pmatrix} = \begin{pmatrix} j_{11}(\boldsymbol{q}) & j_{12}(\boldsymbol{q}) & j_{13}(\boldsymbol{q}) \\ j_{21}(\boldsymbol{q}) & j_{22}(\boldsymbol{q}) & j_{23}(\boldsymbol{q}) \end{pmatrix}.$$

The singularities of the Jacobian correspond to the zeroing of the manipulability index H. In order to obtain this index, we compute

$$\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}) = \begin{pmatrix} j_{11}^2(\boldsymbol{q}) + j_{12}^2(\boldsymbol{q}) + j_{13}^2(\boldsymbol{q}) & j_{11}(\boldsymbol{q})j_{21}(\boldsymbol{q}) + j_{12}(\boldsymbol{q})j_{22}(\boldsymbol{q}) + j_{13}(\boldsymbol{q})j_{23}(\boldsymbol{q}) \\ j_{11}(\boldsymbol{q})j_{21}(\boldsymbol{q}) + j_{12}(\boldsymbol{q})j_{22}(\boldsymbol{q}) + j_{13}(\boldsymbol{q})j_{23}(\boldsymbol{q}) & j_{21}^2(\boldsymbol{q}) + j_{22}^2(\boldsymbol{q}) + j_{23}^2(\boldsymbol{q}) \end{pmatrix}$$

and from this, after some trigonometric simplifications[1],

$$H=\sqrt{\det(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}))}=\sqrt{(\ell_1(\ell_2 s_2+\ell_3 s_{23}))^2+(\ell_3(\ell_1 s_{23}+\ell_2 s_3))^2+(\ell_2\ell_3 s_3)^2}\geq 0, \qquad (1)$$

which is only a function of q_2 and q_3, as expected (and required). A singularity occurs when all three addends under the square root are simultaneously zero[2]. This happens if and only if (proceeding from the last addend to the first)

$$s_3=0 \quad \text{AND} \quad s_{23}=0 \quad \text{AND} \quad s_2=0.$$

As a result, four types of singularities are present (at any q_1) for the pair $\{q_2,q_3\}$: $\{0,0\}$, $\{0,\pm\pi\}$, $\{\pm\pi,0\}$, and $\{\pm\pi,\pm\pi\}$. The robot is in a singular configuration whenever the links are stretched or folded along a ray starting from the origin. Note that the singularities remain the same for all (positive) values of the link lengths ℓ_i, $i=1,2,3$.

Exercise 2

The desired motion task can be formulated using the Cartesian vector $\boldsymbol{r}=\begin{pmatrix} p_x & p_y & \phi\end{pmatrix}^T\in\mathbb{R}^3$. However, the problem is best addressed in the joint space, since the bounds are directly defined in this space and there are no requirements on the Cartesian path to be followed by the end-effector during motion. Therefore, we will plan a joint trajectory such that the robot end-effector moves from the initial Cartesian task vector at time $t=0$,

$$\boldsymbol{r}_A=\boldsymbol{r}(0)=\begin{pmatrix}\boldsymbol{p}_A\\ \phi_A\end{pmatrix}=\begin{pmatrix}1.75\\0\\0\end{pmatrix},$$

to the final Cartesian task vector at time $t=T$,

$$\boldsymbol{r}_B=\boldsymbol{r}(T)=\begin{pmatrix}\boldsymbol{p}_B\\ \phi_B\end{pmatrix}=\begin{pmatrix}1.25\\0.5\\0\end{pmatrix}.$$

We use inverse kinematics to associate initial and final joint configurations to $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$. Note that the planar 3R robot is *not* redundant for a planar position and orientation task ($m=n=3$). With the given link lengths, there is a unique joint configuration $\boldsymbol{q}_A$ associated to $\boldsymbol{r}_A$,

$$\boldsymbol{q}_A=\begin{pmatrix}0\\0\\0\end{pmatrix}=\boldsymbol{q}(0),$$

[1] One can use the *Symbolic Toolbox* of Matlab to obtain this compact expression. However, simplifications by hand are easier in this case in view of the specific recurrent structure of the terms in $\boldsymbol{J}$. A convenient alternative way is to express the Jacobian in the reference frame attached to the first (or even second!) link. Let

$${}^0\boldsymbol{R}_1(q_1)=\begin{pmatrix}c_1 & -s_1\\ s_1 & c_1\end{pmatrix}$$

be the planar rotation matrix characterizing the orientation of the first frame with respect to the zero frame. It is easy to verify that ${}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q})={}^1\boldsymbol{J}(q_2,q_3)$ is independent from q_1. Moreover, this matrix can be used as well for the computation of H since

$$H=\sqrt{\det\left(\boldsymbol{J}\boldsymbol{J}^T\right)}=\sqrt{\det{}^0\boldsymbol{R}_1^T}\cdot\sqrt{\det\left(\boldsymbol{J}\boldsymbol{J}^T\right)}\cdot\sqrt{\det{}^0\boldsymbol{R}_1}=\sqrt{\det\left({}^0\boldsymbol{R}_1^T\boldsymbol{J}\right)\left(\boldsymbol{J}^T\,{}^0\boldsymbol{R}_1\right)}=\sqrt{\det\left({}^1\boldsymbol{J}\;{}^1\boldsymbol{J}^T\right)}.$$

[2] The three terms are the square of the minors $(1,2)$, $(1,3)$, and $(2,3)$ of matrix $\boldsymbol{J}$. In fact, the singularities can be found equivalently by imposing the simultaneous zeroing of these three minors.

i.e., at $t = 0$ the arm is fully stretched along the x_0 axis. On the other hand, there are two inverse kinematic solutions associated to $\boldsymbol{r}_B$, namely[3]

$$\boldsymbol{q}'_B = \begin{pmatrix} 0 \\ 90° \\ -90° \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{q}''_B = \begin{pmatrix} 2\arctan 0.5 \\ -\dfrac{\pi}{2} \\ -\left(2\arctan 0.5 - \dfrac{\pi}{2}\right) \end{pmatrix} [\text{rad}] = \begin{pmatrix} 53.13° \\ -90° \\ 36.87° \end{pmatrix},$$

which can be chosen in alternative as the final desired $\boldsymbol{q}(T)$. Therefore, we should plan two minimum time trajectories, one from $\boldsymbol{q}_A$ to $\boldsymbol{q}'_B$ and another from $\boldsymbol{q}_A$ to $\boldsymbol{q}''_B$, and then choose the fastest one.

The motion task is rest-to-rest, and thus we should have $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \mathbf{0}$. Since joint coordination is not required (the joints can complete their displacements at different time instants), the solution is obtained by planning the fastest possible motion independently for each joint. In view of the presence of joint velocity and acceleration bounds, the optimal solution will have a bang-coast-bang acceleration profile (or it is a sub-case of this) for all joints.

For a generic joint i, let T_i be the minimum motion time, $\Delta_i = |q_i(T_i) - q_i(0)|$ the required displacement (in absolute value), and $T_{s,i}$ the duration of the acceleration (and, symmetrically, of the deceleration) phase. We have the known relations

$$T_{s.i} = \frac{V_i}{A_i}, \qquad T_i = \frac{\Delta_i}{V_i} + \frac{V_i}{A_i} = \frac{\Delta_i A_i + V_i^2}{A_i V_i}, \tag{2}$$

which are valid provided that

$$\Delta_i \geq \frac{V_i^2}{A_i} \qquad \Leftrightarrow \qquad T_I \geq 2T_{s,i}. \tag{3}$$

When strict inequalities hold in (3), we denote this as *case a* (the standard one). When the relations (3) hold as equalities (*case b*), the coast phase collapses and the acceleration profile will be a bang-bang one: the absolute velocity reaches its maximum value V_i only at the midpoint of the trajectory. When the relations (3) are violated (*case c*), a specific motion profile with symmetric phases at maximum acceleration/deceleration and no coast phase can be used: a peak (absolute) velocity $\bar{V}_i$ will be reached only at the midpoint of the trajectory, with $\bar{V}_i < V_i$. The value of $\bar{V}_i$ is found from

$$\Delta_i = \frac{\bar{V}_i^2}{A_i} \qquad \Rightarrow \qquad \bar{V}_i = \sqrt{\Delta_i A_i},$$

which expresses the fact that the area of the triangular velocity profile should be equal to the required displacement Δ_i. In this situation, the formulas (2) should be replaced by

$$T_{s.i} = \frac{\bar{V}_i}{A_i} = \sqrt{\frac{\Delta_i}{A_i}}, \qquad T_i = 2T_{s,i} = 2\sqrt{\frac{\Delta_i}{A_i}}. \tag{4}$$

In particular, the expressions (4) hold also for the limit situation of a joint requiring no displacement ($\Delta_i = 0$): its minimum motion time T_i will be equal to 0 (*case d*). No matter which case occurs, the minimum time T for completing the desired robot motion task will always be the largest of the computed T_i's, or

$$T = \max\{T_1, T_2, T_3\}.$$

[3]These are obtained by applying the formulas of Sect. 2.12.1 of the textbook. In this case, however, both solutions can be found by simple geometric inspection, noting also that it should be $\phi_B = q_{1,B} + q_{2,B} + q_{3,B} = 0$.

For the present problem, the above computations are repeated for the two possible robot displacements in the joint space, leading to T' and T''.

The above developments are applied by taking into account the numerical values of V_i and A_i $(i = 1, 2, 3)$, and the absolute displacements Δ_i $(i = 1, 2, 3)$ for the two different joint trajectories. For the joint trajectory from $\boldsymbol{q}_A$ to $\boldsymbol{q}'_B$, it s

$$T' = \max\{0, 1.5, 1.75\} = 1.75\,[\mathrm{s}].$$

Note that *case d* applies to the motion of joint 1, *case b* to the motion of joint 2, and *case a* to the motion of joint 3. For the joint trajectory from $\boldsymbol{q}_A$ to $\boldsymbol{q}''_B$, it is

$$T'' = \max\{1.1903, 1.5, 0.8645\} = 1.5\,[\mathrm{s}].$$

Here, *case c* applies to the motion of joint 1, *case b* to the motion of joint 2, and *case a* to the motion of joint 3. Therefore, the minimum time trajectory satisfying the task goes from $\boldsymbol{q}_A$ (stretched arm) to $\boldsymbol{q}''_B$ and the associated minimum time is $T = \min\{T', T''\} = 1.5$ s.

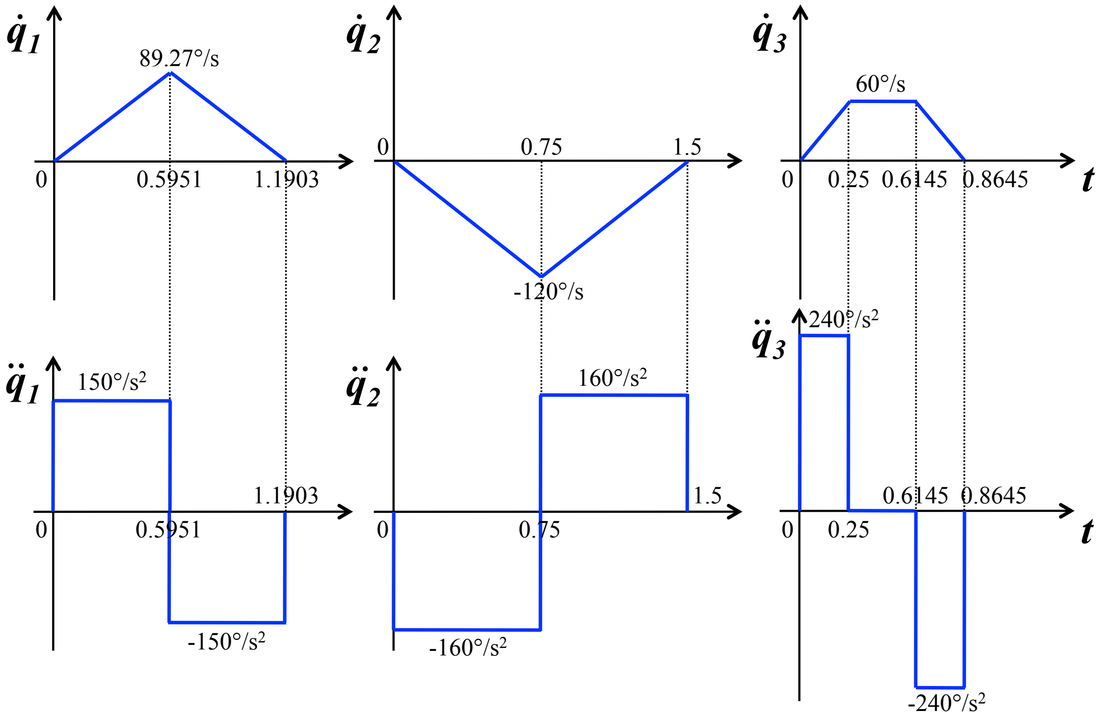


Figure 3: Velocity (top) and acceleration (bottom) profiles for joints 1 to 3 (from left to right)

For the minimum time trajectory thus defined, the velocity and acceleration profiles of the three joints are sketched in Fig. 3. From these, it can be easily checked that the orientation of the end-effector does not remain not constant during the resulting motion (i.e., it is not kept fixed to $\phi = 0$). For the orientation to be constant during motion, the joint velocities should add up to zero at any instant $t \in [0, T]$, which is not the case here.

Finally, the obtained trajectory is only one of the infinitely many that complete the given task in minimum time. In fact, we could add some extra motion to joints 1 and 3, still reaching their desired final values with zero velocity before or at $T = 1.5$ [s].

Bonus Exercise

The following Matlab code plots the 3D profile of the manipulability index H in Fig. 4.

```
% input data (link lengths)

l1=1; l2=0.5; l3=0.25;

% discretization of joint angles (in rads)

delta=0.02; q2=[-pi:delta:pi]; q3=[-pi:delta:pi];

% evaluation of manipulability index H

for J = 1:length(q2), for K = 1:length(q3),
H(J,K)=sqrt((l1*(l2*sin(q2(J))+l3*sin(q2(J)+q3(K))))^2 ...
+(l3*(l1*sin(q2(J)+q3(K))+l2*sin(q3(K))))^2 ...
+(l2*l3*sin(q3(K)))^2 );
end; end;

% mesh plot

[X,Y]=meshgrid(q2,q3);mesh(Y,X,H); % note the reverse order of arguments!
title('Manipulability index H of planar 3R robot');
xlabel('q2');ylabel('q3');zlabel('sqrt (det J*JT)');
```

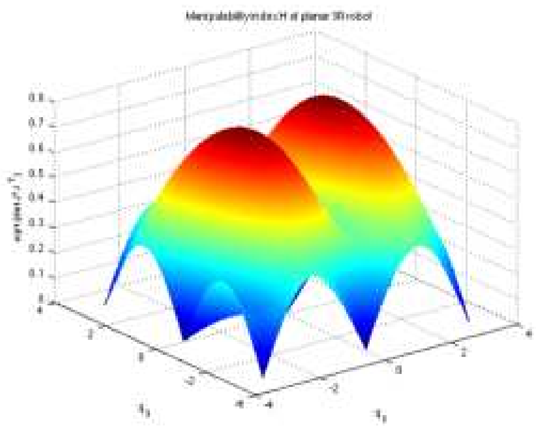


Figure 4: 3D profile of the manipulability index H

An additional instruction provides a plot of the level curves H (see Fig. 5).

```
% contour plot

contour(Y,X,H,30); % with 30 level curves (note the reverse order of arguments!)
title('Manipulability index of planar 3R robot');
xlabel('q2');ylabel('q3');
```

From Figs. 4 and 5, it follows that two (symmetric) configurations provide the maximum value H^*. In fact, it is easy to check from (1) that $H(q_2, q_3) = H(-q_2, -q_3)$ for any pair (q_2, q_3). Also, we can verify graphically the zeroing of H at the singular configurations already found analytically.

The optimal value H^* and the associated configuration (q_2^*, q_3^*) can be computed (up to the chosen discretization of 0.02 rad) by the following piece of code.

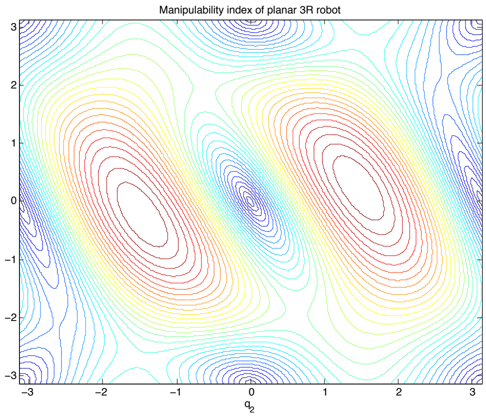


Figure 5: Level curves of the manipulability index H

```
% find global maximum value of H and associated configuration (q2,q3)

maxH=0;indexJ=0; indexK=0;

for J = 1:length(q2),
[maxHJ, Kmax]=max(H(J,:)');
if maxHJ > maxH, maxH=maxHJ; indexJ=J; indexK=Kmax; end;
end;

globalmaxH=maxH;
q2maxH=q2(indexJ); % in rads
q3maxH=q3(indexK); % in rads
```

This provides an (approximately) optimal value $H^* = 0.79$, which is obtained at the configuration $(q_2^*, q_3^*) = (1.48, 0.24)$ [rad] $\approx (84.8°, 13.7°)$. For illustration, this optimal configuration (in red) and the symmetric one (in blue) are shown in Fig. 6, choosing the value $q_1 = 0$ (otherwise arbitrary). We remark that the optimal configuration changes for a different set of link lengths. Only when all link lengths are equal (say, to ℓ), the optimal configuration would not depend on ℓ (while the associated value H^* would have a scaling factor ℓ^2).

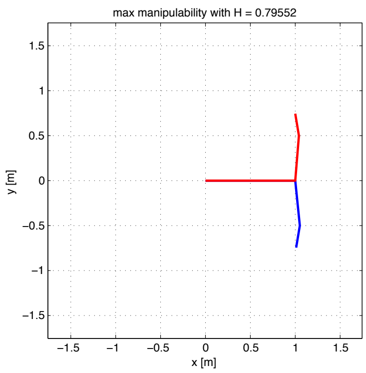


Figure 6: Optimal configurations for the manipulability index H (shown at $q_1 = 0$)

* * * * *

Robotics I

February 9, 2012

Exercise 1

Consider the non-spherical wrist of the Comau Smart5 NJ4 170 robot, i.e., the last three revolute joints of this 6R structure (see Fig. 1). The associated Denavit-Hartenberg parameters are given in the three rows of Tab. 1. Note that α, d_4, and d_5 are all positive constants, and d_6 has been set to zero.

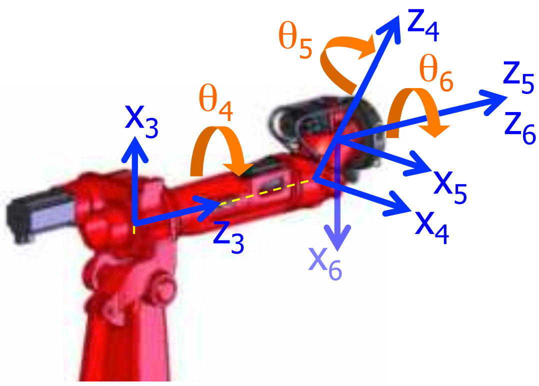


Figure 1: The Comau Smart5 NJ4 170 robot and its last three joints constituting a non-spherical wrist (with DH frames)

i	α_i	a_i	d_i	θ_i
4	α	0	d_4	θ_4
5	$-\alpha$	0	d_5	θ_5
6	0	0	0	θ_6

Table 1: DH parameters of the robot wrist

Provide the explicit relation between the joint velocity $\dot{\boldsymbol{\theta}} = \begin{pmatrix} \dot{\theta}_4 & \dot{\theta}_5 & \dot{\theta}_6 \end{pmatrix}^T$ and the angular velocity ${}^3\boldsymbol{\omega}_e$ of the end-effector frame (labeled as 6 in Fig. 1) expressed in frame 3. Also, analyze the singularities of this differential relation.

Exercise 2

Consider the RRP (polar) robot in Fig. 2, where $d_1 = 1$, and assume that the coordinate q_3 associated to the third (prismatic) joint can only take non-negative values.

- Assign the frames according to the *Denavit-Hartenberg convention* and complete the associated table of parameters. Choose the reference axes so that $\alpha_i \geq 0$, for $i = 1, 2, 3$, and set the origin of the last frame at the end-effector/tip of the robot.

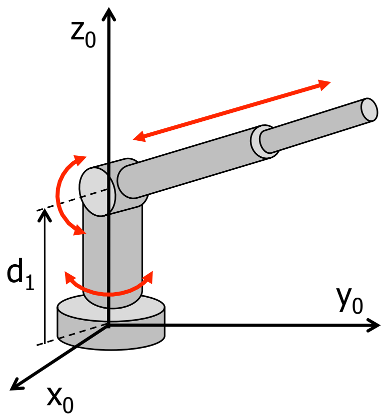


Figure 2: A RRP (polar) robot

- Give the explicit expression of the 3×3 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity $\boldsymbol{v}_e$ of the end-effector

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

  and discuss its singularities.

Exercise 3

Consider the same robot of Exercise 2.

- Define a desired linear path $\boldsymbol{p}_d(\sigma)$, parametrized by its actual length σ, between the initial and final Cartesian points

$$\boldsymbol{p}_{init} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \qquad \boldsymbol{p}_{fin} = \begin{pmatrix} -1 & -1 & 3 \end{pmatrix}^T.$$

  Verify the existence of a value $\sigma = \sigma_s$ at which the desired path encounters a robot singularity. Determine whether or not the desired trajectory $\boldsymbol{p}_d(\sigma(t))$, with $\dot{\sigma} > 0$ at $\sigma = \sigma_s$, can be *perfectly* realized also in that robot configuration. In case it can, provide some reasoning to justify how to execute the desired trajectory; else, explain in detail why this is not possible.

- Consider the same initial point $\boldsymbol{p}_{init}$ and a new final point $\boldsymbol{p}_{new} = \begin{pmatrix} -1 & 1 & 3 \end{pmatrix}^T$, and assume as desired interpolating trajectory $\boldsymbol{p}_d(\sigma(t))$ a linear Cartesian path with constant speed $\dot{\sigma} = 1$. The robot is initially in the configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/3 & \pi/2 & 1 \end{pmatrix}^T$. Design *two* different kinematic control laws that exponentially drive the tracking error to zero, either
  - by keeping the Cartesian error along $\boldsymbol{z}_0$ constantly at zero,

  or, respectively,
  - by keeping the joint error on the second coordinate q_2 constantly at zero.

[210 minutes; open books]

Solutions

February 9, 2012

Exercise 1

A first way to solve the problem is to use the 3×3 geometric Jacobian $\boldsymbol{J}_A(\boldsymbol{\theta})$ associated to the angular velocity $\boldsymbol{\omega}_e$ of the end-effector frame,

$$\boldsymbol{\omega}_e = \boldsymbol{J}_A(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}}.$$

Since the considered last three joints of the robot are all revolute and computations have to be performed w.r.t. the robot frame 3, we have

$$\begin{aligned} {}^3\boldsymbol{J}_A(\boldsymbol{\theta}) &= \begin{pmatrix} {}^3\boldsymbol{z}_3 & {}^3\boldsymbol{z}_4(\theta_4) & {}^3\boldsymbol{z}_5(\theta_4,\theta_5) \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^3\boldsymbol{R}_4(\theta_4) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^3\boldsymbol{R}_4(\theta_4)\,{}^4\boldsymbol{R}_5(\theta_5) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \end{aligned} \tag{1}$$

with rotation matrices obtained from the DH table of parameters (using $s_{-\alpha} = -s_\alpha$ and $c_{-\alpha} = c_\alpha$):

$$\begin{aligned} {}^3\boldsymbol{R}_4(\theta_4) &= \begin{pmatrix} c_4 & -s_4 c_\alpha & s_4 s_\alpha \\ s_4 & c_4 c_\alpha & -c_4 s_\alpha \\ 0 & s_\alpha & c_\alpha \end{pmatrix} \\ {}^4\boldsymbol{R}_5(\theta_5) &= \begin{pmatrix} c_5 & -s_5 c_\alpha & -s_5 s_\alpha \\ s_5 & c_5 c_\alpha & c_5 s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{pmatrix} \\ {}^3\boldsymbol{R}_4(\theta_4)\,{}^4\boldsymbol{R}_5(\theta_5) &= \begin{pmatrix} * & * & (1-c_5)\,s_4 s_\alpha c_\alpha - c_4 s_5 s_\alpha \\ * & * & (c_5-1)\,c_4 s_\alpha c_\alpha - s_4 s_5 s_\alpha \\ * & * & c_5 s_\alpha^2 + c_\alpha^2 \end{pmatrix}. \end{aligned}$$

Above, a $*$ denotes quantities that need not to be computed. Note also that the values of d_4 and d_5 (as well as d_6, if present) are irrelevant. Substituting in (1) yields

$${}^3\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} 0 & s_4 s_\alpha & (1-c_5)\,s_4 s_\alpha c_\alpha - c_4 s_5 s_\alpha \\ 0 & -c_4 s_\alpha & (c_5-1)\,c_4 s_\alpha c_\alpha - s_4 s_5 s_\alpha \\ 1 & c_\alpha & c_5 s_\alpha^2 + c_\alpha^2 \end{pmatrix}.$$

This matrix has determinant

$$\det\left({}^3\boldsymbol{J}_A(\boldsymbol{\theta})\right) = -s_5\, s_\alpha^2$$

being thus singular for $\theta_5 = 0$ (and $\theta_5 = \pi$, but this is likely to be out the admissible range of this joint). Since the rank of

$${}^3\boldsymbol{J}_A(\boldsymbol{\theta})\big|_{\theta_5=0} = \begin{pmatrix} 0 & s_4 s_\alpha & 0 \\ 0 & -c_4 s_\alpha & 0 \\ 1 & c_\alpha & 1 \end{pmatrix}$$

is equal to 2, an angular vector ${}^3\boldsymbol{\omega}_e$ of the form

$${}^3\boldsymbol{\omega}_e = \beta \begin{pmatrix} c_4 \\ s_4 \\ * \end{pmatrix} \notin \mathcal{R}\left({}^3\boldsymbol{J}_A(\boldsymbol{\theta})|_{\theta_5=0}\right), \qquad \beta \neq 0$$

cannot be realized in this configuration by any choice of $\dot{\boldsymbol{\theta}}$.

A second way to address the problem would be to use the relation between the derivative of a rotation matrix and the associated angular velocity

$$\boldsymbol{S}(\boldsymbol{\omega}_e) = {}^3\dot{\boldsymbol{R}}_6\,{}^3\boldsymbol{R}_6^T, \qquad \text{with } {}^3\boldsymbol{R}_6 = {}^3\boldsymbol{R}_4(\theta_4)\,{}^4\boldsymbol{R}_5(\theta_4)\,{}^5\boldsymbol{R}_6(\theta_6)$$

where $\boldsymbol{S}(\cdot)$ is the skew-symmetric matrix built from the components of $\boldsymbol{\omega}_e$. However, this requires much more computations, in particular the evaluation of all three DH rotation matrices, and their complete product and derivation w.r.t. time (to be performed symbolically).

Exercise 2

A DH frame assignment satisfying the stated requirements is shown in Fig. 3, with the associated parameters given in Tab. 2. Note that the third link is horizontal when $q_2 = \pi/2$.

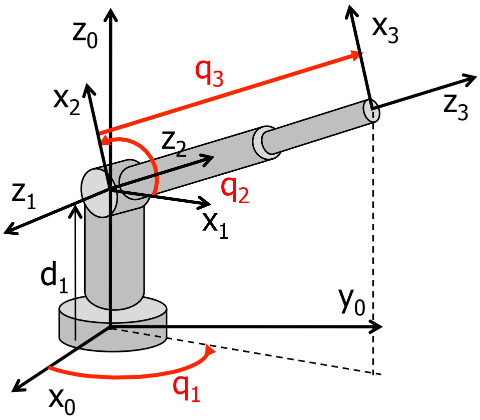


Figure 3: DH frame assignment for the RRP robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	$\pi/2$	0	0	q_2
3	0	0	q_3	0

Table 2: DH parameters for the RRP robot

The end-effector/tip position (i.e, the origin of frame 3) is then obtained as

$$\boldsymbol{p}_e = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} c_1 s_2\, q_3 \\ s_1 s_2\, q_3 \\ d_1 - c_2\, q_3 \end{pmatrix}. \tag{2}$$

The requested Jacobian matrix relating $\dot{\boldsymbol{q}}$ to $\boldsymbol{v}_e = \dot{\boldsymbol{p}}_e$ can be computed either geometrically or by analytic differentiation of (2), yielding the same result in both cases. We have

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 s_2\, q_3 & c_1 c_2\, q_3 & c_1 s_2 \\ c_1 s_2\, q_3 & s_1 c_2\, q_3 & s_1 s_2 \\ 0 & s_2\, q_3 & -c_2 \end{pmatrix}. \tag{3}$$

For later use, we can write the Jacobian in frame 1, using the rotation matrix (from the DH table)

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the expression on the right shows that this is composed by an elementary rotation around the $\boldsymbol{z}_0$ axis and a permutation of axes (preserving the right-hand rule for frames). We compute thus

$${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & c_2\, q_3 & s_2 \\ 0 & s_2\, q_3 & -c_2 \\ -s_2\, q_3 & 0 & 0 \end{pmatrix}.$$

The determinant of $\boldsymbol{J}(\boldsymbol{q})$, equal to that of ${}^1\boldsymbol{J}(\boldsymbol{q})$, is

$$\det \boldsymbol{J}(\boldsymbol{q}) = s_2\, q_3^2.$$

Therefore, the robot is in a singular configuration whenever $q_3 = 0$ or $q_2 \in \{0, \pi\}$. In the first case, the rank of the Jacobian drops to 1, whereas in the second case the rank is 2. When both singularity conditions hold true, the rank of the Jacobian is still equal to 1. In particular, when $s_2 = 0$ the end-effector position $\boldsymbol{p}_e$ is placed on the axis $\boldsymbol{z}_0$ of joint 1 and the Jacobian ${}^1\boldsymbol{J}(\boldsymbol{q})$ becomes

$${}^1\boldsymbol{J}(\boldsymbol{q})\big|_{s_2=0} = \begin{pmatrix} 0 & \pm q_3 & 0 \\ 0 & 0 & \mp 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This clearly shows that, when the end-effector is on the $\boldsymbol{z}_0$ axis, any desired Cartesian velocity vector lying only in the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$, i.e., with zero component along the $\boldsymbol{z}_1$ axis, belongs to the range space of the Jacobian, and thus can be perfectly realized by the robot. Velocity vectors $\boldsymbol{v}_e \in \mathcal{R}\left(\boldsymbol{J}(\boldsymbol{q})|_{s_2=0}\right)$ have the form

$${}^1\boldsymbol{v}_e = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix} \quad \Rightarrow \quad {}^0\boldsymbol{v}_e = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{v}_e = \begin{pmatrix} \beta_1 c_1 \\ \beta_1 s_1 \\ \beta_2 \end{pmatrix} \tag{4}$$

when expressed, respectively, in frame 1 or in frame 0, and for arbitrary β_1 and β_2. Finally, note that also the (double) singularity $q_3 = 0$ corresponds to a situation in which the end-effector is on the $\boldsymbol{z}_0$ axis.

Exercise 3

The linear path is parametrized as follows:

$$\boldsymbol{p}_d(\sigma) = \boldsymbol{p}_{init} + \frac{\sigma}{L}\left(\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right), \qquad L = \left\|\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right\| = \sqrt{12}$$

or

$$\boldsymbol{p}_d(\sigma) = \begin{pmatrix} 1 - \dfrac{2\sigma}{\sqrt{12}} \\ 1 - \dfrac{2\sigma}{\sqrt{12}} \\ 1 + \dfrac{2\sigma}{\sqrt{12}} \end{pmatrix}, \qquad \sigma \in [0, L].$$

It is easy to see that this path will cross the $\boldsymbol{z}_0$ axis for

$$\sigma = \sigma_s = \frac{\sqrt{12}}{2} \qquad \Rightarrow \qquad p_{d,x}(\sigma_s) = p_{d,y}(\sigma_s) = 0$$

so that a singularity of the RRP robot is encountered. The desired velocity along the path has indeed a constant direction

$$\dot{\boldsymbol{p}}_d(t) = \frac{\dot{\sigma}(t)}{L}\left(\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right) = \frac{\dot{\sigma}(t)}{\sqrt{12}} \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}. \tag{5}$$

In particular, at $\sigma = \sigma_s$, this Cartesian velocity is still a feasible one for the RRP robot, since (5) can be written in the form (4) of a vector $\boldsymbol{v}_e \in \mathcal{R}(\boldsymbol{J})$ by setting, e.g.,

$$q_1 = \frac{\pi}{4} \quad \rightarrow \quad c_1 = s_1 = \frac{\sqrt{2}}{2}, \qquad \beta_1 = \frac{-2\,\dot{\sigma}}{\sqrt{6}}, \qquad \beta_2 = \frac{2\,\dot{\sigma}}{\sqrt{12}}.$$

This implies that the first joint should be rotated so that the whole linear path belongs to the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$. If the robot initial configuration is set at $q_1 = \pi/4$, then the entire desired trajectory can be realized by using only joint 2 and 3 of the RRP robot (namely, by the planar RP robot obtained by freezing the first joint) while the first joint does not need to move. This reasoning suggests also that the inversion of the 3×3 Jacobian in (3), which would run into problems close to or crossing the singularity $s_2 = 0$, can be completely avoided.

In fact, consider the 2×2 top right sub-matrix of the Jacobian ${}^1\boldsymbol{J}(\boldsymbol{q})$, i.e.,

$${}^1\bar{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} c_2\,q_3 & s_2 \\ s_2\,q_3 & -c_2 \end{pmatrix}$$

and note that this matrix is never singular, provided that $q_3 \neq 0$ (so, it is independent from the value of q_2). By defining $\dot{\bar{\boldsymbol{q}}} = \begin{pmatrix} \dot{q}_2 & \dot{q}_3 \end{pmatrix}^T$ and ${}^1\dot{\bar{\boldsymbol{p}}} = \begin{pmatrix} {}^1\dot{p}_x & {}^1\dot{p}_y \end{pmatrix}^T$, the following differential relation holds

$${}^1\dot{\bar{\boldsymbol{p}}} = {}^1\bar{\boldsymbol{J}}(\boldsymbol{q})\dot{\bar{\boldsymbol{q}}}.$$

Express now the desired Cartesian velocity in frame 1, for a constant $q_1 = \pi/4$

$${}^1\dot{\boldsymbol{p}}_d = {}^0\boldsymbol{R}_1^T(q_1)\Big|_{q_1=\frac{\pi}{4}} \cdot \dot{\boldsymbol{p}}_d = \frac{\dot{\sigma}}{\sqrt{12}} \begin{pmatrix} \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \frac{\dot{\sigma}}{\sqrt{12}} \begin{pmatrix} -2\sqrt{2} \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} {}^1\dot{\bar{\boldsymbol{p}}}_d \\ 0 \end{pmatrix}.$$

Then, the entire desired Cartesian trajectory will be *perfectly* executed if the robot starts at $t = 0$ in the configuration

$$\boldsymbol{q}(0) = \begin{pmatrix} q_1(0) \\ \bar{\boldsymbol{q}}(0) \end{pmatrix} = \begin{pmatrix} \dfrac{\pi}{4} \\ \dfrac{\pi}{2} \\ \sqrt{2} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0) = \boldsymbol{p}_{init} \tag{6}$$

(which is one of the two solutions found by solving the inverse kinematics problem, taking into account that only $q_3 \geq 0$ is allowed) by setting, for all $t \geq 0$,

$$\begin{aligned} q_1(t) &= \frac{\pi}{4} \\ \bar{\boldsymbol{q}}(t) &= \bar{\boldsymbol{q}}(0) + \int_0^t {}^1\bar{\boldsymbol{J}}^{-1}(\boldsymbol{q}(\tau))\, {}^1\dot{\bar{\boldsymbol{p}}}_d(\sigma(\tau))d\tau\,. \end{aligned} \tag{7}$$

The total motion time T will depend from the time profile of $\sigma(t)$, under the minimal necessary boundary conditions $\sigma(0) = 0$ and $\sigma(T) = L$. Accordingly, we will obtain from (7) a final value $\boldsymbol{q}(T)$ such that $\boldsymbol{f}(\boldsymbol{q}(T)) = \boldsymbol{p}_{fin}$.

For the second part of this Exercise, we have $L = \|\boldsymbol{p}_{new} - \boldsymbol{p}_{init}\| = \sqrt{8}$ and the new desired trajectory $\boldsymbol{p}_d(\sigma(t))$ is given by the linear geometric path

$$\boldsymbol{p}_d(\sigma) = \boldsymbol{p}_{init} + \frac{\sigma}{L}\left(\boldsymbol{p}_{new} - \boldsymbol{p}_{init}\right) = \begin{pmatrix} 1 - \dfrac{2\sigma}{\sqrt{8}} \\ 1 \\ 1 + \dfrac{2\sigma}{\sqrt{8}} \end{pmatrix}, \qquad \sigma \in [0, \sqrt{8}],$$

with the timing law $\sigma = \sigma(t)$. Since $\dot{\sigma}(t) = 1$ (constant) is assigned, we have $\sigma(t) = t$, with $t \in [0, T]$ and $T = L = \sqrt{8}$. At the given initial configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/3 & \pi/2 & 1 \end{pmatrix}^T$, it follows from (2)

$$\boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boldsymbol{p}_{init} = \boldsymbol{p}_d(0)$$

so that there is an initial position error w.r.t. the desired Cartesian trajectory. However, the initial error along the $\boldsymbol{z}_0$ axis is zero. In order to be matched with $\boldsymbol{p}_d(0)$, the initial configuration of the RRP robot should be instead, e.g.,

$$\boldsymbol{q}_d(0) = \begin{pmatrix} \dfrac{\pi}{4} & \dfrac{\pi}{2} & \sqrt{2} \end{pmatrix}^T \tag{8}$$

as verified in (6). Indeed, also the initial joint position error is different from zero, but note that we have now $q_2(0) = q_{d,2}(0)$. Moreover, if the robot starts from the desired initial configuration (8), we could generate an entire desired joint trajectory $\boldsymbol{q}_d(t)$ associated to $\boldsymbol{p}_d(t)$ as

$$\boldsymbol{q}_d(t) = \boldsymbol{q}_d(0) + \int_0^t \dot{\boldsymbol{q}}_d(\tau)\, d\tau = \boldsymbol{q}_d(0) + \int_0^t \boldsymbol{J}^{-1}(\boldsymbol{q}_d(\tau))\, \dot{\boldsymbol{p}}_d(\tau)\, d\tau \tag{9}$$

since the Jacobian never encounters a singularity in this case.

With the above in mind, in order to recover the initial error and asymptotically track the desired Cartesian trajectory, a feedback/feedforward control law has to be designed at the kinematic level (i.e., considering $\dot{\boldsymbol{q}}$ as the control input). Depending on the additional requirements, the first solution is a law driven by the *Cartesian error* $\boldsymbol{e}_c = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$, namely

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_c\,(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}))\big), \qquad \text{with } \boldsymbol{K}_c > 0 \text{ and diagonal.}$$

This will force each component of the Cartesian error $\boldsymbol{e}_c$ to converge exponentially to zero with a rate prescribed by the associated diagonal element of $\boldsymbol{K}_c$, or $e_{c,i}(t) = \exp(-K_{c,i}\,t)\,e_{c,i}(0)$. As a consequence, the error along the $\boldsymbol{z}_0$ component (i.e, for $i = 3$) will remain constantly zero also during the transient phase of the trajectory tracking task ($e_{c,3}(0) = 0 \rightarrow e_{c,3}(t) \equiv 0$), so that the first requested behavior is obtained.

On the other hand, the second solution is a law driven by the *joint error* $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, with $\boldsymbol{q}_d(t)$ given by (9), namely

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_d + \boldsymbol{K}\,(\boldsymbol{q}_d - \boldsymbol{q})\,, \qquad \text{with } \boldsymbol{K} > 0 \text{ and diagonal.}$$

This will force each component of the joint error $\boldsymbol{e}$ to converge exponentially to zero with a rate prescribed by the associated diagonal element of $\boldsymbol{K}$, or $e_i(t) = \exp(-K_i\,t)\,e_i(0)$. Thus, the error on the second component of the joint configuration vector (i.e., on q_2) will remain constantly zero ($e_2(0) = 0 \rightarrow e_2(t) \equiv 0$) and the second requirement is satisfied.

* * * * *

Robotics I

April 26, 2012

Exercise 1

Consider the 3R robot in Fig. 1, with the associated Denavit-Hartenberg parameters of Tab. 1. An extra frame is shown on the robot end-effector, representing the typical frame associated to an eye-in-hand camera.

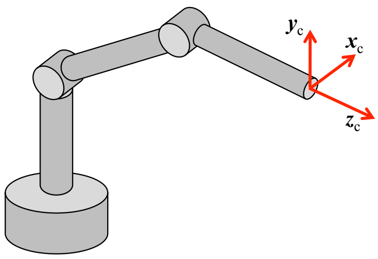


Figure 1: A 3R robot

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	L_1	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Table 1: Table of DH parameters

- Draw on the figure the Denavit-Hartenberg frames specified by Tab. 1.
- Derive the explicit expression of the 3×3 Jacobian ${}^c\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity ${}^c\boldsymbol{v}$ of the origin of the camera frame *expressed in the camera frame* as
$$ {}^c\boldsymbol{v} = {}^c\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. $$

Exercise 2

Let two absolute orientations ${}^0\boldsymbol{R}_i$ (initial) and ${}^0\boldsymbol{R}_f$ (final) be assigned through their minimal representation with the (Z, X, Y) Euler angles:
$$ \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4} & -\frac{\pi}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha_f & \beta_f & \gamma_f \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2} & 0 & \frac{\pi}{2} \end{pmatrix}. $$

- Design a rest-to-rest orientation trajectory that joins ${}^0\boldsymbol{R}_i$ to ${}^0\boldsymbol{R}_f$ in time $T = 1.5$ s using the *axis-angle method* and a cubic polynomial as timing law.
- Provide the expression of the orientation ${}^0\boldsymbol{R}(t)$ at a generic instant $t \in (0, T)$ of the planned motion and the associated angular velocity ${}^0\boldsymbol{\omega}(t)$, both expressed in the absolute reference frame.
- What is the maximum value ω_{max} of the norm of the angular velocity ${}^0\boldsymbol{\omega}(t)$ for $t \in [0, T]$?

[180 minutes; open books & software]

Solution

April 26, 2012

Exercise 1

The correct frame assignment is shown in Fig. 2, where the second and third joint as well as the second link are illustrated in transparency for better clarity.

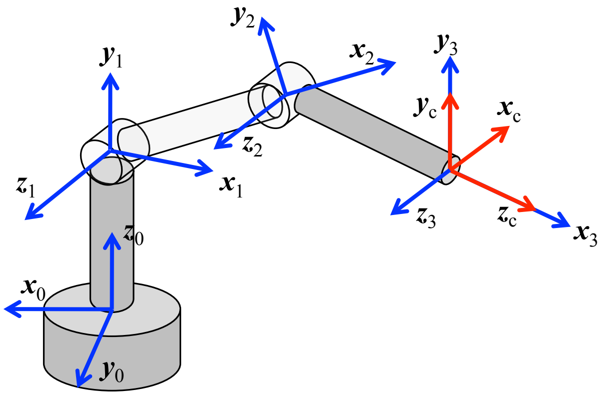


Figure 2: The DH frames for the 3R robot

For later use, we can see that the constant rotation from the end-effector to the camera frame is given by

$$^3\boldsymbol{R}_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Moreover, from the DH table we can build the homogenous transformation matrices $^0\boldsymbol{A}_1(q_1)$, $^1\boldsymbol{A}_2(q_2)$, and $^2\boldsymbol{A}_3(q_3)$ containing the rotation matrices

$$^0\boldsymbol{R}_1 = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix} \; ^1\boldsymbol{R}_2 = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ^2\boldsymbol{R}_3 = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that will be needed in the following.

The position $\boldsymbol{p}$ of the origin O_3 of frame 3 can be computed (in homogeneous coordinates) as

$$\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)^1\boldsymbol{A}_2(q_2)^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

yielding

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ \sin q_1 (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ L_1 + L_2 \sin q_2 + L_3 \sin(q_2 + q_3) \end{pmatrix}.$$

The Jacobian related to the linear velocity ${}^0\boldsymbol{v}$ $(= {}^0\boldsymbol{v}_3)$ of the origin of frame 3 and expressed in the base frame is obtained as

$$
{}^0\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} =
$$

$$
\begin{pmatrix} -\sin q_1(L_2\cos q_2 + L_3\cos(q_2+q_3)) & -\cos q_1(L_2\sin q_2 + L_3\sin(q_2+q_3)) & -L_3\cos q_1\sin(q_2+q_3)) \\ \cos q_1(L_2\cos q_2 + L_3\cos(q_2+q_3)) & -\sin q_1(L_2\sin q_2 + L_3\sin(q_2+q_3)) & -L_3\sin q_1\sin(q_2+q_3)) \\ 0 & L_2\cos q_2 + L_3\cos(q_2+q_3) & L_3\cos(q_2+q_3) \end{pmatrix}.
$$

The requested Jacobian ${}^c\boldsymbol{J}(\boldsymbol{q})$ that relates $\dot{\boldsymbol{q}}$ to ${}^c\boldsymbol{v}$ $(= {}^c\boldsymbol{v}_3)$ is obtained by applying suitable rotation matrices:

$$
\begin{aligned}
{}^c\boldsymbol{J}(\boldsymbol{q}) &= {}^0\boldsymbol{R}_c^T(\boldsymbol{q})\,{}^0\boldsymbol{J}(\boldsymbol{q}) = {}^3\boldsymbol{R}_c^T\left({}^2\boldsymbol{R}_3^T(q_3)\left({}^1\boldsymbol{R}_2^T(q_2)\left({}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{J}(\boldsymbol{q})\right)\right)\right) \\
&= \begin{pmatrix} L_2\cos q_2 + L_3\cos(q_2+q_3) & 0 & 0 \\ 0 & L_3 + L_2\cos q_3 & L_3 \\ 0 & L_2\sin q_3 & 0 \end{pmatrix}.
\end{aligned}
$$

The following is a symbolic Matlab script performing intermediate and final computations.

```
clear all
clc
syms L1 L2 L3 q1 q2 q3 alfa d a theta pi real
% DH parameters
alfa1=pi/2; alfa2=0; alfa3=0; d1=L1; d2=0; d3=0; a1=0; a2=L2; a3=L3;
% DH homogeneous matrix
A=[cos(theta) -sin(theta)*cos(alfa) sin(theta)*sin(alfa) a*cos(theta);
sin(theta) cos(theta)*cos(alfa) -cos(theta)*sin(alfa) a*sin(theta);
0 sin(alfa) cos(alfa) d;
0 0 0 1];
% evaluations
A1=subs(A, {alfa,d,a,theta}, {alfa1,d1,a1,q1})
A2=subs(A, {alfa,d,a,theta}, {alfa2,d2,a2,q2})
A3=subs(A, {alfa,d,a,theta}, {alfa3,d3,a3,q3})
R1=A1(1:3,1:3); R2=A2(1:3,1:3); R3=A3(1:3,1:3);
% camera frame
Rc=[0 0 1; 0 1 0; -1 0 0]
% position of O3
phom=A1*(A2*(A3*[0 0 0 1]'));
p=simplify(phom(1:3))
% Jacobian in frame 0
q=[q1 q2 q3]';
J=jacobian(p,q)
% Jacobian in frame 1,2,3
J1=simplify(R1'*J)
J2=simplify(R2'*J1)
J3=simplify(R3'*J2)
% final Jacobian in camera frame
Jc=simplify(Rc'*J3)
% end
```

Exercise 2

The rotation matrix associated to the (α, β, γ) angles in the (Z, X, Y) Euler representation, i.e., for a sequence of rotations around the axes Z, X' (moved), and Y'' (moved), is obtained from the elementary rotation matrices

$$\boldsymbol{R}_Z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{R}_X(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \boldsymbol{R}_Y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix},$$

as

$$\boldsymbol{R}_{ZXY}(\alpha, \beta, \gamma) = \boldsymbol{R}_Z(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma),$$

or

$$\boldsymbol{R}_{ZXY}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}.$$

Thus, we can compute the rotation matrices associated to the given $(\alpha_i, \beta_i, \gamma_i)$

$$^0\boldsymbol{R}_i = \boldsymbol{R}_{ZXY}(\alpha_i, \beta_i, \gamma_i) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \end{pmatrix},$$

and to $(\alpha_f, \beta_f, \gamma_f)$

$$^0\boldsymbol{R}_f = \boldsymbol{R}_{ZXY}(\alpha_f, \beta_f, \gamma_f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

The relative rotation between the initial and final orientation is thus

$$\boldsymbol{R}_{if} = {^0\boldsymbol{R}_i^T}\, {^0\boldsymbol{R}_f} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Note that this rotation matrix is defined with respect to the initial orientation $^0\boldsymbol{R}_i$ (or $\boldsymbol{R}_{if} = {^i\boldsymbol{R}_{if}}$).

We extract then the angle θ_{if} and the invariant axis $\boldsymbol{r}$ (a unit vector) from the elements R_{ij} of the rotation matrix $\boldsymbol{R}_{if}$:

$$\theta_{if} = \text{ATAN2}\left\{\sqrt{(R_{21} - R_{12})^2 + (R_{31} - R_{13})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\} = 2.5936 \text{ [rad]}$$

(or, in degrees, $\theta_{if} = 148.6°$). Being $\sin\theta_{if} \neq 0$, we have

$$\boldsymbol{r} = \frac{1}{2\sin\theta_{if}} \begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} = \frac{1}{1.042} \begin{pmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \\ 1 - (\sqrt{2}/2) \end{pmatrix} = \begin{pmatrix} -0.6786 \\ -0.6786 \\ 0.2811 \end{pmatrix}.$$

Again, this vector is expressed in the frame defined by the initial orientation ${}^0\boldsymbol{R}_i$ (or $\boldsymbol{r} = {}^i\boldsymbol{r}$).

For the rest-to-rest rotation in time T, the cubic polynomial (in normalized time $\tau = t/T$)

$$\theta(t) = \theta_{if}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right)$$

is such that $\theta(0) = 0$ and $\theta(T) = \theta_{if}$, and its time derivative

$$\dot{\theta}(t) = \frac{\theta_{if}}{T}\left(6\left(\frac{t}{T}\right) - 6\left(\frac{t}{T}\right)^2\right),$$

satisfies $\dot{\theta}(0) = \dot{\theta}(T) = 0$ as required. The maximum rotation speed is attained at $t = T/2$:

$$\dot{\theta}\left(\frac{T}{2}\right) = \frac{3\theta_{if}}{2T} \;\; (> 0) \qquad \Rightarrow \qquad (\text{for } T = 1.5) \quad \dot{\theta}_{max} = \dot{\theta}(0.75) = 2.5936 \text{ [rad/s]}.$$

Using the obtained $\boldsymbol{r}$, the orientation at a generic instant $t \in [0, T]$ is

$$\boldsymbol{R}(\boldsymbol{r}, \theta(t)) =$$

$$\begin{pmatrix} r_x^2(1-\cos\theta(t))+\cos\theta(t) & r_x r_y(1-\cos\theta(t))-r_z\sin\theta(t) & r_x r_z(1-\cos\theta(t))+r_y\sin\theta(t) \\ r_x r_y(1-\cos\theta(t))+r_z\sin\theta(t) & r_y^2(1-\cos\theta(t))+\cos\theta(t) & r_y r_z(1-\cos\theta(t))-r_x\sin\theta(t) \\ r_x r_z(1-\cos\theta(t))-r_y\sin\theta(t) & r_y r_z(1-\cos\theta(t))+r_x\sin\theta(t) & r_z^2(1-\cos\theta(t))+\cos\theta(t) \end{pmatrix} =$$

$$\begin{pmatrix} 0.5395\cos\theta(t)+0.4605 & 0.4605(1-\cos\theta(t))-0.2811\sin\theta(t) & -0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) \\ 0.4605(1-\cos\theta(t))+0.2811\sin\theta(t) & 0.5395\cos\theta(t)+0.4605 & -0.1907(1-\cos\theta(t))+0.6786\sin\theta(t) \\ -0.1907(1-\cos\theta(t))+0.6786\sin\theta(t) & -0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) & 0.9210\cos\theta(t)+0.07901 \end{pmatrix}.$$

Indeed, this orientation is relative to the initial one ${}^0\boldsymbol{R}_i$, or $\boldsymbol{R}(\boldsymbol{r}, \theta(t)) = {}^i\boldsymbol{R}({}^i\boldsymbol{r}, \theta(t))$. For check, it is easy to see that at $t = 0$ ($\theta(0) = 0$) it is $\boldsymbol{R}(\boldsymbol{r}, 0) = \boldsymbol{I}$. Similarly, at $t = T$ ($\theta(T) = \theta_{if}$) it is $\boldsymbol{R}(\boldsymbol{r}, \theta_{if}) = \boldsymbol{R}_{if}$. The absolute orientation is simply obtained as

$${}^0\boldsymbol{R}(\boldsymbol{r}, \theta(t)) = {}^0\boldsymbol{R}_i{}^i\boldsymbol{R}({}^i\boldsymbol{r}, \theta(t)) = \boldsymbol{R}({}^0\boldsymbol{R}_i{}^i\boldsymbol{r}, \theta(t)) = \boldsymbol{R}({}^0\boldsymbol{r}, \theta(t)) =$$

$$\begin{pmatrix} 0.2466\cos\theta(t)-0.4798\sin\theta(t)+0.4605 & 0.4605(1-\cos\theta(t))+0.2811\sin\theta(t) & -0.1907-0.5164\cos\theta(t))-0.4798\sin\theta(t) \\ 0.5164\cos\theta(t)+0.4798\sin\theta(t)+0.1907 & 0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) & 0.7861\cos\theta(t)-0.4798\sin\theta(t)-0.07901 \\ -0.4605(1-\cos\theta(t))-0.2811\sin\theta(t) & -0.5395\cos\theta(t)-0.4605 & 0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) \end{pmatrix}.$$

Finally, the angular velocity associated to the planned motion expressed in the frame ${}^0\boldsymbol{R}_i$ is

$${}^i\boldsymbol{\omega}(t) = {}^i\boldsymbol{r}\,\dot{\theta}(t) = \begin{pmatrix} -0.6786 \\ -0.6786 \\ 0.2811 \end{pmatrix} \dot{\theta}(t),$$

and in the absolute frame

$${}^0\boldsymbol{\omega}(t) = {}^0\boldsymbol{R}_i{}^i\boldsymbol{\omega}(t) = {}^0\boldsymbol{R}_i{}^i\boldsymbol{r}\,\dot{\theta}(t) = {}^0\boldsymbol{r}\,\dot{\theta}(t) = \begin{pmatrix} -0.6786 \\ -0.2811 \\ 0.6786 \end{pmatrix} \dot{\theta}(t).$$

Its maximum value in norm (invariant with respect to the frame of definition) is simply

$$\max_{t\in[0,T]} \left\| {}^0\boldsymbol{\omega}(t) \right\| = \max_{t\in[0,T]} \left\| {}^i\boldsymbol{\omega}(t) \right\| = \left\| {}^i\boldsymbol{r} \right\| \cdot \max_{t\in[0,T]} |\dot{\theta}(t)| = 1 \cdot \dot{\theta}_{max} = 2.5936 \text{ [rad/s]}.$$

A symbolic/numeric Matlab script supporting the computations of Exercise 2 is available.

* * * * *

Robotics I

June 11, 2012

Exercise 1

The time derivative of a rotation matrix can be given the following two alternative expressions:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\Omega}), \qquad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}.$$

Prove the correctness of both expressions and give the physical interpretation of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$.

Exercise 2

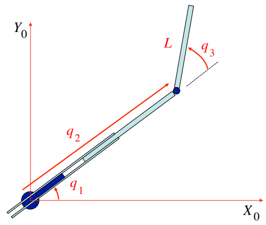


Figure 1: Planar RPR robot

For the planar RPR robot shown in Fig. 1, derive the 2×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the Cartesian velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^2$ of the end effector, and find all its singularities. Keeping q_1 as arbitrary, choose a singular configuration of this robot and denote the Jacobian in this configuration as $\bar{\boldsymbol{J}} = \bar{\boldsymbol{J}}(q_1)$. For each of the following linear subspaces,

$$\mathcal{R}\left(\bar{\boldsymbol{J}}\right) \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}\right) \qquad \mathcal{R}\left(\bar{\boldsymbol{J}}^T\right) \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}^T\right),$$

provide the symbolic expression of a unitary basis (i.e., a set of linearly independent unit vectors spanning the whole subspace).

Exercise 3

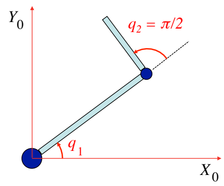


Figure 2: A planar 2R robot with the second link at $q_2 = \pi/2$

Consider a planar 2R robot, with links of length $\ell_1 = 1$ and $\ell_2 = 0.5$ [m], in the configuration shown in Fig. 2. The two motors at the joints are equipped with incremental encoders, respectively providing r_1 and r_2 pulses/turn. The gear ratios of the transmission/reduction systems of the two motors are $N_1 = 100$ and $N_2 = 80$. Determine the minimum resolutions of the two encoders so that they can be used to sense a displacement at the robot end-effector level as small as $\Delta p = 10^{-4}$ [m], alternatively in one of two arbitrary orthogonal directions.

[150 minutes; open books]

Solution

June 11, 2012

Exercise 1

As presented in the lecture slides, we consider first the identity $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$. Taking the time derivative:

$$\frac{d}{dt}\left(\boldsymbol{R}\boldsymbol{R}^T\right) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right) + \left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right)^T = \boldsymbol{O}.$$

Therefore, the matrix $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ is skew symmetric. We can write

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{\omega}) \qquad \Rightarrow \qquad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R},$$

where the angular velocity $\boldsymbol{\omega}$ is expressed in the *base* (unrotated) frame.

Similarly, consider the identity $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$. Taking the time derivative:

$$\frac{d}{dt}\left(\boldsymbol{R}^T\boldsymbol{R}\right) = \dot{\boldsymbol{R}}^T\boldsymbol{R} + \boldsymbol{R}^T\dot{\boldsymbol{R}} = \left(\boldsymbol{R}^T\dot{\boldsymbol{R}}\right) + \left(\boldsymbol{R}^T\dot{\boldsymbol{R}}\right)^T = \boldsymbol{O}.$$

Therefore, the matrix $\boldsymbol{R}^T\dot{\boldsymbol{R}}$ is skew symmetric. We can write

$$\boldsymbol{R}^T\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\Omega}) \qquad \Rightarrow \qquad \dot{\boldsymbol{R}} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}),$$

where the angular velocity $\boldsymbol{\Omega}$ is now expressed in the *body* (rotated) frame.

From this interpretation, it also follows that

$$\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega},$$

and so

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} = \boldsymbol{S}(\boldsymbol{R}\boldsymbol{\Omega})\boldsymbol{R} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}).$$

This implies that (see Exercise 3.1 in the textbook)

$$\boldsymbol{S}(\boldsymbol{R}\boldsymbol{\Omega}) = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega})\boldsymbol{R}^T.$$

Conversely,

$$\boldsymbol{S}(\boldsymbol{R}^T\boldsymbol{\omega}) = \boldsymbol{R}^T\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}.$$

Exercise 2

The direct kinematics of the considered RPR planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2c_1 + Lc_{13} \\ q_2s_1 + Ls_{13} \end{pmatrix}.$$

Therefore, the Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(q_2s_1 + Ls_{13}\right) & c_1 & -Ls_{13} \\ q_2c_1 + Lc_{13} & s_1 & Lc_{13} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}.$$

To check the singularities (i.e., where rank $\boldsymbol{J} < 2$), we consider the three 2×2 minors:

$$\det\begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \end{pmatrix} = (q_2 + Lc_3), \quad \det\begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_3 \end{pmatrix} = Lq_2s_3, \quad \det\begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix} = Lc_3.$$

They are simultaneously zero iff $\{q_2 = 0 \text{ .AND. } c_3 = 0\}$. Let then q_1 be arbitrary, $q_2 = 0$, and choose for instance $q_3 = +\pi/2$. In this configuration,

$$\bar{\boldsymbol{J}} = \bar{\boldsymbol{J}}(q_1) = \begin{pmatrix} -Lc_1 & c_1 & -Lc_1 \\ -Ls_1 & s_1 & -Ls_1 \end{pmatrix}$$

Unitary bases for the range and null spaces of interest are provided as follows:

$$\mathcal{R}\left(\bar{\boldsymbol{J}}\right) = \text{span}\left\{\begin{pmatrix} c_1 \\ s_1 \end{pmatrix}\right\} \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}^T\right) = \text{span}\left\{\begin{pmatrix} s_1 \\ -c_1 \end{pmatrix}\right\},$$

$$\mathcal{R}\left(\bar{\boldsymbol{J}}^T\right) = \text{span}\left\{\frac{1}{\sqrt{1+2L^2}}\begin{pmatrix} L \\ -1 \\ L \end{pmatrix}\right\} \quad \mathcal{N}\left(\bar{\boldsymbol{J}}\right) = \text{span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{1+L^2}}\begin{pmatrix} 1 \\ L \\ 0 \end{pmatrix}\right\}.$$

Exercise 3

Denote by $\boldsymbol{\Delta\theta}$ the vector of motor position variations (the increments measured by the encoders at the motor sides), by $\boldsymbol{\Delta q}$ the associated vector of link position variations, by $\boldsymbol{\Delta p}$ the resulting vector of end-effector position variations, and by $\boldsymbol{N}$ the diagonal matrix of reduction ratios

$$\boldsymbol{N} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} = \begin{pmatrix} 100 & 0 \\ 0 & 80 \end{pmatrix}.$$

We have

$$\Delta\boldsymbol{p} = \boldsymbol{J}(\boldsymbol{q})\Delta\boldsymbol{q} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{N}^{-1}\Delta\boldsymbol{\theta}, \tag{1}$$

with the Jacobian of the 2R planar robot given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\left(\ell_1 s_1 + \ell_2 s_{12}\right) & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix}.$$

To eliminate the appearance of q_1, it is convenient to work in the rotated frame 1 attached to the first link. Since we are working in the plane $(\boldsymbol{x}, \boldsymbol{y})$, it is

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_2 s_2 & -\ell_2 s_2 \\ \ell_1 + \ell_2 c_2 & \ell_2 c_2 \end{pmatrix}.$$

Therefore, we replace eq. (1) by

$${}^1\Delta\boldsymbol{p} = {}^1\boldsymbol{J}(\boldsymbol{q})\boldsymbol{N}^{-1}\Delta\boldsymbol{\theta}. \tag{2}$$

At the given configuration $q_2 = \pi/2$,

$${}^1\boldsymbol{J}\boldsymbol{N}^{-1} = \begin{pmatrix} -\ell_2/N_1 & -\ell_2/N_2 \\ \ell_1/N_1 & 0 \end{pmatrix}.$$

The end-effector displacement Δp that is requested to be sensed in either of two orthogonal directions can be defined using again the coordinate axes of frame 1. Let

$${}^1\Delta\boldsymbol{p}_I = \begin{pmatrix} \Delta p \\ 0 \end{pmatrix}, \qquad {}^1\Delta\boldsymbol{p}_{II} = \begin{pmatrix} 0 \\ \Delta p \end{pmatrix}.$$

In case I, we solve from eq. (2)

$$\Delta\boldsymbol{\theta}_I = \boldsymbol{N} \cdot {}^1\boldsymbol{J}^{-1} \cdot {}^1\Delta\boldsymbol{p}_I = \begin{pmatrix} 0 \\ -N_2\Delta p/\ell_2 \end{pmatrix}.$$

Plugging the data $\ell_2 = 0.5$, $N_2 = 80$, and $\Delta p = 10^{-4}$, we obtain the minimum increment that should be sensed by the encoder at motor 2 in case I:

$$|\Delta\theta_{2,I}| = 16 \cdot 10^{-3} \text{ [rad]}.$$

Since the resolution of this encoder is $|\Delta\theta_2| = 2\pi/r_2$, the minimum number of pulses/turn needed is

$$r_2 = \frac{\pi}{8} \cdot 10^3 \simeq 392.7.$$

Being the number r of pulses/turn typically a power of 2, an incremental encoder with $512 = 2^9$ pulses/turn would be sufficient for joint 2 in this case.

Similarly, in case II

$$\Delta\boldsymbol{\theta}_{II} = \boldsymbol{N} \cdot {}^1\boldsymbol{J}^{-1} \cdot {}^1\Delta\boldsymbol{p}_{II} = \begin{pmatrix} N_1\Delta p/\ell_1 \\ -N_2\Delta p/\ell_1 \end{pmatrix}.$$

Plugging the data $\ell_1 = 1$, $N_1 = 100$, $N_2 = 80$, and $\Delta p = 10^{-4}$, we obtain the minimum increments that should be sensed by the two encoders in case II:

$$|\Delta\theta_{1,II}| = 10^{-2} \text{ [rad]}, \qquad |\Delta\theta_{2,II}| = 8 \cdot 10^{-3} \text{ [rad]}.$$

Note that the obtained condition on the second encoder will be more stringent in case II than in case I. From $|\Delta\theta_1| = 2\pi/r_1$ and $|\Delta\theta_2| = 2\pi/r_2$, the minimum resolutions for the two encoders will be

$$r_1 = 2\pi \cdot 10^2 \simeq 614, \qquad r_2 = \frac{\pi}{4} \cdot 10^3 \simeq 785.4.$$

Being the number of pulses/turn typically a power of 2, two equal incremental encoders with $1024 = 2^{10}$ pulses/turn mounted at the motor sides of the two robot joints would be sufficient to satisfy the requested end-effector sensing accuracy.

* * * * *

Robotics I

July 5, 2012

Huge portal robots are used in the aeronautical industry for working on large parts of aircraft bodies. One such robot, having six actuated joints, is shown in Fig. 1. This robot is used for automatic riveting of the plates that constitutes the body of an aircraft.

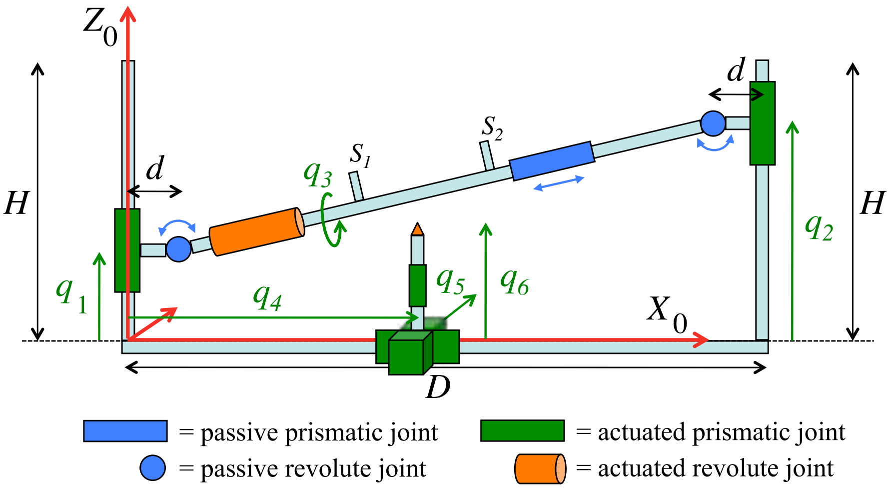


Figure 1: The 6-dof portal robot used for riveting aircraft parts (see the text for definitions and operation)

The portal robot is constituted by two sets of independently actuated axes, the upper section with joints 1 to 3 and the lower section with joints 4 to 6. The upper section carries the part to be worked, while the lower section has the riveting tool on its end effector. In the upper section, two vertical bars of height H are placed at a distance D. Along the two vertical bars, two actuated prismatic joints (with variables $q_1 \in [0, H]$ and $q_2 \in [0, H]$) are used to change the orientation in the vertical plane of a connecting bar (of variable length), which has two supports S_1 and S_2 where the aircraft part will be placed and fixed. The structure contains three passive joints (shown in different colors in Fig. 1), two revolute and one prismatic, that move accordingly to the the actuated prismatic joints. The passive revolute joints, placed at a distance $d << D$ from the vertical bars, transform the linear motions (when different) of q_1 and q_2 in a tilt of the connecting bar (say, by an angle α with respect to the horizontal). The passive prismatic joint accommodates itself so that the connecting bar changes length consistently with the values of q_1 and q_2. Furthermore, the connecting bar can be rotated along its main axis through an actuated revolute joint (with variable $\beta = q_3$). The lower section of the portal robot carries the riveting tool, which can be moved by three actuated prismatic joints with orthogonal axes, horizontally with $q_4 \in [0, D]$ and q_5 (unconstrained in sign), and vertically with $q_6 \in [0, H]$. The riveting tool can operate only along the vertical direction.

Consider now a part of an aircraft body, as shown in Fig. 2. Typically, this is a metallic plate with spatially changing curvature. Each plate is designed at the computer and manufactured with numerically-controlled machines (i.e., through a CAD/CAM system). For this, a reference frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ is attached to the plate for defining its geometry. In order to join two plates together and/or each plate to a stiff supporting structure, a large number of rivets have to be placed in small holes previously drilled on the plate surface (the total number of rivets can go up to three millions for a whole airplane in the Airbus 300 class). The position of a riveting point is specified by vector $\boldsymbol{p}$ and, at this point, the (external) normal to the surface is specified by the unit vector $\boldsymbol{n}$. Both these vectors are expressed in the frame attached to the plate. In order to be successful, the riveting task should be performed as accurately as possible along the normal to the plate surface.

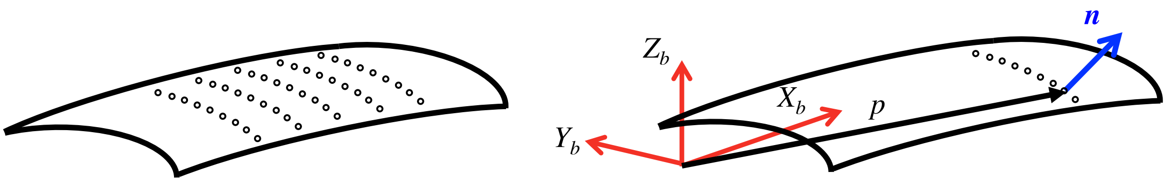


Figure 2: An aircraft plate (left) and its associated reference frame (right): The position vector $\boldsymbol{p}$ locates a riveting point, where the normal to the surface is given by the unit vector $\boldsymbol{n}$

Figure 3 shows the plate mounted on the portal robot (in particular, fixed to the supports S_1 and S_2 of the connecting bar in the upper section of the robot). The correct mounting is done so that the origin of frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ coincides with the passive revolute joint on the left, and the $\boldsymbol{x}_b$ axis is aligned with the connecting bar (i.e., along the axis of the passive prismatic joint). In correspondence to $q_3 = 0$, the $\boldsymbol{z}_b$ axis is in the vertical plane.

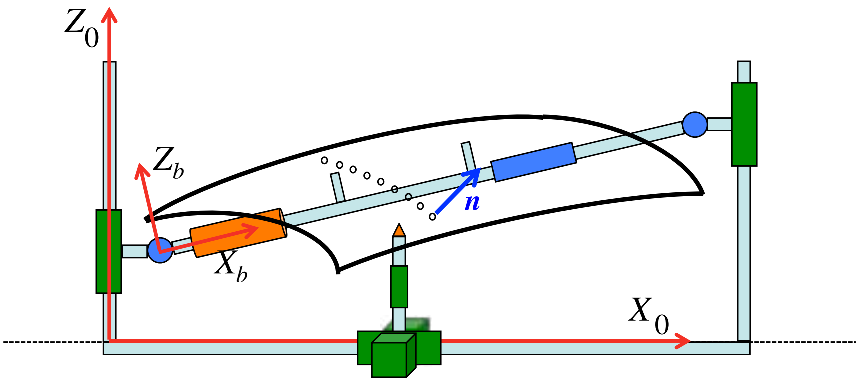


Figure 3: The aircraft plate correctly mounted on the two supports of the connecting bar of the upper section of the portal robot

With the above description in mind, a single riveting task with the portal robot is executed as follows. The upper section of the robot should move the part so that the normal $\boldsymbol{n}$ to the surface at the riveting point is oriented along the upward vertical direction. Then, the end effector carrying the riveting tool on the lower section of the robot is moved so as to position itself at the riveting point.

1. Given the normal ${}^b\boldsymbol{n}$ to the surface plate at the riveting point, find the inverse kinematic solution for the two pointing angles (α, β) in closed symbolic form. Note that two solutions can be found in general, but one may not be feasible because of the limits on the robot joint motions.

2. Each obtained pair (α, β) needs to be realized by suitable $\boldsymbol{q}_u = (q_1, q_2, q_3)$ of the upper section of the portal robot. Since this is an underdetermined problem, an infinite number of solutions exists. Choose a solution to this redundant problem, determining a single value of $\boldsymbol{q}_u$ that allows the pointing task to be realized in the most convenient way.

3. Consider, in addition to ${}^b\boldsymbol{n}$, a given position ${}^b\boldsymbol{p}$ of the riveting point. With the solution found for the pointing subtask, determine the explicit (unique) expression of $\boldsymbol{q}_l = (q_4, q_5, q_6)$ that will position the end-effector riveting tool at the desired point of the plate.

4. Provide a feasible numerical solution $\boldsymbol{q} = (\boldsymbol{q}_u, \boldsymbol{q}_l) = (q_1, q_2, q_3, q_4, q_5, q_6)$, using the robot data
$$D = 10 \text{ [m]}, \qquad H = 5 \text{ [m]}, \qquad d = 0.75 \text{ [m]},$$
for a riveting task specified by
$${}^b\boldsymbol{p} = \begin{pmatrix} 5.5 \\ -0.3 \\ -0.2 \end{pmatrix} \text{ [m]}, \qquad {}^b\boldsymbol{n} = \begin{pmatrix} 0.5 \\ 0.8138 \\ -0.2962 \end{pmatrix}.$$

[Hint]: There is no need of assigning D-H frames/parameters. Work instead with suitable frames.

[240 minutes; open books]

Solution

July 5, 2012

The first part of the problem is a pointing task in which the joint variables $\boldsymbol{q}_u = (q_1, q_2, q_3)$ are used to bring the unit normal vector $\boldsymbol{n}$ to the surface aligned with $\boldsymbol{z}_0$. When expressed in the base frame $(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{z}_0)$, this alignment is represented by

$$ {}^0\boldsymbol{n} = {}^0\boldsymbol{R}_b {}^b\boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(= {}^0\boldsymbol{z}_0\right), \tag{1} $$

where the rotation matrix ${}^0\boldsymbol{R}_b$ depends on $\boldsymbol{q}_u = (q_1, q_2, q_3)$. To describe this pointing task, it is convenient to introduce the pair of angles (α, β) as

$$ \alpha = \arctan\left(\frac{q_1 - q_2}{D - 2d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \beta = q_3. \tag{2} $$

Note that the domain of definition for α is actually contained in the indicated one, but these limits show that we can use directly the arctangent function and not the ATAN2. In particular, due to the limits of joints 1 and 2, we have that

$$ \max|\alpha| = \arctan\left(\frac{H}{D - 2d}\right) > 0. \tag{3} $$

To determine ${}^0\boldsymbol{R}_b$, we can use an intermediate frame $(\boldsymbol{x}_d, \boldsymbol{y}_d, \boldsymbol{z}_d)$, with origin coincident with that of $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ and rotated by α around the $\boldsymbol{y}_0$ axis. Therefore,

$$ {}^0\boldsymbol{R}_b = {}^0\boldsymbol{R}_d(\alpha) {}^d\boldsymbol{R}_b(\beta) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} c_\alpha & s_\alpha s_\beta & s_\alpha c_\beta \\ 0 & c_\beta & -s_\beta \\ -s_\alpha & c_\alpha s_\beta & c_\alpha c_\beta \end{pmatrix}, $$

where a compact notation has been used in the last expression. The $\boldsymbol{n}$ axis can be taken as one (actually, any) of the coordinate axes of a frame $(\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n)$ attached to the riveting point. We choose for instance the $\boldsymbol{x}_n = \boldsymbol{n}$ axis (i.e., ${}^n\boldsymbol{n} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$), so that

$$ {}^b\boldsymbol{R}_n = \begin{pmatrix} {}^b\boldsymbol{n} & {}^b\boldsymbol{s} & {}^b\boldsymbol{a} \end{pmatrix}, \qquad {}^b\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. $$

Our pointing task equation (1), i.e., the direct kinematics of the task, is then

$$ {}^0\boldsymbol{R}_d(\alpha) {}^d\boldsymbol{R}_b(\beta) {}^b\boldsymbol{R}_n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} n_x c_\alpha + n_y s_\alpha s_\beta + n_z s_\alpha c_\beta \\ n_y c_\beta - n_z s_\beta \\ -n_x s_\alpha + n_y c_\alpha s_\beta + n_z c_\alpha c_\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4} $$

Squaring and adding the first and third equations in (4) gives

$$ n_x^2 + n_y^2 s_\beta^2 + n_z^2 c_\beta^2 + 2 n_y n_z s_\beta c_\beta = n_x^2 + \left(n_y s_\beta + n_z c_\beta\right)^2 = 1 $$

or

$$ n_y s_\beta + n_z c_\beta = \pm\sqrt{1 - n_x^2}. \tag{5} $$

From eq. (5) and the second equation in (4), we obtain a linear system in the s_β and c_β unknowns:

$$\begin{pmatrix} n_y & n_z \\ -n_z & n_y \end{pmatrix} \begin{pmatrix} s_\beta \\ c_\beta \end{pmatrix} = \begin{pmatrix} \pm\sqrt{1-n_x^2} \\ 0 \end{pmatrix}. \tag{6}$$

This system can be solved uniquely, provided that the determinant of the coefficient matrix is not zero. Singularity occurs if and only if

$$n_y^2 + n_z^2 = 0 \qquad \Longleftrightarrow \qquad n_y = n_z = 0,\ n_x = \pm 1.$$

It is easy to see from eq. (4) that this happens for $\alpha = -\pi/2$ (if $n_x = 1$) or for $\alpha = \pi/2$ (if $n_x = -1$), with β being undefined in both cases. However, this situation is unfeasible (and never encountered in practice) for the upper section of the portal robot. For $n_y^2 + n_z^2 \neq 0$, solving (6) yields

$$s_\beta = \frac{\pm\, n_y}{n_y^2 + n_z^2}\sqrt{1-n_x^2} \qquad\qquad c_\beta = \frac{\pm\, n_z}{n_y^2 + n_z^2}\sqrt{1-n_x^2} \tag{7}$$

and thus (eliminating common positive terms)

$$\beta = \text{ATAN2}\,\{\pm n_y, \pm n_z\}, \tag{8}$$

which are the two possible solutions (depending on the upper or lower signs chosen in the arguments of ATAN2). Replacing (5) in the first and third equations of (4), we can eliminate the appearance of β and obtain a linear system in the s_α and c_α unknowns

$$\begin{pmatrix} \pm\sqrt{1-n_x^2} & n_x \\ -n_x & \pm\sqrt{1-n_x^2} \end{pmatrix} \begin{pmatrix} s_\alpha \\ c_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9}$$

with nonsingular coefficient matrix (its determinant is 1). Its solution is

$$s_\alpha = -n_x, \qquad c_\alpha = \pm\sqrt{1-n_x^2} \tag{10}$$

and thus

$$\alpha = \text{ATAN2}\left\{-n_x, \pm\sqrt{1-n_x^2}\right\}, \tag{11}$$

with the upper or lower sign chosen in correspondence to the two solutions for β. As a result, two pairs of solutions (α, β) have been found. The value of α in these pairs should be checked against the feasible limit (3): if this is exceeded, the associated solution pair should be discarded.

For each (α, β) pair, we associate now a suitable unique value of $\boldsymbol{q}_u = (q_1, q_2, q_3)$ by resolving the intrinsic redundancy in the following convenient way. Consider a value $h > 0$ for the average of the two prismatic joints q_1 and q_2, i.e.,

$$\frac{q_1 + q_2}{2} = h.$$

Putting this together with the definition (2) of α yields the nonsingular (the determinant is 1) linear system

$$\begin{pmatrix} 1 & -1 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} (D-2d)\tan\alpha \\ h \end{pmatrix},$$

with the unique solution

$$q_1 = h + \frac{(D-2d)\tan\alpha}{2}, \qquad q_2 = h - \frac{(D-2d)\tan\alpha}{2}. \tag{12}$$

It is immediate to see that the value $h = H/2$ guarantees the best use of the available joint ranges $[0, H]$ for q_1 and q_2. Finally, for the third joint variable we set simply

$$q_3 = \beta. \tag{13}$$

As a result, two inverse kinematic solutions for the complete vector $\boldsymbol{q}_u = (q_1, q_2, q_3)$.

Consider now the problem of positioning the riveting tool on the lower section of the robot at the given position ${}^b\boldsymbol{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$, a vector starting from the origin of the frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ (see Fig. 2) and expressed in this frame. Given one of the pairs (α, β) found in the previous pointing subtask, we compute the following positional direct kinematics

$${}^0\boldsymbol{p} = \begin{pmatrix} d \\ 0 \\ q_1 \end{pmatrix} + {}^0\boldsymbol{R}_d(\alpha)\,{}^d\boldsymbol{R}_b(\beta)\,{}^b\boldsymbol{p}. \tag{14}$$

Since the lower section of the portal robot is a Cartesian structure, the (unique) inverse kinematics solution is given simply by

$$\boldsymbol{q}_l = \begin{pmatrix} q_4 \\ q_5 \\ q_6 \end{pmatrix} = {}^0\boldsymbol{p} = \begin{pmatrix} d + p_x c_\alpha + s_\alpha \left(p_y s_\beta + p_z c_\beta\right) \\ p_y c_\beta - p_z s_\beta \\ q_1 - p_x s_\alpha + c_\alpha \left(p_y s_\beta + p_z c_\beta\right) \end{pmatrix}. \tag{15}$$

We evaluate the formulas on the provided numerical data. From eq. (11) we obtain the two values

$$\alpha_1 = -0.5236\,[\text{rad}] = -30°, \qquad \alpha_2 = -2.6180\,[\text{rad}] = -150°,$$

while from eq. (8) we have

$$\beta_1 = 1.9199\,[\text{rad}] = 110°, \qquad \beta_2 = -1.2217\,[\text{rad}] = -70°.$$

From eq. (3), the maximum absolute value of α is 0.5317 [rad]= 30.4655°. Therefore, the solution pair (α_2, β_2) is unfeasible and should be discarded. From the pair (α_1, β_1), using eq. (12) with $h = H/2 = 2.5$ and eq. (13), we find

$$q_1 = 0.0463\,[\text{m}], \qquad q_2 = 4.9537\,[\text{m}], \qquad q_3 = 1.9199\,[\text{rad}].$$

Finally, from eq. (15) we obtain

$$q_4 = 5.6199\,[\text{m}], \qquad q_5 = 0.2905\,[\text{m}], \qquad q_6 = 2.6114\,[\text{m}].$$

All joint variables are in their admissible range. A Matlab code is available for this numerical evaluation.

* * * * *

Robotics I

September 10, 2012

A 3R robot manipulator has the following Denavit-Hartenberg table:

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

Table 1: DH table of a 3R robot

1. Sketch the kinematic structure of the robot and place the D-H frames according to Table 1.
2. Draw the robot in the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$ [rad].

Assume now the numerical data $a_1 = 0.2$, $a_2 = 0.5$, and $a_3 = 0.5$ [m] and let the robot be in the configuration specified at step 2.

3. Given a desired velocity $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 0.5 \end{pmatrix}^T$ [m/s] for the robot end-effector (the origin O_3 of frame 3), determine the instantaneous joint velocity vector $\dot{\boldsymbol{\theta}}$ that realizes $\boldsymbol{v}$.
4. With the solution $\dot{\boldsymbol{\theta}}$ found at step 3, compute the associated angular velocity $\boldsymbol{\omega}$ of the robot end-effector frame.
5. Let the value $\boldsymbol{\omega}$ found at step 4 be the desired angular velocity for the robot end-effector frame. Characterize *all* instantaneous joint velocities $\dot{\boldsymbol{\theta}}$ that realize $\boldsymbol{\omega}$ at the given robot configuration.
6. What is the structure of all feasible $\boldsymbol{\omega}$ that can be realized by this robot in a *generic* configuration $\boldsymbol{\theta}$? What can we say about the differential mapping $\dot{\boldsymbol{\theta}} \to \boldsymbol{\omega}$?

[120 minutes; open books]

Solution

September 10, 2012

The robot has a kinematic structure similar to that of the first three joints of the KUKA KR5 robot (the industrial robot in our Robotics Laboratory). Figures 1 and 2 provide, respectively, a sketch of the kinematic structure, with associated D-H frames, and the robot posture at the specified $\boldsymbol{\theta}$.

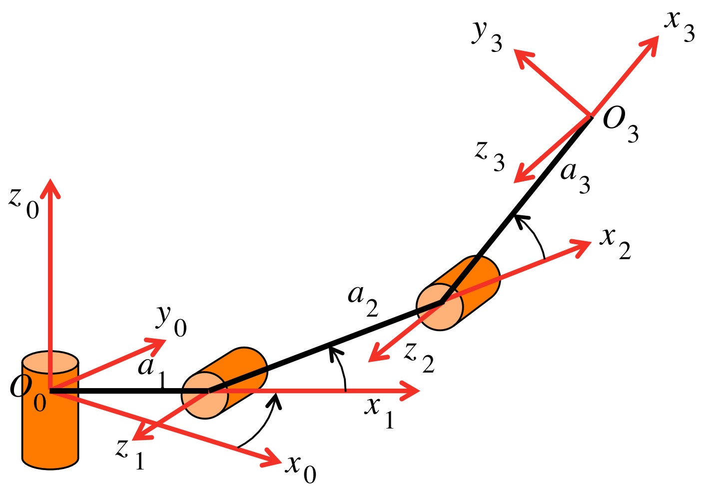


Figure 1: Kinematic structure and D-H frames

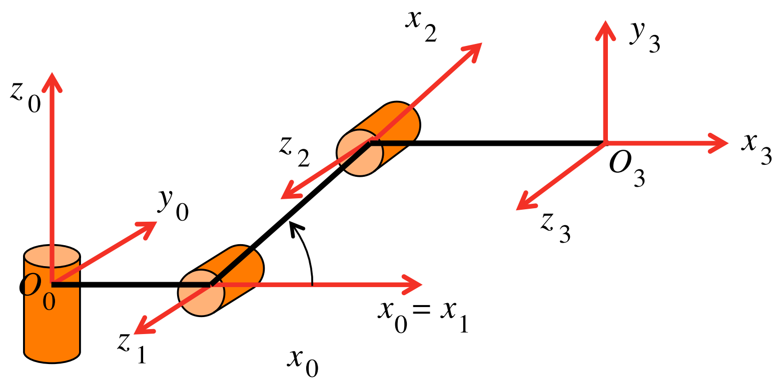


Figure 2: The robot at the configuration $\boldsymbol{\theta} = (0 \quad \pi/4 \quad -\pi/4)^T$

For steps 3-6, we need to compute the robot Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$. For the linear part, $\boldsymbol{J}_L(\boldsymbol{\theta})$, we may use either the vector product computations of the geometric Jacobian or simply differentiate analytically the positional direct kinematics. From the product of the homogeneous matrices associated to the D-H table 1, it follows

$$\boldsymbol{p}_{hom} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(\theta_1)\, {}^1\boldsymbol{A}_2(\theta_2)\, {}^2\boldsymbol{A}_3(\theta_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = \begin{pmatrix} (a_1 + a_2 c_2 + a_3 c_{23})\, c_1 \\ (a_1 + a_2 c_2 + a_3 c_{23})\, s_1 \\ a_2 s_2 + a_3 s_{23} \\ 1 \end{pmatrix}.$$

Therefore,

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}\, \dot{\boldsymbol{\theta}} = \boldsymbol{J}_L(\boldsymbol{\theta})\, \dot{\boldsymbol{\theta}}, \text{ with } \boldsymbol{J}_L(\boldsymbol{\theta}) = \begin{pmatrix} -(a_1 + a_2 c_2 + a_3 c_{23})\, s_1 & -(a_2 s_2 + a_3 s_{23})\, c_1 & -a_3 s_{23} c_1 \\ (a_1 + a_2 c_2 + a_3 c_{23})\, c_1 & -(a_2 s_2 + a_3 s_{23})\, s_1 & -a_3 s_{23} s_1 \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix}.$$

For the angular part, $\boldsymbol{J}_A(\boldsymbol{\theta})$, we have by definition (taking into account that velocity vectors are expressed by default in the 0th frame)

$$\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(\theta_1)\, {}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(\theta_1)\, {}^1\boldsymbol{R}_2(\theta_2)\, {}^2\boldsymbol{z}_2 \end{pmatrix},$$

with ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for $i = 0, 1, 2$. As a result,

$$\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{\theta})\, \dot{\boldsymbol{\theta}}, \quad \text{with } \boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Evaluating the two Jacobians at the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$ with the given numerical data yields

$$\boldsymbol{J}_L = \begin{pmatrix} 0 & -0.3536 & 0 \\ 1.0536 & 0 & 0 \\ 0 & 0.8536 & 0.5 \end{pmatrix}, \qquad \boldsymbol{J}_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2}$$

Therefore, for $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 0.5 \end{pmatrix}^T$,

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}_L^{-1} \boldsymbol{v} = \begin{pmatrix} 0.9492 \\ -2.8284 \\ 5.8284 \end{pmatrix} \text{ [rad/s]} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A \dot{\boldsymbol{\theta}} = \begin{pmatrix} 0 \\ -3 \\ 0.9492 \end{pmatrix} \text{ [rad/s]}. \tag{3}$$

From the general structure of $\boldsymbol{J}_A(\boldsymbol{\theta})$ in (1) we see that this matrix is always singular, having constant rank equal to 2. At a generic configuration (i.e., for a generic value of θ_1), we characterize the following subspaces of interest:

$$\mathcal{R}(\boldsymbol{J}_A(\boldsymbol{\theta})) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}(\boldsymbol{J}_A(\boldsymbol{\theta})) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Therefore, all feasible $\boldsymbol{\omega}$ will have the form

$$\boldsymbol{\omega} \in \mathcal{R}(\boldsymbol{J}_A(\boldsymbol{\theta})) \qquad \Rightarrow \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \beta$$

with $\alpha = \dot{\theta}_1 \in \mathbb{R}$ and $\beta = \dot{\theta}_1 + \dot{\theta}_2 \in \mathbb{R}$. Conversely, given a generic $\dot{\boldsymbol{\theta}}$ generating a $\boldsymbol{\omega}$, the same value of end-effector angular velocity is obtained by adding a joint velocity vector $\dot{\boldsymbol{\theta}}_0 \in \mathcal{N}(\boldsymbol{J}_A(\boldsymbol{\theta}))$, or

$$\dot{\boldsymbol{\theta}} + \gamma\,\dot{\boldsymbol{\theta}}_0 = \dot{\boldsymbol{\theta}} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} = \boldsymbol{J}_A(\boldsymbol{\theta})\left(\dot{\boldsymbol{\theta}} + \gamma\,\dot{\boldsymbol{\theta}}_0\right).$$

for any $\gamma \in \mathbb{R}$.

Particularizing this general result to the specific configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$, with $\boldsymbol{J}_A$ given in (2), all joint velocities that generate the same value $\boldsymbol{\omega}$ as in (3) are given by

$$\dot{\boldsymbol{\theta}}_\gamma = \begin{pmatrix} 0.9492 \\ -2.8284 \\ 5.8284 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \text{for any } \gamma \in \mathbb{R} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A \dot{\boldsymbol{\theta}}_\gamma = \begin{pmatrix} 0 \\ -3 \\ 0.9492 \end{pmatrix}.$$

Note that the minimum norm joint velocity $\dot{\boldsymbol{\theta}}^*$ realizing this value of $\boldsymbol{\omega}$ is obtained by unconstrained minimization of $\|\dot{\boldsymbol{\theta}}_\gamma\|^2$ with respect to γ. This yields

$$\gamma = -\frac{\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}_0}{\dot{\boldsymbol{\theta}}_0^T \dot{\boldsymbol{\theta}}_0} = 4.3284 \qquad \Rightarrow \qquad \dot{\boldsymbol{\theta}}^* = \begin{pmatrix} 0.9492 \\ 1.5 \\ 1.5 \end{pmatrix},$$

with $\|\dot{\boldsymbol{\theta}}^*\| = 2.3240$ —as opposed to $\|\dot{\boldsymbol{\theta}}\| = 6.5476$ for the value $\dot{\boldsymbol{\theta}}$ computed in (3). As could be expected, the minimum norm solution balances the effort between the velocities of joints 2 and 3.

* * * * *

Robotics I

January 9, 2013

Exercise 1

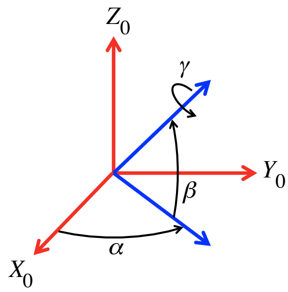


Figure 1: Definition of the three angles α, β, and γ

Consider the orientation obtained through the sequence of three rotations specified by the angles α, β, and γ in Fig. 1. Pay attention to the definition of positive rotations —the figure shows a situation in which α and β have some positive values in $(0, \pi/2)$.

a) Determine the associated rotation matrix $\boldsymbol{R}(\alpha, \beta, \gamma)$ (*direct problem*).

b) When the orientation is expressed by a rotation matrix $\boldsymbol{R}$, find the closed-form expressions for the minimal representation of orientation using the above set of angles α, β, and γ (*inverse problem*). Characterize the cases when two solutions or an infinite number of solutions exist.

c) Obtain the mapping between the time derivatives of the three angles in this minimal representation and the angular velocity vector, i.e.,

$$\boldsymbol{\omega} = \boldsymbol{T}(\alpha, \beta) \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix},$$

and find the singularities of the matrix $\boldsymbol{T}$. In one of these singularities, provide two numerical examples in which a desired $\boldsymbol{\omega}$ can or, respectively, cannot be realized.

Exercise 2

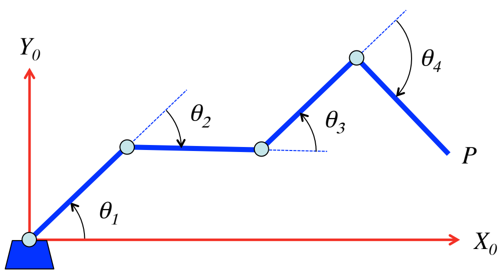


Figure 2: A planar $4R$ robot arm with unitary link lengths

A $4R$ planar robot with links of unitary length is shown in Fig. 2.

a) Provide the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ relating the joint velocity $\dot{\boldsymbol{\theta}} = \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{pmatrix}^T \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} = \begin{pmatrix} v_x & v_y \end{pmatrix}^T \in \mathbb{R}^2$ of the robot end-effector.

b) Find all singular configurations of this Jacobian.

c) In the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & 0 & -\pi/4 & \pi/2 \end{pmatrix}^T$, determine the joint velocity $\dot{\boldsymbol{\theta}}$ of *minimum norm* that realizes the desired end-effector velocity $\boldsymbol{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

Exercise 3

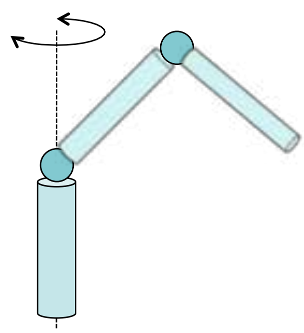


Figure 3: An anthropomorphic $3R$ robot arm

The $3R$ anthropomorphic robot in Fig. 3 is equipped with encoders at the joints for position sensing. Suppose that a small error $\Delta\theta_1$ affects the position measurement of the encoder at the first joint. What is the *maximum* norm of the error $\boldsymbol{\Delta p}$ over the *whole* workspace when estimating the end-effector position $\boldsymbol{p}$ using the encoder readings? Provide a complete explanation of your answer.

[180 minutes; open books]

Solutions

January 9, 2013

Exercise 1

The first rotation is around axis $Z = Z_0$ (by an angle α), and the following ones are around the moving axes $Y' = -Y_1$ (by β) and $X'' = X_2$ (by γ). The only caution is in the definition of the second angle β, which is positive counterclockwise around $-Y_1$. Thus, the associated rotation matrices are

$$\begin{aligned}
\boldsymbol{R}_1(\alpha) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\boldsymbol{R}_2(\beta) &= \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \text{(note the \textit{opposite} signs of the sin terms)} \\
\boldsymbol{R}_3(\gamma) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}.
\end{aligned}$$

The rotation matrix for the direct problem is computed as

$$\begin{aligned}
\boldsymbol{R}(\alpha,\beta,\gamma) &= \boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) \\
&= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\ \sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}. \qquad (1)
\end{aligned}$$

For the inverse problem, let $\boldsymbol{R} = \{R_{ij}\}$. From the expressions of the elements in the last row of $\boldsymbol{R}(\alpha,\beta,\gamma)$, one has

$$\beta = \text{ATAN2}\left\{R_{31}, \pm\sqrt{R_{32}^2 + R_{33}^2}\right\}. \qquad (2)$$

When $R_{32}^2 + R_{33}^2 \neq 0$ (or, $\cos^2\beta \neq 0$), for each of the two values of β obtained from eq. (2) we have an associated solution

$$\alpha = \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}.$$

Instead, when $R_{32} = R_{33} = R_{11} = R_{21} = 0$ (or, $\cos\beta = 0$), it is $\sin\beta = \pm 1$ and thus

$$\boldsymbol{R}(\alpha,\beta,\gamma)|_{\beta=\pm\frac{\pi}{2}} = \begin{pmatrix} 0 & -\sin(\alpha\pm\gamma) & \mp\cos(\alpha\pm\gamma) \\ 0 & \cos(\alpha\pm\gamma) & \mp\sin(\alpha\pm\gamma) \\ \pm 1 & 0 & 0 \end{pmatrix}.$$

Therefore, we can only determine the *sum* or, respectively, the *difference* of the two angles α and γ, leading to an infinite number of inverse solutions. If $R_{33} = 1$, we have

$$\beta = \frac{\pi}{2}, \qquad \alpha + \gamma = \text{ATAN2}\left\{-R_{12}, R_{22}\right\}.$$

If $R_{33} = -1$, we have

$$\beta = -\frac{\pi}{2}, \qquad \alpha - \gamma = \text{ATAN2}\left\{-R_{12}, R_{22}\right\}.$$

Finally, the contributions of the time derivatives $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ to $\boldsymbol{\omega}$ are computed by evaluating the directions of the rotation axes Z_0, $-Y_1$ and X_2 in the original reference frame[1]:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\alpha} + \begin{pmatrix} \sin\alpha \\ -\cos\alpha \\ 0 \end{pmatrix} \dot{\beta} + \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ \sin\beta \end{pmatrix} \dot{\gamma} = \boldsymbol{T}(\alpha,\beta) \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}.$$

We have a singularity when $\det \boldsymbol{T}(\alpha,\beta) = \cos\beta = 0$, or $\beta = \pm\pi/2$. For instance, when $\beta = \pi/2$, the angular velocity

$$\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{R}\left\{\boldsymbol{T}(\alpha,\pi/2)\right\}$$

can be realized in infinite ways by choosing

$$\dot{\beta} = 0, \quad \dot{\alpha} + \dot{\gamma} = 1.$$

On the other hand, the angular velocity $\boldsymbol{\omega} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \end{pmatrix}^T$ will certainly not belong to the range of $\boldsymbol{T}$ at the current α, and therefore cannot be realized by any choice of $\begin{pmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{pmatrix}^T$—such situations are always present when using any minimal representation for the orientation.

Exercise 2

The position $\boldsymbol{p}$ of the robot end-effector is

$$\boldsymbol{p} = \begin{pmatrix} \cos\theta_1 + \cos(\theta_1+\theta_2) + \cos(\theta_1+\theta_2+\theta_3) + \cos(\theta_1+\theta_2+\theta_3+\theta_4) \\ \sin\theta_1 + \sin(\theta_1+\theta_2) + \sin(\theta_1+\theta_2+\theta_3) + \sin(\theta_1+\theta_2+\theta_3+\theta_4) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{\theta}).$$

Therefore, its velocity $\boldsymbol{v}$ is obtained as

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})\, \dot{\boldsymbol{\theta}},$$

where the Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -(s_1+s_{12}+s_{123}+s_{1234}) & -(s_{12}+s_{123}+s_{1234}) & -(s_{123}+s_{1234}) & -s_{1234} \\ c_1+c_{12}+c_{123}+c_{1234} & c_{12}+c_{123}+c_{1234} & c_{123}+c_{1234} & c_{1234} \end{pmatrix}$$

and we used the compact notation $c_{ijk} = \cos(\theta_i+\theta_j+\theta_k)$ and similar.

Note that this matrix can be conveniently rewritten as

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}'(\boldsymbol{\theta})\, \boldsymbol{T}.$$

Being $\boldsymbol{T}$ nonsingular, the analysis of the rank deficiencies of $\boldsymbol{J}$ can be performed on the simplified matrix $\boldsymbol{J}'$. We have a singularity at every configuration where the six (2×2) minors of $\boldsymbol{J}'$ vanish simultaneously. It is easy to see that this occurs if and only if

$$\sin\theta_2 = \sin\theta_3 = \sin\theta_4 = 0,$$

[1]The axis $-Y_1$ is obtained as $\boldsymbol{R}_1(\alpha)\cdot\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}^T$; the axis X_2 is obtained as $\boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\cdot\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. An alternative (but longer) procedure would be to extract $\boldsymbol{\omega}$ from the relation $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T$, with $\boldsymbol{R} = \boldsymbol{R}(\alpha,\beta,\gamma)$ given by eq. (1).

namely when all the links are folded or stretched along a common radial line originating at the robot base.

The configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & 0 & -\pi/4 & \pi/2 \end{pmatrix}^T$ is a regular one, and thus any Cartesian velocity $\boldsymbol{v}$ of the end-effector can be realized (by ∞^2 different joint velocity vectors $\dot{\boldsymbol{\theta}}$). The minimum norm solution is found when using the pseudoinverse of $\boldsymbol{J}$, namely

$$\begin{aligned}\dot{\boldsymbol{\theta}}^* &= \boldsymbol{J}^{\#}(\boldsymbol{\theta})\,\boldsymbol{v} = \boldsymbol{J}^T(\boldsymbol{\theta})\left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}^T(\boldsymbol{\theta})\right)^{-1}\boldsymbol{v} \\ &= \begin{pmatrix} 0 & 0 & 0 & -0.7071 \\ 3.4142 & 2.4142 & 1.4142 & 0.7071 \end{pmatrix}^{\#}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1752 \\ 0.1239 \\ 0.0726 \\ -1.4142 \end{pmatrix}\ \text{[rad/s]},\end{aligned}$$

with $\left\|\dot{\boldsymbol{\theta}}^*\right\| = 0.2265$.

Exercise 3

The presence of a small measurement error at joint 1 affects the computation of the nominal position of the robot end-effector through the direct kinematics. This error can be seen as a displacement of the end-effector position resulting from a small angular variation of the joint. As such, the Cartesian error for sufficiently small variations can be estimated by using differential arguments, i.e., through the robot Jacobian. In the considered case, we only need to evaluate the first column of the geometric Jacobian related to the linear velocity, i.e.,

$$\boldsymbol{\Delta p} = \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \end{pmatrix} = \boldsymbol{J}_{L1}(\boldsymbol{q})\,\Delta\theta_1 = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{0,e} \end{pmatrix}\Delta\theta_1.$$

Computations are simplified when expressing all vectors in the Denavit-Hartenberg frame RF_1 that has the $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ axes in the plane of motion of links 2 and 3. Moreover, what really matters is the distance of the end-effector from the axis $\boldsymbol{z}_0$ of joint 1 (i.e., the component of $\boldsymbol{p}_{0,e}$ along the $\boldsymbol{x}_1$ direction). Therefore

$$\|\boldsymbol{\Delta p}\| = \left\|{}^1\boldsymbol{\Delta p}\right\| = \left\|{}^1\boldsymbol{z}_0 \times {}^1\boldsymbol{p}_{0,e}\right\| \cdot |\Delta\theta_1| = \left\|\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) \\ * \\ 0 \end{pmatrix}\right\| \cdot |\Delta\theta_1|,$$

where a_2 and a_3 are DH parameters (the length of links 2 and 3) and $*$ denotes an irrelevant quantity. As a result, the maximum norm of the position error over the whole workspace is

$$\max\|\boldsymbol{\Delta p}\| = \max_{(\theta_2,\theta_3)} |a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)| \cdot |\Delta\theta_1| = (a_2+a_3)\cdot|\Delta\theta_1|.$$

As intuition suggests, the maximum error is obtained when the robot is stretched horizontally, with its end-effector at the limit of the robot workspace.

* * * * *

Robotics I

February 6, 2013

Exercise 1

Figure 1: A 4-DOF finger of a robotic hand (left) and the kinematics of the full hand (right). The first three DOFs of the index finger are circled in red

Figure 1 shows a picture of the index finger of a prototype robotic hand, as well as the kinematics scheme of the full hand. The index finger has 4 revolute degrees of freedom (DOFs), two at its base, allowing abduction/adduction and flexion/extension, and two other for flexion/extension of the phalanxes. For simplicity, assume that each joint is independently actuated. A reference frame is placed at the base of the finger, attached to the palm of the hand (see Fig. 2).

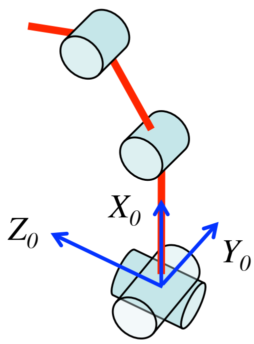


Figure 2: Kinematics scheme of the 4-DOF index finger

a) Assign a set of Denavit-Hartenberg frames to the robotic finger and define the associated table of parameters, introducing symbolic quantities as needed. The origin of the last frame should be located at the tip of the finger.

b) By mimicking the mobility of the index finger of your hand, define reasonable numerical values for the joint ranges of the robotic finger.

c) Derive the 6×4 geometric Jacobian of the robotic finger in symbolic form.

%

Exercise 2

Consider the class of trigonometric functions of time

$$q(t) = \sum_{h=1}^{m} \left[a_h \sin\left((2h-1)\,\frac{\pi}{2}\frac{t}{T} \right) + b_h \cos\left((2h-1)\,\frac{\pi}{2}\frac{t}{T} \right) \right], \qquad t \in [0,T] \tag{1}$$

parametrized by the $2m$ coefficients a_h, b_h, for $h = 1, \ldots, m$, and let a trajectory planning problem be specified by the following boundary conditions to be interpolated

$$q(0) = q_0, \quad q(T) = q_1, \quad \dot{q}(0) = v_0, \quad \dot{q}(T) = v_1, \tag{2}$$

where $T > 0$ is the motion time.

a) Address problem (2) using (1), so that a solution trajectory $q(t)$ always exists and is uniquely specified by the given data. Write a program that computes such solution and plots the resulting position, velocity, and acceleration profiles.

b) Test your program on the two data sets:

i) $q_0 = -40°$, $q_1 = 40°$, $v_0 = v_1 = 0$, $T = 2$ s;

ii) $q_0 = 20°$, $q_1 = 60°$, $v_0 = v_1 = 0$, $T = 2$ s.

Verify that in the second case the resulting trajectory *wanders*[1], while no under- or over-shoot occurs to the position profile in the first case. Which is the source of this different behavior?

c) Expand the class of trajectories $q(t)$ in (1) by adding a constant term, namely

$$q'(t) = \bar{q} + q(t). \tag{3}$$

The previous method can be modified in a simple way so that, at least in the rest-to-rest case ($v_0 = v_1 = 0$), the solution $q'(t)$ will always be a non-wandering trajectory. Show how to obtain this, and test the modified solution again on case *ii)* above.

[180 minutes; open books]

[1] In general, *wandering* in case of a rest-to-rest trajectory between two points q_0 and q_1 means that the position exceeds the interval $[\min\{q_0, q_1\}, \max\{q_0, q_1\}]$ at some instant during motion.

Solutions

February 6, 2013

Exercise 1

Figure 3 shows a possible assignment of Denavit-Hartenberg frames for the robotic finger. Table 1 contains the associated parameters, where L_2, L_3, and L_4 are the lengths of the three phalanxes of the finger.

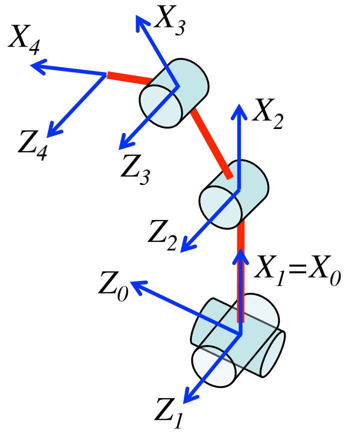


Figure 3: Denavit-Hartenberg frames for the 4-DOF index finger

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	0	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3
4	0	0	L_4	q_4

Table 1: Denavit-Hartenberg parameters associated to Fig. 3

With the above choices, the zero configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T = \boldsymbol{0}$ corresponds to the finger pointing straight upward. Accordingly, possible estimates of the joint ranges that mimic the mobility of a human index finger are

$$q_1 \in [-20°, +20°]\,, \quad q_2 \in [-10°, +80°]\,, \quad q_2 \in [0°, +95°]\,, \quad q_4 \in [0°, +45°]\,.$$

The desired 6×4 Jacobian matrix is most efficiently computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{p}_{04}}{\partial q_1} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_2} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_3} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_4} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix},$$

namely:

- for the first three rows (linear components), by analytic derivation of the finger tip position vector $\boldsymbol{p}_{04}$;
- for the last three rows (angular components), by the standard geometric expressions for revolute joints.

From Table 1, we obtain the homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} \boldsymbol{R}_1(q_1) & \boldsymbol{p}_{01} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{A}_i(q_i) = \begin{pmatrix} \boldsymbol{R}_i(q_i) & \boldsymbol{p}_{i-1,i}(q_i) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_i & -\sin q_i & 0 & L_i \cos q_i \\ \sin q_i & \cos q_i & 0 & L_i \sin q_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for } i = 2, 3, 4.$$

The position of the tip finger is then

$$\begin{aligned} \boldsymbol{p}_{04,\text{hom}}(\boldsymbol{q}) &= \boldsymbol{A}_1(q_1)\left(\boldsymbol{A}_2(q_2)\left(\boldsymbol{A}_3(q_3)\left(\boldsymbol{A}_4(q_4)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)\right) \\ &= \begin{pmatrix} \cos q_1 \left(L_2 \cos q_2 + L_3 \cos(q_2+q_3) + L_4 \cos(q_2+q_3+q_4)\right) \\ \sin q_1 \left(L_2 \cos q_2 + L_3 \cos(q_2+q_3) + L_4 \cos(q_2+q_3+q_4)\right) \\ L_2 \sin q_2 + L_3 \sin(q_2+q_3) + L_4 \sin(q_2+q_3+q_4) \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{q}) \\ 1 \end{pmatrix}. \end{aligned}$$

In the first row above, brackets have been used to indicate the most convenient order of products, especially for symbolic computations.

Using the usual compact notations, we obtain $\boldsymbol{J}_L(\boldsymbol{q})$ as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{q})}{\partial \boldsymbol{q}} =$$

$$\begin{pmatrix} -(L_2c_2 + L_3c_{23} + L_4c_{234})s_1 & -(L_2s_2 + L_3s_{23} + L_4s_{234})c_1 & -(L_3s_{23} + L_4s_{234})c_1 & -L_4s_{234}c_1 \\ (L_2c_2 + L_3c_{23} + L_4c_{234})c_1 & -(L_2s_2 + L_3s_{23} + L_4s_{234})s_1 & -(L_3s_{23} + L_4s_{234})s_1 & -L_4s_{234}s_1 \\ 0 & L_2c_2 + L_3c_{23} + L_4c_{234} & L_3c_{23} + L_4c_{234} & L_4c_{234} \end{pmatrix},$$

while for $\boldsymbol{J}_A(\boldsymbol{q})$ we have

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_1 = \boldsymbol{R}_1(q_1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{z}_2 = \boldsymbol{R}_1(q_1)\boldsymbol{R}_2(q_2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{z}_3 = \boldsymbol{R}_1(q_1)\boldsymbol{R}_2(q_2)\boldsymbol{R}_3(q_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}.$$

Indeed, $\boldsymbol{z}_1 = \boldsymbol{z}_2 = \boldsymbol{z}_3$ since the three joints axes 2, 3, and 4 are always parallel.

Exercise 2

By considering $m = 2$ in eq. (1), the function $q(t)$ will contain the four coefficients a_1, b_1, a_2, and b_2, which are necessary and also sufficient, as we will see, to impose four arbitrary boundary conditions in (2). We have thus

$$q(t) = a_1 \sin\left(\frac{\pi}{2}\frac{t}{T}\right) + a_2 \sin\left(\frac{3\pi}{2}\frac{t}{T}\right) + b_1 \cos\left(\frac{\pi}{2}\frac{t}{T}\right) + b_2 \cos\left(\frac{3\pi}{2}\frac{t}{T}\right),$$

and

$$\dot{q}(t) = \left(\frac{\pi}{2T}\right) \cdot \left(a_1 \cos\left(\frac{\pi}{2}\frac{t}{T}\right) + 3a_2 \cos\left(\frac{3\pi}{2}\frac{t}{T}\right) - b_1 \sin\left(\frac{\pi}{2}\frac{t}{T}\right) - 3b_2 \sin\left(\frac{3\pi}{2}\frac{t}{T}\right)\right).$$

Note that the arguments of the trigonometric terms are properly scaled with respect to the motion time T, so that all these terms take only the values 0, +1 or −1 at the boundaries $t = 0$ and $t = T$. Therefore,

$$q(0) = q_0 \quad \Rightarrow \quad b_1 + b_2 = q_0, \qquad q(T) = q_1 \quad \Rightarrow \quad a_1 - a_2 = q_1,$$

and

$$\dot{q}(0) = v_0 \quad \Rightarrow \quad a_1 + 3a_2 = v_0\,\frac{2T}{\pi}, \qquad \dot{q}(T) = v_1 \quad \Rightarrow \quad -b_1 + 3b_2 = v_1\,\frac{2T}{\pi}.$$

The resulting linear system of equations has a simple block diagonal structure

$$\begin{pmatrix} 1 & -1 & & \\ 1 & 3 & & \\ & & 1 & 1 \\ & & -1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ (2T/\pi)\, v_0 \\ q_0 \\ (2T/\pi)\, v_1 \end{pmatrix}$$

and is always solvable (the determinant of both 2×2 blocks is equal to 4), leading to

$$\begin{aligned} a_1 &= \frac{1}{4}\left(3q_1 + (2T/\pi)\, v_0\right) & a_2 &= \frac{1}{4}\left((2T/\pi)\, v_0 - q_1\right) \\ b_1 &= \frac{1}{4}\left(3q_0 - (2T/\pi)\, v_1\right) & b_2 &= \frac{1}{4}\left(q_0 + (2T/\pi)\, v_1\right). \end{aligned} \tag{4}$$

We note that, by increasing the order m, additional boundary conditions on higher order derivatives could be handled similarly (e.g., with $m = 3$, we can match also desired initial and final accelerations). Just for reference, the expression of the time derivative of (1) for a generic m is

$$\dot{q}(t) = \frac{\pi}{2T} \sum_{h=1}^{m} (2h-1) \left[a_h \cos\left((2h-1)\,\frac{\pi}{2}\frac{t}{T}\right) - b_h \cos\left((2h-1)\,\frac{\pi}{2}\frac{t}{T}\right)\right], \qquad t \in [0, T]$$

and the four boundary conditions (2) at $t = 0$ and $t = T$ are written in general as

$$\sum_{h=1}^{m} b_h = q_0, \quad \sum_{h=1}^{m} (-1)^{h-1} a_h = q_1, \quad \sum_{h=1}^{m} (2h-1) a_h = v_0\,\frac{2T}{\pi}, \quad \sum_{h=1}^{m} (2h-1)(-1)^h b_h = v_1\,\frac{2T}{\pi}.$$

While this flexibility is a nice feature of the considered class of trajectories, there is still an issue concerning the full predictability of the interpolating motion.

For case *i)*, which is a rest-to-rest motion in $T = 2$ s from $q_0 = -40°$ to $q_1 = 40°$, Figure 4 shows the position and velocity profiles of the solution trajectory, obtained for $a_1 = -b_1 = 30$

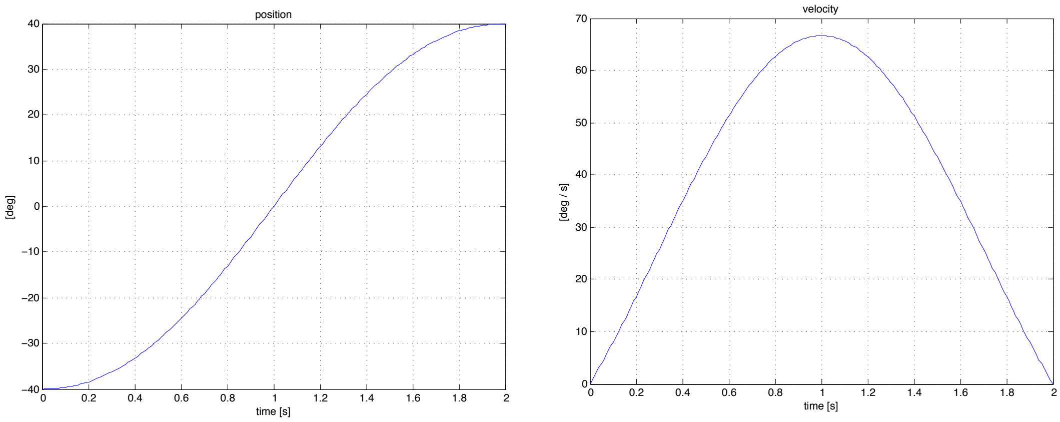


Figure 4: Rest-to-rest motion case *i)*: Balanced data give no wandering motion

and $a_2 = b_2 = -10$. The complete motion is satisfactory, with no wandering (neither under- nor over-shoot in position). We call this a situation with *balanced* data: the initial and final positions have opposite values, and motion occurs symmetrically around the zero average position. As a matter of fact, it can be shown analytically that $q_0 \cdot q_1 \leq 0$ is a sufficient condition for the velocity *not* to change sign in the interval $[0, T]$.

Figure 5 shows the position and velocity profiles obtained for the rest-to-rest case *ii)*, where $q_0 = 20°$ and $q_1 = 60°$. From (4), we have $a_1 = 45$, $b_1 = 15$, $a_2 = -15$, and $b_2 = 5$. Despite the total displacement is now halved with respect to the first case, a wandering behavior is present: the position undershoots the initial value $q_0 = 20°$ during the first 0.6 s (velocity is negative for 0.4 s from the motion start). The only apparent difference is that the initial and final position data are now *unbalanced* with respect to zero.

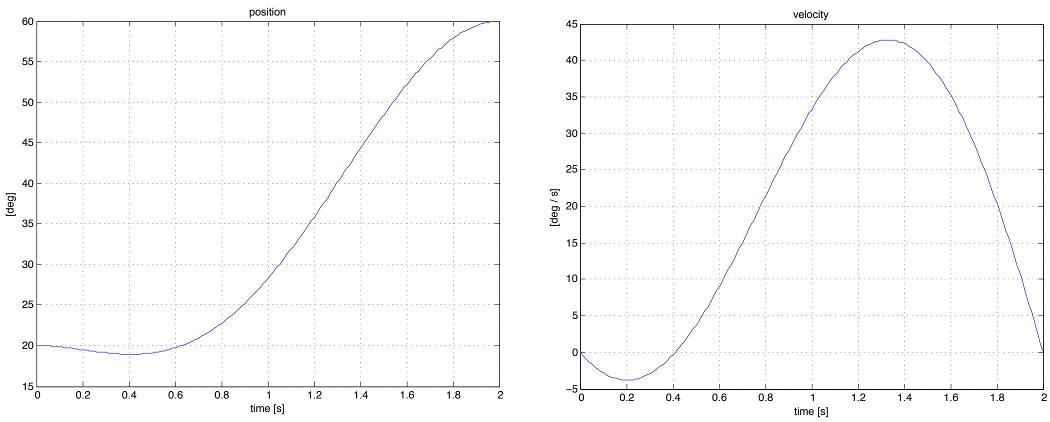


Figure 5: Rest-to-rest motion case *ii)*: Unbalanced data give rise to undershoot in position

With reference to the rest-to-rest case ($v_0 = v_1 = 0$), the above considerations allows to enforce a nice behavior in general (i.e., also in he unbalanced case). The addition of a suitable constant term, as in the modified trajectory (3), overcomes in fact the wandering problem.

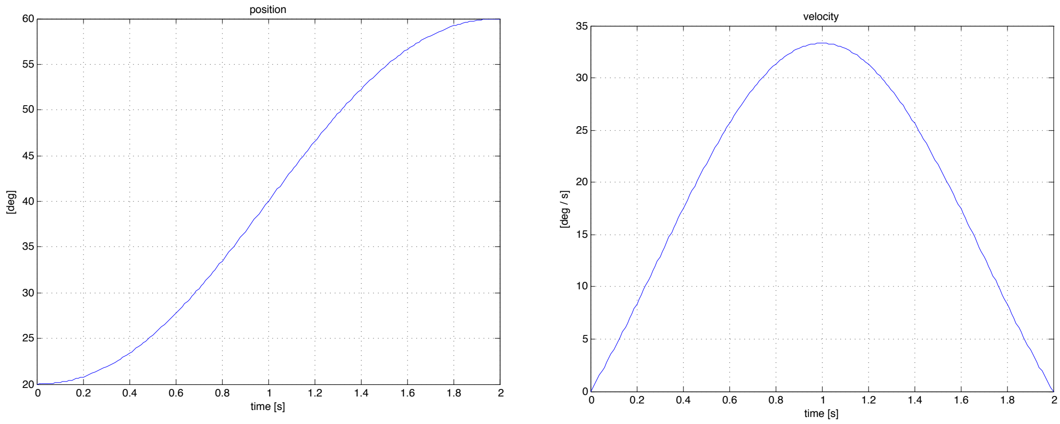


Figure 6: The modified solution for the rest-to-rest motion case *ii)* eliminates wandering, reduces accordingly the peak velocity, and provides a symmetric velocity profile

Denote now the given initial and final position as p_0 and p_1 (this redefinition allows using the previous formulas, with minimal intervention), and compute

$$\bar{q} = \frac{p_0 + p_1}{2}, \qquad q_0 = -\frac{p_1 - p_0}{2}, \qquad q_1 = \frac{p_1 - p_0}{2}. \tag{5}$$

The original data are thus transformed so that motion occurs around the average position $\bar{q}$, with opposite $q_1 = -q_0$ as desired. Taking into account the boundary conditions (2) for $q(t)$, the interpolation problem for $q'(t)$ is correctly formulated since

$$\text{at } t = 0 \quad \Rightarrow \quad q'(0) = \bar{q} + q_0 = p_0 \qquad\qquad \text{at } t = T \quad \Rightarrow \quad q'(T) = \bar{q} + q_1 = p_1.$$

As a result, the modified solution $q'(t)$ will be defined by $\bar{q}$ in (5) and by the same expressions (4) for a_1, a_2, b_1, and b_2, where q_0 and q_1 are now taken from (5).

The obtained trajectory will have a symmetric velocity profile with respect to the mid-motion instant $t = T/2$. We also note that the modified trajectory will provide a natural solution even in the degenerate case of no displacement ($p_0 = p_1 = p$): since we would have then $q_0 = q_1 = 0$, the solution will be a constant trajectory $q'(t) = \bar{q} = p$ (the previous method would generate instead a trajectory that moves out from the value p, returning to it at the final instant). Figure 6 shows the trajectory obtained with the modified method for case *ii)*. Wandering is no longer present. For completeness, Figure 7 reports the acceleration profiles of the various computed trajectories.

A Matlab code follows.

```
clear all; close all; clc
% boundary data
p0=20; % deg
p1=60;
v0=0; % deg/s
v1=0;
T=2; % total motion time (s)
% balanced solution does not wander (symmetric around average position)
pm=(p0+p1)/2;
```

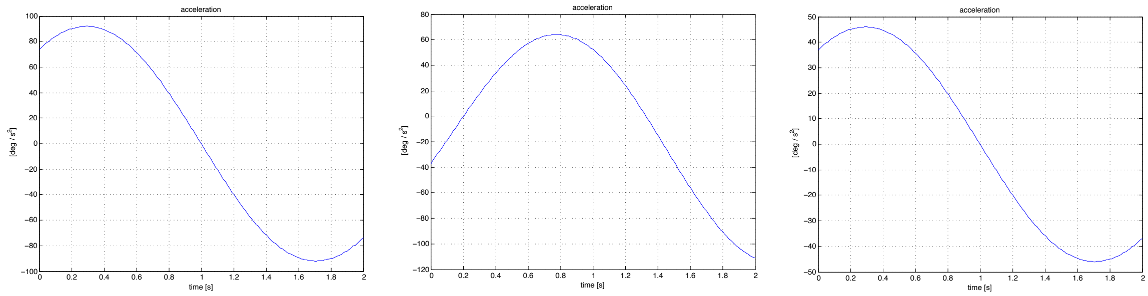


Figure 7: Acceleration profiles for case *i)* (left), case *ii)* with original solution (center), and case *ii)* with modified solution (right)

```
pd=p1-p0;
q0=-pd/2;
q1=pd/2;
% unbalanced solution leads to wandering ---uncomment to see this result
% q0=p0;
% q1=p1;
% pm=0;
% solution
a1=0.25*(3*q1+v0*2*T/pi);
a2=0.25*(v0*2*T/pi-q1);
b1=0.25*(3*q0-v1*2*T/pi);
b2=0.25*(q0+v1*2*T/pi);
t=[0:0.01:T];
tau=(pi/2)*t/T;
p=pm+a1*sin(tau)+a2*sin(3*tau)+b1*cos(tau)+b2*cos(3*tau);
pdot=pi/(2*T)*(a1*cos(tau)+3*a2*cos(3*tau)-b1*sin(tau)-3*b2*sin(3*tau));
pddot=-(pi/(2*T))^2*(a1*sin(tau)+9*a2*sin(3*tau)+b1*cos(tau)+9*b2*cos(3*tau));
plot(t,p);grid;title('position');xlabel('time [s]');ylabel('[deg]'); pause;
plot(t,pdot);grid;title('velocity');xlabel('time [s]');ylabel('[deg / s]'); pause;
plot(t,pddot);grid;title('acceleration');xlabel('time [s]');ylabel('[deg / s^2]')
% end
```

We conclude with a final remark. An alternative way to avoid trajectory wandering in the rest-to-rest case ($v_0 = v_1 = 0$) would be to replace eqs. (5) with

$$\bar{q} = p_0, \qquad q_0 = 0, \qquad q_1 = p_1 - p_0, \tag{6}$$

namely resetting the initial position to zero and working with the total displacement $p_1 - p_0$. When inserted in (4), these choices yield

$$a_1 = \frac{3}{4}\left(p_1 - p_0\right), \qquad a_2 = -\frac{1}{4}\left(p_1 - p_0\right), \qquad b_1 = b_2 = 0,$$

discarding the presence of cosinusoidal functions in (1). Figure 8 shows the resulting motion for case *ii)*. Wandering has been eliminated as expected (being $q_0 \cdot q_1 = 0$, the sufficient condition is in fact satisfied), but the velocity profile is no longer symmetric and has a higher peak (compare with Fig. 6, right). Thus, the modified solution in (5) is to be preferred to (6).

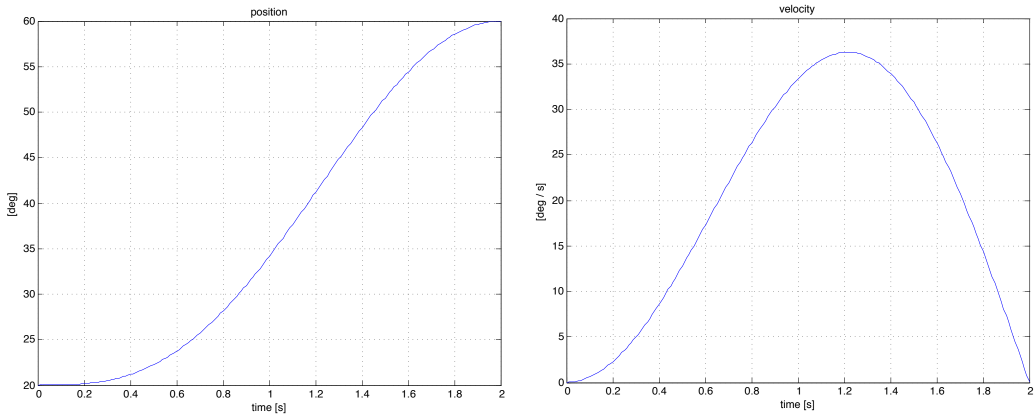


Figure 8: The alternative solution (6) for the rest-to-rest motion case *ii)* eliminates wandering, but yields a non-symmetric velocity profile as opposed to the use of (5)

* * * * *

Robotics I

June 10, 2013

Table 1 contains the Denavit-Hartenberg parameters of a robot with four revolute joints.

i	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	$-\frac{\pi}{2}$	0	d_3	θ_3
4	0	a_4	0	θ_4

Table 1: Denavit-Hartenberg parameters of a 4R robot

1. Draw a kinematic sketch of the robot, including the associated Denavit-Hartenberg frames according to Tab. 1.
2. Draw the two robot configurations corresponding to $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/2 & \pi & 0 \end{pmatrix}^T$ [rad].
3. Find a singular configuration for the 3×4 Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$ relating $\dot{\boldsymbol{\theta}}$ to the linear velocity $\boldsymbol{v}$ of the origin of frame 4.
4. In such a singular configuration $\boldsymbol{\theta}^*$, consider as numerical data $d_3 = a_4 = 0.5$ [m].
   a) Provide the numerical value of a *feasible* $\boldsymbol{v}_f$ and determine a minimum norm joint velocity $\dot{\boldsymbol{\theta}}_f$ such that $\boldsymbol{J}(\boldsymbol{\theta}^*)\dot{\boldsymbol{\theta}}_f = \boldsymbol{v}_f$. Is this minimum norm solution unique?
   b) Provide the numerical value of an *unfeasible* $\boldsymbol{v}_u$ and use the Jacobian pseudoinverse to compute the joint velocity $\dot{\boldsymbol{\theta}}_u = \boldsymbol{J}^{\#}(\boldsymbol{\theta}^*)\boldsymbol{v}_u$. Which are the properties of $\dot{\boldsymbol{\theta}}_u$?

[120 minutes; open books]

Robotics I

July 15, 2013

For a 2R planar robot having links of equal length $\ell_1 = \ell_2 = 0.5$ [m], consider the rest-to-rest motion defined by the joint velocity profiles $\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \dot{q}_1(t) & \dot{q}_2(t) \end{pmatrix}^T$ shown in Fig. 1. The motion starts at $t = 0$ from $\boldsymbol{q}(0) = \begin{pmatrix} -30° & 60° \end{pmatrix}^T$ and ends at $t = T$. The trajectory parameters are:

$$T = 2 \text{ [s]}, \qquad T_{s,1} = 0.5 \text{ [s]}, \qquad T_{s,2} = 1 \text{ [s]}, \qquad V_{max,1} = 50 \text{ [°/s]}, \qquad V_{max,2} = -90 \text{ [°/s]}.$$

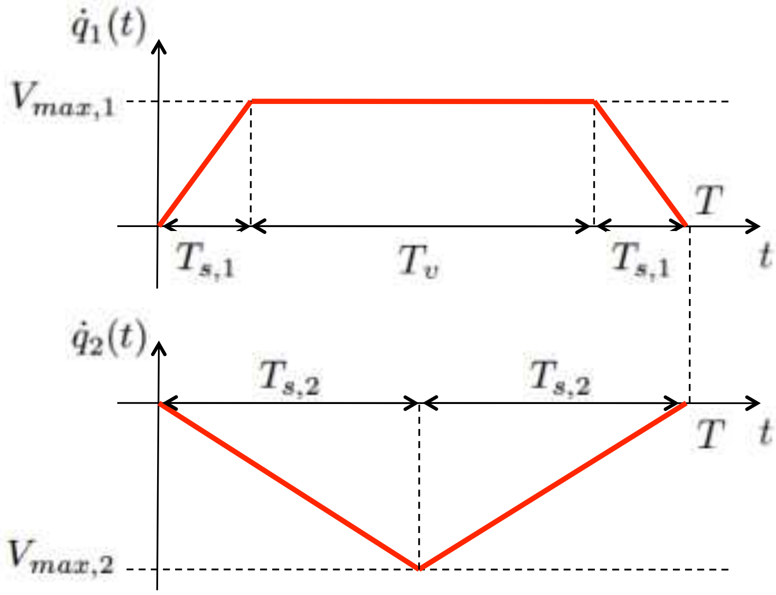


Figure 1: Velocity profiles of joint 1 (top) and joint 2 (bottom)

i) Determine the displacement of both joints at the end of motion and the Cartesian distance between the initial and final position $\boldsymbol{p}$ of the robot end-effector. Does the robot cross a singular configuration?

ii) Compute the velocity $\dot{\boldsymbol{p}}$ and acceleration $\ddot{\boldsymbol{p}}$ of the robot end-effector at $t_1 = T/10$ and $t_2 = T/2$. Sketch the robot configuration and the two vectors at these instants of time.

[120 minutes; open books]

Robotics I

September 19, 2013

Consider the planar RPPR robot in Fig. 1.

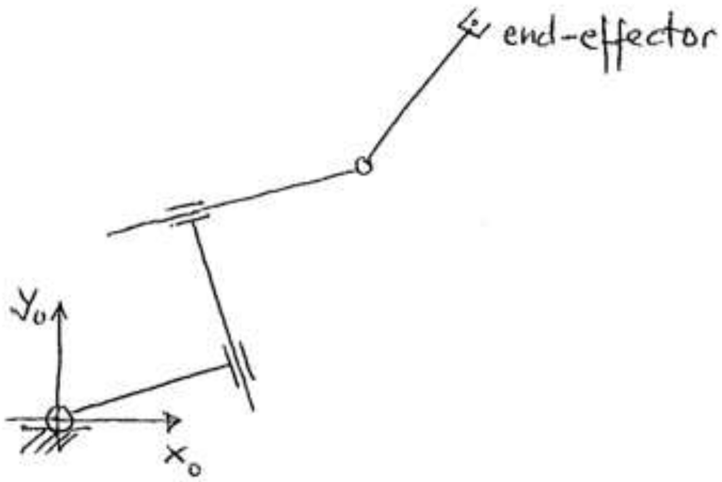


Figure 1: Planar RPPR robot

- Assign the frames according to the Denavit-Hartenberg convention and provide the associated table of parameters.
- Suppose that the two prismatic joints have a limited range: $|q_i| < D$, $i = 2, 3$. Determine the maximum possible distance of the end-effector from the origin of the base frame (*maximum reach*), and the robot configuration(s) q at which this value is attained.

[120 minutes; open books]

Robotics I

Test — November 29, 2013

Exercise 1 [6 points]

A DC motor is used to actuate a single robot link that rotates in the horizontal plane around a joint axis passing through its base. The motor is connected to the link by means of two transmission/reduction elements placed in series: a spur gear (SG) made of two toothed wheels, and a harmonic drive (HD). The output shaft of the motor drives the smaller toothed wheel (of radius r_1). The output axis of the larger wheel (of radius $r_2 > r_1$) is connected to the wave generator of the HD. Finally, the output axis of the flexspline is the joint axis of the robot link. The motor delivers a maximum torque $T_{m,max} = 2.2$ [Nm], while the inertia of its rotor is $J_m = 0.0012$ [kg·m^2]. The smaller wheel of the gear has radius $r_1 = 2$ [cm]. The flexspline of the HD has 70 outer teeth. Finally, the link has an inertia $J_\ell = 5.88$ [kg·m^2] around its rotation axis (at the link base).

a) Neglecting dissipative effects and other inertial loads except rotor and link inertias, determine the radius r_2 of the larger toothed wheel of the spur gear so that the reduction ratio $n > 1$ of the complete transmission is *optimal* in terms of motor torque needed to accelerate the link.

b) With the resulting optimal value n^*, determine the maximum angular acceleration $\ddot{\theta}_{\ell,max}$ of the link that can be realized using this motor/transmission unit.

Exercise 2 [12 points]

The K-1207 robot developed by Robotics Research Co. (USA) is a modular 7-dof manipulator having all revolute joints and *no* spherical wrist or shoulder. Figure 1 shows a picture of the robot (mounted on an inclined base) and a few snaphots of the robot in motion (mounted on a vertical base), in order to illustrate its dexterity.

Figure 1: The Robotics Research K-1207 robot

A drawing of the K-1207 robot for a particular configuration is shown in Fig. 2, with indication of the seven revolute joint axes and the physical sizes (in inches). Origin O_0 and axis $\boldsymbol{x}_0$ of the base frame as well as origin O_7 and axis $\boldsymbol{z}_7$ of the last frame are assigned as in the figure.

a) Assign the link frames and the table of parameters according to the Denavit-Hartenberg convention (use the extra sheet for your sketch of the frames).

b) Provide the numerical values of the constant parameters and of the joint variables associated to the configuration shown in Fig. 2.

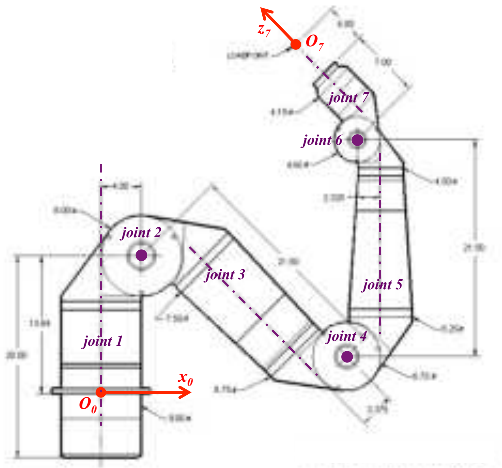


Figure 2: Drawing of the K-1207 robot: In this configuration, joint axes 1, 3, 5, and 7 are on a common plane, while joint axes 2, 4, and 6 are normal to this plane

Exercise 3 [12 points]

Consider the planar RPR robot shown in Fig. 3, and the definition of joint variables $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ given therein. The three-dimensional task vector is $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p}^T & \alpha \end{pmatrix}^T = \boldsymbol{f}(\boldsymbol{q})$, where $\boldsymbol{p} = \begin{pmatrix} p_x & p_y \end{pmatrix}^T$ is the position of the end-effector and α is the orientation angle of the last link w.r.t. the $\boldsymbol{x}_0$ axis. Assume that $q_2 \geq 0$ holds for the prismatic joint variable.

a) Solve the inverse kinematics problem for a given $\boldsymbol{r}_d$, providing the expression of all feasible solutions in closed form.

b) Compute the solutions $\boldsymbol{q}$ associated to $\boldsymbol{r}_d = \begin{pmatrix} -2 & -2 & \pi/2 \end{pmatrix}^T$ [m,m,rad] (i.e., such that $\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{r}_d$) using the data $L_1 = 1$ [m] and $L_3 = 0.7$ [m], and sketch the robot configurations.

c) Draw the primary workspace in the plane of robot motion for generic values of L_1 and L_3, assuming that the prismatic joint range is bounded as $|q_2| \leq D$ (with $D > 0$).

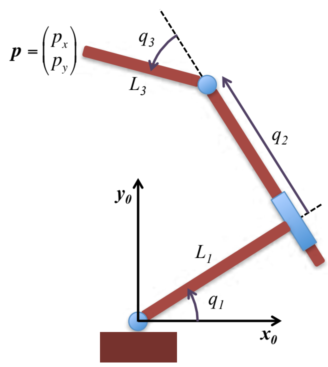


Figure 3: A planar RPR robot with the definition of joint variables

[210 minutes; open books]

Solutions

November 29, 2013

Exercise 1 [6 points]

a) The reduction ratio of the gear is obtained by equating the absolute value of the linear velocities of the two wheels at the point of contact between the meshing teeth (the wheels rotate in opposite directions). Denoting by ω_i the angular velocity of toothed wheel i (1 = input, 2 = output),

$$|\omega_1|\, r_1 = |\omega_2|\, r_2 \qquad \Rightarrow \qquad n_{SG} = \left|\frac{\omega_1}{\omega_2}\right| = \frac{r_2}{r_1} = \frac{r_2}{2}.$$

The reduction ratio of the harmonic drive is

$$n_{HD} = \frac{\#\,\text{teeth of flexspline}}{\#\,\text{teeth of circular spline} - \#\,\text{teeth of flexspline}} = \frac{70}{2} = 35,$$

since the number of (inner) teeth of the circular spline always exceeds that (in the outer side) of the flexspline by 2. The complete transmission has then reduction ratio

$$n = n_{SG} \cdot n_{HD} = \frac{r_2}{2} \cdot 35 = 17.5\, r_2.$$

The optimal value of the reduction ratio that minimizes the motor torque needed to accelerate the link at a desired value $\ddot{\theta}_\ell$ is

$$n^* = \sqrt{\frac{J_\ell}{J_m}} = \sqrt{\frac{5.88}{0.0012}} = \sqrt{4900} = 70.$$

Thus, such reduction ratio is obtained by choosing $r_2 = 70/17.5 = 4$ [cm] ($n_{SG} = 2$).

b) The torque balance for the complete motor/transmission/load system is then

$$T_m = J_m \ddot{\theta}_m + \frac{1}{n^*} J_\ell \ddot{\theta}_\ell = J_m \left(n^* \ddot{\theta}_\ell\right) + \frac{1}{n^*} J_\ell \ddot{\theta}_\ell = \left(J_m \sqrt{\frac{J_\ell}{J_m}} + J_\ell \sqrt{\frac{J_m}{J_\ell}}\right) \ddot{\theta}_\ell = 2\sqrt{J_\ell J_m}\, \ddot{\theta}_\ell$$

$$\left(\text{or also} \ldots = \frac{1}{n^*}\left(n^{*2} J_m + J_\ell\right) \ddot{\theta}_\ell = \frac{2\, J_\ell}{n^*}\, \ddot{\theta}_\ell\right).$$

As a result, the maximum angular acceleration of the link is

$$\ddot{\theta}_{\ell,max} = \frac{T_{m,max}}{2\sqrt{J_\ell J_m}} \left(= \frac{n^* T_{m,max}}{2\, J_\ell}\right) = \frac{2.2}{0.168} = 13.095 \ \ [\text{rad/s}^2].$$

Exercise 2 [12 points]

a) A feasible assignment of the Denavit-Hartenberg frames is shown in Figure 4. The associated parameters are given in Table 1. We note that the mechanical (modular) structure of joints 2, 4, and 6 leads to the kinematic identities $|a_1| = |a_2|$, $|a_3| = |a_4|$, and $|a_5| = |a_6|$ (the absolute value is needed because of different possible choices for the positive directions of the $\boldsymbol{x}_i$ axes).

b) The numerical values of the constant parameters a_i and d_i are specified (in inches) in the same Table, together with the values of the joint variables $q_i = \theta_i \in (-\pi, +\pi]$ when the robot is in the shown configuration. In this configuration, the $\boldsymbol{z}_i$ axes not lying in the plane (i.e., for $i = 2, 4, 6$) are pointing outwards.

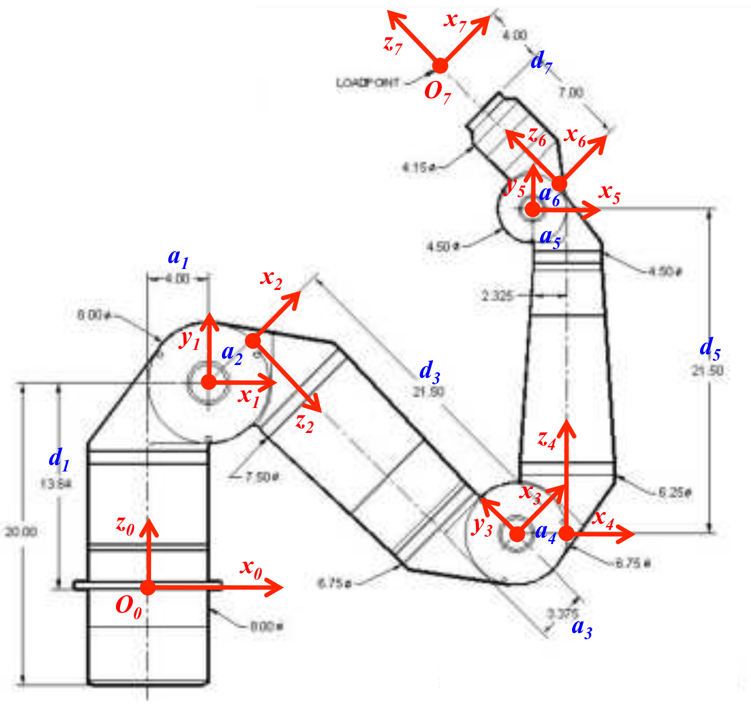


Figure 4: Assignment of Denavit-Hartenberg frames for the K-1207 robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 = 4.00$"	$d_1 = 13.64$"	$q_1 = 0$
2	$\pi/2$	$a_2 = 4.00$"	0	$q_2 = \pi/4$
3	$-\pi/2$	$a_3 = 3.375$"	$d_3 = 21.50$"	$q_3 = 0$
4	$-\pi/2$	$a_4 = 3.375$"	0	$q_4 = -\pi/4$
5	$\pi/2$	$a_5 = -2.25$"	$d_5 = 21.50$"	$q_5 = 0$
6	$-\pi/2$	$a_6 = 2.25$"	0	$q_6 = \pi/4$
7	0	0	$d_7 = 11.00$"	$q_7 = 0$

Table 1: Denavit-Hartenberg parameters for the K-1207 robot (associated to the frames in Fig. 4)

Exercise 3 [12 points]

a) The direct kinematics for the given task is

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \alpha \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} L_1 \cos q_1 + q_2 \cos\left(q_1 + \frac{\pi}{2}\right) + L_3 \cos\left(q_1 + q_3 + \frac{\pi}{2}\right) \\ L_1 \sin q_1 + q_2 \sin\left(q_1 + \frac{\pi}{2}\right) + L_3 \sin\left(q_1 + q_3 + \frac{\pi}{2}\right) \\ q_1 + q_3 + \frac{\pi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} L_1 \cos q_1 - q_2 \sin q_1 - L_3 \sin\left(q_1 + q_3\right) \\ L_1 \sin q_1 + q_2 \cos q_1 + L_3 \cos\left(q_1 + q_3\right) \\ q_1 + q_3 + \frac{\pi}{2} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

From a given $\boldsymbol{r}$ $(= \boldsymbol{r}_d)$, we can easily write the position $\boldsymbol{w}$ of the end-point of the second link as

$$\boldsymbol{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \boldsymbol{p} - L_3 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} L_1 \cos q_1 - q_2 \sin q_1 \\ L_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix}.$$

Squaring and summing the components of $\boldsymbol{w}$ yields

$$q_2 = +\sqrt{w_x^2 + w_y^2 - L_1^2} = +\sqrt{\left(p_x - L_3\cos\alpha\right)^2 + \left(p_y - L_3\sin\alpha\right)^2 - L_1^2}. \tag{1}$$

Only the positive sign has been kept in (1), since we have assumed that only $q_2 \geq 0$ is feasible. Indeed, the value of q_2 is real if and only if $\|\boldsymbol{w}\| \geq L_1$ (namely, when the position $\boldsymbol{w}$ of the tip of the second link is in the workspace of the sub-structure RP made by the first two joints and links). With q_2 from (1), we can always solve the following linear system for $\cos q_1$ and $\sin q_1$

$$\begin{pmatrix} L_1 & -q_2 \\ q_2 & L_1 \end{pmatrix} \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \end{pmatrix},$$

yielding

$$\begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \frac{1}{L_1^2 + q_2^2} \begin{pmatrix} L_1 w_x + q_2 w_y \\ L_1 w_y - q_2 w_x \end{pmatrix}.$$

Being $L_1^2 + q_2^2 > 0$, we can skip division by this quantity when evaluating q_1 with the ATAN2 function. Thus,

$$\begin{aligned} q_1 &= \text{ATAN2}\left\{L_1 w_y - q_2 w_x, L_1 w_x + q_2 w_y\right\} \\ &= \text{ATAN2}\left\{L_1\left(p_y - L_3\sin\alpha\right) - q_2\left(p_x - L_3\cos\alpha\right), L_1\left(p_x - L_3\cos\alpha\right) + q_2\left(p_y - L_3\sin\alpha\right)\right\} \end{aligned} \tag{2}$$

Finally, we have

$$q_3 = \alpha - q_1 - \frac{\pi}{2}. \tag{3}$$

There is only one feasible solution to the inverse kinematics problem, as given by eqs. (1)–(3).

b) With the data $L_1 = 1$ [m], $L_3 = 0.7$ [m], and $\boldsymbol{r}_d = \begin{pmatrix} p_x & p_y & \alpha \end{pmatrix}^T = \begin{pmatrix} -2 & -2 & \pi/2 \end{pmatrix}^T$ [m,m,rad], the above formulas yield (the resulting robot configuration is sketched in Fig. 5)

$$\begin{aligned} \boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T &= \begin{pmatrix} 2.8062 & 3.2078 & -2.8062 \end{pmatrix}^T \quad \text{[rad, m, rad]} \\ &= \begin{pmatrix} 160.7855 & 3.2078 & -160.7855 \end{pmatrix}^T \quad \text{[deg, m, deg]}. \end{aligned} \tag{4}$$

c) In order to determine the robot primary workspace for $|q_2| \leq D$, we have to distinguish two cases. When $L_1 \geq L_3$, the primary workspace is an annulus with inner and outer radius given by

$$R_{in, L_1 \geq L_3} = L_1 - L_3 \geq 0, \qquad R_{out} = \sqrt{L_1^2 + D^2} + L_3 > 0.$$

Figure 6 shows the actual construction of the workspace in the case of a length L_1 strictly larger than L_3. In particular, for equal link lengths $L_1 = L_3$, the workspace is a circle of radius R_{out}. When $L_1 < L_3$, the additional mobility provided by the prismatic joint allows to reduce, at least in part, the 'hole' (of radius $L_3 - L_1$) at the center that would characterize the workspace of a 2R planar arm. It is easy to see that, starting with $q_2 = 0$ and with the third link folded on the first one, the robot end-effector can access this inner part by progressively extending the second joint and *pivoting* with the third link (or with its prolongation) about the origin. When the second joint reaches its limit, the end-effector will be at a distance $|R_{in, L_1 < L_3}|$ from the origin, with

$$R_{in, L_1 < L_3} = L_3 - \sqrt{L_1^2 + D^2}, \qquad \Rightarrow \qquad R_{in, L_1 < L_3} = 0 \iff D^2 = L_3^2 - L_1^2.$$

Therefore, the workspace will be an annulus with inner radius $R_{in, L_1 < L_3} > 0$ and outer radius R_{out} when $D^2 < L_3^2 - L_1^2$, or a circle of radius R_{out} when $D^2 \geq L_3^2 - L_1^2$.

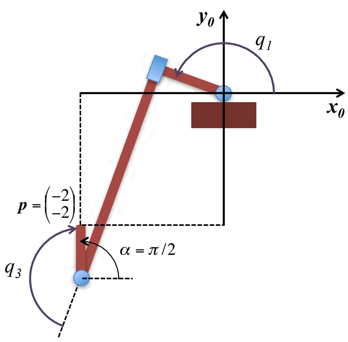


Figure 5: The inverse kinematic solution (4) associated to $\boldsymbol{r}_d = (-2 \ -2 \ \pi/2)^T$ [m, m, rad]

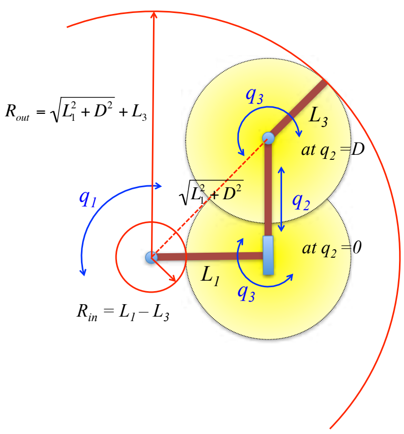


Figure 6: Primary workspace of RPR robot with prismatic joint in range $[-D, +D]$, for $L_1 > L_3$

* * * * *

Robotics I

January 9, 2014

Exercise 1

A planar PPR robot is shown in Fig. 1, together with the axes ($\boldsymbol{x}_w$,$\boldsymbol{y}_w$) of a world reference frame RF_w. The third link of the robot has length L. The position of the end-effector in the plane is given by ${}^w\boldsymbol{p} = \begin{pmatrix} {}^w p_x & {}^w p_y \end{pmatrix}^T$.

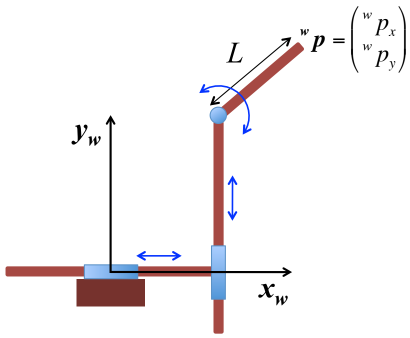


Figure 1: Planar PPR robot

- Assign the frames according to the Denavit-Hartenberg convention and provide the associated table of parameters. Make sure that all constant parameters in the table are *non-negative*.
- Define the homogeneous transformation matrix ${}^w\boldsymbol{T}_0$ between the world reference frame RF_w and the Denavit-Hartenberg frame RF_0 just assigned.
- Assuming that the two prismatic joints have a limited range, $|q_i| < D$ ($i = 1, 2$) with $D > L$:
  - draw the primary and secondary workspaces of the robot, respectively WS_1 and WS_2;
  - for a given end-effector position ${}^w\boldsymbol{p} \in WS_2$, provide all inverse kinematics solutions (q_1, q_2) as parametric functions of q_3.

Exercise 2

Consider a trajectory planning problem for the orientation of the end-effector of a robot. The end-effector should move from an initial orientation, specified by the rotation matrix $\boldsymbol{R}_{in}$, to a final orientation, specified by $\boldsymbol{R}_{fin}$, in time T and with zero initial and final angular velocity ($\boldsymbol{\omega}(0) = \boldsymbol{\omega}(T) = \boldsymbol{0}$). The trajectory has to be designed in terms of the (Y, Z, Y) Euler angles (α, β, γ) of a minimal representation of orientation. The motion time T has to be adjusted so that the norm of the angular velocity $\boldsymbol{\omega}(t)$ does never exceed a constant value $\Omega > 0$, i.e., $\|\boldsymbol{\omega}(t)\| \leq \Omega$ for all $t \in [0, T]$. After sketching the steps of the solution approach, provide a solution to the problem and the associated minimum feasible motion time T^* using as numerical data:

$$\boldsymbol{R}_{in} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{R}_{fin} = \begin{pmatrix} \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right) & \frac{1}{2} & \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -\frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) & -\frac{1}{2} & -\frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right) \end{pmatrix}, \quad \Omega = \pi \text{ [rad/s]}.$$

Exercise 3

For a 6R manipulator with *spherical* wrist, assume that the origin O_6 of the end-effector frame is placed at the center of the wrist. Let the joint velocity vector be partitioned into base and wrist velocities as $\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_b^T & \dot{\boldsymbol{q}}_w^T \end{pmatrix}^T$, with $\dot{\boldsymbol{q}}_b \in \mathbb{R}^3$ and $\dot{\boldsymbol{q}}_w \in \mathbb{R}^3$.

- Provide the symbolic expressions of $\dot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_w$ that assign a desired (zero) angular velocity $\boldsymbol{\omega}_d = \mathbf{0}$ to the end-effector frame and a desired velocity $\boldsymbol{v}_d \neq \mathbf{0}$ to its origin.

- Suppose now that the linear part of the motion task is specified by a desired position trajectory $\boldsymbol{p}_d(t)$, with $\boldsymbol{v}_d(t) = \dot{\boldsymbol{p}}_d(t)$, and that the orientation part is specified by *constant* values of a set of Euler angles $\boldsymbol{\phi}_d = (\alpha_d, \beta_d, \gamma_d)$, with $\boldsymbol{\omega}_d = \boldsymbol{T}(\alpha_d, \beta_d)\dot{\boldsymbol{\phi}}_d = \mathbf{0}$. At time $t = 0$, the robot configuration $\boldsymbol{q}(0)$ is such that the end-effector pose is out of the desired trajectory, i.e., there are initial errors on the task

$$\boldsymbol{e}_p(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) \neq \mathbf{0}, \qquad \boldsymbol{e}_\phi(0) = \boldsymbol{\phi}_d - \boldsymbol{\phi}(0) \neq \mathbf{0}.$$

  Define a kinematic control law for the commands $\dot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_w$ such that each component of the error vectors $\boldsymbol{e}_p(t) \in \mathbb{R}^3$ and $\boldsymbol{e}_\phi(t) \in \mathbb{R}^3$ will exponentially converge to zero in an independent and prescribed way as t increases.

In your answer to each problem, specify which relevant matrices are required to be invertible during the entire motion.

[240 minutes; open books]

Solutions

January 9, 2014

Exercise 1

The DH frame assignment is shown in Fig. 2, with the associated Table 1. Note that all constant non-zero parameters are positive, as requested.

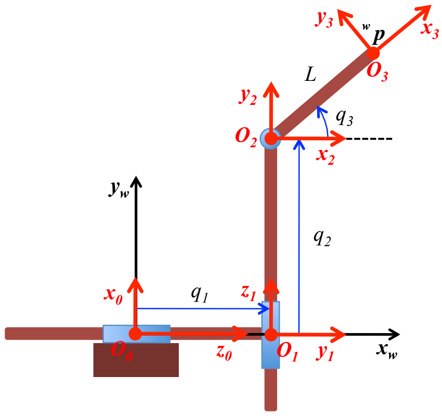


Figure 2: Assignment of Denavit-Hartenberg frames for the planar PPR robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	q_1	$\pi/2$
2	$\pi/2$	0	q_2	$\pi/2$
3	0	L	0	q_3

Table 1: Denavit-Hartenberg parameters for the planar PPR robot

The homogeneous transformation matrix between the (right-handed) frames RF_w and RF_0 is

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}.
$$

The primary (positional) workspace WS_1 and the secondary (dexterous) workspace WS_2 of the planar PPR robot are depicted in Fig. 3. The primary workspace (displayed in light orange) is the set of points in the plane that can be reached by the robot end-effector, independently from its orientation: it consists of the larger square with side $2(D+L)$ and smoothed corners (rounded as circles of radius L). The external boundary of WS_1 can be generated by sliding the center of a circle of radius L along the borders of the inner square having side $2D$, which is the mobility area of the tip of the second link due to the two prismatic joints. The secondary workspace (displayed in deep orange) is the smaller square of side $2(D-L) > 0$: WS_2 contains only those points of WS_1 that can be reached with *all* possible orientations of the end-effector in the plane, i.e., with the third link being able to approach the point from *any* direction (which is obtained by letting q_3 vary in the whole interval $(-\pi, +\pi]$).

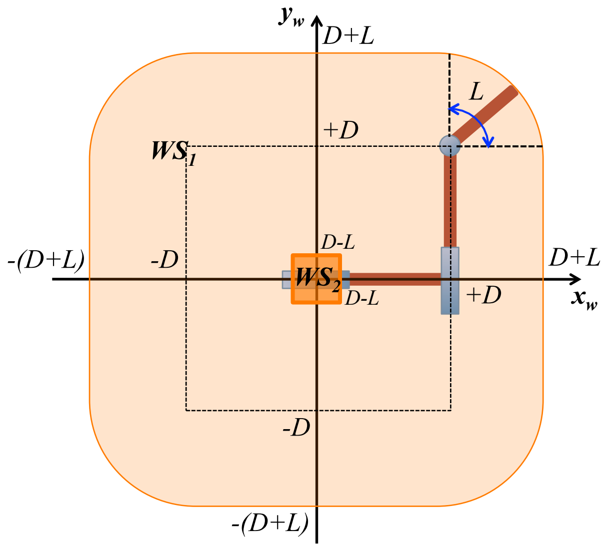


Figure 3: Primary and secondary workspaces of the planar PPR robot

By simple inspection, the direct kinematics of the end-effector position in the plane is

$$^w\boldsymbol{p} = \begin{pmatrix} ^w p_x \\ ^w p_y \end{pmatrix} = \begin{pmatrix} q_1 + L\cos q_3 \\ q_2 + L\sin q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

This result could also be obtained (via lengthy operations) from the first two components of the last column of the matrix product $^w\boldsymbol{T}_0{}^0\boldsymbol{A}_1(q_1){}^1\boldsymbol{A}_2(q_2){}^2\boldsymbol{A}_3(q_3)$. Therefore, for a given $^w\boldsymbol{p} \in WS_2$, all inverse kinematics solutions can be written in parametric form as

$$q_1 = {}^w p_x - L\cos q_3, \qquad q_2 = {}^w p_y - L\sin q_3, \qquad \forall q_3 \in (-\pi, +\pi].$$

Exercise 2

As a preliminary step, we set up the direct and inverse formulas for the (Y, Z, Y) Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ and then the associated differential mapping between $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega}$. Using the elementary rotation matrices

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad \boldsymbol{R}_Z(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{R}_Y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix},$$

the (Y, Z, Y) Euler rotation matrix of the direct problem is obtained as

$$\begin{aligned} \boldsymbol{R}(\alpha, \beta, \gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_Z(\beta)\boldsymbol{R}_Y(\gamma) \\ &= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma \\ \sin\beta\cos\gamma & \cos\beta & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\beta & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma \end{pmatrix}. \end{aligned} \tag{1}$$

For the inverse problem, let $\boldsymbol{R} = \{R_{ij}\}$ be a given rotation matrix. From the expressions of the elements in the second row of $\boldsymbol{R}(\alpha, \beta, \gamma)$, one has

$$\beta = \text{ATAN2}\left\{\pm\sqrt{R_{21}^2 + R_{23}^2}, R_{22}\right\}, \tag{2}$$

providing two values β_1 and $\beta_2 = -\beta_1$. When $R_{21}^2 + R_{23}^2 \neq 0$ (or, $\sin\beta \neq 0$), the problem is regular and for each $\beta = \beta_i$ $(i = 1, 2)$ in eq. (2) we have an associated solution

$$\alpha = \text{ATAN2}\left\{\frac{R_{32}}{\sin\beta}, \frac{-R_{12}}{\sin\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{23}}{\sin\beta}, \frac{R_{21}}{\sin\beta}\right\}. \tag{3}$$

The singular case occurs when $R_{21} = R_{23} = 0$, or $\sin\beta = 0$ (and thus also $R_{12} = R_{32} = 0$). Being $\cos\beta = \pm 1$, it is

$$\boldsymbol{R}(\alpha, \beta, \gamma)|_{\beta=\{0,\pi\}} = \begin{pmatrix} \pm\cos(\alpha \pm \gamma) & 0 & \sin(\alpha \pm \gamma) \\ 0 & \pm 1 & 0 \\ \mp\sin(\alpha \pm \gamma) & 0 & \cos(\alpha \pm \gamma) \end{pmatrix}.$$

Therefore, we can only determine the *sum* or, respectively, the *difference* of the two angles α and γ, leading to an infinite number of inverse solutions. If $R_{22} = 1$, we have

$$\beta = 0, \qquad \alpha + \gamma = \text{ATAN2}\{R_{13}, R_{33}\}.$$

If $R_{22} = -1$, we have

$$\beta = \pi, \qquad \alpha - \gamma = \text{ATAN2}\{R_{13}, R_{33}\}.$$

The differential relationship between $\dot{\boldsymbol{\phi}}$ and $\boldsymbol{\omega}$ is obtained by adding the contributions to the angular velocity of the time derivatives $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$, respectively along the directions of the rotation axes Y_0, Z_1, and Y_2, once these are expressed in the original reference frame. In particular, since the moving axes Z_1 and Y_2 are

$$Z_1 = \boldsymbol{R}_Y(\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}, \qquad Y_2 = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_Z(\beta)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\alpha\sin\beta \\ \cos\beta \\ \sin\alpha\sin\beta \end{pmatrix},$$

we obtain[1]

$$\begin{aligned} \boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\alpha} + \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}\dot{\beta} + \begin{pmatrix} -\cos\alpha\sin\beta \\ \cos\beta \\ \sin\alpha\sin\beta \end{pmatrix}\dot{\gamma} \\ &= \begin{pmatrix} 0 & \sin\alpha & -\cos\alpha\sin\beta \\ 1 & 0 & \cos\beta \\ 0 & \cos\alpha & \sin\alpha\sin\beta \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\alpha, \beta)\,\dot{\boldsymbol{\phi}}. \end{aligned} \tag{4}$$

[1] An alternative, longer procedure would be to extract $\boldsymbol{\omega}$ from the relation $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T$, with $\boldsymbol{R} = \boldsymbol{R}(\alpha, \beta, \gamma)$ given by eq. (1).

Note that a singularity occurs when $\det \boldsymbol{T} = -\sin\beta = 0$, or $\beta = \{0, \pi\}$. Finally, using the fact that the columns of $\boldsymbol{T}$ are unit vectors (though not necessarily orthogonal to each other), from (4) it follows that

$$\|\boldsymbol{\omega}\|^2 = \boldsymbol{\omega}^T\boldsymbol{\omega} = \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\,\dot{\alpha}\dot{\gamma}\cos\beta. \tag{5}$$

Note that $\|\boldsymbol{\omega}\|^2 \neq \|\dot{\boldsymbol{\phi}}\|^2 = \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2$.

The first step for determining a solution to the given problem is to compute the initial and final values of the (Y, Z, Y) Euler angles associated to the rotation matrices $\boldsymbol{R}_{in}$ and $\boldsymbol{R}_{fin}$. Since the inverse problem is regular for the initial and final orientation data, from eqs. (2–3) we obtain two sets of possible initial values[2]

$$(\alpha_{in,1}, \beta_{in,1}, \gamma_{in,1}) = \left(0, \frac{\pi}{4}, -\frac{\pi}{2}\right) \quad \text{or} \quad (\alpha_{in,2}, \beta_{in,2}, \gamma_{in,2}) = \left(\pi, -\frac{\pi}{4}, \frac{\pi}{2}\right) \tag{6}$$

and two sets of possible final values

$$(\alpha_{fin,1}, \beta_{fin,1}, \gamma_{fin,1}) = \left(-\frac{3\pi}{4}, \frac{\pi}{4}, -\frac{3\pi}{4}\right) \quad \text{or} \quad (\alpha_{fin,2}, \beta_{fin,2}, \gamma_{fin,2}) = \left(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}\right). \tag{7}$$

Any combination of these boundary conditions (there are four in total) can be chosen to proceed. While the computational steps are formally the same for all cases, it should be noted that the assigned change of orientation may induce larger or smaller variations of the Euler angles, depending on the chosen sets of boundary conditions. Accordingly, for a given motion time T, the angular velocities associated to the four solutions will also be different. Conversely, given the bound on the norm of the angular velocity, each solution will lead in general to a different minimum feasible motion time.

Although not explicitly requested in the text of the problem, we will determine the best among all four possible solutions, namely the one associated to the smallest minimum time T^*. In the following, the four solution trajectories will be labeled as $\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, where the first index refers to the set of initial values used for the Euler angles and the second to the set of final values from eqs. (6–7).

For each combination of initial and final conditions, we choose cubic polynomials (with common motion time T) as interpolating trajectories for all three Euler angles. In this way, a zero angular velocity $\boldsymbol{\omega}$ (or, equivalently, a zero time derivative $\dot{\boldsymbol{\phi}}$) can also be imposed at the initial and final instants. For $t \in [0, T]$, we have the general expression

$$a(t) = a_{in} + (a_{fin} - a_{in})\left(3\,(t/T)^2 - 2\,(t/T)^3\right), \qquad \text{where } a = \{\alpha, \beta, \gamma\},$$

with

$$\dot{a}(t) = \frac{6\,(a_{fin} - a_{in})}{T}\left((t/T) - (t/T)^2\right), \qquad \ddot{a}(t) = \frac{6\,(a_{fin} - a_{in})}{T^2}\,(1 - 2\,(t/T)).$$

It is easy to see that, at $t = T/2$, the angular variation is the half of the total requested and the absolute value of the velocity reaches its maximum:

$$a(T/2) = a_{in} + \frac{a_{fin} - a_{in}}{2} = \frac{a_{in} + a_{fin}}{2}; \qquad |\dot{a}(T/2)| = \max_{t\in[0,T]} |\dot{a}(t)| = \frac{1.5\,|a_{fin} - a_{in}|}{T}. \tag{8}$$

Note also that, from the boundary conditions (6-7) on β, it follows that either $\beta(t)$ is constant over the whole interval of motion, and so $d = \cos\beta(T/2) = \cos(\pm\pi/4) = \sqrt{2}/2$, or $\beta(t)$ should cross zero at $t = T/2$, and so $d = \cos\beta(T/2) = 1$ is at its maximum.

[2]All angles are assumed to be defined in the interval $(-\pi, \pi]$. Indeed this is only a local representation.

motion solution	$\alpha_{fin} - \alpha_{in}$	$\beta_{fin} - \beta_{in}$	$\gamma_{fin} - \gamma_{in}$	$d = \cos\beta(T/2)$	$\dot{\alpha}(t)\dot{\gamma}(t)$
$\{1,1\}$	$-3\pi/4$	0	$-\pi/4$	$\sqrt{2}/2$	≥ 0
$\{1,2\}$	$\pi/4$	$\pi/2$	$3\pi/4$	1	≥ 0
$\{2,1\}$	$-7\pi/4$	$-\pi/2$	$-5\pi/4$	1	≥ 0
$\{2,2\}$	$-3\pi/4$	0	$-\pi/4$	$\sqrt{2}/2$	≥ 0

Table 2: Quantities used for evaluating the maximum of $\|\boldsymbol{\omega}\|$ in the solution trajectories obtained from the four possible combinations of boundary conditions in (6-7)

Based on eqs. (8) and on the formula (5), Table 2 summarizes the relevant quantities needed for evaluating the maximum of $\|\boldsymbol{\omega}\|$, as associated to the four possible trajectories for the Euler angles. It can be easily verified that in all four combinations of initial and final conditions, one has $\dot{\alpha}(t)\dot{\gamma}(t) \geq 0$ for any $t \in [0, T]$. As a result of this analysis, it can be concluded that the maximum norm of $\boldsymbol{\omega}$ is always attained at the motion midpoint, $t = T/2$, where each of the (positive) terms in the right-hand side of eq. (5) attains its maximum value. Since

$$\|\boldsymbol{\omega}(T/2)\|^2 = \left(\frac{1.5}{T}\right)^2 \left[(\alpha_{fin} - \alpha_{in})^2 + (\beta_{fin} - \beta_{in})^2 + (\gamma_{fin} - \gamma_{in})^2 + 2d\,(\alpha_{fin} - \alpha_{in})\,(\gamma_{fin} - \gamma_{in})\right],$$

the inequality $\|\boldsymbol{\omega}(T/2)\| \leq \Omega$ implies

$$T \geq \frac{1.5}{\Omega}\sqrt{(\alpha_{fin} - \alpha_{in})^2 + (\beta_{fin} - \beta_{in})^2 + (\gamma_{fin} - \gamma_{in})^2 + 2d\,(\alpha_{fin} - \alpha_{in})\,(\gamma_{fin} - \gamma_{in})}. \tag{9}$$

Imposing the equality in (9) and using $\Omega = \pi$ and the values in Tab. 2, we determine and then compare the minimum feasible motion times for all combinations of initial/final conditions. For instance, in the solution trajectory $\{1,2\}$ it is

$$T_{\{1,2\}} = \frac{1.5}{\pi}\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{2}\right)^2 + \left(\frac{3\pi}{4}\right)^2 + 2\cdot 1\cdot\left(\frac{\pi}{4}\right)\left(\frac{3\pi}{4}\right)} = 1.5\sqrt{\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + \frac{6}{16}} = \frac{3\sqrt{5}}{4}.$$

Therefore, the smallest minimum time is

$$\begin{aligned} T^* &= \min\left\{T_{\{1,1\}}, T_{\{1,2\}}, T_{\{2,1\}}, T_{\{2,2\}}\right\} \\ &= \min\left\{\frac{1.5\sqrt{10+3\sqrt{2}}}{4}, \frac{3\sqrt{5}}{4}, \frac{3\sqrt{37}}{4}, \frac{1.5\sqrt{10+3\sqrt{2}}}{4}\right\} \\ &\approx \min\{1.4152, 1.6771, 4.5621, 1.4152\} = 1.4152 \text{ [s]}, \end{aligned}$$

which is attained with the solution trajectory $\{1,1\}$ as well as with $\{2,2\}$.

Choosing for instance the solution $\{1,1\}$ leads to the following trajectories for the Euler angles, with $t \in [0, T^*]$:

$$\begin{aligned} \alpha(t) &= \alpha_{in,1} + (\alpha_{fin,1} - \alpha_{in,1})\left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \\ &= -\frac{3\pi}{4}\left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \end{aligned}$$

$$
\begin{aligned}
\beta(t) &= \beta_{in,1} + (\beta_{fin,1} - \beta_{in,1}) \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3 \right) \\
&= \frac{\pi}{4} \\
\gamma(t) &= \gamma_{in,1} + (\gamma_{fin,1} - \gamma_{in,1}) \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3 \right) \\
&= -\frac{\pi}{2} - \frac{\pi}{4} \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3 \right).
\end{aligned}
$$

Figure 4 shows the evolution of the Euler angles in the minimum time solution $\{1,1\}$. Note that in this case $\beta(t)$ is kept constant at the value $\pi/4$ [rad]. The maximum absolute velocity is attained by the angle $\alpha(t)$ at the trajectory midpoint ($\dot{\alpha}(T^*/2) = -2.5$ [rad/s]). In Fig. 5, the plot of the associated $\|\boldsymbol{\omega}\|$, computed from eq. (5), shows that the given bound $\Omega = \pi$ [rad] is never violated and reached only at the trajectory midpoint $t = T^*/2$, as predicted by our analysis. For comparison, the evolution of the Euler angles in the alternative solution $\{1,2\}$ are shown in Fig. 6. All angles will move in this case, while the minimum feasible motion time T_{12} is about 18% longer than the optimal T^*. The maximum absolute velocity is attained here by the angle $\gamma(t)$ ($\dot{\gamma}(T_{12}/2) \approx 2.15$ [rad/s]). Indeed, also in this case the associated $\|\boldsymbol{\omega}\|$ (not reported) remains always feasible and reaches the bound $\Omega = \pi$ only at the trajectory midpoint.

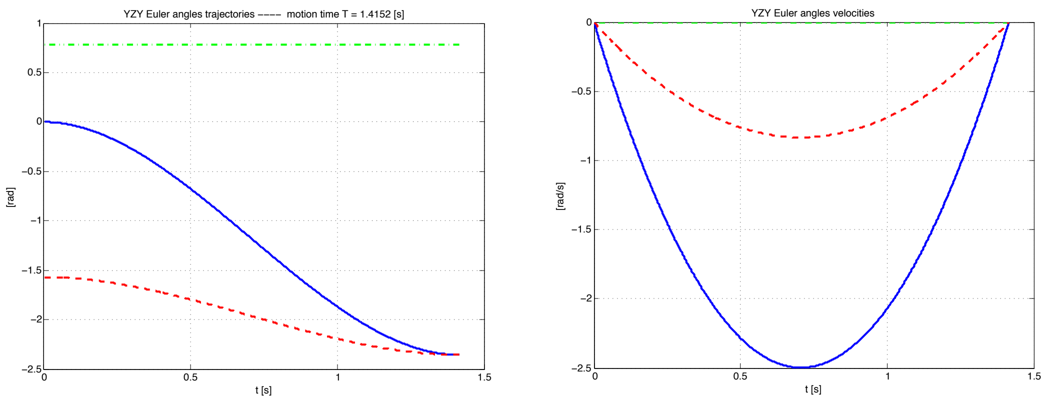


Figure 4: Trajectories (left) and velocities (right) of the Euler angles in the minimum time solution $\{1,1\}$: $\alpha(t)$ and $\dot{\alpha}(t)$ (blue, solid), $\beta(t)$ and $\dot{\beta}(t)$ (green, dashdot), $\gamma(t)$ and $\dot{\gamma}(t)$ (red, dashed)

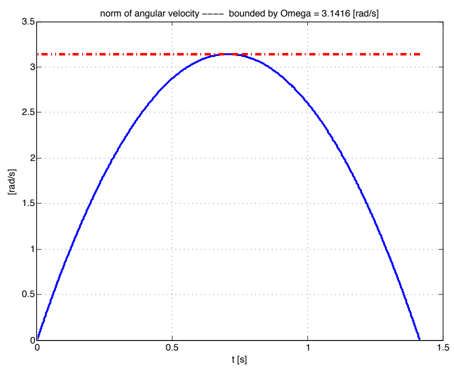


Figure 5: Norm of the angular velocity $\boldsymbol{\omega}$ in the minimum time solution $\{1,1\}$

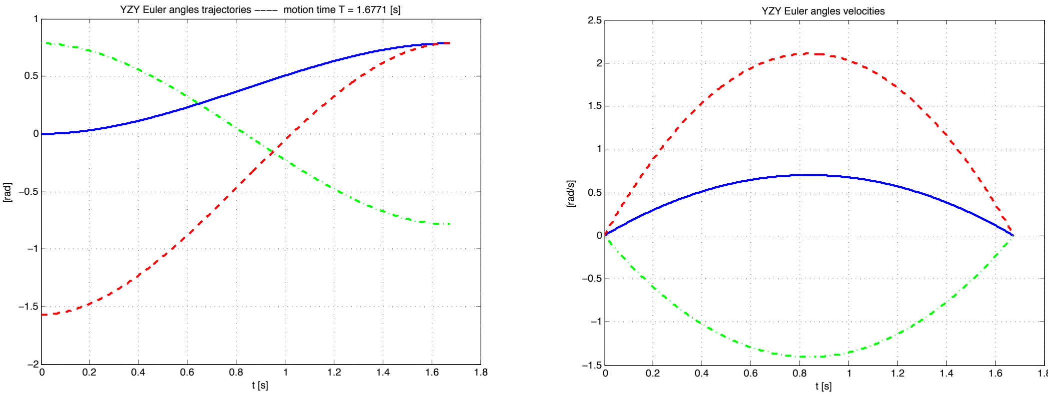


Figure 6: Trajectories (left) and velocities (right) of the Euler angles in the alternative solution $\{1,2\}$: $\alpha(t)$ and $\dot{\alpha}(t)$ (blue, solid), $\beta(t)$ and $\dot{\beta}(t)$ (green, dashdot), $\gamma(t)$ and $\dot{\gamma}(t)$ (red, dashed)

Exercise 3

For a 6R manipulator with a spherical wrist, the (6×6) geometric Jacobian that relates the joint velocity vector $\dot{\boldsymbol{q}}$ to the linear and angular velocity of the end-effector at a configuration $\boldsymbol{q}$ can be written, under the given assumptions, in the partitioned way

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}(\boldsymbol{q}) & \boldsymbol{O} \\ \boldsymbol{J}_{21}(\boldsymbol{q}) & \boldsymbol{J}_{22}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_b \\ \dot{\boldsymbol{q}}_w \end{pmatrix}.$$

Assuming that the Jacobian is non-singular at $\boldsymbol{q}$ (i.e., that both (3×3) diagonal blocks $\boldsymbol{J}_{11}$ and $\boldsymbol{J}_{22}$ are invertible) and dropping dependencies, we have

$$\begin{pmatrix} \dot{\boldsymbol{q}}_b \\ \dot{\boldsymbol{q}}_w \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}^{-1} & \boldsymbol{O} \\ -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\boldsymbol{J}_{11}^{-1} & \boldsymbol{J}_{22}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix}.$$

Thus, by setting $\boldsymbol{v} = \boldsymbol{v}_d \neq \boldsymbol{0}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_d = \boldsymbol{0}$, we have

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{11}^{-1}\boldsymbol{v}_d \\ \dot{\boldsymbol{q}}_w &= -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b = -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\boldsymbol{J}_{11}^{-1}\boldsymbol{v}_d. \end{aligned}$$

In the presence of errors on both the positional (linear) and orientation (angular) task quantities, a suitable kinematic control law can be defined as

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{11}^{-1}(\boldsymbol{q})\left(\boldsymbol{v}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\right)\right) \\ \dot{\boldsymbol{q}}_w &= -\boldsymbol{J}_{22}^{-1}(\boldsymbol{q})\boldsymbol{J}_{21}(\boldsymbol{q})\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}^{-1}(\boldsymbol{q})\boldsymbol{T}(\alpha(\boldsymbol{q}),\beta(\boldsymbol{q}))\boldsymbol{K}_\phi\left(\boldsymbol{\phi}_d - \boldsymbol{\phi}(\boldsymbol{q})\right), \end{aligned} \tag{10}$$

where the two gain matrices $\boldsymbol{K}_p$ and $\boldsymbol{K}_\phi$ are positive definite and diagonal, matrix $\boldsymbol{T}$ relates the time derivative $\dot{\boldsymbol{\phi}}$ of the chosen set of Euler angles to the angular velocity $\boldsymbol{\omega}$, and the functional expressions $\boldsymbol{p}(\boldsymbol{q})$ and $\boldsymbol{\phi}(\boldsymbol{q}) = (\alpha(\boldsymbol{q}), \beta(\boldsymbol{q}), \gamma(\boldsymbol{q}))$ are given by the appropriate direct kinematics mappings.

It is easy to verify that the evolutions of the errors

$$\dot{\boldsymbol{e}}_p = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = \boldsymbol{v}_d - \boldsymbol{J}_{11}\dot{\boldsymbol{q}}_b = \boldsymbol{v}_d - \boldsymbol{J}_{11}\boldsymbol{J}_{11}^{-1}\left(\boldsymbol{v}_d + \boldsymbol{K}_p\boldsymbol{e}_p\right) = -\boldsymbol{K}_p\boldsymbol{e}_p$$

and

$$\begin{aligned}\dot{\boldsymbol{e}}_\phi = \dot{\boldsymbol{\phi}}_d - \dot{\boldsymbol{\phi}} &= -\boldsymbol{T}^{-1}\boldsymbol{\omega} = -\boldsymbol{T}^{-1}\left(\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}\dot{\boldsymbol{q}}_w\right) \\ &= -\boldsymbol{T}^{-1}\left(\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}\left(-\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}^{-1}\boldsymbol{T}\boldsymbol{K}_\phi\boldsymbol{e}_\phi\right)\right) = -\boldsymbol{K}_\phi\boldsymbol{e}_\phi\end{aligned}$$

are exponentially converging to zero with a rate prescribed by the elements of the gain matrices $\boldsymbol{K}_p$ and $\boldsymbol{K}_\phi$, as desired. The independent behavior of the error components is enforced by the choice of diagonal gain matrices. The non-singularity of the blocks $\boldsymbol{J}_{11}$ and $\boldsymbol{J}_{22}$ in the Jacobian matrix is again required for the feasibility of the kinematic control law (10). On the other hand, since matrix $\boldsymbol{T}$ needs not to be inverted in the law (10), its possible rank deficiencies will not lead the control command to grow unbounded. Nonetheless, $\boldsymbol{T}$ should remain invertible (during the transient behavior and along the nominal desired trajectory) in order to guarantee a trajectory tracking behavior with all the features requested in the problem formulation.

* * * * *

Robotics I

February 6, 2014

Exercise 1

A pan-tilt[1] camera sensor, such as the commercial webcams in Fig. 1, is mounted on the fixed base of a robot manipulator and is used for pointing at a (point-wise) target in the 3D Cartesian space. The tilt rotation is typically limited to maximum $\pm 90°$ w.r.t. the vertical axis, or slightly more. The motion of the optical axis of the camera can be described with the Denavit-Hartenberg (DH) formalism, as that of a 2-dof simple robot manipulator.

Figure 1: Pan-tilt cameras, with placement of the reference frame RF_0 (shown for two of them)

- Assign the frames according to the DH convention and provide the associated table of parameters. Use *mandatorily* the reference frame $RF_0 = \{\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{z}_0\}$ as indicated in Fig. 1. One of the axes of the last frame should be coincident with the optical axis of the camera.

- Determine the unit vector of the pointing axis as a function of the joint angles $\boldsymbol{q} \in \mathbb{R}^2$.

- For a given target position ${}^0\boldsymbol{p}_T = \begin{pmatrix} {}^0x_T & {}^0y_T & {}^0z_T \end{pmatrix}^T$, with ${}^0z_T \geq 0$ (and sufficiently large), determine the value(s) of $\boldsymbol{q}$ that solve the pointing task. Is this inverse kinematics problem always solvable or well defined?

Exercise 2

In a pick-and-place task, beside the desired initial Cartesian pose $\boldsymbol{r}_0$ at time $t = t_0$ and the final pose $\boldsymbol{r}_f$ at $t = t_f$ of the robot, it is common to assign two intermediate robot poses, $\boldsymbol{r}_1$ at $t = t_1$ and $\boldsymbol{r}_2$ at $t = t_2$ (with $t_0 < t_1 < t_2 < t_f$), so as to shape the total robot motion in three phases: *Lift off* from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$; *Travel* from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$; and, *Set down* from $\boldsymbol{r}_2$ to $\boldsymbol{r}_f$. Smoothness of the trajectory is requested, with continuity up to the acceleration at any $t \in [t_0, t_f]$. The first and last phases are in 'guarded' move, and should be performed with caution because of the closeness to environmental surfaces. Therefore, we require also zero velocity and acceleration at the initial and final poses. The four Cartesian poses $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_f$ have been transformed into four configurations $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_f$ of a robot with revolute joints by means if its inverse kinematics.

[1]Panning refers to left/right rotations around the vertical axis, tilting to up/down rotations around an horizontal axis.

Define a 4-3-4 trajectory for a generic joint q addressing the given task. Give the expressions used for each polynomial tract, formulate the problem, and provide the value of all coefficients using:

$$t_0 = 0, \qquad t_1 = 2, \qquad t_2 = 4, \qquad t_1 = 6 \qquad [\text{s}]$$

$$q_0 = q(t_0) = 0, \qquad q_1 = q(t_1) = 10, \qquad q_2 = q(t_2) = 80, \qquad q_f = q(t_f) = 90 \qquad [\text{deg}].$$

Hint: You can solve the problem numerically (by a direct method) or analytically (by proper choosing the structure of the polynomials, so as to ease the analysis). Either way is fine. The former approach is rather straightforward, but only if you use a (simple) Matlab code (do not include this, just report the results as requested). The latter is more complex, though quite elegant (you can reduce the problem to the solution of a suitable linear system of two equations in two unknowns, which is indeed solvable in closed form).

Exercise 3

Consider a robot manipulator with n joints, its configuration vector $\boldsymbol{q} \in \mathbb{R}^n$, and a task described by $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{q})$, with $\boldsymbol{r} \in \mathbb{R}^m$. Assume that $m = n$ and a desired task trajectory $\boldsymbol{r}_d(t)$ is given. Prove that the kinematic control law

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{r}}_d + k\,\boldsymbol{J}^T(\boldsymbol{q})\,(\boldsymbol{r}_d - \boldsymbol{r}(\boldsymbol{q}))\,, \tag{1}$$

with the task Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{r}(\boldsymbol{q})/\partial \boldsymbol{q}$ and a scalar $k > 0$, will force any initial error $\boldsymbol{e}(0) = \boldsymbol{r}_d(0) - \boldsymbol{r}(\boldsymbol{q}(0))$ at time $t = 0$ to converge to zero, as long as no singularities of $\boldsymbol{J}(\boldsymbol{q})$ are encountered. Discuss the above controller in terms of off-line versus on-line computations needed.

Next, assume that $m < n$. How can the kinematic control law (1) be modified in order to guarantee the same previous features?

[210 minutes for all exercises; open books]

Solutions

February 6, 2014

Exercise 1

The DH frame assignment is shown in Fig. 2, with the associated parameters in Table 1. Note that a non-zero offset $d_1 > 0$ is present, since the reference frame cannot be assigned arbitrarily, as specified by the text of the problem (otherwise, we could move up the origin O_0 along $\boldsymbol{z}_0$ until intersecting the second joint axis).

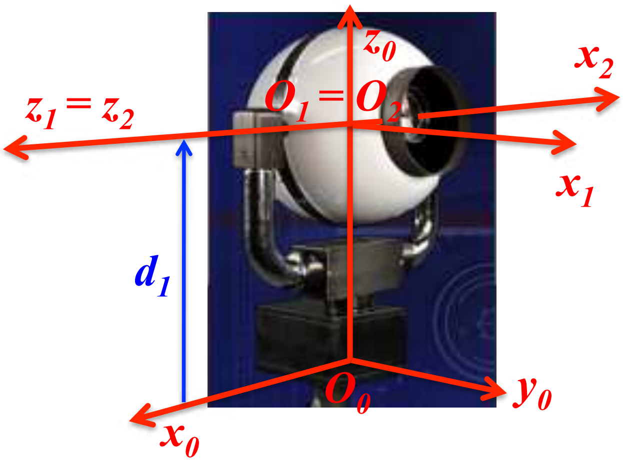


Figure 2: Assignment of Denavit-Hartenberg frames for a pan-tilt camera

i	α_i	a_1	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	0	0	0	q_2

Table 1: Denavit-Hartenberg parameters for the pan-tilt camera

The optical axis of the camera is the unit axis ${}^0\boldsymbol{x}_2$ of last frame, whose expression is given by

$$
{}^0\boldsymbol{x}_2 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos q_1 & 0 & \sin q_1\\ \sin q_1 & 0 & -\cos q_1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\cos q_2 & -\sin q_2 & 0\\ \sin q_2 & \cos q_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}
$$
$$
= \begin{pmatrix}\cos q_1 \cos q_2\\ \sin q_1 \cos q_2\\ \sin q_2\end{pmatrix}.
$$

For the pointing task, observe first that any 3D position ${}^0\boldsymbol{p}_T = \begin{pmatrix} {}^0x_T & {}^0y_T & {}^0z_T \end{pmatrix}^T$ of a target having ${}^0z_T \geq 0$ is mapped into a single value of two task variables, namely the pan angle α and the tilt angle β, as

$$\alpha = \text{ATAN2}\,\{{}^0y_T, {}^0x_T\}, \qquad \beta = \text{ATAN2}\,\{{}^0z_T, \sqrt{{}^0x_T^2 + {}^0y_T^2}\}. \tag{2}$$

When the target is on the (positive) $\boldsymbol{z}_0$ axis, α is not defined[2]. Otherwise, the mapping in (2) is surjective (but not injective, since all Cartesian points on a half-ray from the origin give the same values of (α, β)). If there were no base offset ($d_1 = 0$), these two angles would also coincide with the values (q_1, q_2), in the same order, solution of the inverse kinematics problem. Nevertheless, simple geometry shows that the inverse kinematics solution for the two-dimensional pointing task is given by

$$q_1 = \text{ATAN2}\,\{{}^0y_T, {}^0x_T\}, \qquad q_2 = \text{ATAN2}\,\{{}^0z_T - d_1, \sqrt{{}^0x_T^2 + {}^0y_T^2}\}, \tag{3}$$

except for the singular case ${}^0y_T = {}^0x_T = 0$, where the angle q_1 is not specified (any value of q_1 satisfies the problem). Note that there is no choice of signs for angle q_2, because of the range limitation of this joint (moreover, the target position is assumed to lie in the Cartesian region with positive values of ${}^0z_T - d_1$).

Exercise 2

A numerical procedure for obtaining the solution is detailed in steps **A**–**E** below. We may call this a *brute force* approach.

A. Define the interpolating polynomials for the three phases L = *Lift off* (degree 4), T = *Travel* (degree 3), and S = *Set down* (degree 4):

$$\begin{aligned}
q_L(t) &= a_{L0} + a_{L1}t + a_{L2}t^2 + a_{L3}t^3 + a_{L4}t^4 \qquad && t \in [t_0, t_1] \\
q_T(t) &= a_{T0} + a_{T1}t + a_{T2}t^2 + a_{T3}t^3 && t \in [t_1, t_2] \\
q_S(t) &= a_{S0} + a_{S1}t + a_{S2}t^2 + a_{S3}t^3 + a_{S4}t^4 && t \in [t_1, t_f].
\end{aligned} \tag{4}$$

The 4-3-4 trajectory will be the concatenation of these three polynomials for the motion interval $t \in [t_0, t_f]$:

$$q_{434}(t) = \begin{cases} q_L(t) & \text{for } t \in [t_0, t_1] \\ q_T(t) & \text{for } t \in [t_1, t_2] \\ q_S(t) & \text{for } t \in [t_2, t_f]. \end{cases} \tag{5}$$

B. The 14 unknown coefficients are organized in the vector

$$\boldsymbol{x}^T = \begin{pmatrix} a_{L0} & a_{L1} & a_{L2} & a_{L3} & a_{L4} & a_{T0} & a_{T1} & a_{T2} & a_{T3} & a_{S0} & a_{S1} & a_{S2} & a_{S3} & a_{S4} \end{pmatrix}. \tag{6}$$

C. Impose the requested 14 boundary conditions on the 14 coefficients:

$$\begin{aligned}
&q_L(t_0) = q_0 \qquad \dot{q}_L(t_0) = 0 \qquad \ddot{q}_L(t_0) = 0 \qquad \dot{q}_L(t_1) = q_1 \\
&\dot{q}_L(t_1) = \dot{q}_T(t_1) \qquad \ddot{q}_L(t_1) = \ddot{q}_T(t_1) \quad \text{(continuity of velocity and acceleration at } t_1) \\
&q_T(t_1) = q_1 \qquad q_T(t_2) = q_2 \\
&\dot{q}_T(t_2) = \dot{q}_S(t_2) \qquad \ddot{q}_T(t_2) = \ddot{q}_S(t_2) \quad \text{(continuity of velocity and acceleration at } t_2) \\
&q_S(t_2) = q_2 \qquad q_S(t_f) = q_f \qquad \dot{q}_S(t_f) = 0 \qquad \ddot{q}_S(t_f) = 0.
\end{aligned} \tag{7}$$

[2] In order to be interpreted as trigonometric values (sine and cosine of some angle), the arguments of each of the two ATAN2 functions in the eqs. (2) should be divided, respectively by $\sqrt{{}^0x_T^2 + {}^0y_T^2}$ and by $\sqrt{{}^0x_T^2 + {}^0y_T^2 + {}^0z_T^2}$. When ${}^0x_T^2 +{}^0 y_T^2 \neq 0$, this division by a (positive) value can be avoided. Instead, for ${}^0x_T^2 ={}^0 y_T^2 = 0$, the pan angle α would remain anyway undefined. So, such divisions are skipped altogether in eqs. (2), while introducing a preliminary warning in (or close to) a singularity. Similar arguments hold for eqs. (3).

D. Following the same order, the conditions (7) can be written in matrix form as

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}, \tag{8}$$

with

$$\boldsymbol{A} = \begin{pmatrix}
1 & t_0 & t_0^2 & t_0^3 & t_0^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 6t_0 & 12t_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & t_1 & t_1^2 & t_1^3 & t_1^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2t_1 & 3t_1^2 & 4t_1^3 & 0 & -1 & -2t_1 & -3t_1^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 6t_1 & 12t_1^2 & 0 & 0 & -2 & -6t_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2t_2 & 3t_2^2 & 0 & -1 & -2t_2 & -3t_2^2 & -4t_2^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_2 & 0 & 0 & -2 & -6t_2 & -12t_2^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 & t_2^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_f & t_f^2 & t_f^3 & t_f^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_f & 12t_f^2
\end{pmatrix} \tag{9}$$

and

$$\boldsymbol{b} = \begin{pmatrix} q_0 & 0 & 0 & q_1 & 0 & 0 & q_1 & q_2 & 0 & 0 & q_2 & q_f & 0 & 0 \end{pmatrix}^T.$$

It can be shown that matrix $\boldsymbol{A}$ in (9) is always non-singular as long as $t_0 < t_1 < t_2 < t_f$, a condition which is satisfied by assumption. Using the problem data, we have

$$\boldsymbol{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & 12 & 32 & 0 & -1 & -4 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 12 & 48 & 0 & 0 & -2 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 4 & 16 & 64 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 48 & 0 & -1 & -8 & -48 & -256 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 24 & 0 & 0 & -2 & -24 & -192 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 16 & 64 & 256 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 36 & 216 & 1296 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 108 & 864 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 36 & 432
\end{pmatrix}$$

and

$$\boldsymbol{b} = \begin{pmatrix} 0 & 0 & 0 & 10 & 0 & 0 & 10 & 80 & 0 & 0 & 80 & 90 & 0 & 0 \end{pmatrix}^T.$$

E. By using Matlab, the linear system of equations (8) is solved as

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b},$$

providing the desired coefficients of the three polynomials:

$$\begin{aligned}\boldsymbol{x}^T &= \begin{pmatrix} a_{L0} & a_{L1} & a_{L2} & a_{L3} & a_{L4} & a_{T0} & a_{T1} & a_{T2} & a_{T3} & a_{S0} & a_{S1} & a_{S2} & a_{S3} & a_{S4} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -.625 & 0.9375 & 90 & -125.5 & 56.25 & -6.25 & -990 & 742.5 & -191.25 & 21.875 & -0.9375 \end{pmatrix}.\end{aligned} \tag{10}$$

Figure 3 shows the position, velocity, and acceleration of the obtained 4-3-4 trajectory. It can be noted that the trajectory is fully symmetric because of the specific data distribution. As requested, the solution trajectory $q_{434}(t)$ is continuous up to the acceleration, and has zero initial and final values of the first two time derivatives.

Once the above numerical procedure has been coded (e.g., in Matlab), it can be used for other sets of data. For instance, we can change just the timing through the same knots, i.e.,

$$\begin{array}{ccccc} t_0 = 0, & t_1 = 1, & t_2 = 4, & t_1 = 6 & [\mathrm{s}] \\ q_0 = 0, & q_1 = 10, & q_2 = 80, & q_f = 90 & [\mathrm{deg}]. \end{array} \tag{11}$$

Figure 4 shows the resulting 4-3-4 trajectory. Symmetry of motion is now lost.

Matrix $\boldsymbol{A}$ in (9) is clearly sparse. As a matter of fact, no special structure has been exploited in order to reduce the number of independent equations to be solved numerically. A more careful definition of the interpolating polynomials can lead instead to a complete analytical solution in closed form. For this, normalization of time is used in each motion phase, together with a symmetric definition of the first and last quartic polynomials (reversing time for the latter). Moreover, a similar idea is used as in spline interpolation in order to break the problem in three separate parts.

The analytical solution is found by the procedure **a**–**f** below. For the time being, assume that $q_0 \neq q_1$ and $q_2 \neq q_f$. Special cases will be treated at the end.

a. Define the quartic polynomial in the *Lift off* phase as

$$q_L(\tau_1) = q_0 + (q_1 - q_0)\left(a_{13}\,\tau_1^3 + a_{14}\,\tau_1^4\right), \qquad \tau_1 = \frac{t - t_0}{t_1 - t_0} \in [0, 1]. \tag{12}$$

It is easy to see that, by construction, $q_L(\tau_1)$ satisfies the three initial boundary conditions at $t = t_0$ (or, $\tau_1 = 0$) on position, velocity, and acceleration. Moreover,

$$q_L(1) = q_0 + (q_1 - q_0)\,(a_{13} + a_{14}) = q_1 \qquad \Rightarrow \qquad a_{13} + a_{14} = 1.$$

Therefore, for the derivatives and their values at $t = t_1$ (or, $\tau_1 = 1$) we have

$$\dot{q}_L(\tau_1) = \frac{q_1 - q_0}{t_1 - t_0}\left(3\,a_{13}\,\tau_1^2 + 4\,a_{14}\,\tau_1^3\right) \quad \Rightarrow \quad \dot{q}_L(1) = \frac{q_1 - q_0}{t_1 - t_0}\,(3 + a_{14}) \tag{13}$$

and

$$\ddot{q}_L(\tau_1) = \frac{q_1 - q_0}{(t_1 - t_0)^2}\left(6\,a_{13}\,\tau_1 + 12\,a_{14}\,\tau_1^2\right) \quad \Rightarrow \quad \ddot{q}_L(1) = \frac{6(q_1 - q_0)}{(t_1 - t_0)^2}\,(1 + a_{14})\,. \tag{14}$$

b. Define the quartic polynomial in the *Set down* phase as

$$q_S(\tau_3) = q_f + (q_2 - q_f)\left(a_{33}(1 - \tau_1)^3 + a_{34}(1 - \tau_1)^4\right), \qquad \tau_3 = \frac{t - t_2}{t_f - t_2} \in [0, 1]. \tag{15}$$

By construction, $q_S(\tau_3)$ satisfies the three final boundary conditions at $t = t_f$ (or, $\tau_3 = 1$) on position, velocity, and acceleration. Moreover,

$$q_S(0) = q_f + (q_2 - q_f)\,(a_{33} + a_{34}) = q_2 \qquad \Rightarrow \qquad a_{33} + a_{34} = 1.$$

Therefore, for the derivatives and their values at $t = t_2$ (or, $\tau_3 = 0$) we have

$$\dot{q}_S(\tau_3) = \frac{q_f - q_2}{t_f - t_2}\left(3\,a_{33}(1-\tau_3)^2 + 4\,a_{34}(1-\tau_3)^3\right) \quad \Rightarrow \quad \dot{q}_S(0) = \frac{q_f - q_2}{t_f - t_2}\,(3 + a_{34}) \tag{16}$$

and

$$\ddot{q}_S(\tau_3) = -\frac{q_f - q_2}{(t_f - t_2)^2}\left(6\,a_{33}(1-\tau_3) + 12\,a_{34}(1-\tau_3)^2\right) \quad \Rightarrow \quad \ddot{q}_S(0) = -\frac{6(q_f - q_2)}{(t_f - t_2)^2}\,(1 + a_{34})\,. \tag{17}$$

c. For the cubic polynomial in the *Travel* phase, we choose the symmetric form

$$q_T(\tau_2) = q_2\tau_2 + q_1(1-\tau_2) + a_{21}\,\tau_2(1-\tau_2)^2 + a_{22}\,\tau_2^2(1-\tau_2), \qquad \tau_2 = \frac{t - t_1}{t_2 - t_1} \in [0,1]. \tag{18}$$

By construction, $q_T(\tau_2)$ satisfies the interpolating conditions on position in $t = t_1$ ($\tau_2 = 0$) and $t = t_2$ ($\tau_2 = 1$). Denote at this stage the (yet unknown) velocities at the internal knots as v_1 and v_2, respectively. By imposing

$$\dot{q}_T(0) = v_1, \qquad \dot{q}_T(1) = v_2,$$

the cubic polynomial (18) becomes fully specified as

$$q_T(\tau_2) = q_1 + (q_2 - q_1)\tau_2 + [v_1(t_2 - t_1) + (q_2 - q_1)]\,\tau_2(1-\tau_2)^2 + [(q_2 - q_1) - v_2(t_2 - t_1)]\,\tau_2^2(1-\tau_2), \tag{19}$$

for $\tau_2 \in [0,1]$. For later use, we compute also the expression of its acceleration:

$$\ddot{q}_T(\tau_2) = \frac{6(q_2 - q_1)}{(t_2 - t_1)^2}\,(1 - 2\tau_2) + \frac{1}{t_2 - t_1}\,(6(v_1 + v_2)\,\tau_2 - 2(2v_1 + v_2))\,. \tag{20}$$

d. Using the expressions in (13) and (16), and imposing the equalities

$$\dot{q}_L(1) = v_1, \qquad \dot{q}_S(0) = v_2,$$

we solve for the coefficients of $q_L(\tau_1)$ and $q_S(\tau_3)$ as

$$a_{14} = v_1\,\frac{t_1 - t_0}{q_1 - q_0} - 3, \qquad a_{13} = 1 - a_{14} = 4 - v_1\,\frac{t_1 - t_0}{q_1 - q_0} \tag{21}$$

and

$$a_{34} = v_2\,\frac{t_f - t_2}{q_f - q_2} - 3, \qquad a_{33} = 1 - a_{34} = 4 - v_2\,\frac{t_f - t_2}{q_f - q_2}. \tag{22}$$

e. Having tailored the solution as above, we are left only with the problem of finding the correct values v_1 and v_2. For this, the conditions of continuity of the acceleration at the intermediate knots are used, namely

$$\ddot{q}_L(1) = \ddot{q}_T(0), \qquad \ddot{q}_T(1) = \ddot{q}_S(0).$$

Using eqs. (14) and (17), substituting therein the coefficients given by (21) and (22), and evaluating (20) at $\tau_2 = 0$ and $\tau_2 = 1$ yields

$$\begin{aligned}
\ddot{q}_L(1) = \frac{6v_1}{t_1 - t_0} - \frac{12(q_1 - q_0)}{(t_1 - t_0)^2} &= -\frac{2(v_2 + 2v_1)}{t_2 - t_1} + \frac{6(q_2 - q_1)}{(t_2 - t_1)^2} = \ddot{q}_T(0) \\
\ddot{q}_T(1) = \frac{2(v_1 + 2v_2)}{t_2 - t_1} - \frac{6(q_2 - q_1)}{(t_2 - t_1)^2} &= -\frac{6v_2}{t_f - t_2} + \frac{12(q_f - q_2)}{(t_f - t_2)^2} = \ddot{q}_S(0).
\end{aligned} \tag{23}$$

After some manipulation, the following linear system of *two* equations in *two* unknowns in matrix form is obtained:

$$\begin{aligned}
\boldsymbol{M} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 6\,(t_2 - t_1) + 4\,(t_1 - t_0) & 2\,(t_1 - t_0) \\ 2\,(t_f - t_2) & 6\,(t_2 - t_1) + 4\,(t_f - t_2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} 6(q_2 - q_1)\,\dfrac{t_1 - t_0}{t_2 - t_1} + 12(q_1 - q_0)\,\dfrac{t_2 - t_1}{t_1 - t_0} \\ 6\,(q_2 - q_1)\,\dfrac{t_f - t_2}{t_2 - t_1} + 12\,(q_f - q_2)\,\dfrac{t_2 - t_1}{t_f - t_2} \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \boldsymbol{n}.
\end{aligned} \tag{24}$$

Since

$$\det \boldsymbol{M} = 12\left\{3(t_2 - t_1)^3 + 2(t_2 - t_1)\left[(t_1 - t_0) + (t_f - t_2)\right] + (t_1 - t_0)(t_f - t_2)\right\} \neq 0, \tag{25}$$

its solution is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{n} = \frac{1}{\det \boldsymbol{M}} \begin{pmatrix} 6\,(t_2 - t_1) + 4\,(t_f - t_2) & -2\,(t_1 - t_0) \\ -2\,(t_f - t_2) & 6\,(t_2 - t_1) + 4\,(t_1 - t_0) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{26}$$

where n_1 and n_2 are defined in (24), and eq. (25) is used for the determinant. The closed form solution of the problem is found by replacing the analytical expressions of v_1 and v_2 in eq. (19) for $q_T(\tau_2)$ and in eqs. (21–22) for $q_L(\tau_1)$ and $q_S(\tau_3)$.

f. Using the problem data, we obtain

$$v_1 = v_2 = 22.5 \quad [\text{deg/s}] \tag{27}$$

and so

$$a_{13} = -0.5, \quad a_{14} = 1.5, \quad a_{33} = -0.5, \quad a_{34} = 1.5.$$

Indeed, the resulting trajectory is the same as the one obtained with the numerical method. However, since the symbolic expressions of the used polynomials are different, also the associated numerical values of the coefficients will be different. In Fig. 3, the computed values (27) are shown as magenta dots in the velocity plots. Note also that identical values are obtained for the coefficients of the two quartic polynomials, because of the symmetric nature of the given data. For the alternative data in (11), we obtain instead (see the plots in Fig. 4)

$$v_1 = 21.4184, \qquad v_2 = 14.3972 \qquad [\text{deg/s}] \tag{28}$$

and

$$a_{13} = 1.8582, \quad a_{14} = -0.8582, \quad a_{33} = 1.1206, \quad a_{34} = -0.1206.$$

To complete the analysis, we have to consider the degenerate (or singular) cases, namely when $q_0 = q_1$ and/or $q_2 = q_f$. The numerical approach is totally unaffected by any repetition of values at the knots (i.e., matrix $\boldsymbol{A}$ does not lose rank). On the other hand, in the more tailored analytical approach some changes are needed. In fact, the expressions of the quartic polynomials (12) and (15) would be forced to become constant in such cases (while they should not, unless $q_0 = q_1 = q_2 = q_f$ and no motion is needed). In a singular case, it is sufficient to replace the previous definitions by

$$q_L(\tau_1) = q_0 + v_1(t_1 - t_0)\left(\tau_1^4 - \tau_1^3\right), \qquad \tau_1 = \frac{t - t_0}{t_1 - t_0} \in [0, 1]$$

when $q_1 = q_0$, and/or by

$$q_S(\tau_3) = q_f + v_2(t_f - t_2)\left((1 - \tau_3)^3 - (1 - \tau_3)^4\right), \qquad \tau_3 = \frac{t - t_2}{t_f - t_2} \in [0, 1]$$

when $q_2 = q_f$. In this way, we have in particular

$$\ddot{q}_L(1) = \frac{6v_1}{t_1 - t_0}, \qquad \ddot{q}_S(0) = -\frac{6v_2}{t_f - t_2}$$

that coincide with the expressions in (23) of the accelerations at the intermediate knots, respectively when $q_1 = q_0$ and when $q_2 = q_f$. Therefore, v_1 and v_2 can be found using (26) as before. For instance, consider the double degenerate case

$$\begin{array}{c} t_0 = 0, \qquad t_1 = 2, \qquad t_2 = 4, \qquad t_1 = 6 \qquad [\mathrm{s}] \\ q_0 = q_1 = 0, \qquad q_2 = q_f = 90 \qquad [\mathrm{deg}]. \end{array} \tag{29}$$

Figure 5 shows the result obtained via the analytical method, yielding again $v_1 = v_2 = 22.5$ [deg/s] (the same trajectory would be obtained also with the numerical method). From the position profile, we can see that there is some motion also in the first and last time intervals, which is needed to guarantee continuity up to the acceleration in the intermediate knots. However, an under- and over-shooting is present, which makes this particular situation no longer interesting for a collision-free guarded motion of the robot, when close to environmental surfaces.

Two different Matlab programs for the numerical solution and for the analytical solution (including treatment of degenerate cases) are available upon request.

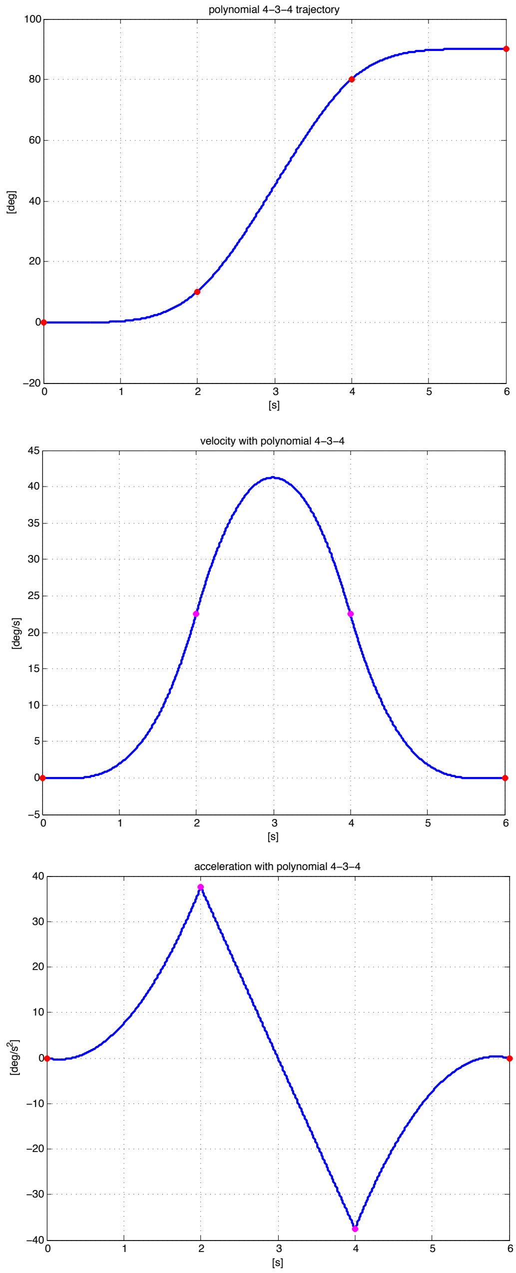


Figure 3: Position (top, the red dots are the interpolated knots), velocity (center, the magenta dots are the computed v_1 and v_2), and acceleration (bottom) of the 4-3-4 solution trajectory

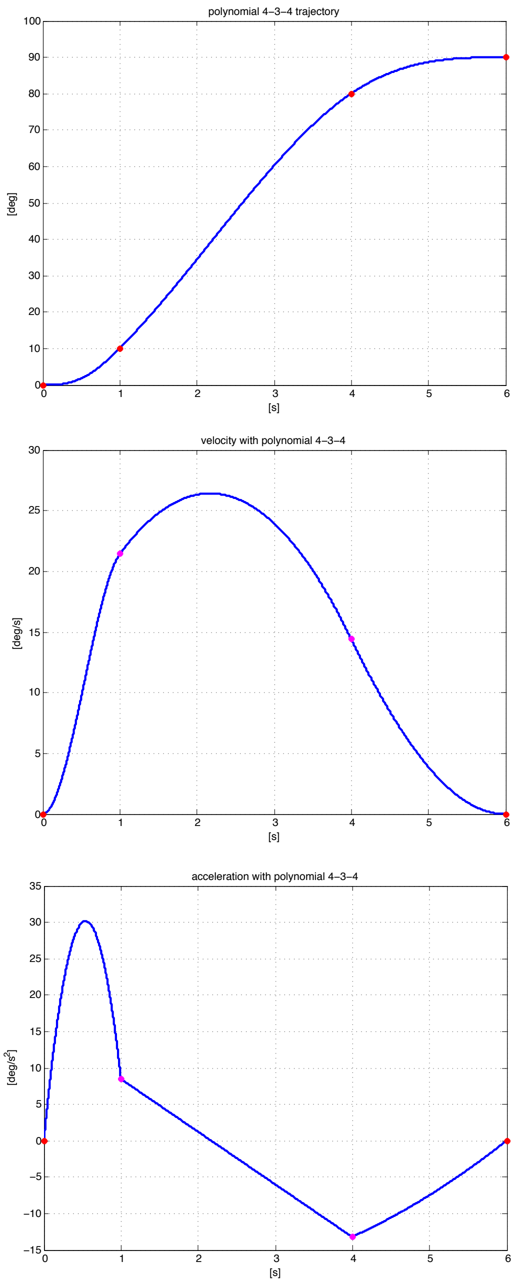


Figure 4: Position (top), velocity (center), and acceleration (bottom) of the 4-3-4 solution trajectory for the motion task in (11): Same knots, but different timing

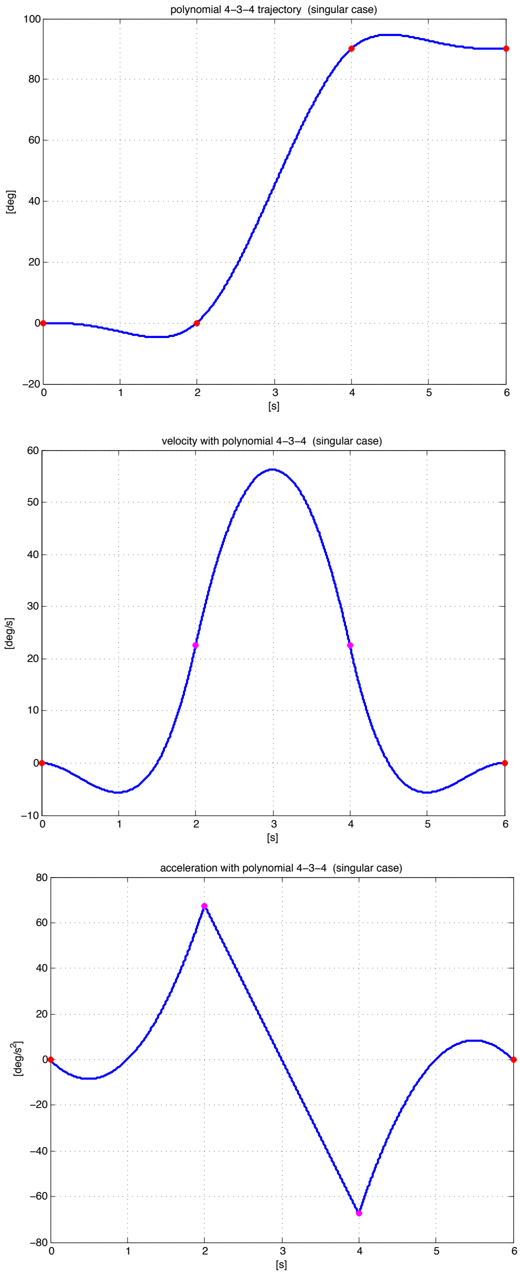


Figure 5: Position (top), velocity (center), and acceleration (bottom) of the 4-3-4 solution trajectory for the motion task in (29): Initial and first intermediate knot, as well as second intermediate and final knot have repeated position values

Exercise 3

The proof is based on Lyapunov stability, namely on the analysis of the time evolution of the error function $V = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{e} = \frac{1}{2}(\boldsymbol{r}_d - \boldsymbol{r})^T(\boldsymbol{r}_d - \boldsymbol{r}) \geq 0$. Using eq. (1) and dropping dependencies, we have

$$\begin{aligned} \dot{V} = \boldsymbol{e}^T\dot{\boldsymbol{e}} = \boldsymbol{e}^T(\dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}) &= \boldsymbol{e}^T\dot{\boldsymbol{r}}_d - \boldsymbol{e}^T\boldsymbol{J}\dot{\boldsymbol{q}} \\ &= \boldsymbol{e}^T\dot{\boldsymbol{r}}_d - \boldsymbol{e}^T\boldsymbol{J}\left[\boldsymbol{J}^{-1}\dot{\boldsymbol{r}}_d + k\,\boldsymbol{J}^T(\boldsymbol{r}_d - \boldsymbol{r})\right] = -k\,\boldsymbol{e}^T\boldsymbol{J}\boldsymbol{J}^T\boldsymbol{e} \leq 0. \end{aligned} \tag{30}$$

As long as $\boldsymbol{J}(\boldsymbol{q})$ is non-singular, we have $\boldsymbol{J}^T\boldsymbol{e} = \boldsymbol{0}$ (and so $\dot{V} = 0$) *if and only if* $\boldsymbol{e} = \boldsymbol{0}$. Therefore, the controlled robot will be an asymptotically stable system, and the error $\boldsymbol{e}(t)$ will converge to zero from any initial condition $\boldsymbol{e}(0)$. Note that, since the closed-loop system is still nonlinear, asymptotic stability (and convergence) will not be exponential in general.

The two terms in the control law (1), the first with the inverse Jacobian and the second with the Jacobian transpose, need both to be computed on line (i.e., all vectors and matrices are evaluated at the current configuration $\boldsymbol{q}$), even if the desired task trajectory $\boldsymbol{r}_d(t)$, $t \in [0, T]$, is completely known in advance for an arbitrary duration T.

When the robot is kinematically redundant for the given task ($m < n$), we can just replace in (1) the inverse of the Jacobian $\boldsymbol{J}$ by its pseudoinverse $\boldsymbol{J}^{\#}$. In fact, as long as the Jacobian is full (row) rank, it is $\boldsymbol{J}\boldsymbol{J}^{\#} = \boldsymbol{J}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} = \boldsymbol{I}$. Therefore, we obtain the same cancellation of terms as in (30).

* * * * *

Robotics I

April 2, 2014

Exercise 1

Consider a robot with four revolute joints, having the Denavit-Hartenberg parameters of Table 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	$d_1 > 0$	q_1
2	$\pi/2$	$a_2 > 0$	0	q_2
3	$-\pi/2$	0	$d_3 > 0$	q_3
4	0	$a_4 > 0$	0	q_4

Table 1: Denavit-Hartenberg parameters of a 4-dof robot

- Sketch the robot and the associated Denavit-Hartenberg frames in two different configurations: *i)* $\boldsymbol{q}_A = \boldsymbol{0}$, and *ii)* $\boldsymbol{q}_B = \begin{pmatrix} 0 & \pi/2 & -\pi/2 & \pi/2 \end{pmatrix}^T$.
- Provide the symbolic expression of the direct kinematics map $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^3$ for the position $\boldsymbol{p}$ of the origin of frame 4.

Exercise 2

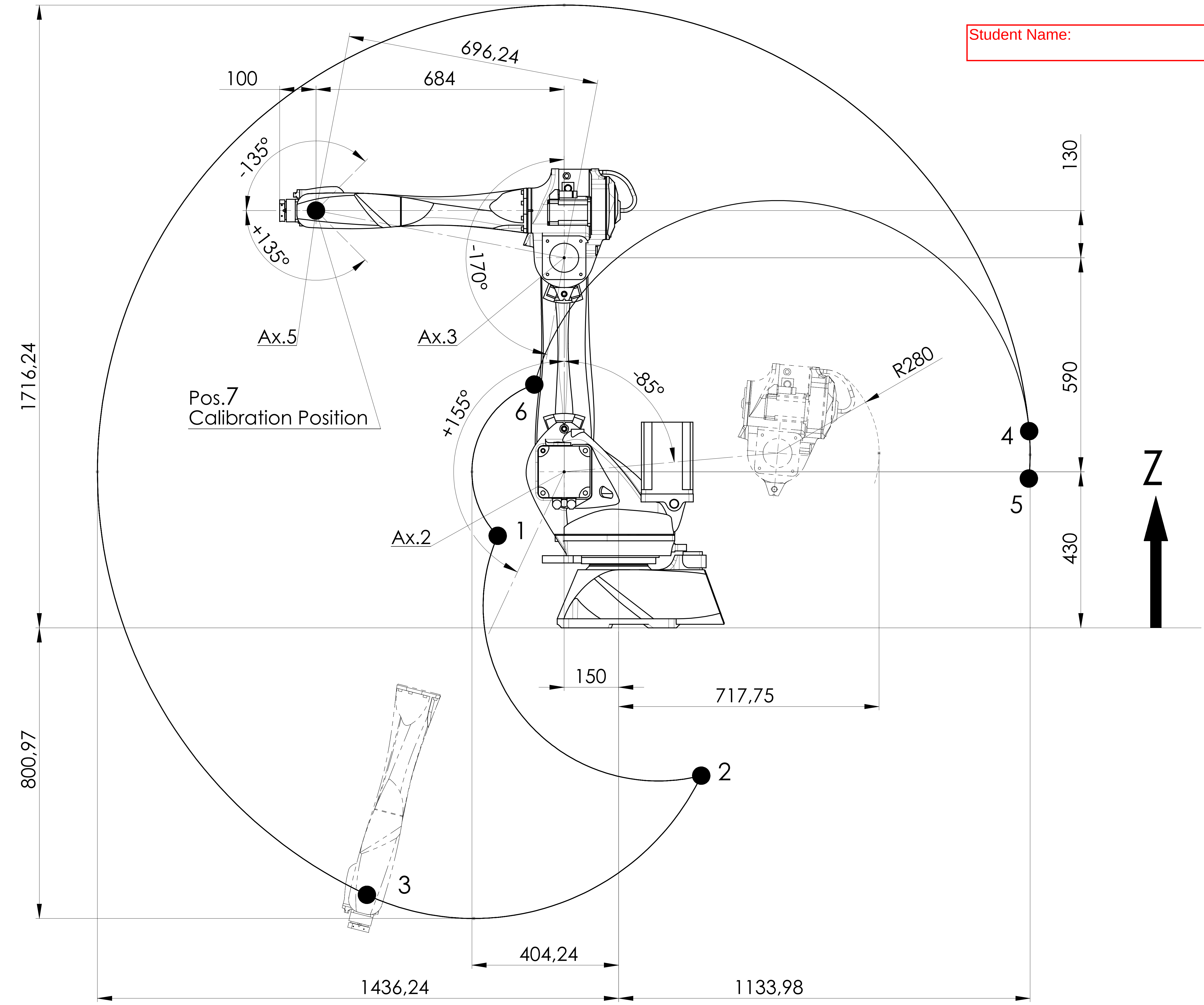
For the robot of Exercise 1, find the joint torque $\boldsymbol{\tau} \in \mathbb{R}^4$ that balances a force ${}^0\boldsymbol{F} = \begin{pmatrix} 0 & 10 & 0 \end{pmatrix}^T$ [N] applied to the origin of frame 4, when the robot is in the configuration $\boldsymbol{q} = \boldsymbol{q}_B$. Keep the symbolic dependence on parameters that are not specified numerically.

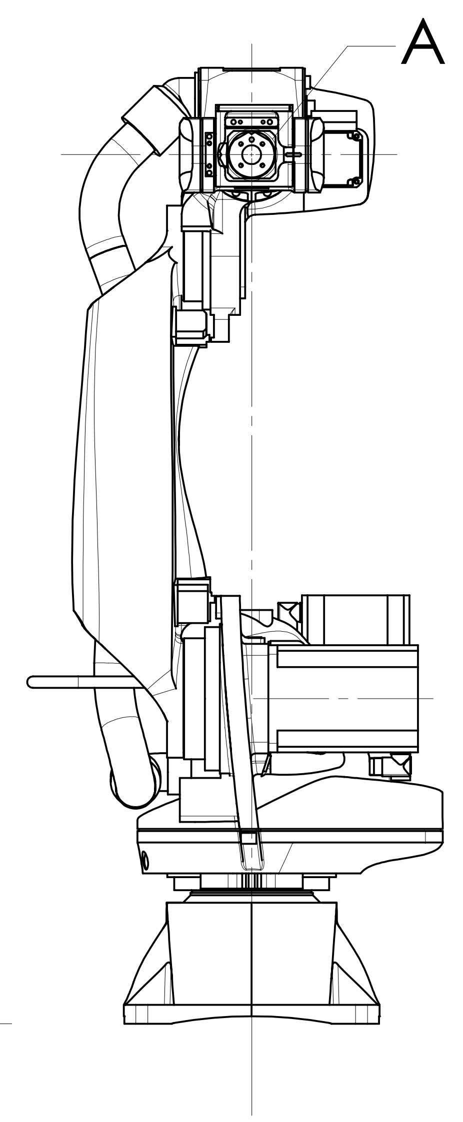
Exercise 3

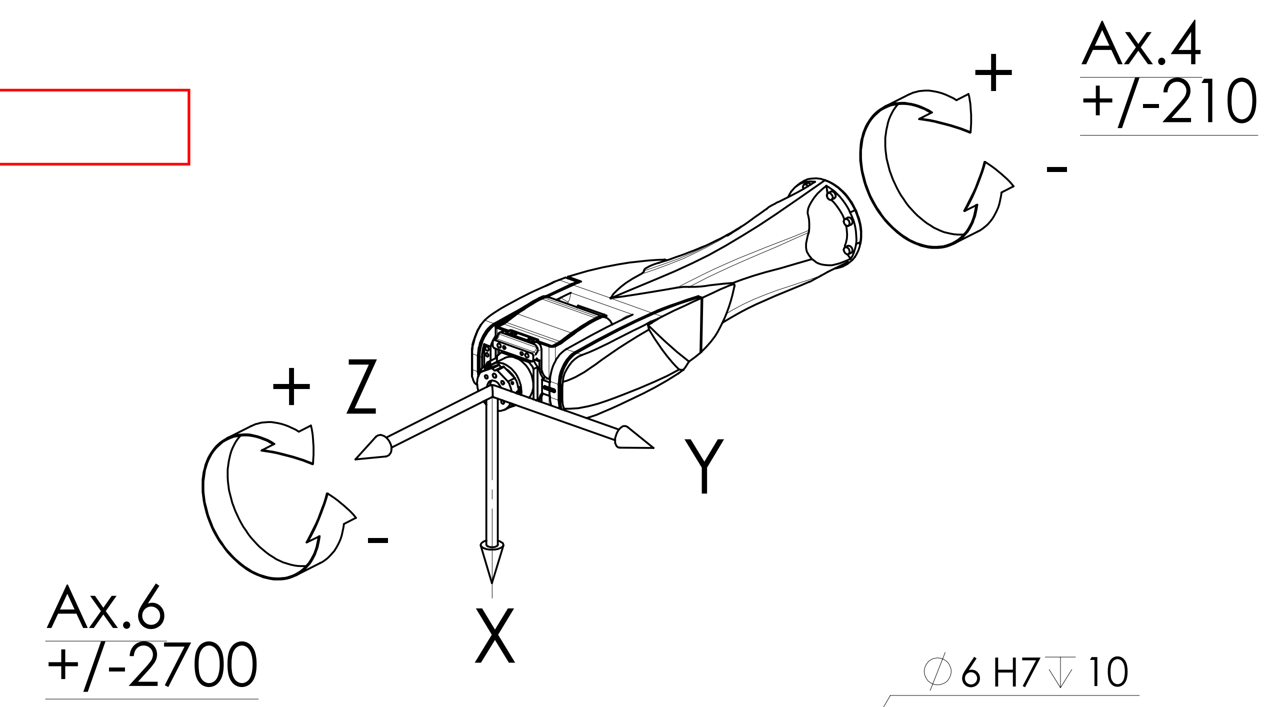
Plan a smooth minimum time trajectory $q_d(t)$ for a robot joint that provides rest-to-rest motion from $q_{in} = 90°$ to $q_{fin} = -90°$, with velocity and acceleration equal to zero at the initial and final instants and satisfying the bounds $|\dot{q}_d(t)| \leq 90°/\text{s}$ and $|\ddot{q}_d(t)| \leq 90°/\text{s}^2$. Give the final expression of $q_d(t)$ and plot approximately this solution trajectory and its first and second time derivatives. Provide also the minimum feasible time T and the maximum absolute value attained by the velocity and by the acceleration.

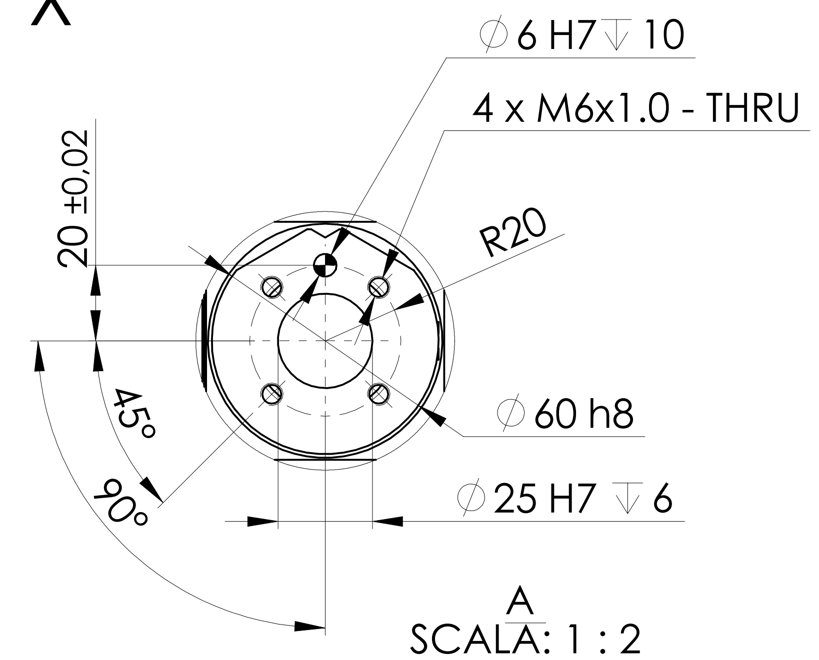
[180 minutes; open books]

Student Name:





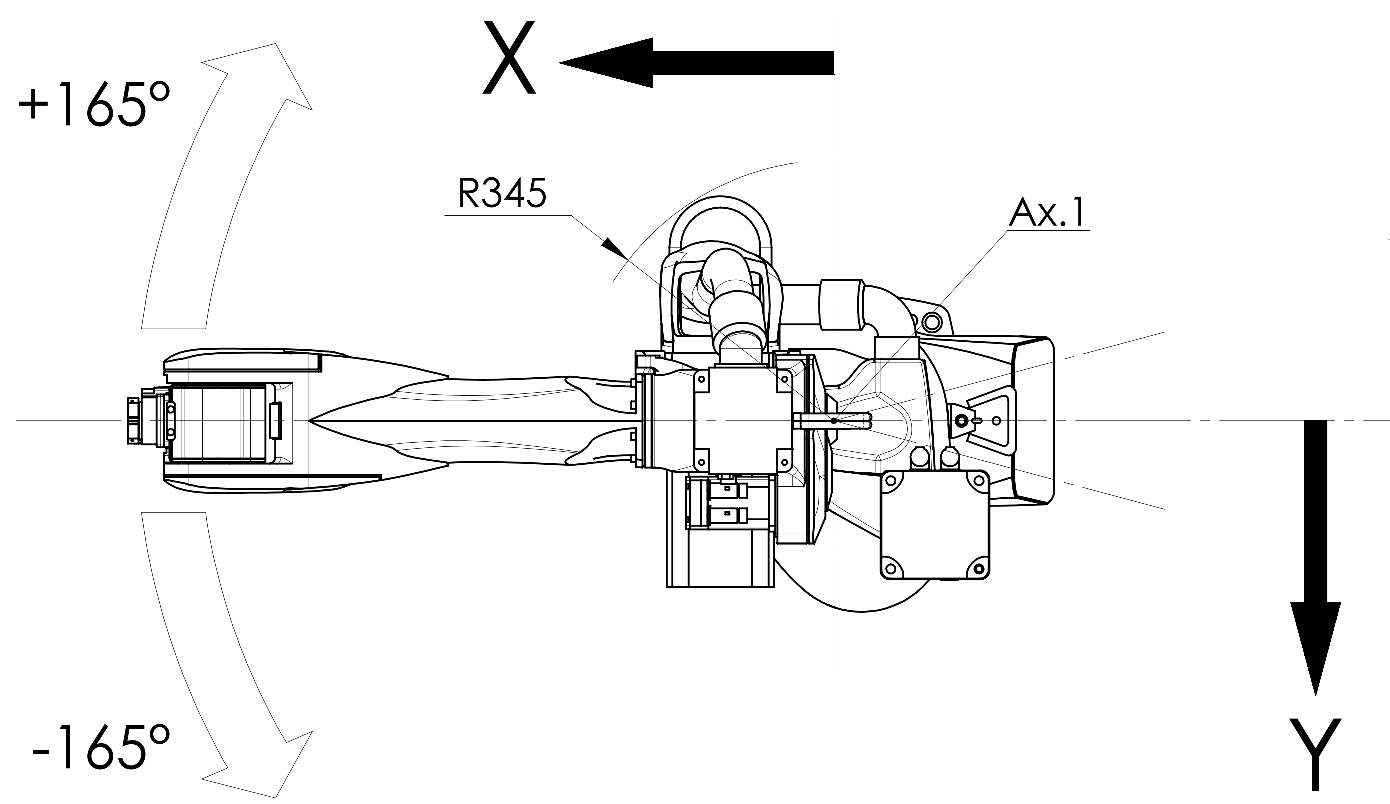


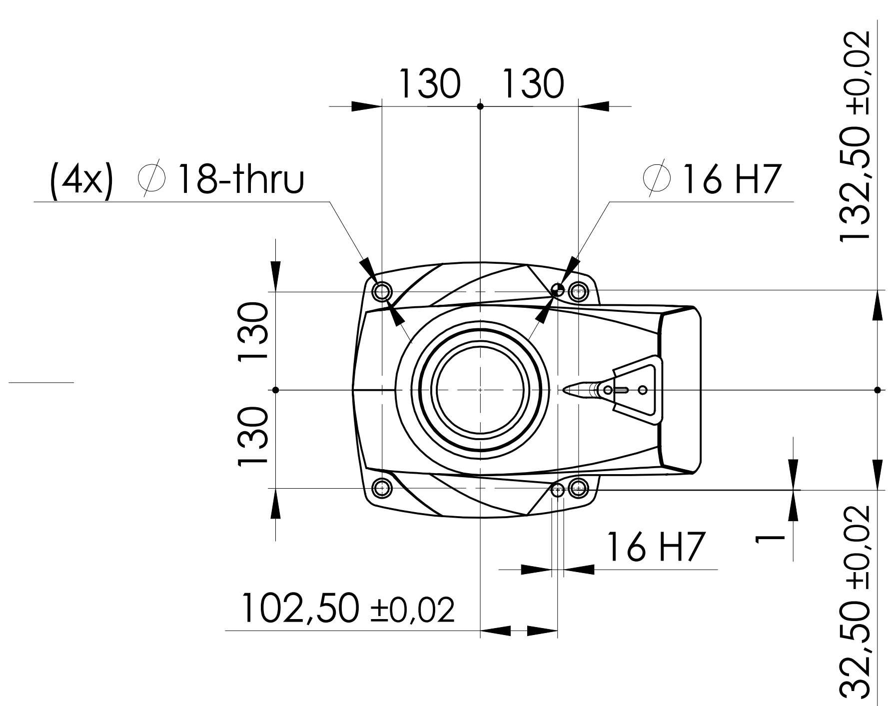


POS.	X	Z	AX.2	AX.3
	[mm]	[mm]	[deg]	[deg]
1	333,22	253,74	+30°	-170°
2	-227,70	-407,32	+155°	-100°
3	693,59	-735,73	+155°	-10,76°
4	-1131,35	542,10	-85°	-10,76°
5	-1130,48	411,53	-85°	0°
6	232,31	670,54	-85°	-170°
7	834	1150	0°	-90°

Giunti in posizione di calibrazione (pos.7)

Ax.1	Ax.2	Ax.3	Ax.4	Ax.5	Ax.6
0°	0°	-90°	0°	0°	0°





N° Complessivo - Assembly Dwgs | Quantita' per Complessivo - Q.ty Assembly | N° Part. - Part. N°

Inizio N° - Sheet first N° | Fine N° - Sheet last N° | Ultimo N° - Last used

Numerazione conduttori - Wires numbering

Tipo - Type | Codice - Code

Materiale - Material

Superficiale - Surface | Termico - Heat

Trattamento - Treatment

No. | Kg. 190 | HRC - | HB

Modello - Model | Peso - Weight | Durezza - Hardness

Quote senza indicazioni di tolleranza : ISO 2768-mK

For dimensions with no tolerance : ISO 2768-mK

Tolleranze generali - General tolerances

N° -No. | Modifiche - Revisions | Data - Date | Visto - Approv.

Commessa - Job	23.07.13 Data - Date	1:10 Scala - Scale	A 2 Formato - Size
	1/1 Foglio - Sheet	Pernechele Disegn. - Drawn	Amparore Visto - Checked

Titolo: Racer 7-1.4 :Title

CR82225005

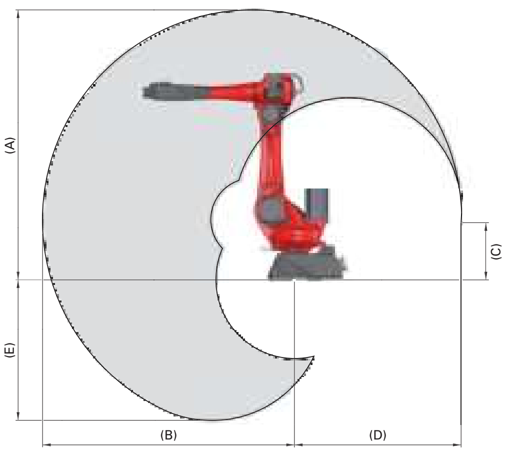
N° Disegno - Drawing No.

COMAU

Robotics I

June 10, 2014

Consider the COMAU RACER 7-1.4 robot in Fig. 1. The robot has six revolute joints and a spherical wrist. As shown in the data sheet, each joint has a limited range (called 'stroke'), which is specified in terms of the *COMAU manufacturer convention* used for defining the joint variables.



MODEL		RACER 7-1.4
Structure / n° axes		Anthropomorphous / 6 axes
Maximum payload		7 kg
Stroke / (Speed)	Axis 1	+/- 165° (220 °/s)
	Axis 2	-85° / + 155° (250 °/s)
	Axis 3	0° / - 170° (300 °/s)
	Axis 4	+/- 210° (550 °/s)
	Axis 5	+/- 135° (550 °/s)
	Axis 6	+/- 2700° (600 °/s)
Maximum horizontal reach		1436 mm
Robot weight		160 kg
Repeatability		+/- 0,05 mm
Motors		AC brushless
Position measurement system		encoder
Total power installed		3KVA
Assembly position		Floor
		Ceiling
		Sloped (45° max)
Operating area	A	1716 mm
	B	1436 mm
	C	412 mm
	D	1134 mm
	E	801 mm

Figure 1: COMAU RACER 7-1.4 robot (taken from the data sheet)

Figure 2 (in the following page) shows a more detailed drawing of the robot workspace, with all relevant metric data (in mm or deg). For seven different Cartesian positions of the center W of the spherical wrist, a table reports the numerical values of the absolute coordinates X and Z of the point W and of the joint variables of axes 1 and 2. For the calibration position #7, the values of the complete joint configuration vector are reported.

- Assign the robot frames according to the *Denavit-Hartenberg convention* and define the numerical values in the associated table of DH parameters. On the provided extra sheet (a replica of Fig. 2), draw clearly the DH frames, put your name, and return the sheet with the rest of your solution.

- When comparing the joint variables according to the DH assignment with those used by the robot manufacturer, there may be some differences. Let $\boldsymbol{\theta} \in \mathbb{R}^6$ be the chosen DH variables, and $\boldsymbol{\theta}_C \in \mathbb{R}^6$ be the variables used by the COMAU manufacturer. Using the information in the data sheet, show that the two set of variables are related by the affine map

$$\boldsymbol{\theta}_C = \boldsymbol{\theta}_{C,0} + \boldsymbol{S}\,\boldsymbol{\theta}, \qquad \boldsymbol{S} = \text{diag}\{s_i,\, i = 1, \ldots, 6\}, \qquad s_i = \{+1, -1\}, \tag{1}$$

  where $\boldsymbol{\theta}_{C,0} \in \mathbb{R}^6$ is the value of the COMAU variables corresponding to $\boldsymbol{\theta} = \mathbf{0}$ (the zero configuration of the DH joint variables). The signs of the unitary elements s_i, $i = 1, \ldots, 6$, on the diagonal of matrix $\boldsymbol{S}$ are used to realign the sense of rotations in the two conventions. Find the actual values of vector $\boldsymbol{\theta}_{C,0}$ and of the diagonal elements of matrix $\boldsymbol{S}$.

[180 minutes; open books]

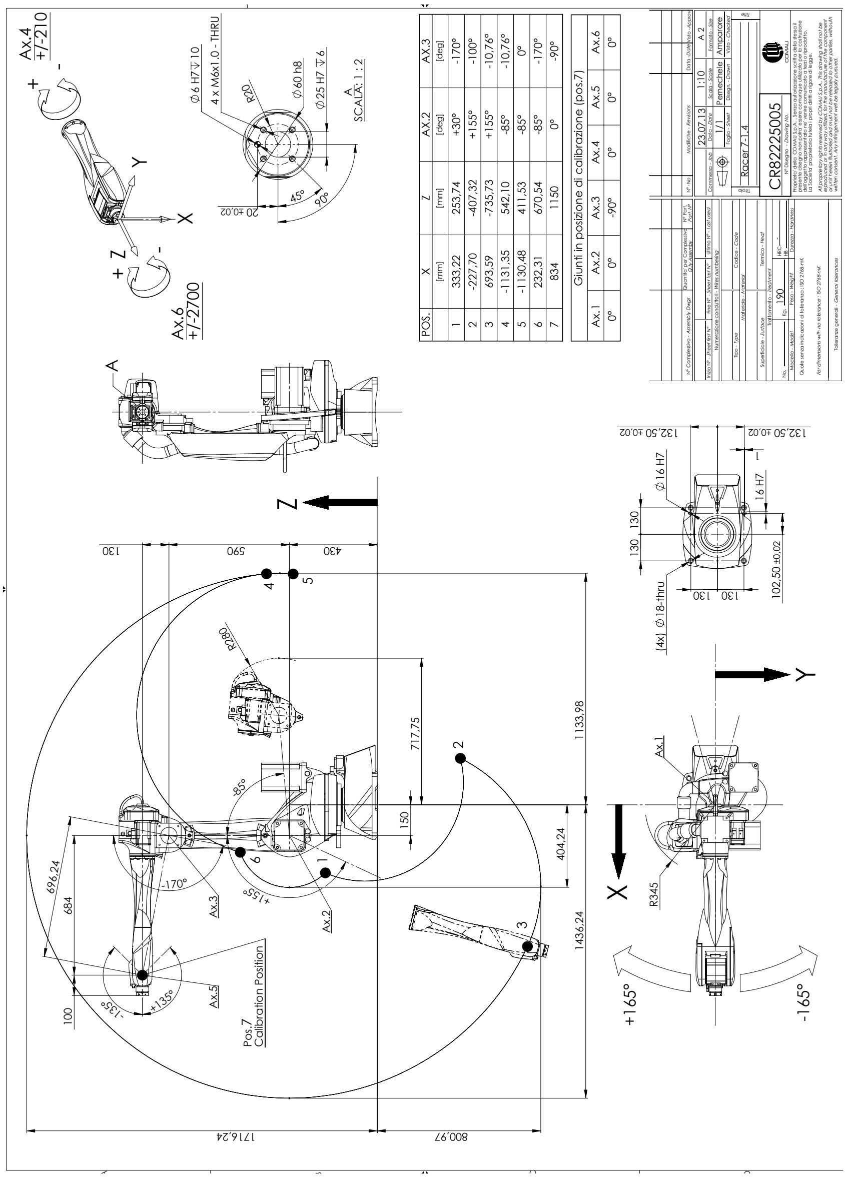


Figure 2: The workspace of the COMAU RACER 7-1.4, with the data associated to the wrist center positions (labeled from 1 to 7)

Solution

June 10, 2014

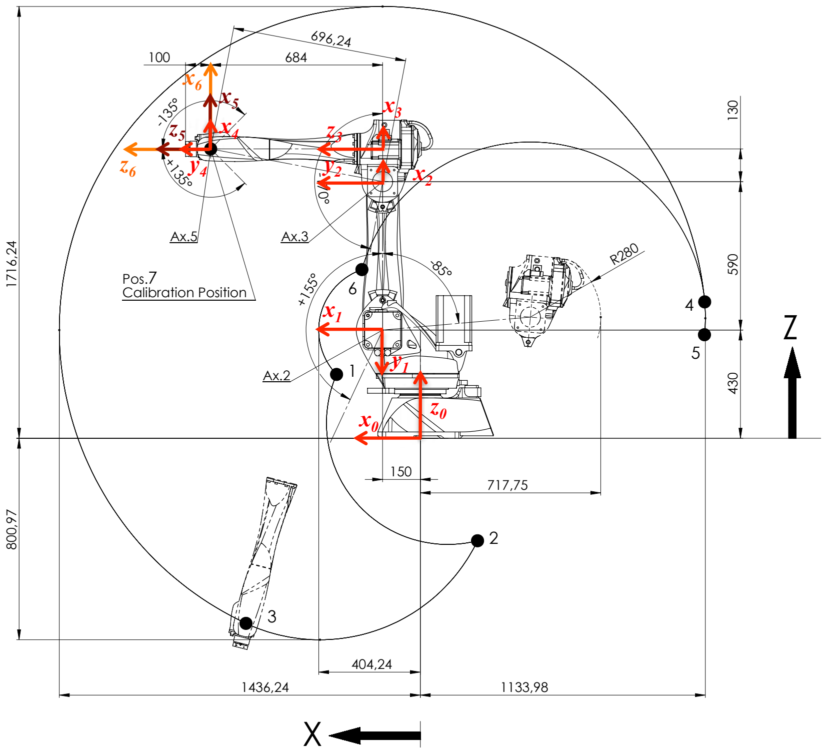


Figure 3: Assignment of Denavit-Hartenberg frames for the COMAU RACER 7-1.4 robot

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 > 0$	$d_1 > 0$	θ_1
2	0	$a_2 > 0$	0	θ_2
3	$-\pi/2$	$a_3 > 0$	0	θ_3
4	$\pi/2$	0	$d_4 > 0$	θ_4
5	$-\pi/2$	0	0	θ_5
6	0	0	0	θ_6

Table 1: Denavit-Hartenberg parameters associated to the frame assignments in Fig. 3

A possible DH frame assignment is shown in Fig. 3, with the associated parameters given in Table 1. The actual numerical values are (in mm):

$$a_1 = 150, \quad d_1 = 430, \quad a_2 = 590, \quad a_3 = 130, \quad d_4 = 684.$$

The configuration $\boldsymbol{\theta} = \mathbf{0}$ of the chosen DH convention corresponds to the COMAU angles (in deg):

$$\boldsymbol{\theta}_{C,0} = \begin{pmatrix} 0 & +90 & -90 & 0 & 0 & 0 \end{pmatrix}^T.$$

The diagonal elements matrix $\boldsymbol{S}$ are:

$$\boldsymbol{S} = \operatorname{diag}\{-1, +1, -1, -1, +1, -1\}.$$

One can also reverse eq. (1), by multiplying it by $\boldsymbol{S}$ and noting that $\boldsymbol{S}^2 = \boldsymbol{I}$. We obtain

$$\boldsymbol{\theta} = -\boldsymbol{S}\,\boldsymbol{\theta}_{C,0} + \boldsymbol{S}\,\boldsymbol{\theta}_C = \boldsymbol{\theta}_0 + \boldsymbol{S}\,\boldsymbol{\theta}_C, \qquad \boldsymbol{S} = \operatorname{diag}\{s_i,\, i = 1,\dots,6\}, \qquad s_i = \{+1, -1\}. \tag{2}$$

Thus, the value $\boldsymbol{\theta}_0 \in \mathbb{R}^6$ of the DH joint variables corresponding to $\boldsymbol{\theta}_C = \mathbf{0}$, the zero configuration of the COMAU variables, is given by:

$$\boldsymbol{\theta}_0 = -\boldsymbol{S}\,\boldsymbol{\theta}_{C,0} = \begin{pmatrix} 0 & -90 & -90 & 0 & 0 & 0 \end{pmatrix}^T. \tag{3}$$

Finally, Table 2 compares the feasible joint ranges (in deg) in the two notations. For $\boldsymbol{\theta}$, instead of reporting the min and max values (in increasing order), we preferred to use the notation $\lim_1$ and $\lim_2$ (corresponding, respectively, to the lower and upper limits of θ_C). In this way, we take into account the possible reverse sense of rotation of the joints (i.e., when $s_i = -1$ for joint i) in the two conventions. Note that almost all limits in the fourth and fifth columns of this table can be computed directly using eqs. (2) and (3). Joint 5 is an exception: a $+90°$ should be added because of the special placement of the $\boldsymbol{x}_5$ axis in the $\theta_5 = 0$ configuration w.r.t. the actual joint range.

i	min $\theta_{C,i}$	max $\theta_{C,i}$	$\lim_1 \theta_i$	$\lim_2 \theta_i$
1	$-165°$	$165°$	$165°$	$-165°$
2	$-85°$	$155°$	$-175°$	$65°$
3	$-170°$	$0°$	$80°$	$-90°$
4	$-210°$	$210°$	$210°$	$-210°$
5	$-135°$	$135°$	$-45°$	$225°$
6	$-2700°$	$2700°$	$2700°$	$-2700°$

Table 2: Joint ranges in the COMAU and DH conventions

* * * * *

Robotics I

July 15, 2014

For a KUKA LWR robot, let $\boldsymbol{\theta} \in \mathbb{R}^7$ be the joint variables and consider a situation in which the last three joints (constituting a spherical wrist with center $W = O_5 = O_6$) are permanently *frozen*. For kinematic analysis, use the DH frame assignment of Fig. 1, where the robot is shown in its configuration $\boldsymbol{\theta} = \mathbf{0}$. Assume $l_1 = l_2 = l_3 = l_4 = l_5 = l$ (while l_0 and l_6 are different). Frame 7 is drawn for clarity in a displaced position, but is actually located on the final flange of the robot at a distance l_6 from W.

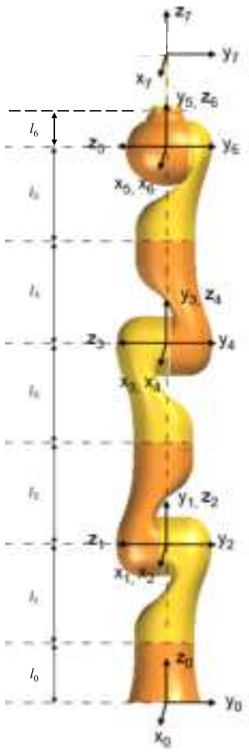


Figure 1: A DH frame assignment for the KUKA LWR robot

- Provide the expression $\boldsymbol{p}_W = \boldsymbol{f}(\boldsymbol{\theta})$ for the position of the robot wrist center W.
- Determine the expression of the 3×4 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ relating the velocity of the *active* joints $\dot{\boldsymbol{\theta}}_a \in \mathbb{R}^4$ to the velocity $\boldsymbol{v}_W = \dot{\boldsymbol{p}}_W$.
- Having set $\theta_3 = 0$, find suitable numerical values for the remaining variables in $\boldsymbol{\theta}_a$ so that point W is on the axis $\boldsymbol{z}_0$ at a generic distance d from the origin O_1 of frame 1. The distance d can be chosen arbitrarily, as long as it satisfies $0 < d < 4l$. In the selected configuration, show that the Jacobian $\boldsymbol{J}$ has full rank and give a basis for its null space $\mathcal{N}\{\boldsymbol{J}\}$.
- In the same configuration, show that if also joint 3 is considered to be *frozen*, then the resulting square Jacobian $\boldsymbol{J}_{/3}$ would be singular. Determine then all independent Cartesian directions $\boldsymbol{w}$ that are not instantaneously accessible by the point W (i.e., $\boldsymbol{w} \notin \mathcal{R}\{\boldsymbol{J}_{/3}\}$).

[180 minutes; open books]

Robotics I

September 22, 2014

The articulated robotic structure in Fig. 1 is used for diagnostic and interventional imaging in cardiology and radiology. The robot has a branched kinematics, with a serial chain of bodies having five revolute joints, followed by two symmetric final ‘branches´. Each branch can be modeled as having two elementary joints (one prismatic and one revolute) that move in perfect coordination with those of the other branch. This kinematic structure provides a convenient positioning and orienting flexibility to the imaging device that operates along the segment between the tips of the two branches.

Figure 1: The Siemens Artis Zeego medical robot and its degrees of freedom

- Consider only one branch of the robot, i.e., a serial kinematic chain with 7 dofs:
  1. assign the robot frames according to the Denavit-Hartenberg convention, and draw the frames on the sheet;
  2. provide the associated table of D-H parameters, and enter also reasonable (approximate) values for the joint variables at the robot configuration shown in the figure.
- Consider now the complete robot and the midpoint P of the (variable length) segment joining the tips of the two robot branches. The motion of which joints does not affect the position of P? Can we state that this robot has a spherical wrist?

[120 minutes; open books]

Robotics I

October 27, 2014

Exercise 1

Consider a minimal representation of the orientation of a rigid body as given by Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes XYZ.

a) Determine the relation

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

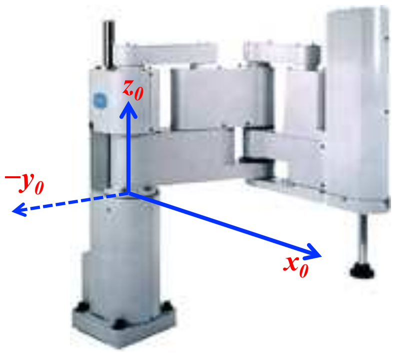
between the time derivatives of the Euler angles and the angular velocity $\boldsymbol{\omega}$ of the rigid body.

b) Find the singularities of $\boldsymbol{T}(\boldsymbol{\phi})$, and provide an example of an angular velocity vector $\boldsymbol{\omega}$ that cannot be represented in a singularity.

c) Given a rotation matrix $\boldsymbol{R} = \{r_{ij}\} \in SO(3)$, provide the analytic solution $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ to the inverse representation problem out of singularities.

Exercise 2

Consider a 4-dof SCARA robot, with base frame and Denavit-Hartenberg table as shown in Fig. 1.



i	α_i	a_i	d_i	θ_i
1	0	a_1	0	q_1
2	π	a_2	0	q_2
3	0	0	q_3	0
4	0	0	0	q_4

Figure 1: A SCARA robot and its DH table

a) Provide the 6×4 geometric Jacobian of this robot in analytic form.

b) In the following, neglect the presence of joint limits. When the robot arm is fully stretched along the $\boldsymbol{x}_0$ axis, determine, if possible, a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that realizes the following desired end-effector generalized velocity:

$$\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}^T \qquad \text{[m/s] or [rad/s].}$$

c) In this configuration, does a solution exist for any desired generalized velocity $\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T$, provided only that $\omega_{d,x} = \omega_{d,y} = 0$? And when a solution does exist, is it unique?

[150 minutes; open books]

Solution

October 27, 2014

Exercise 1

a) The orientation of a rigid body using the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes XYZ is represented by the product of elementary rotation matrices

$$\boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) = \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_Z(\gamma). \tag{1}$$

The angular velocity $\boldsymbol{\omega}$ due to $\dot{\boldsymbol{\phi}}$ can be obtained as the sum of the three angular velocities contributed by, respectively, $\dot{\alpha}$ (along the unit vector $\boldsymbol{X}$), $\dot{\beta}$ (along the rotated $\boldsymbol{Y}'$), and $\dot{\gamma}$ (along the doubly rotated $\boldsymbol{Z}''$), or

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{X}\dot{\alpha} + \boldsymbol{Y}'\dot{\beta} + \boldsymbol{Z}''\dot{\gamma}$$

where the unit vectors $\boldsymbol{X}$, $\boldsymbol{Y}'$ e $\boldsymbol{Z}''$ are expressed with respect to the initial reference frame. It is

$$\boldsymbol{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{Y}' = \boldsymbol{R}_X(\alpha)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{Z}'' = \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, it is sufficient to compute

$$\boldsymbol{R}_X(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \boldsymbol{R}_Y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

and

$$\boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta) = \begin{pmatrix} * & * & \sin\beta \\ * & * & -\sin\alpha\cos\beta \\ * & * & \cos\alpha\cos\beta \end{pmatrix}$$

in order to obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\dot{\alpha} + \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}\dot{\beta} + \begin{pmatrix} \sin\beta \\ -\sin\alpha\cos\beta \\ \cos\alpha\cos\beta \end{pmatrix}\dot{\gamma} = \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

Note also that, as a general property, matrix $\boldsymbol{T}$ depends only on the first two Euler angles.

b) Matrix $\boldsymbol{T}$ is singular when

$$\det\boldsymbol{T} = \cos\beta = 0 \quad \Longleftrightarrow \quad \beta = \pm\frac{\pi}{2}.$$

In this condition, an angular velocity vector (with norm k) of the form

$$\boldsymbol{\omega} = k\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{T}(\alpha, \pm\frac{\pi}{2})\right\}$$

cannot be represented by any choice of $\dot{\boldsymbol{\phi}}$.

c) We need first to determine the complete expression of the rotation matrix in (1). For this, we need also the third elementary rotation matrix

$$\boldsymbol{R}_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Performing the matrix multiplication in (1), we obtain 9 scalar nonlinear equations in matrix form:

$$\begin{aligned} \boldsymbol{R}_{XYZ}(\alpha,\beta,\gamma) &= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma+\sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma-\cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix} \\ &= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}. \end{aligned}$$

Note that only the elements of the first row and last column will be used for the solution. We have

$$\beta = \text{ATAN2}\left\{r_{13},\ \pm\sqrt{r_{11}^2+r_{12}^2}\right\},$$

and, assuming $\cos\beta \neq 0$,

$$\alpha = \text{ATAN2}\left\{-\frac{r_{23}}{\cos\beta},\ \frac{r_{33}}{\cos\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{-\frac{r_{12}}{\cos\beta},\ \frac{r_{11}}{\cos\beta}\right\}.$$

For completeness, we report also the expression of the Euler XYZ rotation matrix in the singular case (this refers to T, since a rotation matrix can never be singular!), which occurs if and only if $r_{11}^2 + r_{12}^2 = 0$, i.e., $\cos\beta = 0$. Then, either we have $\beta = \pi/2$ and

$$\boldsymbol{R}_{XYZ}(\alpha,\pi/2,\gamma) = \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{pmatrix},$$

or $\beta = -\pi/2$ and

$$\boldsymbol{R}_{XYZ}(\alpha,-\pi/2,\gamma) = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\gamma-\alpha) & \cos(\gamma-\alpha) & 0 \\ \cos(\gamma-\alpha) & -\sin(\gamma-\alpha) & 0 \end{pmatrix}.$$

In these cases, we cannot find two distinct solutions to the inverse representation problem. Rather can solve only for the sum $(\alpha+\gamma)$ or, respectively, for the difference $(\gamma-\alpha)$, obtaining thus an infinite number of solutions.

Exercise 2

a) The desired 6×4 Jacobian matrix is most efficiently computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{p}_{04}}{\partial q_1} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_2} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_3} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_4} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix},$$

namely:

- for the first three rows (linear components), by analytic derivation of the position vector $\boldsymbol{p}_{04}$ of the origin of the end-effector frame (# 4);
- for the last three rows (angular components), by the standard geometric expressions for revolute or prismatic joints.

Using the DH table in Fig. 1, we easily obtain from the products with the homogeneous transformation matrices

$$\begin{aligned} \boldsymbol{p}_{04,\text{hom}}(\boldsymbol{q}) &= \boldsymbol{A}_1(q_1)\left(\boldsymbol{A}_2(q_2)\left(\boldsymbol{A}_3(q_3)\left(\boldsymbol{A}_4(q_4)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)\right) \\ &= \begin{pmatrix} a_1 \cos q_1 + a_2 \cos(q_1+q_2) \\ a_1 \sin q_1 + a_2 \sin(q_1+q_2) \\ -q_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{q}) \\ 1 \end{pmatrix}. \end{aligned}$$

In the first row above, brackets have been used to indicate the most convenient order of products, especially for symbolic computations.

The sub-matrix $\boldsymbol{J}_L(\boldsymbol{q})$ is given then by

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where the usual compact notations have been used. Since all joint axes are parallel, for $\boldsymbol{J}_A(\boldsymbol{q})$ we have just to take into account the actual directions of the $\boldsymbol{z}_i$ axes (pointing upwards or downwards). It is

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{z}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Summarizing, the geometric Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}. \tag{2}$$

The fourth and fifth rows are structurally zero, since the end-effector of a SCARA robot cannot rotate out of the horizontal plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. Thus, in order to be feasible, any desired $\boldsymbol{\omega}_d$ should have $\omega_{d,x} = \omega_{d,y} = 0$.

b) When the arm is fully stretched along the $\boldsymbol{x}_0$ axis, it is $q_1 = q_2 = 0$ (the values of q_3 and q_4 are irrelevant). Therefore, we need to solve the following set of linear equations (the link lengths a_1 and a_2 are left parametric):

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{v}_d \\ \boldsymbol{\omega}_d \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 + a_2 & a_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Eliminating the first, fourth, and fifth equations (which are identities of the form $0 = 0$), we are left with three linear independent equations in four unknowns. Therefore, the solutions are infinite in number (more precisely, ∞^1). One possible solution is the joint velocity $\dot{\boldsymbol{q}}$ with components

$$\begin{aligned} \dot{q}_1 &= \frac{a_1 + a_2}{(a_1 + a_2)^2 + a_2^2}, \\ \dot{q}_2 &= \frac{a_2}{(a_1 + a_2)^2 + a_2^2}, \\ \dot{q}_3 &= -1, \\ \dot{q}_4 &= \dot{q}_1 + \dot{q}_2 - 1 = \frac{a_1 + 2a_2}{(a_1 + a_2)^2 + a_2^2} - 1, \end{aligned}$$

which minimizes in norm the velocity of the first two joints. Another solution $\dot{\boldsymbol{q}}$ is given by

$$\begin{aligned} \dot{q}_1 &= \frac{a_2 + 2a_1 - a_1 a_2}{2(a_1^2 + a_1 a_2 + a_2^2)}, \\ \dot{q}_2 &= \frac{a_2 - a_1 + a_1^2 + a_1 a_2}{2(a_1^2 + a_1 a_2 + a_2^2)}, \\ \dot{q}_3 &= -1, \\ \dot{q}_4 &= \frac{a_1^2 + a_1 + 2a_2}{2(a_1^2 + a_1 a_2 + a_2^2)} - 1, \end{aligned}$$

which is the joint velocity of minimum norm (considering all four joints) among all possible solutions.

c) From the previous arguments, it can be concluded that a generic desired generalized velocity, even when it has $\omega_{d,x} = \omega_{d,y} = 0$ may not solve the problem unless it has also $v_{d,x} = 0$. The first two joints of the robot are those responsible for the linear velocity of the end-effector in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. This robot substructure is associated to the upper left 2×2 block in the Jacobian $\boldsymbol{J}$ in (2), which is singular in the considered configuration. However, since the desired planar velocity $(v_{d,x}\ v_{d,y})^T = (0\ 1)^T$ still lies in the range space of this submatrix, then the problem is solvable (as a matter of fact, it has an infinity of solutions). Otherwise, the problem would have no solution.

* * * * *

Robotics I

Classroom Test — November 21, 2014

Exercise 1 [6 points]

In the Unimation Puma 560 robot, the DC motor that drives joint 2 is mounted in the body of link 2 (upper arm) and is connected to the joint axis through a train of transmission elements (see Fig. 1). The output shaft of the motor (code 506-1612) is connected to an idler assembly (code 500-2401) through a *bevel gear*, which changes the axis of rotation by 90° and has a reduction ratio $n_{bg} = 10.88$. In turn, the idler assembly is connected via a *spur gear* (code 500-0941) to the axis of joint 2, moving thus the second link. The two engaged wheels in the spur gear have radius $r_{in} = 1.1$ [cm] and $r_{out} = 10.86$ [cm], respectively.

a) What is the reduction ratio n_r of the complete transmission from motor 2 to link 2?

b) The inertia (of the rotor) of this motor is $J_m = 0.0002$ [kg·m^2]. If the reduction ratio used by Unimation were *optimal*, what should be the target inertia of the load (link 2)?

c) The maximum rated torque produced at the motor shaft is $\tau_m = 0.3$ [Nm]. With the values found in a) and b), neglecting dissipative effects, gravity, and all other dynamic couplings, what would be the maximum angular acceleration $\ddot{\theta}_2$ of link 2 realized by this motor/transmission unit?

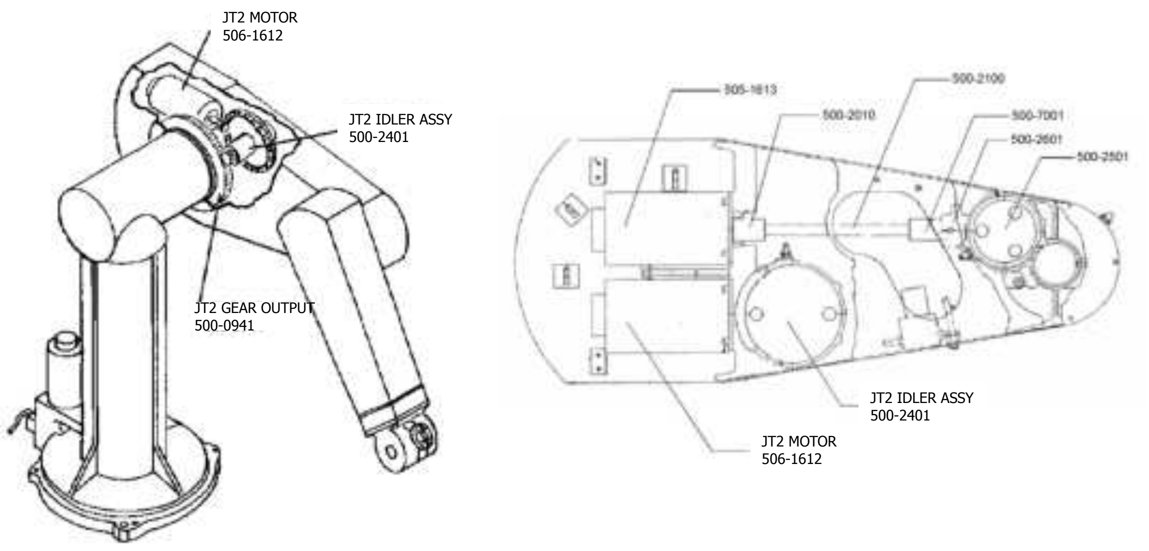


Figure 1: Two inside views of the upper arm of the Unimation Puma 560 robot, showing in particular the motor and gear train driving joint 2

(continues)

Exercise 2 [6 points]

Figure 2 shows again the Unimation Puma 560, a 6R articulated robot having a spherical wrist, with a set of link frames and relevant distances labeled from A to E. For the sake of drawing clarity, frames may be shown displaced from their actual placement; in particular, the origin of frame 5 is at the wrist center, while the origin of frame 6 is midway between the gripper jaws. Verify that the frame assignments are correct according to the Denavit-Hartenberg (DH) convention. If so, determine the associated table of DH parameters, using the distances A to E (with signs) and providing the values of the joint variables in the shown configuration.

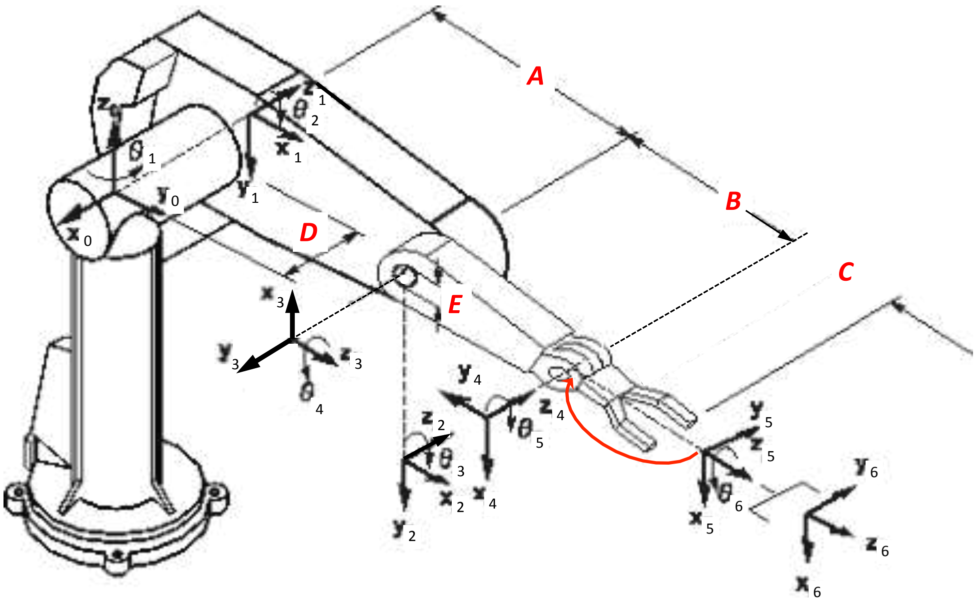


Figure 2: The Unimation Puma 560 robot with assigned link frames

Exercise 3 [6 points]

Consider a 3R planar manipulator with links of *generic* lengths l_1, l_2, and l_3. Assuming that each joint has unlimited rotation, determine the primary workspace for the end-effector of this robot. Verify your analysis by drawing the workspace in the following two numerical cases (both have the same value for the sum of the link lengths):

a) $l_1 = 1$, $l_2 = 0.4$, $l_3 = 0.3$ [m];

b) $l_1 = 0.5$, $l_2 = 0.7$, $l_3 = 0.5$ [m].

(continues)

Exercise 4 [12 points]

A large Cartesian robot has 3 prismatic joints, followed by a spherical wrist with center W. Table 1 provides the DH parameters of this 6-dof robot. A variety of tools can be mounted on the robot end-effector, each having the Tool Center Point (TCP) placed along the approaching axis. The distance of the TCP from the wrist center W is specified by the parameter $d_{TCP} > 0$.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	0	q_3	0
4	$-\pi/2$	0	0	q_4
5	$\pi/2$	0	0	q_5
6	0	0	d_{TCP}	q_6

Table 1: Denavit-Hartenberg parameters of a 3P-3R robot with spherical wrist

a) Given a desired position of the TCP (with its d_{TCP}) and a desired orientation of the robot end-effector frame (i.e., of frame 6), provide an analytic solution in closed form to the inverse kinematics problem. Is the solution unique?

b) Apply your results to the data

$$d_{TCP} = 0.15 \text{ [m]}, \qquad \boldsymbol{p}_d = \begin{pmatrix} 5.0 \\ 2.0 \\ 1.5 \end{pmatrix} \text{ [m]}, \qquad \boldsymbol{R}_d = \begin{pmatrix} -0.5 & 0.5 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0.5 & -0.5 & -\sqrt{2}/2 \end{pmatrix},$$

and provide the numerical value of at least one joint configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix}^T$ solving the inverse kinematics.

[210 minutes; open books]

Solutions

November 21, 2014

Exercise 1 [6 points]

a) The reduction ratio of the spur gear is $n_{sg} = r_{out}/r_{in} = 10.86/1.1 = 9.87$. The complete transmission has then reduction ratio

$$n_r = n_{bg} \cdot n_{sg} = 10.88 \cdot 9.87 = 107.38.$$

b) Assuming that this is the optimal value of the reduction ratio (i.e., the one that minimizes the motor torque needed to accelerate the link at a desired value $\ddot{\theta}_2 = a$), then the inertia J_2 of link 2 around its rotation axis should satisfy

$$n_r^* = 107.38 = \sqrt{\frac{J_2}{J_m}} = \sqrt{\frac{J_2}{0.0002}} \quad \Longrightarrow \quad J_2 = 0.0002 \cdot (107.38)^2 = 2.3061 \text{ [kg·m}^2\text{]}.$$

Note that in this analysis we have considered the inertia of the intermediate gears as negligible w.r.t. motor and link inertias. In the Puma 560, this is reasonable as the inertia of the gear train for joint 2, when reflected to the axis of motor 2, is less than 2% of the rotor inertia of the motor.

c) When using the optimal reduction ratio, there is a balanced partition of the torque produced by the motor:

$$\tau_m = J_m\ddot{\theta}_m + \frac{1}{n_r^*}J_2\ddot{\theta}_2 = \left(J_m n_r^* + \frac{1}{n_r^*}J_2\right)\ddot{\theta}_2 = 2\sqrt{J_m J_2}\,\ddot{\theta}_2 = 2\,J_m n_r^*\,\ddot{\theta}_2.$$

Thus, the maximum acceleration of link 2 (to be intended in absolute value) is

$$\ddot{\theta}_2 = \frac{\tau_m}{2}\frac{1}{J_m n_r^*} = 0.15\,\frac{1}{0.0002 \cdot 107.38} = 6.9845 \text{ [rad/s}^2\text{]}.$$

Exercise 2 [6 points]

The assignment of link frames is feasible according to the (classical) Denavit-Hartenberg convention. The associated DH parameters are given in Table 2.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	$q_1 = \pi/2$
2	0	$A > 0$	$D > 0$	$q_2 = 0$
3	$-\pi/2$	$E > 0$	0	$q_3 = -\pi/2$
4	$-\pi/2$	0	$B > 0$	$q_4 = \pi$
5	$\pi/2$	0	0	$q_5 = 0$
6	0	0	$C > 0$	$q_6 = 0$

Table 2: Denavit-Hartenberg parameters of the Unimation Puma 560 robot associated to the link frames and to the specific configuration shown in Fig. 2

Since the first two joint axes intersect, the origin O_1 *must* be set at the intersection point, which is where also O_0 is, and so $a_1 = d_1 = 0$. Also, the drawing may not be 100% clear on the sign of a_3: here, we took $a_3 = E > 0$, i.e., along the positive direction of $\boldsymbol{x}_3$ (as it is in reality).

Exercise 3 [6 points]

Denote the lengths of the longest and shortest links, respectively with

$$l_{max} = \max\{l_i,\ i = 1,2,3\}, \quad l_{min} = \min\{l_i,\ i = 1,2,3\},$$

and with l_{med} the length of the intermediate link of intermediate length. If two or more links have equal lengths, their relative ordering is irrelevant. No matter how the links of different length are placed within the kinematic chain, the workspace of the planar $3R$ will have as outer boundary a circumference of radius

$$R_{out} = l_{min} + l_{med} + l_{max} = l_1 + l_2 + l_3,$$

and as inner boundary a circumference of radius

$$R_{in} = \max\{0, l_{max} - (l_{med} + l_{min})\}.$$

This last formula means that when $l_{max} \le l_{med} + l_{min}$, there will be no forbidden area internal to the workspace, which is then a circle of radius R_{out} centered at the robot base. Otherwise, there will be a circular 'hole' of radius $l_{max} - (l_{med} + l_{min}) > 0$ in the center of the workspace. Consider next the two case studies.

a) Here, $l_{max} = l_1 = 1$, $l_{med} = l_2 = 0.4$, and $l_{min} = l_3 = 0.3$. Since $l_{max} = 1 > 0.7 = l_{med} + l_{min}$, the workspace will be an annulus with inner radius $R_{in} = 0.3$ and outer radius $R_{out} = 1.7$. Figure 3 shows the geometric construction of the workspace in this case.

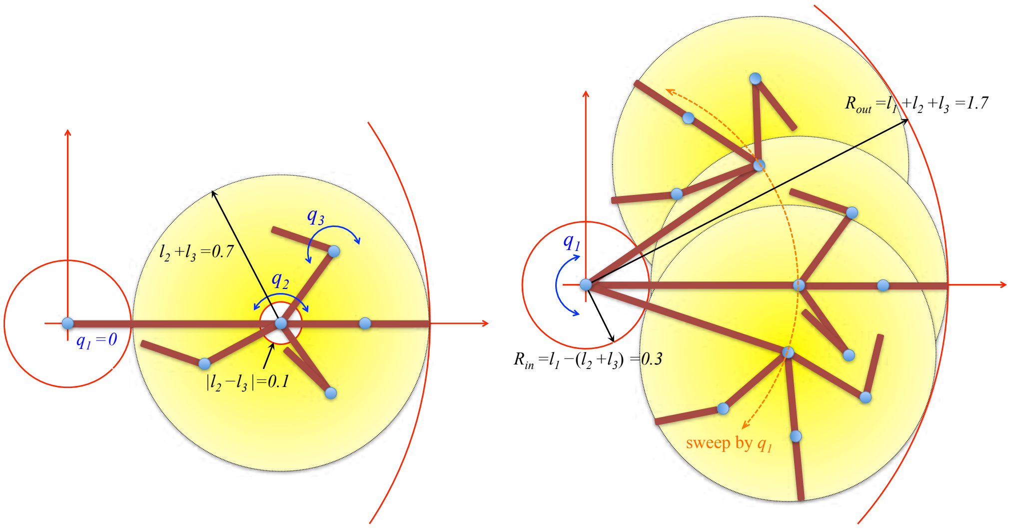


Figure 3: Construction of the primary workspace of a 3R robot with link lengths $l_1 = 1$, $l_2 = 0.4$, and $l_3 = 0.3$: [Left] the workspace of the second and third link (keeping q_1 fixed) contains an unreachable circular area centered at the second joint; [Right] sweeping by q_1 will eliminate this area, although the workspace of the robot still contains a central 'hole' of radius $R_{in} = 0.3$

b) In this case, $l_{max} = l_2 = 0.7$, $l_{med} = l_{min} = l_1(\text{or } l_3) = 0.4$. Since $l_{max} = 0.7 < 0.8 = l_{med} + l_{min}$, the workspace will be a full circle of radius $R_{out} = 1.7$.

Exercise 4 [12 points]

From the DH table, we first compute the direct kinematics. The first three prismatic joints provide a homogeneous transformation matrix

$$
{}^{0}\boldsymbol{T}_3(q_1,q_2,q_3) = \begin{pmatrix} {}^{0}\boldsymbol{R}_3 & {}^{0}\boldsymbol{p}_{03}(q_1,q_2,q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & q_3 \\ 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
$$

It is clear that prismatic joints do not change orientation, up to a (signed) permutation of the Cartesian axes. The position of the wrist center is $\boldsymbol{p}_W = {}^{0}\boldsymbol{p}_{03}(q_1,q_2,q_3)$. Performing the complete computation (in case, by using a straightforward adaptation of the symbolic code `dirkin_SCARA.m` available on the course web page) yields

$$
{}^{0}\boldsymbol{T}_6(\boldsymbol{q}) = \begin{pmatrix} {}^{0}\boldsymbol{R}_6(q_4,q_5,q_6) & {}^{0}\boldsymbol{p}_{06}(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \tag{1}
$$

where

$$
\begin{aligned}
{}^{0}\boldsymbol{R}_6(q_4,q_5,q_6) &= \begin{pmatrix} \boldsymbol{n}(q_4,q_5,q_6) & \boldsymbol{s}(q_4,q_5,q_6) & \boldsymbol{a}(q_4,q_5) \end{pmatrix} \\
&= \begin{pmatrix} -\sin q_5 \cos q_6 & \sin q_5 \sin q_6 & \cos q_5 \\ -\sin q_4 \cos q_5 \cos q_6 - \cos q_4 \sin q_6 & \sin q_4 \cos q_5 \sin q_6 - \cos q_4 \cos q_6 & -\sin q_4 \sin q_5 \\ \cos q_4 \cos q_5 \cos q_6 - \sin q_4 \sin q_6 & -\cos q_4 \cos q_5 \sin q_6 - \sin q_4 \cos q_6 & \cos q_4 \sin q_5 \end{pmatrix}
\end{aligned} \tag{2}
$$

and

$$
{}^{0}\boldsymbol{p}_{06}(\boldsymbol{q}) = \boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} q_3 + d_{TCP}\cos q_5 \\ q_2 - d_{TCP}\sin q_4 \sin q_5 \\ q_1 + d_{TCP}\cos q_4 \sin q_5 \end{pmatrix}. \tag{3}
$$

a) Since the robot has a spherical wrist, we can find the solution to the inverse kinematics problem in a partitioned way, first determining the values of the joint variables q_1, q_2, and q_3 for the main axes, and then (for each solution found in the first step) finding the values of the joint variables q_4, q_5, and q_6 for the spherical wrist. Indeed, things are particularly simple in the first step because this sub-structure is a PPP (Cartesian) robot. From eqs. (1–3), we obtain directly the first part of the solution by means of the expression

$$
\boldsymbol{p}(\boldsymbol{q}) - d_{TCP}\,\boldsymbol{a}(q_4,q_5) = (\boldsymbol{p}_W =) \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} \tag{4}
$$

There is indeed a *unique* solution to step 1 (note also the reordering of the first three variables). In the second step, since the rotation matrix ${}^{0}\boldsymbol{R}_3$ independent from the joint variables q_1, q_2, and q_3, it is not necessary to isolate the rotation matrix

$${}^{3}\boldsymbol{R}_6(q_4,q_5,q_6) = {}^{0}\boldsymbol{R}_3^T\,{}^{0}\boldsymbol{R}_6(q_4,q_5,q_6) = \ldots \text{(this matrix is associated to a ZYZ Euler sequence!).}$$

In fact, we can just equate the expression (2) with that of a generic orientation matrix $\boldsymbol{R} = \{r_{ij}\}$ of the end-effector frame. Using a compact notation, we have

$$\begin{pmatrix} -s_5c_6 & s_5s_6 & c_5 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & -s_4s_5 \\ c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}. \tag{5}$$

Proceeding as in the solution of an inverse problem of minimal representation of orientation by a generic Euler sequence, we can isolate q_5 from the elements in the last column or in the first row of (5) as

$$q_5 = \text{ATAN2}\left\{\pm\sqrt{r_{23}^2 + r_{33}^2}, r_{13}\right\} \qquad \text{or} \qquad q_5 = \text{ATAN2}\left\{\pm\sqrt{r_{11}^2 + r_{12}^2}, r_{13}\right\}. \tag{6}$$

Provided that $s_5 \neq 0$, which should be checked in advance as $r_{23}^2 + r_{33}^2 \neq 0$ (or, equivalently, $r_{11}^2 + r_{12}^2 \neq 0$) on the given data, we can solve for q_4 and q_6 from the last column and first row in (5):

$$q_4 = \text{ATAN2}\{-r_{23}/s_5, r_{33}/s_5\}, \qquad q_6 = \text{ATAN2}\{r_{12}/s_5, -r_{11}/s_5\}. \tag{7}$$

In the regular case, *two solutions* are obtained from eqs. (4), (6), and (7). Instead, when $s_5 = 0$ the robot is in a wrist singularity (occurring for $q_5 = 0$ or π), and two cases arise. If $q_5 = 0$, we can only solve for the sum of the two angles $q_4 + q_6$. In fact, setting $s_5 = 0$ and $c_5 = 1$ in (5) and considering a compatible orientation matrix $\boldsymbol{R}$, eq. (5) becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ -s_{46} & -c_{46} & 0 \\ c_{46} & -s_{46} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix},$$

from which

$$q_4 + q_6 = \text{ATAN2}\{-r_{21}, r_{31}\}.$$

Similarly, if $q_5 = \pi$ we can only solve for the difference $q_4 - q_6$ of the two remaining joint angles of the robot wrist. In both cases, we will have a (simple) infinity of inverse solutions. Note that often the situation $q_5 = \pi$ is ruled out by the presence of a limited range for joint 5 around its zero value.

b) From the given numerical data for the desired pose, with $\boldsymbol{a}_d$ being the third column of $\boldsymbol{R}_d$, we perform the same operation as in eq. (4) and obtain the values (fully specified by the data, without the need to compute q_4 and q_5 first)

$$\begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \boldsymbol{p}_d - d_{TCP}\,\boldsymbol{a}_d = \begin{pmatrix} 5.0 \\ 2.0 \\ 1.5 \end{pmatrix} - 0.15 \begin{pmatrix} -\sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 5.1061 \\ 2.0 \\ 1.6061 \end{pmatrix} \text{ [m]}. \tag{8}$$

Moreover, since $r_{23,d}^2 + r_{33,d}^2 = 0.5 \neq 0$, the robot is not in a wrist singularity. Applying then eqs. (6–7), we obtain

$$q_5^{(+,-)} = \text{ATAN2}\left\{\pm 1/\sqrt{2}, -1/\sqrt{2}\right\} = \pm\frac{3\pi}{4} \text{ [rad]}$$

and

$$q_4^{(+,-)} = \text{ATAN2}\{0, \mp 1\} = \{\pi, 0\} \text{ [rad]}, \quad q_6^{(+,-)} = \text{ATAN2}\left\{\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right\} = \{\pi/4, -3\pi/4\} \text{ [rad]}.$$

As a result, there are two regular solutions for the robot wrist angles. The complete inverse kinematics solutions are then

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4^+ & q_5^+ & q_6^+ \end{pmatrix} = \begin{pmatrix} 1.6061 & 2.0 & 5.1061 & \pi & \dfrac{3\pi}{4} & \dfrac{\pi}{4} \end{pmatrix}$$

and

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4^- & q_5^- & q_6^- \end{pmatrix} = \begin{pmatrix} 1.6061 & 2.0 & 5.1061 & 0 & -\dfrac{3\pi}{4} & -\dfrac{3\pi}{4} \end{pmatrix}.$$

Hint: It is always good practice to feed the obtained solutions into the direct kinematics and check if the results coincide with what should be expected.

* * * * *

Robotics I

January 9, 2015

Exercise 1

A planar 2R robot with links of length $l_1 = 0.1492$ m and $l_2 = 0.1905$ m and actuated by direct-drive motors is equipped at the two joints with incremental encoders, providing respectively 8192 and 4096 pulses per turn. When the robot is in the nominal configuration $\hat{\theta}_1 = 45°$, $\hat{\theta}_2 = -60°$, determine the maximum uncertainty (in norm) that affects the measure of the Cartesian end-effector position.

Exercise 2

Consider a 2-dof planar RP robot with the following kinematic constraints:

$$\begin{array}{rll} \text{joint ranges} & q_1 \in [0, 120°]\,, & q_2 \in [0.5, 1] \;\; \text{[m]}, \\ \text{joint velocity limits} & |\,\dot{q}_1| \leq 40°\!/\text{s}, & |\,\dot{q}_2| \leq 1.5 \; \text{[m/s]}. \end{array} \tag{1}$$

Assume that both joint velocities can switch their value instantaneously (in practice, this simplifying assumption is reasonable when the physical limits on joint accelerations are very high). Plan a straight line trajectory between two points in the Cartesian space (say, $\boldsymbol{A}$ and $\boldsymbol{B}$) such that *i)* the entire path belongs to the robot workspace, *ii)* the path has the maximum possible length, *iii)* the trajectory satisfies the velocity limits in (1), and *iv)* the transfer from $\boldsymbol{A}$ to $\boldsymbol{B}$ is realized in minimum time T (provide this value).

Exercise 3

A 3R anthropomorphic robot is characterized by the D-H parameters given in Tab. 1.

i	α_i	a_i [m]	d_i	θ_i
1	$\pi/2$	0	0	θ_1
2	0	1.5	0	θ_2
3	0	1	0	θ_3

Table 1: Denavit-Hartenberg parameters of the 3R robot

A desired trajectory $\boldsymbol{p}_d(t)$ is specified for the position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta})$ of the robot end effector as a straight line rest-to-rest motion from point $\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 0.5 \end{pmatrix}^T$ to point $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0.5 \end{pmatrix}^T$ [m], with a trapezoidal velocity law having maximum speed $v_{\max} = 0.5$ [m/s] and maximum acceleration $a_{\max} = 5$ [m/s^2]. The initial configuration of the robot is $\boldsymbol{\theta}(0) = \begin{pmatrix} -\pi/2 & 0 & \pi/6 \end{pmatrix}^T$. Let the joint velocity $\dot{\boldsymbol{\theta}}$ be the command input. Design a controller so that the robot asymptotically tracks the desired trajectory. Furthermore, determine also the smallest feedback gains in the control law so that the norm of the Cartesian error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ is brought definitely below 5% of the initial value $\|\boldsymbol{e}(0)\|$ as soon as *one fourth* of the nominal motion time of the desired trajectory has passed. Provide the expressions of all terms involved in the control law. Sketch the time evolution of the three Cartesian error components e_x, e_y and e_z. Does the robot encounter singular configurations during motion? Will all robot joints move while performing this control task?

[210 minutes; open books]

Solution

January 9, 2015

Exercise 1

The limited accuracy in the indirect measure of the end-effector position is due to the resolution of the incremental encoders, and is related to the robot Jacobian in the nominal configuration $\hat{\boldsymbol{\theta}}$. We have (with the usual shorthand notation)

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta}) = \begin{pmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{pmatrix} \Rightarrow \boldsymbol{J}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix}.$$

From the Taylor expansion, it is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta}) \approx \boldsymbol{f}(\hat{\boldsymbol{\theta}}) + \boldsymbol{J}(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) = \hat{\boldsymbol{p}} + \boldsymbol{J}(\hat{\boldsymbol{\theta}})\boldsymbol{\Delta\theta}. \tag{2}$$

From the given data, it is

$$\hat{\boldsymbol{p}} = \begin{pmatrix} 0.2895 \\ 0.0562 \end{pmatrix} \text{[m]}, \qquad \boldsymbol{\Delta\theta} = \begin{pmatrix} \pm\, 2\pi/8192 \\ \pm\, 2\pi/4096 \end{pmatrix} \text{[rad]}.$$

The small joint position uncertainty due to the resolution of the encoders can be applied in two different ways to (2), depending on the choice of signs in the components of $\boldsymbol{\Delta\theta}$ —*never* use degrees here! These signs are either the same (say, positive, leading to $\boldsymbol{\Delta\theta}_1$) or opposite (say, the first positive and second negative, leading to $\boldsymbol{\Delta\theta}_2$). The two other combinations lead to values of $\boldsymbol{\Delta p} = \boldsymbol{p} - \hat{\boldsymbol{p}}$ which are the opposite of what already found, and so with same norms. We have thus

$$\boldsymbol{\Delta p}_1 = \boldsymbol{p}_1 - \hat{\boldsymbol{p}} = \boldsymbol{J}(\hat{\boldsymbol{\theta}})\boldsymbol{\Delta\theta}_1 = \begin{pmatrix} -0.0562 & 0.0493 \\ 0.2895 & 0.1840 \end{pmatrix} \begin{pmatrix} 0.0008 \\ 0.0015 \end{pmatrix} = \begin{pmatrix} 0.0325 \\ 0.5043 \end{pmatrix} \text{[mm]}$$

and

$$\boldsymbol{\Delta p}_2 = \boldsymbol{p}_2 - \hat{\boldsymbol{p}} = \boldsymbol{J}(\hat{\boldsymbol{\theta}})\boldsymbol{\Delta\theta}_2 = \begin{pmatrix} -0.0562 & 0.0493 \\ 0.2895 & 0.1840 \end{pmatrix} \begin{pmatrix} 0.0008 \\ -0.0015 \end{pmatrix} = \begin{pmatrix} -0.1187 \\ -0.0602 \end{pmatrix} \text{[mm]}.$$

Therefore,

$$\max \|\boldsymbol{\Delta p}\| = \max\left\{\|\boldsymbol{\Delta p}_1\|, \|\boldsymbol{\Delta p}_2\|\right\} = \max\left\{0.5054, 0.1331\right\} = 0.5054\,\text{[mm]},$$

i.e., the maximum Cartesian uncertainty is about half a millimeter (which makes sense). Note that the given data are the actual ones for the Quanser underactuated robot (*Pendubot*) available in the Robotics Lab at DIAG.

Exercise 2

Drawing the workspace WS of the planar RP robot based on the joint ranges in (1), we obtain part of a circular sector with inner radius 0.5 m and outer radius 1 m. With reference to Fig. 1, the longest segment contained in this workspace is $\overline{AB}$ (tangent to the inner boundary of WS at point E), which connects two vertices of the admissible area. It is

$$\boldsymbol{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} -0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad L = \|\boldsymbol{B} - \boldsymbol{A}\| = \sqrt{3} \approx 1.7321 \text{ [m]}.$$

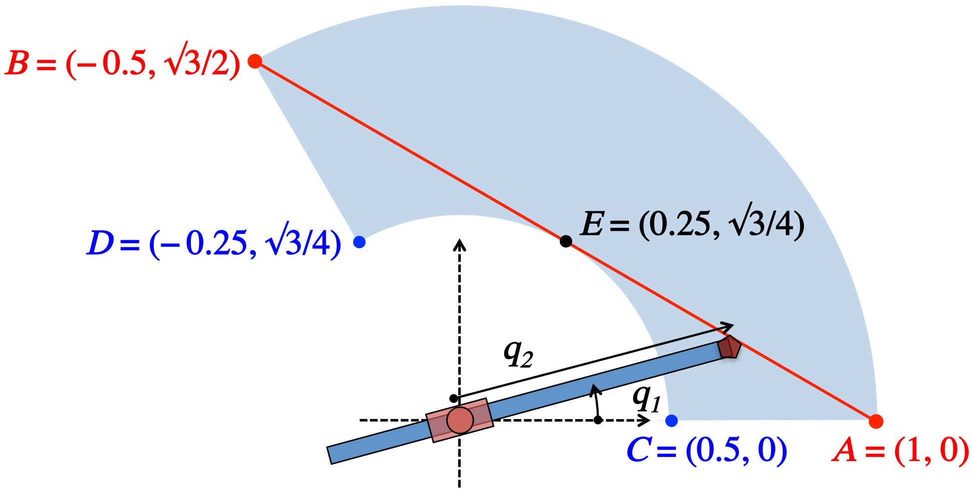


Figure 1: Workspace of the planar RP robot with the segment $\overline{AB}$ of maximum length as path

The desired Cartesian path and velocity can be parametrized as follows:

$$\boldsymbol{p}_d(s) = \boldsymbol{A} + \frac{\boldsymbol{B} - \boldsymbol{A}}{L}\, s = \begin{pmatrix} 1 - \dfrac{1.5\, s}{\sqrt{3}} \\ 0.5\, s \end{pmatrix}, \quad s \in [0, L]; \quad \dot{\boldsymbol{p}}_d(s) = \frac{\boldsymbol{B} - \boldsymbol{A}}{L}\, \dot{s} = \begin{pmatrix} -\dfrac{1.5}{\sqrt{3}} \\ 0.5 \end{pmatrix} \dot{s}. \quad (3)$$

The direct and inverse kinematics of the PR robot are[1]

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \mathrm{ATAN2}\{p_y, p_x\} \\ \sqrt{p_x^2 + p_y^2} \end{pmatrix} = \boldsymbol{f}^{-1}(\boldsymbol{p}), \quad (4)$$

where we have chosen only the positive solution for q_2. Corresponding to points $\boldsymbol{A}$, $\boldsymbol{E}$ (midpoint of the trajectory), and $\boldsymbol{B}$, we have thus

$$\boldsymbol{q}_A = \boldsymbol{f}^{-1}(\boldsymbol{A}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{q}_E = \boldsymbol{f}^{-1}(\boldsymbol{E}) = \begin{pmatrix} 60^\circ \\ 0.5 \end{pmatrix}, \quad \boldsymbol{q}_B = \boldsymbol{f}^{-1}(\boldsymbol{B}) = \begin{pmatrix} 120^\circ \\ 1 \end{pmatrix}.$$

Finally, the differential kinematics of the PR robot is

$$\dot{\boldsymbol{p}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin_1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} -p_y & \cos q_1 \\ p_x & \sin q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \quad (5)$$

From the robot type and the shape of the path, the joint trajectories will display some symmetry in time while moving from $\boldsymbol{A}$ to $\boldsymbol{E}$ and from $\boldsymbol{E}$ to $\boldsymbol{B}$. Moreover, considering the numerical values of the velocity limits, it is clear that the revolute joint will need more time to complete its motion. Joint 1 will thus proceed at maximum positive speed, switching from rest to $V_1 = 40^\circ/\mathrm{s}$ at $t = 0$ and vice versa at the (yet unknown) final time $t = T$. Simultaneously, the prismatic joint will reduce its extension during the first half of the trajectory and reverse this motion during the

[1] We have not used here the standard DH coordinate θ_1 as q_1. In that case, everything would remain the same modulo a clockwise rotation of WS and of the planned path by $\pi/2$ around the Cartesian origin.

second half, so as to *keep the robot end effector on the linear Cartesian path* between $\boldsymbol{A}$ and $\boldsymbol{B}$. In particular, the velocity of joint 2 in the segment from $\boldsymbol{A}$ to $\boldsymbol{E}$ (reached at $t = T/2$) will be negative (but neither at its maximum value nor constant, otherwise the end effector would not travel along the straight Cartesian path). The velocity profile will mirror itself for $t = (T/2, T]$ according to the rule $\dot{q}_2(t) = -\dot{q}_2((T/2) - t)$.

For this intuitively described trajectory to be also the desired *time optimal* solution, we just need to compute the resulting velocity of joint 2 and check its feasibility against the limit $V_2 = 1.5$ m/s during the entire motion interval $[0, T]$. The time profile of the first joint is

$$q_{d1}(t) = q_{d1}(0) + V_1 t, \quad t \in [0, T], \quad \text{with } q_{d1}(0) = 0 \qquad \Rightarrow \qquad T = \frac{\Delta q_1}{V_1} = \frac{120°}{40°/\text{s}} = 3 \text{ s}. \tag{6}$$

A closed-form solution for the time profile $q_{d2}(t)$ of joint 2 and for the timing law $s(t)$ along the Cartesian path are obtained with the following method, which provides also $\dot{q}_{d2}(t)$ and $\dot{s}(t)$:

1. For each instant t (sampling uniformly the interval $[0, T]$, say every $T_c = 1$ ms), equate the desired path position $\boldsymbol{p}_d(s)$, expressed from the task side by (3) as a function of s, with the direct kinematics of the end effector, as given by (4) from the robot side:

$$\begin{pmatrix} 1 - \dfrac{1.5\, s}{\sqrt{3}} \\ 0.5\, s \end{pmatrix} = \begin{pmatrix} q_2 \cos q_{d1} \\ q_2 \sin q_{d1} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \cos q_{d1} & \dfrac{1.5}{\sqrt{3}} \\ \sin q_{d1} & -0.5 \end{pmatrix} \begin{pmatrix} q_2 \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7}$$

2. Solve the linear system (7) for $q_2 = q_{d2}(t)$ and $s = s(t)$, and substitute therein $q_1 = q_{d1} = V_1 t$:

$$\begin{pmatrix} q_{d2}(t) \\ s(t) \end{pmatrix} = \frac{1}{0.5 \cos V_1 t + \frac{1.5}{\sqrt{3}} \sin V_1 t} \begin{pmatrix} 0.5 \\ \sin V_1 t \end{pmatrix}. \tag{8}$$

3. Similarly, equate at the differential level the desired Cartesian velocity on the path, expressed from the task side by the second relation in (3), with the velocity of the end effector, as given by (5) from the robot side, substituting therein $\boldsymbol{p} = \boldsymbol{p}_d(s)$, with $s = s(t)$, and $\dot{q}_{d1} = V_1$:

$$\begin{pmatrix} -\dfrac{1.5}{\sqrt{3}} \\ 0.5 \end{pmatrix} \dot{s} = \begin{pmatrix} -p_{dy}(s) \\ p_{dx}(s) \end{pmatrix} \dot{q}_{d1} + \begin{pmatrix} \cos q_{d1} \\ \sin q_{d1} \end{pmatrix} \dot{q}_2 = \begin{pmatrix} -0.5\, s(t) \\ 1 - \dfrac{1.5\, s(t)}{\sqrt{3}} \end{pmatrix} V_1 + \begin{pmatrix} \cos V_1 t \\ \sin V_1 t \end{pmatrix} \dot{q}_2$$

or

$$\begin{pmatrix} \cos V_1 t & \dfrac{1.5}{\sqrt{3}} \\ \sin V_1 t & -0.5 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 0.5 V_1\, s(t) \\ -V_1 \left(1 - \dfrac{1.5\, s(t)}{\sqrt{3}}\right) \end{pmatrix}. \tag{9}$$

4. Solve the linear system (9) for $\dot{q}_2 = \dot{q}_{d2}(t)$ and $\dot{s} = \dot{s}(t)$:

$$\begin{pmatrix} \dot{q}_{d2}(t) \\ \dot{s}(t) \end{pmatrix} = \frac{V_1}{0.5 \cos V_1 t + \frac{1.5}{\sqrt{3}} \sin V_1 t} \begin{pmatrix} \left(0.25 + \left(\frac{1.5}{\sqrt{3}}\right)^2\right) s(t) - \frac{1.5}{\sqrt{3}} \\ \cos V_1 t + \left(0.5 \sin V_1 t - \frac{1.5}{\sqrt{3}} \cos V_1 t\right) s(t) \end{pmatrix}, \tag{10}$$

where the expression of $s(t)$ from (8) should be be used.

Note that the above steps 3 and 4 can be replaced (approximately) by a numerical derivative of the expressions (8), e.g., by finite differences at the sampling rate $1/T_c$. The final check is indeed

$$|\dot{q}_{d2}(t)| \leq V_2 = 1.5 \text{ [m/s]}, \qquad \forall t \in [0, T]. \tag{11}$$

The following simple Matlab code implements the above method:

```
V1=40*pi/180; T=3;
Tc=0.001; t=[0:Tc:T];
% solution for desired q2 and s
dets=0.5*cos(V1*t)+(1.5/sqrt(3))*sin(V1*t);
qd2=0.5./dets;
sd=sin(V1*t)./dets;
% solution for desired velocity of q2 and s
dotqd2=V1*(0.25*sd+(1.5/sqrt(3))^2*sd-(1.5/sqrt(3)))./dets;
dotsd=V1*((0.5*sin(V1*t)-(1.5/sqrt(3))*cos(V1*t)).*sd+cos(V1*t))./dets;
```

With the obtained values, we can verify that the constraint (11) is indeed always satisfied. Therefore, the optimal solution is given by the joint trajectory $\boldsymbol{q}_d(t)$ already found in (6) and (8). Figure 2 shows the actual Cartesian path that has been planned, while Figs. 3–4 report the time evolution of all the relevant variables. Note in particular that the speed $\dot{s}(t)$ on the linear path is *not* constant.

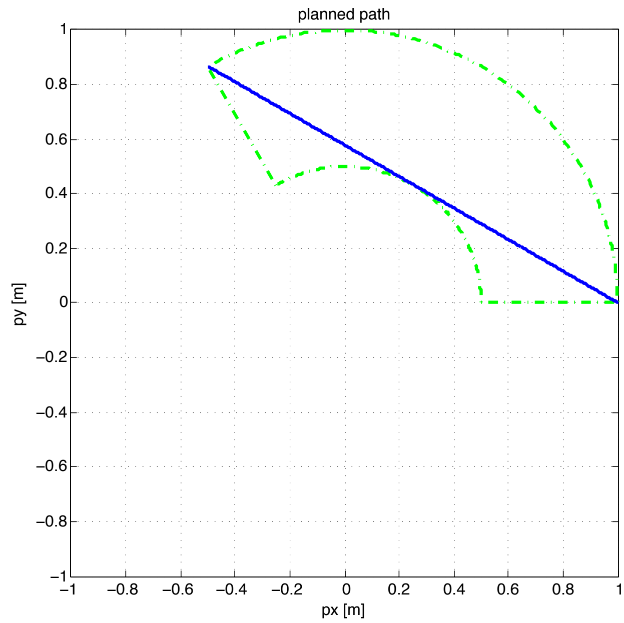


Figure 2: Actual Cartesian path obtained with the planned joint trajectories

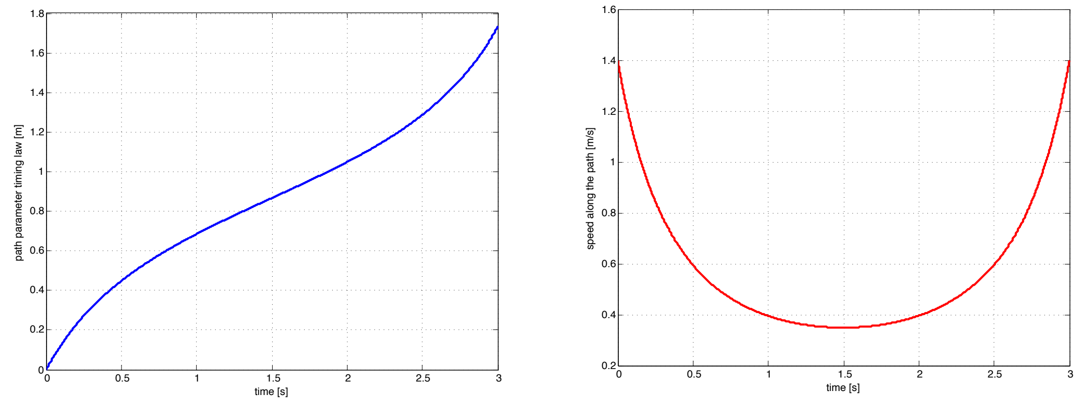


Figure 3: Timing law $s(t)$ for the path parameter in (3) and its speed $\dot{s}(t)$, as computed from (8) and (10). The minimum speed is at point $\boldsymbol{E}$, where the motion of joint 2 is orthogonal to the path and only joint 1 contributes with $\|\boldsymbol{E}\| \cdot V_1 \approx 0.35$ m/s

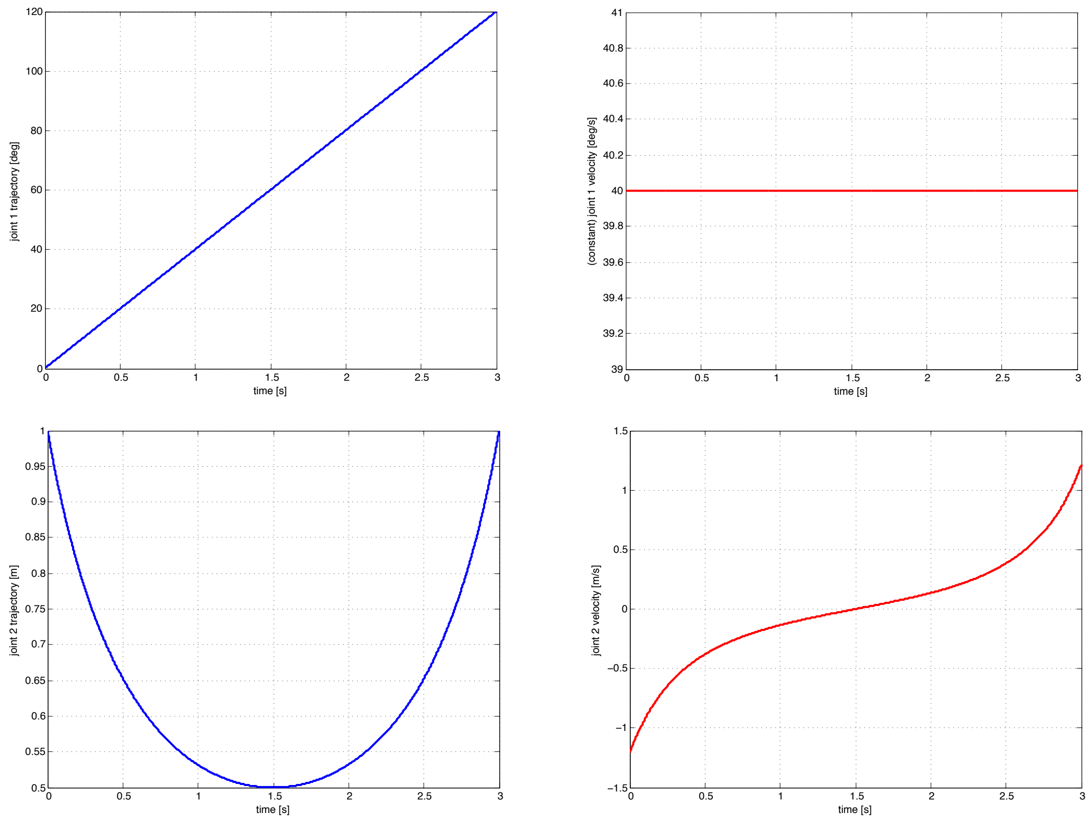


Figure 4: Planned trajectories in position (blue) and velocity (red) for the revolute (top) and prismatic (bottom) joints, as computed from (6), (8), and (10). As anticipated, motion of joint 2 is symmetric vs. the path midpoint $\boldsymbol{E}$, and its velocity is maximum at the initial and final points

Exercise 3

The length of the desired path is $L = \|\boldsymbol{B} - \boldsymbol{A}\| = \sqrt{5} \approx 2.2361$ m. Since

$$L = \sqrt{5} > 0.05 = \frac{v_{\max}^2}{a_{\max}},$$

the existence of a coast phase at constant speed is verified, and the nominal motion time to trace the path with a trapezoidal velocity profile can be computed as

$$T = \frac{L\,a_{\max} + v_{\max}^2}{v_{\max}\,a_{\max}} = 4.5721\,\text{s}.$$

The desired trajectory is written in parametrized form as

$$\boldsymbol{p}_d(t) = \boldsymbol{A} + \frac{\boldsymbol{B} - \boldsymbol{A}}{L}\,s(t), \quad \text{for } t \in [0, T] \rightarrow s(t) \in [0, L], \quad \dot{\boldsymbol{p}}_d(t) = \frac{\boldsymbol{B} - \boldsymbol{A}}{L}\,\dot{s}(t) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \dot{s}(t),$$

with

$$s(t) = \begin{cases} 2.5\,t^2, & t \in [0, 0.1] \\ 0.5\,(t - 0.05), & t \in [0.1, T - 0.1] \\ -2.5\,(t - T)^2 + 0.5\,(T - 0.1), & t \in [T - 0.1, T] \end{cases}$$

and

$$\dot{s}(t) = \begin{cases} 5\,t, & t \in [0, 0.1] \\ 0.5, & t \in [0.1, T - 0.1] \\ -5\,(t - T), & t \in [T - 0.1, T]. \end{cases}$$

The nominal path is internal to the primary workspace and never crosses the axis of joint 1 (the minimum distance to $\boldsymbol{z}_0$ is about 0.89 m) nor reaches the external boundary (where the links 2 and 3 would be stretched). Thus, if the end effector were always on this desired path, the robot would not encounter any kinematic singularity.

Using the values in Tab. 1, we have for the direct kinematics of the robot end-effector position

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta}) = \begin{pmatrix} \cos\theta_1\,(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ \sin\theta_1\,(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ a_2\sin\theta_2 + a_3\sin(\theta_2 + \theta_3) \end{pmatrix}, \quad \text{with } a_2 = 1.5,\ a_3 = 1 \text{ [m]}.$$

The associated Jacobian is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -s_1\,(a_2c_2 + a_3c_{23}) & -c_1\,(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1\,(a_2c_2 + a_3c_{23}) & -s_1\,(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}.$$

In the initial configuration $\boldsymbol{\theta}(0) = \begin{pmatrix} -\pi/2 & 0 & \pi/6 \end{pmatrix}^T$, we have

$$\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{\theta}(0)) = \begin{pmatrix} 0 \\ -2.3660 \\ 0.5 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \boldsymbol{A} - \boldsymbol{p}(0) = \begin{pmatrix} 0 \\ 0.3660 \\ 0 \end{pmatrix},$$

so that only the $e_y(0)$ component is different from zero, while $e_x(0) = e_z(0) = 0$.

The kinematic control law that allows to obtain the desired characteristics has to be designed on the *Cartesian error*, and with a Cartesian velocity feedforward, as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}(\boldsymbol{\theta})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{\theta})\right)\right), \qquad \text{with } \boldsymbol{K} = \operatorname{diag}\{k_x, k_y.k_z\} > 0, \tag{12}$$

where the expressions of the required terms $\boldsymbol{f}(\boldsymbol{\theta})$, $\boldsymbol{J}(\boldsymbol{\theta})$, $\boldsymbol{p}_d(t)$, and $\dot{\boldsymbol{p}}_d(t)$ have already been given. In fact, the law (12) guarantees that the Cartesian tracking error $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t)$ behaves as

$$\dot{\boldsymbol{e}} = -\boldsymbol{K}\boldsymbol{e} \qquad \Rightarrow \qquad e_i(t) = e_i(0)\exp(-k_i t) \to 0 \qquad \text{for } t \geq 0, \quad i = x, y, z.$$

Any choice of strictly positive values for k_x, k_y, and k_z will work. In this case, being the initial errors on two Cartesian components already zero, it will be $e_x(t) = e_z(t) = 0$ for all times —this is a consequence of the Cartesian decoupling achieved by the control law (12). Note also that $\|\boldsymbol{e}(t)\| = |e_y(t)|$ holds for all $t \geq 0$. For the gain k_y, the requested minimum value is found by imposing at $t = T/4 = 1.1430$ s

$$e_y(T/4) = e_y(0)\exp(-k_y T/4) = 0.05\, e_y(0) \qquad \Rightarrow \qquad k_y = -\frac{4}{T}\ln 0.05 = 2.6209.$$

Figure 5 shows the evolution of the norm of the Cartesian tracking error with this choice, and confirms the satisfaction of the error reduction as soon as $t \geq T/4$.

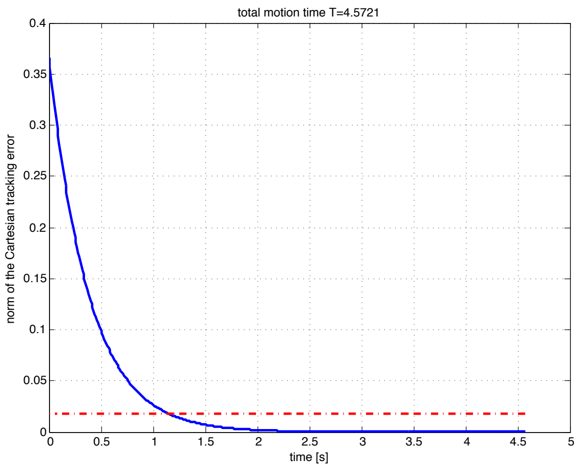


Figure 5: Evolution of the norm of the Cartesian tracking error with $k_y = 2.6209$

Moreover, the y-component of the Cartesian trajectory followed by the robot end effector will not overshoot its initial value and will always be *larger* than that of the nominal trajectory, practically coinciding with the desired one after five times the time constant τ of the exponential trajectory (i.e., for $t \geq 5\tau = 5 \cdot (1/k_y) \approx 1.9$ s). As a consequence, also the *actual* path executed by the robot will never encounter kinematic singularities. Finally, all joints will be simultaneously in motion during the execution of the controlled task.

* * * * *

Robotics I

February 6, 2015

Exercise 1

Consider the 3R robot in Fig. 1 *(this is the same robotic structure of an exercise assigned in September 2007)*. The base frame and an *additional* end-effector frame are already specified.

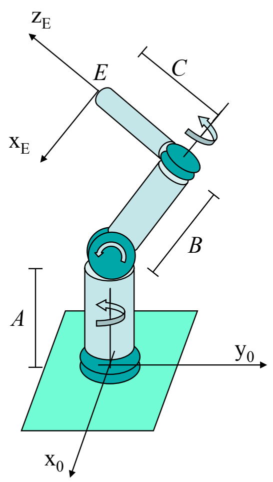


Figure 1: A robot with three revolute joints.

- Given a desired *orientation* $\boldsymbol{R}_d$ of the end-effector frame, solve the inverse kinematics problem in symbolic form. Consider also possible singular cases.
- Apply your result and determine *all* numerical solutions $\boldsymbol{q}$ for the following two sets of data:

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{R}_{d,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}.$$

- Provide for this robot the relation between $\dot{\boldsymbol{q}}$ and the angular velocity $\boldsymbol{\omega}_E$ of the end-effector frame (expressed in the base frame).
- Determine a joint velocity $\dot{\boldsymbol{q}}$ in the configuration $\boldsymbol{q} = \boldsymbol{0}$ that produces the desired angular velocity $\boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}^T$ [rad/s]. Has this problem a solution? If so, is it unique?
- *"This robot is of little use for positioning the end-effector in 3D space."* Do you agree with this statement? Why?

Extra • Based on the analysis you have performed, can this robot realize any *pointing* task with its end-effector axis $\boldsymbol{z}_E$? If so, is there a unique solution in the generic case? Are there singular situations? *(If you reply correctly to the extra questions, you get a bonus)*

Exercise 2

Given the two points

$$\boldsymbol{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \text{[m]} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 0.732 \\ 1 \end{pmatrix} \text{[m]}$$

on the plane, connect them with the arc (of minimum length) of a circle having radius $R = 2$ [m] and parametrize this path by its *arc length* s. Design a timing law $s = s(t)$ with *trapezoidal speed profile* so as to obtain a rest-to-rest circular trajectory $\boldsymbol{p}(t)$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ that performs the transfer in minimum time T under the maximum velocity and acceleration constraints

$$\|\dot{\boldsymbol{p}}(t)\| \leq V_{max}, \qquad \|\ddot{\boldsymbol{p}}(t)\| \leq A_{max}, \qquad t \in [0, T],$$

and the bound on the *normal* acceleration $\ddot{\boldsymbol{p}}_n(t)$ to the path

$$\|\ddot{\boldsymbol{p}}_n(t)\| \leq A_{n,max}, \qquad t \in [0, T].$$

Solve this Cartesian trajectory planning problem with the data

$$V_{max} = 3\,\text{[m/s]}, \qquad A_{max} = 4\,\text{[m/s}^2\text{]}, \qquad A_{n,max} = 2\,\text{[m/s}^2\text{]},$$

providing also the numerical values of the associated minimum time T.

[180 minutes; open books]

Solution

February 6, 2015

Exercise 1

For the assignment of DH frames and the associated table of parameters, see Fig. 2 and Tab. 1. We need also an additional transformation matrix ${}^3\boldsymbol{T}_E$ relating the third DH frame RF_3 to RF_E:

$$
{}^3\boldsymbol{T}_E = \begin{pmatrix} {}^3\boldsymbol{R}_E & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}
$$

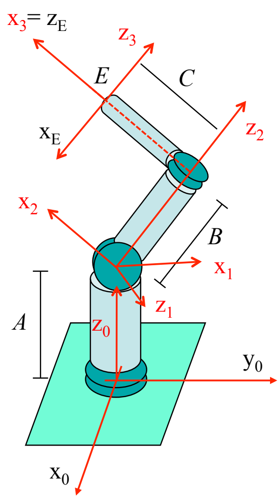


Figure 2: Denavit-Hartenberg frames for the robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	A	q_1
2	$\pi/2$	0	0	q_2
3	0	C	B	q_3

Table 1: Denavit-Hartenberg parameters associated to the frames chosen as in Fig. 2.

Using Tab. 1 and (1), the direct kinematics for the orientation is computed as

$$
\begin{aligned}
{}^0\boldsymbol{R}_E &= {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3)\,{}^3\boldsymbol{R}_E \qquad (2)\\
&= \begin{pmatrix} -\cos q_1 \sin q_2 & \sin q_1 \cos q_3 - \cos q_1 \cos q_2 \sin q_3 & \sin q_1 \sin q_3 + \cos q_1 \cos q_2 \cos q_3 \\ -\sin q_1 \sin q_2 & -\cos q_1 \cos q_3 - \sin q_1 \cos q_2 \sin q_3 & \sin q_1 \cos q_2 \cos q_3 - \cos q_1 \sin q_3 \\ \cos q_2 & -\sin q_2 \sin q_3 & \sin q_2 \cos q_3 \end{pmatrix},
\end{aligned}
$$

which is independent of A, B, and C.

The Jacobian matrix in ${}^0\boldsymbol{\omega}_E \left(= {}^0\boldsymbol{\omega}_3\right) = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \left({}^0\boldsymbol{z}_0 \; {}^0\boldsymbol{z}_1 \; {}^0\boldsymbol{z}_2 \right) \dot{\boldsymbol{q}}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & \sin q_1 & \cos q_1 \sin q_2 \\ 0 & -\cos q_1 & \sin q_1 \sin q_2 \\ 1 & 0 & -\cos q_2 \end{pmatrix},$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = \sin q_2$.

For a desired orientation of the end-effector frame, represented by a given rotation matrix $\boldsymbol{R}_d = \{R_{ij}\}$, one can determine the inverse kinematics solution using the elements in (2). First, compute

$$q_2^I = \text{ATAN2}\left\{ \sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}.$$

If $R_{32}^2 + R_{33}^2 \neq 0$, which means $\sin q_2 \neq 0$, we are in the *regular* case. A second distinct solution for q_2 is computed as

$$q_2^{II} = \text{ATAN2}\left\{ -\sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}.$$

Moreover,

$$q_1^I = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^I}, \frac{-R_{11}}{\sin q_2^I} \right\}, \qquad q_1^{II} = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^{II}}, \frac{-R_{11}}{\sin q_2^{II}} \right\},$$

and

$$q_3^I = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^I}, \frac{R_{33}}{\sin q_2^I} \right\}, \qquad q_3^{II} = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^{II}}, \frac{R_{33}}{\sin q_2^{II}} \right\}.$$

When $R_{32} = R_{33} = 0$, we are in a *singular* situation. This occurs if and only if $\sin q_2 = 0$, thus when either $q_2 = 0$ or $q_2 = \pi$. If $q_2 = 0$, we can solve only for the difference $q_1 - q_3$:

$${}^0\boldsymbol{R}_E\big|_{q_2=0} = \begin{pmatrix} 0 & \sin(q_1 - q_3) & \cos(q_1 - q_3) \\ 0 & -\cos(q_1 - q_3) & \sin(q_1 - q_3) \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1-3} := q_1 - q_3 = \text{ATAN2}\left\{ R_{23}, R_{13} \right\},$$

leading to an infinity of solutions of the form

$$\boldsymbol{q} = \begin{pmatrix} \alpha & 0 & \alpha - q_{1-3} \end{pmatrix}^T, \qquad \forall \alpha \in \mathbb{R}.$$

Similarly, when $q_2 = \pi$ we can solve only for the sum $q_1 + q_3$:

$${}^0\boldsymbol{R}_E\big|_{q_2=\pi} = \begin{pmatrix} 0 & \sin(q_1 + q_3) & -\cos(q_1 + q_3) \\ 0 & -\cos(q_1 + q_3) & -\sin(q_1 + q_3) \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1+3} := q_1 + q_3 = \text{ATAN2}\left\{ R_{12}, -R_{13} \right\},$$

leading to an infinity of solutions of the form

$$\boldsymbol{q} = \begin{pmatrix} \beta & \pi & q_{1+3} - \beta \end{pmatrix}^T, \qquad \forall \beta \in \mathbb{R}.$$

Applying these results to the given data, we have that

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{a singular case with } q_2 = \pi,$$

leading to the solutions $\boldsymbol{q} = \begin{pmatrix} \beta & \pi & \pi/2 - \beta \end{pmatrix}^T$, for any β. On the other hand, $\boldsymbol{R}_{d,2} = \boldsymbol{I}$ is a regular case leading to the pair of solutions

$$\boldsymbol{q}^I = \begin{pmatrix} \pi & \pi/2 & 0 \end{pmatrix}^T, \qquad \boldsymbol{q}^{II} = \begin{pmatrix} 0 & -\pi/2 & \pi \end{pmatrix}^T.$$

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ is singular in the zero configuration $\boldsymbol{q} = \boldsymbol{0}$. However,

$$\boldsymbol{J}(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \in \mathcal{R}\left(\boldsymbol{J}(\boldsymbol{0})\right) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right\}.$$

Therefore, there exists an infinite number of joint velocity solutions $\dot{\boldsymbol{q}}$ providing $\boldsymbol{\omega}_{E,d}$, all having $\dot{q}_2 = 0$ and with $\dot{q}_1 - \dot{q}_3 = 3$ [rad/s]. In particular, $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{0})\, \boldsymbol{\omega}_{E,d} = \begin{pmatrix} 1.5 & 0 & -1.5 \end{pmatrix}^T$ [rad/s] provides the minimum norm solution.

Due to its kinematics this robot has a limited use for positioning tasks, since the primary workspace is very restricted. In fact, it is a thin spherical mantle/surface, placed on top of the surface of the sphere described by the tip position of the second link (a 2R polar sub-structure).

The solution to the last (extra) question is left as an exercise.

Exercise 2

The specified path from $\boldsymbol{A}$ to $\boldsymbol{B}$ can be constructed easily in a geometric way, by defining a circumference of given radius R passing through two points, as illustrated in Fig. 3. The shortest path from $\boldsymbol{A}$ to $\boldsymbol{B}$ on the circle of radius R centered in $\boldsymbol{C}_1$ is shown as a bolded arc (the arrow indicates its clockwise rotation).

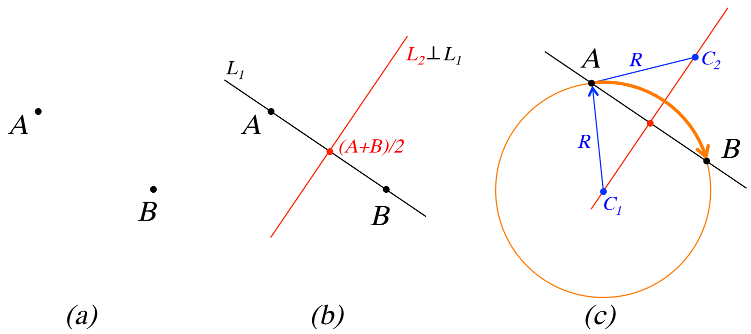


Figure 3: Geometric steps for constructing a circle of radius R through two points $\boldsymbol{A}$ and $\boldsymbol{B}$ (there are two solutions in the regular case).

In general, an *infinitesimal arc length* on a circle of radius R can be written as $ds = R\, d\theta$, where $d\theta$ is the angle spanning the arc from the circle center $\boldsymbol{C}$. Using simple trigonometry, the path parametrization by the arc length is given by

$$\boldsymbol{p}(s) = \boldsymbol{C} + R \begin{pmatrix} \cos\left(\pm\dfrac{s}{R} + \phi\right) \\ \sin\left(\pm\dfrac{s}{R} + \phi\right) \end{pmatrix}, \qquad s \in [0, L], \tag{3}$$

where the sign in $\pm$ is chosen positive if the path is traced counterclockwise, negative otherwise. L is the total arc length (from point $\boldsymbol{A}$ to point $\boldsymbol{B}$), while the phase ϕ is chosen so as to be in $\boldsymbol{A}$ for $s = 0$.

A nice feature of the problem is that one does *not* have to determine the center $\boldsymbol{C}$ of the circle, nor the circle itself, in order to satisfy all the design specifications on the trajectory! Even the path length L can be directly computed from the known formula (see, e.g., wikipedia) relating the distance d of two points $\boldsymbol{A}$ and $\boldsymbol{B}$ with the length L of the (shortest) arc of a circle of radius R passing through the two points:

$$L = R\,\theta_{AB}, \quad d = \|\boldsymbol{B} - \boldsymbol{A}\| = 2R\sin\left(\frac{\theta_{AB}}{2}\right) \quad \Rightarrow \quad L = 2R\arcsin\left(\frac{d}{2R}\right).$$

With the length L and the generic expression (3), we can solve completely the assigned problem.

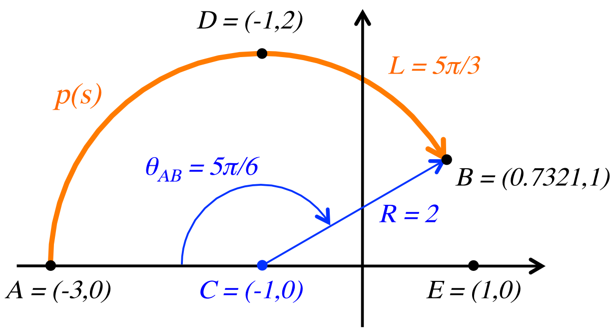


Figure 4: The actual geometric path $\boldsymbol{p}(s)$ constructed with the problem data, using a circle of radius $R = 2$ m with center at $\boldsymbol{C} = (-1, 0)$.

Nonetheless, due to the specific data values that were given, a center $\boldsymbol{C}$ can be found rather immediately by visual inspection —see Fig. 4. By imposing $\boldsymbol{p}(0) = \boldsymbol{A}$, the parametrization of the clockwise circular path becomes

$$\boldsymbol{p}(s) = \boldsymbol{C} - R\begin{pmatrix} \cos\left(-\frac{s}{R}\right) \\ \sin\left(-\frac{s}{R}\right) \end{pmatrix}, \qquad s \in [0, L], \quad L = R\,\frac{5\pi}{6} = \frac{5\pi}{3} = 5.236 \text{ [m]}, \tag{4}$$

where the phase $\phi = \pi$ chosen in the argument of the trigonometric functions in (3) leads to the minus sign in front of the first R. The length L is obtained from the angle θ_{AB} spanning the whole path (equal to 150°, if expressed in degrees) multiplied by the radius $R = 2$.

For a generic $s = s(t)$, the first and second time derivatives of $\boldsymbol{p}(s)$ in (4) are given by

$$\dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{ds}\frac{ds}{dt} = \begin{pmatrix} \sin\left(\frac{s}{R}\right) \\ \cos\left(\frac{s}{R}\right) \end{pmatrix}\dot{s} \tag{5}$$

and

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_t + \ddot{\boldsymbol{p}}_n = \frac{d\boldsymbol{p}}{ds}\,\ddot{s} + \frac{d^2\boldsymbol{p}}{ds^2}\,\dot{s}^2 = \begin{pmatrix} \sin\left(\frac{s}{R}\right) \\ \cos\left(\frac{s}{R}\right) \end{pmatrix} \ddot{s} + \frac{1}{R}\begin{pmatrix} \cos\left(\frac{s}{R}\right) \\ -\sin\left(\frac{s}{R}\right) \end{pmatrix} \dot{s}^2$$
$$= \begin{pmatrix} \cos\left(\frac{s}{R}\right) & \sin\left(\frac{s}{R}\right) \\ -\sin\left(\frac{s}{R}\right) & \cos\left(\frac{s}{R}\right) \end{pmatrix} \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix} = \mathrm{Rot}^T\!\left(\frac{s}{R}\right) \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix}, \tag{6}$$

with a decomposition in tangential and normal acceleration to the path, respectively $\ddot{\boldsymbol{p}}_t$ and $\ddot{\boldsymbol{p}}_n$. The 2×2 matrix $\mathrm{Rot}(\theta)$ is a planar rotation by an angle θ, acting on 2-dimensional vectors. Thanks to the used parametrization by the arc length, we have the following properties for the norms

$$\left\|\frac{d\boldsymbol{p}}{ds}\right\| = 1 \;\Rightarrow\; \|\dot{\boldsymbol{p}}\| = |\dot{s}|,\; \|\ddot{\boldsymbol{p}}_t\| = |\ddot{s}|, \qquad \left\|\frac{d^2\boldsymbol{p}}{ds^2}\right\| = \frac{1}{R} \;\Rightarrow\; \|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R}, \qquad \|\ddot{\boldsymbol{p}}\| = \sqrt{\left(\frac{\dot{s}^2}{R}\right)^2 + \ddot{s}^{\,2}}.$$

We consider now a generic a trapezoidal profile for $\dot{s}(t)$ of duration T, with symmetric initial and final acceleration/deceleration phases of absolute value $\bar{A}$ and equal duration T_s, and a central constant cruising speed $\bar{V} > 0$ to be kept for $T - 2T_s$ seconds. The four quantities $\bar{V}$, $\bar{A}$, T_s, and T have to be determined so as to cover the total path length L, while minimizing T and satisfying the constraints specified by V_{max}, A_{max}, and $A_{n,max}$.

The important thing to note is that the curvature $1/R$ of the path and the bound on the normal acceleration $\ddot{\boldsymbol{p}}_n$

$$\|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R} \leq A_{n,max}$$

may impose a more severe limit on $\dot{s}$ than the bound V_{max} on the norm of $\dot{\boldsymbol{p}}$. In fact, we have that

$$|\dot{s}| \leq \min\left\{V_{max}, \sqrt{R\,A_{n,max}}\right\} = \min\{3, \sqrt{4}\} = 2 =: \bar{V}'. \tag{7}$$

To evaluate the constraint on the total acceleration $\ddot{\boldsymbol{p}}$, we distinguish two situations for the tangential acceleration: constant $\ddot{s} = \pm\bar{A} \neq 0$ (in the initial and final phases) and $\ddot{s} = 0$ (in the cruise phase at constant speed). During the cruise phase, it is

$$\|\ddot{\boldsymbol{p}}\| = \frac{\dot{s}^2}{R} \leq A_{max} \quad\Rightarrow\quad |\dot{s}| \leq \sqrt{R\,A_{max}} = \sqrt{8} =: \bar{V}''. \tag{8}$$

As a result, combining (7) and (8), we have for the maximum constant speed $\dot{s}$ during cruising

$$\dot{s}(t) = \bar{V} = \min\left\{\bar{V}', \bar{V}''\right\} = 2, \qquad t \in [T_s, T - T_s].$$

In the constant acceleration phase (a specular argument applies to the constant deceleration phase), the speed increases linearly from 0 at $t = 0$ (start at rest) to $\bar{V}$ at $t = T_s$. The largest value for the norm of the total acceleration is approached when $t = T_s$. Thus, we impose satisfaction of the constraint in the worst case:

$$\|\ddot{\boldsymbol{p}}(T_s)\| = \sqrt{\left(\frac{\bar{V}^2}{R}\right)^2 + \bar{A}^{\,2}} \leq A_{max} \quad\Rightarrow\quad \bar{A} \leq \sqrt{A_{max}^2 - \left(\frac{\bar{V}^2}{R}\right)^2} = \sqrt{12}.$$

Since a minimum transfer time is requested, we choose the maximum feasible value of the acceleration norm (i.e., $\|\ddot{\boldsymbol{p}}(T_s)\| = A_{max}$), leading to $\bar{A} = \sqrt{12}$.

With the above values for $\bar{V}$ and $\bar{A}$, having already computed the length L of the path, we determine the remaining unknowns with the usual formulas:

$$T_s = \frac{\bar{V}}{\bar{A}} = \frac{2}{\sqrt{12}} = 0.577 \text{ [s]}, \qquad (T - T_s)\bar{V} = L \quad \Rightarrow \quad T = T_s + \frac{L}{\bar{V}} = 0.577 + \frac{5\pi}{6} = 3.195 \text{ [s]}.$$

We obtained $T > 2T_s$, confirming that the actual speed profile is trapezoidal.

* * * * *

Robotics I

April 1, 2015

Consider the planar 2R robot in Fig. 1. Determine a *minimum time* trajectory that brings the robot end-effector from point P_{in} to point P_{fin}, with zero initial and final velocity, keeping into account the following constraints on the velocity and acceleration of the two joints:

$$|\dot{q}_i| \leq V_i, \qquad |\ddot{q}_i| \leq A_i, \qquad i = 1, 2.$$

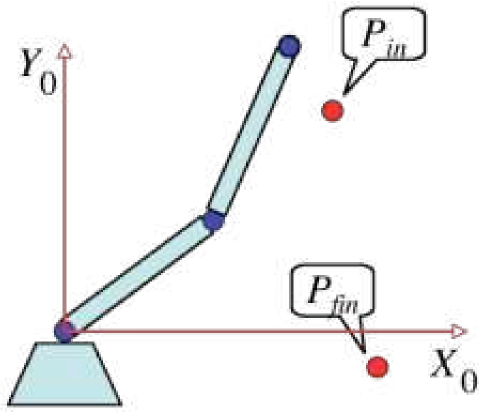


Figure 1: A planar 2R robot performing a point-to-point task

Describe in an algorithmic way the procedure to follow in order to determine the optimal solution. Provide the value of the minimum time and sketch the joint velocity profiles for a robot with the following numerical data

$$\begin{aligned} \ell_1 &= \ell_2 = 2.5 \text{ [m]}, \\ V_1 &= 1 \text{ [rad/s]}, \quad A_1 = 2 \text{ [rad/s}^2\text{]}, \\ V_2 &= 2 \text{ [rad/s]}, \quad A_2 = 1.5 \text{ [rad/s}^2\text{]}, \end{aligned}$$

that needs to execute the following motion task:

$$P_{in} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ [m]} \qquad \Rightarrow \qquad P_{fin} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ [m]}.$$

How would the solution change in case we additionally request a *coordinated* robot joint motion, using the same class of velocity and acceleration time profiles for both joints?

[120 minutes; open books]

Robotics I

June 5, 2015

Exercise 1

Consider a helix path whose parametrization is given by

$$\boldsymbol{p}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} r\,(\cos s - 1) + x_0 \\ r \sin s + y_0 \\ ks + z_0 \end{pmatrix}, \qquad s \in \mathbb{R}, \tag{1}$$

and let two Cartesian points $\boldsymbol{P}_A = \begin{pmatrix} p_{Ax} & p_{Ay} & p_{Az} \end{pmatrix}^T$ and $\boldsymbol{P}_B = \begin{pmatrix} p_{Bx} & p_{By} & p_{Bz} \end{pmatrix}^T$ be assigned. Define an interval $s \in [0, s_{\max}]$ and scalar values r, k, x_0, y_0, and z_0 in (1) such that $\boldsymbol{p}(0) = \boldsymbol{P}_A$ and $\boldsymbol{p}(s_{\max}) = \boldsymbol{P}_B$. Moreover, associate to this path a rest-to-rest timing law given by a cubic polynomial $s = s(t)$, $t \in [0, T]$, where T is the total motion time.

- Does the trajectory interpolation problem always have a solution? Is the solution unique?
- Determine a path (1) that solves the above problem for the numerical data $\boldsymbol{P}_A = \begin{pmatrix} 0 & 2 & -10 \end{pmatrix}^T$ and $\boldsymbol{P}_B = \begin{pmatrix} -2 & 0 & 10 \end{pmatrix}^T$. Compute the expression of the curvature $\kappa(s)$ of this path.
- For the chosen timing law, provide the expressions of $\dot{\boldsymbol{p}}(t)$ and $\ddot{\boldsymbol{p}}(t)$, and determine the minimum time T that realizes the interpolation under the constraint $\|\dot{\boldsymbol{p}}(t)\| \le V_{\max}$.

Exercise 2

Consider a 3R elbow-type robot having its base mounted on the plane $z = 0$. The shoulder joint is at a height $\ell_1 = 5$. The links 2 and 3 have equal lengths $\ell_2 = \ell_3 = 10$.

- Place the robot base at a point (x_b, y_b) on the plane $z = 0$ so that the end-effector is capable of executing the solution path of Exercise 1.
- Find a robot configuration $\boldsymbol{q} = \boldsymbol{q}^*$ at which the end-effector is placed in the (single) point of path (1) where the norm of the Cartesian velocity $\dot{\boldsymbol{p}}$ in the minimum time trajectory of Exercise 1 has its maximum value.
- Compute at $\boldsymbol{q}^*$ the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ of the robot that realizes the desired velocity $\dot{\boldsymbol{p}}$ of the above minimum time trajectory.

[150 minutes; open books]

Robotics I

July 10, 2015

Exercise 1

Consider the timing law $s = s(t)$ defined by means of the bang-bang type profile shown in Fig. 1 for the fourth time derivative $s^{(4)} = d^4s/dt^4$ (called *snap*) of the path parameter s. The boundary conditions at time $t = 0$ and $t = T$ for all lower order time derivatives are zero. Moreover, $s(0) = 0$.

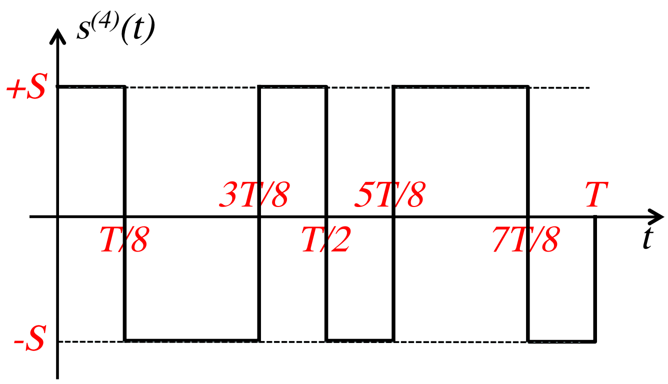


Figure 1: The time profile of the fourth time derivative $s^{(4)}(t)$

- Determine the expressions of the total displacement $\Delta = s(T)$, as well as of the maximum speed $\dot{s}_{\max}$ and maximum (absolute value of) acceleration $\ddot{s}_{\max}$ reached during motion, in terms of motion time T and maximum absolute value S of the snap.
- Sketch the time profiles of $s(t)$, $\dot{s}(t)$, $\ddot{s}(t)$, and $\dddot{s}(t)$, for $t \in [0, T]$.

Exercise 2

Consider a 2R planar robot having link lengths $\ell_1 = 0.8$ and $\ell_2 = 0.4$ [m]. The robot should execute a motion along the straight path from the initial point $A = \begin{pmatrix} 1.42 & 0.6 \end{pmatrix}^T$ [m] to the final point $B = \begin{pmatrix} 1.42 & -1.6 \end{pmatrix}^T$ [m], both expressed in the world reference frame $\mathcal{F}_w$.

- Define a position $\boldsymbol{P}_0 = \begin{pmatrix} x_0 & y_0 \end{pmatrix}^T$ in the plane, expressed in frame $\mathcal{F}_w$, where to place the robot base so that its end-effector is capable of moving along the entire given path.
- Are there any kinematic singularities encountered along this path?
- Find a robot configuration $\boldsymbol{q}^*$ such that the end-effector is at the midpoint of the given path.
- At $\boldsymbol{q} = \boldsymbol{q}^*$, compute an instantaneous joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ that realizes the desired Cartesian motion with a speed $V = 1.5$ [m/s].

[150 minutes; open books]

Robotics I

September 11, 2015

Exercise 1

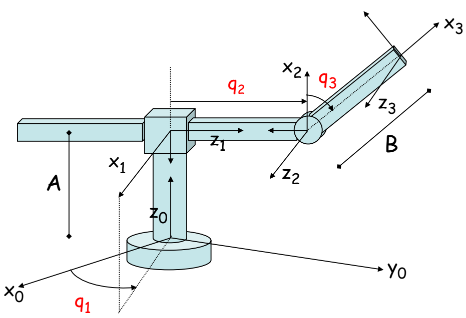
The kinematics of the spherical wrist of a 6R robot is described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
4	$\pi/2$	0	0	q_4
5	$\pi/2$	0	0	q_5
6	0	0	0	q_6

Table 1: Denavit-Hartenberg parameters for a spherical wrist

- Provide the differential mapping between the (wrist) joint velocity $\dot{\boldsymbol{q}}_W = \begin{pmatrix} \dot{q}_4 & \dot{q}_5 & \dot{q}_6 \end{pmatrix}^T$ and the angular velocity of the end-effector $\boldsymbol{\omega} = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T$ when the first three joints of the robot do not move. Vector $\boldsymbol{\omega}$ is expressed in the Denavit-Hartenberg frame 3 of the robot.
- In the wrist configuration $\boldsymbol{q}_W = \begin{pmatrix} q_4 & q_5 & q_6 \end{pmatrix}^T = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$ rad, determine a joint velocity vector $\dot{\boldsymbol{q}}_W$ that generates the desired angular velocity $\boldsymbol{\omega}_d = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}^T$ rad/s.

Exercise 2



i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	A	q_1
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	B	0	q_3

Figure 1: A spatial RPR robot with its Denavit-Hartenberg frames and associated table

For the spatial RPR robot in Fig. 1, the direct kinematics map for the position $\boldsymbol{p}$ of the end-effector (i.e., the origin of the last frame 3) is given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} -(q_2 - B\sin q_3)\sin q_1 \\ (q_2 - B\sin q_3)\cos q_1 \\ A + B\cos q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

Solve the inverse kinematics problem $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p})$ in closed analytical form, assuming unlimited joint ranges. How many inverse kinematics solutions exist in the generic case?

(continues with Exercise 3)

Exercise 3

Consider a 3R planar robot having equal and unitary link lengths, with kinematics described in terms of the standard Denavit-Hartenberg variables. A task is specified at the differential level by a desired $\dot{\boldsymbol{r}} = \begin{pmatrix} v_x & v_y & \omega_z \end{pmatrix}^T$, namely in terms of linear velocity of the robot end-effector on the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ and of the (scalar) angular velocity of the end-effector frame around $\boldsymbol{z}_0$. Find all singular configurations of the mapping from $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to $\dot{\boldsymbol{r}} \in \mathbb{R}^3$. At a singularity, characterize the directions spanning the range space and the null space of the associated Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$.

[210 minutes; open books]

Solution

September 11, 2015

Exercise 1

From Table 1 of DH parameters, we get the following rotation matrices associated to the three joints of the spherical wrist:

$$
{}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 \\ \sin q_4 & 0 & -\cos q_4 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^4\boldsymbol{R}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 \\ \sin q_5 & 0 & -\cos q_5 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} \cos q_6 & -\sin q_4 & 0 \\ \sin q_4 & \cos q_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

With these matrices, one can proceed in two alternative (and equivalent) ways.

The first way is to recognize that the requested mapping is given by part of the geometric Jacobian of the robot, namely the lower right 3×3 matrix in the orientation rows,

$$
\boldsymbol{\omega} = \boldsymbol{J}_O(\boldsymbol{q}_W)\,\dot{\boldsymbol{q}}_W = \begin{pmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{pmatrix} \begin{pmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix},
$$

where $\boldsymbol{z}_{i-1}$ is the unitary vector along joint i, and all vectors should be expressed here in the DH frame 3. We obtain:

$$
\boldsymbol{z}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_4 = {}^3\boldsymbol{R}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_4 \\ -\cos q_4 \\ 0 \end{pmatrix}, \quad \boldsymbol{z}_5 = {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 \sin q_5 \\ \sin q_4 \sin q_5 \\ -\cos q_5 \end{pmatrix}.
$$

The second way uses the time derivative of rotation matrices, and is a bit longer. The orientation of the end-effector (with respect to the DH frame 3) is given by

$$
\begin{aligned}
{}^3\boldsymbol{R}_6 &= {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5)\,{}^5\boldsymbol{R}_6(q_6) \\
&= \begin{pmatrix} \sin q_4\, \sin q_6 + \cos q_4\, \cos q_5\, \cos q_6 & \cos q_6\, \sin q_4 - \cos q_4\, \cos q_5\, \sin q_6 & \cos q_4\, \sin q_5 \\ \cos q_5\, \cos q_6\, \sin q_4 - \cos q_4\, \sin q_6 & -\cos q_4\, \cos q_6 - \cos q_5\, \sin q_4\, \sin q_6 & \sin q_4\, \sin q_5 \\ \cos q_6\, \sin q_5 & -\sin q_5\, \sin q_6 & -\cos q_5 \end{pmatrix}.
\end{aligned}
$$

Using the known differential relation $\dot{\boldsymbol{R}}\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{\omega})$ applied to $\boldsymbol{R} = {}^3\boldsymbol{R}_6$, one obtains the skew-symmetric matrix

$$
\begin{aligned}
\boldsymbol{S}(\boldsymbol{\omega}) &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ \omega_y & -\omega_z & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dot{q}_6\, \cos q_5 - \dot{q}_4 & \dot{q}_6\, \sin q_4\, \sin q_5 - \dot{q}_5\, \cos q_4 \\ \dot{q}_4 - \dot{q}_6\, \cos q_5 & 0 & -\dot{q}_5\, \sin q_4 - \dot{q}_6\, \cos q_4\, \sin q_5 \\ \dot{q}_5\, \cos q_4 - \dot{q}_6\, \sin q_4\, \sin q_5 & \dot{q}_5\, \sin q_4 + \dot{q}_6\, \cos q_4\, \sin q_5 & 0 \end{pmatrix},
\end{aligned}
$$

from which the angular velocity vector (still expressed in the DH frame 3, i.e., as ${}^3\boldsymbol{\omega}$) can be extracted:

$$
\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \sin q_4 & \cos q_4\, \sin q_5 \\ 0 & -\cos q_4 & \sin q_4\, \sin q_5 \\ 1 & 0 & -\cos q_5 \end{pmatrix} \begin{pmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix} = \boldsymbol{J}_O(q_4, q_5)\,\dot{\boldsymbol{q}}_W.
$$

The determinant of matrix $\boldsymbol{J}_O(q_4, q_5)$ is det $\boldsymbol{J}_O = \sin q_5$. When $\boldsymbol{q}_W = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$, the wrist is not in a singular configuration. Therefore, we can determine the unique solution to the requested task as

$$\boldsymbol{J}_O(0, \pi/2)\, \dot{\boldsymbol{q}}_W = \boldsymbol{\omega}_d \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \dot{\boldsymbol{q}}_W = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_W = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Exercise 2

As a first step, we characterize and analyze the singular configurations of the robot. In fact, in a non-redundant situation, singular configurations correspond to Cartesian positions where the number of inverse kinematic solutions differs from the generic one. From the direct kinematics (1), we obtain the analytic Jacobian

$$\boldsymbol{J}_L(q) = \begin{pmatrix} -(q_2 - B\sin q_3)\cos q_1 & -\sin q_1 & B\cos q_3 \sin q_1 \\ -(q_2 - B\sin q_3)\cos q_1 & \cos q_1 & -B\cos q_3 \cos q_1 \\ 0 & 0 & -B\sin q_3 \end{pmatrix},$$

whose determinant is

$$\det \boldsymbol{J}_L(q) = B\sin q_3 (q_2 - B\sin_3).$$

Therefore, the robot is in a singularity either when $\sin q_3 = 0$ ($q_3 = \{0, \pi\}$), or when $q_2 - B\sin q_3 = 0$, or when these two conditions are simultaneously satisfied, i.e., when both $\sin q_3 = 0$ and $q_2 = 0$ hold. In the first case ($\sin q_3 = 0$), the last link is vertical (upwards or downwards) and the robot end-effector is on one of the two (top or bottom) horizontal planes defining the boundaries of its (otherwise unlimited) workspace. In correspondence to these boundary points, we shall see that there is is a drop in the number of inverse kinematic solutions (from four to two). In the second case ($q_2 - B\sin q_3 = 0$), the end-effector is placed on the axis of the first joint. For these Cartesian points, it is apparent that any change of q_1 only, will not change the position of the robot end-effector. As a consequence, any value of q_1 can be part of a solution to the inverse kinematics (the number of solutions becomes infinite). In the combined case, the rank of the Jacobian matrix $\boldsymbol{J}_L$ drops down to 1 and a double singularity is obtained. For all other Cartesian positions of the robot end-effector within the primary workspace, we are in the generic case with a constant, finite number of inverse kinematic solutions (namely, four).

With the above in mind, consider the last equation in the direct kinematics (1). If $A - B < p_z < A + B$, we have two solutions for q_3:

$$q_3^{[+]} = \operatorname{acos}\left(\frac{p_z - A}{B}\right), \quad q_3^{[-]} = -q_3^{[+]}, \qquad \text{with } \left\{q_3^{[+]}, q_3^{[-]}\right\} \in (-\pi, +\pi). \tag{2}$$

Equivalently, we could have used the ATAN2 function as follows:

$$c_3 = \frac{p_z - A}{B}, \quad s_3 = \pm\sqrt{1 - c_3^2} \quad \Rightarrow \quad q_3^{[+/-]} = \text{ATAN2}\left\{\pm\, s_3, c_3\right\}.$$

For $p_z > A$ the two solutions $q_3^{[+]}$ and $q_3^{[-]}$ will both be in the interval $(-\pi/2, \pi/2)$, whereas for $p_z < A$ their absolute values will be in the interval $(\pi/2, \pi)$. When $p_z = A$, it is $q_3^{[+]} = +\pi/2$ and $q_3^{[-]} = -\pi/2$ (the third link is horizontal in both cases). For $p_z = A + B$, the two values collapse into $q_3 = 0$ (the third link points upward); similarly, for $p_z = A - B$, there is a single solution $q_3 = \pi$ (the third link points downward). These two cases correspond to singularities at the two boundaries of the workspace. Outside the above closed interval, i.e., when $p_z \notin [A - B, A + B]$, there is no solution for q_3 and thus to the inverse kinematics problem: the requested height of the end-effector is outside the workspace.

Squaring and summing the first two equations in (1), we obtain also

$$p_x^2 + p_y^2 = (q_2 - Bs_3)^2. \tag{3}$$

When $p_x^2 + p_y^2 \neq 0$ (again, out of the singularity associated to points along the first joint axis), we can solve for q_1 from the first two equations in (1) and find again two solutions. The first solution

$$q_1^{[+]} = \text{ATAN2}\left\{-p_x, p_y\right\} \tag{4}$$

has the robot facing the desired point (p_x, p_y), while the second solution[1]

$$q_1^{[-]} = \text{ATAN2}\{p_x, -p_y\} \quad (= q_1^{[+]} \pm \pi) \tag{5}$$

has the robot back directed toward the desired point (p_x, p_y). For $p_x^2 + p_y^2 = 0$, the angle q_1 remains undefined (singular case).

Finally, the first two equations in (1) can be combined as follows

$$-\sin q_1 p_x + \cos q_1 p_y = q_2 - B \sin q_3 = 0 \qquad \Rightarrow \qquad q_2 = \cos q_1 p_y - \sin q_1 p_x + B \sin q_3.$$

Taking into account all four combinations of solutions for q_1 and q_3 as given by eqs. (2), (4), and (5), we obtain the four associated solutions for q_2 as

$$\begin{aligned} q_2^{[++]} &= \cos q_1^{[+]} p_y - \sin q_1^{[+]} p_x + B \sin q_3^{[+]} \\ q_2^{[+-]} &= \cos q_1^{[+]} p_y - \sin q_1^{[+]} p_x + B \sin q_3^{[-]} \\ q_2^{[-+]} &= \cos q_1^{[-]} p_y - \sin q_1^{[-]} p_x + B \sin q_3^{[+]} \\ q_2^{[--]} &= \cos q_1^{[-]} p_y - \sin q_1^{[-]} p_x + B \sin q_3^{[-]}, \end{aligned} \tag{6}$$

with an obvious choice for the notation of sign labels. Note that these solutions can also be rewritten as

$$\begin{aligned} q_2^{[++]} &= \sqrt{p_x^2 + p_y^2} + B \sin q_3^{[+]} \\ q_2^{[+-]} &= \sqrt{p_x^2 + p_y^2} + B \sin q_3^{[-]} \\ q_2^{[-+]} &= -\sqrt{p_x^2 + p_y^2} + B \sin q_3^{[+]} \\ q_2^{[--]} &= -\sqrt{p_x^2 + p_y^2} + B \sin q_3^{[-]}. \end{aligned}$$

Therefore, in the generic case (i.e., out of singularities and inside the workspace) there is a total of four distinct inverse kinematics solutions:

$$\left\{q_1^{[+]}, q_2^{[++]}, q_3^{[+]}\right\}, \qquad \left\{q_1^{[+]}, q_2^{[+-]}, q_3^{[-]}\right\}, \qquad \left\{q_1^{[-]}, q_2^{[-+]}, q_3^{[+]}\right\}, \qquad \left\{q_1^{[-]}, q_2^{[--]}, q_3^{[-]}\right\}.$$

For example, assume that $A = B = 1$ [m]. We obtain the following joint solutions for two specific desired Cartesian positions $\boldsymbol{p}_d$

$$\boldsymbol{p}_d = \begin{pmatrix} 1.5 \\ 0 \\ 1 + \frac{\sqrt{3}}{2} \end{pmatrix} \text{[m]} \quad \Rightarrow \quad \left\{ \begin{pmatrix} -90° \\ 2 \\ 30° \end{pmatrix}, \begin{pmatrix} -90° \\ 1 \\ -30° \end{pmatrix}, \begin{pmatrix} 90° \\ -1 \\ 30° \end{pmatrix}, \begin{pmatrix} 90° \\ -2 \\ -30° \end{pmatrix} \right\}$$

$$\boldsymbol{p}_d = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \text{[m]} \quad \Rightarrow \quad \left\{ \begin{pmatrix} 180° \\ 3 \\ 90° \end{pmatrix}, \begin{pmatrix} 180° \\ 1 \\ -90° \end{pmatrix}, \begin{pmatrix} 0° \\ -1 \\ 90° \end{pmatrix}, \begin{pmatrix} 0° \\ -3 \\ -90° \end{pmatrix} \right\},$$

where angles q_1 and q_3 have been expressed in degrees, while the translation variable q_2 is in meters.

Exercise 3

Using the standard DH variables and shorthand notations for trigonometric quantities, the task kinematics is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{7}$$

[1] The choice of signs in the additional expression in (5) is made as follows: if $q_1^{[+]}$ is in the first or second quadrant, i.e., $q_1^{[+]} \in (0, \pi)$, then $q_1^{[-]} = q_1^{[+]} - \pi$; if $q_1^{[+]}$ is in the third or fourth quadrant, $q_1^{[+]} \in (-\pi, 0)$, then $q_1^{[-]} = q_1^{[+]} + \pi$.

and thus

$$\dot{\boldsymbol{r}} = \begin{pmatrix} v_x \\ v_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Note that we can factorize the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and redefine the joint velocity as follows

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} \\ c_1 & c_{12} & c_{123} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}_a(\boldsymbol{q})\,\boldsymbol{T}, \qquad \dot{\boldsymbol{q}}_a = \boldsymbol{T}\,\dot{\boldsymbol{q}}.$$

Indeed, $\dot{\boldsymbol{r}} = \boldsymbol{J}_a \dot{\boldsymbol{q}}_a$ and $\boldsymbol{q}_a$ is the vector of *absolute* joint angles w.r.t. the $\boldsymbol{x}_0$ axis of the world frame (the components of $\boldsymbol{q}_a$ appear also as arguments of the trigonometric functions in matrix $\boldsymbol{J}_a$).

It is easy to recognize that all task singularities occur when

$$\det \boldsymbol{J} \,(= \det \boldsymbol{J}_a) = \sin q_2 = 0 \qquad \Leftrightarrow \qquad q_2 = \{0, \pi\}.$$

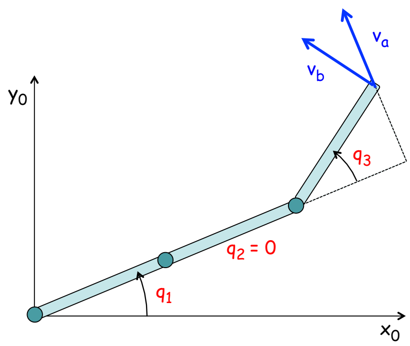


Figure 2: The planar 3R robot in a singular configuration for the three-dimensional task $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ specified in (7), with two range space vectors defined in the text.

Consider for example the case $q_2 = 0$. Then,

$$\bar{\boldsymbol{J}}_a = \boldsymbol{J}_a(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -s_1 & -s_1 & -s_{13} \\ c_1 & c_1 & c_{13} \\ 0 & 0 & 1 \end{pmatrix}.$$

In this singularity, the range of instantaneous motions covered in the task space is characterized by

$$\mathcal{R}(\bar{\boldsymbol{J}}_a) = \text{span} \left\{ \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -s_{13} \\ c_{13} \\ 1 \end{pmatrix} \right\}.$$

With reference to Fig. 2, the first basis vector in the task space is associated to a pure linear motion of the end-effector position in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ —see the unit vector $\boldsymbol{v}_a = \begin{pmatrix} -s_1 & c_1 \end{pmatrix}^T$. The second basis vector implies always a combined roto-translation, with the linear part given by the unitary vector $\boldsymbol{v}_b = \begin{pmatrix} -s_{13} & c_{13} \end{pmatrix}^T$ (projection of the second basis vector on the $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ plane) and with the angular part of unitary value as well.

As for the null space motions in the considered singularity, we have

$$\mathcal{N}(\bar{\boldsymbol{J}}_a) = \alpha\,\dot{q}_{a,N} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

and thus

$$\mathcal{N}(\bar{\boldsymbol{J}}) = \alpha\, \boldsymbol{T}^{-1} \dot{q}_{a,N} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

* * * * *

Robotics I

October 27, 2015

Exercise 1

Consider a planar 2R robot with links of equal length $\ell_1 = \ell_2 = 0.5$ [m] and assume that the robot controller generates a *joint acceleration* command $\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 & \ddot{q}_2 \end{pmatrix}^T$. While in motion, at a given instant the robot end-effector P reaches the position $\boldsymbol{p} = \begin{pmatrix} 0.6 & 0.2 \end{pmatrix}^T$ [m] with a Cartesian velocity $\dot{\boldsymbol{p}} = \begin{pmatrix} -0.5 & 0.5 \end{pmatrix}^T$ [m/s]. In this state, the task requires a desired Cartesian acceleration $\ddot{\boldsymbol{p}}_d = \begin{pmatrix} -0.7 & 1 \end{pmatrix}^T$ [m/s^2].

- Describe in general the procedure for obtaining the command $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_d$ that realizes the desired task acceleration $\ddot{\boldsymbol{p}}_d$ using only the above information. Is the solution unique? Does it always exist? Explain your answers.
- Compute numerically $\ddot{\boldsymbol{q}}_d$ with the given data.

Exercise 2

In the transmission gears shown in Fig. 1, the four toothed wheels W_i, each of radius $r_i > 0$ (for $i = 1, \dots, 4$), are perfectly engaged each to other.

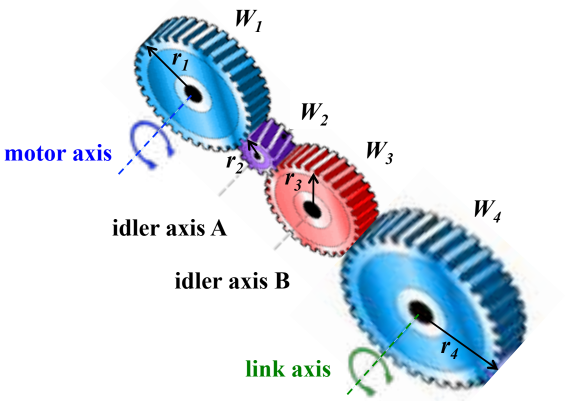


Figure 1: A sequence of four toothed wheels in a motor-to-link transmission

- Using the available information, determine the expression of the total transmission ratio N between the angular speed $\dot{\theta}_1$ of the motor axis and the angular speed $\dot{\theta}_4$ of the link axis. Is the link rotating in the same direction (CW or CCW) of the motor, or in the opposite one? Is there any restriction among the (positive) values of the radiuses r_i in order to have an increase of the torque produced when passing from the motor axis to the link axis?
- Compute the value of N when

$$r_1 = 0.025, \quad r_2 = 0.01, \quad r_3 = 0.02, \quad r_4 = 0.035 \qquad \text{[m]}.$$

What happens to N if we double the radius of both wheels W_2 and W_3 (possibly redesigning the teeth)? Provide any comments you may have on this issue.

[120 minutes; open books]

Solution

October 27, 2015

Exercise 1

In the absence of redundancy and away from kinematic singularities, the solution is found through the following procedure.

1. Solve the inverse kinematics for the current Cartesian position $\boldsymbol{p}$. We shall have multiple (analytic or numerical) solutions, each in the form $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p})$.
2. For each actual joint configuration $\boldsymbol{q}$ found in this way, evaluate the (square and nonsingular) Jacobian $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q})$ and solve for the actual joint velocity $\dot{\boldsymbol{q}}$ from the current Cartesian position $\dot{\boldsymbol{p}}$ as $\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}}$.
3. The second-order differential kinematics is given by $\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Therefore, using the $\boldsymbol{q}$ and associated $\dot{\boldsymbol{q}}$ obtained so far, compute the vector $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ and solve for the joint acceleration command associated to the desired Cartesian acceleration $\ddot{\boldsymbol{p}}_d$ as $\ddot{\boldsymbol{q}}_d = \boldsymbol{J}^{-1}\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}\right)$.

The solution is not unique (because of the multiplicity of solutions in step 1). At least one solution exists, provided that $\boldsymbol{p}$ belongs to the robot workspace. When a singularity is encountered, the problem may degenerate at step 2 or at step 3. When the Cartesian velocity $\dot{\boldsymbol{p}}$ is not in the range of the Jacobian $\boldsymbol{J}$, i.e., $\dot{\boldsymbol{p}} \notin \mathcal{R}\{\boldsymbol{J}\}$, or, similarly, when the modified Cartesian acceleration vector $\ddot{\boldsymbol{p}}_d - \boldsymbol{n} \notin \mathcal{R}\{\boldsymbol{J}\}$, there will be no solution.

Applying the above procedure to the given planar 2R robot and data yields the following solution.

Step 1. From the direct kinematics

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \end{pmatrix}, \tag{1}$$

two inverse kinematic solutions, labeled $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$, are obtained from the analytic formulas

$$q_2 = \text{ATAN2}\left\{s_2, c_2\right\}, \quad \text{with } c_2 = \frac{p_x^2 + p_y^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}, \quad s_2 = \pm\sqrt{1 - c_2^2} \tag{2}$$

and

$$q_1 = \text{ATAN2}\left\{s_1, c_1\right\}, \quad \text{with } c_1 = p_x\left(\ell_1 + \ell_2 c_2\right) + p_y \ell_2 s_2, \quad s_1 = p_y\left(\ell_1 + \ell_2 c_2\right) - p_x \ell_2 s_2.$$

We denote by $\boldsymbol{q}_a = \begin{pmatrix} q_{1a} & q_{2a} \end{pmatrix}^T$ the solution that considers the '+' sign for s_2 in (2). Similarly, $\boldsymbol{q}_b$ will be the solution corresponding to the choice of the '−' sign for s_2.

Using the problem data, we obtain

$$\boldsymbol{q}_a = \begin{pmatrix} -0.5643 \\ 1.7722 \end{pmatrix} \text{[rad]} \qquad \boldsymbol{q}_b = \begin{pmatrix} 1.2078 \\ -1.7722 \end{pmatrix} \text{[rad]},$$

or in degrees

$$\boldsymbol{q}_a = \begin{pmatrix} -32.3335^\circ \\ 101.5370^\circ \end{pmatrix}, \qquad \boldsymbol{q}_b = \begin{pmatrix} 69.2034^\circ \\ -101.5370^\circ \end{pmatrix}.$$

Step 2. The 2×2 robot Jacobian matrix can be written as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_1 \sin q_1 - \ell_2 \sin(q_1+q_2) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1+q_2) & \ell_2 \cos(q_1+q_2) \end{pmatrix} = \begin{pmatrix} -p_y & -\ell_2 \sin(q_1+q_2) \\ p_x & \ell_2 \cos(q_1+q_2) \end{pmatrix},$$

where the direct kinematic relationship (1) has been used. Its evaluation gives

$$\boldsymbol{J}_a = \boldsymbol{J}(\boldsymbol{q}_a) = \begin{pmatrix} -0.2 & -0.4674 \\ 0.6 & 0.1775 \end{pmatrix}, \quad \boldsymbol{J}_b = \boldsymbol{J}(\boldsymbol{q}_b) = \begin{pmatrix} -0.2 & 0.2674 \\ 0.6 & 0.4225 \end{pmatrix}.$$

We obtain then the two joint velocities

$$\dot{\boldsymbol{q}}_a = \boldsymbol{J}_a^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 0.5918 \\ 0.8165 \end{pmatrix} \text{ [rad/s]} \qquad \dot{\boldsymbol{q}}_b = \boldsymbol{J}_b^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 1.4082 \\ -0.8165 \end{pmatrix} \text{ [rad/s]}.$$

Step 3. The time derivative of the robot Jacobian (still a 2×2 matrix) takes the form

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{p}_y & -\ell_2 \cos(q_1+q_2)\cdot(\dot{q}_1+\dot{q}_2) \\ \dot{p}_x & -\ell_2 \sin(q_1+q_2)\cdot(\dot{q}_1+\dot{q}_2) \end{pmatrix}.$$

Evaluating terms leads to

$$\dot{\boldsymbol{J}}_a = \dot{\boldsymbol{J}}(\boldsymbol{q}_a) = \begin{pmatrix} -0.5 & -0.25 \\ -0.5 & -0.6582 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_b = \dot{\boldsymbol{J}}(\boldsymbol{q}_b) = \begin{pmatrix} -0.5 & -0.25 \\ -0.5 & 0.1582 \end{pmatrix},$$

and thus

$$\boldsymbol{n}_a = \dot{\boldsymbol{J}}_a\dot{\boldsymbol{q}}_a = \begin{pmatrix} -0.5 \\ -0.8333 \end{pmatrix}, \qquad \boldsymbol{n}_b = \dot{\boldsymbol{J}}_b\dot{\boldsymbol{q}}_b = \begin{pmatrix} -0.5 \\ -0.8333 \end{pmatrix}.$$

We obtain finally the two joint acceleration commands as

$$\ddot{\boldsymbol{q}}_{d,a} = \boldsymbol{J}_a^{-1}(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}_a) = \begin{pmatrix} 3.3535 \\ -1.0070 \end{pmatrix} \text{ [rad/s}^2\text{]} \qquad \ddot{\boldsymbol{q}}_{d,b} = \boldsymbol{J}_b^{-1}(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}_b) = \begin{pmatrix} 2.3465 \\ 1.0070 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

Exercise 2

Since the velocity of the contact point between two successive wheels W_i and W_{i+1} is the same, we have for each transmission ratio of the sub-gears

$$\dot{\theta}_i r_i = \dot{\theta}_{i+1} r_{i+1} \quad \Rightarrow \quad N_i = \frac{\dot{\theta}_i}{\dot{\theta}_{i+1}} = \frac{r_{i+1}}{r_i}, \qquad i = 1, 2, 3. \tag{3}$$

The total transmission ratio is thus the product of the single transmission ratios N_i:

$$N = \frac{\dot{\theta}_1}{\dot{\theta}_4} = \frac{\dot{\theta}_1}{\dot{\theta}_2} \cdot \frac{\dot{\theta}_2}{\dot{\theta}_3} \cdot \frac{\dot{\theta}_3}{\dot{\theta}_4} = N_1 N_2 N_3. \tag{4}$$

When considering the direction of rotation, each pair of wheels inverts the motion (from CW to CCW, and vice versa). With three pairs of wheels, the link axis will rotate in the opposite direction of the motor axis. Indeed, using the second expression of N_i in (3), we can also write that

$$N = N_1 N_2 N_3 = \frac{r_2}{r_1} \cdot \frac{r_3}{r_2} \cdot \frac{r_4}{r_3} = \frac{r_4}{r_1}. \tag{5}$$

Thus, the total transmission ratio will be independent from the radius of any of the intermediate wheels (each rotating around its own axis, also called *idler* axis). In order to have $N > 1$, i.e., a reduction of the speed by $1/N$ from the motor to the link axis with an associated increase by N of the produced torque, it is necessary and sufficient that $r_4 > r_1$. Using the given data in (5), we have

$$N = \frac{0.035}{0.025} = 1.4\,. \tag{6}$$

Doubling the radius of the two idler wheels will produce no change in N. In general, we should limit the number of intermediate gears because of the additional friction at multiple contacts (dissipating power) and the increased inertia (resulting in a reduced acceleration available at the link axis for a given torque produced on the motor axis).

* * * * *

Robotics I

January 11, 2016

Exercise 1

The 5-dof KUKA KR60 L45 robot is shown in Fig. 1. It has all revolute joints and a spherical wrist. The base has no rotation around the vertical axis (and this makes it a robot with 5-dof only). Assign the Denavit-Hartenberg frames and define the associated table of parameters, complying with the positive sense of joint rotation as shown in the left picture. Use the data in the right picture for the constant parameters.

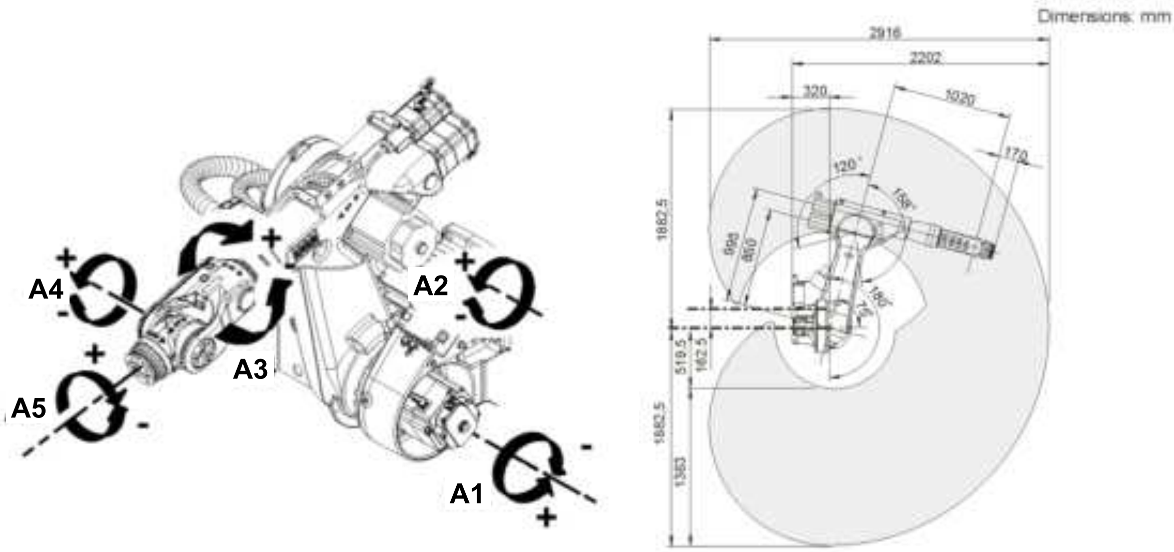


Figure 1: The KUKA KR60 L45 robot and its workspace

Exercise 2

Consider a planar 3R robot with links of equal lengths $\ell_1 = \ell_2 = \ell_3 = 0.5$ [m]. Assuming a Denavit-Hartenberg convention for the definition of the joint angles, consider the robot in the configuration $\boldsymbol{q} = (30°, 30°, 120°)$.

a) Compute a joint velocity vector $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2, \dot{q}_3)$ that realizes, if possible, the robot end-effector instantaneous motion specified by the velocity components

$$v_x = 0, \qquad v_y = 1 \text{ [m/s]}, \qquad \omega_z = 0.$$

In case a solution exists, are there multiple possible solutions to this problem?

b) Compute a joint torque $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ that balances a force $\boldsymbol{F}_e = (F_x, F_y) = (-5, 0)$ [N] applied to the robot end-effector, so that the robot remains in static conditions? Is such a $\boldsymbol{\tau}$ unique?

Exercise 3

Consider a planar 2R robot with links of lengths $\ell_1 = 0.1$ and $\ell_2 = 0.2$ [m]. The end-effector should trace the desired Cartesian trajectory

$$\boldsymbol{p}_d(t) = \begin{pmatrix} 0.15 + 0.05 \cos 5\pi t \\ 0.05 \sin 5\pi t \end{pmatrix}, \qquad t \in [0, T],$$

for some arbitrarily large period of time T.

a) At time $t = 0.2$ [s], which robot configuration $\boldsymbol{q}_d$ and joint velocity $\dot{\boldsymbol{q}}_d$ would instantaneously realize the desired trajectory? Do such numerical values $\boldsymbol{q}_d$ and $\dot{\boldsymbol{q}}_d$ exist? If so, are they unique?

b) Suppose that the robot motion is controlled by the kinematic control law

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\big[\dot{\boldsymbol{p}}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right)\big], \qquad \boldsymbol{K}_p = 10 \cdot \boldsymbol{I}_{2\times 2}, \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{q})$ is the direct kinematics for this task, and that at time $t = 1.8$ [s] the robot is in the configuration $\boldsymbol{q} = (-\pi/2, \pi/2)$. Provide the value of the command $\dot{\boldsymbol{q}}$ given by (1) in such a condition. Compute also the associated end-effector velocity and sketch it on the robot. Where is this end-effector velocity vector pointing?

c) When $\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{p}_d(0.2)$, how can the control law (1) be modified so as to generate $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_d$ as in item a), if this is at all possible for some configuration $\boldsymbol{q}$?

[240 minutes; open books]

Solution

January 11, 2016

Exercise 1

Figures 2–3 show two views of a possible DH frame assignment consistent with the requested sense of joint rotation (counterclockwise = positive). The associated parameters are given in Table 1.

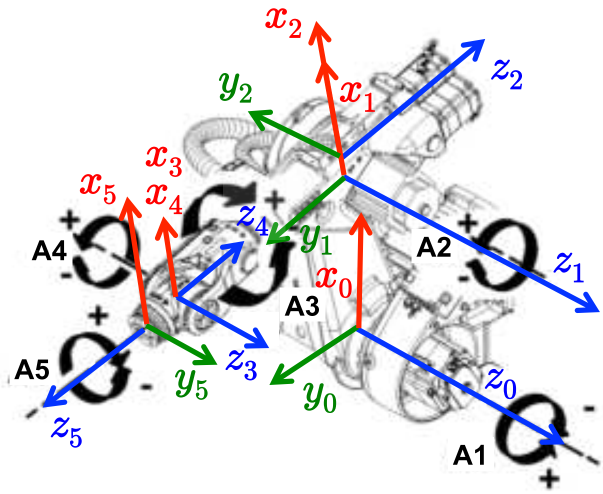


Figure 2: Denavit-Hartenberg frame assignment (perspective view from the right side)

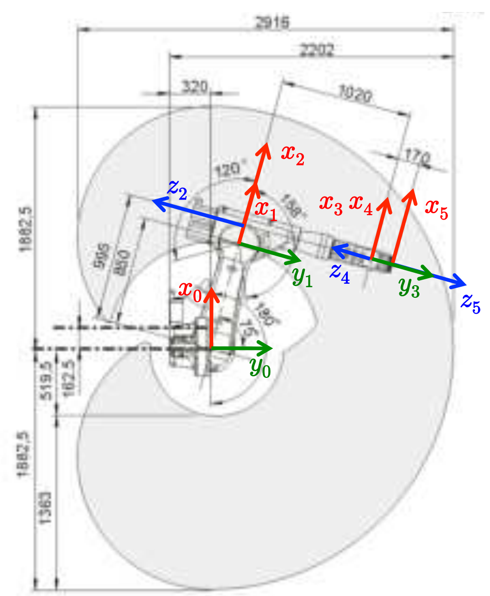


Figure 3: Denavit-Hartenberg frame assignment (lateral view from the left side)

i	α_i	a_i	d_i	θ_i
1	0	a_1	0	q_1
2	$\pi/2$	a_2	0	q_2
3	$-\pi/2$	0	d_3	q_3
4	$\pi/2$	0	0	q_4
5	π	0	d_5	q_5

$$\begin{aligned} a_1 &= 850 \text{ mm} \\ a_2 &= 145 \text{ mm} \\ d_3 &= -1020 \text{ mm} \\ d_5 &= -170 \text{ mm.} \end{aligned}$$

Table 1: Denavit-Hartenberg table of parameters associated with the frame assignment in Figs. 2–3 for the KUKA KR60 L45 robot.

Exercise 2

For a), we consider the direct kinematics for the three-dimensional task vector $\boldsymbol{r}$ associated to the end-effector position (p_x, p_y) in the plane and the angle α_z of the DH end-effector frame with respect to the $\boldsymbol{x}_0$ axis, i.e.,

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{2}$$

where we used the usual shorthand notation $c_{123} = \cos(q_1+q_2+q_3)$ and similar. Differentiating (2), we get

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha}_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ \omega_z \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with the (3×3) Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ expressed by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\left(l_1 s_1 + l_2 s_{12} + l_3 s_{123}\right) & -\left(l_2 s_{12} l_3 s_{123}\right) & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix}. \tag{3}$$

It is easy to see that the Jacobian in (3) is nonsingular at the given configuration ($\det \boldsymbol{J} = 0.125$). Therefore, the numerical solution (computed in Matlab) is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1} \dot{\boldsymbol{r}} = \begin{pmatrix} -0.6830 & -0.4330 & 0.0000 \\ 0.1830 & -0.2500 & -0.5000 \\ 1.0000 & 1.0000 & 1.0000 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.4641 \\ -5.4641 \\ 2.0000 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ -2(1+\sqrt{3}) \\ 2 \end{pmatrix} \text{ rad/s.}$$

Similarly for b), considering that the external is no external moment M_z applied to the end-effector, the joint torque *balancing* (thus, having the opposite sign due to the principle of action and reaction) the external pure force $\boldsymbol{F}_e$ is

$$\boldsymbol{\tau} = -\boldsymbol{J}^T \begin{pmatrix} F_x \\ F_y \\ M_z \end{pmatrix} = -\boldsymbol{J}^T \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.4151 \\ -2.1651 \\ 0.0000 \end{pmatrix} = \begin{pmatrix} -5(1+\sqrt{3})/4 \\ -5\sqrt{3}/4 \\ 0 \end{pmatrix} \text{ Nm.}$$

This torque is unique. Note that the torque component at the third joint is zero, since the line of action of the force $\boldsymbol{F}_e$ in the given robot configuration passes through the axis of this joint.

Exercise 3

The desired end-effector position and velocity are

$$\boldsymbol{p}_d(t) = \begin{pmatrix} 0.15 + 0.05\cos 5\pi t \\ 0.05 \sin 5\pi t \end{pmatrix}, \qquad \boldsymbol{v}_d(t) = \dot{\boldsymbol{p}}_d(t) = \begin{pmatrix} -0.25\pi \sin 5\pi t \\ 0.25\pi \cos 5\pi t \end{pmatrix}.$$

With reference to Fig. 4, we see that the desired path is tangent to the inner boundary of the robot workspace.

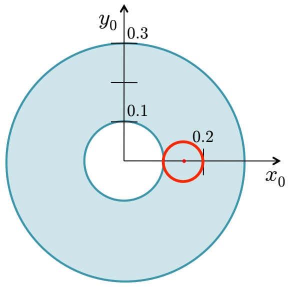


Figure 4: The desired Cartesian path and the robot workspace

In case a), when $t = 0.2$ it is

$$\boldsymbol{p}_d(0.2) = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} \text{[m]}, \qquad \boldsymbol{v}_d(0.2) = \begin{pmatrix} 0 \\ -\pi/4 \end{pmatrix} \text{[m/s]} \tag{4}$$

and the end-effector touches the boundary so that the robot can be there in the *unique* configuration $\boldsymbol{q}_d = (\pi, \pi)$. Moreover, the desired velocity at this point is feasible since it belongs to the range space of the (singular) robot Jacobian at that configuration,

$$\boldsymbol{J}(\boldsymbol{q}_d) = \begin{pmatrix} 0 & 0 \\ 0.1 & 0.2 \end{pmatrix}, \qquad \boldsymbol{v}_d(0.2) = \begin{pmatrix} 0 \\ -\pi/4 \end{pmatrix} \in \mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_d)\right\}.$$

In fact, there is an *infinite* number of combinations for the velocities of the two joints that realize the desired Cartesian velocity. The joint velocity solution with minimum norm is obtained as

$$\dot{\boldsymbol{q}}_{d,min} = \boldsymbol{J}^{\#}(\boldsymbol{q}_d)\boldsymbol{v}_d = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ -\pi/4 \end{pmatrix} = -\begin{pmatrix} \pi/2 \\ \pi \end{pmatrix}.$$

All other solutions can be written as

$$0.1\,\dot{q}_{d1} + 0.2\,\dot{q}_{d2} = v_{d2} = -\pi/4 \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_d = \dot{\boldsymbol{q}}_{d,min} + \begin{pmatrix} -2\alpha \\ \alpha \end{pmatrix}, \text{ for any } \alpha \in \mathbb{R}.$$

For case b), when $t = 1.8$ the position and velocity of the desired trajectory are again as in (4). With the $2R$ robot in the configuration $\boldsymbol{q} = (-\pi/2, \pi/2)$, the Jacobian is out of singularities and can be safely inverted. From the direct kinematics, the robot end-effector position is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix}\Bigg|_{\boldsymbol{q}=(-\pi/2,\pi/2)} = \begin{pmatrix} 0.2 \\ -0.1 \end{pmatrix}$$

with a Cartesian position error

$$\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.2 \\ -0.1 \end{pmatrix} = \begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix}.$$

The Cartesian kinematic controller provides then

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{J}^{-1}(\boldsymbol{q})\left[\dot{\boldsymbol{p}}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right)\right] = \boldsymbol{J}^{-1}(\boldsymbol{q})\left[\boldsymbol{v}_d + \boldsymbol{K}_p\boldsymbol{e}_p\right] \\ &= \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{pmatrix}^{-1}\left[\begin{pmatrix} 0 \\ -\pi/4 \end{pmatrix} + 10\cdot\boldsymbol{I}_{2\times 2}\begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -10.0000 \\ 11.0730 \end{pmatrix} = \begin{pmatrix} -10 \\ 15-5\pi/4 \end{pmatrix} \text{[rad/s]}. \end{aligned} \tag{5}$$

The instantaneous Cartesian velocity associated to (5) is

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -1 \\ 0.2146 \end{pmatrix} \text{[m/s]}.$$

The two situations at $t = 0.2$ s (motion in nominal conditions) and at $t = 1.8$ s (motion with tracking error) are illustrated in Fig. 5.

Finally, the answer to c) is that it is certainly possible to obtain the solution in the conditions of case a) using the control law (1), by replacing the inverse of the Jacobian with its pseudoinverse, possibly specifying also a generic term in the null space of the Jacobian

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left[\boldsymbol{v}_d + \boldsymbol{K}_p\boldsymbol{e}_p\right] + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\dot{\boldsymbol{q}}_0. \tag{6}$$

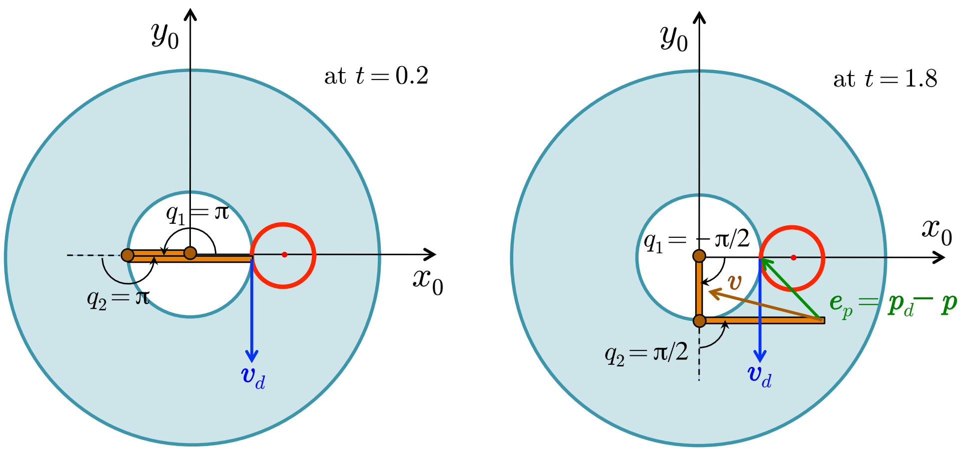


Figure 5: The solution at $t = 0.2$ s [left] and at $t = 1.8$ s [left] (for the sake of illustration, vectors are not represented in scale: only their correct direction is preserved)

* * * * *

Robotics I

February 4, 2016

Exercise 1

We are given an incomplete time-varying rotation matrix from frame 0 to frame 1:

$$
{}^0\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & a(t) & b(t) \\ \sin t & \dfrac{k(t)}{\sqrt{2}}\cos t & c(t) \\ 0 & -\dfrac{k(t)}{\sqrt{2}}\sin t & d(t) \end{pmatrix}.
$$

Determine the expressions of $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $k(t)$ in a consistent way.

Exercise 2

The table of Denavit-Hartenberg parameters of a 2-dof robot is:

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	0	0	q_2	0

The two joints have a range limitation: $|q_1| \leq 120°$ and $|q_2| \leq 2$ [m]. Determine all feasible inverse kinematics solutions, if any, when the origin of frame 2 needs to be placed at ${}^0\boldsymbol{p} = (-1, 1)$ [m].

Exercise 3

Consider a planar 4R robot with links of lengths $\ell_i = 0.25$ [m], $i = 1, \ldots, 4$. The robot performs simultaneously two tasks: moving the end-effector at a desired velocity $\boldsymbol{v}_E$ and moving a midpoint in the structure, i.e., the end of link 2, at another desired velocity $\boldsymbol{v}_M$, as in Fig. 1. Formalize the problem and investigate the conditions for its solvability. When the robot is in the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ [rad], determine if there exists a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ realizing the two Cartesian velocities $\boldsymbol{v}_M = (-0.2, 0.1)$ [m/s] and $\boldsymbol{v}_E = (0.2, 0)$ [m/s]. If so, compute a solution. Is it unique?

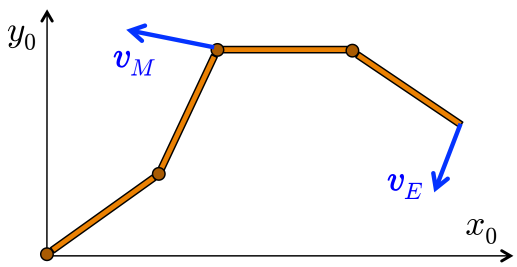


Figure 1: A 4R planar robot with a double motion task

[turn the sheet for next exercise]

Exercise 4

The end-effector of a planar robot moves in a cycle along the rectangular path $ABCD$, having short side M and long side L, placed as in Fig. 2. The robot end-effector should pass through the corner points. The Cartesian speed of the end-effector is limited above by $V_{max} > 0$, while the Cartesian acceleration is bounded in norm as $\|\ddot{\boldsymbol{p}}\| \leq A_{max} > 0$. The trajectory should start at rest from point A and return at rest to the same point at the end. The Cartesian velocity $\dot{\boldsymbol{p}}(t)$ should be continuous everywhere.

a) Determine the minimum feasible motion time T in a parametric way, sketching the speed profile along the entire path.

b) Provide the numerical value of T using the following data:

$$M = 0.4 \text{ [m]}, \qquad L = 1.6 \text{ [m]}, \qquad V_{max} = 1 \text{ [m/s]}, \qquad A_{max} = 2 \text{ [m/s}^2\text{]}.$$

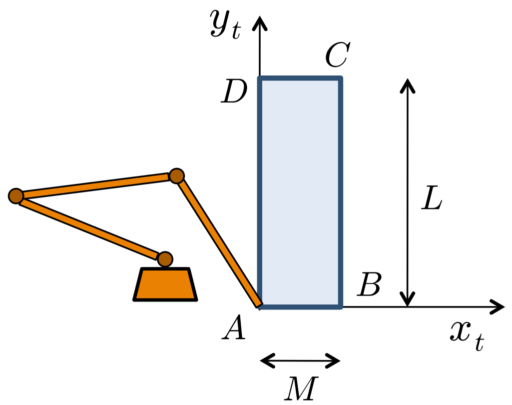


Figure 2: The cyclic rectangular path $A \to B \to C \to D \to A$

[210 minutes; open books]

Solution

February 4, 2016

Exercise 1

We need to impose orthonormality conditions to the columns of ${}^0\boldsymbol{R}_1(t)$ and check finally that $\det {}^0\boldsymbol{R}_1(t) = +1$, for all times t. The first column $\boldsymbol{r}_1$ is already of unitary norm. For the second column $\boldsymbol{r}_2$, we need to impose the unit norm condition

$$\|\boldsymbol{r}_2\|^2 = a^2(t) + \frac{k^2(t)\cos^2 t}{2} + \frac{k^2(t)\sin^2 t}{2} = a^2(t) + \frac{k^2(t)}{2} = 1 \tag{1}$$

and the condition of orthogonality $\boldsymbol{r}_2 \perp \boldsymbol{r}_1$

$$a(t)\cos t + \frac{k(t)\cos t}{\sqrt{2}}\sin t = 0.$$

The latter provides $a(t) = -k(t)\sin t/\sqrt{2}$. Substituting in (1) yields

$$\frac{k^2(t)\sin^2 t}{2} + \frac{k^2(t)}{2} = 1 \qquad \Rightarrow \qquad k(t) = \pm\sqrt{\frac{2}{1+\sin^2 t}}. \tag{2}$$

Therefore, the second column of ${}^0\boldsymbol{R}_1(t)$ is

$$\boldsymbol{r}_2 = \begin{pmatrix} \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{\pm\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} \end{pmatrix}^T. \tag{3}$$

Similarly, for the third column $\boldsymbol{r}_3$, we impose first the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_1$

$$b(t)\cos t + c(t)\sin t = 0 \qquad \Rightarrow \qquad b(t) = \alpha(t)\sin t, \quad c(t) = -\alpha(t)\cos t. \tag{4}$$

Using (3) and (4), we impose next the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_2$ as[1]

$$\alpha(t)\,\frac{\sin^2 t}{\sqrt{1+\sin^2 t}} + \alpha(t)\,\frac{\cos^2 t}{\sqrt{1+\sin^2 t}} + d(t)\,\frac{\sin t}{\sqrt{1+\sin^2 t}} = 0 \quad \Rightarrow \quad \alpha(t) = -d(t)\sin t.$$

Finally, the unit norm condition provides

$$\|\boldsymbol{r}_3\|^2 = 1 \quad \Rightarrow \quad d^2(t)\left(\sin^4 t + \sin^2 t\cos^2 t + 1\right) = 1 \quad \Rightarrow \quad d(t) = \frac{\pm 1}{\sqrt{1+\sin^2 t}}. \tag{5}$$

The uncertainty left in the signs of $k(t)$ and $d(t)$, respectively in eq. (2) and eq. (5), is eliminated by imposing the determinant of ${}^0\boldsymbol{R}_1(t)$ to be equal to $+1$. This holds true when choosing either both positive signs for $k(t)$ and $d(t)$, or both negative. The first solution is

$${}^0\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & -\dfrac{\sin^2 t}{\sqrt{1+\sin^2 t}} \\ \sin t & \dfrac{\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\sin t\cos t}{\sqrt{1+\sin^2 t}} \\ 0 & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{1}{\sqrt{1+\sin^2 t}} \end{pmatrix}, \tag{6}$$

and corresponds to the case when ${}^0\boldsymbol{R}_1(0) = \boldsymbol{I}$. The second solution is as in (6), but with each element of the second and third column having the opposite sign.

[1] The same $\mp$ sign is factored out in all three terms, and thus eliminated as irrelevant in a homogenous equation.

Exercise 2

The given table of parameters refers to the planar RP robot in Fig. 3, where the associated Denavit-Hartenberg frames are also shown. Please note the definition of the first joint angle q_1, which differs from what one may expect (there is an additional $\pi/2$ with respect to the second link orientation).

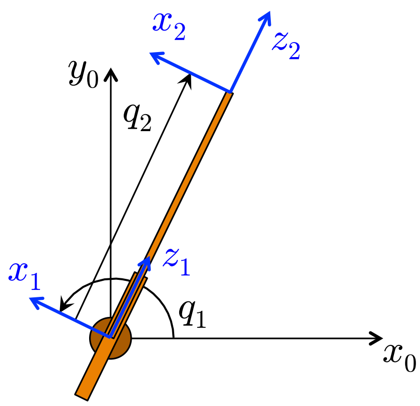


Figure 3: The RP robot, with its Denavit-Hartenberg frames and joint coordinates

The direct kinematics for the position $\boldsymbol{p}$ of the origin of frame 2 is then

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \sin q_1 \\ -q_2 \cos q_1 \end{pmatrix}.$$

Out of the singularity ($q_2 \neq 0 \Leftrightarrow \boldsymbol{p} \neq 0$), the two solutions of the inverse kinematics are analytically found as

$$q_2 = \pm \|\boldsymbol{p}\| = \pm \sqrt{p_x^2 + p_y^2}, \qquad q_1 = \text{ATAN2}\left\{ \frac{p_x}{q_2}, -\frac{p_y}{q_2} \right\}. \tag{7}$$

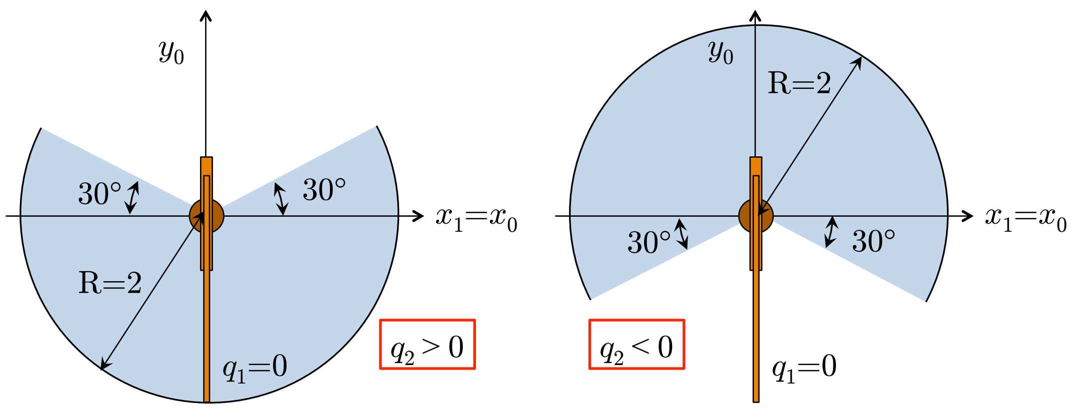


Figure 4: Robot workspace for $|q_1| \leq 120°$, $|q_2| \leq 2$, shown when $q_2 > 0$ (left) and $q_2 < 0$ (right)

For the desired position $\boldsymbol{p} = (-1, 1)$, we obtain

$$\boldsymbol{q}' = \begin{pmatrix} -\dfrac{3\pi}{4} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} -135^\circ \\ \sqrt{2} \end{pmatrix}, \qquad \boldsymbol{q}'' = \begin{pmatrix} \dfrac{\pi}{4} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 45^\circ \\ -\sqrt{2} \end{pmatrix}.$$

Thus, only the solution $\boldsymbol{q}''$ is within the joint range. This end-effector position belongs to the robot workspace shown on the right in Fig. 4.

As a further numerical example, let the desired end-effector position be $\boldsymbol{p} = (0.25, 0.5)$ (a point in the first quadrant). From eqs. (7), we have

$$\boldsymbol{q}' = \begin{pmatrix} 150^\circ \\ \dfrac{\sqrt{3}}{2} \end{pmatrix}, \qquad \boldsymbol{q}'' = \begin{pmatrix} -30^\circ \\ -\dfrac{\sqrt{3}}{2} \end{pmatrix},$$

and the solution $\boldsymbol{q}''$ is again the only feasible one. Indeed, for any $\boldsymbol{p} \in \mathbb{R}^2$ belonging to the intersection of the two 'half' workspaces in Fig. 4 (two cones of 60° around the positive and negative $\boldsymbol{x}_0$ axis), there will be two feasible solutions to the inverse kinematics.

Exercise 3

Consider the position $\boldsymbol{p}_M$ of the midpoint along the robot structure and the position $\boldsymbol{p}_E$ of the end-effector. Use the DH joint angles and partition the four-dimensional joint configuration $\boldsymbol{q}$ into $\boldsymbol{q}_M = (q_1, q_2)$ and $\boldsymbol{q}_E = (q_3, q_4)$. The two relevant direct kinematics maps are

$$\boldsymbol{p}_M = \boldsymbol{f}_M(\boldsymbol{q}_M) = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} \\ \ell_1 s_1 + \ell_2 s_{12} \end{pmatrix} \tag{8}$$

and

$$\boldsymbol{p}_E = \boldsymbol{f}_E(\boldsymbol{q}_M, \boldsymbol{q}_E) = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} \\ \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234} \end{pmatrix} = \boldsymbol{p}_M + \boldsymbol{p}_{ME}, \tag{9}$$

with

$$\boldsymbol{p}_{ME} = \boldsymbol{f}_{ME}(\boldsymbol{q}_M, \boldsymbol{q}_E) = \begin{pmatrix} \ell_3 c_{123} + \ell_4 c_{1234} \\ \ell_3 s_{123} + \ell_4 s_{1234} \end{pmatrix}, \tag{10}$$

and where the usual shorthand notation for trigonometric quantities (e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$) has been used.

Differentiating w.r.t. time eq. (8) and (9) yields

$$\boldsymbol{v}_M = \dot{\boldsymbol{p}}_M = \frac{\partial \boldsymbol{f}_M(\boldsymbol{q}_M)}{\partial \boldsymbol{q}_M}\,\dot{\boldsymbol{q}}_M = \begin{pmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}_{MM}(\boldsymbol{q}_M)\dot{\boldsymbol{q}}_M \tag{11}$$

and

$$
\begin{aligned}
\boldsymbol{v}_E = \dot{\boldsymbol{p}}_E &= \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M,\boldsymbol{q}_E)}{\partial \boldsymbol{q}_M}\,\dot{\boldsymbol{q}}_M + \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M,\boldsymbol{q}_E)}{\partial \boldsymbol{q}_E}\,\dot{\boldsymbol{q}}_E \\
&= \begin{pmatrix} -\left(\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234}\right) & -\left(\ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234}\right) \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} & \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} -\ell_3 s_{123} - \ell_4 s_{1234} & -\ell_4 s_{1234} \\ \ell_3 c_{123} + \ell_4 c_{1234} & \ell_4 c_{1234} \end{pmatrix} \begin{pmatrix} \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} \\
&= \boldsymbol{J}_{EM}(\boldsymbol{q}_M,\boldsymbol{q}_E)\dot{\boldsymbol{q}}_M + \boldsymbol{J}_{EE}(\boldsymbol{q}_M,\boldsymbol{q}_E)\dot{\boldsymbol{q}}_E.
\end{aligned} \tag{12}
$$

Note also that, from (9) and (10),

$$
\boldsymbol{J}_{EE}(\boldsymbol{q}_M,\boldsymbol{q}_E) = \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M,\boldsymbol{q}_E)}{\partial \boldsymbol{q}_E} = \frac{\partial \boldsymbol{f}_{ME}(\boldsymbol{q}_M,\boldsymbol{q}_E)}{\partial \boldsymbol{q}_E}.
$$

The simultaneous execution of the double task is represented by the 4×4 composite Jacobian $\boldsymbol{J}(\boldsymbol{q})$ as

$$
\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_M \\ \boldsymbol{v}_E \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{MM}(\boldsymbol{q}_M) & \boldsymbol{O} \\ \boldsymbol{J}_{EM}(\boldsymbol{q}_M,\boldsymbol{q}_E) & \boldsymbol{J}_{EE}(\boldsymbol{q}_M,\boldsymbol{q}_E) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{13}
$$

The block triangular structure of $\boldsymbol{J}$ indicates that the problem is solvable for any pair of generic desired velocities $\boldsymbol{v}_E \in \mathbb{R}^2$ and $\boldsymbol{v}_M \in \mathbb{R}^2$ if and only if the two diagonal blocks $\boldsymbol{J}_{MM}$ and $\boldsymbol{J}_{EE}$ are both nonsingular. It is easy to see that $\boldsymbol{J}_{MM}$ is the Jacobian of the 2R robot sub-structure made by the first two links. Thus

$$
\det \boldsymbol{J}_{MM}(\boldsymbol{q}_M) = 0 \qquad \Longleftrightarrow \qquad q_2 = 0 \text{ (stretched) or } \pi \text{ (folded)}. \tag{14}
$$

On the other hand, the block $\boldsymbol{J}_{EE}$ can be expressed in the DH frame 2, i.e., premultiplied by the transpose of the 2×2 (planar) rotation matrix ${}^0\boldsymbol{R}_2(\boldsymbol{q}_M)$, resulting in

$$
\begin{aligned}
{}^0\boldsymbol{R}_2^T(\boldsymbol{q}_M)\boldsymbol{J}_{EE}(\boldsymbol{q}_M,\boldsymbol{q}_E) &= \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} -\ell_3 s_{123} - \ell_4 s_{1234} & -\ell_4 s_{1234} \\ \ell_3 c_{123} + \ell_4 c_{1234} & \ell_4 c_{1234} \end{pmatrix} \\
&= \begin{pmatrix} -\ell_3 s_3 - \ell_4 s_{34} & -\ell_4 s_{34} \\ \ell_3 c_3 + \ell_4 c_{34} & \ell_4 c_{34} \end{pmatrix}.
\end{aligned}
$$

Therefore, we recognize that the singularities of $\boldsymbol{J}_{EE}$ are those of the Jacobian of the 2R robot sub-structure made by the last two links, or

$$
\det \boldsymbol{J}_{EE}(\boldsymbol{q}) = 0 \qquad \Longleftrightarrow \qquad q_4 = 0 \text{ (stretched) or } \pi \text{ (folded)}. \tag{15}
$$

When none of the singularity conditions (14) and (15) holds, the solution to (13) is given by blockwise inversion of matrix $\boldsymbol{J}$

$$
\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v} = \begin{pmatrix} \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M) & \boldsymbol{O} \\ -\boldsymbol{J}_{EE}^{-1}(\boldsymbol{q})\boldsymbol{J}_{EM}(\boldsymbol{q})\boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M) & \boldsymbol{J}_{EE}^{-1}(\boldsymbol{q}) \end{pmatrix} \boldsymbol{v} \tag{16}
$$

or

$$
\dot{\boldsymbol{q}}_M = \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M)\boldsymbol{v}_M, \qquad \dot{\boldsymbol{q}}_E = \boldsymbol{J}_{EE}^{-1}(\boldsymbol{q})\left(\boldsymbol{v}_E - \boldsymbol{J}_{EM}(\boldsymbol{q})\dot{\boldsymbol{q}}_M\right). \tag{17}
$$

Note that the term in the last parentheses in (17) represents the part of the desired end-effector velocity that is still missing, once the contribution given by the velocity $\dot{\boldsymbol{q}}_M$ of the first two joints has been taken into account.

Turning now to the numerical evaluation, the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ is shown in Fig. 5 and is clearly nonsingular.

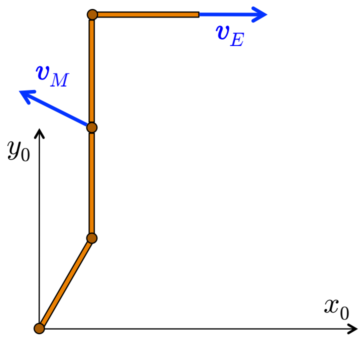


Figure 5: The 4R planar robot in the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ with the prescribed double motion task $\boldsymbol{v}_M = (-0.2, 0.1)$ and $\boldsymbol{v}_E = (0.2, 0)$

Using $\ell_i = 0.25$, $i = 1, \ldots, 4$, the blocks of the complete Jacobian are

$$\boldsymbol{J}_{MM} = \begin{pmatrix} -0.4665 & -0.25 \\ 0.125 & 0 \end{pmatrix}, \quad \boldsymbol{J}_{EM} = \begin{pmatrix} -0.7165 & -0.5 \\ 0.375 & 0.25 \end{pmatrix}, \quad \boldsymbol{J}_{EE} = \begin{pmatrix} -0.25 & 0 \\ 0.25 & 0.25 \end{pmatrix}.$$

The joint velocity $\dot{\boldsymbol{q}}$ realizing the two Cartesian velocities $\boldsymbol{v}_M = (-0.2, 0.1)$ and $\boldsymbol{v}_E = (0.2, 0)$ are computed as in (17), yielding

$$\dot{\boldsymbol{q}}_M = \begin{pmatrix} 0.8 \\ -0.6928 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{q}}_E = \begin{pmatrix} -1.7072 \\ 1.2 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} \in \mathbb{R}^4. \tag{18}$$

This solution is indeed unique.

Final note. A more complex approach to determine the solution would have been the following. Let the solution to the first task be $\dot{\boldsymbol{q}}_M = \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M)\boldsymbol{v}_M$ and consider the second (redundant) task

$$\boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{J}_{EM}(\boldsymbol{q}) & \boldsymbol{J}_{EE}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} = \boldsymbol{v}_E, \tag{19}$$

where the Jacobian $\boldsymbol{J}_E(\boldsymbol{q})$ is a 2×4 matrix. All solutions to (19) can be written as

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{\boldsymbol{q}}_M^* \\ \dot{\boldsymbol{q}}_E^* \end{pmatrix} = \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{v}_E + \left(\boldsymbol{I} - \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{J}_E(\boldsymbol{q})\right)\dot{\boldsymbol{q}}_0, \qquad \text{with arbitrary } \dot{\boldsymbol{q}}_0 \in \mathbb{R}^4. \tag{20}$$

The first term in (20) is the minimum norm joint velocity solution given by the pseudoinverse of the Jacobian $\boldsymbol{J}_E$. The second term is a joint velocity vector belonging to the null space $\mathcal{N}\{\boldsymbol{J}_E\}$ of $\boldsymbol{J}_E$, thanks to the presence of the projection matrix $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}_E^{\#}\boldsymbol{J}_E$. The null space is explored by changing the generic joint velocity $\dot{\boldsymbol{q}}_0$. For $\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$, the upper part $\dot{\boldsymbol{q}}_M^*$ of the minimum norm solution obtained will differ in general from the solution found for the first task, $\dot{\boldsymbol{q}}_M^* \neq \boldsymbol{J}_{MM}^{-1}\boldsymbol{v}_M$, showing an incompatibility at the level of the velocities of the first two joints. This is what happens in fact with the given numerical data:

$$\dot{\boldsymbol{q}}^* = \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{v}_E = \begin{pmatrix} -0.2037 & -0.1591 & 0.1018 & 0.3627 \end{pmatrix}^T,$$

which differs in the first two components from (18). However, there exists indeed a choice of $\dot{\boldsymbol{q}}_0$ in (20) that will provide a fully consistent solution. This is guaranteed by the fact that we found already the solution (18) to our simultaneous double velocity task problem. For the case study, setting for instance

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 1.0037 & -0.5337 & -1.8090 & 0.8373 \end{pmatrix}^T$$

in (20) will provide back the solution (18). We note also that $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}_E\}$, and thus $\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_0$.

Exercise 4

The problem addressed in the Cartesian space. To guarantee continuity of the end-effector velocity $\boldsymbol{p}(t)$ during the entire motion, it is necessary to stop at each of the path corners B, C, and D (because the tangent to the path is discontinuous there). Therefore, we can treat separately each side of the rectangle. The minimum time motion along a side will have either a trapezoidal speed profile or a (degenerate) bang-bang acceleration profile. The type of profile will be identical on two opposite sides, since it depends only on the length of the segment (M or L), once V_{max} and A_{max} are assigned. In order for a 'coast' phase to exist (i.e., the maximum admissible speed is reached, at least for one instant) on each of the four sides, it is necessary and sufficient that

Case I: $M \geq \dfrac{V_{max}^2}{A_{max}}$ (on the short sides) $\Rightarrow$ $L \geq M \geq \dfrac{V_{max}^2}{A_{max}}$ (**also** on the long sides).

Conversely, the profiles on all sides will be of the bang-bang acceleration type if and only if

Case II: $L \leq \dfrac{V_{max}^2}{A_{max}}$ (on the long sides) $\Rightarrow$ $M \leq L \leq \dfrac{V_{max}^2}{A_{max}}$ (**also** on the short sides).

Indeed, a mixed situation occurs when

Case III: $M \leq \dfrac{V_{max}^2}{A_{max}} \leq L$ (bang-bang on short sides, trapezoidal speed on long sides).

From the known expression of the minimum time needed for a rest-to-rest motion along a straight path of length δ with a trapezoidal speed profile

$$T_\delta = \frac{\delta A_{max} + V_{max}^2}{A_{max}V_{max}}, \qquad \text{for } \delta = \{M, L\},$$

the motion time in **Case I** will be:

$$T = 2\left(\frac{MA_{max} + V_{max}^2}{A_{max}V_{max}} + \frac{LA_{max} + V_{max}^2}{A_{max}V_{max}}\right) = \frac{2(M+L)A_{max} + 4V_{max}^2}{A_{max}V_{max}}. \tag{21}$$

For **Case II**, the velocity profile on each side will be triangular, with maximum acceleration and deceleration phases. Let T_Δ be the travel time on one of the sides. At the mid time $t = T_\Delta/2$, the peak speed $A_{max}(T_\Delta/2)$ is reached. The displacement will be equal to $\frac{1}{2}A_{max}(T/2)^2$, where half of the length of the side has been traced. Therefore,

$$\frac{1}{2}A_{max}(T_\Delta/2)^2 = \frac{\Delta}{2} \qquad \Rightarrow \qquad T_\Delta = 2\sqrt{\frac{\Delta}{A_{max}}}, \qquad \text{for } \ \Delta = \{M, L\},$$

and the total motion time will be

$$T = 2\left(2\sqrt{\frac{M}{A_{max}}} + 2\sqrt{\frac{L}{A_{max}}}\right) = 4\,\frac{\sqrt{M}+\sqrt{L}}{\sqrt{A_{max}}}. \tag{22}$$

Finally, **Case III** will be a combination of the two formulas (21) and (22). Thus,

$$T = 2\,\frac{LA_{max} + V_{max}^2}{A_{max}V_{max}} + 4\sqrt{\frac{M}{A_{max}}}. \tag{23}$$

Using the numerical data, we see that **Case III** applies since

$$M = 0.4 < \left(\frac{V_{max}^2}{A_{max}} = \frac{1}{2} = \right) 0.5 < 1.6 = L.$$

From (23), the total travel time is then $T = 5.989$ s.

Note that the total length of the rectangular path is $2(M+L) = 4$ [m]; if we could trace it always at maximum speed $V_{max} = 1$ m/s from the beginning to its end, this would take $T_{ideal} = 4$ s. Because of the limited acceleration and of the required continuity of velocity, motion lasts about 50% longer than in the ideal (but not realizable) limit.

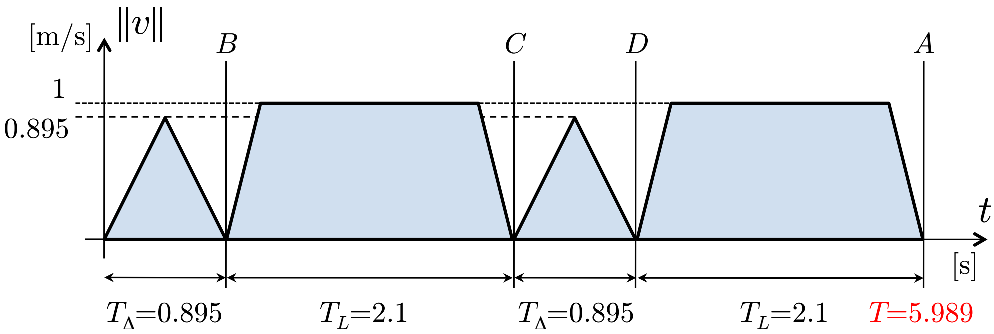


Figure 6: Time profile of the scalar speed along the rectangular path

Figure 6 gives the profile of the (scalar) speed along the entire rectangular path. Note that this speed is always non-negative. Figure 7 reports the associated profiles of the v_x and v_y components of the Cartesian velocity $\boldsymbol{v} = \dot{\boldsymbol{p}}$. Indeed, continuity is enforced at all times.

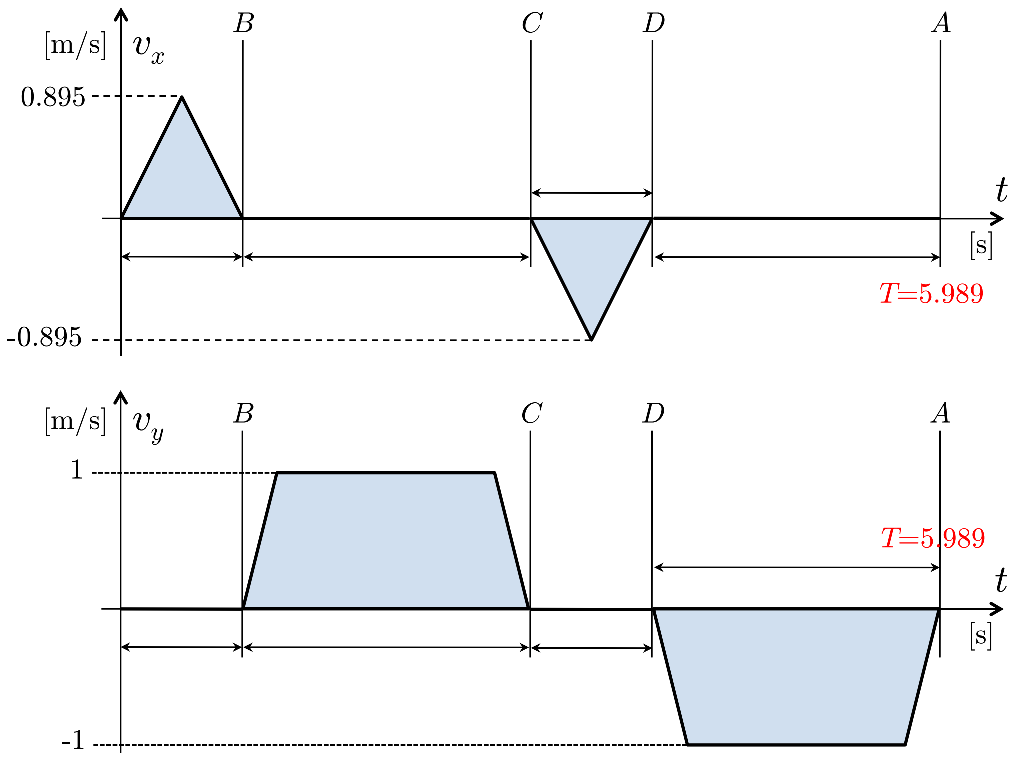


Figure 7: Time profiles of the components of the Cartesian velocity $\boldsymbol{v}$ along the rectangular path of Fig. 2: v_x (top) and v_y (bottom)

* * * * *

Robotics I

April 1, 2016

Consider a planar 2R robot with links of length $\ell_1 = 1$ and $\ell_2 = 0.5$ [m]. The end-effector should move *smoothly* from an initial point $\boldsymbol{p}_{in}$ to a final point $\boldsymbol{p}_{fin}$ in the robot workspace so that

- the motion starts and ends with zero Cartesian velocity and acceleration;
- at the start, the robot is in the 'right arm' inverse kinematics solution (i.e., with positive q_2), and remains in this type of solution throughout the motion;
- coordinated motion is enforced to the joints;
- symmetric limits on joint velocity, acceleration, and jerk are satisfied:

$$|\dot{q}_i| \le V_i, \qquad |\ddot{q}_i| \le A_i, \qquad |\dddot{q}_i| \le J_i, \qquad\qquad i = 1, 2.$$

In order to address this motion task, choose a class of trajectories and determine, within the considered class, a minimum time trajectory, given the following position data

$$\boldsymbol{p}_{in} = \begin{pmatrix} 0.4 \\ 1.2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{p}_{fin} = \begin{pmatrix} -1 \\ -0.2 \end{pmatrix} \text{ [m]}$$

and joint limits

$$\begin{aligned} V_1 &= 1 \text{ [rad/s]}, \quad A_1 = 3 \text{ [rad/s}^2\text{]}, \quad J_1 = 30 \text{ [rad/s}^3\text{]}, \\ V_2 &= 2 \text{ [rad/s]}, \quad A_2 = 7.5 \text{ [rad/s}^2\text{]}, \quad J_2 = 70 \text{ [rad/s}^3\text{]}. \end{aligned}$$

Provide the minimum feasible time T^* obtained and the maximum (absolute) values attained by the velocity and the acceleration at the two joints.

At the trajectory midpoint, $t = T^*/2$, determine the values of the end-effector Cartesian velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$, and draw the robot in its current configuration together with the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$.

[180 minutes; open books]

Solution

April 1, 2016

In view of the nature of the given robot motion limits, it is highly recommended to define the trajectory in the joint space.

The direct and inverse kinematics of the 2R planar robot are given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} \\ \ell_1 s_1 + \ell_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q})$$

$$\Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} {q_1}^{+/-} \\ {q_2}^{+/-} \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\{p_y(\ell_1 + \ell_2 c_2) - p_x \ell_2 s_2, p_x(\ell_1 + \ell_2 c_2) + p_y \ell_2 s_2\} \\ \text{ATAN2}\{s_2, c_2\} \end{pmatrix} = \boldsymbol{f}^{-1}(\boldsymbol{p}),$$

with

$$c_2 = \frac{p_x^2 + p_y^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}, \qquad s_2 = \pm\sqrt{1 - c_2^2},$$

and where the $+/-$ associated as index to the joint angles q_1 and q_2 mean that for their evaluation the $+$ or, respectively, the $-$ sign has been used in the definition of s_2. Substituting the link lengths and the problem data for $\boldsymbol{p} = \boldsymbol{p}_{in}$ and $\boldsymbol{p} = \boldsymbol{p}_{fin}$, and picking up the solution with $q_2^+ > 0$ yields

$$\boldsymbol{q}_{in} = \begin{pmatrix} 49.83° \\ 69.51° \end{pmatrix} = \begin{pmatrix} 0.8697 \\ 1.2132 \end{pmatrix} \text{ [rad]}, \qquad \boldsymbol{q}_{fin} = \begin{pmatrix} 162.66° \\ 102.12° \end{pmatrix} = \begin{pmatrix} 2.8391 \\ 1.7824 \end{pmatrix} \text{ [rad]}.$$

Taking into account the smoothness requirement and the boundary conditions on velocity and acceleration, we choose a polynomial trajectory of degree 5 for each joint. In the double normalized form, its expression is

$$\boldsymbol{q}(\tau) = \boldsymbol{q}_{in} + \boldsymbol{\Delta q}\left(10\tau^3 - 15\tau^4 + 6\tau^5\right), \qquad \boldsymbol{\Delta q} = \boldsymbol{q}_{fin} - \boldsymbol{q}_{in} = \begin{pmatrix} 1.9712 \\ 0.5693 \end{pmatrix} \text{ [rad]}, \quad \tau = \frac{t}{T} \in [0, 1].$$

In order to obtain the maximum values reached along this trajectory by the velocity, acceleration, and jerk, which should satisfy the given limits, we compute the first four time derivatives:

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T}\left(30\tau^2 - 60\tau^3 + 30\tau^4\right) \\ \ddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^2}\left(60\tau - 180\tau^2 + 120\tau^3\right) \\ \dddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^3}\left(60 - 360\tau + 360\tau^2\right) \\ \ddddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^4}\left(-360 + 720\tau\right). \end{aligned}$$

We analyze the constraints imposed by the joint limits starting with the one with highest differential order. We will work now with *scalar* quantities, i.e., joint by joint, dropping for simplicity the joint index. The maximum jerk in the closed interval $\tau \in [0, 1]$ occurs either at the boundaries or where the fourth derivative is zero:

$$\dddot{q}(0) = \dddot{q}(1) = 60\,\frac{\Delta q}{T^3}, \qquad \ddddot{q}(\tau) = 0 \quad @\,\tau^* = 0.5 \quad \Rightarrow \quad \dddot{q}(0.5) = -30\,\frac{\Delta q}{T^3}.$$

Thus, the minimum motion time T that satisfies the jerk limit is given by

$$|\dddot{q}(\tau)| \leq J \quad \Rightarrow \quad T \geq \sqrt[3]{\frac{60|\Delta q|}{J}} =: T_J.$$

The maximum acceleration occurs where the third derivative is zero (no need to check the value at the boundaries, since we have $\ddot{q}(0) = \ddot{q}(1) = 0$ by construction):

$$\dddot{q}(\tau) = 0 \quad \Longleftrightarrow \quad 1 - 6\tau + 6\tau^2 = 0 \quad @\,\tau^* = 0.5 \pm \frac{\sqrt{3}}{6} \quad \Rightarrow \quad \ddot{q}(\tau^*) = \pm\, 5.7735\, \frac{\Delta q}{T^2}.$$

The minimum motion time T that satisfies the acceleration limit is given by

$$|\ddot{q}(\tau)| \leq A \quad \Rightarrow \quad T \geq \sqrt{\frac{5.7735|\Delta q|}{A}} =: T_A.$$

Similarly, the maximum velocity occurs where the second derivative is zero (again, no need to check the value at the boundaries, since $\dot{q}(0) = \dot{q}(1) = 0$):

$$\ddot{q}(\tau) = 0 \quad \Longleftrightarrow \quad \tau(1 - 3\tau + 2\tau^2) = 0 \quad @\,\tau^* = \{0, 0.5, 1\} \quad \Rightarrow \quad \dot{q}(0.5) = \frac{30}{16}\,\frac{\Delta q}{T},$$

and thus

$$|\dot{q}(\tau)| \leq V \quad \Rightarrow \quad T \geq \frac{30}{16}\,\frac{|\Delta q|}{V} =: T_V.$$

As a result, the minimum feasible motion time T^* is obtained as

$$T^* = \max\{T_J, T_A, T_V\} = \max\left\{3.9148\,\sqrt[3]{\frac{|\Delta q|}{J}}, 2.4028\,\sqrt{\frac{|\Delta q|}{A}}, 1.8750\,\frac{|\Delta q|}{V}\right\}.$$

Using the data (all in radians) of the problem at hand, we compute the minimum motion time for the first joint as

$$T_1^* = \max\{T_{J,1}, T_{A,1}, T_{V,1}\} = \max\{1.5792, 1.9468, 3.6926\} = 3.6926 \text{ [s]},$$

where the velocity limit is the most constraining one. Similarly, for the second joint it is

$$T_2^* = \max\{T_{J,2}, T_{A,2}, T_{V,2}\} = \max\{0.7872, 0.6619, 0.5336\} = 0.7872 \text{ [s]}$$

and the jerk will be the variable reaching first its limit. Since coordinated motion of the joints should be enforced, the common minimum motion time will be

$$T^* = \max\{T_1^*, T_2^*\} = T_1^* = 3.6926 \text{ [s]},$$

with the second joint traveling much slower than it could in principle. The trajectory profiles of position, velocity, acceleration, and jerk of the two joints are shown in Figs. 3–2.

The peak velocity of the two joints is reached at $t = T^*/2 = 1.8463$ s

$$\max_{t\in[0,T^*]} \dot{q}_1(t) = \dot{q}_1(1.8463) = 1 \text{ [rad/s]}, \qquad \max_{t\in[0,T^*]} \dot{q}_2(t) = \dot{q}_2(1.8463) = 0.2890 \text{ [rad/s]},$$

while the peak acceleration (in module) is attained at $t = \left(0.5 \pm \sqrt{3}/6\right) T^*$, namely at $t = 0.7803$ s (max positive acceleration) and $t = 2.9123$ s (max negative acceleration = max deceleration)

$$\max_{t\in[0,T^*]} |\ddot{q}_1(t)| = \ddot{q}_1(0.7803) = 0.8339 \text{ [rad/s}^2\text{]}, \qquad \max_{t\in[0,T^*]} |\ddot{q}_2(t)| = \ddot{q}_2(0.7803) = 0.2410 \text{ [rad/s}^2\text{]}.$$

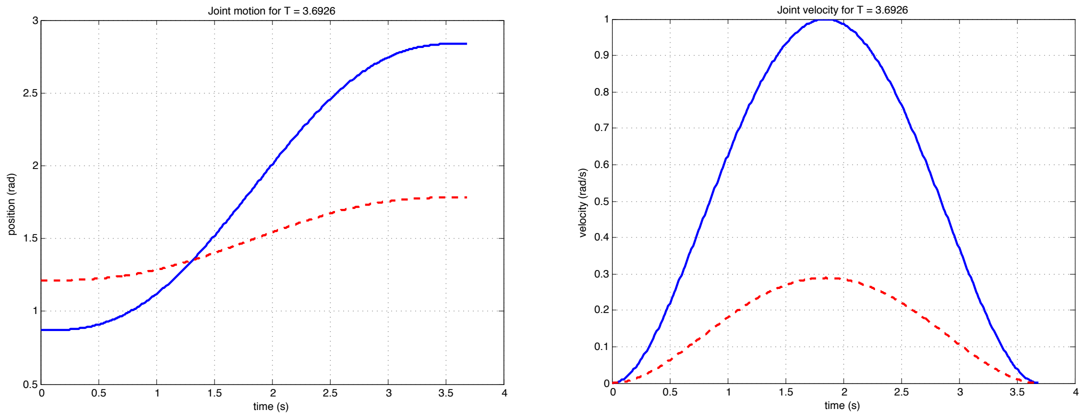


Figure 1: Position [left] and velocity [right] of joint 1 (blue, solid) and joint 2 (red, dashed)

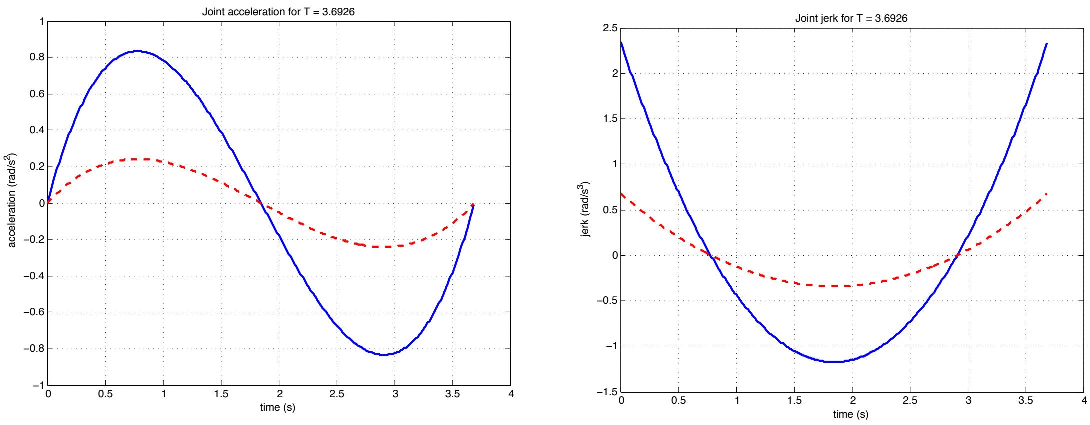


Figure 2: Acceleration [left] and jerk [right] of joint 1 (blue, solid) and joint 2 (red, dashed)

We note also that coordinated motion is symmetric w.r.t. to the total time (for all T, and thus also for T^*). Therefore, the configuration reached at $t = T^*/2$ will simply be

$$\boldsymbol{q}^* := \boldsymbol{q}\left(\frac{T^*}{2}\right) = \boldsymbol{q}_{in} + \frac{\Delta\boldsymbol{q}}{2} = \frac{\boldsymbol{q}_{in} + \boldsymbol{q}_{fin}}{2} = \begin{pmatrix} 106.25^\circ \\ 85.82^\circ \end{pmatrix} = \begin{pmatrix} 1.8544 \\ 1.4978 \end{pmatrix} \text{ [rad]}.$$

Moreover, it is

$$\dot{\boldsymbol{q}}^* := \dot{\boldsymbol{q}}\left(\frac{T^*}{2}\right) = \frac{30}{16}\frac{\Delta\boldsymbol{q}}{T^*} = \begin{pmatrix} 57.30 \\ 16.56 \end{pmatrix} \text{ [°/s]} = \begin{pmatrix} 1 \\ 0.2890 \end{pmatrix} \text{ [rad/s]}, \quad \ddot{\boldsymbol{q}}^* := \ddot{\boldsymbol{q}}\left(\frac{T^*}{2}\right) = \boldsymbol{0}.$$

The robot analytic Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial\boldsymbol{f}(\boldsymbol{q})/\partial\boldsymbol{q})$ and its time derivative $\boldsymbol{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})$,

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 s_1 + \ell_2 s_{12}) & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix}, \quad \boldsymbol{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\begin{pmatrix} \ell_1 c_1 \dot{q}_1 + \ell_2 c_{12}(\dot{q}_1 + \dot{q}_2) & \ell_2 c_{12}(\dot{q}_1 + \dot{q}_2) \\ \ell_1 s_1 \dot{q}_1 + \ell_2 s_{12}(\dot{q}_1 + \dot{q}_2) & \ell_2 s_{12}(\dot{q}_1 + \dot{q}_2) \end{pmatrix},$$

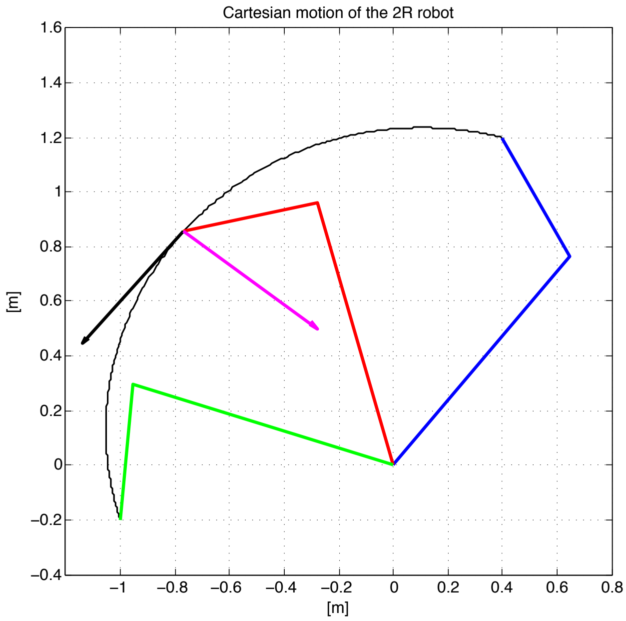


Figure 3: The planar 2R robot arm in its initial configuration (blue), at the midpoint of the trajectory (red), and in the final configuration (green), with the Cartesian path traced by its end-effector (thin blue line). The two arrows placed at the midpoint represent the end-effector velocity (black) and acceleration (magenta), respectively. Their length has been scaled by a factor 2 to fit.

take the numerical values at $\boldsymbol{q} = \boldsymbol{q}^*$, $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^*$

$$\boldsymbol{J}^* := \boldsymbol{J}(\boldsymbol{q}^*) = \begin{pmatrix} -0.8555 & 0.1045 \\ -0.7688 & -0.4890 \end{pmatrix}, \qquad \dot{\boldsymbol{J}}^* := \boldsymbol{H}(\boldsymbol{q}^*, \dot{\boldsymbol{q}}^*) = \begin{pmatrix} 0.9101 & 0.6303 \\ -0.8253 & 0.1347 \end{pmatrix}.$$

Therefore, the required Cartesian velocity and acceleration of the robot end-effector at the trajectory midpoint are

$$\dot{\boldsymbol{p}}^* = \boldsymbol{J}^* \dot{\boldsymbol{q}}^* = \begin{pmatrix} -0.8253 \\ -0.9101 \end{pmatrix} \ [\mathrm{m/s}] \qquad \ddot{\boldsymbol{p}}^* = \boldsymbol{J}^* \ddot{\boldsymbol{q}}^* + \dot{\boldsymbol{J}}^* \dot{\boldsymbol{q}}^* = \dot{\boldsymbol{J}}^* \dot{\boldsymbol{q}}^* = \begin{pmatrix} 1.0922 \\ -0.7864 \end{pmatrix} \ [\mathrm{m/s^2}].$$

Robotics I

June 6, 2016

Exercise 1

Figure 1 shows a drawing and two pictures of the Universal Robot UR5, a 6R lightweight manipulator with 5 kg of payload. The six joints are labeled as base, shoulder, elbow, wrist1, wrist2, and wrist3. The relevant kinematic lengths are reported in the figure.

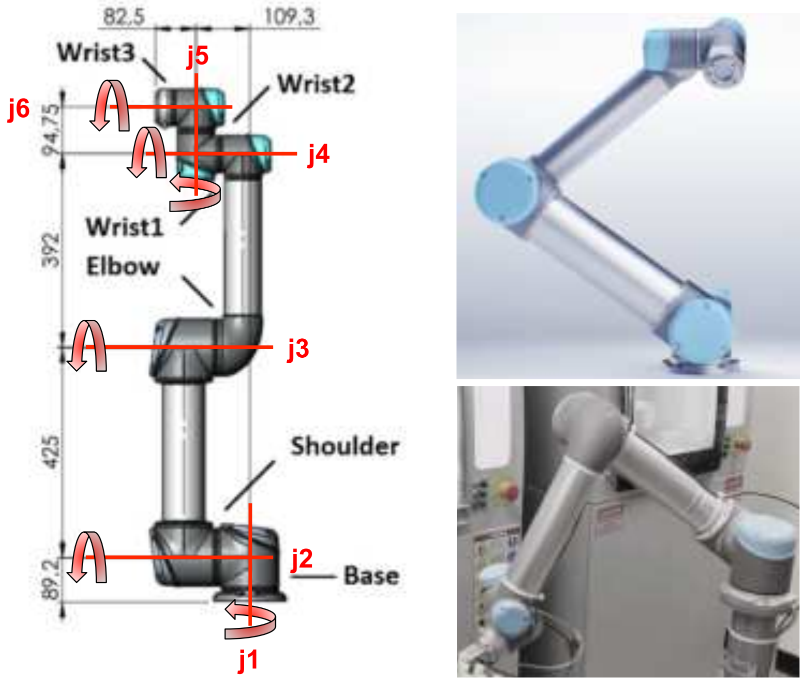


Figure 1: A drawing of the Universal Robot UR5 with the six joint axes indicated, and two views of the manipulator in action.

Assign the link frames according to the classical Denavit-Hartenberg convention and derive the associated table, providing the numerical values of the constant DH parameters. Place the origin of the reference frame (frame 0) at the robot base, and choose the last frame (frame 6) with the origin at the center of the end-effector flange and the $\boldsymbol{z}_6$ axis along the approach direction. Moreover, provide the values of the joint angles θ_i, $i = 1, \ldots, 6$, associated to the robot configuration shown in the drawing of Fig. 1.

Exercise 2

Given a planar 3R robot with equal link lengths, write a program (in pseudo-code or any preferred language) that solves the inverse kinematics in a numerical way, taking as input the desired position $\boldsymbol{p} = (p_x, p_y)$ and providing as output a solution $\boldsymbol{q}$, if one exists. Use appropriate termination/exit conditions for your algorithm. Provide also the robot kinematic functions needed by the program.

[180 minutes; open books]

Solution

June 6, 2016

Exercise 1

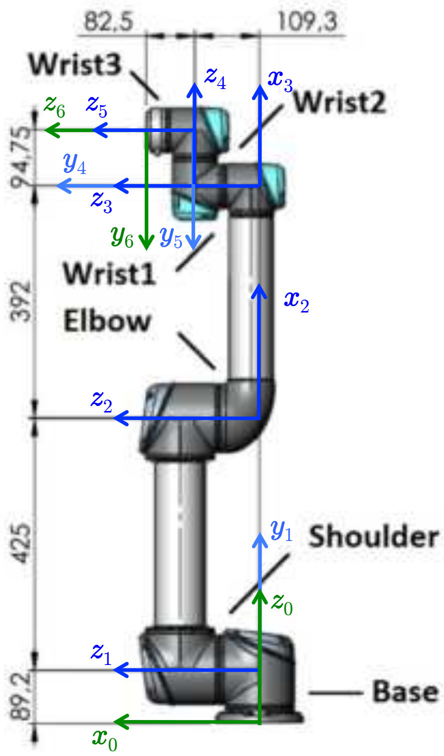


Figure 2: A possible assignment of DH frames for the UR5 robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 89.2$	$\theta_1 = \pi/2$
2	0	$a_2 = 425$	0	$\theta_2 = \pi/2$
3	0	$a_3 = 392$	0	$\theta_3 = 0$
4	$\pi/2$	0	$d_4 = 109.3$	$\theta_4 = \pi/2$
5	$-\pi/2$	0	$d_5 = 94.75$	$\theta_5 = 0$
6	0	0	$d_6 = 82.5$	$\theta_6 = 0$

Table 1: DH parameters (units in mm or rad)

A video of this robot in action can be found on YouTube (e.g., `https://youtu.be/ajCLa3YXDc0`).

* * * * *

Robotics I

July 11, 2016

Exercise 1

A robot joint should be moved between an initial value q_0 at time $t = 0$ and a desired value $q_1 > q_0$ at the final time $t = T$, starting at rest and reaching a velocity $v_1 > 0$ at the final time.

- Provide the expression of a smooth trajectory $q_d(t)$ that solves this interpolation problem.
- For this trajectory, give the conditions on the problem data under which the velocity $\dot{q}_d(t)$ remains confined in the domain $[0, v_1]$ during the whole time interval $[0, T]$.
- Using the data $q_0 = 15°$, $q_1 = 45°$, and $v_1 = 30°/s$, provide the minimum motion time T^* that can be achieved with the chosen trajectory, such that the bound $|\dot{q}_d(t)| \leq V_{max} = 90°/s$ holds for all $t \in [0, T^*]$. Sketch the obtained velocity profile.

Exercise 2

The kinematics of a 3R spatial robot is described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	0	0	0	q_3

Table 1: Denavit-Hartenberg parameters of a 3R robot.

Provide the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ between the joint velocity $\dot{\boldsymbol{q}}$ and the angular velocity $\boldsymbol{\omega}$ of the third (last) robot reference frame and determine its singularities, if any.

Exercise 3

The end effector of a planar 2R robot having links of length $\ell_1 = 1.5$ and $\ell_2 = 2$ [m] should move from point $\boldsymbol{A} = (1, 1)$ [m] to point $\boldsymbol{B} = (0.5, 1.5)$ [m] along a straight line and at constant speed $V = 1$ m/s. Denote with $\boldsymbol{p}_d(t)$ this desired Cartesian trajectory and with $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ the direct kinematics of the robot for this task.

- Determine initial values $\boldsymbol{q}(0) = \boldsymbol{q}_A$ and $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_A$ such that the desired Cartesian trajectory $\boldsymbol{p}_d(t)$ can be traced exactly right from the initial time $t = 0$.
- Assume $\boldsymbol{q}(0) \neq \boldsymbol{q}_A$, so that an initial end-effector position error $\boldsymbol{e}_P(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) \neq \boldsymbol{0}$ results. Provide a kinematic control law at the joint velocity level such that the error $\boldsymbol{e}_P(t)$ is reduced in norm to less than 5% of its initial value before reaching halfway of the nominal Cartesian motion, and eventually vanishes. Disregard the possible presence of singularities.

[210 minutes; open books]

Solution

June 6, 2016

Exercise 1

Choosing the class of cubic polynomial trajectories and imposing the four boundary conditions

$$q_d(0) = q_0, \qquad q_d(0) = q_1, \qquad \dot{q}_d(T) = 0, \qquad \dot{q}_d(T) = v_1,$$

yields the unique trajectory

$$q_d(t) = q_0 + (q_1 - q_0)\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right) + v_1 T\left(\left(\frac{t}{T}\right)^3 - \left(\frac{t}{T}\right)^2\right), \qquad t \in [0, T]. \tag{1}$$

The first and second time derivatives of (1) are

$$\dot{q}_d(t) = \frac{6(q_1 - q_0)}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right) + v_1\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)\right),$$

and

$$\ddot{q}_d(t) = \frac{6(q_1 - q_0)}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right) + \frac{v_1}{T}\left(6\left(\frac{t}{T}\right) - 2\right).$$

In particular, the initial and final accelerations are

$$\ddot{q}_d(0) = \frac{1}{T^2}\left(6(q_1 - q_0) - 2v_1 T\right), \qquad \ddot{q}_d(T) = \frac{1}{T^2}\left(4v_1 T - 6(q_1 - q_0)\right).$$

In order to have $\dot{q}(t) \in [0, v_1]$, for all $t \in [0, T]$, the velocity (a quadratic polynomial) should always be increasing in the closed interval of motion, with its maximum reached at the final boundary instant. As a consequence, the (linear) acceleration profile should always remain positive in the (open) interval $(0, T)$. This occurs if and only if both $\ddot{q}_d(0)$ and $\ddot{q}_d(T)$ are non-negative, i.e.,

$$6(q_1 - q_0) - 2v_1 T \geq 0, \qquad 4v_1 T - 6(q_1 - q_0) \geq 0,$$

or

$$v_1 \leq 3\,\frac{q_1 - q_0}{T} \leq 2v_1, \tag{2}$$

which is the sought condition.

Substituting now the given data values yields the acceleration profile

$$\ddot{q}_d(t) = \frac{180°}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right) + \frac{30°}{T}\left(6\left(\frac{t}{T}\right) - 2\right) = \frac{180°}{T^2}\left((T - 2)\left(\frac{t}{T}\right) + \left(1 - \frac{T}{3}\right)\right),$$

which crosses zero at the instant t^* (conveniently normalized as $\tau^* = t^*/T$)

$$\ddot{q}_d(t^*) = 0 \qquad \Rightarrow \qquad \tau^* = \frac{t^*}{T} = \frac{(T/3) - 1}{T - 2},$$

being $\tau^* \in (0, 1)$ only when $T \notin (1.5, 3)$. The maximum (absolute value) of the velocity in the closed time interval $[0, T]$ occurs either at one of the two boundaries or at t^*, but only when this instant is inside $[0, T]$. Since $\dot{q}_d(0) = 0$ and $\dot{q}_d(T) = v_1 = 30°/\text{s} < 90°/\text{s} = V_{max}$, only the last case is of interest for reaching the maximum velocity bound. From

$$\begin{aligned}\dot{q}_d(t) &= \frac{180°}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right) + 30°\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)\right) \\ &= \frac{180°}{T}\left(\frac{t}{T}\right)\left(\left(\frac{T}{2} - 1\right)\left(\frac{t}{T}\right) + \left(1 - \frac{T}{3}\right)\right),\end{aligned}$$

we have

$$\dot{q}_d(t^*) = \frac{180°}{T}\frac{(T/3) - 1}{T - 2}\left(\left(\frac{T}{2} - 1\right)\frac{(T/3) - 1}{T - 2} + \left(1 - \frac{T}{3}\right)\right) = \frac{90°\,(1 - (T/3))^2}{T(2 - T)}.$$

Setting $|\dot{q}_d(t^*)| = V_{max} = 90°/\text{s}$ leads to

$$\frac{(1 - (T/3))^2}{T\,|2 - T|} = 1.$$

When looking for the minimum feasible value T^* for T, we can assume for the time being that $T < 2$ and eliminate the need for the absolute value. Thus, we solve

$$(1 - (T/3))^2 = T(2 - T) \quad \Rightarrow \quad \frac{10}{9}T^2 - \frac{8}{3}T + 1 = 0 \quad \Rightarrow \quad T_{1,2} = \{0.4652, 1.9348\},$$

obtaining as the minimum feasible motion time

$$T^* = 0.4652 \text{ s.} \tag{3}$$

Indeed, the minimum time found satisfies $T^* \notin (1.5, 3)$, and is also consistent with the assumption made ($T^* < 2$). According to (3), the instant $t^* \in [0, T^*]$ when the maximum velocity $V_{max} = 90°/\text{s}$ is reached is

$$t^* = T^*\,\frac{(T^*/3) - 1}{T^* - 2} = 0.2561 \text{ s.}$$

Figure 1 shows the obtained position, velocity, and acceleration profiles.

Exercise 2

The requested mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ involves the angular part of the geometric Jacobian, usually expressed in the base frame 0. Since the robot has three revolute joints, it is

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2){}^2\boldsymbol{z}_2 \end{pmatrix},$$

with ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for $i = 0, 1, 2$. From Tab. 1, the symbolic expressions of the DH rotation matrices are

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 \\ \sin q_2 & 0 & -\cos q_2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus

$${}^0\boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{z}_1 = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}, \quad {}^0\boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2){}^2\boldsymbol{z}_2\right) = \begin{pmatrix} \cos q_1 \sin q_2 \\ \sin q_1 \sin q_2 \\ -\cos q_2 \end{pmatrix},$$

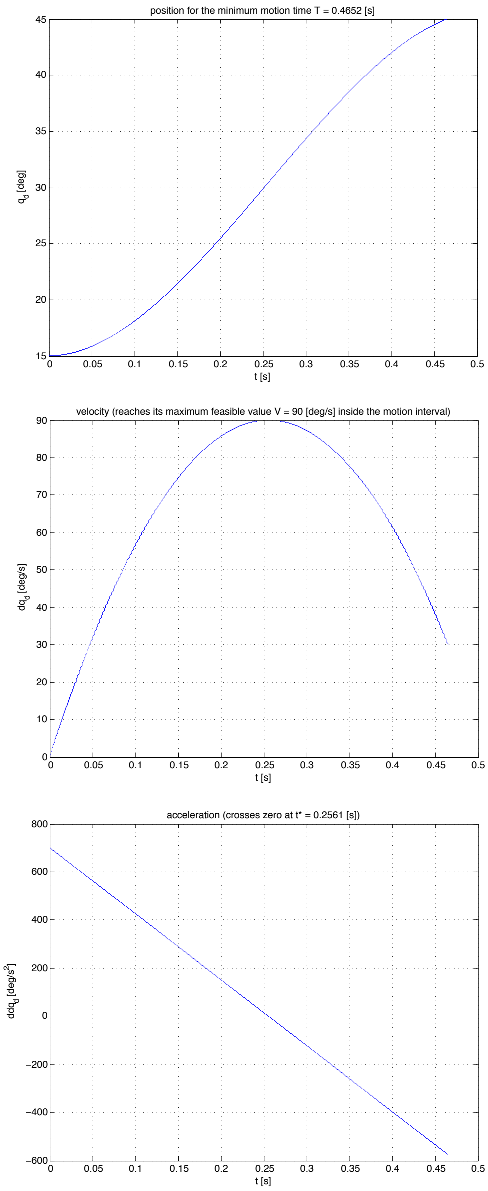


Figure 1: Minimum time position, velocity, and acceleration profiles for Exercise 1.

and the 3×3 Jacobian is

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & \sin q_1 & \cos q_1 \sin q_2 \\ 0 & -\cos q_1 & \sin q_1 \sin q_2 \\ 1 & 0 & -\cos q_2 \end{pmatrix},$$

which is singular when

$$\det \boldsymbol{J}_A(\boldsymbol{q}) = \sin q_2 = 0 \qquad \Rightarrow \qquad q_2 = \{0, \pm\pi\}.$$

Exercise 3

Both points $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to the (interior of the) robot workspace, being

$$\|\boldsymbol{A}\| = \sqrt{2}, \qquad \|\boldsymbol{B}\| = \sqrt{2.5}, \quad \text{and} \quad WS = \left\{\boldsymbol{p} \in \mathbb{R}^2 : 0.5 = |\ell_1 - \ell_2| \leq \|\boldsymbol{p}\| \leq \ell_1 + \ell_2 = 2.5\right\}.$$

It is easy to see that the whole linear path from $\boldsymbol{A}$ to $\boldsymbol{B}$ will also belong to the workspace interior. As a result, no singularities are encountered along the nominal Cartesian trajectory. Being $L = \|\boldsymbol{B} - \boldsymbol{A}\| = 1/sqrt2 = 0.7071$ [m] the length of the path from $\boldsymbol{A}$ to $\boldsymbol{B}$, a motion at the constant speed $V = 1$ [m/s] will reach halfway of this path at $t = T_{mid} = 0.5L/V = 0.3536$ [s]. Moreover, the desired Cartesian velocity at the initial point $\boldsymbol{A}$ is

$$\dot{\boldsymbol{p}}_d(0) = V \frac{\boldsymbol{B} - \boldsymbol{A}}{\|\boldsymbol{B} - \boldsymbol{A}\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}.$$

The direct kinematics of the 2R planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \end{pmatrix}$$

and its Jacobian is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2)\right) & -\ell_2 \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) & \ell_2 \cos(q_1 + q_2) \end{pmatrix}.$$

Solving the inverse kinematics problem —see the textbook— at the point $\boldsymbol{A}$ gives the two solutions (R = right arm solution, L = left arm solution)

$$\boldsymbol{q}_{A,R} = \begin{pmatrix} -0.7264 \\ 2.3579 \end{pmatrix}, \qquad \boldsymbol{q}_{A,L} = \begin{pmatrix} 2.2972 \\ -2.3579 \end{pmatrix}. \tag{4}$$

These are the two robot configurations $\boldsymbol{q}(0)$ at $t = 0$ that can be associated to the starting point of the desired Cartesian trajectory. In these two configurations, the robot Jacobian takes respectively the values

$$\boldsymbol{J}(\boldsymbol{q}_{A,R}) = \begin{pmatrix} -1 & -1.9963 \\ 1 & -0.1213 \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}_{A,L}) = \begin{pmatrix} -1 & 0.1213 \\ 1 & 1.9963 \end{pmatrix},$$

so that the initial joint velocity $\dot{\boldsymbol{q}}(0)$ at $t = 0$ needed for tracing exactly the desired Cartesian trajectory can be computed as

$$\dot{\boldsymbol{q}}_A = \boldsymbol{J}^{-1}(\boldsymbol{q}_{A,R})\, \dot{\boldsymbol{p}}_d(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_{A,L})\, \dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix},$$

with a *single* common value in this very particular case[1].

[1] In general, two different joint velocities $\dot{\boldsymbol{q}}_{A.R}$ and $\dot{\boldsymbol{q}}_{A.L}$ will be found that execute the same Cartesian velocity $\dot{\boldsymbol{p}}_d(0)$ from two different solutions of the inverse kinematics problem. In this case, the uniqueness of the obtained joint velocity depends on the particular direction of the desired Cartesian velocity specified by the problem.

To recover an initial position error of the robot end-effector with respect to the desired trajectory, $\boldsymbol{e}_P(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) \neq \boldsymbol{0}$, the following Cartesian kinematic control law can be used for the joint velocity commands

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right)\right), \qquad \boldsymbol{K}_p = k_p \boldsymbol{I}_{2\times 2} > 0.$$

This imposes the following dynamics to the Cartesian position error:

$$\dot{\boldsymbol{e}}_P = -k_p \boldsymbol{e}_P \quad \Rightarrow \quad \boldsymbol{e}_P(t) = \exp(-k_p t)\, \boldsymbol{e}_P(0) \quad \Rightarrow \quad \|\boldsymbol{e}_P(t)\| = \exp(-k_p t)\, \|\boldsymbol{e}_P(0)\|.$$

In order to have a reduction in norm of the position error to less than 5% of its initial value by the time $t = T_{mid} = 0.3536$ [s], we need to have

$$\exp(-0.3536\, k_p) \leq \exp(-3) \simeq 0.0498 < 0.05 \qquad \Rightarrow \qquad k_p \geq \frac{3}{0.3536} = 8.48. \tag{5}$$

* * * * *

Robotics I

September 12, 2016

Exercise 1

The last three revolute joints (labeled from 4 to 6) of the 6-dof Universal Robot UR10 constitute a non-spherical wrist and are described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i (mm)	θ_i
4	$-\pi/2$	0	$d_4 = 163.9$	q_4
5	$\pi/2$	0	$d_5 = 115.7$	q_5
6	0	0	$d_6 = 92.2$	q_6

Table 1: Denavit-Hartenberg parameters of the non-spherical wrist of the UR10 robot.

- Provide the analytic expressions of the inverse kinematic mapping, which takes as input a desired orientation of the (end-effector) frame 6, as expressed by a rotation matrix $\boldsymbol{R}$, and provides as output *all* solutions for the wrist angles (q_4, q_5, q_6) in the regular case. Characterize also the singular cases, and explain what happens in such situations.
- Apply your formulas to solve the inverse kinematics for the UR10 robot wrist, given the following numerical input:

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Exercise 2

Consider the planar RPR robot in Fig. 1. The prismatic axis of the second joint is skewed by an angle $\beta = 45°$ with respect to the first link.

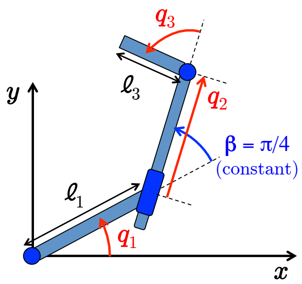


Figure 1: A planar RPR robot with its joint coordinates q_1, q_2 and q_3.

- Using the coordinates shown, provide the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ that relates $\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{pmatrix}^T$ to the velocity $\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{p}_x & \dot{p}_y \end{pmatrix}^T$ of the end effector and find the singularities of this mapping.
- Let the robot be at $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0.2 & -\pi/4 \end{pmatrix}^T$ [rad,m,rad], with kinematic data $\ell_1 = 1$ and $\ell_3 = 0.5$ [m]. For a desired end-effector velocity $\dot{\boldsymbol{p}}_d = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$ [m/s], determine numerically
  - the minimum norm (least squares) solution $\dot{\boldsymbol{q}}_{LS}$;
  - another solution $\dot{\boldsymbol{q}}_0 \neq \dot{\boldsymbol{q}}_{LS}$, such that $\boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{p}}_d$.

[120 minutes; open books]

Solution

September 12, 2016

Exercise 1

From Tab. 1, we build the rotation matrices

$$
{}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 \\ \sin q_4 & 0 & \cos q_4 \\ 0 & -1 & 0 \end{pmatrix}, \qquad {}^4\boldsymbol{R}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 \\ \sin q_5 & 0 & -\cos q_5 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} \cos q_6 & -\sin q_6 & 0 \\ \sin q_6 & \cos q_6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Using the usual compact notation for trigonometric functions, the orientation of the end-effector frame expressed w.r.t. frame 3 of the UR10 robot (which is taken here as reference frame for the wrist kinematics) is given by

$$
{}^4\boldsymbol{R}_6(\boldsymbol{q}) = {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5)\,{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 \\ c_4s_6 + s_4c_5c_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{pmatrix}, \tag{1}
$$

where $\boldsymbol{q} = \begin{pmatrix} q_4 & q_5 & q_6 \end{pmatrix}^T$.

Let R_{ij} $(i, j = 1, 2, 3)$ be the elements of the desired orientation matrix $\boldsymbol{R}$. We solve then the matrix equation ${}^4\boldsymbol{R}_6(\boldsymbol{q}) = \boldsymbol{R}$ by inspecting the structure of the scalar elements in (1). It is easy to see that

$$
q_5 = \text{ATAN2}\left\{\pm\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right\}, \tag{2}
$$

providing in the regular case two solutions q_5^+ and q_5^- (with equal modulus and opposite signs). Provided that $R_{13}^2 + R_{23}^2 = \sin q_5 \neq 0$, namely that $q_5 \neq 0$ and $\neq \pi$ as a result of (2), we can solve for the other two angles in an unique way as

$$
q_4 = \text{ATAN2}\left\{\frac{R_{23}}{\sin q_5^\pm}, \frac{R_{13}}{\sin q_5^\pm}\right\}, \qquad q_6 = \text{ATAN2}\left\{\frac{R_{32}}{\sin q_5^\pm}, \frac{-R_{31}}{\sin q_5^\pm}\right\}, \tag{3}
$$

yielding the two pairs (q_4^+, q_6^+) and (q_4^-, q_6^-), associated respectively to the two choices q_5^+ and q_5^- in (2).

In the singular case, $\sin q_5 = 0, \cos q_5 = \pm 1$, only the sum or the difference of the two other joint angles will be defined.

When the formulas (2–3) are applied to the desired orientation $\boldsymbol{R}$, they yield the two solutions

$$
\boldsymbol{q}^+ = \begin{pmatrix} \pi/2 \\ 3\pi/4 \\ \pi \end{pmatrix} = \begin{pmatrix} 1.5708 \\ 2.3562 \\ 3.1416 \end{pmatrix}, \qquad \boldsymbol{q}^- = \begin{pmatrix} -\pi/2 \\ -3\pi/4 \\ 0 \end{pmatrix}. \tag{4}
$$

Exercise 2

For a generic skew angle β, the direct kinematics of the RPR planar robot in Fig. 1 is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 \cos q_1 + q_2 \cos(\beta + q_1) + \ell_3 \cos(\beta + q_1 + q_3) \\ \ell_1 \sin q_1 + q_2 \sin(\beta + q_1) + \ell_3 \sin(\beta + q_1 + q_3) \end{pmatrix}$$

and its Jacobian $\boldsymbol{J} = \partial \boldsymbol{f} / \partial \boldsymbol{q}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_1 \sin q_1 - q_2 \sin(\beta + q_1) - \ell_3 \sin(\beta + q_1 + q_3) & \cos(\beta + q_1) & -\ell_3 \sin(\beta + q_1 + q_3) \\ \ell_1 \cos q_1 + q_2 \cos(\beta + q_1) + \ell_3 \cos(\beta + q_1 + q_3) & \sin(\beta + q_1) & \ell_3 \cos(\beta + q_1 + q_3) \end{pmatrix}. \tag{5}$$

To find the singularities of the differential kinematics, namely the configurations where the resulting matrix $\boldsymbol{J}(\boldsymbol{q})$ loses rank, we compute the three minors obtained by deleting, respectively, the third, second, or first column of $\boldsymbol{J}(\boldsymbol{q})$. We obtain

$$\begin{aligned} \det \boldsymbol{J}_{[-3]} &= -\left(q_2 + \ell_1 \cos\beta + \ell_3 \cos q_3\right), \\ \det \boldsymbol{J}_{[-2]} &= -\ell_3 \left(\ell_1 \sin(\beta + q_3) + q_2 \sin q_3\right), \\ \det \boldsymbol{J}_{[-1]} &= \ell_3 \cos q_3. \end{aligned}$$

All three determinants are simultaneously equal to zero if and only if

$$\cos q_3 = 0, \qquad q_2 = -\ell_1 \cos\beta.$$

When this happens, the rank of the Jacobian $\boldsymbol{J}$ in (5) falls down to 1. If we plug in now the given value $\beta = \pi/4$, we find the singularity at $q_2 = -\ell_1\sqrt{2}/2$, $q_3 = \pm\pi/2$ (for any value of q_1)[1].

Next, at the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0.2 & -\pi/4 \end{pmatrix}^T$ and with the kinematic data $\beta = \pi/4$, $\ell_1 = 1$, and $\ell_3 = 0.5$, the Jacobian in (5) becomes

$$\boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -1.6414 & -0.7071 & -0.5 \\ -0.1414 & 0.7071 & 0 \end{pmatrix},$$

which is of full rank. For $\dot{\boldsymbol{p}}_d = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$, the minimum norm solution is obtained using the pseudoinverse of the Jacobian[2]

$$\dot{\boldsymbol{q}}_{LS} = \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.5185 \\ 0.1037 \\ 0.1512 \end{pmatrix}. \tag{6}$$

Other solutions can be obtained in many ways. For instance, when 'freezing' the prismatic joint ($\dot{q}_2 = 0$) we would still have a non-singular sub-Jacobian $\boldsymbol{J}_{[-2]}(\boldsymbol{q}_0)$. Thus, by computing

$$\begin{pmatrix} \dot{q}_{0,1} \\ \dot{q}_{0,3} \end{pmatrix} = \boldsymbol{J}_{[-2]}^{-1}(\boldsymbol{q}_0)\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

[1]Another notable case is when $\beta = \pm\pi/2$, i.e., the prismatic joint is orthogonal to the first link. In that case, the singularity occurs when $q_2 = 0$ and $q_3 = \pm\pi/2$.

[2]Note that the units of the solution vector in (6) are non-homogeneous, namely [rad/s] for the first and third joints and [m/s] for the second joint. In this context, the concept of (unweighted) norm is not a properly defined one. Nonetheless, the use of a pseudoinverse solution is still a common practice even in such cases.

a different feasible solution is obtained as

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} \dot{q}_{0,1} \\ 0 \\ \dot{q}_{0,3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}. \tag{7}$$

Note that only the last joint is eventually used in this case in order to realize the desired end-effector motion. Indeed, the norm of this joint velocity, $\|\dot{\boldsymbol{q}}_0\| = 2$, is larger than the one of the pseudoinverse solution, $\|\dot{\boldsymbol{q}}_{LS}\| = 0.55$.

* * * * *

Robotics I

October 28, 2016

Exercise 1

Consider the following matrix

$$
{}^{A}\boldsymbol{R}_B(\rho,\sigma) = \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho\cos\sigma & \cos\rho\cos\sigma & -\sin\sigma \\ \sin\rho\sin\sigma & \cos\rho\sin\sigma & \cos\sigma \end{pmatrix}.
$$

- Prove that this is a rotation matrix (representing thus the orientation of a frame B with respect to a fixed frame A) for any value of the pair of angles (ρ, σ).
- Which is the sequence of two elementary rotations around *fixed* coordinate axes providing ${}^{A}\boldsymbol{R}_B(\rho,\sigma)$?
- Which is the sequence of two elementary rotations around *moving* coordinate axes providing ${}^{A}\boldsymbol{R}_B(\rho,\sigma)$?
- Verify your statements for $\rho = 90°$ and $\sigma = -90°$.

Exercise 2

Consider the planar 2R robot in Fig. 1, having link lengths $\ell_1 = 0.8$ and $\ell_2 = 0.6$ [m], and let the direct kinematic mapping that characterizes the position of its end-effector be defined as $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$. The motion of this robot is controlled by specifying the joint accelerations $\ddot{\boldsymbol{q}}$.

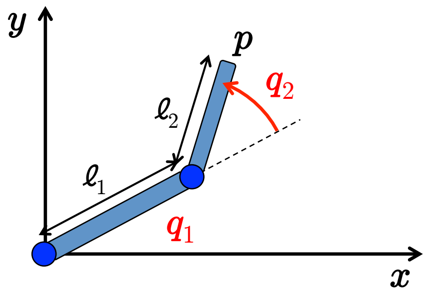


Figure 1: A planar 2R robot.

- What is the expression of the nominal joint acceleration command $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_d$ when the robot is in a state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and its end-effector needs to move instantaneously with a desired acceleration $\ddot{\boldsymbol{p}}_d$? Try out your expression by determining the numerical value of $\ddot{\boldsymbol{q}}_d$ at the time instant $t = \bar{t} = 0.8$ [s], when

$$
\boldsymbol{q}(\bar{t}) = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}(\bar{t}) = \begin{pmatrix} -\pi \\ \pi \end{pmatrix} \text{ [rad/s]}, \qquad \boldsymbol{p}_d(t) = \begin{pmatrix} 0 \\ 0.6\,(t^3 - 1) \end{pmatrix} \text{ [m]}.
$$

- For the same end-effector trajectory specified above, assume now that, at time $t = 0$, the robot is in an initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ such that $\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0)$, but $\dot{\boldsymbol{p}}(0) \neq \dot{\boldsymbol{p}}_d(0)$. What should be the expression of the feedback control law for the joint acceleration $\ddot{\boldsymbol{q}}$ in order to recover the initial Cartesian trajectory error over time, achieving thus asymptotic trajectory tracking? Define all the needed terms and parameters in this second-order kinematic control law, and determine accordingly the initial numerical value $\ddot{\boldsymbol{q}}(0)$ of the control law.

[150 minutes; open books]

Solution

October 28, 2016

Exercise 1

It is easy to verify that the given matrix ${}^A\boldsymbol{R}_B(\rho,\sigma)$ is a rotation matrix: for any pair (ρ,σ), its three columns are of unitary norm and orthogonal each to other, while $\det {}^A\boldsymbol{R}_B(\rho,\sigma) = +1$. Moreover, matrix ${}^A\boldsymbol{R}_B(\rho,\sigma)$ is obtained as the product of two elementary rotation matrices in the form

$$\boldsymbol{R}_{\boldsymbol{x}}(\sigma)\boldsymbol{R}_{\boldsymbol{z}}(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sigma & -\sin\sigma \\ 0 & \sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^A\boldsymbol{R}_B(\rho,\sigma).$$

Therefore, it represents

- either a sequence of two rotations around *fixed* axes: first a rotation by ρ around the $\boldsymbol{z}$-axis, and then a rotation by σ around the original $\boldsymbol{x}$-axis;
- or, a sequence of two rotations around *moving* axes: first a rotation by σ around the $\boldsymbol{x}$-axis, and then a rotation by ρ around the already rotated $\boldsymbol{z}$-axis (i.e., $\boldsymbol{z}'$).

By substituting $\rho = \pi/2$ and $\sigma = -\pi/2$, we obtain

$${}^A\boldsymbol{R}_B(\pi/2,-\pi/2) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{R}_{\boldsymbol{x}}(-\pi/2)\boldsymbol{R}_{\boldsymbol{z}}(\pi/2).$$

Considering for example the case of moving axes, the first (clockwise) rotation by $\sigma = -\pi/2$ keeps the $\boldsymbol{x}$-axis unchanged, $\boldsymbol{x}' \equiv \boldsymbol{x}$, while $\boldsymbol{y}' \equiv -\boldsymbol{z}$ and $\boldsymbol{z}' \equiv \boldsymbol{y}$; the second (counterclockwise) rotation by $\rho = \pi/2$ keeps the current $\boldsymbol{z}'$-axis unchanged, $\boldsymbol{z}'' \equiv \boldsymbol{z}'$, while $\boldsymbol{x}'' \equiv \boldsymbol{y}'$ and $\boldsymbol{y}'' \equiv -\boldsymbol{x}'$; concatenating the two rotations, we obtain $\boldsymbol{x}'' \equiv -\boldsymbol{z}$, $\boldsymbol{y}'' \equiv -\boldsymbol{x}$, and $\boldsymbol{z}'' \equiv \boldsymbol{y}$, which is ${}^A\boldsymbol{R}_B(\pi/2,-\pi/2)$ as expected.

Exercise 2

The following piece of Matlab code summarizes the computations needed to answer to the first question:

```
% first question
tbar=0.8;
ddp1=0;ddp2=3.6*tbar;
ddpd=[ddp1;ddp2]  % outputs the desired Cartesian acceleration at time t=0.8 s
% current state
q1=0;q2=pi/2;
dq1=-pi;dq2=pi;
dq=[dq1; dq2];
% direct kinematics
p=[l1*cos(q1)+l2*cos(q1+q2);
     l1*sin(q1)+l2*sin(q1+q2)];
% Jacobian matrix
J=[-l1*sin(q1)-l2*sin(q1+q2) -l2*sin(q1+q2);
    l1*cos(q1)+l2*cos(q1+q2)  l2*cos(q1+q2)];
% time derivative of the Jacobian
dJ=[-l1*cos(q1)*dq1-l2*cos(q1+q2)*(dq1+dq2) -l2*cos(q1+q2)*(dq1+dq2);
    -l1*sin(q1)*dq1-l2*sin(q1+q2)*(dq1+dq2) -l2*sin(q1+q2)*(dq1+dq2)];
ddqd=inv(J)*(ddpd - dJ*dq) % outputs the requested joint acceleration command
% end
```

The two resulting outputs of this code are

$$\ddot{\boldsymbol{p}}_d(0.8) = \begin{pmatrix} 0 \\ 2.88 \end{pmatrix} \text{ [m/s}^2\text{]}, \qquad \ddot{\boldsymbol{q}}_d(0.8) = \begin{pmatrix} 3.6 \\ -16.7595 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

Similarly, at time $t = 0$ we request

$$\boldsymbol{p}_d(0) = \left. \begin{pmatrix} 0 \\ 0.6\,(t^3 - 1) \end{pmatrix} \right|_{t=0} = \begin{pmatrix} 0 \\ -0.6 \end{pmatrix} \text{ [m]}$$

and

$$\dot{\boldsymbol{p}}_d(0) = \left. \begin{pmatrix} 0 \\ 1.8\,t^2 \end{pmatrix} \right|_{t=0} = \mathbf{0} \text{ [m/s]}, \qquad \ddot{\boldsymbol{p}}_d(0) = \left. \begin{pmatrix} 0 \\ 3.6\,t \end{pmatrix} \right|_{t=0} = \mathbf{0} \text{ [m/s}^2\text{]}.$$

The robot should be in an initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ such that $\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0)$, but $\dot{\boldsymbol{p}}(0) \neq \dot{\boldsymbol{p}}_d(0)$. To determine $\boldsymbol{q}(0)$, we solve the inverse kinematics for $\boldsymbol{p}_d(0)$, picking just one of the two solutions (in an arbitrary way):

```
pd0=[0; -0.6];
% second joint computations
c2=(pd0(1)^2+pd0(2)^2-l1^2-l2^2)/(2*l1*l2);
s2=sqrt(1-c2^2);  %other solution: -sqrt(1-c2^2)
% first joint computations
det=l1^2+l2^2+2*l1*l2*c2;
s1=(pd0(2)*(l1+l2*c2)-pd0(1)*l2*s2)/det;
c1=(pd0(1)*(l1+l2*c2)+pd0(2)*l2*s2)/det;
% output
q01=atan2(s1,c1);
q02=atan2(s2,c2);
q0=[q01; q02]
```

We note that the desired Cartesian position is strictly inside the workspace of the 2R robot, so that we are away from kinematic singularities. The output of the above code gives

$$\boldsymbol{q}(0) = \begin{pmatrix} -2.4119 \\ 2.3005 \end{pmatrix} \text{ [rad]} = \begin{pmatrix} -138.19 \\ 131.81 \end{pmatrix} \text{ [deg]},$$

yielding no initial Cartesian position error at $t = 0$, $\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) = \mathbf{0}$, as desired. In order to be sure that $\dot{\boldsymbol{p}}(0) = \boldsymbol{J}(\boldsymbol{q}(0))\dot{\boldsymbol{q}}(0) \neq \dot{\boldsymbol{p}}_d(0) = \mathbf{0}$, we just need to avoid the specific choice $\dot{\boldsymbol{q}}(0) = \mathbf{0}$. For example, by choosing

$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 0.5 \\ 0.1 \end{pmatrix} \text{ [rad/s]} \qquad \Rightarrow \qquad \dot{\boldsymbol{e}}(0) = \dot{\boldsymbol{p}}_d(0) - \dot{\boldsymbol{p}}(0) = -\boldsymbol{J}(\boldsymbol{q}(0))\,\dot{\boldsymbol{q}}(0) = \begin{pmatrix} -0.3067 \\ -0.0596 \end{pmatrix} \text{ [m/s]}.$$

To recover any initial Cartesian trajectory error (in velocity and/or position) over time and achieve thus asymptotic trajectory tracking, the control law for the joint acceleration input should be chosen as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\ddot{\boldsymbol{p}}_d + \boldsymbol{K}_d \left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \right) + \boldsymbol{K}_p \left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}) \right) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right),$$

with (symmetric) gain matrices $\boldsymbol{K}_p > 0$, $\boldsymbol{K}_d > 0$ (a PD feedback action). By choosing for instance

$$\boldsymbol{K}_p = 100 \cdot \boldsymbol{I}_{2\times 2}, \qquad \boldsymbol{K}_d = 20 \cdot \boldsymbol{I}_{2\times 2},$$

we finally obtain at time $t = 0$

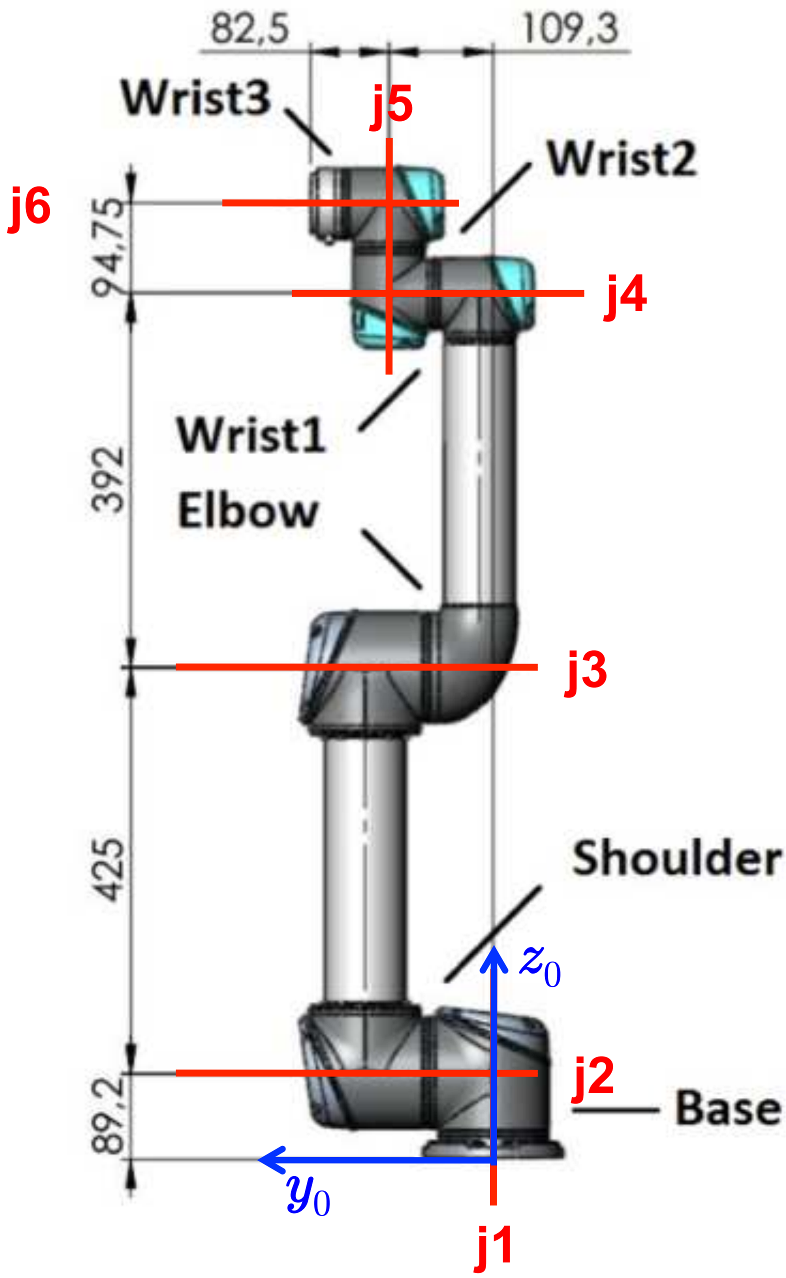
$$\ddot{\boldsymbol{q}}(0) = \begin{pmatrix} -9.8614 \\ -2.2639 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

* * * * *

Robotics I - Sheet for Exercise 1

Midterm test in classroom – November 18, 2016

LAST NAME, First Name ____________________



Draw all the Denavit-Hartenberg frames associated to the robot links according to Tab. 1 in the text.

Robotics I

Midterm test in classroom – November 18, 2016

Exercise 1 [10 points]

Figure 1 shows the 6R Universal Robot UR5, with a non-spherical wrist, and two axes of the reference frame RF_0 placed at the robot base. The Denavit-Hartenberg parameters are given in Tab. 1, together with the numerical values for the constant parameters and the current values that the joint variables assume in the shown configuration.

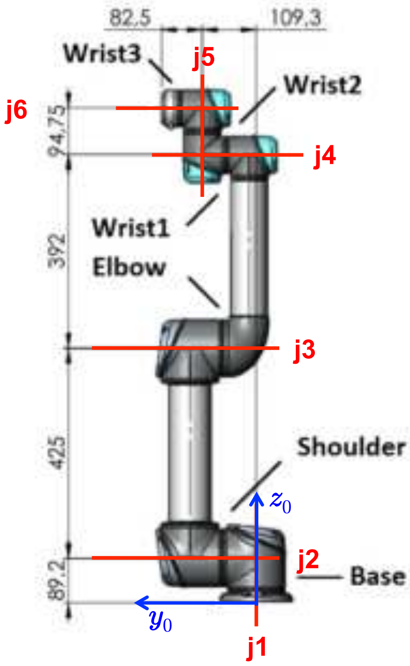


Figure 1: The 6R Universal Robot UR5 and the chosen base frame.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 = 89.2$	$\theta_1 = 0$
2	0	$a_2 = -425$	0	$\theta_2 = \pi/2$
3	0	$a_3 = -392$	0	$\theta_3 = 0$
4	$\pi/2$	0	$d_4 = 109.3$	$\theta_4 = -\pi/2$
5	$-\pi/2$	0	$d_5 = 94.75$	$\theta_5 = 0$
6	0	0	$d_6 = 82.5$	$\theta_6 = 0$

Table 1: DH parameters (in mm or rad), with the value of $\boldsymbol{\theta} \in \mathbb{R}^6$ in the shown configuration.

Using the provided sheet (please write your full name there!), draw all the Denavit-Hartenberg frames associated to the robot links according to Tab. 1.

Exercise 2 [5 points]

A frame $RF_B = \{\boldsymbol{O}_B, \boldsymbol{x}_B, \boldsymbol{y}_B, \boldsymbol{z}_B\}$ is displaced and rotated with respect to a fixed reference frame $RF_A = \{\boldsymbol{O}_A, \boldsymbol{x}_A, \boldsymbol{y}_A, \boldsymbol{z}_A\}$. The displacement is represented by the vector

$$^{A}\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} = \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}^T \qquad \text{[m]},$$

while the orientation of RF_B with respect to RF_A is represented by the following sequence of three Euler $ZY'X''$ angles

$$\alpha = \frac{\pi}{4}, \qquad \beta = -\frac{\pi}{2}, \qquad \gamma = 0 \qquad \text{[rad]}.$$

For a given point P, provide the value of vector $^{A}\boldsymbol{p}_{\boldsymbol{O}_A P}$ knowing that its position with respect to frame RF_B is given by

$$^{B}\boldsymbol{p}_{\boldsymbol{O}_B P} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T \qquad \text{[m]}.$$

Exercise 3 [10 points]

Consider the 2-dof robot in Fig. 2, with two revolute joints having axes (the first vertical and the second horizontal) that do not intercept.

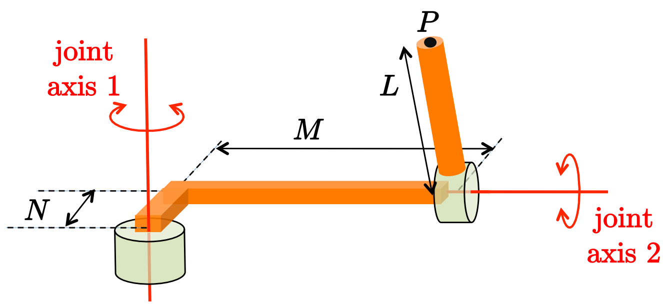


Figure 2: A 2R robot moving in the 3D space.

- Assign the frames according to the Denavit-Hartenberg convention and define the associated table of parameters. Provide the specific expression of the homogenous transformation matrices between the successive frames that you have assigned.
- Determine the symbolic expression of the position vector $^{0}\boldsymbol{p}_{OP}$ of point P in the chosen frame RF_0, and find its numerical value when the kinematic quantities are $L = 1$, $M = 2$, $N = 0.3$ [m] and the robot configuration is $\boldsymbol{q} = \begin{pmatrix} 90° & -45° \end{pmatrix}^T$.

Exercise 4 [5 points]

Given the following matrix

$$\boldsymbol{A} = \begin{pmatrix} -0.5 & -a & 0 \\ 0 & 0 & -1 \\ a & -0.5 & 0 \end{pmatrix}$$

determine, if possible, a value $a > 0$ such that the identity $\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{A}$ holds, where $\boldsymbol{R}(\boldsymbol{r}, \theta)$ is the rotation matrix associated to an axis-angle representation of the orientation. Provide then all unit vectors $\boldsymbol{r}$ and associated angles $\theta \in (-\pi, +\pi]$ that are solutions to this equation.

[180 minutes (open books, but NO computer or internet)]

Solution of Midterm Test

November 18, 2016

Exercise 1

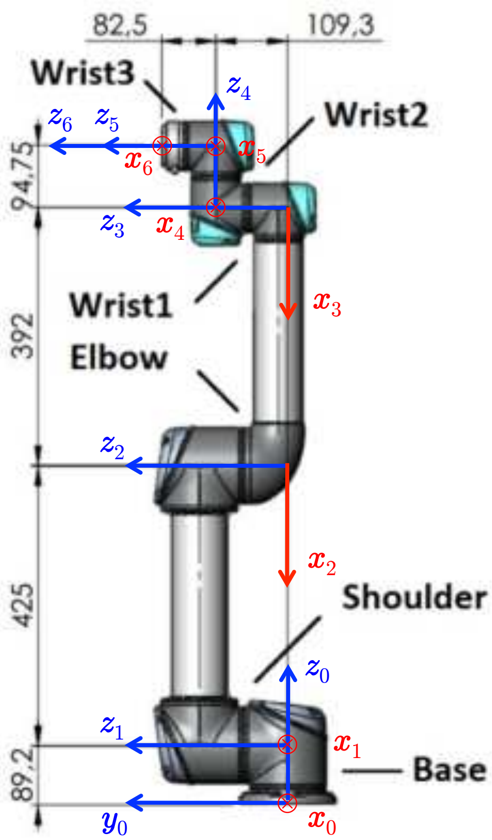


Figure 3: Assignment of DH frames for the UR5 robot associated to Tab. 1. Except for $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, all other $\boldsymbol{x}_i$ point inside the sheet. *Warning: We are not using this type of DH frame assignment for the UR10 available in the DIAG Robotics Lab.*

Exercise 2

We just need to build the homogeneous transformation matrix that relates frame RF_B to frame RF_A. The linear displacement is already represented by the given vector ${}^A\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B}$. As for the angular part, the rotation matrix ${}^A\boldsymbol{R}_B$ is specified from the sequence of three Euler $ZY'X''$ angles. Since these are defined around moving axes, we compute

$$\boldsymbol{R}_z(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix},\quad \boldsymbol{R}_y(\beta)=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix},\quad \boldsymbol{R}_x(\gamma)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\gamma & -\sin\gamma\\ 0 & \sin\gamma & \cos\gamma\end{pmatrix},$$

and multiply them in the suitable order to obtain

$$^A\boldsymbol{R}_B = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_x(\gamma).$$

Replacing the numerical values (with $\boldsymbol{R}_x(\gamma = 0) = \boldsymbol{I}$), we have

$$^A\boldsymbol{T}_B = \begin{pmatrix} ^A\boldsymbol{R}_B & {}^A\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & 3 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 7 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally

$$^A\boldsymbol{p}_{\boldsymbol{O}_AP,h} = {}^A\boldsymbol{T}_B\, {}^B\boldsymbol{p}_{\boldsymbol{O}_BP,h} = {}^A\boldsymbol{T}_B \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - \frac{\sqrt{2}}{2} \\ 7 + \frac{\sqrt{2}}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.2929 \\ 7.7071 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ^A\boldsymbol{p}_{\boldsymbol{O}_AP} \\ 1 \end{pmatrix}.$$

Exercise 3

An assignment of frames and the associated table of Denavit-Hartenberg are given in Fig. 4 and Tab. 2, respectively. The origin of frame RF_2 is conveniently placed at point P.

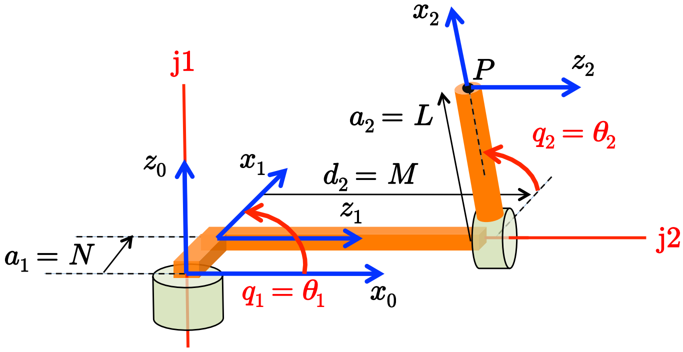


Figure 4: A possible assignment of DH frames for the 2R robot of Fig. 2.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	N	0	q_1
2	0	L	M	q_2

Table 2: Parameters associated to the DH frames in Fig. 4.

From this, the two homogeneous transformation matrices are computed

$$^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & N\cos q_1 \\ \sin q_1 & 0 & -\cos q_1 & N\sin q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & L\cos q_2 \\ \sin q_2 & \cos q_1 & 0 & L\sin q_2 \\ 0 & 0 & 1 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the symbolic expression in frame RF_0 of the position vector associated to point P (in homogeneous coordinates) is

$$
{}^0\boldsymbol{p}_{OP,h}(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix}L\cos q_2\\L\sin q_2\\M\\1\end{pmatrix} = \begin{pmatrix}L\cos q_1\cos q_2 + M\sin q_1 + N\cos q_1\\L\sin q_1\cos q_2 - M\cos q_1 + N\sin q_1\\L\sin q_2\\1\end{pmatrix}
$$

being ${}^0\boldsymbol{p}^T_{OP,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}^T_{OP}(\boldsymbol{q}) & 1 \end{pmatrix}$.

The numerical value of ${}^0\boldsymbol{p}_{OP}(\boldsymbol{q})$ with the data $L = 1$, $M = 2$, $N = 0.3$ [m] and at the requested robot configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/4 \end{pmatrix}^T$ [rad] is

$$
{}^0\boldsymbol{p}_{OP} = \begin{pmatrix} 2 & 0.3 + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}^T = \begin{pmatrix} 2 & 1.0071 & -0.7071 \end{pmatrix}^T.
$$

Exercise 4

One needs first to verify the existence of a scalar $a > 0$ such that $\boldsymbol{A}$ is a rotation matrix (i.e., an orthonormal matrix with determinant $= +1$). The orthogonality among the three columns is already in place (and holds for any value of a). Imposing a unit norm to the first two columns leads to $a = \pm\sqrt{3}/2$, so that the matrix will have $\det \boldsymbol{A} = +1$. Although both choices for the sign of a would work, the + sign is taken in view of the request to find a positive value for a. The matrix equation

$$
\boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{A} = \begin{pmatrix} -0.5 & -\sqrt{3}/2 & 0\\ 0 & 0 & -1\\ \sqrt{3}/2 & -0.5 & 0 \end{pmatrix}
$$

is solved for $\boldsymbol{r}$ and θ, using the inverse mapping of the axis-angle representation. Denoting by A_{ij} the elements of $\boldsymbol{A}$, we find that the problem at hand is a regular one since

$$
\sin\theta = \pm\frac{1}{2}\sqrt{(A_{12}-A_{21})^2 + (A_{13}-A_{31})^2 + (A_{23}-A_{32})^2} = \pm\,0.6614 \neq 0. \tag{1}
$$

Therefore, from

$$
\cos\theta = \frac{1}{2}\left(A_{11} + A_{22} + A_{33} - 1\right) = -0.75,
$$

taking the + sign in (1) we obtain

$$
\theta^{\{1\}} = \text{ATAN2}\left\{0.6614, -0.75\right\} = 2.4189 \text{ [rad]} = 138.59^\circ
$$

and then

$$
\boldsymbol{r}^{\{1\}} = \frac{1}{2\sin\theta^{\{1\}}}\begin{pmatrix} A_{32}-A_{23}\\ A_{13}-A_{31}\\ A_{21}-A_{12} \end{pmatrix} = \begin{pmatrix} 0.3780\\ -0.6547\\ 0.6547 \end{pmatrix}.
$$

The second solution is simply given by $\theta^{\{2\}} = -\theta^{\{1\}}$, $\boldsymbol{r}^{\{2\}} = -\boldsymbol{r}^{\{1\}}$. Indeed, one can check, e.g., that

$$
\boldsymbol{R}(\boldsymbol{r}^{\{2\}},\theta^{\{2\}}) = \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T} + \left(\boldsymbol{I} - \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T}\right)\cos\theta^{\{2\}} + \boldsymbol{S}(\boldsymbol{r}^{\{2\}})\sin\theta^{\{2\}} = \boldsymbol{A}.
$$

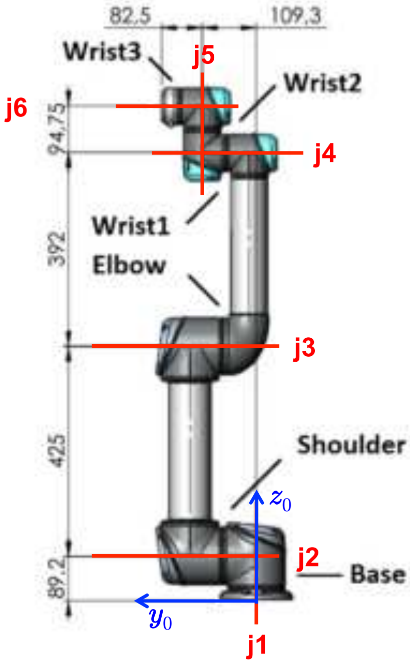
* * * * *

Robotics I

Midterm test in classroom – November 18, 2016

Exercise 1 [10 points]

Figure 1 shows the 6R Universal Robot UR5, with a non-spherical wrist, and two axes of the reference frame RF_0 placed at the robot base. The Denavit-Hartenberg parameters are given in Tab. 1, together with the numerical values for the constant parameters and the current values that the joint variables assume in the shown configuration.



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Using the provided sheet (please write your full name there!), draw all the Denavit-Hartenberg frames associated to the robot links according to Tab. 1.

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$$^A\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} = \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}^T \qquad [\text{m}],$$

while the orientation of RF_B with respect to RF_A is represented by the following sequence of three Euler $ZY'X''$ angles

$$\alpha = \frac{\pi}{4}, \qquad \beta = -\frac{\pi}{2}, \qquad \gamma = 0 \qquad [\text{rad}].$$

For a given point P, provide the value of vector $^A\boldsymbol{p}_{\boldsymbol{O}_A P}$ knowing that its position with respect to frame RF_B is given by

$$^B\boldsymbol{p}_{\boldsymbol{O}_B P} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T \qquad [\text{m}].$$

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Consider the 2-dof robot in Fig. 2, with two revolute joints having axes (the first vertical and the second horizontal) that do not intercept.

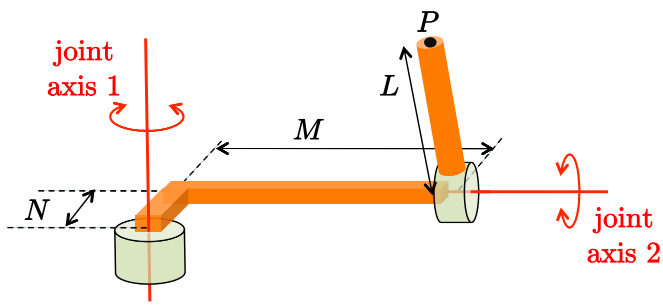

Figure 2: A 2R robot moving in the 3D space.

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Given the following matrix

$$\boldsymbol{A} = \begin{pmatrix} -0.5 & -a & 0 \\ 0 & 0 & -1 \\ a & -0.5 & 0 \end{pmatrix}$$

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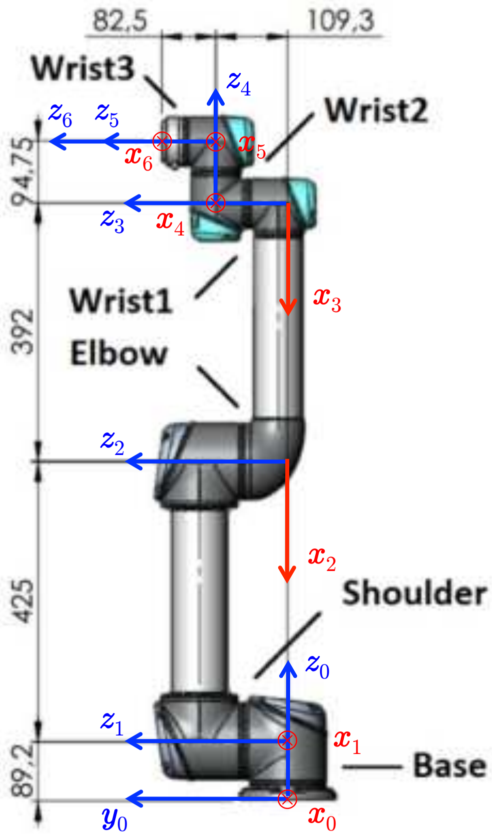
Solution of Midterm Test

November 18, 2016

Exercise 1



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$$\boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \quad \boldsymbol{R}_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix},$$

and multiply them in the suitable order to obtain

$$^A\boldsymbol{R}_B = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_x(\gamma).$$

Replacing the numerical values (with $\boldsymbol{R}_x(\gamma = 0) = \boldsymbol{I}$), we have

$$^A\boldsymbol{T}_B = \begin{pmatrix} ^A\boldsymbol{R}_B & ^A\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & 3 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 7 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally

$$^A\boldsymbol{p}_{\boldsymbol{O}_AP,h} = ^A\boldsymbol{T}_B\, ^B\boldsymbol{p}_{\boldsymbol{O}_BP,h} = ^A\boldsymbol{T}_B \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-\frac{\sqrt{2}}{2} \\ 7+\frac{\sqrt{2}}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.2929 \\ 7.7071 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ^A\boldsymbol{p}_{\boldsymbol{O}_AP} \\ 1 \end{pmatrix}.$$

Exercise 3

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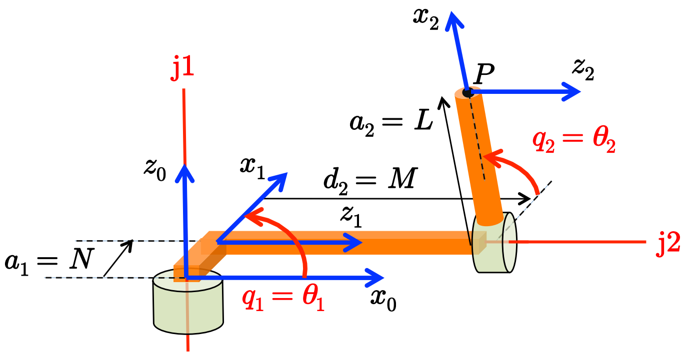

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Table 2: Parameters associated to the DH frames in Fig. 4.

From this, the two homogeneous transformation matrices are computed

$$^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & N\cos q_1 \\ \sin q_1 & 0 & -\cos q_1 & N\sin q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad ^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & L\cos q_2 \\ \sin q_2 & \cos q_1 & 0 & L\sin q_2 \\ 0 & 0 & 1 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the symbolic expression in frame RF_0 of the position vector associated to point P (in homogeneous coordinates) is

$$ {}^0\boldsymbol{p}_{OP,h}(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix}L\cos q_2\\ L\sin q_2\\ M\\ 1\end{pmatrix} = \begin{pmatrix}L\cos q_1\cos q_2 + M\sin q_1 + N\cos q_1\\ L\sin q_1\cos q_2 - M\cos q_1 + N\sin q_1\\ L\sin q_2\\ 1\end{pmatrix} $$

being ${}^0\boldsymbol{p}^T_{OP,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}^T_{OP}(\boldsymbol{q}) & 1 \end{pmatrix}$.

The numerical value of ${}^0\boldsymbol{p}_{OP}(\boldsymbol{q})$ with the data $L = 1$, $M = 2$, $N = 0.3$ [m] and at the requested robot configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/4 \end{pmatrix}^T$ [rad] is

$$ {}^0\boldsymbol{p}_{OP} = \begin{pmatrix} 2 & 0.3 + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}^T = \begin{pmatrix} 2 & 1.0071 & -0.7071 \end{pmatrix}^T. $$

Exercise 4

One needs first to verify the existence of a scalar $a > 0$ such that $\boldsymbol{A}$ is a rotation matrix (i.e., an orthonormal matrix with determinant $= +1$). The orthogonality among the three columns is already in place (and holds for any value of a). Imposing a unit norm to the first two columns leads to $a = \pm\sqrt{3}/2$, so that the matrix will have $\det \boldsymbol{A} = +1$. Although both choices for the sign of a would work, the $+$ sign is taken in view of the request to find a positive value for a. The matrix equation

$$ \boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{A} = \begin{pmatrix} -0.5 & -\sqrt{3}/2 & 0\\ 0 & 0 & -1\\ \sqrt{3}/2 & -0.5 & 0 \end{pmatrix} $$

is solved for $\boldsymbol{r}$ and θ, using the inverse mapping of the axis-angle representation. Denoting by A_{ij} the elements of $\boldsymbol{A}$, we find that the problem at hand is a regular one since

$$ \sin\theta = \pm\frac{1}{2}\sqrt{(A_{12}-A_{21})^2 + (A_{13}-A_{31})^2 + (A_{23}-A_{32})^2} = \pm 0.6614 \neq 0. \tag{1} $$

Therefore, from

$$ \cos\theta = \frac{1}{2}\left(A_{11} + A_{22} + A_{33} - 1\right) = -0.75, $$

taking the $+$ sign in (1) we obtain

$$ \theta^{\{1\}} = \text{ATAN2}\left\{0.6614, -0.75\right\} = 2.4189 \text{ [rad]} = 138.59^\circ $$

and then

$$ \boldsymbol{r}^{\{1\}} = \frac{1}{2\sin\theta^{\{1\}}}\begin{pmatrix} A_{32}-A_{23}\\ A_{13}-A_{31}\\ A_{21}-A_{12}\end{pmatrix} = \begin{pmatrix}0.3780\\ -0.6547\\ 0.6547\end{pmatrix}. $$

The second solution is simply given by $\theta^{\{2\}} = -\theta^{\{1\}}$, $\boldsymbol{r}^{\{2\}} = -\boldsymbol{r}^{\{1\}}$. Indeed, one can check, e.g., that

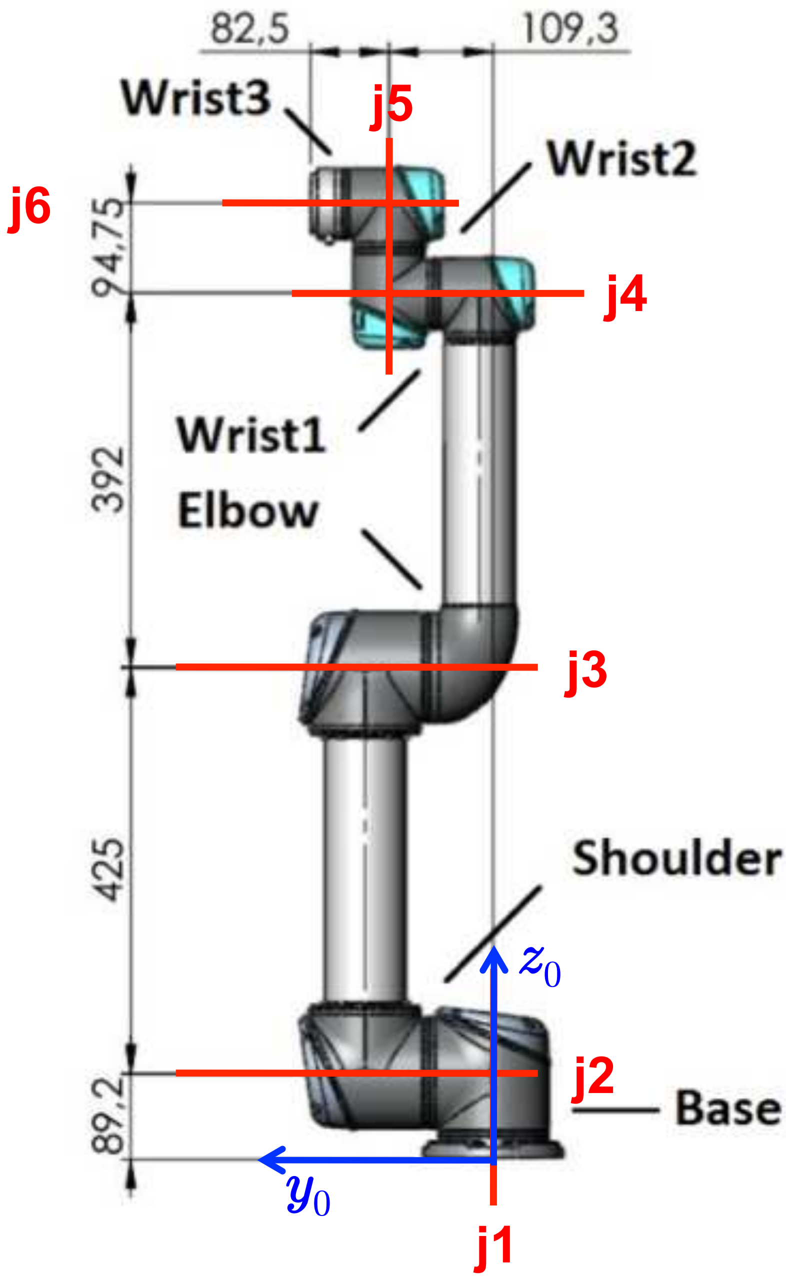
$$ \boldsymbol{R}(\boldsymbol{r}^{\{2\}},\theta^{\{2\}}) = \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T} + \left(\boldsymbol{I} - \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T}\right)\cos\theta^{\{2\}} + \boldsymbol{S}(\boldsymbol{r}^{\{2\}})\sin\theta^{\{2\}} = \boldsymbol{A}. $$

* * * * *

Robotics I - Sheet for Exercise 1

Midterm test in classroom – November 18, 2016

LAST NAME, First Name ______________________________



Draw all the Denavit-Hartenberg frames associated to the robot links according to Tab. 1 in the text.

Robotics I

January 11, 2017

Exercise 1

Consider the 4-dof planar RPRP robot in Fig. 1 and assume that every joint has an unlimited range.

- Assign the link frames according to the Denavit-Hartenberg convention. Place the origin of the last frame coincident with point P. Make the free choices that are available so as to eliminate (i.e., "zeroing out") as many unnecessary constant parameters as possible. Draw the chosen frames directly on the robot in Fig. 1.
- Provide the Denavit-Hartenberg table of parameters associated to the frames that have been assigned. Draw the robot in the configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T$.
- A task requires to place the end-effector frame at a desired position $\boldsymbol{p}_d = \begin{pmatrix} p_{dx} & p_{dy} \end{pmatrix}^T$, with a given orientation α_d of axis $\boldsymbol{x}_4$ w.r.t. axis $\boldsymbol{x}_0$ of the base. For the RPRP robot, define the analytic Jacobian associated to this three-dimensional task and determine all its singular configurations.

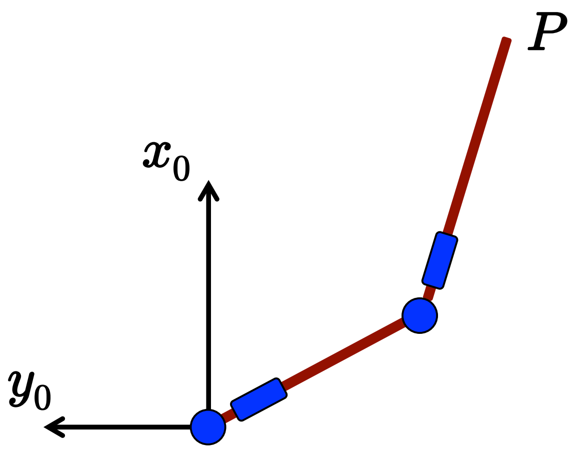


Figure 1: A 4-dof planar RPRP robot.

Exercise 2

A planar 2R robot, with link lengths $\ell_1 = 1$ and $\ell_2 = 0.5$ [m], has its end-effector placed in the Cartesian position $\boldsymbol{p}_0 = \begin{pmatrix} 0.7 & 0.7 \end{pmatrix}^T$ [m] and is at rest at time $t = 0$. Using separation in space and time, plan a Cartesian trajectory for the robot end-effector in order to pick an object in the position $\boldsymbol{p}_d = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [m] at a given time $T > 0$ (to be treated symbolically in this problem). The object is on a conveyor belt, moving with a constant velocity $\boldsymbol{v}_d = V \cdot \begin{pmatrix} -1 & 0 \end{pmatrix}^T$, where $V = 1$ [m/s] is the speed. The robot end effector should match this velocity at the final position. Moreover, the motion task should be executed with joint velocities $\dot{\boldsymbol{q}}(t)$ that are continuous for all $t \in [0, T]$.

- Provide the parametric expression $\boldsymbol{p}(s)$ of the chosen Cartesian path, and of its first and second derivative with respect to the path parameter s.
- Provide the expression of a timing law $s(t)$ that satisfies the required conditions.
- Assuming a motion time $T = 1.6$ [s], compute the joint velocity $\dot{\boldsymbol{q}}_{mid} = \dot{\boldsymbol{q}}(T/2)$ at $t = T/2$, when the robot is executing the planned Cartesian trajectory. How many solutions are there?

Exercise 3

The kinematics of a spatial 3R robot is defined by the Denavit-Hartenberg parameters in Tab. 1, where the three constant parameters d_1, a_2, and a_3 are all strictly positive.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	0	a_2	0	q_2
3	0	a_3	0	q_3

Table 1: DH parameters of a spatial 3R robot.

The 6×3 geometric Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of this robot (expressed in frame 0) has the following expression, which is only partly specified:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\cos q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & J_{13}(\boldsymbol{q}) \\ \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\sin q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & J_{23}(\boldsymbol{q}) \\ 0 & a_2\cos q_2 + a_3\cos(q_2+q_3) & J_{33}(\boldsymbol{q}) \\ & \boldsymbol{J}_A(\boldsymbol{q}) & \end{pmatrix}. \tag{1}$$

- Provide the missing expressions of all remaining terms in eq. (1).
- Show that this geometric Jacobian has always full (column) rank.
- With the robot in the zero configuration, $\boldsymbol{q} = \boldsymbol{0}$, determine the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$ that balances statically a force $\boldsymbol{F}$, applied to the robot tip, and a moment $\boldsymbol{M}$, applied to the third link, given by

  $$\boldsymbol{F} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T \text{ [N]}, \qquad \boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \text{ [Nm]}.$$

  Give the expression of $\boldsymbol{\tau}$, and then its numerical value using the robot kinematic data $d_1 = a_2 = a_3 = 1$ [m].
- Is it possible to apply simultaneously a force $\boldsymbol{F} \neq \boldsymbol{0}$ and a moment $\boldsymbol{M} \neq \boldsymbol{0}$ so that the robot remains in static equilibrium, without the need of an extra joint torque ($\boldsymbol{\tau} = \boldsymbol{0}$) for balancing the force/moment pair? Motivate your answer in general, and illustrate it with a supporting example when the robot is in the configuration $\boldsymbol{q} = \boldsymbol{0}$.

[240 minutes, open books but no computer or smartphone]

Solution

January 11, 2017

Exercise 1

Figure 2 shows a possible assignment of the Denavit-Hartenberg frames, as well as the definition of the joint variables. The associated Tab. 2 highlights that this assignment has zeroed all constant parameters that could be freely chosen. Figure 3 shows the robot in the configuration $\boldsymbol{q} = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T$, as requested.

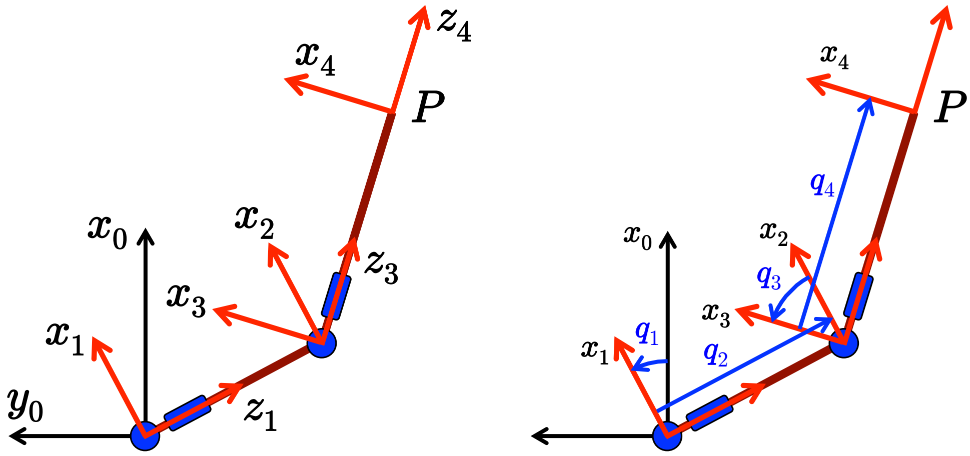


Figure 2: Assigned DH frames for the planar RPRP robot (left) and configuration variables (right).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$-\pi/2$	0	q_2	0
3	$\pi/2$	0	0	q_3
4	0	0	q_4	0

Table 2: DH parameters of the planar RPRP robot.

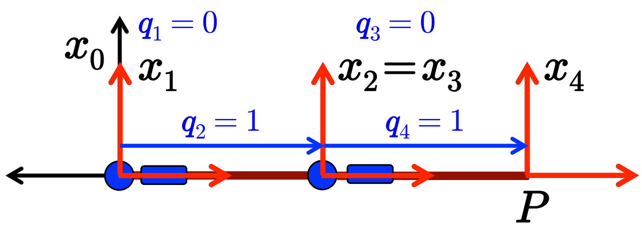


Figure 3: The planar RPRP robot in the configuration $\boldsymbol{q} = (0 \;\; 1 \;\; 0 \;\; 1)^T$.

From Tab. 2, we evaluate the four homogeneous transformation matrices

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & 0 \\ \sin q_3 & 0 & -\cos q_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Performing computations we obtain[1]

$$
{}^0\boldsymbol{T}_4(\boldsymbol{q}) = \begin{pmatrix} \cos(q_1+q_3) & 0 & \sin(q_1+q_3) & q_2 \sin q_1 + q_4 \sin(q_1+q_3) \\ \sin(q_1+q_3) & 0 & -\cos(q_1+q_3) & -(q_2 \cos q_1 + q_4 \cos(q_1+q_3)) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Therefore, the task vector of interest is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 \sin q_1 + q_4 \sin(q_1+q_3) \\ -(q_2 \cos q_1 + q_4 \cos(q_1+q_3)) \\ q_1 + q_3 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}). \tag{2}
$$

The analytic Jacobian for this task is the 3×4 matrix obtained by differentiating (2):

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_r(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} q_2 \cos q_1 + q_4 \cos(q_1+q_3) & \sin q_1 & q_4 \cos(q_1+q_3) & \sin(q_1+q_3) \\ q_2 \sin q_1 + q_4 \sin(q_1+q_3) & -\cos q_1 & q_4 \sin(q_1+q_3) & -\cos(q_1+q_3) \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{3}
$$

The singular configurations of this matrix are those where the rank drops down (to 2 or 1) from its maximum possible value (3). Equivalently, they are defined by the values of $\boldsymbol{q}$ such that all four 3×3 minors extracted from $\boldsymbol{J}(\boldsymbol{q})$ by deleting one column are equal to zero. Let $\boldsymbol{J}_{-i}(\boldsymbol{q})$ be the square matrix obtained from (3) by deleting the ith column, for $i = 1, \ldots, 4$. We have

$$
\det \boldsymbol{J}_{-1}(\boldsymbol{q}) = -\sin q_3, \quad \det \boldsymbol{J}_{-2}(\boldsymbol{q}) = q_2 \cos q_3, \quad \det \boldsymbol{J}_{-3}(\boldsymbol{q}) = \sin q_3, \quad \det \boldsymbol{J}_{-4}(\boldsymbol{q}) = -q_2.
$$

Therefore, the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is singular if and only if $q_2 = \sin q_3 = 0$. For instance, when $q_2 = q_3 = 0$, we obtain

$$
\bar{\boldsymbol{J}}(\boldsymbol{q}) := \boldsymbol{J}(q_1, 0, 0, q_4) = \begin{pmatrix} q_4 \cos q_1 & \sin q_1 & q_4 \cos q_1 & \sin q_1 \\ q_4 \sin q_1 & -\cos q_1 & q_4 \sin q_1 & -\cos q_1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \operatorname{rank}\left(\bar{\boldsymbol{J}}(\boldsymbol{q})\right) = 2.
$$

Exercise 2

We have to interpolate the two Cartesian positions $\boldsymbol{p}_0$ and $\boldsymbol{p}_d$ with a sufficiently smooth path, having also a prescribed tangent direction $\boldsymbol{v}_{du} = \boldsymbol{v}_d / \|\boldsymbol{v}_d\|$ (of unitary norm) at the end position $\boldsymbol{p}_d$. It is required (and also useful) to work in the Cartesian space and to keep separation between space (geometry) and time.

Note first that, because of the required continuity of the joint velocity and since $(\boldsymbol{p}_d - \boldsymbol{p}_0) \nparallel \boldsymbol{v}_d$, the Cartesian path cannot be chosen as a straight segment (linear in the path parameter s), Since the robot end effector has to reach the correct direction of the conveyor belt motion at the end of the transfer, we would have then a discontinuity in the path tangent at $\boldsymbol{p}_d$. This translates into a discontinuity for the Cartesian velocity at the final instant T, since the end effector is not allowed to stop in $\boldsymbol{p}_d$ (rather, it

[1] The expressions in ${}^0\boldsymbol{T}_4(\boldsymbol{q})$ could have been obtained also from a simple inspection of Fig. 2).

should have the speed $V > 0$ of the conveyor belt). Accordingly, a jump in the Cartesian velocity will result in a discontinuity of the joint velocity.

Therefore, when planning the path, we have to approach $\boldsymbol{p}_d$ from the right direction, and this can be done is several ways (without leaving the robot workspace). For instance, one could concatenate two linear segments, from $\boldsymbol{p}_0$ to $\boldsymbol{p}_{int}$ and then from $\boldsymbol{p}_{int}$ to $\boldsymbol{p}_d$, by introducing an intermediate position $\boldsymbol{p}_{int}$ such that $(\boldsymbol{p}_d - \boldsymbol{p}_{int}) \parallel \boldsymbol{v}_d$ and by forcing the motion to stop there (to preserve a continuous velocity in time). Another possibility is to design an arc of a circle (of suitable center and radius) that interpolates $\boldsymbol{p}_0$ with $\boldsymbol{p}_d$, having also the right tangent (in the direction of $\boldsymbol{v}_d$) at the end position $\boldsymbol{p}_d$.

We present here another viable solution that considers a quadratic function of s for the path $\boldsymbol{p}(s)$. Working on the Cartesian (x, y)-plane in vector terms (with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2$), we choose

$$\boldsymbol{p}(s) = \boldsymbol{a} + \boldsymbol{b}s + \boldsymbol{c}s^2, \qquad s \in [0, 1] \tag{4}$$

and impose the boundary conditions

$$\boldsymbol{p}(0) = \boldsymbol{a} = \boldsymbol{p}_0, \qquad \boldsymbol{p}(1) = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{p}_d, \qquad \boldsymbol{p}'(1) = (\boldsymbol{b} + 2\boldsymbol{c}s)_{s=1} = \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{v}_{du}. \tag{5}$$

In this way, the 6 scalar conditions in eq. (5) —3 on the x-coordinates and 3 on the y-coordinates— will be satisfied using the equal number of 6 scalar components in the 3 bi-dimensional vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Solving from (5), we obtain the unique values

$$\boldsymbol{a} = \boldsymbol{p}_0, \qquad \boldsymbol{b} = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right) - \boldsymbol{v}_{du}, \qquad \boldsymbol{c} = \boldsymbol{v}_{du} - \left(\boldsymbol{p}_d - \boldsymbol{p}_0\right), \tag{6}$$

to be replaced in (4). The first and second derivatives of $\boldsymbol{p}(s)$ w.r.t. s are then

$$\boldsymbol{p}'(s) = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right)(1 - s) + \boldsymbol{v}_{du}(2s - 1), \qquad \boldsymbol{p}''(s) = 2\left(\boldsymbol{v}_{du} - \left(\boldsymbol{p}_d - \boldsymbol{p}_0\right)\right). \tag{7}$$

In particular, the path tangent direction at the starting position $\boldsymbol{p}_0$ will be $\boldsymbol{p}'(0) = \boldsymbol{b} = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right) - \boldsymbol{v}_{du}$.

As for the timing law $s(t)$, to be defined for $t \in [0, T]$, we need to satisfy the four boundary conditions

$$s(0) = 0, \qquad s(T) = 1, \qquad \dot{s}(0) = 0, \qquad \dot{s}(T) = V, \tag{8}$$

being $V > 0$ the final speed along the path at the final instant $t = T$. Thus, a cubic polynomial in the normalized time $\tau = t/T$ will be sufficient:

$$s(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3, \qquad \tau = t/T \in [0, 1].$$

Imposing (8), we obtain the timing law

$$s(t) = (3 - TV)\left(\frac{t}{T}\right)^2 + (TV - 2)\left(\frac{t}{T}\right)^3, \qquad t \in [0, T], \tag{9}$$

and its speed profile

$$\dot{s}(t) = \frac{1}{T}\left(2(3 - TV)\left(\frac{t}{T}\right) + 3(TV - 2)\left(\frac{t}{T}\right)^2\right), \qquad t \in [0, T]. \tag{10}$$

Note that at the motion midtime $t = T/2$, it is

$$s\left(\frac{T}{2}\right) = \frac{1}{2} - \frac{TV}{8}, \qquad \dot{s}\left(\frac{T}{2}\right) = \frac{1.5}{T} - \frac{TV}{4}. \tag{11}$$

Replacing now the problem data, we obtain from (4) and (6)

$$\boldsymbol{p}(s) = \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} + \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} s + \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} s^2, \qquad s \in [0, 1],$$

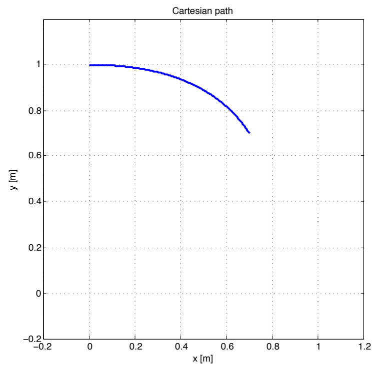


Figure 4: The Cartesian path $\boldsymbol{p}(s)$ interpolating $\boldsymbol{p}_0 = (0.7 \;\; 0.0)^T$ to $\boldsymbol{p}_d = (0 \;\; 1)^T$.

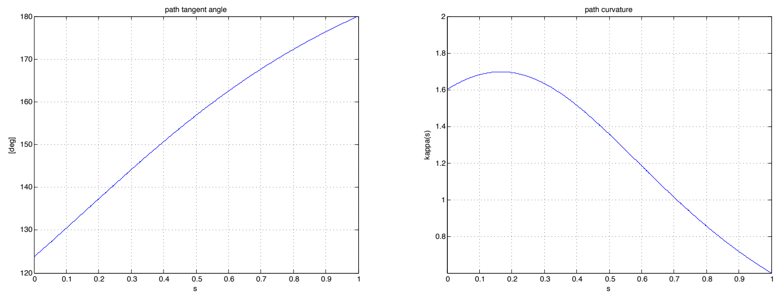


Figure 5: The angle $\alpha(s)$ of the tangent to the path (left) and the path curvature $\kappa(s)$ (right).

and from (7)

$$\boldsymbol{p}'(s) = \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} + \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix} s, \qquad \boldsymbol{p}''(s) = \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix}.$$

The planned Cartesian path $\boldsymbol{p}(s)$ is shown in Fig. 4. Note that the path remains always in the primary workspace of the robot ($0.5 = |\ell_1 - \ell_2| \le \|\boldsymbol{p}\| \le \ell_1 + \ell_2 = 1.5$). Just for completeness of illustration, Figure 5 reports, as functions of the path parameter, also the angle $\alpha(s)$ of the tangent to the path w.r.t. the $\boldsymbol{x}$-axis and the path curvature $\kappa(s)$. These have been evaluated as

$$\alpha(s) = \text{ATAN2}\left\{p_y'(s), p_x'(s)\right\}, \qquad \kappa(s) = \frac{\|\boldsymbol{p}'(s) \times \boldsymbol{p}''(s)\|}{\|\boldsymbol{p}'(s)\|^3}, \qquad s \in [0,1].$$

For the cross product in the second formula, vectors were embedded in 3D (with a zero z-component).

Similarly, using $V = 1$ [m/s] and the given motion time $T = 1.6$ [s], it follows from (9) and (10)

$$s(t) = 1.4\left(\frac{t}{1.6}\right)^2 - 0.4\left(\frac{t}{1.6}\right)^3, \qquad \dot{s}(t) = 1.75\left(\frac{t}{1.6}\right) - 0.75\left(\frac{t}{1.6}\right)^2, \qquad t \in [0, 1.6].$$

The cubic timing law $s(t)$ is shown in Fig. 6, while Figure 7 reports its first and second time derivatives. Note that $\dot{s}(0) = 0$ and $\dot{s}(T) = V = 1$ [m/s].

From the separate profiles in space and time, we can recombine the Cartesian trajectory and its time derivatives as

$$\boldsymbol{p}(t) = \boldsymbol{p}(s(t)), \qquad \dot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s(t))\,\dot{s}(t), \qquad \ddot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s(t))\,\ddot{s}(t) + \boldsymbol{p}''(s(t))\,\dot{s}^2(t), \qquad t \in [0, T].$$

Figure 8 shows the two components of the obtained Cartesian trajectory, in position $\boldsymbol{p}(t)$, velocity $\dot{\boldsymbol{p}}(t)$, and acceleration $\ddot{\boldsymbol{p}}(t)$.

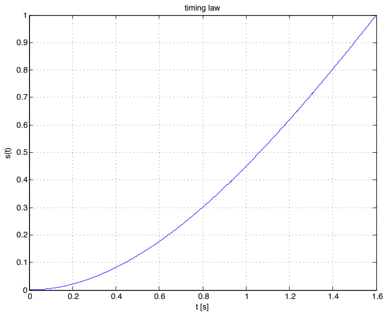


Figure 6: The timing law $s(t)$ for $T = 1.6$ [s].

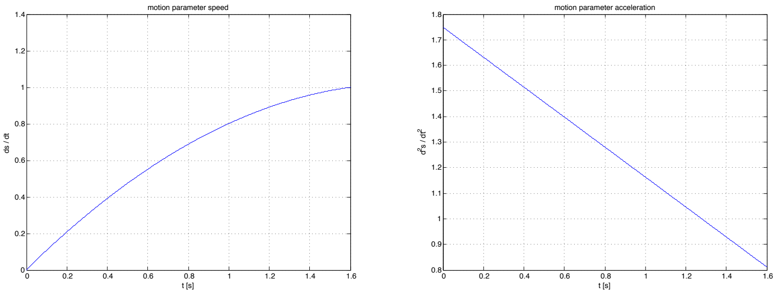


Figure 7: The timing speed $\dot{s}(t)$ (left) and acceleration $\ddot{s}(t)$ (right) for $T = 1.6$ [s].

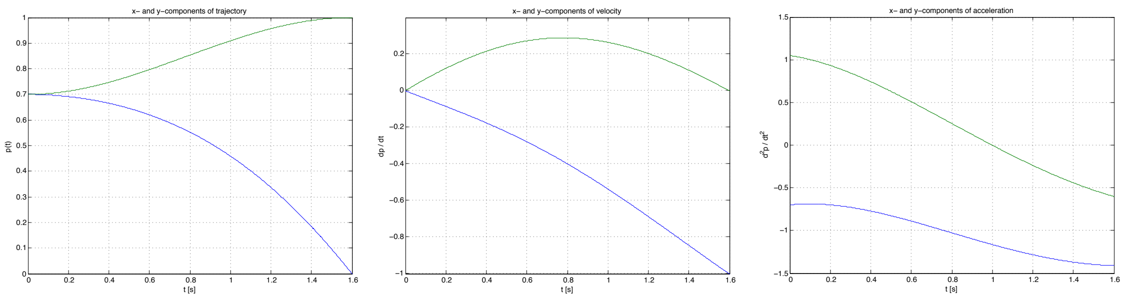


Figure 8: The components (x in blue, y in green) of the Cartesian trajectory $\boldsymbol{p}(t)$ and of its first and second time derivatives $\dot{\boldsymbol{p}}(t)$ and $\ddot{\boldsymbol{p}}(t)$ for $T = 1.6$ [s].

Finally, at the midtime $t = T/2 = 0.8$ [s] of motion, we evaluate from the previous formulas the quantities of interest:

$$s(0.8) = 0.3, \qquad \dot{s}(0.8) = 0.6875, \qquad \Rightarrow \qquad \begin{array}{l} \boldsymbol{p}(t=0.8) = \boldsymbol{p}(s=0.3) = \begin{pmatrix} 0.5530 \\ 0.8530 \end{pmatrix} =: \boldsymbol{p}_{mid} \\ \dot{\boldsymbol{p}}(t=0.8) = \boldsymbol{p}'(s=0.3)\dot{s}(0.8) = \begin{pmatrix} -0.3987 \\ 0.2888 \end{pmatrix} =: \dot{\boldsymbol{p}}_{mid}. \end{array}$$

Up to now, we have not involved the actual robot in the trajectory planning. At this stage, we need to solve an inverse kinematics problem for $\boldsymbol{p}_{mid}$, and then an inverse differential kinematic problem for $\dot{\boldsymbol{p}}_{mid}$.

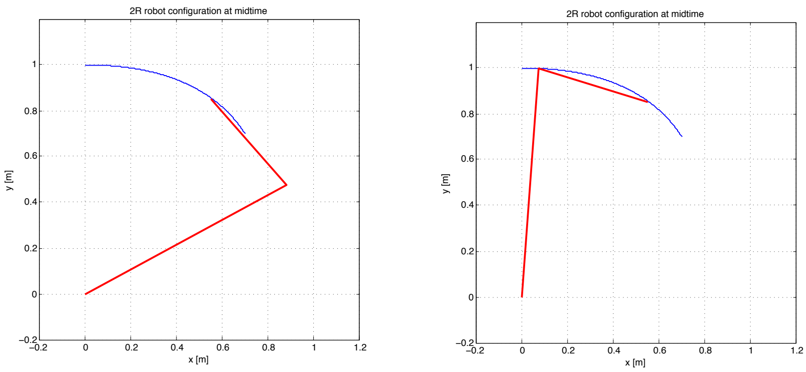


Figure 9: The two solutions to the inverse kinematics at the trajectory midtime $t = T/2 = 0.8$ [s].

By using the usual formulas for a planar 2R robot

$$c_2 = \frac{p_{mid,x}^2 + p_{mid,y}^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}, \quad s_2 = \pm\sqrt{1 - c_2^2} \quad \Rightarrow \quad q_2 = \mathrm{ATAN2}\{s_2, c_2\}$$

and

$$s_1 = (\ell_1 + \ell_2 c_2)\, p_{mid,y} - \ell_2 s_2\, p_{mid,x}, \quad c_1 = (\ell_1 + \ell_2 c_2)\, p_{mid,x} + \ell_2 s_2\, p_{mid,y} \quad \Rightarrow \quad q_1 = \mathrm{ATAN2}\{s_1, c_1\},$$

we obtain the two solutions to the inverse kinematics problem (see Fig. 9)

$$\boldsymbol{q}_{mid,A} = \begin{pmatrix} 0.4948 \\ 1.7891 \end{pmatrix} [\mathrm{rad}] = \begin{pmatrix} 28.35^\circ \\ 102.51^\circ \end{pmatrix}, \qquad \boldsymbol{q}_{mid,B} = \begin{pmatrix} 1.4965 \\ -1.7891 \end{pmatrix} [\mathrm{rad}] = \begin{pmatrix} 85.74^\circ \\ -102.51^\circ \end{pmatrix}.$$

Evaluating the robot Jacobian for the 2R robot

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2)) & -\ell_2 \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2)) & \ell_2 \cos(q_1 + q_2) \end{pmatrix},$$

we have in the first case

$$\boldsymbol{J}(\boldsymbol{q}_{mid,A}) = \begin{pmatrix} 0.8530 & 0.3782 \\ 0.5530 & 0.3271 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{mid.A} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{mid,A})\, \dot{\boldsymbol{p}}_{mid} = \begin{pmatrix} 0.4909 \\ -0.0528 \end{pmatrix} [\mathrm{rad/s}],$$

and in the second case

$$\boldsymbol{J}(\boldsymbol{q}_{mid,B}) = \begin{pmatrix} -0.8530 & 0.1442 \\ 0.5530 & 0.4787 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{mid.B} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{mid,B})\, \dot{\boldsymbol{p}}_{mid} = \begin{pmatrix} 0.4764 \\ 0.0528 \end{pmatrix} [\mathrm{rad/s}].$$

We obtained thus two different (though quite similar) joint velocity solutions to the given problem.

Exercise 3

From Tab. 1, we evaluate the three homogeneous transformation matrices

$${}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$${}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 & a_3 \cos q_3 \\ \sin q_3 & \cos q_3 & 0 & a_3 \sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing computations in an efficient way, we have for the position of the origin of the last frame

$$\begin{aligned}
\boldsymbol{p}_{hom}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{p}(\boldsymbol{q}) \\ \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\right)\right) = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\begin{pmatrix} a_3\cos q_3 \\ a_3 \sin q_3 \\ 0 \\ 1 \end{pmatrix}\right) \\
&= {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix} a_2\cos q_2 + a_3\cos(q_2+q_3) \\ a_2\sin q_2 + a_3\sin(q_2+q_3) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ \sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ d_1 + a_2\sin q_2 + a_3\sin(q_2+q_3) \\ 1 \end{pmatrix}.
\end{aligned}$$

The last column of the 3×3 matrix $\boldsymbol{J}_L(\boldsymbol{q})$ in eq. (1) is then easily computed as

$$\begin{pmatrix} J_{13}(\boldsymbol{q}) \\ J_{23}(\boldsymbol{q}) \\ J_{33}(\boldsymbol{q}) \end{pmatrix} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial q_3} = \begin{pmatrix} -a_3\cos q_1 \sin(q_2+q_3) \\ -a_3 \sin q_1 \sin(q_2+q_3) \\ a_3\cos(q_2+q_3) \end{pmatrix}. \tag{12}$$

As for the 3×3 matrix $\boldsymbol{J}_A(\boldsymbol{q})$, its general expression becomes in the present case

$$\begin{aligned}
\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0\\0\\1 \end{pmatrix} & {}^0\boldsymbol{R}_1(q_1)\begin{pmatrix} 0\\0\\1 \end{pmatrix} & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\begin{pmatrix} 0\\0\\1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin q_1 & \sin q_1 \\ 0 & -\cos q_1 & -\cos q_1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{13}$$

The singularity analysis of $\boldsymbol{J}(\boldsymbol{q})$ can be performed in different ways. We start by observing that matrix $\boldsymbol{J}_A(\boldsymbol{q})$ has always rank exactly equal to 2, since *i)* the second and third columns are identical (thus its rank is always less than 3), and *ii)* sine and cosine of the same angle never vanish simultaneously (thus the rank never drops below 2). The first observation suggests a transformation on the matrix columns, such that the study of the rank of the complete matrix $\boldsymbol{J}(\boldsymbol{q})$ is reduced to the analysis of the last column of $\boldsymbol{J}_L(\boldsymbol{q})$. In fact, substituting to the third column the difference between the second and the third one, we have

$$\begin{aligned}
\boldsymbol{J}'(\boldsymbol{q}) &= \boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{T} = \boldsymbol{J}(\boldsymbol{q})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -\sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\cos q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & -a_2\cos q_1 \sin q_2 \\ \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\sin q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & -a_2\sin q_1\sin q_2 \\ 0 & a_2\cos q_2 + a_3\cos(q_2+q_3) & a_2\cos q_2 \\ 0 & \sin q_1 & 0 \\ 0 & -\cos q_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

From the obtained internal structure, it is easy to see that the rank of matrix $\boldsymbol{J}'(\boldsymbol{q})$, which is the same as the rank ρ of $\boldsymbol{J}(\boldsymbol{q})$, will be full if and only if, in any given configuration, at least one element of its last column is different from zero or, equivalently, if the last column never vanishes. This is immediate to see, again because sine and cosine of the same angle never vanish simultaneously. Thus, $\boldsymbol{J}(\boldsymbol{q})$ has always full rank $\rho = 3$, i.e., is never singular.

In alternative, one could resort to the basic definition of full rank for a matrix that has less columns than rows. Let $\boldsymbol{c}_i$ be the ith column of the matrix, with $i = 1,2,3$, for the present case of our $\boldsymbol{J}(\boldsymbol{q})$. The matrix will have full (column) rank when, for all possible scalars λ_i, $i = 1,2,3$,

$$\lambda_1\boldsymbol{c}_1 + \lambda_2\boldsymbol{c}_2 + \lambda_3\boldsymbol{c}_3 = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}^T = \boldsymbol{0}. \tag{14}$$

Using the obtained expression (12) and (13) inside the Jacobian in eq. (1), the condition on the left-hand side of (14) results in six scalar equations. From the last one, it follows necessarily that $\lambda_1 = 0$. From the fourth and fifth equations, it follows also that $\lambda_2 = -\lambda_3$. Squaring and summing the first three equations in the remaining $\lambda_2 (\boldsymbol{c}_2 - \boldsymbol{c}_3) = \boldsymbol{0}$ yields

$$\lambda_2 a_2^2 \left(\sin^2 q_2 \left(\cos^2 q_1 + \sin^2 q_1\right) + \cos^2 q_2\right) = \lambda_2 a_2^2 = 0 \qquad \Longleftrightarrow \qquad \lambda_2 = 0.$$

As a result, $\boldsymbol{\lambda} = \boldsymbol{0}$ is the only solution, and the matrix $\boldsymbol{J}(\boldsymbol{q})$ has always full rank (for all $\boldsymbol{q}$!).

Another approach, which is however computationally heavier to perform by hand, is to check that the following determinant[2]

$$\begin{aligned}
\det\left[\boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{J}(\boldsymbol{q})\right] &= \det \boldsymbol{T}^T \cdot \det\left[\boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{J}(\boldsymbol{q})\right] \cdot \det \boldsymbol{T} = \det\left[\boldsymbol{J}'^T(\boldsymbol{q})\,\boldsymbol{J}'(\boldsymbol{q})\right] \\
&= \det \begin{pmatrix} 1 + \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right)^2 & 0 & 0 \\ 0 & 1 + a_2^2 + a_3^2 + 2\,a_2 a_3 \cos q_3 & a_2^2 + a_2 a_3 \cos q_3 \\ 0 & a_2^2 + a_2 a_3 \cos q_3 & a_2^2 \end{pmatrix} \\
&= \left(1 + \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right)^2\right) a_2^2 \left(1 + a_3^2 \sin^2 q_3\right)
\end{aligned}$$

is in fact never zero —a necessary and sufficient condition for the 6×3 matrix $\boldsymbol{J}(\boldsymbol{q})$ to have full rank $\rho = 3$.

Evaluating now $\boldsymbol{J}^T(\boldsymbol{q})$ for $\boldsymbol{q} = \boldsymbol{0}$ yields

$$\boldsymbol{J}_0^T := \boldsymbol{J}^T(\boldsymbol{0}) = \begin{pmatrix} 0 & a_2 + a_3 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_2 + a_3 & 0 & -1 & 0 \\ 0 & 0 & a_3 & 0 & -1 & 0 \end{pmatrix}.$$

The joint torque $\boldsymbol{\tau}$ that balances statically in the configuration $\boldsymbol{q} = \boldsymbol{0}$ the assigned force $\boldsymbol{F} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$ and moment $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is given by

$$\boldsymbol{\tau} = -\boldsymbol{J}_0^T \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{pmatrix} = \begin{pmatrix} -\left(1 + a_2 + a_3\right) \\ 1 + a_2 + a_3 \\ 1 + a_3 \end{pmatrix} \qquad \Rightarrow \qquad \text{for } a_2 = a_3 = 1, \quad \boldsymbol{\tau} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$$

In any robot configuration, there will always be some non-zero force/moment pairs that require no balancing joint torque to keep the robot in its equilibrium state. This is independent from having the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ full rank or not. In fact, the 3×6 matrix $\boldsymbol{J}^T(\boldsymbol{q})$ will always have a null space $\mathcal{N}\left\{\boldsymbol{J}^T\right\}$ of dimension $6 - \rho \geq 3$. Since the Jacobian has constant full rank $\rho = 3$ (in all configurations), the null space will be of dimension $6 - 3 = 3$, and there will be always ∞^3 such force/moment pairs at any robot configuration. For example, at $\boldsymbol{q} = \boldsymbol{0}$ the pair

$$\boldsymbol{F}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \qquad \boldsymbol{M}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T,$$

yields

$$\boldsymbol{\tau}_0 = -\boldsymbol{J}_0^T \begin{pmatrix} \boldsymbol{F}_0 \\ \boldsymbol{M}_0 \end{pmatrix} = \boldsymbol{0}.$$

* * * * *

[2] Since $\det \boldsymbol{T} = \det \boldsymbol{T}^T = -1$, the product of these two determinants is equal to 1.

Robotics I

February 3, 2017

Exercise 1

For the NAO humanoid robot in Fig. 1, consider only the *left* arm down to the wrist yaw joint, which is frozen together with the remaining joints of all fingers. The left arm has thus four degrees of freedom (dofs). We provide separately a two-sided technical sheet with the kinematic data of this part of the robot. For describing internal motions, the robot has a global reference frame placed at the torso center.

- Assign the reference frames for the first four dofs of the left arm according to the Denavit-Hartenberg (DH) convention, so that the positive senses of joint rotations match those shown in the technical sheet. Place the origin of the last frame at the end of the Lower Arm (i.e., at the Hand base).
- Draw the torso frame (with axes relabeled as $\boldsymbol{x}_T$, $\boldsymbol{y}_T$, and $\boldsymbol{z}_T$) and the DH frames on one (or on both) of the two distributed extra sheets, which show respectively a CAD view of the torso/left arm and a view of the upper limbs. Provide the 4×4 homogeneous matrix ${}^T\boldsymbol{A}_0$ from the torso to the DH frame 0.
- Complete the Denavit-Hartenberg table of parameters associated to the frames that have been assigned. Determine the values of the joint angles $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T$ when the robot stretches its left arm forward and horizontally, just like in the picture on the left of Fig. 1.

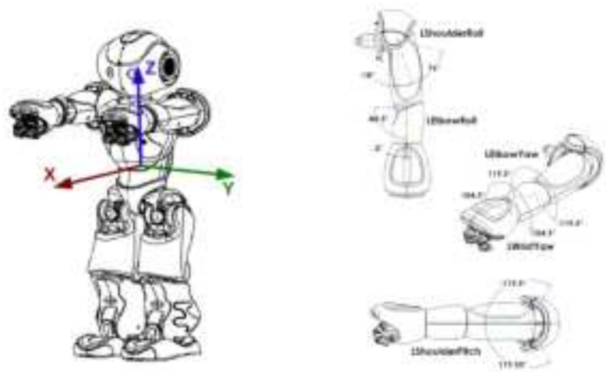


Figure 1: The NAO humanoid robot with the torso frame and three views of its left arm.

Exercise 2

For a planar RP robot with direct kinematics and joint velocity/acceleration limits given respectively by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{V}_{max} = \begin{pmatrix} 2 \\ 2.5 \end{pmatrix} \text{ [rad/s; m/s]}, \quad \boldsymbol{A}_{max} = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix} \text{ [rad/s}^2\text{; m/s}^2\text{]},$$

design a minimum time coordinated trajectory, with the end effector moving rest-to-rest between the initial Cartesian position $\boldsymbol{p}_i = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ and the final position $\boldsymbol{p}_f = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$ [m].

- Provide the minimum motion time T and draw the velocity/acceleration profiles of the two joints.
- For the given data, will the Cartesian path followed by the end-effector be a linear one? Will the robot pass through a singularity during motion? Prove your responses.
- Compute the (velocity) manipulability measure and sketch a plot of its value during the planned motion. Are there Cartesian positions during this motion in which the force and velocity manipulability ellipsoids will coincide? Are there configurations at which the velocity ellipsoid becomes a circle? Explain your responses and comment on these situations.

Exercise 3

A planar 3R robot, having links of equal length ℓ, is being controlled by joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. A desired linear Cartesian trajectory $\boldsymbol{p}_d(t)$ is assigned, which starts from point $\boldsymbol{p}_i$ at time $t = 0$ and reaches point $\boldsymbol{p}_f$ at time $t = T$ and has a rest-to-rest motion profile that is continuous for $t \in [0, T]$ up to the acceleration. With reference to Fig. 2, design a single control law for $\dot{\boldsymbol{q}}$ so that the robot end-effector position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ follows, or at least asymptotically tracks the desired trajectory $\boldsymbol{p}_d(t)$. The following behaviors should be simultaneously enforced.

1. Realize exact trajectory following (i.e., $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t) \equiv \boldsymbol{0}$, $\forall t \in [0, T]$) for suitable initial configurations $\boldsymbol{q}(0) = \boldsymbol{q}_{0,e}$.
2. For any generic initial configuration $\boldsymbol{q}(0) = \boldsymbol{q}_0$, achieve trajectory tracking with the position error $\boldsymbol{e}(t)$ that converges exponentially to zero.
3. The error component $e_n(t)$ along the normal direction to the desired linear path should be reduced three times faster than the error component $e_t(t)$ along the tangential direction.
4. Within half of the nominal motion time $T/2$, the norm of the error $\|\boldsymbol{e}(t)\|$ should be reduced at least to one tenth of its initial value.

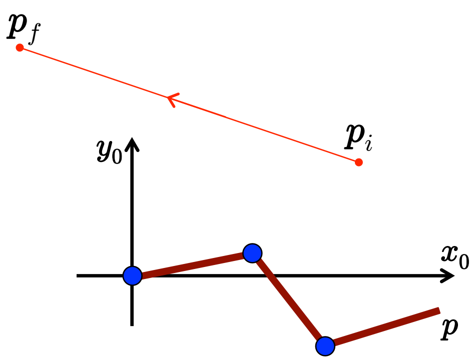


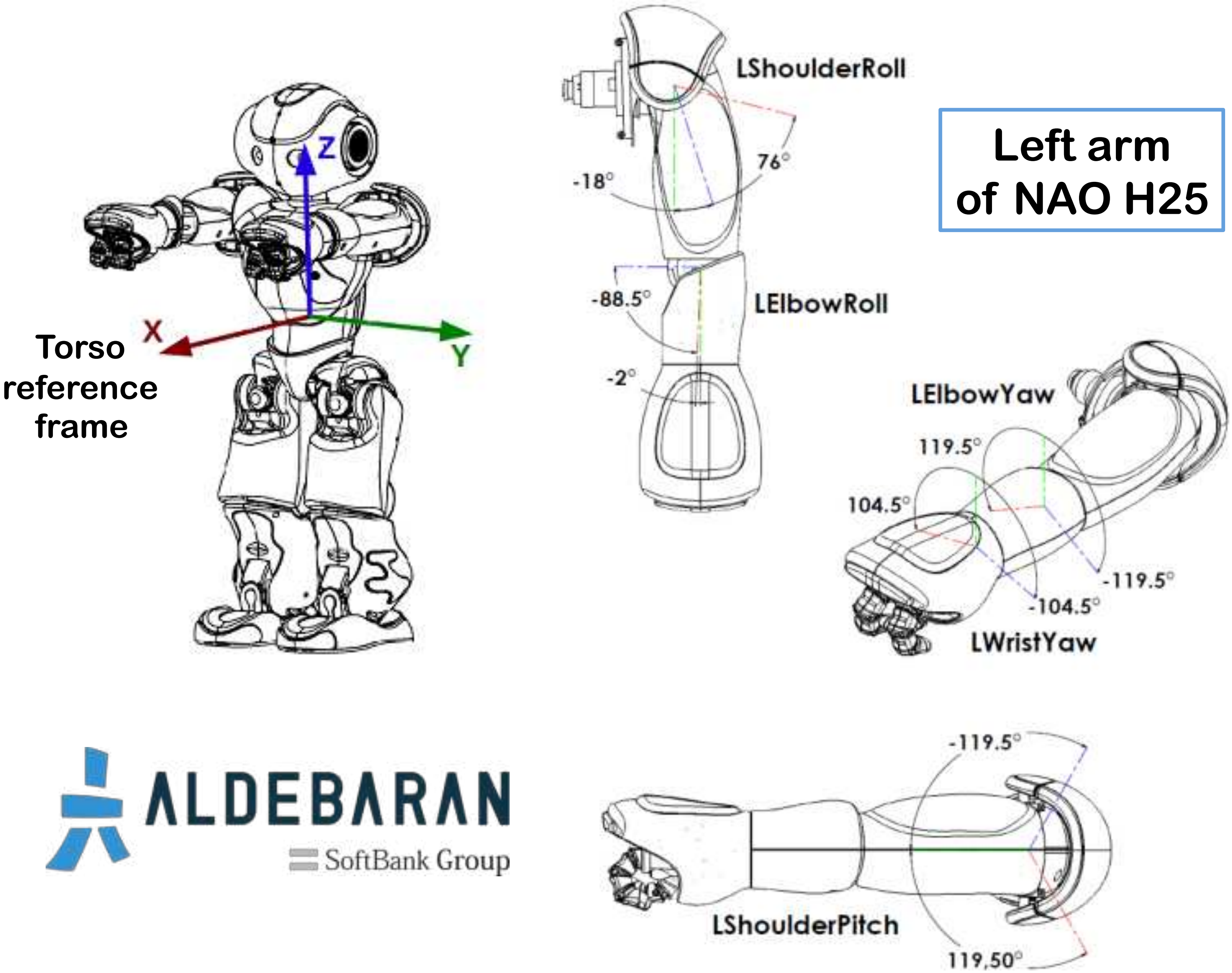
Figure 2: A planar 3R robot and a linear Cartesian trajectory.

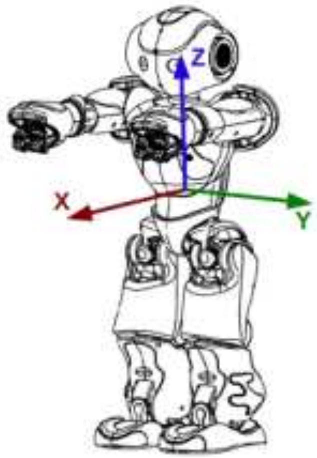
Use the following numerical data:

$$\ell = 2 \text{ [m]}, \qquad \boldsymbol{p}_i = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \boldsymbol{p}_f = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \text{ [m]}, \qquad T = 4 \text{ [s]}.$$

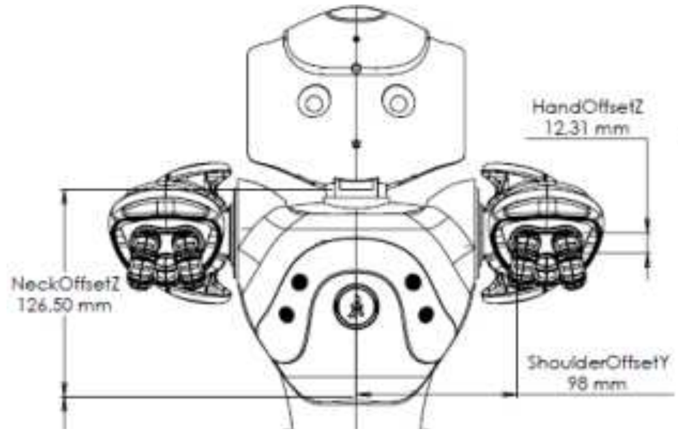
- Determine a possible expression of the desired linear Cartesian trajectory $\boldsymbol{p}_d(t)$.
- Assuming that kinematic singularities are never encountered, provide the explicit symbolic expression of all terms in the control law and the proper numerical values of the control gains.
- Find one possible initial configuration $\boldsymbol{q}_{0,e}$ that leads to exact trajectory following, and explain how to obtain such configurations in general.
- When the robot starts from the configuration $\boldsymbol{q}_0 = \begin{pmatrix} -\pi/2 & 0 & \pi/2 \end{pmatrix}^T$ [rad], compute the numerical value of the joint velocity command $\dot{\boldsymbol{q}}(0)$ at the initial time $t = 0$ with the designed control law.

[240 minutes, open books but no computer or smartphone]

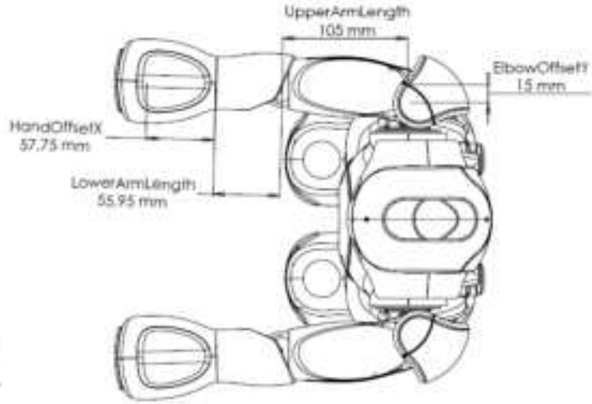

Left arm
of NAO H25
Torso
reference
frame
X
Y
Z
LShoulderRoll
76°
-18°
LElbowRoll
-88.5°
-2°
LElbowYaw
119.5°
104.5°
-119.5°
-104.5°
LWristYaw
LShoulderPitch
-119.5°
119,50°
ALDEBARAN
SoftBank Group



Front view



Top view



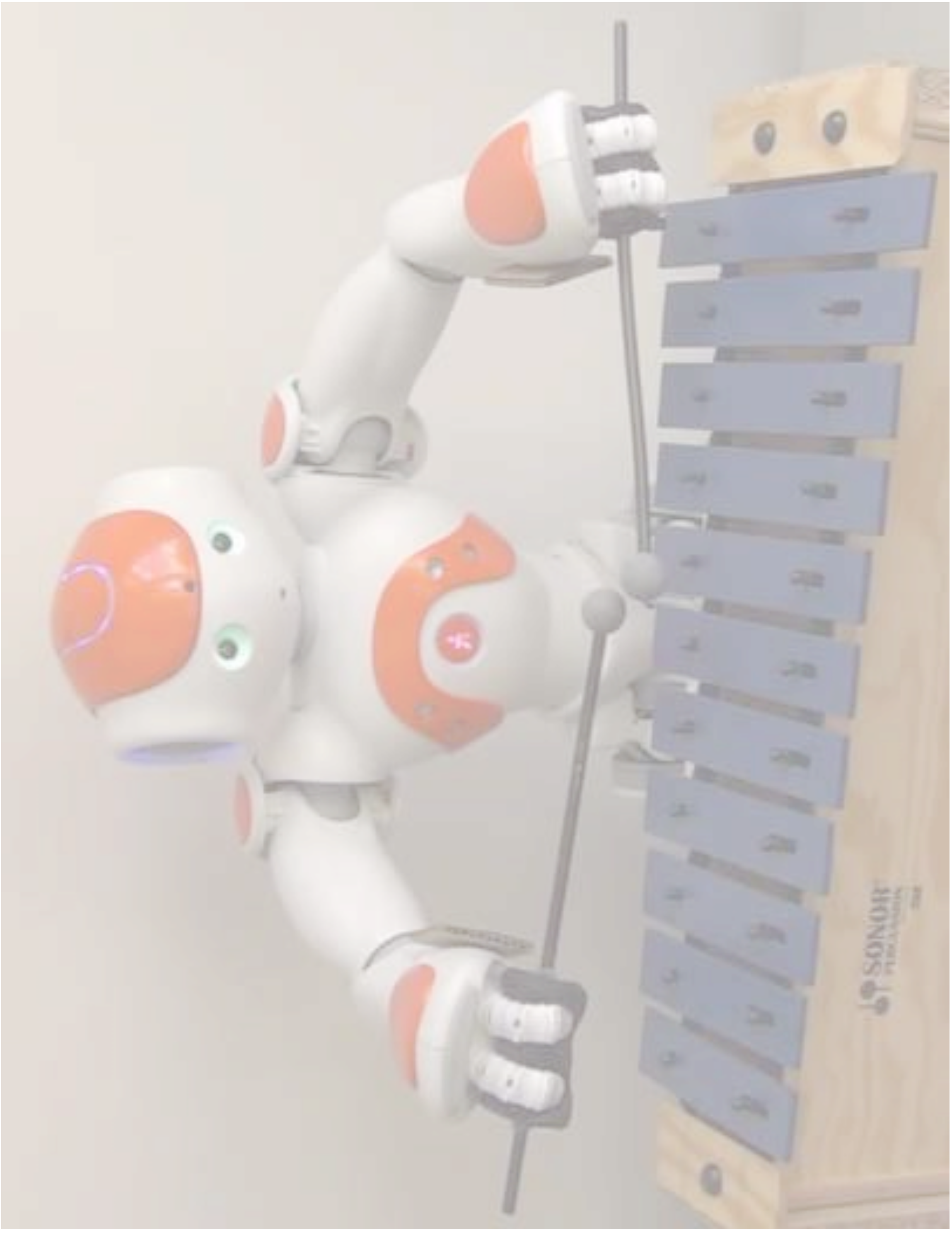
Arms

From ...	To ...	X (mm)	Y (mm)	Z (mm)
Torso	LShoulderPitch	0.00	98.00	100.00
LShoulderPitch	LShoulderRoll	0.00	0.00	0.00
LShoulderRoll	LElbowYaw	105.00	15.00	0.00
LElbowYaw	LElbowRoll	0.00	0.00	0.00
LElbowRoll	LWristYaw	55.95	0.00	0.00

Main length (mm)	
ShoulderOffsetY	98.00
ElbowOffsetY	15.00
UpperArmLength	105.00
LowerArmLength	55.95
ShoulderOffsetZ	100.00
HandOffsetX	57.75
HandOffsetZ	12.31

Name ____________________

Name ____________________



Solution

February 3, 2017

Exercise 1

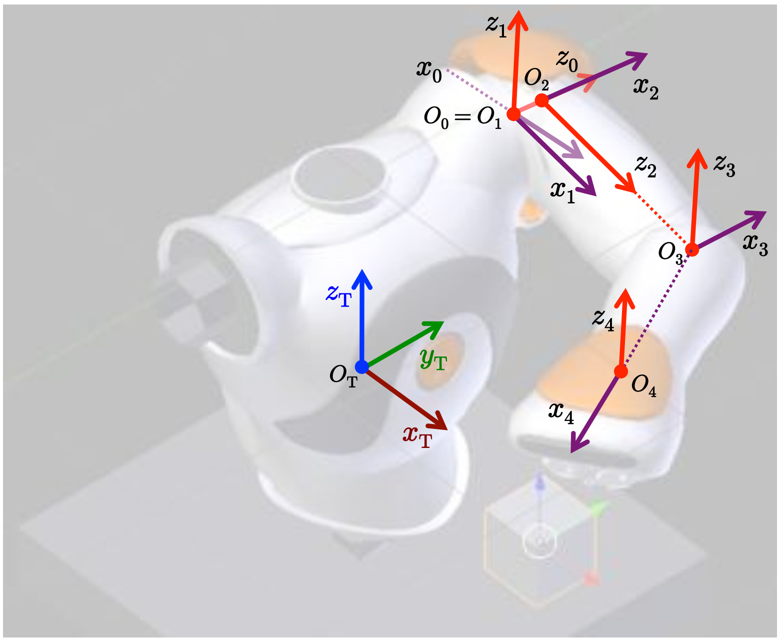


Figure 3: The assigned DH frames on a CAD image of the torso and left arm of the NAO.

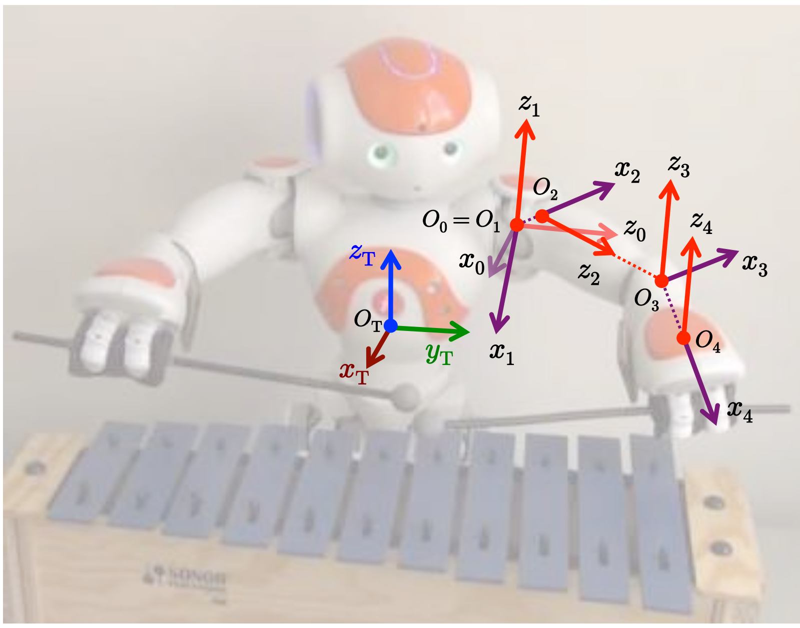
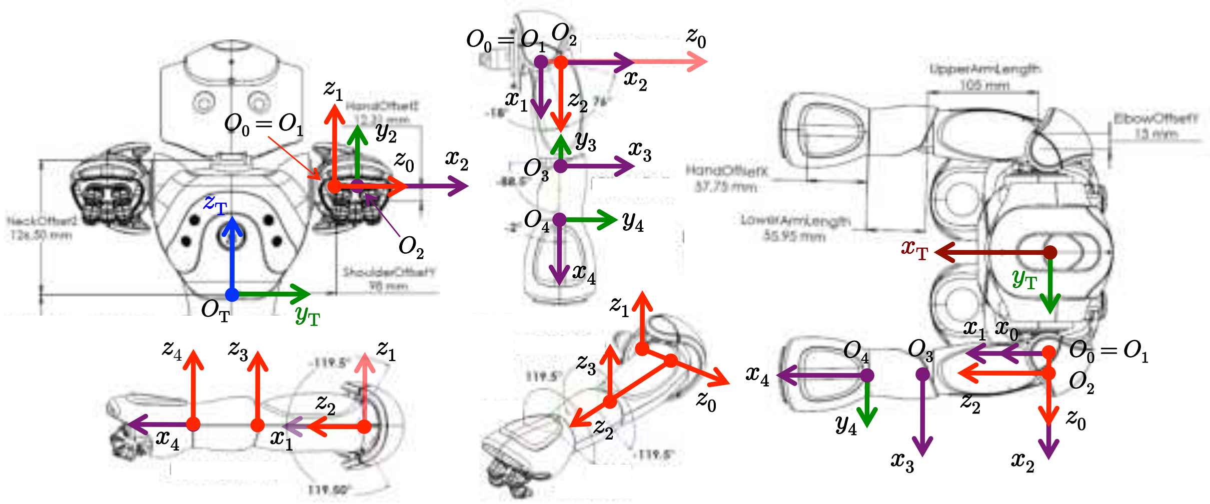


Figure 4: The assigned DH frames displayed on a picture of the upper limbs of the NAO.

The placement of the torso frame and the Denavit-Hartenberg (DH) assignment for the left arm are illustrated in the two Figs. 3 and 4, for different arm configurations and from different perspective views. The first axis of the shoulder is the ShoulderPitch axis $\boldsymbol{z}_0$, followed by the ShoulderRoll axis $\boldsymbol{z}_1$. Please note the small offset (ElbowOffsetY $= 1.5$ cm) between the two incident axes at the robot shoulder and the following axis $\boldsymbol{z}_2$, which provides the ElbowYaw degree of freedom. Without this offset, which would then be added to the value ShoulderOffsetY as a kinematic approximation, the NAO robot would have a spherical shoulder.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	$q_1 = 0$
2	$\pi/2$	ElbowOffsetY $= 15$ mm	0	$q_2 = \pi/2$
3	$-\pi/2$	0	UpperArmLength $= 105$ mm	$q_3 = 0$
4	0	LowerArmLength $= 55.95$ mm	0	$q_4 = -\pi/2$

Figure 5: Front and top views of NAO upper limbs with the arms stretched forward and three views of the left arm [above]; the DH table of parameters [below].

In Fig. 5, the upper limbs of the NAO are shown from the front and top viewpoints, together with three views of the left arm, and the associated DH table is reported. The last column in the table contains the actual values of the joint variables $\boldsymbol{q} = \begin{pmatrix} 0 & \pi/2 & 0 & -\pi/2 \end{pmatrix}^T$ when the left arm is stretched forward and horizontally, as in the picture.

The 4×4 homogeneous matrix ${}^T\!\boldsymbol{A}_0$ from the torso frame to the DH frame 0 is given by

$$
{}^T\!\boldsymbol{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \text{ShoulderOffsetY} \\ 0 & -1 & 0 & \text{ShoulderOffsetZ} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.98 \\ 0 & -1 & 0 & 0.10 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

where lengths are expressed in [m].

Exercise 2

Since the kinematic bounds on motion are imposed on the single joints, it is convenient to plan the interpolating trajectory in the joint space. The minimum transfer time is achieved when each joint executes a bang-bang or bang-coast-bang acceleration trajectory. Coordinated motion of the robot arm is then obtained by uniform time scaling of the faster joint(s), so as to align the final time to that of the joint which has the slowest completion time (due to the values of the velocity and acceleration limits, in combination with the distance to be traveled by that joint).

First, the initial and final Cartesian positions are inverted in the joint space. Considering for simplicity only the inverse solution for the RP robot which has a positive value for the prismatic variable q_2,

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \Rightarrow \qquad q_2 = \sqrt{p_x^2 + p_y^2} > 0, \qquad q_1 = \text{ATAN2}\{p_y, p_x\},$$

we obtain, respectively for $\boldsymbol{p} = \boldsymbol{p}_i$ and $\boldsymbol{p} = \boldsymbol{p}_f$,

$$\boldsymbol{q}_i = \begin{pmatrix} \text{ATAN2}\{3,4\} \\ \sqrt{4^2+3^2} \end{pmatrix} = \begin{pmatrix} 0.6435 \\ 5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q}_f = \begin{pmatrix} \text{ATAN2}\{1,-1\} \\ \sqrt{1^2+(-1)^2} \end{pmatrix} = \begin{pmatrix} 2.3562 \\ \sqrt{2} \end{pmatrix} \quad \text{[rad; m]},$$

yielding

$$\Delta\boldsymbol{q} = \boldsymbol{q}_f - \boldsymbol{q}_i = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} 1.7127 \\ -3.5858 \end{pmatrix} \quad \text{[rad; m]}.$$

From the known formula about the existence of a cruising phase in a trapezoidal velocity profile, since

$$1.7127 = |\Delta q_1| > \frac{V_{max,1}^2}{A_{max,1}} = \frac{4}{3} = 1.3333,$$

we have that the acceleration profile for the first joint is bang-coast-bang, with acceleration/deceleration time and minimum transfer time given by

$$T_{a,1} = \frac{V_{max,1}}{A_{max,1}} = 0.6667 \text{ [s]}, \qquad T_1 = \frac{|\Delta q_1|\, A_{max,1} + V_{max,1}^2}{A_{max,1}\, V_{max,1}} = 1.5230 \text{ [s]}.$$

On the other hand, since

$$3.5858 = |\Delta q_2| < \frac{V_{max,2}^2}{A_{max,2}} = \frac{9}{1.5} = 6,$$

we have that the acceleration profile for the second joint is bang-bang, with acceleration/deceleration time, minimum transfer time, and maximum reached velocity given by

$$T_{a,2} = \sqrt{\frac{|\Delta q_2|}{A_{max,2}}} = 1.5461 \text{ [s]}, \qquad T_2 = 2\,T_{a,2} = 3.0923 \text{ [s]}, \qquad \bar{V}_2 = \frac{|\Delta q_2|}{T_{a,2}} = 2.3192 \text{ [m/s]}.$$

Therefore, the minimum time T for the robot motion will be given by the slowest completion time T_i among all joints (i.e., that of joint 2)

$$T = \max\{T_1, T_2\} = 3.0923 \text{ [s]}.$$

To obtain a coordinated joint trajectory, we need to scale down the motion of joint 1 by the factor

$$k = \frac{T}{T_1} = 2.0304 \qquad \Rightarrow \qquad \dot{q}_{s,1}(t) = \frac{\dot{q}_1(t)}{k}, \qquad \ddot{q}_{s,1}(t) = \frac{\ddot{q}_1(t)}{k^2}.$$

The new trapezoidal velocity profile of joint 1 will have a coordinated motion time $T_{s,1}$ and a reduced cruise velocity $\bar{V}_1$ given by

$$T_{s,1} = k\,T_1 = T = 3.0923 \text{ [s]}, \qquad \bar{V}_1 = \frac{V_{max,1}}{k} = 0.9850 \text{ [rad/s]},$$

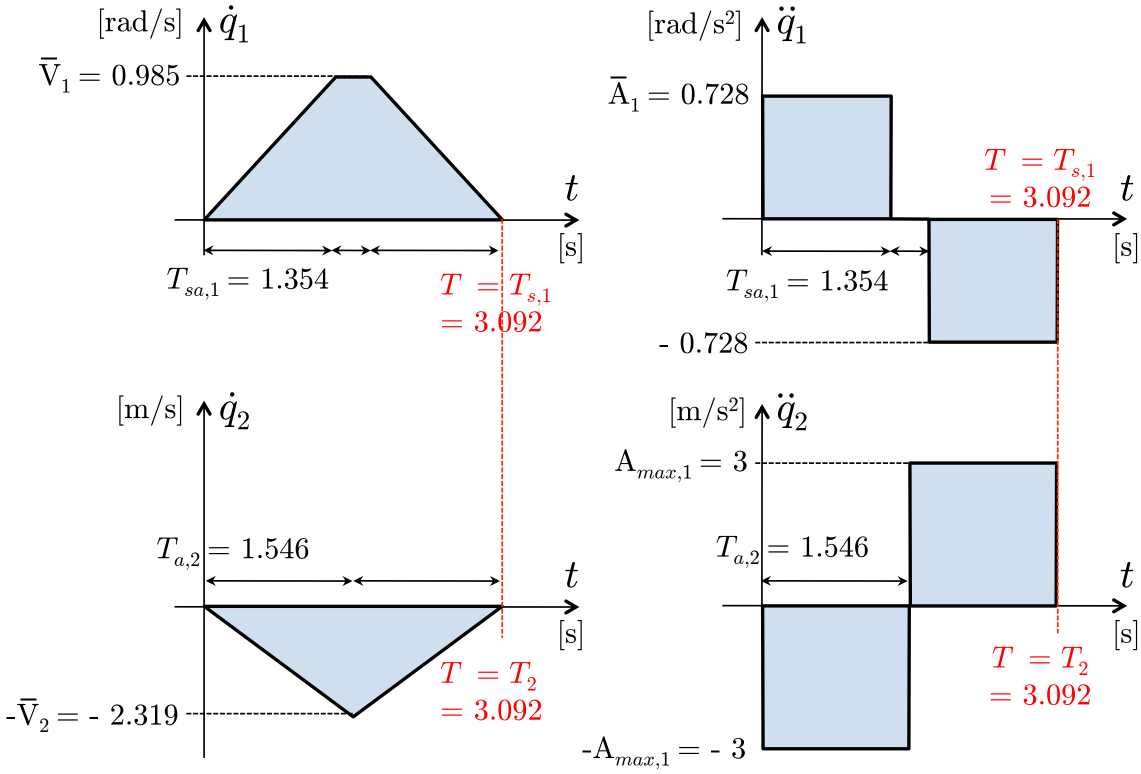


Figure 6: Coordinated minimum time velocity and acceleration profiles of joint 1 [above] and of joint 2 [below].

while the new acceleration profile of joint 1 will have a scaled acceleration/deceleration time $T_{sa,1}$ and a reduced maximum acceleration $\bar{A}_1$ given by

$$T_{sa,1} = k\, T_{a,1} = 1.3536 \text{ [s]}, \qquad \bar{A}_1 = \frac{A_{max,1}}{k^2} = 0.7277 \text{ [rad/s}^2\text{]}.$$

The final velocity and acceleration profiles of the two joints are drawn in Fig. 6. Note that the velocity profile of joint 2 is negative because $\Delta q_2 < 0$.

The Cartesian path corresponding to the designed joint trajectory will *not* be a linear segment joining $\boldsymbol{p}_i$ to $\boldsymbol{p}_f$. To prove this, it is sufficient to show that there exists at least one configuration $\boldsymbol{q}_m = \begin{pmatrix} q_{m1} & q_{m2} \end{pmatrix}^T$ belonging to the joint trajectory that maps (via direct kinematics) into a Cartesian point $\boldsymbol{p}_m = \begin{pmatrix} p_{mx} & p_{my} \end{pmatrix}^T$ which is not on this segment; namely, its Cartesian coordinates will not satisfy

$$\frac{p_x - p_{i,x}}{p_{f,x} - p_{i,x}} = \frac{p_y - p_{i,y}}{p_{f,y} - p_{i,y}} \qquad \Rightarrow \qquad \frac{p_x - 4}{5} = \frac{p_y - 3}{2} \qquad \Leftrightarrow \qquad 2p_x - 5p_y + 7 = 0,$$

which is the equation of a line passing through the two points $\boldsymbol{p}_i = \begin{pmatrix} p_{i,x} & p_{i,y} \end{pmatrix}^T = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ and $\boldsymbol{p}_f = \begin{pmatrix} p_{f,x} & p_{f,y} \end{pmatrix}^T = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$. A simple choice is to pick the midpoint of the joint motion. We have

$$\boldsymbol{q}_m = \frac{\boldsymbol{q}_i + \boldsymbol{q}_f}{2} = \begin{pmatrix} 1.4998 \\ 3.2071 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{p}_m = \begin{pmatrix} q_{m2}\, \cos q_{m1} \\ q_{m2}\, \sin q_{m1} \end{pmatrix} = \begin{pmatrix} 0.2273 \\ 3.1990 \end{pmatrix}$$

$$\Rightarrow \qquad 2 \cdot 0.2273 - 5 \cdot 3.1990 + 7 = -8.5405 \neq 0,$$

which shows that the Cartesian point is not on the linear path from $\boldsymbol{p}_i$ to $\boldsymbol{p}_f$. Moreover, since the analytic Jacobian of the RP robot and its determinant are

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \det \boldsymbol{J}(\boldsymbol{q}) = -q_2,$$

and the motion of the second joint is confined between its initial and final values, i.e., $q_2(t) \in \left[\sqrt{2}, 5\right]$ for all $t \in [0, T]$, then q_2 will never be zero and the robot will not pass through a singularity during motion.

Since the RP robot encounters no singular configurations during motion, the Jacobian will always have full rank and the identity $\boldsymbol{J}^{\#}(\boldsymbol{q})^T \boldsymbol{J}^{\#}(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}$ holds. Thus, the manipulability ellipsoid in velocity and the manipulability measure w are given by

$$\boldsymbol{v}^T \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} \boldsymbol{v} = \boldsymbol{v}^T \begin{pmatrix} q_2^2 \sin^2 q_1 + \cos^2 q_1 & \left(1 - q_2^2\right) \sin q_1 \cos q_1 \\ \left(1 - q_2^2\right) \sin q_1 \cos q_1 & q_2^2 \cos^2 q_1 + \sin^2 q_1 \end{pmatrix} \boldsymbol{v} = 1 \tag{1}$$

and

$$w = \sqrt{\det \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)} = |q_2|\,.$$

As a result, during motion the manipulability w will decrease linearly (no need to plot this!) from the initial value $q_{i,2} = 5$ to the final value $q_{f,2} = \sqrt{2}$. On the other hand, by observing the expression of $\boldsymbol{J}\boldsymbol{J}^T$ in (1), we can immediately see that $\boldsymbol{J}\boldsymbol{J}^T = \boldsymbol{I}$ for $q_2 = 1$, and the manipulability ellipsoid becomes a circle. In this situation, which is not encountered in this particular planned motion, there is an *isotropic* behavior for the transformation of velocities as well as for the transformation of forces.

Exercise 3

The desired Cartesian trajectory $\boldsymbol{p}_d(t)$ can be defined using decomposition in space (tracing a linear path) and time (moving with a quintic polynomial) as

$$\boldsymbol{p}_d(s) = \boldsymbol{p}_i + s\left(\boldsymbol{p}_f - \boldsymbol{p}_i\right), \quad s \in [0,1], \qquad s(t) = 6\left(\frac{t}{T}\right)^5 - 15\left(\frac{t}{T}\right)^4 + 10\left(\frac{t}{T}\right)^3, \quad t \in [0,T], \tag{2}$$

where the six coefficients of a quintic polynomial are necessary and sufficient to impose the required rest-to-rest motion with zero initial velocity and acceleration (which produces a continuous acceleration profile also at the initial and final instants). The desired Cartesian velocity is then

$$\dot{\boldsymbol{p}}_d(t) = \frac{d\boldsymbol{p}_d(s)}{ds}\,\dot{s}(t) = \frac{30\left(\boldsymbol{p}_f - \boldsymbol{p}_i\right)}{T}\left(\left(\frac{t}{T}\right)^4 - 2\left(\frac{t}{T}\right)^3 + \left(\frac{t}{T}\right)^2\right). \tag{3}$$

For later use, we note that the numerical value of $\dot{\boldsymbol{p}}_d(t)$ at the initial time $t = 0$ is indeed $\dot{\boldsymbol{p}}_d(0) = \mathbf{0}$.

In view of the requirements, the control problem should be attacked at the Cartesian level. Since the task is two-dimensional (position tracking in the plane), we consider the following direct kinematics of the 3R robot

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell\left(c_1 + c_{12} + c_{123}\right) \\ \ell\left(s_1 + s_{12} + s_{123}\right) \end{pmatrix}, \tag{4}$$

with the usual shorthand notation, e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$. The associated 2×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell\left(s_1 + s_{12} + s_{123}\right) & -\ell\left(s_{12} + s_{123}\right) & -\ell\, s_{123} \\ \ell\left(c_1 + c_{12} + c_{123}\right) & \ell\left(c_{12} + c_{123}\right) & \ell\, c_{123} \end{pmatrix}. \tag{5}$$

Standard inversion of the non-square Jacobian $\boldsymbol{J}$ is indeed impossible. Rather, we should use the pseudoinverse $\boldsymbol{J}^{\#}$ (or any other form of generalized inversion) in the kinematic control law. Having assumed that the robot Jacobian remains of full rank during the whole execution of the task, we will evaluate the pseudoinverse as

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{6}$$

being $\boldsymbol{J}\boldsymbol{J}^{\#} = \boldsymbol{I}$.

We remind that a Cartesian kinematic control law of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\boldsymbol{e})\,, \qquad \boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad \boldsymbol{K} > 0,\ \text{diagonal}, \tag{7}$$

would satisfy most of the problem requirements, except for the specification of the transient error dynamics along the tangent and normal directions to the Cartesian path. In fact, when setting $\boldsymbol{K} = \text{diag}\{k_i\}$ in the present planar case, the error dynamics along the two orthogonal directions $\boldsymbol{x}$ and $\boldsymbol{y}$ would become linear and decoupled, and the two scalar gains $k_i > 0$, $i = x, y$, in (7) can be chosen so as to yield the desired error decay, i.e.,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\,(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\boldsymbol{e}) = -\boldsymbol{K}\boldsymbol{e} \qquad \Rightarrow \qquad \begin{aligned} \dot{e}_x &= -k_x e_x \\ \dot{e}_y &= -k_y e_y. \end{aligned}$$

However, when using (7), this linear and decoupled dynamics is not displayed along other directions in the plane.

In order to achieve a similar decoupled behavior along the two orthogonal directions $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ which are, respectively, tangent and normal to the linear path, we need to rotate the Cartesian error $\boldsymbol{e} = {}^0\boldsymbol{e}$ into the task frame attached to the path, react to the rotated error ${}^t\boldsymbol{e} = \begin{pmatrix} e_t & e_n \end{pmatrix}^T$ in a decoupled way (so that the two components of ${}^t\boldsymbol{e}$ independently decay at the specified exponential rates), and then map back this control action into a velocity command expressed in the original Cartesian frame (where the robot Jacobian in (5) is also expressed). The kinematic control law (7) becomes then

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t\,\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\boldsymbol{e}\right), \qquad \boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad \boldsymbol{K} = \begin{pmatrix} k_t & 0 \\ 0 & k_n \end{pmatrix} > 0, \tag{8}$$

with the constant 2×2 (planar) rotation matrix ${}^0\boldsymbol{R}_t$ defined from the linear path as

$$\boldsymbol{x}_t = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \boldsymbol{y}_t = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \qquad \Rightarrow \qquad {}^0\boldsymbol{R}_t = \begin{pmatrix} \boldsymbol{x}_t & \boldsymbol{y}_t \end{pmatrix}.$$

To show the resulting closed-loop behavior, let the rotated error be ${}^t\boldsymbol{e} = {}^t\boldsymbol{R}_0\,{}^0\boldsymbol{e} = {}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e}$. Being ${}^0\dot{\boldsymbol{R}}_t = \boldsymbol{0}$ and using (8), the dynamics of ${}^t\boldsymbol{e}$ is

$${}^t\dot{\boldsymbol{e}} = {}^t\boldsymbol{R}_0\,(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) = {}^t\boldsymbol{R}_0\,(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}) = {}^t\boldsymbol{R}_0\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}\boldsymbol{J}^{\#}\left(\dot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t\,\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e}\right)\right) = -\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e} = -\boldsymbol{K}\,{}^t\boldsymbol{e},$$

and thus

$$\begin{aligned} \dot{e}_t &= -k_t e_t \\ \dot{e}_n &= -k_n e_n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} e_t(t) &= e_t(0)\exp(-k_t t) \\ e_n(t) &= e_n(0)\exp(-k_n t). \end{aligned}$$

Having set $T = 4$ [s] for the total motion time, the specification on the transient errors is enforced as follows. Let first $k_t = k_n = k > 0$. Being the norm of a vector invariant w.r.t. rotations ($\|{}^0\boldsymbol{e}\| = \|{}^0\boldsymbol{R}_t\,{}^t\boldsymbol{e}\| = \|{}^t\boldsymbol{e}\|$), it is

$$\|{}^0\boldsymbol{e}(t)\| = \|{}^t\boldsymbol{e}(t)\| = \sqrt{e_t^2(t) + e_n^2(t)} = \exp(-kt)\sqrt{e_t^2(0) + e_n^2(0)} = \exp(-kt)\,\|{}^t\boldsymbol{e}(0)\| = \exp(-kt)\,\|{}^0\boldsymbol{e}(0)\|$$

Thus, from the requested condition at $t = T/2 = 2$,

$$\|{}^0\boldsymbol{e}(2)\| \le \frac{1}{10}\,\|{}^0\boldsymbol{e}(0)\|\,,$$

it follows

$$\exp(-2k)\,\|{}^0\boldsymbol{e}(0)\| \le \frac{1}{10}\,\|{}^0\boldsymbol{e}(0)\| \qquad \Rightarrow \qquad \exp(2k) \ge 10 \qquad \Rightarrow \qquad k \ge 0.5\ln 10 = 1.1513.$$

To complete the design of the control gains, we set then

$$k_t = k = 1.1513, \qquad \text{and} \qquad k_n = 3\,k_t = 3.4539. \tag{9}$$

Finally, from the problem data, we compute

$$\boldsymbol{x}_t = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \begin{pmatrix} \dfrac{-6}{\sqrt{40}} \\ \dfrac{2}{\sqrt{40}} \end{pmatrix} \qquad \Rightarrow \qquad {}^0\boldsymbol{R}_t = \begin{pmatrix} \dfrac{-6}{\sqrt{40}} & \dfrac{-2}{\sqrt{40}} \\ \dfrac{2}{\sqrt{40}} & \dfrac{-6}{\sqrt{40}} \end{pmatrix}. \tag{10}$$

Using the designed position trajectory (2) and its velocity (3), the expression of the direct kinematics (4) and of its Jacobian (5), with the pseudoinverse computed numerically as in (6), the gains (9), and the rotation matrix in (10), the control law (8) can be completely evaluated at every configuration $\boldsymbol{q}$.

The control law (8) achieves exact trajectory following, i.e., $\boldsymbol{e}(t) = \boldsymbol{0}$ for all $t \geq 0$, if and only if the initial configuration $\boldsymbol{q}(0) = \boldsymbol{q}_{0,e}$ of the robot is such that $\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \boldsymbol{p}_i - \boldsymbol{f}(\boldsymbol{q}_{0,e}) = \boldsymbol{0}$. Using (4) with $\ell = 2$ [m], one such solution is immediately found as

$$\boldsymbol{q}_{0,e} = \begin{pmatrix} 0 \\ 0 \\ \pi/2 \end{pmatrix} \text{[rad]} \qquad \Rightarrow \qquad \boldsymbol{f}(\boldsymbol{q}_{0,e}) = \begin{pmatrix} \ell\,(1+1+0) \\ \ell\,(0+0+1) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \boldsymbol{p}_i.$$

More in general, due to the redundancy, finding an initial configuration $\boldsymbol{q}_0$ that matches a desired initial end-effector position $\boldsymbol{p}_0$ for the 3R robot requires the use of an iterative numerical algorithm, such as the gradient or the Newton method. In these two methods, we would need the evaluation of $\boldsymbol{J}^T(\boldsymbol{q})$, which is directly available from (5), or, respectively, of $\boldsymbol{J}^{\#}(\boldsymbol{q})$, which is computed as in (6).

When starting at $\boldsymbol{q}(0) = \boldsymbol{q}_0 = \begin{pmatrix} -\pi/2 & 0 & \pi/2 \end{pmatrix}^T$ [rad], the evaluation of (8) yields as joint velocity control command

$$\begin{aligned}
\dot{\boldsymbol{q}}(0) &= \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\,{}^0\boldsymbol{R}_t\,\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\,(\boldsymbol{p}_i - \boldsymbol{f}(\boldsymbol{q}_0)) \\
&= \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}^{\#} \begin{pmatrix} \frac{-6}{\sqrt{40}} & \frac{-2}{\sqrt{40}} \\ \frac{2}{\sqrt{40}} & \frac{-6}{\sqrt{40}} \end{pmatrix} \begin{pmatrix} 1.1513 & 0 \\ 0 & 3.4539 \end{pmatrix} \begin{pmatrix} \frac{-6}{\sqrt{40}} & \frac{2}{\sqrt{40}} \\ \frac{-2}{\sqrt{40}} & \frac{-6}{\sqrt{40}} \end{pmatrix} \left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0.25 & -0.0833 \\ 0 & 0.1667 \\ -0.25 & 0.4167 \end{pmatrix} \begin{pmatrix} 1.3816 & 0.6908 \\ 0.6908 & 3.2236 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.4539 \\ 6.9078 \end{pmatrix} \text{[rad/s]}.
\end{aligned}$$

* * * * *

Robotics I

April 11, 2017

Exercise 1

The kinematics of a 3R spatial robot is specified by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	L_1	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Table 1: Table of DH parameters of a 3R spatial robot.

- Given a position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin of the end-effector frame, provide the analytic expression of the solution to the inverse kinematics problem.
- For $L_1 = 1$ [m] and $L_2 = L_3 = 1.5$ [m], determine all inverse kinematics solutions in numerical form associated to the end-effector position $\boldsymbol{p} = \begin{pmatrix} -1 & 1 & 1.5 \end{pmatrix}^T$ [m].

Exercise 2

A robot joint should move in minimum time between an initial value q_a and a final value q_b, with an initial velocity $\dot{q}_a$ and a final velocity $\dot{q}_b$, under the bounds $|\dot{q}| \leq V$ and $|\ddot{q}| \leq A$.

- Provide the analytic expression of the minimum feasible motion time T^* when $\Delta q = q_b - q_a > 0$ and the initial and final velocities are arbitrary in sign and magnitude (but both satisfy the velocity bound, i.e., $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$).
- Using the data $q_a = -90°$, $q_b = 30°$, $\dot{q}_a = 45°/\text{s}$, $\dot{q}_b = -45°/\text{s}$, $V = 90°/\text{s}$, $A = 200°/\text{s}^2$, determine the numerical value of the minimum feasible motion time T^* and draw the velocity and acceleration profiles of the joint motion.

[180 minutes, open books but no computer or smartphone]

Solution

April 11, 2017

Exercise 1

From the direct kinematics, using Tab. 1, we obtain for the position of the origin of the end-effector frame

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_2(q_2) \left({}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right) \right)$$

$$\Rightarrow \quad \boldsymbol{p} = \begin{pmatrix} (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \cos q_1 \\ (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \sin q_1 \\ L_1 + L_2 \sin q_2 + L_3 \sin(q_2 + q_3) \end{pmatrix}. \tag{1}$$

The analytic inversion of eq. (1) for $\boldsymbol{p} = \boldsymbol{p}_d = \begin{pmatrix} p_{dx} & p_{dy} & p_{dz} \end{pmatrix}^T$ proceeds as follows. After moving L_1 to the left-hand side of the third equation, squaring and adding the three equations yields the numeric value c_3 (for $\cos q_3$)

$$c_3 = \frac{p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 - L_2^2 - L_3^2}{2L_2L_3}. \tag{2}$$

The desired end-effector position will belong to the robot workspace if and only if $c_3 \in [-1, 1]$. Note that this condition holds no matter if L_2 and L_3 are equal or different. Under such premises, we compute

$$s_3 = \sqrt{1 - c_3^2} \tag{3}$$

and

$$q_3^{\{+\}} = \text{ATAN2}\{s_3, c_3\}, \qquad q_3^{\{-\}} = \text{ATAN2}\{-s_3, c_3\}, \tag{4}$$

yielding by definition two opposite values $q_3^{\{-\}} = -q_3^{\{+\}}$. If $c_3 = \pm 1$, the robot is in a kinematic singularity: the forearm is either stretched or folded, in both cases on the boundary of the workspace. In particular, when $c_3 = 1$, $q_3^{\{+\}}$ and $q_3^{\{-\}}$ are both equal to 0; when $c_3 = -1$, the two solutions will be taken[1] equal to π. Instead, when $c_3 \notin [-1, 1]$, the inverse kinematics algorithm should output a warning message ("desired position is out of workspace") and exit.

When $p_{dx}^2 + p_{dy}^2 > 0$, from the first two equations in (1) we can further compute

$$p_{dx}^2 + p_{dy}^2 = (L_2 \cos q_2 + L_3 \cos(q_2 + q_3))^2 \quad \Rightarrow \quad \cos q_1 = \frac{p_{dx}}{\pm\sqrt{p_{dx}^2 + p_{dy}^2}}, \quad \sin q_1 = \frac{p_{dy}}{\pm\sqrt{p_{dx}^2 + p_{dy}^2}},$$

and thus

$$q_1^{\{+\}} = \text{ATAN2}\{p_{dy}, p_{dx}\}, \qquad q_1^{\{-\}} = \text{ATAN2}\{-p_{dy}, -p_{dx}\}. \tag{5}$$

These two values belong to $(-\pi, \pi]$ and will always differ by π. Instead, when $p_{dx} = p_{dy} = 0$, the first joint angle q_1 remains undefined and the robot will be in a kinematic singularity (with the end-effector placed along the axis of joint 1). The solution algorithm should output a warning message ("singular case: angle q_1 is undefined"), possibly set a flag ($sing_1 = ON$), but continue.

[1] Remember that we use as conventional range $q \in (-\pi, \pi]$, for all angles q. Thus, if the output of a generic computation is $-\pi$, we always replace it with $+\pi$.

At this stage, we can rewrite a suitable combination of the first two equations in (1) as well as the third equation in the following way:

$$\cos q_1 p_{dx} + \sin q_1 p_{dy} = L_2 \cos q_2 + L_3 \cos(q_2 + q_3) = (L_2 + L_3 \cos q_3) \cos q_2 - L_3 \sin q_3 \sin q_2$$

and

$$p_{dz} - L_1 = L_2 \sin q_2 + L_3 \sin(q_2 + q_3) = L_3 \sin q_3 \cos q_2 + (L_2 + L_3 \cos q_3) \sin q_2.$$

Plugging the (multiple) values found so far for q_1 and q_3, we obtain four similar 2×2 linear systems in the trigonometric unknowns $c_2 = \cos q_2$ and $s_2 = \sin q_2$:

$$\begin{pmatrix} L_2 + L_3 c_3 & -L_3 s_3^{\{+,-\}} \\ L_3 s_3^{\{+,-\}} & L_2 + L_3 c_3 \end{pmatrix} \begin{pmatrix} c_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} \cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy} \\ p_{dz} - L_1 \end{pmatrix} \iff \boldsymbol{A}^{\{+,-\}} \boldsymbol{x} = \boldsymbol{b}^{\{+,-\}}. \tag{6}$$

In (6), we should use (2) and the values from (4) and (5). This gives rise to four possible combinations for the matrix/vector pair $\left(\boldsymbol{A}^{\{+,-\}}, \boldsymbol{b}^{\{+,-\}}\right)$, which will eventually lead to four solutions for q_2 that are in general distinct[2]. These will be labeled as

$$q_2^{\{f,u\}} \quad q_2^{\{f,d\}} \quad q_2^{\{b,u\}} \quad q_2^{\{b,d\}} \qquad \Rightarrow \qquad \boldsymbol{q}^{\{f,u\}} \quad \boldsymbol{q}^{\{f,d\}} \quad \boldsymbol{q}^{\{b,u\}} \quad \boldsymbol{q}^{\{b,d\}}$$

depending on whether the robot is facing (f) of backing (b) the desired position quadrant —due to the choice of q_1, and on whether the elbow is up (u) or down (d) —due to the combined choice of q_1 and q_3. If the (common) determinant of the coefficient matrix is different from zero, i.e., using eq. (2),

$$\det \boldsymbol{A}^{\{+,-\}} = (L_2 + L_3 c_3)^2 + L_3^2 \left(s_3^{\{+,-\}}\right)^2 = L_2^2 + L_3^2 + 2L_2L_3c_3 = p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 > 0,$$

the solution for q_2 of each of the above four cases is uniquely determined from

$$\begin{pmatrix} c_2^{\{\{f,b\},\{u,d\}\}} \\ s_2^{\{\{f,b\},\{u,d\}\}} \end{pmatrix} = \begin{pmatrix} (L_2 + L_3 c_3) \left(\cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy}\right) + L_3 s_3^{\{+,-\}} (p_{dz} - L_1) \\ (L_2 + L_3 c_3)(p_{dz} - L_1) - L_3 s_3^{\{+,-\}} \left(\cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy}\right) \end{pmatrix},$$

and henceforth

$$q_2^{\{\{f,b\},\{u,d\}\}} = \text{ATAN2}\left\{ s_2^{\{\{f,b\},\{u,d\}\}}, c_2^{\{\{f,b\},\{u,d\}\}} \right\}. \tag{7}$$

Instead, when $p_{dx} = p_{dy} = 0$ and $p_{dy} = L_1$, the robot will be in a *double* kinematic singularity, with the arm folded and the end-effector placed along the axis of joint 1. Note that this situation can only occur in case the robot has $L_2 = L_3$ (otherwise the singular Cartesian point would be out of the robot workspace). The solution algorithm should output a warning message ("singular case: angle q_2 is undefined"), possibly set a second flag ($sing_2 = ON$), and then exit. In this case, only a single value $q_3 = \pi$ for the third joint angle will be defined.

Moving next to the requested numerical case with $L_1 = 1$, $L_2 = 1.5$, and $L_3 = 1.5$ [m], and for the desired position

$$\boldsymbol{p}_d = \begin{pmatrix} -1 \\ 1 \\ 1.5 \end{pmatrix} \text{ [m]},$$

[2] A special case arises when the joint angle q_1 remains undefined (a singularity with flag $sing_1 = ON$). The first component of the known vector $\boldsymbol{b}$ in (6) will vanish ($p_{dx} = p_{dy} = 0$) and only two solutions would be left for q_2. The case in which these two well-defined solutions collapse into a single value is left to the reader's analysis.

we can see that $\boldsymbol{p}_d$ belongs to the robot workspace and that this is not a singular case since

$$c_3 = -0.5 \in [-1, 1], \qquad p_{dx}^2 + p_{dy}^2 = 2 > 0.$$

We note that the desired position is in the second quadrant ($x < 0, y > 0$). Thus, the four inverse kinematics solutions obtained from (4), (5) and (7) are:

$$\begin{aligned}
\boldsymbol{q}^{\{f,u\}} &= \begin{pmatrix} 2.3562 \\ 1.3870 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 1.3870 \\ -2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 135.00^\circ \\ 79.47^\circ \\ -120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{f,d\}} &= \begin{pmatrix} 2.3562 \\ -0.7074 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 2.3562 \\ 2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 135.00^\circ \\ -40.53^\circ \\ 120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{b,u\}} &= \begin{pmatrix} -0.7854 \\ 1.7546 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ 1.7546 \\ 2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} -45.00^\circ \\ 100.53^\circ \\ 120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{b,d\}} &= \begin{pmatrix} -0.7854 \\ -2.4342 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ -2.4342 \\ -2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} -45.00^\circ \\ -139.47^\circ \\ -120.00^\circ \end{pmatrix}.
\end{aligned} \tag{8}$$

As a double-check of correctness, it is always highly recommended to evaluate the direct kinematics with the obtained solutions (8). In return, one should get every time the desired position $\boldsymbol{p}_d$.

Exercise 2

This exercise is a generalization of the minimum-time trajectory planning problem for a single joint under velocity and acceleration bounds, with zero initial and final velocity (rest-to-rest) as boundary conditions.

It is useful to recap first the solution to the rest-to-rest problem. The minimum-time motion is given by a trapezoidal velocity profile (or a bang-coast-bang profile in acceleration), with minimum motion time T^* and symmetric initial and final acceleration/deceleration phases of duration T_s given by

$$T^* = \frac{|\Delta q|}{V} + \frac{V}{A} > 2T_s, \qquad T_s = \frac{V}{A} > 0. \tag{9}$$

This solution is only valid when the distance $|\Delta q|$ to travel (in absolute value) and the limit velocity and acceleration values $V > 0$ and $A > 0$ satisfy the inequality

$$|\Delta q| \geq \frac{V^2}{A}, \tag{10}$$

namely, when the distance is "sufficiently long" with respect to the ratio of the squared velocity limit to the acceleration limit. When the equality holds in (10), the maximum velocity V is reached only at the single instant $T^*/2 = T_s$, when half of the motion has been completed. Instead, when (10) is violated, the minimum-time motion is given by a bang-bang acceleration profile (i.e., with a triangular velocity profile) having only the acceleration/deceleration phases, each of duration

$$T_s = \sqrt{\frac{|\Delta q|}{A}} \qquad \Rightarrow \qquad T^* = 2T_s. \tag{11}$$

The crusing phase with maximum velocity V is not reached in this case. For all the above cases, when $\Delta q < 0$ the optimal velocity and acceleration profiles are simply changed of sign (flipped over the time axis).

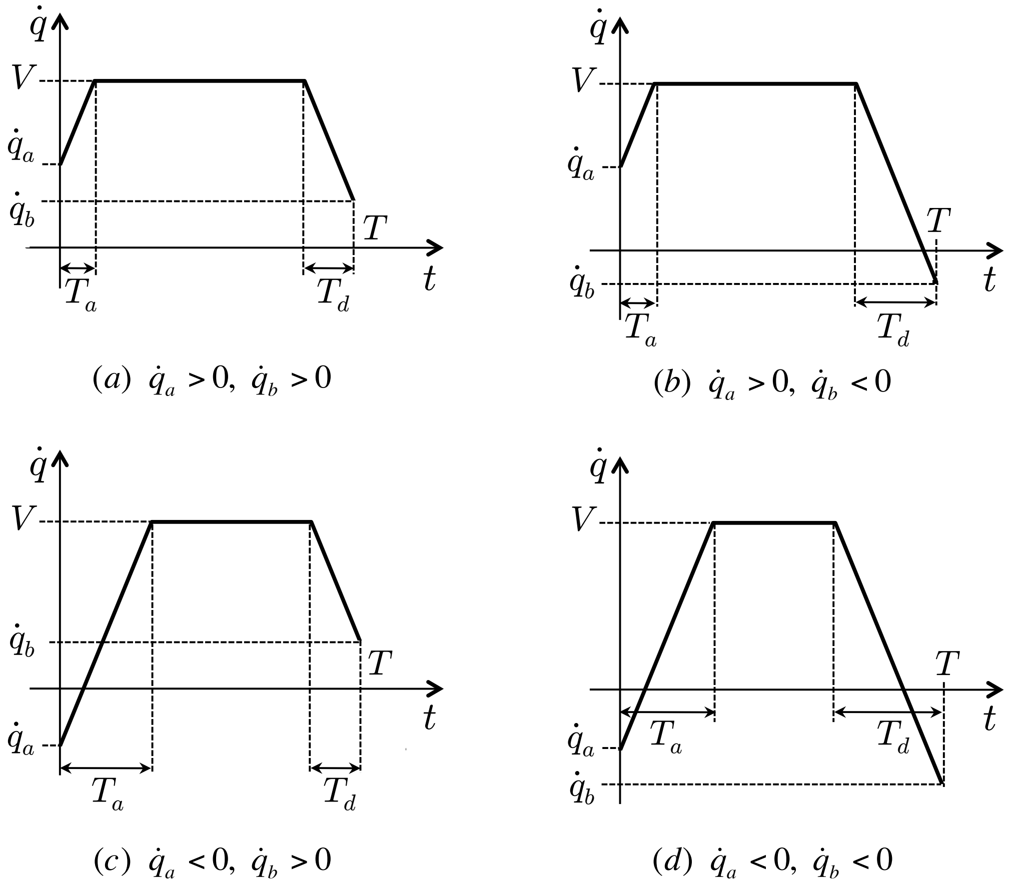


Figure 1: Qualitative asymmetric velocity profiles of the trapezoidal type for the four combinations of signs of the initial and final velocity $\dot{q}_a$ and $\dot{q}_b$. It is assumed that $\Delta q > 0$, and that this distance is sufficiently long so as to have a non-vanishing cruising interval at maximum velocity $\dot{q} = V$.

Consider now the problem of moving in minimum time the joint by a distance $\Delta q = q_b - q_a > 0$, but with generic non-zero boundary conditions $\dot{q}(0) = \dot{q}_a$ and $\dot{q}(T) = \dot{q}_b$ on the initial and final velocity. The requirement that $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$ is obviously mandatory in order to have a feasible solution. With reference to the qualitative trapezoidal velocity profiles sketched in Fig. 1, we see that non-zero initial and final velocities may help in reducing the motion time or work against it. In particular, when both $\dot{q}_a$ and $\dot{q}_b$ are positive (case (a)) it is clear that less time will be needed to ramp up from $\dot{q}_a > 0$ to V, rather than from 0 to V. The same is true for slowing down from V to $\dot{q}_b > 0$, rather than down to 0. On the contrary, when both $\dot{q}_a$ and $\dot{q}_b$ are negative (case (d)), an extra time will be spent for reversing motion from $\dot{q}_a < 0$ to 0 (in this time interval, the joint will continue to move in the opposite direction to the desired one, until it stops), when finally a positive velocity can be achieved, and, similarly, another extra time will be spent toward the end of the trajectory for bringing the velocity from 0 to $\dot{q}_b < 0$ (also in this second interval, the joint will move in the opposite direction to the desired one). Cases (b) and (c) in Fig. 1 are intermediate situations between (a) and (d), and can be analyzed in a similar way.

As a result:

- in general, the acceleration/deceleration phases will have different durations $T_a \geq 0$ and $T_d \geq 0$ (rather than the single $T_s \geq 0$ of the rest-to-rest case);
- the original required distance to travel $\Delta q > 0$ will become in practice longer, since we need to counterbalance the negative displacements introduced during those intervals where the velocity is negative;
- since we need to minimize the total motion time, intervals with negative velocity should be traversed in the least possible time, thus with maximum (positive or negative) acceleration $\ddot{q} = \pm A$.

With the above general considerations in mind, we perform now quantitative calculations. In the (positive) acceleration and (negative) deceleration phases, we have

$$T_a = \frac{V - \dot{q}_a}{A}, \qquad T_d = \frac{V - \dot{q}_b}{A}. \tag{12}$$

We note that both these time intervals will be shorter than $T_s = V/A$ for a positive boundary velocity and longer than T_s for a negative one. The area (with sign) underlying the velocity profile should provide, over the total motion time $T > 0$, the required distance $\Delta q > 0$. We compute this area as the sum of three contributions, using the trapezoidal rule for the two intervals where the velocity is changing linearly over time:

$$T_a \cdot \frac{\dot{q}_a + V}{2} + (T - T_a - T_d) \cdot V + T_d \cdot \frac{V + \dot{q}_b}{2} = \Delta q. \tag{13}$$

Substituting (12) in (13) and rearranging terms gives

$$\frac{(V + \dot{q}_a)(V - \dot{q}_a)}{2A} + \left(T - \frac{2V}{A} + \frac{\dot{q}_a + \dot{q}_b}{A}\right) \cdot V + \frac{(V + \dot{q}_b)(V - \dot{q}_b)}{2A} = \Delta q. \tag{14}$$

Solving for the motion time T, we obtain finally the optimal value

$$T^* = \frac{\Delta q}{V} + \frac{(V - \dot{q}_a)^2 + (V - \dot{q}_b)^2}{2AV}. \tag{15}$$

This is the generalization (for $\Delta q > 0$) of the minimum motion time formula (9) of the rest-to-rest case (which we recover by setting $\dot{q}_a = \dot{q}_b = 0$). This solution is only valid when the distance to travel $\Delta q > 0$, the velocity and acceleration limit values $V > 0$ and $A > 0$, and the boundary velocities $\dot{q}_a$ and $\dot{q}_b$ satisfy the inequality

$$\Delta q \geq \frac{2V^2 - \left(\dot{q}_a^2 + \dot{q}_b^2\right)}{2A} \; (\geq 0), \tag{16}$$

which is again the generalization of condition (10). This inequality is obtained by imposing that the sum of the first and third term in the left-hand side of (14), i.e, the space traveled during the acceleration and deceleration phases, does not exceed Δq (a cruising phase at maximum speed $V > 0$ would no longer be necessary).

It is interesting to note that, for a given triple Δq, V, and A, the inequality (16) would be easier to enforce as soon as $\dot{q}_a \neq 0$ and/or $\dot{q}_b \neq 0$, independently from their signs. The physical reason, however, is slightly different for a positive or negative boundary velocity, say of $\dot{q}_a$. When $\dot{q}_a > 0$, less time is needed in order to reach the maximum velocity $V > 0$; thus, it is more likely that the same problem data will imply a cruising velocity phase. Instead, when $\dot{q}_a < 0$, a negative displacement will occur in the initial phase, which needs to be recovered; thus, it is more likely that a cruising phase at maximum velocity V will be needed later.

Finally, we point out that:

- when inequality (16) is violated, or for special values of $\dot{q}_a$ or $\dot{q}_b$ (e.g., $\dot{q}_a = V$), a number of sub-cases arise; their complete analysis is out of the present scope and is left as an exercise for the reader;
- for $\Delta q < 0$, it is easy to show that the formulas corresponding to (12), (15), and (16) are
$$T_a = \frac{V + \dot{q}_a}{A}, \qquad T_d = \frac{V + \dot{q}_b}{A}, \qquad T^* = \frac{|\Delta q|}{V} + \frac{(V + \dot{q}_a)^2 + (V + \dot{q}_b)^2}{2AV},$$
$$|\Delta q| \geq \frac{2V^2 - \left(\dot{q}_a^2 + \dot{q}_b^2\right)}{2A}.$$
Indeed, the velocity profiles in Fig. 1 will use the value $-V$ as cruising velocity.

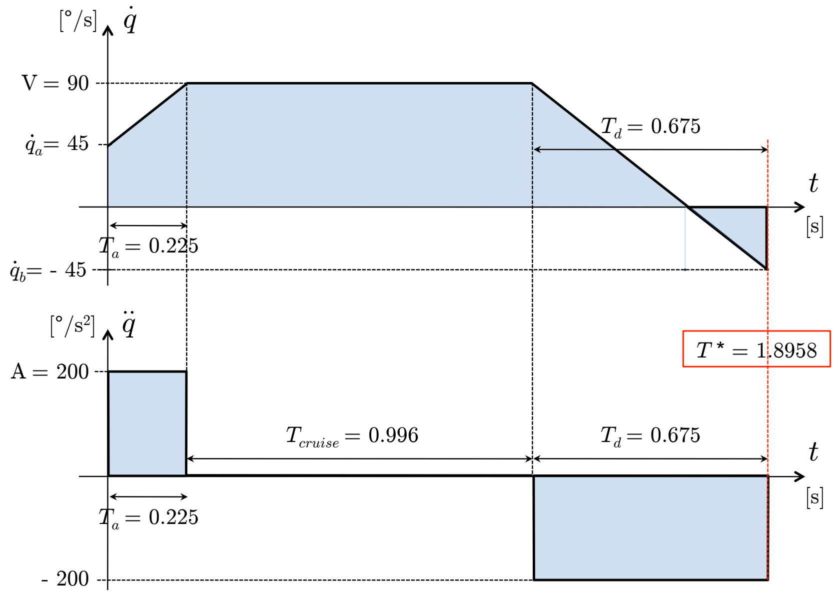


Figure 2: Time-optimal velocity and acceleration profiles for the numerical problem in Exercise 2.

Moving to the given numerical problem, from $\Delta q = q_b - q_a = 30° - (-90°) = 120° > 0$, $\dot{q}_a = 45°/\text{s}$, $\dot{q}_b = -45°/\text{s}$, $V = 90°/\text{s}$, and $A = 200°/\text{s}^2$, we evaluate first the inequality (16) and verify that
$$120 > \frac{2 \cdot 90^2 - \left(45^2 + (-45)^2\right)}{2 \cdot 200} = 30.375,$$
so that the general formula (15) applies. This yields
$$T^* = 1.8958 \text{ [s]},$$
while from (12) we obtain
$$T_a = 0.225 \text{ [s]}, \qquad T_d = 0.675 \text{ [s]},$$
with an interval of duration $T_{cruise} = T^* - T_a - T_d = 0.9958$ [s] in which the joint is cruising at $V = 90°/\text{s}$. The associated time-optimal velocity and acceleration profiles are reported in Fig. 2.

* * * * *

Robotics I

June 6, 2017

Exercise 1

Consider the planar PRPR manipulator in Fig. 1. The joint variables defined therein are those used by the manufacturer and do not correspond necessarily to a Denavit-Hartenberg (DH) convention. The blue arrow indicates in this case the *positive* increments of the joint variable.

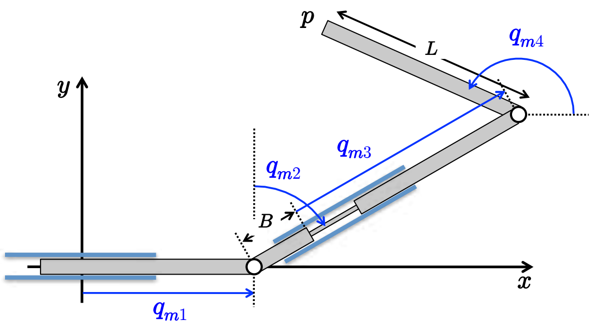


Figure 1: A planar PRPR manipulator with the joint variables $\boldsymbol{q}_m = (q_{m1}, q_{m2}, q_{m3}, q_{m4})$ defined by the manufacturer.

- Determine the direct kinematics $\boldsymbol{f}_m(\boldsymbol{q}_m)$ for the planar position $\boldsymbol{p} \in \mathbb{R}^2$ of the end-effector w.r.t. the given $(\boldsymbol{x}, \boldsymbol{y})$ reference frame, using the joint variables $\boldsymbol{q}_m$ provided by the manufacturer.
- Assign a set of frames according to the DH convention, draw them on the manipulator, and compile the associated table of DH parameters. Choose the origin of frame 0 coincident with that of the manufacturer.
- Determine again the direct kinematics $\boldsymbol{f}(\boldsymbol{q})$ for the planar position $\boldsymbol{p} \in \mathbb{R}^2$ of the end-effector w.r.t. the DH frame 0, using now the joint variables $\boldsymbol{q}$ of your DH notation.
- Draw the manipulator in the two *zero* configurations: the one associated with $\boldsymbol{q}_m = \boldsymbol{0}$ and the one with $\boldsymbol{q} = \boldsymbol{0}$.
- Establish the transformation $\boldsymbol{\phi}(\cdot)$ needed to map one set of variables into the other, say $\boldsymbol{q}_m \to \boldsymbol{q}$, so that for any given $\boldsymbol{q}_m$ the transformed $\boldsymbol{q} = \boldsymbol{\phi}(\boldsymbol{q}_m)$ describes the same physical configuration of the manipulator (in particular, $\boldsymbol{f}_m(\boldsymbol{q}_m) = \boldsymbol{f}(\boldsymbol{\phi}(\boldsymbol{q}_m))$.
- Suppose that a joint velocity command $\dot{\boldsymbol{q}}$ has been determined using routines and control software that operates with the DH variables $\boldsymbol{q}$. What velocity command $\dot{\boldsymbol{q}}_m$ should be transferred to the low-level controller of the robot manufacturer in order to obtain the expected motion behavior?

Exercise 2

With reference to Fig. 2, a laser sensor placed at the base of a 3R robot is monitoring an angular sector of the workspace, scanning it on a horizontal plane at a given height, with minimum and maximum radial sensing distance ρ_{min} and ρ_{max}, respectively. When a human presence is detected by the laser sensor and localized by vector $\boldsymbol{r}_{human}$, the robot end-effector should be controlled for safety, so as to: either *i)* retract with a velocity $\boldsymbol{v}_e^{[i]}$ along the horizontal and centripetal direction of the workspace, with a scalar speed which is inversely proportional to the distance $\|\boldsymbol{r}_{human}\|$; or *ii)* be repulsed with a velocity $\boldsymbol{v}_e^{[ii]}$ along an horizontal direction and away from the detected human, with a scalar speed which is inversely proportional to the relative distance between the human and the robot end-effector. The two alternative reactions occur only when $\|\boldsymbol{r}_{human}\| \in [\rho_{min}, \rho_{max}]$. Beyond ρ_{max} the robot continues its original task, as specified by a joint velocity $\dot{\boldsymbol{q}}_{task}$. Finally, when the human is closer than ρ_{min} to the robot base, the robot should simply stop.

Provide the symbolic expression of the commanded joint velocity $\dot{\boldsymbol{q}}$ that realizes these human-robot coexistence control rules in all the considered cases.

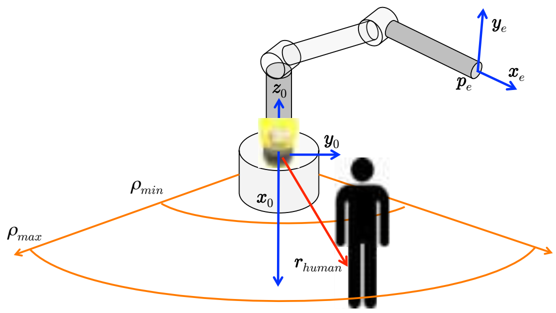


Figure 2: Human-robot coexistence, with a laser scanner measuring the position of a human w.r.t. the base of a spatial 3R robot.

Exercise 3

A generic revolute joint of a robot should move through the sequence of four knots specified by the angular values

$$q_1 = 70°, \quad q_2 = -45°, \quad q_3 = 10°, \quad q_4 = 100°.$$

Define an interpolating path in the form of a cubic spline $q = q(s) \in C^2$, for $s \in [0, 1]$, such that at $s_1 = 0$, $s_2 = 0.25$, $s_3 = 0.5$, and $s_4 = 1$, we have

$$q(0) = q_1, \quad q(0.25) = q_2, \quad q(0.5) = q_3, \quad q(1) = q_4.$$

The path tangent $q'(s)$ at the initial and final knots is given by $(q_2 - q_1)/(s_2 - s_1)$ and $(q_4 - q_3)/(s_4 - s_3)$, respectively.

- Provide the parametric expression and the associated numerical values of the coefficients for the three cubic polynomials in the spline solution.
- Assuming that the path is executed at a constant speed $\dot{s} = 0.4$/s:
  - what is the total traveling time T from q_1 to q_4?
  - which is the value of the joint velocity at the initial and final time, $\dot{q}(0)$ and $\dot{q}(T)$?
  - what is the maximum absolute value of the acceleration $\ddot{q}(t)$ reached during the time interval $[0, T]$, and which is the time instant $t = t_a$ at which this maximum occurs?

[240 minutes, open books but no computer or smartphone]

Solution

June 6, 2017

Exercise 1

A frame assignment for the PRPR manipulator which is consistent with the Denavit-Hartenberg (DH) convention is shown in Fig. 3, and the associated parameters are given in Tab. 1.

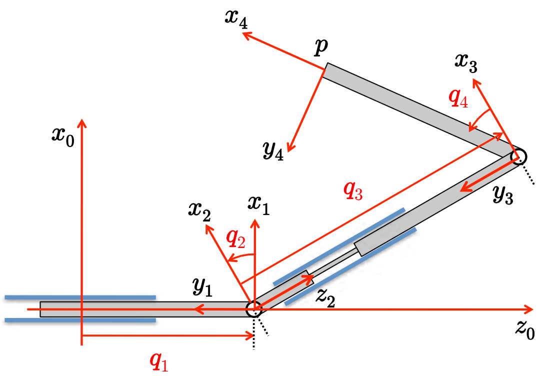


Figure 3: The DH frames assigned to the planar PRPR manipulator, with associated joint variables.

i	α_i	d_i	a_i	θ_i
1	$-\pi/2$	q_1	0	0
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	q_3	0	0
4	0	0	L	q_4

Table 1: Table of DH parameters of the planar PRPR manipulator.

Using the joint variables $\boldsymbol{q}_m = (q_{m1}, q_{m2}, q_{m3}, q_{m4})$ of the manufacturer, by geometric inspection of Fig. 1 the direct kinematics of the PRPR manipulator is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_{m1} + (B + q_{m3}) \sin q_{m2} + L \cos q_{m4} \\ (B + q_{m3}) \cos q_{m2} + L \sin q_{m4} \end{pmatrix} = \boldsymbol{f}_m(\boldsymbol{q}_m). \tag{1}$$

Similarly, using the D-H variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$, by geometric inspection of Fig. 3 (or, computing the sequence of matrix/vector products ${}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T\right)\right)\right)$ from Tab. 1, and then extracting the $\boldsymbol{z}_0$ and $\boldsymbol{x}_0$ components) we have

$$\boldsymbol{p} = \begin{pmatrix} {}^0p_z \\ {}^0p_x \end{pmatrix} = \begin{pmatrix} q_1 + q_3 \cos q_2 - L \sin(q_2 + q_4) \\ q_3 \sin q_2 + L \cos(q_2 + q_4) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{2}$$

The zero configurations of the robot arm when $\boldsymbol{q}_m = \boldsymbol{0}$ and when $\boldsymbol{q} = \boldsymbol{0}$ are obviously different

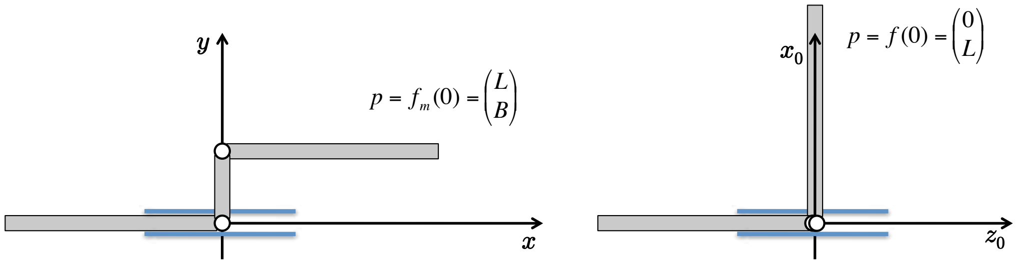


Figure 4: Zero configurations of the robot arm: when $\boldsymbol{q}_m = \boldsymbol{0}$ (left) and when $\boldsymbol{q} = \boldsymbol{0}$ (right).

and are shown in Fig. 4. By comparing eqs. (1) and (2), we find the direct transformation

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} q_{m1} \\ \pi/2 - q_{m2} \\ B + q_{m3} \\ q_{m2} + q_{m4} - \pi \end{pmatrix} = \boldsymbol{\phi}(\boldsymbol{q}_m). \tag{3}$$

From the inverse transformation, we obtain also the linear mapping from the DH velocities $\dot{\boldsymbol{q}}$ to the robot joint velocities $\dot{\boldsymbol{q}}_m$ used by the manufacturer:

$$\boldsymbol{q}_m = \boldsymbol{\phi}^{-1}(\boldsymbol{q}) = \begin{pmatrix} q_1 \\ \pi/2 - q_2 \\ q_3 - B \\ q_2 + q_4 + \pi/2 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}. \tag{4}$$

Exercise 2

Define the projections along the $\boldsymbol{z}$ axis (on an arbitrary plane, say $\boldsymbol{z} = \boldsymbol{0}$) for the two vectors

$$\boldsymbol{p}_e = \begin{pmatrix} p_{e,x} \\ p_{e,y} \\ p_{e,z} \end{pmatrix} \quad \Rightarrow \quad \bar{\boldsymbol{p}}_e = \Pi_z(\boldsymbol{p}_e) = \begin{pmatrix} p_{e,x} \\ p_{e,y} \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{r}_{human} = \begin{pmatrix} r_{human,x} \\ r_{human,y} \\ h \end{pmatrix} \quad \Rightarrow \quad \bar{\boldsymbol{r}}_{human} = \Pi_z(\boldsymbol{r}_{human}) = \begin{pmatrix} r_{human,x} \\ r_{human,y} \\ 0 \end{pmatrix},$$

where h is the (arbitrary) height of the laser scanning plane. The joint velocity command for the requested robot motion is then given by the following cases:

$$\dot{\boldsymbol{q}} = \begin{cases} \boldsymbol{0}\,, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| < \rho_{min} \\ \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}_e\,, \text{ with } \boldsymbol{v}_e = \begin{cases} \boldsymbol{v}_e^{[i]} = -\dfrac{\bar{\boldsymbol{p}}_e}{\|\bar{\boldsymbol{p}}_e\|} \cdot \dfrac{1}{\|\bar{\boldsymbol{r}}_{human}\|} \\ \boldsymbol{v}_e^{[ii]} = \dfrac{\bar{\boldsymbol{p}}_e - \bar{\boldsymbol{r}}_{human}}{\|\bar{\boldsymbol{p}}_e - \bar{\boldsymbol{r}}_{human}\|^2} \end{cases}, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| \in [\rho_{min}, \rho_{max}] \\ \dot{\boldsymbol{q}}_{task}\,, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| > \rho_{max}. \end{cases} \tag{5}$$

Exercise 3

Using path parametrization, the three cubic tracts of the interpolating spline are conveniently defined as

$$q_A(\sigma_A) = q_1 + a_1\sigma_A + a_2\sigma_A^2 + a_3\sigma_A^3, \qquad \sigma_A = \frac{s - s_1}{s_2 - s_1} \in [0,1], \quad s \in [s_1, s_2] \tag{6}$$

$$q_B(\sigma_B) = q_2 + b_1\sigma_B + b_2\sigma_B^2 + b_3\sigma_B^3, \qquad \sigma_B = \frac{s - s_2}{s_3 - s_2} \in [0,1], \quad s \in [s_2, s_3] \tag{7}$$

$$q_C(\sigma_C) = q_3 + c_1\sigma_C + c_2\sigma_C^2 + c_3\sigma_C^3, \qquad \sigma_C = \frac{s - s_3}{s_4 - s_3} \in [0,1], \quad s \in [s_3, s_4], \tag{8}$$

with the nine coefficients $a_1, \ldots, c_3$ determined by satisfying the nine boundary conditions

$$\begin{array}{llll} q_A(1) = q_2, & & & \\ & q'_A(0) = \dfrac{q_2 - q_1}{s_2 - s_1}, & q'_A(1) = q'_B(0)\,[= v_2\,], & q''_A(1) = q''_B(0), \\ q_B(1) = q_3, & & & \\ & q'_C(1) = \dfrac{q_4 - q_3}{s_4 - s_3}, & q'_B(1) = q'_C(0)\,[= v_3\,], & q''_B(1) = q''_C(0). \\ q_C(1) = q_4, & & & \end{array} \tag{9}$$

However, the *divide et impera* approach is more convenient if we assume, for the time being, that we know the value of the first derivatives v_2 and v_3 at the two intermediate knots (at $s = s_2$ and $s = s_3$, respectively). The coefficients of each of the three cubic polynomials would then be completely defined by the four local boundary conditions at the two extremes of their interval of definition. Performing computations for the cubic A yields the coefficients

$$a_1 = q_2 - q_1, \qquad a_2 = (q_2 - q_1) - v_2(s_2 - s_1), \qquad a_3 = v_2(s_2 - s_1) - (q_2 - q_1), \tag{10}$$

and thus

$$q''_A(1) = \frac{2a_2 + 6a_3}{(s_2 - s_1)^2} = \frac{4v_2}{s_2 - s_1} - \frac{4(q_2 - q_1)}{(s_2 - s_1)^2}. \tag{11}$$

Similarly, for the cubic B

$$b_1 = v_2(s_3 - s_2), \quad b_2 = 3(q_3 - q_2) - (2v_2 + v_3)(s_3 - s_2), \quad b_3 = -2(q_3 - q_2) + (v_2 + v_3)(s_3 - s_2), \tag{12}$$

and thus

$$q''_B(0) = \frac{2b_2}{(s_3 - s_2)^2} = \frac{6(q_3 - q_2)}{(s_3 - s_2)^2} - \frac{4v_2 + 2v_3}{s_3 - s_2} \tag{13}$$

and

$$q''_B(1) = \frac{2b_2 + 6b_3}{(s_3 - s_2)^2} = \frac{2v_2 + 4v_3}{s_3 - s_2} - \frac{6(q_3 - q_2)}{(s_3 - s_2)^2}. \tag{14}$$

Finally, for the cubic C

$$c_1 = v_3(s_4 - s_3), \qquad c_2 = 2(q_4 - q_3) - 2v_3(s_4 - s_3), \qquad c_3 = v_3(s_4 - s_3) - (q_4 - q_3), \tag{15}$$

and thus

$$q''_C(0) = \frac{2c_2}{(s_4 - s_3)^2} = \frac{4(q_4 - q_3)}{(s_4 - s_3)^2} - \frac{4v_3}{s_4 - s_3}. \tag{16}$$

Imposing the continuity of the second derivative at the internal knots

$$q''_A(1) = q''_B(0) \qquad q''_B(1) = q''_C(0),$$

and using eqs. (11), (13–14) and (16), leads to the linear system of equations

$$\boldsymbol{A}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{b}$$

with

$$\boldsymbol{A} = \begin{pmatrix} 4(s_3 - s_1) & 2(s_2 - s_1) \\ 2(s_4 - s_3) & 4(s_4 - s_2) \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 6(q_3 - q_2)\,\dfrac{s_2 - s_1}{s_3 - s_2} + 4(q_2 - q_1)\,\dfrac{s_3 - s_2}{s_2 - s_1} \\ 4(q_4 - q_3)\,\dfrac{s_3 - s_2}{s_4 - s_3} + 6(q_3 - q_2)\,\dfrac{s_4 - s_3}{s_3 - s_2}. \end{pmatrix}$$

Replacing the numerical data, the system is solved as

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} -147.2727 \\ 329.0909 \end{pmatrix},$$

and the coefficients (10), (12), and (15) of the three cubic polynomials take then the numerical values

$$\begin{aligned} &a_0 = q_1 = 70, \quad a_1 = -115, \quad a_2 = -78.1818, \quad a_3 = 78.1818, \\ &b_0 = q_2 = -45, \quad b_1 = -36.8182, \quad b_2 = 156.3636, \quad b_3 = -64.5455, \\ &c_0 = q_3 = 10, \quad c_1 = 164.5455, \quad c_2 = -149.0909, \quad c_3 = 74.5455. \end{aligned}$$

The plots of the interpolating geometric spline $q(s)$, for $s \in [0,1]$, and of its first and second derivatives $q'(s)$ and $q''(s)$ are shown in Fig. 5.

If the motion on the geometric path $q(s)$ is at constant speed $\dot{s} = k = 0.4\ \mathrm{s}^{-1}$ ($\ddot{s} = 0$), then the total time for tracing the interval between $s = 0$ and $s = 1$ is simply $T = 1/k = 1/0.4 = 2.5$ s. Moreover, velocity and acceleration of the joint moving at constant speed are $\dot{q} = q'\,\dot{s} = q'\,k$ and $\ddot{q} = q'\,\ddot{s} + q''\,\dot{s}^2 = q''\,k^2$, respectively. Therefore, the initial and final velocity will be

$$\dot{q}(0) = q_A'(0)\,\dot{s}(0) = \frac{q_2 - q_1}{s_2 - s_1}\,k = -184°/\mathrm{s}, \qquad \dot{q}(T) = q_C'(1)\,\dot{s}(T) = \frac{q_4 - q_3}{s_4 - s_3}\,k = 72°/\mathrm{s}.$$

On the other hand, since the second derivative of a geometric cubic spline is made by linear segments between the knots, its maximum necessarily occurs in one of the knots. Being the joint acceleration simply a scaled version of the second spatial derivative, the maximum (absolute) value of the acceleration is found as

$$\begin{aligned} \max_{t\in[0,T]} |\ddot{q}(t)| &= k^2 \max_{s\in[0,1]} |q''(s)| = k^2 \max\{|q_A''(0)|, |q_B''(0)|, |q_C''(0)|, |q_C''(1)|\} \\ &= 0.16\ \max\{|-2501.8|, |5003.6|, |-1192.7|, |596.3|\} \\ &= \max\{|-400.2909|, |800.5818|, |-190.8364|, |95.4182|\} = 800.5818°/\mathrm{s}^2, \end{aligned}$$

occurring at the second knot. The plots of the interpolating trajectory $q(t)$, for $t \in [0,T]$, and of its velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$ are shown in Fig. 6.

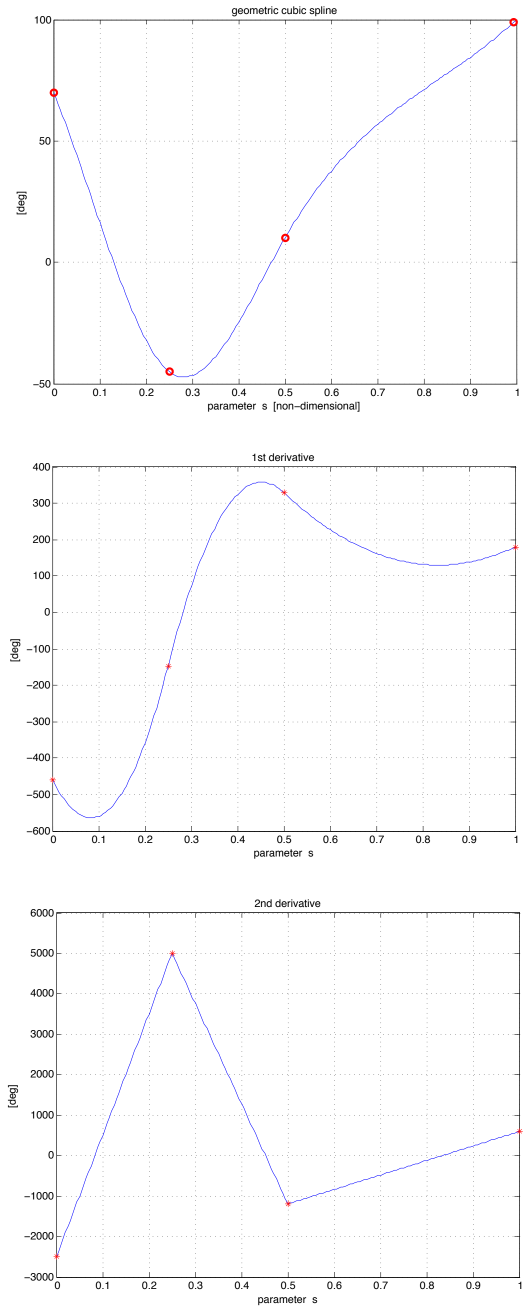


Figure 5: Geometric spline $q(s)$, with first and second derivative.

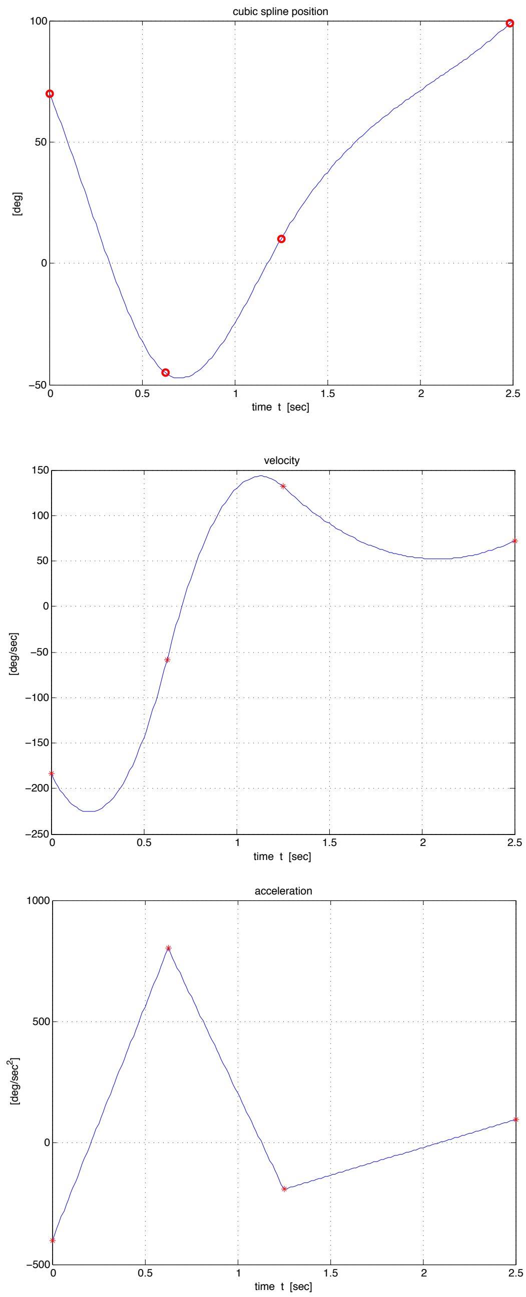


Figure 6: Spline trajectory $q(t)$, with velocity and acceleration.

* * * * *

Robotics I

July 11, 2017

Exercise 1

Consider the 5-dof GMF M-100 manipulator sketched in Fig. 1, having a RPP (cylindric) sequence for the first three main joints and two more revolute joints with intersecting axes at the wrist.

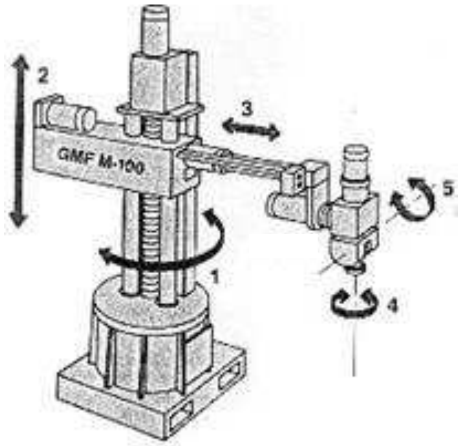


Figure 1: The GMF M-100, a 5-dof manipulator with a RPP-RR sequence of joints.

- Assign the link frames according to the Denavit-Hartenberg convention and derive the associated table of parameters. Place the origin of the reference frame (frame 0) at the robot base on the floor, and choose the origin of the last frame (frame 5) such that $d_5 = 0$.
- Derive the direct kinematics for the position $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) \in \mathbb{R}^3$ of the origin of the last frame. What if we use cylindrical coordinates $\boldsymbol{p}_{cyl} = \begin{pmatrix} \phi & h & r \end{pmatrix}^T$ to describe this Cartesian position?
- Derive the explicit expression of the 6×5 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and analyze if and when this matrix loses full rank.

Exercise 2

For a rest-to-rest motion of a robot joint by a given amount Δq, we plan polynomial trajectories $q_i(t)$ of degree i, with $i = 3, 5, 7$. In each case, we impose all possible derivatives to zero at the initial and final instants $t = 0$ and $t = T$. If there is a limit $|\dot{q}_i(t)| \leq V$ on the velocities, which is the fastest possible trajectory among the three? Which are the ratios between the achievable minimum times T_i, for $i = 3, 5, 7$? What if there is instead only a limit $|\ddot{q}_i(t)| \leq A$ on the accelerations?

Exercise 3

- Shortly present (in the form of a table) which are the pros' and cons' of using incremental vs. absolute encoders as position sensors in a robot manipulator.
- What are the dis-/advantages of mounting an optical encoder on the motor side rather than on the link side of a motion transmission/reduction element in a robot joint?
- Describe as many as possible direct or indirect ways to measure the current position of the tip of a tool mounted as end effector of a robot.

[180 minutes, open books but no computer or smartphone]

Solution

July 11, 2017

Exercise 1

A Denavit-Hartenberg (DH) frame assignment for the 5-dof GMF M-100 manipulator is shown in Fig. 2. The associated parameters are given in Tab. 1.

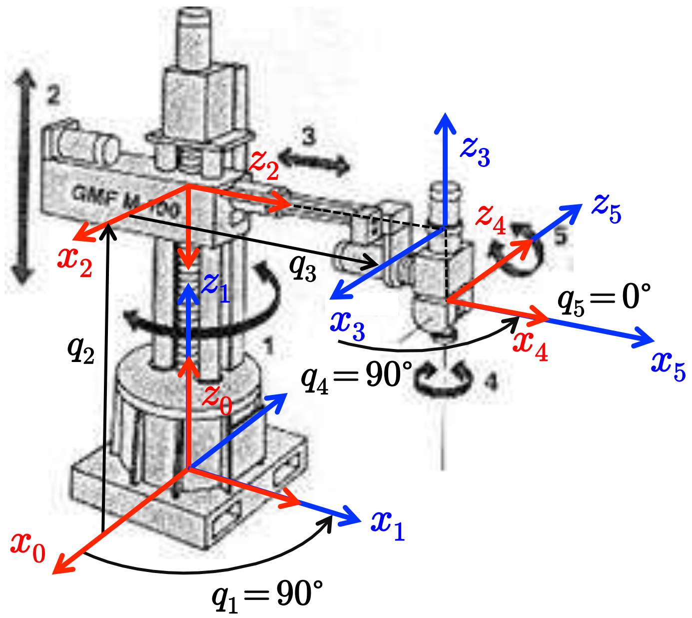


Figure 2: A DH frame assignment for the GMF M-100 manipulator, with the associated joint variables (numerical values are shown for the revolute joints in the this configuration).

i	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$-\pi/2$	0	q_2	$-\pi/2$
3	$\pi/2$	0	q_3	0
4	$-\pi/2$	0	$d_4 < 0$	q_4
5	0	0	0	q_5

Table 1: Table of DH parameters of the frame assignment in Fig. 2 for the GMF M-100 manipulator.

Accordingly, the homogeneous transformation matrices are:

$$
{}^{0}\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^{1}\boldsymbol{A}_2(q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^{2}\boldsymbol{A}_3(q_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^{3}\boldsymbol{A}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 & 0 \\ \sin q_4 & 0 & \cos q_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^{4}\boldsymbol{A}_5(q_5) = \begin{pmatrix} \cos q_5 & -\sin q_5 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

From these, we compute

$$
\begin{aligned}
\boldsymbol{p}_{hom} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} &= {}^{0}\boldsymbol{A}_1(q_1)\left({}^{1}\boldsymbol{A}_2(q_2)\left({}^{2}\boldsymbol{A}_3(q_3)\left({}^{3}\boldsymbol{A}_4(q_4)\left({}^{4}\boldsymbol{A}_5(q_5)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\right)\right)\right) \\
&= {}^{0}\boldsymbol{A}_1(q_1)\left({}^{1}\boldsymbol{A}_2(q_2)\left({}^{2}\boldsymbol{A}_3(q_3)\left({}^{3}\boldsymbol{A}_4(q_4)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\right)\right) \\
&= {}^{0}\boldsymbol{A}_1(q_1)\left({}^{1}\boldsymbol{A}_2(q_2)\left({}^{2}\boldsymbol{A}_3(q_3)\begin{pmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{pmatrix}\right)\right) \\
&= {}^{0}\boldsymbol{A}_1(q_1)\left({}^{1}\boldsymbol{A}_2(q_2)\begin{pmatrix} 0 \\ -d_4 \\ q_3 \\ 1 \end{pmatrix}\right) \\
&= {}^{0}\boldsymbol{A}_1(q_1)\begin{pmatrix} q_3 \\ 0 \\ q_2 + d_4 \\ 1 \end{pmatrix} = \begin{pmatrix} q_3\cos q_1 \\ q_3 \sin q_1 \\ q_2 + d_4 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{p} = \begin{pmatrix} q_3\cos q_1 \\ q_3 \sin q_1 \\ q_2 + d_4 \end{pmatrix}.
\end{aligned}
$$

It is easy to recognize that the expression in cylindrical coordinates is

$$
\boldsymbol{p}_{cyl} = \begin{pmatrix} \phi \\ h \\ r \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 + d_4 \\ q_3 \end{pmatrix}.
$$

The geometric Jacobian is also very simple

$$
\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} & & \dfrac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} & & \\ & & & & \\ \boldsymbol{z}_0 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{z}_3 & \boldsymbol{z}_4(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -q_3 \sin q_1 & 0 & \cos q_1 & 0 & 0 \\ q_3 \cos q_1 & 0 & \sin q_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(q_1+q_3) \\ 0 & 0 & 0 & 0 & \sin(q_1+q_3) \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.
$$

This matrix has full (column) rank equal to 5 if and only if $q_3 \neq 0$. When $q_3 = 0$, the rank drops to 4.

Exercise 2

It is convenient to work with doubly normalized expressions of the three polynomial trajectories interpolating q_0 at $t = 0$ with $q_0 + \Delta q$ at $t = T$. Let $\tau = t/T \in [0, 1]$. For the cubic polynomial, imposing zero velocity at the two boundaries, we have

$$
q_3(\tau) = q_0 + \Delta q \left(3\tau^2 - 2\tau^3\right). \tag{1}
$$

For the quintic polynomial, imposing zero velocity and acceleration at the two boundaries, we have

$$
q_5(\tau) = q_0 + \Delta q \left(10\tau^3 - 15\tau^4 + 6\tau^5\right). \tag{2}
$$

Finally, for the 7th-degree polynomial, imposing zero velocity, acceleration, and jerk at the two boundaries, we have

$$
q_7(\tau) = q_0 + \Delta q \left(35\tau^4 - 84\tau^5 + 70\tau^6 - 20\tau^7\right). \tag{3}
$$

The three velocities take the expressions

$$
\dot{q}_3(\tau) = \frac{\Delta q}{T}\left(6\tau - 6\tau^2\right) = \frac{6\Delta q}{T}\,\tau\,(1-\tau), \tag{4}
$$

$$
\dot{q}_5(\tau) = \frac{\Delta q}{T}\left(30\tau^2 - 60\tau^3 + 30\tau^4\right) = \frac{30\Delta q}{T}\,\tau^2\left(1 - 2\tau + \tau^2\right) = \frac{30\Delta q}{T}\,\tau^2\,(1-\tau)^2, \tag{5}
$$

and

$$
\begin{aligned}
\dot{q}_7(\tau) &= \frac{\Delta q}{T}\left(140\tau^3 - 420\tau^4 + 420\tau^5 - 140\tau^6\right) \\
&= \frac{140\Delta q}{T}\,\tau^3\left(1 - 3\tau + 3\tau^2 - \tau^3\right) = \frac{140\Delta q}{T}\,\tau^3\,(1-\tau)^3,
\end{aligned} \tag{6}
$$

while the accelerations are

$$
\ddot{q}_3(\tau) = \frac{\Delta q}{T^2}\,(6 - 12\tau) = \frac{6\Delta q}{T^2}\,(1-2\tau) \tag{7}
$$

$$
\ddot{q}_5(\tau) = \frac{\Delta q}{T^2}\left(60\tau - 180\tau^2 + 120\tau^3\right) = \frac{60\Delta q}{T^2}\,\tau\left(1 - 3\tau + 2\tau^2\right) = \frac{60\Delta q}{T^2}\,\tau\,(1-\tau)\,(1-2\tau), \tag{8}
$$

and

$$
\begin{aligned}
\ddot{q}_7(\tau) &= \frac{\Delta q}{T^2}\left(420\tau^2 - 1680\tau^3 + 2100\tau^4 - 840\tau^5\right) = \frac{420\Delta q}{T^2}\,\tau^2\left(1 - 4\tau + 5\tau^2 - 2\tau^3\right) \\
&= \frac{420\Delta q}{T^2}\,\tau^2\left(1 - 2\tau + \tau^2\right)(1-2\tau) = \frac{420\Delta q}{T^2}\,\tau^2\,(1-\tau)^2\,(1-2\tau).
\end{aligned} \tag{9}
$$

The maximum velocity is always attained at the trajectory halftime $t = T/2$, or $\tau = 0.5$, where the acceleration is in fact zero in all cases. Indeed, for the cubic trajectory this is the only instant where the acceleration $\ddot{q}_3$ is zero. For the quintic trajectory, the factorization in (8) shows that the acceleration $\ddot{q}_5$ has (by construction) a zero also at the boundaries (where the velocity is anyway zero, together with the acceleration). Similarly, for the 7th-degree trajectory, the factorization in (9) shows that the acceleration $\ddot{q}_7$ has (by construction) a zero of multiplicity 2 also at the boundaries (where the velocity is anyway zero, together with acceleration and jerk). Therefore, in all three cases we evaluate the presence of a symmetric bound V on the velocity as

$$|\dot{q}_3(\tau)| \leq V, \quad \max_{\tau\in[0,1]} |\dot{q}_3(\tau)| = |\dot{q}_3(0.5)| = \frac{3|\Delta q|}{2T} \quad \Rightarrow \quad T_3\,(= \min T_V \text{ for } q_3(\tau)) = \frac{3|\Delta q|}{2V}, \tag{10}$$

$$|\dot{q}_5(\tau)| \leq V, \quad \max_{\tau\in[0,1]} |\dot{q}_5(\tau)| = |\dot{q}_5(0.5)| = \frac{15|\Delta q|}{8T} \quad \Rightarrow \quad T_5\,(= \min T_V \text{ for } q_5(\tau)) = \frac{15|\Delta q|}{8V}, \tag{11}$$

and

$$|\dot{q}_7(\tau)| \leq V, \quad \max_{\tau\in[0,1]} |\dot{q}_7(\tau)| = |\dot{q}_7(0.5)| = \frac{35|\Delta q|}{16T} \quad \Rightarrow \quad T_7\,(= \min T_V \text{ for } q_7(\tau)) = \frac{35|\Delta q|}{16V}. \tag{12}$$

As a result —not really unexpected— the fastest trajectory under a maximum velocity bound is the cubic one, followed by the quintic, and then by the 7th-degree polynomial ($T_3 < T_5 < T_7$). The ratios of the minimum times are independent from Δq and V and equal to:

$$\frac{T_7}{T_5} = \frac{3}{2} = 1.5, \qquad \frac{T_5}{T_3} = \frac{5}{4} = 1.25, \qquad \frac{T_7}{T_3} = \frac{15}{8} = 1.875. \tag{13}$$

In order to solve the same minimum time problem when only a symmetric bound A is set on the acceleration, we compute first the jerk for the three polynomial trajectories:

$$\dddot{q}_3(\tau) = -\frac{12\Delta q}{T^3} \neq 0, \tag{14}$$

$$\dddot{q}_5(\tau) = \frac{60\Delta q}{T^3}\left(1 - 6\tau + 6\tau^2\right) = \frac{60\Delta q}{T^3}\left(1 - \frac{6}{3-\sqrt{3}}\,\tau\right)\left(1 - \frac{6}{3+\sqrt{3}}\,\tau\right), \tag{15}$$

and

$$\begin{aligned}\dddot{q}_7(\tau) &= \frac{840\Delta q}{T^3}\,\tau\left(1 - 6\tau + 10\tau^2 - 5\tau^3\right) = \frac{840\Delta q}{T^3}\,\tau\,(1-\tau)\left(1 - 5\tau + 5\tau^2\right) \\ &= \frac{840\Delta q}{T^3}\,\tau\,(1-\tau)\left(1 - \frac{10}{5-\sqrt{5}}\,\tau\right)\left(1 - \frac{10}{5+\sqrt{5}}\,\tau\right).\end{aligned} \tag{16}$$

The jerk of the cubic trajectory is constant over the motion interval ($\tau \in [0,1]$), so that the maximum acceleration in this closed interval is at its boundaries. Because of the symmetric behavior, we have from $|\ddot{q}_3(\tau)| \leq A$:

$$\max_{\tau\in[0,1]} |\ddot{q}_3(\tau)| = |\ddot{q}_3(0)| = |\ddot{q}_3(1)| = \frac{6\,|\Delta q|}{T^2} \quad \Rightarrow \quad T_3'\,(= \min T_A \text{ for } q_3(\tau)) = \sqrt{\frac{6\,|\Delta q|}{A}}. \tag{17}$$

From (15), the jerk of the quintic trajectory has two roots, i.e.,

$$\tau_{5,1} = 0.5 - \frac{\sqrt{3}}{6} \qquad \text{and} \qquad \tau_{5,2} = 0.5 + \frac{\sqrt{3}}{6}, \tag{18}$$

in the interval $\tau \in [0,1]$, placed in symmetric positions w.r.t. the motion halftime. Since the acceleration is anyway zero at the boundaries, the maximum acceleration in the closed interval occurs only in the two instants specified in (18). Because of the symmetric behavior, we have from $|\ddot{q}_5(\tau)| \leq A$:

$$\max_{\tau\in[0,1]} |\ddot{q}_5(\tau)| = |\ddot{q}_5(\tau_{5,1})| = |\ddot{q}_5(\tau_{5,2})| = \frac{5.7735\,|\Delta q|}{T^2} \quad \Rightarrow \quad T_5' \,(= \min T_A \text{ for } q_5(\tau)) = \sqrt{\frac{5.7735\,|\Delta q|}{A}}. \tag{19}$$

Finally, from (16) the jerk of the 7th-degree trajectory has two internal roots, i.e.,

$$\tau_{7,1} = 0.5 - \frac{\sqrt{5}}{10} \qquad \text{and} \qquad \tau_{7,2} = 0.5 + \frac{\sqrt{5}}{10}, \tag{20}$$

in the interval $\tau \in [0,1]$, placed again in symmetric positions w.r.t. the motion halftime, and two other roots coincident with the boundaries, where the acceleration is anyway zero. Thus, the maximum acceleration in the closed interval occurs in the two instants specified in (20). Because of the symmetric behavior, we have from $|\ddot{q}_7(\tau)| \leq A$:

$$\max_{\tau\in[0,1]} |\ddot{q}_7(\tau)| = |\ddot{q}_7(\tau_{7,1})| = |\ddot{q}_7(\tau_{7,2})| = \frac{3.36\,|\Delta q|}{T^2} \quad \Rightarrow \quad T_7' \,(= \min T_A \text{ for } q_7(\tau)) = \sqrt{\frac{3.36\,|\Delta q|}{A}}. \tag{21}$$

As a result —maybe with a certain surprise— the situation is now reversed with respect to the previous case: the fastest trajectory under a maximum acceleration bound is in fact the 7th-degree polynomial, followed by the quintic, and then by the cubic one ($T_7' < T_5' < T_3'$). The ratios of the minimum times are again independent from Δq and V and equal to:

$$\frac{T_3'}{T_5'} = \sqrt{\frac{6}{5.7735}} = 1.0194, \qquad \frac{T_5'}{T_7'} = \sqrt{\frac{5.7735}{3.36}} = 1.3108, \qquad \frac{T_3'}{T_7'} = \sqrt{\frac{6}{3.36}} = 1.3363. \tag{22}$$

Indeed, limiting the acceleration is already a request targeting increased smoothness of the trajectory. This explains *de facto* why a polynomial of higher degree performs better in this case.

Exercise 3

This exercise asks for a free text. Completeness, technical accuracy, and clarity in writing are evaluated.

* * * * *

Robotics I

September 21, 2017

Exercise 1

Consider the rigid body in Fig. 1, a thin rod of length L. The rod will be rotated by an angle α around the $\boldsymbol{z}$ axis, then by an angle β around the resulting $\boldsymbol{x}$ axis, and finally by an angle γ around the resulting $\boldsymbol{y}$ axis.

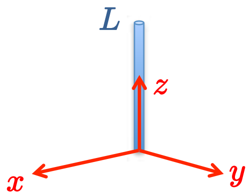


Figure 1: Initial placement of a rigid thin rod of length L in an absolute reference frame.

- Provide the final orientation of the rod, as expressed by a rotation matrix, and the symbolic expression of its angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ in terms of the angles (α, β, γ) and their time derivatives $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$.
- Assuming the numerical values $L = 0.4$ [m] for the rod, $\alpha = 30°$, $\beta = -30°$, and $\gamma = 60°$ for the angles, and $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 45°/\text{s}$ for their time derivatives, determine:
  - the components of the final unit vector $\boldsymbol{y}$ attached to the rod after the three rotations;
  - the absolute coordinates of the position $\boldsymbol{p}$ of the end-point of the rod after the three rotations;
  - the angular velocity $\boldsymbol{\omega}$ of the rod, expressed both in the initial (absolute) reference frame and in the final rotated frame;
  - the absolute velocity $\dot{\boldsymbol{p}}$ of the end-point of the rod.

Exercise 2

Using standard DH angles, a planar 2R robot with links of lengths $\ell_1 = \ell_2 = 0.5$ [m] is in the initial static configuration $\boldsymbol{q}_0 = (45°, -90°)$. Plan a rest-to-rest cubic polynomial trajectory in the joint space so as to reach the final end-effector position $\boldsymbol{p}_{goal} = (0.5, 0.866)$ [m] in minimum time under the following joint velocity limits: $|\dot{q}_1| \leq 30°/\text{s}$, $|\dot{q}_2| \leq 90°/\text{s}$.

- Provide the minimum time T^* and the numerical values of the coefficients of the optimal cubic polynomials.
- If the robot performs the above minimum-time task with a *coordinated* motion in the joint space, what is the maximum velocity reached by each joint during motion?

Exercise 3

Consider the transmission/reduction assembly of a robot joint driving a single robot link, as sketched in Fig. 2. The actuator is an electrical motor with maximum rotation speed $\dot{\theta}_m$ of its output shaft equal to 2000 RPM. The transmission assembly consists of an Harmonic Drive (HD), with its wave generator mounted on the motor shaft ($A \leftrightarrow A'$) and whose circular spline has $n_{CS} = 64$ teeth, followed by a two-wheel toothed gear, with a smaller wheel of radius $r_1 = 0.9$ [cm] mounted on the HD output axis ($B \leftrightarrow B'$) and a larger wheel of radius $r_2 = 1.7$ [cm]. The link rotates with the larger wheel at an angular speed $\dot{\theta}_\ell$ and has a length $L = 0.68$ [m].

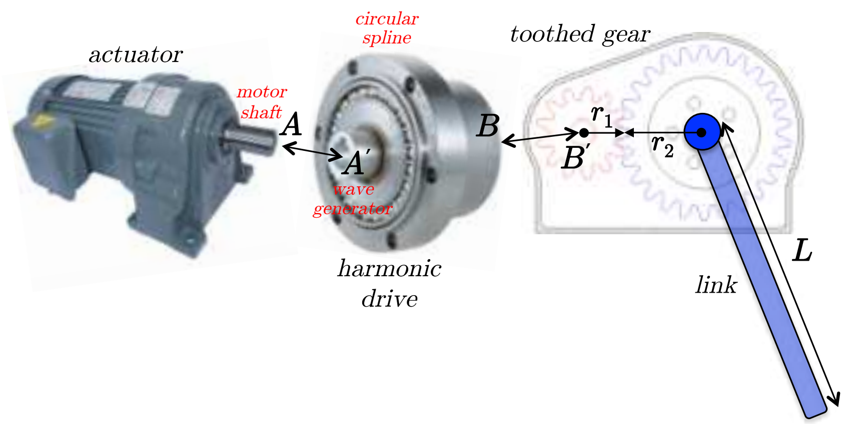


Figure 2: Transmission/reduction assembly of a robot joint.

- What is the reduction ratio $N = |\dot{\theta}_m/\dot{\theta}_\ell|$ of the complete transmission?
- Does the link rotate in the same or in the opposite direction of the motor shaft?
- What is the maximum velocity achievable by the end-point of the link?

[180 minutes, open books but no computer or smartphone]

Solution

September 21, 2017

Exercise 1

The sequence of rotations about the first (fixed) axis Z and the following (moving) axes X' and Y'' provides a ZXY Euler representation of orientation. Therefore, the final orientation of the rod is given by

$$\begin{aligned}
\boldsymbol{R}_{ZXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Z(\alpha)\,\boldsymbol{R}_{X'}(\beta)\,\boldsymbol{R}_{Y''}(\gamma)\\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\beta & -\sin\beta\\ 0 & \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}\\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma\\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma\\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\sin\gamma \end{pmatrix}\\
&= \begin{pmatrix} \boldsymbol{x}'''(\alpha,\beta,\gamma) & \boldsymbol{y}'''(\alpha,\beta) & \boldsymbol{z}'''(\alpha,\beta,\gamma) \end{pmatrix}
\end{aligned}$$

In particular, the position of the end-point of the rod after the three rotations is $\boldsymbol{p} = L\,\boldsymbol{z}'''$.

Using the numerical data, the final rotation matrix is

$$\boldsymbol{R}_{ZXY}\left(\frac{\pi}{6},-\frac{\pi}{6},\frac{\pi}{3}\right) = \begin{pmatrix} \frac{3\sqrt{3}}{8} & -\frac{\sqrt{3}}{4} & \frac{5}{8}\\ -\frac{1}{8} & \frac{3}{4} & \frac{3\sqrt{3}}{8}\\ -\frac{3}{4} & -\frac{1}{2} & \frac{\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} 0.6495 & -0.4330 & 0.6250\\ -0.1250 & 0.7500 & 0.6495\\ -0.7500 & -0.5000 & 0.4330 \end{pmatrix},$$

while the unit vector $\boldsymbol{y}''' = \boldsymbol{y}''$ and the position vector $\boldsymbol{p}$ of the end-point of the rod after the rotations are, respectively,

$$\boldsymbol{y}''\left(\frac{\pi}{6},-\frac{\pi}{6}\right) = \begin{pmatrix} -0.4330\\ 0.7500\\ -0.5000 \end{pmatrix}, \qquad \boldsymbol{p} = 0.4\,\boldsymbol{z}'''\left(\frac{\pi}{6},-\frac{\pi}{6},\frac{\pi}{3}\right) = \begin{pmatrix} \frac{1}{4}\\ \frac{3\sqrt{3}}{20}\\ \frac{\sqrt{3}}{10} \end{pmatrix} = \begin{pmatrix} 0.2500\\ 0.2598\\ 0.1732 \end{pmatrix}\ [\mathrm{m}].$$

The angular velocity $\boldsymbol{\omega}$ of the rod associated to the time derivatives $(\dot\alpha,\dot\beta,\dot\gamma)$ in the configuration specified by the angles (α,β,γ) can be computed either from the relation

$$\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{ZXY}(\alpha,\beta,\gamma,\dot\alpha,\dot\beta,\dot\gamma)\,\boldsymbol{R}^T_{ZXY}(\alpha,\beta,\gamma) = \begin{pmatrix} 0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0 \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\omega}$$

or, more explicitly, as

$$\boldsymbol{\omega} = \boldsymbol{z}\,\dot\alpha + \boldsymbol{x}'\,\dot\beta + \boldsymbol{y}''\,\dot\gamma = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}\dot\alpha + \boldsymbol{R}_Z(\alpha)\begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}\dot\beta + \boldsymbol{R}_Z(\alpha)\,\boldsymbol{R}_{X'}(\beta)\begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}\dot\gamma = \begin{pmatrix} \omega_x\\ \omega_y\\ \omega_z \end{pmatrix}.$$

Performing computations, we obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \\ 1 & 0 & \sin\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \cos\alpha\,\dot{\beta} - \sin\alpha\cos\beta\,\dot{\gamma} \\ \sin\alpha\,\dot{\beta} + \cos\alpha\cos\beta\,\dot{\gamma} \\ \dot{\alpha} + \sin\beta\,\dot{\gamma} \end{pmatrix}.$$

This expression is referenced to the initial frame. In the rotated frame, one has

$${}^{R}\boldsymbol{\omega} = \boldsymbol{R}_{ZXY}^{T}\,\boldsymbol{\omega} = \begin{pmatrix} -\cos\beta\sin\gamma\,\dot{\alpha} + \cos\gamma\,\dot{\beta} \\ \sin\beta\,\dot{\alpha} + \dot{\gamma} \\ \cos\beta\cos\gamma\,\dot{\alpha} + \sin\gamma\,\dot{\beta} \end{pmatrix}.$$

Finally, the linear velocity $\dot{\boldsymbol{p}}$ of the end-point of the rod is given by

$$\dot{\boldsymbol{p}} = \boldsymbol{\omega} \times \boldsymbol{p} = \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{p} = L\left[\begin{pmatrix} \cos\alpha\sin\beta\cos\gamma - \sin\alpha\sin\gamma \\ \sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma \\ 0 \end{pmatrix}\dot{\alpha}\right.$$
$$\left. + \begin{pmatrix} \sin\alpha\cos\beta\cos\gamma \\ -\cos\alpha\cos\beta\cos\gamma \\ -\sin\beta\cos\gamma \end{pmatrix}\dot{\beta} + \begin{pmatrix} -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma \\ \cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma \\ -\cos\beta\sin\gamma \end{pmatrix}\dot{\gamma}\right].$$

Using the numerical data, we obtain

$$\boldsymbol{\omega} = \frac{\pi}{4}\begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{5}{4} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.3401 \\ 0.9817 \\ 0.3927 \end{pmatrix} \text{[rad/s]}, \qquad {}^{R}\boldsymbol{\omega} = \frac{\pi}{4}\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ \frac{3\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} -0.1963 \\ 0.3927 \\ 1.0203 \end{pmatrix} \text{[rad/s]},$$

and

$$\dot{\boldsymbol{p}} = \begin{pmatrix} 0.0680 \\ 0.0393 \\ -0.1571 \end{pmatrix} \text{[m/s]}.$$

Exercise 2

We first solve the inverse kinematics for the final position $\boldsymbol{p}_{goal} = (p_{x,goal}, p_{y,goal}) = (0.5, \sqrt{3}/2)$. Noting that $\|\boldsymbol{p}_{goal}\| = 1 = \ell_1 + \ell_2$, this point is on the external boundary of the robot workspace. Thus, there is a single configuration with stretched arm as solution, given by

$$\boldsymbol{q}_{goal} = \begin{pmatrix} q_{1,goal} \\ q_{2,goal} \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\{p_{y,goal}, p_{x,goal}\} \\ 0 \end{pmatrix} = \begin{pmatrix} \pi/3 \\ 0 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 60^\circ \\ 0^\circ \end{pmatrix}.$$

As a result the joint displacement is

$$\boldsymbol{\Delta q} = \boldsymbol{q}_{goal} - \boldsymbol{q}_0 = \begin{pmatrix} 15^\circ \\ 90^\circ \end{pmatrix} = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix}$$

and the associated rest-to-rest cubic trajectories for the two joints, each in time $T_i > 0$, $i = 1, 2$, will take the form

$$q_i(t) = q_{0,i} + \Delta q_i \left(3\left(\frac{t}{T_i}\right)^2 - 2\left(\frac{t}{T_i}\right)^3\right), \tag{1}$$

with velocities and accelerations

$$\dot{q}_i(t) = \frac{6\Delta q_i}{T_i}\left(\left(\frac{t}{T_i}\right) - \left(\frac{t}{T_i}\right)^2\right), \qquad \ddot{q}_i(t) = \frac{6\Delta q_i}{T_i^2}\left(1 - 2\left(\frac{t}{T_i}\right)\right), \qquad \text{for } i = 1, 2.$$

For each joint, the maximum velocity (in absolute value) will occur at the trajectory midpoint, where the acceleration is zero, i.e.,

$$\ddot{q}_i(t_i^*) = 0 \quad \Rightarrow \quad t_i^* = \frac{T_i}{2} \qquad \Rightarrow \qquad |\dot{q}_i(T_i/2)| = \frac{3\,|\Delta q_i|}{2T_i}.$$

Since both displacements Δq_i are positive, we will discard from now on the absolute value. Imposing that the joint velocities both reach their admissible limits, respectively, $V_1 = 30°/\text{s}$ and $V_2 = 90°/\text{s}$, we have

$$|\dot{q}_i(T_i/2)| = V_i \qquad \Rightarrow \qquad T_i^* = \frac{1.5\,\Delta q_i}{V_i} = \begin{cases} 0.75, & \text{for } i = 1, \\ 1.5, & \text{for } i = 2. \end{cases}$$

Therefore, the minimum motion time will be

$$T^* = \max\{T_1^*, T_2^*\} = T_2^* = 1.5\,\text{s}.$$

The associated numerical coefficients of the optimal cubic polynomials in (1) are

$$\begin{aligned} q_1(t) &= 45 + 80\,t^2 - 71.1111\,t^3, & t \in [0, 0.75], \\ q_2(t) &= -90 + 120\,t^2 - 53.3333\,t^3, & t \in [0, 1.5]. \end{aligned}$$

In order to perform a coordinated motion in the joint space, the fastest joint, namely the first one, should uniformly slow down its motion so that the its final time equals T^*. Since $T^* = 2T_1^*$, the needed scaling is by a factor $k_1 = 2$ and, accordingly, the new maximum velocity of joint 1, reached again at the midpoint of the coordinated trajectory, will be reduced to $V_1/k_1 = 15°/\text{s}$. Indeed, no changes occur in the velocity profile of the second joint.

Exercise 3

The considered harmonic drive has $n_{CS} = 64$ teeth on the (internal side of the) circular spline and thus $n_{FS} = n_{CS} - 2 = 62$ teeth on the external side of the flexspline. Its reduction ratio is thus

$$N_{HD} = \frac{n_{FS}}{n_{CS} - n_{FS}} = \frac{n_{FS}}{2} = \frac{62}{2} = 31.$$

Since the toothed gear has reduction ratio

$$N_{TG} = \frac{r_2}{r_1} = \frac{1.7}{0.9} = 1.8889,$$

the reduction ratio of the transmission is

$$N = \left|\frac{\dot{\theta}_m}{\dot{\theta}_\ell}\right| = N_{HD} \cdot N_{TG} = 58.5556.$$

There is a double inversion of rotations in the overall transmission. Thus, the link will rotate in the same direction (clockwise or counterclockwise) of the motor shaft, i.e., $\text{sign}\{\dot{\theta}_m\} = \text{sign}\{\dot{\theta}_\ell\}$. Finally, the maximum velocity v achievable by the end-point of the link is

$$v = L\,\max\{\dot{\theta}_\ell\} = L\,\frac{\max\{\dot{\theta}_m\}}{N} = 0.68\,[\text{m}]\,\frac{2000\,[\text{RPM}] \cdot (2\pi/60)[\text{rad/s}]}{58.5556} = 2.4322\,[\text{m/s}].$$

* * * * *

Robotics I

October 27, 2017

Consider the 6-dof robot Stäubli RX 160 in Fig. 1, described in the attached technical data sheet.

Figure 1: The Stäubli RX 160 robot.

1. Determine a frame assignment and the table of parameters according to the Denavit-Hartenberg (DH) convention.
2. Using the data sheet, specify the numerical values of the constant DH parameters.
3. Provide the numerical values of the variable DH parameters when the robot is in a stretched configuration pointing upwards.
4. Are the joint limits indicated in the data sheet consistent with the chosen DH convention? If not, explain why and which is the relation between the six DH variables $\boldsymbol{\theta}$ and the six joint angles $\boldsymbol{\theta}_S$ used by the Stäubli manufacturer.
5. How many distinct inverse kinematics solutions do you expect for this robot out of singularities? Would all these nonsingular inverse kinematics solutions be feasible with respect to the physical joint limits of the robot?
6. Derive the symbolic expression of the position $\boldsymbol{p}_{\mathbf{O}_4} = \boldsymbol{f}(\boldsymbol{\theta})$ of the origin $\mathbf{O}_4$ of the DH frame 4.
7. Has the robot a spherical wrist? If so, does $\mathbf{O}_4$ coincide with the center of the spherical wrist?
8. Compute the symbolic expression of the partial 3×3 Jacobian matrix $\boldsymbol{J}_{A,3}(\boldsymbol{\theta}_{\text{main}})$ that relates the velocity $\dot{\boldsymbol{\theta}}_{\text{main}} = \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{pmatrix}^T \in \mathbb{R}^3$ of the first three (main) joint axes of the robot to the angular velocity $\boldsymbol{\omega}_3 \in \mathbb{R}^3$ of the DH frame 3.

[180 minutes, open books but no computer or smartphone]

Robotics I

Midterm classroom test – November 24, 2017

Exercise 1 [8 points]

Consider the 3-dof (RPR) planar robot in Fig. 1, where the joint coordinates $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}^T$ have been defined in a free, arbitrary way, with reference to a base frame RF_b.

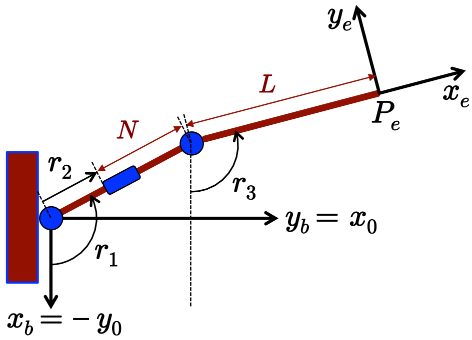


Figure 1: A RPR planar robot.

- Determine position and orientation of the end-effector frame RF_e in terms of the coordinates $\boldsymbol{r}$.
- Assign the Denavit-Hartenberg (DH) frames and define the joint coordinates $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ according to the standard DH convention, starting from the reference frame RF_0. Provide the DH table, the expression of the homogeneous transformation matrices between the successive frames that have been assigned. Determine position and orientation of the end-effector frame RF_e in terms of the coordinates $\boldsymbol{q}$.
- Find the transformation $\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{r})$ between the two sets of coordinates so as to associate the same kinematic configurations of the robot. Does this mapping have singularities?

Exercise 2 [5 points]

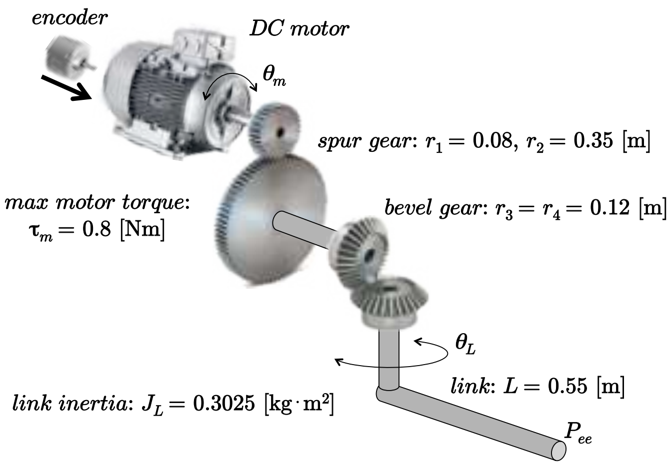
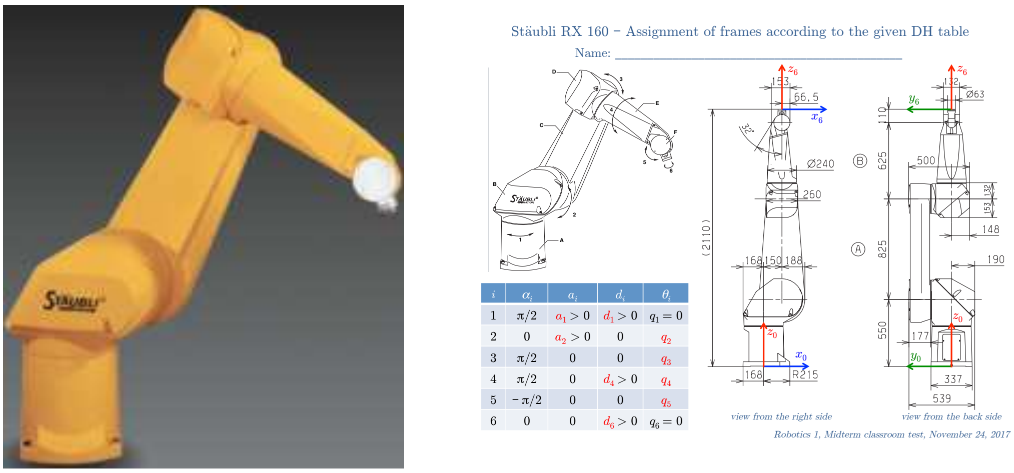


Figure 2: A DC servomotor that drives a robot link through transmissions.

With reference to the servo-drive sketched in Fig. 2 and the data therein, we need to measure the position of the tip point P_{ee} of the link with a resolution of 0.01 [mm]. A suitable incremental encoder with quadrature detection is chosen to measure the angular position θ_m of the DC motor, which is capable of delivering a maximum torque τ_m. How many pulses per turn should generated by the optical disk on each of the channels A and B? How many bits should be used by the digital counter of the encoder? Neglecting dissipative effects, what should be the value of the motor inertia J_m in order to maximize the angular acceleration of the link for a given motor torque? With this choice, what is the maximum linear acceleration achievable by the tip of the link?

Exercise 3 [12 points]

Consider the 6-dof robot Stäubli RX 160 in Fig. 3. In the extra sheet provided separately, the Denavit-Hartenberg (DH) table of parameters is specified, in part numerically and in part symbolically. The two DH frames 0 and 6 are already drawn on the manipulator (in two views). In the shown 'straight upward' robot configuration, the first and last joint variables take the values $q_1 = q_6 = 0$. Draw directly on the extra sheet the remaining DH frames, according to the DH table. Provide all parameters labeled in red in the table, i.e., the missing numerical values of the constant parameters and of the joint variables q_2 to q_5 when the robot is in the 'straight upward' configuration. [Please, make clean drawings and return the sheet with your name written on it.]



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	$d_1 > 0$	$q_1 = 0$
2	0	$a_2 > 0$	0	q_2
3	$\pi/2$	0	0	q_3
4	$\pi/2$	0	$d_4 > 0$	q_4
5	$-\pi/2$	0	0	q_5
6	0	0	$d_6 > 0$	$q_6 = 0$

Figure 3: The Stäubli RX 160 robot manipulator and the extra sheet (provided separately).

Exercise 4 [5 points]

The orientations of two right-handed frames RF_A and RF_B with respect to a third right-handed frame RF_0 (all having the same origin) are specified, respectively, by the rotation matrices

$$
{}^0\boldsymbol{R}_A = \begin{pmatrix} \frac{3}{4} & \sqrt{\frac{3}{8}} & -\frac{1}{4} \\ -\sqrt{\frac{3}{8}} & \frac{1}{2} & -\sqrt{\frac{3}{8}} \\ -\frac{1}{4} & \sqrt{\frac{3}{8}} & \frac{3}{4} \end{pmatrix} \qquad \text{and} \qquad {}^0\boldsymbol{R}_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
$$

Determine, if possible, a unit vector $\boldsymbol{r}$ and an angle $\theta < 0$ such that the axis-angle rotation matrix $\boldsymbol{R}(\boldsymbol{r}, \theta)$ provides the orientation of the frame RF_B with respect to the frame RF_A.

[180 minutes, open books but no computer or smartphone]

Solution of Midterm Test

November 24, 2017

Exercise 1

The direct kinematics in terms of the coordinates $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}^T$ is easily computed as

$$
{}^b\boldsymbol{p}_e(\boldsymbol{r}) = \begin{pmatrix} (r_2+N)\cos r_1 + L\cos r_3 \\ (r_2+N)\sin r_1 + L\sin r_3 \\ 0 \end{pmatrix}, \qquad {}^b\boldsymbol{R}_e(\boldsymbol{r}) = \begin{pmatrix} \cos r_3 & -\sin r_3 & 0 \\ \sin r_3 & \cos r_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Indeed, if these quantities have to be expressed with respect to the (DH) frame 0 indicated in Fig. 1, we need to introduce a constant rotation. Since the two frames have the same origin, we obtain

$$
{}^0\boldsymbol{R}_b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad
\begin{array}{l}
{}^0\boldsymbol{p}_e(\boldsymbol{r}) = {}^0\boldsymbol{R}_b\,{}^b\boldsymbol{p}_e(\boldsymbol{r}) = \begin{pmatrix} (r_2+N)\sin r_1 + L\sin r_3 \\ -(r_2+N)\cos r_1 - L\cos r_3 \\ 0 \end{pmatrix}, \\[2ex]
{}^0\boldsymbol{R}_e(\boldsymbol{r}) = {}^0\boldsymbol{R}_b\,{}^b\boldsymbol{R}_e(\boldsymbol{r}) = \begin{pmatrix} \sin r_3 & \cos r_3 & 0 \\ -\cos r_3 & \sin r_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{array}
$$

Keeping into account the already specified assignment of frame 0 in Fig. 1, a feasible assignment of DH frames for the RPR robot is shown in Fig. 4, with associated parameters given in Tab. 1.

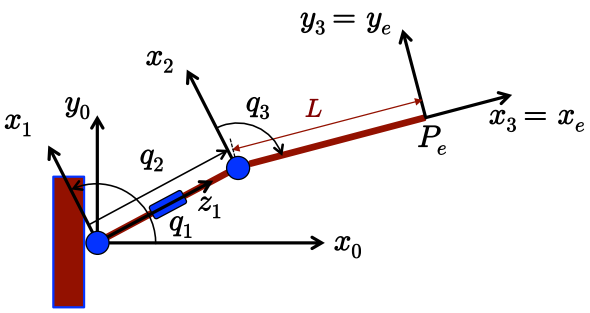


Figure 4: Assignment of DH frames for the RPR planar robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$-\pi/2$	0	q_2	0
3	0	L	0	q_3

Table 1: Parameters associated to the DH frames in Fig. 4.

The DH homogenous transformations take the following expressions:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 & L\cos q_3 \\ \sin q_3 & \cos q_3 & 0 & L\sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The direct kinematics in terms of the coordinates $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ is then

$$
{}^0\boldsymbol{T}_e(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^0\boldsymbol{R}_e(\boldsymbol{q}) & {}^0\boldsymbol{p}_e(\boldsymbol{q}) \\ 0\ 0\ 0 & 1 \end{pmatrix},
$$

with

$$
{}^0\boldsymbol{p}_e(\boldsymbol{q}) = \begin{pmatrix} q_2\sin q_1 + L\cos(q_1+q_3) \\ -q_2\cos q_1 + L\sin(q_1+q_3) \\ 0 \end{pmatrix}, \qquad {}^0\boldsymbol{R}_e(\boldsymbol{q}) = \begin{pmatrix} \cos(q_1+q_3) & -\sin(q_1+q_3) & 0 \\ \sin(q_1+q_3) & \cos(q_1+q_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

The transformation that matches the robot configurations using the different sets of coordinates is

$$
\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{r}) = \begin{pmatrix} r_1 \\ r_2 + N \\ r_3 - r_1 - \dfrac{\pi}{2} \end{pmatrix} \qquad \Longleftrightarrow \qquad \boldsymbol{r} = \boldsymbol{f}^{-1}(\boldsymbol{q}) = \begin{pmatrix} q_1 \\ q_2 - N \\ q_1 + q_3 + \dfrac{\pi}{2} \end{pmatrix},
$$

which is invertible and without singularities. It is easy to check that, e.g.,

$$
{}^0\boldsymbol{p}_e(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{f}(\boldsymbol{r})} = {}^0\boldsymbol{p}_e(\boldsymbol{r}), \qquad {}^0\boldsymbol{R}_e(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{f}(\boldsymbol{r})} = {}^0\boldsymbol{R}_e(\boldsymbol{r}).
$$

Exercise 2

The requested resolution $\Delta = 10^{-5}$ [m] on the linear motion at the tip of the link of length $L = 0.55$ [m] needs to be transformed into an angular one $\delta = \Delta/L = 1.8182 \cdot 10^{-5}$ [rad] at the base of the link and then, via the two transmission gears with reduction ratios $N_{\text{bevel}} = r_4/r_3 = 1$ and $N_{\text{spur}} = r_2/r_1 = 0.35/0.08 = 4.375$ respectively, into an angular resolution on the motor axis

$$
\delta_m = N_{\text{spur}} \cdot N_{\text{bevel}} \cdot \frac{\Delta}{L} = 7.9545 \cdot 10^{-5}\ [\text{rad}] = (4.56 \cdot 10^{-3})^\circ.
$$

Taking into account the factor 4 introduced by the quadrature electronics, the number of pulses per turn N_{ppt} of the optical disk should be at least

$$
N_{\text{ppt}} = \left\lceil \frac{2\pi}{4 \cdot \delta_m} \right\rceil = 19748.
$$

Accordingly, the digital counter should have at least a number of bits

$$
N_{\text{bit}} = \lceil \log_2 19748 \rceil = 15.
$$

Being the link inertia (around its axis of rotation) $J_L = 0.3025$ [kg·m^2] and the reduction ratio of the transmissions $N = N_{\text{spur}} \cdot N_{\text{bevel}} = 4.375$, the optimal value of the motor inertia according to the requested criterion is

$$J_m = \frac{J_L}{N^2} = 0.0158\,[\text{kg}\cdot\text{m}^2].$$

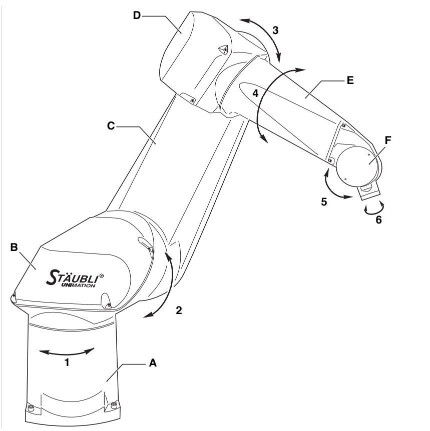
Accordingly, with a maximum available motor torque $\tau_m = 0.8$ [Nm] on the motor axis, the maximum torque at the link base is $\tau_{L,\max} = N \cdot \tau_m = 3.5$ [Nm]. Since $N^2 = J_L/J_m$, the balance of torques on the link side provides

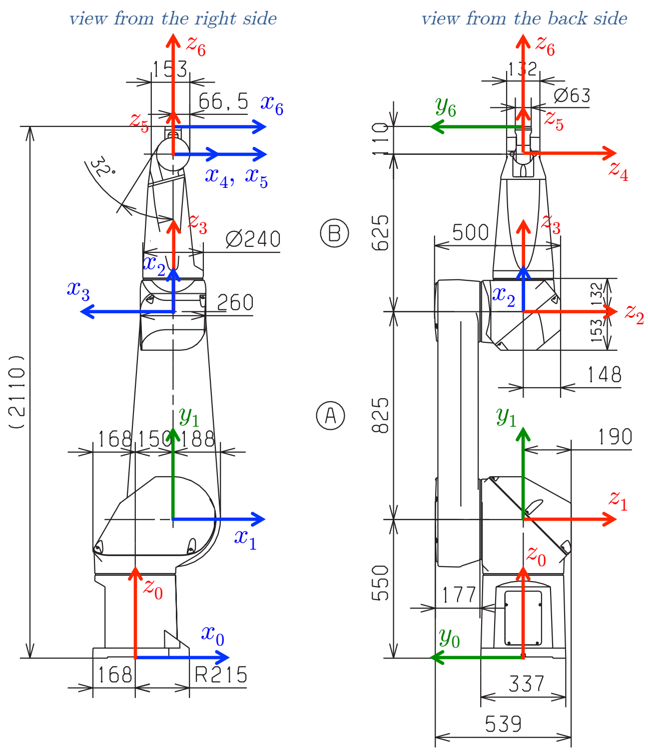
$$\tau_L = J_L \ddot{\theta}_L + N(J_m \ddot{\theta}_m) = (J_L + J_m N^2)\ddot{\theta}_L = 2 J_L \ddot{\theta}_L.$$

The maximum tip acceleration $a_{\max}$ of the link tip will then be

$$a_{\max} = L\,\ddot{\theta}_{L,\max} = L\,\frac{\tau_{L,\max}}{2\,J_L} = 0.55\,[\text{m}] \cdot 5.7851\,[\text{rad}\cdot\text{s}^{-2}] = 3.1818\,[\text{m}\cdot\text{s}^{-2}].$$

Exercise 3





i	α_i	a_i	d_i	θ_i
1	$\pi/2$	150	550	$q_1 = 0$
2	0	825	0	$q_2 = \pi/2$
3	$\pi/2$	0	0	$q_3 = \pi/2$
4	$\pi/2$	0	625	$q_4 = \pi$
5	$-\pi/2$	0	0	$q_5 = 0$
6	0	0	110	$q_6 = 0$

Figure 5: The DH frames and the complete table of parameters for the Stäubli RX 160 robot manipulator. The numerical values of $\boldsymbol{q} = \boldsymbol{\theta}$ refer to the shown 'straight upward' configuration.

The assignment of Denavit-Hartenberg frames for the Stäubli robot RX 160 according to the given table is shown in Fig. 5. The numerical values of all symbolic parameters are reported, with the values of the joint variables $\boldsymbol{q} = \boldsymbol{\theta}$ when the robot is in the 'straight upward' configuration.

Exercise 4

The relative orientation of frame RF_B with respect to frame RF_A is expressed by the rotation matrix

$$^{A}\boldsymbol{R}_B = {}^{0}\boldsymbol{R}_A^T \cdot {}^{0}\boldsymbol{R}_B = \begin{pmatrix} \frac{1}{2\sqrt{2}} & -\sqrt{\frac{3}{8}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & -\sqrt{\frac{3}{8}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.3536 & -0.6124 & -0.7071 \\ 0.8660 & 0.5 & 0 \\ 0.3536 & -0.6124 & 0.7071 \end{pmatrix},$$

whose elements will be denoted by R_{ij}. Therefore, the equation $\boldsymbol{R}(\boldsymbol{r}, \theta) = {}^{A}\boldsymbol{R}_B$ should be solved for $\boldsymbol{r}$ and θ, using the inverse mapping of the axis-angle representation. Since

$$\sin\theta = \pm\frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2} = \pm\, 0.9599 \neq 0, \tag{1}$$

the problem at hand is regular, and two distinct solutions can be found depending on the choice of the $+$ or $-$ sign in the expression of $\sin\theta$. From

$$\cos\theta = \frac{1}{2}\left(R_{11} + R_{22} + R_{33} - 1\right) = 0.2803,$$

taking the $-$ sign in (1) will yield a solution angle $\theta < 0$, as requested. Thus

$$\theta = \text{ATAN2}\left\{-0.9599, 0.2803\right\} = -1.2867 \text{ [rad]} = -73.72°$$

and

$$\boldsymbol{r} = \frac{1}{2\sin\theta}\begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} = \begin{pmatrix} 0.3190 \\ 0.5525 \\ -0.7701 \end{pmatrix}.$$

* * * * *

Robotics I - Sheet for Exercise 2

January 11, 2018

Name: ____

Consider the basic algorithms of the two main numerical methods used for solving inverse kinematics problems, denoted here as **N** (Newton method) and **G** (Gradient method). Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. **N** and **G** always fail at singularities.

   True ☐ **False** ☐

2. **G** stops when a singularity is encountered.

   True ☐ **False** ☐

3. Out of singularities, **N** finds always a solution faster than **G**.

   True ☐ **False** ☐

4. **N** can be used only when there is a single global solution to the problem.

   True ☐ **False** ☐

5. Both **N** and **G** need knowledge of the analytic Jacobian of the task.

   True ☐ **False** ☐

6. For a non-square Jacobian, the pseudoinverse should replace the Jacobian transpose in **G**.

   True ☐ **False** ☐

7. Close to a solution, it is computationally faster to evaluate an iteration of **N** than one of **G**.

   True ☐ **False** ☐

8. **G** works better for linear problems, **N** for quadratic ones.

   True ☐ **False** ☐

9. Neither **N** nor **G** would terminate without the use of a small tolerance on the final error.

   True ☐ **False** ☐

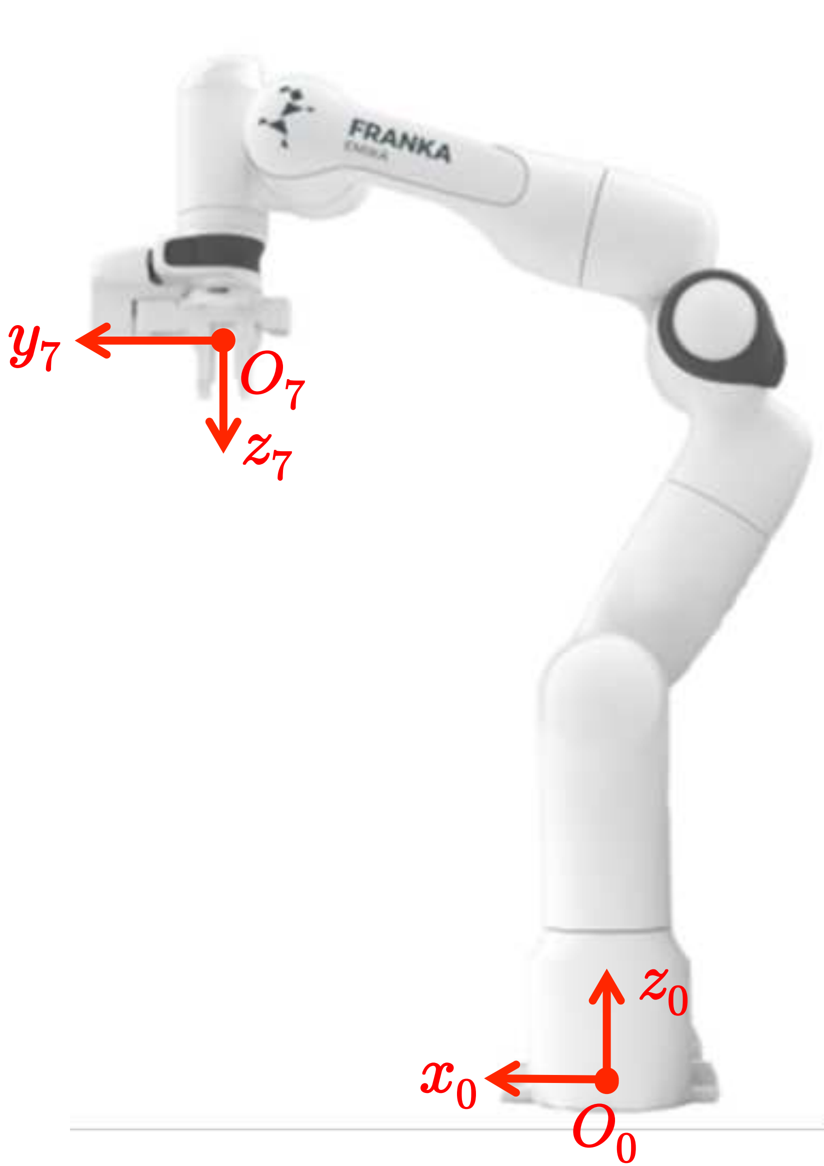
10. Beside matrix operations with the Jacobian and the error, **G** needs an extra choice to be made.

    True ☐ **False** ☐

7R Panda robot by Franka Emika – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				



Robotics I

January 11, 2018

Exercise 1

The Panda by Franka Emika shown in Fig. 1 is an innovative lightweight robot intended for friendly and safe human-robot interaction. The robot has seven revolute joints and its kinematics is characterized by a spherical shoulder, an elbow with two offsets, and a non-spherical wrist. This combination allows eliminating unaccessible 'holes' close to the robot base, thus increasing the robot workspace.

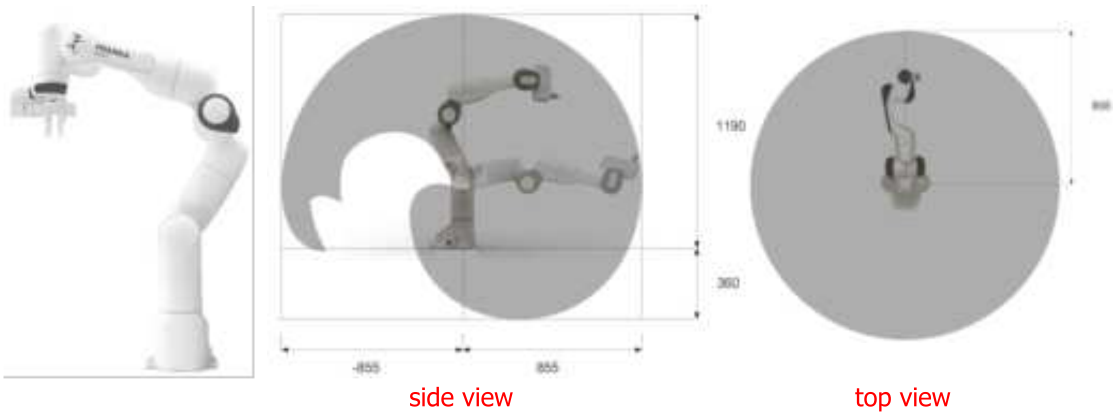


Figure 1: The 7R Panda robot by Franka Emika and two views of its workspace.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated symbolic table of parameters, specifying also the signs of the non-zero constant parameters. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. Therein, the two DH frames 0 and 7 are already assigned and should not be modified. [Please, make clean drawings and return the completed sheet with your name written on it.]
- Write explicitly the seven resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^6\boldsymbol{A}_7(q_7)$. [Do NOT attempt to write the direct kinematics in symbolic form!]
- Assume that all constant DH parameters have been specified numerically. While the robot is moving, the actual position $\boldsymbol{p} \in \mathbb{R}^3$ of the end-effector (i.e., of the origin O_7 of frame 7) should be computed in real time using the measurements of $\boldsymbol{q} \in \mathbb{R}^7$ collected by the encoders at each sampling instant (say, every 400 μs). Provide an efficient scheme for this computation, and determine the total number of elementary operations (evaluation of trigonometric functions, of products $\times$, and of sums $+$) required at each sampling step. You may proceed without exploiting the specific structure of the DH matrices, or customize the procedure avoiding unnecessary elementary operations for this robot.

Exercise 2

A number of statements are reported on the extra sheet #2, regarding the Newton and the Gradient methods for the numerical solution of inverse kinematics problems. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the sheet with your answers and your name written on it.]

Exercise 3

With reference to the setup in Fig. 2, two identical planar 3R manipulators, a master robot M and a slave robot S having link lengths $\ell_1 = \ell_2 = 0.5$ and $\ell_3 = 0.25$ [m], should perform a Cartesian motion task in coordination. The base frames of the robots are displaced by $\boldsymbol{p}_{MS} = (\Delta x, \Delta y, 0) = (1.6, 0.9, 0)$ [m] and rotated by $\alpha_{MS} = \pi$ [rad] around the common $\boldsymbol{z}_0$ axis. The desired Cartesian motion starts at $t = t_0$ from the position $\boldsymbol{p}_M(t_0) \in \mathbb{R}^2$ assumed by the end-effector of the master robot in the configuration $\boldsymbol{q}_M(t_0) = (\pi/2, -\pi/3, 0)$ [rad] and will proceed along a straight line path, which is specified by the initial direction of the end-effector velocity $\boldsymbol{v}_M = \dot{\boldsymbol{p}}_M(t_0) \in \mathbb{R}^2$ resulting from $\dot{\boldsymbol{q}}_M(t_0) = (-\pi/6, 0, -\pi/2)$ [rad/s]. The slave robot should execute the same Cartesian motion in position, while keeping its end-effector always oriented orthogonally to the linear path (more specifically, rotated by a constant angle $\beta = -\pi/2$ [rad] with respect to the vector $\boldsymbol{v}_M$).

- Determine an initial configuration $\boldsymbol{q}_S(t_0)$ of the slave robot such that its end-effector position $\boldsymbol{p}_S(t_0) \in \mathbb{R}^2$ and orientation are initially matched with those required by the motion task.
- Determine the initial joint velocity $\dot{\boldsymbol{q}}_S(t_0) \in \mathbb{R}^3$ of the slave robot, in order to match also the initial desired Cartesian velocity (i.e., $\boldsymbol{v}_S = \dot{\boldsymbol{p}}_S(t_0)$ is equal to $\boldsymbol{v}_M$).
- If the initial configuration of the slave robot is not matched with the desired Cartesian motion, how can this robot still perform the task after an initial transient, with its task error decreasing exponentially to zero?

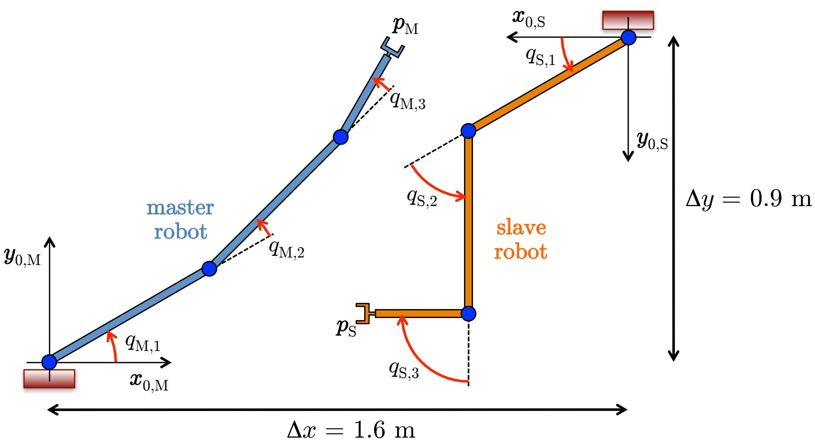


Figure 2: The relative placement of the bases of the two planar 3R robots that should perform a Cartesian motion task in coordination.

Exercise 4

Consider the planning of a smooth trajectory for a planar RP robot (with unlimited joint ranges) between the configurations $\boldsymbol{q}(0) = (\pi/4, -1)$ [rad,m] at $t = 0$ and $\boldsymbol{q}(T) = (-\pi/2, 1)$ [rad,m] at $t = T$. The initial and final joint velocity and acceleration should be zero, and the acceleration should be continuous in the entire time interval $[0, T]$. The following joint velocity and acceleration bounds are also present:

$$|\dot{q}_1| \leq V_1 = 120°/\mathrm{s}, \quad |\dot{q}_2| \leq V_2 = 180 \text{ cm/s}, \quad |\ddot{q}_1| \leq A_1 = 150°/\mathrm{s}^2, \quad |\ddot{q}_2| \leq A_2 = 200 \text{ cm/s}^2. \tag{1}$$

Define a suitable class of trajectories and choose a final time $T = 3$ s. Will the resulting robot motion be feasible with respect to the bounds in (1)? Using uniform time scaling, find the minimum feasible motion time T^* to perform the desired reconfiguration along the chosen trajectory. Sketch a plot of the resulting joint velocity and acceleration profiles. Will the robot cross a singular configuration during its motion?

[240 minutes, open books but no computer or smartphone]

7R Panda robot by Franka Emika – DH frames assignment and table

Name: ______________________________

i	α_i	a_i	d_i	θ_i
1				
2				
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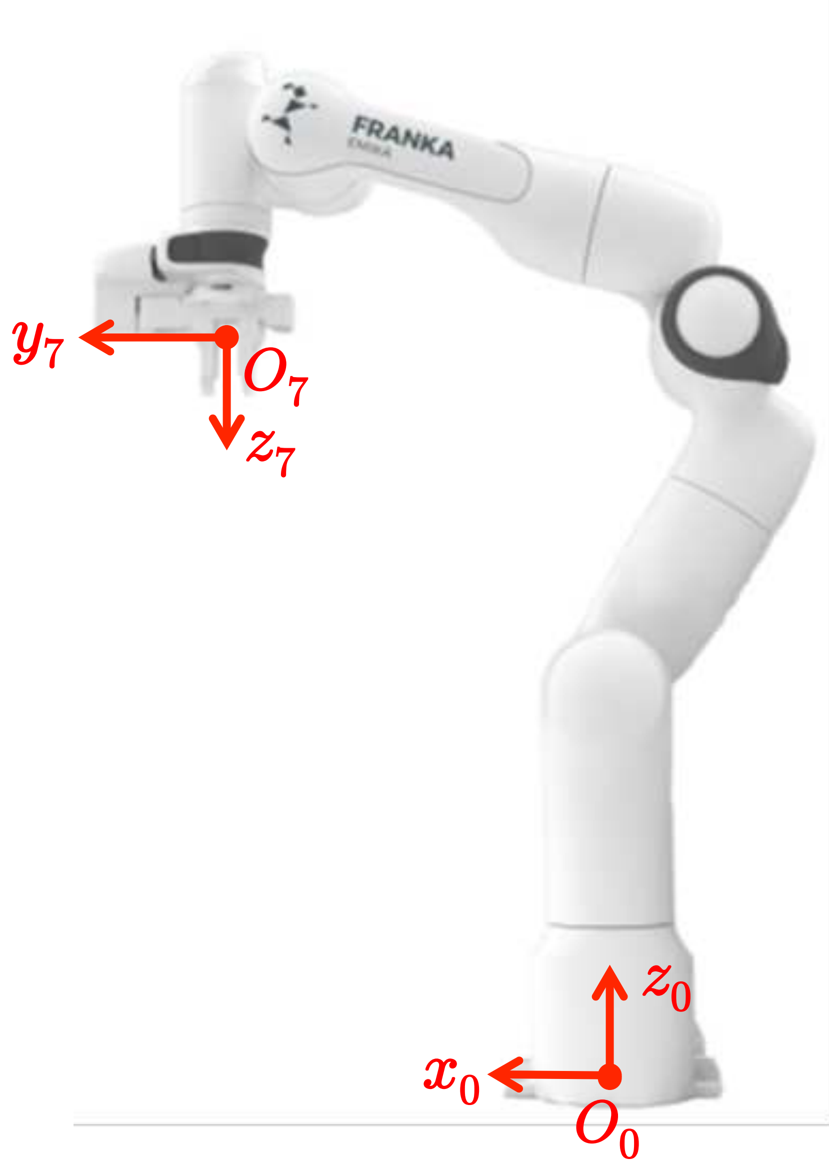


Robotics I – Sheet for Exercise 1, January 11, 2018

Robotics I - Sheet for Exercise 2

January 11, 2018

Name: ______________________________

Consider the basic algorithms of the two main numerical methods used for solving inverse kinematics problems, denoted here as **N** (Newton method) and **G** (Gradient method). Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. **N** and **G** always fail at singularities.

   True ☐ **False** ☐

   ____

2. **G** stops when a singularity is encountered.

   True ☐ **False** ☐

   ____

3. Out of singularities, **N** finds always a solution faster than **G**.

   True ☐ **False** ☐

   ____

4. **N** can be used only when there is a single global solution to the problem.

   True ☐ **False** ☐

   ____

5. Both **N** and **G** need knowledge of the analytic Jacobian of the task.

   True ☐ **False** ☐

   ____

6. For a non-square Jacobian, the pseudoinverse should replace the Jacobian transpose in **G**.

   True ☐ **False** ☐

   ____

7. Close to a solution, it is computationally faster to evaluate an iteration of **N** than one of **G**.

   True ☐ **False** ☐

   ____

8. **G** works better for linear problems, **N** for quadratic ones.

   True ☐ **False** ☐

   ____

9. Neither **N** nor **G** would terminate without the use of a small tolerance on the final error.

   True ☐ **False** ☐

   ____

10. Beside matrix operations with the Jacobian and the error, **G** needs an extra choice to be made.

    True ☐ **False** ☐

    ____

Solution

January 11, 2018

Exercise 1

To help in defining DH frames for the Panda robot by Franka Emika, Figure 3 shows preliminarily the arm decomposition into the series of its links and the definition of the seven joint axes (with a few comments).

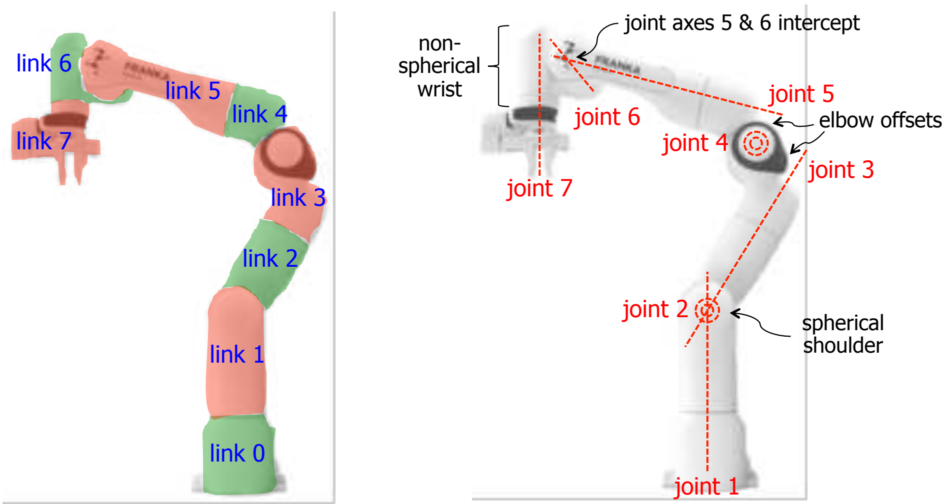


Figure 3: Decomposition of the Panda robot into links and definition of the seven joint axes.

A possible DH frame assignment and the associated table of parameters are reported in Fig. 4 and Tab. 1, respectively, together with their signs. The definition of the constant non-zero DH parameters d_j and a_k and of the DH variables θ_i $(i = 1, \ldots, 7)$ at the current configuration is illustrated in Fig. 5.

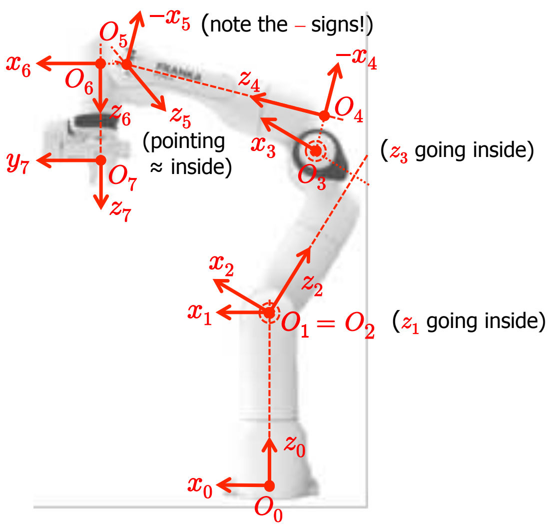


Figure 4: A possible DH frame assignment for the Panda robot by Franka Emika.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$-\pi/2$	0	0	q_2
3	$\pi/2$	$a_3 > 0$	$d_3 > 0$	q_3
4	$-\pi/2$	$a_4 < 0$	0	q_4
5	$\pi/2$	0	$d_5 > 0$	q_5
6	$\pi/2$	$a_6 > 0$	0	q_6
7	0	0	$d_7 > 0$	q_7

Table 1: Parameters associated to the DH frames in Fig. 4.

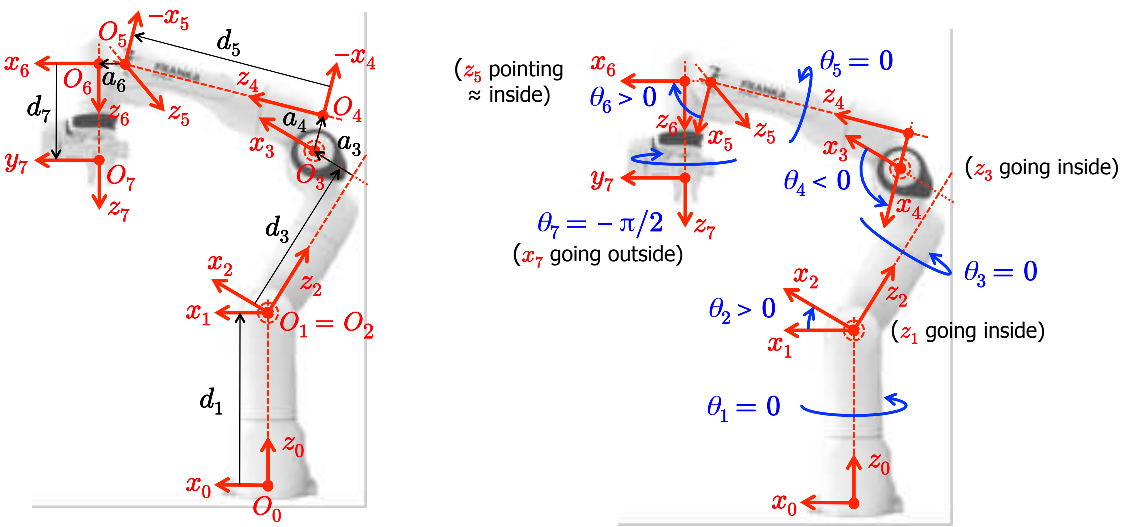


Figure 5: Definition of the (non-zero) constant and variable DH parameters for the Panda robot.

Based on Tab. 1, the seven DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & -\sin q_2 & 0 \\ \sin q_2 & 0 & \cos q_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & a_3 \cos q_3 \\ \sin q_3 & 0 & -\cos q_3 & a_3 \sin q_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 & a_4 \cos q_4 \\ \sin q_4 & 0 & \cos q_4 & a_4 \sin q_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^4\boldsymbol{A}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 & 0 \\ \sin q_5 & 0 & -\cos q_5 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^5\boldsymbol{A}_6(q_6) = \begin{pmatrix} \cos q_6 & 0 & \sin q_6 & a_6 \cos q_6 \\ \sin q_6 & 0 & -\cos q_6 & a_6 \sin q_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

and

$$
{}^6\boldsymbol{A}_7(q_7) = \begin{pmatrix} \cos q_7 & -\sin q_7 & 0 & 0 \\ \sin q_7 & \cos q_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The numerical computation for obtaining the end-effector position $\boldsymbol{p} \in \mathbb{R}^3$ at a given (measured) configuration $\boldsymbol{q} \in \mathbb{R}^7$ is efficiently organized as recursive matrix-vector operations in the form

$$
\boldsymbol{p}_{\text{hom}} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \left[{}^4\boldsymbol{A}_5(q_5) \left[{}^5\boldsymbol{A}_6(q_6) \left[{}^6\boldsymbol{A}_7(q_7) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] \right] \right] \right]. \quad (2)
$$

The number of elementary operations to be performed is evaluated as follows. We need in any case:

- 14 trigonometric evaluations (2 for each of the 7 DH matrices);
- 6 products within the elements of the DH matrices.

Without exploiting the specific structure of the DH matrices, but taking into account that only the first three rows of each matrix are actually involved in the computations, we have additionally:

- at each step, a product of a 3×4 matrix times a 4-dimensional vector $\Rightarrow 12\times, 9+$;
- for the 7 recursive steps $\Rightarrow$ 84 products and 63 sums.

The total number of products and sums can be reduced if we proceed by customization, namely by using the known structure of the specific DH matrices and thus avoiding useless products by 0 or ± 1 and additions of null terms. In this case, we have:

- the first step (the product by ${}^6\boldsymbol{A}_7$) can be skipped, and computations can start with the 4-dimensional vector $\begin{pmatrix} 0 & 0 & d_7 & 1 \end{pmatrix}^T$;
- in the second step (the product by ${}^5\boldsymbol{A}_6$), we have only $4\times$ and $2+$ actual operations;
- proceeding similarly: in the third step (with ${}^4\boldsymbol{A}_5$) $\Rightarrow 3\times, 1+$; in the fourth (etc., ...) $\Rightarrow 6\times, 4+$; in the fifth $\Rightarrow 7\times, 5+$; in the sixth $\Rightarrow 4\times, 2+$; in the seventh and last (with ${}^0\boldsymbol{A}_1$) $\Rightarrow 5\times, 3+$;
- in total $\Rightarrow$ 29 products and 17 sums.

Exercise 2

1. **N** and **G** always fail at singularities.

   False. **G** fails in a singularity only if the error belongs to the null space of $\boldsymbol{J}^T$.

2. **G** stops when a singularity is encountered.

   False. **G** can still move through a singularity if the error $\boldsymbol{e} \notin \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q})\}$.

3. Out of singularities, **N** finds always a solution faster than **G**.

   False. When it converges, **N** is faster than **G**. But **N** may not converge, even out of singularities.

4. **N** can be used only when there is a single global solution to the problem.

   False. **N** (as well as **G**) can be used in any case, finding only one solution at a time.

5. Both **N** and **G** need knowledge of the analytic Jacobian of the task.

   True. Both methods use $\boldsymbol{J}$ (and would converge to a wrong solution with an approximated one).

6. For a non-square Jacobian, the pseudoinverse should replace the Jacobian transpose in **G**.

   False. The pseudoinverse replaces in this case the Jacobian inverse in **N**. **G** needs no changes.

7. Close to a solution, it is computationally faster to evaluate an iteration of **N** than one of **G**.

   False. (Pseudo-)Inverting the Jacobian matrix (in **N**) needs more time than transposing it (in **G**)!

8. **G** works better for linear problems, **N** for quadratic ones.

   False. Relative performance hard to assess in general, but **N** solves a linear problem in one step.

9. Neither **N** nor **G** would terminate without the use of a small tolerance on the final error.

   True. Numerical methods always require a small final tolerance (due to roundings, etc.).

10. Beside matrix operations with the Jacobian and the error, **G** needs an extra choice to be made.

    True. **G** needs the choice of a stepsize in the (negative) gradient direction.

Exercise 3

The solution to the problem requires a combination of direct kinematics and differential kinematics for the master robot and of inverse kinematics and inverse differential kinematics for the slave robot. In addition, one should take into account the roto-translation between the base frames of the two robots. This is represented by a constant homogenous transformation matrix used to change the representations of position vectors starting from the two different origins. Some care should be also used in the treatment of the end-effector orientation of the slave robot as required by the task. Note that the slave robot, with its $n = 3$ joints, is not redundant for the given coordinated task, which in fact specifies $m = 3$ variables.

To begin, frame 0 of the slave robot is related to frame 0 of the master robot by

$$\boldsymbol{T}_{MS} = \begin{pmatrix} \boldsymbol{R}_{MS} & \boldsymbol{p}_{MS} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_{MS} & -\sin\alpha_{MS} & 0 & \Delta x \\ \sin\alpha_{MS} & \cos\alpha_{MS} & 0 & \Delta y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1.6 \\ 0 & -1 & 0 & 0.9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

It follows that

$$\boldsymbol{R}_{SM} = \boldsymbol{R}_{MS}^T = \boldsymbol{R}_{MS}, \qquad \boldsymbol{T}_{SM} = (\boldsymbol{T}_{MS})^{-1} = \boldsymbol{T}_{MS}. \tag{4}$$

Since the problem is planar, we can conveniently work with two-dimensional vectors in the plane (and their three-dimensional homogeneous version), 2×2 rotation matrices (around a unit normal to the motion plane), and associated 3×3 homogeneous transformation matrices. We have in particular

$$\overline{\boldsymbol{T}}_{MS} = \begin{pmatrix} \overline{\boldsymbol{R}}_{MS} & \overline{\boldsymbol{p}}_{MS} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1.6 \\ 0 & -1 & 0.9 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

and similar relations hold for the overlined matrices as those in (4).

With the direct positional kinematics of the master robot

$$\boldsymbol{p}_M(\boldsymbol{q}_M) = \begin{pmatrix} \ell_1 \cos q_{M,1} + \ell_2 \cos(q_{M,1} + q_{M,2}) + \ell_3 \cos(q_{M,1} + q_{M,2} + q_{M,3}) \\ \ell_1 \sin q_{M,1} + \ell_2 \sin(q_{M,1} + q_{M,2}) + \ell_3 \sin(q_{M,1} + q_{M,2} + q_{M,3}) \end{pmatrix}, \tag{6}$$

the end-effector position is evaluated in $\boldsymbol{q}_M(t_0) = (\pi/2, -\pi/3, 0)$ [rad] $= (90°, -60°, 0)$ at time $t = t_0$,

$$\boldsymbol{p}_M(\boldsymbol{q}_M(t_0)) = \begin{pmatrix} 0.6495 \\ 0.8750 \end{pmatrix} = {}^M\boldsymbol{p}_M, \tag{7}$$

where the superscript on the last term reminds us of the frame in which this (planar) vector is expressed. Similarly, using the 2×3 Jacobian of the master robot (written with the usual compact notation)

$$\boldsymbol{J}_M(\boldsymbol{q}_M) = \frac{\partial \boldsymbol{p}_M(\boldsymbol{q}_M)}{\partial \boldsymbol{q}_M} = \begin{pmatrix} -(\ell_1 s_{M,1} + \ell_2 s_{M,12} + \ell_3 s_{M,123}) & -(\ell_2 s_{M,12} + \ell_3 s_{M,123}) & -\ell_3 s_{M,123} \\ \ell_1 c_{M,1} + \ell_2 c_{M,12} + \ell_3 c_{M,123} & \ell_2 c_{M,12} + \ell_3 c_{M,123} & \ell_3 c_{M,123} \end{pmatrix}, \tag{8}$$

the initial linear velocity in the plane of the master end-effector is evaluated as

$$\dot{\boldsymbol{p}}_M(\boldsymbol{q}_M(t_0)) = \boldsymbol{J}_M(\boldsymbol{q}_M(t_0))\,\dot{\boldsymbol{q}}_M(t_0) = \begin{pmatrix} -0.8750 & -0.3750 & -0.1250 \\ 0.6495 & 0.6495 & 0.2165 \end{pmatrix} \begin{pmatrix} -\pi/6 \\ 0 \\ -\pi/2 \end{pmatrix} = \begin{pmatrix} 0.6545 \\ -0.6802 \end{pmatrix} = {}^M\boldsymbol{v}_M. \tag{9}$$

In order to obtain the desired initial position of the end-effector of the slave robot and its desired linear velocity, the vectors in (7) and (9) should be expressed in the reference frame of this second robot. Using (3) and (4), we have

$${}^S\boldsymbol{p}_{S,hom} = \begin{pmatrix} {}^S\boldsymbol{p}_S \\ 1 \end{pmatrix} = \overline{\boldsymbol{T}}_{SM}\,{}^M\boldsymbol{p}_{M,hom} = \overline{\boldsymbol{T}}_{MS} \begin{pmatrix} {}^M\boldsymbol{p}_M \\ 1 \end{pmatrix} \quad \Rightarrow \quad {}^S\boldsymbol{p}_S = \begin{pmatrix} 0.9505 \\ 0.0250 \end{pmatrix} \text{ [m]} \tag{10}$$

and

$${}^S\boldsymbol{v}_S = \overline{\boldsymbol{R}}_{SM}\,{}^M\boldsymbol{v}_M = \begin{pmatrix} -0.6545 \\ 0.6802 \end{pmatrix} \text{ [m/s]}. \tag{11}$$

The angular direction of vector ${}^S\boldsymbol{v}_S$ prescribes also the desired orientation of the slave end-effector as

$$\phi_S = q_{S,1} + q_{S,2} + q_{S,3} = \text{ATAN2}\left\{{}^S\boldsymbol{v}_{S,y}, {}^S\boldsymbol{v}_{S,x}\right\} + \beta = 2.3370 - \frac{\pi}{2} = 0.7662 \text{ [rad]} = 43.89°. \tag{12}$$

The above quantities, together with other parts of the solution, are conveniently illustrated in Fig. 6.

In order to solve the inverse kinematics of the slave robot for the desired ${}^S\boldsymbol{p}_S$ and ϕ_S, given respectively by (10) and (12), we compute first the associated position of the tip of the second link

$$\boldsymbol{p}_{S2} = \begin{pmatrix} p_{S2,x} \\ p_{S2,y} \end{pmatrix} = {}^S\boldsymbol{p}_S - \begin{pmatrix} \ell_3 \cos\phi_S \\ \ell_3 \sin\phi_S \end{pmatrix} = \begin{pmatrix} 0.7703 \\ -0.1483 \end{pmatrix} \text{ [m]}. \tag{13}$$

The solution for the first two joints follows then from the standard inverse formulas for a planar 2R robot:

$$\begin{aligned} c_2 &= \frac{p_{S2,x}^2 + p_{S2,y}^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2} = 0.2308 \quad \text{(should belong to the interval } [-1,+1]) \\ s_2 &= -\sqrt{1-c_2^2} = -0.9730 \quad \text{(the solution with } s_2 < 0 \text{ is chosen here)} \\ q_{S,2} &= \text{ATAN2}\{s_2, c_2\} = 0.4787 \text{ [rad]} = 27.42°, \end{aligned} \tag{14}$$

and

$$\begin{aligned} c_1 &= p_{S2,x}(\ell_1 + \ell_2 c_2) + p_{S2,y}\ell_2 s_2 \\ s_1 &= p_{S2,y}(\ell_1 + \ell_2 c_2) - p_{S2,x}\ell_2 s_2 \\ q_{S,1} &= \text{ATAN2}\{s_1, c_1\} = -1.3378 \text{ [rad]} = -76.65°. \end{aligned} \tag{15}$$

Finally,

$$q_{S,3} = \phi_S - q_{S,1} - q_{S,2} = -1.6253 \text{ [rad]} = 93.12°. \tag{16}$$

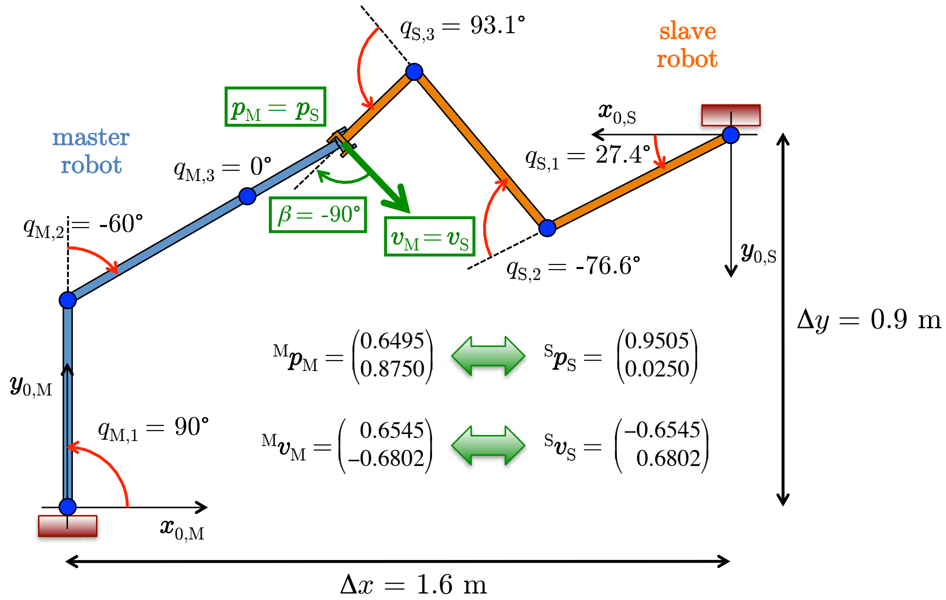


Figure 6: The desired initial configurations of the master and slave robots and the direction of the Cartesian velocity (not in scale) for the coordinated task.

As just seen, the coordination task for the slave robot is specified by three variables which can be jointly described as

$$\boldsymbol{r}_S = \begin{pmatrix} {}^S\boldsymbol{p}_S \\ \phi_S \end{pmatrix} \in \mathbb{R}^3, \qquad \dot{\boldsymbol{r}}_S = \begin{pmatrix} {}^S\dot{\boldsymbol{p}}_S \\ \dot{\phi}_S \end{pmatrix} = \begin{pmatrix} {}^S\boldsymbol{v}_S \\ 0 \end{pmatrix} \in \mathbb{R}^3, \tag{17}$$

where we set $\dot{\phi}_S = 0$ since the absolute orientation of the slave end-effector should remain constant along the fixed linear path. Therefore, since

$$\boldsymbol{r}_S(\boldsymbol{q}_S) = \begin{pmatrix} \ell_1 \cos q_{S,1} + \ell_2 \cos(q_{S,1} + q_{S,2}) + \ell_3 \cos(q_{S,1} + q_{S,2} + q_{S,3}) \\ \ell_1 \sin q_{S,1} + \ell_2 \sin(q_{S,1} + q_{S,2}) + \ell_3 \sin(q_{S,1} + q_{S,2} + q_{S,3}) \\ q_{S,1} + q_{S,2} + q_{S,3} \end{pmatrix}, \tag{18}$$

the 3×3 Jacobian of the slave robot is defined as

$$\boldsymbol{J}_S(\boldsymbol{q}_S) = \frac{\partial \boldsymbol{r}_S(\boldsymbol{q}_S)}{\partial \boldsymbol{q}_S} = \begin{pmatrix} -(\ell_1 s_{S,1} + \ell_2 s_{S,12} + \ell_3 s_{S,123}) & -(\ell_2 s_{S,12} + \ell_3 s_{S,123}) & -\ell_3 s_{S,123} \\ \ell_1 c_{S,1} + \ell_2 c_{S,12} + \ell_3 c_{S,123} & \ell_2 c_{S,12} + \ell_3 c_{S,123} & \ell_3 c_{S,123} \\ 1 & 1 & 1 \end{pmatrix}. \tag{19}$$

Substituting the values of $\boldsymbol{q}_S$ from (14–16) yields

$$\boldsymbol{J}_S = \begin{pmatrix} -0.0250 & 0.2053 & -0.1733 \\ 0.9505 & 0.5067 & 0.1801 \\ 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

The initial joint velocity of the slave robot is then computed as

$$\dot{\boldsymbol{q}}_S = \boldsymbol{J}_S^{-1} \dot{\boldsymbol{r}}_S = \begin{pmatrix} -1.3424 & 1.5566 & -0.5131 \\ 3.1669 & -0.6098 & 0.6588 \\ -1.8245 & -0.9468 & 0.8543 \end{pmatrix} \begin{pmatrix} -0.6545 \\ 0.6802 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.9374 \\ -2.4875 \\ 0.5501 \end{pmatrix} [\text{rad/s}] = \begin{pmatrix} 111.00 \\ -142.52 \\ 31.52 \end{pmatrix} [°/\text{s}]. \tag{21}$$

Finally, if there is an initial mismatch (because the slave robot is not in one of the two possible correct initial configurations, the other being that obtained with $s_2 < 0$ in (14)), a kinematic feedback control law driven by the (Cartesian) task error should be applied. In order to have the task error converge to zero exponentially and in a decoupled way for each component of the task error, the required control law is

$$\dot{\boldsymbol{q}}_S = \boldsymbol{J}_S^{-1}(\boldsymbol{q}_S)\left(\dot{\boldsymbol{r}}_{S,d} + \boldsymbol{K}(\boldsymbol{r}_{S,d} - \boldsymbol{r}_S(\boldsymbol{q}_S))\right), \tag{22}$$

where a subscript d has been added to denote the desired task variables and their first time derivatives, and with $\boldsymbol{K} > 0$ being a diagonal 3×3 gain matrix that specifies the rate of convergence of the errors.

Exercise 4

In view of the smoothness requirement, it is convenient to choose quintic polynomials as the class of motion trajectories for each joint. These polynomials can be used to impose also the initial and final values for the joint velocity and acceleration, in particular all zero as in the present case. Being the required joint displacements

$$\boldsymbol{\Delta q} = \boldsymbol{q}(T) - \boldsymbol{q}(0) = \begin{pmatrix} -\dfrac{3\pi}{4} \\ 2 \end{pmatrix} \text{[rad; m]} = \begin{pmatrix} -135 \\ 200 \end{pmatrix} [^\circ\text{; cm}], \tag{23}$$

the double normalized expression of the trajectory in the joint space is

$$\boldsymbol{q}(\tau) = \boldsymbol{q}(0) + \boldsymbol{\Delta q}\left(6\tau^5 - 15\tau^4 + 10\tau^3\right), \qquad \tau = \frac{t}{T} \in [0,1]. \tag{24}$$

In order to find the maximum velocity and acceleration reached along this trajectory, which should satisfy the bounds (1), the first three time derivatives are needed:

$$\dot{\boldsymbol{q}}(\tau) = 30\,\frac{\boldsymbol{\Delta q}}{T}\left(\tau^4 - 2\tau^3 + \tau^2\right), \quad \ddot{\boldsymbol{q}}(\tau) = 60\,\frac{\boldsymbol{\Delta q}}{T^2}\left(2\tau^3 - 3\tau^2 + \tau\right), \quad \dddot{\boldsymbol{q}}(\tau) = 60\,\frac{\boldsymbol{\Delta q}}{T^3}\left(6\tau^2 - 6\tau + 1\right). \tag{25}$$

The maximum acceleration occurs where the third derivative is zero (no need to check the value at the boundaries $t = \tau = 0$ and $t = T$ ($\tau = 1$), since we have $\ddot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{q}}(1) = \boldsymbol{0}$ by construction):

$$\dddot{\boldsymbol{q}}(\tau) = \boldsymbol{0} \quad \Longleftrightarrow \quad 6\tau^2 - 6\tau + 1 = 0 \quad @\,\tau_a = 0.5 \pm \frac{\sqrt{3}}{6} \quad \Rightarrow \quad \ddot{\boldsymbol{q}}(\tau_a) = \pm\, 5.7735\,\frac{\boldsymbol{\Delta q}}{T^2}. \tag{26}$$

Similarly, the maximum velocity occurs where the second derivative is zero (again, no need to check the value at the boundaries, since $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(1) = 0$):

$$\ddot{\boldsymbol{q}}(\tau) = \boldsymbol{0} \quad \Longleftrightarrow \quad \tau\left(2\tau^2 - 3\tau + 1\right) = 0 \quad @\,\tau_v = \{0, 0.5, 1\} \quad \Rightarrow \quad \dot{\boldsymbol{q}}(0.5) = \frac{30}{16}\,\frac{\boldsymbol{\Delta q}}{T}. \tag{27}$$

For the given motion time $T = 3$ [s], the maximum values are:

$$\begin{aligned} |\dot{q}_1(0.5)| &= |-84.375| < 120 = V_1\ [^\circ/\text{s}], & |\dot{q}_2(0.5)| &= 125 < 180 = V_2\ [\text{cm/s}], \\ |\ddot{q}_1(\tau_a)| &= 86.6025 < 150 = A_1\ [^\circ/\text{s}^2], & |\ddot{q}_2(\tau_a)| &= 128.3 < 200 = A_2\ [\text{cm/s}^2]. \end{aligned} \tag{28}$$

It follows that the original trajectory is feasible. Figure 7 shows the originally planned trajectory profiles.

As a result, we can speed up motion by considering a uniform time scaling. In order to find the minimum motion time T^* that still produces feasible trajectories, we compute

$$k_1 = \min\left\{\frac{V_1}{|\dot{q}_1(0.5)|}, \sqrt{\frac{A_1}{|\ddot{q}_1(\tau_a)|}}\right\} = 1.3161, \qquad k_2 = \min\left\{\frac{V_2}{|\dot{q}_2(0.5)|}, \sqrt{\frac{A_2}{|\ddot{q}_2(\tau_a)|}}\right\} = 1.2485 \tag{29}$$

$$\Rightarrow \qquad k = \min\{k_1, k_2\} = 1.2485 \qquad \Rightarrow \qquad T^* = \frac{T}{k} = \frac{3}{1.2485} = 2.4028 \text{ [s]}. \tag{30}$$

The scaled trajectory and its first two derivatives are shown in Fig. 8. The new maximum values are:

$$\begin{aligned} |\dot{q}_1(0.5)| &= |-105.34| < 120 = V_1\ [^\circ/\text{s}], & |\dot{q}_2(0.5)| &= 156.06 < 180 = V_2\ [\text{cm/s}], \\ |\ddot{q}_1(\tau_a)| &= 135 < 150 = A_1\ [^\circ/\text{s}^2], & |\ddot{q}_2(\tau_a)| &= 200 = A_2\ [\text{cm/s}^2]. \end{aligned} \tag{31}$$

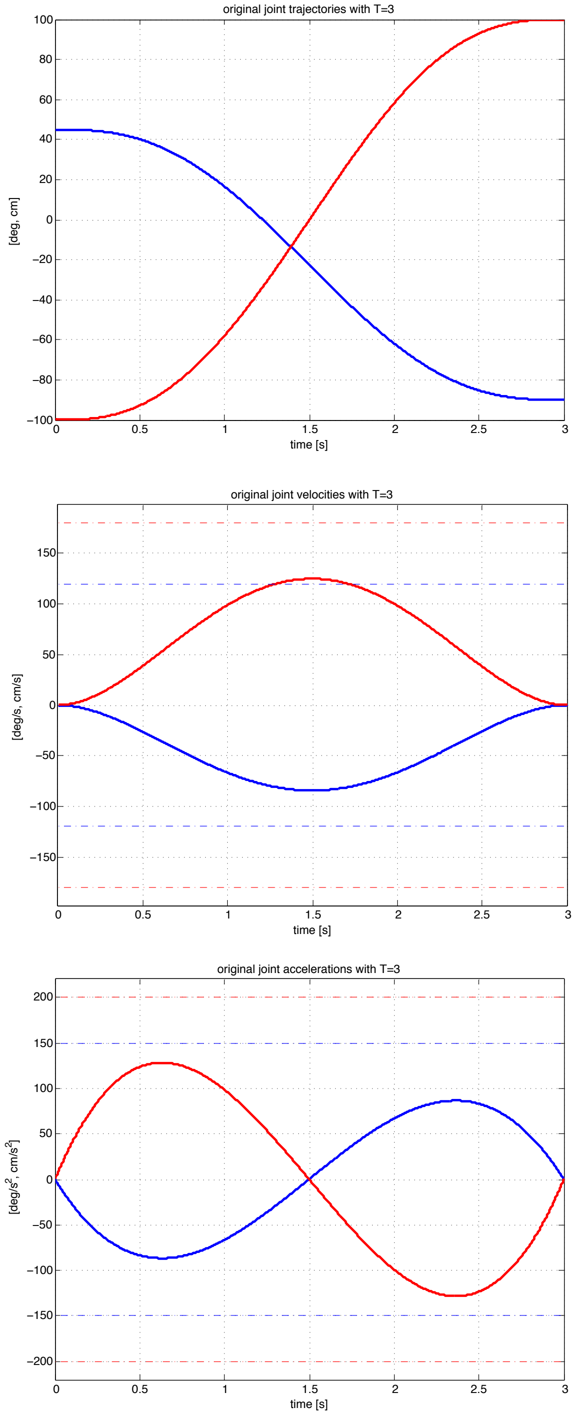


Figure 7: Original joint trajectory for a motion time $T = 3$ [s], with velocity and acceleration profiles (joint 1 = blue, joint 2 = red). The bounds (1), shown by dashed lines ($\cdot-$), are satisfied.

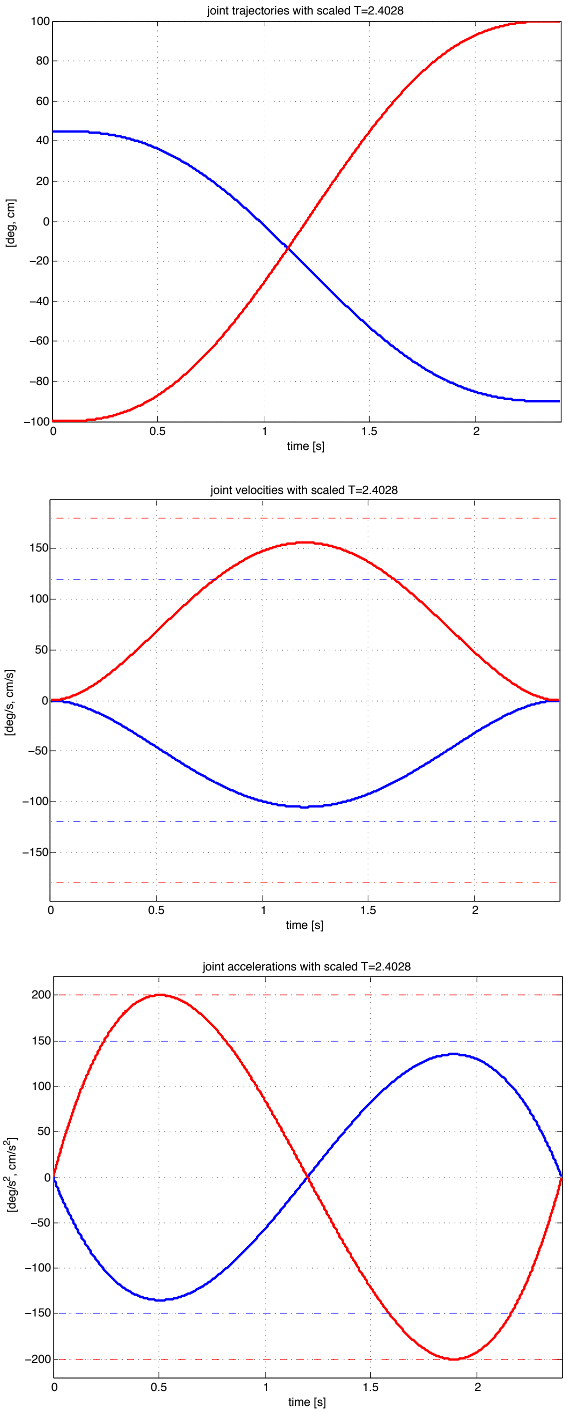


Figure 8: Scaled joint trajectory for a motion time $T^* = 2.4028$ [s], with velocity and acceleration profiles (joint 1 = blue, joint 2 = red). The bound in (1) for the acceleration of the second joint is being reached in two instants that are symmetric w.r.t. the trajectory midpoint.

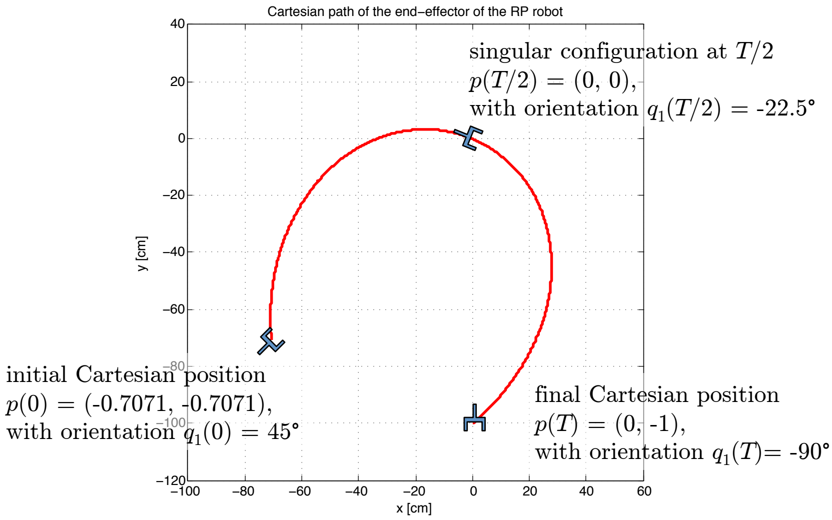


Figure 9: Cartesian path traced by the end-effector of the RP robot. Its orientation is shown at the start and final configurations, and at the intermediate motion instant $t = T/2$ (or $T^*/2$) when the robot crosses the singularity $q_2 = 0$.

During motion, the RP robot will certainly cross a singular configuration with $q_2 = 0$. This will certainly happen for any class of interpolating trajectories, since the initial and final values of q_2 have opposite signs. However, this singularity does not harm when planning (and control) is made directly in the joint space, as in the present case. Figure 9 shows the motion path of the end-effector with the (original or uniformly scaled) quintic polynomial trajectory.

* * * * *

Robotics I - Sheet for Exercise 2

February 5, 2018

Name: ____________________

Consider motion sensing devices available for xed-base robot manipulators and related issues in the measurement process. Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. Encoders of the absolute type cannot be used for estimating joint velocity.

   True ☐ **False** ☐

   ____

2. Encoders should never be mounted beyond the reduction element in motor-link transmission systems.

   True ☐ **False** ☐

   ____

3. Dynamic repeatability of a robot improves when the robot is moving at slow speed.

   True ☐ **False** ☐

   ____

4. Absolute encoders need no calibration before being operative.

   True ☐ **False** ☐

   ____

5. For estimating velocity, integration of accelerometer data outperforms differentiation of encoder data.

   True ☐ **False** ☐

   ____

6. Vision systems are preferred when a direct measure of the robot end-effector position is needed.

   True ☐ **False** ☐

   ____

7. An incremental encoder with 6000 ppt has a better resolution than an absolute encoder with 15 tracks.

   True ☐ **False** ☐

   ____

8. With a sensor mounted on the motor, the larger is the reduction ratio N of the transmission, the better the resolution of the link position estimate is.

   True ☐ **False** ☐

   ____

9. In general, repeatability of a sensor can be improved by calibration, whereas accuracy cannot.

   True ☐ **False** ☐

   ____

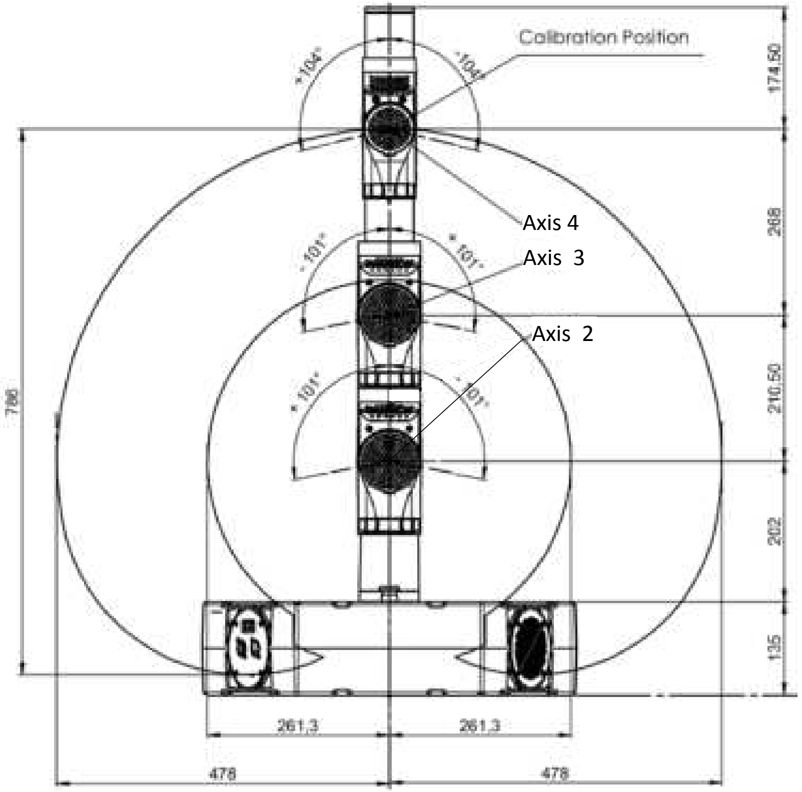
10. Sensor devices should be used only in their domain of linearity (within $2 \div 3\%$ of deviation).

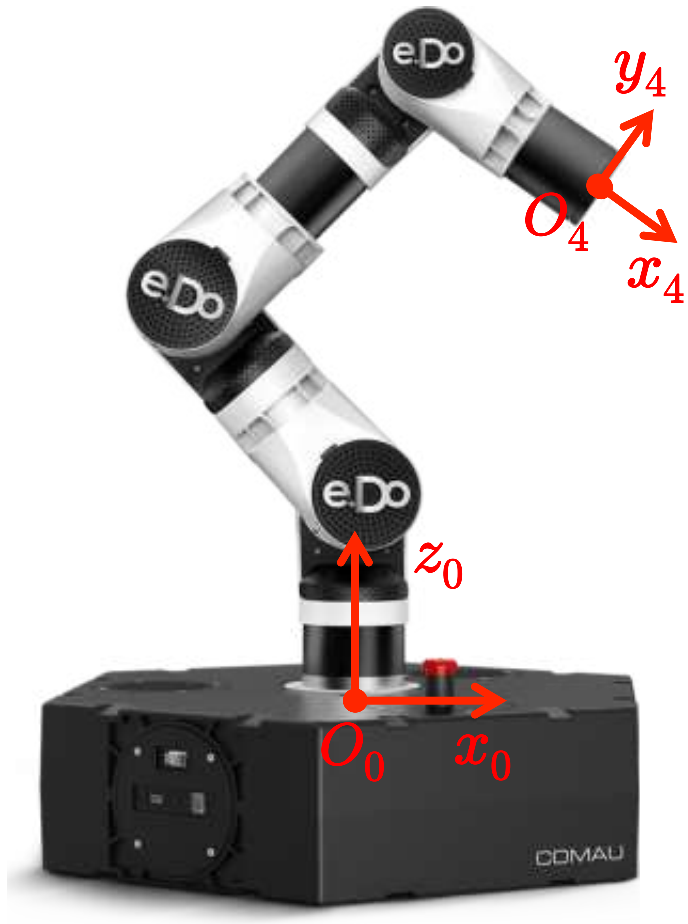
    True ☐ **False** ☐

    ____

4R e.Do robot by Comau – DH frames assignment and table

Name: ______________________________________





i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				

all constant DH parameters should be ≥ 0

Robotics I

February 5, 2018

Exercise 1

The Italian robot manufacturer Comau has recently put on the market two educational manipulators of small size and weight called *e.Do*. The version with four actuated revolute joints is shown in Fig. 1.

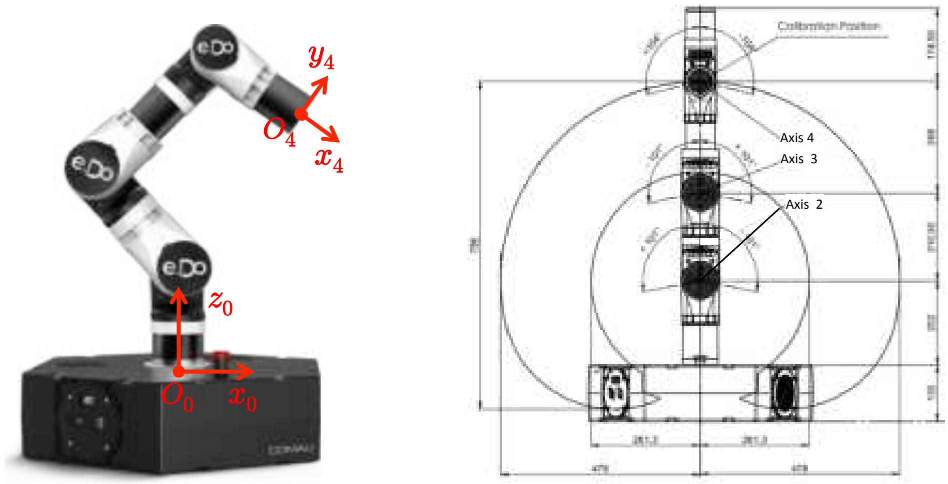


Figure 1: The 4R *e.Do* manipulator by Comau with base and end-effector frames [left] and its relevant dimensions [right].

Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are *non-negative*. Specify also their numerical values. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 4 are already assigned and should not be modified. Finally, write the DH homogeneous transformation matrices. [Please, make clean drawings and return the completed sheet with your name written on it.]

Exercise 2

A number of statements are reported on the extra sheet #2, regarding sensor devices for fixed-base manipulators and related measurement issues. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the completed sheet with your name on it.]

Exercise 3

Determine the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for the robot in Fig. 1 (do not enter numerical values). Partition this matrix in blocks as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{1}$$

- Find all configurations $\boldsymbol{q}_L^*$, if any, where $\boldsymbol{J}_L(\boldsymbol{q})$ loses rank.
- Determine the range space of all feasible angular velocities $\boldsymbol{\omega} \in \mathbb{R}^3$.
- Find all singular configurations $\boldsymbol{q}^*$ of $\boldsymbol{J}(\boldsymbol{q})$, if any.

Choose next a configuration $\boldsymbol{q}_0$ where $\boldsymbol{J}_L$ is full rank, and substitute all the available numerical data in this matrix. Sketch this configuration and compute then a non-zero joint velocity $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^4$ such that the resulting linear velocity $\boldsymbol{v}$ of the robot end-effector at $\boldsymbol{q}_0$ is identically zero.

[turn for next exercise]

Exercise 4

Plan a cubic spline trajectory $q(t)$ that interpolates the following data at given time instants

$$t_1 = 1,\ q(t_1) = 45^\circ, \quad t_2 = 2,\ q(t_2) = 90^\circ, \quad t_3 = 2.5,\ q(t_3) = -45^\circ, \quad t_4 = 4,\ q(t_4) = 45^\circ, \tag{2}$$

starting with $\dot{q}(t_1) = 0$ and arriving with $\dot{q}(t_4) = 0$.

- Give an expression and the associated numerical values of the coefficients of each cubic polynomial.
- Find the maximum (absolute) values attained by the velocity $\dot{q}(t)$ and the acceleration $\ddot{q}(t)$ over the whole motion interval $[t_1, t_4]$, as well as the time instants at which these occur.
- Check if the following bounds are satisfied throughout the motion,

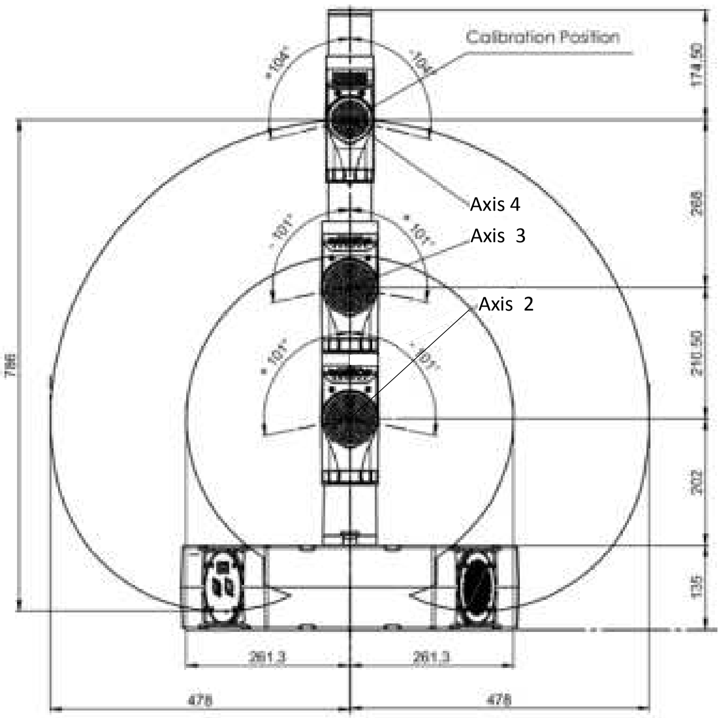
  $$|\dot{q}(t)| \leq V_{\max} = 250^\circ/\mathrm{s}, \qquad |\ddot{q}(t)| \leq A_{\max} = 1000^\circ/\mathrm{s}^2, \tag{3}$$

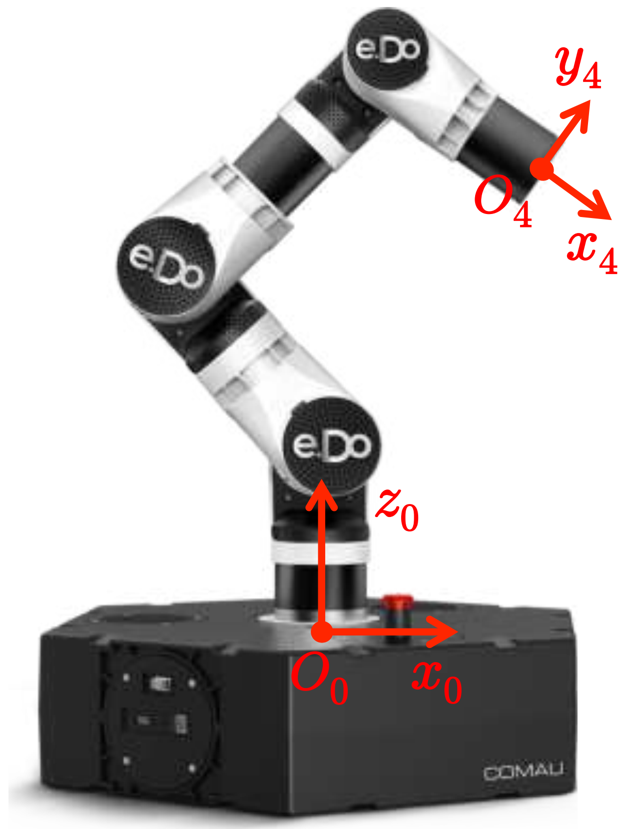
  and, if needed, determine the minimum uniform scaling factor for the trajectory so that feasibility is recovered.
- Provide the total motion time of the feasible trajectory and sketch as accurately as possible the profiles of the resulting velocity and acceleration.

[210 minutes, open books but no computer or smartphone]

4R e.Do robot by Comau – DH frames assignment and table

Name: ______________________________________





i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				

all constant DH parameters should be ≥ 0

Robotics I - Sheet for Exercise 2

February 5, 2018

Name: ____________________

Consider motion sensing devices available for fixed-base robot manipulators and related issues in the measurement process. Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. Encoders of the absolute type cannot be used for estimating joint velocity.

   True ☐ **False** ☐

   ____

2. Encoders should never be mounted beyond the reduction element in motor-link transmission systems.

   True ☐ **False** ☐

   ____

3. Dynamic repeatability of a robot improves when the robot is moving at slow speed.

   True ☐ **False** ☐

   ____

4. Absolute encoders need no calibration before being operative.

   True ☐ **False** ☐

   ____

5. For estimating velocity, integration of accelerometer data outperforms differentiation of encoder data.

   True ☐ **False** ☐

   ____

6. Vision systems are preferred when a direct measure of the robot end-effector position is needed.

   True ☐ **False** ☐

   ____

7. An incremental encoder with 6000 ppt has a better resolution than an absolute encoder with 15 tracks.

   True ☐ **False** ☐

   ____

8. With a sensor mounted on the motor, the larger is the reduction ratio N of the transmission, the better the resolution of the link position estimate is.

   True ☐ **False** ☐

   ____

9. In general, repeatability of a sensor can be improved by calibration, whereas accuracy cannot.

   True ☐ **False** ☐

   ____

10. Sensor devices should be used only in their domain of linearity (within $2 \div 3\%$ of deviation).

    True ☐ **False** ☐

    ____

Solution

February 5, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively, together with the numerical values of the constant parameters (all non-negative, as requested).

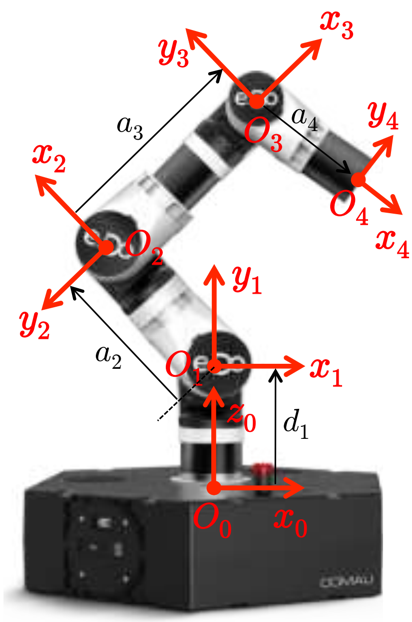


Figure 2: A DH frame assignment for the Comau *e.Do* robot, with associated length parameters.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 202$	q_1
2	0	$a_2 = 210.5$	0	q_2
3	0	$a_3 = 268$	0	q_3
4	0	$a_4 = 174.5$	0	q_4

Table 1: Parameters associated to the DH frames in Fig. 2. Lengths are in [mm].

The robot is equivalent to a planar 3R structure in a vertical plane, mounted on a rotating first axis. Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$ {}^{0}\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{i-1}\boldsymbol{A}_i(q_i) = \begin{pmatrix} \cos q_i & -\sin q_i & 0 & a_i \cos q_i \\ \sin q_i & \cos q_i & 0 & a_i \sin q_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad i = 2,3,4. \tag{4} $$

Exercise 2

1. Encoders of the absolute type cannot be used for estimating joint velocity.

   False. Both types can be used, although incremental encoders output directly Δq (and $\Delta q/\Delta t \simeq \dot{q}$).

2. Encoders should never be mounted beyond the reduction element in motor-link transmission systems.

   False. An encoder on the link side provides a better measure of its position (e.g., for flexible shafts).

3. Dynamic repeatability of a robot improves when the robot is moving at slow speed.

   True. Position errors in the execution of reference trajectories usually increase with larger speeds.

4. Absolute encoders need no calibration before being operative.

   False. A 'homing' at start is not needed, but calibration will recover an erroneous/rotated mounting.

5. For estimating velocity, integration of accelerometer data outperforms differentiation of encoder data.

   False. All the rest being equal, integration of signals is usually subject to drifts over time.

6. Vision systems are preferred when a direct measure of the robot end-effector position is needed.

   True. An external camera senses directly the position, by-passing inaccuracies of robot kinematics.

7. An incremental encoder with 6000 ppt has a better resolution than an absolute encoder with 15 tracks.

   False. $\Delta\theta_{\rm inc} = 360°/6000 = 0.06°$ ($0.015°$ with quadrature). $\Delta\theta_{\rm abs} = 360°/2^{15} = 0.011°$ is better.

8. With a sensor mounted on the motor, the larger is the reduction ratio N of the transmission, the better the resolution of the link position estimate is.

   True. Given a resolution $\Delta\theta_m$ on the motor side, the resolution on the link side is $\Delta\theta_\ell = \Delta\theta_m/N$.

9. In general, repeatability of a sensor can be improved by calibration, whereas accuracy cannot.

   False. Calibration affects accuracy reducing systematic errors. Repeatability relies on sensor quality.

10. Sensor devices should be used only in their domain of linearity (within $2 \div 3\%$ of deviation).

    True. Superposition of physical effects should hold on measurements, possibly after their equalization.

Exercise 3

Using (4), we compute first the direct kinematics of the end-effector position as

$$\boldsymbol{p}_{\rm hom} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left[{}^1\boldsymbol{A}_2(q_2)\left[{}^2\boldsymbol{A}_3(q_3)\left[{}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\right]\right]\right] = \begin{pmatrix} c_1\left(a_2c_2+a_3c_{23}+a_4c_{234}\right) \\ s_1\left(a_2c_2+a_3c_{23}+a_4c_{234}\right) \\ a_2s_2+a_3s_{23}+a_4s_{234} \\ 1 \end{pmatrix} = \begin{pmatrix} p_x\\p_y\\p_z\\1 \end{pmatrix} \tag{5}$$

with the usual shorthand notation for trigonometric terms (e.g., $s_{234} = \sin(q_2+q_3+q_4)$). The linear part of the geometric Jacobian is easily obtained by time differentiation of $\boldsymbol{p}$ in (5) as

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{6}$$

yielding

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1\left(a_2c_2+a_3c_{23}+a_4c_{234}\right) & -c_1\left(a_2s_2+a_3s_{23}+a_4s_{234}\right) & -c_1\left(a_3s_{23}+a_4s_{234}\right) & -a_4c_1s_{234} \\ c_1\left(a_2c_2+a_3c_{23}+a_4c_{234}\right) & -s_1\left(a_2s_2+a_3s_{23}+a_4s_{234}\right) & -s_1\left(a_3s_{23}+a_4s_{234}\right) & -a_4s_1s_{234} \\ 0 & a_2c_2+a_3c_{23}+a_4c_{234} & a_3c_{23}+a_4c_{234} & a_4c_{234} \end{pmatrix}. \tag{7}$$

For analysis purposes, the structure of matrix $\boldsymbol{J}_L$ can be manipulated by (invertible) transformations on the rows and on the columns. It is easy to see that simplifications are obtained by writing the Jacobian in the rotated DH frame 1 (i.e., expressing linear velocity as ${}^1\boldsymbol{v} = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{v}$) and by factoring out recursive expressions in (7). Using the rotational part of matrix ${}^0\boldsymbol{A}_1(q_1)$ in (4), the first step leads in fact to

$$
\begin{aligned}
{}^1\boldsymbol{J}_L(\boldsymbol{q}) \;&=\; {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) = \\
&= \begin{pmatrix} 0 & -\left(a_2 s_2 + a_3 s_{23} + a_4 s_{234}\right) & -\left(a_3 s_{23} + a_4 s_{234}\right) & -a_4 s_{234} \\ 0 & a_2 c_2 + a_3 c_{23} + a_4 c_{234} & a_3 c_{23} + a_4 c_{234} & a_4 c_{234} \\ -\left(a_2 c_2 + a_3 c_{23} + a_4 c_{234}\right) & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{8}
$$

The second step requires post-multiplication of matrix ${}^1\boldsymbol{J}_L$ by a non-singular constant matrix $\boldsymbol{H}$ as follows:

$$
\begin{aligned}
{}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q}) \;&=\; {}^1\boldsymbol{J}_L(\boldsymbol{q})\boldsymbol{H} = {}^1\boldsymbol{J}_L(\boldsymbol{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} 0 & -a_2 s_2 & -a_3 s_{23} & -a_4 s_{234} \\ 0 & a_2 c_2 & a_3 c_{23} & a_4 c_{234} \\ -\left(a_2 c_2 + a_3 c_{23} + a_4 c_{234}\right) & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{9}
$$

The subscript 'abs' is there to remind that the upper two rows and last three columns of the obtained matrix have the same structure of the Jacobian that we would have when considering a planar 3R robot and using absolute coordinates w.r.t. to a horizontal axis.

Analyzing the rank of ${}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q})$ is easy. The last row will become dependent from the other two (and actually vanish) if and only if

$$
\boldsymbol{q}_L^* : \quad a_2 c_2 + a_3 c_{23} + a_4 c_{234} = \sqrt{p_x^2 + p_y^2} = 0, \tag{10}
$$

namely, if the robot end-effector lies on the axis $\boldsymbol{z}_0$ of joint 1. Moreover, the first two rows (deleting their useless, zero first column) will be linearly dependent if and only if the three 2×2 minors that can be extracted are all equal to zero, or

$$
s_3 = 0, \quad s_4 = 0, \quad s_{34} = 0 \qquad \Longleftrightarrow \qquad \boldsymbol{q}_L^* : \quad q_3 = \{0, \pi\}, \quad q_4 = \{0, \pi\}, \tag{11}
$$

namely when the third and fourth link are aligned with the second link, either stretched or folded. Each of the two singularity types (10) and (11) reduces by one the rank of ${}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q})$, which is clearly equal to the rank of $\boldsymbol{J}_L(\boldsymbol{q})$. When the arm is fully aligned with the axis of joint 1, either in a stretched or in a folded configuration, then $\operatorname{rank} \boldsymbol{J}_L(\boldsymbol{q}) = 1$ (i.e., it drops by two at the intersection of the singularities).

Since all joints are revolute, the angular part of the geometric Jacobian is obtained from the general expression

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix}, \tag{12}
$$

where $\boldsymbol{z}_{i-1}$ is the unit vector aligned with the ith joint axis, and expressed in the base frame (of index 0). Using the DH rotation matrices, we have

$$
\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0, \quad \boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0, \quad \boldsymbol{z}_3 = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2){}^2\boldsymbol{R}_3(q_3)\boldsymbol{z}_0, \tag{13}
$$

and performing easy computations

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{z}_3 = \boldsymbol{z}_2 = \boldsymbol{z}_1 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}. \tag{14}$$

Therefore, we obtain

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & s_1 & s_1 & s_1 \\ 0 & -c_1 & -c_1 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

This matrix has constant rank equal to two. All angular velocities that can be generated as $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ take the form

$$\boldsymbol{\omega} = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \qquad \text{for any } \alpha,\ \beta. \tag{16}$$

Vice versa, angular velocities that cannot be generated by this robot take the form

$$\boldsymbol{\omega} = \gamma \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \qquad \text{for any } \gamma \neq 0. \tag{17}$$

For further analysis, matrix $\boldsymbol{J}_A(\boldsymbol{q})$ can be written in the rotated DH frame 1, similarly to (8). We obtain the constant matrix

$${}^1\boldsymbol{J}_A = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{18}$$

Observing that

$$\text{rank } \boldsymbol{J}(\boldsymbol{q}) = \text{rank } \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \text{rank } \begin{pmatrix} {}^1\boldsymbol{J}_{L,\text{abs}}(\boldsymbol{q}) \\ {}^1\boldsymbol{J}_A \end{pmatrix}, \tag{19}$$

we can stack the two matrices in eq. (9) and (18) and analyze the rank of this 6×4 matrix. Deleting the third row (as linearly dependent on the fifth row) and the fourth (null) row, we are left with the 4×4 matrix

$$\boldsymbol{J}_{\text{red}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -a_2 s_2 & -a_3 s_{23} & -a_4 s_{234} \\ 0 & a_2 c_2 & a_3 c_{23} & a_4 c_{234} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

Thus, a singularity of $\boldsymbol{J}(\boldsymbol{q})$ (rank $\boldsymbol{J}(\boldsymbol{q}^*) < 4$) will occur if and only if the determinant of $\boldsymbol{J}_{\text{red}}(\boldsymbol{q})$ vanishes, or

$$\det \boldsymbol{J}_{\text{red}}(\boldsymbol{q}) = 0 \quad \Longleftrightarrow \quad \boldsymbol{q}^*: \quad a_2 a_3 s_3 + a_3 a_4 s_4 - a_2 a_4 s_{34} = 0. \tag{21}$$

We note that when the robot is in a stretched of folded configuration and $\boldsymbol{J}_L(\boldsymbol{q})$ loses rank, as expressed by (11), then condition (21) is also satisfied and $\boldsymbol{J}(\boldsymbol{q})$ is necessarily singular too (i.e., $\boldsymbol{J}_A(\boldsymbol{q})$ does not help in recovering full rank).

Finally, we choose a simple configuration that is nonsingular for $\boldsymbol{J}_L(\boldsymbol{q})$ in (7), say $\boldsymbol{q}_0 = \begin{pmatrix} 0 & 0 & \pi/2 & 0 \end{pmatrix}^T$ (the robot has the second link horizontal and the third and fourth links vertical, pointing upwards: a sketch is left to the reader), and plug in the numerical values for $a_2 = 210.5$, $a_3 = 268$, and $a_4 = 174.5$ [mm], as found in Exercise #1. We obtain the numerical matrix (with rank equal to 3)

$$\boldsymbol{J}_L(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -442.5 & -442.5 & -174.5 \\ 210.5 & 0 & 0 & 0 \\ 0 & 210.5 & 0 & 0 \end{pmatrix}. \tag{22}$$

A velocity vector $\dot{\boldsymbol{q}}_0$ in the null space of $\boldsymbol{J}_L(\boldsymbol{q}_0)$ is

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 & 0 & -0.3669 & 0.9303 \end{pmatrix}^T \text{ [rad/s]},$$

and it is easy to see that $\boldsymbol{J}_L(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$.

Exercise 4

Using time normalization, the three cubic tracts of the interpolating spline are conveniently defined as

$$q_A(\tau_A) = q_1 + a_1\tau_A + a_2\tau_A^2 + a_3\tau_A^3, \qquad \tau_A = \frac{t-t_1}{t_2-t_1} \in [0,1], \quad t \in [t_1, t_2] \tag{23}$$

$$q_B(\tau_B) = q_2 + b_1\tau_B + b_2\tau_B^2 + b_3\tau_B^3, \qquad \tau_B = \frac{t-t_2}{t_3-t_2} \in [0,1], \quad t \in [t_2, t_3] \tag{24}$$

$$q_C(\tau_C) = q_3 + c_1\tau_C + c_2\tau_C^2 + c_3\tau_C^3, \qquad \tau_C = \frac{t-t_3}{t_4-t_3} \in [0,1], \quad t \in [t_3, t_4], \tag{25}$$

with the nine coefficients $a_1, \ldots, c_3$ determined by satisfying the nine boundary conditions

$$\begin{array}{llll} q_A(1) = q_2, & & & \\ & \dot{q}_A(0) = 0, & \dot{q}_A(1) = \dot{q}_B(0)\,[= v_2\,], & \ddot{q}_A(1) = \ddot{q}_B(0), \\ q_B(1) = q_3, & & & \\ & \dot{q}_C(1) = 0, & \dot{q}_B(1) = \dot{q}_C(0)\,[= v_3\,], & \ddot{q}_B(1) = \ddot{q}_C(0). \\ q_C(1) = q_4, & & & \end{array} \tag{26}$$

Assume, for the time being, that we know the value of the velocities v_2 and v_3 in the two intermediate knots (at $t = t_2$ and $t = t_3$, respectively). The coefficients of each of the three cubic polynomials would then be completely defined by the four local boundary conditions on position and velocity at the two extremes of their interval of definition. Performing computations for the cubic A yields the coefficients

$$a_1 = 0, \qquad a_2 = 3(q_2 - q_1) - v_2(t_2 - t_1), \qquad a_3 = v_2(t_2 - t_1) - 2(q_2 - q_1), \tag{27}$$

and thus

$$\ddot{q}_A(1) = \frac{2a_2 + 6a_3}{(t_2 - t_1)^2} = \frac{4v_2}{t_2 - t_1} - \frac{6(q_2 - q_1)}{(t_2 - t_1)^2}. \tag{28}$$

Similarly, for the cubic B

$$b_1 = v_2(t_3 - t_2), \quad b_2 = 3(q_3 - q_2) - (2v_2 + v_3)(t_3 - t_2), \quad b_3 = -2(q_3 - q_2) + (v_2 + v_3)(t_3 - t_2), \tag{29}$$

and thus

$$\ddot{q}_B(0) = \frac{2b_2}{(t_3 - t_2)^2} = \frac{6(q_3 - q_2)}{(t_3 - t_2)^2} - \frac{4v_2 + 2v_3}{t_3 - t_2} \tag{30}$$

and

$$\ddot{q}_B(1) = \frac{2b_2 + 6b_3}{(t_3 - t_2)^2} = \frac{2v_2 + 4v_3}{t_3 - t_2} - \frac{6(q_3 - q_2)}{(t_3 - t_2)^2}. \tag{31}$$

Finally, for the cubic C

$$c_1 = v_3(t_4 - t_3), \qquad c_2 = 3(q_4 - q_3) - 2v_3(t_4 - t_3), \qquad c_3 = v_3(t_4 - t_3) - 2(q_4 - q_3), \tag{32}$$

and thus

$$\ddot{q}_C(0) = \frac{2c_2}{(t_4 - t_3)^2} = \frac{6(q_4 - q_3)}{(t_4 - t_3)^2} - \frac{4v_3}{t_4 - t_3}. \tag{33}$$

Imposing continuity of the acceleration at the internal knots

$$\ddot{q}_A(1) = \ddot{q}_B(0), \qquad \ddot{q}_B(1) = \ddot{q}_C(0),$$

and using eqs. (28), (30–31) and (33), leads to the linear system of equations

$$\boldsymbol{A}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{b}, \tag{34}$$

with[1]

$$\boldsymbol{A} = \begin{pmatrix} 2(t_3 - t_1) & (t_2 - t_1) \\ (t_4 - t_3) & 2(t_4 - t_2) \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 3(q_3 - q_2)\dfrac{t_2 - t_1}{t_3 - t_2} + 3(q_2 - q_1)\dfrac{t_3 - t_2}{t_2 - t_1} \\ 3(q_4 - q_3)\dfrac{t_3 - t_2}{t_4 - t_3} + 3(q_3 - q_2)\dfrac{t_4 - t_3}{t_3 - t_2}. \end{pmatrix}.$$

Replacing the numerical data (degrees are used everywhere here), the system is solved as

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} -175.7143 \\ -215.3571 \end{pmatrix} \; [^\circ/\mathrm{s}],$$

and the coefficients (27), (29), and (32) of the three cubic polynomials take then the numerical values

$$\begin{aligned} a_0 = q_1 = 45, \quad a_1 = 0, \quad a_2 = 310.7143, \quad a_3 = -265.7143, \\ b_0 = q_2 = 90, \quad b_1 = -87.8571, \quad b_2 = -121.6071, \quad b_3 = 74.4643, \\ c_0 = q_3 = -45, \quad c_1 = -323.0357, \quad c_2 = 916.0714, \qquad c_3 = -503.0357. \end{aligned}$$

The plots of the interpolating cubic spline $q(t)$, for $t \in [t_1, t_4] = [1, 4]$, and of its velocity and acceleration are shown in Fig. 3. We can see that velocity is peaking between knots 2 and 3, whereas the maximum (absolute) value of the acceleration is reached at knot 2. Apart from this visualization, we should work indeed analytically in order to check if and how the bounds (3) on the velocity and acceleration are satisfied.

Being piecewise linear, the spline acceleration can assume its maximum values only at the boundaries of each time sub-interval. Therefore, we evaluate the acceleration at the knots (expressed in $[^\circ/\mathrm{s}^2]$):

$$\begin{aligned} A_1 = \ddot{q}(t_1) &= \ddot{q}_A(0) = \frac{2a_2}{(t_2 - t_1)^2} = 621.4286, \\ A_2 = \ddot{q}(t_2) &= \ddot{q}_B(0) = \frac{2b_2}{(t_3 - t_2)^2} = -972.8571, \\ A_3 = \ddot{q}(t_3) &= \ddot{q}_C(0) = \frac{2c_2}{(t_4 - t_3)^2} = 814.2857, \\ A_4 = \ddot{q}(t_4) &= \ddot{q}_C(1) = \frac{2c_2 + 6c_3}{(t_4 - t_3)^2} = -527.1429. \end{aligned} \tag{35}$$

As a result, none of the (absolute) values exceeds the limit of $A_{\max} = 1000\,^\circ/\mathrm{s}^2$.

On the other hand, being piecewise quadratic, the spline velocity can assume its maximum values only where the acceleration is zero, or at the boundaries of each time sub-interval. An instant with zero acceleration occurs inside a given sub-interval if and only if the acceleration changes sign between the boundary knots (i.e., $A_i\,A_{i+1} < 0$). Looking at (35), this happens in fact in all three intervals. We should test then the spline velocity also at the instants $t_{\mathrm{acc_0,i}}$ where acceleration vanishes, or

$$t_{\mathrm{acc_0,i}} = t_i + \frac{|A_i|}{|A_i - A_{i+1}|}(t_{i+1} - t_i), \qquad i = 1, 2, 3.$$

[1]The first equation has been multiplied conveniently by $(t_3 - t_2)(t_2 - t_1)/2$, the second by $(t_4 - t_3)(t_3 - t_2)/2$. This makes $\boldsymbol{A}$ and $\boldsymbol{b}$ in eq. (34) identical to those in the lecture slides, for $N = 4$ and with $v_1 = v_4 = 0$.

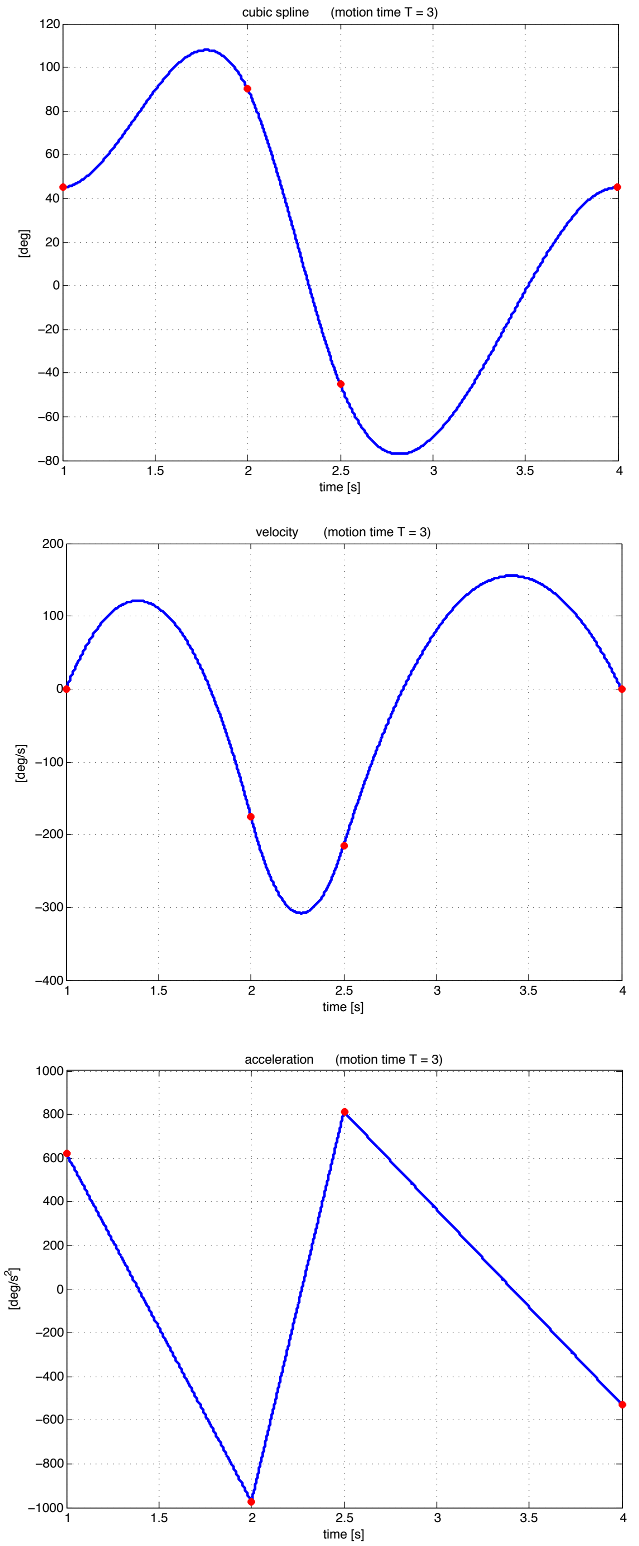


Figure 3: Planned spline trajectory $q(t)$, velocity, and acceleration. Total time $T = t_4 - t_1 = 3$ [s].

Therefore, we first evaluate the velocity at the knots (expressed in [°/s]):

$$\begin{aligned} v_1 &= \dot{q}(t_1) = 0, \\ v_2 &= \dot{q}(t_2) = -175.7143, \\ v_3 &= \dot{q}(t_3) = -215.3571, \\ v_4 &= \dot{q}(t_4) = 0. \end{aligned} \tag{36}$$

Next, having set[2]

$$\tau_{\mathrm{acc}_0,\mathrm{i}} = \frac{t_{\mathrm{acc}_0,\mathrm{i}} - t_i}{t_{i+1} - t_i} = \frac{|A_i|}{|A_i - A_{i+1}|} = \frac{|A_i|}{|A_i| + |A_{i+1}|} \in [0, 1], \qquad i = 1, 2, 3, \tag{37}$$

we evaluate velocity also in the intermediate instants (expressed in [°/s]):

$$\begin{aligned} \dot{q}(t_{\mathrm{acc}_0,1}) &= \dot{q}_A(\tau_{\mathrm{acc}_0,1}) = \frac{a_1 + 2a_2\tau_{\mathrm{acc}_0,1} + 3a_3\tau^2_{\mathrm{acc}_0,1}}{t_2 - t_1} = 121.1118, \\ \dot{q}(t_{\mathrm{acc}_0,2}) &= \dot{q}_B(\tau_{\mathrm{acc}_0,2}) = \frac{b_1 + 2b_2\tau_{\mathrm{acc}_0,2} + 3b_3\tau^2_{\mathrm{acc}_0,2}}{t_3 - t_2} = -308.1115, \\ \dot{q}(t_{\mathrm{acc}_0,3}) &= \dot{q}_C(\tau_{\mathrm{acc}_0,3}) = \frac{c_1 + 2c_2\tau_{\mathrm{acc}_0,3} + 3c_3\tau^2_{\mathrm{acc}_0,3}}{t_4 - t_3} = 155.3640. \end{aligned} \tag{38}$$

As a result, the velocity at the time instant $t_{\mathrm{acc}_0,2} = 2.2722$ [s] (in the second interval) is the only one that violates the bound specified in (3): $V_{\mathrm{peak}} = |\dot{q}(t_{\mathrm{acc}_0,2})| = 308.1115 > 250 = V_{\max}$.

In order to recover feasibility, we should then uniformly scale the total motion time $T = t_4 - t_1 = 3$ s by the factor

$$k = \frac{V_{\mathrm{peak}}}{V_{\max}} = 1.2324 \quad \Rightarrow \quad T_{\mathrm{scaled}} = kT = k(t_4 - t_1) = 3.6973. \tag{39}$$

The spline trajectory $q_{\mathrm{scaled}}(t_{\mathrm{scaled}})$, for $t_{\mathrm{scaled}} \in [t_1, t_1 + T_{\mathrm{scaled}}] = [1, 4.6973]$, is shown in Fig. 4, together with its scaled velocity and acceleration. Some caution should be used to handle a non-zero value for the initial time $t_1 = 1$ in the planned motion. We have $t_{\mathrm{scaled}} = t_1 + k(t - t_1)$ in this case (rather than simply $t_{\mathrm{scaled}} = kt$), and the interpolation of the original knots will be achieved at the new instants

$$t_1 \to t_{\mathrm{scaled},1} = t_1 = 1 \text{ (unchanged)} \Rightarrow q_1, \qquad t_2 \to t_{\mathrm{scaled},2} = t_1 + k(t_2 - t_1) = 2.2324 \Rightarrow q_2,$$

$$t_3 \to t_{\mathrm{scaled},3} = t_1 + k(t_3 - t_1) = 2.8487 \Rightarrow q_3, \qquad t_4 \to t_{\mathrm{scaled},4} = t_1 + k(t_4 - t_1) = 4.6973 \Rightarrow q_4.$$

Accordingly,

$$q_{\mathrm{scaled}}(t_{\mathrm{scaled}}) = q(t), \qquad \dot{q}_{\mathrm{scaled}}(t_{\mathrm{scaled}}) = \frac{\dot{q}(t)}{k}, \qquad \ddot{q}_{\mathrm{scaled}}(t_{\mathrm{scaled}}) = \frac{\dot{q}(t)}{k^2}.$$

The scaled velocity (in absolute value) reaches now its limit $V_{\mathrm{scaled,peak}} = V_{\max} = 250$°/s at the new time instant $t_{\mathrm{scaled,peak}} = t_1 + k\,(t_{\mathrm{acc}_0,2} - t_1) = 1 + 1.2324 \cdot (2.2722 - 1) = 2.5679$ s.

* * * * *

[2]The last equality in (37) holds because of the assumed condition $A_i\, A_{i+1} < 0$, under which the zero acceleration instant occurs inside $[t_i, t_{i+1}]$.

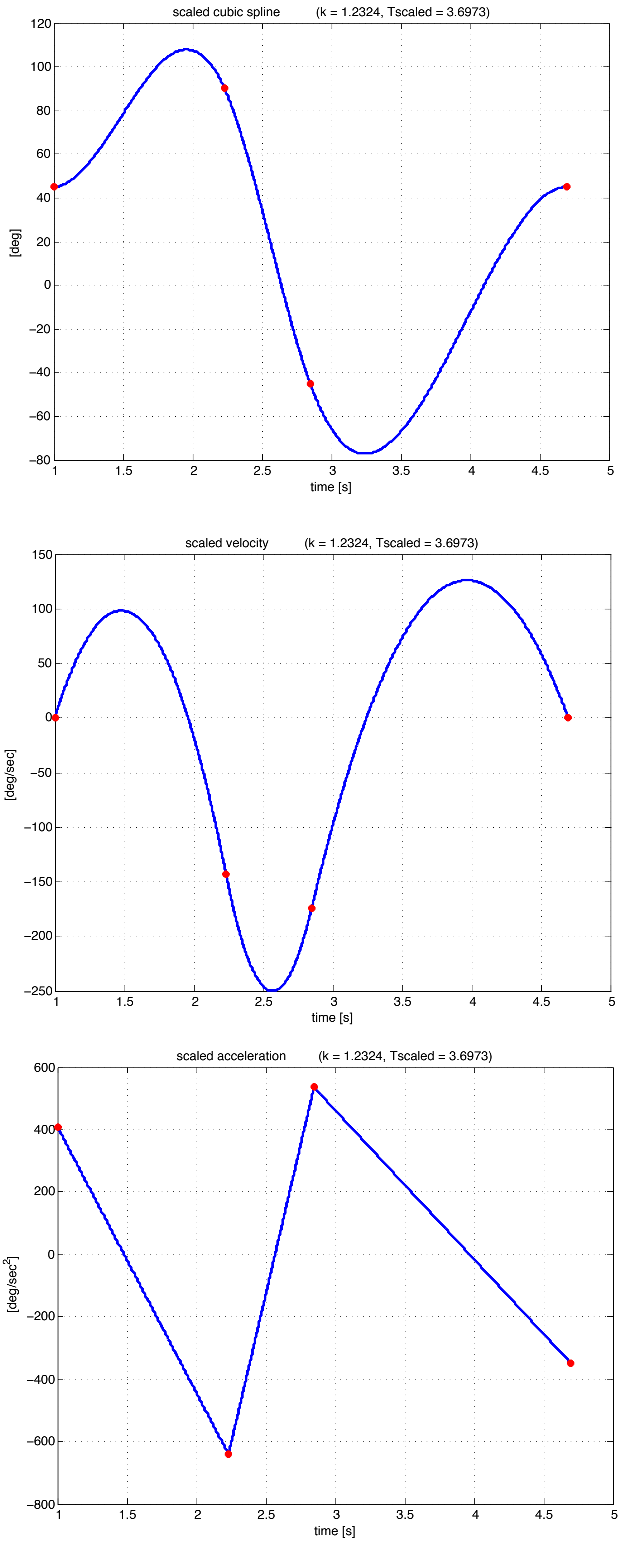


Figure 4: Scaled spline trajectory $q_{\text{scaled}}(t_{\text{scaled}})$, with velocity and acceleration. The scaled total time of motion is $T_{\text{scaled}} = 3.6973$ [s].

Robotics I - Sheet for Exercise 2

March 27, 2018

Name: ____

Consider only serial manipulators having $\boldsymbol{q} \in \mathbb{R}^6$, with direct kinematics expressed by homogenous transformation matrices ${}^0\boldsymbol{T}_6(\boldsymbol{q})$, and their 6×6 geometric Jacobians $\boldsymbol{J}(\boldsymbol{q})$. Check if each of the following statements about singularities is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. In a singular con guration, there may be an in nite number of inverse kinematics solutions.

   True ☐ **False** ☐

   ---

2. In a singularity, the manipulator can access instantaneously any nearby joint con guration.

   True ☐ **False** ☐

   ---

3. Close to a singularity of $\boldsymbol{J}$, some Cartesian directions of motion are not accessible.

   True ☐ **False** ☐

   ---

4. In a singularity, the end-effector angular velocities $\boldsymbol{\omega}$ are linearly dependent on the linear velocities $\boldsymbol{v}$.

   True ☐ **False** ☐

   ---

5. In a singular con guration, $\mathcal{R}\{\boldsymbol{J}^T\} \oplus \mathcal{N}\{\boldsymbol{J}\} \neq \mathbb{R}^6$.

   True ☐ **False** ☐

   ---

6. The linear part $\boldsymbol{J}_L(\boldsymbol{q})$ and the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the Jacobian cannot lose rank simultaneously.

   True ☐ **False** ☐

   ---

7. The lower is the rank of $\boldsymbol{J}$, the larger is the loss of mobility of the end-effector.

   True ☐ **False** ☐

   ---

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q})\}$.

   True ☐ **False** ☐

   ---

9. There cannot be singularities of $\boldsymbol{J}(\boldsymbol{q})$ outside the joint range of the manipulator.

   True ☐ **False** ☐

   ---

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.

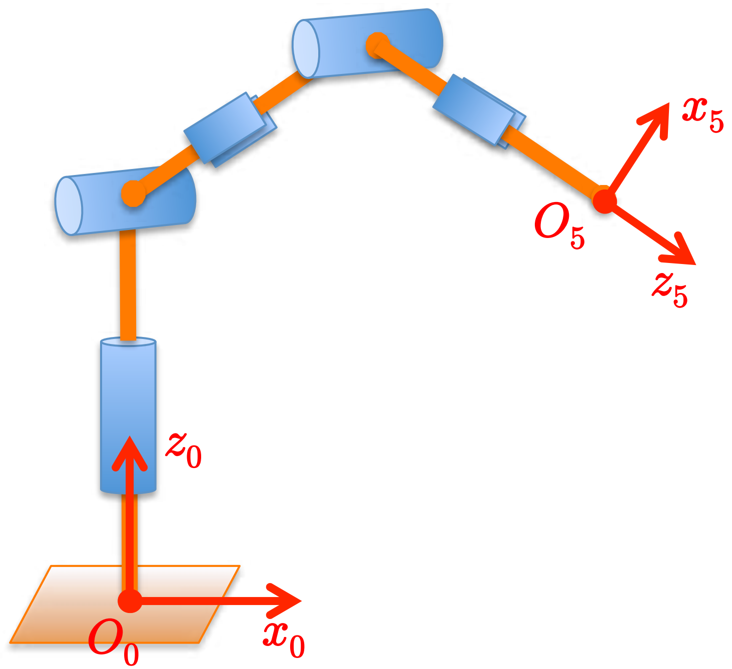
    True ☐ **False** ☐

5-dof spatial robot – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				

all constant DH parameters
should be ≥ 0



Robotics I

March 27, 2018

Exercise 1

Consider the 5-dof spatial robot in Fig. 1, having the third and fifth joints of the prismatic type while the others are revolute.

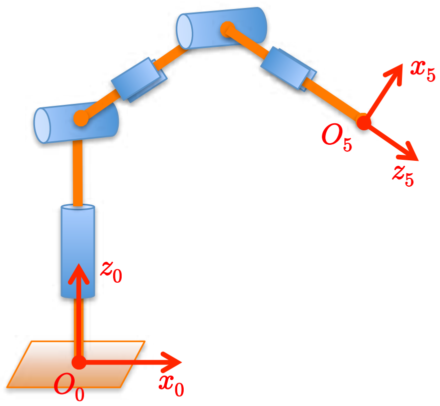


Figure 1: A 5-dof robot, with a RRPRP joint sequence, moving in 3D space.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are *non-negative*. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 5 are already assigned and should not be modified. [Please, make clean drawings and return the completed sheet with your name written on it.]
- Sketch the robot in the configuration $\boldsymbol{q}_a = \begin{pmatrix} 0 & \frac{\pi}{2} & 1 & \frac{\pi}{2} & 1 \end{pmatrix}^T$ [rad, rad, m, rad, m].
- For which value $\boldsymbol{q}_b \in \mathbb{R}^5$ does the robot assume a stretched upward configuration?
- Determine the symbolic expression of the 6×5 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for this robot.
- In the configuration $\boldsymbol{q}_a$, find as many independent *wrench* vectors $\boldsymbol{w} \in \mathbb{R}^6$ (of forces and moments) as possible, with

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} \neq \boldsymbol{0}, \qquad \boldsymbol{f} \in \mathbb{R}^3, \qquad \boldsymbol{m} \in \mathbb{R}^3,$$

  such that when any of these wrenches is applied to the end-effector, the robot remains in static equilibrium without the need of balancing generalized forces at the joints ($\boldsymbol{\tau} = \boldsymbol{0}$, with some components being forces and some torques).

Exercise 2

A number of statements are reported on the extra sheet #2, regarding singularity issues in the direct kinematics of serial manipulators. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the completed sheet with your name on it.]

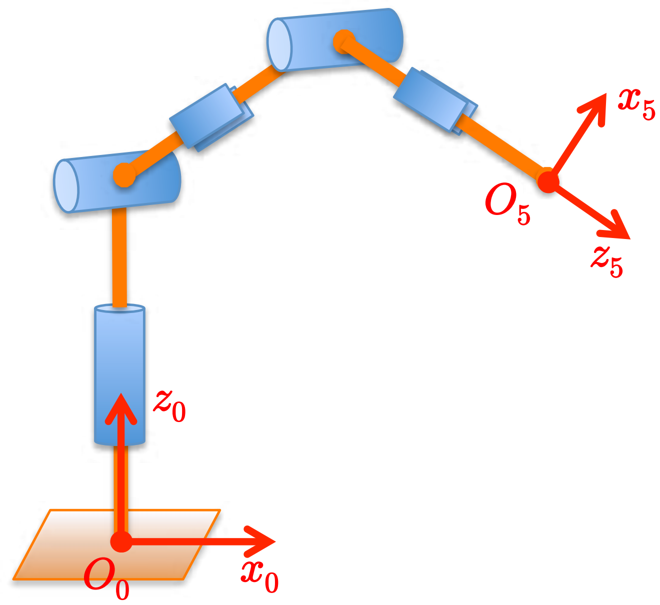
[180 minutes, open books but no computer or smartphone]

5-dof spatial robot – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				

all constant DH parameters
should be ≥ 0



Robotics I – Sheet for Exercise 1, March 27, 2018

Robotics I - Sheet for Exercise 2

March 27, 2018

Name: ______________________________

Consider only serial manipulators having $\boldsymbol{q} \in \mathbb{R}^6$, with direct kinematics expressed by homogenous transformation matrices ${}^0\boldsymbol{T}_6(\boldsymbol{q})$, and their 6×6 geometric Jacobians $\boldsymbol{J}(\boldsymbol{q})$. Check if each of the following statements about singularities is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.

   True ☐ **False** ☐

   ____

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.

   True ☐ **False** ☐

   ____

3. Close to a singularity of $\boldsymbol{J}$, some Cartesian directions of motion are not accessible.

   True ☐ **False** ☐

   ____

4. In a singularity, the end-effector angular velocities $\boldsymbol{\omega}$ are linearly dependent on the linear velocities $\boldsymbol{v}$.

   True ☐ **False** ☐

   ____

5. In a singular configuration, $\mathcal{R}\{\boldsymbol{J}^T\} \oplus \mathcal{N}\{\boldsymbol{J}\} \neq \mathbb{R}^6$.

   True ☐ **False** ☐

   ____

6. The linear part $\boldsymbol{J}_L(\boldsymbol{q})$ and the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the Jacobian cannot lose rank simultaneously.

   True ☐ **False** ☐

   ____

7. The lower is the rank of $\boldsymbol{J}$, the larger is the loss of mobility of the end-effector.

   True ☐ **False** ☐

   ____

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q})\}$.

   True ☐ **False** ☐

   ____

9. There cannot be singularities of $\boldsymbol{J}(\boldsymbol{q})$ outside the joint range of the manipulator.

   True ☐ **False** ☐

   ____

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.

    True ☐ **False** ☐

    ____

Solution

March 27, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively. All constant parameters are non-negative, as requested.

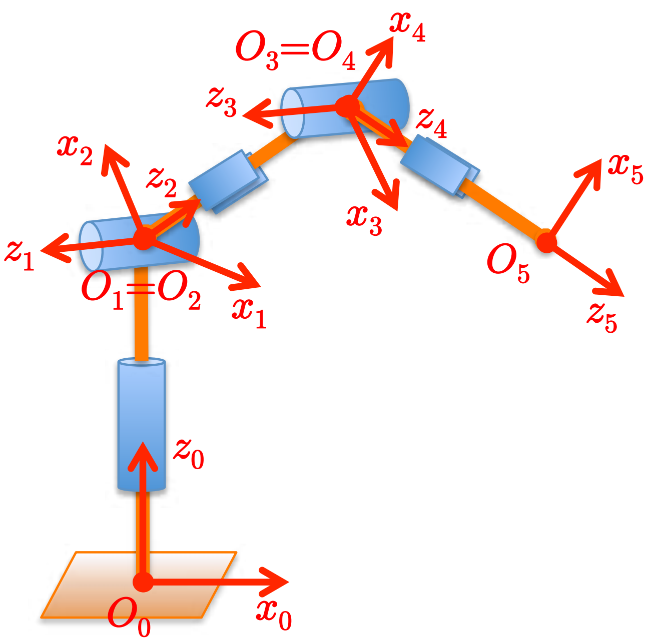


Figure 2: A DH frame assignment for the spatial RRPRP robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$\pi/2$	0	q_3	π
4	$\pi/2$	0	0	q_4
5	0	0	q_5	0

Table 1: Parameters associated to the DH frames in Fig. 2.

For later use, based on Tab. 1, the five DH homogeneous transformation matrices are:

$$
\begin{aligned}
{}^0\boldsymbol{A}_1(q_1) &= \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^1\boldsymbol{A}_2(q_2) &= \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^2\boldsymbol{A}_3(q_3) &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3 & {}^2\boldsymbol{p}_3(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
{}^3\boldsymbol{A}_4(q_4) &= \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^4\boldsymbol{A}_5(q_5) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^4\boldsymbol{R}_5 & {}^4\boldsymbol{p}_5(q_5) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
\end{aligned} \tag{2}
$$

A sketch of the robot in the configuration $\boldsymbol{q}_a = (0, \pi/2, 1, \pi/2, 1)$ is given on the left of Fig. 3, while on the right a stretched upward configuration is shown, corresponding to $\boldsymbol{q}_b = (0, \pi, 1, \pi, 1)$.

In order to compute the linear part $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for this robot, it is convenient to compute first the end-effector position $\boldsymbol{p}$ and then to proceed by symbolic differentiation. For efficiency, we compute this vector (in homogeneous coordinates) using the recursive formula:

$$
\begin{aligned}
\boldsymbol{p}_h(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{p}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_1(q_2) \left({}^2\boldsymbol{A}_3(q_3) \left({}^3\boldsymbol{A}_4(q_4) \left({}^4\boldsymbol{A}_5(q_5) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right)\right)\right)\right) \\
&= \begin{pmatrix} \cos q_1 \, (q_3 \sin q_2 - q_5 \sin(q_2 + q_4)) \\ \sin q_1 \, (q_3 \sin q_2 - q_5 \sin(q_2 + q_4)) \\ d_1 - q_3 \cos q_2 + q_5 \cos(q_2 + q_4) \\ 1 \end{pmatrix}.
\end{aligned}
$$

Therefore, resorting to the usual compact notation, we obtain

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} s_1 \, (q_5 s_{24} - q_3 s_2) & c_1 \, (q_3 c_2 - q_5 c_{24}) & c_1 s_2 & -q_5 c_1 c_{24} & -c_1 s_{24} \\ -c_1 \, (q_5 s_{24} - q_3 s_2) & s_1 \, (q_3 c_2 - q_5 c_{24}) & s_1 s_2 & -q_5 s_1 c_{24} & -s_1 s_{24} \\ 0 & q_3 s_2 - q_5 s_{24} & -c_2 & -q_5 s_{24} & c_{24} \end{pmatrix}.
$$

For the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian, taking into account that the third and fifth

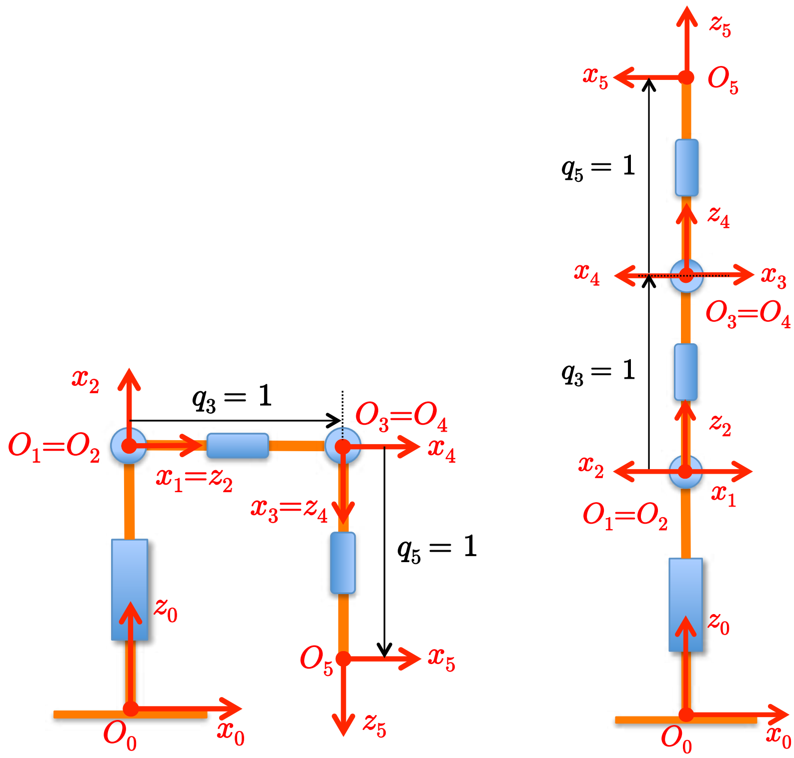


Figure 3: A side view of the RRPRP robot in the configuration $\boldsymbol{q}_a = (0, \pi/2, 1, \pi/2, 1)$ and in a stretched upward configuration with $\boldsymbol{q}_b = (0, \pi, 1, \pi, 1)$.

joints are prismatic, we have

$$
\begin{aligned}
\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{z}_3 & \boldsymbol{0} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)^1\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{R}_1(q_1)^1\boldsymbol{R}_2(q_2)^2\boldsymbol{R}_3(q_3)^3\boldsymbol{z}_3 & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} 0 & s_1 & 0 & s_1 & 0 \\ 0 & -c_1 & 0 & -c_1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T = \boldsymbol{z}_0$ for any i.

The complete Jacobian is then

$$
\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}.
$$

In the assigned configuration $\boldsymbol{q}_a = \begin{pmatrix} 0 & \frac{\pi}{2} & 1 & \frac{\pi}{2} & 1 \end{pmatrix}^T$ the transpose of this Jacobian matrix takes the value

$$
\boldsymbol{J}^T(\boldsymbol{q}_a) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank} \boldsymbol{J}^T(\boldsymbol{q}_a) = 4.
$$

It is easy to see that the null space of $\boldsymbol{J}^T(q_a)$ is spanned, e.g., by the two wrenches

$$\boldsymbol{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The wrench $\boldsymbol{w}_1$ corresponds to a pure moment with $m_x \neq 0$, while $\boldsymbol{w}_2$ is associated to a force $f_y \neq 0$, combined with a moment $m_z \neq 0$. The generalized forces in the joint space needed for balancing any wrench generated by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are indeed

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}_a)\,(\alpha_1 \boldsymbol{w}_1 + \alpha_2 \boldsymbol{w}_2) = \boldsymbol{0}, \qquad \forall \alpha_1, \alpha_2.$$

Exercise 2

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.

   True. The number of solutions changes from the generic case, decreasing or going to infinity.

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.

   True. There is no mobility loss in the joint space commanding motion without inversion of $\boldsymbol{J}$.

3. Close to a singularity of $\boldsymbol{J}$, some Cartesian directions of motion are not accessible.

   False. This is true in a singular configuration, not close to it (though motion effort may increase).

4. In a singularity, the end-effector angular velocities $\boldsymbol{\omega}$ are linearly dependent on the linear velocities $\boldsymbol{v}$.

   False. Not necessarily. It depends on the geometric relation between subspaces $\mathcal{R}\{\boldsymbol{J}_L\}$ and $\mathcal{R}\{\boldsymbol{J}_A\}$.

5. In a singular configuration, $\mathcal{R}\{\boldsymbol{J}^T\} \oplus \mathcal{N}\{\boldsymbol{J}\} \neq \mathbb{R}^6$.

   False. The direct sum of these two subspaces covers always the entire joint space.

6. The linear part $\boldsymbol{J}_L(\boldsymbol{q})$ and the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the Jacobian cannot lose rank simultaneously.

   False. Both ranks of $\boldsymbol{J}_L$ and $\boldsymbol{J}_A$ can be < 3 (when both are full rank, it may still be rank $\boldsymbol{J} < 6$).

7. The lower is the rank of $\boldsymbol{J}$, the larger is the loss of mobility of the end-effector.

   True. For instance, two 6-dim independent Cartesian directions are inaccessible when rank $\boldsymbol{J} = 4$.

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q})\}$.

   True. $\boldsymbol{J}$ is singular iff its null space is $\neq \boldsymbol{0}$ —the condition can be used in the search of singularities.

9. There cannot be singularities of $\boldsymbol{J}(\boldsymbol{q})$ outside the joint range of the manipulator.

   False. Singularities are found without considering the joint range. Those outside are then discarded.

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.

    False. Crossing a singular configuration on a feasible Cartesian cycle can destroy joint-space cyclicity.

* * * * *

Robotics I

June 11, 2018

Exercise 1

Consider the planar 2R robot in Fig. 1, having a L-shaped second link. A frame RF_e is attached to the gripper mounted on the robot end effector.

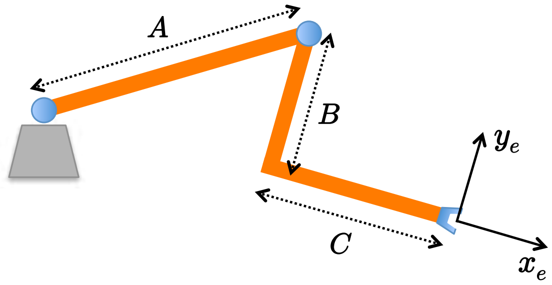


Figure 1: A planar 2R robot with L-shaped second link.

- Assign the link frames and define the joint variables $\boldsymbol{q}$ according to the Denavit-Hartenberg (DH) convention, and complete the associated table of parameters.
- Determine the robot direct kinematics, specifying the position $\boldsymbol{p}_e(\boldsymbol{q}) \in \mathbb{R}^3$ of the origin of frame RF_e and the end-effector orientation as expressed by the rotation matrix ${}^0\boldsymbol{R}_e(\boldsymbol{q}) \in SO(3)$.
- Sketch the robot in the zero configuration ($\boldsymbol{q} = \boldsymbol{0}$); find and sketch also the configuration $\boldsymbol{q}_s$ where the robot gripper is pointing in the direction $\boldsymbol{y}_0$ and is the farthest away from axis $\boldsymbol{x}_0$.
- Draw the primary workspace of the robot using the symbolic values A, B and C for the lengths.
- Provide all closed-form solutions to the inverse kinematics problem, when the end-effector position $\bar{\boldsymbol{p}}_e \in \mathbb{R}^2$ (i.e., reduced to the plane of motion) is given as input.
- Derive the 2×2 Jacobian $\boldsymbol{J}$ in
$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$
where $\boldsymbol{v}_e = \dot{\bar{\boldsymbol{p}}}_e \in \mathbb{R}^2$ is the linear velocity of the end-effector in the plane.
- Determine all singular configurations of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$. In each of these singularities, provide a basis for the null space and the range space of $\boldsymbol{J}$.
- Using the numerical data
$$A = 0.6, \qquad B = 0.3, \qquad C = 0.4 \quad \text{[m]} \tag{1}$$
  - find all joint configurations $\boldsymbol{q}_{sol}$ associated to the end-effector position $\bar{\boldsymbol{p}}_e = (0.6 \;\; -0.5)^T$ [m];
  - for each $\boldsymbol{q}_{sol}$, compute the joint velocity $\dot{\boldsymbol{q}}_{sol}$ that realizes the velocity $\boldsymbol{v}_e = (1 \;\; 0)^T$ [m/s]; are these joint velocities equal or different in norm? why?

Exercise 2

For the robot in Exercise 1, find the minimum time rest-to-rest motion between $\boldsymbol{q}_0 = (1 \;\; -0.5)^T$ and $\boldsymbol{q}_f = (0 \;\; 0.2)^T$ [rad], when the joint velocities and accelerations are subject to the bounds
$$\begin{aligned} |\dot{q}_1| &\le V_1 = 0.5 \text{ [rad/s]}, \qquad & |\ddot{q}_1| &\le A_1 = 0.8 \text{ [rad/s}^2\text{]}, \\ |\dot{q}_2| &\le V_2 = 0.8 \text{ [rad/s]}, \qquad & |\ddot{q}_2| &\le A_2 = 0.5 \text{ [rad/s}^2\text{]}. \end{aligned} \tag{2}$$
Draw accurately the minimum time velocity profiles of the two joints, when a coordinated motion is also required.

[180 minutes, open books but no computer or smartphone]

Solution

June 11, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively.

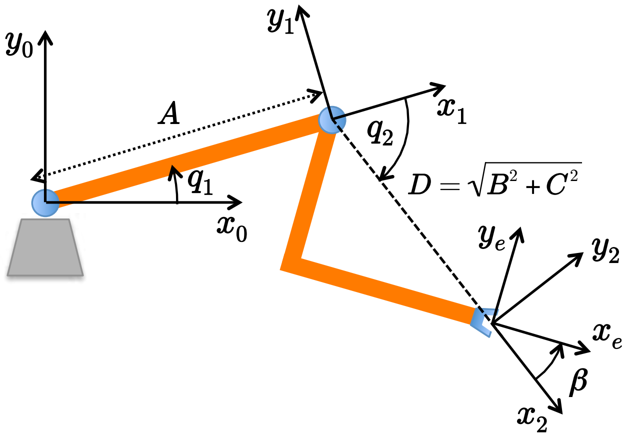


Figure 2: DH frame assignment for the planar 2R robot with a L-shaped second link.

i	α_i	a_i	d_i	θ_i
1	0	A	0	q_1
2	0	D	0	q_2

Table 1: Parameters associated to the DH frames in Fig. 2.

Based on Tab. 1, the DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & A\cos q_1 \\ \sin q_1 & \cos q_1 & 0 & A\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1(q_1) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & D\cos q_2 \\ \sin q_2 & \cos q_2 & 0 & D\sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2(q_2) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

where $D = \sqrt{B^2 + C^2} > 0$. Moreover, the constant homogeneous transformation between the (last) DH frame RF_2 and the end-effector frame RF_e is given by

$$^2\boldsymbol{T}_e = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_e & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}.$$

with $\beta = \arctan(B/C) > 0$. The requested direct kinematics is given by

$$\boldsymbol{p}_e(\boldsymbol{q}) = {}^0\boldsymbol{p}_e(\boldsymbol{q}) = {}^0\boldsymbol{p}_1(q_1) + {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{p}_2(q_2) = \begin{pmatrix} A\cos q_1 + D\cos(q_1+q_2) \\ A\sin q_1 + D\sin(q_1+q_2) \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{p}}_e(\boldsymbol{q}) \\ 0 \end{pmatrix} \quad (3)$$

and

$${}^0\boldsymbol{R}_e(\boldsymbol{q}) = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_e = \begin{pmatrix} \cos(q_1+q_2+\beta) & -\sin(q_1+q_2+\beta) & 0 \\ \sin(q_1+q_2+\beta) & \cos(q_1+q_2+\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

It is evident from (3) that the kinematics of this robot is the same as that of a standard planar 2R arm, once we use the value D as length of an equivalent straight second link. On the other hand, the orientation of the end-effector frame is affected by the constant bias angle β, see (4). A sketch of the robot in the zero configuration is given in Fig. 3a, while Fig. 3b shows the requested configuration $\boldsymbol{q}_s = (\pi/2, -\arctan(B/C))$, with the gripper pointing in the $\boldsymbol{y}_0$ direction and placed the farthest away (at a distance $A+C$) from the $\boldsymbol{x}_0$ axis. The primary workspace of the robot is drawn in Fig. 4, with inner radius and outer radius given, respectively, by

$$r = |A - D|, \qquad R = A + D.$$

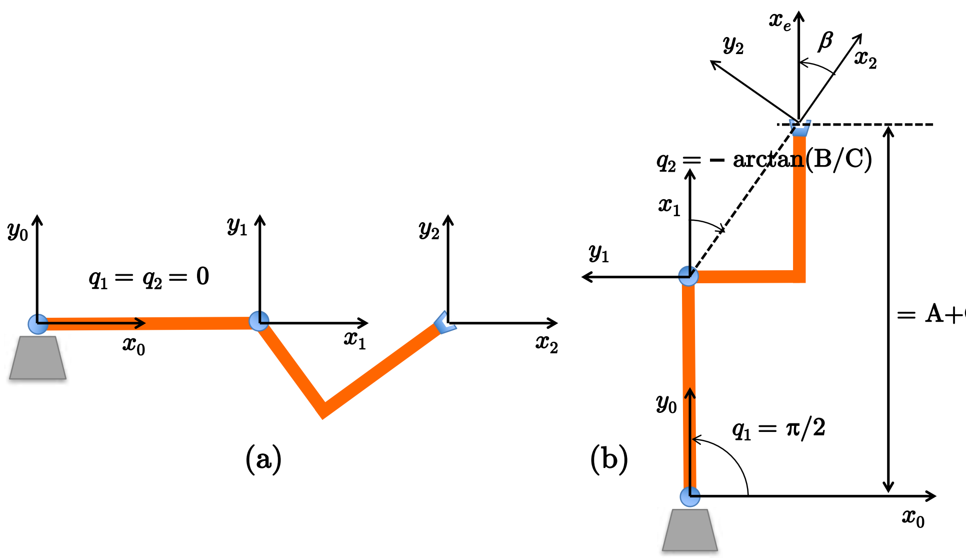


Figure 3: Two special robot postures: (a) the configuration $\boldsymbol{q} = \boldsymbol{0}$, and (b) the configuration $\boldsymbol{q}_s = (\pi/2, -\arctan(B/C))$, in which the robot gripper points in the direction $\boldsymbol{y}_0$ while being the farthest away from the base axis $\boldsymbol{x}_0$.

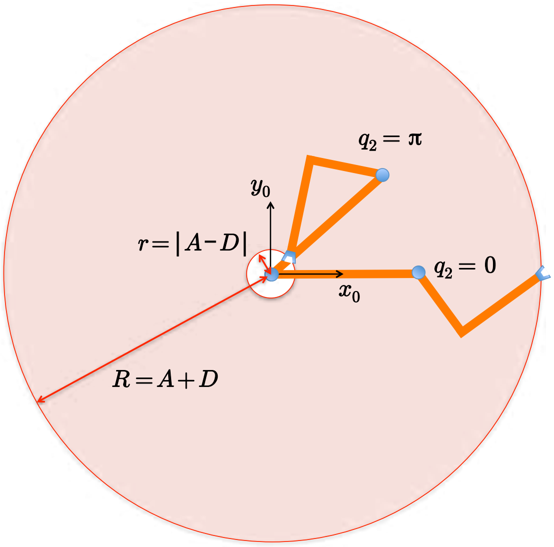


Figure 4: Primary workspace of the planar 2R robot with a L-shaped second link and generic lengths A, B, and C (with $D = \sqrt{B^2 + C^2}$).

From the previous argument, the inverse kinematics problem for this robot has two solutions (out of singularities). Given an end-effector position $\bar{\boldsymbol{p}}_e = (p_x, p_y)$ inside the primary workspace, we have

$$c_2 = \frac{p_x^2 + p_y^2 - A^2 - D^2}{2AD} \in (-1, 1), \qquad s_2 = \pm\sqrt{1 - c_2^2}, \tag{5}$$

and then (for each of the two possible signs of s_2)

$$q_1 = \text{ATAN2}\left\{p_y(A + Dc_2) - p_x D s_2,\, p_x(A + Dc_2) + p_y D s_2)\right\}, \qquad q_2 = \text{ATAN2}\left\{s_2,\, c_2\right\}. \tag{6}$$

The requested (analytic) Jacobian is computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \bar{\boldsymbol{p}}_e(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(A \sin q_1 + D \sin(q_1 + q_2)) & -D \sin(q_1 + q_2) \\ A \cos q_1 + D \cos(q_1 + q_2) & D \cos(q_1 + q_2) \end{pmatrix},$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = AD \sin q_2$. Singularities occur at $q_2 = \{0, \pi\}$ (with arbitrary q_1). For $q_2 = 0$ (and a generic q_1), we have

$$\boldsymbol{J}(q_1, 0) = \begin{pmatrix} -(A + D) \sin q_1 & -D \sin q_1 \\ (A + D) \cos q_1 & D \cos q_1 \end{pmatrix} \quad \Rightarrow \quad \mathcal{N}\{\boldsymbol{J}\} = \nu \begin{pmatrix} -D \\ A + D \end{pmatrix}, \ \mathcal{R}\{\boldsymbol{J}\} = \rho \begin{pmatrix} \sin q_1 \\ -\cos q_1 \end{pmatrix},$$

while at $q_2 = \pi$

$$\boldsymbol{J}(q_1, \pi) = \begin{pmatrix} -(A - D) \sin q_1 & D \sin q_1 \\ (A - D) \cos q_1 & -D \cos q_1 \end{pmatrix} \quad \Rightarrow \quad \mathcal{N}\{\boldsymbol{J}\} = \nu \begin{pmatrix} D \\ A - D \end{pmatrix}, \ \mathcal{R}\{\boldsymbol{J}\} = \rho \begin{pmatrix} \sin q_1 \\ -\cos q_1 \end{pmatrix},$$

where ν and ρ are two scaling factors.

Using the numerical data in (1), we have $A = 0.6$ and $D = 0.5$ [m]. For $\boldsymbol{p}_e = (0.6 \ \ -0.5)^T$ [m], we obtain from (5–6) the two inverse kinematic solutions

$$\boldsymbol{q}_{sol,a} = \begin{pmatrix} 1.3895 \\ \pi/2 \end{pmatrix} \text{[rad]} = \begin{pmatrix} -79.61^\circ \\ 90^\circ \end{pmatrix}, \qquad \boldsymbol{q}_{sol,b} = \begin{pmatrix} 0 \\ -\pi/2 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 0^\circ \\ -90^\circ \end{pmatrix}.$$

In these two (nonsingular) configurations, we solve the inverse differential kinematics problem for $\boldsymbol{v}_e = (1 \;\; 0)^T$ [m/s] as

$$\dot{\boldsymbol{q}}_{sol,a} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{sol,a})\,\boldsymbol{v}_e = \begin{pmatrix} 0.5 & -0.0902 \\ 0.6 & 0.4918 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.6393 \\ -2 \end{pmatrix} \text{ [rad/s]},$$

$$\dot{\boldsymbol{q}}_{sol,b} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{sol,b})\,\boldsymbol{v}_e = \begin{pmatrix} 0.5 & 0.5 \\ 0.6 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ [rad/s]},$$

The norms of these two joint velocity vectors are different, namely

$$\left\|\dot{\boldsymbol{q}}_{sol,a}\right\| = 2.5860, \qquad \left\|\dot{\boldsymbol{q}}_{sol,b}\right\| = 2.$$

This should not come out unexpected because the given problem (including the direction of $\boldsymbol{v}_e$ with the respect to the robot postures) has no special symmetries.

Exercise 2

The time-optimal profile for the desired rest-to-rest motion from $\boldsymbol{q}_0$ to $\boldsymbol{q}_f$ under the bounds given in (2) is bang-coast-bang in acceleration for the first joint and bang-bang only for the second. In fact, the joint displacements are

$$\boldsymbol{\Delta} = \boldsymbol{q}_f - \boldsymbol{q}_0 = \begin{pmatrix} -1 \\ 0.7 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix},$$

and the condition for the existence of a time interval when a joints is cruising at its maximum velocity is satisfied for the first joint, but not for the second:

$$|\Delta_1| = 1 > 0.35 = \frac{0.5^2}{0.8} = \frac{V_1^2}{A_1}, \qquad |\Delta_2| = 0.7 < 1.28 = \frac{0.8^2}{0.5} = \frac{V_2^2}{A_2}.$$

The minimum time for the first joint is then

$$T_1 = \frac{|\Delta_1|\,A_1 + V_1^2}{V_1 A_1} = 2.6250 \text{ [s]}.$$

For the second joint, we have instead a triangular velocity profile that is symmetric w.r.t. the middle instant of motion $T_2/2$. At $t = T_2/2$, the joint reaches its maximum velocity $V_{m,2} = A_2 T_2/2 < V_2$. The area under this velocity profile should be equal to the displacement (in absolute value) for the second joint. Thus,

$$\text{(area =)} \;\; \frac{T_2 V_{m,2}}{2} = \frac{A_2 T_2^2}{4} = |\Delta_2| \qquad \Rightarrow \qquad T_2 = \sqrt{\frac{4\,|\Delta_2|}{A_2}} = 2.3664 \text{ [s]}.$$

The minimum motion time is therefore

$$T_{min} = \max\{T_1, T_2\} = \max\{2.6250, 2.3664\} = 2.6250 \text{ [s]}.$$

and is imposed by the first joint. In order to have a coordinated joint motion, the second joint should then slow down a bit, by using an acceleration/deceleration $\pm\, A_{m,2}$ (whose absolute value is less than A_2) so as to complete its motion exactly at $t = T_{min}$. Thus,

$$\text{(area =)} \;\; \frac{A_{m,2}\, T_{min}^2}{4} = |\Delta_2| \qquad \Rightarrow \qquad A_{m,2} = \frac{4\,|\Delta_2|}{T_{min}^2} = 0.4063 \text{ [rad/s}^2\text{]}.$$

Accordingly, $V_{m,2} = A_{m,2} T_{min}/2 = 0.5333$ [rad/s^2] is the velocity reached by the second joint at the middle instant of motion. The velocity profiles of the two joints for the obtained coordinated motion in minimum time are shown in Fig. 5.

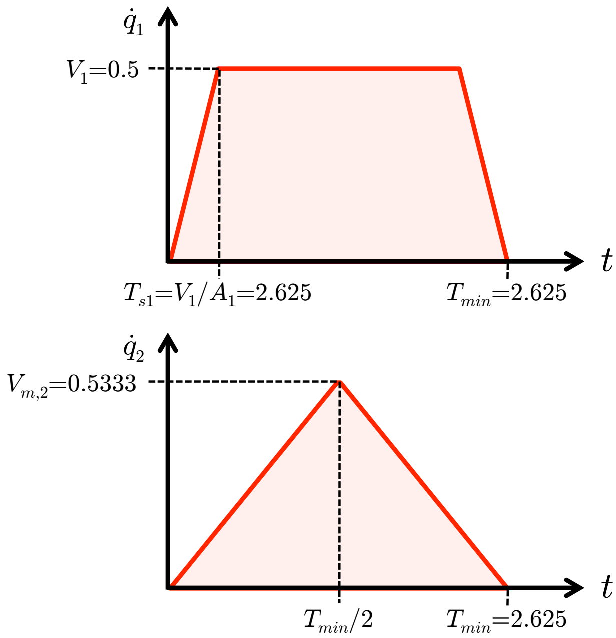


Figure 5: The profiles of the joint velocities achieving the desired coordinated motion in minimum time $T_{min} = 2.6250$ s.

* * * * *

Robotics I

July 11, 2018

Exercise 1

- Define the orientation of a rigid body in the 3D space through three rotations by the angles α, β, and γ around three fixed axes in the sequence Y, X, and Z, and determine the associated rotation matrix $\boldsymbol{R}_{YXZ}(\alpha, \beta, \gamma)$. Check if the determinant of this matrix has the correct value.
- Provide the analytical solution(s) to the inverse representation problem of an orientation specified by a rotation matrix $\boldsymbol{R} = \{R_{ij}\}$, using the above angles $\{\alpha, \beta, \gamma\}$. Discuss singular cases.
- Find the mapping between the time derivative $\dot{\boldsymbol{\phi}} = \begin{pmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{pmatrix}^T$ of the above minimal representation and the angular velocity $\boldsymbol{\omega}$ of the rigid body. Discuss the invertibility of this mapping.
- When the desired orientation $\boldsymbol{R}_d$ and the desired angular velocity $\boldsymbol{\omega}_d$ are

$$\boldsymbol{R}_d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}_d = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ [rad/s]},$$

  determine all associated solutions $\boldsymbol{\phi}_d = \{\alpha_d, \beta_d, \gamma_d\}$ and $\dot{\boldsymbol{\phi}}_d = \begin{pmatrix} \dot{\alpha}_d & \dot{\beta}_d & \dot{\gamma}_d \end{pmatrix}^T$, respectively to the inverse and the inverse differential problem. Check the correctness of the obtained results.

Exercise 2

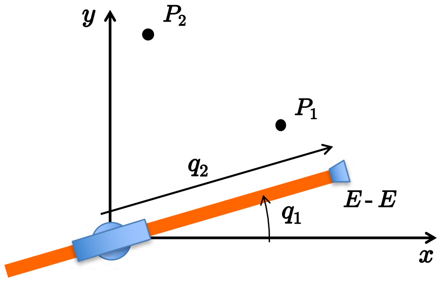


Figure 1: A planar RP robot and the generic setup of a desired motion task.

Consider the planar RP robot in Fig. 1. Define a computational scheme that generates joint-space commands at the acceleration level realizing a cyclic trajectory such that:

- the robot end effector starts at rest from point P_1 at $t = 0$, and returns there at $t = T$ with zero velocity;
- the path traced by the end effector is a circle of suitable radius, passing through the point P_2 and having there its tangent orthogonal to the segment $\overline{P_1 P_2}$;
- the timing law along the Cartesian path is a polynomial of the least possible degree.

Determine, first symbolically and then numerically:

- the Cartesian velocity $\boldsymbol{v} \in \mathbb{R}^2$ and acceleration $\boldsymbol{a} \in \mathbb{R}^2$ when passing through the point P_2;
- the associated joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ and joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^2$ of the RP robot;
- the numerical values of the above four quantities when using the data

$$P_1 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} \text{ [m]}, \qquad P_2 = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \text{ [m]}, \qquad T = 3.2 \text{ [s]}.$$

How can the scheme be made robust w.r.t. disturbances and/or initial trajectory errors?

[180 minutes, open books but no computer or smartphone]

Solution

July 11, 2018

Exercise 1

The elementary rotation matrices around the three coordinate axes are

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \ \boldsymbol{R}_X(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}, \ \boldsymbol{R}_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Being the sequence YXZ of rotations defined around fixed axes (i.e., of the Roll-Pitch-Yaw type), the rotation matrix representing the final orientation of the rigid body is obtained by multiplying the elementary matrices in the following order:

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\alpha) \\ &= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\sin\alpha\cos\beta & \sin\beta & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{1}$$

It is tedious but straightforward to check that $\det \boldsymbol{R}_{YXZ} = +1$.

Given a rotation matrix $\boldsymbol{R} = \{R_{ij}\}$ that uniquely specifies the orientation of a rigid body, the inverse representation problem is solved as follows. The angle β is given by the analytic formula

$$\beta = \text{ATAN2}\left\{R_{32}, \pm\sqrt{R_{31}^2 + R_{33}^2}\right\}. \tag{2}$$

Provided that $R_{31}^2 + R_{33}^2 = \cos^2\beta \neq 0$, this formula provides two different solution values β_1 and β_2, depending on the choice of the sign in the second argument. For each of these, the following formulas provide an associated solution pair (α_i, γ_i), for $i = 1, 2$:

$$\alpha = \text{ATAN2}\left\{-\frac{R_{31}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{-\frac{R_{12}}{\cos\beta}, \frac{R_{22}}{\cos\beta}\right\}. \tag{3}$$

Therefore, in the generic case, two different solution triples are found, $\boldsymbol{\phi}_1 = \{\alpha_1, \beta_1, \gamma_1\}$ and $\boldsymbol{\phi}_2 = \{\alpha_2, \beta_2, \gamma_2\}$.

In the singular case, i.e., when $R_{31} = R_{33} = 0$ and thus $\cos\beta = 0$, the problem reduces to

$$\begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & 0 & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & 0 & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ 0 & \sin\beta & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & 0 & R_{13} \\ R_{21} & 0 & R_{23} \\ 0 & \pm 1 & 0 \end{pmatrix}. \tag{4}$$

When $R_{32} = \sin\beta = 1$, then $\beta = \pi/2$, and the set of equations in (4) allows only to specify the sum of the two other angles as

$$\alpha + \gamma = \text{ATAN2}\{R_{13}, R_{11}\}.$$

Similarly, when $R_{32} = \sin\beta = -1$, then $\beta = -\pi/2$, and the set of equations in (4) allows only to specify the difference of the two other angles as

$$\alpha - \gamma = \text{ATAN2}\{R_{13}, R_{11}\}.$$

The mapping between the time derivative $\dot{\boldsymbol{\phi}} = \begin{pmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{pmatrix}^T$ of the above minimal representation and the angular velocity $\boldsymbol{\omega}$ can be obtained in different ways. The easiest is probably to reinterpret eq. (1) as a ZXY Euler sequence (with reverse order of angles $\{\gamma, \beta, \alpha\}$) and to compute the three contributions to $\boldsymbol{\omega}$ due to the variation of each angle around its current rotation axis. We have

$$
\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{\omega}_{\dot{\gamma}}(Z) + \boldsymbol{\omega}_{\dot{\beta}}(X') + \boldsymbol{\omega}_{\dot{\alpha}}(Y'') \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\gamma} + \boldsymbol{R}_Z(\gamma) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\beta} + \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_X(\beta) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\alpha} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\gamma} + \begin{pmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix} \dot{\beta} + \begin{pmatrix} -\cos\beta\sin\gamma \\ \cos\beta\cos\gamma \\ \sin\beta \end{pmatrix} \dot{\alpha} \\
&= \begin{pmatrix} -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \cos\beta\cos\gamma & \sin\gamma & 0 \\ \sin\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\beta, \gamma)\dot{\boldsymbol{\phi}}.
\end{aligned}
$$

When $\det \boldsymbol{T} = -\cos\beta = 0$, we have a singularity of the transformation. Therefore, when $\beta = \pm\pi/2$, the dimension of the range of $\boldsymbol{T}$ drops to two, and there exists a one-dimensional subspace of angular velocities $\boldsymbol{\omega}$ that cannot be represented by any choice of $\dot{\boldsymbol{\phi}}$. These are all $\boldsymbol{\omega} = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T \in \mathbb{R}^3$ which are orthogonal to the second column of $\boldsymbol{T}$, i.e., such that

$$
\omega_x \cos\gamma + \omega_y \sin\gamma = 0.
$$

With the data

$$
\boldsymbol{R}_d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}_d = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ [rad/s]},
$$

we see that this is not a singular situation ($R_{d,31} \neq 0$). Therefore, from eqs. (2–3), we find two solutions to the inverse representation problem:

$$
\boldsymbol{\phi}_{d1} = \left\{ \alpha_{d1}, \beta_{d1}, \gamma_{d1} \right\} = \left\{ \frac{\pi}{2}, 0, \frac{\pi}{2} \right\}, \qquad \boldsymbol{\phi}_{d2} = \left\{ \alpha_{d2}, \beta_{d2}, \gamma_{d2} \right\} = \left\{ -\frac{\pi}{2}, \pi, \frac{\pi}{2} \right\}.
$$

Accordingly, we obtain two different transformation matrices

$$
\boldsymbol{T}_{d1}(\beta_{d1}, \gamma_{d1}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{T}_{d2}(\beta_{d2}, \gamma_{d2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I},
$$

and therefore two different solutions to the inverse differential problem, namely

$$
\begin{aligned}
\dot{\boldsymbol{\phi}}_{d1} &= \begin{pmatrix} \dot{\alpha}_{d1} \\ \dot{\beta}_{d1} \\ \dot{\gamma}_{d1} \end{pmatrix} = \boldsymbol{T}_{d1}^{-1}(\beta_{d1}, \gamma_{d1})\, \boldsymbol{\omega}_d = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ [rad/s]}, \\
\dot{\boldsymbol{\phi}}_{d2} &= \begin{pmatrix} \dot{\alpha}_{d2} \\ \dot{\beta}_{d2} \\ \dot{\gamma}_{d2} \end{pmatrix} = \boldsymbol{T}_{d2}^{-1}(\beta_{d2}, \gamma_{d2})\, \boldsymbol{\omega}_d = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ [rad/s]}.
\end{aligned}
$$

Indeed, we can immediately check that $\boldsymbol{T}_{d1}(\beta_{d1}, \gamma_{d1})\, \dot{\boldsymbol{\phi}}_{d1} = \boldsymbol{T}_{d2}(\beta_{d2}, \gamma_{d2})\, \dot{\boldsymbol{\phi}}_{d2} = \boldsymbol{\omega}_d$.

Exercise 2

A generic circle with center at $P_0 = (x_0, y_0)$ and radius R is defined by the quadratic equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \tag{5}$$

The unknown quantities in (5) can be determined by imposing the passage through the two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, i.e.,

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = R^2, \qquad (x_2 - x_0)^2 + (y_2 - y_0)^2 = R^2,$$

whose solution for (x_0, y_0, R) generates two families of circles with increasing radius, placed symmetrically with respect to the line passing through P_1 and P_2. The additional condition of having a desired value for the path tangent at some point along the path will specify completely one circle in each family. The situation is particularly simple when requiring that the path tangent in P_2 should be orthogonal to the segment $\overline{P_1P_2}$. In fact, this implies directly that the center P_0 lies on this segment and, therefore, it coincides with its midpoint. A single circle is obtained in this special case, with

$$P_0 = \frac{P_1 + P_2}{2} = \begin{pmatrix} \dfrac{x_1 + x_2}{2} \\ \dfrac{y_1 + y_2}{2} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad R = \frac{\|P_2 - P_1\|}{2} = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{2}. \tag{6}$$

A parametric representation of the circular path (5) is given by

$$\boldsymbol{p}(s) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + R \begin{pmatrix} \cos(s + \phi) \\ \sin(s + \phi) \end{pmatrix}, \qquad s \in [0, 2\pi], \tag{7}$$

where the circular path is traced counterclockwise for increasing s and the angle ϕ characterizes the starting point chosen on the circle. Being P_1 the starting point (i.e., for $s = 0$), we have

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + R \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \Rightarrow \qquad R \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \dfrac{x_1 - x_2}{2} \\ \dfrac{y_1 - y_2}{2} \end{pmatrix} \tag{8}$$

and thus

$$\phi = \text{ATAN2}\{y_1 - y_2, x_1 - x_2\}. \tag{9}$$

The timing law $s(t)$ is given by a cubic polynomial, which has in fact the least possible degree that guarantees satisfaction of the four boundary conditions

$$s(0) = 0, \qquad s(T) = 2\pi, \qquad \dot{s}(0) = 0, \qquad \dot{s}(T) = 0.$$

Therefore,

$$s(t) = 2\pi \left(3 \left(\frac{t}{T} \right)^2 - 2 \left(\frac{t}{T} \right)^3 \right). \tag{10}$$

For $t = T/2$, it is $s(T/2) = \pi$ and thus $\boldsymbol{p}(\pi) = P_2$, as it can be easily checked from (6–8).

Differentiating twice (7) and (10), respectively in space and time, we obtain

$$\boldsymbol{p}'(s) = \frac{d\boldsymbol{p}(s)}{ds} = R \begin{pmatrix} -\sin(s + \phi) \\ \cos(s + \phi) \end{pmatrix}, \qquad \boldsymbol{p}''(s) = \frac{d^2\boldsymbol{p}(s)}{ds^2} = -R \begin{pmatrix} \cos(s + \phi) \\ \sin(s + \phi) \end{pmatrix},$$

and

$$\dot{s}(t) = \frac{ds(t)}{dt} = \frac{12\pi}{T}\left(\frac{t}{T} - \left(\frac{t}{T}\right)^2\right), \qquad \ddot{s}(t) = \frac{d^2s(t)}{dt^2} = \frac{12\pi}{T^2}\left(1 - 2\,\frac{t}{T}\right).$$

Therefore,

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s)\dot{s}(t) = \frac{12R\pi}{T}\left(\frac{t}{T} - \left(\frac{t}{T}\right)^2\right)\begin{pmatrix} -\sin(s+\phi) \\ \cos(s+\phi) \end{pmatrix}, \tag{11}$$

and thus, using also (8),

$$\boldsymbol{v} = \dot{\boldsymbol{p}}\left(\tfrac{T}{2}\right) = \boldsymbol{p}'(\pi)\dot{s}\left(\tfrac{T}{2}\right) = \frac{3R\pi}{T}\begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix} = \frac{3\pi}{2T}\begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix}. \tag{12}$$

Similarly,

$$\begin{aligned}\ddot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s)\ddot{s}(t) + \boldsymbol{p}''(s)\dot{s}^2(t) = {} & \frac{12R\pi}{T^2}\left(1 - 2\frac{t}{T}\right)\begin{pmatrix} -\sin(s+\phi) \\ \cos(s+\phi) \end{pmatrix} \\ & -\frac{144R\pi^2}{T^2}\left(\frac{t}{T} - \left(\frac{t}{T}\right)^2\right)^2\begin{pmatrix} \cos(s+\phi) \\ \sin(s+\phi) \end{pmatrix},\end{aligned} \tag{13}$$

and thus, being $\ddot{s}(T/2) = 0$ and using (8),

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}\left(\tfrac{T}{2}\right) = \boldsymbol{p}''(\pi)\dot{s}^2\left(\tfrac{T}{2}\right) = \frac{9R\pi^2}{T^2}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \frac{9\pi^2}{2T^2}\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}. \tag{14}$$

Having fully specified the Cartesian trajectory, and obtained in particular the requested quantities $\boldsymbol{v}$ and $\boldsymbol{a}$ consider now the execution of this trajectory by the RP robot. At $t = 0$, the robot end effector should be placed in P_1. Solving this inverse kinematics problem, we have

$$\boldsymbol{q}(0) = \begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\{y_1, x_1\} \\ \sqrt{x_1^2 + y_1^2} \end{pmatrix}, \tag{15}$$

where for simplicity we considered only the robot configuration 'facing' point P_1 (i.e., with $q_2 > 0$). The robot should start at rest, and so $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$. From this initial robot state, which is matched with the Cartesian trajectory at the initial time $t = 0$, we generate the desired cyclic trajectory using the acceleration command obtained by solving the second-order inverse differential kinematics, or

$$\ddot{\boldsymbol{q}}(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}(t))\left(\ddot{\boldsymbol{p}}(t) - \dot{\boldsymbol{J}}(\boldsymbol{q}(t))\,\dot{\boldsymbol{q}}(t)\right), \qquad t \in [0, T], \tag{16}$$

where $\ddot{\boldsymbol{p}}(t)$ is given by eq. (13) and we assumed that kinematic singularities are not encountered. The analytic Jacobian $\boldsymbol{J}$ and its time derivative $\dot{\boldsymbol{J}}$ needed within (16) are found by differentiating the direct kinematics of the RP robot

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix},$$

yielding

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix},$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \text{with} \quad \dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_2 \sin q_1 - \dot{q}_1 q_2 \cos q_1 & -\dot{q}_1 \sin q_1 \\ \dot{q}_2 \cos q_1 - \dot{q}_1 q_2 \sin q_1 & \dot{q}_1 \cos q_1 \end{pmatrix}.$$

When passing through the point P_2, the robot configuration is

$$\boldsymbol{q}\left(\tfrac{T}{2}\right) = \begin{pmatrix} q_1\left(\frac{T}{2}\right) \\ q_2\left(\frac{T}{2}\right) \end{pmatrix} = \begin{pmatrix} \mathrm{ATAN2}\{y_2, x_2\} \\ \sqrt{x_2^2 + y_2^2} \end{pmatrix}. \tag{17}$$

Thus, the joint velocity at $t = T/2$ (when the robot end effector is in P_2) will be

$$\begin{aligned} \dot{\boldsymbol{q}}\left(\tfrac{T}{2}\right) &= \boldsymbol{J}^{-1}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\,\dot{\boldsymbol{p}}\left(\tfrac{T}{2}\right) = \boldsymbol{J}^{-1}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\,\boldsymbol{v} \\ &= -\frac{3\pi}{2T}\,\frac{1}{q_2\left(\frac{T}{2}\right)} \begin{pmatrix} \sin q_1\left(\frac{T}{2}\right) & -\cos q_1\left(\frac{T}{2}\right) \\ -q_2\left(\frac{T}{2}\right)\cos q_1\left(\frac{T}{2}\right) & -q_2\left(\frac{T}{2}\right)\sin q_1\left(\frac{T}{2}\right) \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix}, \end{aligned} \tag{18}$$

where the values of $q_1(T/2)$ and $q_2(T/2)$ are given by eq. (17), and we used $\boldsymbol{v}$ as defined in (12). Similarly, the requested joint acceleration at $t = T/2$ is

$$\begin{aligned} \ddot{\boldsymbol{q}}\left(\tfrac{T}{2}\right) &= \boldsymbol{J}^{-1}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\left(\ddot{\boldsymbol{p}}\left(\tfrac{T}{2}\right) - \dot{\boldsymbol{J}}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\,\dot{\boldsymbol{q}}\left(\tfrac{T}{2}\right)\right) = \boldsymbol{J}^{-1}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q}\left(\tfrac{T}{2}\right))\,\dot{\boldsymbol{q}}\left(\tfrac{T}{2}\right)\right) \\ &= -\frac{1}{q_2\left(\frac{T}{2}\right)} \begin{pmatrix} \sin q_1\left(\frac{T}{2}\right) & -\cos q_1\left(\frac{T}{2}\right) \\ -q_2\left(\frac{T}{2}\right)\cos q_1\left(\frac{T}{2}\right) & -q_2\left(\frac{T}{2}\right)\sin q_1\left(\frac{T}{2}\right) \end{pmatrix} \left[\frac{9\pi^2}{2T^2} \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \right. \\ &\qquad \left. - \begin{pmatrix} -\dot{q}_2\left(\frac{T}{2}\right)\sin q_1\left(\frac{T}{2}\right) - \dot{q}_1\left(\frac{T}{2}\right) q_2\left(\frac{T}{2}\right)\cos q_1\left(\frac{T}{2}\right) & -\dot{q}_1\left(\frac{T}{2}\right)\sin q_1\left(\frac{T}{2}\right) \\ \dot{q}_2\left(\frac{T}{2}\right)\cos q_1\left(\frac{T}{2}\right) - \dot{q}_1\left(\frac{T}{2}\right) q_2\left(\frac{T}{2}\right)\sin q_1\left(\frac{T}{2}\right) & \dot{q}_1\left(\frac{T}{2}\right)\cos q_1\left(\frac{T}{2}\right) \end{pmatrix} \dot{\boldsymbol{q}}\left(\tfrac{T}{2}\right) \right] \end{aligned} \tag{19}$$

where the values of $q_1(T/2)$ and $q_2(T/2)$ are given by eq. (17), $\dot{\boldsymbol{q}}(T/2)$ comes from eq. (18), and we used $\boldsymbol{a}$ as defined in (14).

Substituting the numerical values

$$P_1 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} \text{[m]}, \qquad P_2 = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \text{[m]}, \qquad T = 3.2 \text{ [s]},$$

we obtain from (6) and (9)

$$P_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.25 \end{pmatrix} \text{[m]}, \qquad R = \frac{\sqrt{1.25}}{2} = 0.5590 \text{ [m]}, \qquad \phi = -26.565^\circ = -0.4636 \text{ [rad]},$$

and thus from (7)

$$\boldsymbol{p}(s) = \begin{pmatrix} 1 \\ 1.25 \end{pmatrix} + 0.5590 \begin{pmatrix} \cos(s - 0.4636) \\ \sin(s - 0.4636) \end{pmatrix}, \qquad s \in [0, 2\pi].$$

We evaluate then (12) and (14), obtaining

$$\boldsymbol{v} = \begin{pmatrix} -0.7363 \\ -1.4726 \end{pmatrix} \text{[m/s]}, \qquad \boldsymbol{a} = \begin{pmatrix} 4.3372 \\ -2.1686 \end{pmatrix} \text{[m/s}^2\text{]}.$$

The robot configurations at the initial time $t = 0$ and at the halftime $t = T/2 = 1.6$ s are computed from eqs. (15) and (17),

$$\boldsymbol{q}(0) = \begin{pmatrix} 0.5880 \\ 1.8028 \end{pmatrix} \begin{matrix} [\text{rad}] \\ [\text{m}] \end{matrix}, \qquad \boldsymbol{q}(1.6) = \begin{pmatrix} 1.2490 \\ 1.5811 \end{pmatrix} \begin{matrix} [\text{rad}] \\ [\text{m}] \end{matrix},$$

while (18) and (19) yield

$$\dot{\boldsymbol{q}}\,(1.6) = \begin{pmatrix} 0.1473 \\ -1.6299 \end{pmatrix} \begin{matrix} [\text{rad/s}] \\ [\text{m/s}] \end{matrix}, \qquad \ddot{\boldsymbol{q}}\,(1.6) = \begin{pmatrix} -2.7325 \\ -0.6515 \end{pmatrix} \begin{matrix} [\text{rad/s}^2] \\ [\text{m/s}^2] \end{matrix}.$$

The planned Cartesian task is sketched in Fig. 2, while the evolution of the joint positions and velocities associated to the computational scheme (16) are shown in Figs. 3 and 4.

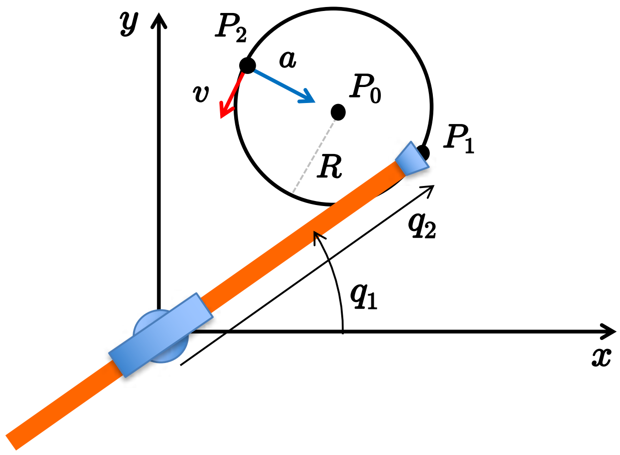


Figure 2: The planned Cartesian motion with the RP robot in the correct initial position at $t = 0$.

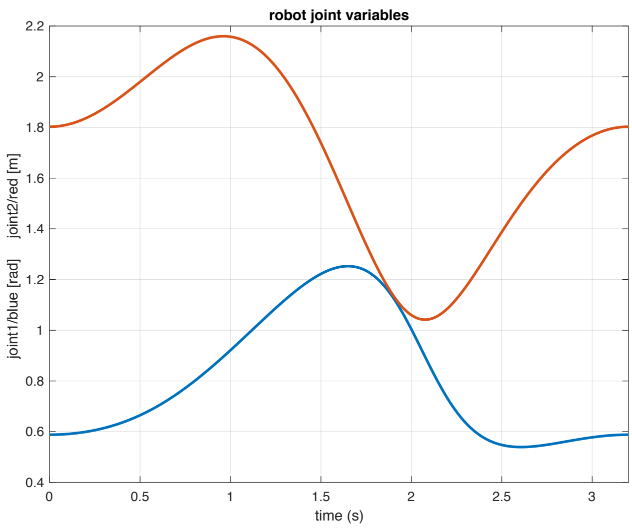


Figure 3: Joint positions of the RP robot while executing the task in Fig. 2.

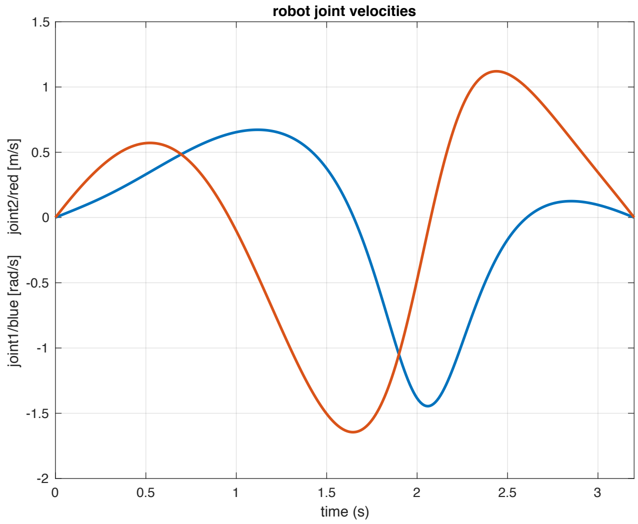


Figure 4: Joint velocities of the RP robot while executing the task in Fig. 2.

Finally, if there is an initial mismatch of the robot state with respect to the desired Cartesian trajectory $\boldsymbol{p}_d(t)$ at $t = 0$, or if an external disturbance will bring the robot end effector out of its nominal trajectory, a feedback modification of the scheme (16) is needed. In order to recover the tracking errors, the following control law should be used

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \dot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} \right), \tag{20}$$

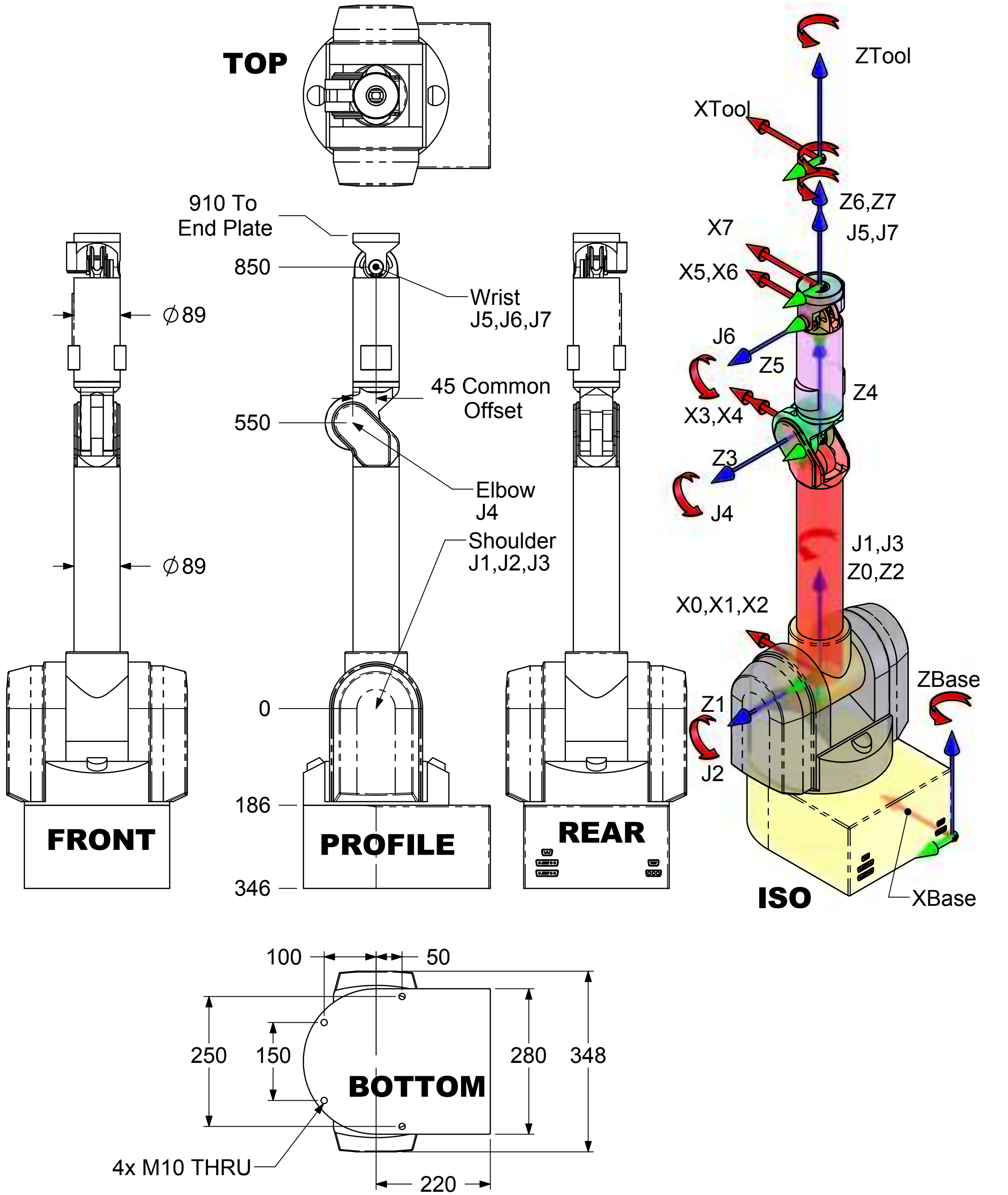
with (diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, and where the desired motion in position, velocity, and acceleration is specified by (7), (11), and (13), respectively. The control law (20) robustifies the robot behavior also with respect to numerical approximations due to a discrete-time implementation of the scheme.

* * * * *

Derivation of DH table from a given assignment of frames for a 7R arm

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				



Robotics I

Midterm classroom test – November 16, 2018

Exercise 1 [6 points]

The orientation of a rigid body is given in terms of the YXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, -\pi/4, \pi/4)$. This orientation is the result of a rotation around the unit vector $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ (expressed in the absolute frame) by an angle $\eta = -30°$. Which was the initial orientation of the body? Is it uniquely defined? Express the solution (or solutions) for the initial orientation by a rotation matrix $\boldsymbol{R}$ and in terms of XYZ Roll-Pitch-Yaw angles (ψ, θ, ϕ) around fixed axes.

Exercise 2 [2 points]

The following 4×4 matrix is given:

$$\boldsymbol{M} = \begin{pmatrix} -0.7071 & 0.5 & -0.5 & -1 \\ -0.7071 & -0.5 & 0.5 & -1 \\ 0 & 0.7071 & 0.7071 & -0.7071 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Is it possible to generate $\boldsymbol{M}$ by a set of four Denavit-Hartenberg parameters (α, a, d, θ)? If so, provide these values. If not, explain why.

Exercise 3 [8 points]

Consider the 7R manipulator in Fig. 1, where Denavit-Hartenberg (DH) frames have been defined already (with axes $\boldsymbol{x}_i$ in red, axes $\boldsymbol{y}_i$ in green, and axes $\boldsymbol{z}_i$ in blue). Ten reference frames are shown in total, 8 of which are DH frames, plus one fixed frame attached with the base platform, and a last one attached with a generic tool.

- On the extra sheet provided separately [*to be returned with your name*], complete the table of DH parameters. Enter in the table only numerical values (expressed in [rad] or [m]), including those of the joint variables $\boldsymbol{q}$ in the configuration shown. In the drawings, all data are given in mm. Note the presence of offsets (equal to 45 mm) at the elbow of the arm.
- Provide numerically the transformation matrix ${}^B\boldsymbol{T}_0$ from the base frame to the DH frame 0.

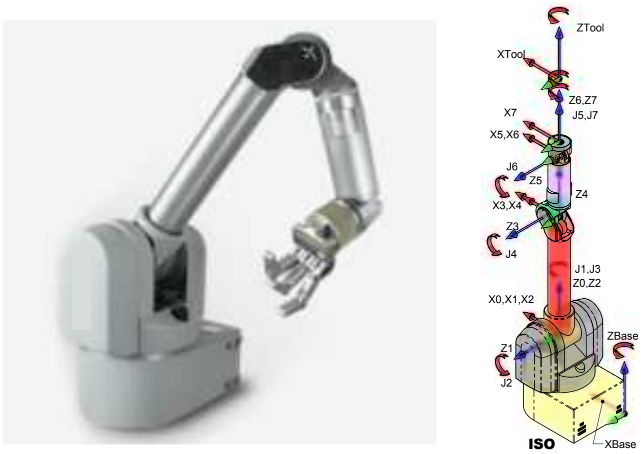


Figure 1: A 7R anthropomorphic manipulator.

Exercise 4 [6 points]

A revolute joint of a robot is actuated by a DC motor driven by a controlled voltage V_a in the armature circuit. We know that the motor is characterized by the following data, respectively for armature resistance, viscous friction, and current-to-torque gain parameters:

$$R_a = 0.309 \text{ [Ohm]}, \qquad F_m = 5 \cdot 10^{-2} \text{ [mNm/(rad/s)]}, \qquad k_t = 7.88 \text{ [mNm/A]}.$$

A harmonic drive reducer with a flexspline having 256 external teeth is used as a transmission element for moving the link. At steady state, the output axis on the link side is rotating at an angular speed $\omega = 180°/\text{s}$ in the counterclockwise direction.

- What is the value of the applied voltage V_a in this situation? Pay attention to the SI units!
- Which is the rotation speed ω_m (with sign, in [rad/s]) of the rotor of the motor?
- If an incremental encoder with quadrature detection is used on the motor side, how many bits should its internal counter use to obtain at least a resolution of $\epsilon = 10^{-3}$ [deg] on the link side? How many pulses/turn should the optical disk have? How many pulses would be counted in total by the counter in one second when the motor is rotating at the steady-state speed ω_m? Would the counter go up or down?

Exercise 5 [8 points]

The kinematics of a planar RP robot is defined by the following Tab. 1 of DH parameters:

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 = 0.2$ [m]	0	q_1
2	$\pi/2$	0	q_2	π [rad]

Table 1: DH parameters for a planar RP robot.

We would like to solve an inverse kinematics problem for this robot using an iterative numerical method, either Newton or Gradient. The desired position the origin of the last frame in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is $\boldsymbol{p}_d = \begin{pmatrix} -2 & -3 \end{pmatrix}^T$ [m]. At some iteration k, the algorithm drives the robot from configuration $\boldsymbol{q}^k$ to $\boldsymbol{q}^{k+1}$, with

$$\boldsymbol{q}^k = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ [rad, m]}, \qquad \boldsymbol{q}^{k+1} = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix} \text{ [rad, m]}.$$

- Which is the solution method being used? Provide the symbolic expressions of each term in its general formula and their numerical values at the given iteration k.
- If the method converges, what is the expected solution $\boldsymbol{q}^*$? In this robot configuration, what will be the orientation of the last DH frame as expressed by the rotation matrix ${}^0\boldsymbol{R}_2(\boldsymbol{q}^*)$?
- In what configuration $\widehat{\boldsymbol{q}}$ would the Gradient method certainly stop with a non-zero position error for the above problem? What happens if we apply the Newton method there?

[240 minutes, open books]

Solution of Midterm Test

November 16, 2018

Exercise 1

The orientation of a rigid body expressed in terms a sequence of YXY Euler angles (α, β, γ) is represented by the rotation matrix

$$\begin{aligned}
\boldsymbol{R}_{YXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma) \\
&= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma \\ \sin\beta\sin\gamma & \cos\beta & -\sin\beta\cos\gamma \\ -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma \end{pmatrix}.
\end{aligned}$$

Thus, the final orientation of the considered body is specified with respect to the absolute reference frame as

$$^0\boldsymbol{R}_f = \boldsymbol{R}_{YXY}\left(\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 & -0.7071 & 0.5 \\ -0.5 & 0.7071 & 0.5 \\ -0.7071 & 0 & -0.7071 \end{pmatrix}.$$

Operatively, to obtain this result one can either evaluate numerically the symbolic matrix $\boldsymbol{R}_{YXY}$, or evaluate numerically the single elementary rotation matrices and then do their products.

The matrix associated to a rotation around an axis $\boldsymbol{r}$ by an angle η is given by

$$\boldsymbol{R}(\boldsymbol{r}, \eta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\eta + \boldsymbol{S}(\boldsymbol{r})\sin\eta.$$

Thus, the rotation that changes the initial to the final orientation of the body is specified as

$$\begin{aligned}
^0\boldsymbol{R}_{i,f} &= \boldsymbol{R}\left(\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T, -\frac{\pi}{6}\right) \\
&= \begin{pmatrix} \frac{\sqrt{3}+1}{3} & \frac{\sqrt{3}-1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{\sqrt{3}+1}{3} & \frac{\sqrt{3}-1}{3} \\ \frac{1-\sqrt{3}}{3} & -\frac{1}{3} & \frac{\sqrt{3}+1}{3} \end{pmatrix} = \begin{pmatrix} 0.9107 & 0.2440 & 0.3333 \\ -0.3333 & 0.9107 & 0.2440 \\ -0.2440 & -0.3333 & 0.9107 \end{pmatrix},
\end{aligned}$$

where the superscript 0 to $\boldsymbol{R}_{i,f}$ is there to indicate that the given rotation axis was expressed in the absolute reference frame (i.e., $^0\boldsymbol{r}$).

Since the two rotation matrices $^0\boldsymbol{R}_f$ and $^0\boldsymbol{R}_{if}$ are both defined with respect to the original reference frame, we have for their composition the product order

$$^0\boldsymbol{R}_f = {}^0\boldsymbol{R}_{i,f}\, {}^0\boldsymbol{R}_i.$$

Thus, the initial orientation expressed by the rotation matrix $^0\boldsymbol{R}_i$ is computed as

$$^0\boldsymbol{R}_i = {}^0\boldsymbol{R}_{i,f}^T\, {}^0\boldsymbol{R}_f = \begin{pmatrix} -0.1161 & -0.8797 & 0.4612 \\ -0.3416 & 0.4714 & 0.8131 \\ -0.9326 & -0.0632 & -0.3553 \end{pmatrix}.$$

This initial orientation is indeed uniquely specified. To express this solution using the XYZ Roll-Pitch-Yaw angles (ψ, θ, ϕ), we first compute symbolically the associated rotation matrix as

$$\begin{aligned}
\boldsymbol{R}_{XYZ}(\psi,\theta,\phi) &= \boldsymbol{R}_Z(\phi)\boldsymbol{R}_Y(\theta)\boldsymbol{R}_X(\psi) \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}.
\end{aligned}$$

Next, we solve the inverse representation problem for

$$\boldsymbol{R}_{XYZ}(\psi,\theta,\phi) = {}^0\boldsymbol{R}_i.$$

Denote by $R_{i,j}$ the elements of ${}^0\boldsymbol{R}_i$. Since $R_{3,2}^2 + R_{3,3}^2 \neq 0$, we are in a regular situation. Therefore, from

$$\cos\theta = \pm\sqrt{R_{3,2}^2 + R_{3,3}^2},$$

there are two solutions given by the angles (all given in [rad])

$$\begin{aligned}
\theta_1 &= \text{ATAN2}\left\{-R_{3,1}, \sqrt{R_{3,2}^2 + R_{3,3}^2}\right\} = 1.2016 \\
\psi_1 &= \text{ATAN2}\left\{\frac{R_{3,2}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}, \frac{R_{3,3}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = -2.9657 \\
\phi_1 &= \text{ATAN2}\left\{\frac{R_{2,1}}{\sqrt{R_{2,1}^2 + R_{3,3}^2}}, \frac{R_{1,1}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = -1.8985
\end{aligned}$$

and

$$\begin{aligned}
\theta_2 &= \text{ATAN2}\left\{-R_{3,1}, -\sqrt{R_{3,2}^2 + R_{3,3}^2}\right\} = 1.9400 \\
\psi_2 &= \text{ATAN2}\left\{\frac{R_{3,2}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}, \frac{R_{3,3}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = 0.1759 \\
\phi_2 &= \text{ATAN2}\left\{\frac{R_{2,1}}{-\sqrt{R_{2,1}^2 + R_{3,3}^2}}, \frac{R_{1,1}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = 1.2431.
\end{aligned}$$

Exercise 2

The given 4×4 matrix $\boldsymbol{M}$ is indeed a homogeneous transformation matrix, since the upper left 3×3 block, say $\boldsymbol{R}$, is a rotation matrix (i.e., it is an orthonormal matrix, with determinant $= +1$). However, this is only a necessary condition for being able to express $\boldsymbol{R}$ only in terms of the two DH angular parameters α and θ. Therefore, we attempt directly to solve the matrix equation $\boldsymbol{A}(\alpha, a, d, \theta) = \boldsymbol{M}$, or

$$\begin{pmatrix} \cos\theta & -\cos\alpha\sin\theta & \sin\alpha\sin\theta & a\cos\theta \\ \sin\theta & \cos\alpha\cos\theta & -\sin\alpha\cos\theta & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.7071 & 0.5 & -0.5 & -1 \\ -0.7071 & -0.5 & 0.5 & -1 \\ 0 & 0.7071 & 0.7071 & -0.7071 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The following four values can be obtained (uniquely) from the analytic expressions

$$\alpha = \text{ATAN2}\{M_{3,2}, M_{3,3}\} = \text{ATAN2}\{0.7071, 0.7071\} = \frac{\pi}{4}$$

$$\theta = \text{ATAN2}\{M_{1,2}, M_{1,1}\} = \text{ATAN2}\{-0.7071, -0.7071\} = -\frac{3\pi}{4}$$

$$d = M_{3,4} = -0.7071 = -\frac{\sqrt{2}}{2}$$

$$a = M_{1,4}\cos\theta + M_{2,4}\sin\theta = (-1)\cdot(-0.7071) + (-1)\cdot(-0.7071) = \sqrt{2}.$$

It is easy to see that this set (α, a, d, θ) satisfies as identities also the remaining equations in (1).

Exercise 3

The robot shown in Fig. 1 is the Barrett Whole-Arm-Manipulator (WAM) with 7 revolute joints. The assigned frames comply with the standard Denavit-Hartenberg convention. The associated parameters are given in Tab. 2, with data expressed in [rad] or [m] and with the numerical values of the joint variables $\boldsymbol{q}$ taken in the configuration shown (the 'straight up' configuration is in fact the *zero* configuration for this arm). See also the compiled extra sheet at the end of the solution.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	$q_1 = 0$
2	$\pi/2$	0	0	$q_2 = 0$
3	$-\pi/2$	0.045	0.55	$q_3 = 0$
4	$\pi/2$	-0.045	0	$q_4 = 0$
5	$-\pi/2$	0	0.3	$q_5 = 0$
6	$\pi/2$	0	0	$q_6 = 0$
7	0	0	0.06	$q_7 = 0$

Table 2: Parameters associated to the DH frames in Fig. 1.

From the sheet, one determines also the transformation matrix from base frame to DH frame 0 as

$${}^{B}\boldsymbol{T}_0 = \begin{pmatrix} 1 & 0 & 0 & 0.220 \\ 0 & 1 & 0 & 0.140 \\ 0 & 0 & 1 & 0.346 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4

The dynamic equations of a DC motor in the time domain are known to be

$$\begin{aligned} L_a\,\frac{di_a}{dt} &= V_a - R_a i_a - V_{emf}, \qquad \text{with } V_{emf} = k_v\,\omega_m, \qquad \text{(electrical balance)} \\ I_m\,\frac{\omega_m}{dt} &= T_m - F_m\omega_m - T_{load}, \qquad \text{with } T_m = k_t\,i_a. \qquad \text{(mechanical balance)} \end{aligned} \tag{2}$$

Assume no disturbance load, $T_{load} = 0$. When the rotor of the motor is rotating at a constant angular velocity[1] $\bar{\omega}_m$, we have the steady-state conditions

$$\bar{V}_a = R_a \bar{i}_a + k_v \bar{\omega}_m, \qquad \bar{T}_m = k_t \bar{i}_a = F_m \bar{\omega}_m, \tag{3}$$

which are obtained by setting to zero the two time derivatives in (2). The motor needs to apply a torque $\bar{T}_m$ (and thus deliver a steady-state armature current $\bar{i}_a$) to compensate for the energy dissipation due to the viscous friction term $F_m \bar{\omega}_m$ at constant angular velocity. The steady-state input voltage $\bar{V}_a$ will then balance the constant back emf $k_v \bar{\omega}_m$ and produce also the required current $\bar{i}_a$ flowing through the armature resistance R_a.

Special care should be taken for the numerical equivalence between the back emf coefficient k_v and the current-to-torque gain k_t of the motor. Based on the conservation of energy principle, we have

$$k_v \text{ [V/(rad/s)]} = k_t \text{ [Nm/A]}$$

only when using the indicated SI units. Therefore, in our case we will have

$$k_t = 7.88 \text{ [mNm/A]} = 7.88 \cdot 10^{-3} \text{ [Nm/A]} \qquad \Rightarrow \qquad k_v = 7.88 \cdot 10^{-3} \text{ [V/(rad/s)]}. \tag{4}$$

The harmonic drive (HD) has a reduction ratio $n > 1$ and transforms input angular velocities ω_m of the motor into output angular velocities ω of the link as

$$n = \frac{N_{flex}}{2} = \frac{256}{2} = 128, \qquad \omega_m = -n\,\omega, \tag{5}$$

where the minus sign denotes the inversion in the rotation direction due to the HD reducer.

When the link is rotating at steady state with an angular velocity $\bar{\omega} = 180°/\text{s} = \pi$ rad/s (positive, being a counterclockwise rotation), combining eqs. (3) and (5) and keeping into account the conversion (4), we obtain the numerical results

$$\begin{aligned}
\bar{\omega}_m &= -n\,\bar{\omega} = -128 \cdot \pi = -402.1239 \text{ [rad/s]} \quad \text{(clockwise!)} \\
\bar{i}_a &= \frac{F_m}{k_t}\,\bar{\omega}_m \left(= -\frac{F_m}{k_t}\,n\,\bar{\omega} \right) = -\frac{5 \cdot 10^{-2}}{7.88}\,128\,\pi = -2.5515 \text{ [A]}, \\
\bar{V}_a &= R_a \bar{i}_a + k_v \bar{\omega}_m \left(= -\left(\frac{R_a F_m}{k_t} + k_v \right) n\,\bar{\omega} \right) \\
&= -0.309 \cdot 2.5515 - 7.88 \cdot 10^{-3} \cdot 402.1239 = -(0.7885 + 3.1687) = -3.9572 \text{ [V]}.
\end{aligned}$$

With a desired resolution $\epsilon = 10^{-3}$ [deg] on the link side of the transmission, we need an output resolution of the incremental encoder on the motor side equal to

$$\epsilon_m = \frac{\Delta\theta_m}{\text{pulse}} = n \cdot \epsilon = 128 \cdot 10^{-3} \text{ [deg]}.$$

Thus, the internal counter of the incremental encoder should be able to count a number N_p of pulses/turn at least equal to

$$N_p = \left\lceil \frac{360}{\epsilon_m} \right\rceil = \left\lceil \frac{360}{128} \cdot 10^3 \right\rceil = 2813,$$

[1] An angular velocity is an angular speed with sign (positive if rotating CCW, as seen from the observation axis).

which requires a minimum number N_b of bits for the digital counter equal to

$$N_b = \lceil \log_2(N_p) \rceil = 12.$$

On the other hand, thanks to the quadrature electronics, the minimum number N_d of pulses/turn of the optical disk that guarantees the desired resolution is

$$N_d = \left\lceil \frac{N_p}{4} \right\rceil = 704.$$

With this incremental encoder, when the motor is spinning at the steady-state angular velocity $\bar{\omega}_m$, the total count of pulses by the digital counter in one second (without considering resets) would then be

$$\text{count} = \left\lfloor |\bar{\omega}_m| \cdot \frac{N_p}{2\pi} \right\rfloor = \left\lfloor 402.1239 \cdot \frac{2813}{2\pi} \right\rfloor = 180032.$$

The counter goes *down*, since the steady-state angular velocity of the motor is negative (the motor rotates CW, while the link rotates CCW having a positive angular velocity).

Exercise 5

The direct kinematics of the planar RP robot is computed from the parameters in Tab. 1, using the DH homogeneous transformation matrices (and keeping for the moment a_1 as a symbolic term):

$$\begin{aligned} {}^0\boldsymbol{A}_2(\boldsymbol{q}) \;=\; {}^0\boldsymbol{A}_1(q_1)\;{}^1\boldsymbol{A}_2(q_2) &= \begin{pmatrix} \cos q_1 & 0 & -\sin q_1 & a_1 \cos q_1 \\ \sin q_1 & 0 & \cos q_1 & a_1 \sin q_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\cos q_1 & -\sin q_1 & 0 & a_1 \cos q_1 - q_2 \sin q_1 \\ -\sin q_1 & \cos q_1 & 0 & a_1 \sin q_1 + q_2 \cos q_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_2(q_1) & {}^0\boldsymbol{p}_2(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \end{aligned} \tag{6}$$

The planar position mapping of interest is given by the first two components of ${}^0\boldsymbol{p}_2(\boldsymbol{q})$ in (6), i.e.,

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} a_1 \cos q_1 - q_2 \sin q_1 \\ a_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix}. \tag{7}$$

Differentiating (7) with respect to $\boldsymbol{q}$ yields the 2×2 Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a_1 \sin q_1 - q_2 \cos q_1 & -\sin q_1 \\ a_1 \cos q_1 - q_2 \sin q_1 & \cos q_1 \end{pmatrix}, \tag{8}$$

which is nonsingular unless $\det \boldsymbol{J}(\boldsymbol{q}) = -q_2 = 0$.

The underlying inverse kinematics problem considered for this robot requires to solve two nonlinear equations, i.e.,

$$\boldsymbol{p}(\boldsymbol{q}) = \boldsymbol{p}_d \qquad \Rightarrow \qquad \begin{pmatrix} a_1 \cos q_1 - q_2 \sin q_1 \\ a_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \tag{9}$$

through the use of an iterative numerical method. A generic iteration step toward the solution is presented, from a configuration $\boldsymbol{q}^k$ to $\boldsymbol{q}^{k+1}$.

Since the Jacobian at $\boldsymbol{q}^k$ is nonsingular ($q_2^k = 2 \neq 0$), the increment provided by the numerical method used at iteration k

$$\Delta \boldsymbol{q}^k = \boldsymbol{q}^{k+1} - \boldsymbol{q}^k = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.7742 \\ -2.6519 \end{pmatrix} \text{ [rad, m]} \tag{10}$$

could have been obtained in principle either by the Newton or by the Gradient method (the latter with some step size $\alpha_k > 0$).

To verify if the Newton method was used, we evaluate the relevant quantities at $\boldsymbol{q}^k$ (with $a_1 = 0.2$):

$$\boldsymbol{e}^k = \boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}^k) = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1.7910 \\ 0.9123 \end{pmatrix} = \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix}, \qquad \left\| \boldsymbol{e}^k \right\| = 5.4477 \tag{11}$$

$$\boldsymbol{J}(\boldsymbol{q}^k) = \begin{pmatrix} -0.9123 & 0.8415 \\ 1.7910 & 0.5403 \end{pmatrix}. \tag{12}$$

It is easy to see that

$$\boldsymbol{q}^k + \boldsymbol{J}^{-1}(\boldsymbol{q}^k)\boldsymbol{e}^k = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -0.2702 & 0.4207 \\ 0.8955 & 0.4562 \end{pmatrix} \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix} = \begin{pmatrix} -1.6219 \\ -3.1795 \end{pmatrix} \neq \boldsymbol{q}^{k+1} = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix}.$$

We conclude that the Newton method was not used. On the other hand, the increment given by the Gradient method is

$$\alpha_k \, \boldsymbol{J}^T(\boldsymbol{q}^k)\boldsymbol{e}^k = \alpha_k \begin{pmatrix} -0.9123 & 1.7910 \\ 0.8415 & 0.5403 \end{pmatrix} \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix} = \alpha_k \begin{pmatrix} -3.5484 \\ -5.3038 \end{pmatrix}.$$

We note that the following two equalities

$$\alpha_k \begin{pmatrix} -3.5484 \\ -5.3038 \end{pmatrix} = \begin{pmatrix} -1.7742 \\ -2.6519 \end{pmatrix} = \Delta \boldsymbol{q}^k$$

are simultaneously satisfied when selecting $\alpha_k = 0.5$ as step size. Thus, the Gradient method was used at iteration k.

The displacement $\Delta \boldsymbol{q}^k$ obtained by the Gradient method leads to a new Cartesian position error $\boldsymbol{e}^{k+1}$ that has a smaller norm than the previous $\boldsymbol{e}^k$ in (11):

$$\boldsymbol{e}^{k+1} = \boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}^{k+1}) = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -0.4208 \\ 0.5366 \end{pmatrix} = \begin{pmatrix} -1.5792 \\ -3.5366 \end{pmatrix}, \quad \left\| \boldsymbol{e}^{k+1} \right\| = 3.8731 < \left\| \boldsymbol{e}^k \right\|. \tag{13}$$

The method is thus converging at this stage, although it may eventually require a reduction of the step size in order to avoid the missing of a solution.

Indeed, we can also determine all solutions to the inverse kinematics problem (9) in a closed form. For this, consider again the two nonlinear equations (7) of the direct kinematics. Squaring each equation and summing yields after simplifications

$$p_x^2 + p_y^2 = a_1^2 + q_2^2,$$

and so

$$q_2^{a,b} = \pm\sqrt{p_x^2 + p_y^2 - a_1^2}. \tag{14}$$

The two solutions (14) for the prismatic joint are real and distinct iff $\|\boldsymbol{p}\|^2 = p_x^2 + p_y^2 > a_1^2$ and collapse into the same one for $\|\boldsymbol{p}\|^2 = a_1^2$. This is in fact a singular case, and provides $q_2 = 0$ as the unique solution. For $\|\boldsymbol{p}\|^2 < a_1^2$, the point in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is out of the workspace (it belongs to an inner circle of radius $|a_1| \geq 0$).

Assume now that $a_1 \neq 0$ and that $\boldsymbol{p}$ belongs to the (primary) workspace of the RP robot. For each solution (14), i.e., two in the regular case or only one, $q_2 = 0$, in the singular case, reorganize the direct kinematics as a linear system in the unknowns $\sin q_1$ and $\cos q_1$:

$$\begin{pmatrix} -q_2^{a,b} & a_1 \\ a_1 & q_2^{a,b} \end{pmatrix} \begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}.$$

A unique solution for q_1 can always be found for each value entered as q_2. We obtain

$$q_1^{a,b} = \text{ATAN2}\left\{a_1 p_y - q_2^{a,b} p_x, a_1 p_x + q_2^{a,b} p_y\right\}. \tag{15}$$

Using the problem data ($\boldsymbol{p} = \boldsymbol{p}_d$ and $a_1 = 0.2$), equations (14) and (15) provide the two solutions

$$\boldsymbol{q}^a = \begin{pmatrix} 2.6091 \\ 3.6 \end{pmatrix}, \qquad \boldsymbol{q}^b = \begin{pmatrix} -0.6435 \\ -3.6 \end{pmatrix} \quad \text{[rad, m]}.$$

At this stage, there is no simple argument that helps us in identifying which inverse kinematic solution would be reached by the Gradient iterative method when continuing its evolution from $\boldsymbol{q}^{k+1}$. Instead of guessing, we just provide the requested orientation of the last DH frame in the two cases, as given by ${}^0\boldsymbol{R}_2(q_1)$ in eq. (6):

$${}^0\boldsymbol{R}_2(q_1^a) = \begin{pmatrix} 0.8615 & -0.5077 & 0 \\ -0.5077 & -0.8615 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad {}^0\boldsymbol{R}_2(q_1^b) = \begin{pmatrix} -0.8 & 0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We determine next the configurations $\widehat{\boldsymbol{q}}$ at which the Gradient method would certainly stop with a non-zero position error for the problem (9) at hand. This requires the Jacobian $\boldsymbol{J}(\widehat{\boldsymbol{q}})$ in (8) to be singular and the position error vector $\widehat{\boldsymbol{e}} = \boldsymbol{p}_d - \boldsymbol{p}(\widehat{\boldsymbol{q}})$ to belong to the null space of $\boldsymbol{J}^T(\widehat{\boldsymbol{q}})$. Therefore, rewrite the Jacobian transpose, the direct kinematics, and the position error in the singularity $\widehat{q}_2 = q_2 = 0$:

$$\boldsymbol{J}_0^T(q_1) = \boldsymbol{J}^T(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -a_1 \sin q_1 & a_1 \cos q_1 \\ -\sin q_1 & \cos q_1 \end{pmatrix},$$

$$\boldsymbol{p}_0(q_1) = \boldsymbol{p}(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} a_1 \cos q_1 \\ a_1 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{e}_0(q_1) = \boldsymbol{e}(\boldsymbol{q})|_{q_2=0} = \boldsymbol{p}_d - \boldsymbol{p}_0 = \begin{pmatrix} -2 - a_1 \cos q_1 \\ -3 - a_1 \sin q_1 \end{pmatrix}.$$

The null space of $\boldsymbol{J}_0^T$ is spanned by a single basis vector

$$\ker\left\{\boldsymbol{J}_0^T(q_1)\right\} = \beta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \qquad \forall \beta.$$

In order to find a suitable value of q_1 such that $\boldsymbol{e}_0 \in \ker\left\{\boldsymbol{J}_0^T\right\}$, we consider the simple linear system in the unknowns $\sin q_1$ and $\cos q_1$, parametrized by the scalar β:

$$\begin{pmatrix} -2 - a_1 \cos q_1 \\ -3 - a_1 \sin q_1 \end{pmatrix} = \beta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \quad \Rightarrow \quad (\beta + a_1) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}.$$

This leads to two solutions (depending on the arbitrary sign —positive or negative— that the factor $(\beta + a_1)$ can assume):

$$\widehat{q}_1^a = \text{ATAN2}\{-3, -2\} = -2.1588, \qquad \widehat{q}_1^b = \text{ATAN2}\{3, 2\} = 0.9828 \quad \text{[rad]}.$$

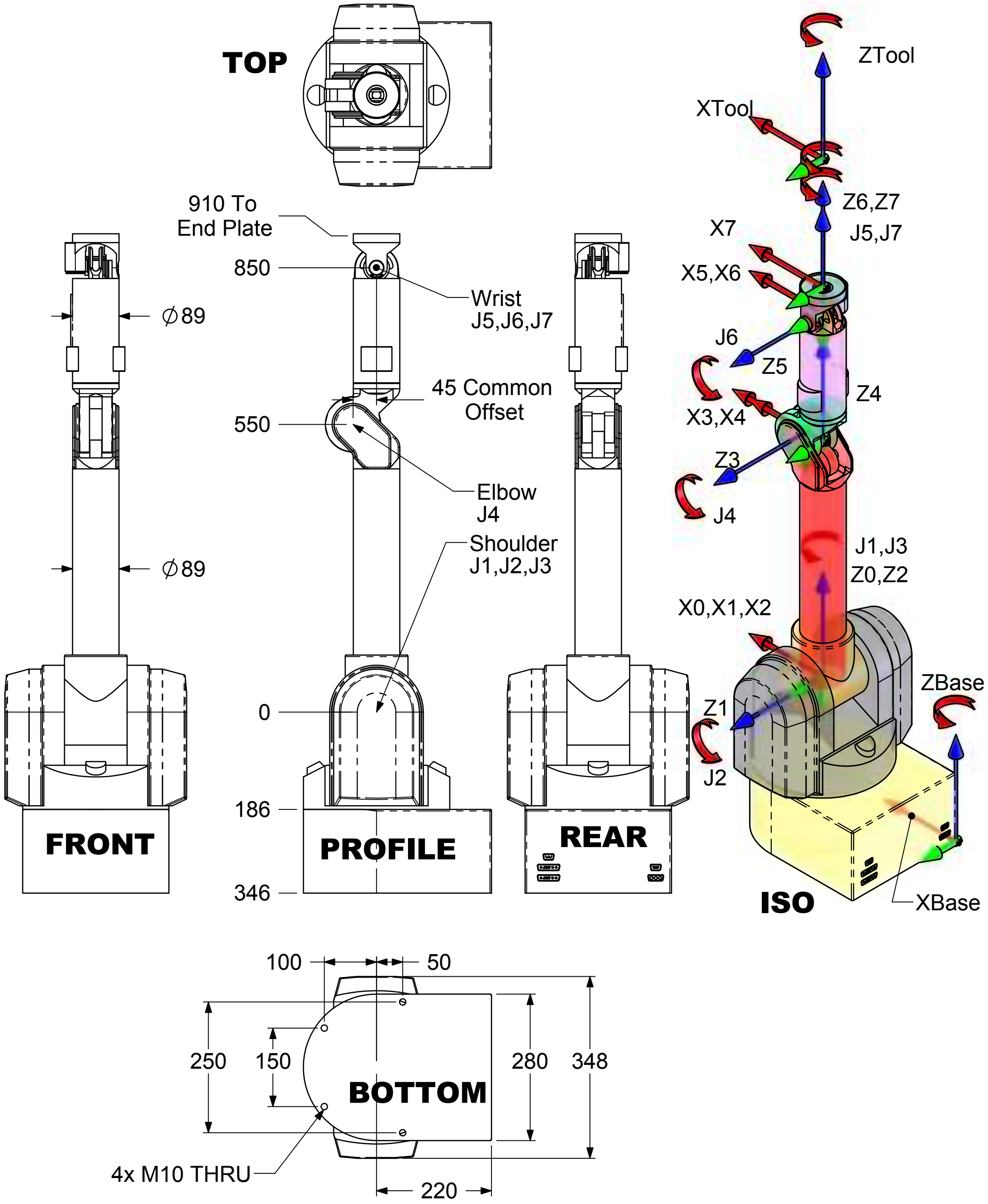
It is left to the reader to check that in both cases $\boldsymbol{J}^T(\widehat{\boldsymbol{q}})\boldsymbol{e}(\widehat{\boldsymbol{q}}) = \mathbf{0}$ (even if $\boldsymbol{e}(\widehat{\boldsymbol{q}}) \neq \mathbf{0}$), resulting in a stopping condition for the Gradient method. It is also very informative at this point to sketch a picture of the robot arm in the configuration $\widehat{\boldsymbol{q}}$, and draw the error $\widehat{\boldsymbol{e}}$ associated to the considered positioning task for the end effector.

When such a situation is encountered, one can force the Gradient method to restart in many possible ways, e.g., by slightly perturbing the current robot configuration so that the position error $\boldsymbol{e}$ exits the null space of $\boldsymbol{J}^T$ or, even better, by momentarily rotating the actual error $\boldsymbol{e}$ (multiplying it by a skew-symmetric matrix $\boldsymbol{K}_s$) so as to obtain the same effect. On the other hand, in a singular configuration (or very close to it) we can never apply the Newton method —at least, not as such.

* * * * *

Assignment of DH frames and table of parameters for the Barrett WAM 7R arm

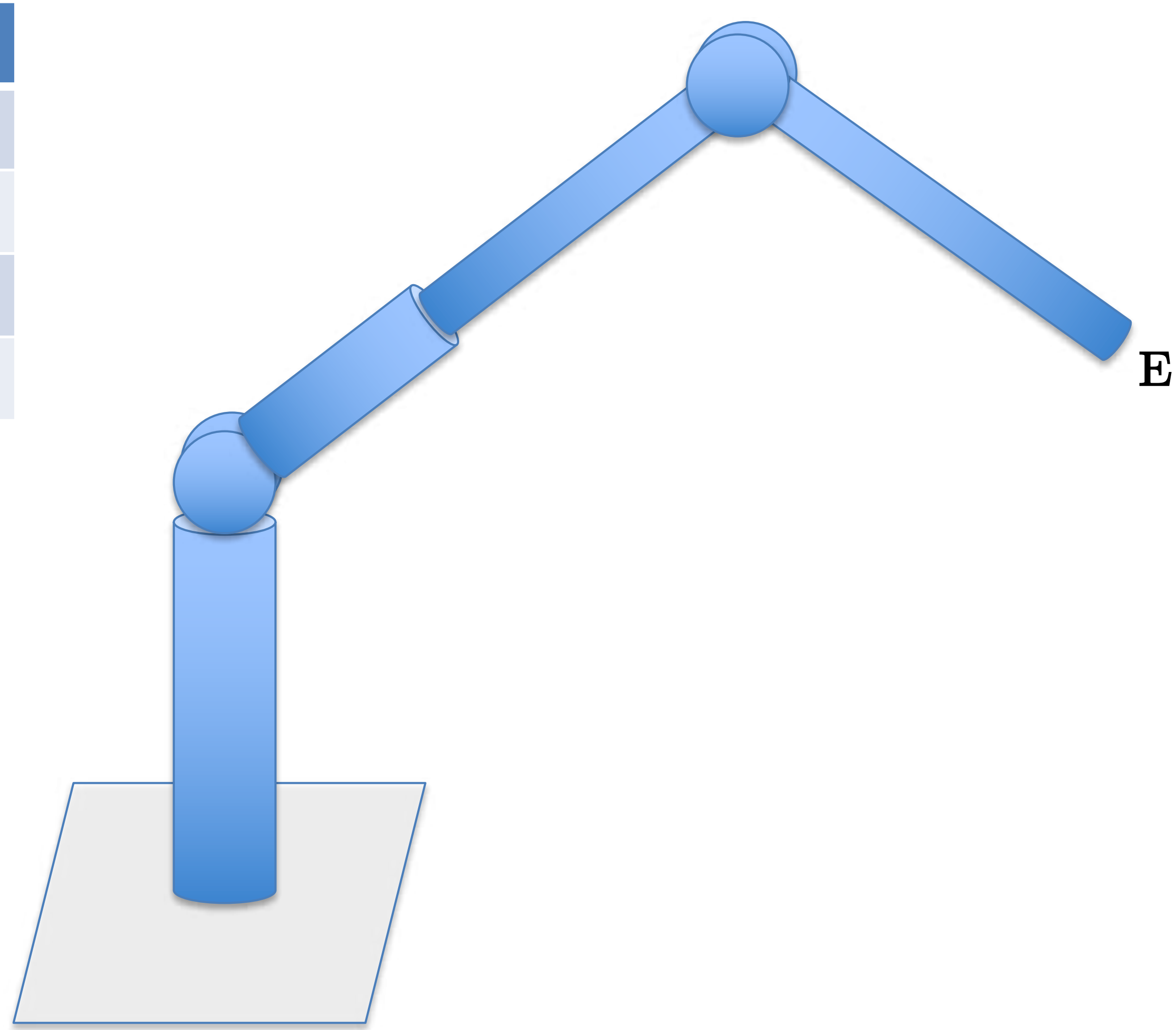
i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	$q_1 = 0$
2	$\pi/2$	0	0	$q_2 = 0$
3	$-\pi/2$	0.045	0.55	$q_3 = 0$
4	$\pi/2$	-0.045	0	$q_4 = 0$
5	$-\pi/2$	0	0.3	$q_5 = 0$
6	$\pi/2$	0	0	$q_6 = 0$
7	0	0	0.06	$q_7 = 0$



RRPR robot – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				



Robotics I

January 11, 2019

Exercise 1

Consider the spatial 4-dof robot with RRPR sequence of joints shown in Fig. 1.

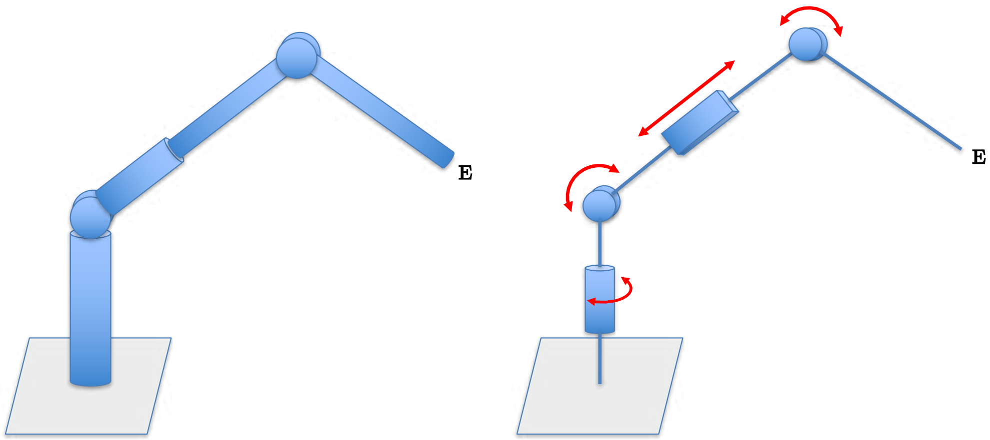


Figure 1: A 4-dof spatial RRPR robot and its kinematic skeleton.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated symbolic table of parameters, choosing only values $\alpha_i \geq 0$ ($i = 1, \ldots, 4$) for the twist angles, and specifying the signs of all other non-zero constant parameters. The base frame (frame 0) should be placed on the ground and the origin of the (last) frame 4 at the end-effector point E. Draw the frames and fill in the table directly on the extra sheet provided separately.
- Write explicitly the four resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^3\boldsymbol{A}_4(q_4)$ and compute in an efficient way the direct kinematics $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q}) \in \mathbb{R}^3$ for the position of the origin O_4 of the last DH frame.
- Draw the robot in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$ for a generic $L > 0$. Compute the position $\boldsymbol{p}_{4,0} = \boldsymbol{p}_4(\boldsymbol{q}_0)$ as a parametric function of L and of the other constant DH parameters in symbolic form.

Exercise 2

Consider again the robot in Exercise 1.

- Derive the expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame. What is the generic rank of the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of this matrix?
- Evaluate $\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0)$, again as a parametric function. At the same previously specified configuration $\boldsymbol{q} = \boldsymbol{q}_0$, provide answers/solutions to the following problems.
  - Find, if possible, a joint velocity $\dot{\boldsymbol{q}}_a \neq \boldsymbol{0}$ that produces no linear velocity ($\boldsymbol{v} = \boldsymbol{0}$) at the end-effector. Would then also $\boldsymbol{\omega} = \boldsymbol{0}$ follow?
  - Determine if the generalized Cartesian velocity $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}^T$ is feasible. If so, provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes it.
  - Find, if possible, a non-zero generalized Cartesian force $\boldsymbol{F}_c = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{m}^T \end{pmatrix}^T \in \mathbb{R}^6$ applied at the end-effector that can be statically balanced by zero joint forces/torques ($\boldsymbol{\tau} = \boldsymbol{0}$, with $\boldsymbol{\tau} \in \mathbb{R}^4$). If such a $\boldsymbol{F}_c \neq \boldsymbol{0}$ does not exist, explain why.

Exercise 3

For a planar RP robot with direct kinematics of the end-effector position given by

$$\boldsymbol{p} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \tag{1}$$

consider the planning of a rest-to-rest motion between an initial and a final Cartesian point, respectively, $\boldsymbol{p}_A = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ [m] at $t = 0$ and $\boldsymbol{p}_B = \begin{pmatrix} -3.5355 & 3.5355 \end{pmatrix}^T$ [m] at $t = T$. Optimization of the motion time T is being sought, in two different operative conditions as follows.

a. Define a joint trajectory $\boldsymbol{q}_a^*(t)$ that minimizes the motion time for this task under the bounds on the joint accelerations,

$$|\ddot{q}_1| \leq A_1 = 200\,°/\mathrm{s}^2, \qquad |\ddot{q}_2| \leq A_2 = 5 \text{ m/s}^2. \tag{2}$$

Find the value of the minimum motion time T_a^* and draw the time profiles of the position, velocity and acceleration of the two robot joints.

b. Consider next the additional Cartesian bound on the norm of the end-effector acceleration,

$$\|\ddot{\boldsymbol{p}}\| \leq A_c = 10 \text{ m/s}^2. \tag{3}$$

Verify whether the previous solution $\boldsymbol{q}_a^*(t)$ satisfies the bound (3) or not. If not, propose a modified joint trajectory $\boldsymbol{q}_b^*(t)$ such that both bounds (2) and (3) will be satisfied, while trying to minimize the new motion time. Discuss the rationale of your choice and the supporting equations, provide the resulting motion T_b^*, and sketch the new time profiles of your solution.

[210 minutes, open books]

Solution

January 11, 2019

Exercise 1

A DH frame assignment that satisfies the condition on the twist angles, $\alpha_i \geq 0$, $i = 1, \ldots, 4$, is shown in Fig. 2, with the associated parameters given in Tab. 1. The signs of the non-zero symbolic constants are also reported in the table.

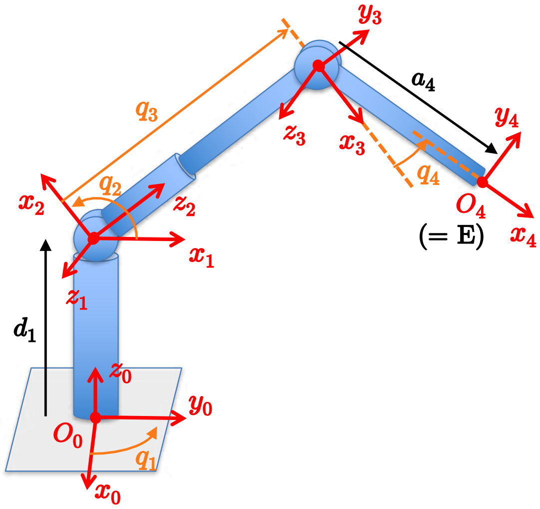


Figure 2: A possible DH frame assignment for the 4-dof spatial RRPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$\pi/2$	0	q_3	π
4	0	$a_4 > 0$	0	q_4

Table 1: Parameters associated to the DH frames in Fig. 2.

Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^2\boldsymbol{R}_3 & {}^2\boldsymbol{p}_3(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4(q_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 & a_4 \cos q_4 \\ \sin q_4 & \cos q_4 & 0 & a_4 \sin q_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$
\begin{pmatrix} \boldsymbol{p}_4(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] = \begin{pmatrix} \cos q_1 \left(q_3 \sin q_2 - a_4 \cos(q_2 + q_4) \right) \\ \sin q_1 \left(q_3 \sin q_2 - a_4 \cos(q_2 + q_4) \right) \\ d_1 - q_3 \cos q_2 - a_4 \sin(q_2 + q_4) \\ 1 \end{pmatrix}. \tag{4}
$$

Figure 3 shows the robot in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$. The end-effector position is evaluated from (4) as

$$
\boldsymbol{p}_4(\boldsymbol{q}_0) = \begin{pmatrix} L \\ 0 \\ d_1 - a_4 \end{pmatrix}.
$$

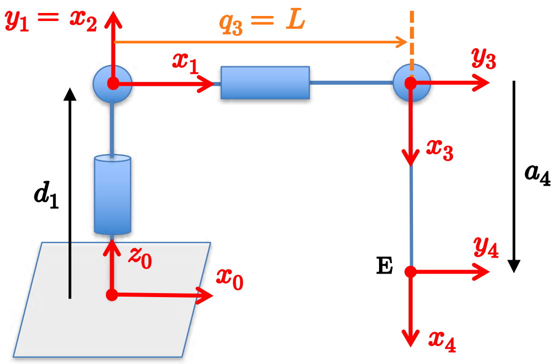


Figure 3: The RRPR robot skeleton in the configuration $\boldsymbol{q}_0 = (0 \;\; \pi/2 \;\; L \;\; 0)^T$.

Exercise 2

The simplest way to derive the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of the spatial PPRP robot in

$$
\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}
$$

is to compute the 3×4 upper block $\boldsymbol{J}_L(\boldsymbol{q})$ by partial differentiation of the position vector $\boldsymbol{p}_4(\boldsymbol{q})$, and the

3×4 lower block $\boldsymbol{J}_A(\boldsymbol{q})$ by using the standard formula. From eq. (4), we obtain

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_4(\boldsymbol{q})}{\partial \boldsymbol{q}} =$$

$$\begin{pmatrix} -\sin q_1(q_3 \sin q_2 - a_4 \cos(q_2+q_4)) & \cos q_1(q_3 \cos q_2 + a_4 \sin(q_2+q_4)) & \cos q_1 \sin q_2 & a_4 \cos q_1 \sin(q_2+q_4) \\ \cos q_1(q_3 \sin q_2 - a_4 \cos(q_2+q_4)) & \sin q_1(q_3 \cos q_2 + a_4 \sin(q_2+q_4)) & \sin q_1 \sin q_2 & a_4 \sin q_1 \sin(q_2+q_4) \\ 0 & q_3 \sin q_2 - a_4 \cos(q_2+q_4) & -\cos q_2 & -a_4 \cos(q_2+q_4) \end{pmatrix}. \tag{5}$$

Further, being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for all i, we have

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)^1\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{R}_1(q_1)^1\boldsymbol{R}_2(q_2)^2\boldsymbol{R}_3{}^3\boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & \sin q_1 & 0 & \sin q_1 \\ 0 & -\cos q_1 & 0 & -\cos q_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

It follows from (6) that the generic rank of matrix $\boldsymbol{J}_A(\boldsymbol{q})$ is equal to 2. Moreover, at the previously specified configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$, we evaluate the geometric Jacobian from (5) and (6) as

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & a_4 & 1 & a_4 \\ L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{L0} \\ \boldsymbol{J}_{A0} \end{pmatrix}. \tag{7}$$

It is easy to see that

$$\rho_0 = \text{rank}\{\boldsymbol{J}_0\} = 4, \qquad \rho_{L0} = \text{rank}\{\boldsymbol{J}_{L0}\} = 3, \qquad \rho_{A0} = \text{rank}\{\boldsymbol{J}_{A0}\} = 2 \ \text{(as expected in general)}. \tag{8}$$

Using (7) and (8), we provide the following answers/solutions when the robot is at the configuration $\boldsymbol{q}_0$.

- Joint velocities $\dot{\boldsymbol{q}}_a \in \mathbb{R}^4$ that produce zero linear velocity at the end-effector, i.e., $\boldsymbol{v} = \boldsymbol{J}_{L0}\,\dot{\boldsymbol{q}}_a = \boldsymbol{0}$, belong to the null space of $\boldsymbol{J}_{L0}$. Since the robot has $n = 4$ joints and $\rho_{L0} = 3$, the null space of $\boldsymbol{J}_{L0}$ has dimension $n - \rho_{L0} = 1$. Thus, there are ∞^1 joint velocities $\dot{\boldsymbol{q}}_a$, all having the form

$$\dot{\boldsymbol{q}}_a = \alpha \begin{pmatrix} 0 \\ 0 \\ -a_4 \\ 1 \end{pmatrix}, \qquad \alpha \lesseqqgtr 0,$$

  that produce zero linear velocity at the end-effector.
- Since

$$\boldsymbol{J}_{A0}\,\dot{\boldsymbol{q}}_a = \alpha \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{\omega}_a,$$

  any non-vanishing ($\alpha \neq 0$) joint velocity $\dot{\boldsymbol{q}}_a$ in the null space of $\boldsymbol{J}_{L0}$ will be associated with a non-zero angular velocity, i.e., $\boldsymbol{\omega}_a \neq \boldsymbol{0}$.
- A generalized Cartesian velocity $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{pmatrix}^T \in \mathbb{R}^6$ will be feasible if and only if it belongs to the range space (or image) of $\boldsymbol{J}_0$. Since $\rho_0 = 4$, the range space of this matrix is given by the linear combinations of all its four columns. A simple test to verify whether or not the Cartesian velocity

$\boldsymbol{V} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}^T$ belongs to the image of $\boldsymbol{J}_0$ is to border the matrix $\boldsymbol{J}_0$ with the column vector $\boldsymbol{V}$ and check the rank of the resulting matrix. Since

$$\text{rank}\{\boldsymbol{J}_0\} = \text{rank}\left\{\begin{pmatrix} 0 & a_4 & 1 & a_4 \\ L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right\} = 4 < 5 = \text{rank}\left\{\begin{pmatrix} 0 & a_4 & 1 & a_4 & 1 \\ L & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}\right\} = \text{rank}\left\{\begin{pmatrix} \boldsymbol{J}_0 & \boldsymbol{V} \end{pmatrix}\right\},$$

the given vector $\boldsymbol{V}$ will not belong to the image of $\boldsymbol{J}_0$. Thus, there is no joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that will instantaneously realize $\boldsymbol{V}$ (i.e., $\boldsymbol{J}_0\dot{\boldsymbol{q}}_b \neq \boldsymbol{V}$, $\forall \dot{\boldsymbol{q}}_b$).

- A generalized Cartesian force $\boldsymbol{F}_c = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{m}^T \end{pmatrix}^T \in \mathbb{R}^6$ applied at the end-effector is statically balanced by zero joint forces/torques $\boldsymbol{\tau} \in \mathbb{R}^4$, i.e., $\boldsymbol{\tau} = \boldsymbol{J}_0^T \boldsymbol{F}_c = \boldsymbol{0}$ if and only if it belongs to the null space of $\boldsymbol{J}_0^T$. Since the Cartesian task has dimension $m = 6$ and $\text{rank}\left\{\boldsymbol{J}_0^T\right\} = \rho_0 = 4$, the null space of $\boldsymbol{J}_0^T$ will have dimension $m - \rho_0 = 2$. A basis for this null space is given by

$$\mathcal{N}\{\boldsymbol{J}_0^T\} = \text{range}\left\{\begin{pmatrix} 0 & 0 \\ 0 & -1/L \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \text{range}\left\{\begin{pmatrix} \boldsymbol{F}_{c1} & \boldsymbol{F}_{c2} \end{pmatrix}\right\}.$$

Therefore, one will obtain $\boldsymbol{\tau} = \boldsymbol{J}_0^T \left(\alpha_1 \boldsymbol{F}_{c1} + \alpha_2 \boldsymbol{F}_{c2}\right) = \boldsymbol{0}$ for any value of the scalars α_1 and α_2.

Exercise 3

The rest-to-rest trajectory planning problem for the planar RP robot can be tackled in the joint space. Through the inverse kinematics of this robot[1]

$$q_1 = \text{ATAN2}\left\{p_y, p_x\right\}, \qquad q_2 = \sqrt{p_x^2 + p_y^2},$$

we obtain for the initial and final Cartesian points, respectively

$$\boldsymbol{p}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{[m]} \qquad \Rightarrow \qquad \boldsymbol{q}_A = \begin{pmatrix} 0.6435 \\ 5 \end{pmatrix} \text{[rad, m]} \left(= \begin{pmatrix} 36.87 \\ 5 \end{pmatrix} [^\circ\text{, m}] \right),$$

to be assumed at the initial time $t = 0$ with zero initial velocity $\dot{\boldsymbol{q}}_A = \boldsymbol{0}$, and

$$\boldsymbol{p}_B = \begin{pmatrix} -3.5355 \\ 3.5355 \end{pmatrix} \text{[m]} \qquad \Rightarrow \qquad \boldsymbol{q}_B = \begin{pmatrix} 2.3562 \\ 5 \end{pmatrix} \text{[rad, m]} \left(= \begin{pmatrix} 135 \\ 5 \end{pmatrix} [^\circ\text{, m}] \right),$$

to be assumed at the final time $t = T$ with zero initial velocity $\dot{\boldsymbol{q}}_B = \boldsymbol{0}$. Note that the same value $q_{A,2} = q_{B,2} = 5$ [m] has been obtained for the prismatic joint.

Case a. When seeking the minimization of the motion time T under the joint acceleration limits (2) only, we can proceed separately for each joint. Since the second (prismatic) joint doesn't need to move

[1] One could have used also the second solution to the inverse kinematics

$$q_1 = \text{ATAN2}\left\{-p_y, -p_x\right\}, \qquad q_2 = -\sqrt{p_x^2 + p_y^2}.$$

The following developments would have been the same, modulo a change of sign for the joint motions. In any event, the objective of minimizing motion time suggests that the same solution class of the inverse kinematics should be used for both Cartesian points $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$.

($q^*_{a,2}(t) \equiv 0$), it does not impose any lower bound on the motion time. The problem is solved by looking just at the first joint motion. In the absence of a joint velocity limit, the time-optimal motion of the first joint will be a bang-bang profile in acceleration and, accordingly, a triangular profile in velocity. Since a positive displacement is requested for q_1, we have

$$\ddot{q}^*_{a,1}(t) = \begin{cases} A_1, & t \in \left[0, \dfrac{T}{2}\right) \\[2ex] -A_1, & t \in \left[\dfrac{T}{2}, T\right] \end{cases},$$

and thus

$$\dot{q}^*_{a,1}(t) = \begin{cases} A_1 t, & t \in \left[0, \dfrac{T}{2}\right) \\[2ex] A_1 \dfrac{T}{2} - A_1 \left(t - \dfrac{T}{2}\right), & t \in \left[\dfrac{T}{2}, T\right] \end{cases}$$

and[2]

$$q^*_{a,1}(t) = \begin{cases} q_{A,1} + \dfrac{1}{2} A_1 t^2, & t \in \left[0, \dfrac{T}{2}\right) \\[2ex] q_{A,1} + \dfrac{A_1 T^2}{8} + A_1 \dfrac{T}{2}\left(t - \dfrac{T}{2}\right) - \dfrac{1}{2} A_1 \left(t - \dfrac{T}{2}\right)^2, & t \in \left[\dfrac{T}{2}, T\right] \end{cases}.$$

By symmetry, half of the total displacement $\Delta q_1 = |q_{B,1} - q_{A,1}|$ will be completed at the midtime of motion. Therefore, from the equality

$$q^*_{a,1}\left(\frac{T}{2}\right) = q_{A,1} + \frac{A_1 T^2}{8} = q_{A,1} + \frac{q_{B,1} - q_{A,1}}{2},$$

we obtain

$$T^*_a = \sqrt{\frac{4\,|q_{B,1} - q_{A,1}|}{A_1}} = 1.401 \text{ [s]}, \tag{9}$$

where $A_1 = 200\,[^\circ/\text{s}^2]$ and $q_{B,1} - q_{A,1} = 98.13^\circ$ have been used. Moreover, the peak velocity of joint 1 will be

$$V_1 = \dot{q}^*_{a,1}\left(\frac{T^*_a}{2}\right) = A_1 \frac{T^*_a}{2} = \sqrt{A_1\,|q_{B,1} - q_{A,1}|} = 140.1\,[^\circ/\text{s}] = 2.4451\,[\text{rad/s}]. \tag{10}$$

The plots of the joint positions, velocities and accelerations are shown in Fig. 4. The resulting Cartesian path traced by the end-effector along the time-optimal joint trajectory $\boldsymbol{q}^*_a$ is indeed an arc of a circle of radius $r = q_{A,2} = 5$ [m], as shown in Fig. 5.

Indeed the solution found for this case is not at all unique. Joint 2 may in fact move in an arbitrary way, as long as it goes back to the same initial position with zero final velocity within the instant of time T^*_a, and without violating its acceleration bound A_2 during the interval $[0, T^*_a]$.

Case b. In order to verify whether the previous solution $\boldsymbol{q}^*_a(t)$ satisfies the additional Cartesian bound (3), we need to compute the end-effector acceleration $\ddot{\boldsymbol{p}}$ differentiating twice the expression (1) of the direct kinematics. We obtain

$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 \cos q_1 - \dot{q}_1 q_2 \sin q_1 \\ \dot{q}_2 \sin q_1 + \dot{q}_1 q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

[2]Formulas are written so as to highlight how they were obtained via integration. The expressions for the velocity and position in in the second half of the motion can be rewritten also as

$$\dot{q}^*_{a,1}(t) = A_1 T - A_1 t, \qquad q^*_{a,1}(t) = q_{A,1} - \frac{A_1 T^2}{4} + A_1 T\, t - \frac{A_1}{2}\, t^2, \qquad t \in \left[\frac{T}{2}, T\right].$$

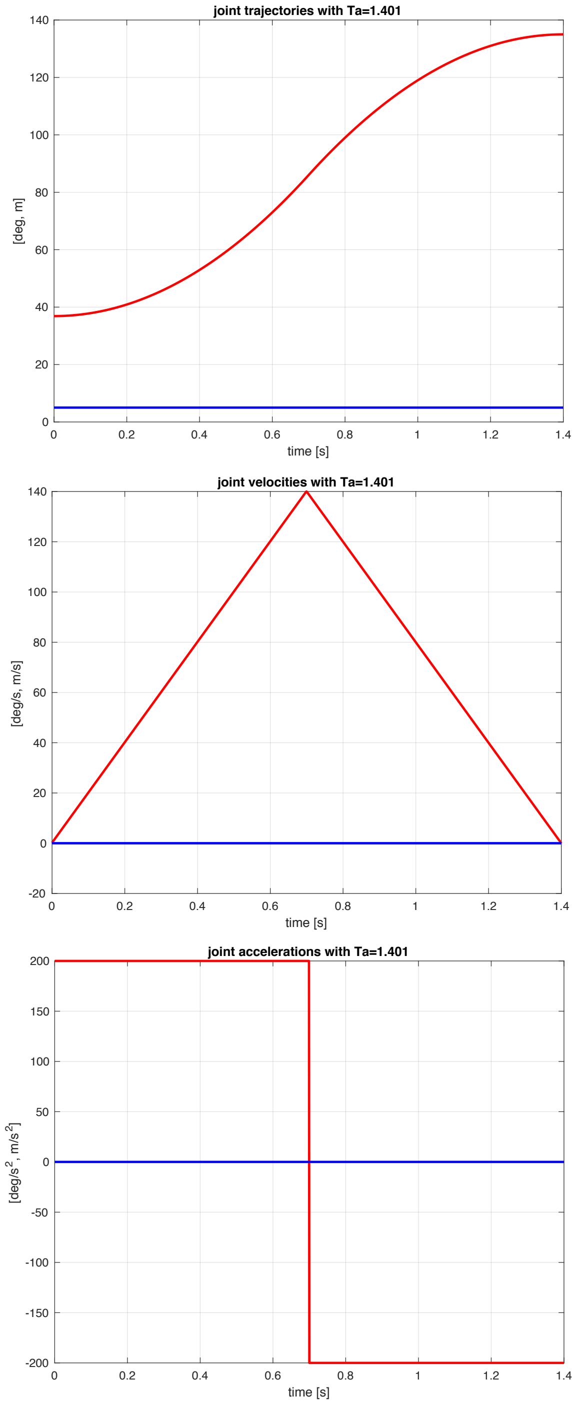


Figure 4: Position, velocity and acceleration profiles of the solution trajectory $\boldsymbol{q}_a^*$ for Case a. Joint 1 = red, joint 2 = blue.

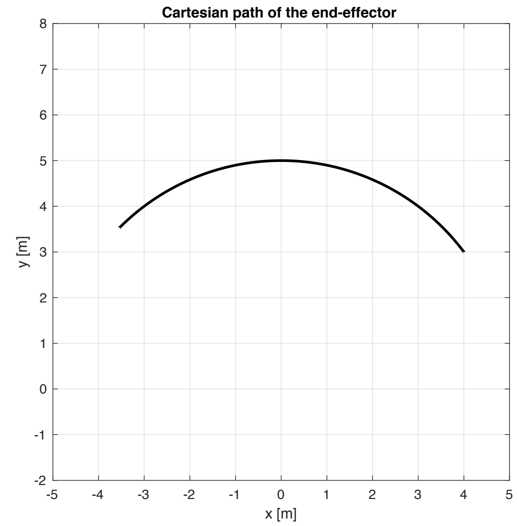


Figure 5: Cartesian path traced by the end-effector along the joint trajectory $\boldsymbol{q}_a^*$ of Fig. 4.

and

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -\left(\dot{q}_1^2\, q_2 \cos q_1 + 2\,\dot{q}_1 \dot{q}_2 \sin q_1\right) \\ -\dot{q}_1^2\, q_2 \sin q_1 + 2\,\dot{q}_1 \dot{q}_2 \cos q_1 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{11}$$

with $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ having a quadratic dependence on the components of $\dot{\boldsymbol{q}}$.

Since the norm of a vector is invariant with respect to a rotation, $\|\ddot{\boldsymbol{p}}\| = \|\boldsymbol{R}\,\ddot{\boldsymbol{p}}\|$, to ease computations it is convenient to express the acceleration (11) in a frame rotated with q_1 on the plane (x, y), namely

$$\begin{aligned} {}^1\ddot{\boldsymbol{p}} &= \begin{pmatrix} \cos q_1 & \sin q_1 \\ -\sin q_1 & \cos q_1 \end{pmatrix} \ddot{\boldsymbol{p}} = \boldsymbol{R}^T(q_1)\, \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{R}^T(q_1)\, \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -\dot{q}_1^2\, q_2 \\ 2\,\dot{q}_1 \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \ddot{q}_2 - \dot{q}_1^2\, q_2 \\ q_2\, \ddot{q}_1 + 2\,\dot{q}_1 \dot{q}_2 \end{pmatrix}. \end{aligned} \tag{12}$$

As a result, we have from (12) the closed-form expression

$$\|\ddot{\boldsymbol{p}}\| = \|{}^1\ddot{\boldsymbol{p}}\| = \sqrt{(\ddot{q}_2 - \dot{q}_1^2\, q_2)^2 + (q_2\, \ddot{q}_1 + 2\,\dot{q}_1 \dot{q}_2)^2}. \tag{13}$$

When (13) is evaluated along the joint trajectory $\boldsymbol{q}_a^*(t)$, since $q_2 = q_{A,2}$ (constant) and $\dot{q}_2 = \ddot{q}_2 = 0$, we have

$$\|\, \ddot{\boldsymbol{p}} \,\|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = \sqrt{q_{A,2}^2\, (\dot{q}_1^4 + \ddot{q}_1^2)}. \tag{14}$$

Taking the maximum of $\dot{q}_1$ and $\ddot{q}_1$ (which occur both at the midtime of motion), using (10), and converting $A_1 = 200 \cdot \pi/180 = 3.4906$ [rad/s^2] yields

$$\max \|\ddot{\boldsymbol{p}}\|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = \sqrt{q_{A,2}^2\, (V_1{}^4 + A_1^2)} = \sqrt{(5)^2\, [(2.4451)^4 + (3.4906)^2]} = 34.58 > 10 = A_c. \tag{15}$$

Thus, $\|\ddot{\boldsymbol{p}}\|$ reaches a peak which is about 3.5 times higher than the bound A_c in (3). The evolution of the norm of the Cartesian acceleration (14) is shown in Fig. 6 (this plot is obtained using Matlab).

Therefore, in the presence of this additional bound, the previous trajectory should be made considerably slower. This can be achieved in many ways, leading to different solutions for the new trajectory $\boldsymbol{q}_b(t)$,

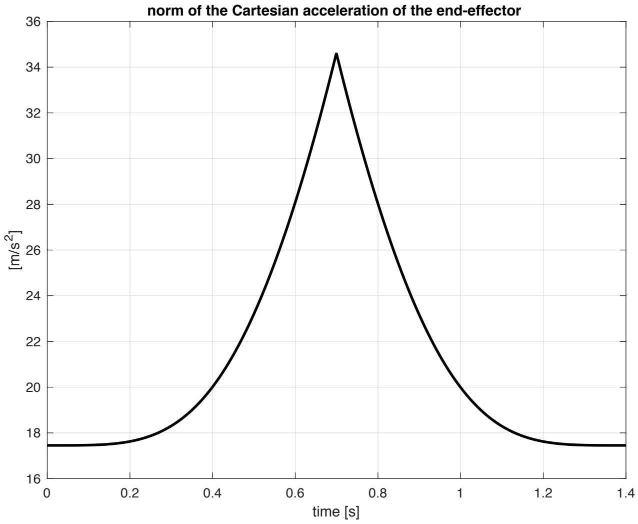


Figure 6: Norm of the end-effector acceleration $\ddot{\boldsymbol{p}}$ along the joint trajectory $\boldsymbol{q}_a^*$ of Fig. 4.

each with a possibly different motion time T_b. Indeed, among all feasible $\boldsymbol{q}_b(t)$, the optimal solution $\boldsymbol{q}_b^*(t)$ will have the least completion time $T_b^* \leq T_b$. However, finding the minimum time in this situation is not straightforward. Below we give some clues on how to proceed, al least for generating trajectories that satisfy all constraints.

1. **Uniform scaling.** The simplest way to recover feasibility is to scale uniformly the motion time by a factor $k > 1$, i.e., $T_{b,1} = kT_a^*$, reducing thus both the joint velocity (by a factor k) and the joint acceleration (by a factor k^2). Using again eqs. (14–15), we find the minimum value k by imposing the equality

$$\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_{b,1}^*} = \sqrt{q_{A,2}^2 \left[\left(\frac{V_1}{k} \right)^4 + \left(\frac{A_1}{k^2} \right)^2 \right]} = \frac{1}{k^2} \sqrt{q_{A,2}^2 \left(V_1^4 + A_1^2 \right)} = \frac{1}{k^2} \max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = A_c, \tag{16}$$

and thus

$$k = \sqrt{\frac{\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_a^*}}{A_c}} = 1.8596 \qquad \Rightarrow \qquad T_{b,1} = kT_a^* = 2.605\,[\text{s}]. \tag{17}$$

The trajectory profile will be the same as before, with the second joint always at rest and the first (revolute) joint having a bang-bang acceleration with $A_{max} = A_1/k^2 = 57.84$ [°/s^2] and a triangular velocity with peak $V_{max} = V_1/k = 75.34$ [°/s]. The plots are similar to those in Fig. 4.

2. **Including a cruise phase.** A different strategy, still following the same Cartesian path as in Fig. 5, would be to apply a smaller acceleration to joint 1 (with $A_{t,max} < A_1$) until reaching some cruise speed $V_{t,max} < V_1$ at $t = T_t$, travel at that speed for a suitable time, and then decelerate for an interval T_t until the final stop at $T_{b,2}$, while complying at all times with the bound on the norm of $\ddot{\boldsymbol{p}}$. The resulting trajectory $\boldsymbol{q}_{b,2}^*$ would be bang-coast-bang in acceleration, i.e., with a symmetric trapezoidal profile in velocity —the reason for the subscript 't' in the above quantities. As usual in such cases, from

$$T_t = \frac{V_{t,max}}{A_{t,max}} \text{ and } \Delta q_1 = (T_{b,2} - T_t)V_{t,max} \quad \Rightarrow \quad T_{b,2} = \frac{\Delta q_1}{V_{t,max}} + \frac{V_{t,max}}{A_{t,max}} = \frac{\Delta q_1 A_{t,max} + V_{t,max}^2}{V_{t,max} A_{t,max}}. \tag{18}$$

On the other hand, the reaching of the Cartesian acceleration limit implies from (14)

$$\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_{b,2}^*} = \sqrt{q_{A,2}^2 \left(V_{t,max}^4 + A_{t,max}^2 \right)} = A_c. \tag{19}$$

This is imposed at $t = T_t$, namely at the end of the acceleration phase, where also the velocity has reached its maximum. Solving for $V_{t,max}$ from (19)

$$V_{t,max} = \sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}, \qquad \text{(where } A_{t,max} < \frac{A_c}{q_{A,2}} \text{ is being assumed)}$$

and substituting this within $T_{b,2}$ in (18) gives

$$T_{b,2} = \frac{\Delta q_1}{\sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}} + \frac{\sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}}{A_{t,max}} = f(A_{t,max}), \qquad \text{for } 0 < A_{t,max} < \frac{A_c}{q_{A,2}}. \tag{20}$$

From the functional dependence in (20), it is clear that the minimum of $T_{b,2}$ will not occur neither for very small values of $A_{t,max}$ nor close to its upper limit, as the function f goes to infinity in both cases. Figure 7 plots the value of $T_{b,2}$ as a function of $A_{t,max}$ in its interval of definition. It can be seen that a minimum is (approximately) found for

$$A_{t,max} = 1.59\,[\text{rad/s}^2] = 91.1\,[°/\text{s}^2] \qquad \Rightarrow \qquad T_{b,2} = 2.2475\,[\text{s}].$$

Thus, the motion time $T_{b,2}$ found when using a trapezoidal profile is smaller that the value $T_{b,1}$ found by uniform scaling by about 14%. This reflects the better adaptability of this new trajectory.

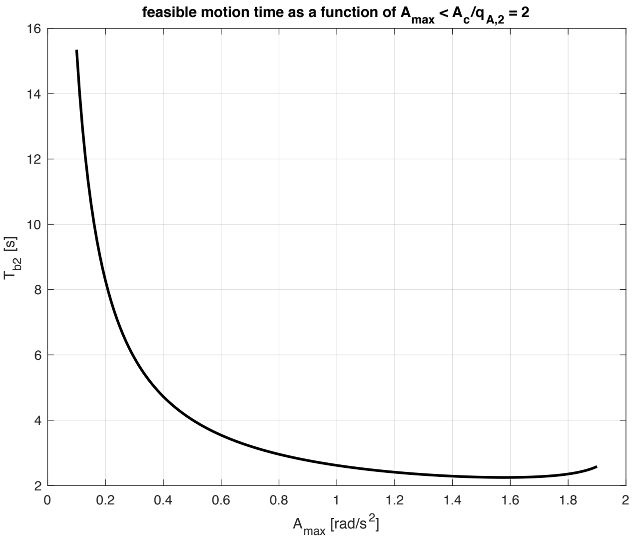


Figure 7: Dependence of the motion time $T_{b,2}$ on $A_{t,max}$ for a feasible trapezoidal velocity profile in Case b.

3. **Moving also joint 2.** Another alternative is to explore the use of an extra motion of the (prismatic) joint 2, which should allow a faster displacement of joint 1 when the Cartesian acceleration bound is limiting the completion time. Note first that, when using (14) in a motion with constant $q_2 = q_{2,0}$, the value of $\|\ddot{\boldsymbol{p}}\|$ will decrease linearly with $q_{2,0}$. Thus, one could try the following three-phase trajectory.

   I. With the first joint kept fixed, retract the second joint to a value $q_{2,0} < q_{A,2}$, using a bang-bang (negative-positive) acceleration profile with maximum acceleration equal to the minimum between A_c and A_2. Let T_I be the time needed for this rest-to-rest motion of joint 2.

II. Perform the displacement Δq_1 with the first joint, just like in Case a. but now with $q_2 = q_{2,0}$. By a judicious choice of $q_{2,0}$, this motion can be executed using a bang-bang acceleration profile with maximum acceleration equal to A_1. In fact, the value of $q_{2,0}$ could be such that the norm of the Cartesian acceleration always satisfies its bound, and reaches the maximum value A_c at least in one instant. Let T_{II} be the needed motion time for this phase.

III. Reverse the motion of phase I so as to move joint 2 from $q_{2,0}$ back to $q_{B,2} = q_{A,2}$, using a bang-bang (now, positive-negative) acceleration profile. Indeed, the time needed for this phase is $T_{III} = T_I$.

The minimum retraction of joint 2 that will guarantee the saturation of the bound (3) mentioned in phase II is evaluated by equating

$$\max \|\ddot{\boldsymbol{p}}\| = \sqrt{q_{20}^2 \left(V_1^{\,4} + A_1^2\right)} = A_c \qquad \Rightarrow \qquad q_{20} = \frac{A_c}{\sqrt{V_1^{\,4} + A_1^2}} = 1.4445\,[\mathrm{m}]. \tag{21}$$

This implies a net displacement $\Delta q_2 = |q_{20} - q_{A,2}| = 3.5555$ [m] for joint 2. Thus, being $A_2 = 5 < 10 = A_c$, the motion times of the three phases are computed as

$$T_I = \sqrt{\frac{4|q_{20} - q_{A,2}|}{A_2}} = 1.6865, \qquad T_{II} = \sqrt{\frac{4|q_{B,1} - q_{A,1}|}{A_1}} = 1.401\ (= T_a^*),$$

and

$$T_{III} = \sqrt{\frac{4|q_{B,2} - q_{20}|}{A_2}} = T_I = 1.6865.$$

Thus, the total motion time is

$$T_{b,3} = 2T_I + T_{II} = 4.774\,[s].$$

As a result, in this case there is no benefit with such approach with respect to the two previous methods. While some time reduction could still be achieved by moving both joints in this way simultaneously (rather than in alternating sequence), the analysis would become far too complex —and perhaps the outcome would still not be worth of it.

* * * * *

4R robot – DH frames assignment and table

Name: ______________________________________

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				

Robotics I

February 5, 2019

Exercise 1

Consider the spatial 4R robot shown in Fig. 1. The second and third joint axes are always horizontal.

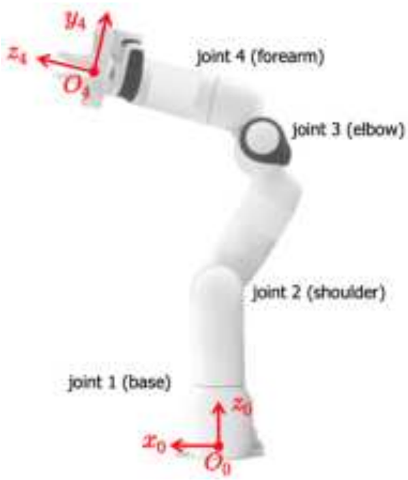


Figure 1: A 4-dof spatial robot with all revolute joints.

a. Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters, specifying the signs of all constant symbolic parameters. Keep base frame (frame 0) and last frame (the end-effector frame 4) as defined in Fig. 1 (these frames satisfy already the DH requirements!). Draw the frames and fill in the table on the extra sheet provided separately.

b. Write explicitly the resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^3\boldsymbol{A}_4(q_4)$ and compute in an efficient way the direct kinematics $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q}) \in \mathbb{R}^3$ for the position of the origin O_4.

c. Discuss if and how the number of symbolic parameters in the direct kinematics of this robot could be reduced. What would be the consequences?

d. Sketch the robot in the stretched upward configuration and specify which is the associated configuration $\boldsymbol{q}_s$ in your DH convention. Compute then $\boldsymbol{p}_s = \boldsymbol{p}_4(\boldsymbol{q}_s)$.

e. In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, determine the expression in the base frame of the absolute position of a Tool Center Point (TCP) which is defined in the end-effector frame by ${}^4\boldsymbol{p}_{4,TCP} = \begin{pmatrix} 0 & 0.1 & 0.2 \end{pmatrix}^T$ [m].

Exercise 2

Make reference to the robot in Exercise 1.

a. Derive the expression of the 6×4 geometric Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of this robot, relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame.

b. Find all configurations at which the upper 3×4 block $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian loses rank.

c. Find all configurations at which the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian loses rank.

d. In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, check if the linear Cartesian velocity $\boldsymbol{v}_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ is feasible. Provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes $\boldsymbol{v}_b$ or, at least, that minimizes the norm of the error w.r.t. the Cartesian velocity $\boldsymbol{v}_b$. If such a joint velocity exists, is it unique?

Exercise 3

Consider the planar 2R robot in Fig. 2, shown together with the geometric data of its desired task. The robot end-effector should follow a desired trajectory made by a circular path of radius R centered at $\boldsymbol{C}_0 = \begin{pmatrix} C_{0,x} & C_{0,y} \end{pmatrix}^T$, to be executed clockwise with a continuous, possibly time-varying desired scalar speed $v(t) > 0$, starting at time $t = 0$ from the path point $\boldsymbol{P}_0 = \begin{pmatrix} P_{0,x} & P_{0,y} \end{pmatrix}^T = \begin{pmatrix} C_{0,x}+R & C_{0,y} \end{pmatrix}^T$.

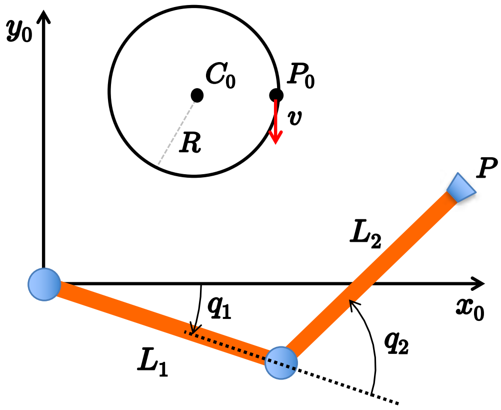


Figure 2: A planar 2R robot and its nominal end-effector trajectory.

Assuming that the robot is commanded by the joint velocity $\dot{\boldsymbol{q}}$, define a single control law that guarantees the following properties:

- when the initial robot configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ at $t = 0$ is matched with the Cartesian point $\boldsymbol{P}_0$, there is a perfect reproduction of the desired trajectory for all $t \geq 0$;
- if there is no such initial matching, the Cartesian trajectory tracking error will converge to zero exponentially and in a decoupled way with respect to its components expressed in a reference frame $RF_r(t) = (\boldsymbol{x}_r(t), \boldsymbol{y}_r(t))$ that is moving with the desired position and has the axis $\boldsymbol{x}_r(t)$ always tangent to the path.

a. Using next the following numerical data

$$L_1 = L_2 = 0.5, \qquad \boldsymbol{C}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \qquad R = 0.15 \ \text{[m]}, \qquad v = 3 \ \text{[m/s]},$$

determine the value $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ of an initial configuration and the value of the commanded velocity $\dot{\boldsymbol{q}}(0)$ at $t = 0$ that are needed for perfect reproduction of the desired trajectory.

b. In addition, with the robot in the initial configuration

$$\boldsymbol{q}_{\text{off}} = \begin{pmatrix} 0 \\ \pi/6 \end{pmatrix} \ \text{[rad]} \ \neq \boldsymbol{q}_0,$$

using the two time constants $\tau_{r,x} = 0.1$ and $\tau_{r,y} = 0.05$ [s] for the desired exponential transients of the trajectory tracking error components in the frame $RF_r(t)$, determine the initial value $\dot{\boldsymbol{q}}(0)$ of the control law that satisfies the above mentioned properties.

[210 minutes, open books]

Solution

February 5, 2019

Exercise 1

The robot is a modified (imaginary) version of the Franka Emika, with 4 degrees of freedom only. A possible DH frame assignment is shown in Fig. 3, with the associated parameters given in Tab. 1. The signs of the non-zero symbolic constants are also reported in the table.

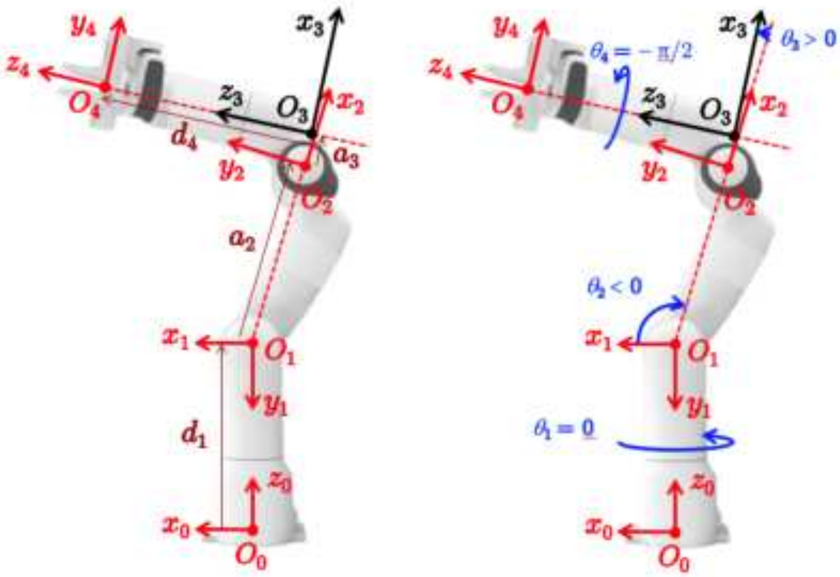


Figure 3: A possible DH frame assignment for the 4-dof robot of Fig. 1. Constant parameters are shown on the left and joint variables shown on the right.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	$-\pi/2$	$a_3 > 0$	0	q_3
4	0	0	$d_4 > 0$	q_4

Table 1: Parameters associated to the DH frames in Fig. 3.

Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & -\sin q_1 & 0 \\ \sin q_1 & 0 & \cos q_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2(q_2) \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2\cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^0\boldsymbol{R}_3(q_3) & {}^2\boldsymbol{p}_3(q_3) \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_3 & 0 & -\sin q_3 & a_3\cos q_3 \\ \sin q_3 & 0 & \cos q_3 & a_3 \sin q_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4 \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 & 0 \\ \sin q_4 & \cos q_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{p}_4(\boldsymbol{q}) \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] \\
&= \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3) - d_4 \sin(q_2+q_3)\right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3) - d_4 \sin(q_2+q_3)\right) \\ d_1 - a_2 \sin q_2 - a_3 \sin(q_2+q_3) - d_4 \cos(q_2+q_3) \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}.
\end{aligned} \tag{1}
$$

From the DH rules of frame assignment, we could eliminate two parameters by setting them to zero, i.e., $d_1 = 0$ and $d_4 = 0$, with simplifications in (1). The consequences would be that:

- all position vectors computed through the direct kinematics would be expressed with respect to a frame $RF_{0'}$ oriented like the original frame 0, but placed at the robot shoulder;
- the position of the origin O_4 of the original frame 4 at the robot end-effector would be given in the new frame $4'$ by
$$
{}^{4'}\boldsymbol{p}_{O_4} = \begin{pmatrix} 0 & 0 & d_4 \end{pmatrix}^T.
$$

Figure 4 shows the robot in the stretched upward configuration $\boldsymbol{q}_s = \begin{pmatrix} * & -\pi/2 & -\pi/2 & * \end{pmatrix}^T$, where '$*$' could be any value. Taking $* = 0$, the end-effector position is evaluated from (1) as

$$
\boldsymbol{p}_s = \boldsymbol{p}_4(\boldsymbol{q}_s) = \begin{pmatrix} -a_3 \\ 0 \\ d_1 + a_2 + d_4 \end{pmatrix}.
$$

Figure 4: The robot in a stretched configuration.

Finally, the absolute position of the TCP in the configuration $\boldsymbol{q}_0 = \mathbf{0}$, given its numerical value ${}^4\boldsymbol{p}_{4,TCP}$ in the end-effector frame, is computed as

$$
\begin{pmatrix} \boldsymbol{p}_{4,TCP}(\boldsymbol{q}_0) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(0) \left[{}^1\boldsymbol{A}_2(0) \left[{}^2\boldsymbol{A}_3(0) \left[{}^3\boldsymbol{A}_4(0) \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 1 \end{pmatrix} \right] \right] \right] = \begin{pmatrix} a_2 + a_3 \\ -0.1 \\ d_1 - d_4 - 0.2 \\ 1 \end{pmatrix} \ [\text{m}].
$$

Exercise 2

In order to derive the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

of the spatial 4-dof robot of Fig. 1, the simplest way is to compute the 3×4 upper block $\boldsymbol{J}_L(\boldsymbol{q})$ by partial differentiation of the position vector $\boldsymbol{p}_4(\boldsymbol{q})$ in eq. (1), and the 3×4 lower block $\boldsymbol{J}_A(\boldsymbol{q})$ by using the standard formulas. From eq. (1), we obtain

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(a_2c_2 + a_3c_{23} - d_4s_{23}) & -c_1(a_2s_2 + a_3s_{23} + d_4c_{23}) & -c_1(a_3s_{23} + d_4c_{23}) & 0 \\ c_1(a_2c_2 + a_3c_{23} - d_4s_{23}) & -s_1(a_2s_2 + a_3s_{23} + d_4c_{23}) & -s_1(a_3s_{23} + d_4c_{23}) & 0 \\ 0 & d_4s_{23} - a_3c_{23} - a_2c_2 & d_4s_{23} - a_3c_{23} & 0 \end{pmatrix}. \tag{2}$$

where the usual compact notation has been used for trigonometric functions (e.g., $c_{23} = \cos(q_2 + q_3)$). For later analysis, it is also convenient to express the Jacobian in the rotated frame 1, or

$${}^1\boldsymbol{J}_L(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -(a_2s_2 + a_3s_{23} + d_4c_{23}) & -(a_3s_{23} + d_4c_{23}) & 0 \\ 0 & a_2c_2 + a_3c_{23} - d_4s_{23} & a_3c_{23} - d_4s_{23} & 0 \\ a_2c_2 + a_3c_{23} - d_4s_{23} & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Further, being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for all i, we have

$$\begin{aligned} \boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_2(q_1,q_2){}^2\boldsymbol{z}_2 & {}^0\boldsymbol{R}_3(q_1,q_2,q_3){}^3\boldsymbol{z}_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -s_1 & -s_1 & -c_1s_{23} \\ 0 & c_1 & c_1 & -s_1s_{23} \\ 1 & 0 & 0 & -c_{23} \end{pmatrix}, \end{aligned} \tag{4}$$

where ${}^0\boldsymbol{R}_j(q_1,\dots,q_j) = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2)\dots{}^{j-1}\boldsymbol{R}_j(q_j)$, for $j \geq 1$.

We immediately see that the last column of the two 3×4 Jacobian matrices in (2) and (3) is identically zero. Thus, the rank of these matrices will drop from the maximum value 3 if and only if the determinant of the first 3×3 square blocks (denoted with an additional bar, i.e., $\bar{\boldsymbol{J}}$) is zero. Using (3), we have

$$\det \bar{\boldsymbol{J}}_L(\boldsymbol{q}) = \det {}^1\bar{\boldsymbol{J}}_L(\boldsymbol{q}) = a_2\,(a_3s_3 + d_4c_3)\,(a_2c_2 + a_3c_{23} - d_4s_{23}). \tag{5}$$

Singularities of $\boldsymbol{J}_L(\boldsymbol{q})$ occur when one (or both) of the factors in the right-hand side of (5) is zero. The first factor depends only on q_3 and vanishes when the forearm is 'almost' stretched or folded[1]. Actually, being in practice $a_3 \ll d_4$, the roots are relatively close to $q_3 = \pm\pi/2$, where $c_3 \simeq 0$). Indeed, according to Fig. 3, only the stretched configuration (corresponding to the negative solution for q_3) is of interest: the other would lead to a self-collision between link 2 and link 3. The second factor in the right-hand side of (5) vanishes when the origin O_4 lies on axis of joint 1. In fact, from eq. (1) we have that

$$|a_2c_2 + a_3c_{23} - d_4s_{23}| = \sqrt{p_x^2 + p_y^2},$$

namely, the distance from the axis $\boldsymbol{z}_0$ of the origin of frame 4 on the robot end-effector.

[1] In general, the solutions of the trigonometric equation $a_3 \sin q_3 + d_4 \cos q_3 = 0$ can be found by the algebraic substitution $q_3 = \tan(x/2)$, which converts the problem into that of finding the roots of a quadratic polynomial (with a number of special cases). For instance, if $a_3 = 0.1$ and $d_4 = 0.5$, we find the two solutions $q_3^+ = 1.768$ and $q_3^- = -1.374$ [rad]. For much smaller values of the ratio a_3/d_4, the two roots converge to $q_3^\pm = \pm 1.57 = \pm\pi/2$ [rad].

Singularities of the lower part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of the robot are simpler to determine. Discarding the third column in (4), which is identical to the second one, a singularity will occur if and only if the determinant of the remaining matrix (denoted again with an additional bar) will vanish, i.e.,

$$\det \bar{\boldsymbol{J}}_A(\boldsymbol{q}) = s_{23} = 0. \tag{6}$$

When this happens, feasible angular velocities of the robot end-effector are characterized by

$$\boldsymbol{\omega} \in \mathcal{R}\{\boldsymbol{J}_A(\boldsymbol{q})\}_{s_{23}=0} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \right\}.$$

In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, the linear part of the geometric Jacobian is evaluated from (2) as

$$\boldsymbol{J}_{L,0} = \boldsymbol{J}_L(\boldsymbol{0}) = \begin{pmatrix} 0 & -d_4 & -d_4 & 0 \\ a_2 + a_3 & 0 & 0 & 0 \\ 0 & -(a_2+a_3) & -a_3 & 0 \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0} & \boldsymbol{0} \end{pmatrix}, \tag{7}$$

where we have partitioned the first three columns from the fourth (zero) column. It is easy to see that $\boldsymbol{J}_{L,0}$ (namely $\bar{\boldsymbol{J}}_{L,0}$) has full rank equal to 3. Therefore, any Cartesian linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ can be instantaneously realized by the robot in the given configuration. Moreover, joint velocities that lie in the null space of $\boldsymbol{J}_{L,0}$ take the form

$$\dot{\boldsymbol{q}}_a = \rho \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for some } \rho \lesseqgtr 0. \tag{8}$$

As a result, the problem of realizing the given linear Cartesian velocity $\boldsymbol{v}_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ has an infinite number of solutions. Among these, the joint velocity solution of minimum norm is given by

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{L,0}^{\#}\boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0} & \boldsymbol{0} \end{pmatrix}^{\#} \boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0}^{\#} \\ \boldsymbol{0}^T \end{pmatrix} \boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0}^{-1} \\ \boldsymbol{0}^T \end{pmatrix} \boldsymbol{v}_b \\ &= \begin{pmatrix} 0 & \dfrac{1}{a_2+a_3} & 0 \\ \dfrac{a_3}{a_2 d_4} & 0 & -\dfrac{1}{a_2} \\ -\dfrac{a_2+a_3}{a_2 d_4} & 0 & \dfrac{1}{a_2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{a_3}{a_2 d_4} - \dfrac{1}{a_2} \\ \dfrac{1}{a_2} - \dfrac{a_2+a_3}{a_2 d_4} \\ 0 \end{pmatrix}, \end{aligned} \tag{9}$$

where we have used the properties of pseudoinverse matrices. All joint velocities that provide zero Cartesian velocity error (i.e., such that $\boldsymbol{J}_{L,0}\,\dot{\boldsymbol{q}} = \boldsymbol{v}_b$) are obtained by adding to the particular solution (9) a null-space joint velocity (8) for some scalar ρ:

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_b + \dot{\boldsymbol{q}}_a = \begin{pmatrix} 0 \\ \dfrac{a_3}{a_2 d_4} - \dfrac{1}{a_2} \\ \dfrac{1}{a_2} - \dfrac{a_2+a_3}{a_2 d_4} \\ \rho \end{pmatrix}.$$

Exercise 3

The reference trajectory $\boldsymbol{p}_d(t)$ in the Cartesian plane is a circular path $\boldsymbol{p}_d(s)$ traced clockwise with a linear speed $v(t) \geq 0$, and is specified thus as

$$\boldsymbol{p}_d(s) = \boldsymbol{C}_0 + R \begin{pmatrix} \cos \dfrac{s}{R} \\ -\sin \dfrac{s}{R} \end{pmatrix}, \quad s \geq 0, \qquad s(t) = \int_0^t v(\tau) d\tau, \quad t \geq 0, \tag{10}$$

satisfying $s(0) = 0$ and $\boldsymbol{p}_d(0) = \boldsymbol{P}_0$. Note that the path is parametrized in this case by the arc length s, and that the minus sign on the second component of the vector in (10) accounts for the clockwise tracing of the circle. From (10), it follows

$$\dot{\boldsymbol{p}}_d(t) = \frac{d\boldsymbol{p}(s)}{ds}\, \dot{s}(t) = -v(t) \begin{pmatrix} \sin \dfrac{s(t)}{R} \\ \cos \dfrac{s(t)}{R} \end{pmatrix}, \quad t \geq 0, \tag{11}$$

providing $\|\dot{\boldsymbol{p}}_d(t)\| = v(t)$.

The robot direct and differential kinematics are given by

$$\boldsymbol{p} = \begin{pmatrix} L_1 \cos q_1 + L_2 \cos(q_1 + q_2) \\ L_1 \sin q_1 + L_2 \sin(q_1 + q_2) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) \tag{12}$$

and

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\, \dot{\boldsymbol{q}} = \begin{pmatrix} -\left(L_1 \sin q_1 + L_2 \sin(q_1 + q_2)\right) & -L_2 \sin(q_1 + q_2) \\ L_1 \cos q_1 + L_2 \cos(q_1 + q_2) & L_2 \cos(q_1 + q_2) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{13}$$

In nominal conditions, when the initial robot configuration $\boldsymbol{q}_0$ is matched with the Cartesian point $\boldsymbol{P}_0$, i.e., $\boldsymbol{f}(\boldsymbol{q}_0) = \boldsymbol{p}_d(0) = \boldsymbol{P}_0$, the joint velocity command that guarantees perfect reproduction of the desired trajectory $\boldsymbol{p}_d(t)$ for all $t \geq 0$ is given just by the feedforward command

$$\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{p}}_d(t), \qquad \text{with} \quad \boldsymbol{q}_d(t) = \boldsymbol{q}_0 + \int_0^t \dot{\boldsymbol{q}}_d(\tau) d\tau. \tag{14}$$

On the other hand, if there is an initial Cartesian error $\boldsymbol{e}_0 = \boldsymbol{P}_0 - \boldsymbol{f}(\boldsymbol{q}_0) \neq \boldsymbol{0}$, a feedback control action is needed. Let the Cartesian trajectory tracking error be

$$\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad t \geq 0, \tag{15}$$

with $\boldsymbol{p}(t) = \boldsymbol{f}(\boldsymbol{q}(t))$ from eq. (12). The error vector $\boldsymbol{e} \in \mathbb{R}^2$ can be expressed in a rotated (planar and right-handed) frame $RF_r(t)$, having the origin attached to the desired Cartesian position $\boldsymbol{p}_d(t)$ of the robot end-effector and the $\boldsymbol{x}_r(t)$-axis pointing along the (positive) tangent direction to the path, see Fig. 5. Define the rotated tracking error as

$${}^{R}\boldsymbol{e}(t) = \boldsymbol{R}(t)\boldsymbol{e}(t), \qquad \text{with} \quad \boldsymbol{R}(t) = \begin{pmatrix} -\sin \dfrac{s(t)}{R} & \cos \dfrac{s(t)}{R} \\ -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_r(t) & \boldsymbol{y}_r(t) \end{pmatrix}. \tag{16}$$

Note that the time derivative of the (planar) rotation matrix $\boldsymbol{R}(t)$ in (16) is

$$\begin{aligned} \dot{\boldsymbol{R}}(t) &= \begin{pmatrix} -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \\ \sin \dfrac{s(t)}{R} & -\cos \dfrac{s(t)}{R} \end{pmatrix} \frac{\dot{s}(t)}{R} = \begin{pmatrix} -\sin \dfrac{s(t)}{R} & \cos \dfrac{s(t)}{R} \\ -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{\dot{s}(t)}{R} \\ -\dfrac{\dot{s}(t)}{R} & 0 \end{pmatrix} \\ &= \boldsymbol{R}(t)\, \boldsymbol{S}(\omega(t)), \qquad \text{with} \quad \omega(t) = \frac{\dot{s}(t)}{R}, \end{aligned} \tag{17}$$

where $\boldsymbol{S}(.)$ is a (planar) skew-symmetric matrix[2].

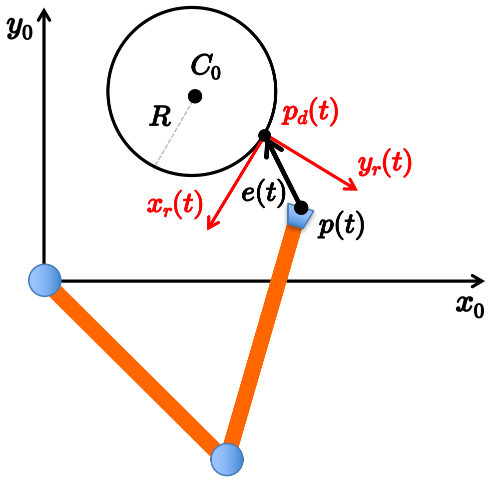


Figure 5: The reference frame $RF_r(t)$ used in the definition of the rotated tracking error.

The target dynamics of the rotated tracking error ${}^R\boldsymbol{e}$ is specified by

$$ {}^R\dot{\boldsymbol{e}} = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} {}^R\boldsymbol{e} = -\boldsymbol{K}\,{}^R\boldsymbol{e}, \qquad \text{with} \quad k_x > 0,\, k_y > 0, \tag{18} $$

namely, as a linear and decoupled behavior along the axes of the rotated frame, with errors exponentially converging to zero:

$$ {}^Re_x(t) = {}^Re_x(0)\exp(-k_x t), \qquad {}^Re_y(t) = {}^Re_y(0)\exp(-k_y t). \tag{19} $$

In these exponential evolutions, the time constants are the inverse of the gains: $\tau_{r,x} = 1/k_x$, $\tau_{r,y} = 1/k_y$. Since

$$ {}^R\dot{\boldsymbol{e}} = \boldsymbol{R}\,\dot{\boldsymbol{e}} + \dot{\boldsymbol{R}}\,\boldsymbol{e} = \boldsymbol{R}\left(\dot{\boldsymbol{e}} + \boldsymbol{S}\boldsymbol{e}\right) = \boldsymbol{R}\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{S}\boldsymbol{e}\right), $$

in order to obtain (18), the control law should be speficied as

$$ \begin{aligned} \dot{\boldsymbol{q}}(t) &= \boldsymbol{J}^{-1}(\boldsymbol{q}(t))\left(\dot{\boldsymbol{p}}_d(t) + \boldsymbol{S}(\omega(t))\boldsymbol{e}(t) + \boldsymbol{R}^T(t)\boldsymbol{K}\,{}^R\boldsymbol{e}(t)\right) \\ &= \boldsymbol{J}^{-1}(\boldsymbol{q}(t))\left(\dot{\boldsymbol{p}}_d(t) + \left(\boldsymbol{S}(\omega(t)) + \boldsymbol{R}^T(t)\boldsymbol{K}\boldsymbol{R}(t)\right)\left(\boldsymbol{p}_d(t) - \boldsymbol{f}(\boldsymbol{q}(t))\right)\right). \end{aligned} \tag{20} $$

We note that the time-varying gain matrix $\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}$ is required in order to obtain a constant and decoupled error dynamics in the rotated frame. Also, when the initial tracking error is zero, one has $\boldsymbol{q}(t) = \boldsymbol{q}_d(t)$ for all $t \geq 0$, and the control law (20) collapses into a feedforward command only, as given by (14).

We move next to the application of the above formulas with the numerical data given for the problem. An initially matched robot configuration $\boldsymbol{q}_0$ is determined by the desired initial Cartesian point $\boldsymbol{P}_0$ and the robot inverse kinematics. From the center $\boldsymbol{C}_0$ of the circular path and its radius R, we have

$$ \boldsymbol{C}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \quad R = 0.15 \qquad \Rightarrow \qquad \boldsymbol{P}_0 = \begin{pmatrix} 0.35 \\ 0.3 \end{pmatrix} \ [\mathrm{m}]. $$

From the inverse kinematics equations of a planar 2R robot,

$$ c_2 = \frac{P_{0,x}^2 + P_{0,y}^2 - L_1^2 - L_2^2}{2L_1L_2}, \qquad s_2 = \pm\sqrt{1 - c_2^2} \qquad \Rightarrow \qquad q_2 = \mathrm{ATAN2}\{s_2, c_2\}, $$

[2]If the path were linear, the orientation of the frame attached to the path would be constant and so $\boldsymbol{R}$. Being $\dot{\boldsymbol{R}} = \boldsymbol{0}$, also $\boldsymbol{S}$ would vanish in the following formulas.

and

$$c_1 = \frac{P_{0,x}(L_1 + L_2 c_2) + P_{0,y} L_2 s_2}{L_1^+ L_2^2 + 2 L_1 L_2 c_2}, \quad s_1 = \frac{P_{0,y}(L_1 + L_2 c_2) - P_{0,x} L_2 s_2}{L_1^+ L_2^2 + 2 L_1 L_2 c_2} \qquad \Rightarrow \qquad q_1 = \mathrm{ATAN2}\{s_1, c_1\},$$

using also the robot link lengths $L_1 = L_2 = 0.5$ [m], we obtain the two solutions

$$\boldsymbol{q}_0^{up} = \begin{pmatrix} q_1^{up} \\ q_2^{up} \end{pmatrix} = \begin{pmatrix} 1.8003 \\ -2.1834 \end{pmatrix} \text{[rad]}, \qquad \boldsymbol{q}_0^{down} = \begin{pmatrix} q_1^{down} \\ q_2^{down} \end{pmatrix} = \begin{pmatrix} -0.3831 \\ 2.1834 \end{pmatrix} \text{[rad]}. \tag{21}$$

With these configurations and the desired initial speed $v = 3$ [m/s], we obtain from (13) and (14) the two alternative initial commanded velocities

$$\dot{\boldsymbol{q}}^{up}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_0^{up})\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 7.6055 \\ -12.2087 \end{pmatrix} \text{[rad/s]}, \quad \dot{\boldsymbol{q}}^{down}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_0^{down})\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -10.7179 \\ -6.6039 \end{pmatrix} \text{[rad/s]}. \tag{22}$$

These joint velocities will achieve perfect reproduction of the desired trajectory at $t = 0$. The values in (22) are relatively large because of the large speed v that is being requested at the Cartesian level.

Instead, when the robot is initially in

$$\boldsymbol{q}_{\text{off}} = \begin{pmatrix} 0 \\ \pi/6 \end{pmatrix} \text{[rad]} \qquad \Rightarrow \qquad \boldsymbol{e}_0 = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}_{\text{off}}) = \begin{pmatrix} -0.583 \\ 0.05 \end{pmatrix} \text{[m]},$$

we need to use the control law (20) at $t = 0$. Therein, considering also the desired time constants $\tau_{r,x} = 0.1$ and $\tau_{r,y} = 0.05$ [s], the diagonal gain matrix $\boldsymbol{K}$, the rotation matrix $\boldsymbol{R}(0)$, the effective gain matrix $\boldsymbol{R}_{\text{eff}}(0) = \boldsymbol{R}^T(0)\boldsymbol{K}\boldsymbol{R}(0)$, and the skew-symmetric matrix $\boldsymbol{S}(\omega(0))$ are evaluated as

$$\boldsymbol{K} = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \qquad \boldsymbol{R}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\boldsymbol{R}_{\text{eff}}(0) = \begin{pmatrix} 20 & 20 \\ -20 & 10 \end{pmatrix}, \qquad \boldsymbol{S}(\omega(0)) = \begin{pmatrix} 0 & 20 \\ -20 & 0 \end{pmatrix}.$$

As a result, the joint velocity control that will be applied at the initial instant will be

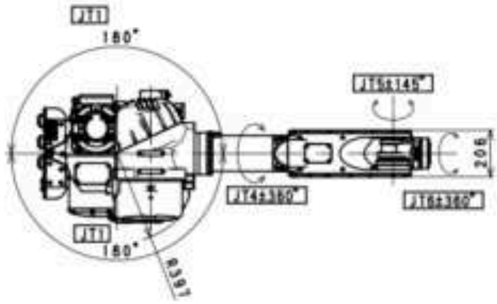
$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 20.2224 \\ -22.4186 \end{pmatrix} \text{[rad/s]}. \tag{23}$$

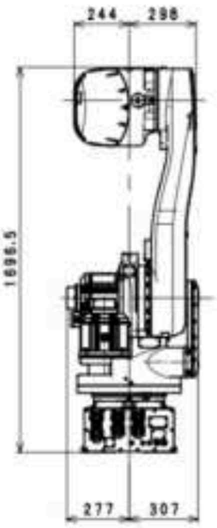
These are indeed extremely large values. However, they are needed in order to obtain the specified fast rate of exponential decay for the trajectory tracking error (the time constants are too small, and could be possibly increased to obtain smaller values in (23)).

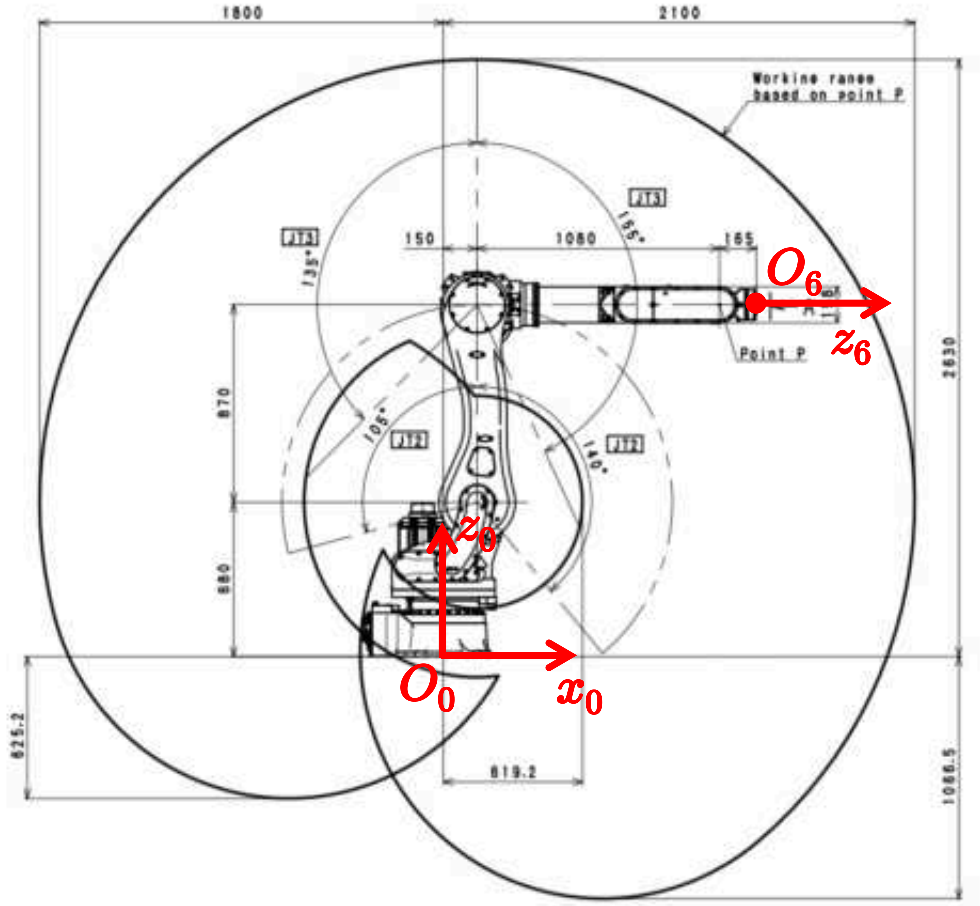
* * * * *

Kawasaki S030 robot – DH frames assignment and table

Name: ____



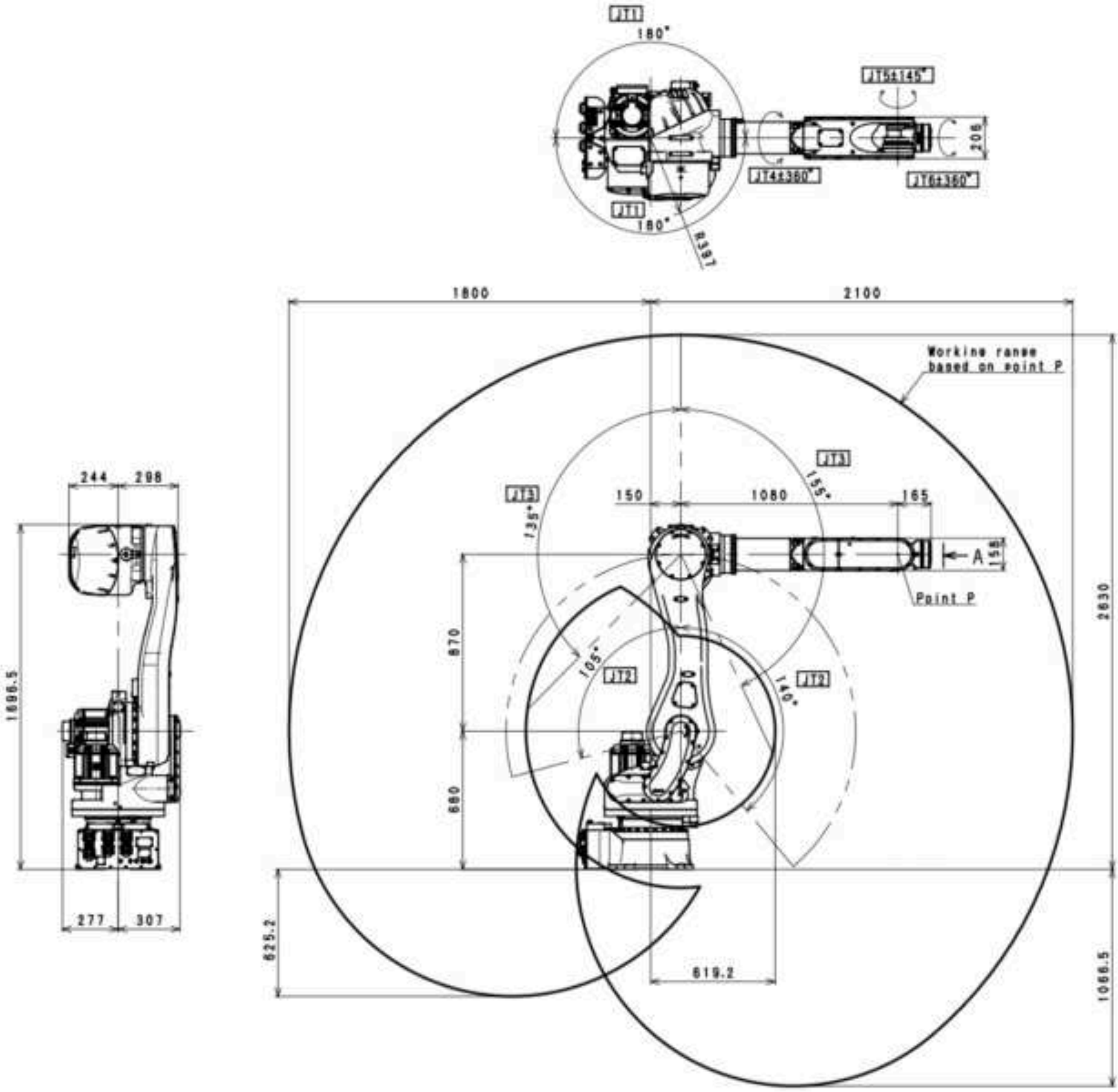




i	α_i	a_i	d_i	θ_i	$\theta_{i,\min}$	$\theta_{i,\max}$
1						
2						
3						
4						
5						
6						

Robotics I – Data sheet
June 17, 2019

Kawasaki S030 robot workspace and data



[1] Robot Arm			
1. Model	RS030N-B		
2. Type	Articulated robot		
3. Degree of freedom	6 axes		
4. Axis specification	Operating axis	Max. operating range	Max. speed
	Arm rotation (JT1)	+180° ～ −180°	180°/s
	Arm out-in (JT2)	+140° ～ −105°	180°/s
	Arm up-down (JT3)	+135° ～ −155°	185°/s
	Wrist swivel (JT4)	+360° ～ −360°	260°/s
	Wrist bend (JT5)	+145° ～ −145°	260°/s
	Wrist twist (JT6)	+360° ～ −360°	360°/s

Robotics I

June 17, 2019

Exercise 1

Consider the Kawasaki robot S030 shown in Fig. 1, having six revolute joints. The geometric dimensions of the robot workspace are reported in the distributed **data** sheet, together with the joint ranges defined according to the manufacturer's convention and the maximum joint speeds.

Figure 1: The 6R Kawasaki S030 robot.

a. Assign the link frames according to a Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all twist angles α_i, for $i = 1, \dots, 6$, are either 0 or $+\pi/2$. Draw the frames and fill in the table on the **reply** sheet provided separately. As shown there, frame 0 is on the floor, with $\boldsymbol{z}_0$ pointing upward, while the sixth frame is at the center of the end-effector flange, with the $\boldsymbol{z}_6$ axis in the approach direction. Specify in the table the numerical value of all constant parameters.

b. Determine the actual joint ranges (lower and upper bounds, $\theta_{i,\min}$ and $\theta_{i,\max}$ for $i = 1, \dots, 6$) according to the DH convention you have defined. Specify also the numerical value $\boldsymbol{\theta}_n \in \mathbb{R}^6$ of the joint variables $\boldsymbol{\theta}$ when the robot is in the configuration shown in the **data** sheet.

c. Compute the symbolic expression of the position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta})$ of point P (center of the spherical wrist) of the robot. Provide its numerical value of $\boldsymbol{p}_n$ when the robot is in the configuration $\boldsymbol{\theta}_n$.

d. For the same value $\boldsymbol{p} = \boldsymbol{p}_n$, determine *all* possible inverse kinematic solutions for the first three joints $\boldsymbol{\theta}_b = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}^T$ of the robot that are feasible with respect to the available joint ranges.

e. Determine the value $\boldsymbol{v}_P \in \mathbb{R}^3$ of the velocity of point P, expressed in the robot base frame, when $\boldsymbol{\theta} = \boldsymbol{\theta}_n$ and the joints have their maximum *positive* speed.

f. Which are the singularities of the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta}_b)$ relating the velocity $\dot{\boldsymbol{\theta}}_b \in \mathbb{R}^3$ of the first three joints ot the linear velocity $\boldsymbol{v}_P \in \mathbb{R}^3$ of point P?

Exercise 2

The desired linear velocity of the end-effector $\boldsymbol{v} \in \mathbb{R}^m$ (with $m = 2$ or 3, in the 2D- or 3D-case) of a robot with n joints is usually defined, at the current configuration, in one of three possible ways: at the joint level, in the base frame B, or in the end-effector/tool frame E. Discuss the pros and cons of these choices and how they relate each to other. Comment also on what may happen when $n < m$, $n = m$, or $n > m$.

With reference to a planar $2R$ robot, with link lengths $l_1 = 0.5$, $l_2 = 0.25$ [m] and in the configuration $\theta_1 = \pi/4$, $\theta_2 = -\pi/2$, provide numerical answers to the following questions:

a. For $\dot{\boldsymbol{\theta}} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ [rad/s], compute ${}^B\boldsymbol{v}$ and ${}^E\boldsymbol{v}$ (both vectors are in $\mathbb{R}^2$);

b. Compute $\dot{\boldsymbol{\theta}} \in \mathbb{R}^2$ when ${}^B\boldsymbol{v}$ or, respectively, ${}^E\boldsymbol{v}$ take the value $\boldsymbol{v} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [m/s].

[180 minutes, open books]

Solution

June 17, 2019

Exercise 1

A possible DH frame assignment for the Kawasaki S030 robot, which satisfies the required conditions on the twist angles α_i, $i = 1, \ldots, 6$, is shown in Fig. 2. The associated parameters are given in Tab. 1, where the numerical values read from the workspace dimensions on the robot data sheet are also reported. The numerical values of the variables q_i refer to the robot configuration shown in Fig. 2. The table gives also the joint ranges obtained from the robot manufacturer's data, once working with the chosen DH convention.

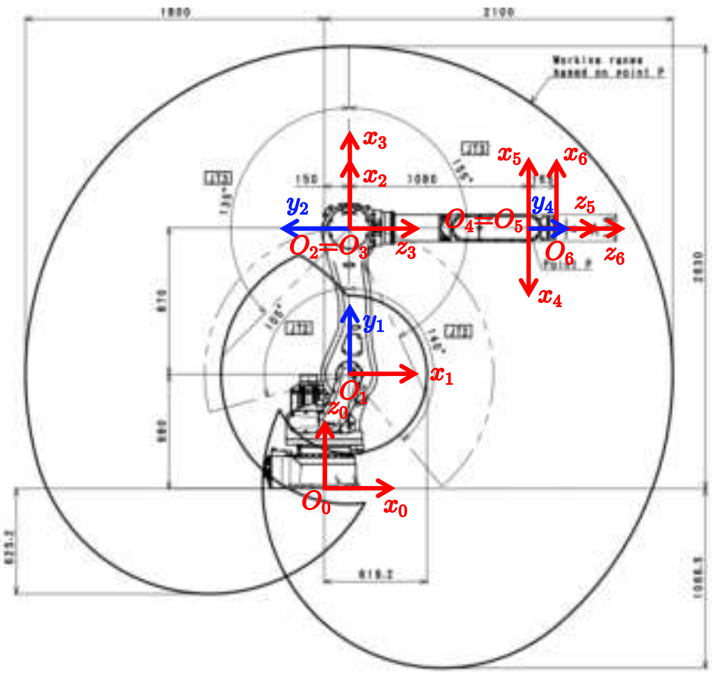


Figure 2: A possible DH frame assignment for the Kawasaki S030 robot.

i	α_i	a_i	d_i	θ_i	$\theta_{i,\min}$	$\theta_{i,\max}$
1	$\pi/2$	$a_1 = 150$	$d_1 = 680$	$q_1 = 0$	$-\pi$	$+\pi$
2	0	$a_2 = 870$	0	$q_2 = \pi/2$	$-50 \cdot (\pi/180)$	$+195 \cdot (\pi/180)$
3	$\pi/2$	0	0	$q_3 = 0$	$-65 \cdot (\pi/180)$	$+225 \cdot (\pi/180)$
4	$\pi/2$	0	$d_4 = 1080$	$q_4 = \pi$	-2π	$+2\pi$
5	$\pi/2$	0	0	$q_5 = \pi$	$+35 \cdot (\pi/180)$	$+325 \cdot (\pi/180)$
6	0	0	$d_4 = 165$	$q_6 = 0$	-2π	$+2\pi$

Table 1: Parameters associated to the DH frames in Fig. 2. Lengths are in [mm], angles in [rad].

Based on Tab. 1, in order to determine the symbolic expression of the position of the center of the spherical wrist (point P), we just need to compute the first four DH homogeneous transformation matrices:

$$ {}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & a_1 \cos q_1 \\ \sin q_1 & 0 & -\cos q_1 & a_1 \sin q_1 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & 0 \\ \sin q_3 & 0 & -\cos q_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

To obtain the position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ (or $= \boldsymbol{f}(\boldsymbol{\theta})$), we make use of the matrix-vector product computations in homogeneous coordinates as

$$ \begin{aligned} \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]\right]\right] \\ &= \begin{pmatrix} \cos q_1 \left(a_1 + a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right) \\ \sin q_1 \left(a_1 + a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right) \\ d_1 + a_2 \sin q_2 - d_4 \cos(q_2 + q_3) \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}. \end{aligned} \tag{1} $$

Plugging in (1) the numerical values from Tab. 1 for $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$, i.e., for the first three components of $\boldsymbol{q}_n = \boldsymbol{\theta}_n$, as well as for a_1, a_2, d_1 and d_4, we obtain

$$ \boldsymbol{p}_n = \boldsymbol{f}(\boldsymbol{q}_n) = \begin{pmatrix} 1.23 \\ 0 \\ 1.55 \end{pmatrix} \ [\text{m}]. \tag{2} $$

For a given desired position $\boldsymbol{p} = \boldsymbol{p}_n$ of point P, the inverse kinematics problem requires solving the nonlinear equations of the direct mapping (1) in terms of the unknown $\boldsymbol{q}_b = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$. This is done by inspection.

First, since $p_y/p_x = \tan q_1$, two solutions are found in the four quadrants for the base joint q_1 by choosing

$$ q_{1,[f]} = \text{ATAN2}\{p_y, p_x\} \qquad \text{and} \qquad q_{1,[b]} = \text{ATAN2}\{-p_y, -p_x\}, \tag{3} $$

corresponding to the robot facing the desired Cartesian position for P with its *front* or with its *back* side.

Next, we sum the first two scalar equations in (1) multiplied by $\sin q_1$ and $\cos q_1$ respectively, reorganize terms and square, and then add this to the third equation, also reorganized and squared, obtaining

$$ \begin{aligned} (p_x \cos q_1 + p_y \sin q_1 - a_1)^2 + (p_z - d_1)^2 &= \left(a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right)^2 + \left(a_2 \sin q_2 - d_4 \cos(q_2 + q_3)\right)^2 \\ &= a_2^2 + d_4^2 + 2 a_2 d_4 \left(\sin(q_2 + q_3) \cos q_2 - \cos(q_2 + q_3) \sin q_2\right) \\ &= a_2^2 + d_4^2 + 2 a_2 d_4 \sin q_3. \end{aligned} $$

From this, for each value substituted from (3), we compute the quantities

$$s_{3,[f,b]} = \frac{(p_x \cos q_{1,[f,b]} + p_y \sin q_{1,[f,b]} - a_1)^2 + (p_z - d_1)^2 - a_2^2 + d_4^2}{2a_2 d_4}$$

and

$$c_{3,[f,b]} = \pm\sqrt{1 - s_{3,[f,b]}^2}\,.$$

Combining these two expressions, once with the sign $+$ and the other with the sign $-$ before the square root (and, in each case, using the two values labeled f or b in the evaluation of q_1), four different solutions are found for the elbow joint q_3, i.e.,

$$q_{3,[f,b;+]} = \text{ATAN2}\left\{s_{3,[f,b]}, +\left|c_{3,[f,b]}\right|\right\} \qquad \text{and} \qquad q_{3,[f,b;-]} = \text{ATAN2}\left\{s_{3,[f,b]}, -\left|c_{3,[f,b]}\right|\right\}. \tag{4}$$

Finally, we consider again the sum of the first two scalar equations in (1) multiplied by $\sin q_1$ and $\cos q_1$, respectively, and the third one rearranged, and expand then $\sin(q_2 + q_3)$ and $\cos(q_2 + q_3)$. We isolate in this way the yet unknown trigonometric expressions $\sin q_2$ and $\cos q_2$ in the two resulting (linear) equations

$$\begin{aligned} a_2 \cos q_2 + d_4(\sin q_2 \cos q_3 + \cos q_2 \sin q_3) &= p_x \cos q_1 + p_y \sin q_1 - a_1 \\ a_2 \sin q_2 - d_4(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) &= p_z - d_1, \end{aligned}$$

or, in matrix form

$$\begin{pmatrix} a_2 + d_4 \sin q_3 & d_4 \cos q_3 \\ -d_4 \cos q_3 & a_2 + d_4 \sin q_3 \end{pmatrix} \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} = \begin{pmatrix} p_x \cos q_1 + p_y \sin q_1 - a_1 \\ p_z - d_1 \end{pmatrix}. \tag{5}$$

Unless $\det \boldsymbol{A} = a_2^2 + d_4^2 + 2a_2 d_4 \sin q_3 = 0$, which happens if and only if $\sin q_3 = -1$ and $a_2 = d_4$ (thus, never in our case), there is a unique solution to (5) given by

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b}. \tag{6}$$

When evaluating the linear system (5) using the previously obtained results for q_1 and q_3, from (6) we obtain also the associated set of four solutions for the elbow joint variable

$$q_{2,[f,b;+,-]} = \text{ATAN2}\left\{x_{2,[f,b;+,-]}, x_{1,[f,b;+,-]}\right\}. \tag{7}$$

When using the numerical value of $\boldsymbol{p}_n$ in (2), application of the above closed-form formulas (3), (4), and (7) for the inverse kinematics problem yields the four joint configurations

$$\boldsymbol{q}_{f,+} = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 0 \\ 90^\circ \\ 0 \end{pmatrix}, \qquad \boldsymbol{q}_{f,-} = \begin{pmatrix} 0 \\ -0.2146 \\ \pi \end{pmatrix} [\text{rad}] = \begin{pmatrix} 0 \\ -12.29^\circ \\ 180^\circ \end{pmatrix},$$

$$\boldsymbol{q}_{b,+} = \begin{pmatrix} \pi \\ -3.0495 \\ 0.4036 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 180^\circ \\ -174.72^\circ \\ 23.12^\circ \end{pmatrix}, \qquad \boldsymbol{q}_{b,-} = \begin{pmatrix} \pi \\ 1.9245 \\ 2.7380 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 180^\circ \\ 110.27^\circ \\ 156.88^\circ \end{pmatrix}.$$

The first solution $\boldsymbol{q}_{f,+}$ is the one shown in Fig. 2, which was used for computing $\boldsymbol{p}_n$. It is indeed a feasible solution with respect to the available joint ranges of the robot. Two out of the three other inverse kinematic solutions, namely $\boldsymbol{q}_{f,-}$ and $\boldsymbol{q}_{b,-}$, are also feasible. On the other hand, $\boldsymbol{q}_{b,+}$ is unfeasible because it is below the lower limit for joint 2.

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}_b)$ (or $\boldsymbol{J}(\boldsymbol{\theta}_b)$) of interest in the forward differential mapping

$$\boldsymbol{v}_P = \boldsymbol{J}(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_b, \qquad \boldsymbol{q}_b = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$$

is obtained from eq. (1) as

$$J(q_b) = \frac{\partial f(q_b)}{\partial q_b} =$$

$$\begin{pmatrix} -\sin q_1\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & \cos q_1\,(d_4\cos(q_2+q_3) - a_2\sin q_2) & d_4\cos(q_2+q_3)\cos q_1 \\ \cos q_1\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & \sin q_1\,(d_4\cos(q_2+q_3) - a_2\sin q_2) & d_4\cos(q_2+q_3)\sin q_1 \\ 0 & a_2\cos q_2 + d_4\sin(q_2+q_3) & d_4\sin(q_2+q_3) \end{pmatrix}. \tag{8}$$

This matrix can be equivalently expressed in the rotated reference frame 1 as

$$\begin{aligned} {}^1J(q_b) \;&=\; {}^0R_1^T(q_1)\,J(q_b) = \\ &\begin{pmatrix} 0 & d_4\cos(q_2+q_3) - a_2\sin q_2 & d_4\cos(q_2+q_3) \\ 0 & d_4\sin(q_2+q_3) + a_2\cos q_2 & d_4\sin(q_2+q_3) \\ -\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & 0 & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

which is simpler for the investigation of its singularities. In fact, the determinant factorizes as

$$\det J(q_b) = \det {}^1J(q_b) = a_2 d_4 \cos q_3\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3))\,,$$

and the singularities are as follows:

$$\cos q_3 = 0 \quad\Longleftrightarrow\quad q_3 = \pm\frac{\pi}{2} \quad\Longleftrightarrow\quad \text{arm stretched (+) or folded } (-\text{, not in feasible range of joint 3!});$$

$$a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3) = 0 \quad\Longleftrightarrow\quad p_x = p_y = 0 \quad\Longleftrightarrow\quad \text{point } P \text{ is on the axis of joint 1.}$$

When the robot is in the configuration q_n, the Jacobian (8) becomes

$$J_n = J(q_n) = \begin{pmatrix} 0 & -0.87 & 0 \\ 1.23 & 0 & 0 \\ 0 & 1.08 & 1.08 \end{pmatrix}.$$

When applying the maximum positive (according to the counterclockwise convention) speed at the first three joints, we obtain

$$v_P = J_n \cdot \begin{pmatrix} 3.1416 \\ 3.1416 \\ 3.2289 \end{pmatrix} \text{ [rad/s]} = \begin{pmatrix} -2.7332 \\ 3.8642 \\ 6.8801 \end{pmatrix} \text{ [m/s]}.$$

Exercise 2

The linear velocity $v \in \mathbb{R}^m$ (with $m = 2$ in 2D or $m = 3$ in 3D) of the end-effector of a robot with n joints is uniquely specified, at a given configuration q, by the joint velocity vector $\dot{q}$, no matter if n is larger, equal, or smaller than m. Indeed, given a desired v, there exists no solution for $\dot{q}$ if $v \notin \mathcal{R}\{J(q)\}$, being J the $m \times n$ (analytic = geometric) robot Jacobian related to the linear motion of the end-effector. This Jacobian is usually (i.e., by default) expressed in the base frame. If $v \in \mathcal{R}\{J(q)\}$, the solution if unique for $n \le m$, or there is an infinity of joint velocity solutions when $n > m$. The (pseudo-)inversion of the Jacobian matrix may run into trouble around or at a singular configuration. Moreover, the end-effector velocity expressions in the base frame B and in the end-effector/tool frame E are related by

$${}^E v = {}^E R_B\, {}^B v = {}^E R_B\, J(q)\dot{q} = {}^E J(q)\dot{q},$$

with matrix ${}^E R_B \in SO(m)$ and where the robot Jacobian ${}^E J$ is expressed now in the end-effector frame. Representing the vector v in the local end-effector frame E is more useful for visualizing the instantaneous direction of the commanded motion, typically in response to a sensory input at the end-effector level. Other than for this, the two representations are fully equivalent.

For a planar $2R$ robot with link lengths $l_1 = 0.5$, $l_2 = 0.25$ [m], the Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -0.5\sin\theta_1 - 0.25\sin(\theta_1+\theta_2) & -0.25\sin(\theta_1+\theta_2) \\ 0.5\cos\theta_1 + 0.25\cos(\theta_1+\theta_2) & 0.25\cos(\theta_1+\theta_2) \end{pmatrix}.$$

In the configuration $\theta_1 = \pi/4$, $\theta_2 = -\pi/2$, the Jacobian becomes

$$\boldsymbol{J} = \begin{pmatrix} -\sqrt{2}/8 & \sqrt{2}/8 \\ 3\sqrt{2}/8 & \sqrt{2}/8 \end{pmatrix} = \begin{pmatrix} -0.1768 & 0.1768 \\ 0.5303 & 0.1768 \end{pmatrix},$$

which is clearly nonsingular ($\det \boldsymbol{J} = -1/8$). The (planar) rotation matrix from the base to the end-effector frame, once evaluated at the desired configuration, is

$${}^B\boldsymbol{R}_E = \left.\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}\right|_{\substack{\theta_1=\pi/4,\\ \theta_2=-\pi/2}} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{pmatrix}.$$

Therefore, we have the following numerical results.

a. For $\dot{\boldsymbol{\theta}} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ [rad/s], the end-effector velocities (in [m/s]) are

$${}^B\boldsymbol{v} = \boldsymbol{J}\dot{\boldsymbol{\theta}} = \begin{pmatrix} -\sqrt{2}/4 \\ -\sqrt{2}/4 \end{pmatrix} = \begin{pmatrix} -0.3536 \\ -0.3536 \end{pmatrix}, \qquad {}^E\boldsymbol{v} = {}^B\boldsymbol{R}_E^T\,\boldsymbol{J}\dot{\boldsymbol{\theta}} = {}^E\boldsymbol{J}\,\dot{\boldsymbol{\theta}} = \begin{pmatrix} -0.5 & 0 \\ 0.25 & 0.25 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}.$$

b. For the inverse differential problem, the two requested joint velocities (in [rad/s]) are

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\,{}^B\boldsymbol{v} = \begin{pmatrix} -1.4142 & 1.4142 \\ 4.2426 & 1.4142 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}, \qquad \dot{\boldsymbol{\theta}} = {}^E\boldsymbol{J}^{-1}\,{}^E\boldsymbol{v} = \begin{pmatrix} -2 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

* * * * *

Robotics I

July 11, 2019

Exercise 1

Consider the 3R planar robot in Fig. 1. The robot is placed in a vertical plane and holds firmly a payload P, modeled as a concentrated mass m, which is off-centered with respect to its tip (the relevant kinematic data are defined in the figure). Determine the symbolic expression of the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$ needed to keep the system in static equilibrium at a configuration $\boldsymbol{q}_0$, when the position ${}^e\boldsymbol{p}_{ep}$ of the payload is known in the end-effector frame. Using the DH convention for the joint variables, compute the numerical value of $\boldsymbol{\tau}$ for the following data:

$$L_1 = 1,\ L_2 = 0.5,\ L_3 = 0.25,\ {}^e\boldsymbol{p}_{ep} = \begin{pmatrix} 0.2 & 0.3 & 0 \end{pmatrix}^T \text{ [m]};\ \boldsymbol{q}_0 = \begin{pmatrix} \pi/3 & -\pi/6 & -\pi/6 \end{pmatrix}^T \text{ [rad]};\ m = 7 \text{ [kg]}.$$

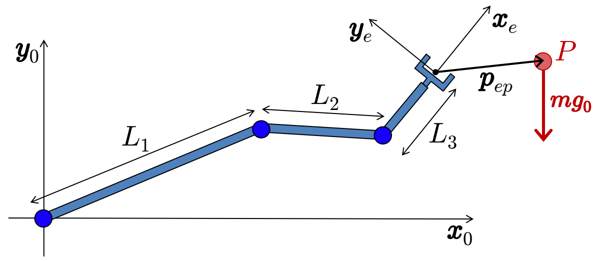


Figure 1: A 3R planar robot holding a payload P of mass m.

Exercise 2

The end effector of the RP planar robot shown in Fig. 2 should trace a linear path between the points $A = \begin{pmatrix} 4.5 & 1.5 \end{pmatrix}^T$ and $B = \begin{pmatrix} 3 & 3 \end{pmatrix}^T$ (units are in [m]), as expressed in a world reference frame ${}^wRF$. The robot has limited joint ranges: $|q_1| \le q_{1,max} = \pi/4$ [rad], $1.5 = q_{2,min} \le q_2 \le q_{2,max} = 3$ [m]. Check if the given task is feasible and, if so, place and orient the robot base frame RF_0 so that the task can be realized. Provide the positions of the two points A and B expressed in the robot base frame and the associated robot configurations.

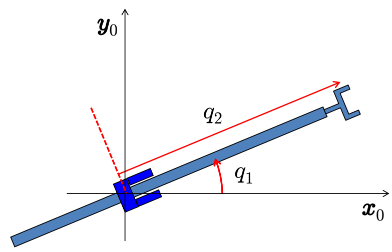


Figure 2: A RP planar robot, with the definition of the joint variables.

Exercise 3

A robot joint should perform a rest-to-rest rotation $\Delta\theta$ in a total time T by using a bang-coast-bang acceleration profile with symmetric acceleration and deceleration phases, each of duration $T_s = T/4$. Given a maximum joint velocity $V_{max} > 0$ and a maximum bound $A_{max} > 0$ for the absolute value of the joint acceleration, find the minimum time T_{min} in this class of trajectories such that the motion is feasible. Provide the general expression of T_{min} in terms of the symbolic parameters of the problem, and then its numerical value for the following data:

$$\Delta\theta = \pi \text{ [rad]}, \qquad V_{max} = 90 \text{ [°/s]}, \qquad A_{max} = 300 \text{ [°/s}^2\text{]}.$$

Sketch the resulting angular position, velocity, and acceleration profiles.

[150 minutes, open books]

Solution

July 11, 2019

Exercise 1

With reference to Fig. 1 (and embedding the problem in 3D), we need to compute the 3×3 Jacobian $\boldsymbol{J}_P(\boldsymbol{q})$ associated to the linear velocity of the payload point P and then use duality to determine the static torque $\boldsymbol{\tau}_g = -\boldsymbol{J}_P^T(\boldsymbol{q}_0)\boldsymbol{F}_g \in \mathbb{R}^3$ that will balance, at the configuration $\boldsymbol{q}_0$, the effect of the gravity force $\boldsymbol{F}_g$ acting on the payload. This force is in the direction of $-\boldsymbol{y}_0$ and has intensity $mg_0 > 0$. In organizing computations, we take into account that the position ${}^e\boldsymbol{p}_{ep} = \begin{pmatrix} p_{ep,x} & p_{ep,x} & 0 \end{pmatrix}^T$ of the payload mass is given in the end-effector frame RF_e (= RF_3 of a DH convention).

Using DH angles, the position of the origin of frame RF_e is

$$\boldsymbol{p}_e(\boldsymbol{q}) = \begin{pmatrix} L_1c_1 + L_2c_{12} + L_3c_{123} \\ L_1s_1 + L_2s_{12} + L_3s_{123} \\ 0 \end{pmatrix},$$

with the usual compact notation for the trigonometric functions (e.g., $c_{12} = \cos(q_1 + q_2)$). The orientation of frame RF_e w.r.t. the robot base frame is expressed by the rotation matrix

$${}^0\boldsymbol{R}_e(\boldsymbol{q}) = \begin{pmatrix} c_{123} & -s_{123} & 0 \\ s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The position of the payload P is then

$$\boldsymbol{p}_P(\boldsymbol{q}) = \boldsymbol{p}_e(\boldsymbol{q}) + {}^0\boldsymbol{R}_e(\boldsymbol{q})\,{}^e\boldsymbol{p}_{ep}.$$

The llnear velocity of the payload is computed as

$$\boldsymbol{v}_P = \dot{\boldsymbol{p}}_P = \frac{\partial \boldsymbol{p}_e(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} + {}^0\dot{\boldsymbol{R}}_e(\boldsymbol{q})^e\,\boldsymbol{p}_{ep} = \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{\omega}_e)\,{}^0\boldsymbol{R}_e(\boldsymbol{q})\,{}^e\boldsymbol{p}_{ep},$$

with the skew-symmetric matrix $\boldsymbol{S}(\boldsymbol{\omega}_e)$ given by

$$\boldsymbol{\omega}_e = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{S}(\boldsymbol{\omega}_e) = \begin{pmatrix} 0 & -(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & 0 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Performing computations gives

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} -(L_1s_1 + L_2s_{12} + L_3s_{123}) & -(L_2s_{12} + L_3s_{123}) & -L_3s_{123} \\ L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{S}(\boldsymbol{\omega}_e)\,{}^0\boldsymbol{R}_e(\boldsymbol{q})\,{}^e\boldsymbol{p}_{ep} = \begin{pmatrix} -\left(p_{ep,x}s_{123} + p_{ep,y}c_{123}\right) \\ p_{ep,x}c_{123} - p_{ep,y}s_{123} \\ 0 \end{pmatrix}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) = \boldsymbol{n}(\boldsymbol{q})\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\dot{\boldsymbol{q}}.$$

Thus

$$\boldsymbol{v}_P = \left(\boldsymbol{J}_e(\boldsymbol{q}) + \boldsymbol{n}(\boldsymbol{q})\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\right)\dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}},$$

and so

$$\boldsymbol{\tau}_g = -\boldsymbol{J}_P^T(\boldsymbol{q}_0)\boldsymbol{F}_g = \boldsymbol{J}_P^T(\boldsymbol{q}_0)\begin{pmatrix} 0 \\ mg_0 \\ 0 \end{pmatrix} = mg_0\begin{pmatrix} L_1c_1 + L_2c_{12} + \left(L_3 + p_{ep,x}\right)c_{123} - p_{ep,y}s_{123} \\ L_2c_{12} + \left(L_3 + p_{ep,x}\right)c_{123} - p_{ep,y}s_{123} \\ \left(L_3 + p_{ep,x}\right)c_{123} - p_{ep,y}s_{123} \end{pmatrix}. \qquad (1)$$

Plugging in (1) the numerical values for the link lengths L_i, $i = 1,2,3$, the configuration $\boldsymbol{q}_0$, the mass m and the coordinates $p_{ep,x}$ and $p_{ep,y}$ of the payload in RF_e, we obtain finally

$$\boldsymbol{\tau}_g = \begin{pmatrix} 94.9715 \\ 60.6365 \\ 30.9015 \end{pmatrix} \text{ [Nm]}.$$

Exercise 2

We should check first whether the linear path from A to B fits in the bounded workspace of the RP planar robot. The workspace is represented in Fig. 3, together with a segment of length $L = \|B - A\| = \sqrt{4.5} = 2.2113$ [m] that joins ${}^0A$ with ${}^0B$, i.e., the two given points A and B as expressed in frame RF_0 (rather than in RF_w). It is immediate to see that the linear path is fully contained in the robot workspace, once the robot base is suitably placed and rotated. In fact, there are infinite ways for doing so. We shall work with the choice made in Fig. 3, which makes derivations easier: the segment $\overline{AB}$ is placed symmetrically w.r.t. the axis $\boldsymbol{x}_0$ and at a distance $D = q_{2,max}/\sqrt{2} = 2.1213$ [m] from the origin of RF_0. As a result, the coordinates of the two points in frame RF_0 are

$${}^0A = \begin{pmatrix} D \\ -L/2 \end{pmatrix} = \begin{pmatrix} 2.1213 \\ -1.0607 \end{pmatrix}, \qquad {}^0B = \begin{pmatrix} D \\ L/2 \end{pmatrix} = \begin{pmatrix} 2.1213 \\ 1.0607 \end{pmatrix} \text{ [m]}.$$

Moreover, the robot configurations corresponding to the initial and final points of the linear path are

$$\boldsymbol{q}_A = \begin{pmatrix} \text{ATAN2}\{-L/2, D\} \\ \|{}^0A\| \end{pmatrix} = \begin{pmatrix} -0.4636 \\ 2.3717 \end{pmatrix}, \quad \boldsymbol{q}_B = \begin{pmatrix} \text{ATAN2}\{L/2, D\} \\ \|{}^0B\| \end{pmatrix} = \begin{pmatrix} 0.4636 \\ 2.3717 \end{pmatrix} \text{ [rad]}.$$

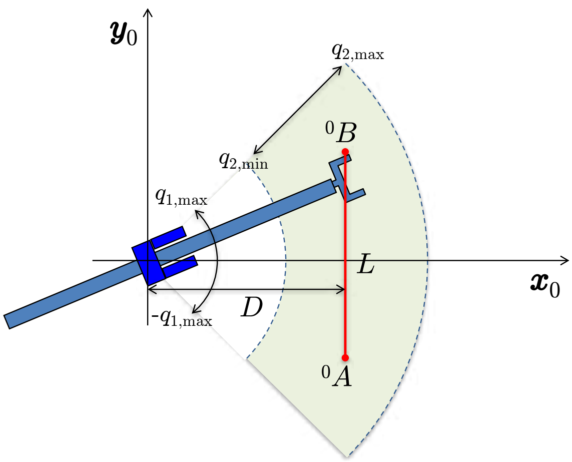


Figure 3: The workspace of the RP robot, as specified by the limited joint ranges.

The position and orientation of the base of the RP planar robot with respect to RF_w can be expressed by a (planar/2D) homogeneous transformation matrix of the form

$${}^w\boldsymbol{A}_0 = \begin{pmatrix} {}^w\boldsymbol{R}_0 & {}^w\boldsymbol{p}_0 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & p_x \\ \sin\theta_0 & \cos\theta_0 & p_y \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, from the kinematic identity for the representations of vectors in homogeneous coordinates we have

$$
{}^{w}A_{hom} = \begin{pmatrix} 4.5 \\ 1.5 \\ 1 \end{pmatrix} = {}^{w}\boldsymbol{A}_0 \, {}^{0}A_{hom} = {}^{w}\boldsymbol{A}_0 \begin{pmatrix} D \\ -L/2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} D\cos\theta_0 + \dfrac{L}{2}\sin\theta_0 + p_x & = & 4.5 \\ D\sin\theta_0 - \dfrac{L}{2}\cos\theta_0 + p_y & = & 1.5 \end{array} \right. \tag{2}
$$

and

$$
{}^{w}B_{hom} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = {}^{w}\boldsymbol{A}_0 \, {}^{0}B_{hom} = {}^{w}\boldsymbol{A}_0 \begin{pmatrix} D \\ L/2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} D\cos\theta_0 - \dfrac{L}{2}\sin\theta_0 + p_x & = & 3 \\ D\sin\theta_0 + \dfrac{L}{2}\cos\theta_0 + p_y & = & 3. \end{array} \right. \tag{3}
$$

The linear system (2–3) of four equations in the four unknowns $\cos\theta_0$, $\sin\theta_0$, p_x and p_y is nonsingular, and thus we have a unique solution to the problem (up to a reflection of the robot placement w.r.t. the line containing the path). The solution is obtained numerically from the given data as

$$
\theta_0 = \frac{\pi}{4} \text{ [rad]} \quad (\sin\theta_0 = \cos\theta_0 = \frac{\sqrt{2}}{2}), \qquad p_x = 2.25, \quad p_y = 0.75 \text{ [m]}. \tag{4}
$$

It is interesting to note that one can obtain this solution also in closed symbolic form. The linear system to be solved is in fact

$$
\begin{pmatrix} D & L/2 & 1 & 0 \\ -L/2 & D & 0 & 1 \\ D & -L/2 & 1 & 0 \\ L/2 & D & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ B_x \\ B_y \end{pmatrix} \qquad \Longleftrightarrow \qquad \boldsymbol{M}\boldsymbol{x} = \boldsymbol{b} = \begin{pmatrix} 4.5 \\ 1.5 \\ 3 \\ 3 \end{pmatrix}.
$$

The determinant of the coefficient matrix is $\det \boldsymbol{M} = L^2 \neq 0$. The unique solution is found then as

$$
\boldsymbol{x} = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ p_x \\ p_y \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{b} = \begin{pmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & 0 & -1/L & 0 \\ 0.5 & D/L & 0.5 & -D/L \\ -D/L & 0.5 & D/L & 0.5 \end{pmatrix} \begin{pmatrix} 4.5 \\ 1.5 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/(2L) \\ 3/(2L) \\ 3.75 - 3D/(2L) \\ 2.25 - 3D/(2L) \end{pmatrix}.
$$

Substituting now $D = 3/\sqrt{2} = 2.1213$ and $L = 1.5\sqrt{2} = 2.1213$ returns the numerical values in (4).

Exercise 3

The acceleration profile for the rest-to-rest motion trajectory $\theta(t)$ is *assigned* to be of the bang-coast-bang type, having symmetric initial and final acceleration/deceleration phases, each of duration $T_s = T/4$ and with $\ddot{\theta} = \pm A$ (to be determined), and a cruising phase that lasts for half of the motion time, i.e., $T/2$, with constant velocity $\dot{\theta} = V$. From this motion structure, choosing as arbitrary initial angle $\theta(0) = 0$, it is easy to compute the following quantities:

$$
V = \dot{\theta}\left(\frac{T}{4}\right) = A\,\frac{T}{4}, \qquad \Delta\theta_s = \theta\left(\frac{T}{4}\right) = \frac{1}{2}A\left(\frac{T}{4}\right)^2 = \frac{AT^2}{32}, \qquad \Delta\theta = \theta(T) = 2\Delta\theta_s + V\,\frac{T}{2} = \frac{3AT^2}{16}.
$$

Thus, for a desired total displacement $\Delta\theta > 0$ and a given motion time T, we have for the acceleration A and cruise velocity V

$$
A = \frac{16\Delta\theta}{3T^2} > 0 \qquad \Rightarrow \qquad V = \frac{4\Delta\theta}{3T} > 0. \tag{5}
$$

Note that, when the acceleration phase ends at time $t = T_s = T/4$, the performed angular displacement is $\Delta\theta_s = \Delta\theta/6$. By symmetry, when the deceleration phase begins at time $t = T - T_s = 3T/4$, the displacement done so far will be $\Delta\theta - \Delta\theta_s = 5\Delta\theta/6$.

Imposing now on (5) the two constraints

$$V \leq V_{max}, \qquad A \leq A_{max},$$

yields the minimum feasible motion time

$$T_{min} = \max\left\{4\sqrt{\frac{\Delta\theta}{3A_{max}}}, \frac{4\Delta\theta}{3V_{max}}\right\}.$$

With the data $\Delta\theta = \pi$ [rad], $V_{max} = 90$ [°/s] $\cdot$ $(\pi/180°) = \pi/2 = 1.5708$ [rad/s], and $A_{max} = 300$ [°/s^2] $\cdot$ $(\pi/180°) = 5\pi/3 = 5.2360$ [rad/s^2], we find the value of the optimal motion time

$$T_{min} = 2.6667 \text{ [s]}.$$

From (5), it follows $A = 2.3562$ [rad/s^2] and $V = V_{max} = 1.5708$ [rad/s]. Saturation of the constraint has occurred in the cruise phase. Figure 4 shows the resulting profiles of the angular position, velocity, and acceleration.

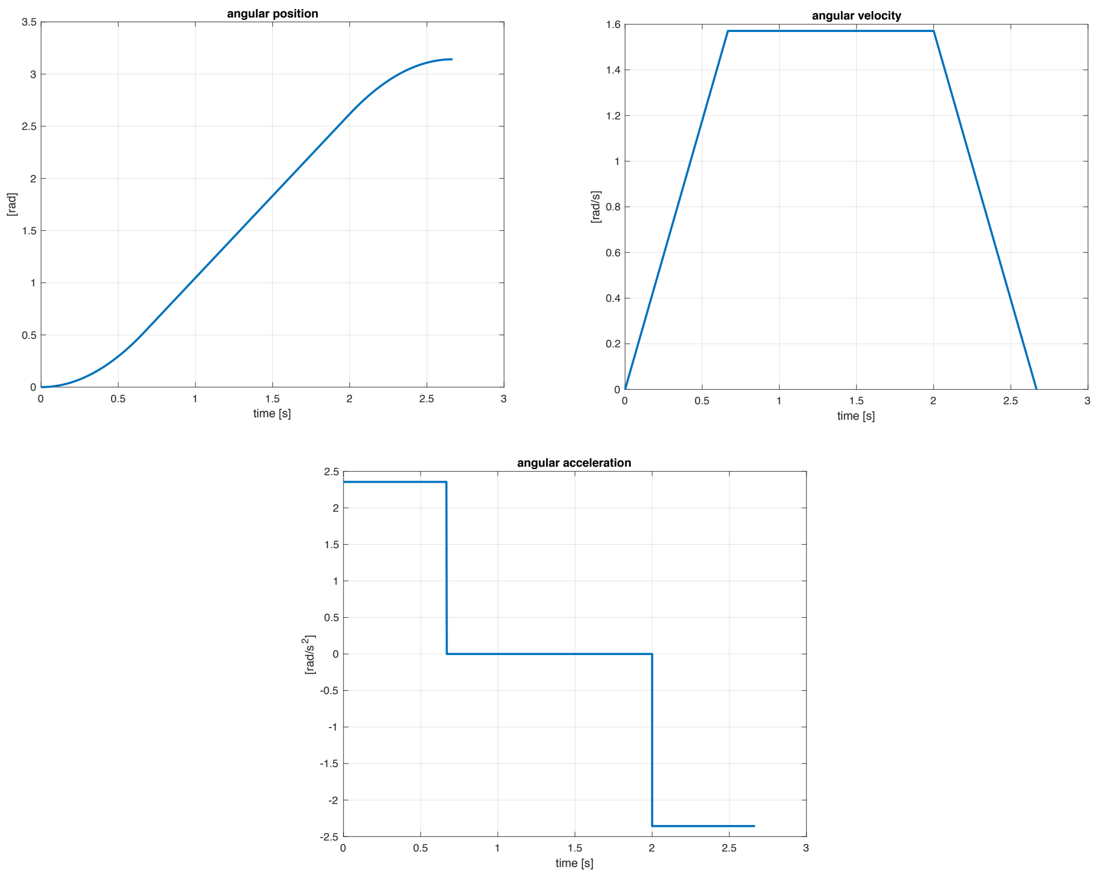


Figure 4: Position, velocity, and acceleration profiles for the considered minimum time rest-to-rest joint rotation.

* * * * *

Robotics I

September 11, 2019

Exercise 1

The kinematics of a 3R robot is defined by the following Denavit-Hartenberg table (units in [m] or [rad]):

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 5$	q_1
2	0	$a_2 = 4$	0	q_2
3	0	$a_3 = 3$	0	q_3

Determine the 3×3 linear part of the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot. When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], determine, if possible, a joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration, i.e., $\ddot{\boldsymbol{p}} = \boldsymbol{0}$. *[Bonus: What if the second link parameter is changed to $a_2 = 3$?]*

Exercise 2

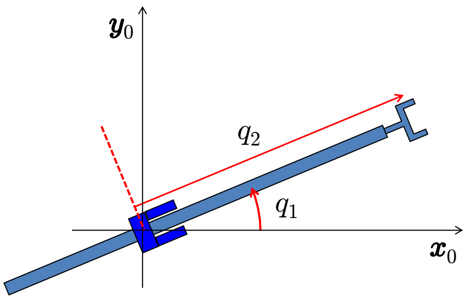


Figure 1: A RP planar robot, with the definition of the joint variables.

The RP robot shown in Fig. 1 starts from rest at time $t = 0$ in the configuration $\boldsymbol{q}(0) = (0, 1)$ [rad; m] and moves under the action of the following discontinuous joint acceleration commands for a time $T = 2$ [s]:

$$\ddot{q}_1(t) = \begin{cases} A_1 = 2 \text{ [rad/s}^2], & t \in [0, T/4], \\ 0, & t \in [T/4, 3T/4], \\ -A_1 = -2 \text{ [rad/s}^2], & t \in [3T/4, T]; \end{cases} \qquad \ddot{q}_2(t) = \begin{cases} -A_2 = -0.5 \text{ [m/s}^2], & t \in [0, T/2], \\ A_2 = 0.5 \text{ [m/s}^2], & t \in [T/2, T]. \end{cases}$$

a. Plot the time profiles of $q_i(t)$, $\dot{q}_i(t)$ and $\ddot{q}_i(t)$, for $i = 1, 2$.
b. Does the robot cross a singularity during this motion?
c. Compute the mid time configuration $\boldsymbol{q}(T/2)$ and the final configuration $\boldsymbol{q}(T)$ reached in this motion. Sketch the robot in these two configurations, as well as in the initial one.
d. Provide the analytic expressions of the end-effector velocity and acceleration norms, i.e., $\|\dot{\boldsymbol{p}}\|$ and $\|\ddot{\boldsymbol{p}}\|$.
e. Draw the end-effector velocity and acceleration vectors $\dot{\boldsymbol{p}}(T/2)$, $\ddot{\boldsymbol{p}}((T/2)^-)$ and $\ddot{\boldsymbol{p}}((T/2)^+)$ on the mid time configuration of the robot sketched at item c. Compute the numerical values of $\|\dot{\boldsymbol{p}}(T/2)\|$, $\left\|\ddot{\boldsymbol{p}}((T/2)^-)\right\|$ and $\left\|\ddot{\boldsymbol{p}}((T/2)^+)\right\|$.

Exercise 3

A link of length $L = 1.5$ m is rotated by a DC motor mounted at the base through a gear with reduction ratio $N_r = 4$. The motor has a quadrature incremental encoder with $N_p = 250$ pulses/turn and a digital counter of $n = 10$ bits. The link carries a laser scanner at its other end. The laser measures distances from the link tip to obstacles (in the same plane of link motion) up to a maximum distance of $d = 5$ m. The laser has a depth resolution $\Delta_\rho = 12$ mm and an angular resolution $\delta_s = 0.2°$ in the range $\alpha = \pm 90°$ (the zero is when the scanning ray is aligned with the link). Sketch the setup and analyze the resolution of this system for measuring the position of objects in the environment. In the worst-case condition, which is the largest possible Cartesian displacement Δ of an object that provides no change in the output readings?

[180 minutes; open books]

Solution

September 11, 2019

Exercise 1

The 3×3 Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that relates the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the linear velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^3$ of the end-effector of this robot can be equivalently computed either by differentiation of the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ or geometrically. Both methods require the information provided by the DH table. We follow here the first method. Note that the given DH table refers to a standard 3R elbow-type manipulator without offsets: this is helpful to know, but not really needed in the computations.

The end-effector position is obtained from the homogenous transformation matrices:

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_2(q_2) \left({}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right)\right)$$

$$\Longrightarrow \qquad \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3)\right) \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2+q_3) \end{pmatrix}. \tag{1}$$

Therefore, using the usual compact notation for trigonometric functions, we have

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1\left(a_2c_2 + a_3c_{23}\right) & -c_1\left(a_2s_2 + a_3s_{23}\right) & -a_3c_1s_{23} \\ c_1\left(a_2c_2 + a_3c_{23}\right) & -s_1\left(a_2s_2 + a_3s_{23}\right) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}. \tag{2}$$

The Jacobian is singular when[1]

$$\det \boldsymbol{J}(\boldsymbol{q}) = -a_2a_3\, s_3\left(a_2c_2 + a_3c_{23}\right) = 0. \tag{3}$$

The end-effector acceleration $\ddot{\boldsymbol{p}}$ is computed as

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{4}$$

Thus, in order to realize a zero end-effector acceleration, we need to set $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in (4) and solve for $\ddot{\boldsymbol{q}}$, or

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{5}$$

Indeed, this solution is valid as long as the robot is out of singularities. To evaluate (5), we need first to derive the time derivative of the robot Jacobian. Let $\boldsymbol{J}_i(\boldsymbol{q})$ be the ith column of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$, for $i = 1,2,3$. We compute [2]

$$\begin{aligned} \dot{\boldsymbol{J}}(\boldsymbol{q}) &= \frac{d\boldsymbol{J}(\boldsymbol{q})}{dt} \quad \left(= \sum_{i=1}^{3} \left(\frac{\partial \boldsymbol{J}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} \right) \boldsymbol{e}_i^T = \sum_{j=1}^{3} \frac{\partial \boldsymbol{J}(\boldsymbol{q})}{\partial q_j}\,\dot{q}_j \right) \\ &= \left(\begin{matrix} -c_1\dot{q}_1\left(a_2c_2 + a_3c_{23}\right) + s_1\left(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) \\ -s_1\dot{q}_1\left(a_2c_2 + a_3c_{23}\right) - c_1\left(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) \\ 0 \end{matrix} \right. \\ &\qquad \left. \begin{matrix} s_1\dot{q}_1\left(a_2s_2 + a_3s_{23}\right) - c_1\left(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)\right) & a_3s_1\dot{q}_1s_{23} - a_3c_1c_{23}(\dot{q}_2+\dot{q}_3) \\ c_1\dot{q}_1\left(a_2s_2 + a_3s_{23}\right) - s_1\left(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)\right) & -a_3c_1\dot{q}_1s_{23} - a_3s_1c_{23}(\dot{q}_2+\dot{q}_3) \\ -\left(a_2s_2\dot{q}_2 - a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) & -a_3s_{23}(\dot{q}_2+\dot{q}_3) \end{matrix} \right). \end{aligned} \tag{6}$$

When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], evaluation of (2) and (6) gives

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -0.7071 & 0 & 0 \\ 0 & -4.9497 & -2.1213 \\ 0 & 0.7071 & -2.1213 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_0 = \dot{\boldsymbol{J}}(\boldsymbol{q}_0)|_{\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}_0} = \begin{pmatrix} 5.6569 & 4.9497 & 2.1213 \\ -0.7071 & -5.6569 & 0 \\ 0 & -5.6569 & 0 \end{pmatrix}.$$

Since $\det \boldsymbol{J}_0 = -8.4853 \neq 0$,we can use eq. (5) for computing the joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration:

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}_0^{-1}\left(\dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0\right) = -\begin{pmatrix} -1.4142 & 0 & 0 \\ 0 & -0.1768 & 0.1768 \\ 0 & -0.0589 & -0.4125 \end{pmatrix} \begin{pmatrix} 11.3137 \\ -12.0208 \\ -11.3137 \end{pmatrix} = \begin{pmatrix} 16 \\ -0.1250 \\ -5.3750 \end{pmatrix} \text{ [rad/s}^2\text{]}. \tag{7}$$

[1] This computation is made easier when expressing the Jacobian in frame 1, i.e., using ${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q})$.

[2] The first expression in the large parenthesis uses the dyadic expansion of a matrix: $\boldsymbol{J}(\boldsymbol{q}) = \sum_{i=1}^{3} \boldsymbol{J}_i(\boldsymbol{q})\,\boldsymbol{e}_i^T$, where $\boldsymbol{e}_i^T$ is the i-th row of the identity matrix.

Bonus part. If we change the second link parameter from $a_2 = 4$ to $a_2 = 3$, a singular configuration will be encountered. This can be recognized already from the direct klnematics (1); in fact, we have in this case

$$\boldsymbol{p}' = \boldsymbol{p}_{|a_2=3} = \begin{pmatrix} 0 & 0 & 9.2426 \end{pmatrix}^T,$$

namely the end-effector is placed on the axis of joint 1: any rotation $\dot{q}_1$ will not move the end-effector —a situation of singularity. Re-evaluating then the Jacobian (2) yields

$$\boldsymbol{J}'_0 = \boldsymbol{J}(\boldsymbol{q}_0)_{|a_2=3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4.2426 & -2.1213 \\ 0 & 0 & -2.1213 \end{pmatrix} \quad \Rightarrow \quad \det \boldsymbol{J}'_0 = 0, \quad \mathcal{R}\left\{\boldsymbol{J}'_0\right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In these cases, compensating with a suitable joint acceleration a drift of the end-effector acceleration due to the current joint velocity is still possible, provided that the Cartesian drift is in the span of the Jacobian, Moreover, the inversion in (5) should be replaced by a pseudoinversion of the Jacobian matrix, namely

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{8}$$

To check if this is the case, we re-evaluate with (6) the time derivative of the Jacobian, and then the drift term:

$$\dot{\boldsymbol{J}}'_0 = \dot{\boldsymbol{J}}_{0|a_2=3} = \begin{pmatrix} 4.2426 & 4.2426 & 2.1213 \\ 0 & -4.2426 & 0 \\ 0 & -4.2426 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{J}}'_0\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{J}}'_0 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8.4853 \\ -8.4853 \\ -8.4853 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}'_0\right\}.$$

Therefore, even with the use of a pseudoinverse, we will not be able to impose the desired (zero) end-effector acceleration: an error (of minimum possible norm) will result. Computing the pseudoinverse and evaluating (8) gives

$$\boldsymbol{J}_0^{\#'} = \boldsymbol{J}^{\#}_{|a_2=3}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.2357 & 0.2357 \\ 0 & 0 & -0.4714 \end{pmatrix} \quad \Longrightarrow \quad \ddot{\boldsymbol{q}} = -\boldsymbol{J}_0^{\#'}\dot{\boldsymbol{J}}'_0\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}.$$

Checking the end-effector acceleration obtained with this joint solution,

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}'_0\dot{\boldsymbol{J}}'_0\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 8.4853 & 0 & 0 \end{pmatrix}^T \neq \boldsymbol{0}^T,$$

confirms that the component that is outside the range of the Jacobian (i.e., in the $\boldsymbol{x}$ direction) is not canceled, while the task is achieved for the remaining part.

Exercise 2

We proceed by integrating twice the joint acceleration commands, taking into account the initial state of the robot at time $t = 0$ ($\boldsymbol{q}(0) = (0, 1)$ [rad; m], $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$) and the total motion time $T = 2$ [s]. For the joint velocities, we have obtain

$$\dot{q}_1(t) = \begin{cases} A_1 t = 2\,t \text{ [rad/s]}, & t \in [0, 0.5], \\ V_1 = 1 \text{ [rad/s]}, & t \in [0.5, 1.5], \\ V_1 - A_1(t - 1.5) = 1 - 2(t - 1.5) = 4 - 2\,t \text{ [rad/s]}, & t \in [1.5, 2] \end{cases}$$

and

$$\dot{q}_2(t) = \begin{cases} -A_2 t = -0.5\,t \text{ [m/s]}, & t \in [0, 1], \\ -V_2 + A_2(t - 1) = -0.5 + 0.5(t - 1) = 0.5\,t - 1 \text{ [m/s]}, & t \in [1, 2]. \end{cases}$$

For the joint positions, we obtain

$$q_1(t) = \begin{cases} q_1(0) + \frac{1}{2}A_1 t^2 = t^2 \text{ [rad]}, & t \in [0, 0.5], \\ q_1(0.5) + V_1(t - 0.5) = 0.25 + (t - 0.5) = t - 0.25 \text{ [rad]}, & t \in [0.5, 1.5], \\ q_1(1.5) + V_1(t - 1.5) - \frac{1}{2}A_1(t - 1.5)^2 = 1.25 + (t - 1.5) - (t - 1.5)^2 \text{ [rad]}, & t \in [1.5, 2] \end{cases}$$

and

$$q_2(t) = \begin{cases} q_2(0) - \frac{1}{2}A_2 t^2 = 1 - 0.25\,t^2 \text{[m]}, & t \in [0, 1], \\ q_2(1) - V_2(t - 1) + \frac{1}{2}A_2(t - 1)^2 = 0.75 - 0.5(t - 1) + 0.25(t - 1)^2 \text{ [m]}, & t \in [1, 2]. \end{cases}$$

The qualitative time profiles of $q_i(t)$, $\dot{q}_i(t)$ and $\ddot{q}_i(t)$, for $i = 1, 2$, are shown in Fig. 2, with joint variations $\Delta q_1 = 1.5$ [rad] and $\Delta q_2 = -0.5$ [m]. The robot never crosses its kinematic singularities, i.e., any

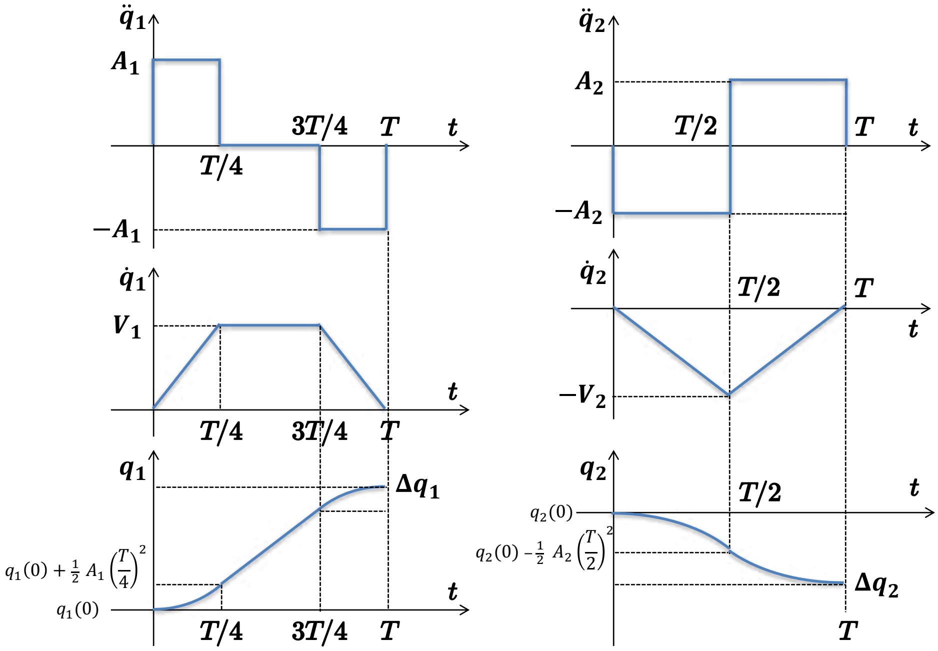


Figure 2: Qualitative plots of $\ddot{\boldsymbol{q}}(t)$, $\dot{\boldsymbol{q}}(t)$ and $\boldsymbol{q}(t)$ for the RP robot, as specified in the text.

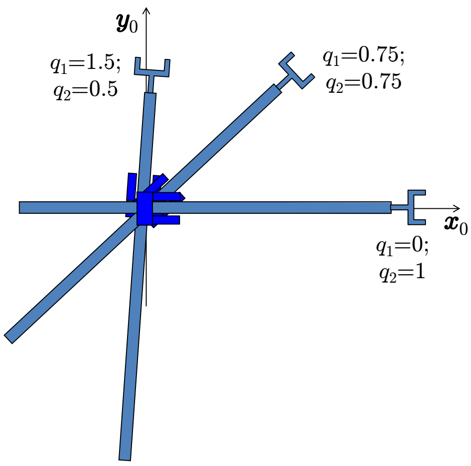


Figure 3: The RP robot in the initial, mid and final time configurations (values in [rad;m])

configuration $\boldsymbol{q}_s = (*, 0)$ with the second link fully retracted. The initial, mid time and final configurations are

$$\boldsymbol{q}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{q}(1) = \begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix}, \qquad \boldsymbol{q}(2) = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix} \qquad \text{[rad; m]}.$$

Fig. 3 sketches the RP robot in these configurations.

The end-effector velocity and acceleration of the RP robot are computed by differentiation of its direct kinematics

$$\boldsymbol{p} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}.$$

We have

$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 \cos q_1 - \dot{q}_1 q_2 \sin q_1 \\ \dot{q}_2 \sin q_1 + \dot{q}_1 q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix}, \tag{9}$$

where a planar 2×2 rotation matrix $\boldsymbol{R}$ by an angle q_1 has been put in evidence. The last vector in (9) is the end-effector velocity expressed in the frame rotated by the angle q_1, i.e., ${}^1\dot{\boldsymbol{p}}$. The norm of vector $\dot{\boldsymbol{p}}$ is computed as follows:

$$\|\dot{\boldsymbol{p}}\|^2 = \dot{\boldsymbol{p}}^T \dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 & q_2 \dot{q}_1 \end{pmatrix} \boldsymbol{R}^T(q_1) \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} = \left\| \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} \right\|^2 = q_2^2\, \dot{q}_1^2 + \dot{q}_2^2 \quad \Rightarrow \quad \|\dot{\boldsymbol{p}}\| = \sqrt{q_2^2\, \dot{q}_1^2 + \dot{q}_2^2}. \tag{10}$$

For computing the acceleration $\ddot{\boldsymbol{p}}$, note first that

$$\dot{\boldsymbol{R}}(q_1) = \begin{pmatrix} -\sin q_1 & -\cos q_1 \\ \cos q_1 & -\sin q_1 \end{pmatrix} \dot{q}_1 = \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{q}_1 = \boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -\dot{q}_1 \\ \dot{q}_1 & 0 \end{pmatrix}.$$

Differentiation of eq. (9) provides

$$\begin{aligned} \ddot{\boldsymbol{p}} &= \boldsymbol{R}(q_1) \begin{pmatrix} \ddot{q}_2 \\ \dot{q}_1\dot{q}_2 + q_2\ddot{q}_1 \end{pmatrix} + \dot{\boldsymbol{R}}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2\dot{q}_1 \end{pmatrix} \\ &= \boldsymbol{R}(q_1) \left(\begin{pmatrix} \ddot{q}_2 \\ \dot{q}_1\dot{q}_2 + q_2\ddot{q}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\dot{q}_1 \\ \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ q_2\dot{q}_1 \end{pmatrix} \right) = \boldsymbol{R}(q_1) \begin{pmatrix} \ddot{q}_2 - q_2\,\dot{q}_1^2 \\ q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2 \end{pmatrix} = \boldsymbol{R}(q_1)\,{}^1\ddot{\boldsymbol{p}}. \end{aligned} \tag{11}$$

Moreover, the norm of this vector is

$$\|\ddot{\boldsymbol{p}}\| = \|{}^1\ddot{\boldsymbol{p}}\| = \sqrt{\left(\ddot{q}_2 - q_2\,\dot{q}_1^2\right)^2 + \left(q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\right)^2}. \tag{12}$$

The evaluation of (9) and (10) at $t = T/2 = 1$ s yields

$$\dot{\boldsymbol{p}}(1) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \dot{q}_2(1) \\ q_2(1)\dot{q}_1(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} -0.8771 \\ 0.2079 \end{pmatrix}$$

and

$$\|\dot{\boldsymbol{p}}(1)\| = \sqrt{\dot{q}_2^2(1) + q_2^2(1)\dot{q}_1^2(1)} = 0.9014.$$

The evaluation of (11) at $t = T/2 = 1$ s should take into account the discontinuity of the acceleration of the second joint at the mid time of motion. Therefore, we should consider the two values just before ($^-$) and just after ($^+$) the mid time instant:

$$\ddot{\boldsymbol{p}}(1^-) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \ddot{q}_2(1^-) - q_2(1)\dot{q}_1^2(1) \\ q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -1.25 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.2330 \\ -1.5837 \end{pmatrix}$$

and

$$\ddot{\boldsymbol{p}}(1^+) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \ddot{q}_2(1^+) - q_2(1)\dot{q}_1^2(1) \\ q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -0.25 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.4987 \\ -0.9021 \end{pmatrix}.$$

Similarly, for the norm (12) we have

$$\left\|\ddot{\boldsymbol{p}}(1^-)\right\| = \sqrt{\left(\ddot{q}_2(1^-) - q_2(1)\dot{q}_1^2(1)\right)^2 + \left(q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1)\right)^2} = 1.6008$$

and

$$\left\|\ddot{\boldsymbol{p}}(1^+)\right\| = \sqrt{\left(\ddot{q}_2(1^+) - q_2(1)\dot{q}_1^2(1)\right)^2 + \left(q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1)\right)^2} = 1.0308.$$

Finally, Fig. 4 shows the vectors $\dot{\boldsymbol{p}}(1)$, $\ddot{\boldsymbol{p}}(1^-)$ and $\ddot{\boldsymbol{p}}(1^+)$ on the RP robot in the mid time configuration. For this picture, it is more convenient to use the vectors expressed in the rotated frame, i.e., to draw for instance ${}^1\dot{\boldsymbol{p}}(1)$ on the rotated second link (rather than attempting directly to draw $\dot{\boldsymbol{p}}(1)$). Note that the relative scales of these vectors are somewhat arbitrary.

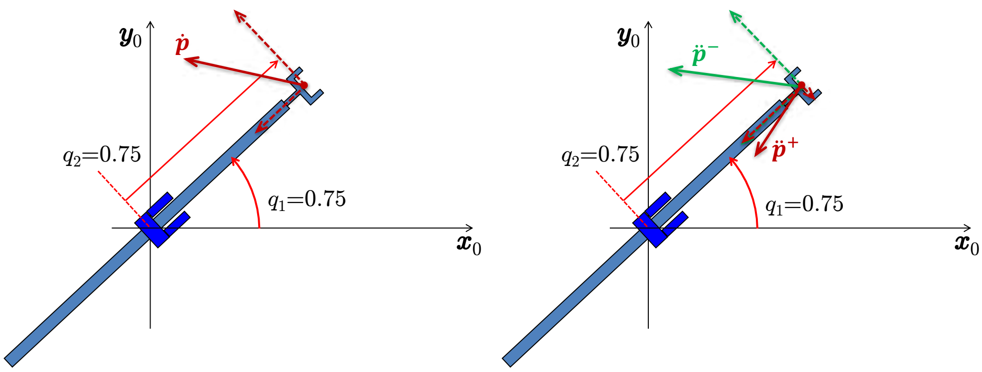


Figure 4: The mid time robot configuration with: $\dot{\boldsymbol{p}}(1)$ [left]; $\ddot{\boldsymbol{p}}(1^-)$ and $\ddot{\boldsymbol{p}}(1^+)$ [right].

Exercise 3

The measurement system is composed by two parts, the laser scanner and the rotating link carrying it, see Fig. 5. The rotation added by the actuated link is useful because it enlarges the angular range of the sensor. On the other hand, uncertainty is added to the laser measurement, due to the angular resolution of the encoder which translates into an uncertain localization of the base of the sensor. To analyze the overall behavior, we consider first the two systems separately.

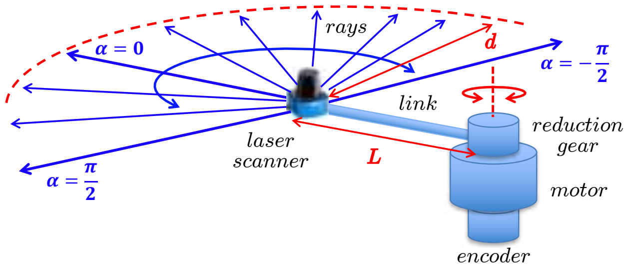


Figure 5: The measurement system made by a rotating link that carries a laser scanner.

For the motor-link assembly, the angular resolution δ_m of the encoder mounted on the motor shaft (after electronic multiplication by 4) and the lateral uncertainty Δ_L at the link end are computed as

$$\delta_m = \frac{360°}{4 \times N_p}\left(\frac{\pi}{180°}\right) = \frac{2\pi}{4 \times 250} = 0.00628\,[\text{rad}], \qquad \Delta_L = \frac{\delta_m}{N_r}\,L = \frac{0.00628}{4}\,1.5 = 0.0024\,[\text{m}] = 2.4\,[\text{mm}].$$

Note that the $n = 10$ bits of the digital counter in the encoder are sufficient to represent the full rotation, since $2^n = 2^{10} = 1024 > 1000$ (the number of electrical pulses per turn).

For the laser scanner, the angular resolution δ_s corresponds to an uncertainty in the lateral positioning (w.r.t. the pointing ray) of a sensed object. The worst-case situation is when the object is placed at the maximum sensing distance d from the laser source. The (Cartesian) width resolution Δ_ϕ in this case is

$$\Delta_\phi = \delta_s\, d = 0.2°\left(\frac{\pi}{180°}\right) 5 = 0.00349 \times 5 = 0.0175\,[\text{m}] = 17.5\,[\text{mm}].$$

Instead, the depth resolution Δ_ρ is rather independent from the distance. Thus, the region of uncertainty in the scanning process, when the base of the laser sensor is in a fixed, known position, can be approximated by a rectangle of size $\Delta_\rho \times \Delta_\phi = 12 \times 17.5$ [mm×mm]. A displacement of an object within this small area will not generate any change in the sensor reading. In particular, if it crosses this area in diagonal, we get

$$\Delta \simeq \sqrt{\Delta_\rho^2 + \Delta_\phi^2} = \sqrt{12^2 + 17.5^2} = 21.2\ [\text{mm}]. \tag{13}$$

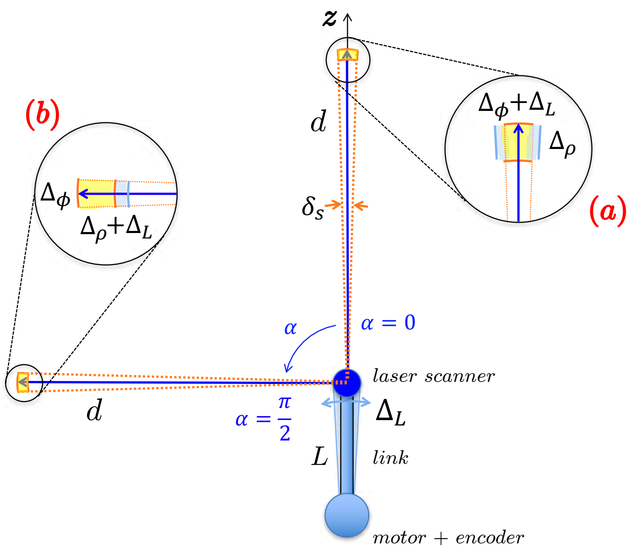


Figure 6: Two (expanded) Cartesian regions of measurement uncertainty: $(a)\ \alpha = 0$; $(b)\ \alpha = \pi/2$.

When combining the scanning process with the variable orientation of the link, the measurement uncertainty on the object position will increase, due to the uncertainty Δ_L on the lateral positioning at the link end where the sensor is placed. However, the outcome of this combination will depend on the relative angle between the direction of the laser ray and the direction of the link that carries the sensor. With reference to Fig. 6, we consider two limit cases: when the ray is aligned with the link, at $\alpha = 0$ [case (a)], and when the ray is at the boundary of its angular sensing range, e.g., at $\alpha = \pi/2$ [case (b)]. In the first case, Δ_L adds to Δ_ϕ while the depth resolution Δ_ρ remains unaffected. In the second case, Δ_L adds to Δ_ρ while the width resolution Δ_ϕ is unaffected. All other feasible values of α lead to intermediate situations. The largest Cartesian displacement of an object that would provide no change in the output reading is again on the diagonal of the rectangle of uncertainly. We have:

$$\Delta_a \simeq \sqrt{\Delta_\rho^2 + (\Delta_\phi + \Delta_L)^2}, \qquad \Delta_b \simeq \sqrt{(\Delta_\rho + \Delta_L)^2 + \Delta_\phi^2}.$$

Since for the given data $\Delta_\phi > \Delta_\rho$, it follows that $\Delta_a > \Delta_b$. Thus, the worst increase in uncertainty will happen in case (a):

$$\Delta = \Delta_a \simeq \sqrt{12^2 + (17.5 + 2.4)^2} = 23.2 \text{ [mm]}.$$

The resolution of the measurement system (or, equivalently, the largest positional uncertainty of the sensed object) in the mobile case has worsened by about 2 mm with respect to the fixed case.

* * * * *

Robotics I - Extra sheet #2 (for Exercise 5)

November 29, 2019

Name: ____

Answer to the questions or comment/complete the statements, providing also a *short* motivation/explanation (within the given lines of text) for each of the 7 items.

1. Are there 3-dof robots with just a single inverse kinematics solution in their primary workspace? If so, which ones? If not, why?

2. In order to measure the joint velocities of a robot, extra dedicated sensors may not be needed since ...

3. A large reduction ratio for a robot joint transmission is good because ..., and is bad because ...

4. Use of link acceleration measurements to generate torques that move the robot may be critical. Why?

5. Compare an incremental encoder with $N = 900$ pulses per turn and quadrature electronics, mounted on a motor connected to the link with a reduction ratio $n_r = 40$, with a 16-bit absolute encoder mounted directly on the link side of the transmission. Which is better in terms of link position resolution?

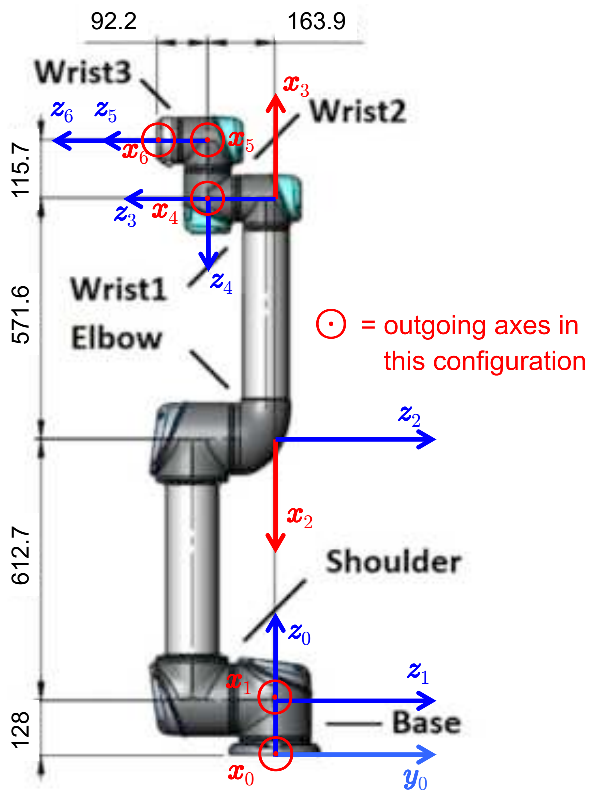
6. An installed 6-dof industrial robot has repeatability $\rho = 0.1$ [mm] and accuracy $\delta = 0.6$ [mm] in a certain region of its workspace. Which of these two parameters can be improved, and how?

7. An object of mass $m = 5$ [kg] is hanging statically to a 6D F/T sensor, whose only non-zero outputs are $f_z = -49.05$ [N], $\mu_x = 7.3575$ [Nm]. Where is the object center of mass located in the sensor frame?

Robotics I - Extra sheet #1 (for Exercise 2)

November 29, 2019

Name: ______________________________



i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				

Insert the constant parameters (mm or rad) and the value of $\boldsymbol{\theta}$ (in rad) in the shown configuration.

Robotics I - Extra sheet #2 (for Exercise 5)

November 29, 2019

Name: ____________________

Answer to the questions or comment/complete the statements, providing also a *short* motivation/explanation (within the given lines of text) for each of the 7 items.

1. Are there 3-dof robots with just a single inverse kinematics solution in their primary workspace? If so, which ones? If not, why?

2. In order to measure the joint velocities of a robot, extra dedicated sensors may not be needed since ...

3. A large reduction ratio for a robot joint transmission is good because ..., and is bad because ...

4. Use of link acceleration measurements to generate torques that move the robot may be critical. Why?

5. Compare an incremental encoder with $N = 900$ pulses per turn and quadrature electronics, mounted on a motor connected to the link with a reduction ratio $n_r = 40$, with a 16-bit absolute encoder mounted directly on the link side of the transmission. Which is better in terms of link position resolution?

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Robotics I

Midterm classroom test – November 29, 2019

Exercise 1 [6 points]

The initial orientation of a rigid body with respect to a basis reference frame is given by the matrix

$$\boldsymbol{R}_i = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

The final desired orientation $\boldsymbol{R}_f$ is expressed in terms of roll-pitch-yaw angles $(\alpha, \beta, \gamma) = (\pi/3, \pi/3, -\pi/2)$ in the sequence ZYX around the fixed axes associated to the initial orientation. Find a pair $(\boldsymbol{r}, \theta)$ such that the relative change of orientation of the body is represented by the axis-angle method associated to the unit vector $\boldsymbol{r}$ and angle θ. Comment on how the same result can be obtained when the unit vector $\boldsymbol{r}$ is expressed in terms of the basis reference frame, rather than in the frame associated to $\boldsymbol{R}_i$.

Exercise 2 [6 points]

Consider the 6R Universal Robots UR10 manipulator in Fig. 1, where a possible set of Denavit-Hartenberg (DH) frames has been defined.

- On the extra sheet #1 provided separately [*to be returned with your name*], complete the table of DH parameters. Enter in the table numerical values (expressed in [rad] or [mm]), including those of the joint variables $\boldsymbol{q} = \boldsymbol{\theta}$ in the configuration shown. In the drawing, all data are given already in mm.
- Provide the numerical value of the position of the origin O_6 of frame 6 in the shown configuration.

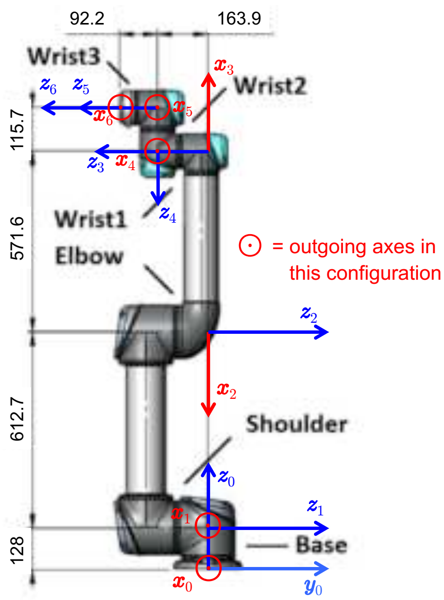


Figure 1: A possible assignment of DH frames for the UR10 robot.

Exercise 3 [6 points]

Consider the planar 2R robot in Fig. 2, with the numerical data $L = 0.4$, $A = 0.4$, and $B = 0.3$ [m]. An end-effector frame RF_e is attached at point P to the gripper, with the $\boldsymbol{z}_e$ axis along the approach direction.

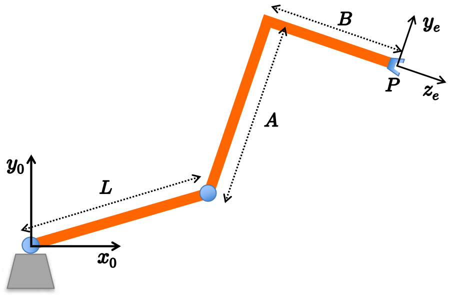


Figure 2: A planar 2R robot, with a L-shaped second link.

- Draw accurately the primary workspace of the robot. Which is its secondary workspace?
- Assign the link frames and define the joint variables $\boldsymbol{q} = (q_1, q_2)$ according to the Denavit-Hartenberg (DH) convention. Let the origin O_2 of the DH frame 2 be placed at point P. Complete the associated table of parameters.
- Determine the matrix ${}^2\boldsymbol{R}_e \in SO(3)$.
- Provide all solutions, if any, to each of the following three inverse kinematics problems, where the end-effector position $\boldsymbol{p}_e \in \mathbb{R}^2$ (i.e., reduced to the plane of motion) is given as input:

$$\boldsymbol{p}_{e,1} = \begin{pmatrix} 0 \\ -0.9 \end{pmatrix}; \qquad \boldsymbol{p}_{e,2} = \begin{pmatrix} -0.4 \\ 0.7 \end{pmatrix}; \qquad \boldsymbol{p}_{e,3} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 4 [6 points]

Write a simple (pseudo-)code that solves the following inverse kinematics problem with the iterative Newton method. The task (direct) kinematics is

$$\boldsymbol{r} = \begin{pmatrix} L_1 \cos q_1 + L_2 \cos q_2 + L_3 \cos q_3 \\ L_1 \sin q_1 + L_2 \sin q_2 + L_3 \sin q_3 \\ q_3 - q_2 \end{pmatrix}.$$

Let $L_1 = 0.4$, $L_2 = 0.3$, and $L_3 = 0.2$ [m]. If the desired task value $\boldsymbol{r}_d$ and the initial guess $\boldsymbol{q}^{(0)}$ for the solution are, respectively,

$$\boldsymbol{r}_d = \begin{pmatrix} 0.7 \\ 0.5 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{q}^{(0)} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi/2 \end{pmatrix},$$

what is the numerical value of the next guess $\boldsymbol{q}^{(1)}$ at the end of iteration 1? Do you think the sequence $\{\boldsymbol{q}^{(0)}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \dots\}$ generated by this method will converge to a solution of the problem? For the given value of $\boldsymbol{r}_d$, how many solutions do you expect to have for this inverse problem?

Exercise 5 [6 points]

A number of questions and statements are reported on the extra sheet #2. Fill in your answers and/or comments on the same sheet [*to be returned with your name*], providing also a *short* motivation/explanation for each item.

[180 minutes, open books]

Solution of Midterm Test

November 29, 2019

Exercise 1

The orientation of a rigid body, as expressed by a ZYX sequence of roll-pitch-yaw angles (α, β, γ), i.e., with respect to a set of fixed axes, is represented by the rotation matrix

$$\begin{aligned}
\boldsymbol{R}_{ZYX}^{RPY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_X(\gamma)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_Z(\alpha) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\beta & \sin\beta \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma \end{pmatrix},
\end{aligned}$$

where the order in the product of elementary rotation matrices is reversed, as required. The final desired orientation of the body with respect to the frame with orientation $\boldsymbol{R}_i = {}^0\boldsymbol{R}_i$ is specified as

$$ {}^i\boldsymbol{R}_f = \boldsymbol{R}_{ZYX}^{RPY}\left(\frac{\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{2}\right) = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0.25 & -0.433 & 0.866 \\ -0.433 & 0.75 & 0.5 \\ -0.866 & -0.5 & 0 \end{pmatrix}. \tag{1}$$

Operatively, to obtain this matrix one can either evaluate numerically the symbolic matrix $\boldsymbol{R}_{ZYX}^{RPY}$, or evaluate numerically the single elementary rotation matrices $\boldsymbol{R}_X$, $\boldsymbol{R}_Y$ and $\boldsymbol{R}_Z$ and then do their products.

To represent this change of orientation by the axis-angle method with a unit vector $\boldsymbol{r}$ and angle θ, we need to solve the equation[1]

$$\boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta = {}^i\boldsymbol{R}_f.$$

Indeed, the unit vector computed in this way will naturally be expressed in the frame associated to ${}^0\boldsymbol{R}_i$, i.e., $\boldsymbol{r} = {}^i\boldsymbol{r}$. Let R_{ij} be the elements of matrix ${}^i\boldsymbol{R}_f$ in (1). Since

$$\sin\theta = \frac{1}{2}\sqrt{(R_{21}-R_{12})^2 + (R_{13}-R_{31})^2 + (R_{32}-R_{23})^2} = 1 \neq 0, \qquad \cos\theta = \frac{\operatorname{trace}\left({}^i\boldsymbol{R}_f\right) - 1}{2} = 0;$$

the problem is regular and the two (specular) solution pairs $(\boldsymbol{r}, \theta)$ are given by

$$\boldsymbol{r}_1 = \frac{1}{2\sin\theta}\begin{pmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.866 \\ 0 \end{pmatrix}, \qquad \theta_1 = \text{ATAN2}\{\sin\theta, \cos\theta\} = \frac{\pi}{2}, \tag{2}$$

and $(\boldsymbol{r}_2, \theta_2) = (-\boldsymbol{r}_1, -\theta_1)$.

In order to obtain the same solution using the unit vector ${}^0\boldsymbol{r}_1$ (what follows apply also to ${}^0\boldsymbol{r}_2$), namely expressed in terms of the basis reference frame, we compute

$${}^0\boldsymbol{r}_1 = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{r}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} -0.5 \\ 0.866 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.3536 \\ -0.866 \\ -0.3536 \end{pmatrix}.$$

[1] A completely different result would be obtained if the problem request was to use the axis-angle method $\boldsymbol{R}(\boldsymbol{r}', \theta')$ to represent the final orientation in terms of the basic reference frame, i.e., $\boldsymbol{R}(\boldsymbol{r}', \theta') = {}^0\boldsymbol{R}_f = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{R}_f$. In that case, the computed unit axis $\boldsymbol{r}'$ would have been expressed directly in the basic frame, i.e., $\boldsymbol{r}' = {}^0\boldsymbol{r}'$.

It is easy to verify (left as an exercise for the reader) that the following identity holds:

$$ {}^0\boldsymbol{R}_f = \boldsymbol{R}({}^0\boldsymbol{r}_1, \theta_1)\,{}^0\boldsymbol{R}_i = {}^0\boldsymbol{R}_i\,\boldsymbol{R}({}^i\boldsymbol{r}_1, \theta_1) = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{R}_f. \tag{3} $$

In fact, the absolute orientation of the final frame w.r.t. the basis (zero) reference frame can be obtained either by two rotations defined both w.r.t. fixed axes (with the reverse order in the product of rotations, as in the first identity in (3)), or by two rotations, the second of which is defined w.r.t. the axes obtained after the first one (chain rule of products with moving axes, as in the second and third identities in (3)).

Exercise 2

The Denavit-Hartenberg parameters uniquely associated to the frames specified for the UR10 robot (see also Fig. 3) are given in Tab. 1. Note that this is NOT the frame assignment used for this robot in the DIAG Robotics Lab. The position of the origin O_6 in the shown configuration is found just by inspection as $\boldsymbol{p}_6 = \begin{pmatrix} 0 & -(163.9+92.2) & (128+612.7+571.6+115.7) \end{pmatrix}^T = \begin{pmatrix} 0 & -256.1 & 1428 \end{pmatrix}^T$ [mm] —there is no need to perform lengthy computations with the DH homogenous transformation matrices!

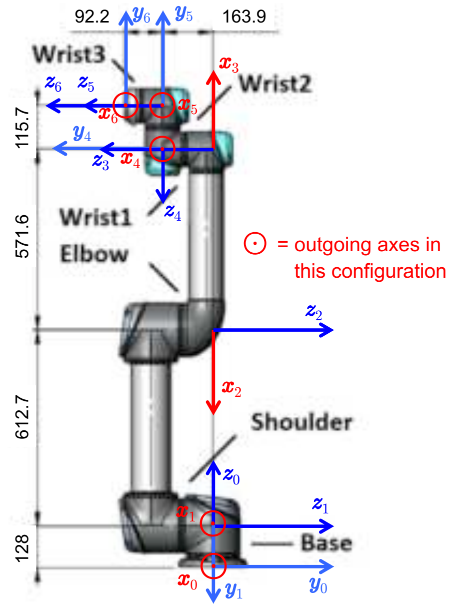


Figure 3: The assignment of DH frames as in Fig. 1, with all the $\boldsymbol{y}_i$ axes shown.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 = 128$	$\theta_1 = 0$
2	0	$a_2 = -612.7$	0	$\theta_2 = \pi/2$
3	π	$a_3 = 571.6$	0	$\theta_3 = \pi$
4	$\pi/2$	0	$d_4 = 163.9$	$\theta_4 = -\pi/2$
5	$-\pi/2$	0	$d_5 = -115.7$	$\theta_5 = 0$
6	0	0	$d_6 = 92.2$	$\theta_6 = 0$

Table 1: DH parameters (in mm or rad), with the value of $\boldsymbol{\theta} \in \mathbb{R}^6$ in the shown 'home' configuration.

Exercise 3

The primary workspace WS_1 of the planar 2R robot with the L-shaped second link of Fig. 2 is shown in Fig. 4, where the numerical data about link geometry have been taken into account. This workspace is that of a standard planar 2R robot having the first link of length $L = 0.4$ [m] and the second of length $D = \sqrt{A^2 + B^2} = \sqrt{0.16 + 0.09} = 0.5$ [m]. Thus, it is a circular annulus with external radius $R = L + D = 0.9$ [m] and internal radius $r = |L - D| = 0.1$ [m]. The secondary workspace WS_2 is empty.

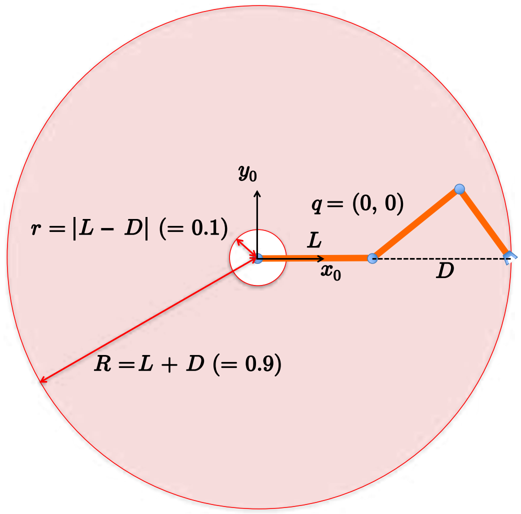
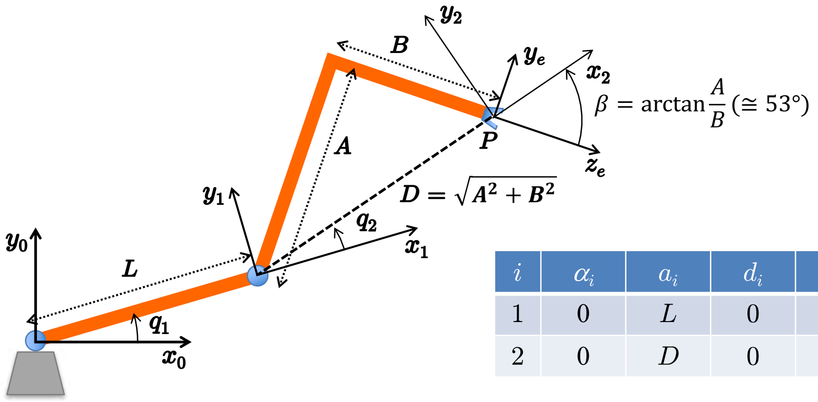


Figure 4: The primary workspace of the planar 2R robot with L-shaped second link of Fig. 2.

The Denavit-Hartenberg frames and the associated table of parameters for this robot are reported in Fig. 5. Note that $\boldsymbol{x}_2$, as required, is incident and orthogonal to the last defined joint axis (i.e., $\boldsymbol{z}_1$ at joint 2). Figure 4 shows also the robot in the configuration $\boldsymbol{q} = \boldsymbol{0}$. The constant rotation matrix ${}^2\boldsymbol{R}_e \in SO(3)$ from the DH frame RF_2 to the end-effector frame RF_e is given by

$$
{}^2\boldsymbol{R}_e = \begin{pmatrix} 0 & \sin\beta & \cos\beta \\ 0 & \cos\beta & -\sin\beta \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \\ -1 & 0 & 0 \end{pmatrix},
$$

with $\beta = \arctan(A/B) = \arctan 1.3333 = 53.13^\circ = 0.9273$ [rad].



i	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	0	D	0	q_2

Figure 5: DH frames and table of parameters for the robot of Fig. 2.

The solutions of two of the assigned inverse kinematics problems are straightforward. For the first problem with $\boldsymbol{p}_{e,1} = \begin{pmatrix} 0 & -0.9 \end{pmatrix}^T$, since $\|\boldsymbol{p}_{e,1}\| = 0.9 = R$, the desired end-effector position is on the external boundary of the workspace on the negative $\boldsymbol{y}_0$ axis: therefore, the only solution is $\boldsymbol{q} = \begin{pmatrix} -\pi/2 & 0 \end{pmatrix}^T$ [rad]. For the third problem, since $\boldsymbol{p}_{e,3} = \boldsymbol{0}$ does not belong to the robot workspace, there will be no solution. Finally, for the second problem with $\boldsymbol{p}_{e,2} = \begin{pmatrix} -0.4 & 0.7 \end{pmatrix}^T$, it is $r = 0.1 < \|\boldsymbol{p}_{e,2}\| = 0.8062 < 0.9 = R$. We are thus in a regular situation, and the known formulas for the inverse kinematics of a planar 2R robot can be applied, using as link lengths $l_1 = L = 0.4$ and $l_2 = D = 0.5$. The two solutions are:

$$\boldsymbol{q}^{[a]} = \begin{pmatrix} 90^\circ \\ 53.13^\circ \end{pmatrix} = \begin{pmatrix} \pi/2 \\ 0.9273 \end{pmatrix} \text{[rad]}, \qquad \boldsymbol{q}^{[b]} = \begin{pmatrix} 149.49^\circ \\ -53.13^\circ \end{pmatrix} = \begin{pmatrix} 2.6091 \\ -0.9273 \end{pmatrix} \text{[rad]}, \tag{4}$$

where, as usual, $q_2^{[b]} = -q_2^{[a]}$. We note that the first solution could have been found also with a simple geometric reasoning about the data of the problem (see Fig. 6).

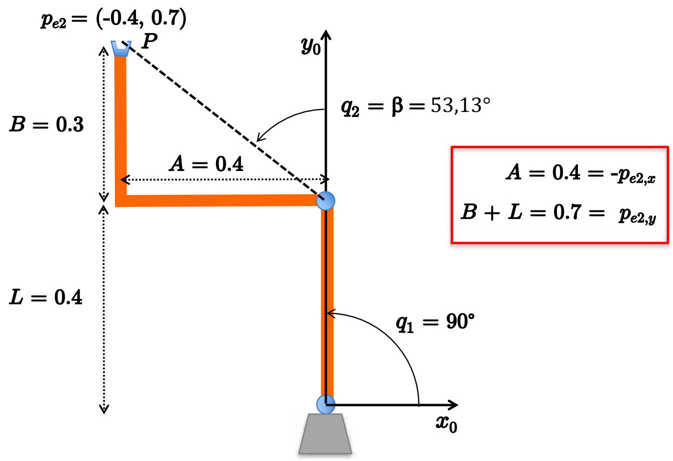


Figure 6: The geometric reasoning for obtaining the inverse kinematics solution $\boldsymbol{q}^{[a]}$ in (4).

Exercise 4

The given mapping

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} L_1 \cos q_1 + L_2 \cos q_2 + L_3 \cos q_3 \\ L_1 \sin q_1 + L_2 \sin q_2 + L_3 \sin q_3 \\ q_3 - q_2 \end{pmatrix} = \boldsymbol{r}(\boldsymbol{q}), \tag{5}$$

with $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$, has indeed a robotic interpretation. The task vector $\boldsymbol{r}$ contains the components of the end-effector position ($r_1 = p_x$, $r_2 = p_y$) and the relative (DH) angle between second and third link ($r_3 = \theta_3$) of a planar 3R robot, when the *absolute* angles q_i ($i = 1, 2, 3$) of the links w.r.t. the $\boldsymbol{x}_0$ axis are used as coordinates (see Fig. 7).

In order to solve the inverse kinematics problem

$$\boldsymbol{r}(\boldsymbol{q}) = \boldsymbol{r}_d = \begin{pmatrix} 0.7 \\ 0.5 \\ 0 \end{pmatrix}, \tag{6}$$

we would like to use the Newton method with the iterative formula for $k = 0, 1, 2, \ldots$

$$\boldsymbol{q}^{(k+1)} = \boldsymbol{q}^{(k)} + \boldsymbol{J}^{-1}\left(\boldsymbol{q}^{(k)}\right)\left(\boldsymbol{r}_d - \boldsymbol{r}(\boldsymbol{q}^{(k)})\right), \tag{7}$$

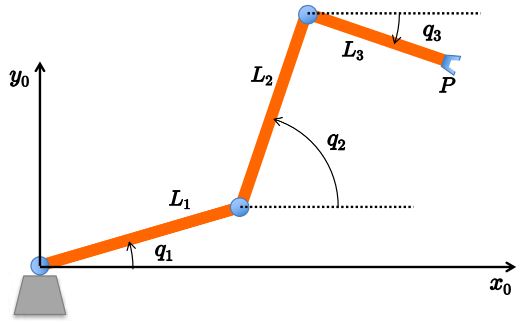


Figure 7: A planar 3R robot, with the definition of the used absolute coordinates.

where $\boldsymbol{J}(\boldsymbol{q})$ is the (analytical) Jacobian of the task kinematics. A pseudo-code should be written to this purpose (and eventually used as actual code in a chosen programming language, in order to compute a solution). This step is left to the reader.

Differentiating (5) with respect to $\boldsymbol{q}$ yields the 3×3 Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{r}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -L_1 \sin q_1 & -L_2 \sin q_2 & -L_3 \sin q_3 \\ L_1 \cos q_1 & L_2 \cos q_2 & L_3 \cos q_3 \\ 0 & -1 & 1 \end{pmatrix}, \tag{8}$$

which is nonsingular unless $\det \boldsymbol{J}(\boldsymbol{q}) = L_1 \left(L_2 \sin(q_2 - q_1) + L_3 \sin(q_3 - q_1) \right) = 0$.

At the initial guess $\boldsymbol{q}^{(0)} = \begin{pmatrix} 0 & \pi/2 & \pi/2 \end{pmatrix}^T$, using also the robot length data, we have a task error

$$\boldsymbol{e}^{(0)} = \boldsymbol{r}_d - \boldsymbol{r}\left(\boldsymbol{q}^{(0)}\right) = \begin{pmatrix} 0.7 \\ 0.5 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0 \\ 0 \end{pmatrix} \neq \boldsymbol{0} \quad \Rightarrow \quad \left\| \boldsymbol{e}^{(0)} \right\| = 0.3.$$

Moreover, since

$$\boldsymbol{J}\left(\boldsymbol{q}^{(0)}\right) = \begin{pmatrix} 0 & -0.3 & -0.2 \\ 0.4 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \det \boldsymbol{J}\left(\boldsymbol{q}^{(0)}\right) = 0.2,$$

we can safely compute the first iteration of (7) for $k = 0$:

$$\boldsymbol{q}^{(1)} = \boldsymbol{q}^{(0)} + \boldsymbol{J}^{-1}\left(\boldsymbol{q}^{(k)}\right) \boldsymbol{e}^{(0)} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi/2 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.6 \\ -0.6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.9708 \\ 0.9708 \end{pmatrix}.$$

The new guess leads to

$$\boldsymbol{r}\left(\boldsymbol{q}^{(1)}\right) = \begin{pmatrix} 0.6823 \\ 0.4127 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e}^{(1)} = \boldsymbol{r}_d - \boldsymbol{r}\left(\boldsymbol{q}^{(1)}\right) = \begin{pmatrix} 0.0177 \\ 0.0873 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\| \boldsymbol{e}^{(1)} \right\| = 0.0891,$$

showing a substantial progress toward a solution. As a matter of fact, thanks to the quadratic convergence rate of Newton method when near to a solution, we obtain (with our Matlab code) convergence of the sequence to a solution in only 4 iterations, with an accuracy on the error norm of less than $\epsilon = 10^{-5}$. The obtained numerical results are summarized in Tab. 2.

k	$\boldsymbol{q}^{(k)}$	$\boldsymbol{r}^{(k)}$
0	$\begin{pmatrix} 0 & 1.5708 & 1.5708 \end{pmatrix}$	$\begin{pmatrix} 0.4 & 0.5 & 0 \end{pmatrix}$
1	$\begin{pmatrix} 0 & 0.9708 & 0.9708 \end{pmatrix}$	$\begin{pmatrix} 0.68232 & 0.41267 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 0.24857 & 0.92796 & 0.92796 \end{pmatrix}$	$\begin{pmatrix} 0.68744 & 0.4986 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0.28297 & 0.88812 & 0.88812 \end{pmatrix}$	$\begin{pmatrix} 0.69953 & 0.49962 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 0.28555 & 0.88616 & 0.88616 \end{pmatrix}$	$\begin{pmatrix} 0.7 & 0.5 & 0 \end{pmatrix}$

k	$\boldsymbol{e}^{(k)}$	$\left\Vert\boldsymbol{e}^{(k)}\right\Vert$
0	$\begin{pmatrix} 0.3 & 0 & 0 \end{pmatrix}$	0.3
1	$\begin{pmatrix} 0.017679 & 0.087332 & 0 \end{pmatrix}$	0.089104
2	$\begin{pmatrix} 0.012558 & 0.0013956 & 0 \end{pmatrix}$	0.012635
3	$\begin{pmatrix} 0.00047072 & 0.00037521 & 0 \end{pmatrix}$	0.00060197
4	$\begin{pmatrix} 1.8871 \cdot 10^{-6} & 1.1168 \cdot 10^{-6} & 0 \end{pmatrix}$	$2.1928 \cdot 10^{-6}$

Table 2: Convergence with the Newton method (7) in solving the nonlinear system (6): Joint configurations and associated task values (top), and task errors and their norms (bottom) during the first four iterations.

The following remarks are in order.

- The first solution guess $\boldsymbol{q}^{(0)}$ already satisfies the constraint of a common absolute orientation for the second and third link, namely $q_3 - q_2 = 0$. The Newton method will keep this constraint satisfied over the iterations ($q_2^{(k)} = q_3^{(k)}$, $\forall k$).
- As a result, the third component of the task error, i.e., $e_3^{(k)}$, the only one with angular units, will remain identically zero. Therefore, the norm reported in the last column of Tab. 2 will have no problem of unit inconsistency, being always made just by the two components e_x and e_y of the position error.
- Enforcing $q_2 = q_3$ through the specific choice of $\boldsymbol{r}_d$ allows us to draw a simple conclusion on the number of solutions to the given inversion problem. Under such premise, this is equivalent to solving an inverse kinematics problem for a planar 2R robot having the first link of length $L_1 = 0.4$ [m] and the second of length $L_2' = L_2 + L_3 = 0.5$ [m]. Since $\|\boldsymbol{r}_d\| = 0.8602$, and this value is provided only by the positional task components, the desired end-effector position will be strictly inside the workspace of the equivalent planar 2R robot (using the notation of Exercise 3, $r = |L_2' - L_1| = 0.1 < 0.8602 < 0.9 = L_1 + L_2' = R$). It follows that the original inversion problem will have exactly two solutions —one of which has been found already with the iterative Newton method ($\boldsymbol{q}^* = \boldsymbol{q}^{(4)}$).

Exercise 5

Answer to the questions or comment/complete the statements, providing also a *short* motivation/explanation (within the given lines of text) for each of the 7 items.

1. Are there 3-dof robots with just a single inverse kinematics solution in their primary workspace? If so, which ones? If not, why?

A: *Yes, indeed. These are all PPP (Cartesian and gantry type) robots, no matter which is the sequence of prismatic joints —and also for non-perpendicular joint axes (twist angles* $\alpha_i \neq \pm\pi/2$*)!*

2. In order to measure the joint velocities of a robot, extra dedicated sensors may not be needed since ...

A: *... a digital position encoder, especially with high resolution and stability at high speed, can be used. Joint velocity is then estimated online by numerical differentiation of position measures (with various BDF = Backward Differentiation Formulas). Use of (kinematic) Kalman filters can also be considered.*

3. A large reduction ratio for a robot joint transmission is good because ..., and is bad because ...

A: *It is good because it amplifies the torque available beyond the transmission for accelerating loads with larger inertia. It is bad because larger reduction ratios are usually accompanied by higher energy dissipation due to friction and possible backlash, both effects reducing efficiency. On the other hand, speed reduction* per se *is not a major problem.*

4. Use of link acceleration measurements to generate torques that move the robot may be critical. Why?

A: *Assume no delays and no extra flexibility effects in a torque command loop based on the acceleration* $\ddot{x}$ *measured on a rotating link, at a distance* d *from its joint axis. Since we have* $\tau = J\ddot{\theta} = (J/d)\ddot{x}$*, torque and acceleration are at the same differential level. If we let* τ *depend on* $\ddot{x}$*, there would be an algebraic loop and thus problems with causality (and stability) of such feedback law.*

5. Compare an incremental encoder with $N = 900$ pulses per turn and quadrature electronics, mounted on a motor connected to the link with a reduction ratio $n_r = 40$, with a 16-bit absolute encoder mounted directly on the link side of the transmission. Which is better in terms of link position resolution?

A: *The resolution in the first case is* $r_1 = 360°/(4N \cdot n_r) = 360°/(4 \cdot 900 \cdot 40) = 2.5 \cdot 10^{-3}$ [deg]. *In the second case, the number of tracks* $N_t = 16$ *equals the bits used, so* $r_2 = 360°/2^{N_t} = 360°/65536 = 5.5 \cdot 10^{-3}$ [deg]. *Since* $r_1 < r_2$*, the considered incremental encoder arrangement gives a slightly better resolution (more than twice better, although still less than an order of magnitude).*

6. An installed 6-dof industrial robot has repeatability $\rho = 0.1$ [mm] and accuracy $\delta = 0.6$ [mm] in a certain region of its workspace. Which of these two parameters can be improved, and how?

A: *Repeatability depends on quality of the components (which cannot be changed on an installed robot). Robot accuracy can be improved instead by calibration (*δ *more than halved). Such procedures (software routines, with some extra sensing) can be used even if the robot is already installed on the factory floor.*

7. An object of mass $m = 5$ [kg] is hanging statically to a 6D F/T sensor, whose only non-zero outputs are $f_z = -49.05$ [N], $\mu_x = 7.3575$ [Nm]. Where is the object center of mass located in the sensor frame?

A: *A frame is placed at the center of the symmetric cylindric body of the F/T sensor, with its* z*-axis going up. In this sensor frame, the center of mass of the object is located on a vertical line passing through point* $(x, y) = (0, -0.15)$ [m]. *In fact, if the gravity force* $f_z = -mg_0 = -5g_0 = -49.05$ [N] *(in the opposite direction of* z*) has a vertical line of action crossing the negative* y*-axis at a distance* $d = 0.15$ [m] *from the origin, it will produce a positive (ccw) momentum around the* x*-axis of the sensor equal to* $\mu_x = f_z \cdot (-d) = (-49.05) \cdot (-0.15) = 7.3575$ [Nm]. *The object is certainly below the F/T sensor (it hangs), but nothing more can be said on the* z*-component of its position from this single measure.*

* * * * *

Robotics I - Extra Sheet #2 (for Exercise 4)

January 7, 2020

Name: ____

Answer to the questions or comment/complete the statements, providing also a *short* motivation/explanation (within the given lines of text) for each of the following 8 items.

1. At the same level of resolution, the cost of incremental encoders is usually less than that of absolute encoders because ...

2. What is the purpose of using Wheatstone bridge configurations in the electronics of strain gages?

3. Compare the link position resolution of an incremental encoder with 600 pulses per revolution (PPR) mounted on the motor having a transmission of reduction ratio $n_r = 30$, with that of an incremental encoder with 4000 PPR and quadrature electronics mounted directly on the link. Which is better?

4. Given a desired end-effector position of a planar 3R robot, the iterative Newton method can find all solutions to the inverse kinematics problem out of singularities. True or false? Why?

5. Which is the relation between the second derivative $\ddot{\boldsymbol{R}}$ of a time-varying rotation matrix $\boldsymbol{R}(t)$ and the associated angular velocity $\boldsymbol{\omega}$ and acceleration $\dot{\boldsymbol{\omega}}$?

6. For a joint that needs to move by $\Delta q > 0$, if the bounds on maximum absolute velocity and acceleration are related by $A_{max} = V_{max}^2/\Delta q$, is the minimum time acceleration profile always bang-coast-bang?

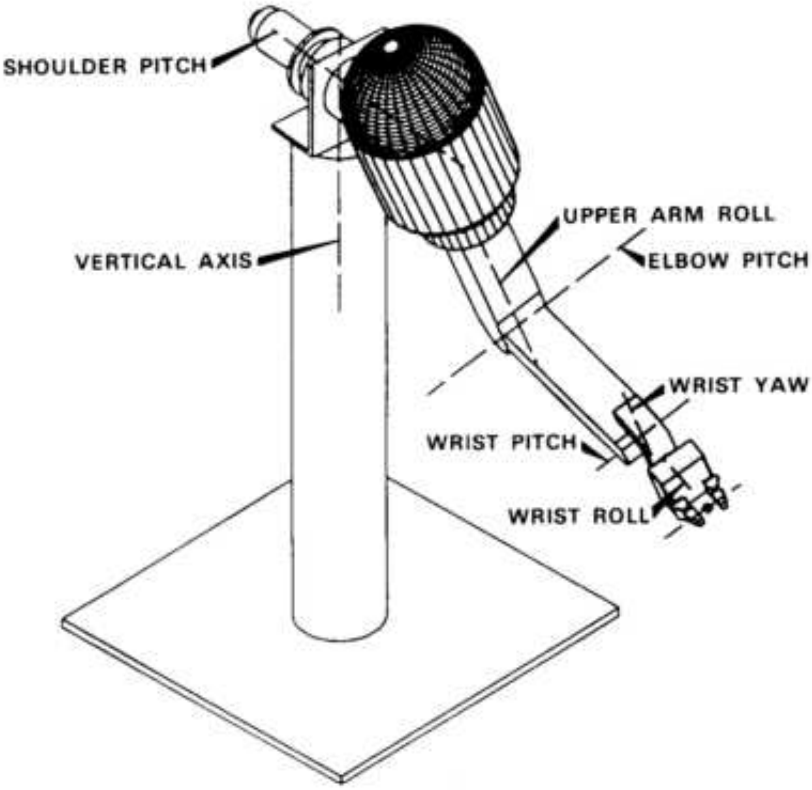
7. The uniform time scaling procedure allows to obtain the minimum motion time along a parametrized path under maximum velocity and acceleration constraints. True or false? Why?

8. Kinematic control laws designed at the Cartesian level are better than those designed at the joint level because ..., and are worse because ...

Robotics I - Extra Sheet #1 (for Exercise 1)

January 7, 2020

Name: ____



i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				

Robotics I

January 7, 2020

Exercise 1

Consider the 7R CESAR research manipulator in Fig. 1, developed at the Oak Ridge National Laboratories, USA. The robot has a spherical wrist, and the ordered sequence of the last three axes is pitch, yaw, and roll (see the naming of joint axes in the figure). Assume that the upper arm roll axis intersects the elbow pitch axis (we neglect here a small existing offset). The geometric dimensions are: upper arm length = 0.635; lower arm length = 0.508; shoulder offset = 0.356; distance from the wrist center to the center of the end-effector gripper jaws = 0.343 (all in [m]). Determine a frame assignment and the associated table of parameters following the Denavit-Hartenberg (DH) convention, and assign the geometric data to the corresponding constant DH parameters. Place the first DH frame so that $a_1 = d_1 = 0$, and the last frame with the origin O_7 at the center of the gripper jaws and with the axis $\boldsymbol{z}_7$ in the approach direction. Use the provided Extra Sheet #1 and return it, with your name added.

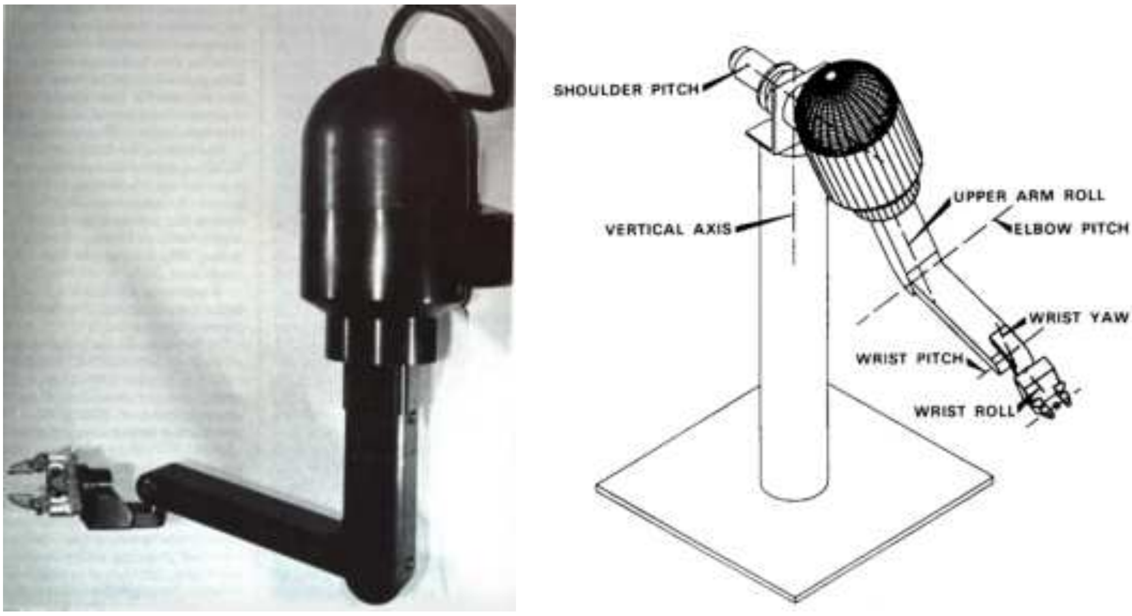


Figure 1: The 7R CESAR manipulator and its drawing with the names of the joint axes.

Exercise 2

For the CESAR manipulator of Exercise 1, determine the Jacobian matrix $\boldsymbol{J}_{L,w}(\boldsymbol{q})$ that relates the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^7$ to the linear velocity $\boldsymbol{v}_w \in \mathbb{R}^3$ of the wrist center. Note that the expression of this Jacobian becomes simpler when expressed in frame 2 or 3 (i.e., ${}^2\boldsymbol{J}_{L,w}$ or ${}^3\boldsymbol{J}_{L,w}$). Sketch the elements used by an iterative numerical scheme based on the Gradient method used to solve the inverse kinematics problem for a given desired position $\boldsymbol{p}_w \in \mathbb{R}^3$ of the center of the wrist, *without* considering unnecessary joint variables. In the generic case, how many inverse kinematic solutions are there for this problem?

Exercise 3

Consider the 3-dof, planar RPR robot in Fig. 2, with an associated base frame RF_0. The robot end-effector should move in contact with the shown surface from point $\boldsymbol{A} = (5.2, 1.5)$ to point $\boldsymbol{B} = (2.2, -2.5)$ (both in [m]). Moreover, the orientation of the end-effector with respect to the normal $\boldsymbol{n}$ to the surface should change continuously from the initial angle $\alpha_A = -60°$ to the final angle $\alpha_B = 30°$.

a) For the (Cartesian) task variables $\boldsymbol{r} = [\boldsymbol{p}^T\ \alpha]^T \in \mathbb{R}^3$, provide a spatial description $\boldsymbol{r} = \boldsymbol{r}(s)$ of the defined task in terms of a normalized parameter $s \in [0, 1]$.

b) At time $t = 0$, the robot end-effector is in point $\boldsymbol{A}$, with the correct orientation α_A, and its initial non-zero velocity is *consistent* with the execution of the desired task, with a linear speed $V = 2.5$ [m/s] and an angular speed $\Omega = 45$ [°/s]. From this state, plan a state-to-rest coordinated Cartesian motion that will complete the task in a given time $T = 2$, with continuity up to the acceleration (including at the two ends of the path).

c) What will be the value of the task velocity $\dot{\boldsymbol{r}}(t)$ at the half-time $t = T/2$?

d) Associate next the DH variables to the RPR robot, and assume that the prismatic joint range is limited to non-negative values of q_2 and that the third link has length $L = 1$ [m]. Show that the parametrized Cartesian task implies also a unique parametrized path in the robot joint space, and provide the analytic expression of $\boldsymbol{q} = \boldsymbol{q}(s)$.

e) What will be the value of the joint velocity $\dot{\boldsymbol{q}}(t)$ at the half-time $t = T/2$?

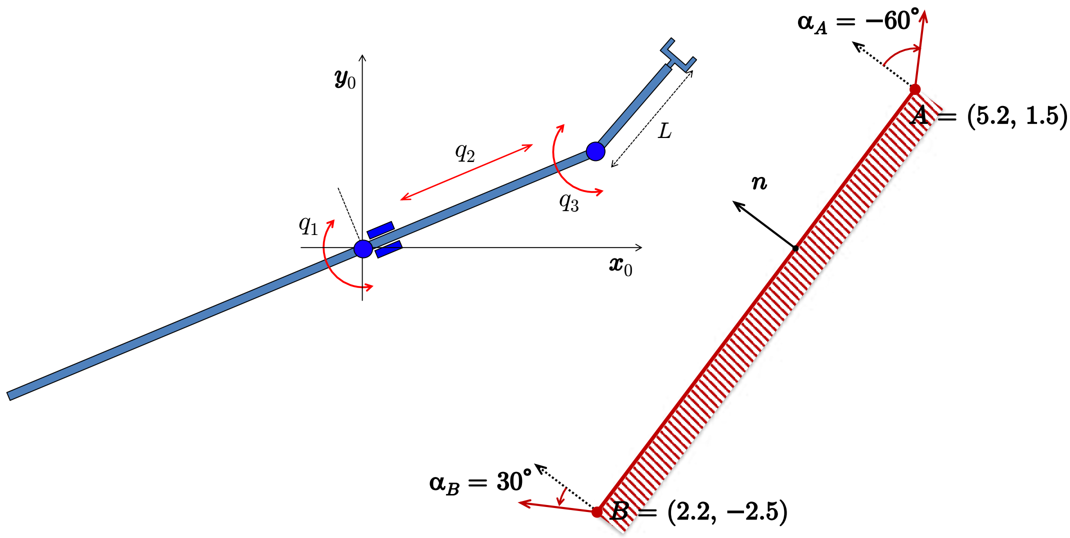


Figure 2: The planar RPR robot and the Cartesian task to be executed.

Exercise 4

A number of questions and statements are reported on the Extra Sheet #2. Fill in your answers and/or comments on the same sheet, providing also a short motivation/explanation for each item. Add your name on the sheet and return it.

[240 minutes, open books]

Solution

January 7, 2020

Exercise 1

A possible Denavit-Hartenberg frame assignment for the 7R CESAR manipulator is shown in Fig. 3, with the associated parameters reported in Tab. 1. In the table, a zero (with an asterisk) is set for a_3, according to the assumption about neglecting the small offset at the elbow joint. In the real robot, it is $a_3 = 0.029$ [m].

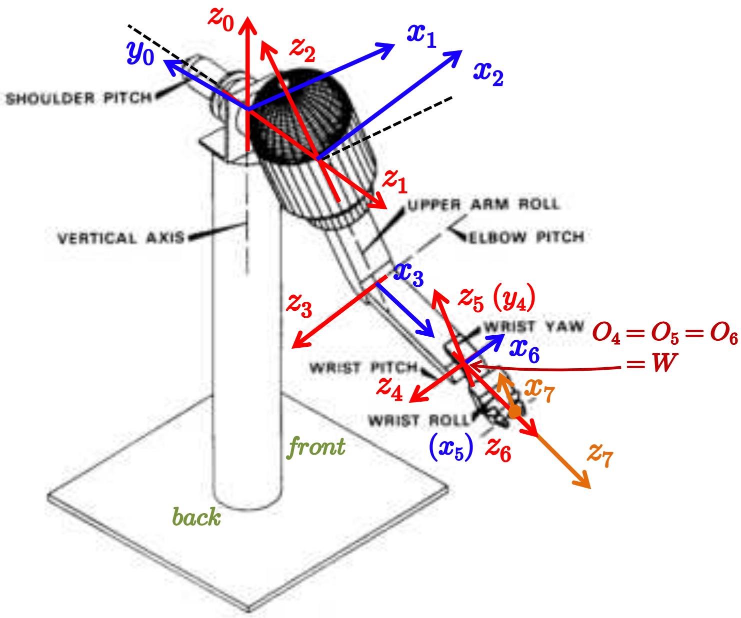


Figure 3: Assignment of the DH frames for the 7R CESAR manipulator (view from the back).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$-\pi/2$	0	$d_2 = 0.356$	q_2
3	$\pi/2$	0^*	$d_3 = -0.635$	q_3
4	0	$a_4 = 0.508$	0	q_4
5	$-\pi/2$	0	0	q_5
6	$\pi/2$	0	0	q_6
7	0	0	$d_7 = 0.343$	q_7

Table 1: The DH table of parameters for the 7R CESAR manipulator corresponding to Fig. 3.

The manipulator is viewed from the front side (and with the same frame assignment) in Fig. 4, which is taken from the original paper:

[1] R.V. Dubey, J.A. Euler, and S.M. Babcock, "Real-time implementation of an optimization scheme for seven-degree-of-freedom redundant manipulators," *IEEE Trans. on Robotics and Automation*, vol. 7, no. 5, pp. 579–588, 1991.

Therein, also the non-zero parameters a_i and d_i are shown. The robot is in a slightly different configuration in the two figures: in Fig. 4, it is $\boldsymbol{q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \pi/2 & \pi/2 \end{pmatrix}^T$ [rad], whereas q_1 is slightly negative, q_2 is positive, and $q_3 \simeq -\pi/2$ in Fig. 3.

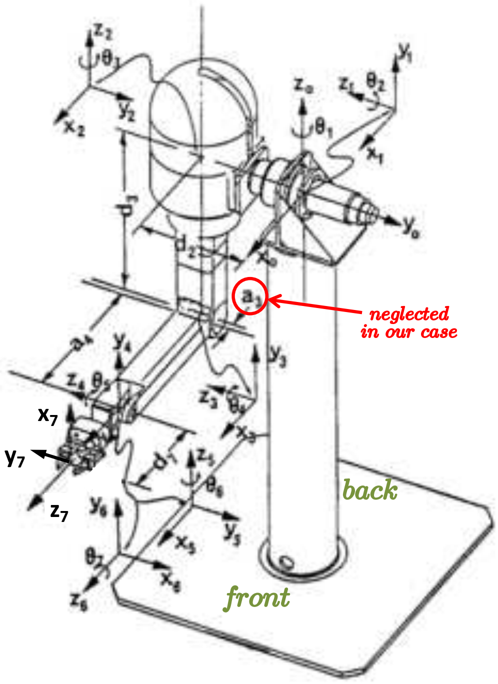


Figure 4: Front view of the DH assignment for the 7R CESAR manipulator (modified from [1]).

Exercise 2

Based on Tab. 1, in order to determine the position $\boldsymbol{p}_w$ of the center of the spherical wrist, i.e., the position of the origin O_4, we need only the following DH homogenous transformation matrices:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_{12} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & -s_4 & 0 & a_4 c_4 \\ s_4 & c_4 & 0 & a_4 s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

where the shorthand notations $s_i = \sin q_i$, $c_i = \cos q_i$ have been used.

The position $\boldsymbol{p}_w$ is computed from

$$\boldsymbol{p}_{w,H} = \begin{pmatrix} \boldsymbol{p}_w \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)\right),$$

giving

$$\boldsymbol{p}_w(\boldsymbol{q}) = \begin{pmatrix} d_2 s_1 - d_3 c_1 s_2 - a_4 c_1 s_2 s_4 - a_4 c_4 (s_1 s_3 - c_1 c_2 c_3) \\ -d_2 c_1 - d_3 s_1 s_2 - a_4 s_1 s_2 s_4 + a_4 c_4 (c_1 s_3 + s_1 c_2 c_3) \\ d_3 c_2 + a_4 c_2 s_4 + a_4 s_2 c_3 c_4 \end{pmatrix}.$$

The 3×7 Jacobian matrix $\boldsymbol{J}_{L,w}$ in $\boldsymbol{v}_w = \dot{\boldsymbol{p}}_w = \boldsymbol{J}_{L,w}(\boldsymbol{q})\dot{\boldsymbol{q}}$ can be computed in two alternative but equivalent ways (i.e., analytically or geometrically) as

$$\boldsymbol{J}_{L,w}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_w(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{0,w} & \boldsymbol{z}_1 \times \boldsymbol{p}_{1,w} & \boldsymbol{z}_2 \times \boldsymbol{p}_{2,w} & \boldsymbol{z}_3 \times \boldsymbol{p}_{3,w} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},$$

where $\boldsymbol{p}_{0,w} = \boldsymbol{p}_w$. Since there is no actual dependence of $\boldsymbol{p}_w(\boldsymbol{q})$ on q_5, q_6, and q_7, the last three columns of $\boldsymbol{J}_{L,w}(\boldsymbol{q})$ are automatically zero and can be skipped. We will denote by $\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q})$ the resulting 3×4 Jacobian matrix. Performing computations (possibly with the Symbolic Toolbox of Matlab) one obtains

$$\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q}) = \begin{pmatrix} d_2 c_1 + d_3 s_1 s_2 + a_4 s_1 s_2 s_4 - a_4 c_4 (c_1 s_3 + s_1 c_2 c_3) & -c_1 (d_3 c_2 + a_4 c_2 s_4 + a_4 s_2 c_3 c_4) \\ d_2 s_1 - d_3 c_1 s_2 - a_4 c_1 s_2 s_4 - a_4 c_4 (s_1 s_3 - c_1 c_2 c_3) & -s_1 (d_3 c_2 + a_4 c_2 s_4 + a_4 s_2 c_3 c_4) \\ 0 & -d_3 s_2 - a_4 s_2 s_4 + a_4 c_2 c_3 c_4 \end{pmatrix.$$
$$\begin{matrix} -a_4 c_4 (s_1 c_3 + c_1 c_2 s_3) & -a_4 c_1 s_2 c_4 + a_4 s_4 (s_1 s_3 - c_1 c_2 c_3) \\ a_4 c_4 (c_1 c_3 - s_1 c_2 s_3) & -a_4 s_1 s_2 c_4 - a_4 s_4 (c_1 s_3 + s_1 c_2 c_3) \\ -a_4 s_2 s_3 c_4 & a_4 c_2 c_4 - a_4 s_2 c_3 s_4 \end{matrix}\Bigg).$$

By noticing the presence of some recurrent trigonometric terms, one may obtain simpler expressions of this Jacobian by expressing it in a rotated reference frame (say, RF_1, RF_2 or RF_3). Following the hint given in the text, the simplest form is in fact obtained when working in RF_2, i.e., expressing the linear velocity of center of the wrist as ${}^2\boldsymbol{v}_w = {}^2\boldsymbol{J}_{L,w}(\boldsymbol{q})\dot{\boldsymbol{q}}$. We have

$$\begin{aligned} {}^2\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q}) &= {}^1\boldsymbol{R}_2^T(q_2)\left({}^0\boldsymbol{R}_1^T(q_1)\,\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q})\right) = {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q}) \\ &= \begin{pmatrix} c_2 (d_2 - a_4 s_3 c_4) & -d_3 - a_4 s_4 & a_4 s_3 c_4 & -a_4 c_3 s_4 \\ -d_2 s_2 - a_4 s_2 s_4 + a_4 c_2 c_3 c_4 & 0 & a_3 c_3 c_4 & -a_4 s_3 s_4 \\ -s_2 (d_2 - a_4 s_3 c_4) & a_4 c_3 c_4 & 0 & a_4 c_4 \end{pmatrix}. \end{aligned} \tag{1}$$

To find a solution to the inverse kinematics problem for a given desired position $\boldsymbol{p}_{w,d}$ of the center of the wrist, we can only resort to an iterative numerical scheme. The problem has in fact an infinite number of solutions that cannot be obtained in closed form. The Gradient method will use the transpose of the Jacobian matrix $\bar{\boldsymbol{J}}_{L,w}(\boldsymbol{q})$, possibly expressed in the simpler form (1). Moreover, the updates of the algorithm will concern only the first four joint variables $\bar{\boldsymbol{q}} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T$ —the remaining ones being irrelevant. Thus, the generic iteration $k \geq 1$ of the Gradient method will be

$$\bar{\boldsymbol{q}}^{[k+1]} = \bar{\boldsymbol{q}}^{[k]} + \alpha^{[k]} \cdot {}^2\bar{\boldsymbol{J}}_{L,w}^T\big(\bar{\boldsymbol{q}}^{[k]}\big)\,{}^0\boldsymbol{R}_2^T(q_1^{[k]}, q_2^{[k]})\Big(\boldsymbol{p}_{w,d} - \boldsymbol{p}_w\big(\bar{\boldsymbol{q}}^{[k]}\big)\Big), \tag{2}$$

with a stepsize $\alpha^{[k]} > 0$ possibly varying over iterations. Note the added rotation matrix needed to express the Cartesian position error $\boldsymbol{e}^{[k]} = \boldsymbol{p}_{w,d} - \boldsymbol{p}_w^{[k]}$ at iteration k in the frame RF_2 as ${}^2\boldsymbol{e}^{[k]}$.

Exercise 3

The first part of the problem, items a) to c), is concerned only with the specification of the task, independently from the robot that has to execute it (in this case a RPR robot).

The desired planar task is three-dimensional, involving the motion in position $\boldsymbol{p}(t) \in \mathbb{R}^2$ along the linear surface from $\boldsymbol{A}$ to $\boldsymbol{B}$ and the simultaneous change of orientation $\alpha(t) \in \mathbb{R}$ from α_A to α_B, with the angle $\alpha(t)$ being defined w.r.t. the constant normal $\boldsymbol{n}$ to the surface as in Fig. 2. We approach the problem by decomposing the definition of the task trajectory in (normalized) space and time, i.e.,

$$\boldsymbol{r}(t) = \begin{pmatrix} \boldsymbol{p}(t) \\ \alpha(t) \end{pmatrix}, \text{ for } t \in [0, T] \quad \Rightarrow \quad \boldsymbol{r}(s) = \begin{pmatrix} \boldsymbol{p}(s) \\ \alpha(s) \end{pmatrix}, \text{ with } s \in [0,1], \quad s = s(t), \text{ for } t \in [0, T].$$

The simplest parametrization of the desired task is through the linear expressions

$$\boldsymbol{p}(s) = \boldsymbol{A} + (\boldsymbol{B} - \boldsymbol{A})\, s \in \mathbb{R}^2, \qquad \alpha(s) = \alpha_A + (\alpha_B - \alpha_A)\, s \in \mathbb{R}, \qquad \text{with } s \in [0,1], \tag{3}$$

or, replacing numerical values,

$$\boldsymbol{r}(s) = \begin{pmatrix} \boldsymbol{A} \\ \alpha_A \end{pmatrix} + \begin{pmatrix} \boldsymbol{B} - \boldsymbol{A} \\ \alpha_B - \alpha_A \end{pmatrix} s = \begin{pmatrix} 5.2 \\ 1.5 \\ -60^\circ \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ 90^\circ \end{pmatrix} s, \qquad s \in [0,1].$$

As a consequence, we have also

$$\dot{\boldsymbol{p}}(t) = \frac{d\boldsymbol{p}(t)}{dt} = \frac{d\boldsymbol{p}(s)}{ds}\, \dot{s}(t) = \boldsymbol{p}'(s)\dot{s}(t) = (\boldsymbol{B} - \boldsymbol{A})\, \dot{s}(t), \qquad \ddot{\boldsymbol{p}}(t) = (\boldsymbol{B} - \boldsymbol{A})\, \ddot{s}(t), \tag{4}$$

and

$$\dot{\alpha}(t) = \frac{d\alpha(t)}{dt} = \frac{d\alpha(s)}{ds}\, \dot{s}(t) = \alpha'(s)\dot{s}(t) = (\alpha_B - \alpha_A)\, \dot{s}(t), \qquad \ddot{\alpha}(t) = (\alpha_B - \alpha_A)\, \ddot{s}(t). \tag{5}$$

The definition of the timing law $s = s(t)$ takes into account the smoothness requirement (continuity up to the acceleration $\ddot{s}(t)$, for $t \in [0, T]$) and the boundary conditions at the initial time $t = 0$ and final time $t = T$. In particular, the linear and angular velocity should satisfy the non-zero initial conditions

$$\dot{\boldsymbol{p}}(0) = (\boldsymbol{B} - \boldsymbol{A})\, \dot{s}(0) = \frac{\boldsymbol{B} - \boldsymbol{A}}{\|\boldsymbol{B} - \boldsymbol{A}\|}\, V \qquad \text{and} \qquad \dot{\alpha}(0) = (\alpha_B - \alpha_A)\, \dot{s}(0) = \Omega,$$

implying the following common conditions on the initial speed

$$\dot{s}(0) = \frac{V}{\|\boldsymbol{B} - \boldsymbol{A}\|} = \frac{2.5\,[\text{m/s}]}{5\,[\text{m}]} = 0.5 = v_i \qquad \text{and} \qquad \dot{s}(0) = \frac{\Omega}{\alpha_B - \alpha_A} = \frac{45\,[^\circ/\text{s}]}{90\,[^\circ]} = 0.5 = v_i. \tag{6}$$

The equality of the two numerical values for the alternative expressions of $\dot{s}(0) = v_i$ in (6) is indeed necessary in order to have an initial velocity that is *consistent* with the desired task. On the other hand, the condition of zero velocity at the final instant (a rest state) implies for the final speed

$$\dot{s}(T) = v_f = 0. \tag{7}$$

For the continuity of the scalar acceleration $\ddot{s}(t)$ also at $t = 0$ and $t = T$, we have to impose boundary accelerations too, though with arbitrary values a_i and a_f, i.e.,

$$\ddot{s}(0) = a_i, \qquad \ddot{s}(T) = a_f. \tag{8}$$

In fact, leaving instead these accelerations unconstrained would result in specific values at the boundaries (as outcome of a lower-order interpolation problem) that may not match those before the start and after the end of the planned motion (i.e., $\ddot{s}(0^-) \neq \ddot{s}(0^+)$ and/or $\ddot{s}(T^-) \neq \ddot{s}(T^+)$). As a result of the boundary conditions (6–8), the function interpolating an initial value $s(0) = s_i$ to a final value $s(T) = s_f$ is chosen to be a quintic polynomial. Its general expression can be given in terms of the normalized time $\tau = t/T$ as (see, e.g., the lecture slides)

$$\begin{aligned} s(\tau) &= (1-\tau)^3 \left(s_i + (3s_i + v_i T)\tau + 0.5\left(12 s_i + 6 v_i T + a_i T^2\right)\tau^2\right) \\ &\quad + \tau^3 \left(s_f + (3s_f - v_f T)(1-\tau) + 0.5\left(12 s_f - 6 v_f T + a_f T^2\right)(1-\tau)^2\right), \quad \tau \in [0,1]. \end{aligned} \tag{9}$$

Specializing (9) to the case at hand ($s_i = 0$, $s_f = 1$, $v_i = 0.5$, $v_f = 0$, and $T = 2$ [s]) and choosing for simplicity zero values for the boundary accelerations, $a_i = a_f = 0$, results in

$$s(\tau) = (1-\tau)^3(\tau + 3\tau^2) + \tau^3(1 + 3(1-\tau) + 6(1-\tau)^2) = 3\tau^5 - 7\tau^4 + 4\tau^3 + \tau, \quad \tau \in [0,1]. \tag{10}$$

Furthermore,

$$\dot{s}(\tau) = \frac{ds(\tau)}{dt} = \frac{ds(\tau)}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{ds(\tau)}{d\tau} = 0.5\left(15\tau^4 - 28\tau^3 + 12\tau^2 + 1\right), \qquad \tau \in [0,1], \tag{11}$$

and

$$\ddot{s}(\tau) = \frac{1}{T^2}\frac{d^2 s(\tau)}{d\tau^2} = 0.25\left(60\tau^3 - 84\tau^2 + 24\tau\right) = \left(15\tau^2 - 21\tau + 6\right)\tau, \qquad \tau \in [0,1]. \tag{12}$$

The plots of (10–12) over the actual time $t \in [0,T] = [0,2]$ are shown in Fig. 5. Note in particular the asymmetry of the speed profile. Also, at the half-time $t = T/2 = 1$ (or $\tau = 0.5$) more than half of the path length $\|\boldsymbol{B} - \boldsymbol{A}\| = 5$ [m] will have been traced, being $s_m = s(0.5) = 0.6562 > 0.5$.

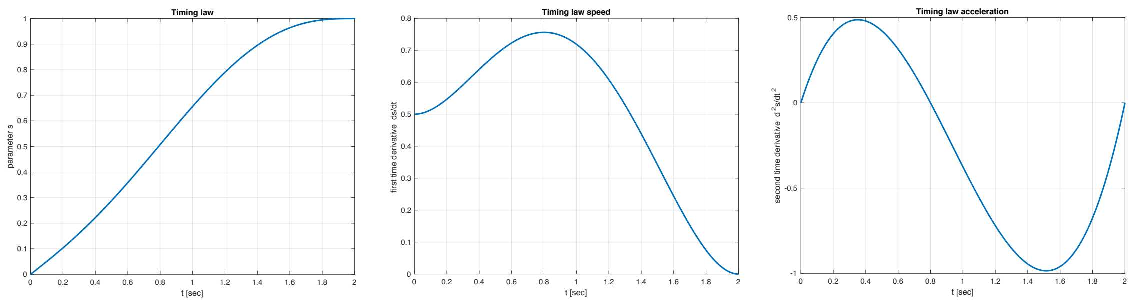


Figure 5: The time evolution of $s(t)$, $\dot{s}(t)$, and $\ddot{s}(t)$, as given by eqs. (10–12).

The task velocity at the half-time is obtained from (4), (5) and (11) as[1]

$$\dot{\boldsymbol{r}}(1) = \begin{pmatrix} \dot{\boldsymbol{p}}(1) \\ \dot{\alpha}(1) \end{pmatrix} = \begin{pmatrix} \boldsymbol{B} - \boldsymbol{A} \\ \alpha_B - \alpha_A \end{pmatrix} \dot{s}(0.5) = \begin{pmatrix} -3 \\ -4 \\ 90^\circ \end{pmatrix} 0.7188 = \begin{pmatrix} -2.1562 \\ -2.8750 \\ 64.69\ [^\circ/\mathrm{s}] \end{pmatrix}.$$

Figures 6–8 show the time evolutions of the two coordinates, respectively, of the Cartesian position, linear velocity, and linear acceleration, placed side by side with the evolution of the angle with respect to the surface normal $\boldsymbol{n}$, its speed and acceleration. As it can be seen, all boundary conditions are satisfied. Moreover, coordinated motion follows from the chosen planning approach, with space-time decomposition and a common timing law for all variables: all quantities start and end their motion at the same time.

[1] In this formula, $\dot{\boldsymbol{p}}$ and $\dot{\alpha}$ are expressed with respect to time t, whereas $\dot{s}$ uses the normalized time τ.

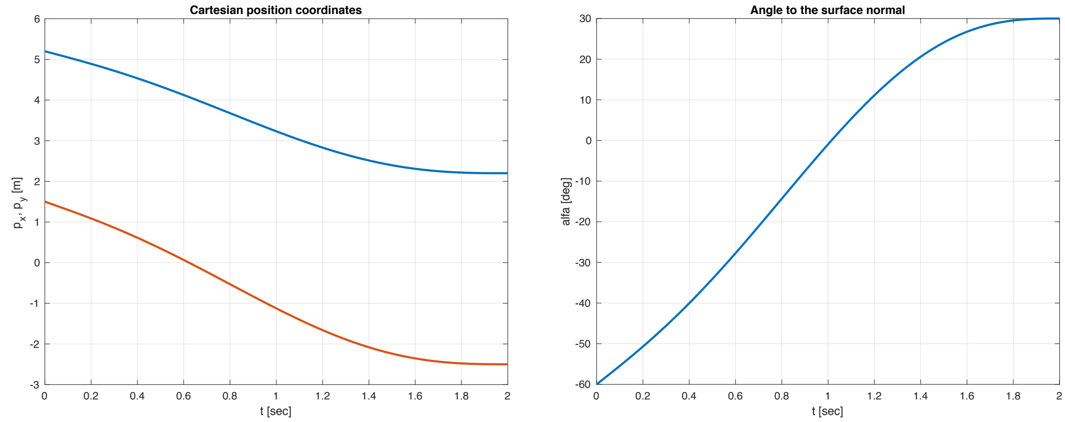


Figure 6: Cartesian position $\boldsymbol{p}(t) = (p_x(t), p_y(t))$ [left] and angle $\alpha(t)$ to the surface normal [right].

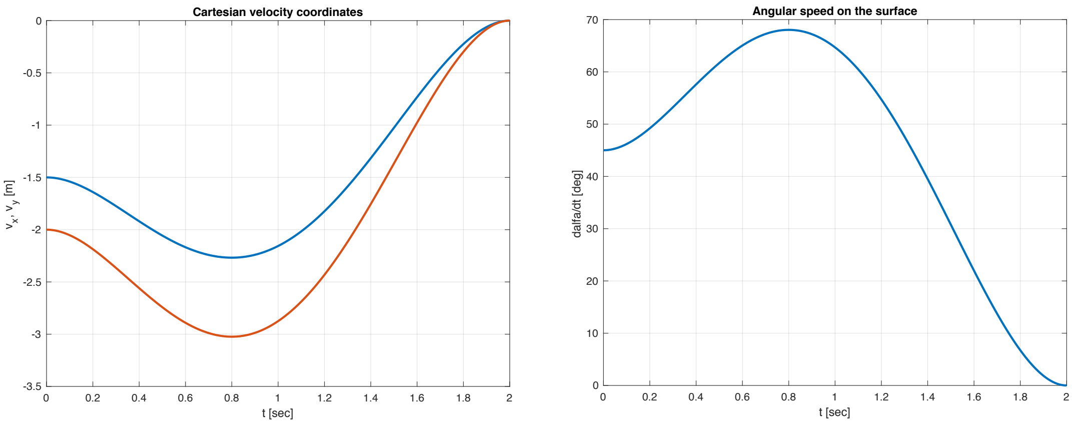


Figure 7: Cartesian velocity $\dot{\boldsymbol{p}}(t) = (v_x(t), v_y(t))$ [left] and angular speed $\dot{\alpha}(t)$ [right].

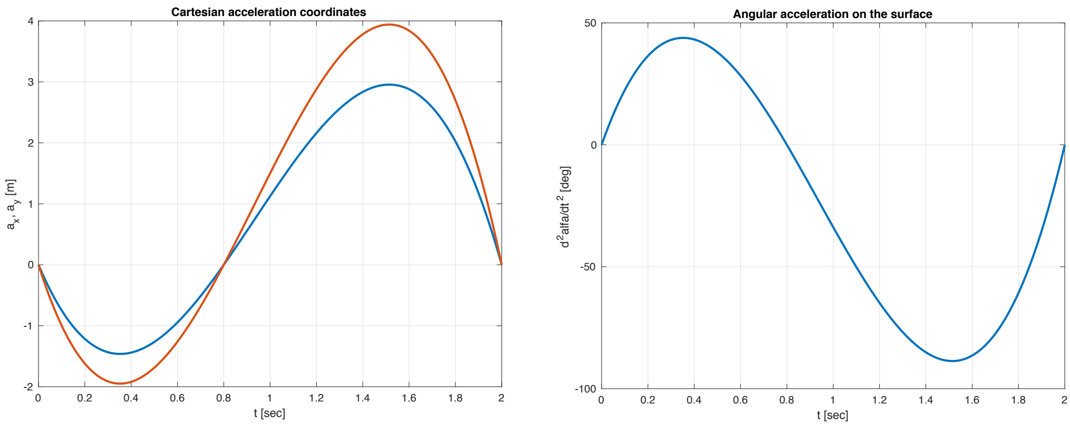


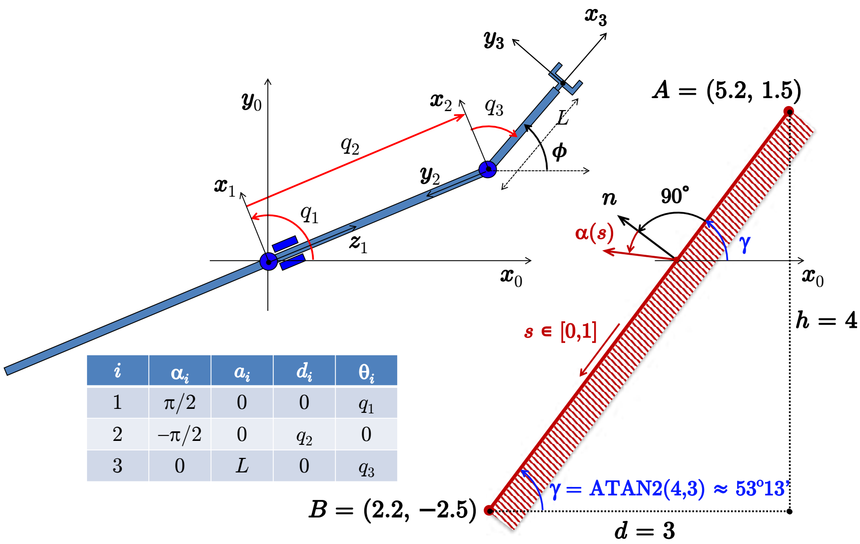
Figure 8: Cartesian acceleration $\ddot{\boldsymbol{p}}(t) = (a_x(t), a_y(t))$ [left] and angular acceleration $\ddot{\alpha}(t)$ [right].

In the second part of the problem, items (d) and (e), we have to map the task planned so far into a joint motion of the planar RPR robot. The assigned DH frames and the associated joint variables are defined in Fig. 9. The direct kinematics for the robot end-effector position $\boldsymbol{p}_e \in \mathbb{R}^2$ and its absolute orientation (as given by an angle $\phi_e \in \mathbb{R}$ defined w.r.t. the axis $\boldsymbol{x}_0$ of the robot base frame) is

$$\begin{pmatrix} \boldsymbol{p}_e \\ \phi_e \end{pmatrix} = \boldsymbol{f}_e(\boldsymbol{q}) = \begin{pmatrix} L\cos(q_1+q_3) + q_2 \sin q_1 \\ L\sin(q_1+q_3) - q_2\cos q_1 \\ q_1 + q_3 \end{pmatrix}, \tag{13}$$

and the associated Jacobian is

$$\boldsymbol{J}_e(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_e(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -L\sin(q_1+q_3) + q_2\cos q_1 & \sin q_1 & -L\sin(q_1+q_3) \\ L\cos(q_1+q_3) + q_2 \sin q_1 & -\cos q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}. \tag{14}$$



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$-\pi/2$	0	q_2	0
3	0	L	0	q_3

Figure 9: DH frames and joint variables for the planar RPR robot.

To match the orientation of the task with the robot end-effector orientation, we should consider the different definition of the two angles ϕ_e (pertaining to the robot) and α (pertaining to the task). Still with reference to Fig. 9, it is easy to see that we have the relation[2]

$$\begin{aligned} \phi_e &= \alpha_{\text{abs}} - 180° = (\text{ATAN2}\{A_y - B_y, A_x - B_x\} + 90° + \alpha) - 180° \\ &= \alpha + \text{ATAN2}\{4,3\} - 90° = \alpha - 36.87°. \end{aligned} \tag{15}$$

The angle α_{abs} is the absolute orientation w.r.t. the axis $\boldsymbol{x}_0$ imposed by the task, based on the desired angle α specified w.r.t. the normal $\boldsymbol{n}$ to the surface. The subtraction (or also the addition) of a half turn is due to the fact that the robot end-effector has to approach the desired orientation from the external side of the surface.

[2]Although most of the angular quantities in this exercise are expressed in degrees (just to match the initial format of the problem data), remember that computations (e.g., arguments of trigonometric functions) are always, more conveniently expressed in radians.

For any desired value $\boldsymbol{r}(s)$ of the parametrized task in (3), we obtain, using also (15), equivalent conditions for the robot end-effector variables as

$$\boldsymbol{p}_{ed}(s) = \begin{pmatrix} p_{ed,x}(s) \\ p_{ed,y}(s) \end{pmatrix} = \boldsymbol{p}(s), \qquad \phi_{ed}(s) = \alpha(s) - 36.87^\circ, \qquad \text{with } s \in [0,1]. \tag{16}$$

The inverse kinematics problem consists in finding one (or more) $\boldsymbol{q}(s) = \begin{pmatrix} q_1(s) & q_2(s) & q_3(s) \end{pmatrix}^T$ in parametrized form, such that

$$\begin{pmatrix} L\cos\left(q_1(s) + q_3(s)\right) + q_2(s)\sin q_1(s) \\ L\sin\left(q_1(s) + q_3(s)\right) - q_2(s)\cos q_1(s) \\ q_1(s) + q_3(s) \end{pmatrix} = \begin{pmatrix} p_{ed,x}(s) \\ p_{ed,y}(s) \\ \phi_{ed}(s) \end{pmatrix}, \qquad \forall s \in [0,1]. \tag{17}$$

Thanks to the non-negative assumption made on the prismatic joint variable, $q_2 \geq 0$, equations 17 admit one and only one solution $\boldsymbol{q}(s)$ for each $s \in [0,1]$. In fact, we have

$$q_1(s) + q_3(s) = \phi_{ed}(s) \quad \Rightarrow \quad \begin{pmatrix} q_2(s)\sin q_1(s) \\ -q_2(s)\cos q_1(s) \end{pmatrix} = \begin{pmatrix} p_{ed,x}(s) - L\cos\phi_{ed}(s) \\ p_{ed,y}(s) - L\sin\phi_{ed}(s) \end{pmatrix},$$

and so

$$\begin{aligned} q_1(s) &= \text{ATAN2}\left\{p_{ed,x}(s) - L\cos\phi_{ed}(s),\ -\big(p_{ed,y}(s) - L\sin\phi_{ed}(s)\big)\right\}, \\ q_2(s) &= \sqrt{\left(p_{ed,x}(s) - L\cos\phi_{ed}(s)\right)^2 + \left(p_{ed,y}(s) - L\sin\phi_{ed}(s)\right)^2} \geq 0, \\ q_3(s) &= \phi_{ed}(s) - q_1(s). \end{aligned} \tag{18}$$

As a result, the parametrized Cartesian task $\boldsymbol{r} = \boldsymbol{r}(s)$ implies also a unique parametrized path $\boldsymbol{q} = \boldsymbol{q}(s)$ in the robot joint space.

At the motion half-time $t = T/2 = 1$ [s], for the desired task expressed in time we have

$$\boldsymbol{r}(1) = \begin{pmatrix} \boldsymbol{p}(1) \\ \alpha(1) \end{pmatrix} = \begin{pmatrix} 3.2313 \\ -1.1250 \\ -0.94^\circ \end{pmatrix}.$$

Evaluating (16) at the corresponding $s = s_m = 0.6562$ gives

$$\boldsymbol{p}_{ed}(s_m) = \begin{pmatrix} 3.2313 \\ -1.1250 \end{pmatrix} \text{ [m]}, \qquad \phi_{ed}(s_m) = -37.80^\circ \ (= -0.6599 \text{ [rad]}),$$

and so, from (18),

$$\boldsymbol{q}_m = \boldsymbol{q}(s_m) = \begin{pmatrix} 78.15^\circ \\ 2.4943 \\ -115.96^\circ \end{pmatrix} = \begin{pmatrix} 1.3641 \\ 2.4943 \\ -2.0239 \end{pmatrix} \text{ [rad, m, rad]}.$$

The configuration of the RPR robot at this stage along the path from $\boldsymbol{A}$ to $\boldsymbol{B}$ is shown in Fig. 10. Note that in practice the end-effector is almost oriented along the normal to the linear surface.

Finally, using $\boldsymbol{q}_m$ to evaluate the Jacobian in (14) and plugging in the link length $L = 1$, we

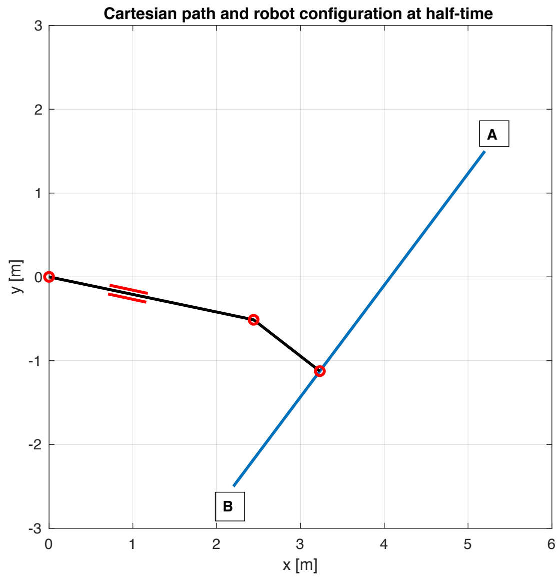


Figure 10: The RPR robot configuration $\boldsymbol{q}_m$ reached at the half-time $t = T/2 = 1$ [s] of the task.

compute the joint velocity $\dot{\boldsymbol{q}}_m$ at the motion half-time from the corresponding task velocity $\dot{\boldsymbol{r}}_m$:

$$\dot{\boldsymbol{q}}_m = \boldsymbol{J}_e^{-1}(\boldsymbol{q}_m)\dot{\boldsymbol{r}}_m = \begin{pmatrix} 1.1250 & 0.9787 & 0.6130 \\ 3.2313 & -0.2053 & 0.7901 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2.1562 \\ -2.8750 \\ 1.1290 \end{pmatrix}$$

$$= \begin{pmatrix} -1.7125 \\ -2.0145 \\ 2.8415 \end{pmatrix} [\text{rad/s, m/s, rad/s}] = \begin{pmatrix} -98.12 \\ -2.0145 \\ 162.81 \end{pmatrix} [°\text{/s, m/s, °/s}].$$

Exercise 4

Answer to the questions or comment/complete the statements, providing also a *short* motivation/explanation for each of the following 8 items.

1. At the same level of resolution, the cost of incremental encoders is usually less than that of absolute encoders because ...

A: ... incremental encoders have a simpler structure for generating pulses, with a regular optical disc and only three channels, while absolute encoders have more electronic components inside (arrays of opaque/transparent LED/sensor pairs), Gray-to-binary code converters, multi-turn option, an optical disc with more complex patterns to be engraved, etc.

2. What is the purpose of using Wheatstone bridge configurations in the electronics of strain gages?

A: Strain gages are configured in Wheatstone bridge circuits (the electrical equivalent of two parallel voltage divider circuits) so as to better detect small changes in resistance (and thus in the applied strain). Moreover, special such configurations (e.g., two strain gages used in a quarter-bridge) help in further minimize undesired effects due to temperature changes.

3. Compare the link position resolution of an incremental encoder with 600 pulses per revolution (PPR) mounted on the motor having a transmission of reduction ratio $n_r = 30$, with that of an incremental encoder with 4000 PPR and quadrature electronics mounted directly on the link. Which is better?

A: *The position resolution at the link level is* $r_1 = 360°/(600 \cdot 30) = 0.02°$ *in the first case, and* $r_2 = 360°/(4000 \cdot 4) = 0.0225°$ *in the second case. Thus, the first setup is (slightly) better.*

4. Given a desired end-effector position of a planar 3R robot, the iterative Newton method can find all solutions to the inverse kinematics problem out of singularities. True or false? Why?

A: *False. The considered problem has in fact an infinite number of solutions, so neither a numerical nor an analytical method can generate all of them.*

5. Which is the relation between the second derivative $\ddot{\boldsymbol{R}}$ of a time-varying rotation matrix $\boldsymbol{R}(t)$ and the associated angular velocity $\boldsymbol{\omega}$ and acceleration $\dot{\boldsymbol{\omega}}$?

A: *We differentiate the known relation* $\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}$ *w.r.t. time. We have thus:*

$$\ddot{\boldsymbol{R}} = \dot{\boldsymbol{S}}(\boldsymbol{\omega})\boldsymbol{R} + \boldsymbol{S}(\boldsymbol{\omega})\dot{\boldsymbol{R}} = \left(\boldsymbol{S}(\dot{\boldsymbol{\omega}}) + \boldsymbol{S}^2(\boldsymbol{\omega})\right)\boldsymbol{R} = \left(\boldsymbol{S}(\dot{\boldsymbol{\omega}}) + \boldsymbol{\omega}\boldsymbol{\omega}^T - \boldsymbol{I}\|\boldsymbol{\omega}\|^2\right)\boldsymbol{R},$$

with the 3×3 *identity matrix* $\boldsymbol{I}$. *Note that the dependence on the angular acceleration* $\dot{\boldsymbol{\omega}}$ *is linear, while* $\boldsymbol{S}^2$ *leads to quadratic terms in the angular velocity* $\boldsymbol{\omega}$.

6. For a joint that needs to move by $\Delta q > 0$, if the bounds on maximum absolute velocity and acceleration are related by $A_{max} = V_{max}^2/\Delta q$, is the minimum time acceleration profile always bang-coast-bang?

A: *No. In a trapezoidal speed profile, the coast phase for the acceleration (the time interval of motion with zero acceleration and maximum speed) exists if and only if* $V_{max}^2/A_{max} < \Delta q$. *The above relation provides instead* $V_{max}^2/A_{max} = \Delta q$. *Thus, the maximum speed is reached in just one instant (at the half-time of motion), not for a finite interval.*

7. The uniform time scaling procedure allows to obtain the minimum motion time along a parametrized path under maximum velocity and acceleration constraints. True or false? Why?

A: *False. By uniformly scaling time, the fastest motion profile that is obtained will have in general just one velocity or acceleration constraint saturated, and in one instant only. On the other hand, we would get faster motions by speeding up where the bounds are largely satisfied and slowing down where they are violated. The minimum motion time along the given parametrized path will be obtained by suitably choosing such a non-uniform time scaling of the original trajectory.*

8. Kinematic control laws designed at the Cartesian level are better than those designed at the joint level because ..., and are worse because ...

A: *Cartesian (or task) kinematic control guarantees asymptotically stable tracking of trajectories that are defined directly in the most relevant space for evaluating robot performance, with errors that will converge to zero exponentially and in a decoupled way. The downside is that the control law is more complex to be implemented in real time. Moreover, singularities that would highly affect robot motion may be encountered at run time and should be carefully handled online.*

* * * * *

Robotics I - Extra Sheet #2 (for Exercise 4)

February 12, 2020

Name: ____

Answer to the questions or reply/comment on/complete the statements, providing a *short* motivation/explanation (within the given lines of text) for each of the following 8 items.

1. Order the three classes of infrared, laser, and ultrasound proximity sensors in terms of their typical range of measurement.

   ____

2. Order infrared, laser, and ultrasound sensors in terms of their typical angular resolution.

   ____

3. Compare the motor-side position resolution of an incremental encoder with 512 pulses per revolution (PPR) and quadrature electronics mounted on the motor with that of an absolute encoder with 16 bits mounted on the link, when the transmission has reduction ratio $n_r = 20$, Which one is better?

   ____

   ____

4. Given a desired end-effector position for a planar PPR robot, the gradient method will always provide a solution to the inverse kinematics problem without need of restarting procedures. True or false? Why?

   ____

   ____

5. What is the so-called overfly in trajectory planning and which are its pros and cons? Can this concept be applied equally well at the joint level and at the Cartesian level or not? Why?

   ____

   ____

6. We have four positional knots to be interpolated in the 3D Cartesian space, plus a number of boundary conditions and continuity requirements. Should we use 4-3-4 polynomials or cubic splines? If both can be used, which choice is better and why?

   ____

   ____

7. For a single robot joint, we have computed a spline trajectory interpolating $n = 10$ given knots at some assigned instants of time $t_1 < t_2 < \cdots < t_{10}$. If we modify only one of such time instants, but still satisfying the sequential order —e.g., the kth instant t_k becomes a new $t'_k \in (t_{k-1}, t_{k+1})$, and then redo the computations, will the trajectory change or not? Why?

   ____

   ____

8. A robot commanded at the joint velocity level has initially zero position and orientation errors with respect to a desired end-effector trajectory, except along the $\boldsymbol{z}$-component in position. If we apply a Cartesian kinematic control law, the robot will move so that ...

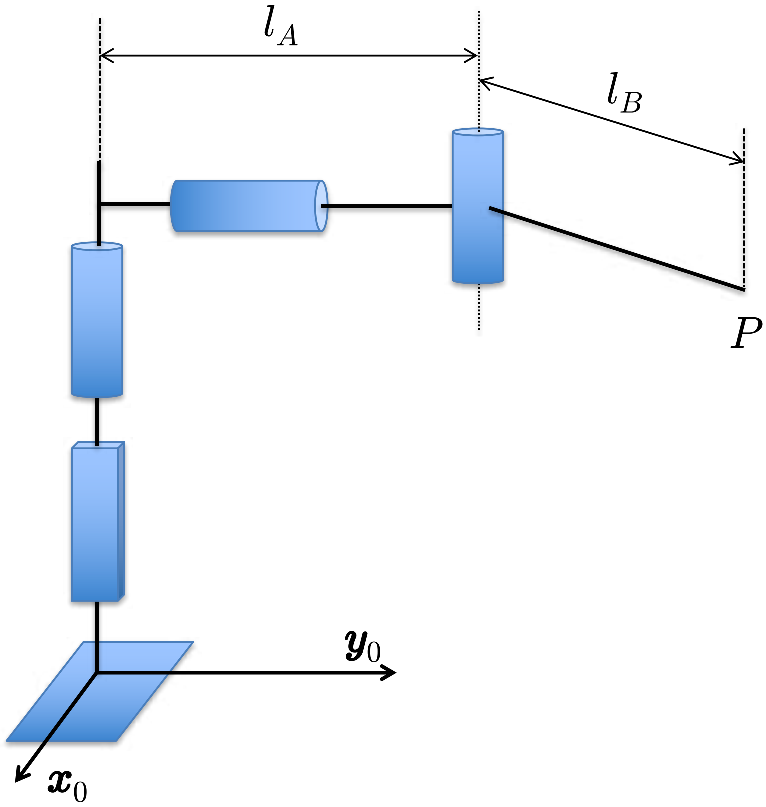
   ____

   ____

Robotics I - Extra Sheet #1 (for Exercise 1)

February 12, 2020

Name: ____



i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				

Robotics I

February 12, 2020

Exercise 1

Consider the 4-dof manipulator in Fig. 1. The robot has the first joint prismatic and the other three revolute. Determine a frame assignment and the associated table of parameters following the Denavit-Hartenberg (DH) convention. Assign the given geometric data l_A and l_B to the corresponding constant DH parameters. The origin of the first DH frame RF_0 is already specified, while the origin of the last frame RF_4 should be placed in P. Use the provided Extra Sheet #1 to draw the frames, and complete the DH table there. Add your name on the sheet and return it.

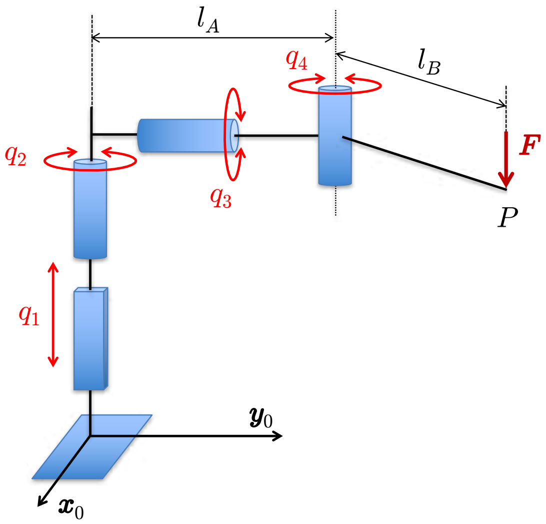


Figure 1: A 4-dof PRRR spatial manipulator.

Exercise 2

For the 4-dof manipulator of Exercise 1:

a) Determine the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ of point P in symbolic form. Using the numerical values $l_A = 0.5$ [m] and $l_B = 0.75$ [m], evaluate the position $\boldsymbol{p}$ in the two configurations $\boldsymbol{q}^I = \boldsymbol{0}$ and $\boldsymbol{q}^{II} = \begin{pmatrix} 1 & 0 & -\pi/2 & \pi/2 \end{pmatrix}^T$.

b) Find the symbolic expression of the generalized joint force/torque $\boldsymbol{\tau} \in \mathbb{R}^4$ that balances the purely vertical, downward force $\boldsymbol{F} \in \mathbb{R}^3$ applied at the point P as shown in Fig. 1, so that the robot remains in a static equilibrium. Using the same previous numerical values for l_A and l_B, evaluate $\boldsymbol{\tau}$ at the two given configurations $\boldsymbol{q}^I$ and $\boldsymbol{q}^{II}$.

c) Determine the angular part of the geometric Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ that relates the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ of the robot to the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of its end-effector frame RF_4. Study the singular configurations of $\boldsymbol{J}_A(\boldsymbol{q})$. Find a basis for all possible $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that produce $\boldsymbol{\omega} = \boldsymbol{0}$ when the Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ looses rank.

Exercise 3

Consider the 3-dof, planar PPR robot in Fig. 2, with a third link of length $L > 0$. The robot end-effector should move at a constant speed $v > 0$, tracing counterclockwise a full circle of radius $R > 0$ centered in $\boldsymbol{P}_c$, and keep its end-effector always aligned with the normal to the surface, pointing toward the circle center. Neglect for simplicity any possible collision between the robot body and the circle (e.g., they may live on two parallel, but different horizontal planes). At time $t = 0$, the robot end-effector is correctly at the initial point $\boldsymbol{A}$ with the right orientation. Suppose that the two prismatic joints have a (common) maximum velocity limit $|\dot{q}_i| \leq V$, $i = 1, 2$, while the revolute joint velocity is limited by $|\dot{q}_3| \leq \Omega$, with $V > 0$ and $\Omega > 0$. Similarly, for the joint accelerations, there are the bounds $|\ddot{q}_i| \leq A$, $i = 1, 2$, and $|\ddot{q}_3| \leq \Psi$, with $A > 0$ and $\Psi > 0$.

a) Provide the expressions of the time evolution of the end-effector position $\boldsymbol{p} \in \mathbb{R}^2$, velocity $\dot{\boldsymbol{p}}$ and acceleration $\ddot{\boldsymbol{p}}$, possibly using separation in space and time, when the given task is perfectly executed. Similarly, provide the expressions of the time evolution of the end-effector absolute orientation angle $\phi \in \mathbb{R}$ and of its derivatives $\dot{\phi}$ and $\ddot{\phi}$. Sketch qualitative plots of all these quantities (keep symbolic values).

b) Provide the associated expressions of the time evolution of the joint position $\boldsymbol{q} \in \mathbb{R}^3$, velocity $\dot{\boldsymbol{q}}$ and acceleration $\ddot{\boldsymbol{q}}$. Sketch qualitative plots also of these quantities.

c) Determine, as a function of the parametric data, the expression of the maximum constant speed v at which the task can be completed without violating any of the physical limits of the robot. Accordingly, give the minimum time T for completing one full round of the circle while remaining feasible.

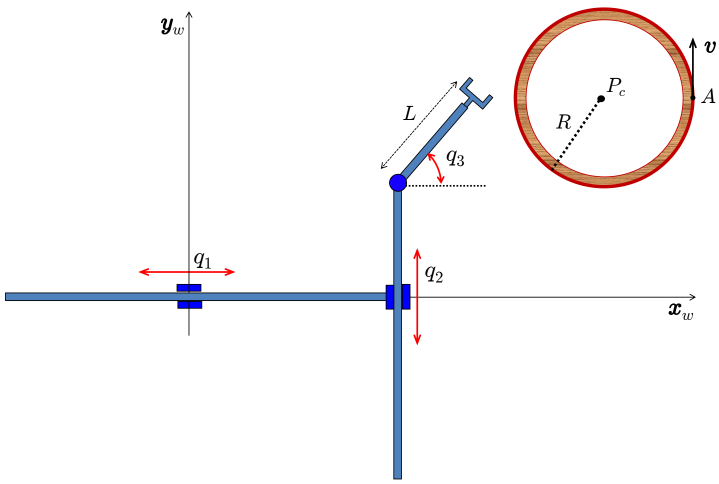


Figure 2: The planar PPR robot and the Cartesian task to be executed.

Exercise 4

A number of questions and statements are reported on the Extra Sheet #2. Fill in your answers on the same sheet, providing also a short motivation/explanation for each item. Add your name on the sheet and return it.

[240 minutes, open books]

Solution

February 12, 2020

Exercise 1

A possible Denavit-Hartenberg frame assignment for the 4-dof PRRR manipulator of Fig. 1 is shown in Fig. 3, with the associated parameters reported in Tab. 1. In the shown picture, the robot is in a configuration with a (generic) $q_1 > 0$, $q_2 = 0$, $q_3 = 0$, and $q_4 > 0$ (about 70°).

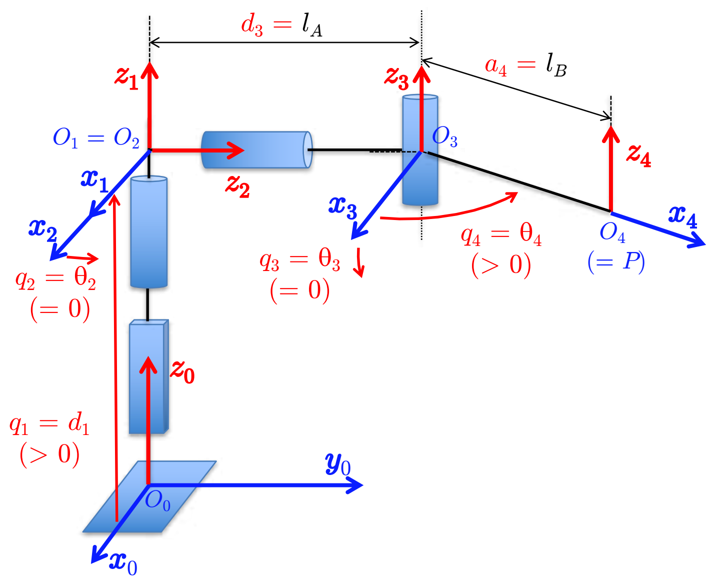


Figure 3: Assignment of DH frames for a PRRR robot.

i	α_i	a_i	d_i	θ_i
1	0	0	q_1	0
2	$-\pi/2$	0	0	q_2
3	$\pi/2$	0	$l_A > 0$	q_3
4	0	$l_B > 0$	0	q_4

Table 1: The DH table of parameters for the frame assignment in Fig. 3.

Exercise 2

Based on Tab. 1, in order to determine the position $\boldsymbol{p}$ of point P, i.e., the position of the origin O_4, we need the following DH homogenous transformation matrices:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & l_A \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3(q_3) & {}^2\boldsymbol{p}_3 \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & -s_4 & 0 & l_B c_4 \\ s_4 & c_4 & 0 & l_B s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4(q_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

where the shorthand notations $s_i = \sin q_i$, $c_i = \cos q_i$ have been used.

The position $\boldsymbol{p}$ is computed from

$$
\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)\right),
$$

giving

$$
\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} -l_A s_2 - l_B s_2 s_4 + l_B c_2 c_3 c_4 \\ l_A c_2 + l_B c_2 s_4 + l_B s_2 c_3 c_4 \\ q_1 - l_B s_3 c_4 \end{pmatrix}. \tag{1}
$$

Using the numerical values $l_A = 0.5$ and $l_B = 0.75$, we evaluate (1) in the two configurations $\boldsymbol{q}^I = \boldsymbol{0}$ and $\boldsymbol{q}^{II} = \begin{pmatrix} 1 & 0 & -\pi/2 & \pi/2 \end{pmatrix}^T$ yielding

$$
\boldsymbol{p}^I = \boldsymbol{f}(\boldsymbol{q}^I) = \begin{pmatrix} 0.75 \\ 0.5 \\ 0 \end{pmatrix} \text{ [m]}, \qquad \boldsymbol{p}^{II} = \boldsymbol{f}(\boldsymbol{q}^{II}) = \begin{pmatrix} 0 \\ 1.25 \\ 1 \end{pmatrix} \text{ [m]}.
$$

For the force balancing problem in static conditions, we need the joint torque

$$
\boldsymbol{\tau} = -\boldsymbol{J}_L^T(\boldsymbol{q})\boldsymbol{F}, \quad \text{with } \boldsymbol{F} = \begin{pmatrix} 0 \\ 0 \\ f_z \end{pmatrix}, \quad f_z < 0.
$$

Because of the structure of the Cartesian force $\boldsymbol{F}$, we just have to compute the last row in the 3×4 Jacobian matrix $\boldsymbol{J}_L$,

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \frac{\partial p_1(\boldsymbol{q})}{\partial \boldsymbol{q}} \\ \frac{\partial p_2(\boldsymbol{q})}{\partial \boldsymbol{q}} \\ \frac{\partial p_3(\boldsymbol{q})}{\partial \boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 1 & 0 & -l_B c_3 c_4 & l_B s_3 s_4 \end{pmatrix},
$$

and thus

$$
\boldsymbol{\tau} = -\boldsymbol{J}_L^T(\boldsymbol{q})\boldsymbol{F} = \begin{pmatrix} 1 \\ 0 \\ -l_B c_3 c_4 \\ l_B s_3 s_4 \end{pmatrix} |f_z|. \tag{2}
$$

Evaluating numerically (2) as before yields

$$
\boldsymbol{\tau}^I = \boldsymbol{\tau}|_{\boldsymbol{q}=\boldsymbol{q}^I} = \begin{pmatrix} 1 \\ 0 \\ -0.75 \\ 0 \end{pmatrix} |f_z|, \qquad \boldsymbol{\tau}^{II} = \boldsymbol{\tau}|_{\boldsymbol{q}=\boldsymbol{q}^{II}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -0.75 \end{pmatrix} |f_z|.
$$

The angular part of the geometric Jacobian is computed as

$$
\begin{aligned}
\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{0} & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{z}_2 & {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_1)\,{}^3\boldsymbol{z}_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -s_2 & c_2 s_3 \\ 0 & 0 & c_2 & s_2 s_3 \\ 0 & 1 & 0 & c_3 \end{pmatrix}.
\end{aligned}
\tag{3}
$$

being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for all i, and ${}^0\boldsymbol{R}_1 = \boldsymbol{I}_{3\times 3}$. It follows from (3) that the rank of matrix $\boldsymbol{J}_A(\boldsymbol{q})$ is equal to 3, except for $\sin q_3 = 0$, or $q_3 = \{0, \pi\}$ [rad]. When the matrix is singular, it becomes then

$$
\boldsymbol{J}_A(q_2)|_{q_3=\{0,\pi\}} = \begin{pmatrix} 0 & 0 & -s_2 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 1 & 0 & \pm 1 \end{pmatrix}, \tag{4}
$$

which has always rank equal to 2, for all q_2. Therefore, its null space (i.e., all vectors $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ such that $\boldsymbol{\omega} = \boldsymbol{J}_A(q_2)|_{q_3=\{0,\pi\}}\dot{\boldsymbol{q}} = \boldsymbol{0}$) is spanned by a two-dimensional basis, e.g., by

$$
\mathcal{N}\big(\boldsymbol{J}_A(q_2)|_{q_3=\{0,\pi\}}\big) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ \pm 1 \end{pmatrix} \right\}.
$$

Exercise 3

We first describe the Cartesian task in terms of absolute position $\boldsymbol{p} \in \mathbb{R}^2$, relative to the world frame RF_w, and orientation angle $\phi \in \mathbb{R}$, defined w.r.t. the $\boldsymbol{x}_w$ axis. For this, we use separation in space[1] and time:

$$
\boldsymbol{p}(s) = P_c + R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \quad \phi(s) = \pi + s, \quad s = [0, 2\pi]; \qquad s = s(t), \quad t = [0, T].
$$

Note that the angle $\phi(s)$ can also be evaluated modulo 2π (if we don't care, as here, about the number of rounds traced on the circle). Taking into account that motion along the circle should be performed at a constant speed v, the time derivative of the above quantities is given by

$$
\dot{\boldsymbol{p}}(t) = \frac{d\boldsymbol{p}}{ds} \cdot \frac{ds}{dt} = \boldsymbol{p}'(s)\,\dot{s}(t) = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \dot{s} \quad \Rightarrow \quad \dot{s} = \frac{v}{R} \text{ (constant, and so } \ddot{s} = 0),
$$

and

$$
\dot{\phi}(t) = \dot{s} = \frac{v}{R}.
$$

As a consequence, $s(t) = (v/R)\,t$, and the time for completing a full circle will be $T = 2\pi R/v$. So, the choice of a maximum speed v that satisfies the robot motion constraints will provide also the minimum feasible time T. Differentiating further w.r.t. time gives

$$
\ddot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s)\,\ddot{s}(t) + \boldsymbol{p}''(s)\,\dot{s}^2(t) = -\begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \frac{v^2}{R}, \qquad \ddot{\phi}(t) = 0.
$$

[1] The used parametrization specifies s as an angle (in [rad]). One could have used equally well a parametrization in terms of the arc length $\sigma = s/R \in [0, 2\pi R]$, in [m], as the argument of the trigonometric functions. The modifications that would follow are trivial, e.g., $\dot{s} = v$ (in [m/s]).

Note that $\|\ddot{\boldsymbol{p}}(t)\| = v^2/R$, $\forall t \in [0,T]$.

With reference to Fig. 2, we have for the direct kinematics of the PPR robot

$$\begin{pmatrix} \boldsymbol{p} \\ \phi \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_1 + L\cos q_3 \\ q_2 + L\sin q_3 \\ q_3 \end{pmatrix}. \tag{5}$$

Accordingly, the first- and second-order differential maps are

$$\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\phi} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 & -L\sin q_3 \\ 0 & 1 & L\cos q_3 \\ 0 & 0 & 1 \end{pmatrix} \dot{\boldsymbol{q}}, \tag{6}$$

and

$$\begin{pmatrix} \ddot{\boldsymbol{p}} \\ \ddot{\phi} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 & -L\sin q_3 \\ 0 & 1 & L\cos q_3 \\ 0 & 0 & 1 \end{pmatrix} \ddot{\boldsymbol{q}} - \begin{pmatrix} L\cos q_3 \\ L\sin q_3 \\ 0 \end{pmatrix} \dot{q}_3^2 = \ddot{\boldsymbol{q}} - \begin{pmatrix} L\cdot \boldsymbol{R}(q_3)\begin{pmatrix} \dot{q}_3^2 \\ \ddot{q}_3 \end{pmatrix} \\ 0 \end{pmatrix}, \tag{7}$$

where $\boldsymbol{R}(q_3)$ is the 2×2 planar rotation matrix by an angle q_3.

Equation (5) can be easily inverted. Taking into account the parametrization of the task, we have:

$$\begin{aligned} \boldsymbol{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = \begin{pmatrix} p_x(t) - L\cos\phi(t) \\ p_y(t) - L\sin\phi(t) \\ \phi(t) \end{pmatrix} &= \begin{pmatrix} P_{c,x} + R\cos s(t) - L\cos(\pi + s(t)) \\ P_{c,y} + R\sin s(t) - L\sin(\pi + s(t)) \\ \pi + s(t) \end{pmatrix} \\ &= \begin{pmatrix} P_{c,x} + (R+L)\cos s(t) \\ P_{c,y} + (R+L)\sin s(t) \\ \pi + s(t) \end{pmatrix}. \end{aligned} \tag{8}$$

Equation (8) shows that the tip of the second link (viz., the base of the third link) of the PPR robot should trace a circle of large radius $R+L$.

Taking into account the timing law, we can invert also eq. (6) as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}(t)) \begin{pmatrix} \dot{\boldsymbol{p}}(t) \\ \dot{\phi}(t) \end{pmatrix} = \begin{pmatrix} -(R+L)\sin s(t) \\ (R+L)\cos s(t) \\ 1 \end{pmatrix} \frac{v}{R}.$$

Therefore, a first upper bound on v follows from the joint velocity limits:

$$|\dot{q}_1| \le V,\ \ |\dot{q}_2| \le V,\ \ |\dot{q}_3| \le \Omega \qquad \Rightarrow \qquad v \le v_v = \min\left\{\frac{VR}{R+L}, \Omega R\right\}.$$

Finally, inversion of eq. (7) yields for the acceleration of the third joint

$$\ddot{q}_3(t) = \ddot{\phi}(t) = 0,$$

and thus, as a whole

$$\ddot{\boldsymbol{q}}(t) = -\begin{pmatrix} (R+L)\cos s(t) \\ (R+L)\sin s(t) \\ 0 \end{pmatrix} \left(\frac{v}{R}\right)^2.$$

A second upper bound on v follows then from the joint acceleration limits:

$$|\ddot{q}_1| \leq A, \;\; |\ddot{q}_2| \leq A, \;\; |\ddot{q}_3| \leq \Psi \qquad \Rightarrow \qquad v \leq v_a = \sqrt{\frac{AR^2}{R+L}}.$$

Accordingly, the maximum (constant) speed of feasible execution of the task and the associated minimum time will be

$$v_{max} = \min\{v_v, v_a\} = \min\left\{\frac{VR}{R+L}, \Omega R, \sqrt{\frac{AR^2}{R+L}}\right\}, \qquad T_{min} = \frac{2\pi R}{v_{max}}.$$

Note that the limit Ψ on the acceleration of joint 3 plays no role for the considered task.

In the following figures, we plot the time evolution of all requested quantities in the Cartesian and in the joint space of the PPR robot. In Matlab, the following data were used:

$$L = 1.5 \text{ [m]}, \quad P_c = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ [m]}, \quad R = 0.5 \text{ [m]}, \quad V = 2 \text{ [m/s]}, \quad \Omega = 2 \text{ [rad/s]}, \quad A = 6 \text{ [m/s}^2\text{]}.$$

These numerical values imply

$$v_{max} = 0.5 \text{ [m/s]}, \qquad T_{min} = \frac{\pi}{v_{max}} \simeq 6.28 \text{ [s]}.$$

Indeed, any other set of values would provide qualitatively similar results, i.e., linear quantities would be have like sine and cosine, while angular quantities would be linear, constant, or zero over time (depending on their differential order). The only difference may be on which of the given bounds would be attained in the minimum time solution.

Figures 4–6 show the time evolutions of the coordinates of the Cartesian position $\boldsymbol{p}$, linear velocity $\dot{\boldsymbol{p}}$, and linear acceleration $\ddot{\boldsymbol{p}}$, together with the evolution of the angle ϕ of the normal to the surface, pointing inward to the center of the circle, its speed $\dot{\phi}$ (here constant and $= 1$ [rad/s]) and acceleration $\ddot{\phi}$ (here $= 0$).

Figures 7–9 show the time evolutions of the joint position $\boldsymbol{q}$, joint velocity $\dot{\boldsymbol{q}}$, and joint acceleration $\ddot{\boldsymbol{q}}$, respectively. Note that the bounds that limit the task speed v to v_{max} are those on the velocities of the two prismatic joints ($\dot{q}_1$ and $\dot{q}_2$).

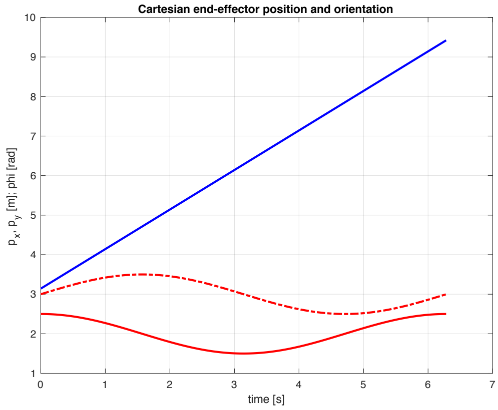


Figure 4: Cartesian position $\boldsymbol{p}(t)$ ($p_x(t)$ [red], $p_y(t)$ [red, dashed]) and angle $\phi(t)$ [blue].

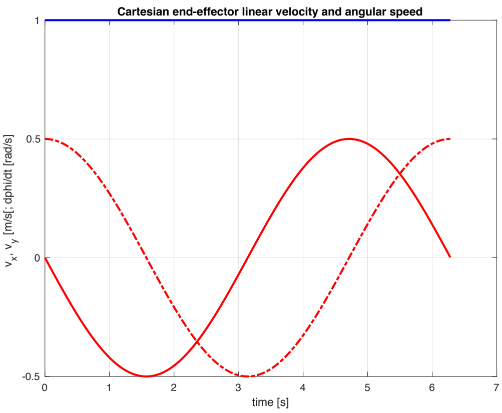


Figure 5: Cartesian linear velocity $\dot{\boldsymbol{p}}(t)$ ($v_x(t)$ [red], $v_y(t)$ [red, dashed]) and angular speed $\dot{\phi}(t)$ [blue].

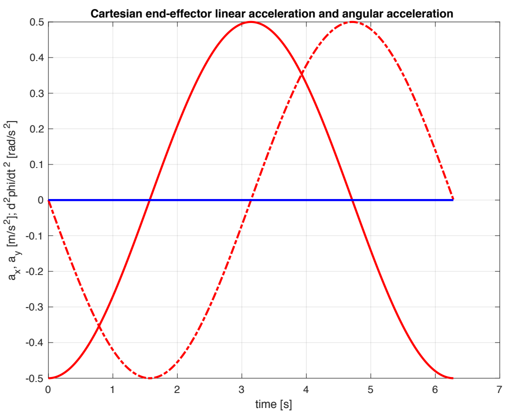


Figure 6: Cartesian linear acceleration $\ddot{\boldsymbol{p}}(t)$ ($a_x(t)$ [red], $a_y(t)$ [red, dashed]) and angular acceleration $\ddot{\phi}(t)$ [blue].

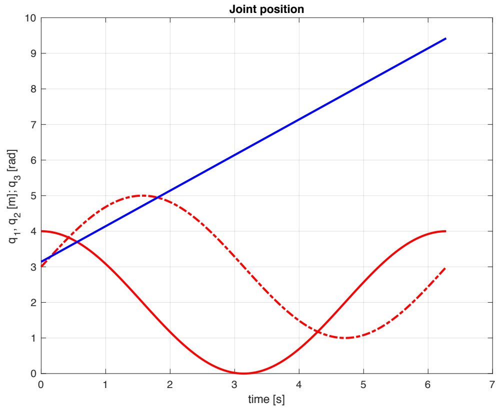


Figure 7: Joint position $\boldsymbol{q}(t)$: $q_1(t)$ [red], $q_2(t)$ [red, dashed], $q_3(t)$ [blue].

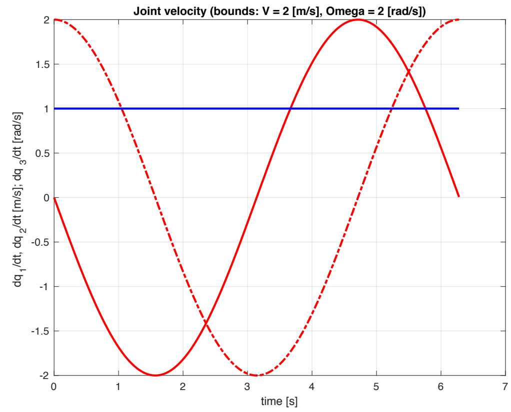


Figure 8: Joint velocity $\dot{\boldsymbol{q}}(t)$: $\dot{q}_1(t)$ [red], $\dot{q}_2(t)$ [red, dashed], $\dot{q}_3(t)$ [blue].

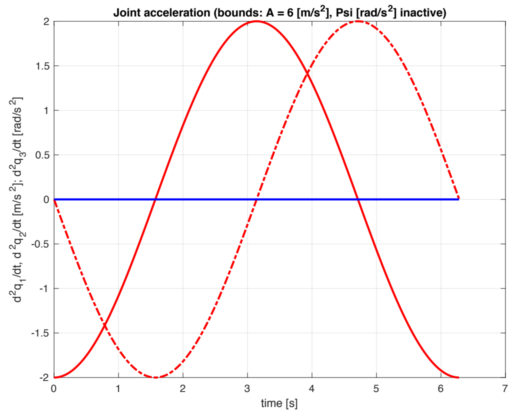


Figure 9: Joint acceleration $\ddot{\boldsymbol{q}}(t)$: $\ddot{q}_1(t)$ [red], $\ddot{q}_2(t)$ [red, dashed], $\ddot{q}_3(t)$ [blue].

Exercise 4

Answer to the questions or reply/comment on/complete the statements, providing a *short* motivation/explanation for each of the following 8 items.

1. Order the three classes of infrared, laser, and ultrasound proximity sensors in terms of their typical range of measurement.

A: *In terms of increasing distance covered: infrared < ultrasound < laser.*

2. Order infrared, laser, and ultrasound sensors in terms of their typical angular resolution.

A: *In terms of angular resolution at the same distance: laser is better than infrared, which is in general better than ultrasound.*

3. Compare the motor-side position resolution of an incremental encoder with 512 pulses per revolution (PPR) and quadrature electronics mounted on the motor with that of an absolute encoder with 16 bits mounted on the link, when the transmission has reduction ratio $n_r = 20$, Which one is better?

A: *The incremental encoder has resolution* $r_{IE} = 360°/(512 \cdot 4) = 0.18°$ *(as evaluated on the motor side). The resolution of the absolute encoder mounted on the link side, once reflected back to the motor side, is* $r_{AE} = (360°/2^{n_{bit}}) \cdot n_r = (360°/2^{16}) \cdot 20 = 0.11°$. *Thus, the absolute encoder has a (almost twice) better resolution.*

4. Given a desired end-effector position for a planar PPR robot, the gradient method will always provide a solution to the inverse kinematics problem without need of restarting procedures. True or false? Why?

A: *True. The* 2×3 *Jacobian of this robot has the structure* $\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{I}_{2\times 2} & \boldsymbol{j}_3(q_3) \end{pmatrix}$, *and is in fact always full rank. So, being* $\mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q})\right\} = \boldsymbol{0}$, *the iterations of the gradient method will always and only end with a zero end-effector position error, in one of the* ∞^1 *possible configurations that are solutions to the given inverse kinematic problem.*

5. What is the so-called overfly in trajectory planning and which are its pros and cons? Can this concept be applied equally well at the joint level and at the Cartesian level or not? Why?

A: *When a sequence of position/orientation knots is assigned (equally well in the Cartesian or in the joint space), performing overfly of a knot allows the robot to getting close to it, yet without passing through it. Continuity of motion is typically gained at the expense of accurate interpolation. If the interpolating path between knots were made of linear segments, passing through the knots would typically require to stop to avoid velocity discontinuity, whereas overfly allows to keep the same motion speed.*

6. We have four positional knots to be interpolated in the 3D Cartesian space, plus a number of boundary conditions and continuity requirements. Should we use 4-3-4 polynomials or cubic splines? If both can be used, which choice is better and why?

A: *Both classes of functions have sufficient parameters to satisfy the boundary conditions and continuity requirements. The 4-3-4 polynomial asks for an initial and final specification of the second derivative (in space or in time, i.e., acceleration). The cubic splines can be suitably modified to handle also this case. However, cubic splines are the functions providing the minimum total curvature among all possible interpolating functions in space.*

7. For a single robot joint, we have computed a spline trajectory interpolating $n = 10$ given knots at some assigned instants of time $t_1 < t_2 < \cdots < t_{10}$. If we modify only one of such time instants, but still satisfying the sequential order —e.g., the kth instant t_k becomes a new $t'_k \in (t_{k-1}, t_{k+1})$, and then redo the computations, will the trajectory change or not? Why?

A: *Yes, it will change. The modification of any single or multiple data (time intervals, knot positions) affects the entire trajectory.*

8. A robot commanded at the joint velocity level has initially zero position and orientation errors with respect to a desired end-effector trajectory, except along the $\boldsymbol{z}$-component in position. If we apply a Cartesian kinematic control law, the robot will move so that ...

A: . . . the initial error on the $\boldsymbol{z}$*-component of the Cartesian position is recovered at an exponential rate, while the errors on all other components remain always zero (at least in nominal conditions). This is thanks to the decoupling and exact linearization properties of the Cartesian error components achieved when using a Cartesian kinematic control law. The same is* not *true if the kinematic control law is defined on the trajectory errors in the joint space.*

* * * * *

Robotics 1

Remote Exam – June 5, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, with all revolute joints. Some axes of a Denavit-Hartenberg (D-H) frame assignment are already given, together with an end-effector frame placed on the gripper. Assuming that all angles defined as usual in the interval $(-\pi, +\pi]$, complete the assignment of the frames so that $\alpha_i \geq 0$, for $i = 1, \ldots, 4$. Provide the associated table of D-H parameters and specify the value of q_1 and the signs of q_2, q_3, and q_4 in the configuration shown. Finally, find the homogeneous transformation between the last D-H frame and the end-effector frame.

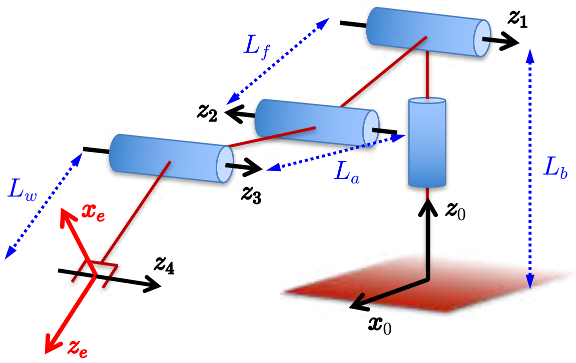


Figure 1: A 4-dof robot with given D-H axes z_i and an end-effector frame on the gripper.

Exercise #2

For the planar RRP robot in Fig. 2, define the direct kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ from the joint variables $\boldsymbol{q} = (q_1, q_2, q_3)$ to the task variables $\boldsymbol{r} = (p_x, p_y, \phi)$, derive the associated Jacobian $\boldsymbol{J}(\boldsymbol{q})$, and find all its kinematic singularities. With $l_1 = 0.5$ [m], compute in static conditions the joint torque/force vector $\boldsymbol{\tau}$ (with units [Nm,Nm,N]) that balances a force/moment vector $\boldsymbol{F} = \begin{pmatrix} 0 & 1.5 & -4.5 \end{pmatrix}^T$ [N,N,Nm] applied to the robot end-effector, first in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0 & 3 \end{pmatrix}^T$ [rad,rad,m] and then in a singular configuration $\boldsymbol{q}_s$ among those found.

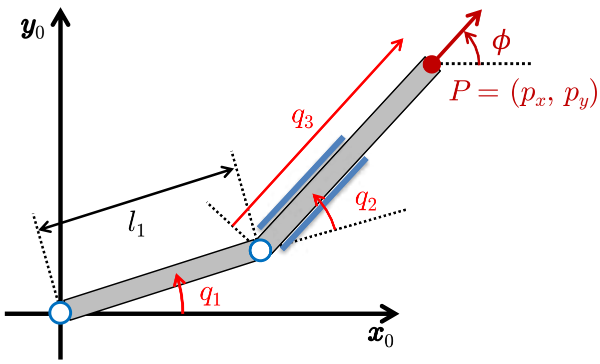


Figure 2: A RRP planar robot.

Exercise #3

The Jacobian of a 3R spatial robot relating $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the velocity $\boldsymbol{v} \in \mathbb{R}^3$ of its end-effector is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(c_2+c_{23}) & -c_1(s_2+s_{23}) & -c_1 s_{23} \\ c_1(c_2+c_{23}) & -s_1(s_2+s_{23}) & -s_1 s_{23} \\ 0 & c_2+c_{23} & c_{23} \end{pmatrix},$$

where the shorthand notation has been used (e.g., $c_{23} = \cos(q_2+q_3)$). This matrix may have rank 1, 2, or 3, depending on the configuration $\boldsymbol{q}$. In each of these cases, define a basis for the null space $\mathcal{N}\{\boldsymbol{J}\}$ and for the range space $\mathcal{R}\{\boldsymbol{J}\}$ of the Jacobian. Find a configuration $\boldsymbol{q}_s$ with rank $\boldsymbol{J}(\boldsymbol{q}_s) = 2$ such that the end-effector velocity $\boldsymbol{v}_s = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$ is feasible. Determine then a joint velocity $\dot{\boldsymbol{q}}_s$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s$. Sketch graphically the situation.

Exercise #4

A 2R planar robot has to perform in a coordinated way a rest-to-rest motion from $\boldsymbol{q}_s = \begin{pmatrix} 0 & -\pi/2 \end{pmatrix}^T$ to $\boldsymbol{q}_g = \begin{pmatrix} -\pi/2 & \pi/2 \end{pmatrix}^T$, while guaranteeing continuity of acceleration at all times. Plan a joint trajectory in the presence of bounds $|\dot{q}_i| \le V_i$ on joint velocities and $|\ddot{q}_i| \le A_i$ on joint accelerations (for $i = 1, 2$), so as to complete the motion task in minimum time T^* within the chosen class of trajectories. Provide the value of T^* for $V_1 = 1$, $V_2 = 2$ [rad/s] and $A_1 = 1.5$, $A_2 = 2$ [rad/s^2].

Exercise #5

This is in the form of a Questionnaire. Please answer with formulas and/or clear and short texts.

A) Which of the following matrices represents a rotation and which not? Motivate your answers.

$$\boldsymbol{R}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{R}_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad \boldsymbol{R}_3 = \begin{pmatrix} -\sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

B) An Harmonic Drive with a circular spline having 150 inner teeth is used as reduction element in a robot joint. An absolute encoder is mounted on the motor side of the joint. How many bits should have this encoder in order to provide an angular resolution better than or equal to 0.0002 rad on the link side of the transmission?

C) A time series $\{q_k\} = \{q(kT_c)\}$ of joint position measurements is collected every time step $T_c = 0.03$ s (about 33 Hz) in the interval $t \in [0, 0.6]$ s from the profile $q(t) = -3\cos\omega t$, with $\omega = 2$. Compute the joint velocity estimate $\dot{q}_k$ for $k = 20$ using 1-step and 4-step backward difference formulas (BDF methods). What is the related percentage error using each method? What is the relation between the time step and the accuracy? Write a short code (e.g., in MATLAB) and comment the obtained numerical results.

[180 minutes (3 hours); open books]

Solution

June 5, 2020

Exercise #1

A completed frame assignment with non-negative values of the twist angles α_i, for $i = 1, \ldots, 4$, is shown in Fig. 3. The associated Denavit-Hartenberg parameters are reported in Tab. 1, together with the signs of the variables $q_i = \theta_i$, for $i = 1, \ldots, 4$, when the robot is in the configuration shown in the figure (where $q_1 = 0$). Note that this solution is not yet unique since the axes from $\boldsymbol{x}_2$ to $\boldsymbol{x}_4$ could have been chosen each in the opposite direction (with no change of the constant parameters α_i's in the table!). However, this one is more natural as it assigns positive lengths to the non-zero parameters a_i. The transformation between D-H frame 4 and end-effector frame is

$$
{}^4\boldsymbol{T}_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

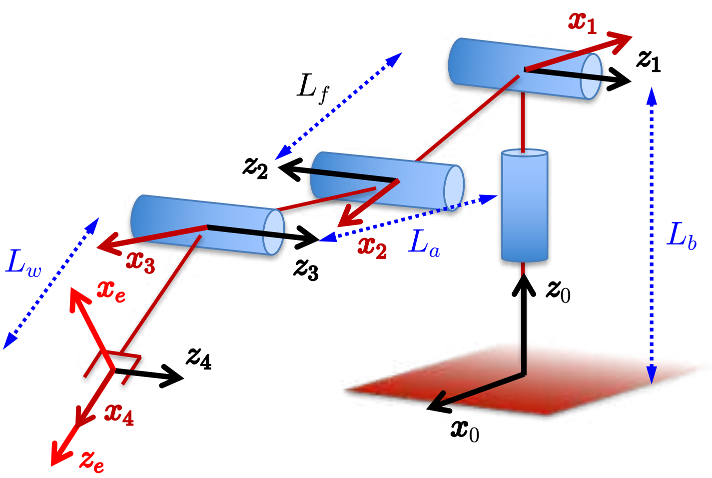


Figure 3: Complete assignment of the D-H frames for the 4-dof robot. All frames are right-handed, so the $\boldsymbol{y}_i$ axes follow automatically (and are not shown).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	L_b	$q_1 = \pi$
2	π	L_f	0	$q_2 < 0$
3	π	L_a	0	$q_3 > 0$
4	0	L_w	0	$q_4 > 0$

Table 1: The D-H table of parameters for the frame assignment in Fig. 3.

Exercise #2

The direct kinematics of this robot for the task at hand is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} l_1 \cos q_1 + q_3 \cos(q_1 + q_2) \\ l_1 \sin q_1 + q_3 \sin(q_1 + q_2) \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

and its 3×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 \sin q_1 - q_3 \sin(q_1 + q_2) & -q_3 \sin(q_1 + q_2) & \cos(q_1 + q_2) \\ l_1 \cos q_1 + q_3 \cos(q_1 + q_2) & q_3 \cos(q_1 + q_2) & \sin(q_1 + q_2) \\ 1 & 1 & 0 \end{pmatrix}.$$

It is easy to check that $\det \boldsymbol{J}(\boldsymbol{q}) = l_1 \cos q_2$ and so a singularity occurs iff $q_2 = \pm\pi/2$. In this singular configurations, the rank of the Jacobian is 2. The vector of torques (first two components) and force (third component, for the prismatic joint) in the joint space that statically balances the Cartesian vector $\boldsymbol{F}$ of forces/moment is given by $\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}$. Setting $l_1 = 0.5$ [m], we have in the (nonsingular) configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0 & 3 \end{pmatrix}^T$

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_0)\boldsymbol{F} = -\begin{pmatrix} -3.5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 4.5 \\ -1.5 \end{pmatrix} \text{ [Nm, Nm, N]}.$$

We can choose for comparison a singular configuration $\boldsymbol{q}_s$ which is similar to $\boldsymbol{q}_0$, namely with the same values for $q_1 = \pi/2$ and $q_3 = 3$, but indeed with $q_2 = \pm\pi/2$. For $\boldsymbol{q}_{s,1} = \begin{pmatrix} \pi/2 & -\pi/2 & 3 \end{pmatrix}^T$, we obtain

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_{s,1})\boldsymbol{F} = -\begin{pmatrix} -0.5 & 3 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm, Nm, N]}.$$

Therefore, no effort by the joint motors is needed[1] to balance the Cartesian vector $\boldsymbol{F}$, which lies in this case in the null space of $\boldsymbol{J}^T(\boldsymbol{q}_{s,1})$. On the other hand, with $\boldsymbol{q}_{s,2} = \begin{pmatrix} \pi/2 & \pi/2 & 3 \end{pmatrix}^T$ we obtain

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_{s,2})\boldsymbol{F} = -\begin{pmatrix} -0.5 & -3 & 1 \\ 0 & -3 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} \text{ [Nm, Nm, N]},$$

and we need torques on the first two joints in order to balance the Cartesian vector $\boldsymbol{F}$. In fact, $\boldsymbol{F} \notin \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}_{s,2})\} = \alpha \begin{pmatrix} 0 & 1 & 3 \end{pmatrix}^T, \forall\alpha$.

Exercise #3

The given Jacobian matrix is associated to the direct kinematics of a 3R spatial robot with unitary lengths of links 2 and 3. In fact, the end-effector position for such robot (with $d_1 = 0$, without loss of generality) is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} c_1(c_2 + c_{23}) \\ s_1(c_2 + c_{23}) \\ s_2 + s_{23} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \qquad (1)$$

[1] It would be useful to draw a picture of this case and to reason geometrically about the balance of moments at the first and the second joint.

and thus

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1(c_2+c_{23}) & -c_1(s_2+s_{23}) & -c_1 s_{23} \\ c_1(c_2+c_{23}) & -s_1(s_2+s_{23}) & -s_1 s_{23} \\ 0 & c_2+c_{23} & c_{23} \end{pmatrix}. \qquad (2)
$$

We compute first the determinant of $\boldsymbol{J}(\boldsymbol{q})$. To simplify the result, it is convenient to premultiply the Jacobian by the (nonsingular) rotation matrix

$$
{}^0\boldsymbol{R}_1^T(q_1) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

which implies expressing the Cartesian end-effector velocity $\boldsymbol{v}$ in the rotated frame 1 (attached to the first link of the robot). This yields the simpler form

$$
{}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & -(s_2+s_{23}) & -s_{23} \\ c_2+c_{23} & 0 & 0 \\ 0 & c_2+c_{23} & c_{23} \end{pmatrix}.
$$

We obtain then[2]

$$
\det \boldsymbol{J}(\boldsymbol{q}) = \det {}^1\boldsymbol{J}(\boldsymbol{q}) = -(c_2+c_{23}) \det \begin{pmatrix} -(s_2+s_{23}) & -s_{23} \\ c_2+c_{23} & c_{23} \end{pmatrix} = -\sin q_3 \left(\cos q_2 + \cos(q_2+q_3)\right). \qquad (3)
$$

Therefore, singularities occur for $\sin q_3 = 0$, i.e., when the forearm is stretched ($q_3 = 0$, type I) or fully folded ($q_3 = \pi$, type II); or for $c_2 + c_{23} = 0$ (type I), which corresponds to $p_x^2 + p_y^2 = 0$ from (1), i.e., when the end-effector lies on the joint axis 1; or at the intersection of the two previous situations (type II). Singularities of type I are associated to a single loss of rank (i.e., in these configurations, rank $\boldsymbol{J}(\boldsymbol{q}) = 2$), whereas singularities of type II are associated to a double loss of rank (i.e., rank $\boldsymbol{J}(\boldsymbol{q}) = 1$). We remark also that, because of the equal length of links 2 and 3, when the forearm is fully folded the robot end-effector will certainly be on the axis of joint 1. This explains why $q_3 = \pi$ is a singularity of type II.

Indeed, at configurations $\boldsymbol{q}$ where the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ has full rank, we will have $\mathcal{N}\{\boldsymbol{J}\} = \boldsymbol{0}$ and $\mathcal{R}\{\boldsymbol{J}\} = \mathbb{R}^3$. Let us now perform the analysis of subspaces for the various singular cases.

a) $q_3 = 0$ (and $c_2 \neq 0$, otherwise this would become case **d)** below —a type-II singularity). The Jacobian (2) takes the form

$$
\boldsymbol{J}_{\mathrm{I},0}(q_1,q_2) = \begin{pmatrix} -2s_1c_2 & -2c_1s_2 & -c_1s_2 \\ 2c_1c_2 & -2s_1s_2 & -s_1s_2 \\ 0 & 2c_2 & c_2 \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -2s_2 & -s_2 \\ 2c_2 & 0 & 0 \\ 0 & 2c_2 & c_2 \end{pmatrix}
$$

with rank $\boldsymbol{J}_{\mathrm{I},0} = 2$ and

$$
\mathcal{N}\{\boldsymbol{J}_{\mathrm{I},0}\} = \left\{ \alpha \begin{pmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}, \quad \mathcal{R}\{\boldsymbol{J}_{\mathrm{I},0}\} = \left\{ \beta_1 \begin{pmatrix} -s_1c_2 \\ c_1c_2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} -c_1s_2 \\ -s_1s_2 \\ c_2 \end{pmatrix} \right\}, \quad \forall \alpha, \beta_1, \beta_2.
$$

[2]When using the Symbolic Toolbox of MATLAB, the straight command `detJ=simplify(det(J))` will produce as output: $-(\cos q_3 + 1)(\sin(q_2+q_3) - \sin q_2)$. This expression is indeed equivalent to (3), but more difficult to analyze. The procedure followed in the text is suggested by the observation of the internal structure of matrix $\boldsymbol{J}(\boldsymbol{q})$ in (2). It also lends itself to a more intuitive interpretation of the singular configurations of this 3R spatial robot.

b) $q_3 = \pi$. The Jacobian (2) becomes

$$\boldsymbol{J}_{\text{II}.\pi}(q_1, q_2) = \begin{pmatrix} 0 & 0 & c_1 s_2 \\ 0 & 0 & s_1 s_2 \\ 0 & 0 & -c_2 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\text{II},\pi} = 1$ and

$$\mathcal{N}\{\boldsymbol{J}_{\text{II},\pi}\} = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{R}\{\boldsymbol{J}_{\text{II},\pi}\} = \left\{ \beta \begin{pmatrix} c_1 s_2 \\ s_1 s_2 \\ -c_2 \end{pmatrix} \right\}, \quad \forall \alpha_1, \alpha_2, \beta.$$

c) $c_2 + c_{23} = 0$ (and $c_2 \neq 0$, otherwise this would also imply either $q_3 = 0$ (case **d)** below) or $q_3 = \pi$ (case **b)** above) —both singularities of type II). The Jacobian (2) becomes

$$\boldsymbol{J}_{\text{I.axis1}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -c_1(s_2 + s_{23}) & -c_1 s_{23} \\ 0 & -s_1(s_2 + s_{23}) & -s_1 s_{23} \\ 0 & 0 & -c_2 \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -(s_2 + s_{23}) & -s_{23} \\ 0 & 0 & 0 \\ 0 & 0 & -c_2 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\text{I,axis1}} = 2$ and

$$\mathcal{N}\{\boldsymbol{J}_{\text{I,axis1}}\} = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \; \mathcal{R}\{\boldsymbol{J}_{\text{II,axis1}}\} = \left\{ \beta_1 \begin{pmatrix} c_1(s_2 + s_{23}) \\ s_1(s_2 + s_{23}) \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} c_1 s_{23} \\ s_1 s_{23} \\ c_2 \end{pmatrix} \right\}, \; \forall \alpha, \beta_1, \beta_2.$$

d) $q_3 = 0$ AND $c_2 + c_{23} = 0 \;\Rightarrow\; q_2 = \pm\pi/2$. The Jacobian (2) becomes

$$\boldsymbol{J}_{\text{II,double}}(q_1) = \begin{pmatrix} 0 & \mp 2c_1 & \mp c_1 \\ 0 & \mp 2s_1 & \mp s_1 \\ 0 & 0 & 0 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\text{II,double}} = 1$ and

$$\mathcal{N}\{\boldsymbol{J}_{\text{II,double}}\} = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}, \; \mathcal{R}\{\boldsymbol{J}_{\text{II,double}}\} = \left\{ \beta \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} \right\}, \; \forall \alpha_1, \alpha_2, \beta.$$

In order to find a joint velocity that realizes the desired end-effector velocity $\boldsymbol{v}_s = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$ when the robot is in a type-I singularity, we should consider only the above two cases **a)** and **c)**. In case **a)**, however, $\boldsymbol{v}_s \notin \mathcal{R}\{\boldsymbol{J}_{\text{I},0}\}$ for any possible pair (q_1, q_2) (with $c_2 \neq 0$). This is checked by organizing a 3×3 matrix with vector $\boldsymbol{v}_s$ next to basis vectors that span $\mathcal{R}\{\boldsymbol{J}_{\text{I},0}\}$, and computing

$$\det \begin{pmatrix} -s_1 c_2 & -c_1 s_2 & -1 \\ c_1 c_2 & -s_1 s_2 & 1 \\ 0 & c_2 & 0 \end{pmatrix} = c_2^2\,(c_1 + s_1) \neq 0, \quad \forall (q_1, q_2), \text{ with } q_2 \neq \pm\frac{\pi}{2}.$$

Therefore $\boldsymbol{v}_s$ is independent from (and thus cannot be generated by) any combination of columns of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ in such configurations. Conversely, in case **c)** the similar check yields

$$\det \begin{pmatrix} c_1(s_2 + s_{23}) & c_1 s_{23} & -1 \\ s_1(s_2 + s_{23}) & s_1 s_{23} & 1 \\ 0 & c_2 & 0 \end{pmatrix} = c_2(s_1 - c_1)(s_2 + s_{23}).$$

This determinant can be zeroed by choosing $s_1 = c_1$, i.e., for $q_1 = -\pi/4$ or for $q_1 = 3\pi/4$, both admissible values in this case. Thus, for these two values of the first joint angle, it follows that $\boldsymbol{v}_s \in \mathcal{R}\{\boldsymbol{J}_{\mathrm{II,axis1}}\}$ in case **c)**, i.e., when the robot end-effector is placed on the axis of joint 1 and its forearm is not stretched ($q_3 \neq 0$) nor folded ($q_3 \neq \pi$). A solution can then be found by pseudoinversion of the Jacobian. Taking for instance $q_1 = -\pi/4$, we have

$$\boldsymbol{J}_s(q_2,q_3) = \boldsymbol{J}(\boldsymbol{q})|_{\{q_1=-\pi/4, c_2+c_{23}=0, c_2\neq 0\}} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}(s_2+s_{23}) & -\frac{1}{\sqrt{2}}s_{23} \\ 0 & \frac{1}{\sqrt{2}}(s_2+s_{23}) & \frac{1}{\sqrt{2}}s_{23} \\ 0 & 0 & -c_2 \end{pmatrix}.$$

Its pseudoinverse can be computed also symbolically in MATLAB (still using the `pinv` function):

$$\boldsymbol{J}_s^{\#}(q_2,q_3) = \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{\sqrt{2}}{2(s_2+s_{23})} & \dfrac{\sqrt{2}}{2(s_2+s_{23})} & \dfrac{s_{23}}{c_2(s_2+s_{23})} \\ 0 & 0 & -\dfrac{1}{c_2} \end{pmatrix}.$$

The joint velocity that solves the problem is then

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}_s^{\#}(q_2,q_3)\boldsymbol{v}_s = \begin{pmatrix} 0 \\ \dfrac{\sqrt{2}}{s_2+s_{23}} \\ 0 \end{pmatrix}. \tag{4}$$

As a result, only joint 2 will move in order to realize the desired $\boldsymbol{v}_s$. Indeed, it is immediate to check that $\boldsymbol{J}_s(q_2,q_3)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s$. Wishing to obtain one of the many possible numerical solutions, we can set, e.g., $\boldsymbol{q} = \begin{pmatrix} -\pi/4 & \pi/4 & \pi/2 \end{pmatrix}^T$ and obtain from (4) the joint velocity $\dot{\boldsymbol{q}}_s = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$. This situation is sketched in Fig. 4.

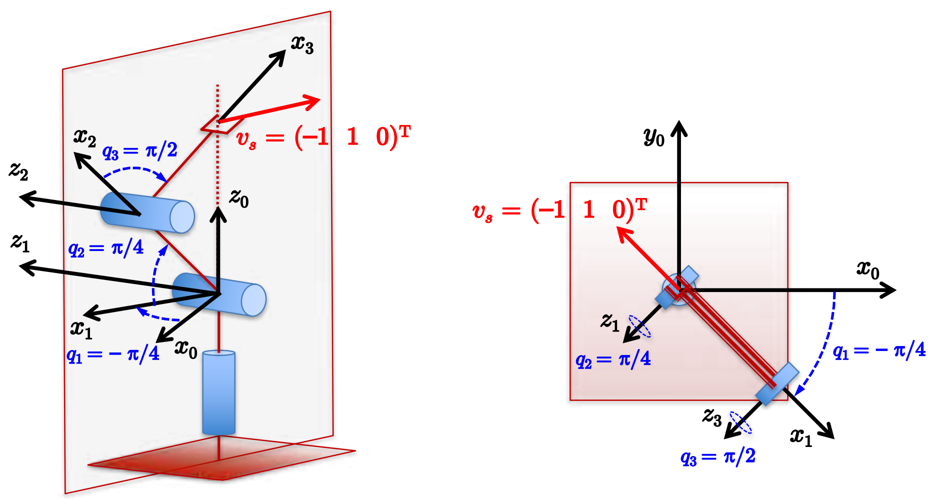


Figure 4: 3D view [left] and top view [right] of a singular configuration of type I for the 3R spatial robot, where the desired end-effector velocity $\boldsymbol{v}_s$ can actually be realized.

Exercise #4

The class of interpolating trajectories that we will consider is given by quintic polynomials, allowing to impose rest-to-rest motion (zero initial and final velocity), but also zero initial and final acceleration and providing thus the required continuity over the entire motion interval, for $t \in [0, T]$. For a generic joint, we will use the doubly normalized quintic polynomial

$$q_i(\tau) = q_i(0) + (q_i(1) - q_i(0)) \left(10\tau^3 - 15\tau^4 + 6\tau^5\right), \qquad i = 1, 2,$$

for $\tau = t/T \in [0, 1]$ and with $\boldsymbol{q}(0) = \boldsymbol{q}_s$, $\boldsymbol{q}(1) = \boldsymbol{q}_g$. For $i = 1, 2$, the associated first and second time derivatives are

$$\dot{q}_i(\tau) = \frac{q_{g,i} - q_{s,i}}{T} \left(30\tau^2 - 60\tau^3 + 30\tau^4\right)$$

and

$$\ddot{q}_i(\tau) = \frac{q_{g,i} - q_{s,i}}{T^2} \left(60\tau - 180\tau^2 + 120\tau^3\right),$$

which are both automatically zero at $\tau = 0$ and $\tau = 1$. For this class of trajectories, the instants of maximum velocity and acceleration (in absolute value) can be found analytically. The maximum acceleration occurs when the jerk (third derivative) is zero, i.e., symmetrically to the trajectory midpoint:

$$\dddot{q}_i(\tau) = \frac{60(q_{g,i} - q_{s,i})}{T^3} \left(1 - 6\tau + 6\tau^2\right) = 0 \quad \Rightarrow \quad \tau_a = 0.5 \pm \frac{\sqrt{3}}{6} \quad (\text{with } \tau_a \in [0, 1]).$$

Therefore[3], we impose the acceleration bounds as

$$\max_{\tau \in [0,1]} |\ddot{q}_i(\tau)| = |\ddot{q}_i(\tau_a)| = \frac{|q_{g,i} - q_{s,i}|}{T^2} \left|60\,\tau_a - 180\,\tau_a^2 + 120\,\tau_a^3\right| \le A_i, \qquad i = 1, 2. \tag{5}$$

Similarly, the maximum velocity occurs when the acceleration is zero. In turn, the acceleration is a cubic function of time for which we already know that two roots are in $\tau = 0$ and $\tau = 1$ (where velocity has its minimum, i.e., zero). It is easy to see that the third root is at $\tau_v = 0.5$ (the trajectory midpoint), where in fact the velocity reaches its maximum (in absolute value). Thus, we have for the velocity bounds

$$\max_{\tau \in [0,1]} |\dot{q}_i(\tau)| = |\dot{q}_i(\tau_v)| = \frac{|q_{g,i} - q_{s.i}|}{T} \left(30\,\tau_v^2 - 60\,\tau_v^3 + 30\,\tau_v^4\right) \le V_i, \qquad i = 1, 2. \tag{6}$$

From (5) and (6), we solve for the minimum feasible motion time. One obtains

$$T^* = \max\left\{T_{\min,V_1}, T_{\min,V_2}, T_{\min,A_1}, T_{\min,A_2}\right\},$$

with

$$T_{\min,V_i} = \frac{|q_{g,i} - q_{s.i}|}{V_i} \left(30\,\tau_v^2 - 60\,\tau_v^3 + 30\,\tau_v^4\right), \qquad i = 1, 2,$$

and

$$T_{\min,A_i} = \sqrt{\frac{|q_{g,i} - q_{s,i}|}{A_i} \left|60\,\tau_a - 180\,\tau_a^2 + 120\,\tau_a^3\right|}, \qquad i = 1, 2.$$

[3] Any of the two signs can be used in the expression of τ_a, provided we recognize that the accelerations in the two instants will be equal in magnitude and opposite in sign. In the first instant (before the midpoint), the acceleration will have the same sign of $(q_{g,i} - q_{s,i})$. However, in order to avoid making distinctions, we will take the absolute value of each term in (5).

Plugging in the numerical data of the problem, we obtain

$$T_{\min,V_1} = 2.9452, \qquad T_{\min,V_2} = 2.9452, \qquad T_{\min,A_1} = 2.4589, \qquad T_{\min,A_2} = 3.0115,$$

so that $T^* = T_{\min,A_2} = 3.0115$ [s]. Accordingly, the saturating quantity is the acceleration of joint 2, which will reach $|\ddot{q}_2(\tau_a)| = A_2 = 2$ [rad/s^2]. The other maximum values attained are indeed all feasible:

$$|\ddot{q}_1(\tau_v)| = 0.9870 < 1 = V_1, \qquad |\ddot{q}_2(\tau_v)| = 1.9560 < 2 = V_2, \qquad |\ddot{q}_1(\tau_a)| = 1 < 1.5 = A_1.$$

The plots of trajectory position, velocity, and acceleration of both joints are shown in Figs. 5–7.

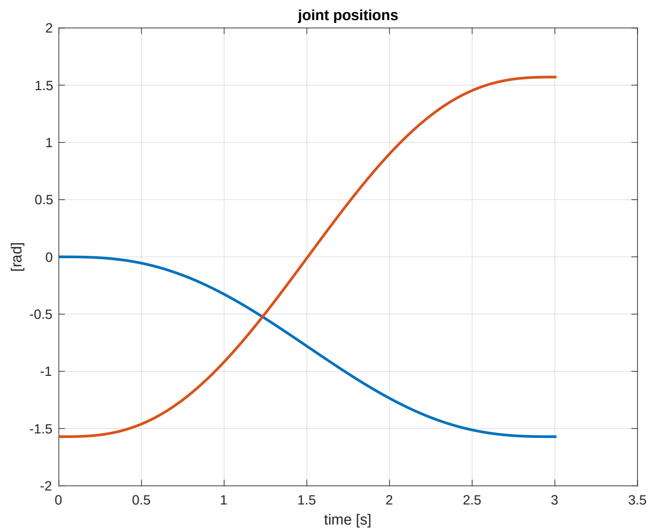


Figure 5: Position of joint 1 (in blue) and 2 (in red).

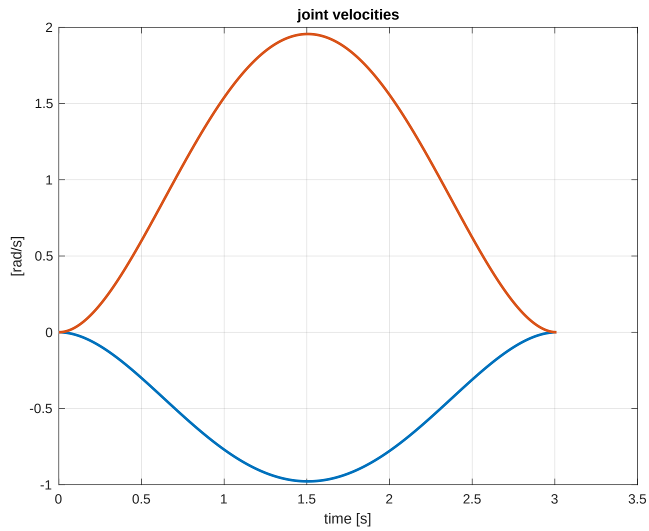


Figure 6: Velocity of joint 1 (in blue) and 2 (in red).

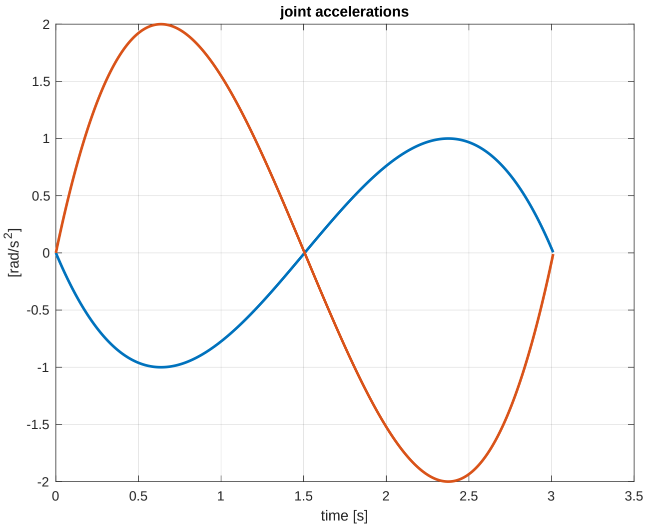


Figure 7: Acceleration of joint 1 (in blue) and 2 (in red). The acceleration of the second joint is the limiting factor in the minimum time solution.

Exercise #5

Three questions were posed.

A) Matrix $\boldsymbol{R}_1$ is not a rotation matrix: its columns are orthonormal, but the determinant is -1. On the other hand, matrices $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ are both elements of $SO(3)$, thus representing rotations.

B) In Harmonic Drives, the Flexspline element has always a number of outer teeth n_{FS} which is two less than the number of inner teeth n_{CS} of the Circular Spline. The reduction ratio is thus

$$n_r = \frac{n_{FS}}{n_{CS} - n_{FS}} = \frac{n_{CS} - 2}{2} = \frac{148}{2} = 74.$$

The angular resolution ρ_m of an absolute encoder mounted on the motor side and having N_t traces ($= n_b$ bits) is related to the angular resolution ρ_l at the link side of the reduction element by

$$\rho_m = \frac{2\pi}{2^{n_b}} = n_r \cdot \rho_l = 74 \cdot 0.0002 = 0.00148 \text{ [rad]}.$$

Therefore

$$n_b = \left\lceil \log_2 \frac{2\pi}{0.00148} \right\rceil = \lceil 8.7298 \rceil = 9 \text{ bits}.$$

C) Given the time evolution of the position profile $q(t) = -3\cos\omega t$, with $\omega = 2$ [rad/s], we have to compare the (known) true value of its analytical time derivative $\dot{q}(t) = 3\,\omega \sin\omega t$, evaluated at $t = t_k = kT_c = 20 \cdot 0.03 = 0.6$ s, with two approximations given by Backward Difference Formulas (BDF) in discrete time[4], namely the 1-step (Euler)

$$\dot{q}_{k,1} = \frac{1}{T_c}\,(q_k - q_{k-1})$$

[4]Note that the coefficients of the combination of samples in BDFs of any order are always alternating in sign and sum up to 0.

and the 4-step one

$$\dot{q}_{k,4} = \frac{1}{T_c}\left(\frac{25}{12}\,q_k - 4\,q_{k-1} + 3\,q_{k-2} - \frac{4}{3}\,q_{k-3} + \frac{1}{4}\,q_{k-4}\right),$$

both evaluated for $k = 20$, i.e., at the time sample $t_k = 0.6$ s. The percentage errors of the two approximations are given by

$$e_f = \left|\frac{\dot{q}_k - \dot{q}_{k,f}}{\dot{q}_k}\right| \cdot 100\ (\%), \qquad f = 1 \text{ or } 4.$$

This simple MATLAB code provides the result:

```
tc=0.03; om=2; k=20; % input data
t=k*tc; % time instant of evaluation = 0.6
t1=t-tc; t2=t1-tc; t3=t2-tc; t4=t3-tc; % backward time instants (up to 4)
qt=-3*cos(om*t); % position at the time instant of evaluation
qt1=-3*cos(om*t1); qt2=-3*cos(om*t2); % positions at previous time instants
qt3=-3*cos(om*t3); qt4=-3*cos(om*t4);
dq=3*om*sin(om*t) % exact value of velocity
dq1=(qt-qt1)/tc % approximation using 1-step BDF (Euler)
dq4=((25/12)*qt-4*qt1+3*qt2-(4/3)*qt3+(1/4)*qt4)/tc % approximation using 4-step BDF
e1=abs((dq-dq1)/dq)*100 % percentage error using 1-step BDF
e4=abs((dq-dq4)/dq)*100 % percentage error using 4-step BDF
```

The output is

$$\texttt{dq} = 5.5922, \qquad \texttt{dq1} = 5.5237, \qquad \texttt{dq4} = 5.5922, \qquad \texttt{e1} = 1.2260\,\% \qquad \texttt{e4} = 2.4770 \cdot 10^{-4}\,\%.$$

The 4-step approximation is much more accurate than the Euler method (in the short format output of MATLAB, it has the same first four decimal digits as the true value). Reducing T_c will reduce the estimation error (absolute or in percentage) for both methods.

* * * * *

Robotics 1

Remote Exam – July 15, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, made by a 3R planar arm mounted on a rail. The world coordinate frame $(\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$ is also shown. Assign the Denavit-Hartenberg (D-H) frames to the robot and provide the associated table of parameters. Place the last D-H frame on the robot gripper with its origin in P. Draw the frames on the robot, together with the joint variables and the non-zero constant parameters. In the configuration shown, specify the signs assumed by the four joint variables (angles are defined as always in the interval $(-\pi, +\pi]$). Finally, find the homogeneous transformation between the world frame and the first D-H frame.

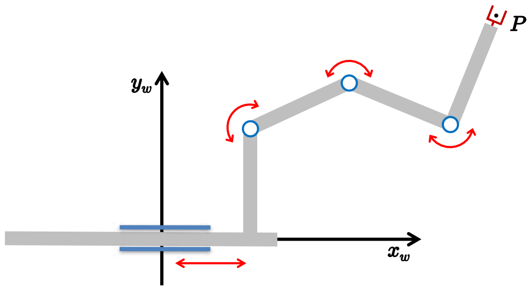


Figure 1: A 4-dof planar robot with the world coordinate frame $(\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$.

Exercise #2

The 2-dof Cartesian robot in Fig. 2 should execute with its end-effector the following desired eight-shaped periodic trajectory

$$\boldsymbol{p}_d(t) = \begin{pmatrix} c + a\sin 2\omega t \\ c + b\sin \omega t \end{pmatrix}, \qquad \text{with } a, b, c, \omega > 0, \text{ for } t \in \left[0, \frac{2\pi}{\omega}\right]. \tag{1}$$

The robot joint velocities and accelerations are bounded as

$$|\dot{q}_i| \leq V_i > 0, \qquad |\ddot{q}_i| \leq A_i > 0, \qquad i = 1, 2,$$

while the velocity along the Cartesian path is bounded in norm as $\|\dot{\boldsymbol{p}}_d(t)\| \leq V_{c,max} > 0$. The robot is commanded by joint accelerations.

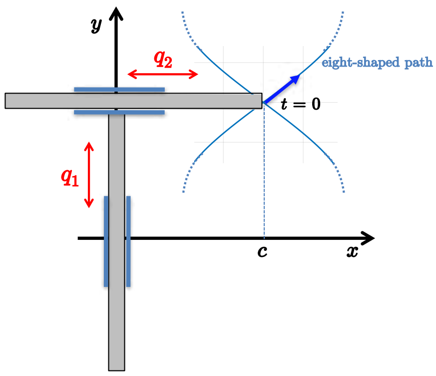


Figure 2: A 2P robot with the end-effector in the initial point of the desired trajectory at $t = 0$.

Give the symbolic expressions of the needed robot joint commands, and determine the maximum value ω_{max} of the angular frequency ω in (1) so that the robot motion satisfies all the constraints. Provide then the numerical value of ω_{max} using the following data: $a = 1$ [m], $b = 1.5$ [m], $c = 3$ [m], $V_1 = V_2 = 2$ [m/s], $V_{c,max} = 1.8$ [m/s], $A_1 = 2$ [m/s^2], $A_2 = 1.5$ [m/s^2].

Exercise #3

For a minimal representation of the orientation of a rigid body given by the YXY sequence of Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, define the instantaneous mapping between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega}$ of the body. Determine also all the singularities of this mapping.

Exercise #4

With reference to Fig. 3, a 3R planar robot with equal link lengths $\ell = 2$ [m] executes a linear Cartesian path from point $\boldsymbol{A} = (3, 2.5)$ [m] (at $t = 0$) to point $\boldsymbol{B} = (0.75, 1.8)$ [m] with constant speed $v = 0.5$ [m/s], while keeping its end-effector always orthogonal to the path. Provide the value of the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ realizing the task at $t = 1$ [s]. Sketch graphically the situation.

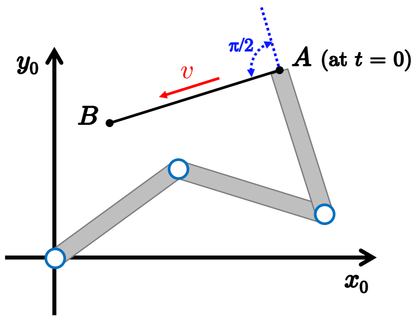


Figure 3: The 3R planar robot and its configuration at the initial point of the desired path.

Exercise #5

This is in the form of a Questionnaire. Please answer with formulas and/or clear and short texts.

A) List all possible Euler sequences of angles around moving axes that can be used to represent the orientation of a rigid body, and associate to each the correct equivalent sequence of Roll-Pitch-Yaw angles around fixed axes.

B) A DC motor has rotor inertia $J_m = 1.2 \cdot 10^{-5}$ [kg·m^2] and maximum speed $\dot{\theta}_{max} = 2060$ rpm. It is connected to the driven link through a rigid transmission with reduction ratio $n_r = 100$. Is the link angular velocity $\dot{q} = 3.5$ [rad/s] a feasible one? In the absence of dissipative effects, if the actual value of the reduction ratio is the one that minimizes the required motor torque for a given link angular acceleration $\ddot{q}$, which is then the value of the link inertia J_l? With this numerical value, if the desired link acceleration is $\ddot{q} = 4$ [rad/s^2], compute the torque τ_m that the motor needs to produce on its axis.

[210 minutes (3.5 hours); open books]

Solution

July 15, 2020

Exercise #1

A Denavit-Hartenberg frame assignment is shown in Fig. 4 and the associated parameters are reported in Tab. 1, together with the signs of the constant non-zero parameters (a_i) and the signs of the variables q_i, for $i = 1, \ldots, 4$, when the robot is in the configuration shown in the figure. The transformation between the world frame and the D-H frame 0 is

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

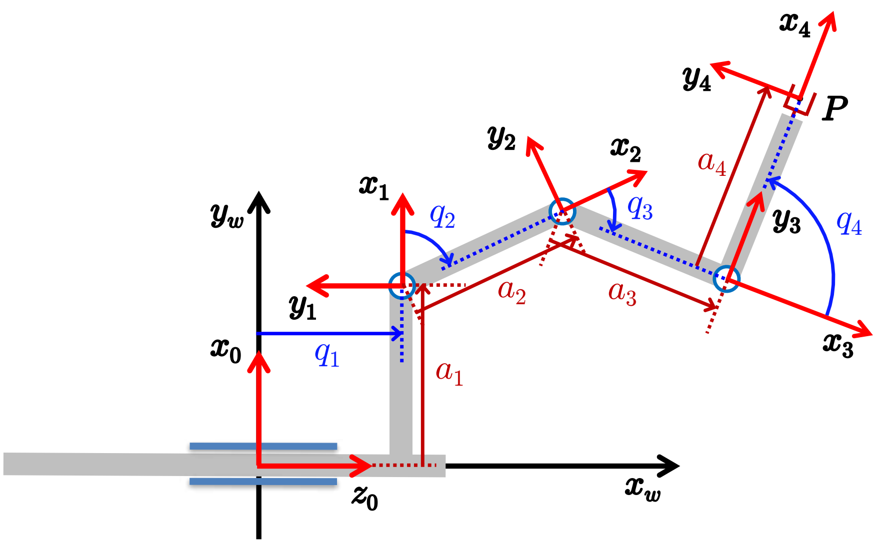


Figure 4: An assignment of D-H frames for the 4-dof robot.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 > 0$	$q_1 > 0$	0
2	0	$a_2 > 0$	0	$q_2 < 0$
3	0	$a_3 > 0$	0	$q_3 < 0$
4	0	$a_4 > 0$	0	$q_4 > 0$

Table 1: The D-H table of parameters for the frame assignment in Fig. 4.

Exercise #2

The eight-shaped Cartesian path is plotted in Fig. 5 using the given parameters $a = 1$, $b = 1.5$, and $c = 3$ [m]. This shape is indeed independent from the time/speed at which the path is being traced by the robot end-effector.

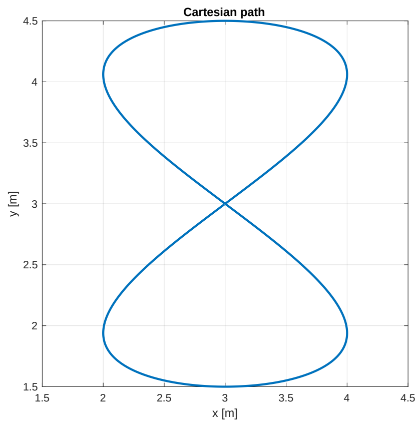


Figure 5: Eight-shaped path traced by the robot end-effector.

The desired Cartesian velocity and acceleration are computed by time derivation of eq. (1). Since the 2P robot has the first joint moving along the y-component and the second joint along the x-component, we have

$$\dot{\boldsymbol{p}}_d(t) = \begin{pmatrix} 2a\omega\cos 2\omega t \\ b\omega\cos\omega t \end{pmatrix} = \begin{pmatrix} \dot{q}_2(t) \\ \dot{q}_1(t) \end{pmatrix}, \tag{2}$$

and

$$\ddot{\boldsymbol{p}}_d(t) = \begin{pmatrix} -4a\omega^2\sin 2\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \begin{pmatrix} \ddot{q}_2(t) \\ \ddot{q}_1(t) \end{pmatrix}, \tag{3}$$

which are also the expressions of the robot joint commands. Moreover, the norm of (2) is

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{4a^2\omega^2\cos^2 2\omega t + b^2\omega^2\cos^2\omega t}. \tag{4}$$

The bounds to be satisfied for all $t \in [0, 2\pi/\omega]$ are then

$$|\dot{q}_1| = |b\omega\cos\omega t| \le V_1 \quad\Rightarrow\quad \omega \le \frac{V_1}{b}, \qquad |\dot{q}_2| = |2a\omega\cos 2\omega t| \le V_2 \quad\Rightarrow\quad \omega \le \frac{V_2}{2a},$$

$$|\ddot{q}_1| = |-b\omega^2\sin\omega t| \le A_1 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_1}{b}}, \qquad |\ddot{q}_2| = |-4a\omega^2\sin 2\omega t| \le A_2 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_2}{4a}},$$

and

$$\|\dot{\boldsymbol{p}}_d(t)\| = \omega\sqrt{4a^2\cos^2 2\omega t + b^2\cos^2\omega t} \le V_{c,max} \quad\Rightarrow\quad \omega \le \frac{V_{c,max}}{\sqrt{4a^2+b^2}}.$$

Therefore, the maximum feasible value of ω is

$$\omega_{max} = \min\left(\frac{V_1}{b}, \frac{V_2}{2a}, \sqrt{\frac{A_1}{b}}, \sqrt{\frac{A_2}{4a}}, \frac{V_{c,max}}{\sqrt{4a^2+b^2}}\right). \tag{5}$$

Substituting in (5) the numerical data, we obtain $\omega_{max} = \sqrt{A_2/(4a)} = 0.6124$, corresponding to the saturation of the acceleration bound at joint 2. Figure 6 shows the resulting joint velocities and

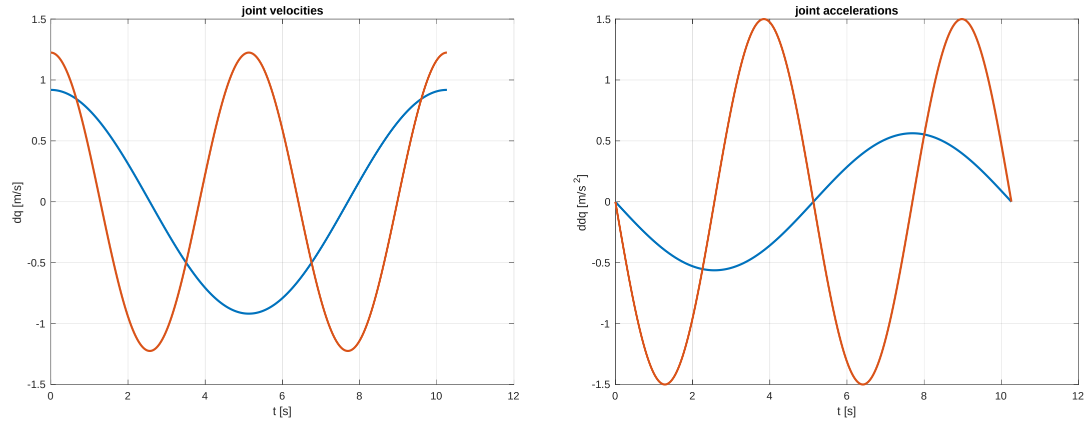


Figure 6: Joint velocities [left] and accelerations [right]. First joint in blue, second joint in red.

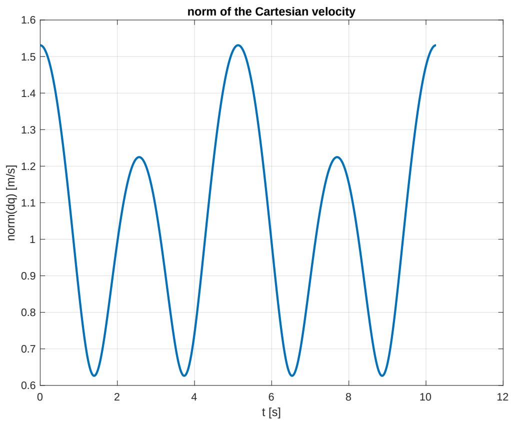


Figure 7: Norm of the Cartesian velocity of the robot end-effector.

accelerations, while the norm of the Cartesian velocity is reported in Fig. 7. Note the (multiple) periodicity of all plots.

Exercise #3

The orientation of a rigid body using the YXY sequence of Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ is given by the rotation matrix

$$
\begin{aligned}
\boldsymbol{R}_{YXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma) \\
&= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma \\ \sin\beta\sin\gamma & \cos\beta & -\sin\beta\cos\gamma \\ -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma \end{pmatrix}.
\end{aligned}
$$

The angular velocity $\boldsymbol{\omega}$ of the body can be obtained from the formula $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{YXY}(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}})\boldsymbol{R}^T_{YXY}(\boldsymbol{\phi})$, where $\boldsymbol{S}$ is a skew-symmetric matrix. Using the shorthand notation for trigonometric functions, taking the time derivative of $\boldsymbol{R}_{YXY}$ and post-multiplying by the transpose of the rotation matrix yields

$$
\begin{aligned}
&\dot{\boldsymbol{R}}_{YXY}(\boldsymbol{\phi},\dot{\boldsymbol{\phi}})\cdot\boldsymbol{R}^T_{YXY}(\boldsymbol{\phi}) \\
&= \begin{pmatrix}
\begin{array}{c}-(s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma)\dot{\alpha} + s_\alpha s_\beta s_\gamma \dot{\beta}\\ -(c_\alpha s_\gamma + s_\alpha c_\beta c_\gamma)\dot{\gamma}\end{array} & c_\alpha s_\beta \dot{\alpha} + s_\alpha c_\beta \dot{\beta} & \begin{array}{c}-(s_\alpha s_\gamma - c_\alpha c_\beta c_\gamma)\dot{\alpha} - s_\alpha s_\beta c_\gamma \dot{\beta}\\ +(c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma)\dot{\gamma}\end{array}\\
c_\beta s_\gamma \dot{\beta} + s_\beta c_\gamma \dot{\gamma} & -s_\beta \dot{\beta} & -c_\beta c_\gamma \dot{\beta} + s_\beta s_\gamma \dot{\gamma}\\
\begin{array}{c}-(c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma)\dot{\alpha} + c_\alpha s_\beta s_\gamma \dot{\beta}\\ +(s_\alpha s_\gamma - c_\alpha c_\beta c_\gamma)\dot{\gamma}\end{array} & -s_\alpha s_\beta \dot{\alpha} + c_\alpha c_\beta \dot{\beta} & \begin{array}{c}-(s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma)\dot{\alpha} - c_\alpha s_\beta c_\gamma \dot{\beta}\\ -(c_\alpha c_\beta s_\gamma + s_\alpha c_\gamma)\dot{\gamma}\end{array}
\end{pmatrix}\\
&\quad\cdot\begin{pmatrix}
c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & s_\beta s_\gamma & -s_\alpha c_\gamma - c_\alpha c_\beta s_\gamma\\
s_\alpha s_\beta & c_\beta & c_\alpha s_\beta\\
c_\alpha s_\gamma + s_\alpha c_\beta c_\gamma & -s_\beta c_\gamma & c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma
\end{pmatrix} \qquad (6)\\
&= \begin{pmatrix}
0 & s_\alpha \dot{\beta} - c_\alpha s_\beta \dot{\gamma} & \dot{\alpha} + c_\beta \dot{\gamma}\\
-s_\alpha \dot{\beta} + c_\alpha s_\beta \dot{\gamma} & 0 & -c_\alpha \dot{\beta} - s_\alpha s_\beta \dot{\gamma}\\
-\dot{\alpha} - c_\beta \dot{\gamma} & c_\alpha \dot{\beta} + s_\alpha s_\beta \dot{\gamma} & 0
\end{pmatrix} = \begin{pmatrix}
0 & -\omega_z & \omega_y\\
\omega_z & 0 & -\omega_x\\
-\omega_y & \omega_x & 0
\end{pmatrix} = \boldsymbol{S}(\boldsymbol{\omega})
\end{aligned}
$$

The above derivation is greatly simplified by using the symbolic calculation in Matlab. Having defined the rotation matrix $\boldsymbol{R}_{YXY}$ and all the other needed quantities as symbolic variables, the $\boldsymbol{S}$ matrix and the angular velocity $\boldsymbol{\omega}$ are obtained by the following three instructions:

```
Rdot=diff(R_YXY,alfa)*dalfa+diff(R_YXY,beta)*dbeta+diff(R_YXY,gamma)*dgamma
S_omega=simplify(Rdot*R_YXY')}
omega=[S_omega(3,2);S_omega(1,3);S_omega(2,1)]
```

The linear mapping $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ is then extracted from the elements of the $\boldsymbol{S}$ matrix in (6) as

$$
\boldsymbol{\omega} = \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \begin{pmatrix}\cos\alpha\,\dot{\beta} + \sin\alpha\sin\beta\,\dot{\gamma}\\ \dot{\alpha} + \cos\beta\,\dot{\gamma}\\ -\sin\alpha\,\dot{\beta} + \cos\alpha\sin\beta\,\dot{\gamma}\end{pmatrix} = \begin{pmatrix}0 & \cos\alpha & \sin\alpha\sin\beta\\ 1 & 0 & \cos\beta\\ 0 & -\sin\alpha & \cos\alpha\sin\beta\end{pmatrix}\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.
$$

The singularities of this mapping occur when $\det \boldsymbol{T}(\boldsymbol{\phi}) = -\sin\beta = 0$, i.e., for $\beta = 0$ and $\beta = \pi$.

In alternative to the above procedure, and perhaps more quickly, we can build the matrix $\boldsymbol{T}(\boldsymbol{\phi})$ by noting the individual contributions to the angular velocity $\boldsymbol{\omega}$ of $\dot{\alpha}$ (a rotation around the initial, fixed Y-axis), $\dot{\beta}$ (a rotation around the X'-axis, i.e., the X-axis after the rotation $\boldsymbol{R}_Y(\alpha)$), and $\dot{\gamma}$ (a rotation around the Y''-axis, i.e., the Y-axis after the first two rotations $\boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)$). We have

$$
\begin{aligned}
\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha},Y} + \boldsymbol{\omega}_{\dot{\beta},X'} + \boldsymbol{\omega}_{\dot{\gamma},Y''} &= \begin{pmatrix}0\\1\\0\end{pmatrix}\dot{\alpha} + \boldsymbol{R}_Y(\alpha)\begin{pmatrix}1\\0\\0\end{pmatrix}\dot{\beta} + \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\begin{pmatrix}0\\1\\0\end{pmatrix}\dot{\gamma}\\
&= \begin{pmatrix}0\\1\\0\end{pmatrix}\dot{\alpha} + \begin{pmatrix}\cos\alpha\\0\\-\sin\alpha\end{pmatrix}\dot{\beta} + \begin{pmatrix}\sin\alpha\sin\beta\\ \cos\beta\\ \cos\alpha\sin\beta\end{pmatrix}\dot{\gamma} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.
\end{aligned}
$$

Note also that, being each contribution to $\boldsymbol{\omega}$ a vector itself, the order in the sum is irrelevant.

Exercise #4

The solution requires to compute the position $\boldsymbol{p}_d(t) \in \mathbb{R}^2$ and orientation $\phi_d(t) \in \mathbb{R}$ of the end-effector at $t = 1$ [s] during the execution of the assigned task, together with the task velocity vector $\dot{\boldsymbol{r}}_d(t) = \begin{pmatrix} \dot{\boldsymbol{p}}_d^T(t) & \dot{\phi}_d(t) \end{pmatrix}^T$. An inverse kinematics problem is solved then analytically to obtain at that time instant a unique value of $\boldsymbol{q}$, which is used to evaluate the 3×3 task Jacobian $\boldsymbol{J}(\boldsymbol{q})$. Finally, inversion of the differential kinematics map provides the commanded joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$.

Note first that the motion trajectory from $\boldsymbol{A}$ to $\boldsymbol{B}$ lasts $T = \|\boldsymbol{B} - \boldsymbol{A}\|/v = 2.3564/0.5 = 4.7127$ [s], so that $t = 1$ [s] corresponds to an instant when the end-effector is actually on the linear path. The Cartesian trajectory and the desired position $\bar{\boldsymbol{p}}_d$ are thus

$$\boldsymbol{p}_d(t) = \boldsymbol{A} + \frac{vt}{\|\boldsymbol{B} - \boldsymbol{A}\|}(\boldsymbol{B} - \boldsymbol{A}) = \begin{pmatrix} 3 \\ 2.5 \end{pmatrix} - \frac{0.5t}{2.3564}\begin{pmatrix} 2.25 \\ 0.7 \end{pmatrix}, \quad \text{at } t = 1 \;\Rightarrow\; \bar{\boldsymbol{p}}_d = \boldsymbol{p}_d(1) = \begin{pmatrix} 2.5226 \\ 2.3515 \end{pmatrix}.$$

The orientation remains instead constant at all times and, according to Fig. 3, is given by[1]

$$\phi_d = \text{ATAN2}\left\{\boldsymbol{A}_y - \boldsymbol{B}_y, \boldsymbol{A}_x - \boldsymbol{B}_x\right\} + \frac{\pi}{2} = \text{ATAN2}\left\{0.7, 2.25\right\} + \frac{\pi}{2} = 1.8724 \text{ [rad]} = 107.28^\circ.$$

Accordingly, the desired task velocity (at $t = 1$, as well as at any other instant) is constant and is specified by

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} v\,\dfrac{\boldsymbol{B} - \boldsymbol{A}}{\|\boldsymbol{B} - \boldsymbol{A}\|} \\ \dot{\phi}_d \end{pmatrix} = \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix}.$$

Using the standard D-H joint variables, the task kinematics of the 3R robot at hand is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} l\left(\cos q_1 + \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3)\right) \\ l\left(\sin q_1 + \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3)\right) \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

with its 3×3 Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l\left(s_1 + s_{12} + s_{123}\right) & -l\left(s_{12} + s_{123}\right) & -l s_{123} \\ l\left(c_1 + c_{12} + c_{123}\right) & l\left(c_{12} + c_{123}\right) & l c_{123} \\ 1 & 1 & 1 \end{pmatrix}, \tag{7}$$

where the shorthand notation for trigonometric functions has been used, e.g., $s_{123} = \sin(q_1+q_2+q_3)$.

With reference to Fig. 8, the inverse kinematics problem can be decomposed in two parts. First, we solve for the two joint variables q_1 and q_2 in order to place the tip of the second link (or, the base of the third link) in the necessary position

$$\bar{\boldsymbol{p}}_{t2} = \bar{\boldsymbol{p}}_d - \ell\begin{pmatrix} \cos\phi_d \\ \sin\phi_d \end{pmatrix} = \begin{pmatrix} 3.1167 \\ 0.4418 \end{pmatrix} \text{ [m]}.$$

Since the robot arm has not crossed a singularity while moving the end-effector from $\boldsymbol{A}$ to $\bar{\boldsymbol{p}}_d$ (this can be easily verified), the configuration of the first two joints should remain the initial one, or *elbow up*. Thus, we find a unique solution for the pair (q_1, q_2) given by

$$c_2 = \frac{\bar{\boldsymbol{p}}_{t2,x}^2 + \bar{\boldsymbol{p}}_{t2,y}^2 - 2\ell^2}{2\ell^2} = 0.2386, \quad s_2 = -\sqrt{1 - c_2^2} = -0.9711$$

$$\Rightarrow\; q_2 = \text{ATAN2}\left\{s_2, c_2\right\} = -1.3298 \text{ [rad]},$$

[1] In the two arguments of the ATAN2 function, we have eliminated the common denominator $\|\boldsymbol{B} - \boldsymbol{A}\| > 0$.

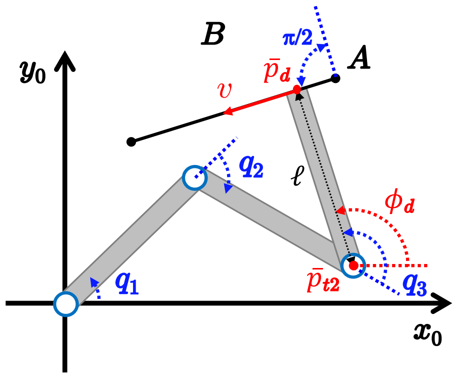


Figure 8: Solution approach to the inverse kinematics for the 3R planar robot.

and[2]

$$s_1 = \frac{\bar{\boldsymbol{p}}_{t2,y}\,(\ell + \ell c_2) - \bar{\boldsymbol{p}}_{t2,x}\ell s_2}{2\ell^2(1+c_2)} = 0.7213, \quad c_1 = \frac{\bar{\boldsymbol{p}}_{t2,x}\,(\ell + \ell c_2) + \bar{\boldsymbol{p}}_{t2,y}\ell s_2}{2\ell^2(1+c_2)} = 0.6926$$

$$\Rightarrow\; q_1 = \text{ATAN2}\{s_1, c_1\} = 0.8057\,[\text{rad}].$$

At this point, with $(q_1, q_2) = (0.8057, -1.3298)$ [rad] $= (46.16°, -76.19°)$, the third joint variable q_3 is recovered from the specification $\phi_d = 1.8724$ [rad] on the end-effector orientation:

$$q_3 = \phi_d - (q_1 + q_2) = 2.3965 \text{ [rad]} = 137.31°.$$

The above solution of the inverse kinematics problem is coded in Matlab by the instructions:

```
p_t2=p_d-l*[cos(phi_d); sin(phi_d)]
px=p_t2(1);
py=p_t2(2);
c2=(px^2+py^2-2*l^2)/(2*l^2)
s2=-sqrt(1-c2^2)  % elbow up solution (as the initial configuration)
q2=atan2(s2,c2)
s1=py*(l+l*c2)-px*l*s2  % denominator (> 0) discarded in s1 and c1
c1=px*(l+l*c2)+py*l*s2
q1=atan2(s1,c1)
q3=phi_d-(q1+q2)
```

Evaluating the Jacobian in (7) for the obtained $\boldsymbol{q} = (q_1, q_2, q_3)$ and inverting the differential mapping yields finally the joint velocity

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{r}}_d = \begin{pmatrix} -2.3515 & -0.9088 & -1.9097 \\ 2.5226 & 1.1374 & -0.5941 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -0.4458 & 0.2577 & -0.6982 \\ 0.8024 & 0.1137 & 1.5998 \\ -0.3566 & -0.3714 & 0.0983 \end{pmatrix} \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1745 \\ -0.4000 \\ 0.2254 \end{pmatrix} \text{ [rad/s]}.$$

[2]The common (positive) denominator $2\ell^2(1 + c_2)$ in the expressions of s_1 and c_1 can be discarded without affecting the final result.

Exercise #5

A) The possible sequences of Euler angles are 12. They are listed in Tab. 2, together with their one-to-one correspondence with RPY-type angles (with the reverse order in the products of elementary rotation matrices). The rotation matrix produced in one or in the corresponding sequence of angles (around the moving or fixed axes) will be the same.

i	Euler sequences	Roll-Pitch-Yaw sequences
1	$X(\alpha)Y'(\beta)X''(\gamma)$	$X(\gamma)Y(\beta)X(\alpha)$
2	$X(\alpha)Y'(\beta)Z''(\gamma)$	$Z(\gamma)Y(\beta)X(\alpha)$
3	$X(\alpha)Z'(\beta)X''(\gamma)$	$X(\gamma)Z(\beta)X(\alpha)$
4	$X(\alpha)Z'(\beta)Y''(\gamma)$	$Y(\gamma)Z(\beta)X(\alpha)$
5	$Y(\alpha)X'(\beta)Y''(\gamma)$	$Y(\gamma)X(\beta)Y(\alpha)$
6	$Y(\alpha)X'(\beta)Z''(\gamma)$	$Z(\gamma)X(\beta)Y(\alpha)$
7	$Y(\alpha)Z'(\beta)X''(\gamma)$	$X(\gamma)Z(\beta)Y(\alpha)$
8	$Y(\alpha)Z'(\beta)Y''(\gamma)$	$Y(\gamma)Z(\beta)Y(\alpha)$
9	$Z(\alpha)X'(\beta)Y''(\gamma)$	$Y(\gamma)X(\beta)Z(\alpha)$
10	$Z(\alpha)X'(\beta)Z''(\gamma)$	$Z(\gamma)X(\beta)Z(\alpha)$
11	$Z(\alpha)Y'(\beta)X''(\gamma)$	$X(\gamma)Y(\beta)Z(\alpha)$
12	$Z(\alpha)Y'(\beta)Z''(\gamma)$	$Z(\gamma)Y(\beta)Z(\alpha)$

Table 2: Correspondence between Euler and RPY minimal representations of orientation.

B) The maximum angular velocity that the driven link can reach is equal to

$$\dot{q}_{max} = \frac{\dot{\theta}_{max}(\text{rpm})}{n_r} \cdot \frac{2\pi}{60} = \frac{4120}{6000}\,\pi = 2.1572\,[\text{rad/s}].$$

Thus, the link velocity $\dot{q} = 3.5$ [rad/s] is unfeasible. The optimal reduction ratio that minimizes the required motor torque for a given link angular acceleration $\ddot{q}$ satisfies the relation $n_r = \sqrt{J_l/J_m}$. If $n_r = 100$ is such an optimal value for the given motor inertia $J_m = 1.2 \cdot 10^{-5}$ [kg·m^2], then the value of the link inertia is

$$J_l = n_r^2\, J_m = 1.2 \cdot 10^{-1}\ [\text{kg}\cdot\text{m}^2].$$

Therefore, the motor torque needed to produce a link angular acceleration $\ddot{q} = 4$ [rad/s^2] is

$$\tau_m = \left(J_m n_r + \frac{J_l}{n_r}\right)\ddot{q} = 2\sqrt{J_m J_l}\,\ddot{q} = 9.6 \cdot 10^{-3}\ [\text{Nm}],$$

where the second (equivalent) expression follows from the optimality of the reduction ratio.

* * * * *

Robotics 1

Remote Exam – September 11, 2020

Exercise #1

Given a smooth time-varying rotation matrix $\boldsymbol{R}(t) \in SO(3)$, provide a formula to determine the associated angular acceleration vector $\dot{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ as a function of $\boldsymbol{R}(t)$ and of the angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$. Apply then this formula to compute $\boldsymbol{\omega}(t)$ and $\dot{\boldsymbol{\omega}}(t)$, given the following rotation matrix:

$$\boldsymbol{R}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ \sin^2 t & \cos t & -\sin t \cos t \\ -\sin t \cos t & \sin t & \cos^2 t \end{pmatrix}.$$

Exercise #2

Consider the 6R Universal Robots UR5 manipulator in Fig. 1, where a feasible set of Denavit-Hartenberg (DH) frames has been assigned. Complete the table of DH parameters and enter also the associated numerical values (expressed in [rad] or [mm]), including those of the joint variables $\boldsymbol{q} = \boldsymbol{\theta}$ in the configuration shown. In the figure, all data are already given in mm.

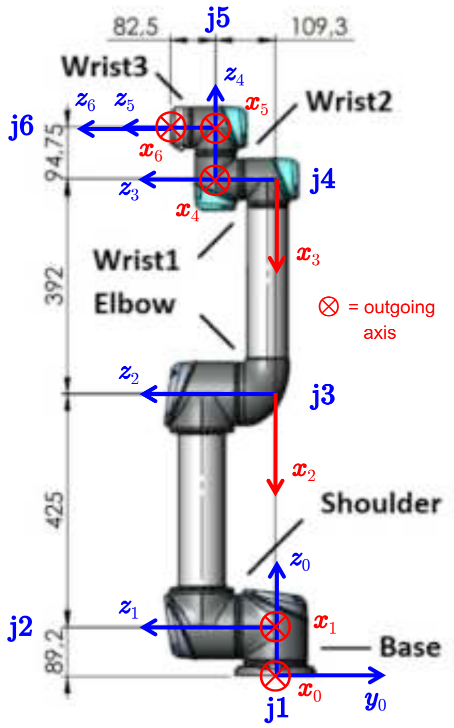


Figure 1: An assignment of DH frames for the UR5 manipulator.

Exercise #3

With reference to Fig. 2, two planar manipulators, a 2R robot (labeled as A) and a 3R robot (labeled as B), both with links of unitary length, should perform a task in cooperation, handing over an object between their end-effector grippers. The base frames of the two robots are positioned with respect to a common world frame by ${}^w\boldsymbol{p}_A = \begin{pmatrix} -2.5 & 1 \end{pmatrix}^T$ and ${}^w\boldsymbol{p}_B = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$. The base of robot B is rotated counterclockwise by an angle $\alpha_B = \pi/6$ [rad] with respect to $\boldsymbol{x}_w$. Robot A holds the object while being in the configuration $\boldsymbol{q}_A = \begin{pmatrix} \pi/3 & -\pi/2 \end{pmatrix}^T$ [rad]. Determine a configuration $\boldsymbol{q}_B$ for robot B such that it can grasp the object held by robot A with the correct orientation.

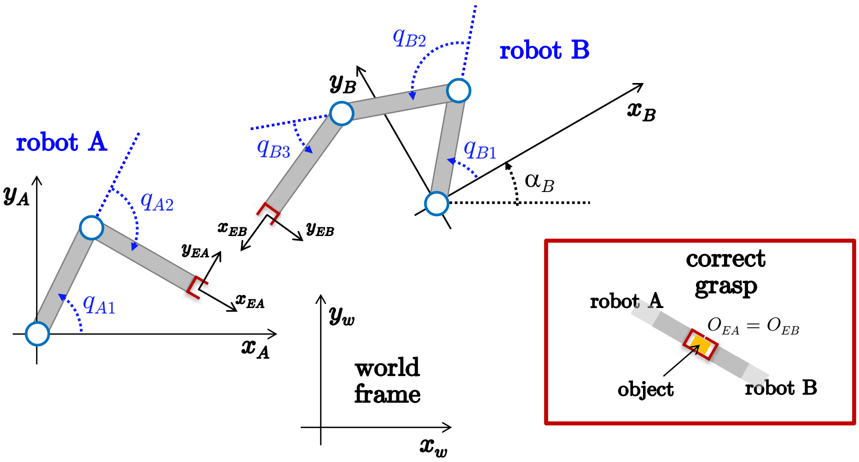


Figure 2: A 2R and a 3R planar manipulators cooperating in a task.

Exercise #4

Consider the 3×3 Jacobian of a 3R spatial robot, with generic link lengths $l_2 > 0$ and $l_3 > 0$:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(l_2c_2 + l_3c_3) & -l_2c_1s_2 & -l_3c_1s_3 \\ c_1(l_2c_2 + l_3c_3) & -l_2s_1s_2 & -l_3s_1s_3 \\ 0 & l_2c_2 & l_3c_3 \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Find all (singular) configurations $\boldsymbol{q}^\diamond$ where the rank of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is equal to 2 and all configurations $\boldsymbol{q}^*$ where the rank is equal to 1. In a singularity with rank 1, determine a basis for each of the subspaces $\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{R}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$, and $\mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$.

Exercise #5

A mass $M = 2$ [kg] moves linearly under a bounded force u, with $|u| \le U_{max} = 8$ [N], according to differential equation $M\ddot{x} = u$. The mass starts at $t = 0$ from $x_i = x(0) = 0$ with a negative velocity $\dot{x}_i = \dot{x}(0) = -2$ [m/s], and has to reach the final position $x_f = x(T) = 3$ [m] at rest (i.e., with $\dot{x}_f = \dot{x}(T) = 0$) in minimum time T. Determine the minimum time T and the associated optimal command $u^*(t)$. Sketch the time evolution of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$.

[240 minutes (4 hours); open books]

Solution

September 11, 2020

Exercise #1

We have that

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}, \quad \text{and thus} \quad \boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T \;\Rightarrow\; \boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{3,2}(\boldsymbol{\omega}) \\ \boldsymbol{S}_{1,3}(\boldsymbol{\omega}) \\ \boldsymbol{S}_{2,1}(\boldsymbol{\omega}) \end{pmatrix}.$$

Differentiating further with respect to time,

$$\ddot{\boldsymbol{R}} = \boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{R} + \boldsymbol{S}(\boldsymbol{\omega})\dot{\boldsymbol{R}} = \boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{R} + \boldsymbol{S}^2(\boldsymbol{\omega})\boldsymbol{R}.$$

Since

$$\begin{aligned} \boldsymbol{S}^2(\boldsymbol{\omega}) &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\left(\omega_y^2 + \omega_z^2\right) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & -\left(\omega_x^2 + \omega_z^2\right) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & -\left(\omega_x^2 + \omega_y^2\right) \end{pmatrix} = \boldsymbol{\omega}\,\boldsymbol{\omega}^T - \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2, \end{aligned}$$

we obtain finally

$$\ddot{\boldsymbol{R}} = \left(\boldsymbol{S}(\dot{\boldsymbol{\omega}}) + \boldsymbol{\omega}\,\boldsymbol{\omega}^T - \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2\right)\boldsymbol{R}, \quad \text{and thus} \qquad \begin{aligned} &\boldsymbol{S}(\dot{\boldsymbol{\omega}}) = \ddot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2 - \boldsymbol{\omega}\,\boldsymbol{\omega}^T \\ &\Rightarrow\; \dot{\boldsymbol{\omega}} = \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{3,2}(\dot{\boldsymbol{\omega}}) \\ \boldsymbol{S}_{1,3}(\dot{\boldsymbol{\omega}}) \\ \boldsymbol{S}_{2,1}(\dot{\boldsymbol{\omega}}) \end{pmatrix}. \end{aligned}$$

For the given time-varying rotation matrix, we obtain

$$\boldsymbol{R}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ \sin^2 t & \cos t & -\sin t\cos t \\ -\sin t\cos t & \sin t & \cos^2 t \end{pmatrix} \;\Rightarrow\; \dot{\boldsymbol{R}}(t) = \begin{pmatrix} -\sin t & 0 & \cos t \\ 2\sin t\cos t & -\sin t & \sin^2 t - \cos^2 t \\ \sin^2 t - \cos^2 t & \cos t & -2\sin t\cos t \end{pmatrix},$$

and thus, after simplifications,

$$\boldsymbol{S}(\boldsymbol{\omega}(t)) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) = \begin{pmatrix} 0 & -\sin t & \cos t \\ \sin t & 0 & -1 \\ -\cos t & 1 & 0 \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\omega}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}.$$

Moreover, one can evaluate

$$\ddot{\boldsymbol{R}}(t) = \begin{pmatrix} -\cos t & 0 & -\sin t \\ 2\left(\cos^2 t - \sin^2 t\right) & -\cos t & 4\sin t\cos t \\ 4\sin t\cos t & -\sin t & 2\left(\sin^2 t - \cos^2 t\right) \end{pmatrix}$$

and then compute

$$\boldsymbol{S}(\dot{\boldsymbol{\omega}}(t)) = \ddot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t)+\boldsymbol{I}\|\boldsymbol{\omega}(t)\|^2-\boldsymbol{\omega}(t)\,\boldsymbol{\omega}^T(t) = \begin{pmatrix} 0 & -\cos t & -\sin t \\ \cos t & 0 & 0 \\ \sin t & 0 & 0 \end{pmatrix} \Rightarrow \dot{\boldsymbol{\omega}}(t) = \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix}.$$

However, as one could have expected, we can also obtain $\dot{\boldsymbol{\omega}}(t) = d\boldsymbol{\omega}(t)/dt$ by direct differentiation (or from $\boldsymbol{S}(\dot{\boldsymbol{\omega}}(t)) = d\boldsymbol{S}(\boldsymbol{\omega}(t))/dt$).

Instead, the analytic formula is strictly required in case $\boldsymbol{R}$, $\boldsymbol{\omega}$, and $\ddot{\boldsymbol{R}}$ are known only numerically at a given instant of time. For example, if we had

$$\boldsymbol{R} = \boldsymbol{I}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \ddot{\boldsymbol{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

we would then compute

$$\boldsymbol{S}(\dot{\boldsymbol{\omega}}) = \ddot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{I}\|\boldsymbol{\omega}\|^2 - \boldsymbol{\omega}\,\boldsymbol{\omega}^T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{\omega}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is nothing else than the considered case for $t = 0$.

Exercise #2

The Denavit-Hartenberg parameters (in mm or rad) of the UR5 manipulator associated to the frames specified in Fig. 1 are given in Tab. 1. Note that both parameters a_2 and a_3 are negative. In fact, to reach O_2 from O_1 we move in the opposite direction of $\boldsymbol{x}_2$, thus $a_2 < 0$. Similarly, to reach O_3 from O_2 we move in the opposite direction of $\boldsymbol{x}_3$, thus $a_3 < 0$.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	89.2	$q_1 = 0$
2	0	-425	0	$q_2 = -\pi/2$
3	0	-392	0	$q_3 = 0$
4	$-\pi/2$	0	109.3	$q_4 = \pi/2$
5	$\pi/2$	0	94.75	$q_5 = 0$
6	0	0	82.5	$q_6 = 0$

Table 1: DH parameters of the UR5 manipulator, with values of $\boldsymbol{q}$ in the configuration of Fig. 1.

Exercise #3

To accomplish the cooperative task we need to find the desired position and orientation of the end-effector of robot B, as expressed in its own base reference frame. For this, we will use the mathematics of 4×4 homogeneous transformations, starting from the definition of the position and orientation of the end-effector of robot A, as computed from the direct kinematics of the task in the world frame. Although the entire problem is planar, with positions in $\mathbb{R}^2$ and scalar orientations expressed by an angle around the normal to the plane $(\boldsymbol{x}_w, \boldsymbol{y}_w)$, we will embed objects in 3D. Once the target pose of the end-effector of robot B is available, the configuration $\boldsymbol{q}_B$ of robot B is found by solving a standard inverse kinematics problem.

With the given data of the problem, the base reference frames of robot A and B are located respectively by

$$
{}^w\boldsymbol{T}_A = \begin{pmatrix} {}^w\boldsymbol{R}_A & {}^w\boldsymbol{p}_A \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} & -2.5 \\ \boldsymbol{I}_{3\times 3} & 1 \\ & 0 \\ \boldsymbol{0}^T & 1 \end{pmatrix}
$$

and

$$
{}^w\boldsymbol{T}_B = \begin{pmatrix} {}^w\boldsymbol{R}_B & {}^w\boldsymbol{p}_B \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_B & -\sin\alpha_B & 0 & 1 \\ \sin\alpha_B & \cos\alpha_B & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5 & 0 & 1 \\ 0.5 & 0.8660 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
$$

The direct kinematics of the planar 2R robot A (from its base to the end-effector frame EA), taking into account the unitary length of the links, is computed as

$$
\begin{aligned}
{}^A\boldsymbol{T}_{EA} &= \begin{pmatrix} {}^A\boldsymbol{R}_{EA} & {}^A\boldsymbol{p}_{EA} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(q_{A1}+q_{A2}) & -\sin(q_{A1}+q_{A2}) & 0 & \cos q_{A1} + \cos(q_{A1}+q_{A2}) \\ \sin(q_{A1}+q_{A2}) & \cos(q_{A1}+q_{A2}) & 0 & \sin q_{A1} + \sin(q_{A1}+q_{A2}) \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.8660 & 0.5 & 0 & 1.3660 \\ -0.5 & 0.8660 & 0 & 0.3660 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
\end{aligned}
$$

Finally, the correct grasping condition by robot B requires that the two end-effector frames have the same origin $(O_{EB} = O_{EA})$ and opposite orientations (i.e., with a relative rotation of π around the common $\boldsymbol{z}_w$ axis). Therefore, the associated homogeneous transformation is

$$
{}^{EA}\boldsymbol{T}_{EB} = \begin{pmatrix} {}^{EA}\boldsymbol{R}_{EB} & {}^{EA}\boldsymbol{p}_{EB} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
$$

We can write now the kinematic equation of the task using the above homogeneous transformation matrices, equating the end-effector pose ${}^w\boldsymbol{T}_{EB}$ of robot B in the world frame, as evaluated from the side of robot A and from the side of robot B:

$$
{}^w\boldsymbol{T}_A {}^A\boldsymbol{T}_{EA} {}^{EA}\boldsymbol{T}_{EB} = {}^w\boldsymbol{T}_B {}^B\boldsymbol{T}_{EB}.
$$

Thus, the desired pose of the end-effector of robot B expressed in the reference frame B is:

$$
\begin{aligned}
{}^{B}\boldsymbol{T}_{EB,d} &= \begin{pmatrix} {}^{B}\boldsymbol{R}_{EB,d} & {}^{B}\boldsymbol{p}_{EB,d} \\ \mathbf{0}^{T} & 1 \end{pmatrix} = \left({}^{w}\boldsymbol{T}_{B}\right)^{-1} {}^{w}\boldsymbol{T}_{A}\,{}^{A}\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB} \\
&= \begin{pmatrix} -0.5 & -0.8660 & 0 & -2.1651 \\ 0.8660 & -0.5 & 0 & 0.5179 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0}^{T} & & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi_{B,d} & -\sin\phi_{B,d} & 0 & {}^{B}\boldsymbol{p}_{EB,d_x} \\ -\sin\phi_{B,d} & \cos\phi_{B,d} & 0 & {}^{B}\boldsymbol{p}_{EB,d_y} \\ 0 & 0 & 1 & 0 \\ & \mathbf{0}^{T} & & 1 \end{pmatrix}
\end{aligned}
$$

The inverse kinematics problem for the planar 3R robot B requires the solution of

$$
\begin{aligned}
&{}^{B}\boldsymbol{T}_{EB,d} = {}^{B}\boldsymbol{T}_{EB}(\boldsymbol{q}_{B}) \\
&= \begin{pmatrix} \cos(q_{B1}+q_{B2}+q_{B3}) & -\sin(q_{B1}+q_{B2}+q_{B3}) & 0 & \cos q_{B1}+\cos(q_{B1}+q_{B2})+\cos(q_{B1}+q_{B2}+q_{B3}) \\ \sin(q_{B1}+q_{B2}+q_{B3}) & \cos(q_{B1}+q_{B2}+q_{B3}) & 0 & \sin q_{B1}+\sin(q_{B1}+q_{B2})+\sin(q_{B1}+q_{B2}+q_{B3}) \\ 0 & 0 & 1 & 0 \\ & \mathbf{0}^{T} & & 1 \end{pmatrix}
\end{aligned}
$$

in terms of the unknown joint variables $\boldsymbol{q}_{B} = (q_{B1}, q_{B2}, q_{B3})$. The desired angle $\phi_{B,d}$ characterizing the orientation in the plane of the end-effector frame of robot B can be extracted from the elements of the rotation matrix ${}^{B}\boldsymbol{R}_{EB,d}$ as

$$
\begin{aligned}
\phi_{B,d} &= \text{ATAN2}\left\{\sin\phi_{B,d}, \cos\phi_{B,d}\right\} = \text{ATAN2}\left\{{}^{B}\boldsymbol{R}_{EB,d_{21}}, {}^{B}\boldsymbol{R}_{EB,d_{11}}\right\} \\
&= \text{ATAN2}\left\{0.8660, -0.5\right\} = 2.0944\ [\text{rad}] = 120^{\circ},
\end{aligned}
$$

the above is equivalent to solving the three nonlinear equations

$$
\begin{pmatrix} \cos q_{B1}+\cos(q_{B1}+q_{B2})+\cos(q_{B1}+q_{B2}+q_{B3}) \\ \sin q_{B1}+\sin(q_{B1}+q_{B2})+\sin(q_{B1}+q_{B2}+q_{B3}) \\ q_{B1}+q_{B2}+q_{B3} \end{pmatrix} = \begin{pmatrix} {}^{B}\boldsymbol{p}_{EB,d_x} \\ {}^{B}\boldsymbol{p}_{EB,d_y} \\ \phi_{B,d} \end{pmatrix} = \begin{pmatrix} -2.1651 \\ 0.5179 \\ 2.0944 \end{pmatrix}.
$$

As usual, this inverse kinematics problem for the planar 3R robot can be decomposed in two parts. First, we solve for the two joint variables q_{B1} and q_{B2} in order to place the tip position $\boldsymbol{p}_{t2}$ of the second link (or, the base of the third link) in the necessary position. Taking again into account the unitary length of the robot links, we have

$$
\boldsymbol{p}_{t2} = \begin{pmatrix} {}^{B}\boldsymbol{p}_{EB,d_x} \\ {}^{B}\boldsymbol{p}_{EB,d_x} \end{pmatrix} - \begin{pmatrix} \cos\phi_{B,d} \\ \sin\phi_{B,d} \end{pmatrix} = \begin{pmatrix} -2.1651 \\ 0.5179 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 0.8660 \end{pmatrix} = \begin{pmatrix} -1.6651 \\ -0.3481 \end{pmatrix}\ [\text{m}].
$$

Thus, a solution for the pair (q_{B1}, q_{B2}) is given by

$$
\begin{aligned}
&c_2 = \frac{\boldsymbol{p}_{t2,x}^2 + \boldsymbol{p}_{t2,y}^2 - 2}{2} = 0.4468, \quad s_2 = \sqrt{1 - c_2^2} = 0.8946 \\
&\Rightarrow\ q_{B2} = \text{ATAN2}\left\{s_2, c_2\right\} = 1.1076\,[\text{rad}] = 63.46^{\circ},
\end{aligned}
$$

and[1]

$$
\begin{aligned}
&s_1 = \frac{\boldsymbol{p}_{t2,y}\,(1+c_2) - \boldsymbol{p}_{t2,x} s_2}{2(1+c_2)} = 0.3408, \quad c_1 = \frac{\boldsymbol{p}_{t2,x}\,(1+c_2) + \boldsymbol{p}_{t2,y} s_2}{2(1+c_2)} = -0.9401 \\
&\Rightarrow\ q_{B1} = \text{ATAN2}\left\{s_1, c_1\right\} = 2.7939\,[\text{rad}] = 160.08^{\circ}.
\end{aligned}
$$

[1] The common denominator $2(1+c_2) > 0$ in the expressions of s_1 and c_1 can be discarded without affecting the final result in the evaluation of ATAN2.

The (arbitrary) choice of the + sign for the square root in s_2 results here in an *elbow up* solution for the first two links of the 3R robot. Next, with $(q_{B1}, q_{B2}) = (2.7939, 1.1076)$ [rad], the third joint variable q_{B3} is recovered from the specification $\phi_{B,d} = 2.0944$ [rad] on the end-effector orientation:

$$q_{B3} = \phi_{B,d} - (q_{B1} + q_{B2}) = -1.8071 \text{ [rad]} = -103.54°.$$

The above solution of the inverse kinematics problem is coded in Matlab by the instructions (for unitary lenghts):

```
p_t2=p_Bd-[cos(phi_Bd); sin(phi_Bd)]
px=p_t2(1);
py=p_t2(2);
c2=(px^2+py^2-2)/2
s2=sqrt(1-c2^2)  % sign + on sqrt results in elbow up solution (arbitrary choice)
q_B2=atan2(s2,c2)
s1=py*(1+c2)-px*s2  % denominator (> 0) discarded in s1 and c1
c1=px*(1+c2)+py*s2
q_B1=atan2(s1,c1)
q_B3=phi_Bd-(q_B1+q_B2)
```

Exercise #4

This exercise can be solved with ease either by hand or using the symbolic instructions of Matlab (with caution on simplifications)[2]. To determine the singularities of $\boldsymbol{J}(\boldsymbol{q})$, it is useful to get rid of the dependence of the Jacobian on q_1, by expressing the velocity $\boldsymbol{v}$ in the rotated frame 1 as[3]

$${}^1\boldsymbol{v} = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{v} = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = {}^1\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Thus, we obtain

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & -l_2 s_2 & -l_3 s_3 \\ l_2 c_2 + l_3 c_3 & 0 & 0 \\ 0 & l_2 c_2 & l_3 c_3 \end{pmatrix}.$$

The determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = \det {}^1\boldsymbol{J}(\boldsymbol{q}) = l_2 l_3 s_{2-3} \left(l_2 c_2 + l_3 c_3\right).$$

Therefore, the singularities occur when

$$\sin(q_2 - q_3) = 0 \iff q_3 = \{q_2, q_2 \pm \pi\} \quad \text{(third link stretched or folded w.r.t. the second link)}^4,$$

or when

$$l_2 c_2 + l_3 c_3 = 0 \qquad \text{(end-effector located along the axis of the first joint)},$$

[2]The robot considered in this exercise is similar to the one in Ex. #3 of June 5, 2020. However, absolute angles w.r.t. the horizontal are used here for joints 2 and 3, and the lengths of links 2 and 3 are generic rather than unitary.

[3]Because of the arbitrary definition of frame 0, we know that the variable q_1 will never enter in the definition of singularities of a serial robot manipulator —in this case in the expression of $\det \boldsymbol{J}(\boldsymbol{q})$.

[4]This comment and the next one follow from the fact that the given Jacobian is associated to a 3R spatial robot of the elbow type, with q_2 and q_3 defined as absolute link angles w.r.t. the horizontal plane.

or when both situations occur. In the first two cases, the rank of $\boldsymbol{J}$ drops by one unit. We have[5]

$$\boldsymbol{J}(\boldsymbol{q}^\diamond) = \boldsymbol{J}(\boldsymbol{q})|_{\sin(q_2-q_3)=0} = \begin{pmatrix} -(l_2 \pm l_3)s_1c_2 & -l_2c_1s_2 & \mp l_3c_1s_2 \\ (l_2 \pm l_3)c_1c_2 & -l_2s_1s_2 & \mp l_3s_1s_2 \\ 0 & l_2c_2 & \pm l_3c_2 \end{pmatrix}, \qquad \text{rank}\,\boldsymbol{J}(\boldsymbol{q}^\diamond) = 2,$$

where $c_2 \neq 0$, otherwise also $l_2c_2 + l_3c_3 = 0$ would follow. Similarly, we have

$$\boldsymbol{J}(\boldsymbol{q}^\diamond) = \boldsymbol{J}(\boldsymbol{q})|_{l_2c_2+l_3c_3=0} = \begin{pmatrix} 0 & -l_2c_1s_2 & -l_3c_1s_3 \\ 0 & -l_2s_1s_2 & -l_3s_1s_3 \\ 0 & l_2c_2 & l_3c_3 \end{pmatrix}, \qquad \text{rank}\,\boldsymbol{J}(\boldsymbol{q}^\diamond) = 2.$$

On the other hand, when both situations occur simultaneously

$$\boldsymbol{J}(\boldsymbol{q}^*) = \boldsymbol{J}(\boldsymbol{q})|_{\sin(q_2-q_3)=0, l_2c_2+l_3c_3=0} = \begin{pmatrix} 0 & -l_2c_1s_2 & \mp l_3c_1s_2 \\ 0 & -l_2s_1s_2 & \mp l_3s_1s_2 \\ 0 & l_2c_2 & \pm l_3c_2 \end{pmatrix}, \qquad \text{rank}\,\boldsymbol{J}(\boldsymbol{q}^*) = 1.$$

Choosing for instance the rank 1 singular configuration $\boldsymbol{q}^*$ with $q_2 = q_3 = \pi/2$ (and with an arbitrary q_1)[6], we have

$$\boldsymbol{J}(\boldsymbol{q}^*) = \boldsymbol{J}(\boldsymbol{q})|_{q_2=q_3=\pi/2} = \begin{pmatrix} 0 & -l_2c_1 & -l_3c_1 \\ 0 & -l_2s_1 & -l_3s_1 \\ 0 & 0 & 0 \end{pmatrix},$$

We obtain the following subspaces:

$$\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}\right\}, \qquad \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ -l_3 \\ l_2 \end{pmatrix}\right\},$$

$$\mathcal{R}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} 0 \\ l_2 \\ l_3 \end{pmatrix}\right\}, \qquad \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} -s_1 \\ c_1 \\ * \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Exercise #5

The structure of the optimal command $u^*(t)$ for this state-to-rest minimum time motion problem is found rather intuitively, observing that the net desired displacement is $x_f - x_i = x_f - x(0) = x_f > 0$ and that the mass has an initial velocity in the opposite direction, $\dot{x}_i = \dot{x}(0) < 0$. Thus, we have to apply first the maximum positive feasible force $U_{max} > 0$ in order to stop as soon as possible the motion in the negative direction. This will happen in a finite time T_d. Then, from the reached position $x_d = x(T_d) < 0$, with $\dot{x}(T_d) = 0$, we have a standard rest-to-rest minimum time motion problem for a displacement $x_f - x_d > x_f > 0$. Since there is no velocity limitation in the problem formulation, this second problem is solved by a symmetric bang-bang force (and acceleration) profile in a time T_{bb}. In particular, we will continue to apply the maximum positive force U_{max} for half of the residual motion, switching then to $-U_{max} < 0$ so as to decelerate and stop at the final instant $t = T = T_d + T_{bb}$.

[5] The upper signs in the expression of $\boldsymbol{J}(\boldsymbol{q}^\diamond)$ apply when $q_3 = q_2$, the lower when $q_3 = q_2 + \pi$. The same situation happens later also in the expression of $\boldsymbol{J}(\boldsymbol{q}^*)$.

[6] The spatial 3R robot will then be fully stretched along the axis of joint 1. Similar computations can be done for $q_2 = q_3 = -\pi/2$, for $q_2 = \pi/2$ and $q_3 = -\pi/2$, or for $q_2 = -\pi/2$ and $q_3 = \pi/2$.

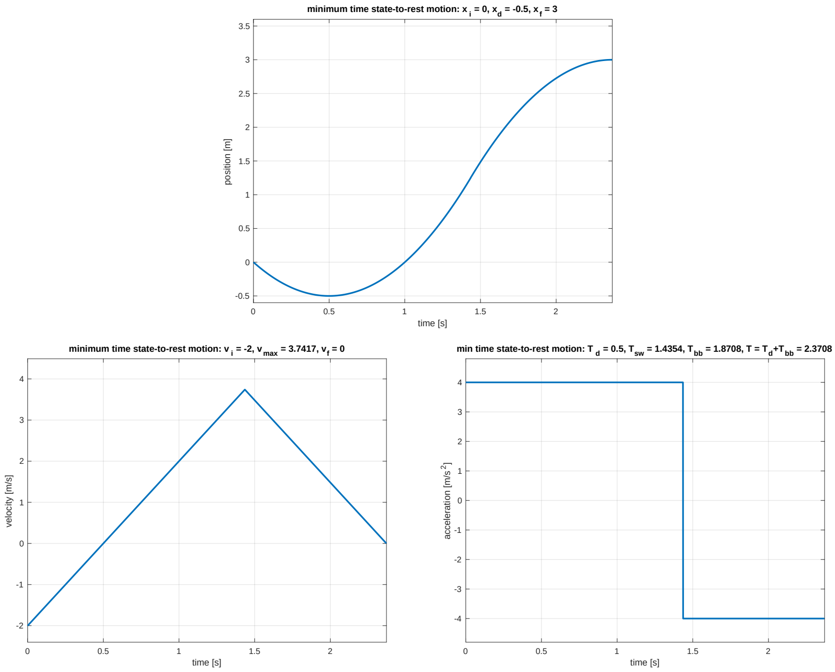


Figure 3: Minimum time state-to-rest motion: mass position, velocity, and acceleration.

Let $A_{max} = U_{max}/M = 8/2 = 4$ [m/s^2] be the maximum feasible acceleration. Applying this from $t = 0$ gives the resulting velocity profile

$$\dot{x}(t) = \dot{x}(0) + A_{max}\, t = -2 + 4\, t \overset{\downarrow}{=} 0 \qquad \Rightarrow \qquad t = T_d = -\frac{\dot{x}(0)}{A_{max}} = 0.5\,[\text{s}].$$

In the interval $t \in [0, T_d]$, the position of the mass evolves as

$$x(t) = x(0) + \dot{x}(0)\, t + A_{max}\frac{t^2}{2} = 0 - 2\, t + 4\,\frac{t^2}{2} = 2\, t\,(t-1) \qquad \Rightarrow \qquad x_d = x(T_d) = -0.5\,[\text{m}].$$

Therefore, the rest-to-rest motion should displace the mass by $L = x_f - x_d = 3 - (-0.5) = 3.5$ [m]. With a symmetric bang-bang acceleration profile, the minimum motion time for this second part of the task is

$$T_{bb} = 2\sqrt{\frac{L}{A_{max}}} = 1.8708\,[\text{s}]$$

and the switching of the command will occur at the middle point $x_d + (L/2) = 1.25$ [m] of this motion, after $T_{bb}/2 = 0.9354$ [s]; in absolute terms, at the instant $t = T_{sw} = T_d + T_{bb}/2 = 1.4354$ [s]. The peak velocity reached at this instant is $V_{max} = A_{max}T_{bb}/2 = 3.7417$ [m/s]. Finally, the minimum motion time is

$$T = T_d + T_{bb} = 2.3708\,[\text{s}].$$

The optimal force command will be

$$u^*(t) = \begin{cases} U_{max} = 8\,[\text{N}], & 0 \le t < T_{sw} = 1.4354\,[\text{s}], \\ -U_{max} = -8\,[\text{N}], & T_{sw} \le t < T = 2.3708\,[\text{s}]. \end{cases}$$

The profiles of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ in the interval $t \in [0, T]$ are shown in Fig. 3. One can clearly appreciate the asymmetry of the bang-bang acceleration profile.

* * * * *

Robotics I

Remote Exam — October 27, 2020

Exercise 1

Consider the spatial 4-dof robot with RRPR sequence of joints shown in Fig. 1. In the following, use **only** the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ defined therein. Note that these are **not** the joint variables of a Denavit-Hartenberg convention!

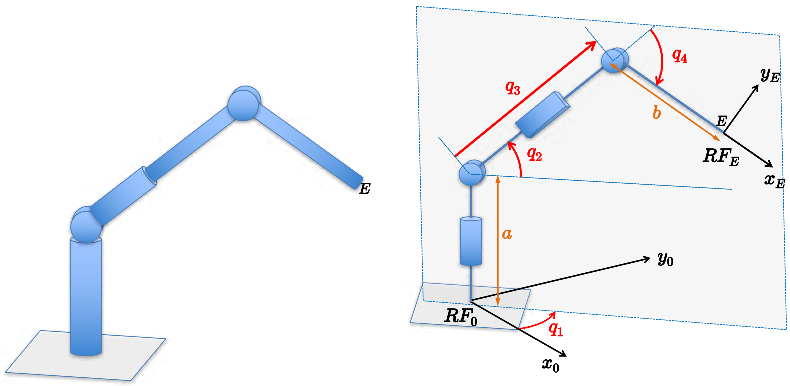


Figure 1: A RRPR robot and its kinematic skeleton, with definition of the joint coordinates $\boldsymbol{q}$.

Exercise 1a

- Determine the direct kinematics, namely the position ${}^0\boldsymbol{p}_E(\boldsymbol{q})$ of the origin and the orientation ${}^0\boldsymbol{R}_E(\boldsymbol{q})$ of the end-effector frame RF_E as functions of the joint variables $\boldsymbol{q}$.

Exercise 1b

- Let $a = 1$ and $b = 0.5$. Assuming that the prismatic joint takes only non-negative values $q_3 \geq 0$, solve the inverse kinematics problem when the (feasible) end-effector pose is given by

$$
{}^0\boldsymbol{A}_E = \begin{pmatrix} 0.5 & 0.5 & \frac{\sqrt{2}}{2} & 0.5 \\ 0.5 & 0.5 & -\frac{\sqrt{2}}{2} & 0.5 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Exercise 1c

- Compute the (6×4) geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$

$$
\begin{pmatrix} \boldsymbol{v}_E \\ \boldsymbol{\omega}_E \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.
$$

Exercise 1d

- Find all singular configurations of the linear part $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian.

Exercise 1e

- Give the symbolic expression (as a function of the configuration $\boldsymbol{q}$) of a non-trivial joint velocity $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$ such that $\boldsymbol{v}_E = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$ for all possible $\boldsymbol{q}$.

Exercise 2

Consider the motion profile in Fig. 2 for a generic robot joint, parametrized by the amplitude $J > 0$ and the duration $T > 0$. This time profile represents the motion jerk, namely the third time derivative of the joint position $q(t)$, for $t \in [0, T]$.

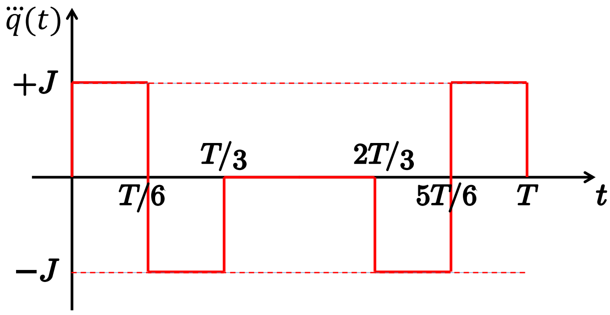


Figure 2: The jerk profile $\dddot{q}(t)$ of the joint motion.

Exercise 2a

- For a (rest-to-rest) motion with zero boundary conditions on velocity and acceleration, determine the value of the net displacement $\Delta = q(T) - q(0)$ as a function of J and T.

Exercise 2b

- Assume now that the initial velocity is $\dot{q}(0) = V > 0$, while $\ddot{q}(0) = 0$ is being kept. What will be then the displacement Δ? Will the final velocity and acceleration be zero at $t = T$?

Exercise 2c

- Assume instead that the initial acceleration is $\ddot{q}(0) = A > 0$, while $\dot{q}(0) = 0$. What will be the displacement Δ in this case? Will the final acceleration be zero at $t = T$?

Exercise 2d

- Let the initial acceleration be $\ddot{q}(0) = A > 0$. What value V should have the initial velocity $\dot{q}(0)$ so that the final velocity $\dot{q}(T)$ is zero? Will the final acceleration be zero at $t = T$?

[180 minutes, open books]

Robotics 1

Remote Midterm Test – November 20, 2020

The test has the form of a Questionnaire with 10 questions. Provide as many answers as you can, with short but significant texts and formulas/tables/pictures. Please write clearly. If you wish, you may use the 'Reply Sheet' in the `Exam.net` *environment to type in some answers. Take a picture of each page of your handwritten answers and upload them before submitting. Try to follow the same order of the questions. Number your answers accordingly (don't repeat the text of the questions).*

Question #1

Given three rotations around the sequence of fixed axes ZYX by the angles $\alpha_1 = -\pi/2$, $\alpha_2 = -\pi/4$, and $\alpha_3 = \pi/4$ [rad], provide the rotation matrix $\boldsymbol{R}$ that specifies the final orientation. Compute then a vector $\boldsymbol{r} \in \mathbb{R}^3$, with $\|\boldsymbol{r}\| = 1$, that will not be rotated by $\boldsymbol{R}$.

Question #2

A rigid body rotates from an initial orientation $\boldsymbol{R}_i$ to a final orientation $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find an axis/angle representation $(\boldsymbol{r}, \theta)$ of the rotation. Is the solution unique in this case?

Question #3

The pose of a rigid body $\mathcal{B}$ in 3D space w.r.t. a reference frame is expressed by 6 independent parameters, 3 for its position and 3 for its orientation when using a minimal representation. Why do we need then *only* 4 Denavit-Hartenberg parameters to characterize the pose of a link in a serial manipulator w.r.t. the frame associated to the previous link?

Question #4

For generic $m \geq 1$ and $n > 1$, give the total number of elementary products $N_\times$ and additions N_+ in evaluating, through operations with rotation matrices ${}^{j-1}\boldsymbol{R}_j$, the vectors ${}^0\boldsymbol{v}_i \in \mathbb{R}^3$ by the expression

$${}^0\boldsymbol{v}_i = \left({}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2\, \ldots\, {}^{n-1}\boldsymbol{R}_n\right) {}^0\boldsymbol{v}_i, \qquad \text{for } i = 1, \ldots, m, \tag{1}$$

or by successive matrix-vector products as

$${}^0\boldsymbol{v}_i = {}^0\boldsymbol{R}_1 \left({}^1\boldsymbol{R}_2 \left(\ldots \left(\ldots \left({}^{n-1}\boldsymbol{R}_n\, {}^0\boldsymbol{v}_i\right)\right) \ldots\right)\right), \qquad \text{for } i = 1, \ldots, m. \tag{2}$$

Given a value $n > 1$, which is the break-even value of m at which the number of evaluations $N_\times$ using (1) becomes advantageous (or disadvantageous) w.r.t. that using (2)?

Question #5

Robots are multi-body electromechanical systems driven by the torques $\boldsymbol{\tau}$ produced by the motors at the joints. In which sense are we allowed to say that one can move them by commanding just a desired joint velocity $\dot{\boldsymbol{q}}$ (or a joint position $\boldsymbol{q}$)?

Question #6

Consider the 4-dof PRPR robot sketched in Fig. 1, where the base frame RF_0 and the end-effector frame RF_4 are already assigned. The robot has a shoulder offset given by the constant $N > 0$.

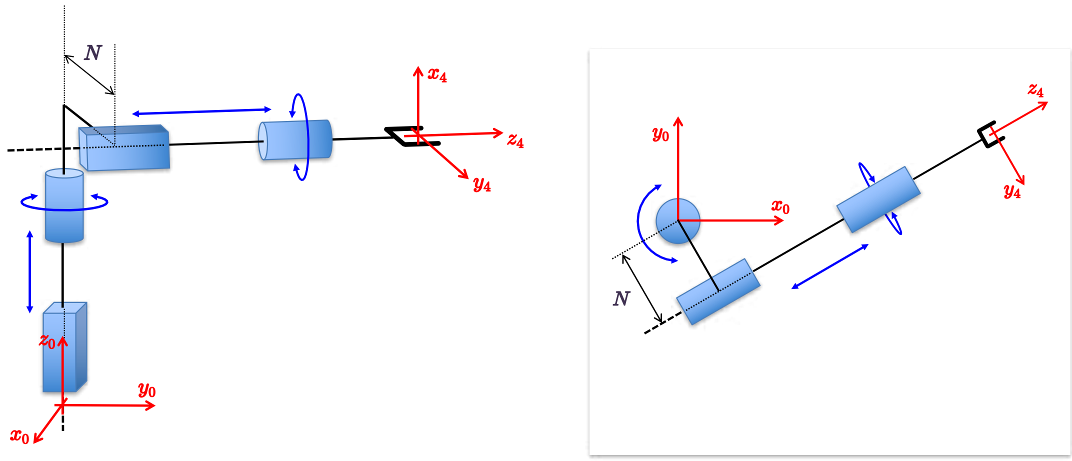


Figure 1: Kinematic skeleton of a PRPR robot. A perspective [left] and a top view (right).

Assign the other frames according to the Denavit-Hartenberg convention and build the associated table of parameters so that the position of the origin O_4 of the end-effector frame will be given by

$$
{}^0\boldsymbol{p}_4(\boldsymbol{q}) = \begin{pmatrix} N\cos q_2 - q_3 \sin q_2 \\ N\sin q_2 + q_3\cos q_2 \\ q_1 \end{pmatrix}. \tag{3}
$$

Determine the symbolic expression of ${}^0\boldsymbol{R}_4(\boldsymbol{q})$ in the direct kinematics. Further, provide a numerical matrix $\boldsymbol{R} \in SO(3)$ representing an orientation that the end-effector of this robot can never assume.

Question #7

Given a desired $\boldsymbol{p} \in \mathbb{R}^3$ for ${}^0\boldsymbol{p}_4(\boldsymbol{q})$ in (3), find all the analytical solutions $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ to the associated inverse kinematics problem in the regular case. Assuming there are no joint limits, sketch also the primary workspace WS_1 of the 4-dof PRPR robot. Finally, compute the numerical solutions to this inverse kinematics problem for $\boldsymbol{p} = \begin{pmatrix} 0 & 2 & 1.5 \end{pmatrix}^T$ with $N = 0.3$ [m].

Question #8

What are the pros and cons in estimating online or offline the velocity of a joint from position data measured by an encoder using numerical differentiation formulas. Write a simple code that uses the 1-step BDF (Euler) formula to provide online estimates $\dot{y}_k^e = \dot{y}^e(t_k)$, for $k = 1, \ldots, 10$, of the velocity from the following series of ten position data (noisy and with only 4 significant digits), collected with a sampling frequency of 40 Hz from $t_1 = 0$ on:

$$
\{y_k\} = \{0.0007 \quad 0.1251 \quad 0.2500 \quad 0.3741 \quad 0.4977 \quad 0.6187 \quad 0.7397 \quad 0.8579 \quad 0.9739 \quad 1.0876\} \text{ [rad]}
$$

Compute also the average value $\bar{\dot{y}}^e$ of the obtained samples of velocity estimates (for comparison, the average value of the true velocity samples $\dot{y}_k$, for $k = 1, \ldots, 10$, is $\bar{\dot{y}} = 4.8239$ [rad/s]).

Question #9

With reference to Fig. 2, the second joint of a 2R planar arm having link length $L_1 = 0.45$ and $L_2 = 0.35$ [m] is actuated by a motor M located at the first joint through a toothed transmission belt inside the body of link 1 (this may represent the situation of the first two dof of a SCARA robot). The belt connects a toothed disk of radius $r_1 = 5$ [cm], placed on the output shaft of motor M, with a second one of radius $r_2 = 0.25$ [m], connected to the axis of joint 2. An incremental encoder with 700 pulses/turn and electronic multiplication by a factor 4 is mounted on the back of motor M, for measuring its angular position θ_M.

a) Suppose that the optical disk of the encoder has generated 300 light pulses while rotating in the CCW direction in a time interval $T = 1.2$ [s]. How large is the rotation $\Delta\theta_2$ (in [rad]) performed by the second link? And what is the average angular speed $\bar{\dot{\theta}}_2$ (in [rad/s]) during T?

b) With the robot in the configuration $\boldsymbol{\theta} = \mathbf{0}$ (stretched arm) and keeping joint 1 at rest, what is the minimal lateral displacement (along the $\boldsymbol{y}_0$ direction) of the tip of link 2 that can be sensed by the encoder?

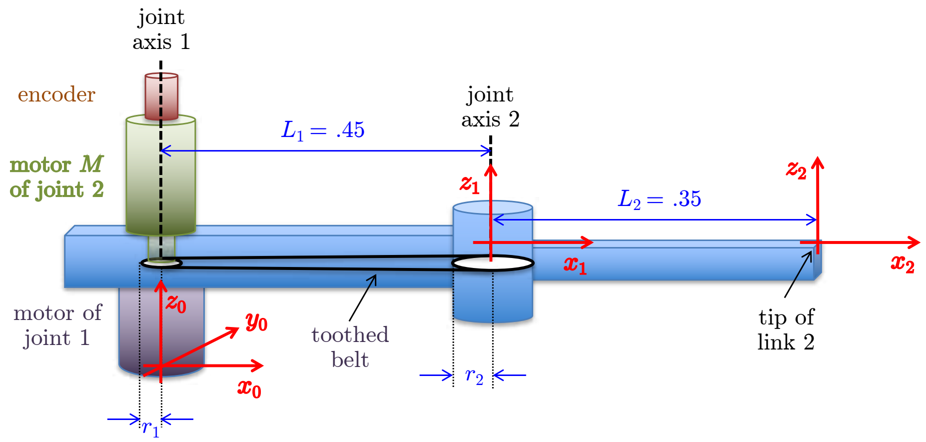


Figure 2: The transmission arrangement for moving joint 2 with a motor M placed at joint 1.

Question #10

The base frame RF_0 of a robot has its origin placed in the position ${}^W\boldsymbol{p}_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ and is rotated by an angle $\beta = \pi/2$ [rad] around the $\boldsymbol{z}_w$ axis of the world frame RF_W. In a given configuration, the end-effector pose of the robot is given by

$$ {}^0\boldsymbol{T}_E = \begin{pmatrix} 0 & 0.5 & -\dfrac{\sqrt{3}}{2} & 1 \\ 1 & 0 & 0 & -0.75 \\ 0 & -\dfrac{\sqrt{3}}{2} & -0.5 & 1.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

The position of the tip of a tool mounted on the end-effector is ${}^E\boldsymbol{p}_{tool} = \begin{pmatrix} 0 & 0.3 & 0.3 \end{pmatrix}^T$ [m]. Moreover, the tool frame RF_{tool} associated to this point is rotated by an angle $\gamma = -\pi/2$ [rad] around the $\boldsymbol{x}_E$ axis of the end-effector frame RF_E. Compute the position of the tip of the tool in the world frame and the absolute orientation of the tool frame w.r.t. RF_W.

[180 minutes (3 hours); open books]

Solution

November 20, 2020

Question #1

Given three rotations around the sequence of fixed axes ZYX by the angles $\alpha_1 = -\pi/2$, $\alpha_2 = -\pi/4$, and $\alpha_3 = \pi/4$ [rad], provide the rotation matrix $\boldsymbol{R}$ that specifies the final orientation. Compute then a vector $\boldsymbol{r} \in \mathbb{R}^3$, with $\|\boldsymbol{r}\| = 1$, that will not be rotated by $\boldsymbol{R}$.

Reply #1

The assigned sequence is of the Roll-Pitch-Yaw type, with

$$\boldsymbol{R}_Z = \left.\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|_{\alpha_1 = -\pi/2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{R}_Y = \left.\begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}\right|_{\alpha_2 = -\pi/4} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{R}_X = \left.\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_3 & -\sin\alpha_3 \\ 0 & \sin\alpha_3 & \cos\alpha_3 \end{pmatrix}\right|_{\alpha_3 = \pi/4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

The final orientation is computed by the product of these matrices in the *reverse* order of definition (rotations around fixed axes) as

$$\boldsymbol{R} = \boldsymbol{R}_X \boldsymbol{R}_Y \boldsymbol{R}_Z = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -0.5 & -0.5 \\ -1/\sqrt{2} & 0.5 & 0.5 \end{pmatrix}.$$

The unit vector $\boldsymbol{r} \in \mathbb{R}^3$ that is not rotated (nor scaled) by $\boldsymbol{R}$ is the eigenvector of $\boldsymbol{R}$ associated to its real eigenvalue $\lambda = +1$, i.e., such that

$$\boldsymbol{R}\boldsymbol{r} = \boldsymbol{r} \qquad \Rightarrow \quad \text{normalizing } \boldsymbol{r}\text{, up to the sign} \quad \Rightarrow \qquad \boldsymbol{r} = \pm \begin{pmatrix} -0.5774 \\ 0 \\ 0.8165 \end{pmatrix}.$$

This can be computed, e.g., with the Matlab instruction `[V,D]=eig(R)`, extracting then the (only) real eigenvector from the columns of the matrix `V`. ■

Question #2

A rigid body rotates from an initial orientation $\boldsymbol{R}_i$ to a final orientation $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find an axis/angle representation $(\boldsymbol{r}, \theta)$ of the rotation. Is the solution unique in this case?

Reply #2

One has to solve the inverse problem for the axis/angle representation $(\boldsymbol{r},\theta)$ of a rotation matrix

$$\boldsymbol{R}(\boldsymbol{r},\theta) = {}^{i}\boldsymbol{R}_f = \boldsymbol{R}_i^T \boldsymbol{R}_f = \begin{pmatrix} 0 & -0.5 & -0.8660 \\ 1 & 0 & 0 \\ 0 & -0.8660 & 0.5 \end{pmatrix}, \tag{4}$$

where ${}^{i}\boldsymbol{R}_f$ is the relative rotation from the initial to the final orientation. Denoting by R_{hk} the elements of the ${}^{i}\boldsymbol{R}_f$ matrix, from the inverse formulas we have

$$\sin\theta = \pm\frac{1}{2}\sqrt{(R_{12}-R_{21})^2+(R_{23}-R_{32})^2+(R_{13}-R_{31})^2} = 0.9682 \neq 0. \tag{5}$$

Therefore, this is a *regular* case and there will be two opposite solutions $(\boldsymbol{r},\theta)$ and $(-\boldsymbol{r},-\theta)$. The solution corresponding to the choice of the + sign in (5) is computed by the four Matlab instructions

```
ctheta=(R(1,1)+R(2,2)+R(3,3)-1)/2
stheta=sqrt((R(1,2)-R(2,1))^2+(R(2,3)-R(3,2))^2+(R(1,3)-R(3,1))^2)/2
ri=1/(2*stheta)*[R(3,2)-R(2,3); R(1,3)-R(3,1); R(2,1)-R(1,2)]
theta=atan2(stheta,ctheta)
```

yielding

$${}^{i}\boldsymbol{r} = \begin{pmatrix} -0.4472 \\ -0.4472 \\ 0.7746 \end{pmatrix}, \qquad \theta = 1.8235 \text{ [rad]} = 104.48^\circ.$$

Note that the unit axis $\boldsymbol{r}$ obtained with this procedure is naturally expressed in the coordinates of the initial frame oriented as $\boldsymbol{R}_i$. From (4), the final orientation is in fact computed by concatenating

$${}^{0}\boldsymbol{R}_i\,\boldsymbol{R}\left({}^{i}\boldsymbol{r},\theta\right) = {}^{0}\boldsymbol{R}_i\,{}^{i}\boldsymbol{R}_f = {}^{0}\boldsymbol{R}_f,$$

where the coordinate frame of definition of each vector/matrix term has been explicitly indicated by the use of superscripts. Thus, the expression of the invariant axis in the reference frame RF_0 is

$${}^{0}\boldsymbol{r} = \boldsymbol{R}_i\,{}^{i}\boldsymbol{r} = \begin{pmatrix} -0.4472 \\ 0.4472 \\ -0.7746 \end{pmatrix}. \qquad \blacksquare$$

Question #3

The pose of a rigid body $\mathcal{B}$ in 3D space w.r.t. a reference frame is expressed by 6 independent parameters, 3 for its position and 3 for its orientation when using a minimal representation. Why do we need then only *4 Denavit-Hartenberg parameters to characterize the pose of a link in a serial manipulator w.r.t. the frame associated to the previous link?*

Reply #3

This reduction follows from the fact that link i (and so, its associated frame RF_i) is not free to be placed in the 3D space w.r.t. link $i-1$ (and its associated frame RF_{i-1}). The two links are connected at a joint that sets 2 scalar geometric constraints on the 6-dimensional relative pose between link $i-1$ and link i, leaving its characterization specified by only 4 residual parameters. The Denavit-Hartenberg convention is a clever choice of the origins and coordinate axes of the link frames, which shows how to cut down the number of relative pose parameters from 6 to 4. One of these parameters is variable, allowing the motion of frame RF_i around or along the axis, of the 1-dof joint (respectively, revolute or prismatic). ■

Question #4

For generic $m \geq 1$ and $n > 1$, give the total number of elementary products $N_\times$ and additions N_+ in evaluating, through operations with rotation matrices ${}^{j-1}\boldsymbol{R}_j$, the vectors ${}^0\boldsymbol{v}_i \in \mathbb{R}^3$ by the expression

$$ {}^0\boldsymbol{v}_i = \left({}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2 \,\ldots\, {}^{n-1}\boldsymbol{R}_n\right){}^0\boldsymbol{v}_i, \qquad \textit{for } i = 1,\ldots,m, \tag{6} $$

or by successive matrix-vector products as

$$ {}^0\boldsymbol{v}_i = {}^0\boldsymbol{R}_1\left({}^1\boldsymbol{R}_2\left(\ldots\left(\ldots\left({}^{n-1}\boldsymbol{R}_n\,{}^0\boldsymbol{v}_i\right)\right)\ldots\right)\right), \qquad \textit{for } i = 1,\ldots,m. \tag{7} $$

Given a value $n > 1$, which is the break-even value of m at which the number of evaluations $N_\times$ using (6) becomes advantageous (or disadvantageous) w.r.t. that using (7)?

Reply #4

The product of a 3×3 matrix by a 3-dimensional vector needs 9 multiplications and 6 additions, whereas the product between two 3×3 matrices needs three times as many, namely 27 multiplications and 18 additions. When doing the $(n-1)$ products of the n rotation matrices first, and then applying it to the m vectors $\boldsymbol{v}_i \in \mathbb{R}^3$ as in the first method (6), one obtains

$$ N_{\times,1} = 27(n-1) + 9\,m \qquad \text{and} \qquad N_{+,1} = 18(n-1) + 6\,m. $$

Using instead the recursive matrix-vector product n times for each of the m vectors v_i as in the second method (7), one has

$$ N_{\times,2} = 9\,mn \qquad \text{and} \qquad N_{+,2} = 6\,mn. $$

Comparing the number of elementary products, gives

$$ N_{\times,1} = 27(n-1) + 9\,m \lesseqgtr 9\,mn = N_{\times,2} \quad \Longleftrightarrow \quad 27(n-1) \lesseqgtr 9\,m(n-1) \quad \Longleftrightarrow \quad 27 \lesseqgtr 9\,m. $$

Thus, the break-even is obtained exactly at $m = 3$, with the first method (6) becoming more convenient when more than 3 vectors $\boldsymbol{v}_i$ have to be transformed. Note that this result is *independent* of n. Moreover, it can be generalized to matrix/vector computations in any dimension $p \geq 2$ (e.g., with $p = 4$, for 4×4 homogeneous matrices) leading to $m = p$ as break-even value. ■

Question #5

Robots are multi-body electromechanical systems driven by the torques $\boldsymbol{\tau}$ produced by the motors at the joints. In which sense are we allowed to say that one can move them by commanding just a desired joint velocity $\dot{\boldsymbol{q}}$ (or a joint position $\boldsymbol{q}$)?

Reply #5

This statement is correct in so far we assume that a low-level servo system with a feedback loop is present on each actuator —often, an electrical motor— at the robot joints. The velocity command $\dot{\boldsymbol{q}}$ (or the position $\boldsymbol{q}$) will be the reference input for these controllers. For the generic j-th servomotor, with $j = 1,\ldots,n$, a voltage V_j and a current i_j are generated that make the motor produce a torque τ_j on its output shaft. This torque will move the driven link i, possibly through transmission/reduction elements, until the reference velocity $\dot{q}_i$ (or position q_i) will be reached, i.e., when the error between the desired and the measured output variable is zero. ■

Question #6

Consider the 4-dof PRPR robot sketched in Fig. 1, where the base frame RF_0 and the end-effector frame RF_4 are already assigned. The robot has a shoulder offset given by the constant $N > 0$. Assign the other frames according to the Denavit-Hartenberg convention and build the associated table of parameters so that the position of the origin O_4 of the end-effector frame will be given by

$$
{}^0\boldsymbol{p}_4(\boldsymbol{q}) = \begin{pmatrix} N\cos q_2 - q_3 \sin q_2 \\ N \sin q_2 + q_3 \cos q_2 \\ q_1 \end{pmatrix}. \tag{8}
$$

Determine the symbolic expression of ${}^0\boldsymbol{R}_4(\boldsymbol{q})$ in the direct kinematics. Further, provide a numerical matrix $\boldsymbol{R} \in SO(3)$ representing an orientation that the end-effector of this robot can never assume.

Reply #6

The *unique* assignment of the remaining Denavit-Hartenberg (DH) frames that is consistent with the positional direct kinematics (8) is illustrated by the two views in Fig. 3 and Fig. 4. The associated set of DH parameters is given in Table 1, with the joint variables $\boldsymbol{q}$ taking the values (or just the signs) according to the configuration shown in Fig. 4.

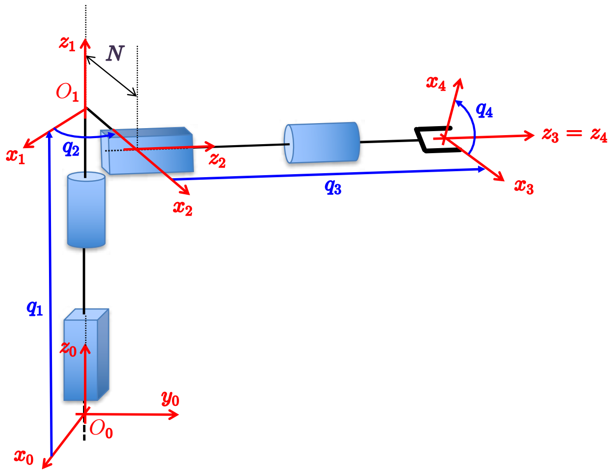


Figure 3: A perspective view of the DH frame assignment for the PRPR robot of Fig. 1.

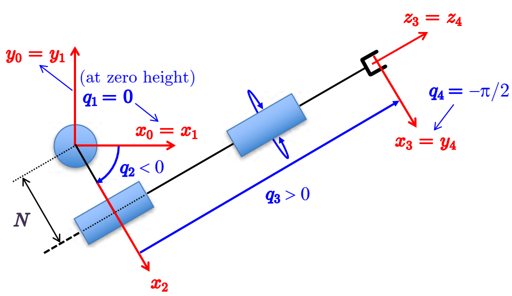


Figure 4: Top view of the frame assignment in Fig. 3, with the robot in a different configuration.

i	α_i	a_i	d_i	θ_i
1	0	0	$q_1 = 0$	0
2	$-\pi/2$	$N > 0$	0	$q_2 < 0$
3	0	0	$q_3 > 0$	0
4	0	0	0	$q_4 = -\pi/2$

Table 1: Table of DH parameters of the PRPR robot: $\boldsymbol{q}$ is associated to the configuration in Fig. 4.

Constructing the DH homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, \ldots, 4$, it is immediate to compute the orientation of the end-effector frame RF_4 as

$${}^0\boldsymbol{R}_4(\boldsymbol{q}) = {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2(q_2)\, {}^2\boldsymbol{R}_3\, {}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_2 \cos q_4 & -\cos q_2 \sin q_4 & -\sin q_2 \\ \sin q_2 \cos q_4 & -\sin q_2 \sin q_4 & \cos q_2 \\ -\sin q_4 & -\cos q_4 & 0 \end{pmatrix}.$$

This matrix is parametrized by the two rotational joint variables q_2 and q_4 only. It is easy to conclude that the end effector has no sufficient mobility to assume an arbitrary orientation in the 3D space. In particular, the end-effector approach axis $\boldsymbol{z}_4$ can never point out of the horizontal plane. This is revealed by the structural 0 in position $(3,3)$ of matrix ${}^0\boldsymbol{R}_4$. Therefore, unfeasible orientations for the robot end-effector are given, e.g., by the one-dimensional family of rotation matrices

$$\boldsymbol{R} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \forall \beta \in \mathbb{R}.$$

Indeed, any matrix $\boldsymbol{R} \in SO(3)$ having $R_{3,3} \neq 0$ will represent an unfeasible end-effector orientation for this 4-dof robot. ■

Question #7

Given a desired $\boldsymbol{p} \in \mathbb{R}^3$ for ${}^0\boldsymbol{p}_4(\boldsymbol{q})$ in (8), find all the analytical solutions $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ to the associated inverse kinematics problem in the regular case. Assuming there are no joint limits, sketch also the primary workspace WS_1 of the 4-dof PRPR robot. Finally, compute the numerical solutions to this inverse kinematics problem for $\boldsymbol{p} = \begin{pmatrix} 0 & 2 & 1.5 \end{pmatrix}^T$ with $N = 0.3$ [m].

Reply #7

The closed-form solution to the inverse kinematics for the end-effector position of the PRP robot (the last rotational joint is irrelevant here) is found as follows. Let ${}^0\boldsymbol{p}_4(\boldsymbol{q}) = \boldsymbol{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$. Using this in (8), by squaring and summing the first two equations we obtain

$$(N\cos q_2 - q_3 \sin q_2)^2 + (N \sin q_2 + q_3 \cos q_2)^2 = N^2 + q_3^2 = p_x^2 + p_y^2 \quad \Rightarrow \quad q_3 = \pm\sqrt{p_x^2 + p_y^2 - N^2}.$$

The argument of the square root should not be negative, which sets in fact the only limitation on the primary workspace $WS_1 = \left\{\boldsymbol{p} \in \mathbb{R}^3 : p_x^2 + p_y^2 \geq N^2\right\}$. In the regular case ($q_3 \neq 0$), for each of the two values $q_3 = q_3^+$ and $q_3 = q_3^-$ we solve the following linear system (whose determinant is always $N^2 + q_3^2 > 0$) in the unknowns $c_2 = \cos q_2$ and $s_2 = \sin q_2$ as

$$\begin{pmatrix} N & -q_3^{\{+,-\}} \\ q_3^{\{+,-\}} & N \end{pmatrix} = \begin{pmatrix} c_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \quad \Rightarrow \quad q_2 = \operatorname{atan2}\left\{ Np_y - q_3^{\{+,-\}} p_x, Np_x + q_3^{\{+,-\}} p_y \right\}.$$

Finally, we have the unique value for the first prismatic joint $q_1 = p_z$.
Summarizing, in the regular case ($p_x^2 + p_y^2 > N^2$) we have the two solutions:

$$\boldsymbol{q}^I = \begin{pmatrix} p_z \\ \text{atan2}\left\{p_y N - p_x\sqrt{p_x^2+p_y^2-N^2},\, p_x N + p_y\sqrt{p_x^2+p_y^2-N^2}\right\} \\ \sqrt{p_x^2+p_y^2-N^2} \end{pmatrix},$$

$$\boldsymbol{q}^{II} = \begin{pmatrix} p_z \\ \text{atan2}\left\{p_y N + p_x\sqrt{p_x^2+p_y^2-N^2}, p_x N - p_y\sqrt{p_x^2+p_y^2-N^2}\right\} \\ -\sqrt{p_x^2+p_y^2-N^2} \end{pmatrix}.$$

At the (inner) boundary of the primary workspace ($p_x^2 + p_y^2 = N^2$), the two solutions $\boldsymbol{q}^I$ and $\boldsymbol{q}^{II}$ collapse into a single one. Being $N > 0$, we can write this singular solution as

$$\boldsymbol{q}^s = \begin{pmatrix} p_z \\ \text{atan2}\left\{p_y, p_x\right\} \\ 0 \end{pmatrix}$$

Finally, there is no solution for $p_x^2 + p_y^2 < N^2$.
The primary workspace[1] WS_1 is obtained by subtracting from the entire Euclidean space $\mathbb{R}^3$ an infinite cylinder of radius N having its axis coincident with the axis $\boldsymbol{z}_0$. For the numerical input data provided, being $\sqrt{p_x^2+p_y^2} = 4 > 0.3 = N$, we will have two regular solutions to the inverse kinematics, namely

$$\boldsymbol{q}^I = \begin{pmatrix} 1.5 \\ 0.1506 \\ 1.9774 \end{pmatrix}, \qquad \boldsymbol{q}^{II} = \begin{pmatrix} 1.5 \\ 2.9910 \\ -1.9774 \end{pmatrix} \quad \text{[m; rad; m]}. \qquad \blacksquare$$

Question #8

What are the pros and cons in estimating online or offline the velocity of a joint from position data measured by an encoder using numerical differentiation formulas. Write a simple code that uses the 1-step BDF (Euler) formula to provide online estimates $\dot{y}_k^e = \dot{y}^e(t_k)$, for $k = 1, \ldots, 10$, of the velocity from the following series of ten position data (noisy and with only 4 significant digits), collected with a sampling frequency of 40 Hz from $t_1 = 0$ on:

$$\{y_k\} = \{0.0007 \;\; 0.1251 \;\; 0.2500 \;\; 0.3741 \;\; 0.4977 \;\; 0.6187 \;\; 0.7397 \;\; 0.8579 \;\; 0.9739 \;\; 1.0876\} \;\; [rad]$$

Compute also the average value $\bar{\dot{y}}^e$ of the obtained samples of velocity estimates (for comparison, the average value of the true velocity samples $\dot{y}_k$, for $k = 1, \ldots, 10$, is $\bar{\dot{y}} = 4.8239$ [rad/s]).

Reply #8

The results are obtained using, e.g., the following segment of Matlab code (with comments).

[1]Sorry, no figure! It is rather awkward to draw such a workspace when there is no limit to the ranges of the two prismatic joints.

```
\% position data as input
yN=[0.0007 0.1251 0.2500 0.3741 0.4977 0.6187 0.7397 0.8579 0.9739 1.0876];
\%
ns=length(yN);       \% number of samples in yN
Tc=0.025;            \% sampling interval for a 40 Hz frequency
t=[0:Tc:(ns-1)*Tc]; \% sampled instants of time
\% this initialization step is commented further in the text
yprec=0;             \% alternatives: yprec=yN(1); or yprec=2*yN(1)-yN(2);
for i=1:length(t)
     yd1(i)=(yN(i)-yprec)/Tc; \% 1-step (Euler) BDF
     yprec=yN(i);
end
\% output results
disp('1-step BDF (Euler) velocity estimates and average')
yd1
avgyd1=mean(yd1)
```

The output is

$$\{\dot{y}_k^e\} = \left\{0.0280 \;\; 4.9760 \;\; 4.9960 \;\; 4.9640 \;\; 4.9440 \;\; 4.8400 \;\; 4.8400 \;\; 4.7280 \;\; 4.6400 \;\; 4.5480\right\} \text{ [rad/s]}$$

with an average value $\bar{\dot{y}}^e = 4.3504$ [rad/s].

The initialization of the 1-step BDF method (`yprec` in the command line before the `for` loop) is needed for computing the first sample $\dot{y}_1^e$ of the velocity estimate. In fact, there is no 'previous' position sample to be used in the BDF formula $\dot{y}_k^e = (y_k - y_{k-1}))/T_c$ when $k = 1$. Any choice for y_0 (i.e., for initializing `yprec` in the code) is feasible. Since the Euler method is a one-step approximation, this will affect only the first sample of the produced output. Here, we took $y_0 = 0$ as a neutral value. Another reasonable choice is to set $y_0 = y_1 = 0.0007$, i.e., repeating the same first position sample of the data series. This leads to $\dot{y}_1^e = (y_1 - y_0)/T_c = 0$ (as opposed to $\dot{y}_1^e = 0.0280$), with just a slightly larger average $\bar{\dot{y}}^e = 4.3476$ [rad/s]. It is easy to obtain a better approximation of the derivative $\dot{y}_1^e$ at $k = 1$ by using the next position sample y_2. This future knowledge would give for the initialization `yprec`

$$y_0 = y_1 - \left(\frac{y_2 - y_1}{T_c}\right) T_c = 2y_1 - y_2,$$

leading to $\dot{y}_1^e = (y_1 - y_0)/T_c = (y_2 - y_1)/T_c = \dot{y}_2^e = 4.9760$, The average of the output series grows then to $\bar{\dot{y}}^e = 4.8452$, which is much closer to the true value $\bar{\dot{y}} = 4.8239$ [rad/s]. ■

Question #9

With reference to Fig. 2, the second joint of a 2R planar arm having link length $L_1 = 0.45$ and $L_2 = 0.35$ [m] is actuated by a motor M located at the first joint through a toothed transmission belt inside the body of link 1 (this may represent the situation of the first two dof of a SCARA robot). The belt connects a toothed disk of radius $r_1 = 5$ [cm], placed on the output shaft of motor M, with a second one of radius $r_2 = 0.25$ [m], connected to the axis of joint 2. An incremental encoder with 700 pulses/turn and electronic multiplication by a factor 4 is mounted on the back of motor M, for measuring its angular position θ_M.

a) *Suppose that the optical disk of the encoder has generated 300 light pulses while rotating in the CCW direction in a time interval $T = 1.2$ [s]. How large is the rotation $\Delta\theta_2$ (in [rad]) performed by the second link? And what is the average angular speed $\bar{\dot{\theta}}_2$ (in [rad/s]) during T?*

b) *With the robot in the configuration $\boldsymbol{\theta} = \mathbf{0}$ (stretched arm) and keeping joint 1 at rest, what is the minimal lateral displacement (along the $\boldsymbol{y}_0$ direction) of the tip of link 2 that can be sensed by the encoder?*

Reply #9

a) Since the reduction ratio of the toothed belt transmission is $N_r = r_2/r_1 = 0.25/0.05 = 5$, the answers are

$$\Delta\theta_2 = \frac{\Delta\theta_M}{N_r} = \frac{\#\text{ pulses}}{\#\text{ pulses per turn}} \cdot 2\pi \cdot \frac{1}{N_r} = \frac{300}{700} \cdot \frac{2\pi}{5} = 0.5386 \text{ [rad]},$$

$$\bar{\dot{\theta}}_2 = \frac{\Delta\theta_2}{T} = \frac{0.5386}{1.2} = 0.4488 \text{ [rad/s]}.$$

In the first formula, only the fraction of a full turn rotation performed by the motor matters (not the resolution of the position sensor).

b) On the other hand, the minimal displacement of the tip of the second link that can be sensed depends on the actual resolution of the digital encoder mounted on its driving motor. In the given situation (arm stretched along the $\boldsymbol{x}_0$-axis and joint 1 not moving), only the length L_2 of the second link of the 2R robot is involved in this evaluation. We have

$$\Delta \boldsymbol{p}_{tip,y} = L_2 \cdot \Delta\theta_{2,res} = L_2 \cdot \frac{2\pi}{4 \times \#\text{ pulses per turn}} \cdot \frac{1}{N_r} = \frac{2\pi}{2800} \cdot \frac{0.35}{5} = 0.157 \text{ [mm]}. \quad \blacksquare$$

Question #10

The base frame RF_0 of a robot has its origin placed in the position ${}^W\boldsymbol{p}_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ and is rotated by an angle $\beta = \pi/2$ [rad] around the $\boldsymbol{z}_w$ axis of the world frame RF_W. In a given configuration, the end-effector pose of the robot is given by

$${}^0\boldsymbol{T}_E = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} & 1 \\ 1 & 0 & 0 & -0.75 \\ 0 & -\frac{\sqrt{3}}{2} & -0.5 & 1.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The position of the tip of a tool mounted on the end-effector is ${}^E\boldsymbol{p}_{tool} = \begin{pmatrix} 0 & 0.3 & 0.3 \end{pmatrix}^T$ [m]. Moreover, the tool frame RF_{tool} associated to this point is rotated by an angle $\gamma = -\pi/2$ [rad] around the $\boldsymbol{x}_E$ axis of the end-effector frame RF_E. Compute the position of the tip of the tool in the world frame and the absolute orientation of the tool frame w.r.t. RF_W.

Reply #10

The result is obtained by multiplying three homogeneous 4×4 matrices, the given ${}^0\boldsymbol{T}_E$ associated to the robot direct kinematics expressed in its base frame, and the two world-to-base and end effector-to-tool transformations

$${}^W\boldsymbol{T}_0 = \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{z}_w}(\beta = \frac{\pi}{2}) & {}^W\boldsymbol{p}_0 \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and, respectively,

$$
{}^{E}\boldsymbol{T}_{tool} = \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{x}_E}(\gamma = -\frac{\pi}{2}) & {}^{E}\boldsymbol{p}_{tool} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.3 \\ 0 & -1 & 0 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Therefore

$$
{}^{W}\boldsymbol{T}_{tool} = {}^{W}\boldsymbol{T}_0\, {}^{0}\boldsymbol{T}_E\, {}^{E}\boldsymbol{T}_{tool} = \begin{pmatrix} -1 & 0 & 0 & 1.75 \\ 0 & 0.8660 & 0.5 & 1.8902 \\ 0 & 0.5 & -0.8660 & 1.0902 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^{W}\boldsymbol{R}_{tool} & {}^{W}\boldsymbol{p}_{tool} \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \quad \blacksquare
$$

* * * * *

Robotics 1

January 12, 2021

There are 10 questions. Provide answers with short texts, completed with drawings and derivations needed for the solutions. Students with confirmed midterm grade should do only the second set of 5 questions.

Question #1 [students without midterm]

The orientation of a rigid body $\mathcal{B}$ is defined by the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}.$$

Determine the angles (α, β, γ) of a YXZ Euler sequence providing the same orientation. Check the correctness of the obtained result by a direct computation. Find also the singular cases for this minimal representation and provide an example of a rotation matrix $\boldsymbol{R}_s$ that falls in this class.

Question #2 [students without midterm]

Let matrix $\boldsymbol{R}$ of Question #1 be the current orientation of body $\mathcal{B}$. If $\boldsymbol{\Omega} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$ is the instantaneous angular velocity of $\mathcal{B}$ expressed in the *body* frame, compute the time derivative $\dot{\boldsymbol{R}}$.

Question #3 [students without midterm]

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

What is this matrix? Is it correct? Provide a convincing explanation.

Question #4 [students without midterm]

Two views of a spatial 3R robot are shown in Fig. 1, together with the world reference frame RF_w. Assign the Denavit-Hartenberg frames and provide the associated table of parameters so that the configuration in Fig. 1(a) is $\boldsymbol{q}_a = (-\pi/2, \pi/2, 0)$ [rad] and the configuration in Fig. 1(b) is $\boldsymbol{q}_b = (0, 0, \pi/2)$ [rad]. The frame assignment must also include all four lengths L_0, L_1, L_2, and L_3 defined in the figure. Compute the symbolic expression ${}^w\boldsymbol{p}(\boldsymbol{q})$ of the end-effector position P.

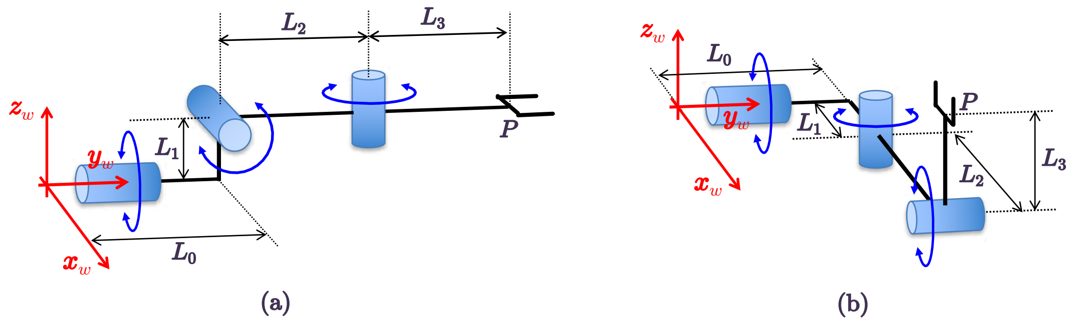


Figure 1: A spatial 3R robot in two different configurations $\boldsymbol{q}_a$ (a) and $\boldsymbol{q}_b$ (b).

Question #5 [students without midterm]

An electrical motor mounts on its axis a multi-turn absolute encoder with 11 bits. The first 3 bits are used for counting turns, while the following 8 bits measure a single turn. The motor drives a robot link through an harmonic drive having a flexspline with 120 external teeth. What is the angular resolution of this equipment at the link side? What is the maximum unidirectional angular displacement at the motor side that can be measured by the encoder? Which motor angle $\theta_m \in [0, 2\pi)$ corresponds to the Gray code [000|01100001]? Express all results in radians.

Question #6 [all students]

For the 4-dof planar RRPR robot in Fig. 2, with the joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ defined therein, derive the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ associated to the 3-dimensional task vector $\boldsymbol{r} = (p_x, p_y, \alpha)$, where $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ gives the position of the final flange center P and $\alpha \in \mathbb{R}$ is the orientation of the last robot link w.r.t. the axis $\boldsymbol{x}_0$. Find all singular configurations $\boldsymbol{q}_s$ of this task Jacobian matrix. For one such $\boldsymbol{q}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ and determine a basis for $\mathcal{R}\{\boldsymbol{J}_s\}$ and one for $\mathcal{N}\{\boldsymbol{J}_s\}$.

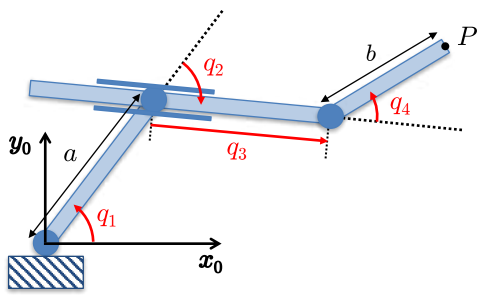


Figure 2: A 4-dof RRPR robot, with joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$.

Question #7 [all students]

Two planar 2R robots, named A and B and having both unitary link lengths, are in the static equilibrium shown in Fig. 3. The two D-H configurations w.r.t. their base frames are, respectively, $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ [rad] and $\boldsymbol{q}_B = (\pi/2, -\pi/2)$ [rad]. Robot A pushes against robot B as in the figure, with a force $\boldsymbol{F} \in \mathbb{R}^2$ having norm $\|\boldsymbol{F}\| = 10$ [N]. Compute the joint torques $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ (both in [Nm]) that keep the two robots in equilibrium.

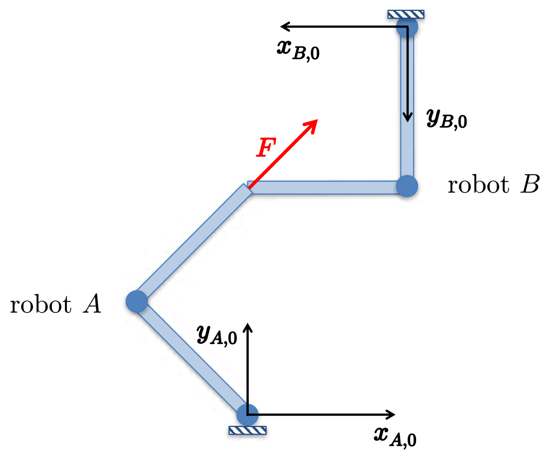


Figure 3: A static equilibrium condition for two planar 2R robots pushing against each other.

Question #8 [all students]

With reference to Fig. 4, a planar 2R robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ [m] should intercept and follow a target that moves at constant speed $v = 0.3$ [m/sec] along a line passing through the point $P_0 = (-0.8, 1.1)$ [m] and making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. The robot starts at rest from the configuration $\boldsymbol{q}_s = (\pi, 0)$ [rad] (in DH terms) as soon as the target enters the workspace. The rendez-vous occurs after $T = 2$ s, with the robot end effector and the target having the same final velocity. Plan a coordinated joint space trajectory for this task.

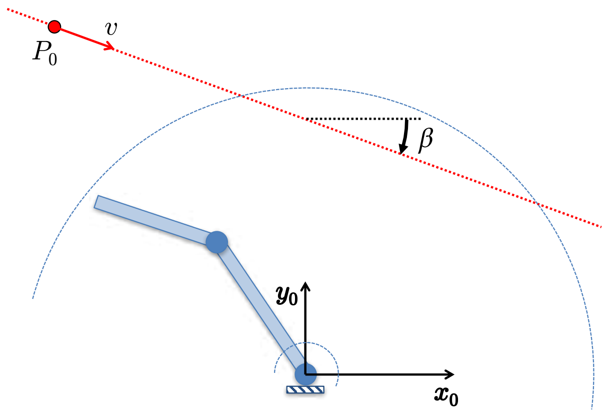


Figure 4: A rendez-vous task for a planar 2R robot and a target in uniform linear motion.

Question #9 [all students]

Consider the following trajectories for the two revolute joints of a robot:

$$q_1(t) = \frac{\pi}{4} + \frac{\pi}{4}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad q_2(t) = -\frac{\pi}{2}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right), \qquad t \in [0, T].$$

Compute the boundary values for the position, velocity, and acceleration at $t = 0$ and $t = T$, and the instants and values of maximum absolute velocity and maximum absolute acceleration for both joints. Assume that the robot motion is bounded by $|\dot{q}_i| \le V_i$ and $|\ddot{q}_i| \le A_i$, for $i = 1, 2$, with

$$V_1 = 4 \text{ [rad/s]}, \qquad V_2 = 8 \text{ [rad/s]}, \qquad A_1 = 20 \text{ [rad/s}^2\text{]}, \qquad A_2 = 40 \text{ [rad/s}^2\text{]}.$$

Determine the minimum feasible motion time T. Sketch the associated time profiles of the position, velocity and acceleration for the two joints.

Question #10 [all students]

Consider again the task in Question #8. The robot is commanded by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. Once the rendez-vous has been accomplished, design a feedback control law that will let the robot follow the moving target and react to position errors e_t and e_n that may occur along the tangent and normal directions to the linear path, respectively with the prescribed decoupled dynamics $\dot{e}_t = -3\, e_t$ and $\dot{e}_n = -10\, e_n$. Provide the explicit expression of all terms in the control law.

[240 minutes (4 hours) for the full exam; open books]
[150 minutes (2.5 hours) for students with midterm; open books]

Solution

January 12, 2021

Question #1 [students without midterm]

The orientation of a rigid body $\mathcal{B}$ is defined by the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}.$$

Determine the angles (α, β, γ) of a YXZ Euler sequence providing the same orientation. Check the correctness of the obtained result by a direct computation. Find also the singular cases for this minimal representation and provide an example of a rotation matrix $\boldsymbol{R}_s$ that falls in this class.

Reply #1

The rotation matrix obtained with the angles (α, β, γ) of a *YXZ* Euler sequence is computed from

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \ \boldsymbol{R}_X(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}, \ \boldsymbol{R}_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Z(\gamma) \\ &= \begin{pmatrix} \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma - \cos\alpha\sin\gamma & \sin\alpha\cos\beta \\ \cos\beta\sin\gamma & \cos\beta\cos\gamma & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{1}$$

We solve the inverse problem for this minimal representation,

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \boldsymbol{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix},$$

by using first the elements of the last column in this matrix equality. We obtain

$$\beta = \text{ATAN2}\left\{\sin\beta, \cos\beta\right\} = \text{ATAN2}\left\{-R_{23}, \pm\sqrt{R_{13}^2 + R_{33}^2}\right\}. \tag{2}$$

Provided that $|\cos\beta| = \sqrt{R_{13}^2 + R_{33}^2} \neq 0$ (regular case), we solve for the other two angles as

$$\alpha = \text{ATAN2}\left\{\frac{R_{13}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{22}}{\cos\beta}\right\},$$

obtaining a pair of solutions, one for each sign chosen for $\cos\beta$ in (2). For the given matrix $\boldsymbol{R}$, we have

$$\sqrt{R_{13}^2 + R_{33}^2} = 0.5 = |\cos\beta| \neq 0,$$

and thus a regular case. The two solutions are

$$(\alpha_1, \beta_1, \gamma_1) = (\pi,\ -1.0472,\ \pi/2), \qquad (\alpha_2, \beta_2, \gamma_2) = (0,\ -2.0944,\ -\pi/2) \qquad \text{[rad]}.$$

To verify the result, use (1) to obtain indeed $\boldsymbol{R}_{YXZ}(\alpha_1, \beta_1, \gamma_1) = \boldsymbol{R}_{YXZ}(\alpha_2, \beta_2, \gamma_2) = \boldsymbol{R}$. Finally, a singular case is encountered, e.g., for the rotation matrix

$$\boldsymbol{R}_s = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = \{R_{s,ij}\} \qquad \Rightarrow \qquad \cos\beta_s = 0.$$

In this case $\beta_s = \text{ATAN2}\,\{-R_{s,23}, 0\} = \pi/2$ and only the difference $\alpha - \gamma$ of the two other angles is defined as

$$\alpha_s - \gamma_s = \text{ATAN2}\,\{R_{s.12}, R_{s,11}\} = -\pi/2,$$

leading to an infinity of solutions $(\alpha, \beta, \gamma) = (\alpha_s, \pi/2, \alpha_s + \pi/2)$, $\forall \alpha_s$. ■

Question #2 [students without midterm]

Let matrix $\boldsymbol{R}$ *of Question #1 be the current orientation of body* $\mathcal{B}$*. If* $\boldsymbol{\Omega} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$ *is the instantaneous angular velocity of* $\mathcal{B}$ *expressed in the* body *frame, compute the time derivative* $\dot{\boldsymbol{R}}$*.*

Reply #2

The result can be obtained in two equivalent ways, using a skew symmetric matrix $\boldsymbol{S}$ built with the angular velocity of the body $\mathcal{B}$. Either we express the angular velocity in the base frame as $\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega}$ and then use $\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}$ (as in the lecture slides). Or we use directly the alternative form $\dot{\boldsymbol{R}} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega})$ (as in Exercise #1 in the June 11, 2012 exam). Being

$$\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega} = \begin{pmatrix} -1 \\ 0.5 \\ \dfrac{\sqrt{3}}{2} \end{pmatrix},$$

we obtain in both cases

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} = \begin{pmatrix} 0 & -\dfrac{\sqrt{3}}{2} & 0.5 \\ \dfrac{\sqrt{3}}{2} & 0 & 1 \\ -0.5 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}$$

$$= \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}) = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ \dfrac{\sqrt{3}}{2} & \dfrac{\sqrt{3}}{2} & -0.5 \\ -0.5 & -0.5 & -\dfrac{\sqrt{3}}{2} \end{pmatrix}.$$ ■

Question #3 [students without midterm]

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

What is this matrix? Is it correct? Provide a convincing explanation.

Reply #3

Matrix $\boldsymbol{M}$ simply represents the inverse of the generic Denavit-Hartenberg homogeneous transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} \cos\theta & -\sin\theta\cos\alpha & \sin\theta\sin\alpha & a\cos\theta \\ \sin\theta & \cos\theta\cos\alpha & -\cos\theta\sin\alpha & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}(\alpha,\theta) & \boldsymbol{p}(a,d,\theta) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.$$

In fact,

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \boldsymbol{R}^T(\alpha,\theta) & -\boldsymbol{R}^T(\alpha,\theta)\boldsymbol{p}(a,d,\theta) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \boldsymbol{M}. \qquad \blacksquare$$

Question #4 [students without midterm]

Two views of a spatial 3R robot are shown in Fig. 1, together with the world reference frame RF_w. Assign the Denavit-Hartenberg frames and provide the associated table of parameters so that the configuration in Fig. 1(a) is $\boldsymbol{q}_a = (-\pi/2, \pi/2, 0)$ [rad] and the configuration in Fig. 1(b) is $\boldsymbol{q}_b = (0, 0, \pi/2)$ [rad]. The frame assignment must also include all four lengths L_0, L_1, L_2, and L_3 defined in the figure. Compute the symbolic expression ${}^w\boldsymbol{p}(\boldsymbol{q})$ of the end-effector position P.

Reply #4

An assignment of the Denavit-Hartenberg (DH) frames is illustrated in the two configurations shown in Fig. 5. This assignment is consistent with the values $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ that joint variables should take in the configurations of Fig. 1(a) and (b). The associated set of DH parameters is given in Table 1. The origins of the DH frames 0 and 3 have been chosen coincident with the origin O_w of the world frame and with the end-effector position P, respectively. In this way, all four kinematic lengths L_i, $i = 0, 1, 2, 3$, appear in the DH table. The fourth and fifth columns in the table return the values of the joint variables for the two configurations $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$.

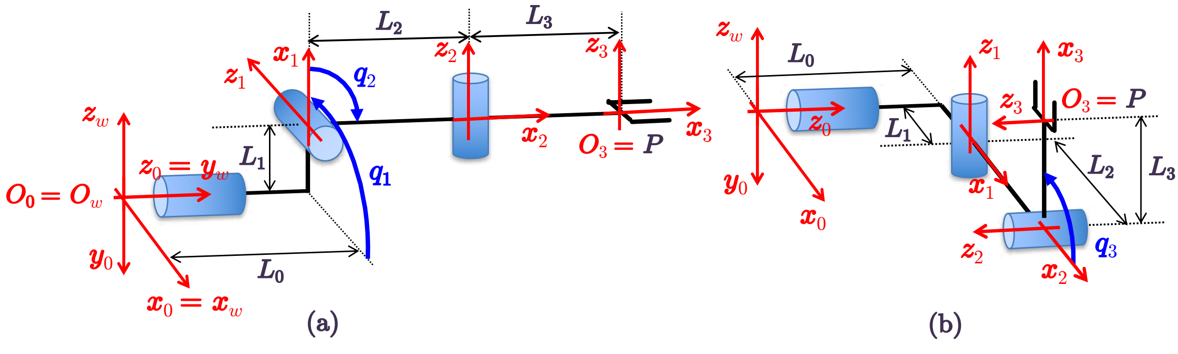


Figure 5: Assignment of the DH frames for the spatial 3R robot, shown here for the two configurations $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ of Fig. 1.

i	α_i	a_i	d_i	θ_i (a)	θ_i (b)
1	$\pi/2$	L_1	L_0	$q_{a1} = -\pi/2$	$q_{b1} = 0$
2	$\pi/2$	L_2	0	$q_{a2} = \pi/2$	$q_{b2} = 0$
3	0	L_3	0	$q_{a3} = 0$	$q_{b3} = \pi/2$

Table 1: Table of DH parameters for the spatial 3R robot.

By building the DH homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, 2, 3$, from Table 1, it is straightforward to compute the position of the origin of the end-effector frame $O_3 = P$ as

$$
{}^0\boldsymbol{p}_H(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\, {}^1\boldsymbol{A}_1(q_2)\, {}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos q_1\,(L_1 + (L_2 + L_3 \cos q_3)\cos q_2) + L_3 \sin q_1 \sin q_3 \\ \sin q_1\,(L_1 + (L_2 + L_3 \cos q_3)\cos q_2) - L_3 \cos q_1 \sin q_3 \\ L_0 + (L_2 + L_3 \cos q_3)\sin q_2 \\ 1 \end{pmatrix}.
$$

In order to change the expression of this position vector from the base frame of the robot (the 0th DH frame) to the world frame, we need the additional rotation matrix

$$
{}^w\boldsymbol{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
$$

Since the world frame and 0th DH frame have the same origin, we have

$$
{}^w\boldsymbol{p} = {}^w\boldsymbol{R}_0\, {}^0\boldsymbol{p} = \begin{pmatrix} \cos q_1\,(L_1 + (L_2 + L_3 \cos q_3)\cos q_2) + L_3 \sin q_1 \sin q_3 \\ L_0 + (L_2 + L_3 \cos q_3)\sin q_2 \\ -\sin q_1\,(L_1 + (L_2 + L_3 \cos q_3)\cos q_2) + L_3 \cos q_1 \sin q_3 \end{pmatrix}. \qquad \blacksquare
$$

Question #5 [students without midterm]

An electrical motor mounts on its axis a multi-turn absolute encoder with 11 bits. The first 3 bits are used for counting turns, while the following 8 bits measure a single turn. The motor drives a robot link through an harmonic drive having a flexspline with 120 external teeth. What is the angular resolution of this equipment at the link side? What is the maximum unidirectional angular displacement at the motor side that can be measured by the encoder? Which motor angle $\theta_m \in [0, 2\pi)$ *corresponds to the Gray code* [000|01101001]*? Express all results in radians.*

Reply #5

Being $N_b = 8$ bits devoted to a single turn of the absolute encoder, its angular resolution on the motor side is $\Delta_m = 2\pi/2^{N_b} = 2\pi/256 = 0.0245$ [rad]. The reduction ratio of an harmonic drive with $N_f = 120$ teeth on the flexspline is $N_r = N_f/2 = 60$. Thus, the angular resolution on the link side is $\Delta_l = \Delta_m/N_r = 4.0906 \cdot 10^{-4}$ [rad] (about 2 hundreds of a degree). Being $N_t = 3$ bits devoted to the counting of full motor turns, when the motor rotates in a single direction (say, starting from $\theta_{m,init} = 0$ and counterclockwise), the maximum angular displacement that will be measured is $\Delta_{max} = 2\pi \cdot 2^{N_t} - \Delta_m = 50.2409$ [rad] (the Δ_m can also be neglected, leading to $\Delta_{max} = 50.2655$ [rad]). Finally, the given Gray code refers to the first motor turn (the $N_t = 3$ most significant bits are zero). The conversion to binary of the least significant $N_b = 8$ bits (a byte), $x_{gray} = [01101001]$, can be done using logical exclusive-or operations (as shown in the lecture slides). We obtain $x_{bin} = [01001110]$ and then $x_{dec} = 2^7 + 2^4 + 2^3 + 2^2 = 78$. The following simple Matlab code does the conversions:

```
xgray=[0 1 1 0 1 0 0 1] \% from MSB to LSB
\% Gray to binary
xbin(1)=xgray(1);
for i=1:Nbits-1
  xbin(i+1)=xor(xbin(i),xgray(i+1));
end
\% binary to decimal
xdec=xbin(Nbits);
for i=1:Nbits-1
  xdec=xdec+xbin(Nbits-i)*2^i;
end
```

The measured motor angle is thus $\theta_m = x_{dec} \cdot \Delta_m = 1.9144$ [rad] (about 109°). ■

Question #6 [all students]

For the 4-dof planar RRPR robot in Fig. 2, with the joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ defined therein, derive the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ associated to the 3-dimensional task vector $\boldsymbol{r} = (p_x, p_y, \alpha)$, where $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ gives the position of the final flange center P and $\alpha \in \mathbb{R}$ is the orientation of the last robot link w.r.t. the axis $\boldsymbol{x}_0$. Find all singular configurations $\boldsymbol{q}_s$ of this task Jacobian matrix. For one such $\boldsymbol{q}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ and determine a basis for $\mathcal{R}\{\boldsymbol{J}_s\}$ and one for $\mathcal{N}\{\boldsymbol{J}_s\}$.

Reply #6

The task kinematics of this robot is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} a\cos q_1 + q_3\cos(q_1+q_2) + b\cos(q_1+q_2+q_4) \\ a\sin q_1 + q_3\sin(q_1+q_2) + b\sin(q_1+q_2+q_4) \\ q_1+q_2+q_4 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

The associated 3×4 task Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a\,s_1 - q_3 s_{12} - b\,s_{124} & -q_3 s_{12} - b\,s_{124} & c_{12} & -b\,s_{124} \\ a\,c_1 + q_3 c_{12} + b\,c_{124} & q_3 c_{12} + b\,c_{124} & s_{12} & b\,c_{124} \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

where we have used the trigonometric shorthand notation (e.g., $s_{124} = \sin(q_1+q_2+q_4)$) for compactness. As usual, in order to perform a singularity analysis of the Jacobian, it is convenient to get rid of the angle q_1 from its expression[1]. This is obtained by premultiplying $\boldsymbol{J}$ by the transpose of the rotation matrix $\boldsymbol{R}(q_1)$:

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \boldsymbol{R}^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_3 s_2 - b\,s_{24} & -q_3 s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ a + q_3 c_2 + b\,c_{24} & q_3 c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

[1] An intrinsic property of the manipulator, like the loss of mobility at the task level in certain configurations, will never depend on the arbitrary choice of the base (zero-th) frame of the robot, and thus on the value of q_1.

To study the rank of matrix $\boldsymbol{J}$ (or, equivalently, of ${}^1\boldsymbol{J}$), we have two possible ways. The first, and more cumbersome, is to compute the determinant of the square 3×3 matrix $\boldsymbol{J}\boldsymbol{J}^T$. Performing computations in Matlab yields

$$\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right) = \det\left({}^1\boldsymbol{J}(\boldsymbol{q})\,{}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = 2a^2c_2^2 + 2q_3^2 + a^2q_3^2 + 2aq_3c_2 - a^2q_3^2c_2^2.$$

Because of the undefined sign of the fourth addend and of the minus sign in the last term, it is not immediate to conclude on necessary and sufficient conditions for zeroing this determinant. The second way is to analyze the four minors obtained by deleting each time one column from the Jacobian. This can be done either on $\boldsymbol{J}$ or on ${}^1\boldsymbol{J}$, leading to identical results in both cases (also when using the symbolic code in Matlab). For compactness, we illustrate the method on the ${}^1\boldsymbol{J}$ matrix only. We have

$$\begin{aligned}
{}^1\boldsymbol{J}_{-1}(\boldsymbol{q}) &= \begin{pmatrix} -q_3s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ q_3c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 0 & 1 \end{pmatrix} &&\Rightarrow \qquad \det {}^1\boldsymbol{J}_{-1}(\boldsymbol{q}) = -q_3 \\
{}^1\boldsymbol{J}_{-2}(\boldsymbol{q}) &= \begin{pmatrix} -q_3s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ a + q_3c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 0 & 1 \end{pmatrix} &&\Rightarrow \qquad \det {}^1\boldsymbol{J}_{-2}(\boldsymbol{q}) = -q_3 - a\,c_2 \\
{}^1\boldsymbol{J}_{-3}(\boldsymbol{q}) &= \begin{pmatrix} -q_3s_2 - b\,s_{24} & -q_3s_2 - b\,s_{24} & -b\,s_{24} \\ a + q_3c_2 + b\,c_{24} & q_3c_2 + b\,c_{24} & b\,c_{24} \\ 1 & 1 & 1 \end{pmatrix} &&\Rightarrow \qquad \det {}^1\boldsymbol{J}_{-3}(\boldsymbol{q}) = a\,q_3s_2 \\
{}^1\boldsymbol{J}_{-4}(\boldsymbol{q}) &= \begin{pmatrix} -q_3s_2 - b\,s_{24} & -q_3s_2 - b\,s_{24} & c_2 \\ a + q_3c_2 + b\,c_{24} & q_3c_2 + b\,c_{24} & s_2 \\ 1 & 1 & 0 \end{pmatrix} &&\Rightarrow \qquad \det {}^1\boldsymbol{J}_{-4}(\boldsymbol{q}) = a\,c_2.
\end{aligned}$$

In order for the Jacobian to be singular, all four determinants above should simultaneously be zero. This occurs if and only if

$$q_3 = \cos q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \pm\frac{\pi}{2},\ q_3 = 0.$$

Thus, the prismatic joint should be fully retracted (so that joint axes 2 and 4 coincide) and the second link should be orthogonal to the first one. Choosing for instance the configuration $\boldsymbol{q}_s = (q_1, \pi/2, 0, q_4)$, with arbitrary q_1 and q_4, leads to

$${}^1\boldsymbol{J}_s = {}^1\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -b\,c_4 & -b\,c_4 & 0 & -b\,c_4 \\ a - b\,s_4 & -b\,s_4 & 1 & -b\,s_4 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank} {}^1\boldsymbol{J}_s = 2.$$

A basis for the null space of the Jacobian $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ can be computed using directly ${}^1\boldsymbol{J}_s$:

$$\mathcal{N}\{\boldsymbol{J}_s\} = \mathcal{N}\left\{{}^1\boldsymbol{J}_s\right\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ a \\ 0 \end{pmatrix} \right\}.$$

Note in particular that the first vector prescribes an equal and opposite velocity to joints 2 and 4. The second basis vector involves instead also the third, prismatic joint. To provide a basis for the range space of $\boldsymbol{J}_s$, we first pick two independent columns of ${}^1\boldsymbol{J}_s$

$$\mathcal{R}\left\{{}^1\boldsymbol{J}_s\right\} = \left\{ \begin{pmatrix} -b\,c_4 \\ -b\,s_4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and then obtain, from $\boldsymbol{J}_s = \boldsymbol{R}(q_1)\,{}^1\boldsymbol{J}_s$,

$$\mathcal{R}\left\{\boldsymbol{J}_s\right\} = \left\{ \boldsymbol{R}(q_1) \begin{pmatrix} -b\,c_4 \\ -b\,s_4 \\ 1 \end{pmatrix}, \boldsymbol{R}(q_1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -b\,c_{14} \\ -b\,s_{14} \\ 1 \end{pmatrix}, \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \right\}. \qquad \blacksquare$$

Question #7 [all students]

Two planar 2R robots, named A and B and having both unitary link lengths, are in the static equilibrium shown in Fig. 3. The two D-H configurations w.r.t. their base frames are, respectively, $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ *[rad] and* $\boldsymbol{q}_B = (\pi/2, -\pi/2)$ *[rad]. Robot A pushes against robot B as in the figure, with a force* $\boldsymbol{F} \in \mathbb{R}^2$ *having norm* $\|\boldsymbol{F}\| = 10$ *[N]. Compute the joint torques* $\boldsymbol{\tau}_A \in \mathbb{R}^2$ *and* $\boldsymbol{\tau}_B \in \mathbb{R}^2$ *(both in [Nm]) that keep the two robots in equilibrium.*

Reply #7

Evaluate the 2×2 Jacobians of the two 2R robots, respectively at $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ [rad] and $\boldsymbol{q}_B = (\pi/2, -\pi/2)$, each expressed in its own DH base frame:

$$\boldsymbol{J}_A(\boldsymbol{q}_A) = \left.\begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} -\sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$\boldsymbol{J}_B(\boldsymbol{q}_B) = \left.\begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The force vector $\boldsymbol{F}_A$ applied by robot A to robot B is oriented as the second link of robot A. When expressed in the base frame of robot A, it is

$${}^A\boldsymbol{F}_A = \|\boldsymbol{F}\| \cdot \left.\begin{pmatrix} \cos(q_1+q_2) \\ \sin(q_1+q_2) \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}.$$

To obtain this Cartesian force at the end effector, robot A should produce a joint torque given by

$$\boldsymbol{\tau}_A = \boldsymbol{J}_A^T(\boldsymbol{q}_A)\,{}^A\boldsymbol{F}_A = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

On the other hand, robot B should balance the force applied by robot A at its end effector by reacting with an equal and opposite force $\boldsymbol{F}_B = -\boldsymbol{F}$, namely ${}^A\boldsymbol{F}_B = -{}^A\boldsymbol{F}_A$ when these forces are both expressed in the same base frame of robot A. However, when expressing the exchanged force in the base frame of robot B, we can easily see that[2]

$${}^B\boldsymbol{F}_B = {}^B\boldsymbol{R}_A\,{}^A\boldsymbol{F}_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \left(-{}^A\boldsymbol{F}_A\right) = {}^A\boldsymbol{F}_A = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}.$$

[2]Here, we use planar 2×2 rotation matrices, i.e., $\boldsymbol{R} \in SO(2)$.

Therefore, to obtain this Cartesian force at the end-effector, robot B should produce a joint torque given by

$$\boldsymbol{\tau}_B = \boldsymbol{J}_B^T(\boldsymbol{q}_B)\,{}^B\boldsymbol{F}_B = \begin{pmatrix} 0 \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 7.0711 \end{pmatrix} \text{ [Nm]}.$$

It is also worth reasoning on the two zeros that appear in $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$, in relation with the geometry of this static interaction task. Such analysis is left to the reader. ■

Question #8 [all students]

With reference to Fig. 4, a planar 2R robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ [m] should intercept and follow a target that moves at constant speed $v = 0.3$ [m/sec] along a line passing through the point $P_0 = (-0.8, 1.1)$ [m] and making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. The robot starts at rest from the configuration $\boldsymbol{q}_s = (\pi, 0)$ [rad] (in DH terms) as soon as the target enters the workspace. The rendez-vous occurs after $T = 2$ s, with the robot end effector and the target having the same final velocity. Plan a coordinated joint space trajectory for this task.

Reply #8

In this trajectory planning problem, we need first to define the boundary conditions for the rendez-vous between the moving target and the robot end-effector. The robot workspace has an external (circular) boundary of radius $R = l_1 + l_2 = 0.9$ [m] (while the internal boundary has radius $R_{min} = |l_1 - l_2| = 0.1$ [m]). The target moves on a line that intercepts the external boundary in two points P_1 and P_2, the first of which is of interest. These points are found by solving the system of equations

$$\begin{cases} (x - x_0)\sin\beta - (y - y_0)\cos\beta = 0 & \text{[line through } P_0 = (x_0, y_0) \text{ with angular coefficient } \beta] \\ x^2 + y^2 = R^2 & \text{[circle with center in the origin and radius } R] \end{cases} \tag{3}$$

The solution is obtained using the two (real) roots x_1 and x_2 of the following second-order polynomial equation[3] derived from (3):

$$x^2 - 2(x_0\sin\beta - y_0\cos\beta)\sin\beta\, x + (x_0\sin\beta - y_0\cos\beta)^2 - R^2\cos^2\beta = 0.$$

With each of these roots, we compute also the y-coordinate of the intercepting points:

$$y_i = y_0 + (x_i - x_0)\tan\beta, \qquad i = 1, 2.$$

From the given data, we get

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -0.1930 \\ 0.8791 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.7129 \\ 0.5494 \end{pmatrix} \quad \text{[m]}.$$

Note that $\|P_1\| = \|P_2\| = R = 0.9$. Clearly, the target enters the workspace in P_1 (and will exit in P_2). We shall set the initial time $t = 0$ of the robot motion at the instant of target entrance. Moreover, at $T = 2$ s, the target will be in the planned rendez-vous point

$$P_{rv} = P_1 + vT\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \begin{pmatrix} 0.3708 \\ 0.6739 \end{pmatrix} \text{ [m]}.$$

Since $\|P_{rv}\| = 0.7692 < R$, the rendez-vouz will occur well inside the robot workspace. This Cartesian point specifies, via kinematic inversion, the goal configuration that the robot should reach at time $t = T$. From the usual inverse kinematics of the 2R robot, coded in Matlab as follows

[3] Indeed, this second-order equation may have either two real roots or two complex conjugate roots. In the latter case, no intercepting points exist. When P_0 is inside the circle ($\|P_0\| < R$), there are always two real roots.

```
p_x=P_rv(1);p_y=P_rv(2);
c2=(p_x^2+p_y^2-(l_1^2+l_2^2))/(2*l_1*l_2)
s2=-sqrt(1-c2^2) \% choose the minus sign to have the elbow-up solution as goal,
                 \% being this ‘closer’ to the start configuration qs
q2_g=atan2(s2,c2)
q1_g=atan2(p_y*(l_1+l_2*c2)-p_x*l_2*s2,p_x*(l_1+l_2*c2)+p_y*l_2*s2)
```

we obtain

$$\boldsymbol{q}_g = \boldsymbol{q}(T) = \begin{pmatrix} 1.5495 \\ -1.0996 \end{pmatrix} \text{[rad]} \quad \left(= \begin{pmatrix} 88.78^\circ \\ -63.00^\circ \end{pmatrix} \right).$$

In addition, the Cartesian velocity of the target at the goal instant $t = T$ of rendez-vous (actually, also at any other instant) is

$$\boldsymbol{v}_g = \dot{\boldsymbol{p}}(T) = v \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = 0.3 \begin{pmatrix} 0.9397 \\ -0.3420 \end{pmatrix} = \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} \text{[m/s]}.$$

Therefore, by inverting the robot Jacobian at the rendez-vous configuration, the joint velocity at the goal is computed as

$$\dot{\boldsymbol{q}}_g = \dot{\boldsymbol{q}}(T) = \boldsymbol{J}^{-1}(\boldsymbol{q}_g)\,\boldsymbol{v}_g = \left.\begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1+q_2) & -l_2 \sin(q_1+q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1+q_2) & l_2 \cos(q_1+q_2) \end{pmatrix}^{-1}\right|_{\boldsymbol{q}=\boldsymbol{q}_g} \boldsymbol{v}_g$$

$$= \begin{pmatrix} -0.6739 & -0.1740 \\ 0.3708 & 0.3602 \end{pmatrix}^{-1} \boldsymbol{v}_g = \begin{pmatrix} -2.0212 & -0.9762 \\ 2.0809 & 3.7814 \end{pmatrix} \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} = \begin{pmatrix} -0.4696 \\ 0.1986 \end{pmatrix} \text{[rad/s]}.$$

At this stage, there are 4 boundary conditions to be interpolated between $t = 0$ and $t = T = 2$ s for each joint. For the first joint, we have

$$q_{s,1} = q_1(0) = \pi, \quad \dot{q}_{s,1} = \dot{q}_1(0) = 0, \quad q_{g,1} = q_1(T) = 1.5495, \quad \dot{q}_{g,1} = \dot{q}_1(T) = -0.4696,$$

while for the second joint

$$q_{s,2} = q_2(0) = 0, \quad \dot{q}_{s,2} = \dot{q}_2(0) = 0, \quad q_{g,2} = q_2(T) = -1.0996, \quad \dot{q}_{g,2} = \dot{q}_2(T) = 0.1986.$$

Since there are no further conditions specified, we choose a cubic polynomial for each joint as interpolating function —the simplest solution with enough parameters to satisfy all boundary conditions. The motion of the robot joints should be coordinated. Thus, we will use a common parametrized time τ to define the joint trajectories $\boldsymbol{q}_d(\tau)$. Let the joint displacement vector be

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \boldsymbol{q}_g - \boldsymbol{q}_s = \begin{pmatrix} -1.5921 \\ -1.0996 \end{pmatrix}.$$

For $\tau = t/T \in [0, 1]$, we compute the desired trajectories as

$$q_{d,i}(\tau) = q_{s,i} + \Delta_i \left(\left(\frac{\dot{q}_{g,i}T}{\Delta_i} - 2 \right) \tau^3 + \left(3 - \frac{\dot{q}_{g,i}T}{\Delta_i} \right) \tau^2 \right), \qquad i = 1, 2, \tag{4}$$

with velocity profiles

$$\dot{q}_{d,i}(\tau) = \frac{\Delta_i}{T} \left(3 \left(\frac{\dot{q}_{g,i}T}{\Delta_i} - 2 \right) \tau^2 + 2 \left(3 - \frac{\dot{q}_{g,i}T}{\Delta_i} \right) \tau \right), \qquad i = 1, 2.$$

Plugging the data in (4), we obtain

$$q_{d,1}(\tau) = 2.2449\,\tau^3 - 3.8370\,\tau^2 + \pi, \qquad \tau = t/T \in [0,1],$$

and

$$q_{d,2}(\tau) = 2.5964\,\tau^3 - 3.6960\,\tau^2, \qquad \tau = t/T \in [0,1].$$

The resulting position and velocity profiles are shown in Fig. 8. In Fig. 7, we illustrate with a stroboscopic view the approaching phase of the robot to the moving target during a time interval of $T = 2$ s. ■

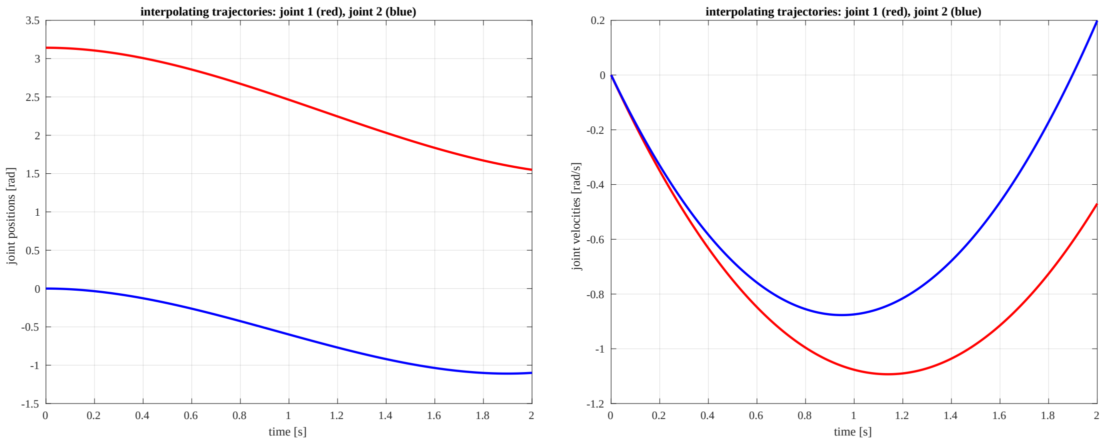


Figure 6: Joint position and velocity profiles of the planned interpolating trajectories for the rendez-vous between the robot end effector and the moving target.

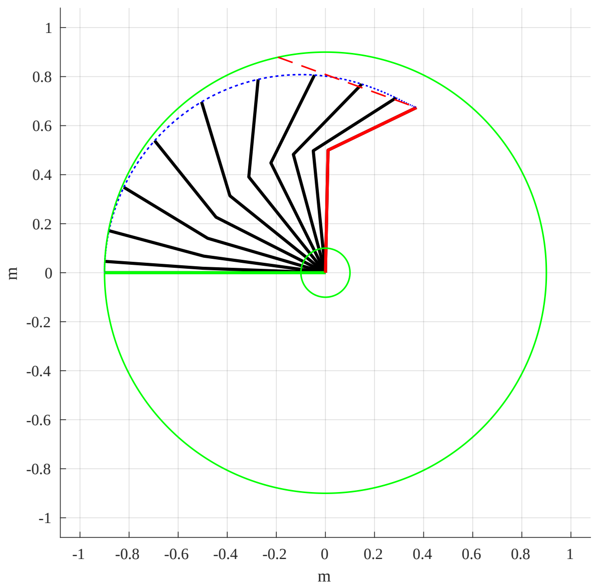


Figure 7: Stroboscopic view of robot and target motion in the rendez-vous task.

Question #9 [all students]

Consider the following trajectories for the two revolute joints of a robot:

$$q_1(t) = \frac{\pi}{4} + \frac{\pi}{4}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad q_2(t) = -\frac{\pi}{2}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right), \qquad t \in [0, T].$$

Compute the boundary values for the position, velocity, and acceleration at $t = 0$ and $t = T$, and the instants and values of maximum absolute velocity and maximum absolute acceleration for both joints. Assume that the robot motion is bounded by $|\dot{q}_i| \le V_i$ and $|\ddot{q}_i| \le A_i$, for $i = 1, 2$, with

$$V_1 = 4 \text{ [rad/s]}, \qquad V_2 = 8 \text{ [rad/s]}, \qquad A_1 = 20 \text{ [rad/s}^2\text{]}, \qquad A_2 = 40 \text{ [rad/s}^2\text{]}.$$

Determine the minimum feasible motion time T. Sketch the associated time profiles of the position, velocity and acceleration for the two joints.

Reply #9

Differentiating w.r.t. time the trajectories of the two joints yields

$$\dot{q}_1(t) = \frac{3\pi}{2T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right), \qquad \dot{q}_2(t) = -\frac{\pi^2}{2T}\sin\left(\frac{\pi t}{T}\right), \qquad t \in [0, T]$$

and further

$$\ddot{q}_1(t) = \frac{3\pi}{2T^2}\left(1 - 2\left(\frac{t}{T}\right)\right), \qquad \ddot{q}_2(t) = -\frac{\pi^3}{2T^2}\cos\left(\frac{\pi t}{T}\right), \qquad t \in [0, T].$$

By direct evaluation, we obtain the boundary values for the first joint

$$\begin{aligned} q_1(0) &= \frac{\pi}{4} & \qquad q_1(T) &= \frac{\pi}{2} \\ \dot{q}_1(0) &= 0 & \dot{q}_1(T) &= 0 \\ \ddot{q}_1(0) &= \frac{3\pi}{2T^2} & \ddot{q}_1(T) &= -\frac{3\pi}{2T^2}, \end{aligned}$$

and for the second joint

$$\begin{aligned} q_2(0) &= 0 & \qquad q_2(T) &= -\pi \\ \dot{q}_2(0) &= 0 & \dot{q}_2(T) &= 0 \\ \ddot{q}_2(0) &= -\frac{\pi^3}{2T^2} & \ddot{q}_2(T) &= \frac{\pi^3}{2T^2}. \end{aligned}$$

Moreover, it is easy to see that the following maximum absolute values are attained for the velocities

$$\max_{t\in[0,T]} |\dot{q}_1(t)| = |\dot{q}_1(T/2)| = \frac{3\pi}{8T}, \qquad \max_{t\in[0,T]} |\dot{q}_2(t)| = |\dot{q}_2(T/2)| = \frac{\pi^2}{2T},$$

and for the accelerations

$$\max_{t\in[0,T]} |\ddot{q}_1(t)| = |\ddot{q}_1(0)| = |\ddot{q}_1(T)| = \frac{3\pi}{2T^2}, \qquad \max_{t\in[0,T]} |\ddot{q}_2(t)| = |\ddot{q}_2(0)| = |\ddot{q}_2(T)| = \frac{\pi^3}{2T^2}.$$

In order to satisfy all the given bounds, the minimum value of the motion time T is determined as

$$T = \max\left\{\frac{3\pi}{8V_1}, \sqrt{\frac{3\pi}{2A_1}}, \frac{\pi^2}{2V_2}, \sqrt{\frac{\pi^3}{2A_2}}\right\} = \max\left\{0.2945,\ 0.4854,\ 0.6169,\ 0.6226\right\} = 0.6226 \text{ [s]},$$

which is enforced by the acceleration limit $A_2 = 40$ [rad/s^2] on the second joint. With $T = 0.6226$ s, the maximum values reached by the absolute velocities and accelerations are

$$\max_{t\in[0,T]} |\dot{q}_1(t)| = 1.8923, \quad \max_{t\in[0,T]} |\dot{q}_2(t)| = 7.9267, \quad \max_{t\in[0,T]} |\ddot{q}_1(t)| = 12.1585, \quad \max_{t\in[0,T]} |\ddot{q}_2(t)| = 40 = A_2.$$

The plots of the minimum time trajectories are reported in Fig. 8. ■

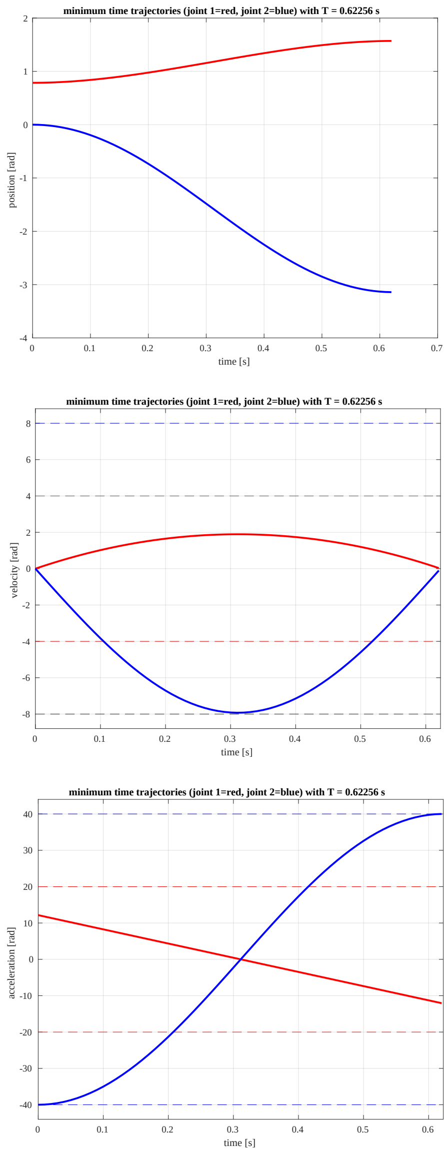


Figure 8: Minimum time solution for the considered class of trajectories under velocity/acceleration bounds: position, velocity, and acceleration profiles (continuous) and their limits (dashed).

Question #10 [all students]

Consider again the task in Question #8. The robot is commanded by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. Once the rendez-vous has been accomplished, design a feedback control law that will let the robot follow the moving target and react to position errors e_t and e_n that may occur along the tangent and normal directions to the linear path, respectively with the prescribed decoupled dynamics $\dot{e}_t = -3\,e_t$ and $\dot{e}_n = -10\,e_n$. Provide the explicit expression of all terms in the control law.

Reply #10

The control law contains a velocity feedforward term, in order to follow the moving target in nominal conditions, and a feedback action on the Cartesian error, which is rotated in the (Frenet) frame associated to the path. The target moves at constant speed $v = 0.3$ [m/sec] along a linear path making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. Thus, we have

$$\dot{\boldsymbol{p}}_d = v \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = 0.3 \begin{pmatrix} 0.9397 \\ -0.3420 \end{pmatrix} = \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} \text{ [m/s]}.$$

The Cartesian error $\boldsymbol{e}_p \in \mathbb{R}^2$ is rotated into the tangential and normal components to the path (i.e., in the task frame) as

$$\boldsymbol{e}_p = \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{e}_{task} = \begin{pmatrix} e_t \\ e_n \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \boldsymbol{R}^T(\beta)\,\boldsymbol{e}_p, \tag{5}$$

where

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{pmatrix}$$

is the direct kinematics of the 2R robot. The complete control law is then

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta)\boldsymbol{K}_{task}\,\boldsymbol{e}_{task}\right), = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta)\boldsymbol{K}_{task}\,\boldsymbol{R}^T(\beta)\,\boldsymbol{e}_p\right), \tag{6}$$

where the robot Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and the (diagonal) task gain matrix $\boldsymbol{K}_{task} > 0$ are given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{pmatrix}, \qquad \boldsymbol{K}_{task} = \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}.$$

By replacing eq. (6) in the differential kinematics, we obtain

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta)\boldsymbol{K}_{task}\,\boldsymbol{R}^T(\beta)\,\boldsymbol{e}_p\right)$$
$$\Rightarrow \quad \dot{\boldsymbol{e}}_p = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = -\boldsymbol{R}(\beta)\boldsymbol{K}_{task}\,\boldsymbol{R}^T(\beta)\,\boldsymbol{e}_p = -\boldsymbol{K}_p\,\boldsymbol{e}_p,$$

having defined the (full and symmetric) Cartesian gain matrix $\boldsymbol{K}_p = \boldsymbol{R}(\beta)\boldsymbol{K}_{task}\,\boldsymbol{R}^T(\beta) > 0$. Being $\boldsymbol{R}(\beta)$ a constant matrix, we immediately see that

$$\dot{\boldsymbol{e}}_{task} = \boldsymbol{R}^T(\beta)\,\dot{\boldsymbol{e}}_p = -\boldsymbol{R}^T(\beta)\boldsymbol{K}_p\,\boldsymbol{e}_p = -\boldsymbol{R}^T(\beta)\boldsymbol{K}_p\,\boldsymbol{R}(\beta)\,\boldsymbol{e}_{task} = -\boldsymbol{K}_{task}\,\boldsymbol{e}_{task},$$

or in scalar terms

$$\begin{pmatrix} \dot{e}_t \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} -3\,e_t \\ -10\,e_n \end{pmatrix},$$

which is exactly the desired decoupled and linear error dynamics. ■

* * * * *

Robotics 1

February 4, 2021

There are 8 questions. Provide answers with short texts, completed with drawings and derivations needed for the solutions. Students with confirmed midterm grade should do only the second set of 4 questions.

Question #1 [students without midterm]

The orientation of a rigid body is defined by the axis-angle pair $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ and $\theta = \pi/6$ [rad]. Determine the angles (α, β, γ) of the Roll-Pitch-Yaw sequence XYZ of fixed axes that provide the same orientation. Check the correctness of the obtained result. Find the singular cases for this RPY representation and provide an example of an axis-angle pair $(\boldsymbol{r}_s, \theta_s)$ that would fall in this class.

Question #2 [students without midterm]

Figure 1 shows a top view of a planar two-jaw articulated gripper. This robotic gripper has a revolute joint at its base, followed by two independent revolute joints for each jaw (the first joints in the two jaws share the same axis). This 5-dof robotic system has a tree structure for which the usual Denavit-Hartenberg frame assignment can also be applied (to each branch). Define the joint coordinates accordingly, together with the two DH tables. Provide then the symbolic expression of some task variables that are relevant for gripping operations, defined as follows:

- position of the midpoint P_c between the tips of the two jaws;
- distance d between the two tips;
- relative angle α_{rel} of the left jaw w.r.t. the right jaw;
- orientation angle β w.r.t. the $\boldsymbol{x}_0$ axis of the jaw pair (from the right jaw tip to the left one).

When the gripper links have all the same length $L = 0.05$ [m], compute the numerical value of such task variables in the configuration $\boldsymbol{q} = (q_1, q_{r2}, q_{r3}, q_{l2}, q_{l3}) = (-\pi/2, -\pi/2, 3\pi/4, \pi/2, -3\pi/4)$. Subscripts r and l stand respectively for DH variables pertaining to the right or left jaw only.

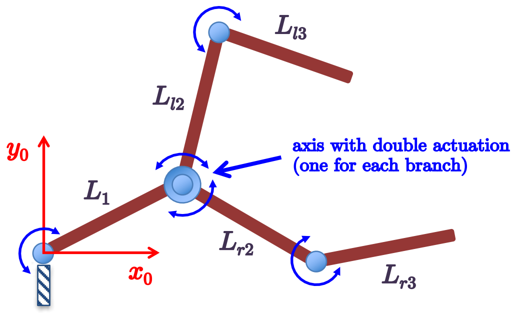


Figure 1: A planar 5-dof two-jaw gripper.

Question #3 [students without midterm]

A planar 2R robot has incremental encoders at the joints measuring the configuration $\boldsymbol{\theta} = (\theta_1, \theta_2)$ used in the computation of its direct kinematics. Because of a bad mounting of the encoders, the two measures are affected by (very) small angular errors δ_1 and δ_2. When using these readings, which of the following statements is correct in terms of Cartesian accuracy of the end-effector position? *A)* there is always an error; *B)* there are configurations at which there may be no error; *C)* the error is always negligible (e.g., below the sensor resolution). Provide a detailed explanation of your answer!

Question #4 [students without midterm]

The prismatic joints of the planar PPR robot in Fig. 2 have bounded ranges, $q_{i,min} \le q_i \le q_{i,max}$, for $i = 1, 2$, while the revolute joint q_3 has an unlimited motion range. Draw accurately the primary workspace WS_1 and the secondary workspace WS_2 of this robot, under the following assumption for the third link length: $L < \min\{(q_{1,max} - q_{1,min})/2, (q_{2,max} - q_{2,min})/2\}$.

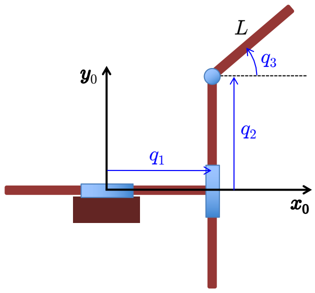


Figure 2: A planar PPR robot.

Question #5 [all students]

The direct kinematics and the initial configuration of a planar RP robot are given by

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{q}^{\{0\}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix},$$

where $0 < \epsilon \ll 1$ is a very small number. Given the following desired end-effector positions,

$$\boldsymbol{p}_{d,\mathrm{I}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{p}_{d,\mathrm{II}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

compute the first iteration (i.e., $\boldsymbol{q}^{\{1\}}$) of a Newton method and of a Gradient method for solving the two inverse kinematics problems. Discuss what happens in each of the four cases when $\varepsilon \to 0$.

Question #6 [all students]

The 3R robotic device in Fig. 3 has joint axes that intersect two by two. The second joint axis is inclined by an angle $\delta \approx 20°$. This structure is mainly intended for pointing the final axis $\boldsymbol{n}$ at a moving target in 3D. Provide the explicit expression of the square angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of this robot. Find the singularities, if any, of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$. Compute the relation between $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the time derivative $\dot{\boldsymbol{n}}$ of the pointing axis.

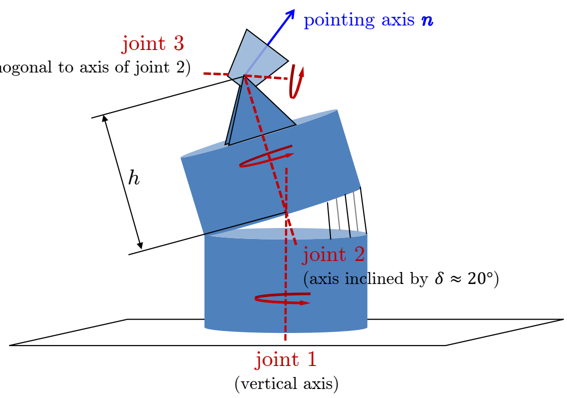


Figure 3: A 3-dof robotic pointing device.

Question #7 [all students]

The desired initial and final orientation of the end effector of a certain robot are specified at time $t = 0$ and $t = T$, respectively by

$$\boldsymbol{R}(0) = \boldsymbol{R}_{in} = \begin{pmatrix} 0.5 & 0 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & -0.5 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{R}(T) = \boldsymbol{R}_{fin} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ -0.5 & -0.5 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -\sqrt{2}/2 \end{pmatrix}.$$

The end effector should start with zero angular velocity and acceleration ($\boldsymbol{\omega}_{in} = \dot{\boldsymbol{\omega}}_{in} = \mathbf{0}$, at $t = 0$) and reach the final orientation with angular velocity and acceleration given by

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_{fin} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \text{ [rad/s]}, \qquad \dot{\boldsymbol{\omega}}(T) = \dot{\boldsymbol{\omega}}_{fin} = \mathbf{0}.$$

Plan a smooth and coordinated trajectory for the end-effector orientation that satisfies all the given boundary conditions for a generic motion time $T > 0$. Setting next $T = 1$, compute at the mid-time instant $t = T/2$ the numerical values of the resulting orientation $\boldsymbol{R}(T/2)$ and angular velocity $\boldsymbol{\omega}(T/2)$ of the robot end effector.

Question #8 [all students]

The planar 3R robot with unitary link lengths shown in Fig. 4 is initially in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\, 11\pi/18,\, -\pi/4)$. Commanded by a joint velocity $\dot{\boldsymbol{q}}(t)$ that uses feedback from the current $\boldsymbol{q}(t)$, the robot should perform a self-motion so as to reach asymptotically the final value $q_{3,fin} = -\pi/2$ for the third joint, while keeping the position of its end-effector always at the same initial point P_{in}. Verify first that such task is feasible. Design then a control scheme that completes the task in a robust way, i.e., by rejecting also possible transient errors and without encountering any singular situation in which the control law is ill conditioned. *Hint: Use an approach based on joint space decomposition.*

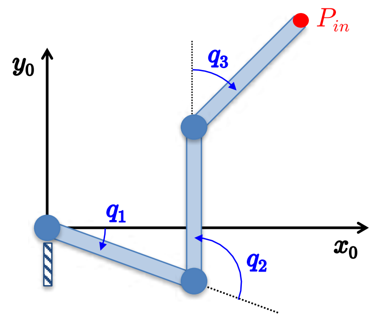


Figure 4: A 3R robot that should perform a self-motion task with constant end-effector position.

[240 minutes (4 hours) for the full exam; open books]

[120 minutes (2 hours) for students with midterm; open books]

Solution

February 4, 2021

Question #1 [students without midterm]

The orientation of a rigid body is defined by the axis-angle pair $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ *and* $\theta = \pi/6$ *[rad]. Determine the angles* (α, β, γ) *of the Roll-Pitch-Yaw sequence* XYZ *of fixed axes that provide the same orientation. Check the correctness of the obtained result. Find the singular cases for this RPY representation and provide an example of an axis-angle pair* $(\boldsymbol{r}_s, \theta_s)$ *that would fall in this class.*

Reply #1

The solution requires back and forth transformations among different representations of orientation. From the given axis-angle pair $(\boldsymbol{r}, \theta)$, we compute the rotation matrix

$$\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta = \begin{pmatrix} 0.9107 & -0.3333 & -0.2440 \\ 0.2440 & 0.9107 & -0.3333 \\ 0.3333 & 0.2440 & 0.9107 \end{pmatrix} = \boldsymbol{R}_d. \tag{1}$$

Three rotations by the angles (α, β, γ) around the sequence of XYZ fixed axes (i.e., of the RPY type) are associated to the rotation matrix

$$\begin{aligned} \boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) &= \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_X(\alpha) \\ &= \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma \\ \cos\beta\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{2}$$

For the nonsingular case, when

$$\cos\beta = \pm\sqrt{R_{32}^2 + R_{33}^2} \neq 0,$$

solving $\boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) = \boldsymbol{R}_d = \{R_{ij}\}$ for (α, β, γ) yields a pair of solutions (one for each sign chosen inside $\cos\beta$):

$$\alpha = \text{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \beta = \text{ATAN2}\left\{-R_{31}, \cos\beta\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}.$$

Using the data, we have $\cos\beta = \pm\, 0.9428 \neq 0$ and find the two numerical solutions

$$\alpha' = 0.2618, \qquad \beta' = -0.3398, \qquad \gamma' = 0.2618 \qquad \text{[rad]}$$

and

$$\alpha'' = -2.8798, \qquad \beta'' = -2.8018, \qquad \gamma'' = -2.8798 \qquad \text{[rad]}.$$

Plugging any of these triples into (2) returns indeed $\boldsymbol{R}_d$. The singularity of this minimal representation of orientation occurs for $\cos\beta = 0$ (i.e., $\beta = \pm\pi/2$), namely when $R_{32}^2 + R_{33}^2 = 0$. In this case, we can further solve only for the sum $\alpha + \gamma$ (when $\sin\beta = -R_{11} = 1$) or the difference $\alpha - \gamma$ (for $\sin\beta = -1$) of the remaining two angles. For instance, a rotation matrix in this class is

$$\boldsymbol{R}_s = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3}$$

for which $\beta = \pi/2$ and $\alpha - \gamma = -\pi/2$. The rotation matrix $\boldsymbol{R}_s = \{R_{ij}\}$ is in turn nonsingular for the transformation with an axis-angle representation, being

$$\sin\theta = \frac{1}{2}\sqrt{(R_{12}-R_{21})^2 + (R_{13}-R_{31})^2 + (R_{23}-R_{32})^2} = 0.8660 \neq 0.$$

Therefore, $\boldsymbol{R}_s$ can be generated by the pair $(\boldsymbol{r}'_s, \theta'_s)$ computed as

$$\begin{aligned} \theta'_s &= \text{ATAN2}\left\{\sin\theta, \frac{R_{11}+R_{22}+R_{33}-1}{2}\right\} = 2.0944 \text{ [rad]} \\ \boldsymbol{r}'_s &= \frac{1}{2\sin\theta}\begin{pmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \end{aligned}$$

as well as by the pair $(\boldsymbol{r}''_s, \theta''_s) = (-\boldsymbol{r}'_s, -\theta'_s)$. Note that $\boldsymbol{r}'_s = \boldsymbol{r}$, the same unit axis given in the text as a starting point. This coincidence occurs by pure chance. It is indeed $\theta'_s \neq \theta$, otherwise we would have generated $\boldsymbol{R}_s$ in (3) instead of $\boldsymbol{R}_d$ in (1), obtaining a singular case for the chosen RPY representation. ■

Question #2 [students without midterm]

Figure 1 shows a top view of a planar two-jaw articulated gripper. This robotic gripper has a revolute joint at its base, followed by two independent revolute joints for each jaw (the first joints in the two jaws share the same axis). This 5-dof robotic system has a tree structure for which the usual Denavit-Hartenberg frame assignment can also be applied (to each branch). Define the joint coordinates accordingly, together with the two DH tables. Provide then the symbolic expression of some task variables that are relevant for gripping operations, defined as follows:

- *position of the midpoint P_c between the tips of the two jaws;*
- *distance d between the two tips;*
- *relative angle α_{rel} of the left jaw w.r.t. the right jaw;*
- *orientation angle β w.r.t. the $\boldsymbol{x}_0$ axis of the jaw pair (from the right jaw tip to the left one).*

When the gripper links have all the same length $L = 0.05$ [m], compute the numerical value of such task variables in the configuration $\boldsymbol{q} = (q_1, q_{r2}, q_{r3}, q_{l2}, q_{l3}) = (-\pi/2, -\pi/2, 3\pi/4, \pi/2, -3\pi/4)$. Subscripts r and l stand respectively for DH variables pertaining to the right or left jaw only.

Reply #2

Figure 5 shows the $\boldsymbol{x}_i$ axes of the frames assigned to the articulated gripper according to the Denavit-Hartenberg convention, together with the associated joint variables and all the relevant quantities for defining the task variables. The two DH tables, respectively for the right and the left jaws, are given in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	0	L_1	0	q_1
2	0	L_{r2}	0	q_{r2}
3	0	L_{r3}	0	q_{r3}

i	α_i	a_i	d_i	θ_i
1	0	L_1	0	q_1
2	0	L_{l2}	0	q_{l2}
3	0	L_{l3}	0	q_{l3}

Table 1: Tables of DH parameters for the right and left jaws of the gripper.

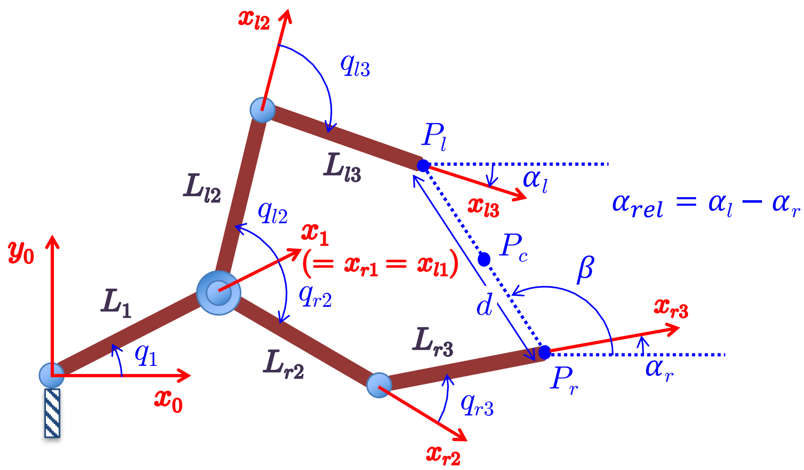


Figure 5: DH frames, with definition of coordinates and task variables for the two-jaw gripper.

Based on these definitions, we compute the direct kinematics of the two tip points P_r and P_l and the final orientations α_r and α_l of the two jaws:

$$\boldsymbol{p}_r = \begin{pmatrix} L_1 \cos q_1 + L_{r2}\cos(q_1+q_{r2}) + L_{r3}\cos(q_1+q_{r2}+q_{r3}) \\ L_1 \sin q_1 + L_{r2}\sin(q_1+q_{r2}) + L_{r3}\sin(q_1+q_{r2}+q_{r3}) \end{pmatrix}$$

$$\boldsymbol{p}_l = \begin{pmatrix} L_1 \cos q_1 + L_{l2}\cos(q_1+q_{l2}) + L_{l3}\cos(q_1+q_{l2}+q_{l3}) \\ L_1 \sin q_1 + L_{l2}\sin(q_1+q_{l2}) + L_{l3}\sin(q_1+q_{l2}+q_{l3}) \end{pmatrix}$$

$$\alpha_r = q_1 + q_{r2} + q_{r3}$$

$$\alpha_l = q_1 + q_{l2} + q_{l3}.$$

The considered task variables are then defined as

$$\boldsymbol{p}_c = \frac{\boldsymbol{p}_l + \boldsymbol{p}_r}{2}$$

$$d = \|\boldsymbol{p}_l - \boldsymbol{p}_r\|$$

$$\alpha_{rel} = \alpha_l - \alpha_r = q_{l2} + q_{l3} - q_{r2} - q_{r3}$$

$$\beta = \text{ATAN2}\left\{p_{ly} - p_{ry}, p_{lx} - p_{rx}\right\}.$$

With the given data for link lengths and current configuration, we obtain the numerical values

$$\boldsymbol{p}_c = \begin{pmatrix} 0 \\ -0.08536 \end{pmatrix} \text{ [m]}, \qquad d = 0.02929 \text{ [m]}, \qquad \alpha_{rel} = -\frac{\pi}{2} \text{ [rad]}, \qquad \beta = 0. \qquad \blacksquare$$

Question #3 [students without midterm]

A planar 2R robot has incremental encoders at the joints measuring the configuration $\boldsymbol{\theta} = (\theta_1, \theta_2)$ *used in the computation of its direct kinematics. Because of a bad mounting of the encoders, the two measures are affected by (very) small angular errors* δ_1 *and* δ_2*. When using these readings, which of the following statements is correct in terms of Cartesian accuracy of the end-effector position?* A) *there is always an error;* B) *there are configurations at which there may be no error;*

C) *the error is always negligible (e.g., below the sensor resolution). Provide a detailed explanation of your answer!*

Reply #3

The correct answer is B. Statement A will automatically be false (because of the word 'always'), once we confirm the correctness of B. Statement C seems ambiguous, since no information is provided on the sensor resolution nor on the link lengths, which may both be arbitrary small or large. Therefore, for a given amount of error δ one can consider a robot with sufficiently long links so that the Cartesian accuracy becomes unacceptably large[1]. To show the validity of B, consider the 2R robot in a singular configuration, say the stretched one $\boldsymbol{\theta}^* = (\theta_1, 0)$ for an arbitrary θ_1. The robot Jacobian would become

$$\boldsymbol{J}(\boldsymbol{\theta}^*) = \begin{pmatrix} -(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{pmatrix}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} = \begin{pmatrix} -(l_1+l_2)\sin\theta_1 & -l_2\sin\theta_1 \\ (l_1+l_2)\cos\theta_1 & l_2\cos\theta_1 \end{pmatrix},$$

with rank $\boldsymbol{J}(\boldsymbol{\theta}^*) = 1$. In view of the assumption of small errors $\boldsymbol{\delta}$ for the sensor readings in the joint space, we can use the differential mapping to evaluate their effect on the Cartesian error displacements, i.e., $\Delta\boldsymbol{p} = \boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{\delta}$. Therefore, at $\boldsymbol{\theta}^*$, all sensing errors $\boldsymbol{\delta}^* \in \mathbb{R}^2$ that are in the null space of $\boldsymbol{J}(\boldsymbol{\theta}^*)$ will produce no the Cartesian errors:

$$\boldsymbol{\delta}^* = \epsilon \begin{pmatrix} -l_1 \\ l_1 + l_2 \end{pmatrix}, \quad |\epsilon| \ll 1 \qquad \Rightarrow \qquad \boldsymbol{J}(\boldsymbol{\theta}^*)\boldsymbol{\delta}^* = \boldsymbol{0}. \qquad \blacksquare$$

Question #4 [students without midterm]

The prismatic joints of the planar PPR robot in Fig. 2 have bounded ranges, $q_{i,min} \le q_i \le q_{i,max}$, for $i = 1, 2$, while the revolute joint q_3 has an unlimited motion range. Draw accurately the primary workspace WS_1 and the secondary workspace WS_2 of this robot, under the following assumption for the third link length: $L < \min\{(q_{1,max} - q_{1,min})/2, (q_{2,max} - q_{2,min})/2\}$.

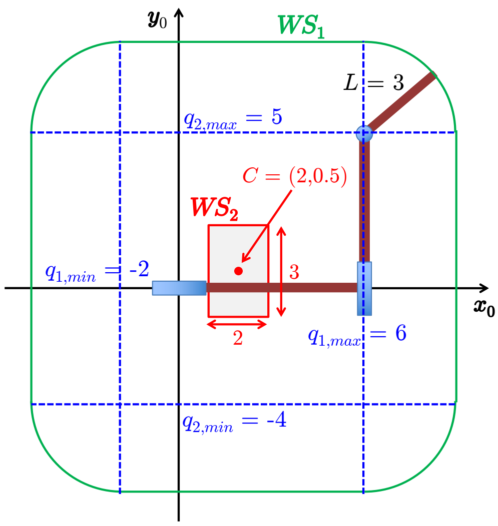


Figure 6: Primary and secondary workspaces for a planar PPR robot with bounded range of the prismatic joints.

[1]Indeed, the definition of a normalized accuracy would scale to the size of the robot. In that case, this argument would not work. But again, the word 'always' largely weakens statement C, just as in case A.

Reply #4
The primary and secondary workspaces of the PPR robot are drawn in Fig. 6, using the following set of representative values:

$$q_{1,min} = -2, \quad q_{1,max} = 6; \qquad q_{2,min} = -4, \quad q_{2,max} = 5; \qquad L = 3 \quad [\mathrm{m}].$$

The ranges of the prismatic joints need not to be symmetric. The assumption on the length L of the third link is satisfied here, allowing the presence of a non-vanishing WS_2, where the robot end-effector can assume any orientation angle in the plane. The outer boundary of WS_1 is a rectangle of side lengths $q_{i,max} - q_{i,min} + 2L$, $i = 1, 2$, with corners smoothed by circles of radius L. The outer boundary of WS_2 is a rectangle with sides $q_{i,max} - q_{i,min} - 2L > 0$, $i = 1, 2$. ■

Question #5 [all students]
The direct kinematics and the initial configuration of a planar RP robot are given by

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{q}^{\{0\}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix},$$

where $0 < \epsilon \ll 1$ *is a very small number. Given the following desired end-effector positions,*

$$\boldsymbol{p}_{d,\mathrm{I}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{p}_{d,\mathrm{II}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

compute the first iteration (i.e., $\boldsymbol{q}^{\{1\}}$*) of a Newton method and of a Gradient method for solving the two inverse kinematics problems. Discuss what happens in each of the four cases when* $\varepsilon \to 0$*.*

Reply #5
Evaluating the Jacobian of the RP robot at $\boldsymbol{q}^{\{0\}}$, we have

$$\boldsymbol{J}(\boldsymbol{q}^{\{0\}}) = \left.\begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}^{\{0\}}} = \begin{pmatrix} -\varepsilon \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \\ \varepsilon \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix}.$$

It is $\det \boldsymbol{J}(\boldsymbol{q}^{\{0\}}) = -\varepsilon$, and thus a singularity is approached when $\varepsilon \to 0$. For the desired end-effector position of case I, with the Newton method we have at the first iteration

$$\begin{aligned} \boldsymbol{q}^{\{1\}}_{Newton,\mathrm{I}} &= \boldsymbol{q}^{\{0\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{0\}}) \left(\boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\dfrac{\sqrt{2}}{2\varepsilon} & \dfrac{\sqrt{2}}{2\varepsilon} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon \dfrac{\sqrt{2}}{2} \\ \varepsilon \dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} + \dfrac{\sqrt{2}}{\varepsilon} \\ 0 \end{pmatrix}, \end{aligned}$$

whereas with the Gradient method (for a generic step size $\alpha > 0$) it is

$$\begin{aligned} \boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{I}} &= \boldsymbol{q}^{\{0\}} + \alpha\, \boldsymbol{J}^{T}(\boldsymbol{q}^{\{0\}}) \left(\boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \alpha \begin{pmatrix} -\varepsilon \dfrac{\sqrt{2}}{2} & \varepsilon \dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon \dfrac{\sqrt{2}}{2} \\ \varepsilon \dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} + \alpha \sqrt{2}\, \varepsilon \\ \varepsilon\,(1 - \alpha) \end{pmatrix}. \end{aligned}$$

For $\varepsilon \to 0$, it is easy to see that

$$\boldsymbol{q}^{\{1\}}_{Newton,\mathrm{I}} \to \begin{pmatrix} \infty \\ 0 \end{pmatrix}, \qquad \boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{I}} \to \begin{pmatrix} \dfrac{\pi}{4} \\ 0 \end{pmatrix} = \boldsymbol{q}^{\{0\}},$$

illustrating how the Newton method diverges while approaching a singularity, while the Gradient method simply stops. In fact, when $\varepsilon = 0$, the position error $\boldsymbol{e}^{\{0\}} = \boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}}) = \boldsymbol{p}_{d,\mathrm{I}} = (-1, 1)$ belongs to the null space of $\boldsymbol{J}^T(\boldsymbol{q}^{\{0\}})|_{\varepsilon=0}$. For case II, with the Newton method one has

$$\begin{aligned} \boldsymbol{q}^{\{1\}}_{Newton,\mathrm{II}} &= \boldsymbol{q}^{\{0\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{0\}}) \left(\boldsymbol{p}_{d,\mathrm{II}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\dfrac{\sqrt{2}}{2\varepsilon} & \dfrac{\sqrt{2}}{2\varepsilon} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\dfrac{\sqrt{2}}{2} \\ \varepsilon\dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} \\ \sqrt{2} \end{pmatrix} = \boldsymbol{q}^*. \end{aligned}$$

Being $\boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{p}_{d,\mathrm{II}}$, we have found a solution $\boldsymbol{q}^*$ to the inverse kinematics problem in just one iteration (thanks to the simple structure of this specific problem). Moreover, this holds true independently from the value of ε, which in fact cancels out. On the other hand, with the Gradient method, and for a generic step size $\alpha > 0$, it is

$$\begin{aligned} \boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{II}} &= \boldsymbol{q}^{\{0\}} + \alpha\, \boldsymbol{J}^{T}(\boldsymbol{q}^{\{0\}}) \left(\boldsymbol{p}_{d,\mathrm{II}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \alpha \begin{pmatrix} -\varepsilon\dfrac{\sqrt{2}}{2} & \varepsilon\dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\dfrac{\sqrt{2}}{2} \\ \varepsilon\dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} \\ \alpha\sqrt{2} + \varepsilon\,(1-\alpha) \end{pmatrix}. \end{aligned}$$

When $\varepsilon \to 0$, we have

$$\boldsymbol{q}^{\{1\}}_{Gradient.\mathrm{II}} \to \begin{pmatrix} \dfrac{\pi}{4} \\ \alpha\sqrt{2} \end{pmatrix},$$

so that the Gradient method will escape the singularity (here, $\boldsymbol{e}^{\{0\}} = \boldsymbol{p}_{d,\mathrm{II}} \notin \mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q}^{\{0\}})|_{\varepsilon=0}\right\}$). One can easily see that also the Gradient method may find the solution $\boldsymbol{q}^*$ in just one iteration, but only if the step size is chosen as $\alpha = 1$ (and then, once again, this would then occur independently from the actual value of ε). However, α is usually chosen smaller than unitary in the final iterations in order to avoid missing a close solution. Therefore, the Gradient method will typically approach the solution $\boldsymbol{q}^*$ at a slower rate than the Newton method. ■

Question #6 [all students]

The 3R robotic device in Fig. 3 has joint axes that intersect two by two. The second joint axis is inclined by an angle $\delta \approx 20°$. This structure is mainly intended for pointing the final axis $\boldsymbol{n}$ at a moving target in 3D. Provide the explicit expression of the square angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of this robot. Find the singularities, if any, of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$. Compute the relation between $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the time derivative $\dot{\boldsymbol{n}}$ of the pointing axis.

Reply #6

The angular part of the geometric Jacobian for this 3R robotic pointing device is the 3×3 matrix

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{z}_2 \end{pmatrix}$$

where $\boldsymbol{z}_{i-1}$ is a unit vector along the axis of joint i, for $i = 1,2,3$, expressed by default in the robot base frame RF_0, and ${}^j\boldsymbol{z}_j = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for any index j. In order to provide the explicit expression of the elements in this matrix, it is convenient to define a set of frames according to the DH convention, e.g., as in Fig. 7. From the DH table reported therein[2], one has

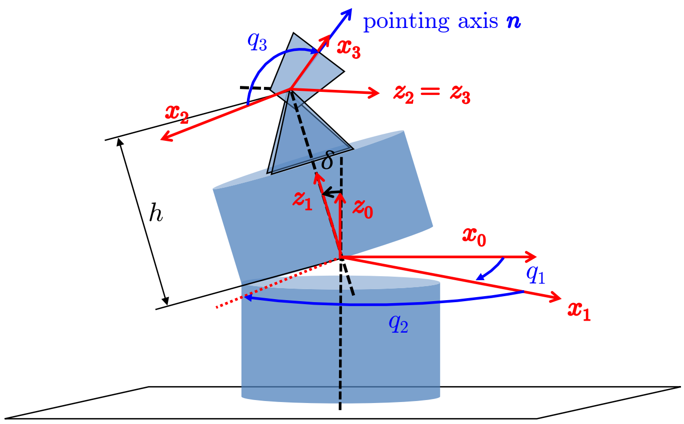
$$
{}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & -\cos\delta \sin q_1 & \sin\delta \sin q_1 \\ \sin q_1 & \cos\delta \cos q_1 & -\sin\delta \cos q_1 \\ 0 & \sin\delta & \cos\delta \end{pmatrix}, \quad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & -\sin q_2 \\ \sin q_2 & 0 & \cos q_2 \\ 0 & -1 & 0 \end{pmatrix},
$$

and so

$$
{}^0\boldsymbol{R}_2(q_1,q_2) = \begin{pmatrix} \cos q_1 \cos q_2 - \cos\delta \sin q_1 \sin q_2 & -\sin\delta \sin q_1 & -\cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ \sin q_1 \cos q_2 + \cos\delta \cos q_1 \sin q_2 & \sin\delta \cos q_1 & \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ \sin\delta \sin q_2 & -\cos\delta & \sin\delta \cos q_2 \end{pmatrix}.
$$

As a result,

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & \sin\delta \sin q_1 & -\cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ 0 & -\sin\delta \cos q_1 & \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ 1 & \cos\delta & \sin\delta \cos q_2 \end{pmatrix}.
$$



i	α_i	a_i	d_i	θ_i
1	δ	0	0	q_1
2	$-\pi/2$	0	h	q_2
3	0	0	0	q_3

Figure 7: DH frames and table used for defining the Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ of the pointing device.

The singularities of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ occur when

$$
\det \boldsymbol{J}_A(\boldsymbol{q}) = -\sin\delta \sin q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \{0, \pi\},
$$

[2]Since $\boldsymbol{z}_3$ does not enter in the Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$, the choice of a particular final twist α_3 is irrelevant here.

namely when the axis $\boldsymbol{x}_1$ and the projection of the axis $\boldsymbol{x}_2$ on the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ are aligned. In this situation, the following instantaneous joint velocities

$$\dot{\boldsymbol{q}} = \lambda \begin{pmatrix} -1 \\ \cos\delta \\ \sin\delta \end{pmatrix}, \text{ for } q_2 = 0 \qquad \text{or} \qquad \dot{\boldsymbol{q}} = \lambda \begin{pmatrix} -1 \\ \cos\delta \\ -\sin\delta \end{pmatrix}, \text{ for } q_2 = \pi,$$

lie in the null space of $\boldsymbol{J}$ and will produce thus $\boldsymbol{\omega} = \boldsymbol{0}$, $\forall\lambda$. Finally, the pointing axis $\boldsymbol{n}$ is given by

$$\begin{aligned} \boldsymbol{n} &= {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (\cos q_1 \cos q_2 - \cos\delta \sin q_1 \sin q_2)\cos q_3 - \sin\delta \sin q_1 \sin q_3 \\ (\sin q_1 \cos q_2 + \cos\delta \cos q_1 \sin q_2)\cos q_3 + \sin\delta \cos q_1 \sin q_3 \\ \sin\delta \sin q_2 \cos q_3 - \cos\delta \sin q_3 \end{pmatrix} = \begin{pmatrix} n_1(\boldsymbol{q}) \\ n_2(\boldsymbol{q}) \\ n_3(\boldsymbol{q}) \end{pmatrix}. \end{aligned}$$

Let $\boldsymbol{R}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} \end{pmatrix}$. The time derivative of the unit vector $\boldsymbol{n}$ is computed then as

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} \quad \Rightarrow \quad \dot{\boldsymbol{n}} = \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{n} = \boldsymbol{\omega} \times \boldsymbol{n} = (\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}) \times \boldsymbol{n} = (\boldsymbol{z}_0 \times \boldsymbol{n})\,\dot{q}_1 + (\boldsymbol{z}_1 \times \boldsymbol{n})\,\dot{q}_2 + (\boldsymbol{z}_3 \times \boldsymbol{n})\,\dot{q}_3.$$

The following computations can be conveniently performed with a symbolic code in Matlab:

$$\boldsymbol{z}_0 \times \boldsymbol{n} = \begin{pmatrix} -n_2(\boldsymbol{q}) \\ n_1(\boldsymbol{q}) \\ 0 \end{pmatrix},$$

$$\boldsymbol{z}_1 \times \boldsymbol{n} = \left({}^0\boldsymbol{R}_1\,{}^1\boldsymbol{z}_1\right) \times \boldsymbol{n} = \begin{pmatrix} \sin\delta \sin q_1 \\ -\sin\delta \cos q_1 \\ \cos\delta \end{pmatrix} \times \boldsymbol{n} = \begin{pmatrix} -(\cos q_1 \sin q_2 + \cos\delta \sin q_1 \cos q_2)\cos q_3 \\ -(\sin q_1 \sin q_2 - \cos\delta \cos q_1 \cos q_2)\cos q_3 \\ \sin\delta \cos q_2 \cos q_3 \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{z}_2 \times \boldsymbol{n} &= \left({}^0\boldsymbol{R}_2\,{}^2\boldsymbol{z}_2\right) \times \boldsymbol{n} = \begin{pmatrix} \cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ \sin\delta \cos q_2 \end{pmatrix} \times \boldsymbol{n} \\ &= \begin{pmatrix} \cos\delta \sin q_1 \sin q_2 \sin q_3 - \sin\delta \sin q_1 \cos q_3 - \cos q_1 \cos q_2 \sin q_3 \\ \sin\delta \cos q_1 \cos q_3 - \sin q_1 \cos q_2 \sin q_3 - \cos\delta \cos q_1 \sin q_2 \sin q_3 \\ -\cos\delta \cos q_3 - \sin\delta \sin q_2 \sin q_3 \end{pmatrix}. \end{aligned}$$

■

Question #7 [all students]

The desired initial and final orientation of the end effector of a certain robot are specified at time $t = 0$ and $t = T$, respectively by

$$\boldsymbol{R}(0) = \boldsymbol{R}_{in} = \begin{pmatrix} 0.5 & 0 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & -0.5 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{R}(T) = \boldsymbol{R}_{fin} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ -0.5 & -0.5 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -\sqrt{2}/2 \end{pmatrix}.$$

The end effector should start with zero angular velocity and acceleration ($\boldsymbol{\omega}_{in} = \dot{\boldsymbol{\omega}}_{in} = \mathbf{0}$, at $t = 0$) and reach the final orientation with angular velocity and acceleration given by

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_{fin} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \text{ [rad/s]}, \qquad \dot{\boldsymbol{\omega}}(T) = \dot{\boldsymbol{\omega}}_{fin} = \mathbf{0}.$$

Plan a smooth and coordinated trajectory for the end-effector orientation that satisfies all the given boundary conditions for a generic motion time $T > 0$. Setting next $T = 1$, compute at the mid-time instant $t = T/2$ the numerical values of the resulting orientation $\boldsymbol{R}(T/2)$ and angular velocity $\boldsymbol{\omega}(T/2)$ of the robot end effector.

Reply #7

This trajectory planning problem for the end-effector orientation has the special feature of requiring a desired non-zero angular velocity at the final instant of motion. So, it is very unlikely that addressing the reorientation using the axis-angle method will work (in fact, this has zero probability to occur!). The reason is that, if we extract a unit axis $\boldsymbol{r}$ and an angle θ_{if} from the relative rotation $\boldsymbol{R}_{if} = \boldsymbol{R}_{in}^T \boldsymbol{R}_{fin}$ and then perform the rotation around $\boldsymbol{r}$ with any possible timing $\theta(t)$, the associated angular velocity $\boldsymbol{\omega}_r(t) = \boldsymbol{r}\dot{\theta}(t)$ will always be aligned with $\boldsymbol{r}$. In particular, at $t = T$, it will be $\boldsymbol{\omega}_r(T) \neq \boldsymbol{\omega}_{fin}$, and we cannot satisfy such equality for an arbitrary $\boldsymbol{\omega}_{fin}$ (one can check that this in fact the case here too). Therefore, we pursue a solution by planning the motion for the three angles of a minimal representation of orientation, and imposing suitable boundary conditions. Indeed, many choices are possible and the only precaution is to avoid singularities of the representation during the entire reorientation. In the following, we shall use the sequence of XYZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$.

For this Euler representation of orientation, we have the rotation matrix

$$\begin{aligned}\boldsymbol{R}_{E,XYZ}(\boldsymbol{\phi}) &= \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_Z(\gamma) \\ &= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}.\end{aligned} \tag{4}$$

At the differential level, we have also the relationship between $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$:

$$\begin{aligned}\boldsymbol{\omega} &= \begin{pmatrix} {}^0\boldsymbol{x}_0 & \boldsymbol{R}_X(\alpha)\,{}^1\boldsymbol{y}_1 & \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\,{}^2\boldsymbol{z}_2 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \begin{matrix} \dot{\alpha} \\ 0 \\ 0 \end{matrix} & \boldsymbol{R}_X(\alpha)\begin{pmatrix} 0 \\ \dot{\beta} \\ 0 \end{pmatrix} & \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\,\dot{\boldsymbol{\phi}}.\end{aligned} \tag{5}$$

Matrix $\boldsymbol{T}$ has $\det \boldsymbol{T}(\boldsymbol{\phi}) = \cos\beta$. This is indeed the singularity of the chosen Euler representation. One should make sure that the condition $\beta = \pm\pi/2$ is never crossed in the solution of the problem (otherwise, we should change representation and restart from scratch). With the expression (4) at hand, one can convert the given initial and final rotation matrices $\boldsymbol{R}_{in}$ and $\boldsymbol{R}_{fin}$ into XYZ Euler angles by using general inversion formulas that hold in the non-singular case only, i.e., when

$$\cos\beta = \pm\sqrt{R_{11}^2 + R_{12}^2} \neq 0.$$

This happens to be the case for both rotation matrices. Then, from

$$\alpha = \text{ATAN2}\left\{\frac{-R_{23}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \beta = \text{ATAN2}\left\{R_{13}, \cos\beta\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{-R_{12}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}, \tag{6}$$

we obtain the following pairs of solutions (depending on the sign chosen in the expression of $\cos\beta$)

$$\boldsymbol{R}_{in} \quad \Rightarrow \quad \cos\beta = \pm\, 0.5 \qquad \Rightarrow \qquad \boldsymbol{\phi}_{in} = (\alpha_{in}, \beta_{in}, \gamma_{in}) = \begin{cases} \boldsymbol{\phi}_{in}^{I} = (\frac{\pi}{2}, -\frac{\pi}{3}, 0) \\ \boldsymbol{\phi}_{in}^{II} = (-\frac{\pi}{2}, -\frac{2\pi}{3}, \pi) \end{cases}$$

$$\boldsymbol{R}_{fin} \quad \Rightarrow \quad \cos\beta = \pm\, 1 \qquad \Rightarrow \qquad \boldsymbol{\phi}_{fin} = (\alpha_{fin}, \beta_{fin}, \gamma_{fin}) = \begin{cases} \boldsymbol{\phi}_{fin}^{I} = (\frac{3\pi}{4}, 0, \frac{\pi}{4}) \\ \boldsymbol{\phi}_{fin}^{II} = (-\frac{\pi}{4}, \pi, -\frac{3\pi}{4}). \end{cases}$$

Keeping into account the need to avoid the crossing of a value $\beta = \pm\pi/2$, we choose the combination (out of four possible) $\boldsymbol{\phi}_{in}^{I}$ (with $\beta_{in} = -\pi/3$) and $\boldsymbol{\phi}_{fin}^{I}$ (with $\beta_{fin} = 0$) as boundary conditions for the Euler angles trajectories[3]. In this way, a smooth interpolation should also guarantee that $\beta(t)$ remains in the singularity-free interval $[-\pi/3, 0]$ for all $t \in [0, T]$. We use finally eq. (5) to convert the given angular velocity $\boldsymbol{\omega}_{fin}$ into a boundary condition for the first derivative of the Euler angles at the final time $t = T$. In view of the non-singularity of $\boldsymbol{T}(\boldsymbol{\phi}_{fin})$, we obtain

$$\dot{\boldsymbol{\phi}}_{fin} = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_{fin})\,\boldsymbol{\omega}_{fin} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2.1213 \\ \sqrt{2}/2 \end{pmatrix} \text{ [rad/s]}.$$

The interpolation problem for the Euler angles requires generating a $\boldsymbol{\phi}(t)$ for $t \in [0, T]$ such that

$$\boldsymbol{\phi}(0) = \boldsymbol{\phi}_{in}, \quad \dot{\boldsymbol{\phi}}(0) = \mathbf{0}, \quad \ddot{\boldsymbol{\phi}}(0) = \mathbf{0}, \qquad \boldsymbol{\phi}(T) = \boldsymbol{\phi}_{fin}, \quad \dot{\boldsymbol{\phi}}(T) = \dot{\boldsymbol{\phi}}_{fin}, \quad \ddot{\boldsymbol{\phi}}(T) = \mathbf{0}.$$

We use then quintic polynomials in normalized time of the form

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_{in} + \boldsymbol{a}_3\tau^3 + \boldsymbol{a}_4\tau^4 + \boldsymbol{a}_5\tau^5, \qquad \tau = \frac{t}{T} \in [0, 1], \tag{7}$$

where the vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5 \in \mathbb{R}^3$ contain the non-vanishing coefficients of the three quintic polynomials for $\alpha(\tau)$, $\beta(\tau)$, and $\gamma(\tau)$. The solution is illustrated next for a generic scalar component $\phi_i(\tau)$ of $\boldsymbol{\phi}(\tau)$, with $i = 1, 2, 3$, dropping also the index. The chosen structure of the polynomial in (7) already satisfies the three boundary conditions at $\tau = 0$. For the remaining three conditions at $\tau = 1$, we have:

$$\begin{aligned} \phi(1) &= \phi_{in} + a_3 + a_4 + a_5 = \phi_{fin} \\ \dot{\phi}(1) &= \left.\frac{d\phi}{d\tau}\right|_{\tau=1} \cdot \frac{d\tau}{dt} = (3a_3 + 4a_4 + 5a_5)\,\frac{1}{T} = \dot{\phi}_{fin} \\ \ddot{\phi}(1) &= \left.\frac{d^2\phi}{d\tau^2}\right|_{\tau=1} \cdot \left(\frac{d\tau}{dt}\right)^2 = (6a_3 + 12a_4 + 20a_5)\,\frac{1}{T^2} = 0, \end{aligned}$$

[3] In the following, we shall drop for conciseness the index I from $\boldsymbol{\phi}_{in}$ and $\boldsymbol{\phi}_{fin}$.

or

$$\boldsymbol{M}\boldsymbol{a} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} \phi_{fin} - \phi_{in} \\ \dot{\phi}_{fin} T \\ 0 \end{pmatrix} = \boldsymbol{b},$$

with $\det \boldsymbol{M} = 2$. Solving this linear system yields

$$\boldsymbol{a} = \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{b} = \begin{pmatrix} 10 & -4 & 0.5 \\ -15 & 7 & -1 \\ 6 & -3 & 0.5 \end{pmatrix} \begin{pmatrix} \phi_{fin} - \phi_{in} \\ \dot{\phi}_{fin} T \\ 0 \end{pmatrix},$$

or

$$a_3 = 10(\phi_{fin} - \phi_{in}) - 4\dot{\phi}_{fin}T, \quad a_4 = -15(\phi_{fin} - \phi_{in}) + 7\dot{\phi}_{fin}T, \quad a_5 = 6(\phi_{fin} - \phi_{in}) - 3\dot{\phi}_{fin}T.$$

In vector form, the solution can be rewritten as

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_{in} + \left(\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}\right)\left(10\tau^3 - 15\tau^4 + 6\tau^5\right) + \dot{\boldsymbol{\phi}}_{fin} T \left(-4\tau^3 + 7\tau^4 - 3\tau^5\right), \quad \tau \in [0,1].$$

Moreover, the first and second time derivatives are

$$\dot{\boldsymbol{\phi}}(\tau) = \frac{\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}}{T}\left(30\tau^2 - 60\tau^3 + 30\tau^4\right) + \dot{\boldsymbol{\phi}}_{fin}\left(-12\tau^2 + 28\tau^3 - 15\tau^4\right),$$

$$\ddot{\boldsymbol{\phi}}(\tau) = \frac{\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}}{T^2}\left(60\tau - 180\tau^2 + 120\tau^3\right) + \frac{\dot{\boldsymbol{\phi}}_{fin}}{T}\left(-24\tau + 84\tau^2 - 60\tau^3\right).$$

By setting now $T = 1$ [s] as motion time, we can fully evaluate the coefficients of the three quintic polynomials. The result is

$$\boldsymbol{\phi}(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix} = \begin{pmatrix} 1.5708 - 4.1460\,t^3 + 9.2190\,t^4 - 4.2876\,t^5 \\ -1.0472 + 1.9867\,t^3 - 0.8587\,t^4 - 0.0808\,t^5 \\ 5.0256\,t^3 - 6.8312\,t^4 + 2.5911\,t^5 \end{pmatrix}, \quad t \in [0,1].$$

Figure 8 shows the trajectories of these interpolating XYZ Euler angles, together with the evolutions of their first and second time derivatives. At the motion mid-time $t = 0.5$ [s], we have

$$\boldsymbol{\phi}_m = \boldsymbol{\phi}(0.5) = \begin{pmatrix} 1.4947 \\ -0.8551 \\ 0.2822 \end{pmatrix} \text{[rad]} \quad \Rightarrow \quad \boldsymbol{R}_{E,XYZ}(\boldsymbol{\phi}_m) = \begin{pmatrix} 0.6302 & -0.1827 & -0.7546 \\ -0.7015 & 0.2825 & -0.6543 \\ 0.3327 & 0.9417 & 0.04985 \end{pmatrix}.$$

Moreover, from the first derivative of the Euler angles trajectory evaluated at the motion mid-time $t = 0.5$ [s],

$$\dot{\boldsymbol{\phi}}_m = \dot{\boldsymbol{\phi}}(0.5) = \begin{pmatrix} 0.3202 \\ 2.0708 \\ 2.3265 \end{pmatrix} \text{[rad/s]},$$

we also obtain

$$\boldsymbol{\omega}_m = \boldsymbol{\omega}(0.5) = \boldsymbol{T}(\boldsymbol{\phi}_m)\dot{\boldsymbol{\phi}}_m = \begin{pmatrix} 1 & 0 & -0.7546 \\ 0 & 0.0760 & -0.6543 \\ 0 & 0.9971 & 0.04985 \end{pmatrix} \dot{\boldsymbol{\phi}}_m = \begin{pmatrix} -1.8511 \\ 2.4771 \\ -2.8774 \end{pmatrix} \text{[rad/s]}. \qquad \blacksquare$$

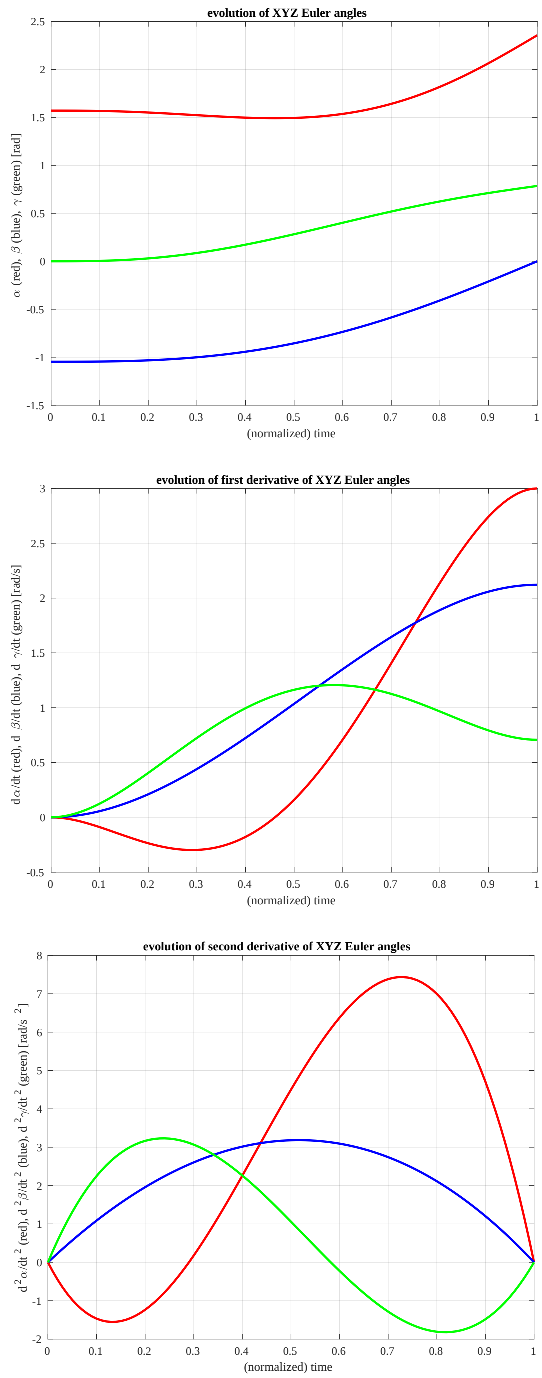


Figure 8: Trajectories of the interpolating XYZ Euler angles (top), with their velocities (center) and accelerations (bottom).

Question #8 [all students]

The planar 3R robot with unitary link lengths shown in Fig. 4 is initially in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\, 11\pi/18,\, -\pi/4)$. *Commanded by a joint velocity* $\dot{\boldsymbol{q}}(t)$ *that uses feedback from the current* $\boldsymbol{q}(t)$, *the robot should perform a self-motion so as to reach asymptotically the final value* $q_{3,fin} = -\pi/2$ *for the third joint, while keeping the position of its end-effector always at the same initial point* P_{in}. *Verify first that such task is feasible. Design then a control scheme that completes the task in a robust way, i.e., by rejecting also possible transient errors and without encountering any singular situation in which the control law is ill conditioned.* Hint: Use an approach based on joint space decomposition.

Reply #8

The planar robot has $n = 3$ joints and is kinematically redundant for the positioning of its end effector in the plane ($m = 2$). The requested task requires the use of this redundancy, exploring the null space motions so as to get to the desired joint configuration while keeping the end-effector at rest in the initial position. Before proposing a control solution, we have to verify whether or not the end effector of the 3R robot with the third link at $q_3 = -\pi/2$ can still reach the point P_{in}. We shall call the 3R robot with the third joint at this angle, a 'reduced 2R' robot having its second equivalent link of length $\sqrt{l_2 + l_3} = \sqrt{2}$. Using the initial configuration $\boldsymbol{q}_{in}$, we compute first the position of point P_{in} as

$$\boldsymbol{p}_{in} = \boldsymbol{f}(\boldsymbol{q}_{in}) = \begin{pmatrix} \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \\ \sin q_1 + \sin(q_1+q_2) + \sin(q_1+q_2+q_3) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_{in}} = \begin{pmatrix} 1.6468 \\ 1.3651 \end{pmatrix},$$

obtaining $\|\boldsymbol{p}_{in}\| = 2.1390$ [m] as distance from the origin. The workspace of the reduced 2R robot is a circular annulus with inner and outer radius given by $R_{min} = \left|1 - \sqrt{2}\right| = 0.4142$ and $R_{max} = 1 + \sqrt{2} = 2.4142$, respectively. Therefore, the point P_{in} will be inside the workspace of the reduced 2R robot when at destination. In principle, we should be able to keep the end effector at P_{in} during the entire self-motion task —see also Fig. 9.

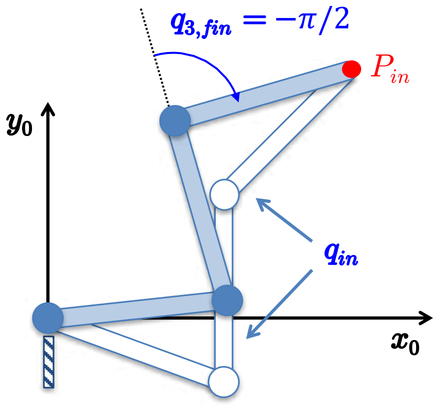


Figure 9: Configuration reached by the 3R robot at the end of the controlled self-motion task.

A possible approach would be to define the joint velocity $\dot{\boldsymbol{q}}$ as the projection of a suitable command $\boldsymbol{\xi} \in \mathbb{R}^3$ in the null space of the 2×3 task Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})$, namely[4]

$$\dot{\boldsymbol{q}} = \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\boldsymbol{\xi}.$$

[4] See also slide #19 in the block 12_InverseDiffKinStatics.pdf (later revised as slide #22 in the lecture block 12a_InverseDifferentialKinematics.pdf).

However, in this way we would have no control on which configuration would be eventually reached by the robot. Therefore, we opt for a more tailored solution that focuses on the third joint, the only one that has a special target, leaving to the other two joints the task of keeping the end effector at the desired position P_{in}. This is also called a joint space decomposition approach. Indeed, it could be used in a planning fashion, but here a feedback solution is required. With this in mind, we set

$$\dot{q}_3 = k_3 \left(q_{3,fin} - q_3 \right), \qquad k_3 > 0. \tag{8}$$

This guarantees that q_3 will converge exponentially from any position to the desired $q_{3,fin}$. Moreover, the natural motion of q_3 will always remain in the interval $[q_{3,fin}, q_{3,in}] = [-\pi/2, -\pi/4]$. Next, decompose the differential kinematics as follows:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_{12}(\boldsymbol{q})\dot{\boldsymbol{q}}_{12} + \boldsymbol{J}_3(\boldsymbol{q})\,\dot{q}_3 \\ &= \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) - \sin(q_1+q_2+q_3) & -\sin(q_1+q_2) - \sin(q_1+q_2+q_3) \\ \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) & \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} -\sin(q_1+q_2+q_3) \\ \cos(q_1+q_2+q_3) \end{pmatrix} \dot{q}_3. \end{aligned} \tag{9}$$

The square sub-Jacobian made by the first two columns of $\boldsymbol{J}$ has $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = \sin q_2 + \sin(q_2+q_3)$. As long as this determinant is different from zero, we can set $\dot{\boldsymbol{p}} = \boldsymbol{0}$ in (9) and solve for $\dot{\boldsymbol{q}}_{12}$ so as to realize our self-motion task by

$$\dot{\boldsymbol{q}}_{12} = -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\boldsymbol{J}_3(\boldsymbol{q})\,\dot{q}_3, \tag{10}$$

for any motion $\dot{q}_3$, in particular that given by (8). To introduce more robustness in the task of keeping the end-effector position at $\boldsymbol{p}_{in}$, we replace

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_{in} = \boldsymbol{0} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_{in} + \boldsymbol{K}_P\left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right) = \boldsymbol{K}_P\left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right), \tag{11}$$

with a (diagonal) 2×2 gain matrix $\boldsymbol{K}_P > 0$ weighting the position error. The final control solution is obtained by using (8) and (11) in eq. (9) and solving again for $\dot{\boldsymbol{q}}_{12}$:

$$\dot{\boldsymbol{q}}_{12} = \boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\left(\boldsymbol{K}_P\left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right) - k_3\,\boldsymbol{J}_3(\boldsymbol{q})\left(q_{3,fin} - q_3\right)\right). \tag{12}$$

We can also combine (8) and (12) in a single formula as

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_{12} \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\,\boldsymbol{J}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P\left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right) \\ k_3\left(q_{3,fin} - q_3\right) \end{pmatrix}. \tag{13}$$

The last thing to check is the absence of singularities for $\boldsymbol{J}_{12}(\boldsymbol{q})$ during the self-motion under the feedback control law (13). It can be shown that $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = \sin q_2 + \sin(q_2+q_3) = 0$ if and only if the end-effector of the 3R robot finds itself aligned with the first link of the structure. From the illustration in Fig. 9, it is rather evident that such condition is not encountered in this task. ■

* * * * *

Robotics 1

June 11, 2021

Exercise #1

Consider the 6-dof robot in Fig. 1. The robot has three prismatic joints in a portal arrangement and a spherical wrist. Assign a set of Denavit-Hartenberg (DH) frames and provide the associated table of parameters. Give plausible values for the joint variables $\boldsymbol{q}$ at the configuration shown in the figure. Define the homogeneous transformation matrix ${}^{w}\boldsymbol{T}_0$ relating the DH reference frame RF_0 to the world frame RF_w (for this, introduce geometric quantities as needed).

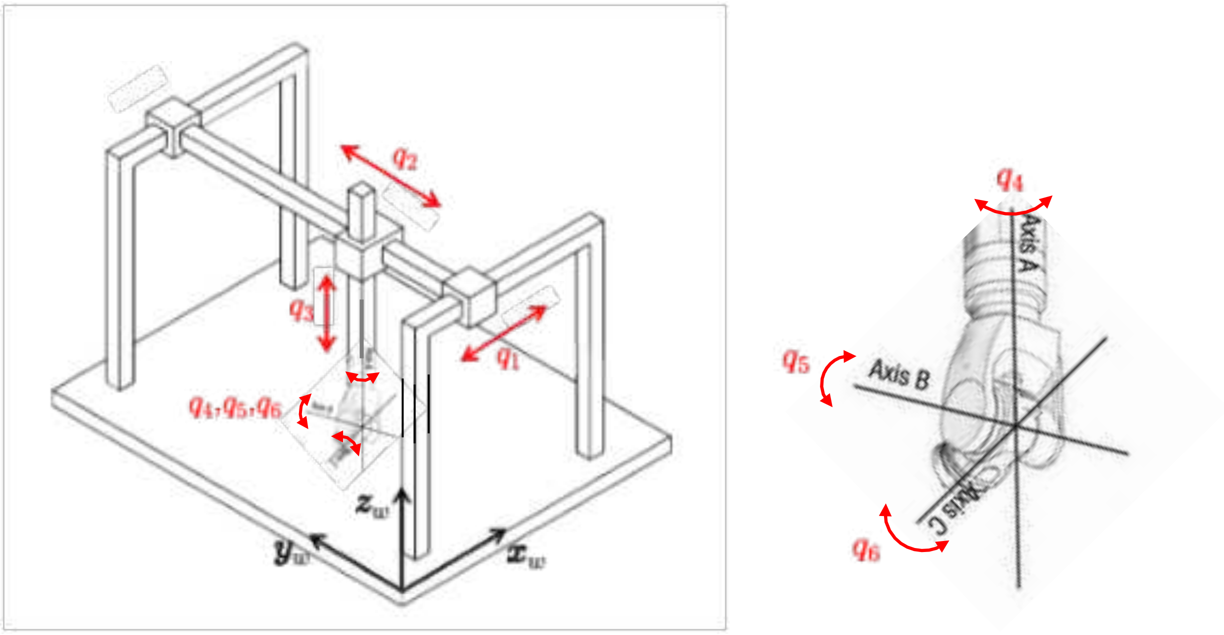


Figure 1: A 6-dof robot with a portal structure (3P) carrying a spherical (3R) wrist. An enlarged view of the wrist is shown on the right.

Exercise #2

A planar RPR robot is shown in Fig. 2, together with the definition of the joint coordinates[1]. The third link has length $L = 0.6$ [m]. The robot has to execute two different tasks, with the end effector placed at the point $P_d = (2, 0.4)$ [m] and pointing downward.

a) In the first task, the robot end effector should start moving inside a tube with a vertical speed $v = -2.5$ [m/s]. Determine the initial joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that realizes this instantaneous motion.

b) In the second task, the robot should keep its initial configuration in the presence of an horizontal force $f = 15$ [N] and a torque $\mu = 6$ [Nm] applied to its end effector. Determine the joint commands $\boldsymbol{\tau} \in \mathbb{R}^3$ (two torques and a force) needed for static balance.

Comments that justify intuitively some of the obtained results are welcome!

[1]Use these coordinates in your developments. Note that q_1 and q_3 are *not* DH variables.

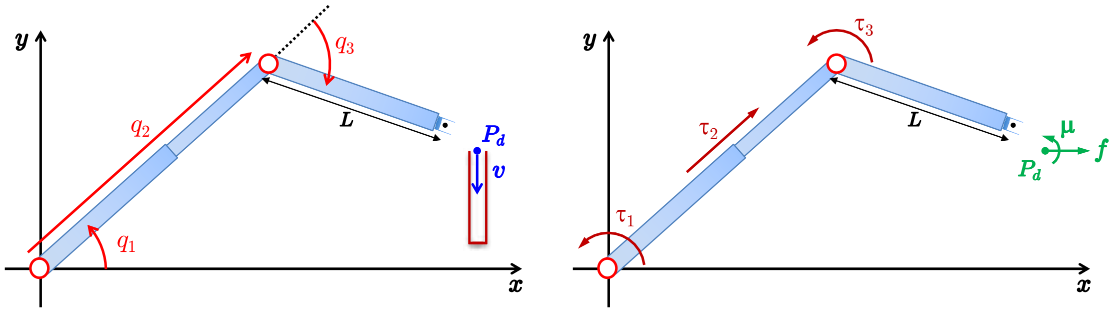


Figure 2: A 3-dof RPR robot, with the definition of a Cartesian motion task [left] and of a static balancing task in the presence of an external force/torque [right]. Both tasks should be executed at point P_d, with the robot end effector pointing downward.

Exercise #3

Plan a smooth rest-to-rest trajectory along a linear path from point $A = (1, 1, 1)$ [m] to point $B = (-1, 5, 0)$ [m], with simultaneous and coordinated change of orientation from

$$\boldsymbol{R}_A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

to

$$\boldsymbol{R}_B = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The total motion time is $T = 2.5$ [s]. The trajectory should be continuous up to the acceleration for all $t \in [0, T]$. Determine the velocity $\boldsymbol{v}_M \in \mathbb{R}^3$, acceleration $\boldsymbol{a}_M \in \mathbb{R}^3$, angular velocity $\boldsymbol{\omega}_M \in \mathbb{R}^3$, and angular acceleration $\dot{\boldsymbol{\omega}}_M \in \mathbb{R}^3$ attained at the time instant(s) when these four vectors assume, respectively, their maximum value in norm. Compute also the absolute orientation $\boldsymbol{R}_{mid} \in SO(3)$ at the midpoint of the planned trajectory.

[180 minutes (3 hours); open books]

Solution

June 11, 2021

Exercise #1

One of the (many) possible assignments of Denavit-Hartenberg frames for the 6-dof portal robot with spherical wrist is shown in Fig. 3. The associated parameters are reported in Tab. 1.

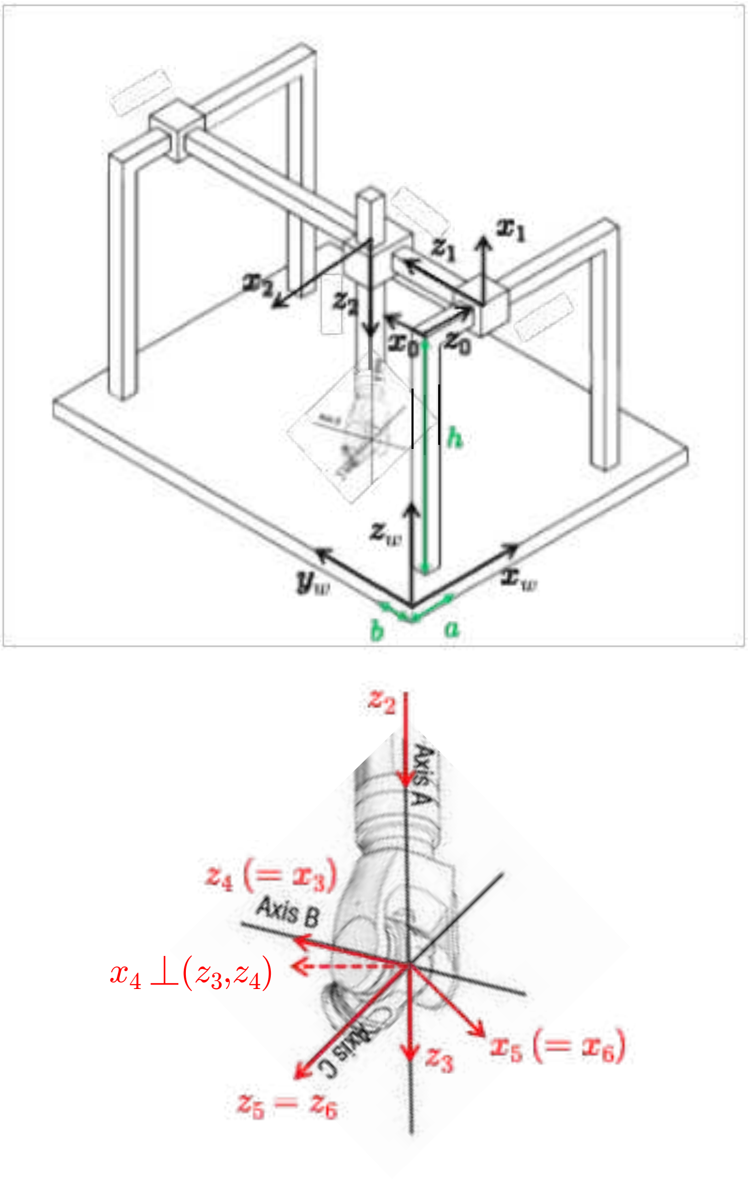


Figure 3: A possible assignment of DH frames for the 6-dof robot of Fig. 1. In the top figure, the world frame RF_w and the first three frames RF_0 to RF_2 of the portal structure are drawn in black, while the extra quantities a, b and h introduced for defining ${}^{w}\boldsymbol{T}_0$ are shown in green. In the bottom figure, the last four frames RF_3 to RF_6 for the spherical wrist are drawn in red.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$q_1 > 0$	$\pi/2$
2	$\pi/2$	0	$q_2 > 0$	$-\pi/2$
3	0	0	$q_3 > 0$	0
4	$\pi/2$	0	0	$q_4 = 0$
5	$\pi/2$	0	0	$q_5 = 3\pi/4$
6	0	0	0	$q_6 = 0$

Table 1: DH table of parameters corresponding to Fig. 3. The joint variables q_i (in red) take values associated to the configuration shown in the same figure. We have assumed $O_6 \equiv O_5$ ($d_6 = 0$).

With the geometric quantities introduced in Fig. 3, the homogenous matrix that locates the DH base frame RF_0 in the world frame is

$$
{}^{w}\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Exercise #2

The direct kinematics of interest for the RPR robot in Fig. 2 is

$$
\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \alpha \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 + L\cos(q_1+q_3) \\ q_2 \sin q_1 + L\sin(q_1+q_3) \\ q_1 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}
$$

where $\boldsymbol{p}$ is the planar position of the robot tip and α is the absolute angle of its end-effector w.r.t. the $\boldsymbol{x}$ axis. The associated 3×3 analytic Jacobian is

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 - L\sin(q_1+q_3) & \cos q_1 & -L\sin(q_1+q_3) \\ q_2 \cos q_1 + L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}, \tag{2}
$$

with $\det J(\boldsymbol{q}) = -q_2$.

To find an initial robot configuration $\boldsymbol{q}_d = (q_{1d}, q_{2d}, q_{3d})$ associated to the desired end-effector pose $\boldsymbol{r}_d = (\boldsymbol{p}_d, \alpha_d) = (p_{xd}, p_{yd}, \alpha_d) = (2, 0.4, -\pi/2)$, we solve the inverse kinematics problem in general. From the third equation in (1), we have

$$
q_{1d} + q_{3d} = \alpha_d,
$$

which, substituted in the first two equations, yields

$$
p_{xd} - L\cos\alpha_d = q_{2d}\cos q_{1d}, \qquad p_{yd} - L\sin\alpha_d = q_{2d}\sin q_{1d}. \tag{3}
$$

By squaring and summing the two equations in (3), we find the value q_{2d} as

$$q_{2d} = +\sqrt{(p_{xd} - L\cos\alpha_d)^2 + (p_{yd} - L\sin\alpha_d)^2}, \tag{4}$$

where the positive sign has been chosen for simplicity. Dividing by $q_{2d} > 0$ the two equations in (3), we also obtain

$$q_{1d} = \text{ATAN2}\left\{p_{y,d} - L\sin\alpha_d, p_{x,d} - L\cos\alpha_d\right\}, \tag{5}$$

and finally

$$q_{3d} = \alpha_d - q_{1d}. \tag{6}$$

For the specific data of the problem, note that eqs. (4) and (5) simplify to an intuitive geometric solution. In fact, when the desired point P_d is in the first quadrant and the end effector points vertically and downward ($\alpha_d = -\pi/2$), the base of the third link should be placed at the 'higher' point $P'_d = P_d + (0, L)$, whose position is

$$\boldsymbol{p}'_d = \begin{pmatrix} p_{xd} \\ p_{yd} + L \end{pmatrix}.$$

Therefore, the solution for the first two joints follows immediately as

$$q_{1d} = \arctan\left(\frac{p_{yd} + L}{p_{xd}}\right), \qquad q_{2d} = \sqrt{p_{xd}^2 + (p_{yd} + L)^2}, \tag{7}$$

while q_{3d} is found again by (6).

With the given data, we obtain

$$\boldsymbol{q}_d = \begin{pmatrix} 0.4636 \\ 2.2361 \\ -2.0344 \end{pmatrix} \text{[rad/m/rad]} = \begin{pmatrix} 26.565 \\ 2.2361 \\ -116.565 \end{pmatrix} [°/\text{m}/°].$$

Thus, the Jacobian in this configuration becomes

$$\boldsymbol{J}_d = \boldsymbol{J}(\boldsymbol{q}_d) = \begin{pmatrix} -0.4000 & 0.8944 & 0.6000 \\ 2.0000 & 0.4472 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $\det \boldsymbol{J}_d = -2.2361$, this matrix is invertible. For the motion task, we have that

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} \dot{p}_{xd} \\ \dot{p}_{yd} \\ \dot{\alpha}_d \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix},$$

because the insertion in the tube is feasible only by keeping the vertical, downward orientation of the end effector. For the static balancing task, it is

$$\boldsymbol{F}_d = \begin{pmatrix} f_{xd} \\ f_{yd} \\ \mu_{zd} \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ \mu \end{pmatrix}.$$

Using the numerical data, the solutions for the required tasks are

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_d^{-1}\,\dot{\boldsymbol{r}}_d = \boldsymbol{J}_d^{-1}\begin{pmatrix} 0 \\ -2.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1.1180 \\ 1 \end{pmatrix} \text{ [rad/s, m/s, rad/s]} \tag{8}$$

and, respectively,

$$\boldsymbol{\tau} = -\boldsymbol{J}_d^T \boldsymbol{F}_d = -\boldsymbol{J}_d^T \begin{pmatrix} 15 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -13.4164 \\ -15 \end{pmatrix} \text{ [Nm, N, Nm].} \tag{9}$$

We note that the first and third (revolute) joints compensate their motion in (8), in order not to change the end-effector orientation. Similarly, in (9) the torque at the third joint directly annihilates the presence of the torque μ at the end effector (both acting on the same link), while the force at the second joint is the only one responsible for compensating the horizontal force component f (as projected along the direction of the prismatic joint). The fact that $\tau_1 = 0$ is just an occasional result here, due to the particular combination of input data[2].

Exercise #3

The trajectory is determined in parametric form in terms of a normalized scalar $s \in [0,1]$ for both the linear and the angular parts, in order to achieve coordinated motion. Then, a sufficiently smooth timing law $s = s(t)$ for $t \in [0,T]$ is assigned, so as to guarantee rest-to-rest motion with continuity up to the acceleration (as a consequence, also the acceleration should be zero at the initial and final instants).

For the linear motion along a straight line from point A to point B, we have

$$\boldsymbol{p}(s) = \boldsymbol{p}_A + (\boldsymbol{p}_B - \boldsymbol{p}_A)\,s = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} s, \qquad s \in [0,1],$$

with path length $L = \|\boldsymbol{p}_B - \boldsymbol{p}_A\| = 4.5826$ [m]. The velocity and the acceleration are then

$$\dot{\boldsymbol{p}} = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\dot{s}, \qquad \ddot{\boldsymbol{p}} = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\ddot{s},$$

and their maximum values in norm will be attained at the instants where, respectively, $\dot{s}$ or $\ddot{s}$ have a maximum (in absolute value for the latter), with

$$\boldsymbol{v}_M = (\boldsymbol{p}_B - \boldsymbol{p}_A) \cdot \max_{t\in[0,T]} \dot{s}(t) = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\dot{s}_{max} \tag{10}$$

and

$$\boldsymbol{a}_M = (\boldsymbol{p}_B - \boldsymbol{p}_A) \cdot \max_{t\in[0,T]} |\ddot{s}(t)| = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\ddot{s}_{max}. \tag{11}$$

For the angular motion, it is convenient to choose an axis-angle planning method[3]. In this way, it is immediate to find the resulting angular velocity and acceleration. First, we compute the relative

[2] It is instructive to look at the outcome of the balancing $\boldsymbol{\tau}$ for $f = 15$ only, for $\mu = 6$ only, or for a slight perturbation of one of these two w.r.t. the given input data, e.g., for $f = 15.1$, $\mu = 6$.

[3] Indeed, one may also choose to convert the initial and final rotation matrices $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ into some minimal set of Euler or RPY-type angles, and then plan a trajectory for these angles in a coordinated way. However, the complete procedure would be more lengthy.

orientation between $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$:

$$\boldsymbol{R}_{AB} = \boldsymbol{R}_A^T \boldsymbol{R}_B = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

From this, we extract the axis-angle solution $(\boldsymbol{r}, \theta_{AB})$ as

$$\begin{aligned} \theta_{AB} &= \text{ATAN2}\left\{\sqrt{(R_{12}-R_{21})^2 + (R_{13}-R_{31})^2 + (R_{23}-R_{32})^2}, R_{11}+R_{22}+R_{33}-1\right\} \\ &= 2.5936 \text{ [rad]}, \\ \boldsymbol{r} &= \frac{1}{2\sin\theta_{AB}} \begin{pmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{pmatrix} = \begin{pmatrix} -0.6786 \\ 0.6786 \\ 0.2811 \end{pmatrix}, \end{aligned}$$

being in a regular condition ($\sin\theta_{aB} \neq 0$). The profile of the rotation angle around the unit vector $\boldsymbol{r}$ is then defined parametrically as

$$\theta(s) = \theta_{AB}\, s, \qquad s \in [0, 1].$$

The angular velocity and acceleration vectors are aligned with the unit vector $\boldsymbol{r} \in \mathbb{R}^3$, with profiles

$$\boldsymbol{\omega} = \theta_{AB}\dot{s}\ \boldsymbol{r}, \qquad \dot{\boldsymbol{\omega}} = \theta_{AB}\ddot{s}\ \boldsymbol{r}.$$

As before, their maximum values in norm will be attained at the instants where, respectively, $\dot{s}$ or $\ddot{s}$ have a maximum (in absolute value), with

$$\boldsymbol{\omega}_M = \theta_{AB}\boldsymbol{r} \cdot \max_{t\in[0,T]} \dot{s}(t) = (\theta_{AB}\boldsymbol{r})\, \dot{s}_{max} \tag{12}$$

and

$$\dot{\boldsymbol{\omega}}_M = \theta_{AB}\boldsymbol{r} \cdot \max_{t\in[0,T]} |\ddot{s}(t)| = (\theta_{AB}\boldsymbol{r})\, \ddot{s}_{max}. \tag{13}$$

The simplest timing law that guarantees a rest-to-rest motion in time T with continuous acceleration in the whole interval $[0, T]$ is given by the doubly-normalized quintic polynomial

$$s(t) = 6\left(\frac{t}{T}\right)^5 - 15\left(\frac{t}{T}\right)^4 + 10\left(\frac{t}{T}\right)^3 \qquad \Rightarrow \quad s(0) = 0,\ s(T) = 1,$$

with first time derivative

$$\dot{s}(t) = \frac{1}{T}\left(30\left(\frac{t}{T}\right)^4 - 60\left(\frac{t}{T}\right)^3 + 30\left(\frac{t}{T}\right)^2\right) \qquad \Rightarrow \quad \dot{s}(0) = \dot{s}(T) = 0,$$

and second time derivative

$$\ddot{s}(t) = \frac{1}{T^2}\left(120\left(\frac{t}{T}\right)^3 - 180\left(\frac{t}{T}\right)^2 + 60\left(\frac{t}{T}\right)\right) \qquad \Rightarrow \quad \ddot{s}(0) = \ddot{s}(T) = 0.$$

It is easy to see that, apart from the two boundary instants $t = 0$ and $t = T$, $\ddot{s}(t) = 0$ has a root also at $t = T_{mid} = T/2$, where the pseudo-velocity $\dot{s}$ has a maximum. We obtain

$$\dot{s}_{max} = \dot{s}\left(T_{mid}\right) = \frac{1}{T}\left(30\left(\frac{1}{2}\right)^4 - 60\left(\frac{1}{2}\right)^3 + 30\left(\frac{1}{2}\right)^2\right) = \frac{7.5}{4T} = 0.75, \tag{14}$$

where $T = 2.5$ [s] has been used. On the other hand, the maximum value (in module) for the pseudo-acceleration $\ddot{s}$ is found by solving for the roots of $\dddot{s}(t) = 0$, or

$$6\left(\frac{t}{T}\right)^2 - 6\left(\frac{t}{T}\right) + 1 = 0$$

$$\Rightarrow \qquad t = T_{a1} = \left(0.5 - \frac{\sqrt{3}}{6}\right) T = 0.2113\,T, \qquad t = T_{a2} = \left(0.5 + \frac{\sqrt{3}}{6}\right) T = 0.7887\,T.$$

We obtain

$$\ddot{s}_{max} = |\ddot{s}\,(T_{a1})| = |\ddot{s}\,(T_{a2})| = \frac{60}{T^2}\left| 2\left(0.5 \pm \frac{\sqrt{3}}{6}\right)^3 - 3\left(0.5 \pm \frac{\sqrt{3}}{6}\right)^2 + \left(0.5 \pm \frac{\sqrt{3}}{6}\right)\right| = 0.9238, \tag{15}$$

where $T = 2.5$ [s] was used again. The time behaviors of $s(t)$, $\dot{s}(t)$ and $\ddot{s}(t)$ are shown in Fig. 4.

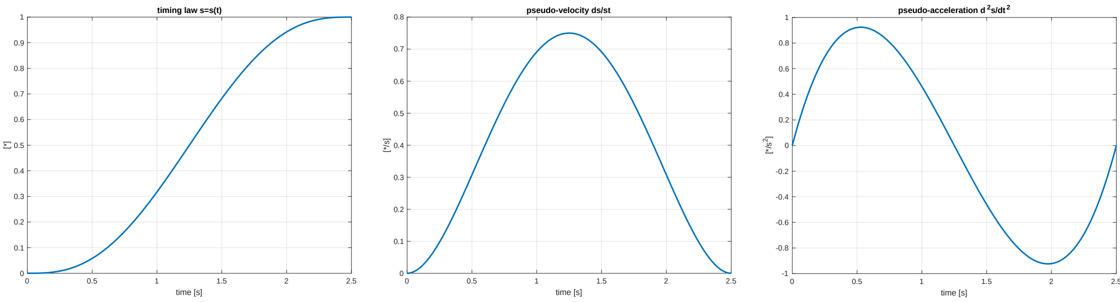


Figure 4: Profile of the timing law $s = s(t)$, with first and second time derivatives. The maximum values of the latter two are $\dot{s}_{max} = 0.75$ and $\ddot{s}_{max} = 0.9238$.

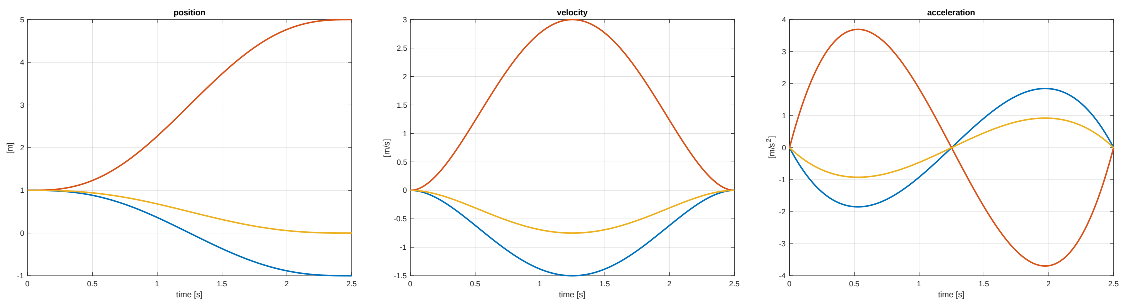


Figure 5: Planned position $\boldsymbol{p}(t)$, velocity $\boldsymbol{v}(t)$, and acceleration $\boldsymbol{a}(t)$ (x-components in blue, y in yellow, z in red).

According to (10–11) and (12–13) and by using (14) and (15), the values of the velocity and acceleration vectors attained at the time instant(s) when they assume their maximum value in norm are

$$\boldsymbol{v}_M = \dot{\boldsymbol{p}}(T_{mid}) = \begin{pmatrix} -1.50 \\ 3.00 \\ -0.75 \end{pmatrix} \text{ [m/s]}, \qquad \boldsymbol{a}_M = \ddot{\boldsymbol{p}}(T_{a1}) = -\ddot{\boldsymbol{p}}(T_{a2}) = \begin{pmatrix} -1.8475 \\ 3.6950 \\ -0.9238 \end{pmatrix} \text{ [m/s}^2\text{]},$$

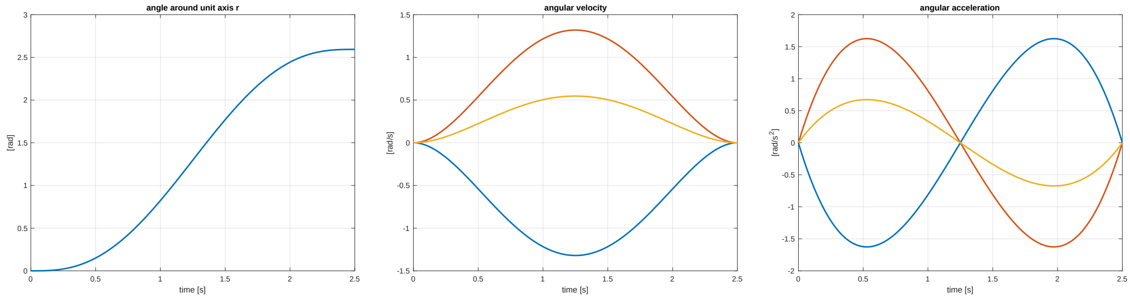


Figure 6: Planned angle $\theta(t) = \theta_{AB}s(t)$, angular velocity $\boldsymbol{\omega}(t) = \theta_{AB}\dot{s}(t)\boldsymbol{r}$, and angular acceleration $\dot{\boldsymbol{\omega}}(t) = \theta_{AB}\ddot{s}(t)\boldsymbol{r}$ (for vectors: x-components in blue, y in yellow, z in red).

see also Fig. 5 for the time behaviors of the single components. Similarly, for the angular velocity and acceleration vectors,

$$\boldsymbol{\omega}_M = \boldsymbol{\omega}(T_{mid}) = \begin{pmatrix} -1.3200 \\ 1.3200 \\ -0.5468 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{\omega}}_M = \dot{\boldsymbol{\omega}}(T_{a1}) = -\dot{\boldsymbol{\omega}}(T_{a2}) = \begin{pmatrix} -1.6258 \\ 1.6258 \\ 0.6734 \end{pmatrix} \text{[rad/s}^2\text{]},$$

see also Fig. 6.

Finally, the absolute orientation at the midpoint of the planned motion (namely, at $t = T_{mid}$, where $\theta_{mid} = \theta(T_{mid}) = \theta_{AB}/2$) is expressed using the rotation matrix of the axis-angle method as

$$\boldsymbol{R}_{mid} = \boldsymbol{R}_A \left(\boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\left(\theta_{AB}/2\right) + \boldsymbol{S}(\boldsymbol{r})\sin\left(\theta_{AB}/2\right)\right) = \begin{pmatrix} -0.0653 & 0.6065 & 0.7924 \\ 0.6065 & -0.6065 & 0.5142 \\ 0.7924 & 0.5142 & -0.3282 \end{pmatrix},$$

where $\boldsymbol{S}(\boldsymbol{r})$ is the skew-symmetric matrix built with $\boldsymbol{r}$.

* * * * *

Robotics 1

July 12, 2021

Exercise #1

Consider the 4-dof spatial RRRP robot in Fig. 1. The robot has a shoulder and an elbow offset.

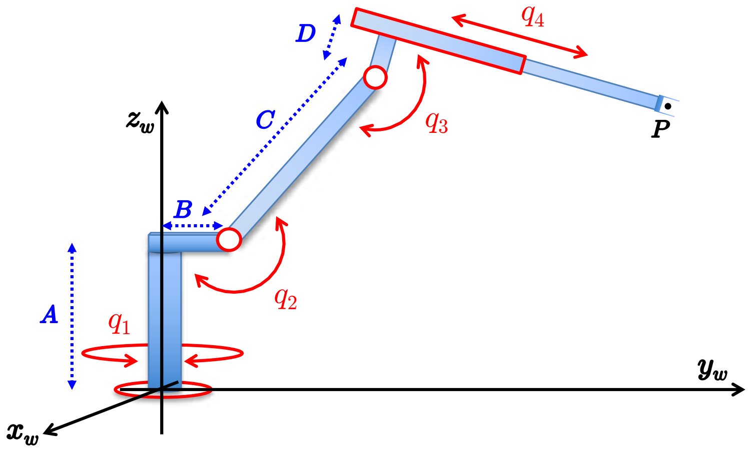


Figure 1: A spatial RRRP robot.

a) Assign a set of Denavit-Hartenberg (DH) frames and derive the associated table of parameters. Place the 0-th DH frame coincident with the world frame RF_w and the last DH frame with the origin in P and the $\boldsymbol{z}$ axis in the approach direction. Draw all the DH frames on the robot. Provide also the (approximate) values of the robot coordinates $\boldsymbol{q}$ in the shown configuration.

b) Compute in symbolic form the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of the end-effector. Derive the analytical Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this map.

c) Neglect from now on the shoulder and elbow offsets (i.e., set $B = D = 0$). For the resulting reduced Jacobian $\boldsymbol{J}_{red}(\boldsymbol{q})$, find (at least) a singular configuration $\boldsymbol{q}^*$. In such a configuration, define a *feasible* end-effector velocity $\boldsymbol{v}^* \in \mathbb{R}^3$ and find a joint velocity $\dot{\boldsymbol{q}}^* \in \mathbb{R}^4$ that realizes it.

Exercise #2

A robot link is actuated by a DC motor with rotor inertia $I_m = 1.5 \cdot 10^{-4}$ [kgm^2] via a double gearbox and a transmission shaft. An incremental encoder with $N = 2000$ pulses/turn is mounted on the motor axis (without extra electronics for quadrature count). A first gearbox with reduction ratio $n_{r1} = 10:1$ is placed at the motor output. This drives a long transmission shaft having rotational inertia $I_t = 0.5 \cdot 10^{-2}$ [kgm^2]. A second gearbox is placed at the end of the shaft with a reduction ratio $n_{r2} \geq 1$ which has *to be defined*. The final payload is the robot link, with an inertia around its rotation axis $I_\ell = 0.8 \cdot 10^{-1}$ [kgm^2]. Neglecting all dissipative effects, model this robot joint structure and find the value n_{r2} that minimizes the motor torque τ_m needed to accelerate the link by $\ddot{\theta}_\ell = a > 0$. Accordingly, determine the angular resolution $\Delta\theta_\ell$ of the link position provided by the encoder measurement on the motor side. For a bang-bang, rest-to-rest trajectory rotating the link by $\theta_{\ell,d} = -\pi/4$ in $T = 0.5$ s, find the maximum absolute value $\tau_{m,max}$ (in [Nm]) of the torque that the motor needs to produce.

Exercise #3

Plan a smooth rest-to-rest trajectory using a minimal representation of the orientation by means of ZYX Euler angles (α, β, γ) from the initial orientation

$$\boldsymbol{R}_1 = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

to the final orientation

$$\boldsymbol{R}_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

passing through the intermediate orientation

$$\boldsymbol{R}_{via} = \begin{pmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}.$$

In the inverse problems, use always the solution with the '+' sign when there is such an option. The time durations for the two subintervals are $T_1 = 2.5$ [s] (from $\boldsymbol{R}_1$ to $\boldsymbol{R}_{via}$) and $T_2 = 1$ [s] (from $\boldsymbol{R}_{via}$ to $\boldsymbol{R}_2$), with total motion time $T = T_1 + T_2$. The planned orientation trajectory should be continuous up to the acceleration for all $t \in (0, T)$ (so, everywhere except at the initial and final instants). At the end of the computations, sketch the time evolution of the three Euler angles $(\alpha(t), \beta(t), \gamma(t))$ and check whether or not a representation singularity is encountered during the planned motion.

[180 minutes (3 hours); open books]

Solution

July 12, 2021

Exercise #1

A possible assignment of Denavit-Hartenberg frames for the 4-dof RRRP robot is shown in Fig. 2. The associated parameters are reported in Tab. 1.

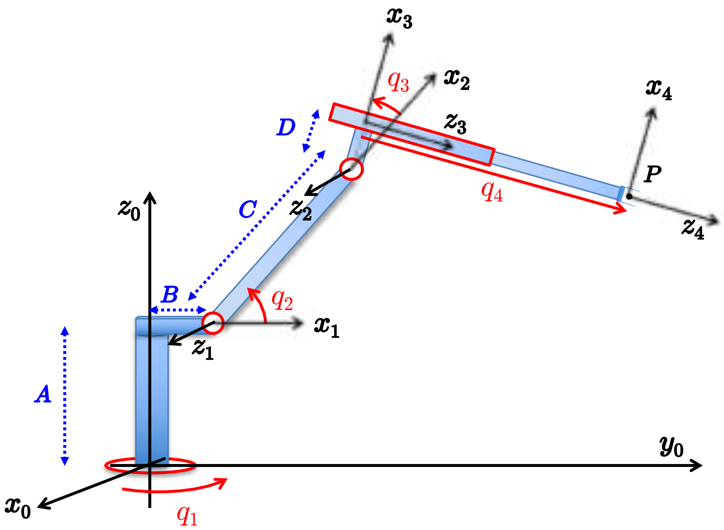


Figure 2: A possible assignment of DH frames for the 4-dof RRRP robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	B	A	$q_1 = \pi/2$
2	0	C	0	$q_2 \simeq \pi/4$
3	$\pi/2$	D	0	$q_3 > 0$
4	0	0	$q_4 > 0$	0

Table 1: DH table of parameters corresponding to Fig. 2. The joint variables q_i (in red) take values associated to the robot configuration shown in the same figure.

With the data in Tab. 1, we construct the homogenous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, \ldots, 4$. The position of the robot end-effector in homogeneous coordinates is efficiently computed as

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)\,{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

yielding

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \left(B + C\cos q_2 + D\cos(q_2+q_3) + q_4 \sin(q_2+q_3)\right) \\ \sin q_1 \left(B + C\cos q_2 + D\cos(q_2+q_3) + q_4 \sin(q_2+q_3)\right) \\ A + C\sin q_2 + D\sin(q_2+q_3) - q_4\cos(q_2+q_3) \end{pmatrix}.$$

Using the usual shorthand notation, the analytic Jacobian is thus

$$
\begin{aligned}
\boldsymbol{J}(\boldsymbol{q}) &= \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \\
&= \begin{pmatrix} -s_1\left(B + Cc_2 + Dc_{23} + q_4 s_{23}\right) & -c_1\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -c_1\left(Ds_{23} - q_4 c_{23}\right) & c_1 s_{23} \\ c_1\left(B + Cc_2 + Dc_{23} + q_4 s_{23}\right) & -s_1\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -s_1\left(Ds_{23} - q_4 c_{23}\right) & s_1 s_{23} \\ 0 & Cc_2 + Dc_{23} + q_4 s_{23} & Dc_{23} + q_4 s_{23} & -c_{23} \end{pmatrix}.
\end{aligned}
\tag{1}
$$

For singularity analysis, it is also very convenient to work with the Jacobian matrix expressed in the (rotated) first DH frame:

$$
\begin{aligned}
{}^1\boldsymbol{J}(\boldsymbol{q}) &= {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}) \\
&= \begin{pmatrix} 0 & -\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -\left(Ds_{23} - q_4 c_{23}\right) & s_{23} \\ B + Cc_2 + Dc_{23} + q_4 s_{23} & 0 & 0 & 0 \\ 0 & Cc_2 + Dc_{23} + q_4 s_{23} & Dc_{23} + q_4 s_{23} & -c_{23} \end{pmatrix}.
\end{aligned}
\tag{2}
$$

By neglecting now the shoulder offset ($B = 0$) and the elbow offset ($D = 0$), we obtain from (1) and (2)

$$
\boldsymbol{J}_{red}(\boldsymbol{q}) = \begin{pmatrix} -s_1\left(Cc_2 + q_4 s_{23}\right) & -c_1\left(Cs_2 - q_4 c_{23}\right) & q_4 c_1 c_{23} & c_1 s_{23} \\ c_1\left(Cc_2 + q_4 s_{23}\right) & -s_1\left(Cs_2 - q_4 c_{23}\right) & q_4 s_1 c_{23} & s_1 s_{23} \\ 0 & Cc_2 + q_4 s_{23} & q_4 s_{23} & -c_{23} \end{pmatrix},
$$

and, respectively,

$$
{}^1\boldsymbol{J}_{red}(\boldsymbol{q}) = \begin{pmatrix} 0 & -\left(Cs_2 - q_4 c_{23}\right) & q_4 c_{23} & s_{23} \\ Cc_2 + q_4 s_{23} & 0 & 0 & 0 \\ 0 & Cc_2 + q_4 s_{23} & q_4 s_{23} & -c_{23} \end{pmatrix}. \tag{3}
$$

The kinematic singularities of the reduced Jacobian are characterized by

$$
\operatorname{rank}\left\{\boldsymbol{J}_{red}(\boldsymbol{q})\right\} = \operatorname{rank}\left\{{}^1\boldsymbol{J}_{red}(\boldsymbol{q})\right\} < 3.
$$

It is easy to see from (3) that the singularities are of two kinds:

(I)	$Cc_2 + q_4 s_{23} = 0$	$\iff$	the end-effector point P is on the axis of joint 1;
(II)	$q_4 = 0$ and $s_3 = 0$	$\iff$	the prismatic joint is fully retracted and link 2 and 4 are orthogonal to each other.

The last question is on generating a joint velocity solution that realizes a desired feasible end-effector velocity in a singular configuration. Two examples are provided next for illustration.

- $\boldsymbol{q}^* = (\pi/2, 0, 0, 0)$ (simple singularity of type II)

  The robot is in the configuration sketched in Fig. 3 [left]. The reduced Jacobian is

$$
\boldsymbol{J}^*_{red} = \boldsymbol{J}_{red}(\boldsymbol{q}^*) = \begin{pmatrix} -C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C & 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank}\left\{\boldsymbol{J}^*_{red}\right\} = 2.
$$

A feasible end-effector velocity $\boldsymbol{v}^* \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}^* \in \mathbb{R}^4$ that will realize it are

$$\boldsymbol{v}^* = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}^*_{red}\} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}^* = \boldsymbol{J}^{\#}_{red}(\boldsymbol{q}^*)\, \boldsymbol{v}^* = \begin{pmatrix} 0 \\ -\frac{C}{C^2+1} \\ 0 \\ \frac{1}{C^2+1} \end{pmatrix},$$

where the pseudoinverse $\boldsymbol{J}^{\#}_{red}$ of $\boldsymbol{J}_{red}$ has been computed numerically using MATLAB (but it is easy to guess its full expression also by inspection). Another possible solution is given simply by $\dot{\boldsymbol{q}}^{*\prime} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$.

- $\boldsymbol{q}^{**} = (\pi/2, \pi/2, 0, 0)$ (double singularity: type I and II together)

  The robot is in the configuration sketched in Fig. 3 [right]. The reduced Jacobian is

$$\boldsymbol{J}^{**}_{red} = \boldsymbol{J}_{red}(\boldsymbol{q}^{**}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\{\boldsymbol{J}^{**}_{red}\} = 1.$$

  A feasible end-effector velocity $\boldsymbol{v}^{**} \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}^{**} \in \mathbb{R}^4$ that will realize it are then

$$\boldsymbol{v}^{**} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}^{**}_{red}\} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}^{**} = \boldsymbol{J}^{\#}_{red}(\boldsymbol{q}^{**})\, \boldsymbol{v}^{**} = \begin{pmatrix} 0 \\ -\frac{C}{C^2+1} \\ 0 \\ \frac{1}{C^2+1} \end{pmatrix} (= \dot{\boldsymbol{q}}^*!),$$

  Another possible solution is given simply by $\dot{\boldsymbol{q}}^{**\prime} = \begin{pmatrix} 0 & -1/C & 0 & 0 \end{pmatrix}^T$.

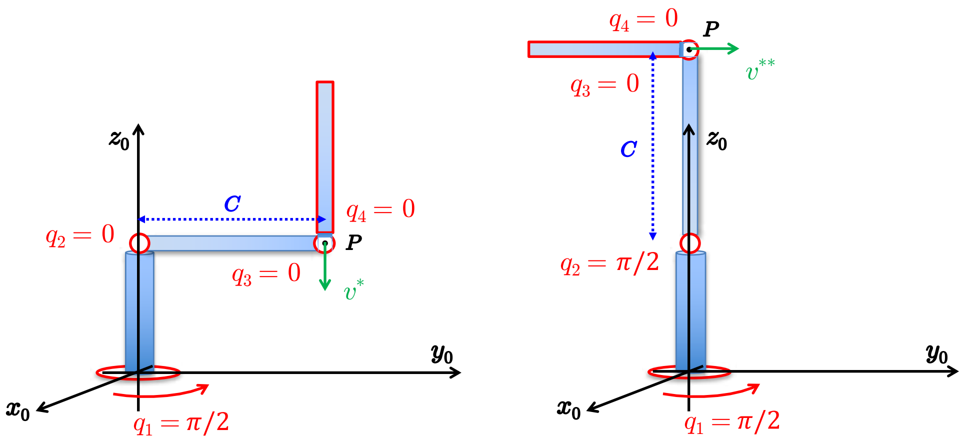


Figure 3: Two singular configurations of the 4-dof RRRP robot with two associated feasible end-effector velocities $\boldsymbol{v}_d$: [left] simple singularity; [right] double singularity.

Exercise #2

Figure 4 shows the motor/sensor/transmission/link arrangement of the considered robot joint.

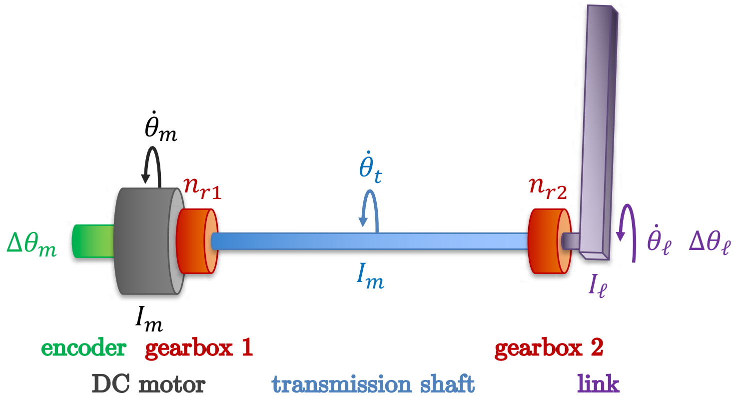


Figure 4: The robot joint with definition of quantities.

The torque balance on the motor axis is given by

$$\tau_m = I_m\ddot\theta_m + \frac{1}{n_{r1}}\left(I_t\ddot\theta_t\right) + \frac{1}{n_{r1}n_{r2}}\left(I_\ell\ddot\theta_\ell\right),$$

whereas the angular velocities are related by

$$\dot\theta_t = n_{r2}\,\dot\theta_\ell, \qquad \dot\theta_m = n_{r1}\,\dot\theta_t = n_{r1}n_{r2}\,\dot\theta_\ell,$$

and similarly for the angular accelerations. Setting now a generic desired acceleration $\ddot\theta_\ell = a > 0$, we obtain

$$\tau_m = I_m\left(n_{r1}n_{r2}\,a\right) + \frac{1}{n_{r1}} I_t\left(n_{r2}\,a\right) + \frac{1}{n_{r1}n_{r2}} I_\ell\,a = \left(\left(I_m\,n_{r1} + \frac{I_t}{n_{r1}}\right)n_{r2} + \frac{I_\ell}{n_{r1}}\,\frac{1}{n_{r2}}\right)a. \qquad (4)$$

The necessary condition for a minimum of τ_m as a function of the unknown n_{r2} is

$$\frac{\partial\,\tau_m}{\partial\,n_{r2}} = \left(\left(I_m\,n_{r1} + \frac{I_t}{n_{r1}}\right) - \frac{I_\ell}{n_{r1}}\,\frac{1}{n_{r2}^2}\right)a = 0,$$

or

$$\left(I_m\,n_{r1} + \frac{I_t}{n_{r1}}\right) - \frac{I_\ell}{n_{r1}}\,\frac{1}{n_{r2}^2} = 0 \qquad \Rightarrow \qquad n_{r2} = \sqrt{\frac{I_\ell}{I_t + I_m\,n_{r1}^2}}.$$

This is indeed also a sufficient condition for a minimum since

$$\frac{\partial^2\tau_m}{\partial\,{n_{r2}}^2} = \frac{2I_\ell}{n_{r1}}\,\frac{1}{n_{r2}^3} > 0.$$

Plugging in the numerical data, we obtain $n_{r2} = 2$. As for the resolution of the angular position of link, we have

$$\Delta\theta_\ell = \frac{1}{n_{r1}n_{r2}}\,\Delta\theta_m = \frac{1}{10\cdot 2}\,\frac{2\pi}{2000} = 1.5708\cdot 10^{-4}\ [\mathrm{rad}] = 0.009°.$$

Finally, the bang-bang angular acceleration $\pm A_{max}$ for a link that moves from rest to rest by $\theta_{\ell,d}$ in time T is obtained from the triangular velocity profile (with peak absolute speed V_{max} at the halftime $t = T/2$) as

$$V_{max} \cdot \frac{T}{2} = \theta_{\ell,d}, \qquad V_{max} = A_{max} \cdot \frac{T}{2} \qquad \Rightarrow \qquad A_{max} = \frac{4\,|\theta_{\ell,d}|}{T^2}.$$

The associated motor torque will also be bang-bang, $\pm\,\tau_{m,max}$, with the maximum absolute value computed setting $a = A_{max}$ in (4). With the numerical data, one obtains

$$A_{max} = 4\pi = 12.5664 \text{ [rad/s}^2\text{]}, \qquad \tau_{m,max} = 0.008 \cdot A_{max} = 0.1005 \text{ [Nm]}.$$

Exercise #3

The three given rotation matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_{via}$, and $\boldsymbol{R}_2$ are first converted into their minimal representation of the orientation by means of ZYX Euler angles (α, β, γ). From the direct mapping

$$\begin{aligned}\boldsymbol{R}_{ZY'X''}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Z(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_X(\gamma)\\ &= \begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix},\end{aligned}$$

we have the inverse solutions, for a given rotation matrix $\boldsymbol{R} = \{R_{ij}\}$ in the regular case, computed in the order

$$\begin{aligned}\beta &= \text{ATAN2}\left\{-R_{31}, +\sqrt{R_{32}^2 + R_{33}^2}\right\},\\ \alpha &= \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\},\\ \gamma &= \text{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\},\end{aligned} \tag{5}$$

where the '+' sign has been used in the expression of β, as requested. Applying (5) to the initial, intermediate, and final rotation matrices yields

$$\begin{array}{lllll} \boldsymbol{R}_1 & \Rightarrow & \alpha_1 = \dfrac{\pi}{2}, & \beta_1 = 0, & \gamma_1 = \dfrac{\pi}{4}, \\ \boldsymbol{R}_{via} & \Rightarrow & \alpha_v = -\dfrac{\pi}{4}, & \beta_v = \dfrac{\pi}{6}, & \gamma_v = \dfrac{\pi}{2}, \\ \boldsymbol{R}_2 & \Rightarrow & \alpha_2 = 0, & \beta_2 = \dfrac{\pi}{4}, & \gamma_2 = \dfrac{3\pi}{4}. \end{array} \tag{6}$$

No singular case ($\cos\beta = 0$, or $\beta = \pm\pi/2$) was found at these orientations.

The problem is to define a smooth interpolating function in time for each of the three Euler angles, with zero boundary velocities. We need thus a simple spline, namely one constituted by only two cubic polynomials. We rewrite these for a generic angle $\theta(t)$ in the two intervals $t \in [0, T_1]$ and $t \in [T_1, T_1 + T_2] = [T_1, T]$, using conveniently the normalized times $\tau_i = t/T_i$, for $i = 1, 2$:

$$\theta(t) = \begin{cases} \theta_1(\tau_1) = a_1\tau_1^3 + b_1\tau_1^2 + \theta_1, & \tau_1 \in [0,1] \\ \theta_2(\tau_2) = a_2\,(\tau_2 - 1)^3 + b_2\,(\tau_2 - 1)^2 + \theta_2, & \tau_2 \in [0,1]. \end{cases} \tag{7}$$

Its first and second derivatives are

$$\dot{\theta}(t) = \begin{cases} \dot{\theta}_1(\tau_1) = \dfrac{1}{T_1}\left(3a_1\tau_1^2 + 2b_1\tau_1\right), & \tau_1 \in [0,1] \\ \dot{\theta}_2(\tau_2) = \dfrac{1}{T_2}\Big(3a_2\,(\tau_2-1)^2 + 2b_2\,(\tau_2-1)\Big), & \tau_2 \in [0,1] \end{cases} \tag{8}$$

and

$$\ddot{\theta}(t) = \begin{cases} \ddot{\theta}_1(\tau_1) = \dfrac{1}{T_1^2}\left(6a_1\tau_1 + 2b_1\right), & \tau_1 \in [0,1] \\ \ddot{\theta}_2(\tau_2) = \dfrac{1}{T_2^2}\left(6a_2\,(\tau_2-1) + 2b_2\right), & \tau_2 \in [0,1]. \end{cases} \tag{9}$$

The cubics in (7) and the quadratics in (8) automatically satisfy the boundary conditions at $t = 0$ and $t = T$, respectively in position ($\theta_1(0) = \theta_1$ and $\theta_2(1) = \theta_2$) and velocity ($\dot{\theta}_1(0) = \dot{\theta}_2(1) = 0$). Considering also the intermediate passage at θ_v and introducing the common (yet to be defined) velocity v at the via point, we impose four more conditions as

$$\begin{aligned} \theta_1(1) &= a_1 + b_1 + \theta_1 = \theta_v \\ \dot{\theta}_1(1) &= \frac{1}{T_1}\left(3a_1 + 2b_1\right) = v \\ \theta_2(0) &= -a_2 + b_2 + \theta_2 = \theta_v \\ \dot{\theta}_2(0) &= \frac{1}{T_2}\left(3a_2 - 2b_2\right) = v, \end{aligned}$$

and solve the resulting (2×2 decoupled) linear system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \theta_v - \theta_1 \\ v\,T_1 \\ \theta_v - \theta_2 \\ v\,T_2 \end{pmatrix}$$

in terms of the spline coefficients

$$\begin{aligned} a_1 &= v\,T_1 - 2\,(\theta_v - \theta_1) \\ b_1 &= 3\,(\theta_v - \theta_1) - v\,T_1 \\ a_2 &= 2\,(\theta_v - \theta_2) + v\,T_2 \\ b_2 &= 3\,(\theta_v - \theta_2) + v\,T_2. \end{aligned} \tag{10}$$

The remaining unknown v is obtained by imposing continuity of acceleration at the via point, i.e.,

$$\ddot{\theta}_1(1) = \frac{1}{T_1^2}\left(6a_1 + 2b_1\right) = \frac{1}{T_2^2}\left(-6a_2 + 2b_2\right) = \ddot{\theta}_2(0),$$

or

$$4\left(\frac{1}{T_1} + \frac{1}{T_2}\right) v = \frac{6\,(\theta_v - \theta_1)}{T_1^2} - \frac{6\,(\theta_v - \theta_2)}{T_2^2}$$

yielding

$$v = \frac{3}{2\,(T_1 + T_2)}\left(\frac{T_2}{T_1}\,(\theta_v - \theta_1) - \frac{T_1}{T_2}\,(\theta_v - \theta_2)\right). \tag{11}$$

By replacing (11) in eqs. (10), and these in (7), we get finally the general solution.

Substituting for θ_1, θ_v, and θ_2 the specific numerical values assigned respectively to α, β, and γ, and using the time intervals $T_1 = 2.5$ and $T_2 = 1$ [s], the following three splines are obtained[1] is for the Euler angles $\alpha(t)$, $\beta(t)$, and $\gamma(t)$:

$$\alpha(t) = \begin{cases} \alpha_1(\tau_1) = \dfrac{207\pi}{112}\,\tau_1^3 - \dfrac{291\pi}{112}\,\tau_1^2 + \dfrac{\pi}{2}, & \tau_1 \in [0,1] \\[2ex] \alpha_2(\tau_2) = -\dfrac{101\pi}{280}\,(\tau_2 - 1)^3 - \dfrac{171\pi}{280}\,(\tau_2 - 1)^2, & \tau_2 \in [0,1], \end{cases}$$

$$\beta(t) = \begin{cases} \beta_1(\tau_1) = -\dfrac{13\pi}{336}\,\tau_1^3 + \dfrac{23\pi}{112}\,\tau_1^2, & \tau_1 \in [0,1] \\[2ex] \beta_2(\tau_2) = -\dfrac{41\pi}{840}\,(\tau_2 - 1)^3 - \dfrac{37\pi}{280}\,(\tau_2 - 1)^2 + \dfrac{\pi}{4}, & \tau_2 \in [0,1], \end{cases}$$

$$\gamma(t) = \begin{cases} \gamma_1(\tau_1) = \dfrac{31\pi}{112}\,\tau_1^3 - \dfrac{3\pi}{112}\,\tau_1^2 + \dfrac{\pi}{4}, & \tau_1 \in [0,1] \\[2ex] \gamma_2(\tau_2) = -\dfrac{53\pi}{280}\,(\tau_2 - 1)^3 - \dfrac{123\pi}{280}\,(\tau_2 - 1)^2 + \dfrac{3\pi}{4}, & \tau_2 \in [0,1]. \end{cases}$$

The plots of these three splines are reported in Fig. 5. The cubic polynomials in the two intervals are drawn in different colors (blue for the first, red for the second). From the evolution of $\beta(t)$ it is clear that the singularity $\beta = \pm\pi/2 \simeq \pm 1.57$ [rad] of the ZYX Euler representation is never encountered during the planned motion.

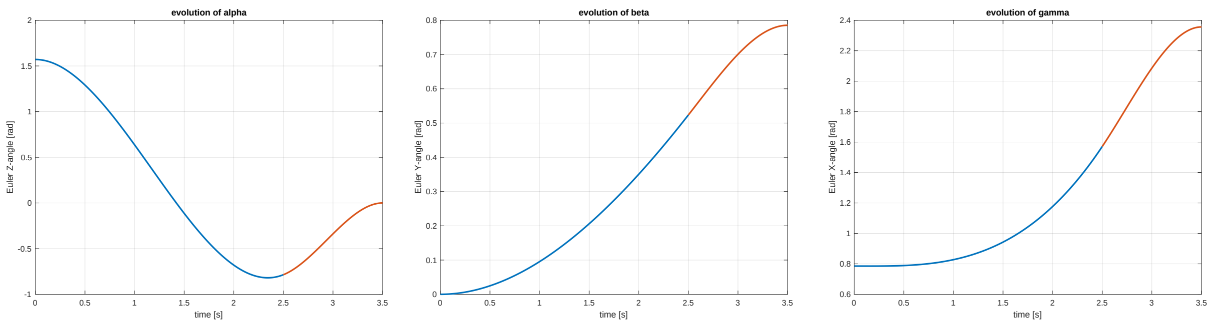


Figure 5: The time evolution of the three interpolating Euler angles $\alpha(t)$, $\beta(t)$, and $\gamma(t)$.

* * * * *

[1] This result has been generated by a symbolic code in MATLAB. Therefore, infinite precision arithmetic is used. Indeed, the same formulas are obtained by a purely numerical code.

Robotics 1

September 10, 2021

Exercise #1

Consider the 3-dof planar PRR robot in Fig. 1, with the joint coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The second and third links have a common length $L > 0$. The robot performs three-dimensional tasks that involve the position $\boldsymbol{p} = (p_x, p_y)$ of its end-effector point $\boldsymbol{P}$ and the orientation angle α of the end-effector w.r.t. the axis $\boldsymbol{x}_0$.

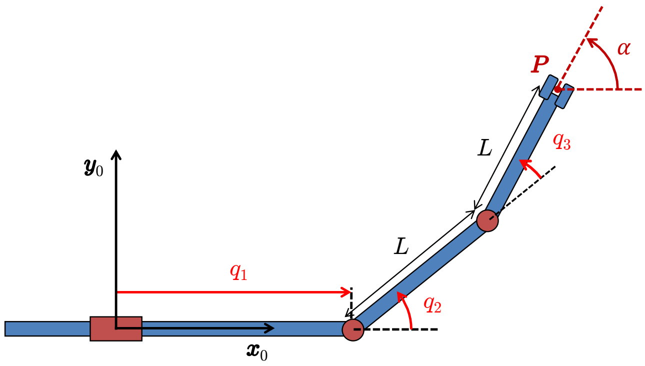


Figure 1: A planar PRR robot.

a) Determine the direct task kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ between $\boldsymbol{q} = (q_1, q_2, q_3)$ and $\boldsymbol{r} = (p_x, p_y, \alpha)$. Derive the task Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of the map $\boldsymbol{f}(\boldsymbol{q})$ and find all singularities $\boldsymbol{q}_s$ of this 3×3 matrix.

b) When the robot is in a singular configuration $\boldsymbol{q}_s$ (choose one at will), determine:

- a null-space joint velocity $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$;
- a task velocity $\dot{\boldsymbol{r}}_1 \in \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ and an associated joint velocity $\dot{\boldsymbol{q}}$ that realizes it;
- an unfeasible task velocity $\dot{\boldsymbol{r}}_2 \notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$;
- a generalized task force $\boldsymbol{F}_0 = (F_x, F_y, M_z)$ applied at the end effector that is statically balanced by joint forces/torques $\boldsymbol{\tau} = \boldsymbol{0}$.

c) Find a closed-form expression for the inverse task kinematics $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{r}_d)$, whenever at least a solution exist. Compute then the numerical value of all inverse solutions for $L = 0.5$ [m] and when $\boldsymbol{r}_d = (0.3, 0.7, \pi/3)$ [m,m,rad].

d) Draw the primary and secondary workspaces for this robot, when the prismatic joint has a finite range $q_1 \in [0, L]$ while the revolute joints have unlimited range.

Exercise #2

For the same PRR robot in Fig. 1 (with a generic value L for link lengths), determine a smooth, coordinated rest-to-rest joint trajectory $\boldsymbol{q}_d(t)$ that will move the robot in T seconds from the initial value $\boldsymbol{r}_i = (2L, 0, \pi/4)$ of the task vector to the final value $\boldsymbol{r}_f = (2L, 0, -\pi/4)$, *without* ever changing the position $\boldsymbol{p}_d = (2L, 0)$ of the point $\boldsymbol{P}$. Sketch a plot of the obtained joint trajectory $\boldsymbol{q}_d(t) = (q_{1d}(t), q_{2d}(t), q_{3d}(t))$.

[180 minutes (3 hours); open books]

Solution

September 10, 2021

Exercise #1

The direct kinematics of the task is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_1 + L\left(\cos q_2 + \cos(q_2+q_3)\right) \\ L\left(\sin q_2 + \sin(q_2+q_3)\right) \\ q_2+q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

The task Jacobian is thus

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -L\left(\sin q_2 + \sin(q_2+q_3)\right) & -L\sin(q_2+q_3) \\ 0 & L\left(\cos q_2 + \cos(q_2+q_3)\right) & L\cos(q_2+q_3) \\ 0 & 1 & 1 \end{pmatrix}. \tag{2}$$

The singularities occur when

$$\det \boldsymbol{J}(\boldsymbol{q}) = L\cos q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \pm\frac{\pi}{2}. \tag{3}$$

The condition (3) is easy to interpret in terms of loss of mobility. When the second link is orthogonal to the first one, the linear motion of the prismatic joint and the rotation of the second joint both produce linear contributions to the end-effector motion restricted to the $\boldsymbol{x}_0$ direction. If the third joint is used to impose a desired rotation of the end effector around the $\boldsymbol{z}_0$ axis, there is no remaining freedom for achieving instantaneously also a non-zero velocity along $\boldsymbol{y}_0$. The robot end effector has lost its full mobility in the task space and we are thus in a singularity.

We set now $\boldsymbol{q}_s = (*, \pi/2, q_3)$, where $*$ denotes an arbitrary value. The task Jacobian becomes

$$\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} 1 & -L\left(1+\cos q_3\right) & -L\cos q_3 \\ 0 & -L\sin q_3 & -L\sin q_3 \\ 0 & 1 & 1 \end{pmatrix}, \tag{4}$$

with $\operatorname{rank}\{\boldsymbol{J}(\boldsymbol{q}_s)\} = 2$. All joint velocities in the null space of $\boldsymbol{J}(\boldsymbol{q}_s)$ are expressed as

$$\dot{\boldsymbol{q}}_0 = \beta \begin{pmatrix} L \\ 1 \\ -1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}, \ \forall\beta \qquad \Longleftrightarrow \qquad \boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_0 = \boldsymbol{0}.$$

Thus, null-space motions always involve all three joints. A basis for the two-dimensional range space of $\boldsymbol{J}(\boldsymbol{q}_s)$ is

$$\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L\sin q_3 \\ 1 \end{pmatrix} \right\}. \tag{5}$$

The complementary space to $\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}$ in $\mathbb{R}^3$ is the one-dimensional subspace

$$\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}^{\perp} = \mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q}_s)\right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ L\sin q_3 \end{pmatrix} \right\}. \tag{6}$$

Note that the three basis vectors in (5) and (6) are linearly independent for all $\boldsymbol{q}$.

A task velocity vector $\dot{\boldsymbol{r}}$ that belongs to the subspace in (5) and an associated joint velocity $\dot{\boldsymbol{q}}$ that realizes it are given by

$$\dot{\boldsymbol{r}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_1 = \boldsymbol{J}^{\#}(\boldsymbol{q}_s)\dot{\boldsymbol{r}}_1 = \frac{1}{L^2+2}\begin{pmatrix} 2 \\ -L \\ L \end{pmatrix},$$

where the minimum norm solution was obtained by using the pseudoinverse of $\boldsymbol{J}(\boldsymbol{q}_s)$. Indeed, it is easy to verify that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_1 = \dot{\boldsymbol{r}}_1$. We provide also a second example where, for simplicity, a numerical value is specified also for q_3. Choose, e.g., $\boldsymbol{q}_{ss} = (*, \pi/2, -\pi/2)$. Then

$$\boldsymbol{J}(\boldsymbol{q}_{ss}) = \begin{pmatrix} 1 & -L & 0 \\ 0 & L & L \\ 0 & 1 & 1 \end{pmatrix}, \qquad \operatorname{rank}\left\{\boldsymbol{J}(\boldsymbol{q}_{ss})\right\} = 2, \tag{7}$$

and

$$\dot{\boldsymbol{r}}_{11} = \alpha\begin{pmatrix} 0 \\ L \\ 1 \end{pmatrix} \in \mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_{ss})\right\},\ \forall\alpha \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_{11} = \boldsymbol{J}^{\#}(\boldsymbol{q}_{ss})\dot{\boldsymbol{r}}_{11} = \frac{\alpha}{L^2+2}\begin{pmatrix} L \\ 1 \\ L^2+1 \end{pmatrix}.$$

Again, $\boldsymbol{J}(\boldsymbol{q}_{ss})\dot{\boldsymbol{q}}_{11} = \dot{\boldsymbol{r}}_{11}$. On the other hand, a task velocity $\dot{\boldsymbol{r}}$ that is always unfeasible in the configuration $\boldsymbol{q}_{ss}$ is given by

$$\dot{\boldsymbol{r}}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_{ss})\right\}.$$

In this case, the minimum norm solution given by the pseudoinverse of $\boldsymbol{J}(\boldsymbol{q}_{ss})$,

$$\dot{\boldsymbol{q}}_2 = \boldsymbol{J}^{\#}(\boldsymbol{q}_{ss})\dot{\boldsymbol{r}}_2 = \begin{pmatrix} \dfrac{2L^2+L+2}{L^4+3L^2+2} \\ -\dfrac{L^3+L-1}{L^4+3L^2+2} \\ \dfrac{L+1}{L^2+2} \end{pmatrix},$$

does never return the original task vector:

$$\boldsymbol{J}(\boldsymbol{q}_{ss})\dot{\boldsymbol{q}}_2 = \frac{1}{L^2+1}\begin{pmatrix} 1 \\ L \\ 1 \end{pmatrix} \neq \dot{\boldsymbol{r}}_2.$$

As another example, consider the task velocity

$$\dot{\boldsymbol{r}}_{22} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_{ss})\right\}, \quad \text{if } L \neq 1.$$

This velocity vector is also unfeasible at $\boldsymbol{q}_{ss}$, unless the link lengths are unitary ($L = 1$). In fact,

$$\dot{\boldsymbol{q}}_{22} = \boldsymbol{J}^{\#}(\boldsymbol{q}_{ss})\dot{\boldsymbol{r}}_{22} = \begin{pmatrix} \dfrac{L(L+1)}{L^4+3L^2+2} \\ -\dfrac{L+1}{L^4+3L^2+2} \\ \dfrac{L+1}{L^2+2} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}(\boldsymbol{q}_{ss})\dot{\boldsymbol{q}}_{22} = \frac{1}{L^2+1}\begin{pmatrix} 0 \\ L(L+1) \\ L+1 \end{pmatrix} = \dot{\boldsymbol{r}}_2|_{L=1}.$$

Finally, a generalized task force $\boldsymbol{F} = (F_x, F_y, M_z)$ that is statically balanced by $\boldsymbol{\tau} = \boldsymbol{0}$ at the joint level belongs to the null space of $\boldsymbol{J}^T(\boldsymbol{q}_s)$ (or of $\boldsymbol{J}^T(\boldsymbol{q}_{ss})$, if we assign also a numerical value to q_3). From (6), we have

$$\boldsymbol{F}_0 = \gamma \begin{pmatrix} 0 \\ 1 \\ L\sin q_3 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q}_s)\right\}, \ \forall\gamma \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}_s)\boldsymbol{F}_0 = \boldsymbol{0}.$$

In fact, for a generic q_3, the momentum $M_z = L\sin q_3$ applied to the last robot link is balanced at joint 3 by the torque produced there by the force $F_y = 1$ applied at the tip, resulting in $\tau_3 = 0$[1]. Moreover, the force F_y produces no torque τ_2 at joint 2, since the second link is vertical, and no force τ_1 at joint 1, being orthogonal to it.

Consider next the inverse kinematics problem for the PRR robot when performing the specified three-dimensional task. Given a desired $\boldsymbol{r} = \boldsymbol{r}_d = (p_{xd}, p_{yd}, \alpha_d)$, we set in (1)

$$q_2 + q_3 = \alpha_d. \tag{8}$$

By reorganizing, squaring and summing the first two equations in (1), we obtain

$$(p_{xd} - q_1 - L\cos\alpha_d)^2 + (p_{yd} - L\sin\alpha_d)^2 = (L\cos q_2)^2 + (L\sin q_2)^2 = L^2.$$

Expanding the left-hand side and simplifying, we get a second order polynomial equation in q_1:

$$q_1^2 - 2\left(p_{xd} - L\cos\alpha_d\right)q_1 + \left(p_{xd}^2 + p_{yd}^2 - 2L\left(p_{xd}\cos\alpha_d + p_{yd}\sin\alpha_d\right)\right) = 0.$$

The two solutions of this equation are

$$q_{1d} = p_{xd} - L\cos\alpha_d \pm \sqrt{L^2\cos^2\alpha_d + 2L\sin\alpha_d\, p_{yd} - p_{yd}^2}. \tag{9}$$

Indeed, a (real) solution q_{1d} exists if and only if the argument of the square root in (9) is non-negative. This argument vanishes for $p_{yd} = L\sin\alpha_d \pm L$ (i.e., the two solutions of a second, auxiliary quadratic equation in p_{yd}) and is (strictly) positive for

$$p_{yd} \in (L\sin\alpha_d - L, L\sin\alpha_d + L). \tag{10}$$

At the boundaries of this interval, the two values of q_{1d} collapse into a single solution. Not surprisingly, the existence of a solution depends on a relation between the desired orientation α_d and the y-position p_{yd} of the end-effector. For instance, if $\alpha_d = \pi/2$, then (at least) a solution exists for $p_{yd} \in [0, 2L]$; if $\alpha_d = -\pi/2$, a solution exists for $p_{yd} \in [-2L, 0]$. The value of p_{xd} plays no role in this analysis, as long as there is no limit to the range of the prismatic joint q_1 (see also the

[1] For $q_3 \in (0, \pi)$, M_z is positive (counterclockwise) and the torque at joint 3 produced by F_y is negative (clockwise).

following workspace analysis). For each solution q_{1d} in (9), consider again the first two equations in (1) and, by using (8), solve for q_2 as

$$q_{2d} = \text{ATAN2}\left\{\frac{p_{yd}}{L} - \sin\alpha_d, \frac{p_{xd} - q_{1d}}{L} - \cos\alpha_d\right\}. \tag{11}$$

Finally,

$$q_{3d} = \alpha_d - q_{2d}. \tag{12}$$

Therefore, (at most) two solutions $\boldsymbol{q}_d$ are found in closed form by using eqs. (9), (11) and (12). Evaluating the inverse kinematics with the data $L = 0.5$ and $\boldsymbol{r}_d = (0.3, 0.7, \pi/3)$ provides the two regular solutions

$$\boldsymbol{q}_d^{(i)} = \begin{pmatrix} q_{1d}^{(i)} \\ q_{2d}^{(i)} \\ q_{3d}^{(i)} \end{pmatrix} = \begin{pmatrix} 0.4728 \\ 2.5783 \\ -1.5311 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q}_d^{(ii)} = \begin{pmatrix} q_{1d}^{(ii)} \\ q_{2d}^{(ii)} \\ q_{3d}^{(ii)} \end{pmatrix} = \begin{pmatrix} -0.3728 \\ 0.5633 \\ 0.4839 \end{pmatrix} \text{[m,rad,rad]}.$$

At last, Fig. 2 shows the primary and secondary workspaces for this robot, taking into account the finite range $q_1 \in [0, L]$ of the prismatic joint. As usual, the primary workspace WS_1 is the set of points in $\mathbb{R}^2$ that can be reached with at least one of the admissible orientations (in the plane) of the robot end effector. A point $P \in WS_1$ belongs also to the secondary workspace WS_2 if it can be reached with *all* the admissible orientations of the end effector. In the present case, this happens only for points on the (green) segment OD in Fig. 2. If there were no bounds on the range of q_1, both WS_1 and WS_2 would expand limitless along the positive and negative $\boldsymbol{x}_0$ direction (WS_1 would be an infinite horizontal stripe of height $4L$).

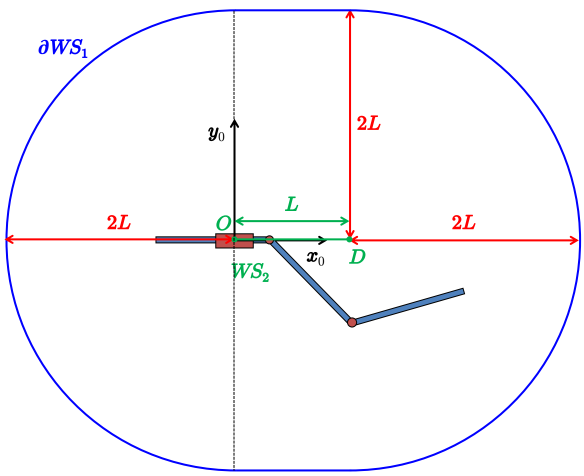


Figure 2: The primary workspace WS_1 (with boundary ∂WS_1 in blue) and the secondary workspace WS_2 (the line from O to D in green) of the PRR robot in Fig. 1.

A remark is in order on the relation between the definition of the above robot workspaces and the number of solutions to the inverse kinematics of the considered task. Although in general these are two different problems (e.g., the task of a robot may or may not involve the end-effector orientation), few simple observations can be made in the present setting:

- outside WS_1 there is no solution for the task $\boldsymbol{r} = \boldsymbol{r}_d$;

- on the boundary ∂WS_1, there is at most a single solution to the task (this happens when the desired orientation α_d takes a single special value at each $\boldsymbol{p}_d \in \partial WS_1$);
- in the interior $\overline{WS}_1$, there are at most two solutions to the task, depending on the satisfaction of the relation (10) between p_{yd} and α_d;
- when $\boldsymbol{p}_d \in WS_2$, there is always at least a solution to the task, for any value of α_d;
- in any case, solutions may be discarded by the presence of a limited range for the prismatic joint (i.e., if $q_{1d} \notin [0, L]$, as computed by eq. (9)), as well as by finite ranges of the revolute joints.

Exercise #2

This trajectory planning problem in the joint space of the PRR robot takes advantage of the availability of a closed form solution for the inverse task kinematics, as obtained in Exercise #1, but it is also greatly simplified by the particular symmetry of the data in the given problem. With reference to Fig. 3, we shall plan first a smooth trajectory for $\alpha_d(t)$ which, according to (8), will also be the trajectory for the sum of the two joint angles $q_{2d}(t) + q_{3d}(t)$. However, by the symmetries of the task, $q_{2d}(t) = -\alpha_d(t)$ and so $q_{3d}(t) = 2\alpha_d(t)$. Since the robot end effector point P has to remain at rest in the constant position $\boldsymbol{p}_d = \boldsymbol{p}_i = \boldsymbol{p}_f$, for all $t \in [0, T]$, the tip position $\boldsymbol{p}_2$ of the second link will trace an arc of a circle (with an absolute speed equal to $|\dot{\alpha}_d(t)|$). Taking into account the obtained trajectory $q_{2d}(t)$, this motion is realized by an oscillatory motion of $q_{1d}(t)$ that will move the base of link 2 accordingly. Note that all joint trajectories will behave symmetrically w.r.t. to the midtime $T/2$. Obviously, the same behavior is obtained from the closed-form solution of the inverse task kinematics in Exercise #1, but the previous analysis is simpler and does not presume the availability of such expressions.

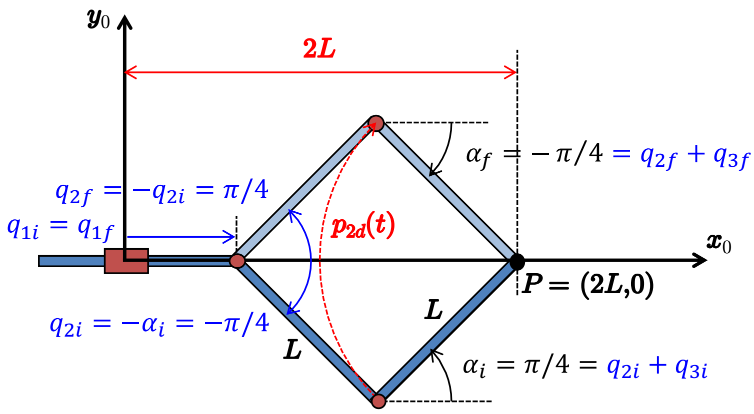


Figure 3: The given trajectory planning problem has symmetries in space (in particular, w.r.t. the axis $\boldsymbol{x}_0$) and in time.

With the above in mind, we plan a cubic[2] rest-to-rest trajectory for α:

$$\alpha_d(t) = \alpha_i + (\alpha_f - \alpha_i)\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad t \in [0, T],$$

with

$$\dot{\alpha}_d(t) = \frac{\alpha_f - \alpha_i}{T}\left(6\left(\frac{t}{T}\right) - 6\left(\frac{t}{T}\right)^2\right), \qquad \ddot{\alpha}_d(t) = \frac{\alpha_f - \alpha_i}{T^2}\left(6 - 12\left(\frac{t}{T}\right)\right).$$

[2]Also a quintic polynomial could have been used, wishing to start and end the motion with zero acceleration.

Substituting the initial and final values for α, we have

$$\alpha_d(t) = \frac{\pi}{4} - \frac{\pi}{2}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad t \in [0, T],$$

The desired trajectory of the tip of the second link is

$$\boldsymbol{p}_{2d}(t) = \boldsymbol{p}_d - L\begin{pmatrix}\cos\alpha_d(t)\\ \sin\alpha_d(t)\end{pmatrix} = \begin{pmatrix}2L\\ 0\end{pmatrix} - L\begin{pmatrix}\cos\alpha_d(t)\\ \sin\alpha_d(t)\end{pmatrix}, \qquad t \in [0, T].$$

Taking advantage of the symmetries, we obtain then

$$\begin{aligned} q_{2d}(t) &= -\alpha_d(t),\\ q_{3d}(t) &= \alpha_d(t) - q_{2d}(t) = 2\alpha_d(t), \qquad\qquad t \in [0, T].\\ q_{1d}(t) &= p_{xd} - L\left(\cos\alpha_d(t) + \cos q_{2d}(t)\right) = 2L\left(1 - \cos\alpha_d(t)\right), \end{aligned} \tag{13}$$

Figure 4 shows the evolution in normalized time $\tau = t/T \in [0, 1]$ of the components of the planned joint trajectory $\boldsymbol{q}_d(\tau)$ obtained by (13) and of those of the resulting task trajectory $\boldsymbol{r}_d(\tau)$, as computed by the direct kinematics (1). For these plots, a link length $L = 0.5$ [m] has been chosen. Note that p_{xd} and p_{yd} remain constant at their initial value, as desired.

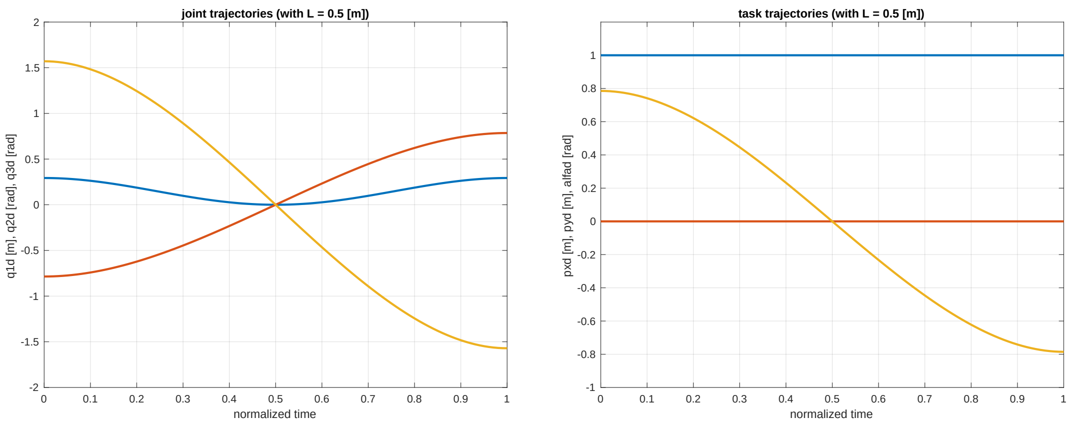


Figure 4: Joint trajectory $\boldsymbol{q}_d(\tau) = (q_{1d}(\tau), q_{2d}(\tau), q_{3d}(\tau))$ [blue, red, yellow] from (13) and associated task trajectory $\boldsymbol{r}_d(\tau) = (p_{xd}(\tau), p_{yd}(\tau), \alpha_d(\tau))$ [blue, red, yellow] in normalized time.

For comparison, use the given data in the closed-form expressions (9), (11) and (12) of the inverse task kinematics. These yield:

$$\begin{aligned} q_{1d}(t) &= 2L - L\cos\alpha_d \pm \sqrt{L^2\cos^2\alpha_d} = \begin{cases} 2L\\ 2L\left(1 - \cos\alpha_d(t)\right), \end{cases}\\ q_{2d}(t) &= \text{ATAN2}\left\{-\sin\alpha_d(t), \frac{2L - q_{1d}(t)}{L} - \cos\alpha_d(t)\right\}\\ &= \text{ATAN2}\left\{-\sin\alpha_d(t), \mp\cos\alpha_d(t)\right\} = \begin{cases} \alpha_d(t) - \pi\\ -\alpha_d(t), \end{cases} \qquad t \in [0, T].\\ q_{3d}(t) &= \alpha_d(t) - q_{2d}(t) = \begin{cases} \pi\\ 2\alpha_d(t), \end{cases} \end{aligned} \tag{14}$$

It is apparent that a second, alternative solution is available: the first joint remains at rest, placing the base of the second link in P; the second joint rotates as $\alpha_d(t)$, modulo an angular displacement of $-\pi$; the third joint is also fixed, with the third link folded on the second, so that the position of the robot end effector is always constant and equal to $\boldsymbol{p}_d$. Figure 5 shows the results when using for $\boldsymbol{q}_d(\tau)$ the alternative solution in (14) (and again, with $L = 0.5$ [m]). Indeed, the resulting task trajectory $\boldsymbol{r}_d(\tau)$ is the same.

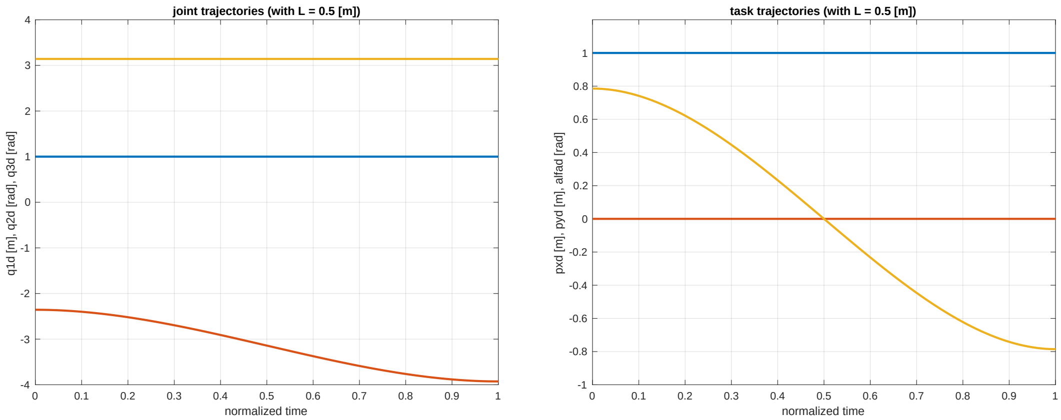


Figure 5: Alternative joint trajectory $\boldsymbol{q}_d(\tau) = (q_{1d}(\tau), q_{2d}(\tau), q_{3d}(\tau))$ [blue, red, yellow] from (14) and associated task trajectory $\boldsymbol{r}_d(\tau) = (p_{xd}(\tau), p_{yd}(\tau), \alpha_d(\tau))$ [blue, red, yellow] in normalized time.

* * * * *

Robotics 1

October 19, 2021

Exercise #1

Consider the 3-dof PPR robot in Fig. 1, with a jaw gripper mounted on the end effector.

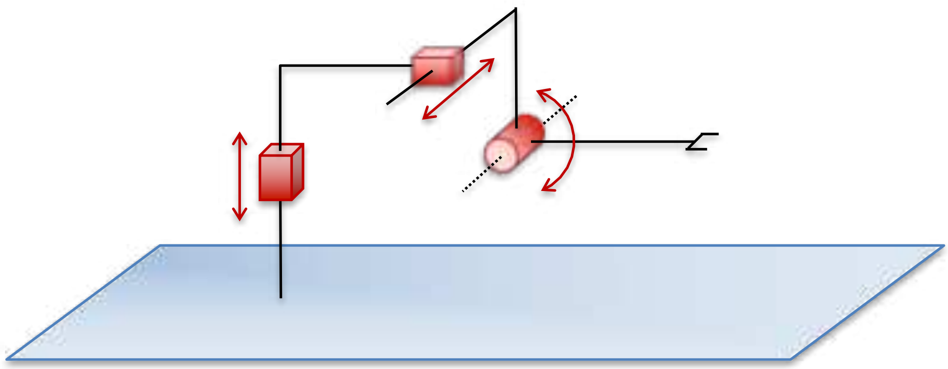


Figure 1: A 3-dof PPR robot.

a) Assign and draw the robot frames according to the Denavit-Hartenberg (DH) convention. Place the origin of frame 0 on the floor and the origin of the last frame at the center of the gripper. Compile the associated table of DH parameters.

b) Check whether the last DH frame assigned coincides in orientation with the definition of the standard frame $(\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{a})$ attached to a jaw gripper. If not, determine the rotation matrix ${}^3\boldsymbol{R}_g$ needed to align the two frames.

c) Provide the expression of the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ between $\boldsymbol{q} = (q_1, q_2, q_3)$ and the position $\boldsymbol{p} = (p_x, p_y, p_z)$ of the center of the gripper.

d) Derive the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating $\dot{\boldsymbol{q}}$ to the linear velocity $\boldsymbol{v} = \dot{\boldsymbol{p}}$ in two different ways, as part of the geometric Jacobian of the robot and using differentiation w.r.t. time.

e) Find all the singular configurations of matrix $\boldsymbol{J}(\boldsymbol{q})$. In one of such configurations $\boldsymbol{q}_s$, characterize which Cartesian directions are instantaneously accessible by the robot gripper and which not.

Exercise #2

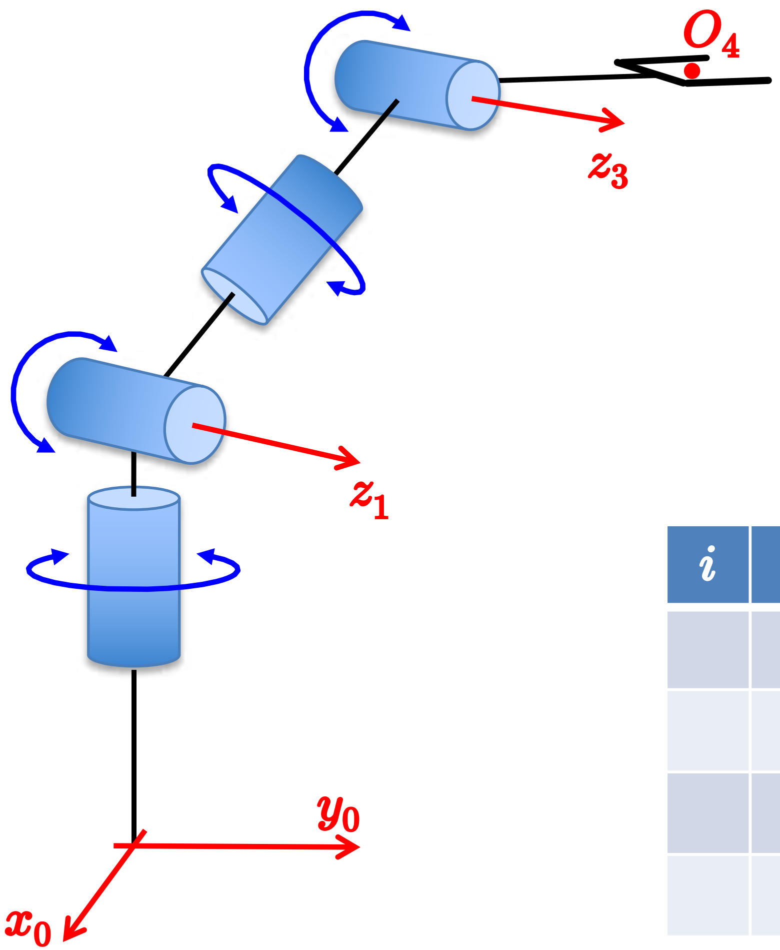
For the robot in Fig. 1, using the associated symbolic DH parameters, determine a smooth and coordinated rest-to-rest joint trajectory that will move in T seconds the robot gripper from the initial position $\boldsymbol{p}_i = (a_1 + a_3, 0, 0)$ to the final position $\boldsymbol{p}_f = (a_1, -\delta, 0)$, with $\delta > 0$. Sketch a plot of the obtained joint trajectory $\boldsymbol{q}_d(t) = (q_{1d}(t), q_{2d}(t), q_{3d}(t))$. What will be the maximum value of the norm of the joint velocity $\|\dot{\boldsymbol{q}}_d(t)\|$ during the interval $[0, T]$?

[120 minutes (2 hours); open books]

Robotics 1 – Extra sheet for Question 3

November 19, 2021

Last name: ....



i	α_i	a_i	d_i	θ_i

Robotics 1

Midterm Test – November 19, 2021

The test has 9 questions. Provide as many answers as you can, with short but significant texts and formulas/tables/pictures. Please write clearly. Take a picture of each of your handwritten answers and upload them to Exam.net *before submitting. Try to follow the same order of the questions. Number your answers accordingly (don't repeat the text of the questions).*

Question #1

A rigid body is rotated first by an angle $\theta = \pi/3$ around the unit vector $\boldsymbol{r} = (1/\sqrt{3}) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and then by an angle $\phi = -\pi/3$ around the fixed $\boldsymbol{y}$-axis. What is the final orientation of the body?

Question #2

An initial orientation $\boldsymbol{R}_i$ and a final orientation $\boldsymbol{R}_f$ are defined by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \boldsymbol{I}.$$

Find the two sequences of ZYZ Euler angles that represent the rotation from $\boldsymbol{R}_i$ to $\boldsymbol{R}_f$.

Question #3

For the 4R robot with a spherical shoulder of Fig. 1, complete the assignment of Denavit-Hartenberg (D-H) frames and fill in the associated table of parameters *[for this, use the extra sheet distributed]*. Keep the quantities that are already defined in the figure unchanged. If needed, provide the transformation between the last D-H frame and the standard frame of an end-effector gripper.

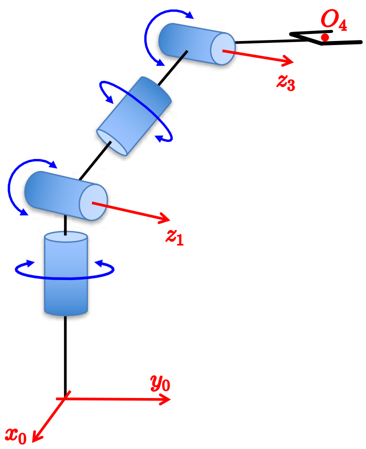


Figure 1: A 4R spatial robot arm with a spherical shoulder.

Question #4

A 2R planar robot with links of equal length L has limited joint ranges as follows: $q_1 \in [-\pi/2, \pi/2]$, $q_2 \in [0, \pi/2]$. Draw the primary workspace $WS_1 \in \mathbb{R}^2$. For $L = 1.4$ [m], is the point $P = (1.6, -0.2)$ reachable by the robot end effector?

Question #5

A branched two-arm planar robot having 5 dofs is sketched in Fig. 2, with generic labels for the link lengths and the actual definition of the joint angles. The sign convention for angles is the usual one (i.e., positive if counterclockwise). Determine the relative pose of the end-effector frame of the left arm with respect to that of the right arm, as expressed by the 4×4 homogeneous matrix ${}^{rE}\boldsymbol{T}_{lE}(\boldsymbol{q})$ with $\boldsymbol{q} = (\theta_0, \theta_{r1}, \theta_{r2}, \theta_{l1}, \theta_{l2})$. Check numerically the obtained symbolic expression when all the links have equal and unitary length and the two-arm robot is in the configuration $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$ —the right arm is horizontal and the left one is vertical and upward.

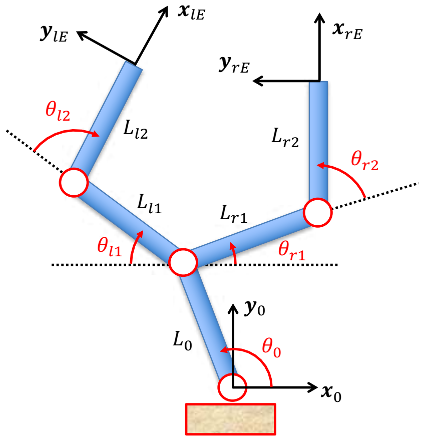


Figure 2: A two-arm robot with 5 dofs.

Question #6

Figure 3 shows a planar RRP robot, with the definition of its joint variables. The task of interest is specified by the position $\boldsymbol{p} = (p_x, p_y)$ of the robot end effector and by the orientation α of the forearm w.r.t. the $\boldsymbol{x}$-axis. The associated direct kinematics is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}).$$

Determine the analytic solutions to the inverse kinematics problem. Disregard any situation that is unfeasible or singular. Provide at least one solution for the following (feasible) input data: $l_1 = 1$ [m], $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad].

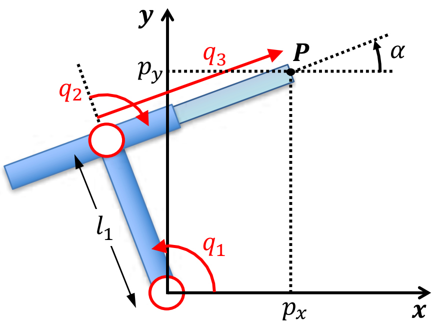


Figure 3: A planar RRP robot.

Question #7

With reference to Fig. 4, a motor with inertia J_M drives a link through a gear with toothed wheels (a photo of this is also shown in the figure). The wheel on the motor shaft (aka, the *pinion*) has radius $r_M = 2$ [cm], while the radius of the wheel on the link rotation axis is $r_L = 10$ [cm]. The link has inertia $J_L = 0.3$ [kgm^2] around its rotation axis. Assuming that an optimal inertia matching is realized by the reduction ratio of this transmission, determine the torque τ_M that the motor needs to produce around its $\boldsymbol{z}_M$ axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2]. Neglect dissipative effects as well as the inertia of the transmission components (and of the encoder).

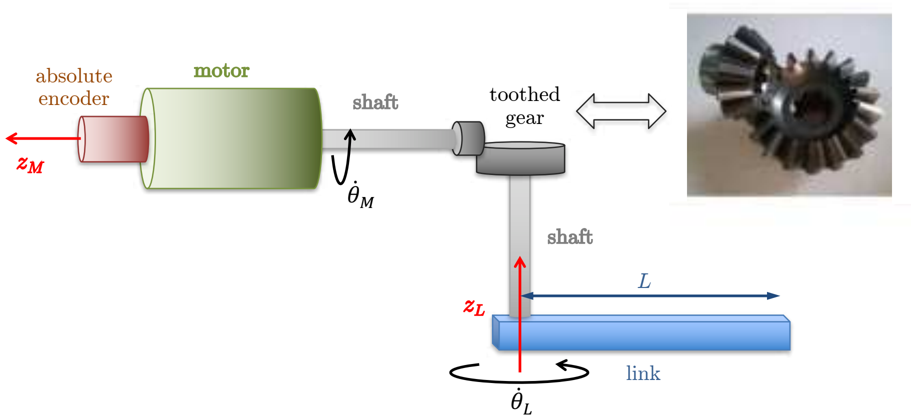


Figure 4: Set up of a motor-transmission-link system using a toothed gear.

Question #8

An absolute encoder is mounted on the motor of the system shown in Fig. 4. If the link length is $L = 0.5$ [m], determine the minimum number of tracks n_t that the encoder needs to have in order to achieve at least a resolution of $\delta = 0.1$ [mm] at the link tip.

Question #9

Explain in exactly three short sentences the specific feature of a SCARA-type robot, its common technical implementation, and the significance in industrial applications.

[150 minutes (2.5 hours); open books]

Solution

November 19, 2021

Question #1

A rigid body is rotated first by an angle $\theta = \pi/3$ around the unit vector $\boldsymbol{r} = (1/\sqrt{3}) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and then by an angle $\phi = -\pi/3$ around the fixed $\boldsymbol{y}$-axis. What is the final orientation of the body?

Reply #1

The rotation matrix associated to an axis/angle representation $(\boldsymbol{r}, \theta)$ is

$$\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{r}^T\boldsymbol{r} + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta,$$

while the elementary rotation by an angle ϕ around the coordinate axis Y is represented by

$$\boldsymbol{R}_Y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}.$$

Being the sequence of two rotations defined around the fixed axes $\boldsymbol{r}$ and $\boldsymbol{y}$, the final orientation is given by product (in the reverse order)

$$\boldsymbol{R}_{\boldsymbol{r},\boldsymbol{y}} = \boldsymbol{R}_Y\left(-\frac{\pi}{3}\right)\boldsymbol{R}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{\pi}{3}\right) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6220 & -0.7440 & -0.2440 \\ 0.6667 & 0.6667 & -0.3333 \\ 0.4107 & 0.0447 & 0.9107 \end{pmatrix}. \qquad \blacksquare$$

Question #2

An initial orientation $\boldsymbol{R}_i$ and a final orientation $\boldsymbol{R}_f$ are defined by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \boldsymbol{I}.$$

Find the two sequences of ZYZ Euler angles that represent the rotation from $\boldsymbol{R}_i$ to $\boldsymbol{R}_f$.

Reply #2

One has to solve the inverse problem of the ZYZ Euler representation with angles $(\alpha_1, \alpha_2, \alpha_3)$ for the relative rotation matrix

$${}^i\boldsymbol{R}_f = \boldsymbol{R}_i^T\boldsymbol{R}_f = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}^T \cdot \boldsymbol{I} = \begin{pmatrix} 0 & -1 & 0 \\ 0.5 & 0 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & 0.5 \end{pmatrix}. \qquad (1)$$

The symbolic expression of the ZYZ Euler rotation matrix is

$$
\begin{aligned}
&\boldsymbol{R}_{ZYZ}(\alpha_1,\alpha_2,\alpha_3) = \boldsymbol{R}_Z(\alpha_1)\boldsymbol{R}_Y(\alpha_2)\boldsymbol{R}_Z(\alpha_3)\\
&= \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0\\ \sin\alpha_1 & \cos\alpha_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2\\ 0 & 1 & 0\\ -\sin\alpha_2 & 0 & \cos\alpha_2\end{pmatrix}\begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0\\ \sin\alpha_3 & \cos\alpha_3 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \begin{pmatrix} \cos\alpha_1\cos\alpha_2\cos\alpha_3-\sin\alpha_1\sin\alpha_3 & -\cos\alpha_1\cos\alpha_2\sin\alpha_3-\sin\alpha_1\cos\alpha_3 & \cos\alpha_1\sin\alpha_2\\ \sin\alpha_1\cos\alpha_2\cos\alpha_3+\cos\alpha_1\sin\alpha_3 & \cos\alpha_1\cos\alpha_3-\sin\alpha_1\cos\alpha_2\sin\alpha_3 & \sin\alpha_1\sin\alpha_2\\ -\sin\alpha_2\cos\alpha_3 & \sin\alpha_2\sin\alpha_3 & \cos\alpha_2\end{pmatrix}.
\end{aligned}
\tag{2}
$$

Denote by R_{hk} the elements of the ${}^i\boldsymbol{R}_f$ matrix in (1). The inverse formulas for the ZYZ Euler representation can be extracted from the simpler elements in the last row and column of the $\boldsymbol{R}_{ZYZ}$ matrix in (2). Since

$$\sin^2\alpha_2 = R_{31}^2 + R_{32}^2 = 0.75 > 0,$$

this is a regular case and there are two solutions. These are computed, e.g., by the Matlab code

```
alfa2=atan2(sqrt(R(3,1)^2+R(3,2)^2),R(3,3))
alfa2bis=-alfa2
alfa1=atan2(R(2,3)/sin(alfa2),R(1,3)/sin(alfa2))
alfa1bis=atan2(R(2,3)/sin(alfa2bis),R(1,3)/sin(alfa2bis))
alfa3=atan2(R(3,2)/sin(alfa2),-R(3,1)/sin(alfa2))
alfa3bis=atan2(R(3,2)/sin(alfa2bis),-R(3,1)/sin(alfa2bis))
```

yielding

$$(\alpha_1,\alpha_2,\alpha_3) = (1.5708, 1.0472, 0) \qquad \text{and} \qquad (\alpha_1',\alpha_2',\alpha_3') = (-1.5708, -1.0472, 3.1416).$$

It is always good to check the result by plugging each of these two triples into (2) and verifying that the obtained rotation matrix is equal to ${}^i\boldsymbol{R}_f$. ■

Question #3

For the 4R robot with a spherical shoulder of Fig. 1, complete the assignment of Denavit-Hartenberg (D-H) frames and fill in the associated table of parameters [for this, use the extra sheet distributed]. *Keep the quantities that are already defined in the figure unchanged. If needed, provide the transformation between the last D-H frame and the standard frame of an end-effector gripper.*

Reply #3

A possible complete assignment of D-H frames for the robot of Fig. 1 is shown in Fig. 5. The associated set of parameters is given in Table 1, where the sign of the constant parameters is also indicated. The last D-H frame does not (and cannot) have its axis $\boldsymbol{z}_4$ along the approach direction of the gripper. Therefore, an extra rotation matrix

$${}^4\boldsymbol{R}_E = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

is needed to align RF_4 with the standard frame RF_E of an end-effector gripper. ■

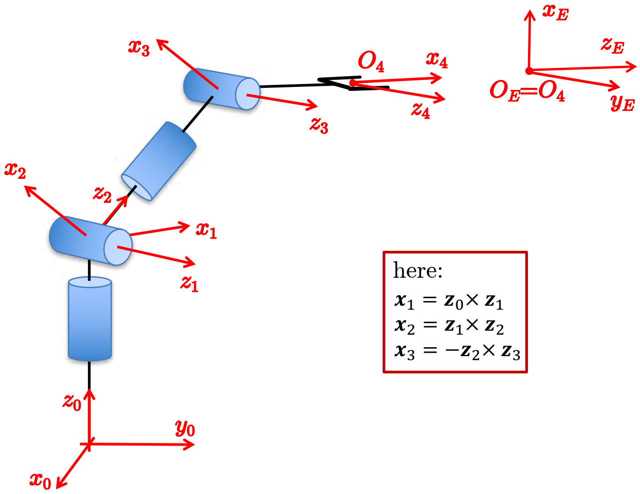


Figure 5: A possible assignment of D-H frames for the 4R robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	$d_3 > 0$	q_3
4	0	$a_4 > 0$	0	q_4

Table 1: Table of D-H parameters associated to the frame assignment in Fig. 5.

Question #4

A 2R planar robot with links of equal length L has limited joint ranges as follows: $q_1 \in [-\pi/2, \pi/2]$, $q_2 \in [0, \pi/2]$. Draw the primary workspace $WS_1 \in \mathbb{R}^2$. For $L = 1.4$ [m], is the point $P = (1.6, -0.2)$ reachable by the robot end effector?

Reply #4

For the given joint ranges, the primary workspace of a 2R robot with equal links of length L is shown in Fig. 6. Note that q_1 and q_2 are defined (by default, if not specified otherwise) according to the D-H convention. The boundaries of WS_1 are drawn with red dotted lines. The inner boundary is made by two parts, a half circumference of radius $L\sqrt{2}$ and a quarter circumference of radius L. Also the outer boundary is made by two parts, a quarter circumference of radius L and a half circumference of radius $2L$. If the link length is set to $L = 1.4$, the point P is out of WS_1. This is shown as well (in the proper scale) in Fig. 6. ■

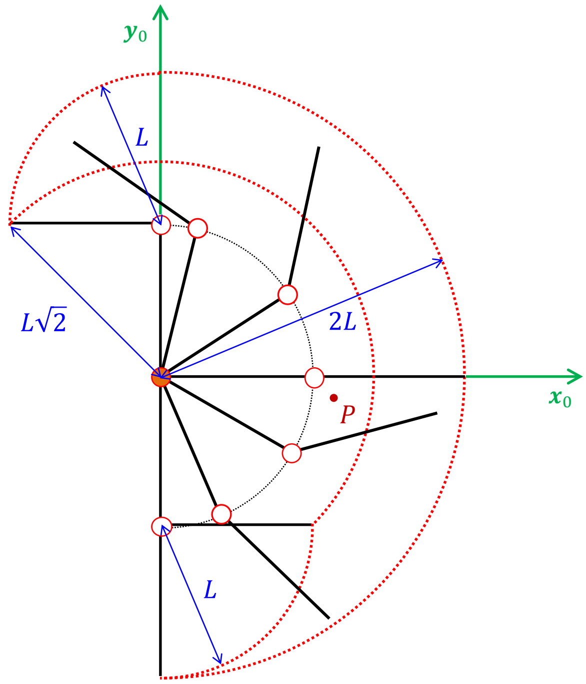


Figure 6: The primary workspace WS_1 of the 2R robot for the given joint limits.

Question #5

A branched two-arm planar robot having 5 dofs is sketched in Fig. 2, with generic labels for the link lengths and the actual definition of the joint angles. The sign convention for angles is the usual one (i.e., positive if counterclockwise). Determine the relative pose of the end-effector frame of the left arm with respect to that of the right arm, as expressed by the 4×4 *homogeneous matrix* ${}^{rE}\boldsymbol{T}_{lE}(\boldsymbol{q})$ *with* $\boldsymbol{q} = (\theta_0, \theta_{r1}, \theta_{r2}, \theta_{l1}, \theta_{l2})$*. Check numerically the obtained symbolic expression when all the links have equal and unitary length and the two-arm robot is in the configuration* $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$ *—the right arm is horizontal and the left one is vertical and upward.*

Reply #5

The result is obtained by computing the direct kinematics of each branch using the 4×4 homogeneous transformation matrices, and then finding the relative pose. The problem is planar (in the plane $z_0 = 0$), and so all rotations will be defined by an angle around the $\boldsymbol{z}_0$-axis normal to the plane. Note also that the joint angles in Fig. 2 are not necessarily part of a D-H convention, but they should be used as defined. For instance, the orientation angles of the right and left forearms are defined w.r.t. the positive and, respectively, negative $\boldsymbol{x}_0$-axis. For the right arm, the position of the end-effector frame RF_{rE} is then given by

$$
{}^0\boldsymbol{p}_r = L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} + L_{r1} \begin{pmatrix} \cos\theta_{r1} \\ \sin\theta_{r1} \\ 0 \end{pmatrix} + L_{r2} \begin{pmatrix} \cos(\theta_{r1} + \theta_{r2}) \\ \sin(\theta_{r1} + \theta_{r2}) \\ 0 \end{pmatrix},
$$

while its orientation is parametrized by the angle

$$
\phi_r = \theta_{r1} + \theta_{r2}.
$$

Similarly, for the left arm we have

$$
\begin{aligned}
{}^0\boldsymbol{p}_l &= L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} + L_{l1} \begin{pmatrix} \cos(\pi+\theta_{l1}) \\ \sin(\pi+\theta_{l1}) \\ 0 \end{pmatrix} + L_{l2} \begin{pmatrix} \cos(\pi+\theta_{l1}+\theta_{l2}) \\ \sin(\pi+\theta_{l1}+\theta_{l2}) \\ 0 \end{pmatrix} \\
&= L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} - L_{l1} \begin{pmatrix} \cos\theta_{l1} \\ \sin\theta_{l1} \\ 0 \end{pmatrix} - L_{l2} \begin{pmatrix} \cos(\theta_{l1}+\theta_{l2}) \\ \sin(\theta_{l1}+\theta_{l2}) \\ 0 \end{pmatrix}
\end{aligned}
$$

and

$$\phi_l = \pi + \theta_{r1} + \theta_{r2}.$$

The addition of π in the angular expressions pertaining to the left arm is needed in order to express the quantities in terms of the common base reference frame RF_0. As a result, from

$$
{}^0\boldsymbol{T}_r = \begin{pmatrix} {}^0\boldsymbol{R}_r(\phi_r) & {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} \cos(\theta_{r1}+\theta_{r2}) & -\sin(\theta_{r1}+\theta_{r2}) & 0 \\ \sin(\theta_{r1}+\theta_{r2}) & \cos(\theta_{r1}+\theta_{r2}) & 0 \\ 0 & 0 & 1 \end{matrix} & {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix}
$$

and

$$
{}^0\boldsymbol{T}_l = \begin{pmatrix} {}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{p}_r(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} -\cos(\theta_{l1}+\theta_{l2}) & \sin(\theta_{l1}+\theta_{l2}) & 0 \\ -\sin(\theta_{l1}+\theta_{l2}) & -\cos(\theta_{l1}+\theta_{l2}) & 0 \\ 0 & 0 & 1 \end{matrix} & {}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

one obtains

$$
\begin{aligned}
{}^r\boldsymbol{T}_l = {}^0\boldsymbol{T}_r^{-1} \cdot {}^0\boldsymbol{T}_l &= \begin{pmatrix} {}^0\boldsymbol{R}_r^T(\phi_r) & -{}^0\boldsymbol{R}_r^T(\phi_r){}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} {}^0\boldsymbol{R}_r^T(\phi_r)\,{}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{R}_r^T(\phi_r)\left({}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) - {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2})\right) \\ \boldsymbol{0}^T & 1 \end{pmatrix}
\end{aligned}
$$

or[1]

$$
{}^r\boldsymbol{T}_l(\boldsymbol{q}) =
$$

$$
\begin{pmatrix}
-\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & \sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & 0 & \begin{matrix} -L_{r2} - L_{r1}\cos\theta_{r2} \\ -L_{l1}\cos(\theta_{l1}-\theta_{r1}-\theta_{r2}) \\ -L_{l2}\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) \end{matrix} \\
-\sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & -\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & 0 & \begin{matrix} L_{r1}\sin\theta_{r2} - L_{l1}\sin(\theta_{l1}-\theta_{r1}-\theta_{r2}) \\ -L_{l2}\sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) \end{matrix} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

[1] A MATLAB program yielding the simplified final output is reported in Appendix 1 at the end of the solution.

When the homogeneous transformation matrix ${}^{r}\boldsymbol{T}_l(\boldsymbol{q})$ is evaluated at $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$, using the numerical data of the problem (all links of equal and unitary length), we obtain

$$
{}^{r}\boldsymbol{T}_l\left(\boldsymbol{q}^*\right) = \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

It is easy to check that this corresponds to the robot configuration with the right arm horizontal (pointing to the right) and the left arm vertical (pointing upward). The distance between the end effectors is $d = \|\boldsymbol{p}_l - \boldsymbol{p}_r\| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$. Similarly, at $\boldsymbol{q}^{**} = (\pi/2, \pi/4, \pi/4, -\pi/4, -\pi/4)$ —a robot posture that is fully symmetric w.r.t. the $\boldsymbol{y}_0$-axis, we have

$$
{}^{r}\boldsymbol{T}_l\left(\boldsymbol{q}^{**}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

In this case, there is a displacement $d = \sqrt{2}$ between the two end effectors just along the (now, horizontal) direction $\boldsymbol{y}_{rE}$, and no relative rotation between the two end-effector frames. ■

Question #6

Figure 3 shows a planar RRP robot, with the definition of its joint variables. The task of interest is specified by the position $\boldsymbol{p} = (p_x, p_y)$ of the robot end effector and by the orientation α of the forearm w.r.t. the $\boldsymbol{x}$-axis. The associated direct kinematics is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}). \tag{3}
$$

Determine the analytic solutions to the inverse kinematics problem. Disregard any situation that is unfeasible or singular. Provide at least one solution for the following (feasible) input data: $l_1 = 1$ [m], $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad].

Reply #6

The closed-form solution to the inverse kinematics of the RRP robot in Fig. 3 is found as follows. Let $\boldsymbol{r} = \boldsymbol{r}_d = \begin{pmatrix} p_{xd} & p_{yd} & \alpha_d \end{pmatrix}^T$ in (3). Substituting $q_1 + q_2 = \alpha_d$ from the last equation in (3) as argument of the two functions c_{12} and s_{12} in the first two equations, rearranging terms, and then squaring and summing, we obtain

$$
\left(p_{xd} - q_3 \cos\alpha_d\right)^2 + \left(p_{yd} - q_3 \sin\alpha_d\right)^2 = l_1^2\left(c_1^2 + s_1^2\right) = l_1^2.
$$

Developing the squares, we find a second-order polynomial equation in the single unknown q_3:

$$
q_3^2 - 2\left(p_{xd}\cos\alpha_d + p_{yd}\sin\alpha_d\right) q_3 + \left(p_{xd}^2 + p_{xd}^2 - l_1^2\right) = 0. \tag{4}
$$

Equation (4) has two real roots (regular case) if and only if its discriminant is

$$
\Delta = \left(p_{xd}\cos\alpha_d + p_{yd}\sin\alpha_d\right)^2 - \left(p_{xd}^2 + p_{xd}^2 - l_1^2\right) \geq 0.
$$

Note that Δ can be rewritten also in the following two equivalent forms

$$
\Delta = l_1^2 - p_{xd}^2 \sin^2\alpha_d - p_{yd}^2 \cos^2\alpha_d + p_{xd} p_{yd} \sin 2\alpha_d = l_1^2 - \left(p_{xd}\sin\alpha_d - p_{yd}\cos\alpha_d\right)^2.
$$

If $\Delta = 0$, the two real roots are indeed coincident (one inverse kinematics solution only, a singular case); if $\Delta < 0$, the two roots of (4) are complex conjugate and there is no solution to the inverse kinematics problem[2]. In the regular case, the two solutions for q_3 are given by

$$q_3^{\pm} = p_{xd} \cos \alpha_d + p_{yd} \sin \alpha_d \pm \sqrt{\Delta}. \tag{5}$$

The associated values for q_1 and q_2 are found from the second and third equation in (3) and, respectively, from the third equation as

$$q_1^{\pm} = \text{atan2}\left\{p_{yd} - q_3^{\pm} \sin \alpha_d, p_{xd} - q_3^{\pm} \cos \alpha_d\right\}, \qquad q_2^{\pm} = \alpha_d - q_1^{\pm}. \tag{6}$$

With a length $l_1 = 1$ of the first link and for the (regular) input data $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad], we obtain from eqs. (5–6)

$$(q_1, q_2, q_3)^+ = (-2.4836, 3.0072, 3.2230) \quad \text{[rad,rad,m]}$$

and

$$(q_1, q_2, q_3)^- = (0.3892, 0.1344, 1.2411) \quad \text{[rad,rad,m]}.$$

The two solutions are shown in Fig. 7 (with the angular values approximated in degrees). ■

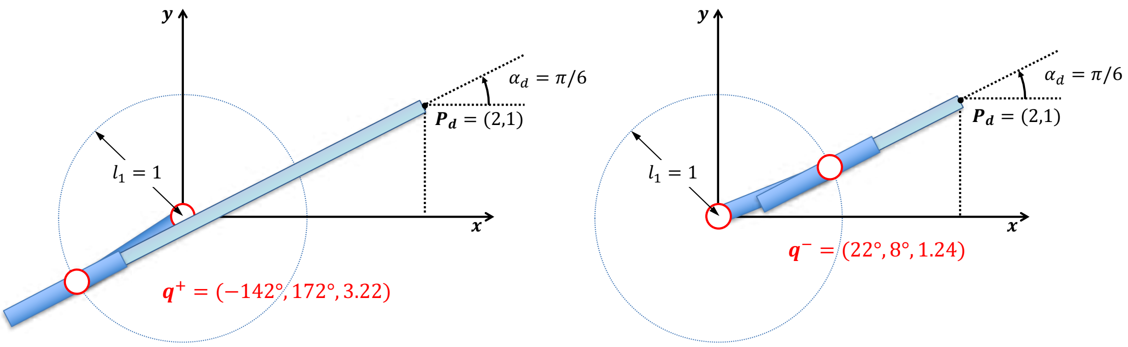


Figure 7: The two inverse kinematics solutions of the RRP robot for the given data.

Question #7

With reference to Fig. 4, a motor with inertia J_M drives a link through a gear with toothed wheels (a photo of this is also shown in the figure). The wheel on the motor shaft (aka, the pinion*) has radius $r_M = 2$ [cm], while the radius of the wheel on the link rotation axis is $r_L = 10$ [cm]. The link has inertia $J_L = 0.3$ [kgm^2] around its rotation axis. Assuming that an optimal inertia matching is realized by the reduction ratio of this transmission, determine the torque τ_M that the motor needs to produce around its $\boldsymbol{z}_M$ axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2]. Neglect dissipative effects as well as the inertia of the transmission components (and of the encoder).*

Reply #7

The reduction ratio of the transmission with toothed wheels is $n_r = r_L/r_M = 10/2 = 5$. On the other hand, the missing information about the motor inertia J_M is recovered from the assumed optimal inertia matching of the transmission:

$$n_r = n_r^* = \sqrt{\frac{J_L}{J_M}} \qquad \Rightarrow \qquad J_M = \frac{J_L}{n_r^2} = 0.012 \quad [\text{kgm}^2].$$

[2]More on these two cases is reported in Appendix 2 at the end of the solution (as additional material not requested in the solution answer).

Therefore, the torque that the motor needs to deliver on its axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2] is given by

$$\tau_M = J_M \left(\ddot{\theta}_L n_r\right) + \frac{1}{n_r} J_L \ddot{\theta}_L = -0.6 \text{ [Nm]}.$$

Note that the motor and the link rotate in the same direction (positive if CCW) as seen from $\boldsymbol{z}_M$ and $\boldsymbol{z}_L$, respectively. So, there is no inversion of the sense of rotation in this toothed gear (as opposed to the planar case). ■

Question #8

An absolute encoder is mounted on the motor of the system shown in Fig. 4. If the link length is $L = 0.5$ [m], determine the minimum number of tracks n_t that the encoder needs to have in order to achieve at least a resolution of $\delta = 0.1$ [mm] at the link tip.

Reply #8

The requested angular resolution at the link base and on the motor side of the transmission are

$$\delta_L = \arctan\left(\frac{\delta}{L}\right) \simeq \frac{\delta}{L} = \frac{0.1}{500} = 2 \cdot 10^{-4} \text{ [rad]} \qquad \Rightarrow \qquad \delta_M = \delta_L n_r = 0.001 \text{ [rad]}.$$

Thus, the minimum number of tracks n_t of the absolute encoder in order to get the required resolution is

$$n_t = \left\lceil \log_2\left(\frac{2\pi}{\delta_M}\right)\right\rceil = \lceil 12.6173 \rceil = 13 \text{ tracks.}$$ ■

Question #9

Explain in exactly three short sentences the specific feature of a SCARA-type robot, its common technical implementation, and the significance in industrial applications.

Reply #9

[Sample reply] SCARA stands for Selective Compliance Arm for Robotic Assembly, a robot having 4 joints with vertical axes, the third one being prismatic and the others revolute. The end-effector compliance is present only along horizontal directions, and is usually provided by an harmonic drive on the motor axis of the first joint and by a transmission belt for driving the second joint by another motor mounted also on the first joint. This allows to accommodate in a passive way the lateral forces that may arise in assembly tasks, when the vertical insertion direction is not sufficiently accurate. ■

Appendix 1

A MATLAB code for Question #5

```
clear all; clc;
syms th0 thr1 thr2 thl1 thl2 L0 Lr1 Lr2 Ll1 Ll2 real
disp('right arm')
pr=L0*[cos(th0);sin(th0);0]+Lr1*[cos(thr1);sin(thr1);0]...
   +Lr2*[cos(thr1+thr2);sin(thr1+thr2);0]
phir=thr1+thr2
```

```
Rr=[cos(phir) -sin(phir) 0
    sin(phir)  cos(phir) 0
      0          0       1]
Tr=[  Rr       pr;
    zeros(1,3)  1]
disp('left arm')
pl=L0*[cos(th0);sin(th0);0]+Ll1*[cos(pi+thl1);sin(pi+thl1);0]...
   +Ll2*[cos(pi+thl1+thl2);sin(pi+thl1+thl2);0];
pl=simplify(pl)
phil=pi+thl1+thl2
Rl=[cos(phil) -sin(phil) 0
    sin(phil)  cos(phil) 0
      0          0       1];
Rl=simplify(Rl);
Tl=[  Rl       pl;
    zeros(1,3)  1]
disp('relative homogeneous transformation')
T_rl=simplify(inv(Tr)*Tl)
% data
L0=1;Lr1=1;Lr2=1;Ll1=1;Ll2=1;
disp('numerical evaluation for the given data')
th0=pi/2;thr1=0;thr2=0;thl1=-pi/2;thl2=0;
T_r_l=subs(T_rl)
disp('numerical evaluation in symmetric conditions')
th0=pi/2;thr1=pi/4;thr2=pi/4;thl1=-pi/4;thl2=-pi/4;
T_r_l=subs(T_rl)
% end
```

Appendix 2

Geometric view on the solution of the IK problem for the RRP robot

We present here an extra analysis that pursues more in depth the answer to Question #6.

It can be recognized that, in general, the inverse kinematics (IK) problem for our RRP robot has one, two, or no solution depending on the existence or not of intersections between a line L (with orientation) and a circumference C in the $\mathbb{R}^2$ plane. In fact, this geometric view is equivalent to the algebraic analysis of the roots of the second-order polynomial equation (4). With reference to Fig. 8 (pay attention also to the color codes used), the line L is defined by the point P_d and by the desired direction $\alpha_d \in (-\pi, \pi)$ of the third robot link, as computed from the horizontal $\boldsymbol{x}$-axis (positive if CCW). The circumference C is centered at the origin and has radius equal to the length l_1 of the first link.

The circumference C characterizes a transition from one type of solutions to another. If the point P_d is outside the circumference C (i.e., for $p_{xd}^2 + p_{yd}^2 > l_1^2$), there may be two, one or no intersection between L and C. Figure 8(a) shows a *regular* case with two intersections, and thus two inverse kinematics solutions $\boldsymbol{q}'$ and $\boldsymbol{q}''$, both having q_3' and q_3'' positive. The same figure shows also two *singular* cases, when the line L is tangent to C: there is only one IK solution in each case, but again with a positive q_3. Figure 8(b) shows the same cases considered in (a), but now with desired orientations α_d that differ by $\pm\pi$ from before. The situation is specular w.r.t. case (a), and the values of q_3 are now negative in all (single or double) solutions. Whenever $q_3 < 0$, note that we adopted a dashed line to represent the (retracted) forearm of the RRP robot.

When P_d is outside C, the line L may also not intersect C (*no solution* to the IK problem). Figure 8(c) shows one such instance. The other two cases in the figure refer to when P_d is inside ($p_{xd}^2 + p_{yd}^2 < l_1^2$) or on ($p_{xd}^2 + p_{yd}^2 = l_1^2$) the circumference C: both situations lead to two *regular* solutions to the IK problem. When the point P_d is *strictly* inside the circumference C, there are *always* two intersections of L with C, and thus two inverse solutions. However, in this case the two solution values of q_3 will have different signs. Finally, Fig. 8(d) suggests that some caution should be used when P_d is on C. In fact, two distinct (regular) inverse solutions will exist unless the orientation α_d is tangent to C, in which case they collapse into a single one with $q_3 = 0$ (again, a singular case). Figure 8 summarizes geometrically all the above possible situations.

Note that all singular solutions (i.e., when the cardinality of the solution set drops to 1 for a given input $\boldsymbol{r}_d$) will occur for $\cos q_2 = 0$. This corresponds to a singularity of the 3×3 analytic Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}_r(\boldsymbol{q})/\partial \boldsymbol{q})$ associated to the direct kinematics (3). The Jacobian matrix arises when considering the differential kinematics of a robot, namely the mapping from joint to task velocities (and vice versa).

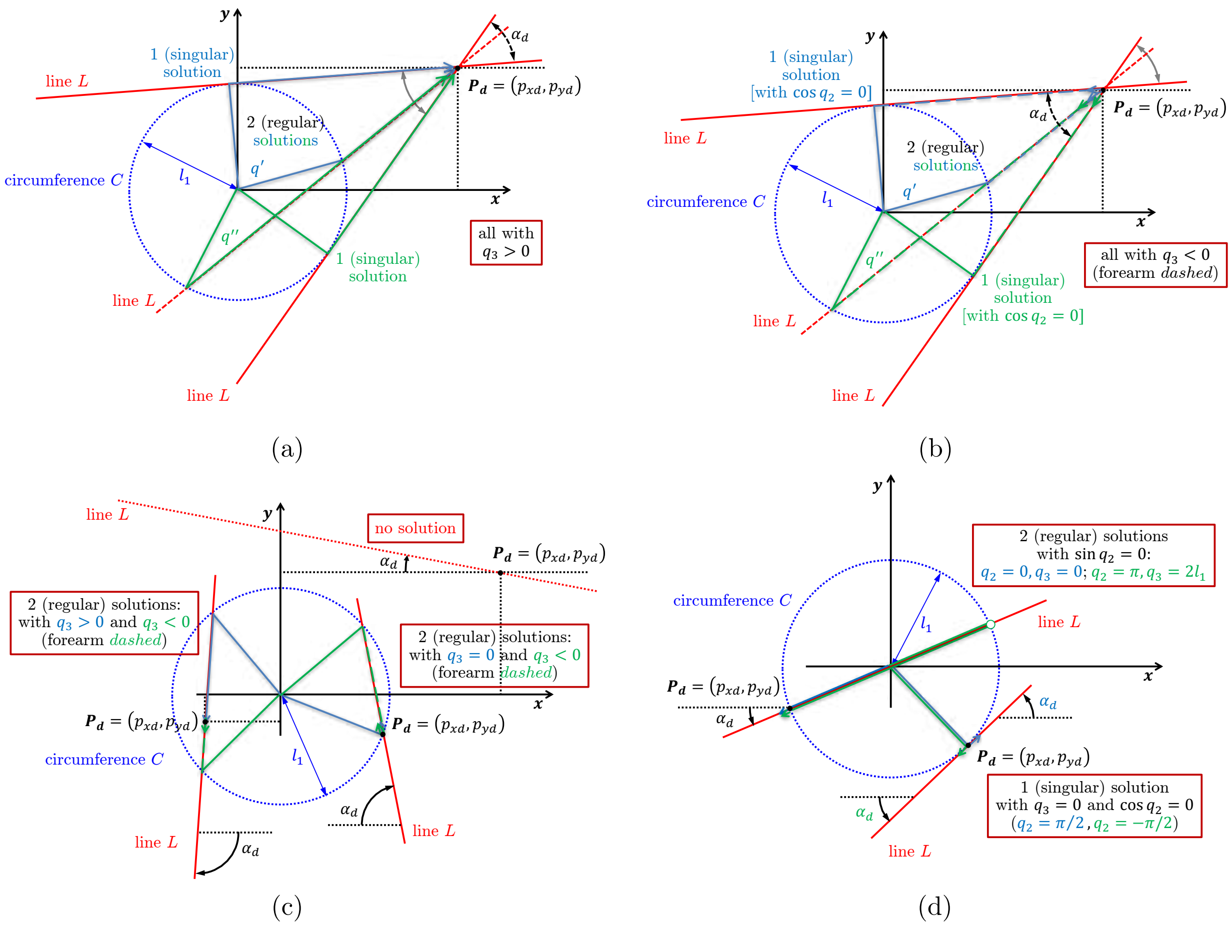


Figure 8: A geometric view of possible situations for the inverse kinematics of the RRP robot. Please refer to the text for a discussion of the cases (a) to (d).

* * * * *

Robotics 1

January 11, 2022

Exercise #1

Consider the planar RPR robot with L-shaped forearm in Fig. 1, shown with the base reference frame RF_0 and the end-effector frame RF_e attached to the gripper.

i. Assign a set of Denavit-Hartenberg (D-H) frames to the robot. The origin of the last D-H frame should be at the point P.

ii. Fill in the associated table of parameters.

iii. Draw the robot in the configuration $\boldsymbol{q} = \mathbf{0}$.

iv. Give the expression of the position $\boldsymbol{p}$ of point P and of the orientation ${}^0\boldsymbol{R}_3$ of the D-H frame RF_3 when the robot is in the configuration $\boldsymbol{q} = \mathbf{0}$.

v. Determine the constant homogeneous matrix ${}^3\boldsymbol{T}_e$.

vi. Give the symbolic expression of all triples $(\alpha_1, \alpha_2, \alpha_3)$ of XYX Euler angles that realize the rotation matrix ${}^3\boldsymbol{R}_e$. Provide the numerical values of these Euler angles when $L = M = 1$.

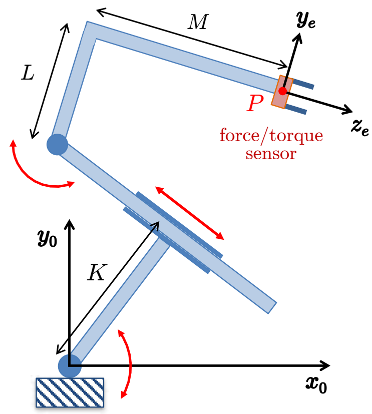


Figure 1: A RPR robot with L-shaped forearm. A force/torque sensor is mounted at the gripper.

Exercise #2

Let a task vector associated to the RPR robot of Fig. 1 be defined as

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \boldsymbol{t}(\boldsymbol{q}) \in \mathbb{R}^3,$$

with the Cartesian coordinates (p_x, p_y) of the point P in the plane and the orientation angle ϕ of the D-H axis $\boldsymbol{x}_3$ w.r.t. the base axis $\boldsymbol{x}_0$.

i. Determine the closed-form expression of the inverse kinematics for a given $\boldsymbol{r}_d = \begin{pmatrix} p_{xd} & p_{yd} & \phi_d \end{pmatrix}^T$.

ii. Provide the numerical values of all inverse solutions for the following data: $K = L = M = 1$ [m]; $p_{xd} = 2$, $p_{yd} = 1$ [m]; $\phi = -\pi/6$ [rad].

Exercise #3

i. Compute the 3×3 task Jacobian $\boldsymbol{J}_t(\boldsymbol{q})$ associated to the task vector function $\boldsymbol{r} = \boldsymbol{t}(\boldsymbol{q})$ defined in Exercise #2.

ii. Find all the singularities of the matrix $\boldsymbol{J}_t(\boldsymbol{q})$.

iii. In a singular configuration $\boldsymbol{q}_s$, determine a basis for the null space $\mathcal{N}\{\boldsymbol{J}_t(\boldsymbol{q}_s)\}$ and a basis for the range space $\mathcal{R}\{\boldsymbol{J}_t(\boldsymbol{q}_s)\}$. Both bases should be *globally* defined, namely they should have a constant dimension for all possible $\boldsymbol{q}$ such that $\boldsymbol{J}_t(\boldsymbol{q})$ is singular.

iv. Set now $K = L = M = 1$ [m]. Find a task velocity $\dot{\boldsymbol{r}}_f \in \mathcal{R}\{\boldsymbol{J}_t(\boldsymbol{q}_s)\}$ and an associated joint velocity $\dot{\boldsymbol{q}}_f \in \mathbb{R}^3$ realizing it, i.e., such that $\boldsymbol{J}_t(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_f = \dot{\boldsymbol{r}}_f$. Is this $\dot{\boldsymbol{q}}_f$ unique?

Exercise #4

Make again reference to the RPR robot shown in Fig. 1. The robot has a force/torque sensor mounted at the gripper which measures in the reference frame RF_e the two linear components ${}^e f_y$ and ${}^e f_z$ of the force ${}^e\boldsymbol{f} \in \mathbb{R}^3$ and the angular component ${}^e m_x$ of the torque ${}^e\boldsymbol{m} \in \mathbb{R}^3$. The other force/torque components are zero. Define the gripper *wrench* as ${}^e\boldsymbol{F} = \left({}^e\boldsymbol{f}^T \; {}^e\boldsymbol{m}^T \right)^T \in \mathbb{R}^6$, when expressed in frame RF_e. Assume again $K = L = M = 1$ [m] and that the robot is in the configuration $\bar{\boldsymbol{q}} = (\pi/2, -1, 0)$ [rad,m,rad], with the gripper in contact with an external environment.

i. If the sensor measures

$$ {}^e f_y = -1 \text{ N}, \qquad {}^e f_z = -2 \text{ N}, \qquad {}^e m_x = 2 \text{ Nm}, $$

what is the value of the gripper wrench $\boldsymbol{F} = \left(\boldsymbol{f}^T \; \boldsymbol{m}^T \right)^T \in \mathbb{R}^6$, as expressed in the absolute frame RF_0?

ii. Compute the vector $\boldsymbol{\tau} \in \mathbb{R}^3$ of forces/torques at the three joints that balances in static conditions the gripper wrench measured by the sensor.

Hint: It is convenient here to work with the complete geometric *Jacobian of the robot.*

Exercise #5

Consider the elliptic path shown in Fig. 2, with major (horizontal) semi-axis of length $a > 0$ and minor (vertical) semi-axis of length $b < a$.

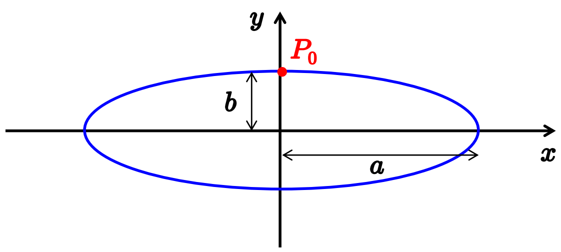


Figure 2: An elliptic path to be parametrized by $\boldsymbol{p}_d(s)$.

i. Choose a smooth parametrization $\boldsymbol{p}_d(s) \in \mathbb{R}^2$, with $s \in [0, 1]$, of the full elliptic path starting at $P_0 = (0, b)$.

ii. Provide a timing law $s = s(t)$ that traces the path counterclockwise with a constant speed $v > 0$ on the path. What will be the motion time T for completing the full ellipse?

iii. The following bounds on the norms of the velocity and of the acceleration should be satisfied along the resulting trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, for all $t \in [0, T]$:

$$\|\dot{\boldsymbol{p}}_d(t)\| \leq V_{max}, \qquad \|\ddot{\boldsymbol{p}}_d(t)\| \leq A_{max}, \qquad \text{with } V_{max} > 0 \text{ and } A_{max} > 0.$$

Accordingly, what will be the maximum feasible speed v_f for this motion?

iv. Provide the numerical values of the maximum feasible speed v_f and of the resulting motion time T_f for the following data: $a = 1$, $b = 0.3$ [m]; $V_{max} = 3$ [m/s]; $A_{max} = 6$ [m/s^2].

Exercise #6

A planar 2R robot has its base placed at the center of the ellipse of Fig. 2, as shown in Fig. 3. The robot has the first link of length a and the second link of length $b < a$, the *same* values of the semi-axes of the ellipse. The position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ of its end effector (point P) should follow the trajectory $\boldsymbol{p}_d(t)$ defined in a parametric way in Exercise #5, with a path speed $v = 0.4$ [s^{-1}].

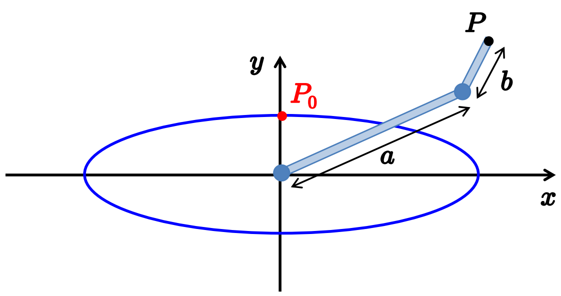


Figure 3: The placement of the 2R robot with respect to the ellipse of Fig. 2.

i. What are the conditions on $a > b > 0$ in order for the robot to be able to reach all points of the desired trajectory $\boldsymbol{p}_d(t)$ while avoiding any robot singularity? Choose numerical values for a and for $b < a$ that satisfy these conditions and keep these values for the rest of this exercise.

ii. Choose an initial robot configuration $\boldsymbol{q}_n(0)$ so as to match the desired trajectory $\boldsymbol{p}_d(t)$ at time $t = 0$, i.e., with initial Cartesian error $\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}_n(0)) = \boldsymbol{0}$.

iii. What nominal joint velocity command $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_n(t)$ should be given for $t \in [0, T]$ in order to execute perfectly the entire trajectory $\boldsymbol{p}_d(t)$ with matched initial conditions?

iv. Choose another initial configuration $\boldsymbol{q}(0)$ such that $\boldsymbol{e}(0) \neq \boldsymbol{0}$, but with the y-component of the error $e_y(0) = 0$. Design a joint velocity control law $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_c(\boldsymbol{q}, t)$, with a feedback term depending on the current configuration $\boldsymbol{q}$, that will let $e_x(t)$ converge to zero with exponential decaying rate $r = 5$ and keep $e_y(t) = 0$ for all $t \geq 0$.

v. With the available data, compute the numerical values of the initial nominal joint velocity command $\dot{\boldsymbol{q}}_n(0) \in \mathbb{R}^2$ and of the initial joint velocity control law $\dot{\boldsymbol{q}}_c(\boldsymbol{q}(0), 0) \in \mathbb{R}^2$.

Exercise #7

The joint of the final flange of a 6R robot has a range of 700°. The driving motor is connected to the joint through a transmission with reduction ratio $n_r = 30$ and mounts a multi-turn absolute encoder. If we want to count the motor revolutions needed to cover the entire joint range and obtain an angular resolution of the final flange of less than 0.02°, how many bits should the multi-turn absolute encoder have at least?

[270 minutes (4.5 hours); open books]

Solution

January 11, 2022

Exercise #1

A possible assignment of Denavit-Hartenberg (D-H) frames is shown in Fig. 4. The associated D-H parameters are given in Table 1. The signs of the q_i's correspond to the robot configuration shown in the figure. Note that the L-shaped form of the forearm is equivalent from a kinematic point of view to a straight link of length $D = \sqrt{L^2 + M^2}$ connecting the origin O_2 of frame RF_2 with the point P, where the origin O_3 of the last D-H frame had to be placed. Accordingly, $\boldsymbol{x}_3$ is chosen along the direction of this equivalent link.

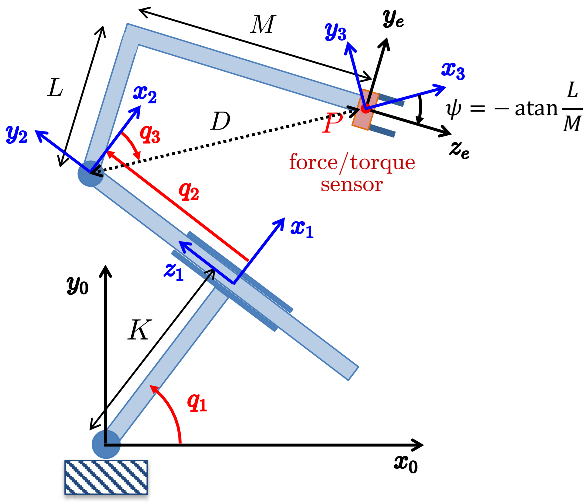


Figure 4: Assignment of D-H frames for the planar RPR robot.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	K	0	$q_1 > 0$
2	$\pi/2$	0	$q_2 > 0$	0
3	0	$D = \sqrt{L^2 + M^2}$	0	$q_3 < 0$

Table 1: Table of D-H parameters for the planar RPR robot.

Figure 5 shows the robot in the configuration $\boldsymbol{q} = \boldsymbol{0}$. In this configuration, the position of the point $P = O_3$ and the orientation of the D-H frame RF_3, as computed from the direct kinematics using the D-H homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, are

$$\boldsymbol{p} = \begin{pmatrix} K + D \\ 0 \\ 0 \end{pmatrix}, \qquad {}^0\boldsymbol{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

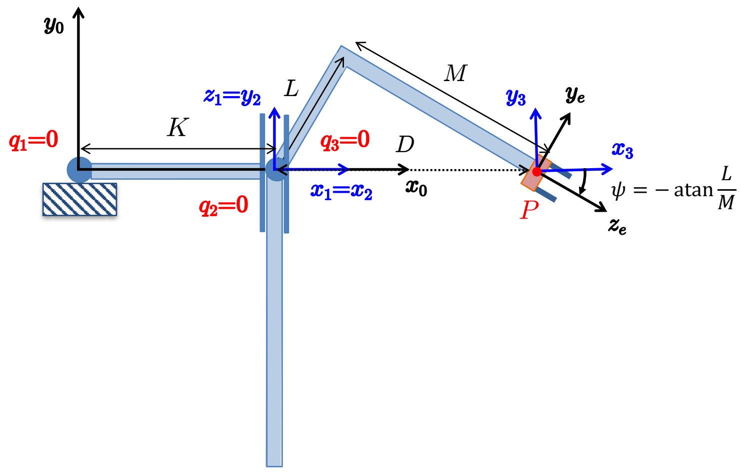


Figure 5: The RPR robot in the configuration $\boldsymbol{q} = \mathbf{0}$.

The constant homogeneous transformation matrix ${}^3\boldsymbol{T}_e$ that aligns the last D-H frame RF_3 with the end-effector (sensor) frame RF_e is given by

$$
{}^3\boldsymbol{T}_e = \begin{pmatrix} {}^3\boldsymbol{R}_e & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \text{with} \quad {}^3\boldsymbol{R}_e = \begin{pmatrix} 0 & L/D & M/D \\ 0 & M/D & -L/D \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\psi & \cos\psi \\ 0 & \cos\psi & \sin\psi \\ -1 & 0 & 0 \end{pmatrix}, \quad (1)
$$

where $\psi = -\arctan(L/M) < 0$.

The XYX Euler angles $(\alpha_1, \alpha_2, \alpha_3)$ define the rotation matrix

$$
\begin{aligned}
\boldsymbol{R}_{XYX} &= \boldsymbol{R}_X(\alpha_1)\boldsymbol{R}_Y(\alpha_2)\boldsymbol{R}_X(\alpha_3) \\
&= \begin{pmatrix} \cos\alpha_2 & \sin\alpha_2\sin\alpha_3 & \sin\alpha_2\cos\alpha_3 \\ \sin\alpha_1\sin\alpha_2 & \cos\alpha_1\cos\alpha_3 - \sin\alpha_1\cos\alpha_2\sin\alpha_3 & -\cos\alpha_1\sin\alpha_3 - \sin\alpha_1\cos\alpha_2\cos\alpha_3 \\ -\cos\alpha_1\sin\alpha_2 & \sin\alpha_1\cos\alpha_3 + \cos\alpha_1\cos\alpha_2\sin\alpha_3 & \cos\alpha_1\cos\alpha_2\cos\alpha_3 - \sin\alpha_1\sin\alpha_3 \end{pmatrix}.
\end{aligned}
$$

We need to solve the inverse orientation problem for this minimal representation of Euler angles:

$$
\boldsymbol{R}_{XYX}(\alpha_1, \alpha_2, \alpha_3) = {}^3\boldsymbol{R}_e(\psi).
$$

Since the two elements in the first and second row of the first column of ${}^3\boldsymbol{R}_e$ are not simultaneously zero, two *regular* solutions for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are obtained in symbolic form as:

$$
\boldsymbol{\alpha}^+ = \begin{pmatrix} 0 \\ \frac{\pi}{2} \\ -\psi \end{pmatrix}, \qquad \boldsymbol{\alpha}^- = \begin{pmatrix} \pi \\ -\frac{\pi}{2} \\ -\psi + \pi \end{pmatrix}.
$$

With the numerical values $L = M = 1$ [m], we have $\psi = -45° = -0.7854$ [rad] and thus

$$
\boldsymbol{\alpha}^+ = \begin{pmatrix} 0 \\ 1.5708 \\ -0.7854 \end{pmatrix}, \qquad \boldsymbol{\alpha}^- = \begin{pmatrix} 3.1416 \\ -1.5708 \\ 2.3562 \end{pmatrix} \quad \text{[rad]}.
$$

Exercise #2

The requested task kinematics for the RPR robot in Fig. 1 is easily obtained as[1]

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} K\cos q_1 - q_2 \sin q_1 + D\cos(q_1+q_3) \\ K\sin q_1 + q_2\cos q_1 + D\sin(q_1+q_3) \\ q_1+q_3 \end{pmatrix} = \boldsymbol{t}(\boldsymbol{q}), \tag{2}$$

with $D = \sqrt{L^2+M^2}$. The closed-form expression of the inverse kinematics

$$\boldsymbol{q} = \boldsymbol{t}^{-1}(\boldsymbol{r}_d), \qquad \text{for a given } \boldsymbol{r}_d = \begin{pmatrix} p_{xd} \\ p_{yd} \\ \phi_d \end{pmatrix}, \tag{3}$$

is found from (2) and (3) as follows. Substituting the third relation $q_1 + q_3 = \phi_d$ in the first two leads to

$$\begin{aligned} K\cos q_1 - q_2\sin q_1 &= p_{xd} - D\cos\phi_d \\ K\sin q_1 + q_2\cos q_1 &= p_{yd} - D\sin\phi_d. \end{aligned} \tag{4}$$

Squaring each equation in (4) and summing, we obtain after simplifications

$$q_2^2 = p_{xd}^2 + p_{yd}^2 + D^2 - 2D\left(p_{xd}\cos\phi_d + p_{yd}\sin\phi_d\right) - K^2 \triangleq A \qquad \Rightarrow \qquad q_2^{\pm} = \pm\sqrt{A}. \tag{5}$$

When $A > 0$, we get two (real) solutions for q_2. If $A = 0$, the two solutions collapse into the single value $q_2 = 0$ (singular case). When $A < 0$,, the inverse problem has no solution because the input data are not compatible with the *secondary* workspace of the robot. When a solution exists (either two or only one), substituting in (4) each value of q_2 from (5), we obtain a linear system of two equations in the two unknowns $s_1 = \sin q_1$ and $c_1 = \cos q_1$:

$$\begin{pmatrix} K & -q_2^{\pm} \\ q_2^{\pm} & K \end{pmatrix}\begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} p_{xd} - D\cos\phi_d \\ p_{yd} - D\sin\phi_d \end{pmatrix}. \tag{6}$$

The determinant of the coefficient matrix is $\det = K^2 + |A| > 0$ (in the assumed situation). Solving (6) provides a value for each $q_2^{\pm}$

$$\begin{aligned} q_1^{\pm} &= \text{atan2}\left\{s_1, c_1\right\} \\ &= \text{atan2}\left\{-q_2^{\pm}\left(p_{x_d} - D\cos\phi_d\right) + K\left(p_{yd} - D\sin\phi_d\right), K\left(p_{xd} - D\cos\phi_d\right) + q_2^{\pm}\left(p_{yd} - D\sin\phi_d\right)\right\}, \end{aligned} \tag{7}$$

and finally

$$q_3^{\pm} = \phi_d - q_1^{\pm}. \tag{8}$$

For the following data

$$K = L = M = 1 \text{ [m]} \quad \Rightarrow \quad D = \sqrt{2} \text{ [m]} \qquad \text{and} \qquad \boldsymbol{r}_d = \begin{pmatrix} p_{xd} \\ p_{yd} \\ \phi_d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -\pi/6 \end{pmatrix} \text{ [m,m,rad]},$$

we have that $A = 2.5152$ (regular case) and the two inverse solutions are

$$\boldsymbol{q}^+ = \begin{pmatrix} 7.81^\circ \\ 1.5859 \\ -37.81^\circ \end{pmatrix} = \begin{pmatrix} 0.1363 \\ 1.5859 \\ -0.6599 \end{pmatrix} \text{ [rad,m,rad]} \quad \boldsymbol{q}^- = \begin{pmatrix} 123.34^\circ \\ -1.5859 \\ -153.34^\circ \end{pmatrix} = \begin{pmatrix} 2.1527 \\ -1.5859 \\ -2.6763 \end{pmatrix} \text{ [rad,m,rad]}.$$

[1] Extract the expressions in (2) from ${}^0\boldsymbol{T}_3(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)$ or just use direct inspection of the figure.

Exercise #3

The (3×3) Jacobian matrix associated to the task (2) is

$$\boldsymbol{J}_t(\boldsymbol{q}) = \frac{\partial \boldsymbol{t}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -K\sin q_1 - q_2\cos q_1 - D\sin(q_1+q_3) & -\sin q_1 & -D\sin(q_1+q_3) \\ K\cos q_1 - q_2\sin q_1 + D\cos(q_1+q_3) & \cos q_1 & D\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}. \quad (9)$$

Its determinant is $\det \boldsymbol{J}_t(\boldsymbol{q}) = -q_2$. When the robot is in a task singularity $\boldsymbol{q}_s = (q_1, 0, q_3)$, with q_1 and q_3 being arbitrary, the Jacobian becomes

$$\boldsymbol{J}_s = \boldsymbol{J}_t(\boldsymbol{q}_s) = \begin{pmatrix} -K\sin q_1 - D\sin(q_1+q_3) & -\sin q_1 & -D\sin(q_1+q_3) \\ K\cos q_1 + D\cos(q_1+q_3) & \cos q_1 & D\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}. \quad (10)$$

It is evident that its first column $\boldsymbol{J}_1$ is a linear combination of the other two: $\boldsymbol{J}_1 = K\boldsymbol{J}_2 + \boldsymbol{J}_3$. Moreover, $\operatorname{rank}\{\boldsymbol{J}_s\} = \operatorname{rank}\left\{\begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}\right\} = 2$, constant for all (q_1, q_3). Therefore, the requested subspaces $\mathcal{N}\{\boldsymbol{J}_s\}$ and $\mathcal{R}\{\boldsymbol{J}_s\}$ associated to the singular matrix $\boldsymbol{J}_s$ are one-dimensional and, respectively, two-dimensional, with *global* bases given by

$$\mathcal{N}\{\boldsymbol{J}_s\} = \operatorname{span}\left\{\begin{pmatrix} -1 \\ K \\ 1 \end{pmatrix}\right\}, \qquad \mathcal{R}\{\boldsymbol{J}_s\} = \operatorname{span}\left\{\begin{pmatrix} -\sin q_1 \\ \cos q_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(q_1+q_3) \\ \cos(q_1+q_3) \\ 1 \end{pmatrix}\right\}.$$

Set now $K = L = M = 1$ (and thus $D = \sqrt{2}$) in (10). A simple choice of a feasible task velocity is

$$\dot{\boldsymbol{r}}_f = \gamma \begin{pmatrix} -\sin q_1 \\ \cos q_1 \\ 0 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}_s\}.$$

There is indeed an infinity of joint velocities $\dot{\boldsymbol{q}}_f \in \mathbb{R}^3$ realizing $\dot{\boldsymbol{r}}_f$. Two possible solutions are

$$\dot{\boldsymbol{q}}_f' = \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} \quad \text{or} \quad \dot{\boldsymbol{q}}_f'' = \begin{pmatrix} \gamma \\ 0 \\ -\gamma \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}_s\dot{\boldsymbol{q}}_f' = \boldsymbol{J}_s\dot{\boldsymbol{q}}_f'' = \dot{\boldsymbol{r}}_f.$$

The first solution uses only the prismatic joint of the robot, while the second uses only the two revolute joints. Note that it is useless to ask which solution has a smaller norm, i.e., it involves a smaller motion in the joint space (apparently, $\left\|\dot{\boldsymbol{q}}_f'\right\| < \left\|\dot{\boldsymbol{q}}_f''\right\|$). In fact, the first solution has [m] as units while the other uses [rad]. These units are not commensurable[2], and the straightforward norm minimization would not be unit-independent. The problem arises because of the different nature (prismatic and revolute) of the robot joints. For this reason, the pseudoinverse solution

$$\dot{\boldsymbol{q}}_f^{\#} = \boldsymbol{J}_s^{\#}\dot{\boldsymbol{r}}_f = \frac{\gamma}{K^2+2}\begin{pmatrix} K \\ 2 \\ -K \end{pmatrix} \quad \text{[rad/s,m/s,rad/s]}$$

makes little sense here.

[2]This is a typical ‘adding apples and oranges’ issue: which is larger, 1 radiant or 1 meter? 1 radiant or 100 centimeters?

Exercise #4

A main issue here is the expression of forces/torques from one reference frame to another: in particular, from the sensor frame RF_e at the robot gripper, where measures of the wrench ${}^e\boldsymbol{F}$ (i.e., forces $\boldsymbol{f} \in \mathbb{R}^3$ and torques $\boldsymbol{m} \in \mathbb{R}^3$) are collected, to the absolute frame RF_0. Because of the set up of the axes of these two reference frames, the problem is naturally embedded in 3D. Moreover, this change of representation is needed also when using the (transpose of the) *geometric* Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for computing the joint torques $\boldsymbol{\tau} \in \mathbb{R}^n$ associated to a wrench at the end-effector gripper. In fact, with the $(6 \times n)$ geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ we usually express the end-effector linear and angular velocities $\boldsymbol{v} \in \mathbb{R}^3$ and $\boldsymbol{\omega} \in \mathbb{R}^3$ directly in frame RF_0. The dual map requires then also wrenches to be expressed in the same frame. In the following, quantities expressed in RF_e carry a preceding superscript e, whereas quantities without a preceding superscript are expressed (by default) in the absolute frame RF_0.

With the above in mind, we have in general

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} &= \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} \\
\Rightarrow \begin{pmatrix} {}^e\boldsymbol{v} \\ {}^e\boldsymbol{\omega} \end{pmatrix} &= \begin{pmatrix} {}^e\boldsymbol{R}_0(\boldsymbol{q})\,\boldsymbol{v} \\ {}^e\boldsymbol{R}_0(\boldsymbol{q})\,\boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} {}^e\boldsymbol{R}_0(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^e\boldsymbol{R}_0(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \begin{pmatrix} {}^e\boldsymbol{J}_L(\boldsymbol{q}) \\ {}^e\boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}}.
\end{aligned}
\tag{11}
$$

On the other hand

$$
\boldsymbol{F} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_e(\boldsymbol{q})\,{}^e\boldsymbol{f} \\ {}^0\boldsymbol{R}_e(\boldsymbol{q})\,{}^e\boldsymbol{m} \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^0\boldsymbol{R}_e(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} {}^e\boldsymbol{f} \\ {}^e\boldsymbol{m} \end{pmatrix}. \tag{12}
$$

Thus, the map from end-effector wrenches to joint torques can be written in equivalent ways as

$$
\begin{aligned}
\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} &= \begin{pmatrix} \boldsymbol{J}_L^T(\boldsymbol{q}) & \boldsymbol{J}_A^T(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{J}_L^T(\boldsymbol{q}) & \boldsymbol{J}_A^T(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} {}^e\boldsymbol{R}_0(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^e\boldsymbol{R}_0(\boldsymbol{q}) \end{pmatrix}^T \begin{pmatrix} {}^e\boldsymbol{R}_0(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^e\boldsymbol{R}_0(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} \\
&= \left(\begin{pmatrix} {}^e\boldsymbol{R}_0(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^e\boldsymbol{R}_0(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \right)^T \begin{pmatrix} {}^e\boldsymbol{f} \\ {}^e\boldsymbol{m} \end{pmatrix} \\
&= \begin{pmatrix} {}^e\boldsymbol{J}_L^T(\boldsymbol{q}) & {}^e\boldsymbol{J}_A^T(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} {}^e\boldsymbol{f} \\ {}^e\boldsymbol{m} \end{pmatrix} \\
&= {}^e\boldsymbol{J}^T(\boldsymbol{q})\,{}^e\boldsymbol{F}.
\end{aligned}
\tag{13}
$$

In the above computations, one needs ${}^0\boldsymbol{R}_3(\boldsymbol{q})$ from the robot direct kinematics and ${}^3\boldsymbol{R}_e$ from (1). Further, we can use conveniently the task Jacobian $\boldsymbol{J}_t(\boldsymbol{q})$ in (9) to build the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$. These quantities are evaluated when the robot is in $\boldsymbol{q} = \bar{\boldsymbol{q}} = (\pi/2, -1, 0)$ [rad,m,rad], using the data $K = L = M = 1$ [m] (thus, $D = \sqrt{2}$ [m] and $\psi = -\pi/4$ [rad]). We have

$$
\begin{aligned}
{}^0\boldsymbol{R}_e(\bar{\boldsymbol{q}}) = {}^0\boldsymbol{R}_3(\bar{\boldsymbol{q}})\,{}^3\boldsymbol{R}_e &= \begin{pmatrix} \cos(\bar{q}_1 + \bar{q}_3) & -\sin(\bar{q}_1 + \bar{q}_3) & 0 \\ \sin(\bar{q}_1 + \bar{q}_3) & \cos(\bar{q}_1 + \bar{q}_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\psi & \cos\psi \\ 0 & \cos\psi & \sin\psi \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.7071 & 0.7071 \\ 0 & 0.7071 & 0.7071 \\ -1 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{14}
$$

and

$$\boldsymbol{J}_t(\bar{\boldsymbol{q}}) = \begin{pmatrix} -\sin\bar{q}_1 - \bar{q}_2\cos\bar{q}_1 - \sqrt{2}\,\sin(\bar{q}_1+\bar{q}_3) & -\sin\bar{q}_1 & -\sqrt{2}\,\sin(\bar{q}_1+\bar{q}_3) \\ \cos\bar{q}_1 - \bar{q}_2\sin\bar{q}_1 + \sqrt{2}\,\cos(\bar{q}_1+\bar{q}_3) & \cos\bar{q}_1 & \sqrt{2}\,\cos(\bar{q}_1+\bar{q}_3) \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1-\sqrt{2} & -1 & -\sqrt{2} \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} -2.4142 & -1 & -1.4142 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{array}{l} \leftarrow\ \dot{p}_x \\ \leftarrow\ \dot{p}_y \\ \leftarrow\ \dot{\phi}_z \end{array} \end{array}$$

Therefore, the mapping

$$\dot{\boldsymbol{q}} \in \mathbb{R}^3 \quad \longrightarrow \quad \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

is given by the (6×3) geometric Jacobian, expressed in the frames RF_0 and RF_e respectively as[3]

$$\boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} \boldsymbol{J}_L(\bar{\boldsymbol{q}}) \\ \boldsymbol{J}_A(\bar{\boldsymbol{q}}) \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} -2.4142 & -1 & -1.4142 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{array}{l} \leftarrow\ v_x = \dot{p}_x \\ \leftarrow\ v_y = \dot{p}_y \\ \leftarrow\ v_z = 0 \\ \leftarrow\ \omega_x = 0 \\ \leftarrow\ \omega_y = 0 \\ \leftarrow\ \omega_z = \dot{\phi}_z \end{array} \end{array}$$

and, using the transpose of (14),

$${}^e\boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} {}^e\boldsymbol{J}_L(\bar{\boldsymbol{q}}) \\ {}^e\boldsymbol{J}_A(\bar{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_e^T(\bar{\boldsymbol{q}})\,\boldsymbol{J}_L(\bar{\boldsymbol{q}}) \\ {}^0\boldsymbol{R}_e^T(\bar{\boldsymbol{q}})\,\boldsymbol{J}_A(\bar{\boldsymbol{q}}) \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 \\ 2.4142 & 0.7071 & 1 \\ -1 & -0.7071 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow\ {}^ev_x = 0 \\ \leftarrow\ {}^ev_y \\ \leftarrow\ {}^ev_z \\ \leftarrow\ {}^e\omega_x = -\dot{\phi}_z \\ \leftarrow\ {}^e\omega_y = 0 \\ \leftarrow\ {}^z\omega_y = 0. \end{array} \end{array}$$

The two posed problems i. and ii. have then the following answers. From the measured data

$${}^e\boldsymbol{F} = \begin{pmatrix} {}^e\boldsymbol{f}^T & {}^e\boldsymbol{m}^T \end{pmatrix}^T = \begin{pmatrix} {}^ef_x & {}^ef_y & {}^ef_z & {}^em_x & {}^em_y & {}^em_z \end{pmatrix}^T = \begin{pmatrix} 0 & -1 & -2 & 2 & 0 & 0 \end{pmatrix}^T,$$

we compute the gripper wrench expressed in the base frame using (12) and (14):

$$\boldsymbol{F} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_e(\bar{\boldsymbol{q}})\,{}^e\boldsymbol{f} \\ {}^0\boldsymbol{R}_e(\bar{\boldsymbol{q}})\,{}^e\boldsymbol{m} \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_e(\bar{\boldsymbol{q}}) & \boldsymbol{0} \\ \boldsymbol{0} & {}^0\boldsymbol{R}_e(\bar{\boldsymbol{q}}) \end{pmatrix} {}^e\boldsymbol{F} = \begin{array}{cc} \begin{pmatrix} -0.7071 \\ -2.1213 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} & \begin{array}{l} \leftarrow\ f_x\ [\mathrm{N}] \\ \leftarrow\ f_y\ [\mathrm{N}] \\ \\ \\ \\ \leftarrow\ m_z\ [\mathrm{Nm}]. \end{array} \end{array}$$

[3]We simply embed here the rows of $\boldsymbol{J}_t(\bar{\boldsymbol{q}})$ in the correct rows of $\boldsymbol{J}(\bar{\boldsymbol{q}})$.

From (13), the joint torques needed to balance in static conditions the gripper wrench (applied by the environment and measured by the sensor) are given by

$$\boldsymbol{\tau} = -\,{}^{e}\boldsymbol{J}^{T}(\bar{\boldsymbol{q}})\,{}^{e}\boldsymbol{F} = -\boldsymbol{J}^{T}(\bar{\boldsymbol{q}})\,\boldsymbol{F} = \begin{pmatrix} 2.4142 \\ -0.7071 \\ 1 \end{pmatrix} \text{ [Nm,N,Nm]}.$$

Note here the minus sign!

Exercise #5

The elliptic path in Fig. 2 can be smoothly parametrized by

$$\boldsymbol{p}_d(s) = \begin{pmatrix} p_{dx}(s) \\ p_{dy}(s) \end{pmatrix} = \begin{pmatrix} -a\sin 2\pi s \\ b\cos 2\pi s \end{pmatrix}, \qquad s \in [0,1].$$

In this way we have $\boldsymbol{p}_d(0) = (0, b)$, the coordinates of the point P_0, and the path is traced counterclockwise for increasing values of the parameter s. The first and second path derivatives are

$$\boldsymbol{p}'_d(s) = \frac{d\boldsymbol{p}_d(s)}{ds} = -2\pi \begin{pmatrix} a\cos 2\pi s \\ b\sin 2\pi s \end{pmatrix}, \qquad \boldsymbol{p}''_d(s) = \frac{d^2\boldsymbol{p}_d(s)}{ds^2} = 4\pi^2 \begin{pmatrix} a\sin 2\pi s \\ -b\cos 2\pi s \end{pmatrix}, \qquad s \in [0,1].$$

Figure 6 shows the plots of the x and y components of $\boldsymbol{p}_d(s)$, $\boldsymbol{p}'_d(s)$, and $\boldsymbol{p}''_d(s)$, when choosing $a = 1$ and $b = 0.3$ [m] as lengths for the semi-axes of the ellipse.

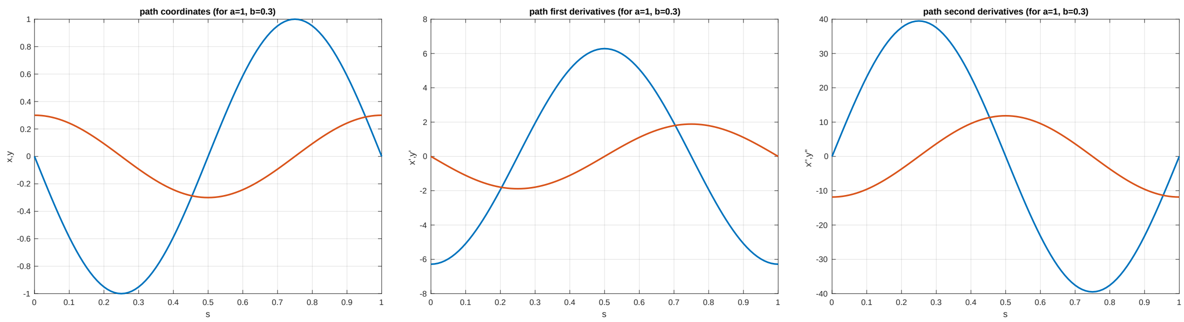


Figure 6: Components of $\boldsymbol{p}_d(s)$, $\boldsymbol{p}'_d(s)$, and $\boldsymbol{p}''_d(s)$ (x in blue, y in red).

The desired timing on the path is simply

$$s = s(t) = vt, \qquad \dot{s}(t) = v > 0, \qquad \ddot{s}(t) = 0, \qquad t \in [0, T],$$

where $T = 1/v$ is the motion time needed to trace a full ellipse with constant speed v. Accordingly, the velocity and the acceleration along the trajectory $\boldsymbol{p}_d(t)$ are

$$\dot{\boldsymbol{p}}_d(t) = \boldsymbol{p}'_d\,\dot{s} = -2\pi v \begin{pmatrix} a\cos 2\pi vt \\ b\sin 2\pi vt \end{pmatrix} \qquad \ddot{\boldsymbol{p}}_d(t) = \boldsymbol{p}'_d\,\ddot{s} + \boldsymbol{p}''_d\,\dot{s}^2 = 4\pi^2 v^2 \begin{pmatrix} a\sin 2\pi vt \\ -b\cos 2\pi vt \end{pmatrix},$$

with associated norms

$$\|\dot{\boldsymbol{p}}_d(t)\| = 2\pi v\,\sqrt{a^2\cos^2 2\pi vt + b^2\sin^2 2\pi vt}, \qquad \|\ddot{\boldsymbol{p}}_d(t)\| = 4\pi^2 v^2\,\sqrt{a^2\sin^2 2\pi vt + b^2\cos^2 2\pi vt}.$$

It is easy to see that, being $a > b$, the maximum values of these norms are

$$\max_{t\in[0,T]} \|\dot{\boldsymbol{p}}_d(t)\| = 2\pi v a, \qquad \text{attained at } t = \{0,\, T/2,\, T\}$$

and, respectively,

$$\max_{t\in[0,T]} \|\ddot{\boldsymbol{p}}_d(t)\| = 4\pi^2 v^2 a, \qquad \text{attained at } t = \{T/4,\, 3T/4\}.$$

From the required bounds on these norms

$$2\pi v a \le V_{max}, \qquad 4\pi^2 v^2 a \le A_{max},$$

we obtain the maximum feasible speed v_f for this motion as

$$v_f = \min\left\{\frac{V_{max}}{2\pi a}, \sqrt{\frac{A_{max}}{4\pi^2 a}}\right\} = \frac{1}{2\pi}\min\left\{\frac{V_{max}}{a}, \sqrt{\frac{A_{max}}{a}}\right\}.$$

Using the given numerical data $a = 1$, $b = 0.3$ [m], $V_{max} = 3$ [m/s] and $A_{max} = 6$ [m/s^2], we obtain for the speed and the motion time

$$v_f = 0.3898 \text{ [s}^{-1}] \qquad \Rightarrow \qquad T_f = 2.5651 \text{ [s]}.$$

The norm of the acceleration saturates the value $A_{max} = 6$ [m/s^2] while the maximum norm of the velocity equals 2.4495 [m/s], remaining below the limit V_{max}. Figure 7 shows the resulting evolution of the norms of $\dot{\boldsymbol{p}}_d(t)$ and $\ddot{\boldsymbol{p}}_d(t)$ along the trajectory.

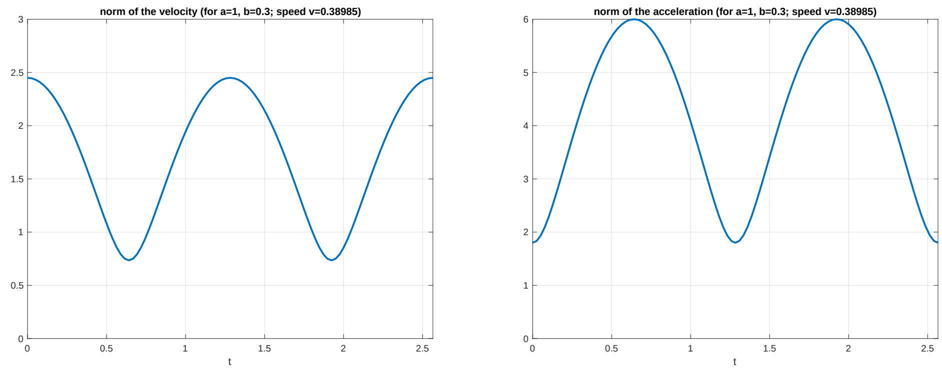


Figure 7: Evolution of $\|\dot{\boldsymbol{p}}_d(t)\|$ and $\|\ddot{\boldsymbol{p}}_d(t)\|$ for $v_f = 0.3898$ [s^{-1}].

Exercise #6

For the planar 2R robot shown in Fig. 3 to be able to execute the task, the elliptic path defined in Exercise #5 should entirely belong to its primary workspace. Since the robot has strictly different link lengths $l_1 = a$ and $l_2 = b < a$, the workspace is a circular annulus with internal radius $\rho_{min} = a - b > 0$ and external radius $\rho_{min} = a + b > 0$. Therefore, the lengths a and b of the semi-axes of the ellipse should satisfy the inequalities

$$\rho_{min} = a - b \le a \le a + b = \rho_{max}, \quad \rho_{min} = a - b \le b \le a + b = \rho_{max} \qquad \Rightarrow \qquad b < a \le 2b.$$

However, the limit value $a = 2b$ would certainly lead to a singularity when the robot end effector is placed at $P_0 = (0, b)$, i.e., at the trajectory start (on the inner boundary of the workspace). In this case, the only inverse kinematics solution is $\boldsymbol{q}_s = (\pi/2, \pi)$, a singular configuration with the second link folded on the first one. The Jacobian of the 2R robot is then

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -a\sin q_1 - b\sin(q_1+q_2) & -b\sin(q_1+q_2) \\ a\cos q_1 + b\cos(q_1+q_2) & b\cos(q_1+q_2) \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -a+b & b \\ 0 & 0 \end{pmatrix}$$

so that $\det \boldsymbol{J}(\boldsymbol{q}_s) = 0$. The same happens also at the opposite point of the ellipse, $P_{-0} = (-b, 0)$. Therefore, a has to belong to the *open* interval $a \in (b, 2b)$ in order to avoid singularities[4]. For illustration, we choose the numerical values $a = 1$ and $b = 0.6$. The following results will be qualitatively similar for other admissible choices of these two geometric parameters. The position and the velocity of the desired Cartesian trajectory for $v = 0.4$ [s^{-1}] are shown in Fig. 8. The motion time is $T = 1/v = 2.5$ [s].

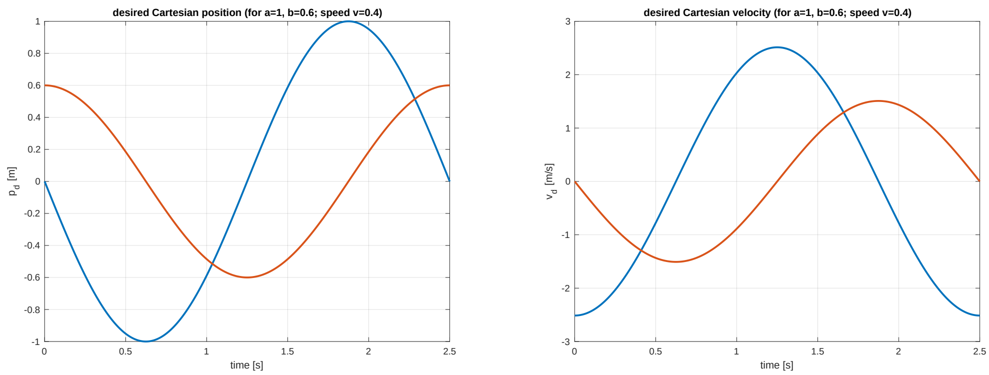


Figure 8: Components of $\boldsymbol{p}_d(t)$ and $\dot{\boldsymbol{p}}_d(t)$ (x in blue, y in red).

To proceed, we solve first the inverse kinematics for $P_0 = (p_{0x}, p_{0y}) = (0, b) = (0, 0.6)$, yielding two initial (regular) configurations. From the known formulas for the 2R robot, we have

$$c_2 = \frac{p_{0x}^2 + p_{0y}^2 - \left(a^2 + b^2\right)}{2ab} = -0.8333, \qquad s_2 = \sqrt{1 - c_2^2} = 0.5528,$$

to be used in

$$\begin{aligned} \boldsymbol{q}_0^+ &= \begin{pmatrix} \text{atan2}\left\{p_{0y}\left(a + bc_2\right) - p_{0x}b\, s_2, p_{0x}\left(a + bc_2\right) + p_{0y}b\, s_2\right\} \\ \text{atan2}\left\{s_2, c_2\right\} \end{pmatrix} \\ &= \begin{pmatrix} 56.44° \\ 146.44° \end{pmatrix} = \begin{pmatrix} 0.9851 \\ 2.5559 \end{pmatrix} \text{[rad]} \qquad \text{(right arm solution)} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{q}_0^- &= \begin{pmatrix} \text{atan2}\left\{p_{0y}\left(a + bc_2\right) + p_{0x}b\, s_2, p_{0x}\left(a + bc_2\right) - p_{0y}b\, s_2\right\} \\ \text{atan2}\left\{-s_2, c_2\right\} \end{pmatrix} \\ &= \begin{pmatrix} 123.56° \\ -146.44° \end{pmatrix} = \begin{pmatrix} 2.1565 \\ -2.5559 \end{pmatrix} \text{[rad]} \qquad \text{(left arm solution).} \end{aligned}$$

[4]It should be noted that the velocity vector $\dot{\boldsymbol{p}}_d$ is actually feasible even in the two singular situations and can be obtained by the use of the pseudoinverse of $\boldsymbol{J}$. However, too large joint velocities would be generated in that case while approaching a singularity.

With both choices, the position of the robot end effector will be matched with the desired trajectory $\boldsymbol{p}_d(t)$ at time $t = 0$ ($\boldsymbol{p}_d(0) = P_0$). With such an initialization, the nominal joint velocity command $\dot{\boldsymbol{q}}_n(t)$ that will execute perfectly the entire trajectory $\boldsymbol{p}_d(t)$, for $t \in [0, T]$, is given by

$$\dot{\boldsymbol{q}}_n = \boldsymbol{J}^{-1}(\boldsymbol{q}_n)\dot{\boldsymbol{p}}_d, \qquad \boldsymbol{q}_n(0) = \boldsymbol{q}_0^{\pm}. \tag{15}$$

Note that the robot Jacobian $\boldsymbol{J}(\boldsymbol{q}_n(t))$ will never become singular because the end-effector path remains always *strictly* inside the robot workspace. Therefore, the right or left arm configuration chosen at the start will be kept throughout the entire trajectory. Choosing, e.g., the right arm solution at start, $\boldsymbol{q}_n(0) = \boldsymbol{q}_0^{+}$, yields the joint velocity command $\dot{\boldsymbol{q}}_n(t)$ and the associated joint evolution $\boldsymbol{q}_n(t)$ shown in Fig. 9. It is apparent that the velocity commands and the motion of the joints are cyclic (modulo 2π for $q_{n1}(t)$). The initial value of the joint velocity command (15) is $\dot{\boldsymbol{q}}_n(0) = \begin{pmatrix} 4.1888 & 0 \end{pmatrix}^T$ [rad/s].

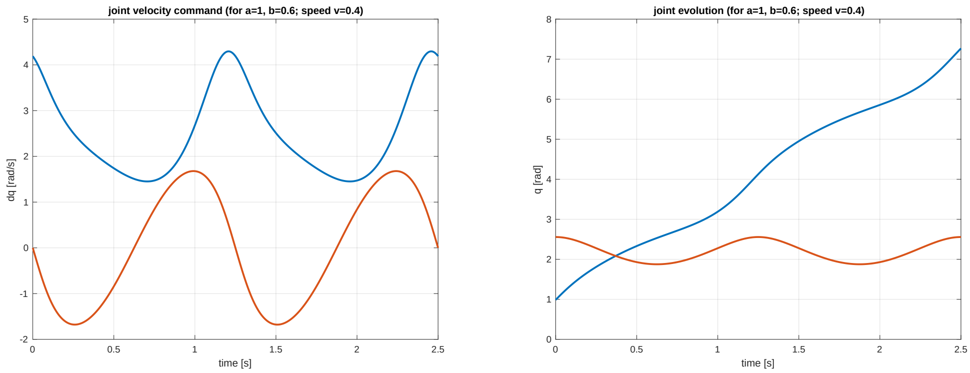


Figure 9: Nominal velocity command $\dot{\boldsymbol{q}}_n(t)$ and resulting evolution $\boldsymbol{q}_n(t)$ (joint 1 in blue, 2 in red).

Next, let the robot start from another initial configuration $\boldsymbol{q}(0)$ such that the end-effector position error is $\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) \neq \boldsymbol{0}$, with $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ being the direct kinematics of the 2R robot, but $e_y(0) = 0$. For instance, we choose $\boldsymbol{q}(0) = (0, \pi/2)$ (still a 'right arm' configuration), corresponding to an error $e_x(0) = -a = -1$ [m] only along the x-direction. In order to obtain asymptotic tracking of the desired trajectory $\boldsymbol{p}_d(t)$ together with the requested performance during the initial transient, the joint velocity control law $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_c(\boldsymbol{q}, t)$ is designed using feedback from the *Cartesian* error. The control law is then

$$\dot{\boldsymbol{q}}_c = \boldsymbol{J}^{-1}(\boldsymbol{q})\big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right)\big), \qquad \boldsymbol{K}_P = r \cdot \boldsymbol{I}_{2\times 2} > 0, \tag{16}$$

with the rate $r = 5$ introduced in the diagonal, positive definite gain matrix $\boldsymbol{K}_P$. This choice guarantees that, in the absence of further disturbances, we have

$$e_x(t) = e_x(0)\exp\left(-5t\right), \qquad e_y(t) \equiv 0, \qquad \forall t \geq 0.$$

Figure 10 shows the desired and the actually executed Cartesian trajectory, together with the Cartesian position error, the feedback control commands, and the resulting motion of the robot joints. Finally, the initial value of the joint velocity control law (16) is $\dot{\boldsymbol{q}}_c(0) = \begin{pmatrix} 0 & 12.5221 \end{pmatrix}^T$ [rad/s].

Exercise #7

The following self-explanatory Matlab code computes the minimum number of bits of the multi-turn absolute encoder which satisfies the given specifications. This number is `bits` $= 16$, namely:

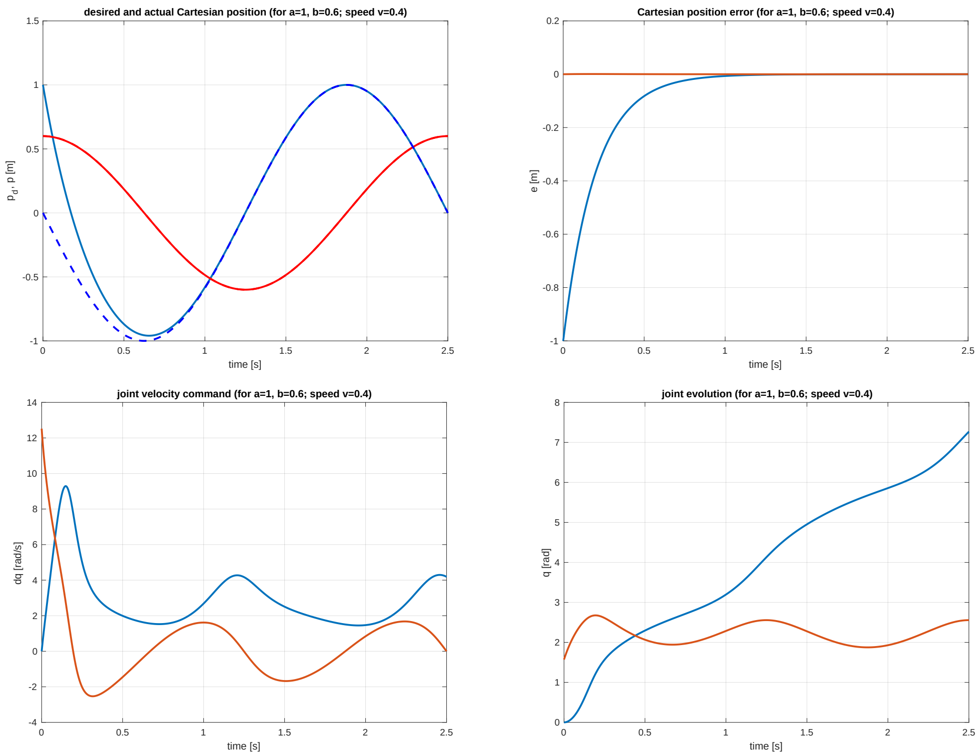


Figure 10: [Top] Components of the desired $\boldsymbol{p}_d(t)$ (dashed) and actually obtained $\boldsymbol{p}(t)$ (continuous) and those of the associated error $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t)$ (x in blue, y in red). [Bottom] Velocity control law $\dot{\boldsymbol{q}}_c(t)$ and resulting evolution $\boldsymbol{q}_c(t)$ (joint 1 in blue, 2 in red).

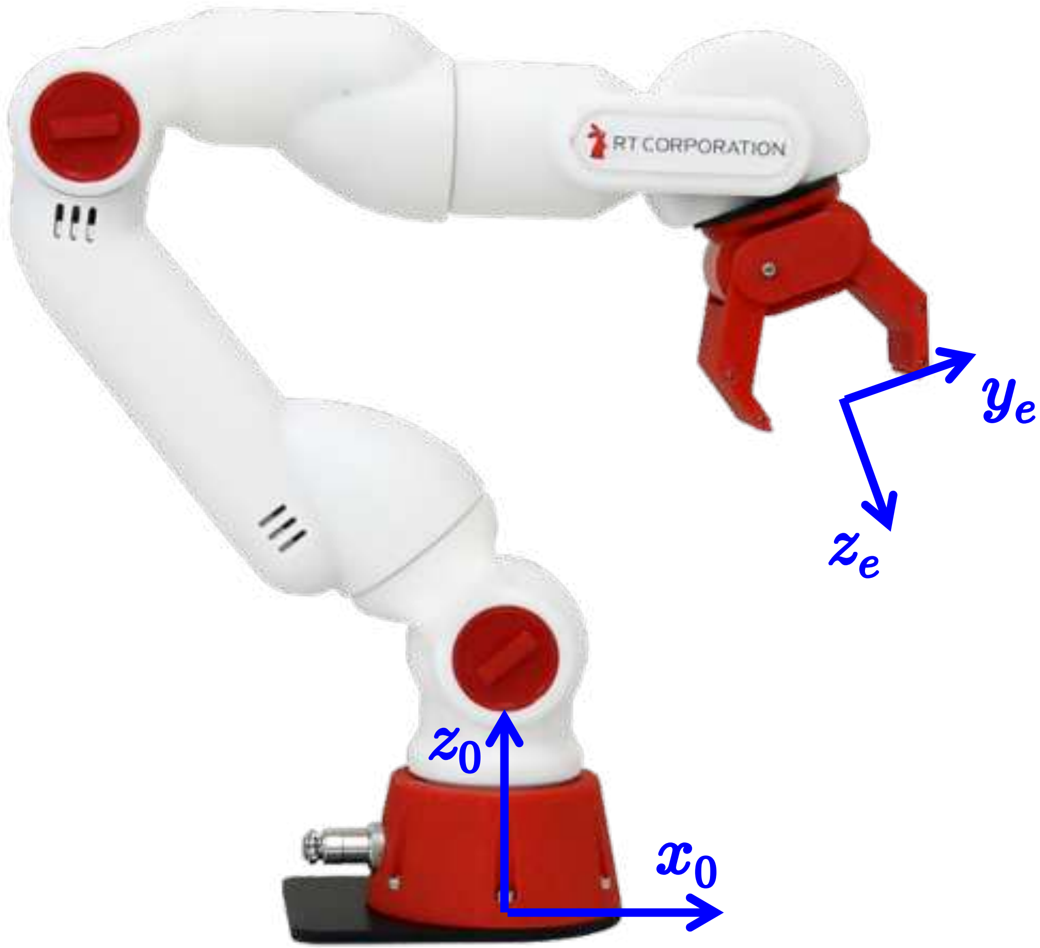
6 bits count separately the number of motor turns that covers the entire joint range of the flange, while 10 bits (equal to the number of tracks of the main encoder wheel) allow to achieve the desired angular resolution on the flange side of the transmission.

```
% data
joint_range=700 %[deg]      % range of the flange rotation
nr=30                       % reduction ratio
res_joint=0.02  %[deg]      % desired resolution at the flange side
% computation
disp(‘all angles are in degrees’)
turns_joint=joint_range/360
turns_motor=nr*turns_joint
bits_turn=ceil(log2(turns_motor))
sectors_joint=360/res_joint
tracks_motor=sectors_joint/nr
res_motor=sectors_motor/360
bits_res=ceil(log2(sectors_motor))
bits=bits_turn+bits_res
% end
```

* * * * *

Robotics 1 – Extra sheet for Exercise 1
February 3, 2022

Student name: ....



$$^7R_e = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				

Robotics 1

February 3, 2022

Exercise #1

Figure 1 shows two 3D views, together with top and side views with geometric data, of the Crane-X7 robot (by RT Corporation, Japan), a 7-dof arm with all revolute joints. The base frame RF_0 and the end-effector frame RF_e attached to the gripper are already assigned as in the figure.

i. Define a set of Denavit-Hartenberg (D-H) frames for the robot. The origin of the last D-H frame should coincide with the origin O_e of frame RF_e.

ii. Draw clearly the relevant axes of the D-H frames and fill in the associated table of parameters. Specify therein the signs of the variables q_i, $i = 1, \ldots, 7$, in the shown robot configuration.

iii. Provide the constant rotation matrix ${}^7\boldsymbol{R}_e$.

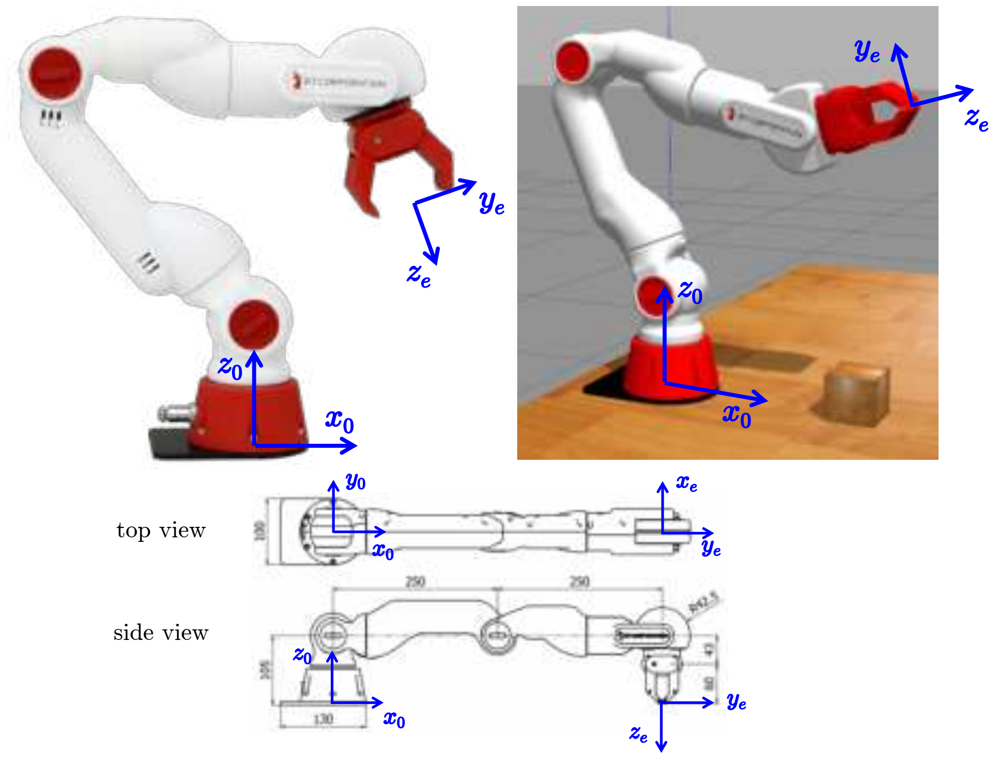


Figure 1: Views of the Crane-X7 robot, with geometric data (in [mm]) and frames RF_0 and RF_e.

Use the Extra Sheet to complete this exercise. Fill in there also the elements of the matrix ${}^7\boldsymbol{R}_e$.

Exercise #2

The absolute initial orientation of the end effector of a 6R robot with a spherical wrist is specified by the YXY sequence of Euler angles $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (45°, -45°, 120°)$. A different orientation is expressed instead by the rotation matrix

$$ {}^0\boldsymbol{R}_f = \begin{pmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi & -\sin\phi \\ -1 & 0 & 0 \end{pmatrix}, \qquad \text{with } \phi = \frac{\pi}{3}. $$

Find an axis-angle representation $(\boldsymbol{r}, \theta)$ of the relative rotation between these two end-effector orientations. Further, if a motion is imposed to the end effector with constant angular velocity $\boldsymbol{\omega} = 1.1 \cdot \boldsymbol{r}$ [rad/s], what will be the time T_ω needed to accomplish this change of orientation?

Exercise #3

Assume that the motion of a 3R planar robot having equal links of unitary length is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^3$. With reference to Fig. 2, the robot end effector should follow a desired smooth trajectory $\boldsymbol{p}_d(t) = \begin{pmatrix} p_{x,d}(t) & p_{y,d}(t) \end{pmatrix}^T \in \mathbb{R}^2$ in position, while keeping constant its angular speed at some value $\omega_{z,d} \in \mathbb{R}$ (perhaps, after an initial transient).

i. Provide the general form of the command $\ddot{\boldsymbol{q}}$ that executes the full task in nominal conditions.

ii. Study the singularities that may be encountered during the execution of the task.

iii. Compute the numerical value of $\ddot{\boldsymbol{q}}$ when the robot is in the nominal state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) \in \mathbb{R}^6$ and for a desired $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$, as given by

$$ \boldsymbol{q}_d = \begin{pmatrix} \pi/4 \\ \pi/3 \\ -\pi/2 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}_d = \begin{pmatrix} -0.8 \\ 1 \\ 0.2 \end{pmatrix} \text{ [rad/s]}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s}^2\text{]}. $$

What are the values of $\boldsymbol{p}_d$, $\dot{\boldsymbol{p}}_d$, and of $\omega_{z,d}$ in this nominal robot state?

iv. If at some time $t \geq 0$, there is a position and/or a velocity error in the execution of the desired end-effector trajectory $\boldsymbol{p}_d(t)$, how would you modify the commanded acceleration $\ddot{\boldsymbol{q}}(t)$ so as to recover exponentially[1] the error to zero, both in position and velocity? And what if also the angular velocity $\omega_z(t)$ is not the desired one?

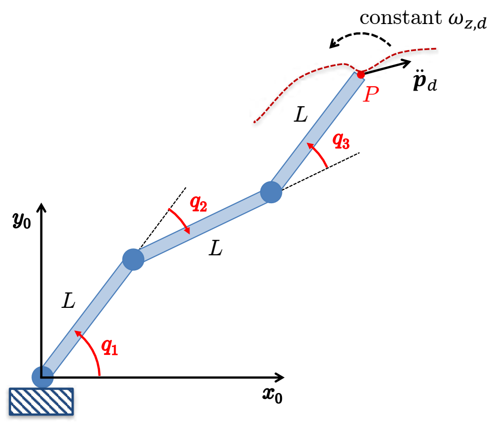


Figure 2: A 3R planar robot executing the desired Cartesian task.

[1] This rate is a property that automatically follows from the linearity of an asymptotically stable error dynamics.

Exercise #4

Consider the situation in Fig. 3, with all data defined therein in symbolic form. The PR robot starts at rest with its end effector placed in $P_{start} = (S, L)$ and should move the end effector to $P_{goal} = (S + \Delta, L)$ in a given time T and stop there, without colliding with the obstacle $\mathcal{O}_{obs}$ located at $(S + (\Delta/2), L/2)$. Design a joint trajectory $\boldsymbol{q}_d(t) \in \mathbb{R}^2$, $t \in [0, T]$, that realizes the task with continuous acceleration $\ddot{\boldsymbol{q}}_d(t)$ and no instant of zero velocity in the open interval $(0, T)$. The solution should be parametric with respect to $L > 0$ (length of the second link of the robot), $S > 0$ (x-coordinate of P_{start}), $\Delta > L/2$ (distance of the two Cartesian points in the x-direction), and T (motion time). Provide then a numerical example, sketching the plot of $\boldsymbol{q}_d(t)$.

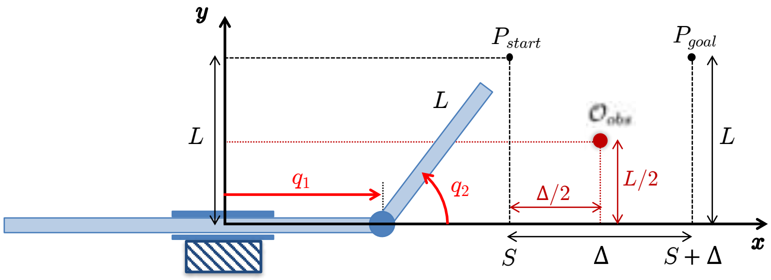


Figure 3: A PR robot should move its end effector from P_{start} to P_{goal}, avoiding the obstacle $\mathcal{O}_{obs}$.

Exercise #5

A transmission/reduction system that displaces rotary motion from the motor axis to the joint axis of a link of length L is sketched in Fig. 4. The system involves two toothed gears and two pulleys, connected by a belt at a distance D. The radius of each of the two gear wheels and of the two pulleys is denoted as r_i, $i = 1, \ldots, 4$. At $t = 0$, the link is in the position shown in the figure. If the motor spins on its axis $\boldsymbol{z}_m$ with a constant angular speed $\dot{\theta}_m > 0$, how much time T_θ will it take for the link to rotate by 90°? Will the link rotate clockwise (CW) or counterclockwise (CCW) w.r.t. its joint axis $\boldsymbol{z}_j$? Evaluate then T_θ using the following data:

$$\dot{\theta}_m = 10 \ \text{[rad/s]}, \qquad r_1 = 20,\ r_2 = 60,\ r_3 = 8,\ r_4 = 32 \ \text{[mm]}, \qquad D = 0.15,\ L = 0.3 \ \text{[m]}.$$

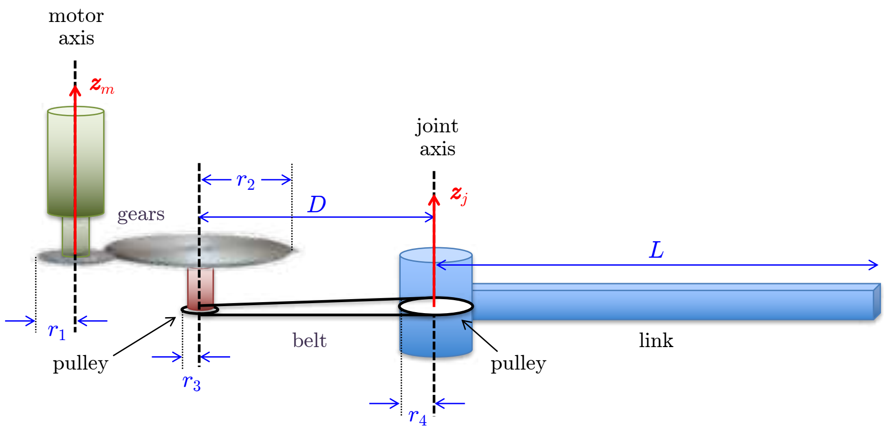


Figure 4: A transmission/reduction system for a motor/link pair.

[210 minutes (3.5 hours); open books]

Solution

February 3, 2022

Exercise #1

A possible assignment of Denavit-Hartenberg (D-H) frames is shown in Fig. 5. The three $\boldsymbol{z}$ axes with a double arrow are coming out of the plane. The associated D-H parameters are given in Table 1. The signs of the q_i's in the table correspond to the robot configuration shown in the figure. The Crane-X7 robot has no offsets, it has both shoulder and wrist spherical, and a kinematics equivalent to that of the KUKA LWR IV robot.

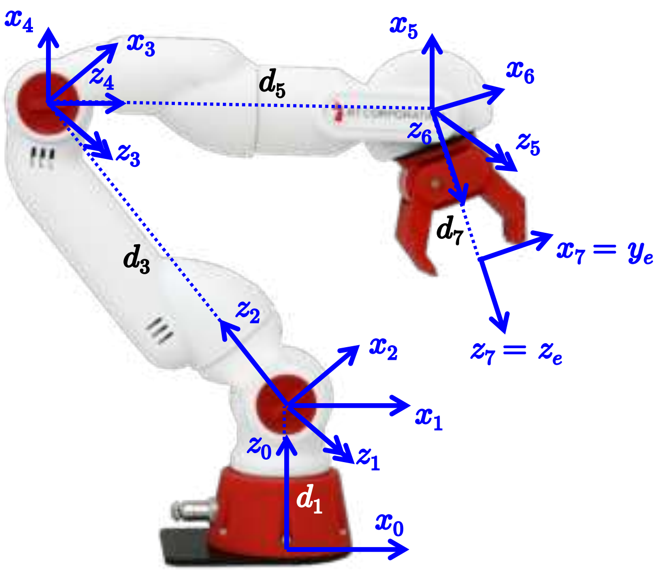


Figure 5: Assignment of D-H frames for the Crane-X7 robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 105$	$q_1 = 0$
2	$-\pi/2$	0	0	$q_2 > 0$
3	$\pi/2$	0	$d_3 = 250$	$q_3 = 0$
4	$\pi/2$	0	0	$q_4 > 0$
5	$-\pi/2$	0	$d_5 = 250$	$q_5 = 0$
6	$\pi/2$	0	0	$q_6 < 0$
7	0	0	$d_7 = 103$	$q_7 = 0$

Table 1: Table of D-H parameters for the frame assignment of Fig. 5. Lengths are in [mm].

The constant rotation matrix from the seventh D-H frame to the end-effector frame is

$$ {}^7\boldsymbol{R}_e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Exercise #2

The YXY sequence of Euler angles $(\alpha_1, \alpha_2, \alpha_3)$ defines the following rotation matrix:

$$ \begin{aligned} \boldsymbol{R}_{YXY} &= \boldsymbol{R}_Y(\alpha_1)\boldsymbol{R}_X(\alpha_2)\boldsymbol{R}_Y(\alpha_3) \\ &= \begin{pmatrix} \cos\alpha_1\cos\alpha_3 - \sin\alpha_1\cos\alpha_2\sin\alpha_3 & \sin\alpha_1\sin\alpha_2 & \cos\alpha_1\sin\alpha_3 + \sin\alpha_1\cos\alpha_2\cos\alpha_3 \\ \sin\alpha_2\sin\alpha_3 & \cos\alpha_2 & -\sin\alpha_2\cos\alpha_3 \\ -\sin\alpha_1\cos\alpha_3 - \cos\alpha_1\cos\alpha_2\sin\alpha_3 & \cos\alpha_1\sin\alpha_2 & \cos\alpha_1\cos\alpha_2\cos\alpha_3 - \sin\alpha_1\sin\alpha_3 \end{pmatrix}. \end{aligned} $$

Therefore, we evaluate the initial orientation by the rotation matrix

$$ \boldsymbol{R}_i = \boldsymbol{R}_{YXY}\left(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{3}\right) = \begin{pmatrix} -0.7866 & -0.5 & 0.3624 \\ -0.6124 & 0.7071 & -0.3536 \\ -0.0795 & -0.5 & -0.8624 \end{pmatrix}. $$

From the final rotation matrix

$$ \boldsymbol{R}_f = \begin{pmatrix} 0 & 0.8660 & 0.5 \\ 0 & 0.5 & -0.8660 \\ -1 & 0 & 0 \end{pmatrix}, $$

we compute the relative rotation matrix as

$$ {}^i\boldsymbol{R}_f = \boldsymbol{R}_i^T\boldsymbol{R}_f = \begin{pmatrix} 0.0795 & -0.9874 & 0.1370 \\ 0.5 & -0.0795 & -0.8624 \\ 0.8624 & 0.1370 & 0.4874 \end{pmatrix}. $$

Denoting by R_{ij} the elements of ${}^i\boldsymbol{R}_f$, we use the solution formulas of the inverse axis/angle representation problem. This is not a singular case since

$$ \sin\theta = \frac{1}{2}\sqrt{(R_{21}-R_{12})^2 + (R_{13}-R_{31})^2 + (R_{32}-R_{23})^2} = 0.9666 \neq 0. $$

Using $\cos\theta = \frac{1}{2}\left(\text{trace}\left\{{}^i\boldsymbol{R}_f\right\} - 1\right) = -0.2563$ and the atan2 function, we obtain the two solutions

$$ \theta' = 1.83 \text{ [rad]}, \quad \boldsymbol{r}' = \begin{pmatrix} 0.5170 \\ -0.3752 \\ 0.7694 \end{pmatrix} \qquad \text{and} \qquad \theta'' = -\theta', \quad \boldsymbol{r}'' = -\boldsymbol{r}'. $$

With a constant angular velocity $\boldsymbol{\omega} = 1.1 \cdot \boldsymbol{r}'$ [rad/s], one traces the total angle θ' in a time

$$ T_\omega = \frac{1.83}{1.1} = 1.6636 \text{ [s]}. $$

Exercise #3

We need to formulate the complete task at the second-order differential level, i.e., in terms of accelerations. For the positional task of the end-effector, we set the common length of the links to $L = 1$ and compute

$$\boldsymbol{p} = \begin{pmatrix} \cos q_1 + \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ \sin q_1 + \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{pmatrix} = \boldsymbol{f}_p(\boldsymbol{q}).$$

Differentiating once, we build the Jacobian for the positional task

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}_p(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} = \boldsymbol{J}_p(\boldsymbol{q})\, \dot{\boldsymbol{q}},$$

where the shorthand notation has been used for trigonometric quantities (e.g., $s_{12} = \sin(q_1 + q_2)$). Differentiating again, we have

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}_p(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_p(\boldsymbol{q})\, \dot{\boldsymbol{q}} = \boldsymbol{J}_p(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

with

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}_p(\boldsymbol{q})\, \dot{\boldsymbol{q}} = -\begin{pmatrix} c_1\, \dot{q}_1^2 + c_{12}\, (\dot{q}_1 + \dot{q}_2)^2 + c_{123}\, (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ s_1\, \dot{q}_1^2 + s_{12}\, (\dot{q}_1 + \dot{q}_2)^2 + s_{123}\, (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \end{pmatrix}. \tag{1}$$

As for the angular velocity of the end effector, in this planar case it is

$$\omega_z = \dot{q}_1 + \dot{q}_2 + \dot{q}_3,$$

and thus

$$\dot{\omega}_z = \ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \ddot{\boldsymbol{q}} = \boldsymbol{J}_\omega\, \ddot{\boldsymbol{q}}.$$

The requested task is executed by imposing the desired linear acceleration $\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d$ to the end effector, while zeroing its angular acceleration $\dot{\omega}_z = \dot{\omega}_{z,d} = 0$ in order to keep $\omega_z = \omega_{z,d} = \text{constant}$ (whatever this value may be). Thus, the Jacobian of the complete task will be the (3×3) matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_p(\boldsymbol{q}) \\ \boldsymbol{J}_\omega \end{pmatrix} = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ (c_1 + c_{12} + c_{123}) & (c_{12} + c_{123}) & c_{123} \\ 1 & 1 & 1 \end{pmatrix}, \tag{2}$$

which is singular if and only if $\det \boldsymbol{J}(\boldsymbol{q}) = \sin q_2 = 0$. As long as the robot is away from the singularity $q_2 = 0$ or π, we can solve the relation for the task accelerations

$$\begin{pmatrix} \ddot{\boldsymbol{p}} \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_p(\boldsymbol{q}) \\ \boldsymbol{J}_\omega \end{pmatrix} \ddot{\boldsymbol{q}} + \begin{pmatrix} \dot{\boldsymbol{J}}_p(\boldsymbol{q})\dot{\boldsymbol{q}} \\ 0 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \begin{pmatrix} \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ 0 \end{pmatrix} \tag{3}$$

in nominal conditions (i.e., for $\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d$ and $\dot{\omega}_z = \dot{\omega}_{z,d} = 0$) in terms of the joint acceleration command as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{pmatrix} \ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ 0 \end{pmatrix}. \tag{4}$$

At the current robot state, in the nominal conditions

$$\boldsymbol{q}_d = \begin{pmatrix} \pi/4 \\ \pi/3 \\ -\pi/2 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}_d = \begin{pmatrix} -0.8 \\ 1 \\ 0.2 \end{pmatrix} \text{ [rad/s]},$$

the position of the end effector and its linear velocity are, respectively,

$$\boldsymbol{p}_d = \boldsymbol{f}_p(\boldsymbol{q}_d) = \begin{pmatrix} 1.4142 \\ 1.9319 \end{pmatrix} \text{ [m]} \qquad \text{and} \qquad \dot{\boldsymbol{p}}_d = \boldsymbol{J}_p(\boldsymbol{q}_d)\,\dot{\boldsymbol{q}}_d = \begin{pmatrix} 0.2690 \\ -0.2311 \end{pmatrix} \text{ [m/s]},$$

while the end-effector angular velocity is

$$\omega_{z,d} = \dot{q}_{1,d} + \dot{q}_{2,d} + \dot{q}_{3,d} = 0.4 \text{ [rad/s]}.$$

For a desired linear acceleration of the end effector

$$\ddot{\boldsymbol{p}}_d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s}^2\text{]},$$

the joint acceleration command (4) is evaluated, using (1) and (2), as

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{pmatrix} \ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1.9319 & -1.2247 & -0.2588 \\ 1.4142 & 0.7071 & 0.9659 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -0.5967 \\ -0.5326 \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} 1.2322 \\ -3.7873 \\ 2.5551 \end{pmatrix} \text{ [rad/s}^2\text{]}. \end{aligned}$$

To correct an error (in any component) that may arise during the execution of the complete task by the robot end effector, feedback terms should be added to the nominal command (4). Due to the task structure, a proportional-derivative (PD) action is used on the error along the positional trajectory tracking task and a simpler proportional (P) action is used on the error in the regulation task of the angular velocity. The resulting law is

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{pmatrix} \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D\,(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) + \boldsymbol{K}_P\,(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ k_\omega\,(\omega_{z,d} - \omega_z) \end{pmatrix} \\ &= \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{pmatrix} \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D\,(\dot{\boldsymbol{p}}_d - \boldsymbol{J}_p(\boldsymbol{q})\,\dot{\boldsymbol{q}}) + \boldsymbol{K}_P\,\big(\boldsymbol{p}_d - \boldsymbol{f}_p(\boldsymbol{q})\big) - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ k_\omega\,(\omega_{z,d} - \boldsymbol{J}_\omega\,\dot{\boldsymbol{q}}) \end{pmatrix}, \end{aligned} \tag{5}$$

with diagonal gain matrices $K_P > 0$ and $K_D > 0$, and a scalar gain $k_\omega > 0$. In fact, define the task errors as

$$\boldsymbol{e}_p(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t) \quad \Rightarrow \quad \dot{\boldsymbol{e}}_p(t) = \boldsymbol{e}_v(t) = \dot{\boldsymbol{p}}_d(t) - \dot{\boldsymbol{p}}(t), \qquad e_\omega(t) = \omega_{z,d} - \omega_z(t).$$

Plugging (5) into (3) and simplifying terms, yields the two (decoupled) error dynamics

$$\ddot{\boldsymbol{e}}_p + \boldsymbol{K}_D \dot{\boldsymbol{e}}_p + \boldsymbol{K}_P \boldsymbol{e}_p = 0, \qquad \dot{e}_\omega + k_\omega e_\omega = 0,$$

whose evolutions will exponentially converge to zero thanks to the choice of the gains. Indeed, if only some of the errors are present, the same control law (5) will work accordingly.

Exercise #4

The problem is addressed by adding a via point P_{mid} in the Cartesian space so as to avoid the obstacle, converting then start, via, and goal points into the joint space of the PR robot, and fitting a spline trajectory to this to guarantee the desired smoothness. It is convenient to add the via point at the middle of the x-displacement D, sufficiently below the obstacle (but not too far away, so as to keep the robot travel limited). In the following, we choose $P_{mid} = (S + \Delta/2, L/4)$.

Note also that the start and goal positions correspond to a singularity for the PR robot (with $q_2 = \pi/2$, its Jacobian $\boldsymbol{J}_s = \boldsymbol{J}(q_2 = 0)$ has rank one). A simple path planning with straight lines joining P_{start} to P_{mid} and P_{mid} to P_{goal} would be unfeasible because the directions of these lines would not belong to $\mathcal{R}\{\boldsymbol{J}_s\}$ at P_{start} and P_{goal}. Moreover, the tangent discontinuity of that path at P_{mid} would force a stop of the robot, contrary to the desired motion requirements.

With the above in mind, we convert first the three positions in the joint space. We have immediately

$$P_{start} = \begin{pmatrix} S \\ L \end{pmatrix} \;\rightarrow\; \boldsymbol{q}_s = \begin{pmatrix} S \\ \pi/2 \end{pmatrix}, \qquad P_{goal} = \begin{pmatrix} S+\Delta \\ L \end{pmatrix} \;\rightarrow\; \boldsymbol{q}_g = \begin{pmatrix} S+\Delta \\ \pi/2 \end{pmatrix}.$$

For the via point, we use the closed-form expression of the inverse kinematics of the PR robot[2],

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_1 + L\cos q_2 \\ L\sin q_2 \end{pmatrix} \;\rightarrow\; \boldsymbol{q}^{[+/-]} = \begin{pmatrix} q_1^{[+/-]} \\ q_2^{[+/-]} \end{pmatrix} = \begin{pmatrix} p_x \pm \sqrt{L^2 - p_y^2} \\ \operatorname{atan2}\left\{p_y, p_x - q_1^{[+/-]}\right\} \end{pmatrix},$$

and choose the '$-$' solution (i.e., with the base of the second link on the left of P_{mid})

$$P_{mid} = \begin{pmatrix} S + \Delta/2 \\ L/4 \end{pmatrix} \;\rightarrow\; \boldsymbol{q}_m = \begin{pmatrix} S + \Delta/2 - L\sqrt{15}/4 \\ \operatorname{atan2}\left\{1, \sqrt{15}\right\} \end{pmatrix}.$$

Moreover, again because of the Cartesian symmetry, we will impose the passage through the via point P_{mid} (or, equivalently, through the joint configuration $\boldsymbol{q}_m$) at the mid motion time $t = T/2$.

The construction of the spline trajectory can be performed in a straightforward way, being composed by only two cubic polynomials (for each joint component), namely $\boldsymbol{q}_A(t)$, for $t \in [0, T/2]$, and $\boldsymbol{q}_B(t)$, for $t \in [T/2, T]$. Both $\boldsymbol{q}_A$ and $\boldsymbol{q}_B$ are (timed) vectors in $\mathbb{R}^2$. As in the general case of multiple cubics, we introduce the joint velocity at the mid knot as the (unknown) vector parameter $\boldsymbol{v}_m = \dot{\boldsymbol{q}}_A(T/2) = \dot{\boldsymbol{q}}_B(T/2) \in \mathbb{R}^2$. The two cubics are expressed in normalized times as

$$\boldsymbol{q}_A(\tau_A) = \boldsymbol{q}_s + \boldsymbol{c}_{A,1}\,\tau_A + \boldsymbol{c}_{A,2}\,\tau_A^2 + \boldsymbol{c}_{A,3}\,\tau_A^3, \quad \tau_A = \frac{t}{T/2} = \frac{2t}{T} \in [0,1] \tag{6}$$

and[3]

$$\boldsymbol{q}_B(\tau_B) = \boldsymbol{q}_g + \boldsymbol{c}_{B,1}\,(\tau_B - 1) + \boldsymbol{c}_{B,2}\,(\tau_B - 1)^2 + \boldsymbol{c}_{B,3}\,(\tau_B - 1)^3, \quad \tau_B = \frac{t - (T/2)}{T/2} = \frac{2t}{T} - 1 \in [0,1], \tag{7}$$

Next, we impose the boundary conditions. For $\boldsymbol{q}_A(\tau_A)$ in (6), we have

$$\begin{array}{lllll} t = 0 & \rightarrow \; \tau_A = 0: & \dot{\boldsymbol{q}}_A(0) = \boldsymbol{0} & \rightarrow & \boldsymbol{c}_{A,1} = \boldsymbol{0} \\ t = \dfrac{T}{2} & \rightarrow \; \tau_A = 1: & \boldsymbol{q}_A(1) = \boldsymbol{q}_m & \rightarrow & \boldsymbol{c}_{A,2} + \boldsymbol{c}_{A,3} = \boldsymbol{q}_m - \boldsymbol{q}_s \\ t = \dfrac{T}{2} & \rightarrow \; \tau_A = 1: & \dot{\boldsymbol{q}}_A(1) = \boldsymbol{v}_m & \rightarrow & 4\,\boldsymbol{c}_{A,2} + 6\,\boldsymbol{c}_{A,3} = \boldsymbol{v}_m\,T, \end{array}$$

[2]Remember that the two arguments of the atan2 function can be arbitrarily scaled by a positive factor.

[3]One could use also a cubic with powers of τ_B, rather than of $(\tau_B - 1)$. Indeed, the final result would be the same. The choice (7) gives time specularity to the treatment, introducing the goal value $\boldsymbol{q}_g$ as constant in the cubic.

yielding

$$\boldsymbol{c}_{A,2} = 3\,(\boldsymbol{q}_m - \boldsymbol{q}_s) - \frac{T}{2}\,\boldsymbol{v}_m, \qquad \boldsymbol{c}_{A,3} = 2\,(\boldsymbol{q}_s - \boldsymbol{q}_m) + \frac{T}{2}\,\boldsymbol{v}_m.$$

Similarly, for $\boldsymbol{q}_B(\tau_B)$ in (7), we have

$$\begin{array}{llllll} t = T & \rightarrow & \tau_B = 1: & \dot{\boldsymbol{q}}_B(1) = \boldsymbol{0} & \rightarrow & \boldsymbol{c}_{B,1} = \boldsymbol{0} \\ t = \dfrac{T}{2} & \rightarrow & \tau_B = 0: & \boldsymbol{q}_B(0) = \boldsymbol{q}_m & \rightarrow & \boldsymbol{c}_{B,2} - \boldsymbol{c}_{B,3} = \boldsymbol{q}_m - \boldsymbol{q}_g \\ t = \dfrac{T}{2} & \rightarrow & \tau_B = 0: & \dot{\boldsymbol{q}}_B(0) = \boldsymbol{v}_m & \rightarrow & -4\,\boldsymbol{c}_{B,2} + 6\,\boldsymbol{c}_{B,3} = \boldsymbol{v}_m\,T, \end{array}$$

yielding in this case

$$\boldsymbol{c}_{B,2} = 3\,\left(\boldsymbol{q}_m - \boldsymbol{q}_g\right) + \frac{T}{2}\,\boldsymbol{v}_m, \qquad \boldsymbol{c}_{B,3} = 2\,\left(\boldsymbol{q}_m - \boldsymbol{q}_g\right) + \frac{T}{2}\,\boldsymbol{v}_m.$$

Finally, we find the unknown value $\boldsymbol{v}_m$ by imposing continuity of the accelerations at the mid point instant $t = T/2$ (i.e., at $\tau_A = 1$ and $\tau_B = 0$). Since

$$\ddot{\boldsymbol{q}}_A(\tau_A) = \frac{8}{T^2}\,\left(\boldsymbol{c}_{A,2} + 3\,\boldsymbol{c}_{A,3}\,\tau_A\right), \qquad \ddot{\boldsymbol{q}}_B(\tau_B) = \frac{8}{T^2}\,\left(\boldsymbol{c}_{B,2} + 3\,\boldsymbol{c}_{B,3}\,(\tau_B - 1)\right),$$

we have

$$\ddot{\boldsymbol{q}}_A(1) = \ddot{\boldsymbol{q}}_B(0) \quad \Rightarrow \quad \boldsymbol{c}_{A,2} + 3\,\boldsymbol{c}_{A,3} = \boldsymbol{c}_{B,2} - 3\,\boldsymbol{c}_{B,3} \quad \Rightarrow \quad \boldsymbol{v}_m = \frac{3}{2T}\left(\boldsymbol{q}_g - \boldsymbol{q}_s\right).$$

Note that the value of $\boldsymbol{q}_m$ plays no role in the definition of the (unique) midpoint velocity $\boldsymbol{v}_m$. Moreover, it is clear that $\boldsymbol{v}_m \neq \boldsymbol{0}$, satisfying the condition of no stops during motion.

As a result, replacing the coefficients of the cubics with the symbolic expression that have been found, we obtain from (6) and (7)

$$\boldsymbol{q}_A(\tau_A) = \boldsymbol{q}_s + \frac{1}{4}\left(12\,\boldsymbol{q}_m - 3\,\boldsymbol{q}_g - 9\,\boldsymbol{q}_s\right)\tau_A^2 + \frac{1}{4}\left(5\,\boldsymbol{q}_s + 3\,\boldsymbol{q}_g - 8\,\boldsymbol{q}_m\right)\tau_A^3, \tag{8}$$

and

$$\boldsymbol{q}_B(\tau_B) = \boldsymbol{q}_g + \frac{1}{4}\left(12\,\boldsymbol{q}_m - 9\,\boldsymbol{q}_g - 3\,\boldsymbol{q}_s\right)(\tau_B - 1)^2 + \frac{1}{4}\left(8\,\boldsymbol{q}_m - 5\,\boldsymbol{q}_g - 3\,\boldsymbol{q}_s\right)(\tau_B - 1)^3. \tag{9}$$

Finally, we provide and example using the following numerical data:

$$L = 1, \qquad S = 0, \qquad \Delta = 3\ [\mathrm{m}], \qquad T = 4\ [\mathrm{s}]. \tag{10}$$

These lead to the following values in the previous formulas

$$P_{start} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad P_{mid} = \begin{pmatrix} 1.5 \\ 0.25 \end{pmatrix}, \qquad P_{goal} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

$$\boldsymbol{q}_s = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}, \qquad \boldsymbol{q}_m = \begin{pmatrix} 0.5318 \\ 0.2527 \end{pmatrix}, \qquad \boldsymbol{q}_g = \begin{pmatrix} 3 \\ \pi/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_m = \begin{pmatrix} 1.1250 \\ 0 \end{pmatrix},$$

from which

$$\boldsymbol{q}_A(\tau_A) = \begin{pmatrix} 0 \\ 1.5708 \end{pmatrix} - \begin{pmatrix} 0.6547 \\ 3.9543 \end{pmatrix}\tau_A^2 + \begin{pmatrix} 1.1865 \\ 2.6362 \end{pmatrix}\tau_A^3$$

and

$$\boldsymbol{q}_B(\tau_B) = \begin{pmatrix} 3 \\ 1.5708 \end{pmatrix} - \begin{pmatrix} 5.1547 \\ 3.9543 \end{pmatrix} (\tau_B - 1)^2 - \begin{pmatrix} 2.6865 \\ 2.6362 \end{pmatrix} (\tau_B - 1)^3 .$$

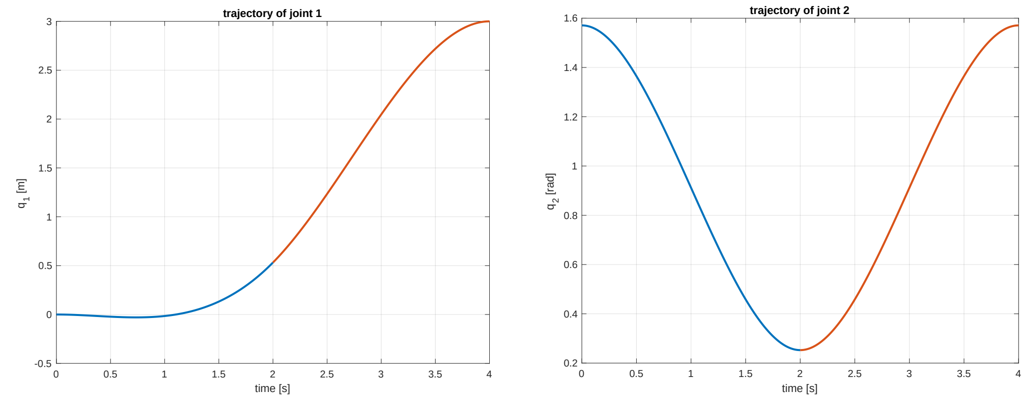


Figure 6: Spline joint trajectories of the PR robot.

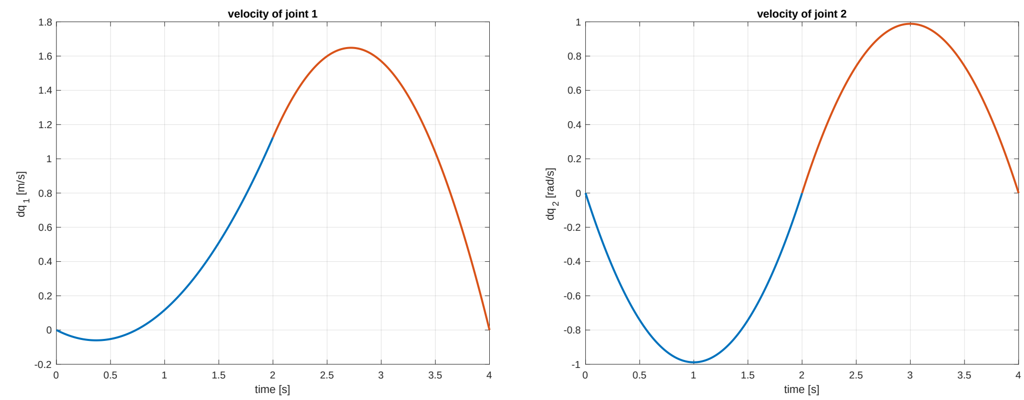


Figure 7: Joint velocities of the PR robot.

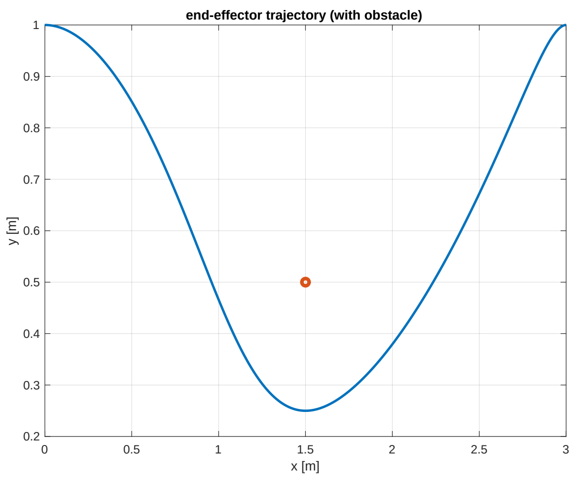


Figure 8: Cartesian path of the end effector of the PR robot. The obstacle (circle in red) is avoided.

Figures 6–8 show the results of the trajectory planning. The Cartesian path avoids the obstacle, although it is slightly asymmetric w.r.t. the via point P_{mid}. The first joint retracts a bit in the initial phase of the motion (when $\dot{q}_1 < 0$), before increasing constantly until the goal is reached. Note that the Cartesian path starts and ends with an horizontal tangent, being this the only admissible direction in the range of the singular Jacobian $\boldsymbol{J}$ at $\boldsymbol{q}_s$ and $\boldsymbol{q}_g$. Finally, there is no instant $t \in (0, T) = (0, 3)$ (excluding obviously the interval boundaries) such that $\dot{\boldsymbol{q}}(t) = \boldsymbol{0}$.

As an additional comment, the path will remain the same when scaling the motion time. Figure 9 shows the same instance of motion planning for $T = 1$ and $T = 10$. Indeed, the joint velocities have a scaled profile (by a factor $k = 4/1 = 4$ in the first case and $k = 4/10 = 0.4$ in the second).

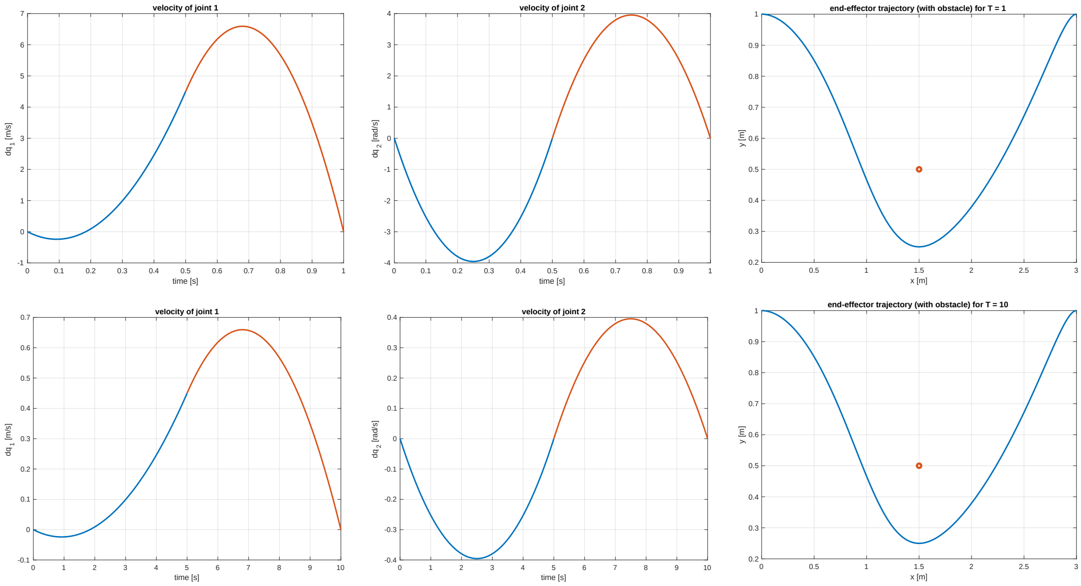


Figure 9: Trajectory planning for the same problem, with $T = 1$ [s] (top) and $T = 10$ [s] (bottom): joint velocities scale linearly while the Cartesian path remains the same.

Note in conclusion that the assumed condition $\Delta > L/2$ is quite stringent. For shorter displacements D along the x-direction between P_{start} and P_{goal}, one via point will not be sufficient to obtain obstacle avoidance of the Cartesian path, at least with the chosen joint space planning method. On the other hand, longer displacements Δ for a given link length L will provide more symmetric solutions. These two aspects are illustrated in Fig. 10 for $L = 1$. The Cartesian path hits the obstacle when $\Delta = 0.5L = 0.5$, while its is practically symmetric with $\Delta = 10L = 10$.

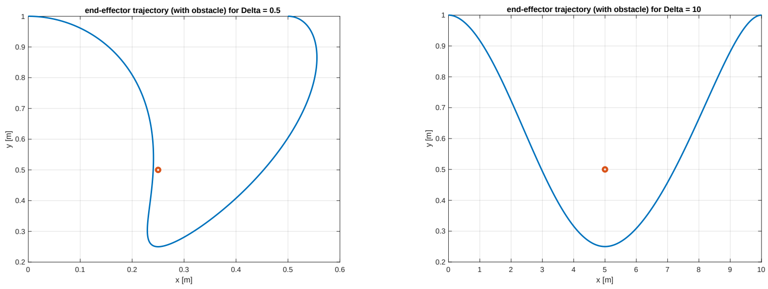


Figure 10: Cartesian paths resulting from the used trajectory planning method for the same problem, but with $D = 0.5$ [m] (left) and $D = 10$ [m] (right).

Final remark. An alternative solution can be developed when assuming that the range of the second (revolute) joint is unlimited. In this case, one could replace the final target configuration $\boldsymbol{q}_g = \begin{pmatrix} S+\Delta & \pi/2 \end{pmatrix}^T$ by $\boldsymbol{q}'_g = \begin{pmatrix} S+\Delta & \pi/2+2\pi \end{pmatrix}^T$, and let a single smooth trajectory (a cubic or quintic polynomial) connect $\boldsymbol{q}_s$ to $\boldsymbol{q}'_g$. The second link would make then a complete counter-clockwise rotation of 360° while the first joint is translating forward, avoiding thus the obstacle. However, even in the absence of joint limits, the price to pay with this strategy is a much higher speed reached by joint 2 at the trajectory midpoint (the motion should be coordinated, i.e., started and completed at the same instants of time for both joints). With the numerical values in (10), there would be a peak $\dot{q}_{2,max} = \dot{q}_2(T/2) = 3\pi/T \simeq 2.36$ [rad/s] for a rest-to-rest cubic trajectory and a peak $\dot{q}_{2,max} = \dot{q}_2(T/2) = 3.75\pi/T \simeq 2.95$ [rad/s] for a quintic trajectory (with zero acceleration at the boundaries), as opposed to $|\dot{q}_{2,max}| = 1$ [rad/s] with the solution in Fig. 7.

Exercise #5

The reduction ratio of the complete transmission is the product of the ratio $n_{r,g} = r_1/r_2$ of the radiuses of the two gear wheels, times the ratio $n_{r,p} = r_3/r_4$ of the radiuses of the two pulleys connected by the belt. While the gears invert the rotation direction, the pulleys preserve the same direction. Thus, the link will rotate in the opposite way of the motor (around their respective axes, $\boldsymbol{z}_m$ and $\boldsymbol{z}_j$). The following (symbolic/numeric) Matlab code computes the complete reduction ratio

$$n_r = \left|\frac{\dot{\theta}_m}{\dot{\theta}_l}\right| \geq 1$$

of the transmission and, accordingly, the time $T_\theta > 0$ needed for the link to rotate by 90°. With the given numerical values, when the motor spins at $\dot{\theta}_m = 10$ [rad/s], it is

$$n_r = 12, \qquad \dot{\theta}_l = -\frac{\dot{\theta}_m}{n_r} = -0.8333 \text{ [rad/s]}, \qquad T_\theta = \frac{\pi/2}{|\dot{\theta}_l|} = \frac{\pi/2}{|\dot{\theta}_m|/n_r} = 1.8850 \text{ [s]},$$

and the link rotates clockwise (CW).

```
syms r1 r2 r3 r4 dtheta_m real  % D and L are irrelevant...

% reduction ratio (symbolic)

dtheta_g=-(r1/r2)*dtheta_m
dtheta_l=(r3/r4)*dtheta_g
nr=abs(dtheta_m/dtheta_l)

% time for 90 [deg] of link rotation

T_th=(pi/2)/abs(dtheta_l)

% numerical values (radiuses in [mm])

dtheta_m=10     % rad/s
dtheta_g=subs(dtheta_g,{r1,r2},{20,60});
dtheta_g=eval(dtheta_g)
dtheta_l=subs(dtheta_l,{r1,r2,r3,r4},{20,60,8,32});
dtheta_l=eval(dtheta_l)
```

```
nr=subs(nr,{r1,r2,r3,r4},{20,60,8,32});
nr=eval(nr)
if dtheta_l > 0
   disp('link rotation is CCW')
else
   disp('link rotation is CW')
end
T_th=subs(T_th,{r1,r2,r3,r4},{20,60,8,32});
T_th=eval(T_th)

% end
```

* * * * *

Robotics I

April 5, 2022

Exercise 1

Consider the spatial 4R robot shown in Fig. 1.

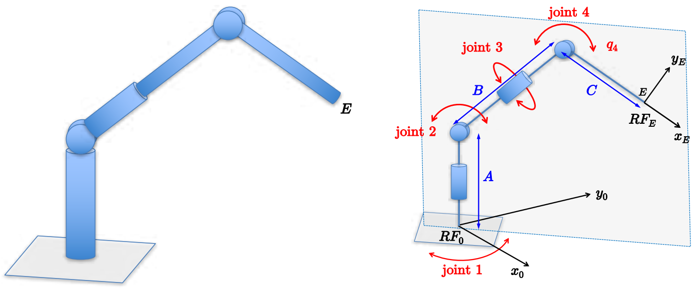


Figure 1: A 4R robot and its kinematic skeleton.

- Assign a set of frames to the robot according to the Denavit-Hartenberg convention and provide the associated table of parameters. Keep the reference frame RF_0 as shown in the figure.
- Determine the homogeneous transformation ${}^4\boldsymbol{T}_E$ from the assigned Denavit-Hartenberg frame RF_4 to the end-effector frame RF_E shown in the figure.
- Compute the symbolic expression of the position ${}^0\boldsymbol{p}_E(\boldsymbol{q})$ of the origin of the end-effector frame by using the minimum amount of operations. Show all intermediate passages. For $A = B = C = 1$, give the numerical value of the position ${}^0\boldsymbol{p}_E$ when $\boldsymbol{q} = (\pi/2, \pi/2, 0, 0)$.
- Compute the angular part of the geometric Jacobian, namely the 3×4 matrix $\boldsymbol{J}_A(\boldsymbol{q})$ such that
$$\boldsymbol{\omega}_E = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}},$$
and find all its singularities.
- Find the symbolic expression (as a function of the configuration $\boldsymbol{q}$) of a non-trivial joint velocity $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$ such that $\boldsymbol{\omega}_E = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$ for all possible $\boldsymbol{q}$.

Exercise 2

Consider the motion profile in Fig. 2 for a generic robot joint, parametrized by the amplitude $J > 0$ and the duration $T > 0$. This time profile represents the motion jerk, namely the third time derivative of the joint position $q(t)$, for $t \in [0, T]$.

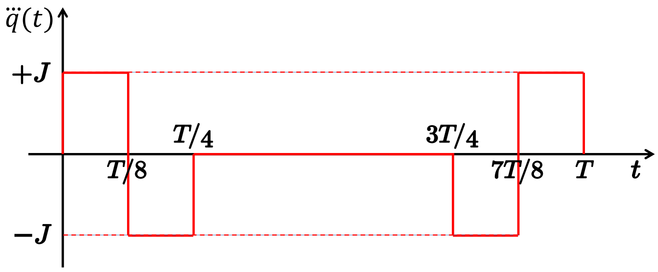


Figure 2: The jerk profile $\dddot{q}(t)$ of the joint motion.

- For a (rest-to-rest) motion with zero boundary conditions on velocity and acceleration, determine the value of the net displacement $\Delta = q(T) - q(0)$ as a function of J and T.
- For $J = 100$ [rad/s^3] and $T = 2$ [s], provide the numerical value of Δ. If we wish to have a displacement $\Delta = -2$ [rad] in $T = 4$ [s], what should be the numerical value of J?

[180 minutes, open books]

Robotics I

June 10, 2022

Exercise 1

Consider the spatial 3R robot in Fig. 1.

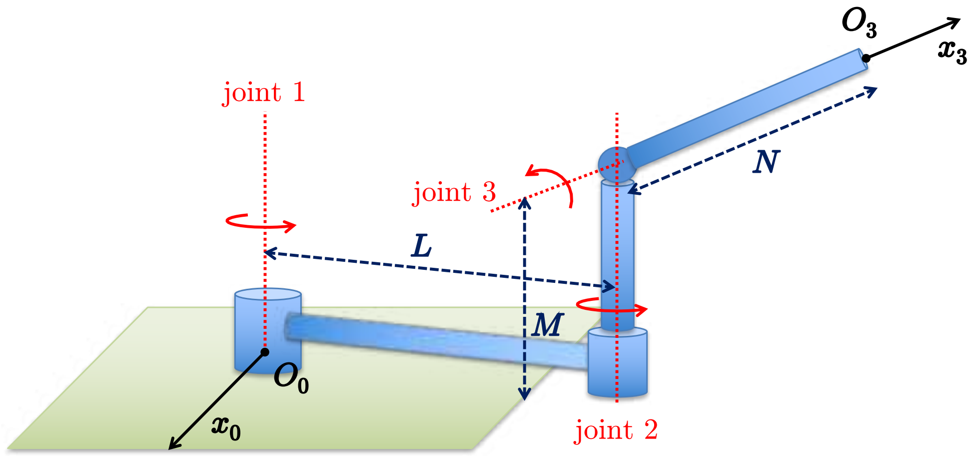


Figure 1: A spatial 3R robot.

- Assign a set of frames to this robot according to the Denavit-Hartenberg (D-H) convention and provide the associated table of parameters. Keep the origins O_0 and O_3 and the axes x_0 and x_3 as shown in the figure, respectively in frame RF_0 and frame RF_3. Indicate also the signs taken by the joint variables q_i, $i = 1, 2, 3$, in the robot configuration shown in Fig. 1.
- Compute the direct kinematics for the position $\boldsymbol{p} = \boldsymbol{p}_3$ of the end effector, i.e., the point O_3.
- Draw accurately the primary workspace of this robot.
- Provide the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of the robot in

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

  and determine all the kinematic singularities, each with the associated rank of $\boldsymbol{J}(\boldsymbol{q})$.
- In a singularity $\boldsymbol{q}_s$ where rank $\boldsymbol{J}(\boldsymbol{q}_s) = 1$, find an admissible end-effector velocity $\boldsymbol{v}_s \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}_s \in \mathbb{R}^3$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s \neq \boldsymbol{0}$. Is such $\dot{\boldsymbol{q}}_s$ unique for a given admissible end-effector velocity $\boldsymbol{v}_s$?

Exercise 2

A planar RP robot is commanded at the acceleration level. Its end-effector position is given by

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}. \tag{1}$$

If the robot is in a generic nonsingular configuration $\boldsymbol{q}$ and with non-zero velocities for both joints, determine the explicit expression of a command $\ddot{\boldsymbol{q}}$ such that the end-effector acceleration is instantaneously $\ddot{\boldsymbol{p}} = \boldsymbol{0}$. Is this command unique?

Exercise 3

Consider again the same RP robot of Exercise #2. Suppose that the generalized forces $\boldsymbol{\tau} \in \mathbb{R}^2$ that the robot actuators can provide at the two joints are bounded componentwise as

$$|\tau_1| \leq \tau_{max,1} = 10\,[\text{Nm}], \qquad |\tau_2| \leq \tau_{max,2} = 5\,[\text{N}].$$

In the configuration $\boldsymbol{q} = (\pi/3, 1.5)$ [rad,m], find the set of feasible Cartesian forces $\boldsymbol{F} = (F_x, F_y) \in \mathbb{R}^2$ (expressed in [N]) which can applied to the end effector and that the robot can sustain while in static equilibrium.

Exercise 4

The end-effector of a 2R planar robot with unitary link lengths has to track a linear path with constant speed $v_d = 0.5$ [m/s] between $\boldsymbol{P}_1 = (1, 0.5)$ and $\boldsymbol{P}_2 = (1, 1.5)$ [m]. However, at the initial time $t = 0$, the end effector is positioned in $\boldsymbol{P}_0 = (0.5, 0.5)$ [m]. The robot is commanded by a joint velocity $\dot{\boldsymbol{q}}$ that is limited componentwise as

$$|\dot{q}_1| \leq V_{max,1} = 3\,[\text{rad/s}], \qquad |\dot{q}_2| \leq V_{max,2} = 2\,[\text{rad/s}].$$

Design a kinematic control law that is able to achieve the fastest exponential convergence to zero of the trajectory tracking error, uniformly in all Cartesian directions, while being still feasible in terms of robot commands at time $t = 0$ for the given task. Provide some discussion on where/how fast the return to the original trajectory will be achieved.

[180 minutes, open books]

Solution

June 10, 2022

Exercise 1

A D-H frame assignment for the spatial 3R robot is shown in Fig. 2, with the associated table of D-H parameters given in Tab. 1.

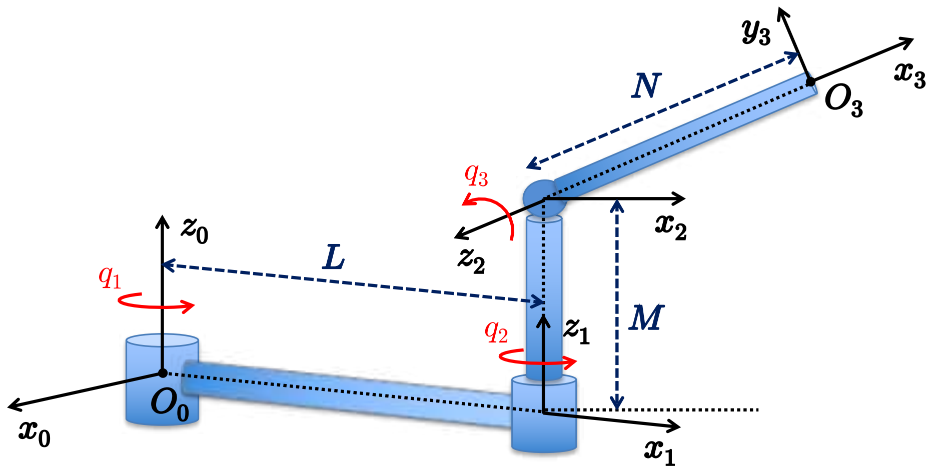


Figure 2: A D-H frame assignment for the spatial 3R robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	0	L	0	$q_1 > 0$
2	$\pi/2$	0	M	$q_2 > 0$
3	0	N	0	$q_3 > 0$

Table 1: Table of D-H parameters associated to Fig. 2.

Based on Tab. 1, one can evaluate the D-H homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, 2, 3$. An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p} = \boldsymbol{p}_3(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$\begin{pmatrix} \boldsymbol{p}_3(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right) = \begin{pmatrix} L\cos q_1 + N\cos(q_1+q_2)\cos q_3 \\ L\sin q_1 + N\sin(q_1+q_2)\cos q_3 \\ M + N\sin q_3 \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}. \tag{2}$$

It is easy to verify that the following inequalities on the components of the position of the end effector should *necessarily* hold

$$|L-N| \leq \sqrt{p_x^2 + p_y^2 + (p_z - M)^2} \leq L + N, \qquad M - N \leq p_z \leq M + N$$

in order for $\boldsymbol{p}$ to belong to the primary (or reachable) workspace WS_1 of the robot, namely the set of all points in $\mathbb{R}^3$ that can be reached by the end-effector position. These inequalities are also helpful for sketching WS_1. As shown in Fig. 3, the workspace is in fact a solid torus parallel to the (x_0, y_0) plane, with center at $(0, 0, M)$, inner radius $R_{in} = |L - N|$ and outer radius $R_{out} = L + N$. Any vertical section of the 3D object with a plane passing though the origin is a circle of radius $r = N$.

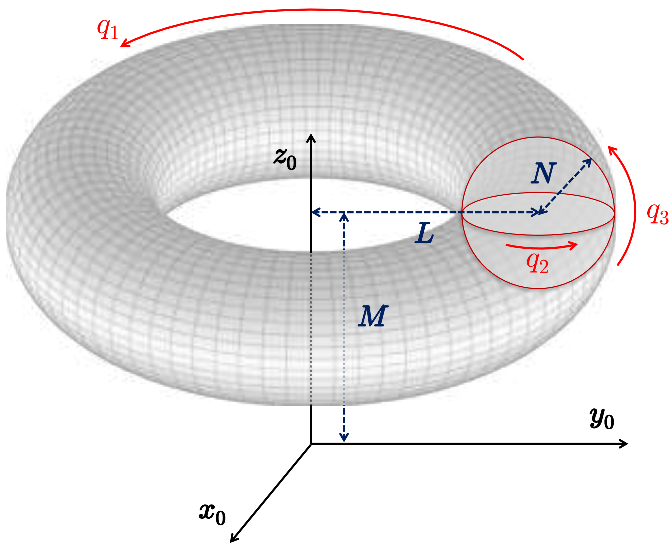


Figure 3: The primary workspace of the spatial 3R robot of Fig. 1.

Differentiating the first three components in (2), we obtain $\boldsymbol{v} = \dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ with the Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 - Ns_{12}c_3 & -Ns_{12}c_3 & -Nc_{12}s_3 \\ Lc_1 + Nc_{12}c_3 & Nc_{12}c_3 & -Ns_{12}s_3 \\ 0 & 0 & Nc_3 \end{pmatrix}, \tag{3}$$

where the compact notation for trigonometric functions has been used (e.g., $s_{12} = \sin(q_1 + q_2)$). The determinant of $\boldsymbol{J}(\boldsymbol{q})$ is

$$\det \boldsymbol{J}(\boldsymbol{q}) = L\, N^2 s_2\, c_3^2,$$

which is independent from q_1 as it should be. Therefore, singularities occur when:

- $s_2 = 0 \iff q_2 = 0$ or $q_2 = \pi$: the three links live in the vertical plane (x_1, z_0).

  The rank of the Jacobian is then always $\rho(\boldsymbol{J}) = 2$, for all q_3. This can be seen also more clearly expressing the Jacobian in the rotated frame RF_1. For instance, when $q_2 = 0$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=0} = \begin{pmatrix} -s_1(L + Nc_3) & -Ns_1c_3 & -Nc_1s_3 \\ c_1(L + Nc_3) & Nc_1c_3 & -Ns_1s_3 \\ 0 & 0 & Nc_3 \end{pmatrix},$$

$${}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=0} = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q})\Big|_{q_2=0} = \begin{pmatrix} 0 & 0 & -Ns_3 \\ L + Nc_3 & Nc_3 & 0 \\ 0 & 0 & Nc_3 \end{pmatrix}.$$

- $c_3 = 0 \iff q_3 = \pi/2$ or $q_3 = -\pi/2$: the third link is straight vertical. In this case, $\rho(\boldsymbol{J}) = 2$, for all $q_2 \neq \pm\pi/2$. For instance, when $q_3 = \pi/2$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=\pi/2} = \begin{pmatrix} -Ls_1 & 0 & -Nc_{12} \\ Lc_1 & 0 & -Ns_{12} \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=\pi/2} = \begin{pmatrix} 0 & 0 & -Nc_2 \\ L & 0 & -Ns_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

- In particular[1], when $c_3 = 0$ and $c_2 = 0$, the rank drops further to $\rho(\boldsymbol{J}) = 1$. For instance, when $q_2 = q_3 = \pi/2$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=q_3=\pi/2} = \begin{pmatrix} -Ls_1 & 0 & Ns_1 \\ Lc_1 & 0 & -Nc_1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}\left\{ \boldsymbol{J}(\boldsymbol{q})\big|_{q_2=q_3=\pi/2} \right\} = \text{span}\left\{ \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}.$$

Consider now this last case, with $\boldsymbol{q}_s$ such that $q_2 = q_3 = \pi/2$. In this singularity, any admissible end-effector velocity $\boldsymbol{v}_s$, as well as the infinite set of joint velocities $\dot{\boldsymbol{q}}_s$ that will realize them, will be of the form

$$\boldsymbol{v}_s = \alpha \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \quad \forall \alpha \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_s = \begin{pmatrix} \beta \\ 0 \\ \gamma \end{pmatrix}, \quad \text{with } \gamma N - \beta L = \alpha.$$

Thus, for a given α, there will be infinite possible solutions $\dot{\boldsymbol{q}}_s$. For instance, for $\alpha = 1$, the joint velocity solution with minimum norm[2] and a generic second solution are

$$\dot{\boldsymbol{q}}_{s,1} = \boldsymbol{J}^{\#}(\boldsymbol{q})\Big|_{q_2=q_3=\pi/2} \boldsymbol{v}_s = \frac{1}{L^2+N^2} \begin{pmatrix} -L \\ 0 \\ N \end{pmatrix}, \qquad \dot{\boldsymbol{q}}_{s,2} = \frac{1}{L} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Exercise 2

Differentiating eq. (1) once and twice w.r.t. time gives

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} -q_2 \cos q_1\, \dot{q}_1^2 - 2\sin q_1\, \dot{q}_1\dot{q}_2 \\ -q_2 \sin q_1\, \dot{q}_1^2 + 2\cos q_1\, \dot{q}_1\dot{q}_2 \end{pmatrix}.$$

Therefore, in order to obtain $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ out of a singular configuration ($q_2 \neq 0$), the *unique* choice for the joint acceleration is

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = -\frac{1}{q_2} \begin{pmatrix} -2\dot{q}_1\dot{q}_2 \\ q_2^2\dot{q}_1^2 \end{pmatrix}.$$

We note also that it will never be possible to obtain $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in a singularity, when the product $\dot{q}_1\dot{q}_2 \neq 0$ (i.e., in the generic case for $\dot{\boldsymbol{q}} \neq \boldsymbol{0}$).

[1] The further exploration of what happens in the singularity $c_3 = 0$ is also suggested by the fact that this factor appears as squared in the symbolic expression of the determinant of the Jacobian.

[2] The pseudoinverse can be computed symbolically with MATLAB in this simple case.

Exercise 3

The mapping between Cartesian forces $\boldsymbol{F} \in \mathbb{R}^2$ applied at the end effector of the RP robot and balancing joint torques $\boldsymbol{\tau} \in \mathbb{R}^2$ guaranteeing static equilibrium is given by

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} = -\begin{pmatrix} -q_2 \sin q_1 & q_2 \cos q_1 \\ \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}, \tag{4}$$

thus being linear at a given configuration $\boldsymbol{q}$. Vice versa, balancing joint torques map into Cartesian forces as

$$\boldsymbol{F} = -\boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{\tau} = \frac{1}{q_2} \begin{pmatrix} \sin q_1 & -q_2 \cos q_1 \\ -\cos q_1 & -q_2 \sin q_1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}.$$

This mapping will transform the rectangular region of feasible joint torques (whose vertices are given by the four combinations of signs in $\boldsymbol{\tau} = (\pm\tau_{max,1}, \pm\tau_{max,2})$ into a polytope (here, a convex polygon) of admissible Cartesian forces $\boldsymbol{F} = (F_x, F_y)$ that can be applied at the robot end effector and effectively balanced. At $\boldsymbol{q} = (\pi/3, 1.5)$ [rad,m], the inverse of the Jacobian transpose is

$$\bar{\boldsymbol{J}}^{-T} = \boldsymbol{J}^{-T}(\boldsymbol{q})\Big|_{\boldsymbol{q}=(\pi/3,1.5)} = \begin{pmatrix} -0.5774 & 0.5000 \\ 0.3333 & 0.8660 \end{pmatrix},$$

and the four vertices of this Cartesian region are computed as

$$\boldsymbol{F}_{++} = -\bar{\boldsymbol{J}}^{-T} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 3.2735 \\ -7.6635 \end{pmatrix} \qquad \boldsymbol{F}_{+-} = -\bar{\boldsymbol{J}}^{-T} \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 8.2735 \\ 0.9968 \end{pmatrix}$$

$$\boldsymbol{F}_{--} = -\bar{\boldsymbol{J}}^{-T} \begin{pmatrix} -10 \\ -5 \end{pmatrix} = \begin{pmatrix} -3.2735 \\ 7.6635 \end{pmatrix} \qquad \boldsymbol{F}_{-+} = -\bar{\boldsymbol{J}}^{-T} \begin{pmatrix} -10 \\ +5 \end{pmatrix} = \begin{pmatrix} -8.2735 \\ -0.9968 \end{pmatrix}.$$

The resulting admissible region is shown (in blue) in Fig. 4 (try to verify the correspondence between the vertices).

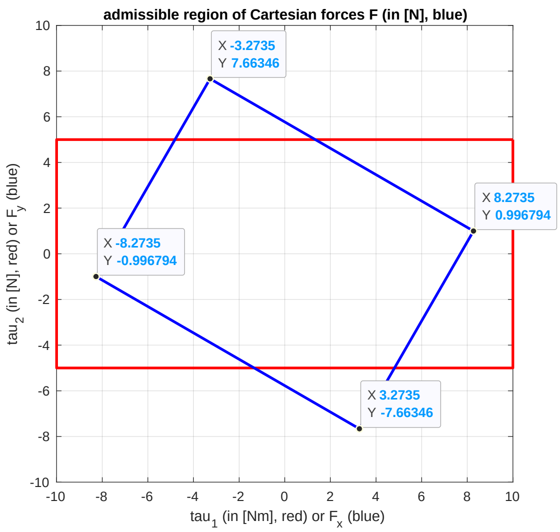


Figure 4: The set of feasible joint torques (rectangle in red) and the region of associated admissibile Cartesian forces (skewed rectangle in blue) that can be statically balanced by the RP robot.

For an additional check, take one Cartesian force that belongs to the blue region and is close to a boundary, and compute the balancing torque by (4) to verify its feasibility. For instance, with

$$\boldsymbol{F} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \text{ [N]} \qquad \Rightarrow \qquad \boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\Big|_{\boldsymbol{q}=(\pi/3,1.5)} \boldsymbol{F} = \begin{pmatrix} -9.6962 \\ -3.1962 \end{pmatrix} \text{ [Nm,N]},$$

the obtained $\boldsymbol{\tau}$ is feasible.

Exercise 4

To address the problem one applies the following Cartesian kinematic control,

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\Big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\right)\Big), \qquad \text{with } \boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2} > 0, \tag{5}$$

where the common scalar gain k_p i s used in both Cartesian directions because of the requested uniformity of error behavior. For the given 2R planar robot and motion task, we have

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix},$$

$$\boldsymbol{p}_d(t) = \boldsymbol{P}_1 + v_d t\big(\boldsymbol{P}_2 - \boldsymbol{P}_1\big) = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} + 0.5\, t \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_d = v_d\big(\boldsymbol{P}_2 - \boldsymbol{P}_1\big) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}.$$

The initial position of the end effector $\boldsymbol{P}_0 = (0.5, 0.5)$ [m] corresponds to an initial Cartesian error at $t = 0$ that is non-zero only along the x-direction

$$\boldsymbol{e}_p(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(\boldsymbol{q}(0)) = \boldsymbol{P}_1 - \boldsymbol{P}_0 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} e_{p,x}(0) \\ e_{p,y}(0) \end{pmatrix}.$$

Moreover, from (5) it follows that $\dot{\boldsymbol{e}}_p = -\boldsymbol{K}_P\, \boldsymbol{e}_p$ and so

$$\boldsymbol{e}_p(t) = \exp\left(-\boldsymbol{K}_p t\right) \boldsymbol{e}_p(0) \qquad \Rightarrow \qquad \begin{cases} e_{p,x}(t) = \exp\left(-k_P t\right) e_{p,x}(0) \\ e_{p,y}(t) = 0, \end{cases} \qquad \forall t \geq 0.$$

The initial configuration of the robot at time $t = 0$ is found by the standard inverse kinematics of a 2R robot (choosing the elbow down solution[3]:

$$\boldsymbol{q}(0) = \text{invkin}\left(\boldsymbol{P}_{in}\right) = \begin{pmatrix} -0.4240 \\ 2.4189 \end{pmatrix} \text{ [rad]}.$$

Plugging all the above information in (5) yields at time $t = 0$

$$\dot{\boldsymbol{q}}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}(0))\left(\begin{pmatrix} 0 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.5\, k_P \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -0.5 & -0.9114 \\ 0.5 & -0.4114 \end{pmatrix}^{-1} \begin{pmatrix} 0.5\, k_P \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} -0.6220 & 1.3780 \\ -0.7559 & -0.7559 \end{pmatrix} \begin{pmatrix} 0.5\, k_P \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6890 \\ -0.3780 \end{pmatrix} + k_P \begin{pmatrix} -0.3110 \\ -0.3780 \end{pmatrix}.$$

[3] The choice of the elbow up solution would lead exactly to the same final result in this case, although passing through different numerical values in intermediate passages.

Therefore, the largest (positive) proportional control gain that can be used to speed up the decrease of the transient error along the x-direction while satisfying the joint velocity bounds on $\dot{\boldsymbol{q}}(0)$,

$$-V_{max,1} = -3 \leq 0.6890 - 0.3110\, k_P \leq 3 = V_{max,1},$$

$$-V_{max,2} = -2 \leq 0.3780 - 0.3780\, k_P \leq 2 = V_{max,2},$$

is computed as follows:

$$k_P^* = \min\left\{\frac{V_{max,1} + 0.6890}{0.3110}, \frac{V_{max,2} + 0.3780}{0.3780}\right\} = \min\{11.8610, 4.2915\} = 4.2915.$$

This choice will saturate the initial velocity of joint 2 to its largest negative value $\dot{q}_2(0) = -V_{max,2} = -2$ rad/s. The solution is in fact

$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} -0.6458 \\ -2 \end{pmatrix} \text{[rad/s]} \qquad \Rightarrow \qquad \boldsymbol{v}(0) = \boldsymbol{J}(\boldsymbol{q}(0))\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 2.1458 \\ 0.5000 \end{pmatrix} \text{[rad/s]},$$

with the end-effector velocity pointing up and toward the path. See also the sketch of the initial situation in Fig. 5.
The time constant of the exponential decrease of the tracking error is $\tau_P = 1/k_P^* = 0.233$ [s]. This means that the error will be practically zero (i.e., reduced to less than 5% of its initial value) in about $3\tau_p \simeq 0.7$ [s], namely when the nominal trajectory is still at 1/3 of its total travel time ($T = \|\boldsymbol{P}_2 - \boldsymbol{P}_1\|/v_d = 2$ [s]).

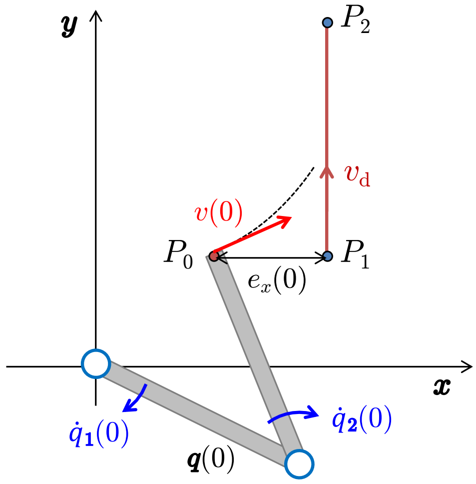


Figure 5: The 2R robot in the initial configuration, recovering the tracking error w.r.t. the desired trajectory.

* * * * *

Robotics I

July 8, 2022

Exercise 1

Assume that the time-varying orientation of a robot end-effector is expressed by the triplet of angles $\boldsymbol{\phi}(t) = (\alpha(t), \beta(t), \gamma(t))$ defined around the sequence of fixed axes XZY (an RPY-type representation). Determine the relationship between $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector and find the representation singularities of this mapping. In one such singularity, determine all vectors $\boldsymbol{\omega}$ that cannot be represented by a choice of $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and, conversely, find all non-trivial values for $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ that are associated to $\boldsymbol{\omega} = \mathbf{0}$.

Exercise 2

Figure 1 shows a spatial 3R robot with its Denavit-Hartenberg (D-H) frames and the definition of joint variables and kinematic parameters. The direct kinematics for the end-effector position (point O_3) of this robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} L\cos q_1 + N\cos(q_1+q_2)\cos q_3 \\ L\sin q_1 + N\sin(q_1+q_2)\cos q_3 \\ M + N\sin q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

Provide the closed-form expression of all inverse kinematics solutions $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p}_d)$ that realize a desired end-effector position $\boldsymbol{p}_d \in WS_1$. Apply your formulas with the following numerical data

$$L = M = N = 0.5 \text{ [m]}, \qquad \boldsymbol{p}_d = \begin{pmatrix} 0.3 & -0.3 & 0.7 \end{pmatrix}^T \text{ [m]},$$

and check at the end the correctness of the obtained numerical solutions by using the direct kinematics (1).

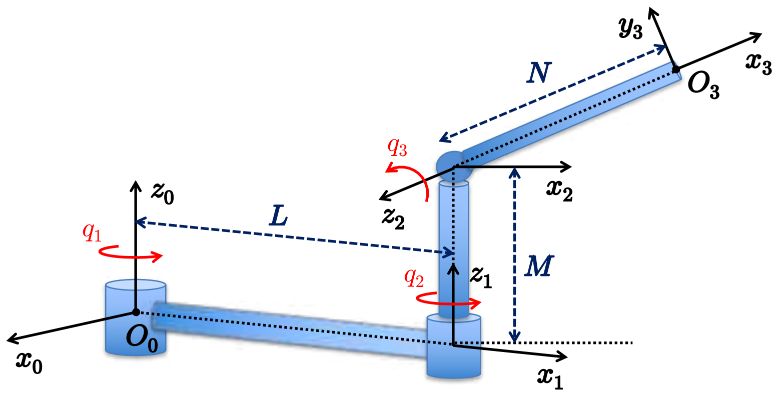


Figure 1: A spatial 3R robot with its D-H frames.

Exercise 3

Using the same numerical data of Exercise 2, solve the inverse kinematics problem with an iterative scheme based on Newton method. Start with the initial guess $\boldsymbol{q}^{\{0\}} = (-\pi/4, \pi/4, \pi/4)$ [rad] and list the values $\boldsymbol{q}^{\{k\}}$, $k = 1, 2, \ldots$, of the first few iterations, until the error $\boldsymbol{e}^{\{k\}} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{\{k\}})$ is such that $\left\|\boldsymbol{e}^{\{k\}}\right\| \leq \epsilon = 10^{-3}$ [m] (i.e., small enough, meaning that convergence has been achieved). How would you search for another inverse kinematics solution using this iterative method?

Exercise 4

Consider a trajectory planning problem for the 3R robot in Fig. 1. The robot should move from the start configuration $\boldsymbol{q}_s = (-\pi/4, \pi/4, \pi/4)$ [rad] to the goal configuration $\boldsymbol{q}_g = (0, 0, \pi/4)$ [rad] in a time $T = 2$ s, with continuity up to the acceleration over the whole interval $t \in [0, T]$. The initial joint velocity is chosen so that the end-effector velocity starts with $\dot{\boldsymbol{p}}(0) = (1, -1, 0)$ [m/s], while the final velocity should be zero. Provide the values of the coefficients of the *doubly normalized* joint trajectories satisfying all the given conditions. Sketch the plots of joint position, velocity and acceleration.

[240 minutes, open books]

Solution

July 8, 2022

Exercise 1

When using the sequence of RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ defined around the fixed axes XZY, the orientation of the robot end-effector is given by the rotation matrix

$$\begin{aligned}
\boldsymbol{R}_{XZY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta)\boldsymbol{R}_X(\alpha) \\
&= \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\beta & \cos\alpha\cos\beta & -\sin\alpha\cos\beta \\ -\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma \end{pmatrix}.
\end{aligned}$$

Note that this is the same orientation obtained by using the sequence of Euler angles (γ, β, α) defined around the moving axes YZ′X″.

The angular velocity $\boldsymbol{\omega}$ of the body can be obtained from the formula $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{XZY}(\boldsymbol{\phi})\boldsymbol{R}^T_{XZY}(\boldsymbol{\phi})$, where $\boldsymbol{S}$ is a skew-symmetric matrix. With the shorthand notation for trigonometric functions, taking the time derivative of $\boldsymbol{R}_{XZY}$ and post-multiplying by the transpose of the same rotation matrix yields

$$\begin{aligned}
&\dot{\boldsymbol{R}}_{XZY}(\boldsymbol{\phi}) \cdot \boldsymbol{R}^T_{XZY}(\boldsymbol{\phi}) \\
&= \begin{pmatrix} -s_\beta c_\gamma\dot\beta - c_\alpha s_\beta\dot\gamma & \begin{array}{r}(c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma)\dot\alpha - c_\alpha c_\beta c_\gamma\,\dot\beta \\ +\,(s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma)\dot\gamma\end{array} & \begin{array}{r}(c_\alpha s_\beta c_\gamma - s_\alpha s_\gamma)\dot\alpha + s_\alpha c_\beta c_\gamma\,\dot\beta \\ +\,(c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma)\dot\gamma\end{array} \\ c_\beta\dot\beta & -s_\alpha c_\beta\dot\alpha - c_\alpha s_\beta\dot\beta & -c_\alpha c_\beta\dot\alpha + s_\alpha s_\beta\dot\beta \\ s_\beta s_\gamma\dot\beta - c_\beta c_\gamma\dot\gamma & \begin{array}{r}(c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma)\dot\alpha + c_\alpha c_\beta s_\gamma\,\dot\beta \\ +\,(c_\alpha s_\beta c_\gamma - s_\alpha s_\gamma)\dot\gamma\end{array} & \begin{array}{r}-(s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma)\dot\alpha - s_\alpha c_\beta s_\gamma\,\dot\beta \\ -\,(s_\alpha s_\beta c_\gamma + c_\alpha s_\gamma)\dot\gamma\end{array} \end{pmatrix} \\
&\quad\cdot \begin{pmatrix} c_\beta c_\gamma & s_\beta & -c_\beta s_\gamma \\ s_\alpha s_\gamma - c_\alpha s_\beta c_\gamma & c_\alpha c_\beta & s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma \\ c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma & -s_\alpha c_\beta & c_\alpha c_\beta - s_\alpha s_\beta s_\gamma \end{pmatrix} \qquad (2) \\
&= \begin{pmatrix} 0 & c_\beta s_\gamma\,\dot\alpha - c_\gamma\,\dot\beta & s_\beta\,\dot\alpha + \dot\gamma \\ -c_\beta s_\gamma\,\dot\alpha + c_\gamma\,\dot\beta & 0 & -c_\beta c_\gamma\,\dot\alpha - s_\gamma\,\dot\beta \\ -s_\beta\,\dot\alpha - \dot\gamma & c_\beta c_\gamma\,\dot\alpha + s_\gamma\,\dot\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \boldsymbol{S}(\boldsymbol{\omega}).
\end{aligned}$$

The above derivation is greatly simplified by using symbolic computation in MATLAB. The linear mapping $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ is then extracted from the elements of matrix $\boldsymbol{S}$ in (2) as

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} c_\beta c_\gamma\,\dot\alpha + s_\gamma\,\dot\beta \\ s_\beta\,\dot\alpha + \dot\gamma \\ -c_\beta s_\gamma\,\dot\alpha + c_\gamma\,\dot\beta \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ \sin\beta & 0 & 1 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \end{pmatrix} \begin{pmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

The singularities of this mapping occur when $\det \boldsymbol{T}(\boldsymbol{\phi}) = -\cos\beta = 0$, i.e., for $\beta = \pm\pi/2$.

In alternative to the above procedure, and perhaps more quickly, one can build the matrix $\boldsymbol{T}(\boldsymbol{\phi})$ by noting the individual contributions to the angular velocity $\boldsymbol{\omega}$ in the Euler interpretation of the rotation matrix $\boldsymbol{R}_{XZY}$: $\dot{\gamma}$ is a rotation around the initial (fixed) Y-axis; $\dot{\beta}$ is a rotation around the Z'-axis, i.e., the Z-axis after the rotation $\boldsymbol{R}_Y(\gamma)$; and $\dot{\alpha}$ is a rotation around the X''-axis, i.e., the X-axis after the first two rotations $\boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta)$. Thus, we have

$$\begin{aligned}\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\gamma},Y} + \boldsymbol{\omega}_{\dot{\beta},Z'} + \boldsymbol{\omega}_{\dot{\alpha},X''} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\gamma} + \boldsymbol{R}_Y(\gamma) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\beta} + \boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\alpha} \\ &= \begin{pmatrix} \cos\beta\cos\gamma \\ \sin\beta \\ -\cos\beta\sin\gamma \end{pmatrix} \dot{\alpha} + \begin{pmatrix} \sin\gamma \\ 0 \\ \cos\gamma \end{pmatrix} \dot{\beta} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\gamma} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}},\end{aligned}$$

where, being each contribution to $\boldsymbol{\omega}$ a vector itself, the order in the sum is irrelevant.

Choose now, for instance, $\beta = \pi/2$. Thus

$$\bar{\boldsymbol{T}}(\gamma) = \boldsymbol{T}(\boldsymbol{\phi})|_{\beta=\pi/2} = \begin{pmatrix} 0 & \sin\gamma & 0 \\ 1 & 0 & 1 \\ 0 & \cos\gamma & 0 \end{pmatrix}, \qquad \operatorname{rank} \bar{\boldsymbol{T}}(\gamma) = 2.$$

In this representation singularity (for any value of α and γ), one has that angular velocities $\boldsymbol{\omega}$ of the form

$$\boldsymbol{\omega} = \rho \begin{pmatrix} \cos\gamma \\ 0 \\ -\sin\gamma \end{pmatrix} \notin \mathcal{R}\left(\bar{\boldsymbol{T}}(\gamma)\right), \qquad \forall \rho \neq 0$$

are not realizable by any possible choice of $\dot{\boldsymbol{\phi}}$. Moreover, time derivatives of $\boldsymbol{\phi}$ of the form

$$\dot{\boldsymbol{\phi}} = \sigma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{N}\left(\bar{\boldsymbol{T}}(\gamma)\right), \qquad \forall \sigma \in \mathbb{R}$$

generate zero angular velocity.

Exercise 2

The problem has four inverse kinematics solutions, provided the desired end-effector position $\boldsymbol{p}_d$ is inside the primary (or, reachable) workspace WS_1 of the robot[1]. For this to happen, the following inequalities should *necessarily* hold for the components p_{dx}, p_{dy} and p_{dz} of the desired position $\boldsymbol{p}_d$ of the end effector:

$$|L - N| \leq \sqrt{p_{dx}^2 + p_{dy}^2 + (p_{dz} - M)^2} \leq L + N, \qquad M - N \leq p_{dz} \leq M + N, \tag{3}$$

with strict inequalities enforced for the interior of WS_1. For the given data, these inequalities are satisfied being, respectively,

$$0 < 0.4690 < 1, \qquad 0 < 0.7 < 1.$$

[1]This workspace is in fact a *solid torus*, see Fig. 3 in the solution of Exercise #1 of the exam of June 10, 2022.

Even if this check is not done (or even known!) in advance, these inequalities will appear as necessary conditions to be satisfied in order to proceed with the computation of analytic inverse kinematics solutions.

From the third equation in the direct kinematics (1), one has

$$M + N \sin q_3 = p_{dz} \quad \Rightarrow \quad s_3 = \frac{p_{dz} - M}{N} \in [-1, 1] \quad \text{—and also } c_3 = \pm\sqrt{1 - s_3^2} \in [-1, 1],$$

with the admissible interval for the trigonometric function s_3 leading to the second pair of inequalities in (3). The two solutions for q_3 are then

$$q_3^{[+]} = \text{atan2}\left\{s_3, +\sqrt{1 - s_3^2}\right\}, \qquad q_3^{[-]} = \text{atan2}\left\{s_3, -\sqrt{1 - s_3^2}\right\}. \tag{4}$$

Each of these will branch in two solution pairs for (q_1, q_2). In fact, the first two equations in (1) can be interpreted as the direct kinematics of a planar 2R arm with link lengths

$$l_1 = L, \qquad l_2 = N \cos q_3^{[+]} \quad \text{or} \quad l_2 = N \cos q_3^{[-]}. \tag{5}$$

For each resulting value of l_2, one can use the solution for the 2R arm, which is obtained though the standard formulas. First, evaluate

$$c_2 = \frac{p_{dx}^2 + p_{dy}^2 - (l_1^2 + l_2^2)}{2 l_1 l_2} \in [-1, 1], \qquad s_2 = \pm\sqrt{1 - c_2^2}.$$

It can be shown that the admissible interval for the trigonometric function c_2 leads to the necessity of the first pair of inequalities in (3). For each value of l_2 in (5), the two solutions for q_2 are then

$$q_2^{[+]} = \text{atan2}\left\{\sqrt{1 - c_2^2}, c_2\right\}, \qquad q_2^{[-]} = \text{atan2}\left\{-\sqrt{1 - c_2^2}, c_2\right\}. \tag{6}$$

By the property of the atan2 function, it follows that $q_2^{[-]} = -q_2^{[+]}$. Finally, for each solution pair (q_2, q_3), a 2×2 linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ in the unknowns $\boldsymbol{x} = (s_1, c_1)$ can be set up, whose solution is found provided the determinant of the system matrix

$$\det \boldsymbol{A} = l_1^2 + l_2^2 + 2 l_1 l_2 c_2 = L^2 + N^2 \cos^2 q_3 + 2LN \cos q_2 \cos q_3 \neq 0.$$

In this case, we have

$$s_1 = \frac{p_{dy}\,(l_1 + l_2 c_2) - p_{dx} l_2 s_2)}{\det \boldsymbol{A}}, \qquad c_1 = \frac{p_{dx}\,(l_1 + l_2 c_2) + p_{dy} l_2 s_2)}{\det \boldsymbol{A}},$$

and the associated (unique) solution for q_1 is

$$q_1 = \text{atan2}\left\{s_1, c_1\right\}. \tag{7}$$

Note that in this case the determinant cannot be eliminated from the denominator of the two arguments of this atan2 function; in fact, when the determinant is different from zero, its sign may change depending on the particular solution for the pair (q_2, q_3) inserted in the linear system.

Summarizing, we have found the following four inverse kinematics solutions (all distinct in the regular case):

$$\begin{pmatrix} q_1^{[+,+,+]} & q_2^{[+,+]} & q_3^{[+]} \end{pmatrix}^T, \quad \begin{pmatrix} q_1^{[+,-,+]} & q_2^{[+.-]} & q_3^{[+]} \end{pmatrix}^T,$$
$$\begin{pmatrix} q_1^{[-,+,-]} & q_2^{[-.+]} & q_3^{[-]} \end{pmatrix}^T, \quad \begin{pmatrix} q_1^{[-,-,-]} & q_2^{[-.+]} & q_3^{[-]} \end{pmatrix}^T.$$

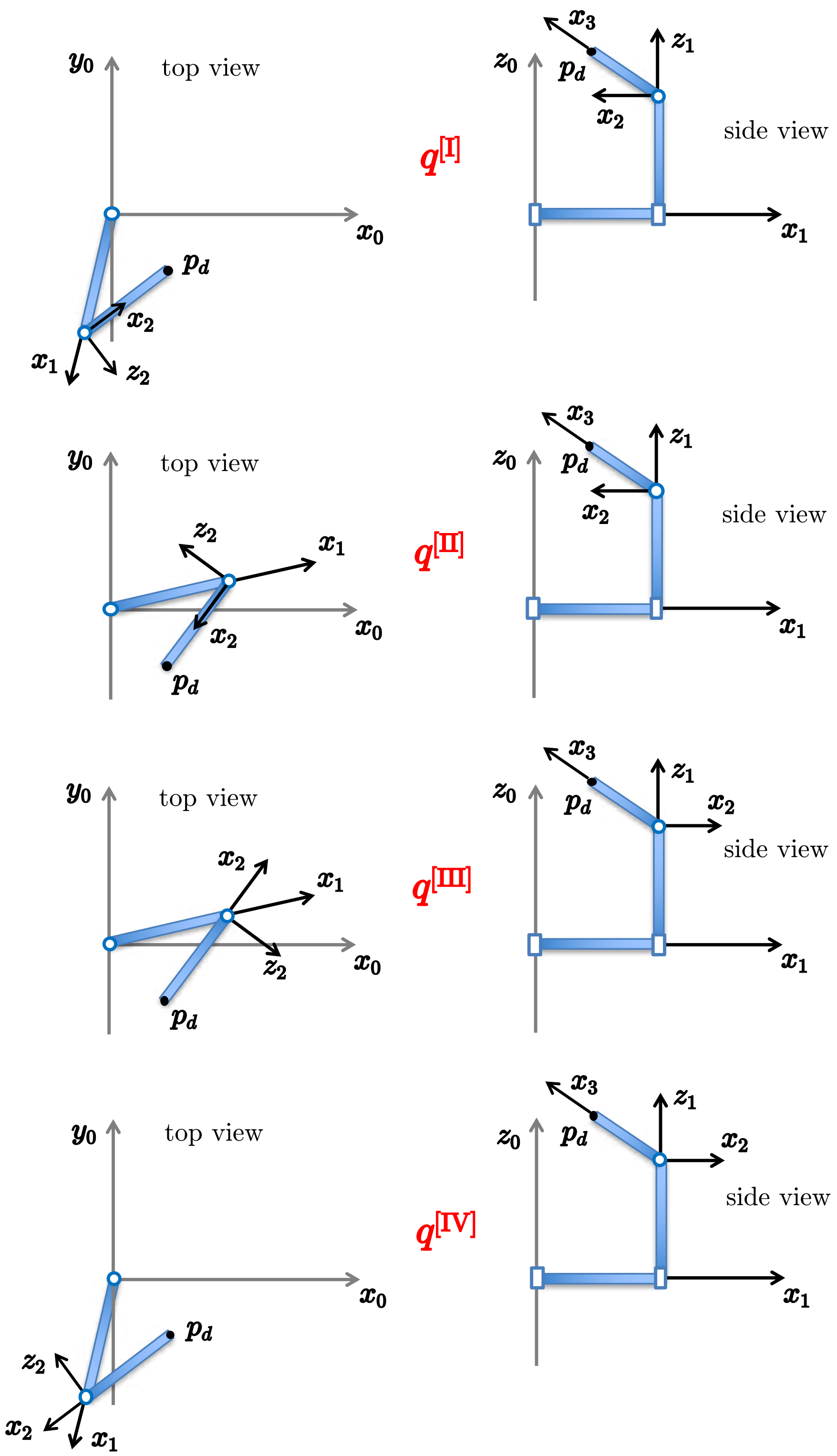


Figure 2: The four inverse kinematics solutions for the spatial 3R robot (top and side views).

Plugging the numerical data in eqs. (4)–(7), we obtain (in [rad])

$$\boldsymbol{q}^{[I]} = \begin{pmatrix} -1.8110 & 2.2281 & 0.4115 \end{pmatrix}^T, \quad \boldsymbol{q}^{[II]} = \begin{pmatrix} 0.2402 & -2.2281 & 0.4115 \end{pmatrix}^T,$$
$$\boldsymbol{q}^{[III]} = \begin{pmatrix} 0.2402 & 0.9135 & 2.7301 \end{pmatrix}^T, \quad \boldsymbol{q}^{[IV]} = \begin{pmatrix} -1.8110 & -0.9135 & 2.7301 \end{pmatrix}^T,$$

or (in degrees)

$$\boldsymbol{q}^{[I]} = \begin{pmatrix} -103.7^\circ & 127.66^\circ & 23.58^\circ \end{pmatrix}^T, \quad \boldsymbol{q}^{[II]} = \begin{pmatrix} 13.77^\circ & -127.66^\circ & 23.58^\circ \end{pmatrix}^T,$$
$$\boldsymbol{q}^{[III]} = \begin{pmatrix} 13.77^\circ & 52.34^\circ & 156.42^\circ \end{pmatrix}^T, \quad \boldsymbol{q}^{[IV]} = \begin{pmatrix} -103.77^\circ & -52.34^\circ & 156.42^\circ \end{pmatrix}^T,$$

Indeed, evaluation of (1) with these solutions returns always the desired $\boldsymbol{p}_d$ (it is worth to do this check!). For each of these four inverse kinematics solutions, Figure 2 sketches two views of the resulting configuration of the spatial 3R robot.

Exercise 3

One should set up a simple code that implements the following Newton iteration

$$\boldsymbol{q}^{\{k\}} = \boldsymbol{q}^{\{k-1\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{k-1\}}) \left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{\{k-1\}})\right), \qquad k = 1, 2, \ldots, \tag{8}$$

starting from an initial guess $\boldsymbol{q}^{\{0\}}$, until convergence is achieved or some other stopping criterion is reached. To obtain a reliable code, it is important to include a limit on the maximum number of iterations, signifying that no convergence is being achieved, and a warning (with exit) when a singularity of the Jacobian is being met.
For the implementation of (8), we need the direct kinematics function $\boldsymbol{f}(\boldsymbol{q})$ in (1) and the associated (analytic) Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 - Ns_{12}c_3 & -Ns_{12}c_3 & -Nc_{12}s_3 \\ Lc_1 + Nc_{12}c_3 & Nc_{12}c_3 & -Ns_{12}s_3 \\ 0 & 0 & Nc_3 \end{pmatrix}, \tag{9}$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = LN^2 s_2 c_3^2$. As long as this determinant is non-zero (or is far enough from it), iteration (8) is well defined. Nonetheless, convergence can be guaranteed *only* when starting sufficiently close to a solution (although the convergence rate is then the fastest possible, namely quadratic).
In the present case, for $L = M = N = 0.5$ [m] and $\boldsymbol{p}_d = (0.3, -0.3, 0.7)$ [m], when starting from $\boldsymbol{q}^{\{0\}} = (-\pi/4, \pi/4, \pi/4)$ [rad], the method converges within the error tolerance $\epsilon = 10^{-3}$ [m] in $k^* = 5$ iteration, generating

$$\boldsymbol{q}^{\{0\}} = \begin{pmatrix} -0.7854 \\ 0.7854 \\ 0.7854 \end{pmatrix} \to \boldsymbol{q}^{\{1\}} = \begin{pmatrix} -2.3712 \\ 4.1084 \\ 0.3511 \end{pmatrix} \to \boldsymbol{q}^{\{2\}} = \begin{pmatrix} -1.1056 \\ 2.2074 \\ 0.4108 \end{pmatrix} \to$$
$$\to \boldsymbol{q}^{\{3\}} = \begin{pmatrix} -1.8344 \\ 2.4611 \\ 0.4115 \end{pmatrix} \to \boldsymbol{q}^{\{4\}} = \begin{pmatrix} -1.8426 \\ 2.2346 \\ 0.4115 \end{pmatrix} \to \boldsymbol{q}^{\{5\}} = \begin{pmatrix} -1.8110 \\ 2.2286 \\ 0.4115 \end{pmatrix},$$

with a final error norm $\left\|\boldsymbol{e}^{\{5\}}\right\| = 1.97 \cdot 10^{-4} < \epsilon = 10^{-3}$. The evolution of this norm is shown in Fig. 3. Note that the method initially increases the norm, but then starts converging fast when

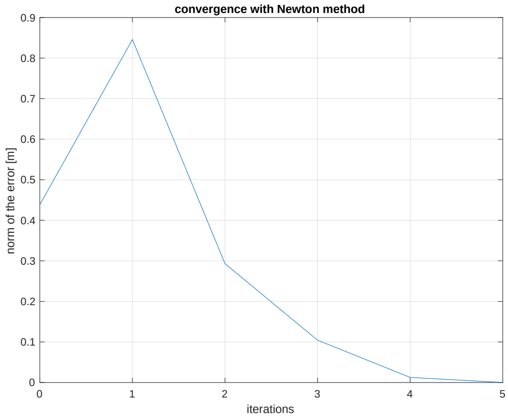


Figure 3: Evolution of the error norm $\left\|\boldsymbol{e}^{\{k\}}\right\|$ during the iterations of Newton method.

closer to a solution. In particular, in the last two iterations, the error norm decreases by one order, and eventually by two orders of magnitude: from 0.104391 at $k = 3$, to 0.012584 at $k = 4$, stopping with $0.000197 < \epsilon = 10^{-3}$ at $k^* = 5$.
The obtained solution corresponds to the analytic solution $\boldsymbol{q}^{[I]}$ found in Exercise #2. To obtain a more accurate numeric value, one should set a tighter error tolerance: with $\epsilon = 10^{-4}$, Newton method runs for two more iterations and stops at $k^* = 7$, returning $\boldsymbol{q}^{[I]}$ with the same four-digit accuracy on all joint variables.

Finally, in order to obtain another inverse kinematics solution, the method has to be restarted from another initial guess, hopefully closer to a different solution. For instance, when starting with $\boldsymbol{q}^{\{0\}} = (\pi/10, \pi/3, 3\pi/4)$ [rad], the method will converge to $\boldsymbol{q}^{[III]}$ in $k^* = 3$ iterations (with the original tolerance $\epsilon = 10^{-3}$) or in $k^* = 6$ iterations (with the tighter tolerance $\epsilon = 10^{-4}$).

Exercise 4

The stated trajectory planning problem is solved by using a single quintic polynomial for each joint, with suitable boundary conditions. For the generic joint i, with $i = 1, 2, 3$, we consider the desired smooth trajectory written as

$$q_i(t) = q_{s,i} + \Delta q_i \cdot q_{n,i}(\tau), \qquad \tau = \frac{t}{T}, \tag{10}$$

with $\Delta q_i = q_{g,i} - q_{s,i}$ and the doubly normalized quintic polynomial

$$q_{n,i}(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5, \qquad \tau \in [0, 1],$$

ranging in values between 0 and 1. The first and second time derivatives of (10) are

$$\dot{q}_i(t) = \frac{\Delta q_i}{T} \cdot q'_{n,i}(\tau) = \frac{\Delta q_i}{T} \cdot \left(a_1 + 2a_2\tau + 3a_3\tau^2 + 4a_4\tau^3 + 5a_5\tau^4\right) \tag{11}$$

and

$$\ddot{q}_i(t) = \frac{\Delta q_i}{T^2} \cdot q''_{n,i}(\tau) = \frac{\Delta q_i}{T^2} \cdot \left(2a_2 + 6a_3\tau + 12a_4\tau^2 + 20a_5\tau^3\right), \tag{12}$$

where $'$ is used to denote a derivative with respect to τ. The boundary conditions to be imposed are

$$
\begin{array}{llllll}
q_i(0) = q_{s,i} & \Rightarrow & q_{n,i}(0) = 0 & \Rightarrow & a_0 = 0; \\
q_i(T) = q_{g,i} & \Rightarrow & q_{n,i}(1) = 1 & \Rightarrow & a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1; \\
\dot{q}_i(0) = \dot{q}_{s,i} \neq 0 & \Rightarrow & q'_{n,i}(0) = \dot{q}_{s,i} \dfrac{T}{\Delta q_i} & \Rightarrow & a_1 = \dot{q}_{s,i} \dfrac{T}{\Delta q_i}; \\
\dot{q}_i(T) = 0 & \Rightarrow & q'_{n,i}(0) = 0 & \Rightarrow & a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0; \\
\ddot{q}_i(0) = 0 & \Rightarrow & q''_{n,i}(0) = 0 & \Rightarrow & a_2 = 0; \\
\ddot{q}_i(T) = 0 & \Rightarrow & q''_{n,i}(1) = 0 & \Rightarrow & 2a_2 + 6a_3 + 12a_4 + 20a_5 = 0,
\end{array}
\tag{13}
$$

where the scalar component $\dot{q}_{s,i}$ of the initial velocity $\dot{\boldsymbol{q}}_s$ at the start configuration is determined by inversion of the differential kinematics

$$
\dot{\boldsymbol{q}}_s = \dot{\boldsymbol{q}}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_s)\dot{\boldsymbol{p}}(0).
$$

Being

$$
a_0 = a_2 = 0, \qquad a_1 = \dot{q}_{s,i} \frac{T}{\Delta q_i},
$$

we need to solve the remaining three linear equations in (13) as

$$
\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 - \dot{q}_{s,i}\, T/\Delta q_i \\ -\,\dot{q}_{s,i}\, T/\Delta q_i \\ 0 \end{pmatrix}
$$

or

$$
\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 10 & -4 & 0.5 \\ -15 & 7 & -1 \\ 6 & -3 & 0.5 \end{pmatrix} \begin{pmatrix} 1 - \dot{q}_{s,i}\, T/\Delta q_i \\ -\,\dot{q}_{s,i}\, T/\Delta q_i \\ 0 \end{pmatrix} = \begin{pmatrix} 10 - 6\dot{q}_{s,i}\, T/\Delta q_i \\ -15 + 8\dot{q}_{s,i}\, T/\Delta q_i \\ 6 - 3\dot{q}_{s,i}\, T/\Delta q_i \end{pmatrix}.
$$

Next, we compute the required data for the problem at hand. For the initial joint velocity at $\boldsymbol{q}_s = (-\pi/4, \pi/4, \pi/4)$, we obtain

$$
\dot{\boldsymbol{q}}_s = \boldsymbol{J}^{-1}(\boldsymbol{q}_s)\, \dot{\boldsymbol{p}}(0) = \begin{pmatrix} 0.3536 & 0 & -0.3536 \\ 0.7071 & 0.3536 & 0 \\ 0 & 0 & 0.3536 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.8284 \\ -8.4853 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{q}_{s,1} \\ \dot{q}_{s,2} \\ \dot{q}_{s,3} \end{pmatrix}.
$$

Moreover,

$$
T = 2, \qquad \boldsymbol{q}_s = \begin{pmatrix} -0.7854 \\ 0.7854 \\ 0.7854 \end{pmatrix} = \begin{pmatrix} q_{s,1} \\ q_{s,2} \\ q_{s,3} \end{pmatrix}, \qquad \boldsymbol{\Delta q} = \boldsymbol{q}_g - \boldsymbol{q}_s = \begin{pmatrix} 0.7854 \\ -0.7854 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix}.
$$

Evaluating then the coefficients a_i's in the solution, we obtain the following quintic joint trajectories, written using the doubly normalized polynomials $q_{n,1}(\tau)$ and $q_{n,2}(\tau)$ (($q_{n3}(\tau)$ is also present, but actually irrelevant):

$$
\begin{aligned}
q_1(t) &= -0.7854 + 0.7854 \left(7.2025\, \tau - 33.2152\, \tau^3 + 42.6202\, \tau^4 - 15.6076\, \tau^5\right) \\
q_2(t) &= 0.7854 - 0.7854 \left(21.6076\, \tau - 119.6455\, \tau^3 + 157.8607\, \tau^4 - 58.8228\, \tau^5\right) \\
q_3(t) &= 0.7854 + 0 \left(10\, \tau^3 - 15\, \tau^4 + 6\, \tau^5\right) = 0.7854 \qquad \text{(no motion needed for this joint!)}
\end{aligned}
\tag{14}
$$

The plots of the joint trajectories, with their velocity and acceleration as given by (11) and (12), are reported in Fig. 4.

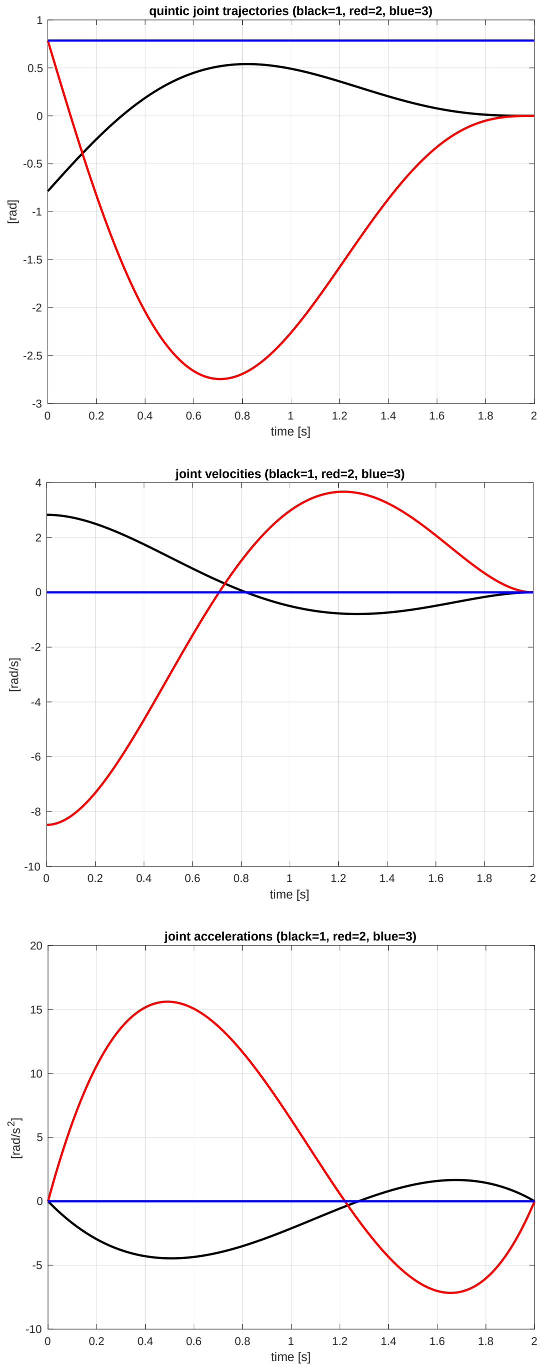


Figure 4: Solution of the trajectory planning problem: joint position (top), velocity (center) and acceleration (bottom).

* * * * *

Robotics 1 - Sheet for Exercise 2

September 9, 2022

With reference to the inverse kinematics problem of robot manipulators, check if each of the following statements is **True** or **False**, and provide mandatorily a *very short* motivation/explanation sentence.

1. When the robot is in a singularity, there is always an infinite number of inverse solutions.

   True ☐ **False** ☐

2. A 6-dof Cartesian robot with a spherical wrist has two inverse solutions, out of singularities.

   True ☐ **False** ☐

3. If a closed-form inverse solution is not known in advance, a numerical method cannot provide one.

   True ☐ **False** ☐

4. A 6R industrial robot may have sixteen inverse solutions in its workspace, out of singularities.

   True ☐ **False** ☐

5. A planar manipulator with $n \geq 3$ revolute joints has up to n inverse solutions for a positioning task.

   True ☐ **False** ☐

6. At workspace boundaries, there is never an analytic solution to the inverse kinematics.

   True ☐ **False** ☐

7. A 3R robot with twist angles α_i different from 0, $\pm\pi/2$, or $\pm\pi$ has no closed-form inverse solution.

   True ☐ **False** ☐

8. The number of inverse solutions under joint limits is always strictly less than that without limits.

   True ☐ **False** ☐

9. A 6R spatial robot without spherical wrist or spherical shoulder has no closed-form inverse solution.

   True ☐ **False** ☐

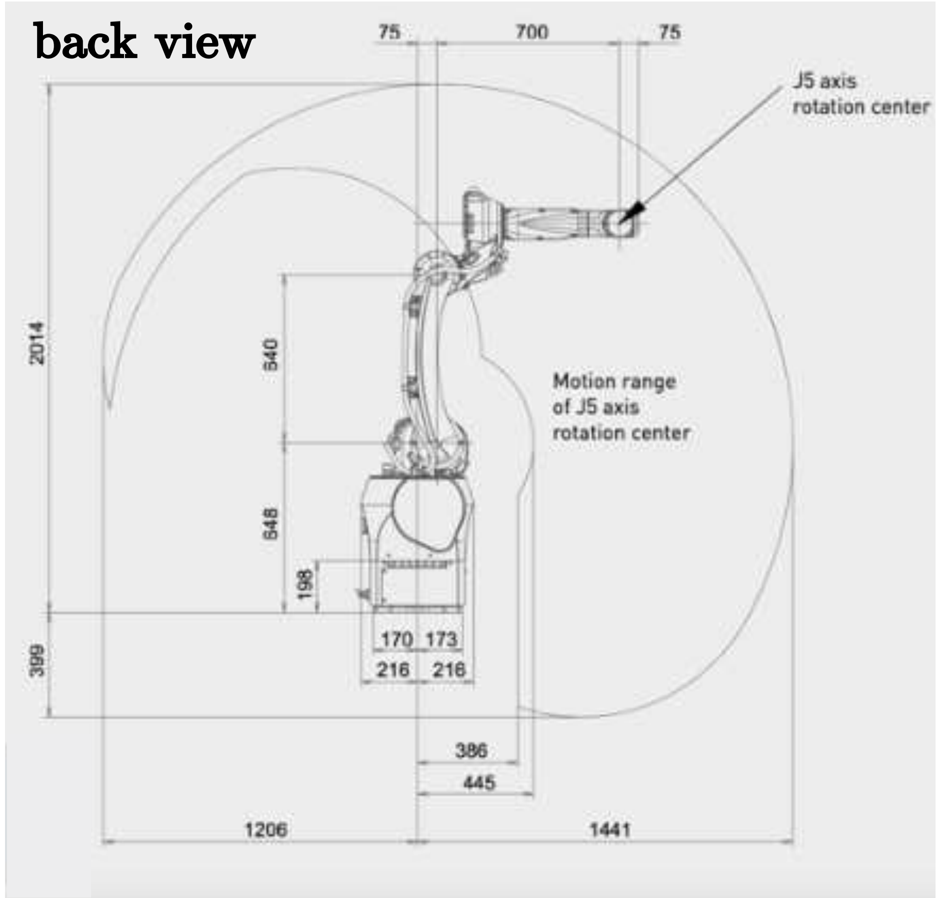
10. A 3-dof gantry-type robot has only one inverse kinematic solution in its workspace.

    True ☐ **False** ☐

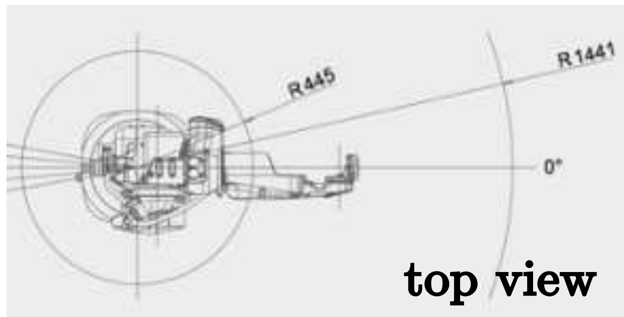
Fanuc CR15ia collaborative robot – DH frames assignment and table

Name: ____

front view



i	1	2	3	4	5	6
α_i						
a_i						
d_i						
θ_i						



Robotics 1

September 9, 2022

Exercise 1

The Fanuc cr15ia is a collaborative robot with six revolute joints and a spherical wrist. Two views are shown in Fig. 1. The drawing with a back view contains the numerical values (in [mm]) of all geometric lengths that are needed for describing the robot kinematics.

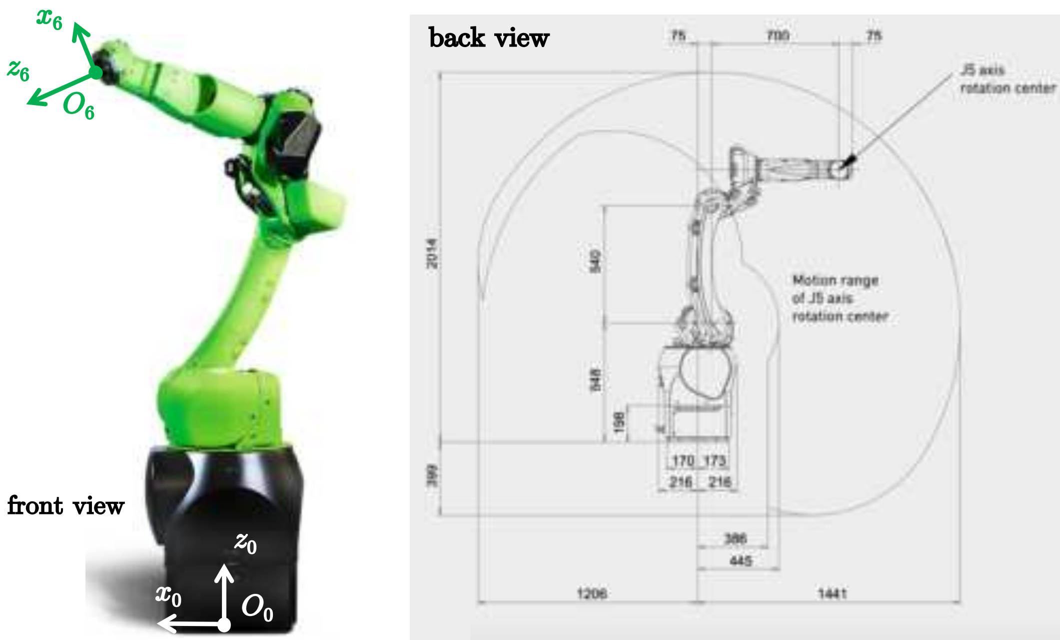


Figure 1: A front view and a drawing from the back of the 6R Fanuc cr15ia collaborative robot.

Assign the link frames according to the Denavit-Hartenberg (DH) convention and fill in the associated table of parameters, specifying the numerical values of the constant parameters (given directly in the drawing of the robot or derived from those data). Moreover, provide the values of the joint variables when the robot is in the configuration shown in the back view. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 (at the robot base) and 6 (at the center of the final flange) are assigned and should not be modified.

Exercise 2

A number of statements are reported on the extra sheet #2, regarding the inverse kinematics problem of robot manipulators. Check if each of the statements is **True** or **False**. Each answer will be considered **only if** you provide also a *very short* motivation/explanation sentence.

Exercise 3

For a 3-dof robot, the task kinematics is given by

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 + L \cos(q_1 + q_3) \\ q_2 \sin q_1 + L \sin(q_1 + q_3) \\ q_1 + q_3 \end{pmatrix},$$

with a constant $L > 0$.

- Find the singularities of the mapping from $\dot{\boldsymbol{q}}$ to $\dot{\boldsymbol{r}}$.
- Determine all possible task velocities $\dot{\boldsymbol{r}}$ that can be realized when the robot is in a singularity.
- When the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), is it possible to obtain a task acceleration $\ddot{\boldsymbol{r}} = \boldsymbol{0}$ by commanding a non-zero joint acceleration $\ddot{\boldsymbol{q}}$? Support your answer with one or more numerical examples.
- Set now $L = 1$. At $\boldsymbol{q} = (\pi/2, 1, 0)$, with the robot having a joint velocity $\dot{\boldsymbol{q}} = (1, -1, -1)$, determine a joint acceleration $\ddot{\boldsymbol{q}}$ that realizes $\ddot{\boldsymbol{r}} = \boldsymbol{0}$. Is this joint acceleration unique?

Exercise 4

A single revolute joint of a robot needs to move between $q_i = \pi/2$ [rad] and $q_f = 0$, under the velocity and acceleration bounds

$$|\,\dot{q}\,| \leq V = 2 \text{ [rad/s]}, \qquad |\,\ddot{q}\,| \leq A = 4 \text{ [rad/s}^2\text{]}.$$

Determine:

- the minimum time T_0 for a rest-to-rest motion;
- the minimum time T_1 for a motion from $\dot{q}_i = 1.5$ [rad/s] to $\dot{q}_f = 0$.

Sketch the position, velocity and acceleration profiles of the two resulting time-optimal motions.

[180 minutes, open books]

Solution

September 9, 2022

Exercise 1

A possible DH frame assignment for the Fanuc CR15ia robot is shown in Fig. 2, in the front and back views. The associated DH parameters are reported in Tab. 1.

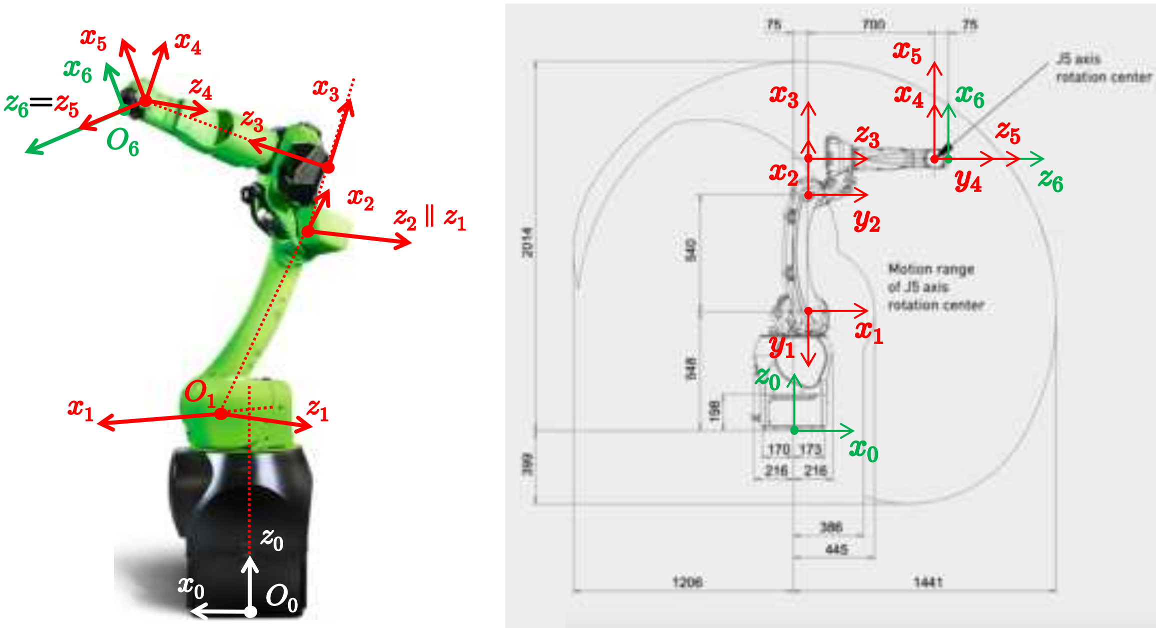


Figure 2: DH frames for the Fanuc CR15ia robot: front view (left) and back view (right).

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	75	648	q_1
2	0	640	0	q_2
3	$-\pi/2$	a_3	0	q_3
4	$\pi/2$	0	700	q_4
5	$-\pi/2$	0	0	q_5
6	0	0	75	q_6

Table 1: Parameters associated to the DH frames of Fig. 2. Lengths are in [mm].

Parameter a_3 is the only one not directly given in the data sheet. By geometric reasoning one has

$$a_3 = \sqrt{(2014 - (648 + 640))^2 - 700^2} \simeq 192.55 \text{ [mm]},$$

which is best evaluated when the forearm is pointing upward and reaches the top of the workspace.

When the robot is in the configuration shown in the back view, the values of the joint variables are:

$$q_1 = 0, \qquad q_2 = -\frac{\pi}{2} \text{ [rad]}, \qquad q_3 = q_4 = q_5 = q_6 = 0.$$

On the other hand, one can approximately guess the joint values (for convenience, expressed in degree) also when the robot is in the configuration shown in the front view:

$$q_1 = 15°, \quad q_2 = -110°, \quad q_3 = 5°, \quad q_4 = 0°, \quad q_5 = 40°, \quad q_1 = 0°.$$

Exercise 2

1. When the robot is in a singularity, there is always an infinite number of inverse solutions.

   False A planar 2R robot is singular at the outer boundary, with only one inverse solution.

2. A 6-dof Cartesian robot with a spherical wrist has two inverse solutions, out of singularities.

   True For such PPP-3R robot, these are the two orientation solutions of the spherical wrist.

3. If a closed-form inverse solution is not known in advance, a numerical method cannot provide one.

   False This is exactly one of the main reasons for using a numerical method for inversion.

4. A 6R industrial robot may have sixteen inverse solutions in its workspace, out of singularities.

   True This maximum number of solutions has been actually reached by a 6R robot.

5. A planar manipulator with $n \geq 3$ revolute joints has up to n inverse solutions for a positioning task.

   False The robot is redundant for the task and can have an infinity of inverse solutions.

6. At workspace boundaries, there is never an analytic solution to the inverse kinematics.

   False For a stretched planar 2R robot: $q_1 = \text{atan2}\{p_y, p_x\}$, $q_2 = 0$.

7. A 3R robot with twist angles α_i different from 0, $\pm\pi/2$, or $\pm\pi$ has no closed-form inverse solution.

   False Though more complex, closed-form inverse solutions can be found in other 3R cases.

8. The number of inverse solutions under joint limits is always strictly less than that without limits.

   False Not always, though this is often the case.

9. A 6R spatial robot without spherical wrist or spherical shoulder has no closed-form inverse solution.

   False Another sufficient condition is having three parallel joint axes, as in the UR10 robot.

10. A 3-dof gantry-type robot has only one inverse kinematic solution in its workspace.

    True This is a PPP robot and there is a unique solution, say, $q_1 = p_x$, $q_2 = p_y$, $q_3 = p_z$.

Exercise 3

Differentiating the given task kinematics[1] gives $\dot{\boldsymbol{r}} = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$, with the task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 - L\sin(q_1+q_3) & \cos q_1 & -L\sin(q_1+q_3) \\ q_2\cos q_1 + L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}.$$

Its determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = -q_2,$$

so that the only singularity occurs when $q_2 = 0$. Substituting this value in the Jacobian yields

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -L\sin(q_1+q_3) & \cos q_1 & L\sin(q_1+q_3) \\ L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix},$$

having rank 2. Thus, all task velocities that can be realized in a singularity by any possible choice of joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ span a two-dimensional subspace, namely $\mathcal{R}(\boldsymbol{J}_s)$, and are of the form

$$\dot{\boldsymbol{r}} = \begin{pmatrix} -L\sin(q_1+q_3) \\ L\cos(q_1+q_3) \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{pmatrix} \beta, \qquad \text{with } \alpha = \dot{q}_1 + \dot{q}_3,\ \beta = \dot{q}_2.$$

Differentiating further $\dot{\boldsymbol{r}}$, we obtain the task acceleration

$$\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}),$$

where the term $\boldsymbol{h}$ is quadratic in $\dot{\boldsymbol{q}}$ and is given by

$$\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -2\sin q_1\,\dot{q}_1\dot{q}_2 - q_2\cos q_1\,\dot{q}_1^2 - L\cos(q_1+q_3)\,(\dot{q}_1+\dot{q}_3)^2 \\ 2\cos q_1\,\dot{q}_1\dot{q}_2 - q_2\sin q_1\,\dot{q}_1^2 - L\sin(q_1+q_3)\,(\dot{q}_1+\dot{q}_3)^2 \\ 0 \end{pmatrix}.$$

Suppose now that the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), so that $\boldsymbol{h} = \boldsymbol{0}$. Then, we can obtain $\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{0}$ for a joint acceleration $\ddot{\boldsymbol{q}} \neq \boldsymbol{0}$ if and only if the task Jacobian $\boldsymbol{J}$ is singular, i.e., it is $\boldsymbol{J}_s$. In this case, any non-zero acceleration $\ddot{\boldsymbol{q}}$ that lies in the null space of $\boldsymbol{J}_s$ solves the requested problem:

$$\ddot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}_s\} = \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \forall\gamma \qquad \Rightarrow \qquad \boldsymbol{J}_s\ddot{\boldsymbol{q}}_0 = \boldsymbol{0}.$$

Note that the same acceleration applied at a generic nonsingular configuration and with zero joint velocity would produce instead

$$\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}_0 = \gamma \begin{pmatrix} q_2\sin q_1 \\ -q_2\cos q_1 \\ 0 \end{pmatrix} \neq \boldsymbol{0}.$$

[1] The robot is a planar RPR arm with the third link of length L, while the task is the position and orientation of its end-effector. All requested derivations are done analytically, so this information is of limited use.

When the task Jacobian is nonsingular, the unique joint acceleration $\ddot{\boldsymbol{q}}$ that produces $\ddot{\boldsymbol{r}} = \mathbf{0}$ is given by

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\,\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{1}$$

Since $\boldsymbol{q} = (\pi/2, 1, 0)$ is a regular configuration, plugging these values of joint position into (1), together with $L = 1$ and $\dot{\boldsymbol{q}} = (1, -1, -1)$, leads to

$$\ddot{\boldsymbol{q}} = -\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

Exercise 4

The case of rest-to-rest motion is standard. Since

$$L = |q_f - q_i| = \frac{\pi}{2} = 1.570 > 1 = \frac{V^2}{A}$$

there will be a coast phase at maximum (negative) velocity $\dot{q} = -V = -2$ [m/s] during motion. Applying then the known formulas for bang-coast-bang acceleration profiles, we have

$$T_s = \frac{V}{A} = 0.5 \text{ [s]}, \qquad T_0 = \frac{LA + V^2}{AV} = \frac{2\pi + 4}{8} = \frac{\pi}{4} + 0.5 = 1.285 \text{ [s]}.$$

Thus, the cruise speed is held for $T - 2T_s = 0.285$ [s]. The resulting position, velocity and acceleration profiles are shown in Fig. 3. Note the negative trapezoidal velocity profile, since the position is being reduced (rotated clockwise!) from $q_i = \pi/2$ to $q_f = 0$.

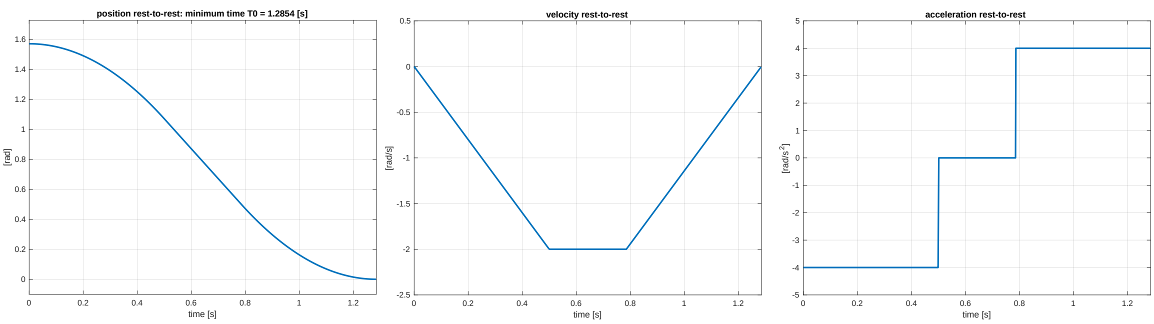


Figure 3: Motion profiles for the rest-to-rest case.

When the initial velocity is $\dot{q}_i = 1.5$ [rad/s] (state-to-rest case), the joint is moving initially in the wrong direction: thus, it needs to reverse its motion, i.e., first decelerate and stop and then move back to q_f. However, while the joint is being brought to a first stop in a time T_d, the position has progressed from $q_i = \pi/2$ to a larger positive value $q_d > q_i$. Applying the maximum negative acceleration $\ddot{q} = -A$ to stop the motion in the shortest possible time, these two quantities are then computed as

$$T_d = \frac{\dot{q}_i}{A} = 0.375 \text{ [s]}, \qquad q_d = q_i + \frac{1}{2}\dot{q}_i T_d = q_i + \frac{\dot{q}_i^2}{2A} = \frac{\pi}{2} + 0.281 = 1.852 \text{ [rad]}.$$

At this point, the remaining part of the motion is similar to the rest-to-rest case, but with the longer displacement to travel

$$L_d = |q_f - q_d| = 1.852 > L.$$

The joint will first continue with the same negative acceleration $\ddot{q} = -A$, until reaching the cruise velocity $\dot{q} = -V$ and so on. Therefore, the total minimum time in this case will be

$$T_1 = T_d + \frac{L_d A + V^2}{AV} = 0.375 + 1.426 = 1.801 \text{ [s]}.$$

The resulting position, velocity and acceleration profiles are shown in Fig. 4. Note that the overall motion is no longer symmetric.

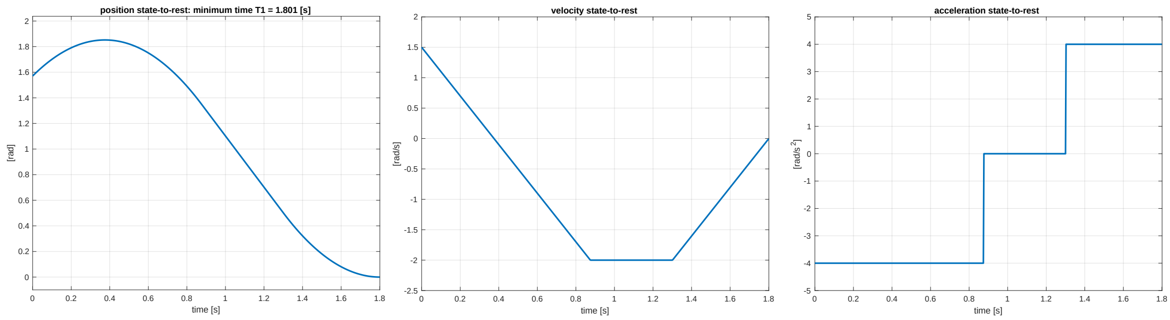


Figure 4: Motion profiles for the state-to-rest case.

* * * * *

Robotics 1

October 21, 2022

Exercise 1a

For the spatial RPR robot of Fig. 1, complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. The origin of the last frame should be placed at the point P. Moreover, the frame assignment should be such that all constant DH parameters are non-negative and the value of the joint variables q_i, $i = 1,2,3$, are strictly positive in the shown configuration. Compute then the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of the point P.

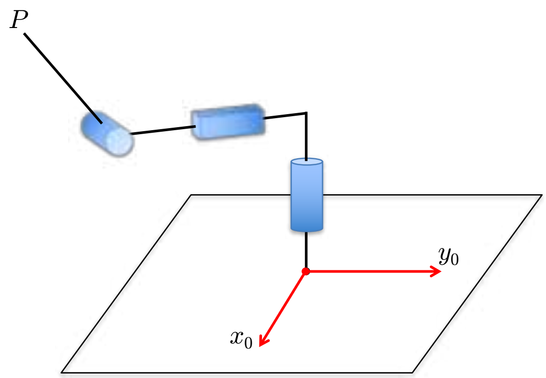


Figure 1: A spatial RPR robot.

Exercise 1b

Provide the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the velocity $\boldsymbol{v} = \dot{\boldsymbol{p}} \in \mathbb{R}^3$ of P and determine all its singularities. For each singularity, determine the rank of $\boldsymbol{J}$, a basis for the null space motion, and the Cartesian direction(s) where instantaneous mobility of P is lost.

Exercise 1c

Determine a joint velocity control law that will eventually bring the robot end-effector to a generic desired position $\boldsymbol{p}_d \in \mathbb{R}^3$ in the reachable workspace, starting from any initial position $\boldsymbol{p}(0)$ and moving the end-effector always along a straight line without the need of planning a trajectory.

Exercise 2

A planar 2R robot having link lengths $L_1 = 2$ [m] and $L_2 = 1$ [m] is commanded by joint accelerations $\ddot{\boldsymbol{q}}$ with a bang-bang profile, under the joint velocity limits $|\dot{q}_1| \leq V_{max,1} = 2$ [rad/s] and $|\dot{q}_2| \leq V_{max,2} = 1.5$ [rad/s]. The robot should move its end-effector between the two points

$$P_{in} = \begin{pmatrix} 2 + 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \text{ [m]} \qquad \rightarrow \qquad P_{fin} = \begin{pmatrix} 3/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \text{ [m]},$$

i) with zero initial and final velocity, *ii)* in minimum time, *iii)* in a coordinated way, with both joints starting and ending their motion at the same instant, and *iv)* without crossing any singular configuration. Provide the minimum time T and the maximum absolute values $A_i > 0$, $i = 1,2$, of the joint accelerations. Draw the time-optimal profiles of $\ddot{q}_1(t)$ and $\ddot{q}_2(t)$, for $t \in [0,T]$.

[180 minutes, open books]

Solution

October 21, 2022

Exercise 1

The correct (and unique) DH frame assignment for the RPR robot of Fig. 1 satisfying all requests is shown in Fig. 2. The associated DH parameters are reported in Tab. 1.

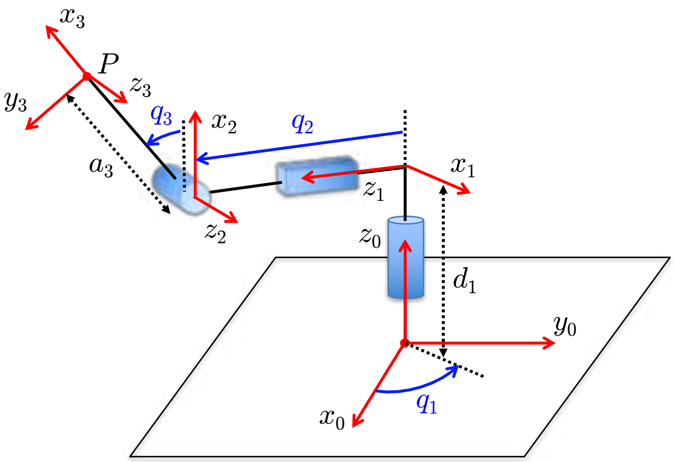


Figure 2: DH frames for the spatial RPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	$q_1 > 0$
2	$\pi/2$	0	$q_2 > 0$	$\pi/2$
3	0	$a_3 > 0$	0	$q_3 > 0$

Table 1: DH parameters corresponding to the frames of Fig. 2.

From the associated homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2(q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = \boldsymbol{A}_1(q_1)\left(\boldsymbol{A}_2(q_2)\left(\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)$$

yielding the direct kinematics of the position of point P as

$$\boldsymbol{p} = \begin{pmatrix} s_1\,(q_2 + a_3 s_3) \\ -c_1\,(q_2 + a_3 s_3) \\ d_1 + a_3 c_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

Exercise 1b

Differentiating the direct kinematics yields the 3×3 Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} c_1 (q_2 + a_3 s_3) & s_1 & a_3 s_1 c_3 \\ s_1 (q_2 + a_3 s_3) & -c_1 & -a_3 c_1 c_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix}.$$

Its determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = a_3 s_3 (q_2 + a_3 s_3)$$

so that the singularities occur when

$$s_3 = 0 \quad (q_3 = \{0, \pi\}) \qquad \text{or} \qquad q_2 = -a_2 s_3.$$

In the first case, setting $q_3 = 0$ for illustration (and for $q_2 \neq 0$), we have

$$\boldsymbol{J}_I = \boldsymbol{J}(\boldsymbol{q})|_{q_3=0} = \begin{pmatrix} q_2 c_1 & s_1 & a_3 s_1 \\ q_2 s_1 & -c_1 & -a_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_I) = 2.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are

$$\mathcal{N}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 0 \\ -a_3 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{R}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{R}^{\perp}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where $\mathcal{R}^{\perp}$ is the complementary subspace to $\mathcal{R}$ in $\mathbb{R}^3$. In this singular configuration, the third link is vertical so that point P is at the boundary of the reachable workspace. Thus, it cannot move along the vertical $\boldsymbol{z}_0$ direction.

In the second singular case, we have (for $q_3 \neq 0$ or π)

$$\boldsymbol{J}_{II} = \boldsymbol{J}(\boldsymbol{q})|_{q_2+a_3 s_3=0} = \begin{pmatrix} 0 & s_1 & a_3 s_1 c_3 \\ 0 & -c_1 & -a_3 c_1 c_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_{II}) = 2.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are in this case

$$\mathcal{N}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{R}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{R}^{\perp}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} \right\}.$$

In this singular configuration, point P is placed on the axis $\boldsymbol{z}_0$ and cannot move along the normal direction to the vertical plane being defined by the links 2 and 3.

Finally, at the intersection of the two singularities, e.g., for $q_2 = q_3 = 0$, we obtain

$$\boldsymbol{J}_{I+II} = \boldsymbol{J}(\boldsymbol{q})|_{q_2=q_3=0} = \begin{pmatrix} 0 & s_1 & a_3 s_1 \\ 0 & -c_1 & -a_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_{I+II}) = 1.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are then

$$\mathcal{N}\left(\boldsymbol{J}_{I+II}\right)=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\-a_3\\1\end{pmatrix}\right\},\ \mathcal{R}\left(\boldsymbol{J}_{I+II}\right)=\left\{\begin{pmatrix}s_1\\-c_1\\0\end{pmatrix}\right\},\ \mathcal{R}^{\perp}\left(\boldsymbol{J}_{I+II}\right)=\left\{\begin{pmatrix}c_1\\s_1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

As a result, the third link is vertical and point P is on the axis $\boldsymbol{z}_0$ at the boundary of the reachable workspace. Thus, it cannot move neither vertically nor along the normal direction to the plane defined by link 2 and 3.

Exercise 1c

Out of singularities, the required joint velocity control law is

$$\dot{\boldsymbol{q}}=\boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{K}\left(\boldsymbol{p}_d-\boldsymbol{f}(\boldsymbol{q})\right), \tag{1}$$

using a diagonal and uniform gain matrix $\boldsymbol{K}=k\boldsymbol{I}$, with $k>0$. Note that no trajectory is being planned between the initial position $\boldsymbol{p}(0)=\boldsymbol{f}(\boldsymbol{q}(0))$ and the constant desired Cartesian position $\boldsymbol{p}_d$, so that this is a pure feedback law (of the nonlinear type). The position error $\boldsymbol{e}=\boldsymbol{p}_d-\boldsymbol{p}$ will evolve as

$$\dot{\boldsymbol{e}}=-\dot{\boldsymbol{p}}=-\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{q}}\,\dot{\boldsymbol{q}}=-\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}=-k\,\boldsymbol{e},$$

yielding the solution

$$e_i(t)=\exp(-kt)\,e_i(0),\qquad i=x,y,z.$$

Thus, the robot end-effector will exponentially converge to the desired position $\boldsymbol{p}_d\in\mathbb{R}^3$ in the reachable workspace, unless it encounters a singularity where the control law (1) is not defined. Moreover, starting from the initial position $\boldsymbol{p}(0)$, the error $\boldsymbol{e}(t)=\boldsymbol{p}_d-\boldsymbol{p}(t)$ will always be aligned to $\boldsymbol{e}(0)=\boldsymbol{p}_d-\boldsymbol{p}(0)$. Hence, in the absence of perturbations, $\boldsymbol{p}(t)$ remains along the straight line going through $\boldsymbol{p}(0)$ and $\boldsymbol{p}_d$.

Exercise 2

The assigned motion task has to be converted in the joint space, where the command input is defined together with the velocity bounds. Through the standard inverse kinematics of the planar 2R robot we obtain

$$\boldsymbol{q}_{in}=\boldsymbol{f}^{-1}(P_{in})=\begin{pmatrix}0\\\frac{\pi}{4}\end{pmatrix}\ \text{[rad]},\qquad \boldsymbol{q}_{fin}=\boldsymbol{f}^{-1}(P_{fin})=\begin{pmatrix}-\frac{\pi}{4}\\\frac{\pi}{2}\end{pmatrix}\ \text{[rad]}, \tag{2}$$

where the right arm solution (with the '+' sign) has been chosen, both at the initial and final points (so as to avoid a singularity crossing). Thus, the required displacement in the joint space is

$$\Delta\boldsymbol{q}=\boldsymbol{q}_{fin}-\boldsymbol{q}_{in}=\begin{pmatrix}-\frac{\pi}{4}\\\frac{\pi}{4}\end{pmatrix}=\begin{pmatrix}-0.7854\\0.7854\end{pmatrix}\ \text{[rad]}.$$

In order to perform the required rest-to-rest motion in minimum time and in a coordinated way, first we compute separately the minimum-time motion for each joint (i.e., for $i=1,2$): joint i

will have a symmetric bang-bang acceleration profile $[A_{max,i}, -A_{max,i}]$, where the sign of $A_{max,i}$ depends on the sign of Δq_i, its intensity is defined so as to reach the maximum velocity $\pm V_{max,i}$ at the trajectory midpoint $t = T_i/2$, and the total motion time T_i will be such to complete the displacement Δq_i. Thus,

$$A_{max,i} = \frac{V_{max,i}^2}{\Delta q_i}, \qquad T_i = \sqrt{\frac{4\Delta q_i}{A_{max,i}}} > 0.$$

With the given data, it is

$$A_{max,1} = -5.0930 \text{ [rad/s}^2\text{]}, \quad T_1 = 0.7854 \text{ [s]}, \quad A_{max,2} = 2.8648 \text{ [rad/s}^2\text{]}, \quad T_2 = 1.0472 \text{ [s]}.$$

However, since the motion has to be coordinated, the total motion time will be dictated by the slowest joint:

$$T = \max\{T_1, T_2\} \qquad \Rightarrow \qquad T = T_2 = 1.0472 \text{ [s]}.$$

Accordingly, the faster joint should be slowed down and its actual peak velocity V_i and constant acceleration A_i in the first motion half recomputed based on the total motion time T. In the present case, joint 1 will be slowed down with

$$V_1 = \frac{2\Delta q_1}{T} = -1.5 \text{ [rad/s]}, \qquad A_1 = \frac{V_1^2}{\Delta q_1} = -2.8648 \text{ [rad/s}^2\text{]},$$

whereas it is still $A_2 = A_{max,2} = 2.8648$ [rad/s^2] and $V_2 = V_{max,2} = 1.5$ [rad/s] for joint 2. Note that, after the scaling, we have opposite values for the two joints ($A_1 = -A_2$ and $V_1 = -V_2$) simply because in this case the displacement are opposite ($\Delta_1 = -\Delta_2$). Figure 3 shows the coordinated, time-optimal profiles of $\ddot{q}_1(t)$ and $\ddot{q}_2(t)$.

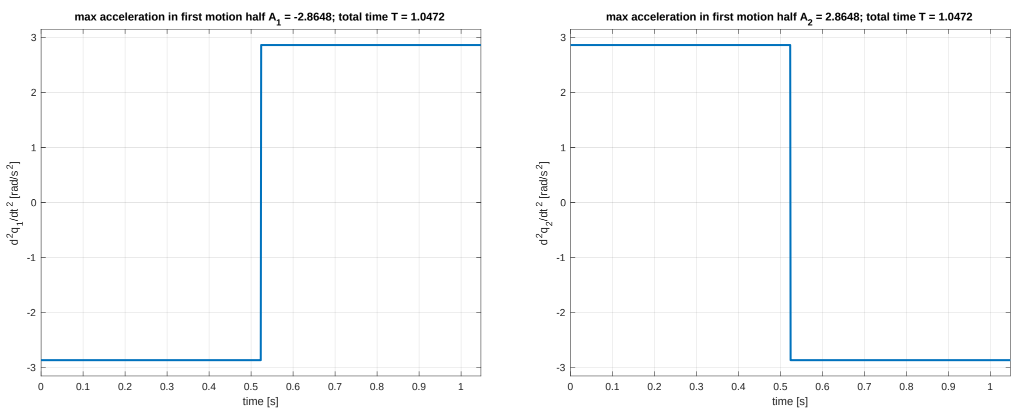


Figure 3: Final acceleration profiles of the two joints.

Note finally that choosing instead the left arm solution in place of (2),

$$\boldsymbol{q}_{in}^- = \boldsymbol{f}^{-1}(P_{in}) = \begin{pmatrix} 0.5110 \\ -\dfrac{\pi}{4} \end{pmatrix} \text{ [rad]}, \qquad \boldsymbol{q}_{fin}^- = \boldsymbol{f}^{-1}(P_{fin}) = \begin{pmatrix} 0.1419 \\ -\dfrac{\pi}{2} \end{pmatrix} \text{ [rad]}, \qquad (3)$$

would have lead to different acceleration profiles, but still to the same coordinated motion time $T = 1.0472$ [s] in this particular case. This is due to the fact that the limiting joint is the second, with a displacement $\Delta q_2 = -\pi/4$ which is the same (in absolute value) as before.

* * * * *

Robotics 1

Midterm Test — November 18, 2022

Exercise 1

Consider the rotation matrix

$$\boldsymbol{R}_d = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix}.$$

Find, if possible, all angle-axis pairs $(\theta, \boldsymbol{r})$ that provide the desired orientation $\boldsymbol{R}_d$. At the end, check your results by verifying that $\boldsymbol{R}(\theta, \boldsymbol{r}) = \boldsymbol{R}_d$.

Exercise 2

The end-effector of a robot undergoes a change of orientation between an initial $\boldsymbol{R}_i$ and a final $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} \\ -1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Provide a minimal representation of the relative rotation between the initial and the final orientation using YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$. At the end, check your solutions by performing the direct computation.

Exercise 3

A DC motor is used to move a link of length $L = 0.7$ [m], as shown in Fig. 1. The motor mounts on its axis an absolute encoder and uses as transmission elements an Harmonic Drive having a flexspline with $N_{FS} = 160$ teeth and a gear with two toothed wheels of radius $r_1 = 2$ and $r_2 = 4$ [cm], respectively.

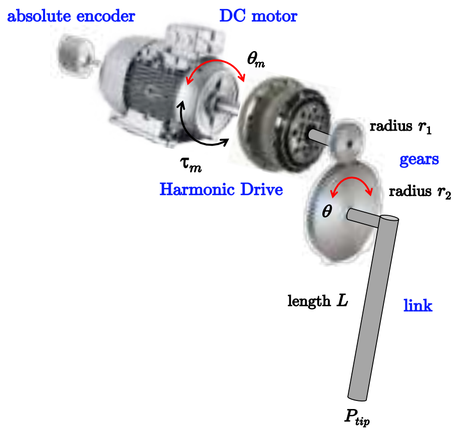


Figure 1: The actuation arrangement of a single link.

- Compute the reduction ratio $n_r > 1$ of the transmission system. Which is the direction of rotation of the link when the motor angular position θ_m is turning counterclockwise?
- Determine the resolution of the absolute encoder that allows distinguishing two link tip positions that are $\Delta r = 0.1$ [mm] away. What should be the minimum number of tracks N_t of the encoder?
- If the link has an angular range $\Delta\theta_{max} = 180°$, how many turns of the motor are needed to cover the entire range? With a multi-turn absolute encoder, what is the minimum number of bits for counting all these turns?
- If the motor inertia is $J_m = 1.2 \cdot 10^{-4}$ [kgm^2], determine the optimal value of the link inertia J_l around the axis at its base which minimizes the motor torque τ_m needed for a desired link acceleration $\ddot{\theta}$. What is then the value of τ_m (in [Nm]) for $\ddot{\theta} = 7$ [rad/s^2]?

Exercise 4

A large 6R robot manipulator is mounted on the ceiling of an industrial cell and holds firmly a cylindric object in its jaw gripper. The world frame RF_w of the cell is placed on the floor, at about the cell center. The robot base frame RF_0 is defined by ${}^w\boldsymbol{T}_0$, while its end-effector frame RF_e has the origin O_e at the center of the grasped object. The robot direct kinematics is expressed in symbolic form by ${}^0\boldsymbol{T}_e(\boldsymbol{q})$, in terms of the joint variables $\boldsymbol{q}$. A camera is placed in the cell and its frame RF_c, having the origin O_c at the center of the image plane and the $\boldsymbol{z}_c$ unit vector along the focal axis of the camera, is defined by ${}^w\boldsymbol{T}_c$.

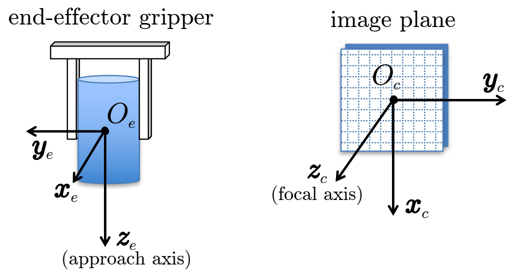


Figure 2: Definition of frames RF_e and RF_c for the considered task.

Figure 2 details the placement of the end-effector frame RF_e and of the camera frame RF_c. The robot should hold the object in front of the camera, with the major axis of the cylinder aligned to the camera focal axis and its center at a distance $d > 0$ from O_c. Define the task kinematics equation, to be solved for the joint variables $\boldsymbol{q}$, when the transformation matrices and the object-camera offset are given by

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^w\boldsymbol{T}_c = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad d = 1 \text{ [m]}.
$$

Discuss also whether the robot is kinematically redundant for the task or not.

Exercise 5

For the spatial RPR robot of Fig. 3, complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. The origin of the last frame should be placed at the point P. Moreover, the frame assignment should be such that all constant DH parameters are *non-negative* and the value of the joint variables q_i, $i = 1, 2, 3$, are *strictly positive* in the shown configuration. Compute then the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of point P.

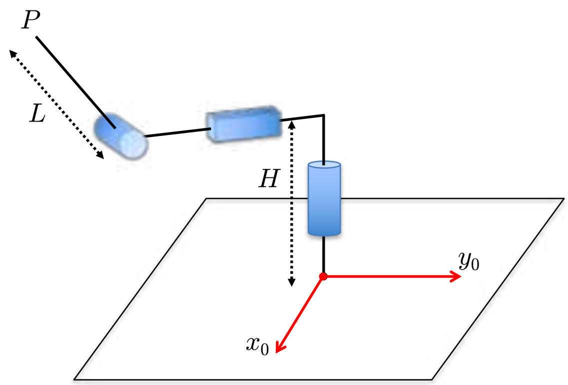


Figure 3: A spatial RPR robot.

Exercise 6

For the spatial RPR robot of Fig. 3, provide the closed-form expression of the inverse kinematics for the position $\boldsymbol{p}$ of point P. Assuming for simplicity that the joints have unlimited ranges, how many inverse kinematics solutions are there in the regular case? Compute the numerical values of all inverse solutions $\boldsymbol{q}$ when $\boldsymbol{p} = (3, 4, 1.5)$ [m] and the geometric parameters of the robot are $H = L = 1$ [m]. Check the solutions!

[180 minutes, open books]

Solution

November 18, 2022

Exercise 1

It is easy to verify that $\boldsymbol{R}_d \in SO(3)$. Denoting by r_{ij} the elements of $\boldsymbol{R}_d$, since the matrix is symmetric, it is

$$\sin\theta = \frac{1}{2}\sqrt{(r_{12}-r_{21})^2 + (r_{13}-r_{31})^2 + (r_{23}-r_{32})^2} = 0.$$

We are in a singular case for the inverse problem of extracting an angle and axis from a rotation matrix. Moreover,

$$\cos\theta = \frac{\text{trace}\,\{\boldsymbol{R}_d\} - 1}{2} = -1 \qquad \Rightarrow \qquad \theta = \pi \quad (\text{or } -\pi, \text{ which is the same angle}).$$

Therefore, a solution exists for $\boldsymbol{r}$ and we shall use the special formulas

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{\dfrac{r_{11}+1}{2}} \\ \pm\sqrt{\dfrac{r_{22}+1}{2}} \\ \pm\sqrt{\dfrac{r_{33}+1}{2}} \end{pmatrix} = \begin{pmatrix} \pm\dfrac{1}{\sqrt{6}} \\ \pm\dfrac{2}{\sqrt{6}} \\ \pm\dfrac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \pm 0.4082 \\ \pm 0.8165 \\ \pm 0.4082 \end{pmatrix},$$

where the correct combinations of signs (among the 8 possibilities) should be determined so as to guarantee that the remaining three equalities in $\boldsymbol{R}_d = 2\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{I}$ hold:

$$2\,r_x r_y = r_{12} = \frac{2}{3}, \qquad 2\,r_x r_z = r_{13} = -\frac{1}{3}, \qquad 2\,r_y r_z = r_{23} = -\frac{2}{3}.$$

By coding this logic, one obtains the two solutions

$$\boldsymbol{r}_1 = \begin{pmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{pmatrix}, \qquad \boldsymbol{r}_2 = \begin{pmatrix} -0.4082 \\ -0.8165 \\ 0.4082 \end{pmatrix} = -\boldsymbol{r}_1.$$

Using

$$\boldsymbol{R}(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta,$$

we can check that $\boldsymbol{R}(\theta,\boldsymbol{r}_1) = \boldsymbol{R}(\theta,\boldsymbol{r}_2) = \boldsymbol{R}_d$ is satisfied.

Exercise 2

The relative rotation ${}^i\boldsymbol{R}_f$ between the initial orientation $\boldsymbol{R}_i$ and the final orientation $\boldsymbol{R}_f$ is computed as

$${}^i\boldsymbol{R}_f = \boldsymbol{R}_i^T \boldsymbol{R}_f = \begin{pmatrix} 0 & 0 & -1 \\ 0.5 & -\dfrac{\sqrt{3}}{2} & 0 \\ -\dfrac{\sqrt{3}}{2} & -0.5 & 0 \end{pmatrix}.$$

On the other hand, the rotation matrix associated to a minimal representation with YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\begin{aligned} \boldsymbol{R}_{YXY}(\alpha_1,\alpha_2,\alpha_3) &= \boldsymbol{R}_Y(\alpha_1)\boldsymbol{R}_X(\alpha_2)\boldsymbol{R}_Y(\alpha_3) = \\ &= \begin{pmatrix} \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 1 & 0 \\ -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \cos\alpha_3 & 0 & \sin\alpha_3 \\ 0 & 1 & 0 \\ -\sin\alpha_3 & 0 & \cos\alpha_3 \end{pmatrix} \\ &= \begin{pmatrix} c_1c_3 - s_1c_2s_3 & s_1s_2 & c_1s_3 + s_1c_2c_3 \\ s_2s_3 & c_2 & -s_2c_3 \\ -s_1c_3 - c_1c_2s_3 & c_1s_2 & c_1c_2c_3 - s_1s_3 \end{pmatrix}, \end{aligned}$$

where the usual shorthand notation has been used (e.g., $c_i = \cos\alpha_i$). The inverse representation problem, namely finding all triples $(\alpha_1, \alpha_2, \alpha_3)$ of YXY Euler angles such that

$$\boldsymbol{R}_{YXY}(\alpha_1, \alpha_2, \alpha_3) = {}^i\boldsymbol{R}_f, \tag{1}$$

can be solved in closed form (up to singular cases). Denote by r_{ij} the elements of ${}^i\boldsymbol{R}_f$. Taking advantage of the simpler expressions in the second column (viz., second row) of $\boldsymbol{R}_{YXY}$, one has from eq. (1)

$$c_2 = r_{22}, \qquad s_2 = \pm\sqrt{r_{12}^2 + r_{32}^2} \qquad \Rightarrow \qquad \alpha_2 = \text{ATAN2}\left\{s_2, c_2\right\},$$

yielding the two (symmetric) values $\alpha_2^{(I),(II)} = \pm 2.6180$ [rad]. Since $s_2 = \pm\, 0.5 \neq 0$, the problem at hand is regular and computations can be carried out also for the other two angles. We have:

$$s_1 = \frac{r_{12}}{s_2}, \qquad c_1 = \frac{r_{32}}{s_2} \qquad \Rightarrow \qquad \alpha_1 = \text{ATAN2}\left\{s_1, c_1\right\},$$

and

$$s_3 = \frac{r_{21}}{s_2}, \qquad c_3 = \frac{-r_{23}}{s_2} \qquad \Rightarrow \qquad \alpha_3 = \text{ATAN2}\left\{s_3, c_3\right\}.$$

Depending on the sign chosen for s_2, there are again two solutions for each angle. We obtain

$$\alpha_1^{(I)} = \pi, \quad \alpha_1^{(II)} = 0 \qquad \text{and} \qquad \alpha_3^{(I)} = \frac{\pi}{2}, \quad \alpha_3^{(II)} = -\frac{\pi}{2} \;\; \text{[rad]}.$$

As a result, the two (regular) solutions of the problem are:

$$\boldsymbol{\alpha}^{(I)} = \begin{pmatrix} \pi \\ \dfrac{5\pi}{6} \\ \dfrac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 3.1416 \\ 2.6180 \\ 1.5708 \end{pmatrix}, \qquad \boldsymbol{\alpha}^{(II)} = \begin{pmatrix} 0 \\ -\dfrac{5\pi}{6} \\ -\dfrac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -2.6180 \\ -1.5708 \end{pmatrix} \;\; \text{[rad]}.$$

It is easy to check that

$$\boldsymbol{R}_i \boldsymbol{R}_{YXY}(\boldsymbol{\alpha}^{(I)}) = \boldsymbol{R}_i \boldsymbol{R}_{YXY}(\boldsymbol{\alpha}^{(II)}) = \boldsymbol{R}_f.$$

Exercise 3

The reduction ratio n_r of the entire transmission is the product of the reduction ratios n_{HD} of the Harmonic Drive and n_g of the spur gear:

$$n_r = n_{HD} \cdot n_g = \frac{N_{FS}}{2} \cdot \frac{r_2}{r_1} = 80 \cdot 2 = 160.$$

Both transmission elements invert on the output axis the direction of rotation of their input axis. As a result, the angular position θ of the link is turning is the same direction (positive counterclockwise) of the angular position θ_m of the motor.

A linear variation $\Delta r = 1 \cdot 10^{-4}$ [m] in position at the tip of the link corresponds to an angular variation $\Delta\theta$ at the base. Therefore, the needed resolution $\Delta\theta_m$ at the motor side (where the absolute encoder is mounted) is

$$\Delta\theta_m = \Delta\theta \cdot n_r = \frac{\Delta r}{L} \cdot n_r = 1.4286 \cdot 10^{-4} \cdot 160 = 0.0229 \text{ [rad] } (= 1.31^\circ).$$

Being the resolution of an absolute encoder equal to $\Delta = 2\pi/2^{N_t}$, the request $\Delta \leq \Delta\theta_m$ implies that the minimum number of tracks N_t is the integer

$$N_t = \left\lceil \log_2 \left(\frac{2\pi}{\Delta\theta_m} \right) \right\rceil = \lceil 8.1027 \rceil = 9.$$

In order to cover the entire range $\Delta\theta_{max}$ (in degrees) of link angular motion, the number of motor turns is

$$n_{turns} = \frac{\Delta\theta_{max} \cdot n_r}{360^\circ} = \frac{180^\circ \cdot 160}{360^\circ} = 80.$$

For counting this number of turns, the minimum number of devoted bits N_{mt} in a multi-turn absolute encoder should be

$$N_{mt} = \lceil \log_2 80 \rceil = 7.$$

Finally, the optimal value of the link inertia J_l is computed from the optimal value of the reduction ratio:

$$n_r = \sqrt{\frac{J_l}{J_m}} \qquad \Rightarrow \qquad J_l = J_m \cdot n_r^2 = 1.2 \cdot 10^{-4} \cdot 160^2 = 3.0720 \text{ [kgm}^2\text{]}.$$

The motor torque τ_m needed for obtaining a desired link acceleration $\ddot{\theta} = 7$ [rad/s^2] is then

$$\tau_m = J_m \ddot{\theta}_m + \frac{1}{n_r} J_l \ddot{\theta} = \left(J_m n_r + \frac{J_l}{n_r} \right) \ddot{\theta} = (2 J_m n_r) \ddot{\theta} = 0.0384 \cdot 7 = 0.2688 \text{ [Nm]}.$$

Exercise 4

The kinematic identity describing the task is given by

$$^w\boldsymbol{T}_0 {}^0\boldsymbol{T}_e(\boldsymbol{q}) = {}^w\boldsymbol{T}_c {}^c\boldsymbol{T}_e, \tag{2}$$

in which the desired pose of the robot end-effector in the world frame is equivalently expressed passing through the robot or through the camera, respectively the left-hand side or the right-hand side of (2). Since the unit axes $\boldsymbol{z}_e$ and $\boldsymbol{z}_c$ should be aligned and in the opposite direction ($\boldsymbol{z}_c = -\boldsymbol{z}_e$) and the offset between O_c and O_e should be only along $\boldsymbol{z}_c$, an homogeneous matrix that defines the correct pose of the end-effector, as seen from the camera frame[1], is given by

$$^c\boldsymbol{T}_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } d = 1 \text{ [m]}. \tag{3}$$

Note that this choice is not unique: it corresponds to aligning also the $\boldsymbol{x}_e$ unit vector of the end-effector frame with the unit vector $\boldsymbol{x}_c$ of the camera frame. However, such alignment is not necessary and one may choose to have an arbitrary angle $\alpha \in (\pi, \pi]$ between these two vectors. As a result, also the more general homogeneous matrix

$$^c\boldsymbol{T}_e(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ -\sin\alpha & -\cos\alpha & 0 & 0 \\ 0 & 0 & -1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } d = 1 \text{ [m]}, \tag{4}$$

satisfies the task[2]. Since there is one parameter left free of choice in defining a desired 3D pose, the task is 5-dimensional and the 6R robot has one degree of redundancy in realizing this task (in fact, the task involves *positioning* and *pointing* in 3D).

Given $^w\boldsymbol{T}_0$ and $^w\boldsymbol{T}_c$, one obtains from (2) and (3)

$$^0\boldsymbol{T}_e(\boldsymbol{q}) = \left({}^w\boldsymbol{T}_0\right)^{-1} {}^w\boldsymbol{T}_c {}^c\boldsymbol{T}_e = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 2.2929 \\ 0 & -1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2.2071 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

[1]The same description holds as seen from the end-effector frame since in this case $^e\boldsymbol{T}_c = \left({}^c\boldsymbol{T}_e\right)^{-1} = {}^c\boldsymbol{T}_e$, due to the task symmetry.

[2]With $\alpha = \pi$, the unit vectors $\boldsymbol{y}_e$ and $\boldsymbol{y}_c$ would be aligned.

which is the requested task kinematics equation to be solved for $\boldsymbol{q}$ (i,e., the formulation of the inverse kinematics problem for the 6R robot). A similar equation is found when using (4) in place of (3).

Exercise 5

The (unique) DH frame assignment for the RPR robot of Fig. 3 satisfying all requests is shown in Fig. 4. The corresponding DH parameters are reported in Tab. 1.

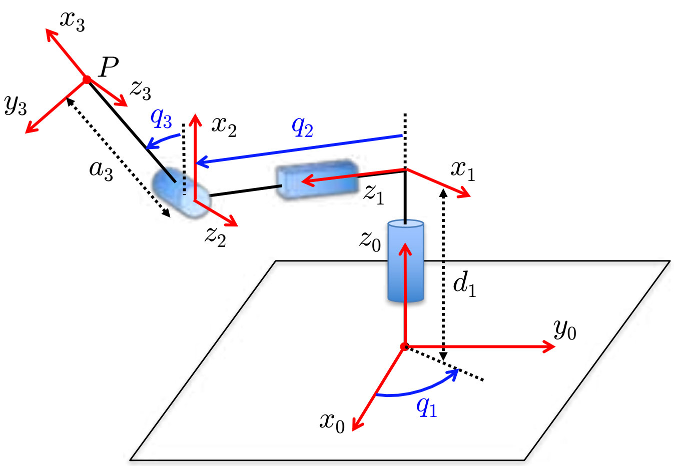


Figure 4: DH frames for the spatial RPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = H > 0$	$q_1 > 0$
2	$\pi/2$	0	$q_2 > 0$	$\pi/2$
3	0	$a_3 = L > 0$	0	$q_3 > 0$

Table 1: DH parameters corresponding to the frames in Fig. 4. The signs attributed to the joint variables refer to the shown robot configuration.

From the associated homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2(q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\boldsymbol{p}_{hom} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = \boldsymbol{A}_1(q_1) \left(\boldsymbol{A}_2(q_2) \left(\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right) \right),$$

yielding the direct kinematics of the position of point P as

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} s_1 (q_2 + a_3 s_3) \\ -c_1 (q_2 + a_3 s_3) \\ d_1 + a_3 c_3 \end{pmatrix}. \tag{5}$$

Exercise 6

Consider the direct kinematics (5), with assigned desired values for the components p_x, p_y, and p_z for the position vector $\boldsymbol{p}$ on the left-hand side. From the third equation, one has

$$c_3 = \frac{p_z - d_1}{a_3} \qquad \Rightarrow \qquad s_3 = \pm\sqrt{1 - c_3^2}.$$

Provided that $c_3 \in [-1, 1]$, two symmetric solutions are found for q_3, each corresponding to a sign chosen for s_3:

$$q_3^{(o)} = \text{ATAN2}\left\{|s_3|, c_3\right\}, \qquad q_3^{(i)} = \text{ATAN2}\left\{-|s_3|, c_3\right\} = -q_3^{(o)}. \tag{6}$$

The solution $q_3^{(o)}$ has the forearm (link 3) bent *outward* from the base joint axis, while with $q_3^{(i)}$ the forearm is bent *inward*. When $c_3 = \pm 1$, the two solutions in (6) collapse into a singleton $q_3 = 0$ (for $c_3 = 1$, link 3 is vertical and points *upward*) or $q_3 = \pi$ (for $c_3 = -1$, link 3 is vertical and points *downward*). These two situations are a *singularity* for the solution q_3. When $|c_3| > 1$, the inverse kinematics problem has no solution because the desired position $\boldsymbol{p}$ of point P is outside the reachable workspace of the robot.

Next, squaring and summing the first two equations in (5) yields

$$p_x^2 + p_y^2 = (q_2 + a_3 s_3)^2 \geq 0.$$

If this quantity is strictly positive, we can extract the root and substitute it in place of the common factor in the right-hand side of the first two kinematic equations in (5) so as to obtain

$$p_x = \pm s_1\sqrt{p_x^2 + p_y^2}, \qquad -p_y = \pm c_1\sqrt{p_x^2 + p_y^2},$$

which involve only the unknown q_1 and the input data. Then, two solutions are obtained for q_1,

$$q_1 = \text{ATAN2}\left\{\frac{p_x}{\pm\sqrt{p_x^2 + p_y^2}}, \frac{-p_y}{\pm\sqrt{p_x^2 + p_y^2}}\right\},$$

depending on the upper or lower sign chosen for the square root in both arguments (and independently from the signs in the solution (6) for q_3). Actually, since this computation is performed only when $p_x^2 + p_y^2 > 0$, one can simplify the expression of the solutions as

$$q_1^{(f)} = \text{ATAN2}\left\{p_x, -p_y\right\}, \qquad q_1^{(b)} = \text{ATAN2}\left\{-p_x, +p_y\right\}. \tag{7}$$

In the solution $q_1^{(f)}$ the base of the robot *faces* point P, whereas with $q_1^{(b)}$ the base is rotated by π and the robot is giving the *back* to point P. If $p_x^2 + p_y^2 = 0$, i.e., the desired position of point P is on the axis of joint 1, q_1 is undefined and there are infinite solutions to the inverse kinematics problem (*singular* case).

Two possible ways can be followed to determine the variable q_2 of the prismatic joint.

First method. Add the first two equations in (5), weighted respectively by s_1 and $-c_1$:

$$s_1 p_x - c_1 p_y = q_2 + a_3 s_3.$$

From this, using the previously obtained results for s_1, c_1 and s_3, we have

$$q_2 = s_1 p_x - c_1 p_y \mp a_3\sqrt{1 - c_3^2} = \pm\sqrt{p_x^2 + p_y^2} \mp \sqrt{a_3^2 - (p_z - d_1)^2}. \tag{8}$$

Note that the argument of the last square root in (8) is always non-negative (otherwise the desired position $\boldsymbol{p}$ of point P would be outside the reachable workspace, as already noted). There are four combinations of possible signs to be chosen in eq. (8), resulting in four solutions for q_2 in the regular case, each corresponding to one of the alternative solutions for q_1 and for q_3. When the solution for q_3 is in singularity, meaning that $a_3^2 = (p_z - d_1)^2$, only two solutions are left for q_2. The same occurs when the solution for q_1 is in singularity ($p_x = p_y = 0$). At the intersection of the singularities, there is only one solution, namely $q_2 = 0$.

Second method. Square and sum all three equations in (5), after having moved d_1 to the left in the third one. This leads to

$$p_x^2 + p_y^2 + (p_z - d_1)^2 = (q_2 + a_3 s_3)^2 + (a_3 c_3)^2 = q_2^2 + a_3^2 + 2 a_3 s_3 q_2.$$

This is a polynomial equation of second degree in the unknown q_2, which can be rewritten in the form

$$q_2^2 + 2b\, q_2 - c = 0,$$

with

$$b = a_3 s_3 = \pm\sqrt{a_3^2 - (p_z - d_1)^2}, \qquad c = p_x^2 + p_y^2 + (p_z - d_1)^2 - a_3^2.$$

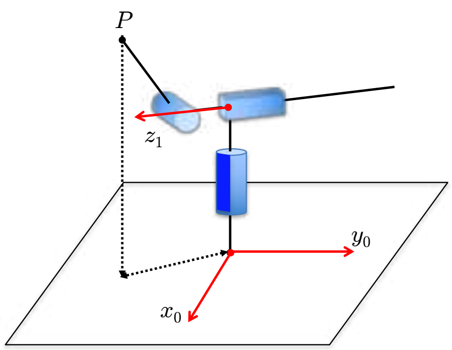
Accordingly, we obtain two pairs of solutions (one pair for each sign chosen for b)

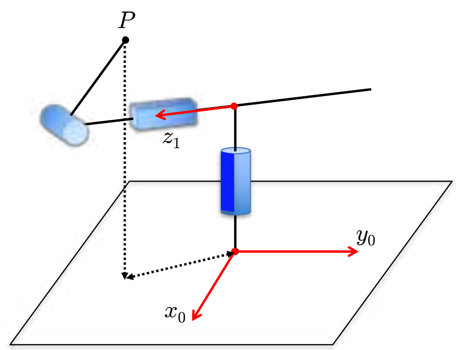
$$\begin{aligned} q_2^{(++/+-)} &= b \pm \sqrt{b^2 + c} = \sqrt{a_3^2 - (p_z - d_1)^2} \pm \sqrt{p_x^2 + p_y^2} \\ q_2^{(-+/--)} &= -b \pm \sqrt{b^2 + c} = -\sqrt{a_3^2 - (p_z - d_1)^2} \pm \sqrt{p_x^2 + p_y^2}. \end{aligned} \tag{9}$$

The two eqs. (9) are clearly equivalent to eq. (8). When $b = 0$, only two solutions are left. When $b = c = 0$ simultaneously, $q_2 = 0$ is the only solution.

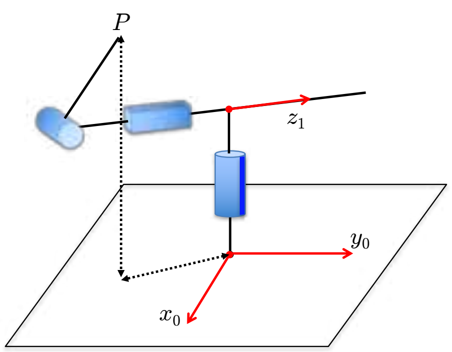
The four generic solutions in the regular case are summarized below, each having a sketch of the associated robot configuration (the front part of the robot base, where $\boldsymbol{z}_1$ is pointing, is shown in dark blue).

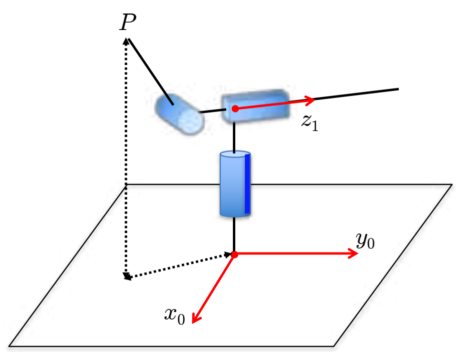
$$\boldsymbol{q}^{(1)} = \begin{pmatrix} q_1^{(f)} \\ q_2^{(+-)} \\ q_3^{(o)} \end{pmatrix} \quad \text{(base facing, forearm outward)} \qquad \boldsymbol{q}^{(2)} = \begin{pmatrix} q_1^{(f)} \\ q_2^{(++)} \\ q_3^{(i)} \end{pmatrix} \quad \text{(base facing, forearm inward)}$$





$$\boldsymbol{q}^{(3)} = \begin{pmatrix} q_1^{(b)} \\ q_2^{(--)} \\ q_3^{(o)} \end{pmatrix} \quad \text{(base backing, forearm outward)} \qquad \boldsymbol{q}^{(4)} = \begin{pmatrix} q_1^{(b)} \\ q_2^{(-+)} \\ q_3^{(i)} \end{pmatrix} \quad \text{(base backing, forearm inward)}$$





Consider now the given numerical data. Since $d_1 = H = 1$ and $a_3 = L = 1$ [m], the four (regular) solutions for $\boldsymbol{p} = (3, 4, 1.5)$ are:

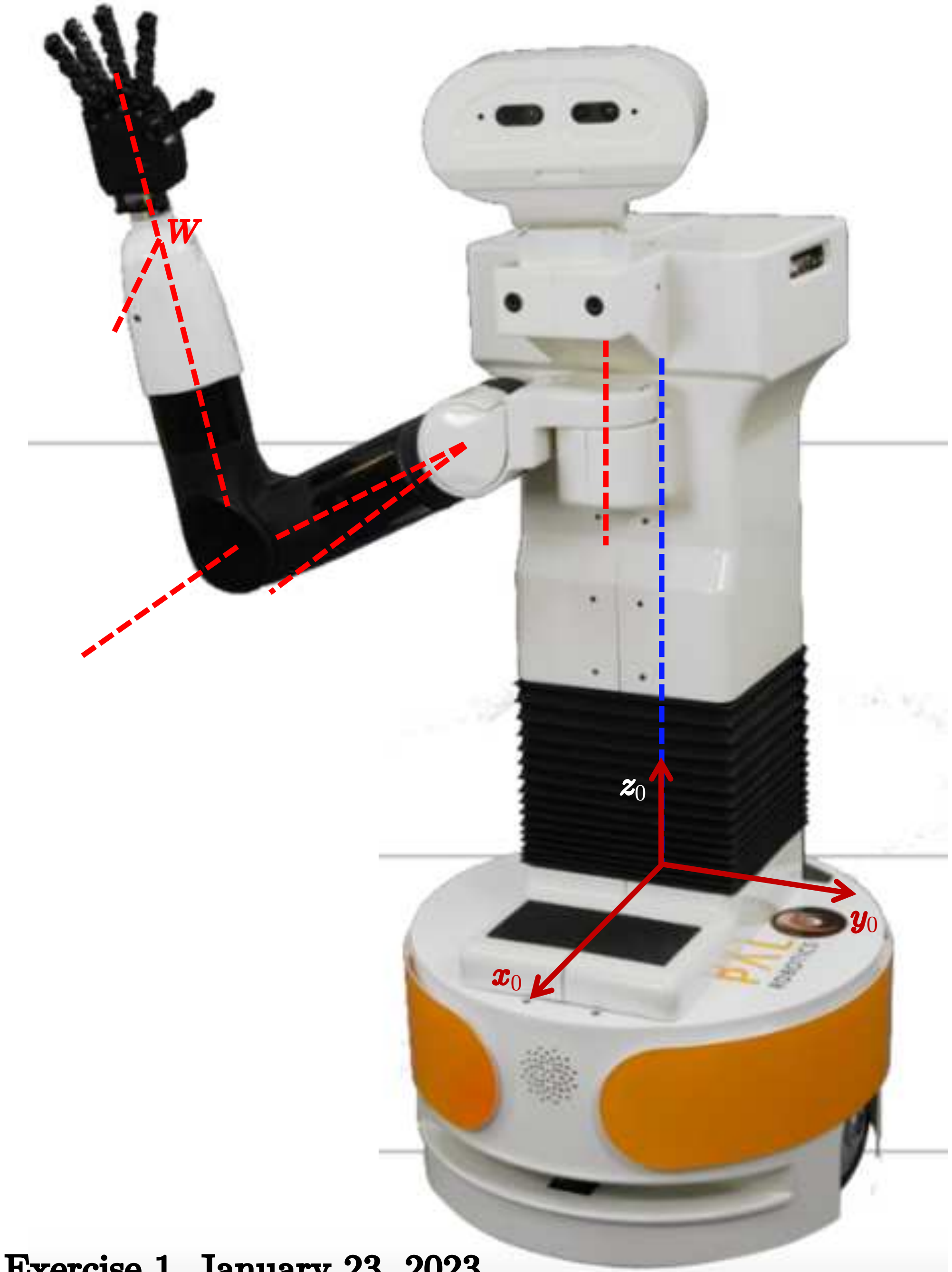
$$\boldsymbol{q}^{(1)} = \begin{pmatrix} 2.4981 \\ 4.1340 \\ 1.0472 \end{pmatrix}, \; \boldsymbol{q}^{(2)} = \begin{pmatrix} 2.4981 \\ 5.8660 \\ -1.0472 \end{pmatrix}, \; \boldsymbol{q}^{(3)} = \begin{pmatrix} -0.6435 \\ -5.8660 \\ 1.0472 \end{pmatrix}, \; \boldsymbol{q}^{(4)} = \begin{pmatrix} -0.6435 \\ -4.1340 \\ -1.0472 \end{pmatrix} \text{[rad/m/rad]}.$$

* * * * *

TIAGo by PAL Robotics – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				
8				



TIAGo by PAL Robotics – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				
8				

Robotics 1

January 23, 2023

Exercise 1

Figure 1 shows two views of TIAGo, a mobile manipulator by PAL Robotics. Disregard the wheeled base and consider only the motion of the 8-dof robotic arm with respect to the on-board reference frame RF_0 as described hereafter. The first prismatic joint of the arm (with axis in blue/dashed) provides elevation to the rest of the structure and is followed by 7 revolute joints (with axes in red/dashed): the axes of joints #1 and #2 are parallel, joints #3 and #4 intersect at the shoulder, joints #4 to #6 intersect at the elbow, while the last three joints (#6 to #8) constitute a spherical wrist with center at W. Videos showing the TIAGo arm mobility can be seen on YouTube[1].

Assign the kinematic frames to the arm links, following the classical Denavit-Hartenberg (DH) convention. Draw clearly the frames and fill in the associated table of parameters using one (or both) of the two extra sheets that have been distributed. Keep the frame RF_0 as first DH frame, and place the origin of the last DH frame at the wrist center point W.

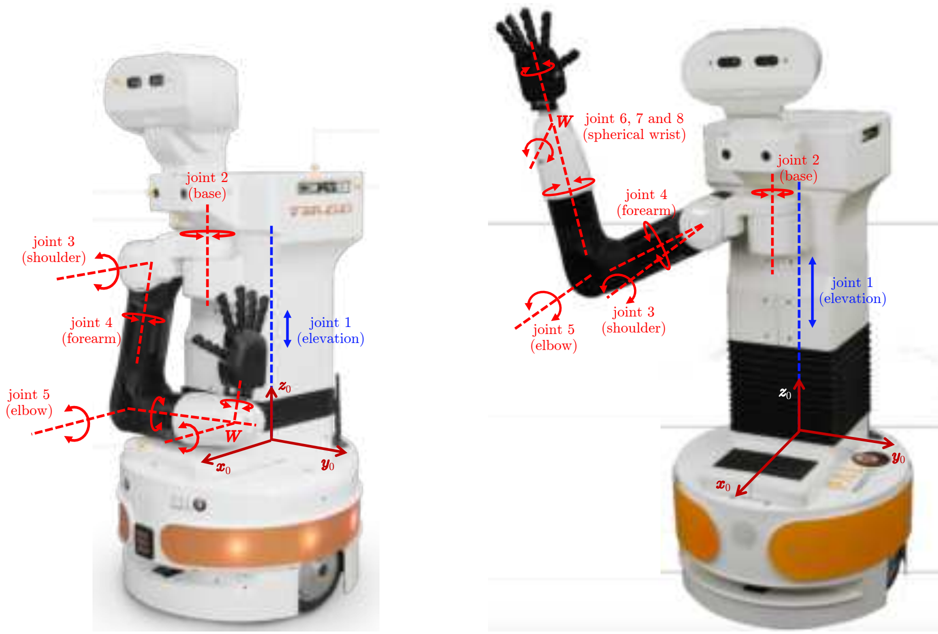


Figure 1: The TIAGo mobile manipulator, with the 8-dof arm shown in two configurations.

Exercise 2

Let the origin of frame RF_0 of the TIAGo robotic arm be at a position ${}^w\boldsymbol{p}_0 = (1.5, -4.5, 0.3)$ [m] with respect to a world frame RF_w placed horizontally on the floor surface. Moreover, let the angle between $\boldsymbol{y}_w$ and $\boldsymbol{y}_0$ axes be $\phi = -45°$. With the TIAGo robotic arm in a generic configuration $\boldsymbol{q}$, use the formula that evaluates the position ${}^w\boldsymbol{p}_W$ of the wrist center W with the minimum number of elementary operations. Provide then the symbolic expression of ${}^w\boldsymbol{p}_W(\boldsymbol{q})$ in explicit form.

[1] TIAGo - Robot Workspace versatility: `https://youtu.be/6BwRqwD066g` (1'08"); TIAGo - Gravity compensation: `https://youtu.be/EjIggPKy0T0` (1'23").

Exercise 3

For which interval of values of the angle $\theta_2 \in (-\pi, \pi]$ does the transcendental equation

$$\sin\theta_1 + 2\cos(\theta_1 + \theta_2) = 2 \tag{1}$$

have real solutions for the angle θ_1?

Exercise 4

For a 4-dof robot, consider the task vector

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 + q_4 \cos(q_1 + q_3) \\ q_2 \sin q_1 + q_4 \sin(q_1 + q_3) \\ q_1 + q_3 \end{pmatrix}. \tag{2}$$

Determine all singular configurations for the corresponding analytic robot Jacobian $\boldsymbol{J}(\boldsymbol{q})$. Moreover, find *if possible*:

- a joint velocity $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$ such that $\dot{\boldsymbol{r}} = \boldsymbol{0}$ when the robot is in a regular configuration;
- all joint velocities $\dot{\boldsymbol{q}}$ such that $\dot{\boldsymbol{r}} = \boldsymbol{0}$ when the robot is in a singular configuration;
- the direction(s) along which no task velocity can be realized when the robot is in the chosen singular configuration;
- a generalized task force $\boldsymbol{f}_0 \neq \boldsymbol{0}$ that is statically balanced by the joint torque $\boldsymbol{\tau} = \boldsymbol{0}$ when the robot is in a regular configuration;
- all generalized task forces $\boldsymbol{f}$ that can be statically balanced by zero joint torque when the robot is in the chosen singular configuration.

Exercise 5

The end-effector of a robot manipulator should follow an helical path $\boldsymbol{p} = \boldsymbol{p}(s)$, parametrized by the scalar $s \geq 0$. The helix is right-handed, with radius $r = 0.4$ m and pitch $2\pi h$, with $h = 0.3$ m, starting from the position $\boldsymbol{p}_0 = (0, 0, 2r)$ at $s = 0$. Its axis passes through the point $C = (0, 0, r)$ and is parallel to the $\boldsymbol{y}$-axis. In the time interval $t \in [0, T]$, the robot end-effector should trace two complete turns of the helix, starting and ending its (rest-to-rest) motion with zero velocity, i.e., with $\dot{\boldsymbol{p}}(0) = \dot{\boldsymbol{p}}(T) = \boldsymbol{0}$.

Plan a timing law $s = s(t)$ that minimizes the motion time T under the following bounds on the norm of the velocity and on the (absolute) tangential and normal accelerations,

$$\|\dot{\boldsymbol{p}}\| \leq V, \qquad \left|\ddot{\boldsymbol{p}}^T \boldsymbol{t}\right| \leq A, \qquad \left|\ddot{\boldsymbol{p}}^T \boldsymbol{n}\right| \leq A, \tag{3}$$

where $\boldsymbol{t} = \boldsymbol{t}(s)$ and $\boldsymbol{n} = \boldsymbol{n}(s)$ are the unit axes of the Frenet frame tangent and normal to the path. Determine the minimum time T^* when $V = 2$ m/s and $A = 4.5$ m/s^2. Sketch the profiles of $s(t)$, $\dot{s}(t)$ and $\ddot{s}(t)$ in the obtained time-optimal solution.

Consider next a spatial elbow-type 3R robot manipulator with its base on the plane $z = 0$, height $L_1 = 0.8$ m between base and shoulder, and length of the second and third link $L_2 = L_3 = 1.5$ m. Determine a good placement (x_b, y_b) of the robot base, such that the complete helical path belongs to the primary workspace of the robot and kinematic singularities are not encountered while the end-effector is tracing the path.

[270 minutes, open books]

Solution

January 23, 2023

Exercise 1

A possible conventional DH frame assignment for the TIAGo robotic arm is shown in Fig. 2. The two views are used in order to better illustrate the assignment (which is indeed consistent in the two pictures). The corresponding DH parameters are reported in Tab. 1. Note that $\boldsymbol{x}_0 \| \boldsymbol{x}_1$ ($\theta_1 = 0$).

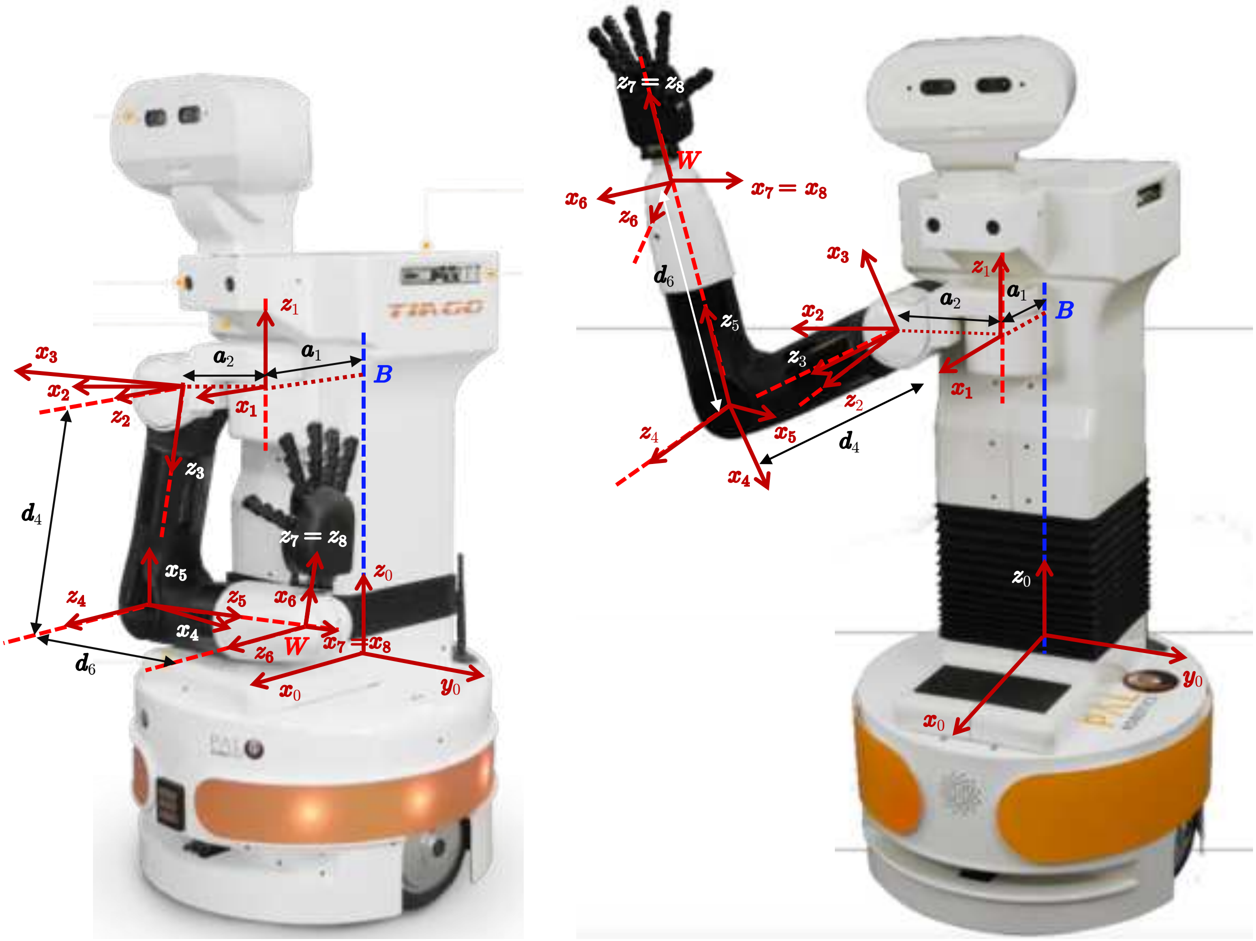


Figure 2: Two views of the DH frames assigned to the TIAGo robotic arm.

The actual values of the constant DH parameters are:

$$a_1 = \overrightarrow{BO_1} = 0.1557, \quad a_2 = \overrightarrow{O_1O_2} = 0.125, \quad d_4 = \overrightarrow{O_3O_4} = 0.3115, \quad d_6 = \overrightarrow{O_5O_6} = 0.312 \ \ [\mathrm{m}].$$

Note that the above assignment may not correspond to the one used by the manufacturer (or in URDF models). Also, very minor offsets exist at the elbow and at the shoulder of the robotic arm; these offsets have been neglected in this exercise.

i	α_i	a_i	d_i	θ_i
1	0	a_1	q_1	0
2	$-\pi/2$	a_2	0	q_2
3	$-\pi/2$	0	0	q_3
4	$-\pi/2$	0	d_4	q_4
5	$\pi/2$	0	0	q_5
6	$-\pi/2$	0	d_6	q_6
7	$-\pi/2$	0	0	q_7
8	0	0	0	q_8

Table 1: Table of DH parameters corresponding to the frames in Fig. 2.

Exercise 2

The direct kinematics of the robotic arm from RF_w to RF_8 using homogeneous transformation matrices is

$$ {}^w\boldsymbol{T}_8\left({}^w\boldsymbol{p}_0, \phi, \boldsymbol{q}\right) = {}^w\boldsymbol{T}_0\left({}^w\boldsymbol{p}_0, \phi\right)\, {}^0\boldsymbol{T}_8(\boldsymbol{q}), \tag{4} $$

with

$$ {}^w\boldsymbol{T}_0\left({}^w\boldsymbol{p}_0, \phi\right) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & p_{0,x} \\ \sin\phi & \cos\phi & 0 & p_{0,y} \\ 0 & 0 & 1 & p_{0,z} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1.5 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -4.5 \\ 0 & 0 & 1 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

In general, the most efficient way for computing only the position of the origin O_8 of the last DH frame expressed in the reference frame RF_0 is by the nested matrix-vector product

$$ {}^0\boldsymbol{p}_{8,hom} = \begin{pmatrix} {}^0\boldsymbol{p}_8 \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left(\ldots\left({}^7\boldsymbol{A}_8(q_8)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\ldots\right)\right). \tag{5} $$

In the present case, it is $W = O_8 = O_7 = O_6$. Thus, the last column of the two matrices ${}^6\boldsymbol{A}_7(q_7)$ and ${}^7\boldsymbol{A}_8(q_8)$ is simply $\begin{pmatrix} \boldsymbol{0}^T & 1 \end{pmatrix}^T$. Moreover, being the last column of matrix ${}^5\boldsymbol{A}_6(q_6)$ equal to $\begin{pmatrix} 0 & 0 & d_6 & 1 \end{pmatrix}^T$, equation (5) simplifies to

$$ {}^0\boldsymbol{p}_{W,hom} = \begin{pmatrix} {}^0\boldsymbol{p}_W \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\left({}^4\boldsymbol{A}_5(q_5)\begin{pmatrix} 0 \\ 0 \\ d_6 \\ 1 \end{pmatrix}\right)\right)\right)\right). \tag{6} $$

The symbolic outcome of (6) can be easily obtained adapting the MATLAB code for the direct kinematics of a robot manipulator `dirkin.m` that is available on the web site of the course. Since

$$ {}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & -s_2 & a_2c_2 \\ s_2 & 0 & c_2 & a_2s_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$
{}^{3}\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{4}\boldsymbol{A}_5(q_5) = \begin{pmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

one obtains

$$
{}^{0}\boldsymbol{p}_W = \begin{pmatrix} a_1 + a_2 c_2 - d_4 c_2 s_3 + d_6 \left(s_2 s_4 s_5 - c_2 s_3 c_5 - c_3 c_4 s_5\right) \\ a_2 s_2 - d_4 s_2 s_3 - d_6 \left(s_2 s_3 c_5 + c_2 s_4 s_5 + c_3 c_4 s_5\right) \\ q_1 - d_4 c_3 - d_6 \left(c_3 c_5 + s_3 c_4 s_5\right) \end{pmatrix}. \tag{7}
$$

Finally, using (4) and keeping only the first three components of the result yields

$$
{}^{w}\boldsymbol{p}_W = \begin{pmatrix} 1.5 + \dfrac{1}{\sqrt{2}}\Big(a_1 + (s_2 + c_2)(a_2 - d_4 s_3) + d_6 \left((s_2 - c_2) s_4 s_5 - (s_2 + c_2) s_3 c_5 - 2 c_3 c_4 s_5\right)\Big) \\ -4.5 + \dfrac{1}{\sqrt{2}}\Big(a_1 + (c_2 - s_2)(a_2 - d_4 s_3) + d_6 \left((s_2 + c_2) s_4 s_5 + (s_2 - c_2) s_3 c_5\right)\Big) \\ 0.3 + q_1 - d_4 c_3 - d_6 \left(c_3 c_5 + s_3 c_4 s_5\right) \end{pmatrix}. \tag{8}
$$

Exercise 3

This problem is solved by the algebraic transformation method used in inverse kinematics. Expand the cosine function in (1) to get

$$
\sin\theta_1 + 2\left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2\right) = 2,
$$

which is of the form

$$
a \sin\theta_1 + b \cos\theta_1 = c, \tag{9}
$$

with

$$
a = 1 - 2\sin\theta_2, \qquad b = 2\cos\theta_2, \qquad c = 2.
$$

The transcendental eq. (9) has already been studied in the lecture slides (InverseKinematics.pdf, slide #13). From there, we know that this equation has (one or two) real solutions if and only if

$$
a^2 + b^2 \geq c^2 \qquad \Rightarrow \qquad (1 - 2\sin\theta_2)^2 + 4\cos^2\theta_2 \geq 4,
$$

or

$$
\sin\theta_2 \leq 0.25 \qquad \Rightarrow \qquad \theta_2 \in (-\pi, 0.2526] \cup [\pi - 0.2526, \pi] \;\; \text{[rad]}.
$$

The range of admissible solutions for θ_2, i.e., those providing a real solution θ_1 to (1), is shown in Fig. 3.

For instance, when $\theta_2 = 0$, eq. (1) becomes

$$
\sin\theta_1 + 2\cos\theta_1 = 2,
$$

which has the two real solutions

$$
\begin{aligned}
\theta_1^{+} &= 2\arctan\left(\frac{a + \sqrt{a^2 + b^2 - c^2}}{b + c}\right) = 2\arctan\left(\frac{1+1}{4}\right) = 2\arctan 0.5 = 0.9273 \text{ [rad]}, \\
\theta_1^{-} &= 2\arctan\left(\frac{a - \sqrt{a^2 + b^2 - c^2}}{b + c}\right) = 2\arctan\left(\frac{1-1}{4}\right) = 2\arctan 0 = 0.
\end{aligned}
$$

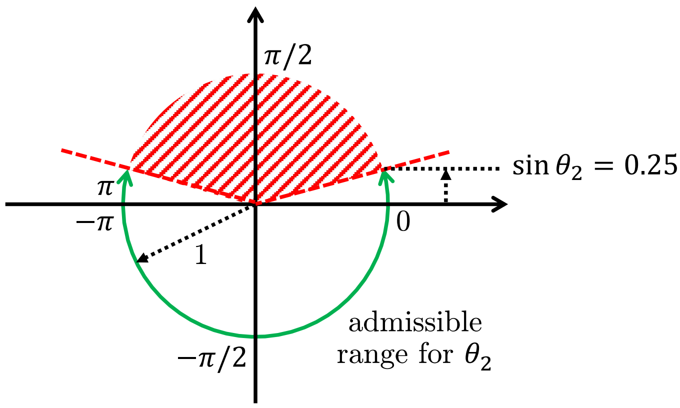


Figure 3: Admissible range for θ_2.

On the other hand, eq. (1) has a single solution when $\sin\theta_2 = 0.25$. In particular, for $\theta_2 = 0.2526$ the equation becomes

$$0.5\sin\theta_1 + 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = 0.2499 \text{ [rad]}.$$

Similarly, for $\theta_2 = \pi - 0.2526$ the equation becomes

$$0.5\sin\theta_1 - 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = \pi - 0.2499 = 2.8917 \text{ [rad]}.$$

Exercise 4

The robot is a planar RPRP arm. The task vector $\boldsymbol{r}$ contains the (x, y) position of the end-effector and the angle ϕ of the last link w.r.t. the $\boldsymbol{x}$-axis. The analytic 3×4 Jacobian associated to the task function in (2) is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 s_1 - q_4 s_{13} & c_1 & -q_4 s_{13} & c_{13} \\ q_2 c_1 + q_4 c_{13} & s_1 & q_4 c_{13} & s_{13} \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

Its singular configurations (corresponding to a loss of rank) are determined by computing

$$\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right) = 2q_2^2 + 2s_3^2 - q_2^2 s_3^2. \tag{11}$$

This determinant is zero if and only if $q_2 = 0$ **and** $s_3 = 0$ ($q_3 = 0$ or π) simultaneously, because cancelation between addends in (11) is ruled out. In fact, let $a = s_3^2 \in (0, 1]$; setting the determinant to zero would correspond to a value $q_2^2 = -2a/(2-a) < 0$, which is impossible.

An alternative method for finding the singularities of the Jacobian would be to check the four minors obtained by deleting one of its columns. Let $\boldsymbol{J}_{-i}(\boldsymbol{q})$ be the 3×3 matrix obtained by deleting column i from $\boldsymbol{J}(\boldsymbol{q})$, for $i = 1, 2, 3, 4$. Then

$$\det \boldsymbol{J}_{-1}(\boldsymbol{q}) = -s_3, \qquad \det \boldsymbol{J}_{-2}(\boldsymbol{q}) = q_2 c_3, \qquad \det \boldsymbol{J}_{-3}(\boldsymbol{q}) = s_3, \qquad \det \boldsymbol{J}_{-4}(\boldsymbol{q}) = -q_2.$$

All minors should vanish at the same time, and this happens again if and only if $q_2 = 0$ and $s_3 = 0$.

In a generic regular configuration, the null space of $\boldsymbol{J}(\boldsymbol{q})$ is one-dimensional, i.e., it is generated by a single vector $\dot{\boldsymbol{q}}_0$ (scaled with any factor $\alpha \in \mathbb{R}$):

$$\dot{\boldsymbol{q}}_0 \in \mathcal{N}(\boldsymbol{J}(\boldsymbol{q})) = \operatorname{span}\left\{\begin{pmatrix} -s_3 \\ -q_2 c_3 \\ s_3 \\ q_2 \end{pmatrix}\right\} \qquad \Rightarrow \qquad \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_0 = \mathbf{0}.$$

In a singular configuration $\boldsymbol{q}_s$, the Jacobian becomes[2]

$$\boldsymbol{J}(\boldsymbol{q}_s) = \left.\boldsymbol{J}(\boldsymbol{q})\right|_{q_2=q_3=0} = \begin{pmatrix} -q_4 s_1 & c_1 & -q_4 s_1 & c_1 \\ q_4 c_1 & s_1 & q_4 c_1 & s_1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

which has rank equal to 2: the first two columns are independent for all $\boldsymbol{q}$, the other two columns are simply duplications. Thus, the null space of $\boldsymbol{J}(\boldsymbol{q}_s)$ is two-dimensional and a basis for all $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}} = \mathbf{0}$ is

$$\mathcal{N}(\boldsymbol{J}(\boldsymbol{q}_s)) = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Looking at the structure of the singular matrix in (12), the complementary space to the range space $\mathcal{R}(\boldsymbol{J}(\boldsymbol{q}_s))$ along which no task velocity can be realized at $\boldsymbol{q}_s$ is given by the single direction

$$\dot{\boldsymbol{r}}^{\perp} = \begin{pmatrix} s_1 \\ -c_1 \\ q_4 \end{pmatrix} \in \mathcal{R}^{\perp}(\boldsymbol{J}(\boldsymbol{q}_s)) = \mathcal{N}\Big(\boldsymbol{J}^T(\boldsymbol{q}_s)\Big),$$

where the latter equality between subspaces follows from the decomposition of the three-dimensional task space. In fact, all generalized task forces that can be statically balanced at $\boldsymbol{q}_s$ by a zero joint torque have the form

$$\boldsymbol{f} = \alpha \begin{pmatrix} s_1 \\ -c_1 \\ q_4 \end{pmatrix}, \quad \forall \alpha \in \mathbb{R} \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}_s)\boldsymbol{f} = \mathbf{0}.$$

Note that the unit of measure for the scalar α is in this case [N]. On the other hand, when the Jacobian has full rank, i.e., has the form (10), the subspace $\mathcal{N}(\boldsymbol{J}^T(\boldsymbol{q}))$ contains only the null vector; so, there is no $\boldsymbol{f}_0 \neq \mathbf{0}$ in the null space of $\boldsymbol{J}^T(\boldsymbol{q})$ in the regular case.

[2] A similar analysis holds for the singularity $q_2 = 0$, $q_3 = \pi$.

Exercise 5

A parametrized description of the assigned helical path is given by [3]

$$\boldsymbol{p} = \boldsymbol{p}(s) = C + \begin{pmatrix} r \sin s \\ h\, s \\ r \cos s \end{pmatrix} = \begin{pmatrix} r \sin s \\ h\, s \\ r\,(1 + \cos s) \end{pmatrix}, \qquad s \in [0, L], \tag{13}$$

so that $\boldsymbol{p}(0) = \boldsymbol{p}_0 = (0, 0, r)$. Each of the two full turns is obtained by a variation of 2π for s. Thus, the upper limit of the interval for s is $L = 4\pi$. Given this path, we need to specify a rest-to-rest timing law $s = s(t)$, with $t \in [0, T]$, that will trace it in minimum time T under the bounds (3).

From (13), by time differentiation (and use of the chain rule) we get

$$\dot{\boldsymbol{p}} = \boldsymbol{p}'\, \dot{s} = \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix} \dot{s}, \qquad \|\boldsymbol{p}'\| = \sqrt{r^2 + h^2}, \tag{14}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{p}'\, \ddot{s} + \boldsymbol{p}''\, \dot{s}^2 = \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix} \ddot{s} - \begin{pmatrix} r \sin s \\ 0 \\ r \cos s \end{pmatrix} \dot{s}^2, \tag{15}$$

where a prime ($'$) denotes differentiation with respect to the parameter s. From (14), we obtain the tangent axis $\boldsymbol{t}$ of the Frenet frame associated to the path,

$$\boldsymbol{t} = \frac{\boldsymbol{p}'}{\|\boldsymbol{p}'\|} = \frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix}. \tag{16}$$

Differentiating $\boldsymbol{t}$ with respect to the parameter s gives

$$\boldsymbol{t}' = -\frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r \sin s \\ 0 \\ r \cos s \end{pmatrix}, \qquad \|\boldsymbol{t}'\| = \frac{r}{\sqrt{r^2 + h^2}}, \tag{17}$$

so that the normal axis $\boldsymbol{n}$ of the Frenet frame associated to the path is

$$\boldsymbol{n} = \frac{\boldsymbol{t}'}{\|\boldsymbol{t}'\|} = -\begin{pmatrix} \sin s \\ 0 \\ \cos s \end{pmatrix}. \tag{18}$$

For checking the bounds on the components of the acceleration (15), we need to evaluate

$$\ddot{\boldsymbol{p}}^T \boldsymbol{t} = \left(\boldsymbol{p}'^T \boldsymbol{t}\right) \ddot{s} + \left(\boldsymbol{p}''^T \boldsymbol{t}\right) \dot{s}^2 \qquad \text{and} \qquad \ddot{\boldsymbol{p}}^T \boldsymbol{n} = \left(\boldsymbol{p}'^T \boldsymbol{n}\right) \ddot{s} + \left(\boldsymbol{p}''^T \boldsymbol{n}\right) \dot{s}^2. \tag{19}$$

[3] Equation (13) is not the only possible parametrization of this helix. For instance, one can also define

$$\boldsymbol{p}(s) = \begin{pmatrix} r \sin 2\pi s \\ 2\pi h\, s \\ r\,(1 + \cos 2\pi s) \end{pmatrix}, \qquad s \in [0, 2],$$

with the parameter s being scaled down by 2π. The new final value $s = 2$ of the interval of definition corresponds again to two full turns along the helix. Indeed, although the expressions are slightly different, all the following results on trajectory planning remain the same.

Being

$$\boldsymbol{p}'^T\boldsymbol{t} = \sqrt{r^2+h^2}, \qquad \boldsymbol{p}''^T\boldsymbol{t} = 0, \qquad \boldsymbol{p}'^T\boldsymbol{n} = 0, \qquad \boldsymbol{p}''^T\boldsymbol{n} = r,$$

from (14) and (19), we have the bounds

$$\|\dot{\boldsymbol{p}}\| = \sqrt{r^2+h^2}\,|\dot{s}| \le V, \qquad \left|\ddot{\boldsymbol{p}}^T\boldsymbol{t}\right| = \sqrt{r^2+h^2}\,|\ddot{s}| \le A, \qquad \left|\ddot{\boldsymbol{p}}^T\boldsymbol{n}\right| = r\,\dot{s}^2 \le A,$$

or

$$|\dot{s}| \le v_{max} = \min\left\{\frac{V}{\sqrt{r^2+h^2}}, \sqrt{\frac{A}{r}}\right\}, \qquad |\ddot{s}| \le a_{max} = \frac{A}{\sqrt{r^2+h^2}}. \tag{20}$$

Using the numerical data, it is

$$v_{max} = \min\left\{\frac{2}{\sqrt{0.25}}, \sqrt{\frac{4.5}{0.4}}\right\} = \sqrt{\frac{4.5}{0.4}} = 3.3541 \text{ m/s}, \qquad a_{max} = 9 \text{ m/s}^2.$$

The minimum-time rest-to-rest motion for moving the scalar path parameter s between $s=0$ and $s = L = 4\pi$ under the bounds $v_{max} > 0$ for the speed and $a_{max} > 0$ for the acceleration is a bang-coast-bang (or bang-bang) acceleration profile. In the present case, there will be a rather long coast phase since, by replacing the problem data,

$$12.5664 \approx 4\pi = L > \frac{v_{max}^2}{a_{max}} = \frac{11.25}{9} = 1.25.$$

Therefore, the minimum time is

$$T^* = \frac{L\,a_{max} + v_{max}^2}{a_{max}v_{max}} = 4.1192 \text{ s},$$

with acceleration/deceleration phases lasting $T_s = v_{max}/a_{max} = 0.3727$ s. The time-optimal profile of the parameter s and of its first two derivatives is shown in Fig. 4.

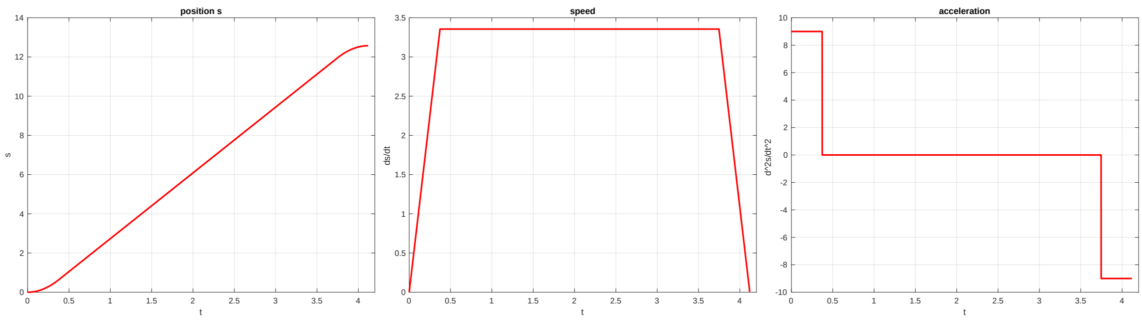


Figure 4: The optimal timing law $s(t)$, with speed and acceleration profiles.

As for the suitable positioning of the base of the spatial 3R manipulator, consider the bounding box (a parallelepiped) containing the complete helical path shown in Fig. 5. The box size is $(\Delta x, \Delta y, \Delta z) = (2r \times 4\pi h \times 2r) = (0.8 \times 3.77 \times 0.8)$ [m], with one of the four large faces lying on the plane $z = 0$ and one of the two (vertical) bases lying on the plane $y = 0$. The shoulder of the robot is at the same level of the top face of the box.

The robot base will be conveniently placed at the midpoint P_b along one of the long sides of the box, e.g., with $(x_b, y_b) = (r, 2\pi h) = (0.4, 1.885)$ [m] (and $z_b = 0$). In this way, the outreach of the second and third robot links ($L_t = L_2 + L_3 = 3$ m) starting from the shoulder point $P_s = (r, 2\pi h, L_1) = (0.4, 1.885, 0.8)$ [m] will cover the entire bounding box. In fact, even the farthest vertex $P_v = (-r, 4\pi h, 0) = (-0.4, 3.77, 0)$ [m] of the box will have a distance from P_s that is reachable, being $\|P_v - P_s\| = 2.198 < 3 = L_t$. The robot will certainly not encounter any kinematic singularity during its motion: the forearm is never outstretched or folded and the base joint axis is out of the box. Moreover, the path will never interfere with the base link of the robot, which is outside the box.

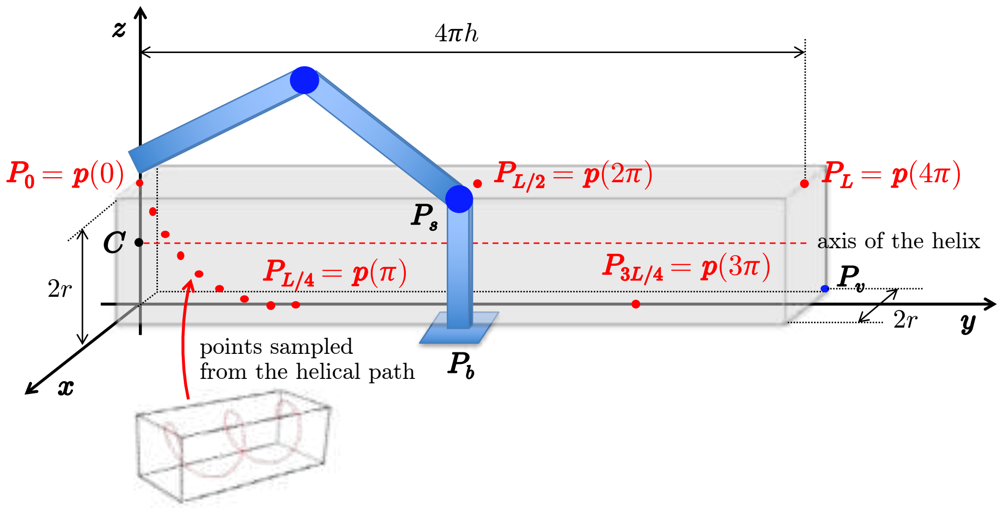


Figure 5: A box containing the helical path and a good placement of the 3R manipulator.

* * * * *

Robotics 1

February 13, 2023

Exercise 1

Consider the planar RPPR robot in Fig. 1.

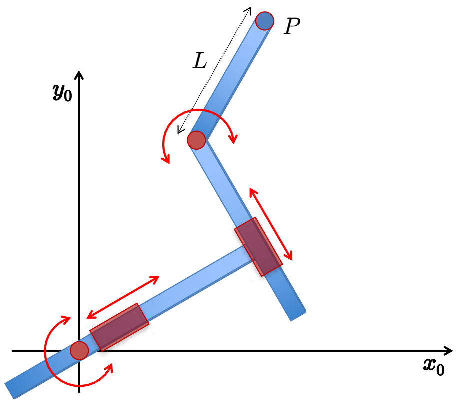


Figure 1: A planar RPPR robot

- Assign the frames according to the standard Denavit-Hartenberg (DH) convention and provide the corresponding table of parameters.
- Suppose that the two prismatic joints have a limited range: $|q_i| < D$, $i = 2, 3$. Determine the maximum possible distance Δ of the end-effector point P from the origin of the base frame and the robot configuration(s) $\boldsymbol{q}$ at which this value is attained.

Exercise 2

Given two different rotation matrices ${}^0\boldsymbol{R}_c$ and ${}^0\boldsymbol{R}_d$, suppose that a minimal representation with a set of ZYZ Euler angles $\boldsymbol{\alpha} \in I\!R^3$ has been extracted from each matrix, i.e., $\boldsymbol{\alpha}_c$ and $\boldsymbol{\alpha}_d$. Then, the relative error between the two orientations can be defined as $\boldsymbol{e}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_d - \boldsymbol{\alpha}_c$, i.e., the difference between the values of these two sets of Euler angles. As an alternative, one can define the relative rotation matrix ${}^c\boldsymbol{R}_d$ and extract from this matrix the same set of ZYZ Euler angles $\boldsymbol{\alpha}_{cd} \in I\!R^3$.

Is it true that $\boldsymbol{e}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_{cd}$ holds? If you believe so, provide a simple proof of this result. If you don't, provide then a numerical counterexample (without any representation singularity).

Exercise 3

A planar 2R robot has its direct kinematics defined as

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}$$

with link lengths $l_1 = 0.5$, $l_2 = 0.4$ [m]. Write a code that solves numerically the inverse kinematics problem for this robot using Newton iterative method. For a desired position $\boldsymbol{p}_d = (0.4, -0.3)$, determine two different initial configurations $\boldsymbol{q}^{[0]}$ so that the method converges in no more than $k_{\max} = 3$ iterations to the two inverse kinematics solutions, respectively $\boldsymbol{q}^a$ and, $\boldsymbol{q}^b$, with a final accuracy of at least $\varepsilon = 10^{-4}$ on the norm of the Cartesian error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$. Provide the values of $\boldsymbol{q}^{[k]}$ for $k = 0, 1, 2, 3$ in the two situations, as well as the final values of the error norm $\|\boldsymbol{e}\|$.

Exercise 4

The kinematics of a 4-dof robot manipulator is characterized by the DH parameters in Tab. 1. Build the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that relates the joint velocities $\dot{\boldsymbol{q}} \in I\!R^4$ to the six-dimensional *twist* vector composed by a velocity $\boldsymbol{v} = \boldsymbol{v}_4 \in I\!R^3$ of the origin of the last (end-effector) DH frame and by an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_4 \in I\!R^3$ of the same frame:

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Correspondingly, the transpose of this matrix relates the six-dimensional end-effector *wrench* vector composed by a force $\boldsymbol{f} = \boldsymbol{f}_4 \in I\!R^3$ applied at the origin of the last (end-effector) DH frame and by a moment $\boldsymbol{\mu} = \boldsymbol{\mu}_4 \in I\!R^3$ applied on the same frame to the joint forces/torques $\boldsymbol{\tau} \in I\!R^4$:

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{\mu} \end{pmatrix}.$$

Find all the singular configurations of this Jacobian, i.e., all $\boldsymbol{q}_s$ such that rank $\boldsymbol{J}(\boldsymbol{q}_s) < 4$. At a singular configuration $\boldsymbol{q}_s$, determine:

i) a basis for the joint velocities $\dot{\boldsymbol{q}} \in I\!R^4$ that produce no end-effector twists;

ii) a basis for the end-effector twists $\boldsymbol{t} \in I\!R^6$ that are not realizable;

iii) all non-zero end-effector wrenches $\boldsymbol{w} \in I\!R^6$ that are statically balanced by $\boldsymbol{\tau} = \boldsymbol{0} \in I\!R^4$.

Hint: It is convenient to work by expressing the geometric Jacobian in the DH frame RF_1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	q_3	0
4	0	a_4	0	q_4

Table 1: Table of DH parameters of a 4-dof robot.

[240 minutes, open books]

Solution

February 13, 2023

Exercise 1

A possible assignment of standard DH frames for the considered RPPR robot arm is shown in Fig. 2. The corresponding DH parameters are reported in Tab. 2.

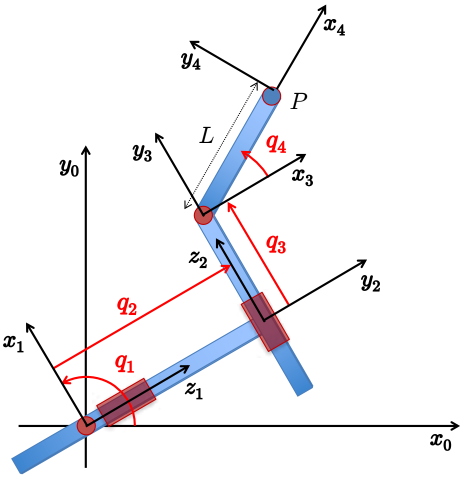


Figure 2: Assignment of DH frames for the RPPR robot in Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	$\pi/2$	0	q_3	$\pi/2$
4	0	L	0	q_4

Table 2: Table of DH parameters for the RPPR robot with frames assigned as in Fig. 2.

The position of point P in the plane (x_0, y_0) is

$$\boldsymbol{p} = \begin{pmatrix} q_2 s_1 + q_3 c_1 + L s_{14} \\ -q_2 c_1 + q_3 s_1 - L c_{14} \end{pmatrix} = \begin{pmatrix} s_1 & c_1 \\ -c_1 & s_1 \end{pmatrix} \begin{pmatrix} q_2 + L c_4 \\ q_3 + L s_4 \end{pmatrix}.$$

Thus, its distance from the origin of the base frame is

$$\|\boldsymbol{p}\| = \sqrt{q_2^2 + q_3^2 + L^2 + 2L\,(q_2 c_4 + q_3 s_4)}.$$

For $|q_2| \leq D$ and $|q_3| \leq D$, the maximum distance is then easily evaluated as

$$\Delta = \max_{\boldsymbol{q}\in I\!R^4:\, |q_i|\leq D,\, i=1,2} \|\boldsymbol{p}\| = \sqrt{D^2 + D^2 + L^2 + 2L\left(D\,\frac{\sqrt{2}}{2} + D\,\frac{\sqrt{2}}{2}\right)} = \sqrt{2}D + L,$$

which is attained for

$$q_2 = \pm D, \qquad q_3 = \pm D, \qquad q_4 = \operatorname{atan2}\{q_3, q_2\} \quad \left(= \left\{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\right\}\right),$$

with an arbitrary value of q_1. Four possible classes of solutions are obtained depending on the combination of signs: $\boldsymbol{q} = (q_1, D, D, \pi/4)$, $\boldsymbol{q} = (q_1, D, -D, -\pi/4)$, $\boldsymbol{q} = (q_1, -D, D, 3\pi/4)$, and $\boldsymbol{q} = (q_1, -D, -D, -3\pi/4)$.

Exercise 2

In general, the difference between the set of angles $\boldsymbol{\alpha}_c$ and $\boldsymbol{\alpha}_d$ of any minimal representation that one can extract from two rotation matrices, respectively $\boldsymbol{R}_c$ and $\boldsymbol{R}_d$, is different from the set of angles $\boldsymbol{\alpha}_{cd}$ of the same minimal representation that are extracted from the relative rotation matrix ${}^c\boldsymbol{R}_d = \boldsymbol{R}_c^T \boldsymbol{R}_d$. This is indeed true for any choice of angles $\boldsymbol{\alpha} \in \mathbb{R}^3$ used for the minimal representation of orientation. This result is due to the fact that the extraction of a minimal representation from a rotation matrix is a nonlinear operation.

The choice of a counterexample in which $\boldsymbol{e}_{\boldsymbol{\alpha}} \neq \boldsymbol{\alpha}_{cd}$ with the ZYZ Euler angles is arbitrary, but should keep in mind that the representation must not run into a singularity for any of the involved rotation matrices. This means that the two elements (1,3) and (2,3) in last column of the matrices $\boldsymbol{R}_c$, $\boldsymbol{R}_d$ and ${}^c\boldsymbol{R}_d$ should not be simultaneously zero.

Consider for example the two elementary rotation matrices by $\pi/4$ around the x and z axes,

$$\boldsymbol{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \boldsymbol{R}_z = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let

$$\boldsymbol{R}_c = \boldsymbol{R}_x \boldsymbol{R}_z = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -0.5 & 0.5 & \frac{\sqrt{2}}{2} \\ 0.5 & -0.5 & \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \boldsymbol{R}_d = \boldsymbol{R}_z \boldsymbol{R}_x = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0.5 & 0.5 \\ -\frac{\sqrt{2}}{2} & 0.5 & 0.5 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

From these, we obtain the relative orientation

$${}^c\boldsymbol{R}_d = \boldsymbol{R}_c^T \boldsymbol{R}_d = \begin{pmatrix} 0.8536 & -0.25 & 0.4571 \\ 0.1464 & 0.9571 & 0.25 \\ -0.5 & -0.1464 & 0.8536 \end{pmatrix}.$$

All three matrices $\boldsymbol{R}_c$, $\boldsymbol{R}_d$ and ${}^c\boldsymbol{R}_d$ satisfy the condition for not having a singularity in their ZYZ Euler representation, i.e.,

$$\sin\theta = \pm\sqrt{R_{13}^2 + R_{23}^2} \neq 0,$$

where R_{ij} denotes an element of the various rotation matrices. Thus, we can extract the set of angles $\boldsymbol{\alpha} = (\phi, \theta, \psi)$ using the inverse relationships in the regular case:

$$\theta = \operatorname{atan2}\{\sin\theta, R_{33}\}, \qquad \phi = \operatorname{atan2}\left\{\frac{R_{31}}{\sin\theta}, \frac{R_{32}}{\sin\theta}\right\}, \qquad \psi = \operatorname{atan2}\left\{\frac{R_{13}}{\sin\theta}, \frac{-R_{23}}{\sin\theta}\right\}.$$

As a result, for each rotation matrix we obtain two regular solutions, namely

$$\boldsymbol{\alpha}_c^{I} = \begin{pmatrix} 2.3562 \\ 0.7854 \\ 3.1416 \end{pmatrix}, \quad \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} -0.7854 \\ -0.7854 \\ 0 \end{pmatrix}; \qquad \boldsymbol{\alpha}_d^{I} = \begin{pmatrix} 3.1416 \\ 0.7137 \\ 2.3562 \end{pmatrix}, \quad \boldsymbol{\alpha}_d^{II} = \begin{pmatrix} 0 \\ -0.7137 \\ -0.7854 \end{pmatrix};$$

and

$$\boldsymbol{\alpha}_{cd}^{I} = \begin{pmatrix} 1.8557 \\ 0.5121 \\ 3.1416 \end{pmatrix}, \qquad \boldsymbol{\alpha}_{cd}^{II} = \begin{pmatrix} -1.2859 \\ -0.5121 \\ 0 \end{pmatrix}.$$

The four possible errors between the Euler angles are[1]

$$\boldsymbol{e}_{\boldsymbol{\alpha}}^{I,I} = \boldsymbol{\alpha}_d^{I} - \boldsymbol{\alpha}_c^{I} = \begin{pmatrix} 0.7854 \\ -0.0717 \\ -0.7854 \end{pmatrix}, \qquad \boldsymbol{e}_{\boldsymbol{\alpha}}^{I,II} = \boldsymbol{\alpha}_d^{II} - \boldsymbol{\alpha}_c^{I} = \begin{pmatrix} -2.3562 \\ -1.4991 \\ -3.9270 \end{pmatrix},$$

$$\boldsymbol{e}_{\boldsymbol{\alpha}}^{II,I} = \boldsymbol{\alpha}_d^{I} - \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} 3.9270 \\ 1.4991 \\ 2.3562 \end{pmatrix}, \qquad \boldsymbol{e}_{\boldsymbol{\alpha}}^{II,II} = \boldsymbol{\alpha}_d^{II} - \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} 0.7854 \\ 0.0717 \\ -0.7854 \end{pmatrix}.$$

As anticipated, none of these angular errors coincide with the two possible values of ZYZ Euler angles $\boldsymbol{\alpha}_{cd}^{I}$ and $\boldsymbol{\alpha}_{cd}^{II}$ extracted from the relative rotation matrix ${}^{c}\boldsymbol{R}_d$.

Exercise 3

From (1), the analytic Jacobian of the planar 2R robot is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix}. \tag{2}$$

The basic step of Newton method at the k-th iteration is

$$\boldsymbol{q}^{[k+1]} = \boldsymbol{q}^{[k]} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{[k]}) \left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[k]})\right), \tag{3}$$

with inversion of the Jacobian and multiplication by the current position error $\boldsymbol{e}^{[k]} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[k]})$. In order to guarantee convergence, the method needs to be initialized with a configuration $\boldsymbol{q}^{[0]}$ that is close enough to a solution. A MATLAB code for the solution of the given inverse kinematics (IK) problem using Newton method is reported further below (without output instructions).

For the initialization, based on the desired $\boldsymbol{p}_d$ and on the link lengths of this robot, one can use intuition to guess a configuration that is close enough to the 'elbow up' IK solution. For instance, with the initial guess

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} 40^\circ \\ -90^\circ \end{pmatrix} = \begin{pmatrix} 0.6981 \\ -1.5708 \end{pmatrix} \text{ [rad]},$$

the method fails to converge with the desired error accuracy $\varepsilon = 10^{-4}$ within the requested $k_{\max} = 3$ iterations (i.e., after three evaluations of the basic step (3)). The final configuration at $k = 3$ is

$$\boldsymbol{q}^{[3]} = \begin{pmatrix} 0.1837 \\ -1.9858 \end{pmatrix} \text{ [rad]} \quad \Rightarrow \quad \boldsymbol{f}(\boldsymbol{q}^{[3]}) = \begin{pmatrix} 0.3999 \\ -0.2980 \end{pmatrix} \neq \boldsymbol{p}_d \quad \Rightarrow \quad \|\boldsymbol{e}^{[3]}\| = 2 \cdot 10^{-3} \text{ [m]}.$$

However, the final configuration that was reached gives a clue for a good new guess. With

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} 20^\circ \\ -120^\circ \end{pmatrix} = \begin{pmatrix} 0.3491 \\ -2.0944 \end{pmatrix} \text{ [rad]},$$

[1] While most of these angles take values in $(-\pi, \pi]$, there are two angular errors exceeding this range, namely $3.9270 = \pi + 0.7854$ and $-3.9270 = -\pi - 0.7854$. Indeed, when defining an angular quantities over a 2π range, one should organize error computations so as to lead to the smallest difference (in this case, ± 0.7854). Even in this way, the results for $\boldsymbol{e}_{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}_{cd}$ are different.

the method converges in fact in $k = 2$ iterations, generating the solution

$$\Rightarrow \quad \boldsymbol{q}^{[1]} = \begin{pmatrix} 0.1736 \\ -1.9961 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[a]} = \boldsymbol{q}^{[2]} = \begin{pmatrix} 0.1797 \\ -1.9824 \end{pmatrix} \text{ [rad]},$$

with a final norm of the Cartesian error $\|\boldsymbol{e}\| = \left\|\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[a]})\right\| = 7 \cdot 10^{-5}$ m.

As for the 'elbow down' IK solution, the initial guess

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} -70° \\ 100° \end{pmatrix} = \begin{pmatrix} -1.2217 \\ 1.7453 \end{pmatrix} \text{ [rad]},$$

leads to convergence in exactly $k = k_{\max} = 3$ iterations, generating the solution

$$\Rightarrow \quad \boldsymbol{q}^{[1]} = \begin{pmatrix} -1.4589 \\ 2.0125 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[2]} = \begin{pmatrix} -1.4672 \\ 1.9826 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[b]} = \boldsymbol{q}^{[3]} = \begin{pmatrix} -1.4665 \\ 1.9823 \end{pmatrix} \text{ [rad]},$$

with a final norm of the Cartesian error $\|\boldsymbol{e}\| = \left\|\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[b]})\right\| = 9 \cdot 10^{-8}$ m. Fig. 3 illustrates the fast convergence rate (in fact, quadratic) of the method for the case of the 'elbow down' IK solution: the norm of the error is plotted in logarithmic scale over iterations.

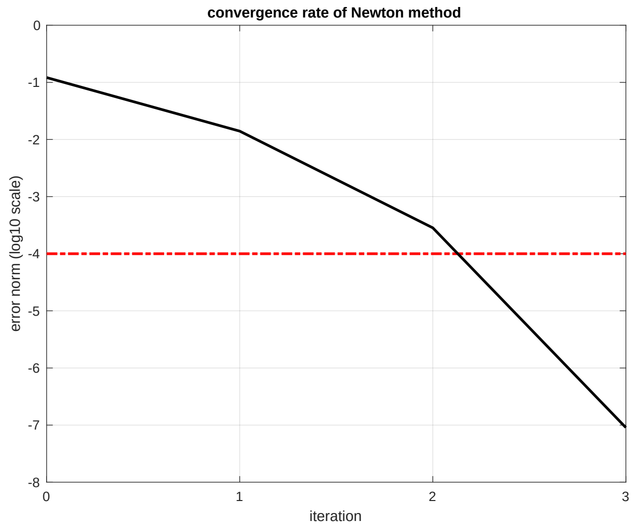


Figure 3: Error convergence of Newton method for the 'elbow down' IK solution.

MATLAB code for Newton method

```
% robot data
l1=0.5;l2=0.4;   %[m]
% desired task (end-effector position)
pd=[0.4;-0.3];   %[m]
% parameters in Newton method
% (final error tolerance and max number of iterations)
eps=0.0001; k_max=3;
% two (alternative) initial guesses
q0=[20;-120]*pi/180;    % for elbow up IK solution
%q0=[-70;100]*pi/180;   % for elbow down IK solution
```

```
% initialization
flag_sol=0;
k=1;
q(:,k)=q0;
% main loop
while k<=k_max,
  f=[l1*cos(q(1,k))+l2*cos(q(1,k)+q(2,k));
     l1*sin(q(1,k))+l2*sin(q(1,k)+q(2,k))];
  e(:,k)=p_d-f;
  norm_e(k)=norm(e(:,k));
  if norm_e(k)<=eps,
    k_sol=k;
    q_sol=q(:,k);
    e_sol=e(:,k);
    norm_e_sol=norm(e(:,k));
    flag_sol=1;
    break
  else
    J=[-(l1*sin(q(1,k))+l2*sin(q(1,k)+q(2,k)))  -l2*sin(q(1,k)+q(2,k));
        l1*cos(q(1,k))+l2*cos(q(1,k)+q(2,k))    l2*cos(q(1,k)+q(2,k))];
    % core Newton step
    q(:,k+1)=q(:,k)+inv(J)*e(:,k);
  end
  k=k+1;
end
```

Exercise 4

From Tab. 1, we compute the DH homogeneous transformations of this RRPR robot:

$$ {}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

The position of the end-effector (the origin of frame 4) follows as

$$ \boldsymbol{p}_{4,hom} = \begin{pmatrix} \boldsymbol{p}_4 \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_2(q_2) \left({}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} a_4c_4 \\ a_4s_4 \\ 0 \\ 1 \end{pmatrix} \right) \right) \quad \Rightarrow \quad \boldsymbol{p}_4 = \begin{pmatrix} c_1\,(q_3s_2 + a_4c_{24}) \\ s_1\,(q_3s_2 + a_4c_{24}) \\ -q_3c_2 + a_4s_{24} \end{pmatrix}. \tag{4} $$

Since $\boldsymbol{v} = \boldsymbol{v}_4 = \dot{\boldsymbol{p}}_4$, the linear part of the geometric Jacobian can be obtained by differentiation as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_4}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1\left(q_3 s_2 + a_4 c_{24}\right) & c_1\left(q_3 c_2 - a_4 s_{24}\right) & c_1 s_2 & -c_1 a_4 s_{24} \\ c_1\left(q_3 s_2 + a_4 c_{24}\right) & s_1\left(q_3 c_2 - a_4 s_{24}\right) & s_1 s_2 & -s_1 a_4 s_{24} \\ 0 & q_3 s_2 + a_4 c_{24} & -c_2 & a_4 c_{24} \end{pmatrix}.$$

Setting $\boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, the angular part of the geometric Jacobian is computed as

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 & \boldsymbol{0} & {}^0\boldsymbol{R}_3(q_1,q_2,q_3)\boldsymbol{z}_0 \end{pmatrix} = \begin{pmatrix} 0 & s_1 & 0 & s_1 \\ 0 & -c_1 & 0 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the 6×4 geometric Jacobian in the base frame is

$${}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}.$$

To simplify the following analysis, it is convenient to express the Jacobian in the rotated frame RF_1. We have

$$\begin{aligned} {}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\bar{\boldsymbol{R}}_1^T(q_1)\,{}^0\boldsymbol{J}(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{R}_1^T(q_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(q_1) \end{pmatrix} {}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}_L(\boldsymbol{q}) \\ {}^0\boldsymbol{R}_1^v(q_1)\,\boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & q_3 c_2 - a_4 s_{24} & s_2 & -a_4 s_{24} \\ 0 & q_3 s_2 + a_4 c_{24} & -c_2 & a_4 c_{24} \\ -\left(q_3 s_2 + a_4 c_{24}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{5}$$

Even when the 6×4 geometric Jacobian is full (column) rank, there exist always directions along which no end-effector twists can be realized. When the range space of the Jacobian has dimension

$$\dim \mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) = \dim \mathcal{R}\left({}^0\boldsymbol{J}(\boldsymbol{q})\right) = 4,$$

being

$$\mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) \oplus \mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = I\!R^6,$$

it follows that the dimension of the complementary space is

$$\dim \mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = 6 - \dim \mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) = 2.$$

In particular, a basis for such unfeasible twists is given by

$$\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = \operatorname{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2\right\} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ q_3 s_2 + a_4 c_{24} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}. \tag{6}$$

In order to evaluate the singular configurations of the robot manipulator, we compute

$$\det\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\,{}^1\boldsymbol{J}(\boldsymbol{q})\right) = q_3^2\left((q_3 s_2 + a_4 c_{24})^2 + 1\right).$$

Therefore, the singularities of the geometric Jacobian occur only when $q_3 = 0$. Setting this value in the rotated Jacobian, one has

$${}^1\boldsymbol{J}(\boldsymbol{q}_s) = {}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=0} = \begin{pmatrix} 0 & -a_4 s_{24} & s_2 & -a_4 s_{24} \\ 0 & a_4 c_{24} & -c_2 & a_4 c_{24} \\ -a_4 c_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank}{}^1\boldsymbol{J}(\boldsymbol{q}_s) = 3$. Thus, the null space of ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$ is one-dimensional and coincides with the null space of ${}^0\boldsymbol{J}(\boldsymbol{q}_s)$ (because we are operating on the columns of the matrix, i.e., in the joint space, while products by $\boldsymbol{R}_1^T(q_1)$ affect the rows). This null space is spanned by

$$\mathcal{N}\left({}^0\boldsymbol{J}(\boldsymbol{q}_s)\right) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Indeed, for any joint velocity $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\left({}^0\boldsymbol{J}(\boldsymbol{q}_s)\right)$, we have

$${}^0\boldsymbol{J}(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_0 = {}^1\boldsymbol{J}(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_0 = {}^1\boldsymbol{J}(\boldsymbol{q}_s)\begin{pmatrix} 0 \\ -\alpha \\ 0 \\ \alpha \end{pmatrix} = \boldsymbol{0}, \qquad \forall \alpha \in I\!R.$$

In order to determine all end-effector twists $\boldsymbol{t} \in I\!R^6$ that are not realizable at a singular configuration $\boldsymbol{q}_s$, we should find $6 - \dim\mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q}_s)\right) = 3$ independent columns to be appended to the geometric Jacobian so that its rank will increase to its maximum possible value, namely 6. Working in the rotated frame, and following the same previous consideration about complementarity spaces, it is easy to see that we can use the two twist directions in (6) —which are never realizable, neither in a regular configuration nor in a singular configuration (where $q_3 = 0$) — and add a third independent column as follows

$$\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\right) = \operatorname{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2, {}^1\boldsymbol{t}_3\right\} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ a_4 c_{24} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ s_2 \\ 0 \\ 0 \\ 0 \\ a_4 s_4 \end{pmatrix}\right\}. \tag{7}$$

Being outside the range space of ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$, the three directions in (7) represent a basis for all generalized end-effector twists that are not realizable at $\boldsymbol{q}_s$. When expressed in the base frame, these become

$$
{}^0\boldsymbol{t}_1 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_1 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \\ 0 \\ 0 \\ a_4 c_{24} \end{pmatrix}, \quad {}^0\boldsymbol{t}_2 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_1 \\ s_1 \\ 0 \end{pmatrix}, \quad {}^0\boldsymbol{t}_3 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_3 = \begin{pmatrix} c_1 c_2 \\ s_1 c_2 \\ s_2 \\ a_4 s_1 s_4 \\ -a_4 c_1 s_4 \\ 0 \end{pmatrix}.
$$

To determine all end-effector wrenches $\boldsymbol{w} \in I\!R^6$ for which the manipulator is statically balanced at $\boldsymbol{q}_s$ without the need of forces/torques $\boldsymbol{\tau} \in I\!R^4$ at the joints, we need to determine a basis for the null space of the transpose of the geometric Jacobian. However, such a basis has already been computed. Therefore, when working in the rotated frame, we have

$$
\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\right) = \text{span}\left\{{}^1\boldsymbol{w}_1, {}^1\boldsymbol{w}_2, {}^1\boldsymbol{w}_3\right\} = \text{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2, {}^1\boldsymbol{t}_3\right\}.
$$

Similarly, when expressed in the base frame as ${}^0\boldsymbol{w}_i = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{w}_i$, for $i = 1, 2, 3$, these end-effector wrenches are

$$
{}^0\boldsymbol{w}_1 = {}^0\boldsymbol{t}_1, \qquad {}^0\boldsymbol{w}_2 = {}^0\boldsymbol{t}_2, \qquad {}^0\boldsymbol{w}_3 = {}^0\boldsymbol{t}_3.
$$

Indeed, for any end-effector wrench $\boldsymbol{w}_0 \in \mathcal{N}(\boldsymbol{J}^T(\boldsymbol{q}_s))$, the balancing forces/torques at the joints is

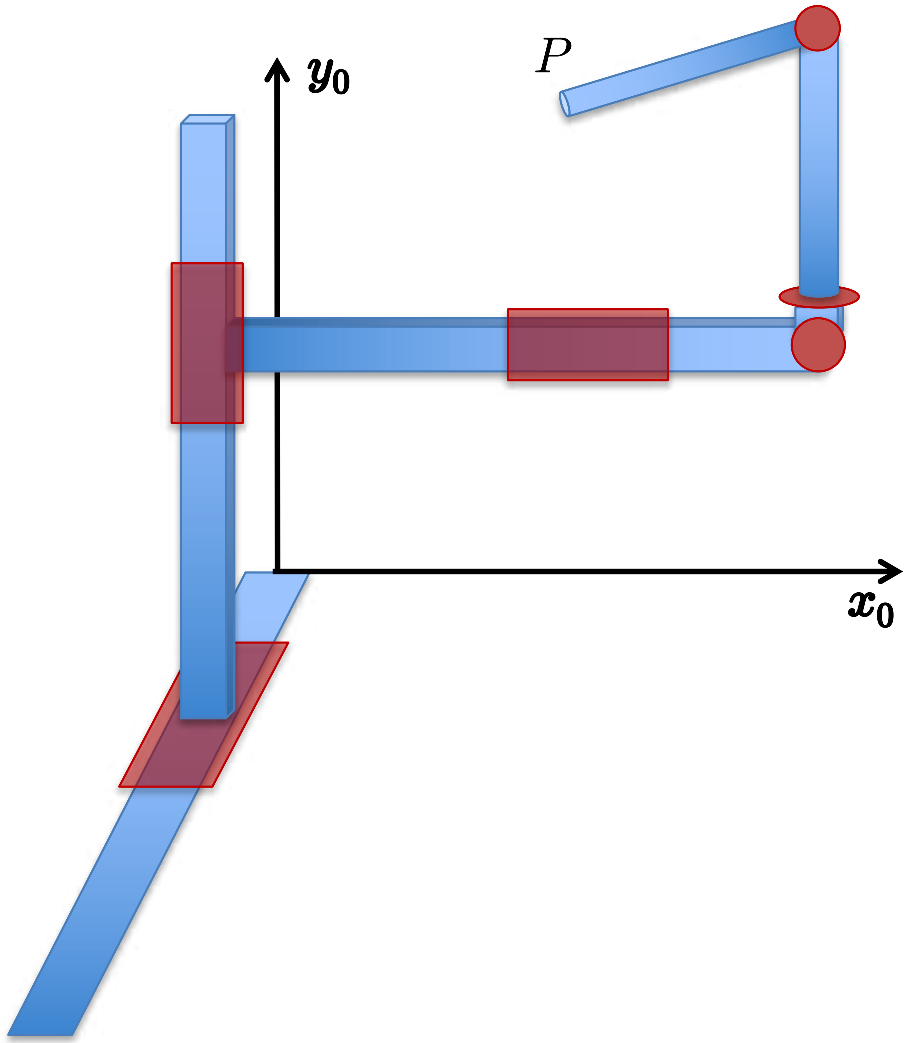
$$
\boldsymbol{\tau} = {}^0\boldsymbol{J}^T(\boldsymbol{q}_s)\,{}^0\boldsymbol{w}_0 = {}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\,{}^1\boldsymbol{w}_0 = \boldsymbol{0}.
$$

* * * * *

DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				



Robotics 1

March 24, 2023

Exercise 1

Consider the spatial 6-dof robot in Fig. 1.

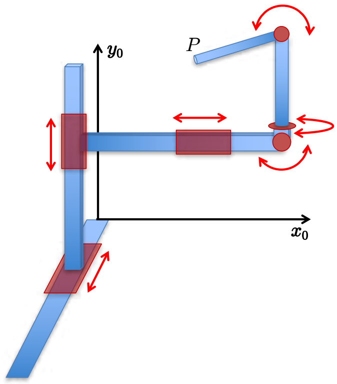


Figure 1: A spatial 6-dof robot, with three prismatic joints followed by three revolute joints.

- Assign the frames according to the standard Denavit-Hartenberg (DH) convention and provide the corresponding table of parameters. The origin of the last DH frame should coincide with point P. Specify the signs of the linear DH parameters that are constant and non-zero, as well as the signs of the joint variables q_i, $i = 1, \dots, 6$, in the shown configuration.
- Determine the symbolic expression of all elements in the 6×6 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot and check that $\boldsymbol{q}_0 = (1, 1, 1, -\pi/2, -\pi/2, -\pi/2)$ is a nonsingular configuration.
- At $\boldsymbol{q}_0$, find the position of point P. Moreover, compute a joint velocity $\dot{\boldsymbol{q}} \in I\!R^6$ that produces the velocity ${}^0\boldsymbol{v} = (0.5, 2, -2)$ [m/s] of P, while the end-effector has an angular velocity ${}^0\boldsymbol{\omega} = (0, 3, 0)$ [rad/s].

Exercise 2

Consider the planar 2P (Cartesian) robot in Fig. 2, where m_1 and m_2 are the masses of the two links in the serial chain. Each input force F_i is bounded in absolute value by $F_{i,max} > 0$, for $i = 1, 2$. Find the expression of the minimum feasible time T for a rest-to-rest robot motion from a start configuration $\boldsymbol{q}_s$ to a goal configuration $\boldsymbol{q}_g$. Compute the numerical value of T with the following data: $m_1 = 5$, $m_2 = 2$ [kg]; $F_{1,max} = 10$, $F_{2,max} = 5$ [N]; $\boldsymbol{q}_s = (0.3, -0.3)$, $\boldsymbol{q}_g = (-0.3, 0.3)$ [m]. Plot the evolutions of $F_i(t)$, $\dot{q}_i(t)$, and $q_i(t)$, for $i = 1, 2$. In your solution, does the mass m_2 trace a linear path during the time-optimal motion?

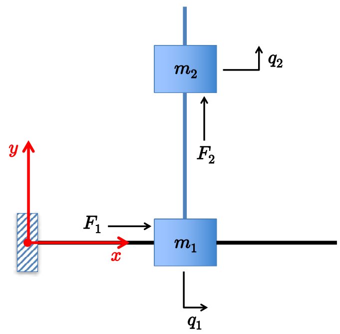


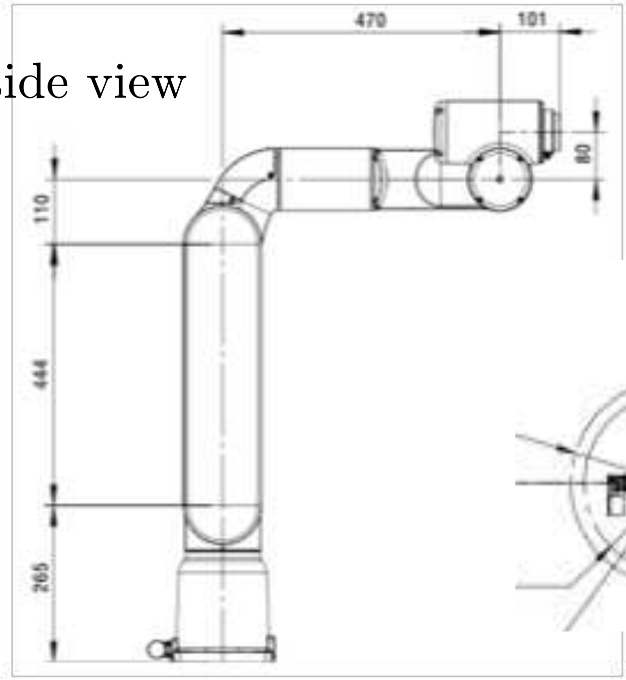
Figure 2: A planar Cartesian robot.

[180 minutes, open books]

DH frames assignment and table for the ABB CRB 15000 robot

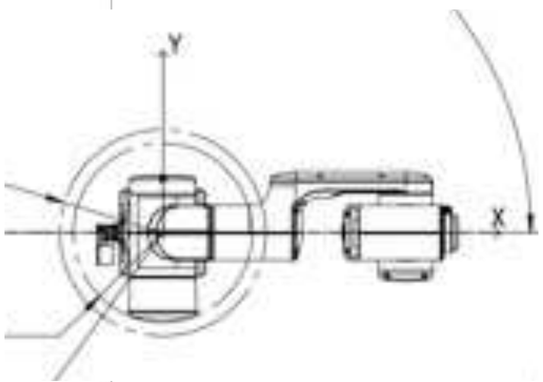
Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				



side view

top view



Robotics 1

June 12, 2023

Exercise 1

Consider the ABB CBR 15000 collaborative robot in Fig. 1, with six revolute joints. This robot has offsets at the elbow and at the wrist. More geometric information is available in the accompanying extra sheet.

Figure 1: The ABB CBR 15000 collaborative robot.

Assign the frames according to the standard Denavit-Hartenberg (DH) convention and fill in the corresponding table of parameters. The origin of the first DH frame is placed on the floor and that of the last frame should coincide with the center of the final flange. The assignment has to be consistent with the positive rotations of the joint variables, as specified by the manufacturer (see again the extra sheet). Moreover, none of the linear DH parameters should be negative (specify also their actual numerical value). Provide the values of the joint variables q_i, $i = 1, \dots, 6$, in the configuration shown in the extra sheet.

Exercise 2

A unitary mass moves along a circular path centered at the origin of the (x, y) plane and having radius $R > 0$. At the initial time $t = 0$, the mass is in $A = (R, 0)$ while at the final time $t = T$ it should be in $B = (-R, 0)$. The timing law is chosen as a cubic rest-to-rest profile. If the norm of the Cartesian acceleration $\|\ddot{\boldsymbol{p}}\|$ is bounded by $A > 0$, what is the minimum feasible time T to execute the desired trajectory? At which time instant(s) is the bound attained? Provide a closed-form solution to the problem in symbolic form, and then evaluate it with the data $R = 1.5$ [m], $A = 3$ [m/s^2]. Sketch the time profile of the norm $\|\ddot{\boldsymbol{p}}(t)\|$ and of the components $\ddot{p}_x(t)$ and $\ddot{p}_y(t)$ of the obtained Cartesian acceleration $\ddot{\boldsymbol{p}}(t)$.

Exercise 3

- For the 4R spatial robot in Fig. 2, compute the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and find all its singular configurations $\boldsymbol{q}_s$, i.e., where rank $\boldsymbol{J}(\boldsymbol{q}_s) < 4$.
- Verify that $\boldsymbol{q}_0 = \boldsymbol{0}$ is NOT a singular configuration. With the robot at $\boldsymbol{q}_0$, show that one of the two following six-dimensional end-effector velocities

$$\boldsymbol{V}_a = \begin{pmatrix} \boldsymbol{v}_a \\ \boldsymbol{\omega}_a \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{V}_b = \begin{pmatrix} \boldsymbol{v}_b \\ \boldsymbol{\omega}_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

  is admissible while the other is not, being $\boldsymbol{v} \in \mathbb{R}^3$ the velocity of point $P = O_4$ and $\boldsymbol{\omega} \in \mathbb{R}^3$ the angular velocity of the DH reference frame RF_4.
- For the admissible end-effector velocity, determine a joint velocity $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^4$ that realizes it, i.e., such that $\boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \boldsymbol{V}_i$, for either $i = a$ or $i = b$.

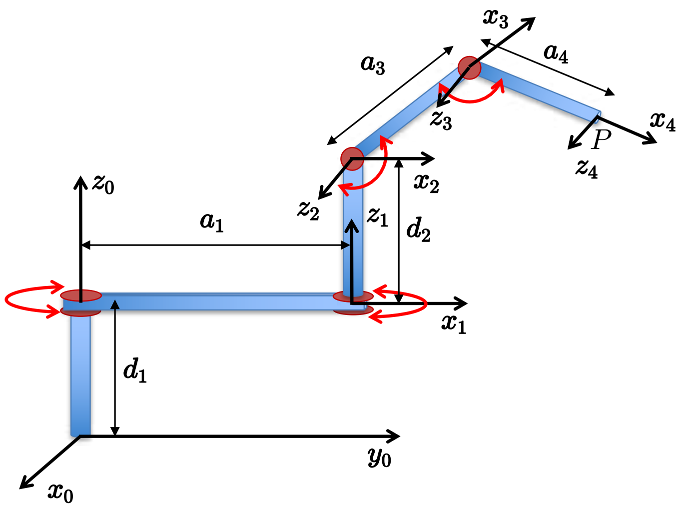


Figure 2: A 4R spatial robot, with DH frames and non-zero linear parameters shown.

[210 minutes, open books]

Solution

June 12, 2023

Exercise 1

An assignment of DH frames for the ABB robot consistent with the positive joint rotations specified by the manufacturer is shown in Fig. 3. The associated parameters are reported in Tab. 1. The numerical values of the linear DH parameters, expressed in [mm], are taken from the side view of the robot (see the extra sheet). The angular values of the joint variables correspond to the configuration shown in Fig. 3. Figure 4 shows the same frame assignment drawn on the side view picture of the robot.[1]

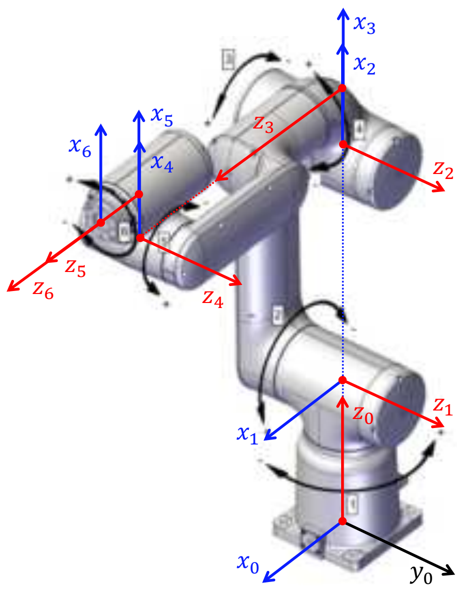


Figure 3: Assignment of DH frames for the ABB CBR 15000 robot.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 = 265$	$q_1 = 0$
2	0	$a_2 = 444$	0	$q_2 = -\pi/2$
3	$-\pi/2$	$a_3 = 110$	0	$q_3 = 0$
4	$\pi/2$	0	$d_4 = 470$	$q_4 = 0$
5	$-\pi/2$	$a_5 = 80$	0	$q_5 = 0$
6	0	0	$d_6 = 101$	$q_6 = 0$

Table 1: Table of DH parameters for the frame assignment in Fig. 3. Lengths are expressed in [mm]. The values of the joint variables (in blue) correspond to the configuration shown in Fig. 3.

[1] When compared to Fig. 3, this view is seen from the oppoite side of the robot: the axes $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ and $\boldsymbol{z}_4$, which are not shown, are entering the page.

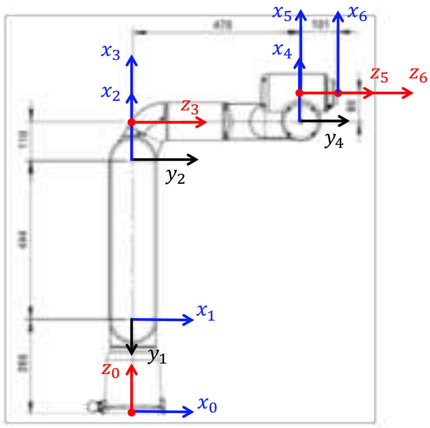


Figure 4: Another view of the DH frames assigned in Fig. 3.

Exercise 2

The given circular path from A to B in the plane (x, y) can be parametrized by

$$\boldsymbol{p} = \boldsymbol{p}(s) = R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \qquad \text{with } s \in [0, \Delta], \quad \Delta = \pi > 0.$$

The first and second spatial derivatives of $\boldsymbol{p}(s)$ are

$$\boldsymbol{p}' = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix}, \qquad \boldsymbol{p}'' = -R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}.$$

Further, the rest-to-rest cubic timing law is

$$s = s(t) = \Delta \left(3\tau^2 - 2\tau^3\right), \qquad \text{with } t \in [0, T], \quad \tau = \frac{t}{T} \in [0, 1],$$

where the total motion time T is to be determined. The first and second time derivatives of $s(t)$ are

$$\dot{s} = \frac{6\Delta}{T} \left(\tau - \tau^2\right), \qquad \ddot{s} = \frac{6\Delta}{T^2} \left(1 - 2\tau\right).$$

Accordingly, the Cartesian velocity and acceleration of the unitary mass will be

$$\dot{\boldsymbol{p}} = \boldsymbol{p}'\dot{s} = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \dot{s},$$

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \boldsymbol{p}'\ddot{s} + \boldsymbol{p}''\dot{s}^2 = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \ddot{s} - R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \dot{s}^2 = R \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} -\dot{s}^2 \\ \ddot{s} \end{pmatrix}.$$

Therefore, the norm of the acceleration is computed as

$$\|\ddot{\boldsymbol{p}}\| = \sqrt{\ddot{\boldsymbol{p}}^T \ddot{\boldsymbol{p}}} = R\sqrt{\dot{s}^4 + \ddot{s}^2} = \frac{6R\,\Delta}{T^2} \sqrt{36\,\Delta^2 \left(\tau(1-\tau)\right)^4 + (1-2\tau)^2} = \frac{6R\,\Delta}{T^2} \sqrt{\alpha(\tau)}. \tag{1}$$

For given Δ and R, this norm is only a function of the total motion time T, which has to be minimized while satisfying the bound $\|\ddot{\boldsymbol{p}}\| \leq A$. Thus, we proceed with the analysis of the functional behavior of the acceleration norm.

The maximum of the norm (1) occurs when the argument $\alpha(\tau)$ of the square root has its maximum. This occurs either at the boundaries of the closed interval $[0, 1]$ of definition for τ or when the time derivative of $\alpha(\tau)$ vanishes. At the boundaries, we have

$$\alpha(0) = \alpha(1) = 1 \qquad \Rightarrow \qquad \|\ddot{\boldsymbol{p}}(t = 0)\| = \|\ddot{\boldsymbol{p}}(t = T)\| = \frac{6R\,\Delta}{T^2} = \frac{6R\,\pi}{T^2}.$$

On the other hand, by zeroing the time derivative

$$\frac{d\alpha}{d\tau} = 144\,\Delta^2\,(\tau(1-\tau))^3\,(1-2\tau) - 4(1-2\tau) = \left(144\,\Delta^2\tau^3(1-\tau)^3 - 4\right)(1-2\tau) = 0, \tag{2}$$

we see that a first root is at $\tau = 0.5$ (i.e., $t = T/2$), in correspondence to which the norm takes the value

$$\|\ddot{\boldsymbol{p}}(t = T/2)\| = \frac{6R\,\Delta}{T^2}\cdot\frac{3\Delta}{8} = \frac{6R\,\pi}{T^2}\cdot\frac{3\pi}{8} > \frac{6R\,\pi}{T^2}.$$

Note that the acceleration norm at $t = T/2$ is larger than at the boundaries ($t = 0$ and $t = T$) because the path length to travel (as parametrized by the angle $\Delta = \pi$) is sufficiently long[2]. Next, when deleting the factor $(1-2\tau) \neq 0$ from (2), any other root $\tau = \tau^* \in [0,1]$ should satisfy

$$\tau^3(1-\tau)^3 = \frac{1}{36\Delta^2}. \tag{3}$$

However, by substituting (3) in the expression (1) of $\|\ddot{\boldsymbol{p}}\|$ and simplifying, it is easy to see that

$$\|\ddot{\boldsymbol{p}}(\tau = \tau^*)\| = \frac{6R\,\Delta}{T^2}\sqrt{\tau^*(1-\tau^*) + (1-2\tau^*)^2} = \frac{6R\,\Delta}{T^2}\sqrt{3\tau^{*2} - 3\tau^* + 1} \leq \frac{6R\,\Delta}{T^2},$$

where the last inequality holds for any $\tau^* \in [0,1]$. Thus, also in the stationary points of $\alpha(\tau)$ at the instants $\tau = \tau^*$, the acceleration norm is not larger than at $\tau = 0.5$.

In summary, we have shown that, for the given $\Delta = \pi$, the maximum acceleration norm occurs at $t = T/2$ and its value is

$$\max_{t\in[0,T]} \|\ddot{\boldsymbol{p}}(t)\| = \|\ddot{\boldsymbol{p}}(t = T/2)\| = \frac{9R\,\pi^2}{4T^2}.$$

Thus, the minimum feasible time T is found by equating the maximum acceleration norm to its bound A:

$$\frac{9R\,\pi^2}{4T^2} = A \qquad \Rightarrow \qquad T = \frac{3\pi}{2}\sqrt{\frac{R}{A}}.$$

Finally, substituting the numerical data $R = 1.5$ [m] and $A = 3$ [m/s^2], we obtain $T = 3.3322$ [s]. Figures 5–6 show the time evolution of $\|\ddot{\boldsymbol{p}}(t)\|$ and of its Cartesian components $\ddot{p}_x(t)$ and $\ddot{p}_y(t)$ for the obtained optimal solution.

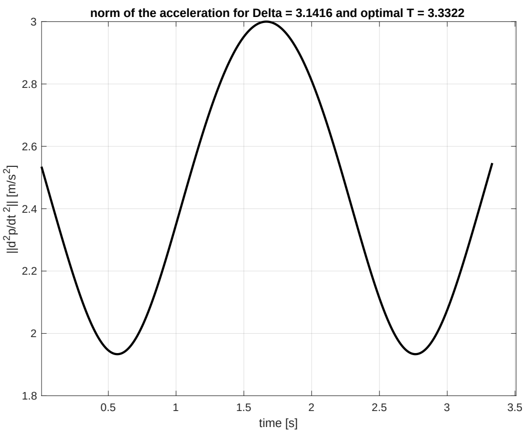


Figure 5: Norm of the optimal Cartesian acceleration $\ddot{\boldsymbol{p}}(t)$ for the given data.

[2]The crossover point is at $\Delta = 8/3 \approx 2.6666$. For smaller values, the path would be too short for the peak velocity $\dot{s}$ at $t = T/2$ to become dominant and the maximum norm would then occur at the boundaries of the time interval, where $\ddot{s}$ is maximum.

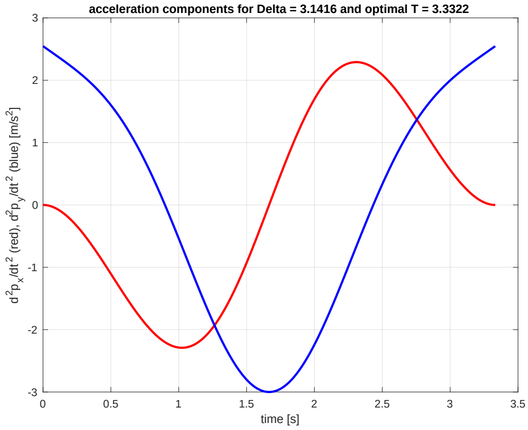


Figure 6: Cartesian components $\ddot{p}_x(t)$ (in red) and $\ddot{p}_y(t)$ (in blue) of the optimal acceleration $\ddot{\boldsymbol{p}}(t)$.

Exercise 3

The 6×4 geometric Jacobian of the 4R spatial robot in Fig. 2 is defined as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{04} & \boldsymbol{z}_1 \times \boldsymbol{p}_{14} & \boldsymbol{z}_2 \times \boldsymbol{p}_{24} & \boldsymbol{z}_3 \times \boldsymbol{p}_{34} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix}, \tag{4}$$

where $\boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and the other elements are determined through the computation of the direct kinematics of the robot. In alternative, the 3×4 linear (upper) block of the Jacobian in (4) can also be computed, perhaps more directly, as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}, \qquad \text{with } \boldsymbol{p} = \boldsymbol{p}_{04}(\boldsymbol{q}), \tag{5}$$

i.e., using only the positional part of the direct kinematics for the origin O_4 of the DH frame RF_4.

Table 2 reports the DH parameters associated to the frames shown in Fig. 2 for the 4R spatial robot.

i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	q_1
2	$\pi/2$	0	d_2	q_2
3	0	a_3	0	q_3
4	0	a_4	0	q_4

Table 2: Table of DH parameters corresponding to the frames in Fig. 2.

From this, we compute (e.g., with the MATLAB code for the standard DH direct kinematics available in the course material)

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_{01}(q_1) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_{12} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^2\boldsymbol{R}_3(q_3) & {}^2\boldsymbol{p}_{23}(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_{34}(q_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Based on these homogeneous transformation matrices, one obtains

$$
\boldsymbol{p} = \begin{pmatrix} a_1c_1 + c_{12}\left(a_3c_3 + a_4c_{34}\right) \\ a_1s_1 + s_{12}\left(a_3c_3 + a_4c_{34}\right) \\ d_1 + d_2 + a_3s_3 + a_4s_{34} \end{pmatrix} \tag{6}
$$

and

$$
\begin{aligned}
\boldsymbol{z}_1 &= {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
\boldsymbol{z}_2 &= {}^0\boldsymbol{R}_2(q_1,q_2)\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0\right) = \begin{pmatrix} s_{12} \\ -c_{12} \\ 0 \end{pmatrix}, \\
\boldsymbol{z}_3 &= {}^0\boldsymbol{R}_3(q_1,q_2,q_3)\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2)\left({}^2\boldsymbol{R}_3(q_3)\boldsymbol{z}_0\right)\right) = \begin{pmatrix} s_{12} \\ -c_{12} \\ 0 \end{pmatrix}.
\end{aligned} \tag{7}
$$

By differentiation of the positional direct kinematics in (6), one has

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -a_1s_1 - s_{12}\left(a_3c_3 + a_4c_{34}\right) & -s_{12}\left(a_3c_3 + a_4c_{34}\right) & -c_{12}\left(a_3s_3 + a_4s_{34}\right) & -a_4c_{12}s_{34} \\ a_1c_1 + c_{12}\left(a_3c_3 + a_4c_{34}\right) & c_{12}\left(a_3c_3 + a_4c_{34}\right) & -s_{12}\left(a_3s_3 + a_4s_{34}\right) & -a_4s_{12}s_{34} \\ 0 & 0 & a_3c_3 + a_4c_{34} & a_4c_{34} \end{pmatrix},
$$

while from the unit vectors in (7) it follows

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & s_{12} & s_{12} \\ 0 & 0 & -c_{12} & -c_{12} \\ 1 & 1 & 0 & 0 \end{pmatrix}.
$$

The obtained geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is expressed in frame RF_0 (or, ${}^0\boldsymbol{J}(\boldsymbol{q})$). Attempting to determine the singularities of this matrix by computing symbolically the determinant of the 4×4 matrix $\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})$

and setting it to zero may be too cumbersome for a standard symbolic manipulation program (such as MATLAB or Mathematica). On the other hand, because of the structure of the first two joints (having parallel axes), for singularity analysis it is definitely more convenient to express the Jacobian in the rotated frame RF_2:

$$
\begin{aligned}
{}^2\boldsymbol{J}(\boldsymbol{q}) &= {}^0\bar{\boldsymbol{R}}_2^T(q_1,q_2)\,{}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_2^T(q_1,q_2) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_2^T(q_1,q_2) \end{pmatrix} \begin{pmatrix} {}^0\boldsymbol{J}_L(\boldsymbol{q}) \\ {}^0\boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \\
&= \begin{pmatrix} a_1 s_2 & 0 & -(a_3 s_3 + a_4 s_{34}) & -a_4 s_{34} \\ 0 & 0 & a_3 c_3 + a_4 c_{34} & a_4 c_{34} \\ -(a_1 c_2 + a_3 c_3 + a_4 c_{34}) & -(a_3 c_3 + a_4 c_{34}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\end{aligned}
\tag{8}
$$

Further, a combination of the columns of ${}^2\boldsymbol{J}(\boldsymbol{q})$ in (8) simplifies even more the analysis:

$$
{}^2\bar{\boldsymbol{J}}(\boldsymbol{q}) = {}^2\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{T} = {}^2\boldsymbol{J}(\boldsymbol{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 s_2 & 0 & -a_3 s_3 & -a_4 s_{34} \\ 0 & 0 & a_3 c_3 & a_4 c_{34} \\ -a_1 c_2 & -(a_3 c_3 + a_4 c_{34}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}
$$

Indeed, it is

$$
\text{rank } {}^0\boldsymbol{J}(\boldsymbol{q}) = \text{rank } {}^2\boldsymbol{J}(\boldsymbol{q}) = \text{rank } {}^2\bar{\boldsymbol{J}}(\boldsymbol{q}).
$$

At this stage, we could evaluate $\det\left({}^2\bar{\boldsymbol{J}}^T(\boldsymbol{q})^2\bar{\boldsymbol{J}}(\boldsymbol{q})\right) = 0$ and find its solutions. However, we pursue here an alternative method which is even simpler. The rank of the 6×4 matrix in (9) will drop below 4 (singularity) if and only if all its 4×4 minors will simultaneously vanish. Since the fourth row is zero, there are only five minors that matter. Denote by ${}^2\bar{\boldsymbol{J}}_{-\{i,4\}}$ the 4×4 matrix obtained by deleting row 4 and row $i \neq 4$ from ${}^2\bar{\boldsymbol{J}}$. We impose then the following equalities:

$$
\begin{aligned}
\det {}^2\bar{\boldsymbol{J}}_{-\{1,4\}} &= -a_1 a_3 c_2 c_3 && = 0 \\
\det {}^2\bar{\boldsymbol{J}}_{-\{2,4\}} &= a_1 a_3 c_2 s_3 && = 0 \\
\det {}^2\bar{\boldsymbol{J}}_{-\{3,4\}} &= -a_1 a_3 s_2 c_3 && = 0 \\
\det {}^2\bar{\boldsymbol{J}}_{-\{5,4\}} &= a_1 a_3 s_2 c_3 \left(a_3 c_3 + a_4 c_{34}\right) && = 0 \\
\det {}^2\bar{\boldsymbol{J}}_{-\{6,4\}} &= -a_1 a_3 a_4 c_2 s_4 && = 0.
\end{aligned}
\tag{10}
$$

It is easy to see[3] that the system of five nonlinear equations (10) has a solution if and only if

$$
q_2 = \pm\frac{\pi}{2} \qquad \textbf{and} \qquad q_3 = \pm\frac{\pi}{2}
$$

(while q_4 does not matter), which characterize then all the singularities of the geometric Jacobian.

[3]This result is obtained by simple inspection of the equations. It can also be found by using MATLAB, with the two instructions:

```
eqn = [det(J_14) == 0; det(J_24) == 0; det(J_34) == 0; det(J_54) == 0; det(J_64) == 0];
q_sing = solve(eqn, [q_2 q_3],'Real', true)
```

.

For $\boldsymbol{q}_0 = \mathbf{0}$, the geometric Jacobian

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 + a_3 + a_4 & a_3 + a_4 & 0 & 0 \\ 0 & 0 & a_3 + a_4 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is clearly nonsingular (rank $\boldsymbol{J}_0 = 4$, for non-vanishing a_1 and a_3). From the structure of the matrix $\boldsymbol{J}_0$ it follows that a component $\omega_x \neq 0$ can never be generated: thus, having $\omega_{b,x} = 1$, $\boldsymbol{V}_b$ is not admissible in this configuration,. On the other hand, $\boldsymbol{V}_a = \begin{pmatrix} 0 & 3 & -3 & 0 & 0 & 1 \end{pmatrix}^T$ is admissible since

$$\text{rank} \begin{pmatrix} \boldsymbol{J}_0 & \boldsymbol{V}_a \end{pmatrix} = \text{rank}\, \boldsymbol{J}_0 = 4 \qquad \Rightarrow \qquad \boldsymbol{V}_a \in \mathcal{R}\{\boldsymbol{J}_0\}.$$

A joint velocity that realizes $\boldsymbol{V}_a$ can be obtained by pseudoinversion of $\boldsymbol{J}_0$:

$$\begin{aligned} \dot{\boldsymbol{q}}_0 &= \boldsymbol{J}_0^{\#} \boldsymbol{V}_a = \left(\boldsymbol{J}_0^T \boldsymbol{J}_0\right)^{-1} \boldsymbol{J}_0^T \boldsymbol{V}_a \\ &= \begin{pmatrix} 0 & \dfrac{1}{a_1} & 0 & 0 & 0 & -\dfrac{a_3+a_4}{a_1} \\ 0 & -\dfrac{1}{a_1} & 0 & 0 & 0 & \dfrac{a_1+a_3+a_4}{a_1} \\ 0 & 0 & \dfrac{1}{a_3} & 0 & \dfrac{a_4}{a_3} & 0 \\ 0 & 0 & -\dfrac{1}{a_3} & 0 & -\dfrac{a_3+a_4}{a_3} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{a_3+a_4-3}{a_1} \\ \dfrac{a_1+a_3+a_4-3}{a_1} \\ -\dfrac{3}{a_3} \\ \dfrac{3}{a_3} \end{pmatrix}. \end{aligned}$$

It can be immediately check that $\boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \boldsymbol{V}_a$. Moreover, being $\boldsymbol{J}_0$ full column rank, the joint velocity $\dot{\boldsymbol{q}}_0$ is the unique solution.

Finally, note that one can also compute $\dot{\boldsymbol{q}}_b = \boldsymbol{J}_0^{\#} \boldsymbol{V}_b = \begin{pmatrix} -\frac{a_3+a_4}{a_1} & \frac{a_1+a_3+a_4}{a_1} & \frac{1}{a_3} & -\frac{1}{a_3} \end{pmatrix}^T$, but this joint velocity will not return the desired end-effector velocity, being $\dot{\boldsymbol{e}}_b = \boldsymbol{V}_b - \boldsymbol{J}_0 \dot{\boldsymbol{q}}_b = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T$. We immediately see that the velocity error is restricted to the inadmissible component ω_x. Moreover, thanks to the property of the pseudoinverse, $\dot{\boldsymbol{q}}_b$ is the approximate solution that minimizes the norm of the task velocity error $\dot{\boldsymbol{e}}_b$ among all possible joint velocities $\dot{\boldsymbol{q}} \in I\!R^4$.

* * * * *

Robotics 1

July 10, 2023

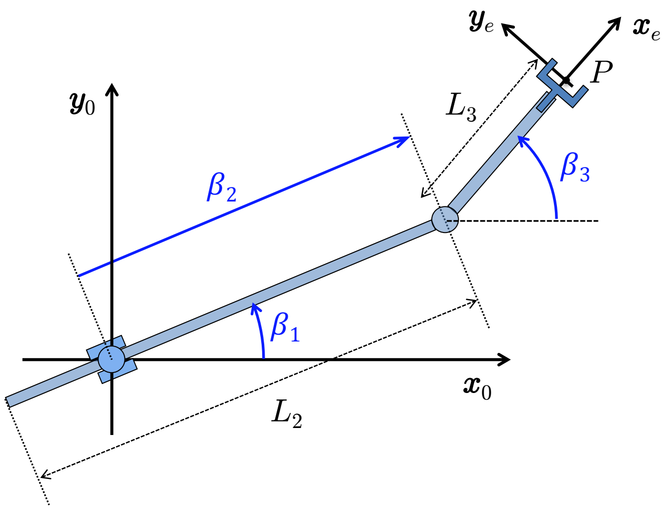


Figure 1: A 3-dof planar robot, with the definition of the joint variables β_i, $i = 1, 2, 3$.

Consider the 3-dof planar robot with one prismatic and two revolute joints shown in Fig. 1. The joint variables $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ are defined therein. The prismatic joint has a limited range, with $\beta_2 \in [-L_2, L_2]$, while the revolute joints are unlimited.

1. Sketch the primary workspace of this robot.
2. Compute the direct kinematics $\boldsymbol{r} = (\boldsymbol{p}, \alpha) = \boldsymbol{f}(\boldsymbol{\beta})$ for the position $\boldsymbol{p} \in \mathbb{R}^2$ of point P and the orientation $\alpha \in \mathbb{R}$ of the end-effector frame w.r.t. the $\boldsymbol{x}_0$ axis.
3. Given a value of $\boldsymbol{r} \in \mathbb{R}^3$ solve the inverse kinematics problem in analytic form, taking into account the limited range of the prismatic joint.
4. Assign the frames for this robot according to the standard Denavit-Hartenberg (DH) convention and fill in the corresponding table of parameters. Denote by $\boldsymbol{q} = (q_1, q_2, q_3)$ the DH joint variables.
5. Compute the direct kinematics of $\boldsymbol{r}$ as a function of $\boldsymbol{q}$, i.e., $\boldsymbol{r} = \boldsymbol{k}(\boldsymbol{q})$. Find the transformation between the two sets of joint variables, in its direct form $\boldsymbol{q} = \boldsymbol{t}(\boldsymbol{\beta})$ and inverse form $\boldsymbol{\beta} = \boldsymbol{t}^{-1}(\boldsymbol{q})$, such that $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{\beta}) = \boldsymbol{k}(\boldsymbol{t}(\boldsymbol{\beta}))$ or, equivalently, $\boldsymbol{r} = \boldsymbol{k}(\boldsymbol{q}) = \boldsymbol{f}\left(\boldsymbol{t}^{-1}(\boldsymbol{q})\right)$.
6. Determine the singularities of the 3×3 Jacobian $\boldsymbol{J}(\boldsymbol{q})$ in $\dot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$.
7. In a singular configuration $\boldsymbol{q}_s$, determine a basis for each of the following four subspaces of $\mathbb{R}^3$: $\mathcal{R}(\boldsymbol{J}(\boldsymbol{q}_s))$, $\mathcal{N}(\boldsymbol{J}(\boldsymbol{q}_s))$, $\mathcal{R}(\boldsymbol{J}^T(\boldsymbol{q}_s))$, and $\mathcal{N}(\boldsymbol{J}^T(\boldsymbol{q}_s))$.
8. Let $L_2 = L_3 = L$. Plan a rest-to-rest trajectory in time T between $\boldsymbol{r}(0) = (L/2, L/2, \pi/2)$ and $\boldsymbol{r}(T) = (-L/2, -L/2, -\pi/2)(= -\boldsymbol{r}(0))$ without violating the joint limits. Is it possible to follow a linear Cartesian path in this case?

[180 minutes, open books]

Robotics 1

September 11, 2023

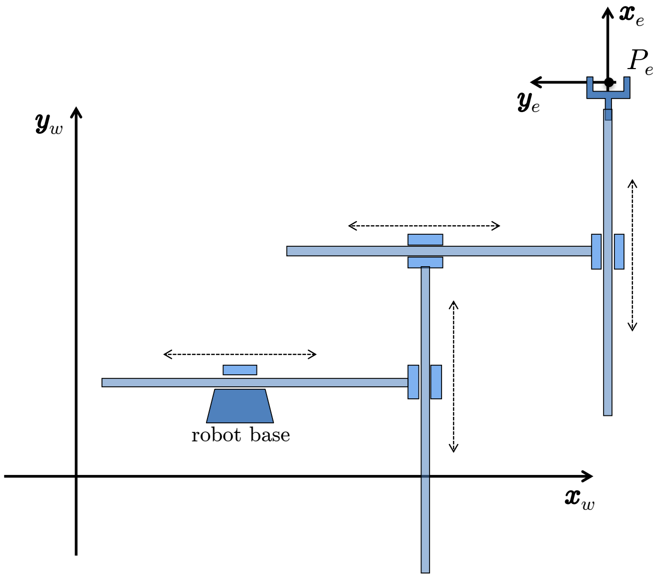


Figure 1: A 4P planar robot, with the world frame RF_w and of the end-effector frame RF_e.

Consider the 4-dof planar robot with a fixed base shown in Fig. 1. All robot joints are prismatic.

1. Draw the Denavit-Hartenberg (DH) frames and fill in the corresponding table of DH parameters.
2. Provide the two constant homogeneous transformations ${}^w\boldsymbol{T}_0$ and ${}^4\boldsymbol{T}_e$, relating respectively the world frame RF_w to the 0-th DH frame and the 4-th DH frame to the end-effector frame RF_e.
3. Compute the direct kinematics as expressed by the homogeneous transformation matrix

$$ {}^w\boldsymbol{T}_e(\boldsymbol{q}) = \begin{pmatrix} {}^w\boldsymbol{R}_e(\boldsymbol{q}) & {}^w\boldsymbol{p}_{we}(\boldsymbol{q}) \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad {}^w\boldsymbol{p}_{we}(\boldsymbol{q}) = \begin{pmatrix} p_x(\boldsymbol{q}) \\ p_y(\boldsymbol{q}) \\ p_z(\boldsymbol{q}) \end{pmatrix}. $$

4. Let the task vector be $\boldsymbol{r} = \boldsymbol{f}_r(\boldsymbol{q}) = (p_x(\boldsymbol{q}), p_y(\boldsymbol{q})) \in \mathbb{R}^2$. Compute the associated task Jacobian $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{f}_r / \partial \boldsymbol{q}$ and find its singularities.
5. At a given nonsingular $\boldsymbol{q}$, compute a basis for each of the two subspaces $\mathcal{N}(\boldsymbol{J})$ and $\mathcal{R}(\boldsymbol{J}^T)$.
6. Determine the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ with minimum norm that realizes a desired $\dot{\boldsymbol{r}} = (3, -2)$ [m/s].
7. Determine a joint torque $\boldsymbol{\tau} \in \mathbb{R}^4$ that statically balances a Cartesian force $\boldsymbol{F} = (2, 1)$ [N] applied at the robot end-effector. Is this $\boldsymbol{\tau}$ unique in the present case?
8. Plan a linear Cartesian trajectory between $\boldsymbol{r}_{\text{in}} = (-1, 1)$ and $\boldsymbol{r}_{\text{fin}} = (3, 7)$ and determine the minimum rest-to-rest motion time T when the joint velocity and acceleration limits are

$$ |\dot{q}_i| \le 2 \text{ [m/s]}, \qquad |\ddot{q}_i| \le 5 \text{ [m/s}^2\text{]}, \qquad \text{for all } i \in \{1, 2, 3, 4\}. $$

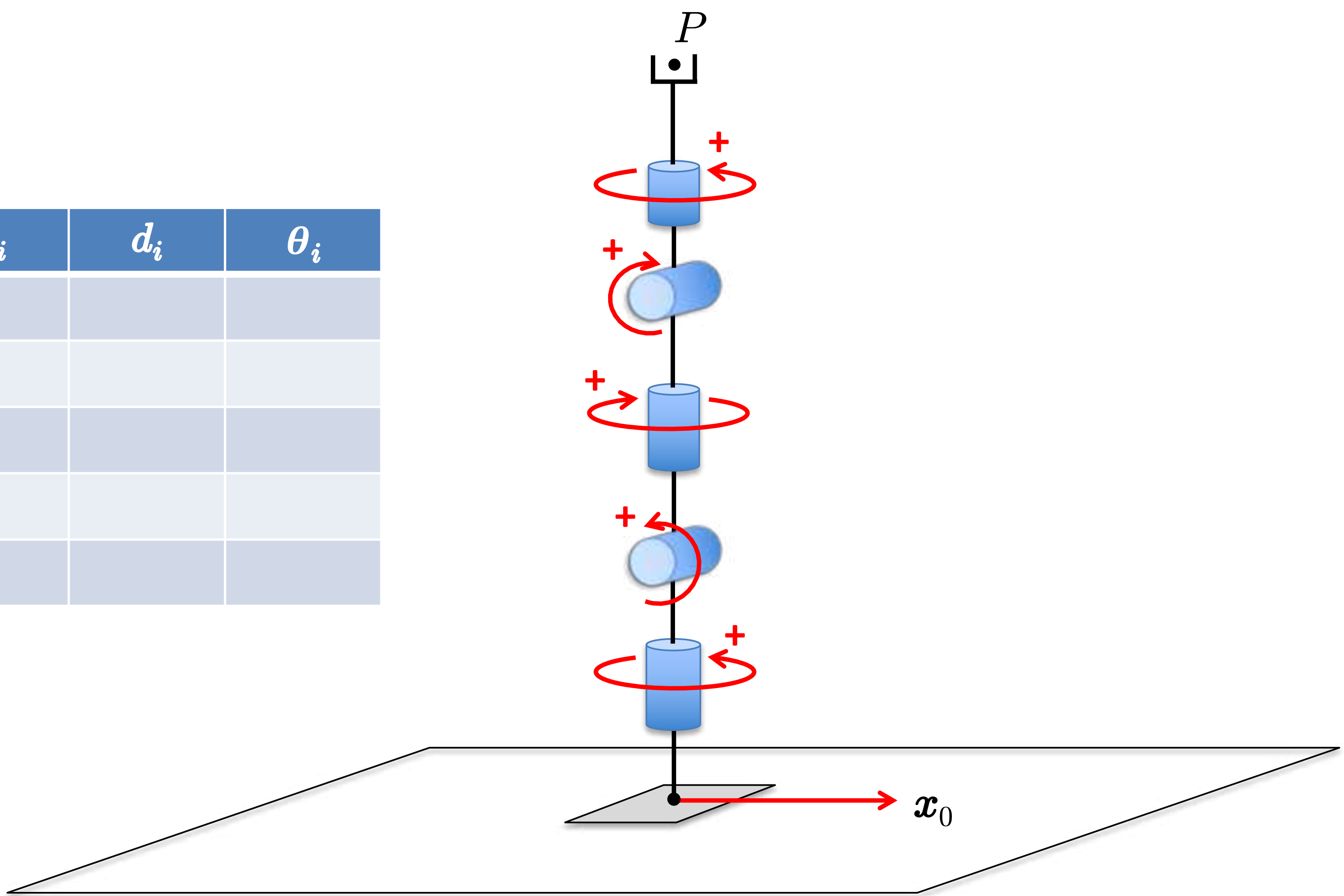
   Motion of all joints should be coordinated. Discontinuous joint accelerations are admissible.

[150 minutes, open books]

DH frames assignment and table for a 5R robot

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				



Robotics 1

Midterm Test — November 15, 2023 [total 100 points]

Exercise 1 [10 points]

Consider the orientation obtained by a (partial) Euler sequence with a rotation of an angle α around $\boldsymbol{z}$, followed by a rotation of an angle β around the current $\boldsymbol{y}$. Find three angles ϕ, χ, and ψ such that the product $\boldsymbol{R_x}(\phi)\boldsymbol{R_y}(\chi)\boldsymbol{R_z}(\psi)$ returns the same final orientation. Give the procedure for solving this problem in general, determine the singular cases, and provide then a numerical value of the sought triple of angles when $\alpha = \pi/4$, $\beta = -\pi/3$ [rad]. Check the result.

Exercise 2 [10 points]

Let a first rotation be defined by an angle γ around $\boldsymbol{x}$, followed by a rotation of an angle δ around the unit vector $\boldsymbol{v} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ expressed in the original frame. Determine the resulting rotation matrix $\boldsymbol{R}(\gamma, \delta)$ in symbolic form. For a numerical case with $\gamma = -\pi/2$, $\delta = \pi/3$ [rad], extract the invariant axis $\boldsymbol{r}$ of the total rotation and the corresponding angle θ. Check the result.

Exercise 3 [10 points]

Consider the 2R planar robot in Fig. 1, with $L_1 = 1$, $L_2 = 0.5$ [m]. The joint variables have a limited range: $\theta_1 \in [0, \pi/2]$, $\theta_2 \in [-\pi/2, \pi/2]$ [rad].

- Sketch the primary workspace of this robot, localizing the relevant points on its boundary.
- Indicate the region of the workspace where two inverse kinematics solutions exist.
- For each of the following five points, specify whether there are 0, 1, 2, or ∞ inverse kinematics solutions: $P_1 = (0.1, 1.5)$, $P_2 = (0.5, 1.3)$, $P_3 = (-0.4, 1.1)$, $P_4 = (1.0, 1.0)$, $P_5 = (1.0, -0.3)$ [m].

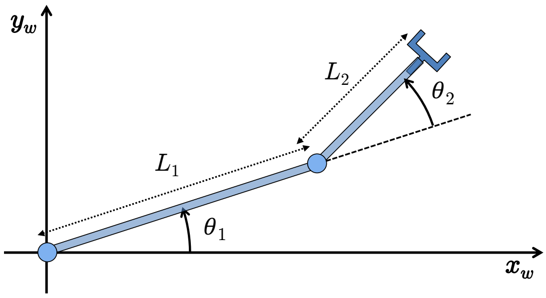


Figure 1: A 2R planar robot.

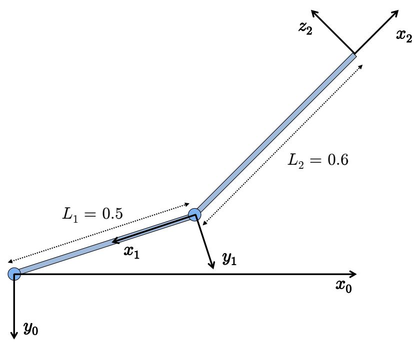


Figure 2: A 2R planar robot with D-H frames.

Exercise 4 [10 points]

Figure 2 shows an unusual but feasible choice of Denavit-Hartenberg (D-H) frames for a 2R planar robot. Provide the corresponding D-H table of parameters and the direct kinematics of this robot as an homogeneous transformation matrix ${}^0\boldsymbol{T}_2(\boldsymbol{q})$. Evaluate then this matrix in numerical form at $\boldsymbol{q}^* = (\pi/2, -\pi/2)$ [rad] and draw the robot in this configuration.

Exercise 5 [10 points]

The differential equations of a DC motor are given in slide #14 of the block 03_CompsActuators.pdf. With the motor unloaded and starting from rest, if we apply a constant armature voltage $\bar{v}_a$, the motor will start rotating and then reach a steady-state condition, with a constant angular velocity $\bar{\omega}$ and a constant produced torque $\bar{\tau}$. What are the expressions of $\bar{\omega}$ and $\bar{\tau}$ in terms of the system parameters and $\bar{v}_a$? If we attach a load with inertia $I_L > 0$ to the motor shaft through a transmission with reduction ratio $n_r > 1$ and assume no dissipative terms on the load side, will the steady-state velocity of the motor change? And what will be the velocity ω_L of the load at steady state?

Exercise 6 [20 points]

The 5R robot in Fig. 3 is shown in its zero configuration (i.e., for $\boldsymbol{q} = \boldsymbol{0}$), with indication of the positive joint rotations. Assign the D-H frames consistently with these specifications and fill the corresponding table of parameters (specifying also the signs of the non-zero constant parameters). The origin of the last D-H frame should be at point P. Evaluate then numerically the position and the orientation of the last frame at $\boldsymbol{q} = \boldsymbol{0}$, when all the non-zero kinematic lengths of the links are unitary.

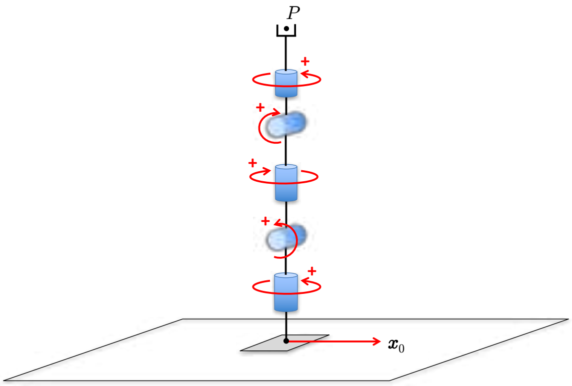


Figure 3: A 5R spatial robot at $\boldsymbol{q} = \boldsymbol{0}$.

Exercise 7 [30 points]

Consider the planar RPR robot in Fig. 4, with the first and third joint revolute and the second prismatic.

a. Determine the task kinematics $\boldsymbol{r} = \boldsymbol{f_r}(\boldsymbol{q})$ for $\boldsymbol{r} = (\boldsymbol{p}, \phi)$, being $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ the position of the end-effector and $\phi \in (-\pi, \pi]$ its orientation angle with respect to $\boldsymbol{x}_0$. [*Hint: Use D-H joint variables.*]

b. Solve analytically the inverse kinematics problem for $\boldsymbol{r}_d = (p_{dx}, p_{dy}, \phi_d)$ in the regular case only.

c. Let the RPR robot have the first and third links of unitary length. The pose of its base frame RF_0 with respect to the world frame RF_w placed at the base of the 2R robot defined in Ex. 3 and shown in Fig. 1 is given by the homogeneous matrix

$$ {}^w\boldsymbol{T}_0 = \begin{pmatrix} {}^w\boldsymbol{R}_0 & {}^w\boldsymbol{p}_0 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.8660 & 0 & 1 \\ 0.8660 & 0.5 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1} $$

When the 2R robot is at $\boldsymbol{\theta} = (0, \pi/2)$, find a configuration of the RPR robot with prismatic joint variable $q_2 > 0$ such that the end-effector of this robot has its position coincident with that of the 2R robot and the approach direction of its gripper is specified by the unit vector ${}^w\boldsymbol{a}_d = (0, -1, 0)$.

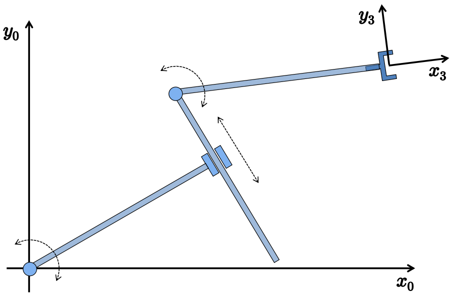


Figure 4: An RPR planar robot.

[240 minutes, open books]

Solution

November 15, 2023

Exercise 1 [10 points]

The orientation $\boldsymbol{R}(\alpha,\beta)$ obtained by the first two rotations is given by

$$\boldsymbol{R}(\alpha,\beta)=\boldsymbol{R_z}(\alpha)\boldsymbol{R_y}(\beta)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}=\begin{pmatrix}c_\alpha c_\beta & -s_\alpha & c_\alpha s_\beta\\ s_\alpha c_\beta & c_\alpha & s_\alpha s_\beta\\ -s_\beta & 0 & c_\beta\end{pmatrix}.$$

On the other hand, the orientation obtained by the Euler sequence XYZ with angles ϕ, χ, and ψ is

$$\begin{aligned}\boldsymbol{R}_{XYZ}(\phi,\chi,\psi) &= \boldsymbol{R_x}(\phi)\boldsymbol{R_y}(\chi)\boldsymbol{R_z}(\psi)\\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\cos\chi & 0 & \sin\chi\\ 0 & 1 & 0\\ -\sin\chi & \cos\chi & 0\end{pmatrix}\begin{pmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}c_\chi c_\psi & -c_\chi s_\psi & s_\chi\\ c_\phi s_\psi + s_\phi s_\chi c_\psi & c_\phi c_\psi - s_\phi s_\chi s_\psi & -s_\phi c_\chi\\ s_\phi s_\psi - c_\phi s_\chi c_\psi & s_\phi c_\psi + c_\phi s_\chi s_\psi & c_\phi c_\chi\end{pmatrix}.\end{aligned}$$

We have to solve a standard inverse problem for this Euler sequence of angles to represent the rotation matrix $\boldsymbol{R}(\alpha,\beta)$:

$$\boldsymbol{R}_{XYZ}(\phi,\chi,\psi)=\boldsymbol{R}(\alpha,\beta). \tag{2}$$

The only peculiarity is that the assigned rotation matrix is not (yet) given in numerical form at this stage, but is parametrized by the two angles α and β. Denote the elements of matrix $\boldsymbol{R}(\alpha,\beta)$ simply by R_{ij}. From the identities in the first row and last column of the matrices in (2) one obtains

$$\chi=\text{ATAN2}\left\{R_{13},\pm\sqrt{R_{11}^2+R_{12}^2}\right\}=\text{ATAN2}\left\{c_\alpha s_\beta,\pm\sqrt{c_\alpha^2c_\beta^2+s_\alpha^2}\right\}$$

We can solve then for the other two angles provided that $R_{11}^2+R_{12}^2=c_\alpha^2c_\beta^2+s_\alpha^2\neq 0$, i.e., excluding singular cases. Taking directly the $+$ sign in the second argument of the above ATAN2 function (so that $c_\chi>0$), one has

$$\phi=\text{ATAN2}\left\{-\frac{R_{23}}{c_\chi},\frac{R_{33}}{c_\chi}\right\}=\text{ATAN2}\left\{-s_\alpha s_\beta,c_\beta\right\}$$

and

$$\psi=\text{ATAN2}\left\{-\frac{R_{12}}{c_\chi},\frac{R_{11}}{c_\chi}\right\}=\text{ATAN2}\left\{s_\alpha,c_\alpha c_\beta\right\}.$$

When substituting the numerical values $\alpha=\pi/4$ and $\beta=-\pi/3$, it is $c_\alpha^2c_\beta^2+s_\alpha^2=0.625\neq 0$; thus, we arein a regular case. The values of the three Euler angles are found then from the above expressions as

$$\phi=0.8861,\qquad \chi=-0.6591,\qquad \psi=1.1071\qquad [\text{rad}].$$

Plugging these into (2), we verify that

$$\boldsymbol{R}_{XYZ}(\phi=0.8861,\chi=-0.6591,\psi=1.1071)=\boldsymbol{R}(\alpha=\pi/4,\beta=-\pi/3)=\begin{pmatrix}0.3536 & -0.7071 & -0.6124\\ 0.3536 & 0.7071 & -0.6124\\ 0.8660 & 0 & 0.5\end{pmatrix}.$$

Exercise 2 [10 points]

The orientation obtained by the two rotations around $\boldsymbol{x}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{R}(\gamma,\delta,\boldsymbol{v}) = \boldsymbol{R}_{\boldsymbol{v}}(\delta)\boldsymbol{R}_{\boldsymbol{x}}(\gamma) = \left(\boldsymbol{v}\boldsymbol{v}^T + \left(\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}^T\right)\cos\delta + \boldsymbol{S}(\boldsymbol{v})\sin\delta\right)\boldsymbol{R}_{\boldsymbol{x}}(\gamma),$$

where the reverse order of the matrix product follows from the fact that both rotations are defined with respect to fixed axes. Using the unit vector $\boldsymbol{v} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$, one obtains the (semi-)symbolic matrix

$$\boldsymbol{R}(\gamma,\delta) = \begin{pmatrix} \dfrac{c_\delta+1}{2} & \dfrac{c_\delta-1}{2}\,c_\gamma - \dfrac{\sqrt{2}}{2}\,s_\gamma s_\delta & -\dfrac{c_\delta-1}{2}\,s_\gamma - \dfrac{\sqrt{2}}{2}\,c_\gamma s_\delta \\ \dfrac{c_\delta-1}{2} & \dfrac{c_\delta+1}{2}\,c_\gamma - \dfrac{\sqrt{2}}{2}\,s_\gamma s_\delta & -\dfrac{c_\delta+1}{2}\,s_\gamma - \dfrac{\sqrt{2}}{2}\,c_\gamma s_\delta \\ \dfrac{\sqrt{2}}{2}\,s_\delta & s_\gamma c_\delta + \dfrac{\sqrt{2}}{2}\,c_\gamma s_\delta & c_\gamma c_\delta - \dfrac{\sqrt{2}}{2}\,s_\gamma s_\delta \end{pmatrix}.$$

For the considered numerical case, this matrix becomes

$$\boldsymbol{R}_s = \boldsymbol{R}(\gamma = -\pi/2, \delta = \pi/3) = \begin{pmatrix} 0.75 & 0.6124 & -0.25 \\ -0.25 & 0.6124 & 0.75 \\ 0.6124 & -0.5 & 0.6124 \end{pmatrix}.$$

Being $(R_{s,12} - R_{s,21})^2 + (R_{s,13} - R_{s,31})^2 + (R_{s,23} - R_{s,32})^2 = 3.0499 \neq 0$, we are in a regular case and the inverse relationships for the axis/angle representation of this matrix yield

$$\boldsymbol{r} = \begin{pmatrix} -0.7158 \\ -0.4938 \\ -0.4938 \end{pmatrix}, \qquad \theta = 1.0617 \quad \text{[rad]}.$$

and its opposite pair $(-\boldsymbol{r}, -\theta)$. It is easy to check that $\boldsymbol{R}_{\boldsymbol{r}}(\theta) = \boldsymbol{R}_{-\boldsymbol{r}}(-\theta) = \boldsymbol{R}_s$.

Exercise 3 [10 points]

Figure 5 shows the primary workspace of the 2R planar robot with the given limits of the joint ranges.

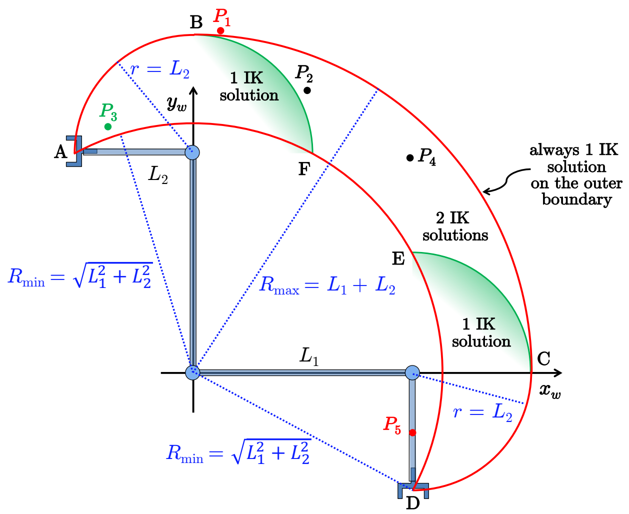


Figure 5: Primary workspace of the given 2R robot, with relevant points of interest.

The robot is shown in its two limit configurations, at $\theta = (\pi/2, \pi/2)$ and at $\theta = (0, -\pi/2)$ [rad]. The 'banana'-like workspace is limited by the four points A $= (-L_2, L_1) = (-0.5, 1)$, $B = (0, L_1 + L_2) = (0, 1.5)$, C $= (L_1 + L_2, 0) = (1.5, 0)$, and D $= (L_1, -L_2) = (1, -0.5)$ (all point coordinates are all expressed in [m]). The inner boundary is an arc of a circle of radius $R_{\min} = \sqrt{L_1^2 + L_2^2} = \sqrt{1.25} = 1.1180$ [m], centered at the origin. The outer boundary is composed by three arcs of circles, two of radius $r = L_2 = 0.5$ [m], centered respectively at $(0, 1)$ (arc AB) and at $(1, 0)$ (arc CD), and one of radius $R_{\max} = L_1 + L_2 = 1.5$ [m], centered again at the origin. The workspace is symmetrically divided in three regions: there is only one solution to the inverse kinematics (IK) in the regions ABF (right arm) and ECD (left arm), including their parts of the workspace boundary, while there are two solutions (right and left arm) in the central area BCEF, with E $= (L_1, L_2) = (1, 0.5)$ and F $= (L_2, L_1) = (0.5, 1)$, including the inner arc EF on the workspace boundary and the two internal arcs BF and CE that limit this area. Finally, there is only one IK solution on the outer boundary, including the arc BC (where the arm is outstretched).

As for the points P_i, $i = 1, \ldots, 5$, it is easy to check that:

- the two points (marked in red) $P_1 = (0.1, 1.5)$ and $P_5 = (1.0, -0.3)$ are out of the workspace, since $\|\boldsymbol{p}_1\|^2 = 2.26 > 2.25 = R^2_{\max}$ and $\|\boldsymbol{p}_2\|^2 = 1.09 < 1.25 = R^2_{\min}$;
- in $P_2 = (0.5, 1.3)$ and $P_4 = (1.0, 1.0)$ (marked in black) there are two IK solutions;
- there is only one IK solution in $P_3 = (-0.4, 1.1)$ (marked in green) —the right arm solution.

Exercise 4 [10 points]

The D-H parameters corresponding to the frame assignment for the 2R planar robot shown in Fig. 2 are given in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	π	$-L_1 = -0.5$	0	q_1
2	$-\pi/2$	$L_2 = 0.6$	0	q_2

Table 1: D-H parameters corresponding to the frames in Fig. 2.

From the associated homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & \sin q_1 & 0 & -L_1 \cos q_1 \\ \sin q_1 & -\cos q_1 & 0 & -L_1 \sin q_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & -\sin q_2 & L_2 \cos q_2 \\ \sin q_2 & 0 & \cos q_2 & L_2 \sin q_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

one obtains

$${}^0\boldsymbol{T}_2(\boldsymbol{q}) = \boldsymbol{A}_1(q_1)\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos(q_1 - q_2) & 0 & \sin(q_1 - q_2) & -L_1 \cos q_1 + L_2 \cos(q_1 - q_2) \\ \sin(q_1 - q_2) & 0 & -\cos(q_1 - q_2) & -L_1 \sin q_1 + L_2 \sin(q_1 - q_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When $\boldsymbol{q} = \boldsymbol{q}^* = (\pi/2, -\pi/2)$ [rad], for $L_1 = 0.5$ and $L_2 = 0.6$ [m], we have

$${}^0\boldsymbol{T}_2(\boldsymbol{q}^*) = \begin{pmatrix} -1 & 0 & 0 & -0.6 \\ 0 & 0 & 1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This configuration of the 2R robot is shown in Fig. 6.

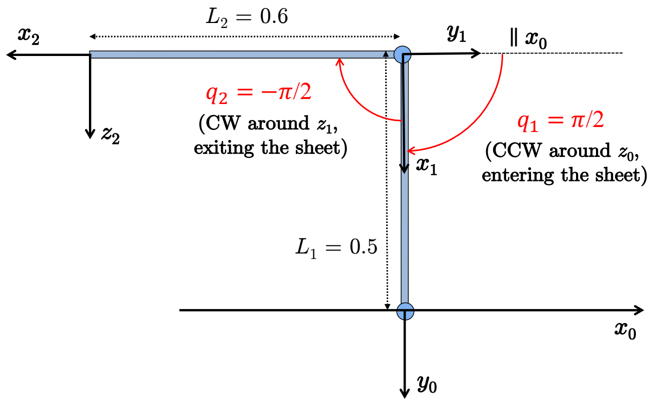


Figure 6: The 2R planar robot of Fig. 2, shown in the configuration $\boldsymbol{q}^* = (\pi/2, -\pi/2)$.

Exercise 5 [10 points]

The differential equations of a DC motor driven by an armature voltage v_a can be written in state-space format, with the two state components $\boldsymbol{x} = (i_a, \omega)$ and the input $u = v_a$, as

$$\begin{aligned} \frac{di_a}{dt} &= -\frac{R_a}{L_a}\, i_a - \frac{k_v}{L_a}\, \omega + \frac{1}{L_a}\, u \\ \frac{d\omega}{dt} &= \frac{k_t}{I_m}\, i_a - \frac{F_m}{I_m}\, \omega - \frac{1}{I_m}\, \tau_{load} \end{aligned} \tag{3}$$

having used[1] the expressions of the back electromagnetic force $v_{emf} = k_v \omega$ and of the output torque produced by the motor $\tau_m = k_t i_a$. When there is no load attached to the motor shaft ($\tau_{load} = 0$), equations (3) become in matrix form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u, \qquad \boldsymbol{A} = \begin{pmatrix} -R_a/L_a & -k_v/L_a \\ k_t/I_m & -F_m/I_m \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1/L_a \\ 0 \end{pmatrix}.$$

From

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \det \begin{pmatrix} \lambda + R_a/L_a & k_v/L_a \\ -k_t/I_m & \lambda + F_m/I_m \end{pmatrix} = \lambda^2 + \left(\frac{R_a}{L_a} + \frac{F_m}{I_m}\right)\lambda + \frac{1}{L_a I_m}\,(R_a F_m + k_v k_t),$$

the two eigenvalues of $\boldsymbol{A}$

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{R_a}{L_a} + \frac{F_m}{I_m}\right) \pm \frac{1}{2}\sqrt{\left(\frac{R_a}{L_a} + \frac{F_m}{I_m}\right)^2 - \frac{4(R_a F_m + k_v k_t)}{L_a I_m}}$$

have negative real part since all physical constants are positive. Thus, the system is asymptotically stable and admits, in response to a constant input $u = \bar{v}_a$, a steady-state condition in which the angular velocity ω and the armature current i_a (and thus also the motor torque τ_m) are constant. The steady state $\bar{\boldsymbol{x}} = (\bar{i}_a, \bar{\omega})$ is computed by setting $u = \bar{v}_a$ and $\dot{\boldsymbol{x}} = \boldsymbol{0}$:

$$\boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{b}\bar{v}_a = \boldsymbol{0} \quad \Rightarrow \quad \bar{\boldsymbol{x}} = -\boldsymbol{A}^{-1}\boldsymbol{b}\,\bar{v}_a \quad \Rightarrow \quad \left\{ \begin{aligned} \bar{i}_a &= \frac{F_m}{R_a F_m + k_v k_t}\, \bar{v}_a \\ \bar{\omega} &= \frac{k_t}{R_a F_m + k_v k_t}\, \bar{v}_a. \end{aligned} \right. \tag{4}$$

[1] The two constants k_v and k_t are numerically equal when using SI units ($k_v = k_t$). They have been kept distinct here for better clarity, also because we are working only symbolically.

Accordingly, the torque produced by the motor at steady state is

$$\bar{\tau}_m = k_t \bar{i}_a = F_m \bar{\omega} = \frac{F_m k_t}{R_a F_m + k_v k_t}\, \bar{v}_a.$$

Consider now an inertial load I_L attached to the motor through a transmission with reduction ratio $n_r > 1$. Since the reflected load torque at the motor shaft is

$$\tau_{load} = \frac{1}{n_r}\left(I_L \dot{\omega}_L\right) = \frac{1}{n_r}\left(I_L \frac{\dot{\omega}}{n_r}\right) = \frac{I_L}{n_r^2}\, \dot{\omega},$$

the second differential equation in (3) becomes

$$\frac{d\omega}{dt} = \frac{k_t}{I'_m}\, i_a - \frac{F_m}{I'_m}\, \omega, \qquad \text{with } I'_m = I_m + \frac{I_L}{n_r^2}.$$

Therefore, the motor dynamics will have the same previous structure, now with a larger equivalent motor inertia I'_m. The system is still asymptotically stable, with the two eigenvalues having a negative real part smaller than before. In response to a constant $\bar{v}_a$, this implies a slower transient before reaching the steady state. However, the steady-state velocity $\bar{\omega}$ of the motor will remain the same, as apparent from its expression in (4) which is independent of I_m (and thus of I'_m). Accordingly, the steady-state velocity of the load will be

$$\bar{\omega}_L = \frac{\bar{\omega}}{n_r} = \frac{k_t}{n_r}\, \frac{\bar{v}_a}{R_a F_m + k_v k_t}.$$

Exercise 6 [20 points]

The D-H frames for the 5R robot in Fig. 3 are uniquely specified as in Fig. 7, up to the arbitrary choice of the direction of $\boldsymbol{z}_5$ (chosen here so that the last twist angle is $\alpha_5 = 0$).

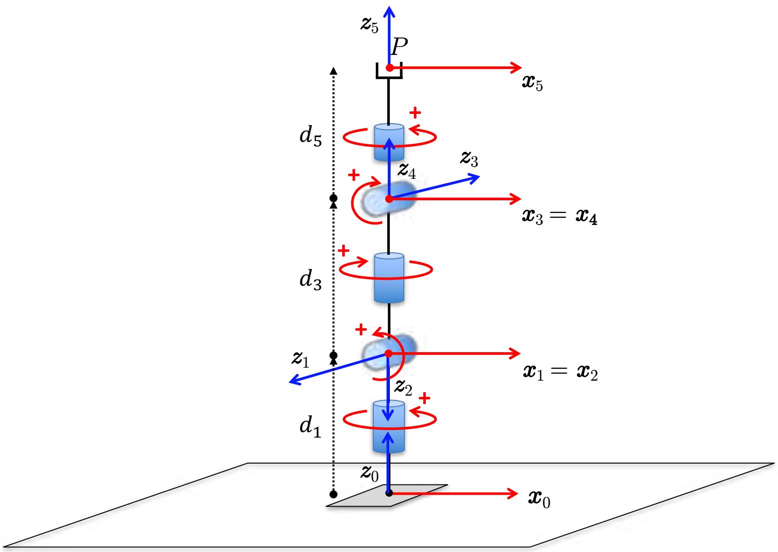


Figure 7: D-H frame assignment for the 5R robot of Fig. 3.

For $i = 0, \ldots, 4$, the directions of the $\boldsymbol{z}_i$ axes should guarantee a positive counterclockwise (CCW) rotation that is consistent with the specifications in Fig. 3. Moreover, since the robot is shown at $\boldsymbol{q} = \boldsymbol{0}$, all $\boldsymbol{x}_i$ axes, for $i = 1, \ldots, 5$, should be aligned with $\boldsymbol{x}_0$ in this configuration. The only non-zero linear parameters are $d_1 > 0$ (displacement from O_0 to O_1 along $\boldsymbol{z}_0$), $d_3 < 0$ (displacement from O_2 to O_3 along z_2), and $d_5 > 0$ (displacement from O_4 to $O_5 = P$ along z_4). The D-H parameters corresponding to this frame assignment for the 5R robot are given in Tab. 2, for the shown configuration $\boldsymbol{q} = \boldsymbol{0}$.

The (long) symbolic expression of the pose of the end-effector frame RF_5 is not requested explicitly by the text of this exercise, although it can be easily obtained using the available symbolic manipulation codes from the D-H table as

$$^0\boldsymbol{T}_5(\boldsymbol{q}) = \boldsymbol{A}_1(q_1)\boldsymbol{A}_2(q_2)\boldsymbol{A}_3(q_3)\boldsymbol{A}_4(q_4)\boldsymbol{A}_5(q_5) = \prod_{i=1}^{5} \boldsymbol{A}_i(q_i).$$

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	$q_1 = 0$
2	$\pi/2$	0	0	$q_2 = 0$
3	$\pi/2$	0	$d_3 < 0$	$q_3 = 0$
4	$\pi/2$	0	0	$q_4 = 0$
5	0	0	$d_5 > 0$	$q_5 = 0$

Table 2: D-H parameters corresponding to the frames in Fig. 7.

In any event, its numerical evaluation at $\boldsymbol{q} = \boldsymbol{0}$ for unitary lengths $d_1 = d_5 = 1$ and $d_3 = -1$ (note this!) leads to

$$^0\boldsymbol{T}_5(\boldsymbol{0}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrix $^0\boldsymbol{T}_5(\boldsymbol{0})$ could have been found also by visual inspection of Fig. 7. In fact, frame RF_5 is oriented as frame RF_0 (thus $^0\boldsymbol{R}_5(\boldsymbol{0}) = \boldsymbol{I}$), while its origin is on the $\boldsymbol{z}_0$ axis at a distance $D = d_1 + |d_3| + d_5 = 3$ [m] from the origin O_0.

Exercise 7 [30 points]

a. For describing the task kinematics associated to the end-effector of the RPR planar robot of Fig. 4, we can use the natural definition of joint coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ shown in Fig. 8.

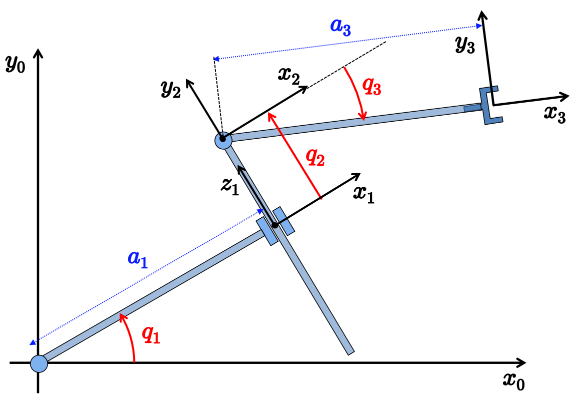


Figure 8: A natural D-H frame assignment for the RPR planar robot of Fig. 4.

We obtain

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} a_1 \cos q_1 + q_2 \cos(q_1 + \frac{\pi}{2}) + a_3 \cos(q_1 + q_3) \\ a_1 \sin q_1 + q_2 \sin(q_1 + \frac{\pi}{2}) + a_3 \sin(q_1 + q_3) \\ q_1 + q_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} a_1 \cos q_1 - q_2 \sin q_1 + a_3 \cos(q_1 + q_3) \\ a_1 \sin q_1 + q_2 \cos q_1 + a_3 \sin(q_1 + q_3) \\ q_1 + q_3 \end{pmatrix} = \boldsymbol{f_r}(\boldsymbol{q}). \tag{5}
$$

The same expression (5) can also be derived by following the D-H convention. The D-H parameters associated to the frames defined in Fig. 8 are reported in Tab. 3.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 > 0$	0	q_1
2	$\pi/2$	0	q_2	0
3	0	$a_3 > 0$	0	q_3

Table 3: D-H parameters corresponding to the frames in Fig. 8.

From the D-H table, computing the homogeneous transformations

$$
\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & -s_1 & a_1 c_1 \\ s_1 & 0 & c_1 & a_1 s_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2(q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

one obtains

$$
{}^0\boldsymbol{T}_3(\boldsymbol{q}) = \boldsymbol{A}_1(q_1)\boldsymbol{A}_2(q_2)\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_{13} & -s_{13} & 0 & a_1 c_1 - q_2 s_1 + a_3 c_{13} \\ s_{13} & c_{13} & 0 & a_1 s_1 + q_2 c_1 + a_3 s_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

which is consistent with (5), once the absolute angle $\phi = q_1 + q_3$ with respect to the $\boldsymbol{x}_0$ axis is extracted from the rotation matrix ${}^0\boldsymbol{R}_3(\boldsymbol{q})$.

b. The inverse kinematics problem for $\boldsymbol{r} = \boldsymbol{r}_d = (p_{dx}, p_{dy}, \phi_d)$ is solved as follows. From the third equation in (5) one has $q_1 + q_3 = \phi_d$. Substituting this argument in the trigonometric functions within the first two equations, one obtains

$$
\begin{pmatrix} p_{dx} - a_3 c_{\phi_d} \\ p_{dy} - a_3 s_{\phi_d} \end{pmatrix} = \begin{pmatrix} a_1 c_1 - q_2 s_1 \\ a_1 s_1 + q_2 c_1 \end{pmatrix}. \tag{6}
$$

Squaring and summing these two equations yields

$$
\left(p_{dx} - a_3 c_{\phi_d}\right)^2 + \left(p_{dy} - a_3 s_{\phi_d}\right)^2 = a_1^2 + q_2^2,
$$

from which we have the two solutions for the prismatic joint

$$
q_2^{[+,-]} = \pm\sqrt{(p_{dx} - a_3 c_{\phi_d})^2 + (p_{dy} - a_3 s_{\phi_d})^2 - a_1^2} = \pm\sqrt{p_{dx}^2 + p_{dy}^2 + a_3^2 - 2a_3(p_{dx} c_{\phi_d} + p_{dy} s_{\phi_d}) - a_1^2}, \tag{7}
$$

provided that the argument of the square root is strictly positive (regular case). If this argument is zero, the two solutions collapse into one (singular case); if it is negative, there is no solution to the inverse kinematics problem. In the regular case, for each of the two solutions $q_2^{[+]}$and $q_2^{[-]}$ in (7) we proceed with finding q_1 through the solution of a linear system in the unknowns c_1 and s_1 obtained from (6):

$$\begin{pmatrix} a_1 & -q_2 \\ q_2 & a_1 \end{pmatrix} \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} p_{dx} - a_3 c_{\phi_d} \\ p_{dy} - a_3 s_{\phi_d} \end{pmatrix}. \tag{8}$$

Since the determinant of the matrix in (8) is $a_1^2 + q_2^2 > 0$, a solution (c_1, s_1) always exists and is unique. Therefore, we have

$$q_1^{[+,-]} = \text{ATAN2}\left\{a_1(p_{dy} - a_3 s_{\phi_d}) - q_2^{[+,-]}(p_{dx} - a_3 c_{\phi_d}),\, a_1(p_{dx} - a_3 c_{\phi_d}) + q_2^{[+,-]}(p_{dy} - a_3 s_{\phi_d})\right\}. \tag{9}$$

Finally, the third joint variable is obtained as

$$q_3^{[+,-]} = \phi_d - q_1^{[+,-]}. \tag{10}$$

Each of the two results (10) should be properly mapped into the interval $(-\pi, \pi]$.

c. We have to solve an inverse kinematics problem for the RPR planar robot, where the input is partly defined by the position of the end-effector of the 2R planar robot in Fig. 1. For this, in order to use the results of the previous section **b**, one has to specify the input data $\boldsymbol{r}_d = (p_{dx}, p_{dy}, \phi_d)$ in the base frame RF_0 of the RPR robot (thus, ${}^0\boldsymbol{r}_d ={}^0 \boldsymbol{p}_d^T\, {}^0\phi_d)^T$). Note that, from the rotational part of matrix ${}^w\boldsymbol{T}_0$ in (1), the frame RF_0 is rotated by an angle $\beta = \pi/3$ around $\boldsymbol{z}_0$ with respect to frame RF_w.

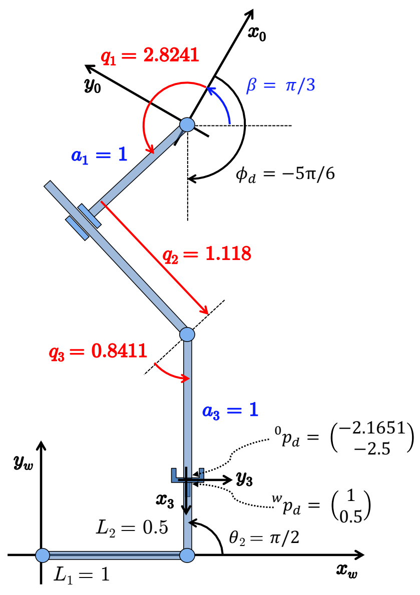


Figure 9: Solution of the inverse kinematics for the RPR robot performing the desired task.

Consider then the following two kinematic identities that hold for the required task.

1) Coincidence of the end-effector positions of the two robots RPR and 2R. This is expressed as

$${}^w\boldsymbol{T}_0\, {}^0\boldsymbol{T}_3^{\text{RPR}}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = {}^w\boldsymbol{T}_2^{\text{2R}}(\boldsymbol{\theta}) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}.$$

Plugging in the given data for the 2R robot ($\boldsymbol{\theta} = (0, \pi/2)$ [rad], $L_1 = 1$ and $L_2 = 0.5$ [m]), and using eq. (1), we obtain

$$\begin{pmatrix} {}^0\boldsymbol{p}_3(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{T}_3^{\rm RPR}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = {}^w\boldsymbol{T}_0^{-1}\, {}^w\boldsymbol{T}_2^{\rm 2R}(0,\pi/2) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = {}^w\boldsymbol{T}_0^{-1} \begin{pmatrix} 1 \\ 0.5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2.1651 \\ -1.25 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{dx} \\ p_{dy} \\ 0 \\ 1 \end{pmatrix}.$$

2) Orientation in the plane of the approach vector ${}^0\boldsymbol{x}_3^{\rm RPR}$ of the gripper (see Fig. 8). The angle ϕ_d is extracted from

$$^w\boldsymbol{R}_0\, {}^0\boldsymbol{R}_3^{\rm RPR}(\boldsymbol{q}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = {}^w\boldsymbol{a}_d.$$

Using ${}^w\boldsymbol{a}_d = (0,-1,0)$, one has

$$^0\boldsymbol{x}_3(\boldsymbol{q}) = {}^0\boldsymbol{R}_3^{\rm RPR}(\boldsymbol{q}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = {}^w\boldsymbol{R}_0^T\, {}^w\boldsymbol{a}_d = \begin{pmatrix} -0.8660 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi_d \\ \sin\phi_d \\ 0 \end{pmatrix}.$$

Thus, the orientation angle of ${}^0\boldsymbol{x}_3$ is given by $\phi_d = {\rm ATAN2}\,\{-0.5, -0.8660\} = -5\pi/6 = -2.6180$ [rad].

The (two) configurations of the RPR robot that solve the task are obtained by substituting the obtained data p_{dx}, p_{dy}, and ϕ_d into eqs. (7), (9), and (10), using also $a_1 = a_3 = 1$ [m]. The solution with a positive value for the prismatic joint variable q_2 is

$$q_1 = 2.8241, \qquad q_2 = 1.1180, \qquad q_3 = 0.8411 \qquad \text{[rad,m,rad]}. \tag{11}$$

The configurations of the RPR robot and of the 2R robot associated to this solution is shown in Fig. 9.

Note finally that a direct computation of the solution angle for the third joint as

$$q_3 = \phi_d - q_1 = -2.6180 - 2.8241 = -5.4421 \text{ [rad]}$$

returns a value outside the interval $(-\pi,\pi]$. Instead, the correct value $q_3 \in (-\pi,\pi]$ in (11) is obtained, e.g., using the MATLAB function below.

```
% This function yields an angle diff in the interval (−π, π]
% from the difference between two angles th_d and th
% both defined in the interval (−π, π].

function diff=min_angle(th_d,th)
  n_d=[cos(th_d), sin(th_d), 0];
  n=[cos(th), sin(th), 0];
  n_d3=[n_d(1) n_d(2) 0]; n3=[n(1) n(2) 0];
  diff_abs=acos(n*n_d');  % always a positive angle between 0 and π
  diff_sign=cross(n,n_d);  % the third component gives the sign
  if diff_sign(3)>0,
     diff=diff_abs;
  else
     diff=-diff_abs;
  end
end
```

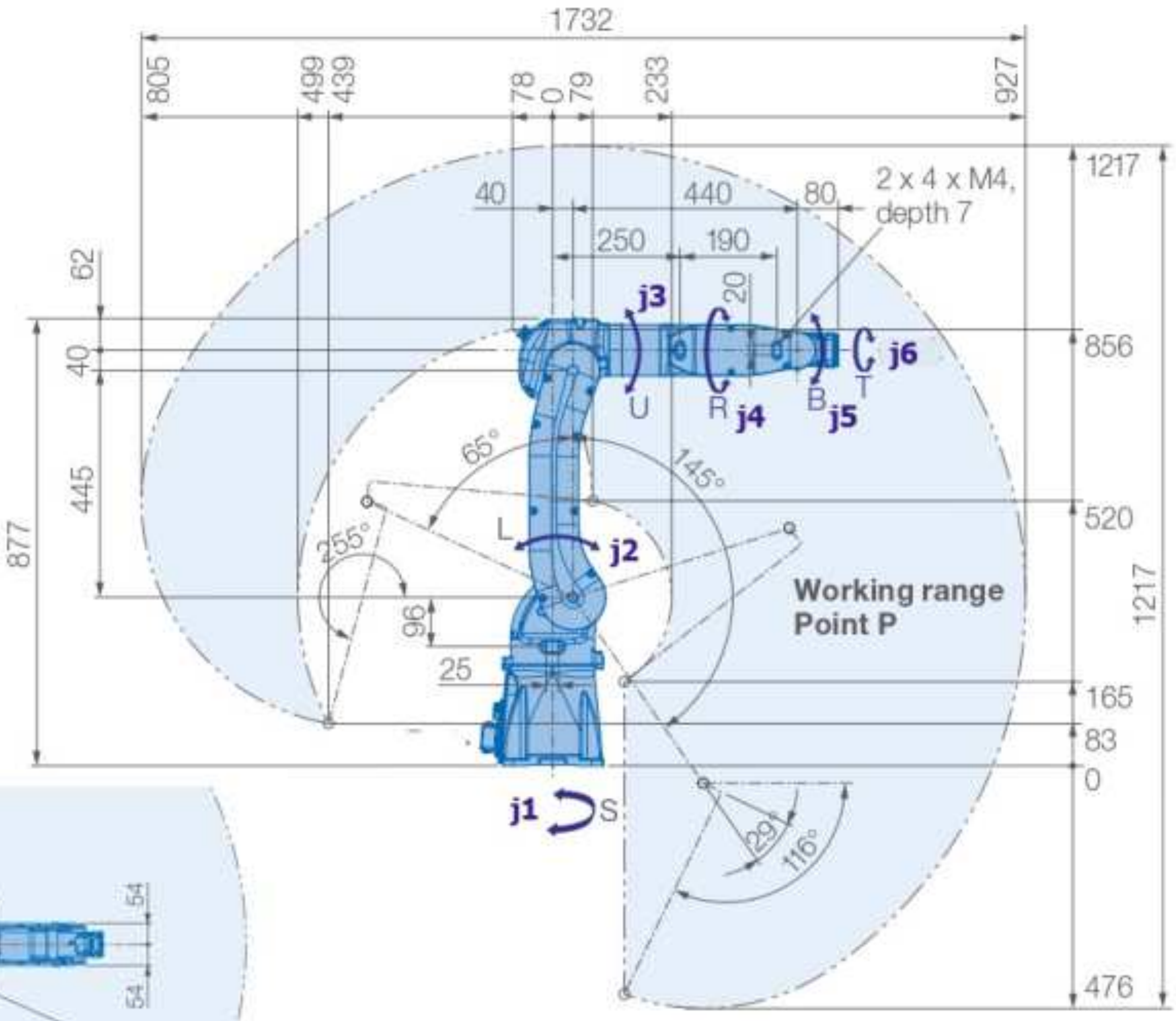
* * * * *

DH frames assignment and table for the YASKAWA Motoman GP7 robot

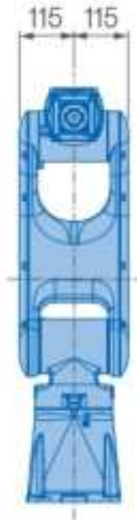
Name: ____________________________________

i	α_i	a_i	d_i	θ_i
1 (S)				
2 (L)				
3 (U)				
4 (R)				
5 (B)				
6 (T)				

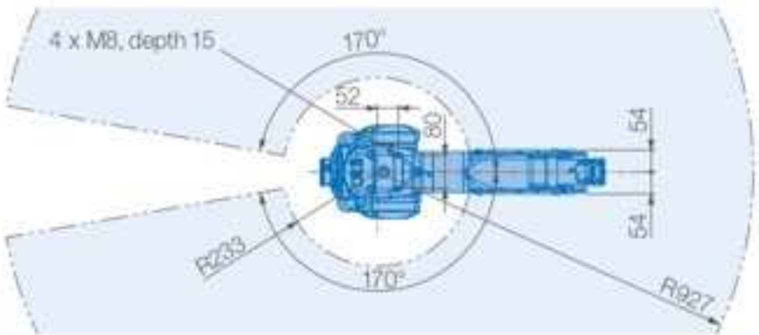
[lengths in mm]



front view



top view



side view

Robotics 1

January 24, 2024

Exercise 1

Consider a sequence of three rotations by the angles α, β, and γ around the fixed axes ZXY.

- Provide in symbolic form the rotation matrix $\boldsymbol{R}(\alpha, \beta, \gamma)$ representing the obtained final orientation.
- Solve the inverse problem in closed form for a generic $\boldsymbol{R} \in SO(3)$, including also singular situations.
- Compute all numerical solutions $\{\alpha, \beta, \gamma\}$ of the inverse problem (and check the results!) when

$$\boldsymbol{R} = \begin{pmatrix} \frac{5\sqrt{2}}{8} & \frac{\sqrt{2}}{8} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{6}}{8} & -\frac{3\sqrt{6}}{8} & \frac{1}{4} \end{pmatrix}. \tag{1}$$

- Express the orientation (1) with respect to the frame defined by a sequence of two rotations around the fixed axes ZX obtained using one pair $\{\alpha, \beta\}$ from the solution angles of the above inverse problem.

Exercise 2

The Yaskawa Motoman GP7 shown in Fig. 1 and in the accompanying Extra Sheet is a 6R robot manipulator with a spherical wrist and some offsets. The six joints are also labeled by the manufacturer (in sequence): S, L, U, R, B, T. Define a set of Denavit-Hartenberg (D-H) frames and compute the corresponding table of parameters. The D-H frame RF_0 should be placed on the floor at the robot base, while the origin O_6 of RF_6 is at the center of the end-effector flange and axis $\boldsymbol{z}_6$ is in the approach direction.

Draw the frames on the Extra Sheet, using the side, front, and top views (for better clarity, draw on each view only those DH axes that lie in the associated plane). Provide the numerical values of the constant parameters and the values of the joint variables $\boldsymbol{q}$ in the configuration shown in the sheet. Compute then numerically the position of O_6 in this configuration, as well as in the configuration $\boldsymbol{q} = \boldsymbol{0}$.

Figure 1: Three views of the Yaskawa Motoman GP7 robot.

Exercise 3

For which values of θ_2 in the interval $(-\pi, \pi]$ has this equation real solutions in terms of the angle θ_1?

$$\sin\theta_1 + 2\cos(\theta_1 + \theta_2) = 2 \tag{2}$$

Exercise 4

The kinematics of a 3R spatial robot is defined through the D-H parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	$\pi/2$	$a_3 > 0$	0	q_3

Table 1: D-H parameters of a 3R spatial robot.

- Determine the 6×3 geometric Jacobian matrix $\boldsymbol{J}_g(\boldsymbol{q})$ in symbolic form of this robot.
- Find at least one singularity of its linear part (i.e., of the upper 3×3 matrix $\boldsymbol{J}_L(\boldsymbol{q})$).
- Determine all feasible directions for the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ (using the lower 3×3 matrix $\boldsymbol{J}_A(\boldsymbol{q})$).
- For a point whose position ${}^3\boldsymbol{p}_D = \begin{pmatrix} 0 & 0 & D \end{pmatrix}^T$ is known and constant in the last D-H frame, compute the expression of its position ${}^0\boldsymbol{p}_D(\boldsymbol{q})$ in the base frame.
- Using the data $a_1 = a_3 = 0.04$, $a_2 = 0.445$, $d_1 = 0.33$, and $D = 0.52$ (all expressed in [m]), compute the velocity $\boldsymbol{v}_D = \dot{\boldsymbol{p}}_D \in \mathbb{R}^3$ in the base frame at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad] for $\dot{\boldsymbol{q}} = (0, \pi/4, \pi/2)$ [rad/s].

Exercise 5

Consider a 3R planar robot with links lengths $l_i > 0$ ($i = 1, 2, 3$) and D-H joint variables q_1 q_2, and q_3. For the task vector $\boldsymbol{r} = (\boldsymbol{p}, \alpha) = (p_x, p_y, \alpha) \in \mathbb{R}^3$, with $\boldsymbol{p}$ being the position of the end-effector and α its orientation angle with respect to the ${}^0\boldsymbol{x}$ axis, the following desired task trajectory $\boldsymbol{r}_d(t)$ is assigned:

$$\begin{aligned} p_{x,d}(t) &= x_0 + R\cos\alpha_d(t) \\ p_{y,d}(t) &= y_0 + R\sin\alpha_d(t) \\ \alpha_d(t) &= \omega\, t, \end{aligned} \tag{3}$$

with $R > 0$, $\omega > 0$, and $t \in [0, \infty)$.

- Determine the analytic expressions of the associated desired joint trajectory $\boldsymbol{q}_d(t)$, and of its velocity $\dot{\boldsymbol{q}}_d(t)$ and acceleration $\ddot{\boldsymbol{q}}_d(t)$.
- Using the data $l_1 = l_2 = l_3 = 1$ [m], $x_0 = y_0 = 1$ [m], $R = 0.5$ m, and $\omega = 2\pi$ rad/s, determine the numerical values of $\boldsymbol{q}_d(\bar{t})$, $\dot{\boldsymbol{q}}_d(\bar{t})$, and $\ddot{\boldsymbol{q}}_d(\bar{t})$ at time $\bar{t} = 0.25$ s.
- Check that the obtained results are consistent with those of the desired task trajectory $\boldsymbol{r}_d(t)$ and of its first and second derivatives at the same time instant.

Exercise 6

- Define a trajectory $q(t)$ for a robot joint that should start at rest from q_i at a given time t_i and arrive at time t_f in q_f with a final velocity $v_f \neq 0$. All the symbolic values are here generic.
- For a motion time $T = t_f - t_i$, find $v_{max} = \max_{t\in[t_i,t_f]} |\dot{q}(t)|$, i.e., the maximum absolute value of the joint velocity, and the instant of time $t^* \in [t_i, t_f]$ at which this value is attained.
- Compute the value of v_{max} (in [rad/s]) and the instant t^* (in [s]) for the following set of data: $t_i = 1.5$ s, $t_f = 2$ s, $q_i = \pi/2$ rad, $q_f = \pi$ rad, $v_f = -4$ rad/s.

[5 hours; open books]

Solution

January 24, 2024

Exercise 1

The orientation obtained with three rotations around the sequence of *fixed* axes ZXY with angles α, β, and γ is given by

$$
\begin{aligned}
\boldsymbol{R}_{ZXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_{\boldsymbol{y}}(\gamma)\boldsymbol{R}_{\boldsymbol{x}}(\beta)\boldsymbol{R}_{\boldsymbol{z}}(\alpha) \\
&= \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\beta s_\gamma \\ s_\alpha c_\beta & c_\alpha c_\beta & -s_\beta \\ s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma & c_\beta c_\gamma \end{pmatrix}.
\end{aligned} \tag{4}
$$

Given a matrix $\boldsymbol{R} \in SO(3)$, with elements denoted by R_{ij}, the inverse problem for this RPY-type sequence of angles is solved in closed form as follows. The second angle is obtained comparing the elements of the second row in (4) with those in $\boldsymbol{R}$:

$$
\beta^{+,-} = \text{ATAN2}\left\{-R_{23}, \pm\sqrt{R_{21}^2 + R_{22}^2}\right\} \tag{5}
$$

If $R_{21}^2 + R_{22}^2 = \cos^2\beta \neq 0$, we are in the regular case (two solution triples). The remaining angles are then found as

$$
\alpha^+ = \text{ATAN2}\left\{R_{21}, R_{22}\right\}, \qquad \gamma^+ = \text{ATAN2}\left\{R_{13}, R_{33}\right\}, \tag{6}
$$

when the $+$ sign ($\cos\beta > 0$) has been chosen in (5), and as

$$
\alpha^- = \text{ATAN2}\left\{-R_{21}, -R_{22}\right\}, \qquad \gamma^- = \text{ATAN2}\left\{-R_{13}, -R_{33}\right\}, \tag{7}
$$

when the $-$ sign ($\cos\beta < 0$) has been chosen in (5). From eqs. (5)–(7), the two solution triples are $\{\alpha^+, \beta^+, \gamma^+\}$ and $\{\alpha^-, \beta^-, \gamma^-\}$.

In the singular case, $R_{21} = R_{22} = \cos\beta = 0$, one has the identity[1]

$$
\begin{pmatrix} c_\alpha c_\gamma \pm s_\alpha s_\gamma & \pm c_\alpha s_\gamma - s_\alpha c_\gamma & 0 \\ 0 & 0 & -s_\beta \\ \pm s_\alpha c_\gamma - c_\alpha s_\gamma & s_\alpha s_\gamma \pm c_\alpha c_\gamma & 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha\mp\gamma) & -\sin(\alpha\mp\gamma) & 0 \\ 0 & 0 & -s_\beta \\ \pm\sin(\alpha\mp\gamma) & \pm\cos(\alpha\mp\gamma) & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 \\ 0 & 0 & \mp 1 \\ R_{31} & R_{32} & 0 \end{pmatrix}.
$$

If $R_{23} = -1$, then it is $\beta = \pi/2$ and one can solve only for the difference $\alpha - \gamma = \text{ATAN2}\left\{-R_{12}, R_{11}\right\}$. If instead $R_{23} = 1$, then it is $\beta = -\pi/2$ and one can solve only for the sum $\alpha + \gamma = \text{ATAN2}\left\{-R_{12}, R_{11}\right\}$.

Considering the rotation matrix $\boldsymbol{R}$ in (1), it is easy to see that we are in the regular case (with $c_\beta = 0.5$). Thus, the two solution triples are

$$
\{\alpha^+, \beta^+, \gamma^+\} = \{0.7854, -1.0472, -1.0472\} = \{\pi/4, -\pi/3, -\pi/3\} \text{ [rad]} \tag{8}
$$

and

$$
\{\alpha^-, \beta^-, \gamma^-\} = \{-2.3562, -2.0944, 2.0944\} = \{-3\pi/4, -2\pi/3, 2\pi/3\} \text{ [rad]}. \tag{9}
$$

When inserted in (4) as a check, both solutions return as expected the given $\boldsymbol{R}$.

The orientation obtained with two rotations by some angles α and β around the sequence of *fixed* axes ZX is given by $\boldsymbol{R}_{ZX} = \boldsymbol{R}_{\boldsymbol{x}}(\beta)\boldsymbol{R}_{\boldsymbol{z}}(\alpha)$. The expression of a generic orientation $\boldsymbol{R}$ with respect to such rotated

[1] Use of the $\mp$ signs: take always either the upper sign or the lower sign in *all* terms.

frame[2] is given then by ${}^{ZX}\boldsymbol{R} = \boldsymbol{R}_{ZX}^{T}\,\boldsymbol{R}$. For the matrix in (1), using the elementary rotation matrices in (4) and the values (α^{+}, β^{+}) from (8) and, respectively, (α^{-}, β^{-}) from (9) gives

$$
{}^{ZX}\boldsymbol{R}^{+} = \begin{pmatrix} 0.5625 & 0.8125 & -0.1531 \\ -0.6875 & 0.5625 & 0.4593 \\ 0.4593 & -0.1531 & 0.8750 \end{pmatrix} \quad \text{and} \quad {}^{ZX}\boldsymbol{R}^{-} = \begin{pmatrix} -0.3125 & -0.5625 & 0.7655 \\ 0.9375 & -0.3125 & 0.1531 \\ 0.1531 & 0.7655 & 0.6250 \end{pmatrix}.
$$

Exercise 2

Views of a possible assignment of D-H frames for the 6R Yaskawa robot of Fig. 1 are shown in Fig. 2.

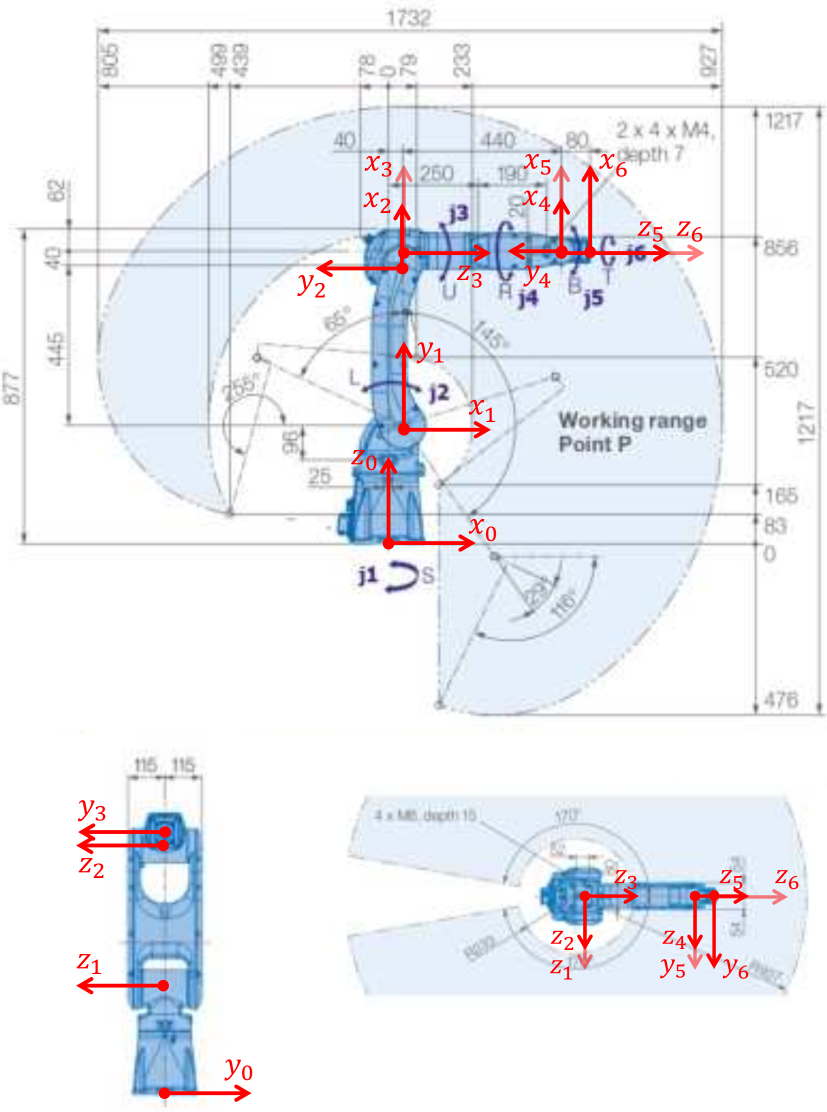


Figure 2: Side, front, and top views of the D-H frame assignment for the Yaskawa robot of Fig. 1.

[2] As usual, when no left superscript is present in a vector or matrix, the default is that this quantity is expressed in an absolute reference frame, say frame 0.

The corresponding D-H parameters are reported in Tab. 2. Note that the parameter d_1 is computed from the data sheet as follows $d_1 = 877 - (445 + 40 + 62) = 330$ mm.

i	α_i	a_i	d_i	θ_i
1 (S)	$\pi/2$	$a_1 = 40$	$d_1 = 330$	$q_1 = 0$
2 (L)	0	$a_2 = 445$	0	$q_2 = \pi/2$
3 (U)	$\pi/2$	$a_3 = 40$	0	$q_3 = 0$
4 (R)	$-\pi/2$	0	$d_4 = 440$	$q_4 = 0$
5 (B)	$\pi/2$	0	0	$q_5 = 0$
6 (T)	0	0	$d_6 = 80$	$q_6 = 0$

Table 2: Table of D-H parameters corresponding to the frames of Fig. 2 for the Yaskawa robot (angles are in [rad], lengths in [mm]). The numerical values of $\boldsymbol{q}$ refer to the configuration shown in the Extra Sheet.

The numerical values in the last column in the table correspond to the robot configuration $\boldsymbol{q}_s$ shown in Fig. 2. In this configuration, the position of the origin O_6 (as well as the orientation of the D-H frame 6 — which was not requested) are computed through the direct kinematics of the robot as

$$
{}^0\boldsymbol{p}_6(\boldsymbol{q}_s) = \begin{pmatrix} 560 \\ 0 \\ 815 \end{pmatrix} \text{[mm]}, \qquad {}^0\boldsymbol{R}_6(\boldsymbol{q}_s) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Similarly, in $\boldsymbol{q} = \boldsymbol{0}$ we have

$$
{}^0\boldsymbol{p}_6(\boldsymbol{0}) = \begin{pmatrix} 525 \\ 0 \\ -190 \end{pmatrix} \text{[mm]}, \qquad {}^0\boldsymbol{R}_6(\boldsymbol{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
$$

Exercise 3

Expand the cosine function in (2) to get

$$
\sin\theta_1 + 2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) = 2,
$$

which is of the form

$$
a\sin\theta_1 + b\cos\theta_1 = c, \tag{10}
$$

with

$$
a = 1 - 2\sin\theta_2, \qquad b = 2\cos\theta_2, \qquad c = 2.
$$

The transcendental eq. (10) has already been studied in the lecture slides (InverseKinematics.pdf, slide #13). From there, we know that this equation has (one or two) real solutions if and only if

$$
a^2 + b^2 \geq c^2 \qquad \Rightarrow \qquad (1 - 2\sin\theta_2)^2 + 4\cos^2\theta_2 \geq 4, \tag{11}
$$

or

$$
\sin\theta_2 \leq 0.25 \qquad \Rightarrow \qquad \theta_2 \in (-\pi, 0.2526] \cup [\pi - 0.2526, \pi] \text{ rad}. \tag{12}
$$

Under the condition (11), viz. (12), the solutions to (10) are computed as

$$
\theta_1^{+/-} = 2\arctan\frac{a \pm \sqrt{a^2 + b^2 - c^2}}{b + c} = 2\arctan\frac{1 - 2\sin\theta_2 \pm \sqrt{1 - 4\sin\theta_2}}{2(1 + \cos\theta_2)}. \tag{13}
$$

For instance, when $\theta_2 = 0$, eq. (2) becomes

$$\sin\theta_1 + 2\cos\theta_1 = 2,$$

which has the two real solutions

$$\begin{aligned}
\theta_1^+ &= 2\arctan\left(\frac{1+1}{4}\right) = 2\arctan 0.5 = 0.9273 \text{ rad},\\
\theta_1^- &= 2\arctan\left(\frac{1-1}{4}\right) = 2\arctan 0 = 0.
\end{aligned}$$

On the other hand, eq. (2) has a single solution when $\sin\theta_2 = 0.25$. In particular, for $\theta_2 = 0.2526$, the equation becomes

$$0.5\sin\theta_1 + 1.9365\cos\theta_1 = 2,$$

and has the single solution

$$\theta_1 = 0.2499 \text{ rad};$$

similarly, for $\theta_2 = \pi - 0.2526$, the equation becomes

$$0.5\sin\theta_1 - 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = \pi - 0.2499 = 2.8917 \text{ rad}.$$

Exercise 4

From Tab. 1, we compute the D-H homogeneous transformation matrices of this 3R spatial robot:

$$^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & a_1c_1\\ s_1 & 0 & -c_1 & a_1s_1\\ 0 & 1 & 0 & d_1\\ 0 & 0 & 0 & 1 \end{pmatrix},\quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2\\ s_2 & c_2 & 0 & a_2s_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},\quad {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & s_3 & a_3c_3\\ s_3 & 0 & -c_3 & a_3s_3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From these, being the end-effector position coincident with the origin O_3 of frame 3, we obtain all the quantities needed for computing the geometric Jacobian $\boldsymbol{J}_g(\boldsymbol{q})$ as

$$\boldsymbol{J}_g(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q})\\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{03} & \boldsymbol{z}_1 \times \boldsymbol{p}_{13} & \boldsymbol{z}_2 \times \boldsymbol{p}_{23}\\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix}, \tag{14}$$

with all quantities being expressed by default in frame 0. For better clarity, we will insert left superscripts in the following. We have

$$\boldsymbol{z}_0 = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix},\quad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 = \begin{pmatrix} s_1\\ -c_1\\ 0 \end{pmatrix},\quad \boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0 = \begin{pmatrix} s_1\\ -c_1\\ 0 \end{pmatrix}, \tag{15}$$

and for the position vectors

$$^0\boldsymbol{p}_{01,h}(\boldsymbol{q}) = \begin{pmatrix} ^0\boldsymbol{p}_{01}(\boldsymbol{q})\\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix} \boldsymbol{0}\\ 1 \end{pmatrix} = \begin{pmatrix} a_1c_1\\ a_1s_1\\ d_1\\ 1 \end{pmatrix},$$

$$^0\boldsymbol{p}_{02,h}(\boldsymbol{q}) = \begin{pmatrix} ^0\boldsymbol{p}_{02}(\boldsymbol{q})\\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\begin{pmatrix} \boldsymbol{0}\\ 1 \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_1c_2\\ a_1s_1 + a_2s_1c_2\\ d_1 + a_2s_2\\ 1 \end{pmatrix},$$

$$
{}^0\boldsymbol{p}_{03,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\, {}^1\boldsymbol{A}_2(q_2)\, {}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(a_1 + a_2c_2 + a_3c_{23}) \\ s_1(a_1 + a_2c_2 + a_3c_{23}) \\ d_1 + a_2s_2 + a_3s_{23} \\ 1 \end{pmatrix}. \tag{16}
$$

From these, we get

$$
{}^0\boldsymbol{p}_{13}(\boldsymbol{q}) = {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) - {}^0\boldsymbol{p}_{01}(\boldsymbol{q}) = \begin{pmatrix} c_1(a_2c_2 + a_3c_{23}) \\ s_1(a_2c_2 + a_3c_{23}) \\ a_2s_2 + a_3s_{23} \end{pmatrix}, \quad {}^0\boldsymbol{p}_{23}(\boldsymbol{q}) = {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) - {}^0\boldsymbol{p}_{02}(\boldsymbol{q}) = \begin{pmatrix} a_3c_1c_{23} \\ a_3s_1c_{23} \\ a_3s_{23} \end{pmatrix}.
$$

Performing now the cross products in (14), we obtain

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(a_1 + a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_1 + a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}. \tag{17}
$$

Indeed, this matrix could have been equivalently obtained by analytic differentiation as $\boldsymbol{J}_L(\boldsymbol{q}) = \partial \boldsymbol{p}_{03}/\partial \boldsymbol{q}$. Moreover, from (15)

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{18}
$$

The determinant of the linear part of the Jacobian is

$$
\det \boldsymbol{J}_L(\boldsymbol{q}) = -a_2a_3s_3\,(a_1 + a_2c_2 + a_3c_{23})\,.
$$

Thus, matrix $\boldsymbol{J}_L(\boldsymbol{q})$ in (17) is singular when $s_3 = 0$ ($q_3 = 0$ or π) and/or when $a_1 + a_2c_2 + a_3c_{23} = 0$. In view of the expressions in (16), the latter corresponds to $p_{03,x} = p_{03,y} = 0$, namely to a situation in which the origin O_3 of frame 3 is placed on the axis $\boldsymbol{z}_0$ of the first joint.

On the other hand, matrix $\boldsymbol{J}_A(\boldsymbol{q})$ in (18) is always singular, with constant rank $\rho = 2$. Being $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$, all feasible directions at $\boldsymbol{q}$ for the angular velocity $\boldsymbol{\omega}$ of the third (last) D-H frame belong to the subspace

$$
\mathcal{R}\left\{\boldsymbol{J}_A(\boldsymbol{q})\right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}.
$$

Finally, the position ${}^3\boldsymbol{p}_D = \begin{pmatrix} 0 & 0 & D \end{pmatrix}^T$ of a fixed point in the third D-H frame is a (constant) vector from the origin O_3 to this point. To compute the vector ${}^0\boldsymbol{p}_D$ starting from the origin O_0 of the base frame and ending at the same point, and expressed in the 0-th frame, we resort to homogeneous transformations:

$$
{}^0\boldsymbol{p}_{D,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_D(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\, {}^1\boldsymbol{A}_2(q_2)\, {}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} {}^3\boldsymbol{p}_D \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(a_1 + a_2c_2 + a_3c_{23} + D\,s_{23}) \\ s_1(a_1 + a_2c_2 + a_3c_{23} + D\,s_{23}) \\ d_1 + a_2s_2 + a_3s_{23} - D\,c_{23} \\ 1 \end{pmatrix}. \tag{19}
$$

Using the kinematic data of the robot at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad] and $D = 0.52$ m, we obtain

$$
{}^0\boldsymbol{p}_{03} = \begin{pmatrix} 0.04 \\ 0 \\ 0.815 \end{pmatrix} \qquad \text{and} \qquad {}^0\boldsymbol{p}_D = \begin{pmatrix} 0.56 \\ 0 \\ 0.815 \end{pmatrix} \quad [\text{m}].
$$

The velocity of the point defined in (19) is easily obtained by time differentiation

$$
\begin{aligned}
\boldsymbol{v}_D = {}^0\dot{\boldsymbol{p}}_D(\boldsymbol{q}) &= \frac{\partial\, {}^0\boldsymbol{p}_D}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}_D(\boldsymbol{q})\,\dot{\boldsymbol{q}} \\
&= \begin{pmatrix} -s_1(a_1+a_2c_2+a_3c_{23}+D\,s_{23}) & -c_1(a_2s_2+a_3s_{23}+D\,c_{23}) & -a_3c_1s_{23}+D\,c_{23} \\ c_1(a_1+a_2c_2+a_3c_{23}+D\,s_{23}) & -s_1(a_2s_2+a_3s_{23}+D\,c_{23}) & -a_3s_1s_{23}+D\,c_{23} \\ 0 & a_2c_2+a_3c_{23}+D\,s_{23} & a_3c_{23}+D\,s_{23} \end{pmatrix} \dot{\boldsymbol{q}}.
\end{aligned} \tag{20}
$$

Plugging in the robot data, evaluating the Jacobian $\boldsymbol{J}_D(\boldsymbol{q})$ at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad], and commanding the joint velocity $\dot{\boldsymbol{q}} = (0, \pi/4, \pi/2)$ [rad/s], we obtain from eq. (20)

$$
\boldsymbol{v}_D = \begin{pmatrix} 0 & -0.4850 & -0.0400 \\ 0.5600 & 0 & 0 \\ 0 & 0.5200 & 0.5200 \end{pmatrix} \begin{pmatrix} 0 \\ 0.7854 \\ 1.5708 \end{pmatrix} = \begin{pmatrix} -0.4437 \\ 0 \\ 1.2252 \end{pmatrix} \text{ [m/s]}.
$$

Exercise 5

In the first place, we have to solve the inverse kinematics problem for this 3R planar robot, repeatedly and parametrically with respect to the desired trajectory $\boldsymbol{r}_d(t) = (p_{x,d}(t), p_{y,d}(t), \alpha_d(t))$. This is a standard problem at each instant of time t (dropped for compactness in the following).

The direct task kinematics is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \\ q_1+q_2+q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{21}
$$

Setting $\boldsymbol{r} = \boldsymbol{r}_d$ in (21) and using the third equation in the first two, one has

$$
\begin{pmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \end{pmatrix} = \begin{pmatrix} p_{x,d} - l_3\, c\alpha_d \\ p_{y,d} - l_3\, s\alpha_d \end{pmatrix}, \tag{22}
$$

with the shorthand notations $c_1 = \cos q_1$, $c_{12} = \cos(q_1+q_2)$, $c\alpha_d = \cos\alpha_d$ — similarly for the sines. By squaring and summing the two equations in (22), we get

$$
c_{2,d} = \frac{p_{x,d}^2 + p_{y,d}^2 + l_3^2 - 2l_3(p_{x,d}\, c\alpha_d + p_{y,d}\, s\alpha_d) - l_1^2 - l_2^2}{2l_1l_2}, \qquad s_{2,d} = \sqrt{1 - c_{2,d}^2}, \tag{23}
$$

where only the + sign has been considered for $s_{2,d}$ (similar developments hold for the choice $s_{2,d} < 0$). Thus,

$$
q_{2,d} = \text{ATAN2}\left\{s_{2,d}, c_{2,d}\right\}. \tag{24}
$$

This corresponds to an 'elbow down' solution for the first two joints. Using the expressions in (23), eq. (22) is expanded as a linear system in the remaining unknowns c_1 and s_1:

$$
\begin{pmatrix} l_1 + l_2\, c_{2,d} & -l_2\, s_{2,d} \\ l_2\, s_{2,d} & l_1 + l_2\, c_{2,d} \end{pmatrix} \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} p_{x,d} - l_3\, c\alpha_d \\ p_{y,d} - l_3\, s\alpha_d \end{pmatrix}. \tag{25}
$$

Unless the determinant of the coefficient matrix in (25) is zero, namely excluding when $l_1^2 + l_2^2 + 2l_1l_2c_{2,d} = 0$ (which happens if and only if $l_1 = l_2$ and $c_{2,d} = -\pi$, being the determinant always positive otherwise), one can solve for

$$
\begin{aligned}
c_{1,d} &= (l_1 + l_2c_{2,d})(p_{x,d} - l_3\, c\alpha_d) + l_2s_{2,d}(p_{y,d} - l_3\, s\alpha_d) \\
s_{1,d} &= (l_1 + l_2c_{2,d})(p_{y,d} - l_3\, s\alpha_d) - l_2s_{2,d}(p_{x,d} - l_3\, c\alpha_d),
\end{aligned} \tag{26}
$$

and then

$$
q_{1,d} = \text{ATAN2}\left\{s_{1,d}, c_{1,d}\right\}. \tag{27}
$$

Finally,

$$q_{3,d} = \alpha_d - (q_{1,d} + q_{2,d}). \tag{28}$$

The analytic expressions of the components of the desired joint trajectory $\boldsymbol{q}_d(t)$ at any instant of time are obtained from eqs. (23)–(24), (26)–(27) and (28), by plugging the desired task trajectory values from (3).

Moving to the differential level, the task Jacobian associated to (21) is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & -(l_2 s_{12} + l_3 s_{123}) & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix}. \tag{29}$$

Therefore, the joint velocity $\dot{\boldsymbol{q}}_d(t)$ along the desired task trajectory is computed as

$$\dot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{r}}_d(t), \qquad \text{with } \dot{\boldsymbol{r}}_d(t) = \begin{pmatrix} -\omega R \sin \omega t \\ \omega R \cos \omega t \\ \omega \end{pmatrix}, \tag{30}$$

where the Jacobian (29) is first evaluated at $\boldsymbol{q}_d(t)$ and then inverted numerically, provided it is away from its singularities ($\det \boldsymbol{J}(\boldsymbol{q}) = l_1 l_2 \sin q_2 = 0$).

Similarly, at the acceleration level one has

$$\ddot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t)) \left(\ddot{\boldsymbol{r}}_d(t) - \dot{\boldsymbol{J}}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{q}}_d(t) \right), \qquad \text{with } \ddot{\boldsymbol{r}}_d(t) = -\begin{pmatrix} \omega^2 R \cos \omega t \\ \omega^2 R \sin \omega t \\ 0 \end{pmatrix}, \tag{31}$$

where the term

$$\begin{aligned} \dot{\boldsymbol{J}}(\boldsymbol{q})\, \dot{\boldsymbol{q}} &= -\begin{pmatrix} l_1 c_1 \dot{q}_1 + l_2 c_{12}(\dot{q}_1+\dot{q}_2) + l_3 c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & l_2 c_{12}(\dot{q}_1+\dot{q}_2) + l_3 c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & l_3 c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) \\ l_1 s_1 \dot{q}_1 + l_2 s_{12}(\dot{q}_1+\dot{q}_2) + l_3 s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & l_2 s_{12}(\dot{q}_1+\dot{q}_2) + l_3 s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & l_3 s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) \\ 0 & 0 & 0 \end{pmatrix} \dot{\boldsymbol{q}} \\ &= -\begin{pmatrix} l_1 c_1 \dot{q}_1^2 + l_2 c_{12}(\dot{q}_1 + \dot{q}_2)^2 + l_3 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ l_1 s_1 \dot{q}_1^2 + l_2 s_{12}(\dot{q}_1 + \dot{q}_2)^2 + l_3 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ 0 \end{pmatrix} \end{aligned}$$

is evaluated using $\boldsymbol{q}_d(t)$, as well as $\dot{\boldsymbol{q}}_d(t)$ from (30).

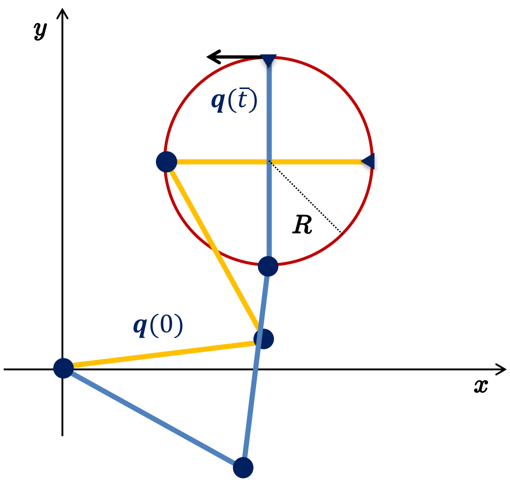


Figure 3: The 3R planar robot executing the desired task trajectory at $t = \bar{t} = 0.25$ s (in blue) and in the initial configuration $\boldsymbol{q}_d(0)$ (in orange).

With the problem data, being for the desired task trajectory at $t = \bar{t} = 0.25$ s

$$\boldsymbol{r}_d(\bar{t}) = \begin{pmatrix} 1 \\ 1.5 \\ \pi/2 \end{pmatrix} \text{[m,m,rad]}, \quad \dot{\boldsymbol{r}}_d(\bar{t}) = \begin{pmatrix} -\pi \\ 0 \\ 2\pi \end{pmatrix} \text{[m/s,m/s,rad/s]}, \quad \ddot{\boldsymbol{r}}_d(\bar{t}) = \begin{pmatrix} 0 \\ -2\pi^2 \\ 0 \end{pmatrix} \text{[m/s}^2\text{,m/s}^2\text{,rad/s}^2\text{]},$$

we obtain the following numerical values

$$\boldsymbol{q}_d(\bar{t}) = \begin{pmatrix} -0.5139 \\ 1.9552 \\ 0.1296 \end{pmatrix} \text{[rad]}, \qquad \dot{\boldsymbol{q}}_d(\bar{t}) = \begin{pmatrix} 0.4378 \\ -3.3889 \\ 9.2343 \end{pmatrix} \text{[rad/s]}, \qquad \ddot{\boldsymbol{q}}_d(\bar{t}) = \begin{pmatrix} 30.4316 \\ -16.6473 \\ -13.7843 \end{pmatrix} \text{[rad/s}^2\text{]}.$$

Figure 3 shows the robot configuration $\boldsymbol{q}_d(\bar{t})$ at the chosen time along the desired circular path, together with the initial (elbow down) configuration $\boldsymbol{q}_d(0) = (0.1296, 1.9552, -2.0847)$ [rad].

Exercise 6

The desired trajectory is found by solving a 'rest-to-move' interpolation problem between two configurations assigned at two given time instants. A cubic polynomial of the form $q(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ has enough free coefficients to satisfy all boundary conditions at the initial and final time. As customary, however, it is more convenient to use a normalized time

$$\tau = \frac{t - t_i}{t_f - t_i} = \frac{t - t_i}{T}, \qquad \text{with } \tau \in [0,1] \text{ when } t \in [t_i, t_f],$$

and define the cubic as

$$q(\tau) = q_i + \Delta q \left(a\tau^2 + b\tau^3 \right), \qquad \Delta q = q_f - q_i, \tag{32}$$

which already satisfies the boundary conditions at the initial time $\tau = 0$ ($t = t_i$) on position and (zero) velocity. Imposing the other two boundary conditions on the normalized cubic polynomial (32)

$$\begin{array}{l} q(t_f) = q_f \\ \dot{q}(t_f) = v_f \end{array} \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ v_f T/\Delta q \end{pmatrix},$$

and solving for (a, b) leads to the planned trajectory

$$q(\tau) = q_i + \Delta q \left((3 - \frac{v_f T}{\Delta q})\, \tau^2 + (-2 + \frac{v_f T}{\Delta q})\, \tau^3 \right), \tag{33}$$

with velocity and acceleration given respectively by

$$\dot{q}(\tau) = \frac{\Delta q}{T} \left(2\,(3 - \frac{v_f T}{\Delta q})\, \tau + 3\,(-2 + \frac{v_f T}{\Delta q})\, \tau^2 \right) \tag{34}$$

and

$$\ddot{q}(\tau) = \frac{\Delta q}{T^2} \left(2\,(3 - \frac{v_f T}{\Delta q}) + 6\,(-2 + \frac{v_f T}{\Delta q})\, \tau \right). \tag{35}$$

The maximum velocity (in absolute value) is reached either at the final instant $\tau^* = 1$, being $v_{max} = |v_f|$, or when the acceleration is zero, i.e.,

$$\ddot{q}(\tau^*) = 0 \qquad \Rightarrow \qquad \tau^* = \frac{3 - \dfrac{v_f T}{\Delta q}}{6 - 3\,\dfrac{v_f T}{\Delta q}} \tag{36}$$

as long $\tau^* \in (0, 1)$. Since $T = t_f - t_i > 0$, this condition is *always* satisfied if $v_f/\Delta q \leq 0$, namely when the displacement Δq and the final velocity v_f have opposite signs[3] (in particular, when $v_f = 0$, it is $\tau^* = 0.5$

[3] A solution to $\ddot{q}(\tau) = 0$ inside the interval of definition for τ exists also when $v_f/\Delta q > 0$, provided v_f is not too large in modulus. The details are left to the reader.

— at the trajectory midpoint). The velocity associated to the zero acceleration condition is found by substituting τ^* from (36) in (34), obtaining

$$\dot{q}(\tau^*) = \frac{\Delta q}{T}\,\tau^* \left((6 - 2\frac{v_f T}{\Delta q}) - (6 - 3\frac{v_f T}{\Delta q})\,\tau^* \right) = \frac{\Delta q}{T}\,\frac{\left(3 - \dfrac{v_f T}{\Delta q}\right)^2}{6 - 3\,\dfrac{v_f T}{\Delta q}}. \tag{37}$$

As a result, the maximum absolute velocity will be

$$v_{max} = \max\left\{|\dot{q}(\tau^*)|, |v_f|\right\}.$$

Note that the actual time instant of maximum velocity will be $t^* = t_i + \tau^* T \in [t_i, t_f]$.

With the data of the problem, one has $T = t_f - t_i = 2 - 1.5 = 0.5$ s, $\Delta q = q_f - q_i = \pi - \pi/2 = \pi/2$ rad, and $v_f = -4$ rad/s. Being $v_f/\Delta q < 0$, the above analysis applies and we obtain

$$\tau^* = 0.4352, \qquad t^* = 1.7176\,\text{s}, \qquad v_{max} = 5.8421\,\text{rad/s}.$$

The resulting cubic trajectory is shown in Fig. 4, together with the associated velocity and acceleration. Note the asymmetric profiles with respect to the middle instant of the motion trajectory, as well as the slight overshoot in position close to the final instant.

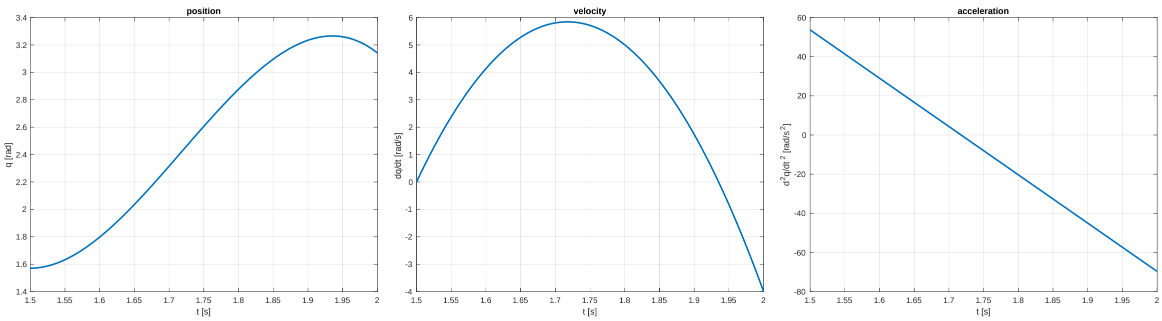


Figure 4: The planned cubic trajectory for the given data (position, velocity, and acceleration).

* * * * *

Robotics 1

February 16, 2024

Exercise 1

With reference to Fig. 1, a PPR planar robot with a third link of length ℓ has to pick up with its end-effector gripper an object that is localized by the views of two co-planar cameras C_1 and C_2. Although living in 3D, this visual localization problem can be essentially restricted to the plane $(\boldsymbol{x}_b, \boldsymbol{y}_b)$ of the reference frame placed at the robot base. Each camera $(i = 1, 2)$ is represented by a pinhole model with focal length f_i and a reference frame placed on its image plane, having the $\boldsymbol{z}_{ci}$ axis along the optical axis and the $\boldsymbol{x}_{ci}$ axis in the same plane $(\boldsymbol{x}_b, \boldsymbol{y}_b)$. The pose of these two camera frames is known with respect to the base frame of the robot, using the geometric parameters L, H, α_1, and α_2 defined in the figure.

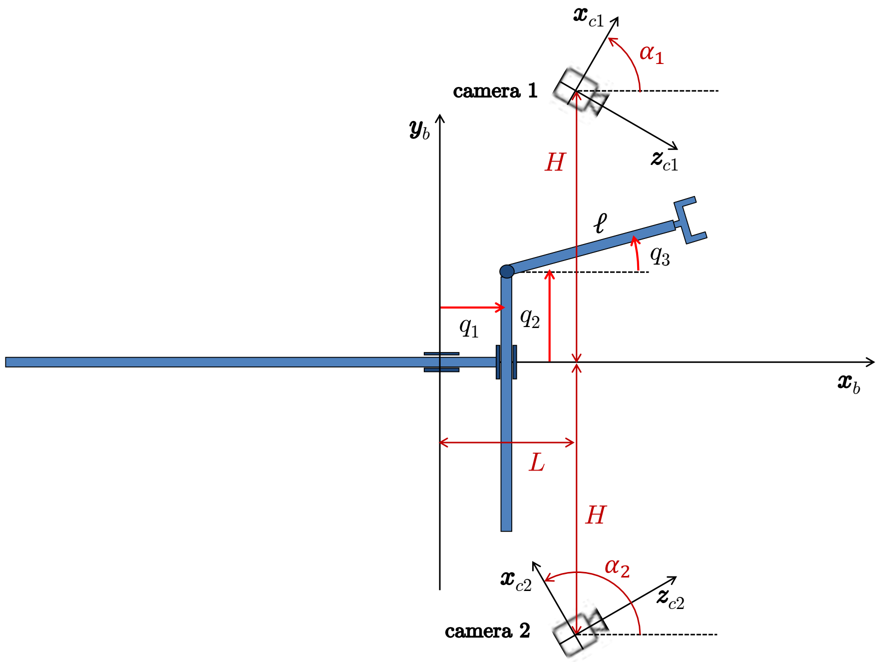


Figure 1: The planar setup of the visual localization problem, with two cameras and a PPR robot.

A point feature P on the object is seen in the image plane of camera C_1 at a signed distance d_1 along its $\boldsymbol{x}_{c1}$ axis, while the same feature is seen by camera C_2 at a signed distance d_2 along its $\boldsymbol{x}_{c2}$ axis.[1] Both d_1 and d_2 are expressed in metric units (i.e., neglecting pixelation).

a) Find the position ${}^{ci}\boldsymbol{p}$ of the point feature P in one of the camera frames — any, $i = 1$ or 2 (or in both).
b) Determine the position of P as ${}^{b}\boldsymbol{p}$, i.e., with respect to the reference frame at the base of the robot.
c) Provide the numerical value of ${}^{b}\boldsymbol{p}$ when using the following data: $\ell = 0.5$, $L = 0.4$, and $H = 0.8$ [m]; $\alpha_1 = \pi/3$ and $\alpha_2 = 2\pi/3$ [rad]; $f_1 = 10$ and $f_2 = 12$ [mm]; $d_1 = 6$ and $d_2 = -2$ [mm].
d) Find a closed form solution of the inverse kinematics problem for the PPR robot, when the planar position and orientation of its end-effector are given. Is the solution unique?
e) Provide a numerical value of the joint variable vector $\boldsymbol{q} = (q_1, q_2, q_3)$ for the position ${}^{b}\boldsymbol{p}$ and the data in item c), when the robot end-effector points toward the origin of the camera frame in C_2.

[1]We assume that the correspondence problem between the two images has been solved already.

Exercise 2

A 3-dof cylindrical robot is shown in Fig. 2.

a) Assign the frames according to the Denavit-Hartenberg (D-H) convention, so that all constant D-H parameters are non-negative. Provide the table with the corresponding parameters. The first frame should be placed on the floor at the robot base and the origin of the last frame should be at point P. Compute position and orientation of the end-effector frame as given by the homogeneous matrix ${}^0\boldsymbol{T}_3(\boldsymbol{q})$.

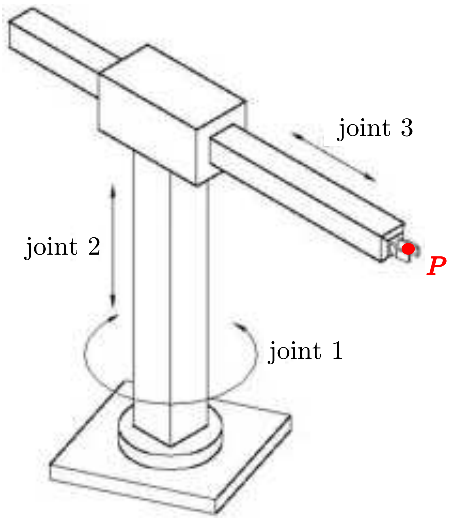


Figure 2: A 3-dof cylindrical robot.

b) For a desired position $\boldsymbol{p}_d \in \mathbb{R}^3$ of the end-effector, find all solutions to the inverse kinematics problem.

c) Sketch the primary workspace of this robot, when $q_1 \in [-5\pi/6, 5\pi/6]$, $q_2 \in [0, H]$, and $q_3 \in [D_{min}, D_{max}]$, with $D_{min} > 0$. How many inverse kinematics solutions may comply with these bounds?

d) Compute the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating $\dot{\boldsymbol{q}}$ to the velocity $\boldsymbol{v}$ of point P and find its singularities.

e) When the Jacobian is evaluated at a generic singular configuration $\boldsymbol{q}_s$, find a basis for the two subspaces $\mathcal{N}(\boldsymbol{J}_s)$ and $\mathcal{N}(\boldsymbol{J}_s^T)$, where $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$. Provide a physical interpretation of the vectors belonging to each of these two subspaces.

Exercise 3

Consider the following trajectory planning problem for a 2R planar robot with links of length $L_1 = 2$ and $L_2 = 1$ [m]. The robot should perform a rest-to-rest motion in a given time $T = 2$ s, moving its end-effector from point $A = (0, 1)$ to point $B = (3, 0)$ [m]. The initial direction of the end-effector motion from point A is specified by the tangent vector in Cartesian space $d\boldsymbol{p}/ds|_A = (5, 0)$, where s is a suitable scalar that parametrizes the path. Similarly, the final approach direction to point B is specified by $d\boldsymbol{p}/ds|_B = (0, -1)$.

a) Design a joint trajectory that solves this problem, providing the analytic expression of the various terms in the complete solution and some plots that help illustrating it.

b) Suppose now that the joint velocities are limited: $|\dot{q}_1| \leq V_1 = 2$, $|\dot{q}_2| \leq V_2 = 3$ [rad/s]. Verify whether the trajectory that has been planned in item a) is feasible. If this is not the case, determine a convenient, possibly minimum, motion time T^* that solves the same problem and complies also with these bounds.

Exercise 4

The kinematics of a 3R spatial robot is specified through the D-H parameters given in Tab. 1. The robot has its base mounted on the floor, defined by the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0.7	q_1
2	0	0.5	0	q_2
3	0	0.5	0	q_3

Table 1: D-H parameters of a 3R spatial robot (lengths are in [m]).

The desired trajectory of the robot end-effector is an helical path to be traced, for $t \in [0, 3]$, with constant parametric speed $\dot{s} = v$:

$$\begin{aligned} p_{x,d}(t) &= r \cos 2\pi s(t) \\ p_{y,d}(t) &= r \sin 2\pi s(t) \\ p_{z,d}(t) &= h_0 + h s(t), \end{aligned} \tag{1}$$

with $r = 0.5$, $h_0 = 0.2$, and $h = 0.4$ [m], and $v = 1$ [s^{-1}]. The robot is commanded by the joint velocity vector $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and has unlimited joint rotations.

a) Show that the entire desired trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^3$ belongs to the primary workspace of the robot and that it can be traced without encountering any singular configuration.

b) With the robot starting at $t = 0$ from the initial configuration $\boldsymbol{q}(0) = (0, \pi/6, -\pi/2)$ [rad], verify whether the end-effector is on the desired trajectory or not.

c) Design a kinematic control law that achieves tracking of the desired end-effector trajectory, with the possible position error $\boldsymbol{e} \in \mathbb{R}^3$ decaying exponentially, so that the error components $e_i(t)$, $i = t, n, b$, are decoupled in the Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ associated to the desired position along the path.

d) Wishing to impose the error dynamics $\dot{e}_t = -2e_t$, $\dot{e}_n = -5e_n$, $\dot{e}_b = -5e_b$, compute at $t = 0$ the numerical value of the command $\dot{\boldsymbol{q}}(0)$ when the robot starts from the configuration of item b) and the control law in item c) is being used.

[5 hours; open books]

Solution

February 16, 2024

Exercise 1

We first define the relationships between the different frames of interest. The two camera frames are related to the base frame of the robot, respectively by

$$
{}^{b}\boldsymbol{T}_{c1} = \begin{pmatrix} \cos\alpha_1 & 0 & \sin\alpha_1 & L \\ \sin\alpha_1 & 0 & -\cos\alpha_1 & H \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^{b}\boldsymbol{T}_{c2} = \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 & L \\ \sin\alpha_2 & 0 & -\cos\alpha_2 & -H \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}
$$

so that the pose of the second camera is expressed in the first camera frame by

$$
{}^{c1}\boldsymbol{T}_{c2} = {}^{b}\boldsymbol{T}_{c1}^{-1}\,{}^{b}\boldsymbol{T}_{c2} = \begin{pmatrix} \cos(\alpha_1-\alpha_2) & 0 & -\sin(\alpha_1-\alpha_2) & -2H\sin\alpha_1 \\ 0 & 1 & 0 & 0 \\ \sin(\alpha_1-\alpha_2) & 0 & \cos(\alpha_1-\alpha_2) & 2H\cos\alpha_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Note that the parameter L disappears from this matrix.

Let ${}^{ci}P = (X_i, Z_i)$ be the coordinates (in the plane) of the feature point P as seen in the i-th camera frame, for $i = 1, 2$. Using the pinhole model, and taking into account that the camera frames are placed on their respective image plane, we have the perspective relations

$$
\frac{f_i}{d_i} = -\frac{Z_i - f_i}{X_i} \qquad \Rightarrow \qquad f_i X_i + d_i Z_i = d_i f_i, \qquad i = 1, 2. \tag{3}
$$

Embedding the point P in homogeneous coordinates, we have

$$
{}^{c1}\boldsymbol{p}_{hom} = \begin{pmatrix} {}^{c1}\boldsymbol{p} \\ 1 \end{pmatrix} = \begin{pmatrix} X_1 \\ 0 \\ Z_1 \\ 1 \end{pmatrix} \qquad {}^{c2}\boldsymbol{p}_{hom} = \begin{pmatrix} {}^{c2}\boldsymbol{p} \\ 1 \end{pmatrix} = \begin{pmatrix} X_2 \\ 0 \\ Z_2 \\ 1 \end{pmatrix}.
$$

Thus, from

$$
{}^{c1}\boldsymbol{p}_{hom} = {}^{c1}\boldsymbol{T}_{c2}\,{}^{c2}\boldsymbol{p}_{hom}
$$

one has in particular

$$
\begin{aligned} X_1 &= X_2\cos(\alpha_1-\alpha_2) - Z_2\sin(\alpha_1-\alpha_2) - 2H\sin\alpha_1 \\ Z_1 &= X_2\sin(\alpha_1-\alpha_2) + Z_2\cos(\alpha_1-\alpha_2) + 2H\cos\alpha_1. \end{aligned} \tag{4}
$$

Substituting (4) in the first of the two perspective relations (3), we obtain the following 2×2 system of linear equations in the unknowns X_2 and Z_2:

$$
\boldsymbol{A}\begin{pmatrix} X_2 \\ Z_2 \end{pmatrix} = \boldsymbol{b}, \tag{5}
$$

with

$$
\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} f_1\cos(\alpha_1-\alpha_2) + d_1\sin(\alpha_1-\alpha_2) & -f_1\sin(\alpha_1-\alpha_2) + d_1\cos(\alpha_1-\alpha_2) \\ f_2 & d_2 \end{pmatrix} \\ \boldsymbol{b} &= \begin{pmatrix} d_1 f_1 + 2H(f_1\sin\alpha_1 - d_1\cos\alpha_1) \\ d_2 f_2 \end{pmatrix}. \end{aligned}
$$

Except for singular cases (here ruled out by the placement of the two cameras), the 'triangulation' system (5) is solvable and provides the localization of the point feature P in the frame of the second camera. Substituting the data,[2] we have

$$\boldsymbol{A} = \begin{pmatrix} -0.000196 & 0.011660 \\ 0.012 & -0.002 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 0.009116 \\ -0.000024 \end{pmatrix},$$

and so

$${}^{c2}P = \begin{pmatrix} X_2 \\ Z_2 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} 0.1287 \\ 0.7840 \end{pmatrix} \text{ [m]}. \tag{6}$$

From eq. (4) we get also the localization of the point feature P in the frame of the second camera

$${}^{c1}P = \begin{pmatrix} X_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} -0.6423 \\ 1.0806 \end{pmatrix} \text{ [m]}. \tag{7}$$

Finally, we localize the point feature P in the base frame of the robot, i.e., compute ${}^bP = (X_b, Y_b)$, using any of the two camera transformations in (2), together with (6) or (7);

$${}^b\boldsymbol{p}_{hom} = {}^b\boldsymbol{T}_{c1}\,{}^{c1}\boldsymbol{p}_{hom} = {}^b\boldsymbol{T}_{c2}\,{}^{c2}\boldsymbol{p}_{hom} \qquad \Rightarrow \qquad {}^bP = \begin{pmatrix} X_b \\ Y_b \end{pmatrix} = \begin{pmatrix} 1.0146 \\ -0.2966 \end{pmatrix} \text{ [m]}. \tag{8}$$

The last step requires solving the inverse kinematics of the PPR planar robot for a desired $\boldsymbol{r}_d = (p_{xd}, p_{yd}, \alpha_d)$. From the direct kinematics, one easily finds the unique solution:

$$\begin{aligned} p_x &= q_1 + \ell\cos q_3 \\ p_y &= q_2 + \ell\sin q_3 \\ \alpha &= q_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} q_1 &= p_{xd} - \ell\cos\alpha_d \\ q_2 &= p_{yd} - \ell\sin\alpha_d \\ q_3 &= \alpha_d. \end{aligned} \;.$$

Setting $(p_{xd}, p_{yd}) = (X_b, Y_b)$ from (8), the desired orientation for pointing toward the origin of the camera frame 2 is computed as

$$\alpha_d = (\alpha_2 - \frac{\pi}{2}) + \text{ATAN2}\{X_2, Z_2\} - \pi = -2.4553 \text{ [rad]} = -140.68^\circ,$$

and thus $\boldsymbol{q} = (q_1, q_2, q_3) = (1.4014, 0.0902, -2.4553)$ [m, m, rad]. The solution is illustrated in Fig. 3.

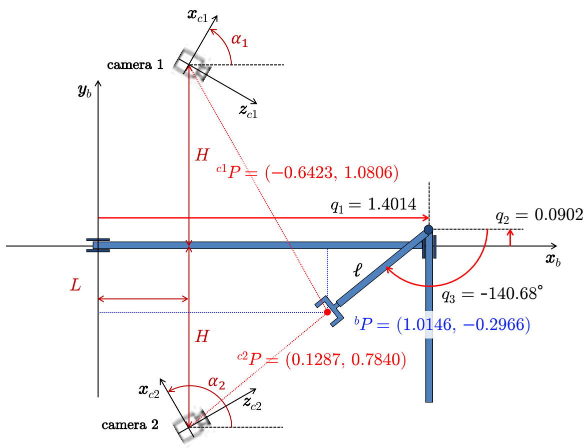


Figure 3: A graphical illustration of the solution to the visual localization and pick-up problem.

[2]The long format of MATLAB has been used to display 15 digits after the decimal point for $\boldsymbol{A}$ and $\boldsymbol{b}$. Only the first 6 are shown here.

Exercise 2

Figure 4 shows an assignment of D-H frames for the cylindrical robot that satisfies all requirements. The corresponding D-H parameters are reported in Tab. 2.

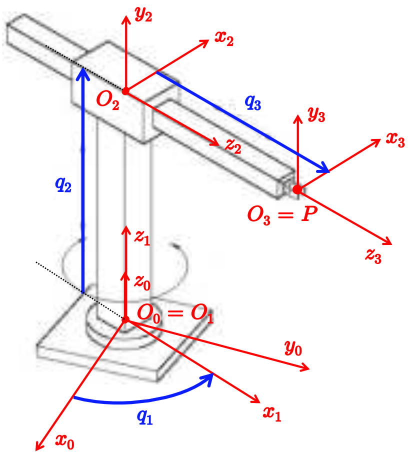


Figure 4: D-H frame assignment for the 3-dof cylindrical robot.

i	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	0	0	q_3	0

Table 2: D-H parameters associated to the frames in Fig. 4.

The direct kinematics is computed using the elements of the D-H table as

$$
{}^0\boldsymbol{T}_3(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} -s_1 & 0 & c_1 & q_3\,c_1 \\ c_1 & 0 & s_1 & q_3\,s_1 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_3(q_1) & {}^0\boldsymbol{p}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
$$

The inverse kinematics for ${}^0\boldsymbol{p}_3(\boldsymbol{q}) = \boldsymbol{p}_d = (p_{xd}, p_{yd}, p_{zd})$ is easily solved. From the third equation, one has directly

$$
q_2 = p_{zd}.
$$

Squaring and summing the first two components yields the two opposite values

$$
q_3 = \pm\sqrt{p_{xd}^2 + p_{yd}^2}.
$$

For each of this, provided that $q_3 \neq 0$, we obtain the other joint variable as

$$
q_1 = \text{ATAN2}\left\{\frac{p_{yd}}{q_3}, \frac{p_{xd}}{q_3}\right\},
$$

with two results that differ by π. Instead, when $q_3 = 0$ one has a singular situation with q_1 remaining undefined (infinite solutions).

In the presence of the given joint limits, Fig. 5 shows the primary workspace WS_1 of the cylindrical robot. Indeed, since the third joint q_3 can never take negative values, for each $\boldsymbol{p} \in WS_1$ there is always one and only one inverse solution (no singular configurations in WS_1).

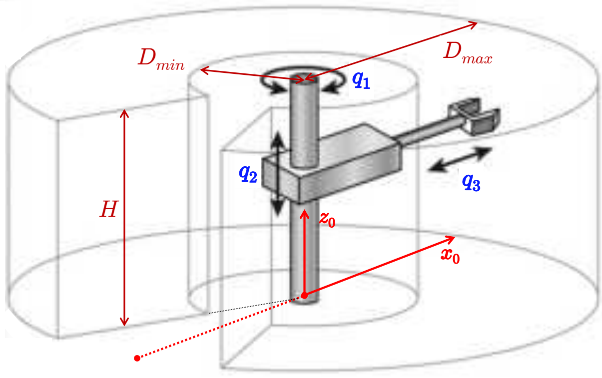


Figure 5: The primary workspace of the 3-dof cylindrical robot under the given joint limits.

The requested Jacobian is obtained by analytical differentiation of $\boldsymbol{p} = {}^0\boldsymbol{p}_3(\boldsymbol{q})$:

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_3\, s_1 & 0 & c_1 \\ q_3\, c_1 & 0 & s_1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{9}$$

As expected, being $\det \boldsymbol{J}(\boldsymbol{q}) = q_3$, the Jacobian is singular if and only if $q_3 = 0$. In a generic singular configuration $\boldsymbol{q}_s = (q_1, q_2, 0)$, the Jacobian in (9) becomes

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & s_1 \\ 0 & 1 & 0 \end{pmatrix}$$

and suitable bases for the two relevant subspaces are given by

$$\mathcal{N}(\boldsymbol{J}_s) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}(\boldsymbol{J}_s^T) = \text{span}\left\{ \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \right\}. \tag{10}$$

In the first case, the physical interpretation is that any velocity of the first joint will not move the end-effector. Moreover, this property holds not only instantaneously but also over time, since a $\dot{\boldsymbol{q}}$ having only $\dot{q}_1 \neq 0$ will not let the robot escape from the singularity $q_3 = 0$. As for the null space of the Jacobian transpose in (10), when applying in the given direction a Cartesian force to the end-effector, no robot motion will result and this does not need any active counteracting joint torque by the motors; such a Cartesian force is statically balanced by the internal reaction forces of the rigid structure.

Exercise 3

Figure 6 shows the setup of the considered trajectory planning problem. The Cartesian points A and B are on the boundary (respectively, the inner and the outer) of the robot workspace. Thus, the corresponding initial and final robot configurations are singular and unique:[3] $\boldsymbol{q}_A = (\pi/2, \pi)$ and $\boldsymbol{q}_B = (0, 0)$. Nonetheless, the initial departure direction from A and the final approaching direction to B are feasible — these

[3] Joint angles are conventionally defined in the interval $(-\pi, \pi]$, open on the left and closed on the right. Therefore, we discarded the otherwise equivalent configuration $\boldsymbol{q}_A$ having $q_{A,2} = -\pi$.

directions belong to the range space of the robot Jacobian, respectively in $\boldsymbol{q}_A$ and $\boldsymbol{q}_B$. Because of these singularities, it is convenient to plan the trajectory in the joint space.

In addition, the problem requires a specific tangent direction to the joint motion in A and B, but also zero velocity at start and end (a rest-to-rest motion). Therefore, it is convenient (here, even necessary!) to split the trajectory planning problem in space (a joint-level path $\boldsymbol{q}(s)$, satisfying geometric boundary conditions) and time (a suitable rest-to-rest timing law $s(t)$).

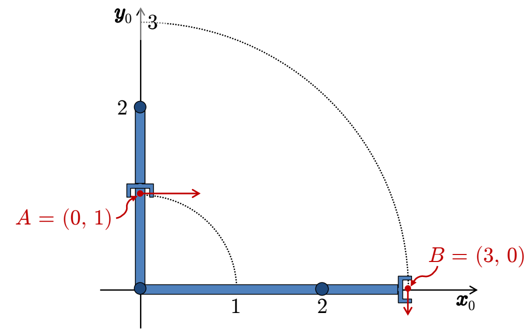


Figure 6: The considered trajectory planning problem for a 2R planar robot.

In fact, when attempting a direct solution in the time domain, we would simply use a rest-to-rest cubic trajectory for both joints, which is defined for $t \in [0, T]$ (with $T = 2$ s) in the doubly normalized form

$$\boldsymbol{q}(\tau) = \boldsymbol{q}_A + (\boldsymbol{q}_B - \boldsymbol{q}_A)\left(3\tau^2 - 2\tau^3\right), \qquad \tau = \frac{t}{T} \in [0, 1]. \tag{11}$$

The resulting trajectory and velocity of the two joints are shown in Fig. 7. Joint velocities are always negative because the two links rotate clockwise. It can be seen that both joint velocities satisfy also their feasibility bounds $|\dot{q}_1(t)| \le V_1 = 2$ and $|\dot{q}_2(t)| \le V_2 = 3$ [rad/s].

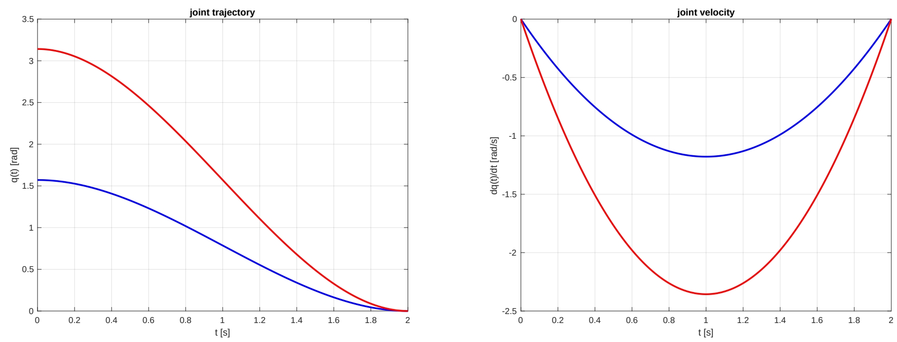


Figure 7: Rest-to-rest joint trajectory directly planned in time for $T = 2$ s, without caring about initial and final tangents to the path: joint 1 (blue), joint 2 (red).

However, when mapping the obtained joint trajectory (11) in the Cartesian space by means of the direct kinematics

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{pmatrix}, \tag{12}$$

the corresponding motion of the end-effector follows the path shown in Fig. 8. As it can be seen, the tangent to the Cartesian path at the initial point A is horizontal, but goes in the *opposite* direction[4] to the one desired, i.e., $d\boldsymbol{p}/ds|_A = (-\alpha, 0)$, for some $\alpha > 0$.

[4]When the 2R planar robot is in a singular configuration, there is only one admissible direction of instantaneous end-effector motion in the Cartesian space, characterized by a (scalable) vector in the null space of the Jacobian; nonetheless, one may exit (or enter) the singularity moving along the positive or negative direction of this vector.

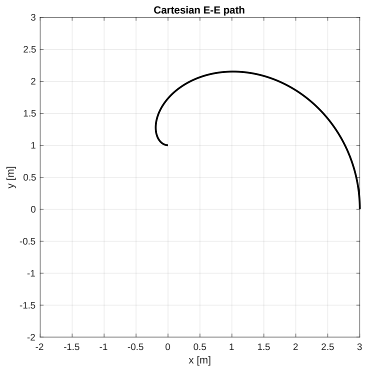


Figure 8: The Cartesian trajectory corresponding to the joint trajectory in Fig. 7.

With the above in mind, we have to the required tangent vector to the joint-space path at $\boldsymbol{q}_A$ and $\boldsymbol{q}_B$ from the given $d\boldsymbol{p}/ds|_A$ and $d\boldsymbol{p}/ds|_B$. For this, the Jacobian of the 2R planar robot

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{pmatrix} \tag{13}$$

is evaluated at $\boldsymbol{q}_A$ and $\boldsymbol{q}_B$, yielding

$$\boldsymbol{J}(\boldsymbol{q}_A) = \begin{pmatrix} l_2 - l_1 & l_2 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}_B) = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}.$$

As anticipated, the two system of equations

$$\boldsymbol{J}(\boldsymbol{q}_A) \left.\frac{d\boldsymbol{q}}{ds}\right|_A = \left.\frac{d\boldsymbol{p}}{ds}\right|_A = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}_B) \left.\frac{d\boldsymbol{q}}{ds}\right|_B = \left.\frac{d\boldsymbol{p}}{ds}\right|_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{14}$$

are solvable, but have an infinity of possible solutions. It is convenient to choose solutions with minimum norm, because in general the larger are the initial and final values along the tangent direction to the path, the longer will be the resulting path. To obtain these minimum norm solutions, we use the pseudoinverse of the Jacobian matrices, taking into account that one has a closed-form solution for this. In fact, the following general result holds (modulo an exchange of rows):

$$\boldsymbol{J} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}^{\#} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$

Plugging the numerical values in (14) leads to

$$\boldsymbol{q}'_A = \left.\frac{d\boldsymbol{q}}{ds}\right|_A = \boldsymbol{J}^{\#}(\boldsymbol{q}_A) \left.\frac{d\boldsymbol{p}}{ds}\right|_A = \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix}, \qquad \boldsymbol{q}'_A = \left.\frac{d\boldsymbol{q}}{ds}\right|_A = \boldsymbol{J}^{\#}(\boldsymbol{q}_A) \left.\frac{d\boldsymbol{p}}{ds}\right|_A = \begin{pmatrix} -0.3 \\ -0.1 \end{pmatrix},$$

where a prime ($'$) denotes differentiation with respect to the parameter s. Having a total of four boundary conditions for each joint, we can choose again cubic polynomials in space to solve the interpolation problem. In vector format, the joint path is defined as

$$\boldsymbol{q}(s) = \boldsymbol{a}_0 + \boldsymbol{a}_1 s + \boldsymbol{a}_2 s^2 + \boldsymbol{a}_3 s^3, \qquad s \in [0, 1], \tag{15}$$

with the $\boldsymbol{a}_i \in \mathbb{R}^2$, $i = 0, \ldots, 3$, used to satisfy the boundary conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_A, \qquad \boldsymbol{q}(1) = \boldsymbol{q}_B, \qquad \boldsymbol{q}'(0) = \boldsymbol{q}'_A, \qquad \boldsymbol{q}'(1) = \boldsymbol{q}'_B. \tag{16}$$

Imposing (16), the solution is found in closed form as

$$\boldsymbol{q}(s) = \boldsymbol{q}_A + \boldsymbol{q}'_A s + \left(3(\boldsymbol{q}_B - \boldsymbol{q}_A) - (2\boldsymbol{q}'_A + \boldsymbol{q}'_B)\right) s^2 + \left(-2(\boldsymbol{q}_B - \boldsymbol{q}_A) + (\boldsymbol{q}'_A + \boldsymbol{q}'_B)\right) s^3, \tag{17}$$

with first spatial derivative

$$\boldsymbol{q}'(s) = \boldsymbol{q}'_A + \left(6(\boldsymbol{q}_B - \boldsymbol{q}_A) - 2(2\boldsymbol{q}'_A + \boldsymbol{q}'_B)\right) s + \left(-6(\boldsymbol{q}_B - \boldsymbol{q}_A) + 3(\boldsymbol{q}'_A + \boldsymbol{q}'_B)\right) s^2. \tag{18}$$

When evaluated with the given data, the general formulas (17) and (18) provide

$$\boldsymbol{q}(s) = \begin{pmatrix} 1.5708 \\ 3.1416 \end{pmatrix} + \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix} s + \begin{pmatrix} 0.5876 \\ -14.3248 \end{pmatrix} s^2 + \begin{pmatrix} 0.3416 \\ 8.6832 \end{pmatrix} s^3 \tag{19}$$

and

$$\boldsymbol{q}'(s) = \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix} + \begin{pmatrix} 1.1752 \\ -28.6496 \end{pmatrix} s + \begin{pmatrix} 1.0248 \\ 26.0496 \end{pmatrix} s^2. \tag{20}$$

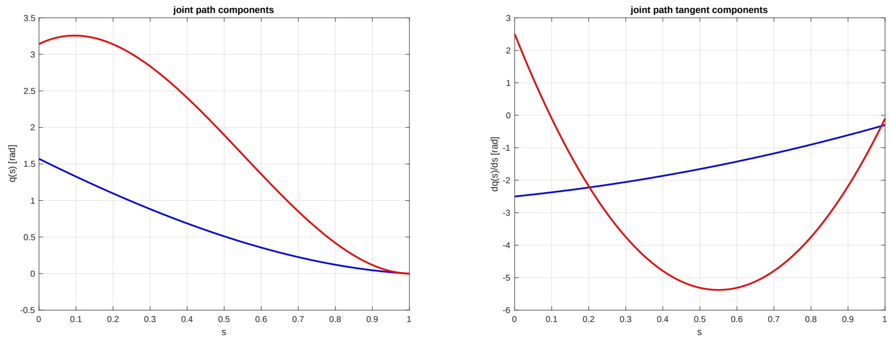


Figure 9: Components of the joint path $\boldsymbol{q}(s)$ and of its tangent vector $\boldsymbol{q}'(s)$: joint 1 (blue), joint 2 (red).

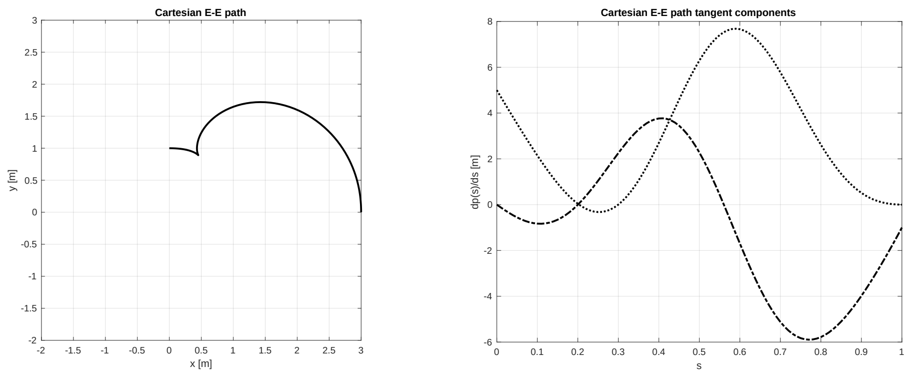


Figure 10: Path of the robot end-effector and components of its tangent vector: $p'_x(s)$ $(\cdots\cdot)$, $p'_y(s)$ $(-\cdot-\cdot)$.

Figure 9 shows the resulting path components in the joint space, as well as the components of its tangent vector. On the other hand, Fig. 10 shows the Cartesian path traced from A to B by the robot end-effector and the components of its tangent vector, as computed from the direct kinematics

$$\boldsymbol{p}(s) = \begin{pmatrix} p_x(s) \\ p_y(s) \end{pmatrix} = \begin{pmatrix} l_1 \cos q_1(s) + l_2 \cos\left(q_1(s) + q_2(s)\right) \\ l_1 \sin q_1(s) + l_2 \sin\left(q_1(s) + q_2(s)\right) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}(s))$$

and from

$$\boldsymbol{p}'(s) = \begin{pmatrix} p'_x(s) \\ p'_y(s) \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q}(s))\boldsymbol{q}'(s)$$

using the Jacobian in (13). As planned, the Cartesian path starts from A and arrives in B with the desired directions of the tangent vectors. The path has a cusp at $s = 0.2$, where $\boldsymbol{p}'(s) = \boldsymbol{0}$. Note that this occurs even if $\boldsymbol{q}'(0.2) \neq \mathbf{0}$, because there the robot is again in a folded configuration ($q_2 = 0$ at $s = 0.2$, see Fig. 9) where the Jacobian is singular. Apparently, the vector $\boldsymbol{q}'(0.2)$ belongs to the null space of $\boldsymbol{J}(\boldsymbol{q}(0.2))$.

In order to obtain a rest-to-rest trajectory from A to B that traces the path $\boldsymbol{q}(s)$ from $s = 0$ to $s = 1$ in a given time T, we need to define a suitable timing law $s(t)$. Also in this case, the choice of a (scalar) cubic polynomial, here in doubly normalized form,

$$s(\tau) = 3\tau^2 - 2\tau^3, \qquad \tau = \frac{t}{T} \in [0, 1], \tag{21}$$

satisfies the four boundary conditions $s(0) = 0$, $s(1) = 1$, and $\dot{s}(0) = \dot{s}(1) = 0$. Figure 11 shows the profiles of the timing law $s(t)$ and of its speed $\dot{s}(t) = s'(\tau)/T = 6\tau(1-\tau)/T$ for the motion time $T = 2$ s.

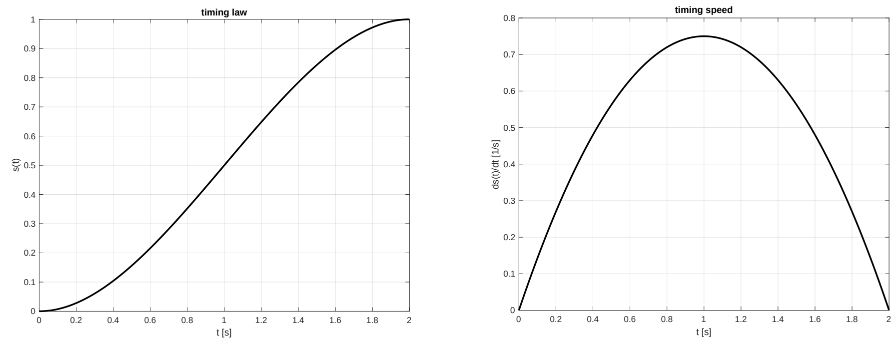


Figure 11: Rest-to-rest timing law $s(t)$ and its time derivative $\dot{s}(t)$ for a motion time $T = 2$ s.

By combining the path geometry and the timing law, we obtain the desired joint trajectory $\boldsymbol{q}_d(t) = \boldsymbol{q}(s(t))$ and its velocity $\dot{\boldsymbol{q}}_d(t) = \boldsymbol{q}'(s(t))\dot{s}(t)$, as shown in Fig. 12.

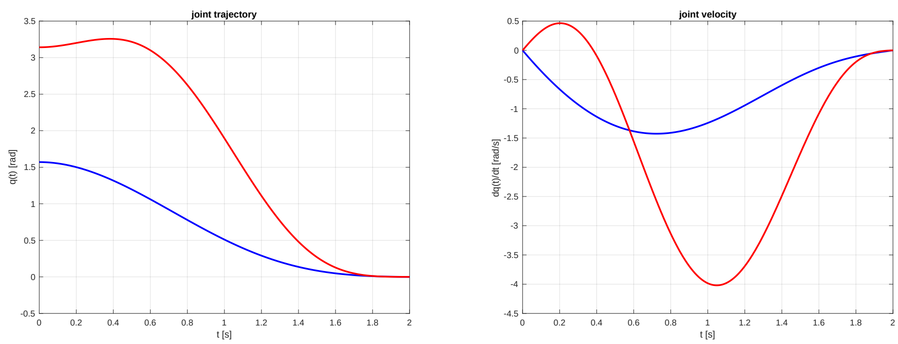


Figure 12: Joint trajectory $\boldsymbol{q}_d(t)$ and velocity $\dot{\boldsymbol{q}}_d(t)$ for motion time $T = 2$ s: joint 1 (blue), joint 2 (red).

Turning to item b), consider now the presence of the joint velocities bounds $|\dot{q}_1| \leq V_1 = 2$ and $|\dot{q}_2| \leq V_2 = 3$. From the velocity profiles in the right plots of Fig. 12, it is easy to see that both bounds are violated. However, we need an analytic verification in order to proceed (in case one has no plots).

Consider first the geometric part of the trajectory (see again Fig. 9) and compute the first and second spatial derivatives of the cubic path (15):

$$\boldsymbol{q}'(s) = \boldsymbol{a}_1 + 2\boldsymbol{a}_2 s + 3\boldsymbol{a}_3 s^2, \qquad \boldsymbol{q}''(s) = 2\boldsymbol{a}_2 + 3\boldsymbol{a}_3 s. \tag{22}$$

To find the maximum of $\boldsymbol{q}'(s)$ for each joint i, we impose

$$q_i''(s) = 0 \quad \Rightarrow \quad s_i^* = -\frac{2a_{2i}}{3a_{3i}} \quad \Rightarrow \quad q_i'(s_i^*) = a_{1i} + 2a_{2i}s_i^* + 3a_{3i}(s_i^*)^2 = a_{1i} - \frac{a_{2i}^2}{3a_{3i}},$$

or

$$q_i'(s_i^*) = q_{Ai}' - \frac{\left(3(q_{Bi} - q_{Ai}) - (2q_{Ai}' + q_{Bi}')\right)^2}{-2(q_{Bi} - q_{Ai}) + (q_{Ai}' + q_{Bi}')}.$$

Being $\boldsymbol{q}'(s)$ quadratic in s, the maximum absolute value of its components in the closed interval $[0,1]$ is either in $s = s_i^*$, provided that $s_i^* \in [0,1]$, or at one of the two boundaries $s = 0$ and $s = 1$. Using the numerical data, the following simple lines of MATLAB code provide the answers:

```
s_ast=-a2./(3*a3);  % here, a2 and a3 are vectors
max_dq=abs(a1-(a2.^2)./(3*a3));
for i=1,2;
  if s_ast(i) <= 0 | s_ast(i) >= 1
      max_dq(i)=max(abs(dq_A(i)), abs(dq_B(i)));
     if abs(dq_A(i)) >= abs(dq_B(i));
       s_ast(i)=0;
     else
       s_ast(i)=1;
     end
  end
end
s_ast
max_dq
```

We obtain

$$\max_{s\in[0,1]} \left|q_1'(s)\right| = |q_1'(0)| = 2.5, \qquad \max_{s\in[0,1]} \left|q_2'(s)\right| = |q_2'(0.5499)| = 5.3773,$$

in agreement with the left plots in Fig. 9.

Next, to verify the bounds on the joint velocity $\dot{\boldsymbol{q}}(t) = \boldsymbol{q}'(s(t))\dot{s}(t)$, the effect of the rest-to-rest quadratic speed $\dot{s}(t)$ of the timing law has to be included. The value $\dot{s}(0) = 0$ discards the relevance of the maximum value $q_1'(0)$ for the first joint; for this joint, one should look for the worst combination of $|q_1'(s)| \leq 2.5$ in the space interval $s \in [0,1]$ with $\dot{s}(t) \leq \dot{s}_{max} = 1.5/T\,(= 0.75)$ in the time interval $t \in [0,T]$, and then compare it with the bound V_1. On the other hand, unfeasibility for the second joint can be confirmed in a more direct way. In fact, the parameter value $s_2^* = 0.5499$ at which the maximum of $|q_2'(s)|$ occurs is obtained[5] at the normalized time $\tau^* = 0.5333$ in (21), thus for $t^* = \tau^* T = 1.0666$ s; the following value violates the bound:

$$|\dot{q}_2(t^*)| = |q_2'(s_2^*)| \cdot \dot{s}(t^*) = |q_2'(0.5499)| \cdot \dot{s}(1.0666) = 5.3773 \cdot 0.7467 = 4.0150 > V_2 = 3.$$

Therefore, the motion time $T = 2$ is certainly unfeasible and the trajectory has to be slowed down.

After this (rather tedious!) verification, a convenient feasible motion time $T^* > T$ is found more directly by uniform time scaling. The only approximation introduced with respect to the minimum possible value of a feasible motion time is to attribute the maximum speed of the timing law

$$\dot{s}_{max} = \max_{t\in[0,T]} \dot{s}(t) = \dot{s}\,(T/2) = \frac{s'(0.5)}{T} = \frac{1.5}{T}$$

to the maximum absolute values of $q_1'(s)$ and $q_2'(s)$. Having already all the necessary data, we obtain

$$\begin{aligned} T^* &= 1.5 \cdot \max\left\{ \frac{\max_{s\in[0,1]} |q_1'(s)|}{V(1)}, \frac{\max_{s\in[0,1]} |q_2'(s)|}{V(2)} \right\} \\ &= 1.5 \cdot \max\left\{ \frac{|q_1'(0)|}{V_1}, \frac{|q_2'(s_2^*)|}{V(2)} \right\} = 1.5 \cdot \max\left\{ \frac{2.5}{2}, \frac{5.3773}{3} \right\} = 2.6886\,[\mathrm{s}], \end{aligned} \tag{23}$$

with $\dot{s}_{max} = 1.5/T^* = 0.5579$. The final timing law and the joint trajectory are shown respectively in Fig. 13 and Fig. 14, together with their time derivatives. While the velocity of joint 1 is well within its

[5] The cubic equation $-2\tau^3 + 3\tau^2 - 0.5499 = 0$ has just a single root $\tau^* = 0.5333$ in the interval $[0,1]$.

bound, the maximum absolute velocity of joint 2 (which is the limiting factor) is $\dot{q}_{2,max} = 2.9903 < 3 = V_2$ — less than 1% away from the theoretical optimum. Note finally that the Cartesian path of the end-effector is still the one in Fig. 10, since time scaling does not change the geometry of the path.

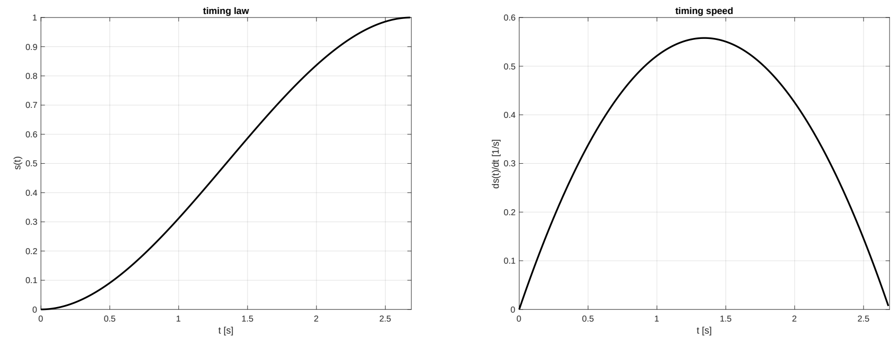


Figure 13: Rest-to-rest timing law $s(t)$ and its time derivative $\dot{s}(t)$ for a motion time $T^* = 2.6886$ s.

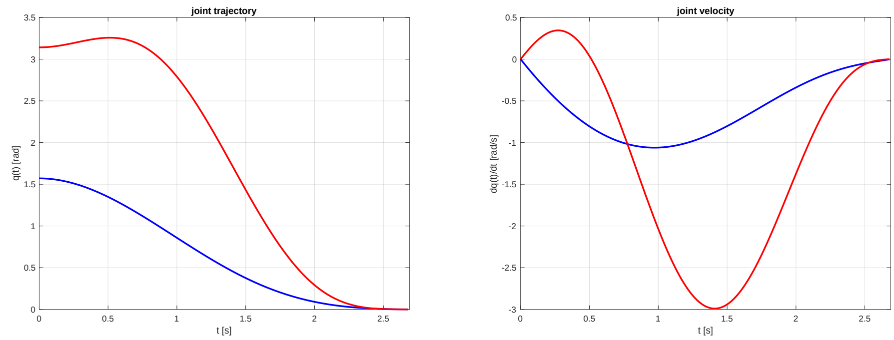


Figure 14: Joint trajectory $\boldsymbol{q}_d(t)$ and velocity $\dot{\boldsymbol{q}}_d(t)$ for motion time $T^* = 2.688$ s: joint 1 (blue), joint 2 (red).

Exercise 4

From Tab. 1, one can compute the D-H homogeneous transformation matrices of the 3R spatial robot, and from these the direct kinematics for the position of the end-effector. One has

$$\boldsymbol{p} = \begin{pmatrix} c_1(a_2c_2 + a_3c_{23}) \\ s_1(a_2c_2 + a_3c_{23}) \\ d_1 + a_2s_2 + a_3s_{23} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{24}$$

with the corresponding Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}. \tag{25}$$

The determinant of $\boldsymbol{J}(\boldsymbol{q})$ is

$$\det \boldsymbol{J}(\boldsymbol{q}) = -a_2a_3s_3(a_2c_2 + a_3c_{23}).$$

Setting $d_1 = 0.7$ and $a_2 = a_3 = 0.5$ [m], the primary workspace of this robot is a full sphere of radius $R = a_2 + a_3 = 1$ [m], centered at $C = (0, 0, d_1) = (0, 0, 0.7)$ [m]. Singularities occur on the boundary of the sphere (where $s_3 = 0$) and along the axis of joint 1 (where $a_2c_2 + a_3c_{23} = 0$, or $p_x = p_y = 0$).

The parametric description of the desired helical path (traced for three full turns) is

$$\boldsymbol{p}_d(s) = \begin{pmatrix} r\cos 2\pi s \\ r\sin 2\pi s \\ h_0 + hs \end{pmatrix}, \qquad s \in [0,3], \tag{26}$$

with $r = 0.5$, $h_0 = 0.2$, and $h = 0.4$ [m]. The lowest point on the path is $\boldsymbol{p}_d(0) = (r, 0, h_0) = (0.5, 0, 0.2)$ [m], at a distance $\|\boldsymbol{p}_d(0) - C\| = 0.7071$ m from the sphere center C: thus, this point is inside the workspace. Similarly, the highest point $\boldsymbol{p}_d(3) = (r, 0, h_0 + 3h) = (0.5, 0, 1.4)$ is at a distance $\|\boldsymbol{p}_d(3) - C\| = 0.8602$ m from the center C of the sphere, and thus also this point belong to the reachable workspace. Moreover, being $r = 0.5 < 1 = R$, also the rest of the helical path is inside the robot workspace and no singularities are encountered.

At the initial time $t = 0$, the robot configuration and the actual end-effector position computed from (24) are, respectively,

$$\boldsymbol{q}(0) = \begin{pmatrix} 0 \\ \pi/6 \\ -\pi/2 \end{pmatrix} \text{ [rad]} \qquad \Rightarrow \qquad \boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} 0.6830 \\ 0 \\ 0.5170 \end{pmatrix} \text{ [m]},$$

while the desired initial position on the path (26) is

$$\boldsymbol{p}_(0) = \begin{pmatrix} 0.5 \\ 0 \\ 0.2 \end{pmatrix} \text{ [m]}.$$

We have thus an initial error

$$\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \begin{pmatrix} -0.1830 \\ 0 \\ -0.3170 \end{pmatrix} \text{ [m]}$$

that should be counteracted by the feedback action in the control law.

On the other hand, for a perfect tracking of the desired trajectory, the required nominal velocity command $\dot{\boldsymbol{q}}_d(t)$ (known as feedforward) is obtained by differentiating (26) with respect to the parameter s to get

$$\boldsymbol{p}'_d(s) = \begin{pmatrix} -2\pi r \sin 2\pi s \\ 2\pi r \cos 2\pi s \\ h \end{pmatrix}, \tag{27}$$

setting $s = s_d(t) = vt$, and using then $\dot{\boldsymbol{p}}_d(t) = \boldsymbol{p}'_d(s(t))\,\dot{s}(t) = \boldsymbol{p}'_d(vt)\,v$ in

$$\dot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t))\,\dot{\boldsymbol{p}}_d(t), \qquad \text{with } \boldsymbol{q}_d(t) = \boldsymbol{q}_d(0) + \int_0^t \dot{\boldsymbol{q}}_d(\tau)d\tau,$$

provided that the robot starts from an inverse kinematics solution $\boldsymbol{q}_d(0) = \boldsymbol{f}^{-1}(\boldsymbol{p}_d(0))$.

Since the feedback part of the control law has to react on the tracking errors expressed in the Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ associated to the desired position along the path, we need to define such frame. From (27), dropping explicit dependence of terms on s, we obtain the tangent axis $\boldsymbol{t}$ of the Frenet frame as

$$\boldsymbol{t} = \frac{\boldsymbol{p}'_d}{\|\boldsymbol{p}'_d\|} = \frac{1}{\sqrt{(2\pi r)^2 + h^2}} \begin{pmatrix} -2\pi r \sin 2\pi s \\ 2\pi r \cos 2\pi s \\ h \end{pmatrix}. \tag{28}$$

Differentiating $\boldsymbol{t}$ with respect to the parameter s gives

$$\boldsymbol{t}' = \frac{\boldsymbol{p}''_d}{\|\boldsymbol{p}'_d\|} = -\frac{1}{\sqrt{(2\pi r)^2 + h^2}} \begin{pmatrix} 4\pi^2 r \cos 2\pi s \\ 4\pi^2 r \sin 2\pi s \\ 0 \end{pmatrix}, \qquad \text{with } \|\boldsymbol{t}'\| = \frac{\|\boldsymbol{p}''_d\|}{\|\boldsymbol{p}'_d\|} = \frac{4\pi^2 r}{\sqrt{(2\pi r)^2 + h^2}},$$

so that the normal axis $\boldsymbol{n}$ of the Frenet frame is

$$\boldsymbol{n} = \frac{\boldsymbol{t}'}{\|\boldsymbol{t}'\|} = \frac{\boldsymbol{p}_d''}{\|\boldsymbol{p}_d''\|} = -\begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \\ 0 \end{pmatrix}. \tag{29}$$

Finally, the third axis $\boldsymbol{b}$ of the Frenet frame is

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \frac{\boldsymbol{p}_d' \times \boldsymbol{p}_d''}{\|\boldsymbol{p}_d'\| \cdot \|\boldsymbol{p}_d''\|} = \frac{1}{\sqrt{(2\pi r)^2 + h^2}} \begin{pmatrix} h \sin 2\pi s \\ -h \cos 2\pi s \\ 2\pi r \end{pmatrix}. \tag{30}$$

Note that the Jacobian matrix and all Cartesian vectors introduced so far were implicitly defined in the base frame, but without the use of a leading superscript 0. For better clarity, from now we will use where appropriate such superscripts for the reference frame of definition.

From (28)–(30), the rotation matrix that characterizes the orientation of the Frenet frame along the desired path with respect to the base frame of the robot (the 0-th frame) is

$${}^0\boldsymbol{R}_F(s) = \begin{pmatrix} \boldsymbol{t}(s) & \boldsymbol{n}(s) & \boldsymbol{b}(s) \end{pmatrix}, \qquad s \in [0, 3].$$

Let the position error with respect to the desired trajectory be expressed in the base frame and in the Frenet frame as

$${}^0\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = {}^0\boldsymbol{p}_d - {}^0\boldsymbol{p} = {}^0\boldsymbol{p}_d - {}^0\boldsymbol{f}(\boldsymbol{q}), \qquad {}^F\boldsymbol{e} = \begin{pmatrix} e_t \\ e_n \\ e_b \end{pmatrix} = {}^0\boldsymbol{R}_F^T\, {}^0\boldsymbol{e}.$$

The dynamics of the position error in the Frenet frame is derived as

$$\begin{aligned} {}^F\dot{\boldsymbol{e}} &= {}^0\boldsymbol{R}_F^T\, {}^0\dot{\boldsymbol{e}} + {}^0\dot{\boldsymbol{R}}_F^T\, {}^0\boldsymbol{e} = {}^0\boldsymbol{R}_F^T\, {}^0\dot{\boldsymbol{e}} + \left(\boldsymbol{S}({}^0\boldsymbol{\omega}){}^0\boldsymbol{R}_F\right)^T {}^0\boldsymbol{e} = {}^0\boldsymbol{R}_F^T \left({}^0\dot{\boldsymbol{e}} + \boldsymbol{S}^T({}^0\boldsymbol{\omega})\, {}^0\boldsymbol{e}\right) \\ &= {}^0\boldsymbol{R}_F^T \left({}^0\dot{\boldsymbol{p}}_d - {}^0\dot{\boldsymbol{p}} - \boldsymbol{S}({}^0\boldsymbol{\omega})\, {}^0\boldsymbol{e}\right) = {}^0\boldsymbol{R}_F^T \left({}^0\dot{\boldsymbol{p}}_d - {}^0\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{S}({}^0\boldsymbol{\omega})\, {}^0\boldsymbol{R}_F^T\, {}^F\boldsymbol{e}\right), \end{aligned} \tag{31}$$

where $\boldsymbol{S}(\cdot)$ is the usual skew-symmetric matrix built with the components of its argument vector and ${}^0\boldsymbol{\omega}$ is the angular velocity of the Frenet frame moving along the path, which is extracted from

$$\boldsymbol{S}({}^0\boldsymbol{\omega}) = {}^0\dot{\boldsymbol{R}}_F\, {}^0\boldsymbol{R}_F = \begin{pmatrix} 0 & -2\pi\, v & 0 \\ 2\pi\, v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad {}^0\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ 2\pi\, v \end{pmatrix}.$$

Observing eq. (31) suggests the design of the following kinematic control law

$$\dot{\boldsymbol{q}} = {}^0\boldsymbol{J}^{-1}(\boldsymbol{q}) \left({}^0\dot{\boldsymbol{p}}_d + \boldsymbol{S}({}^0\boldsymbol{\omega})\, {}^0\boldsymbol{e} - {}^0\boldsymbol{R}_F \boldsymbol{K}_F\, {}^F\boldsymbol{e}\right) = {}^0\boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\cdots - {}^0\boldsymbol{R}_F \boldsymbol{K}_F\, {}^0\boldsymbol{R}_F^T\, {}^0\boldsymbol{e}\right), \tag{32}$$

with a suitable diagonal matrix $\boldsymbol{K}_F > 0$. The law (32) cancels all nonlinearities and couplings in the dynamics of the scalar components e_t, e_n and e_b of the error vector ${}^F\boldsymbol{e}$ in the Frenet frame, yielding

$${}^F\dot{\boldsymbol{e}} = -\boldsymbol{K}_F\, {}^F\boldsymbol{e}, \qquad \text{with } \boldsymbol{K}_F = \operatorname{diag}\{2, 5, 5\} > 0,$$

so that we obtain as requested the decoupled dynamics

$$\dot{e}_t = -2e_t, \qquad \dot{e}_n = -5e_n, \qquad \dot{e}_b = -5e_b,$$

namely with exponentially converging transients.[6]

[6]The differential equation $\dot{e} = -ke$ with $k > 0$ has the solution $e(t) = e(0)\exp(-kt)$ converging to zero.

Finally, we evaluate numerically the control law (32) at $t = 0$ with the data of the problem and starting from $\boldsymbol{q}(0) = (0, \pi/6, -\pi/2)$ [rad]. The individual terms (all expressed implicitly in the base frame) are:

$$\boldsymbol{J}(\boldsymbol{q}(0)) = \begin{pmatrix} 0 & 0.1830 & 0.4330 \\ 0.6830 & 0 & 0 \\ 0 & 0.6830 & 0.2500 \end{pmatrix} = \boldsymbol{J}_0 \qquad \Rightarrow \qquad \boldsymbol{J}_0^{-1} = \begin{pmatrix} 0 & 1.4641 & 0 \\ -1.0000 & 0 & 1.7321 \\ 2.7321 & 0 & -0.7321 \end{pmatrix},$$

$$\dot{\boldsymbol{p}}_d(0) = \boldsymbol{p}_d'(0)\,\dot{s}(0) = \begin{pmatrix} 0 \\ 2\pi r \\ h \end{pmatrix} v = \begin{pmatrix} 0 \\ 3.1416 \\ 0.4 \end{pmatrix}, \qquad \boldsymbol{e}(0) = \begin{pmatrix} -0.1830 \\ 0 \\ -0.3170 \end{pmatrix}, \qquad \boldsymbol{\omega}(0) = \begin{pmatrix} 0 \\ 0 \\ 6.2832 \end{pmatrix},$$

$$\boldsymbol{S}({}^0\boldsymbol{\omega}(0))\,{}^0\boldsymbol{e}(0) = \begin{pmatrix} 0 \\ -1.1499 \\ 0 \end{pmatrix}, \qquad {}^0\boldsymbol{R}_F(0) = \begin{pmatrix} 0 & -1 & 0 \\ 0.9920 & 0 & -0.1263 \\ 0.1263 & 0 & 0.99200 \end{pmatrix},$$

and the control gain matrix

$${}^0\boldsymbol{R}_F \boldsymbol{K}_F\, {}^0\boldsymbol{R}_F^T = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2.0479 & -0.3759 \\ 0 & -0.3759 & 4.9521 \end{pmatrix}.$$

Thus, the initial value of the control command (32) is

$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 2.7416 \\ 2.4967 \\ 1.0580 \end{pmatrix} \text{ [rad/s]}.$$

* * * * *

Robotics 1

June 12, 2024

Exercise 1

Let an initial and a final orientation be specified by the two matrices

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{R}_f = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using the ZXZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, generate a trajectory $\boldsymbol{\phi}(t)$, with $t = [0, T]$, which interpolates these two orientations in $T = 1.5$ s, with initial and final angular velocity given by

$$\boldsymbol{\omega}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{\omega}_f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{[rad/s]}.$$

Provide the value of the resulting angular velocity $\boldsymbol{\omega}(t)$ at the midtime $t = T/2$. Does the found trajectory $\boldsymbol{\phi}(t)$ cross any representation singularity? Is it the unique solution to this problem in the class of interpolating trajectories using ZXZ Euler angles?

Exercise 2

Table 1 contains the Denavit-Hartenberg (D-H) parameters of a robot with three revolute joints.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

Table 1: D-H parameters of a 3R robot

- Sketch a skeleton of this robot and of its reachable workspace.
- Assign the frames to the robot links according to the above table.
- Compute the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ associated to the velocity of the origin of the last D-H frame and determine all its singularities.
- In a singular configuration $\boldsymbol{\theta}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{\theta}_s)$; find a basis for the subspaces $\mathcal{N}(\boldsymbol{J}_s)$ and $\mathcal{R}(\boldsymbol{J}_s)$ and provide an interpretation in terms of robot motion.

Exercise 3

Consider a digital camera with a number of pixel $W \times H = 720 \times 524$ on the image plane and having a lens with focus $f = 8$ mm. Each pixel is square with size $d = 7\mu$m. Assuming a pinhole camera model, which are the horizontal and vertical spatial resolutions (expressed in mm) of points on a plane parallel to the image plane and placed at a distance $L = 2$ m from the lens center? What is the angular field of view (in degrees) on the horizontal plane of this camera-lens system?

Exercise 4

For a 3R planar robot with unit length links, can you define a circular path of diameter $d = 1$ m for its end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ so that the robot will certainly trace such path without crossing a singular configuration? If so, provide an example. If not, explain why.

Exercise 5

Consider a point-to-point path planning problem for a 2R planar robot with unit length links. The robot should move its end-effector between the two Cartesian positions $\boldsymbol{p}_i = (0.6, -0.4)$ and $\boldsymbol{p}_f = (1, 1)$ [m]. Moreover, the Cartesian path should have tangent direction at the start and at the end specified respectively by the vectors $\boldsymbol{p}'_i = (-2, 0)$ and $\boldsymbol{p}'_f = (2, 2)$, where $\boldsymbol{p}' = d\boldsymbol{p}/ds$.

- Define a solution path $\boldsymbol{q}(s)$ directly in the joint space.
- Within the chosen class of interpolating functions, how many solution paths exists? Does any of these paths cross a kinematic singularity?
- On the chosen solution path, define a rest-to-rest timing law $s(t)$ that completes the motion in $T = 3$ s and has continuous acceleration $\ddot{s}(t)$ in the (open) time interval $(0, T)$.
- What is the value of the resulting joint velocity $\dot{\boldsymbol{q}}(t)$ at the midtime $t = T/2$?
- What is the value of the resulting end-effector velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{p}}(t)$ at the midtime $t = T/2$?

[240 minutes (4 hours); open books]

Solution

June 12, 2024

Exercise 1

Using the inverse relationships for the representation of orientation with the ZXZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, each of the two rotation matrices $\boldsymbol{R}_i$ and $\boldsymbol{R}_f$ can be transformed in a pair of solution triples, being the regularity condition $R_{13}^2 + R_{23}^2 \neq 0$ on the elements R_{ij} of both matrices satisfied. Therefore, there are four possible combinations for the boundary conditions of the interpolation problem when using this minimal representation of orientation.

The regularity condition is equivalent to having $\sin\beta \neq 0$, or $\beta \neq \{0, \pm\pi\}$, in the triple. Thus, each pair of solutions is characterized by a positive or a negative value for β, depending on the choice of the sign in the four-quadrant arctangent function that is used to compute this value. As a result, a trajectory that interpolates the initial and final values of β, in the usual domain $(-\pi, \pi]$ of definition, will avoid crossing a singularity *if and only if* both values β_i and β_f will have the same sign. Moreover, the amplitude of the needed change for the Euler angles is likely to be smaller when $\beta(t)$ does not change sign along the trajectory. These remarks reduce the interesting combinations of boundary conditions to two; the following solution will present just one of them.

From

$$\beta = \operatorname{atan2}\left\{+\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right\} \qquad \alpha = \operatorname{atan2}\left\{\frac{R_{31}}{\sin\beta}, \frac{R_{32}}{\sin\beta}\right\} \qquad \gamma = \operatorname{atan2}\left\{\frac{R_{13}}{\sin\beta}, -\frac{R_{23}}{\sin\beta}\right\},$$

one obtains

$$\boldsymbol{\phi}_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \pi/2 \\ -\pi/2 \end{pmatrix} \qquad \boldsymbol{\phi}_f = \begin{pmatrix} \alpha_f \\ \beta_f \\ \gamma_f \end{pmatrix} = \begin{pmatrix} \pi/4 \\ \pi/2 \\ 0 \end{pmatrix}.$$

The interpolating trajectory has to satisfy also boundary conditions on the initial and final angular velocity. The transformation between $\dot{\boldsymbol{\phi}}$ and $\boldsymbol{\omega}$ is found to be

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\,\dot{\boldsymbol{\phi}} = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha\sin\beta \\ 0 & \sin\alpha & -\cos\alpha\sin\beta \\ 1 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}.$$

In fact, since the rotation matrix associated to the ZXZ Euler angles is $\boldsymbol{R} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)$, the three individual contributions of $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ to the angular velocity $\boldsymbol{\omega}$ are given by

$$\boldsymbol{\omega}_{\dot{\alpha}} = \boldsymbol{z}\,\dot{\alpha} \qquad \boldsymbol{\omega}_{\dot{\beta}} = \boldsymbol{R}_z(\alpha)\,\boldsymbol{x}\,\dot{\beta} \qquad \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\,\boldsymbol{z}\,\dot{\gamma},$$

which build the three columns of $\boldsymbol{T}(\boldsymbol{\phi})$ (above, we used $\boldsymbol{z} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $\boldsymbol{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$). Evaluating $\boldsymbol{T}$ with the obtained $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_f$, one has

$$\boldsymbol{T}(\boldsymbol{\phi}_i) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{T}(\boldsymbol{\phi}_f) = \begin{pmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & 0 & 0 \end{pmatrix},$$

and thus

$$\dot{\boldsymbol{\phi}}_i = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_i)\,\boldsymbol{\omega}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \dot{\boldsymbol{\phi}}_f = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_f)\,\boldsymbol{\omega}_f = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

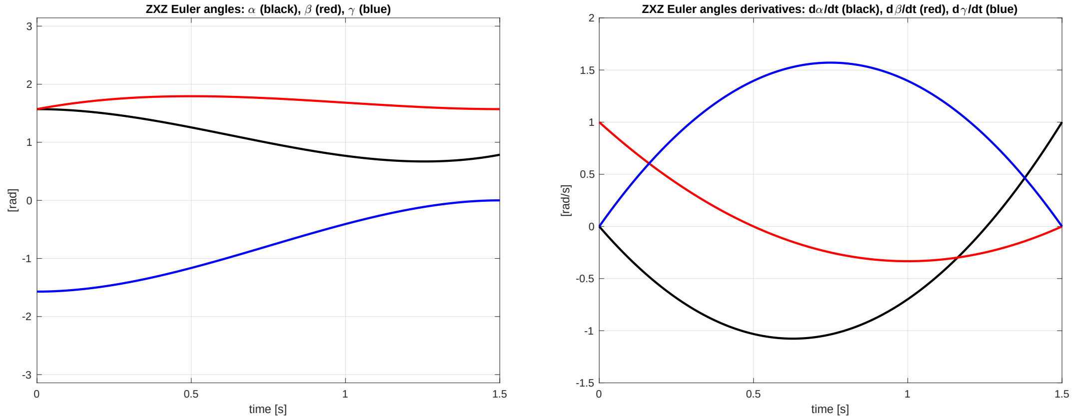


Figure 1: The components of the interpolation trajectory $\boldsymbol{\phi}(t)$ and of its derivative $\dot{\boldsymbol{\phi}}(t)$.

The interpolation problem has the four boundary conditions

$$\boldsymbol{\phi}(0) = \boldsymbol{\phi}_i \qquad \dot{\boldsymbol{\phi}}(0) = \dot{\boldsymbol{\phi}}_i \qquad \boldsymbol{\phi}(T) = \boldsymbol{\phi}_f \qquad \dot{\boldsymbol{\phi}}(T) = \dot{\boldsymbol{\phi}}_f,$$

with $T = 1.5$ s, and can be solved in a unique way when using a cubic polynomial. In its normalized form with $\tau = t/T$, the interpolating trajectory for $\tau \in [0, 1]$ is

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_i + \left(\dot{\boldsymbol{\phi}}_i T\right)\tau + \left(3\,\Delta\boldsymbol{\phi} - (\dot{\boldsymbol{\phi}}_f + 2\dot{\boldsymbol{\phi}}_i)T\right)\tau^2 + \left(-2\,\Delta\boldsymbol{\phi} + (\dot{\boldsymbol{\phi}}_f + \dot{\boldsymbol{\phi}}_i)T\right)\tau^3,$$

where

$$\Delta\boldsymbol{\phi} = \boldsymbol{\phi}_f - \boldsymbol{\phi}_i = \begin{pmatrix} -\pi/4 \\ 0 \\ \pi/2 \end{pmatrix}.$$

Substituting the numerical data, the trajectories of the Euler angles for $\tau \in [0, 1]$ are (in [rad]):

$$\begin{aligned} \alpha(\tau) &= 1.5708 - 3.8562\,\tau^2 + 3.0708\,\tau^3 \\ \beta(\tau) &= 1.5708 + 1.5\,\tau - 3\,\tau^2 - 1.5\,\tau^3 \\ \gamma(\tau) &= -1.5708 + 4.7124\,\tau^2 - 3.1416\,\tau^3. \end{aligned}$$

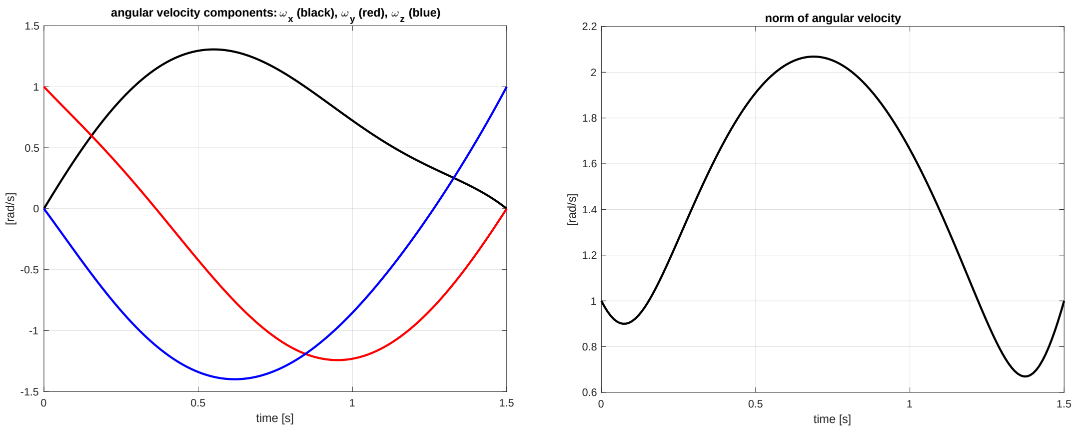


Figure 2: The components of the angular velocity vector $\boldsymbol{\omega}(t)$ and its norm.

The time evolutions of the three components of the trajectory $\boldsymbol{\phi}(t)$ and of its derivative $\dot{\boldsymbol{\phi}}(t)$ are shown in Fig. 1. The angular velocity along the trajectory is computed as $\boldsymbol{\omega}(t) = \boldsymbol{T}(\boldsymbol{\phi}(t))\,\dot{\boldsymbol{\phi}}(t)$ and is shown in Fig. 2, together with its norm. At $t = T/2 = 0.75$ s, it is

$$\boldsymbol{\omega}(T/2) = \begin{pmatrix} 1.1537 \\ -1.0551 \\ -1.3282 \end{pmatrix} \text{ [rad/s]}.$$

Exercise 2

A possible structure described by Tab. 1 is a 3R spatial robot, as sketched in Fig. 3 with the associated D-H frames. Since the size/length of the robot base is irrelevant for the kinematics, the first joint could also be placed on the ground; this is also the meaning of having $a_1 = d_1 = 0$ in the D-H table.

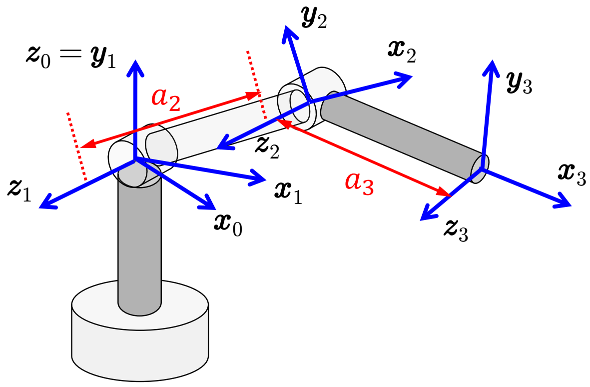


Figure 3: A 3R spatial robot with D-H frame assignment according to Tab. 1.

The reachable workspace is a sphere centered at the robot shoulder (i.e., at $O_0 = O_1$) and with radius $R = a_2 + a_3 > 0$. If $a_2 \neq a_3$, an inner sphere of radius $r = |a_2 - a_3| > 0$ is removed from the workspace.

The direct kinematics in position is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta}) = \begin{pmatrix} c_1\,(a_2c_2 + a_3c_{23}) \\ s_1\,(a_2c_2 + a_3c_{23}) \\ a_2s_2 + a_3s_{23} \end{pmatrix}.$$

The associated 3×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -s_1\,(a_2c_2 + a_3c_{23}) & -c_1\,(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1\,(a_2c_2 + a_3c_{23}) & -s_1\,(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}.$$

To simplify the study of the singularities of $\boldsymbol{J}$, it is convenient to express the Jacobian in the rotated frame 1, namely

$${}^1\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{R}_1^T(\theta_1)\,\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{pmatrix} \boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & -\,(a_2s_2 + a_3s_{23}) & -a_3s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ -\,(a_2c_2 + a_3c_{23}) & 0 & 0 \end{pmatrix}.$$

Thus, for the determinant we have

$$\det \boldsymbol{J}(\boldsymbol{\theta}) = \det {}^1\boldsymbol{J}(\boldsymbol{\theta}) = a_2 a_3\, s_3 \left(a_2 c_2 + a_3 c_{23}\right),$$

and singularities occur when:

a) $s_3 = 0$: the forearm is fully stretched ($\theta_3 = 0$) or fully folded ($\theta_3 = \pm\pi$);

b) $a_2 c_2 + a_3 c_{23} = 0$: being this equal to $\sqrt{p_x^2 + p_y^2} = 0$, the end-effector is located on the $\boldsymbol{z}_0$ axis;

c) both above situations hold: the robot is stretched or folded vertically along the $\boldsymbol{z}_0$ axis.

Consider for instance case a) and set $\boldsymbol{\theta}_s = (\theta_1, \theta_2, 0)$. Then

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{\theta}_s) = \begin{pmatrix} -\left(a_2+a_3\right) s_1 c_2 & -\left(a_2+a_3\right) c_1 s_2 & -a_3 c_1 s_2 \\ -\left(a_2+a_3\right) c_1 c_2 & -\left(a_2+a_3\right) s_1 s_2 & -a_3 s_1 s_2 \\ 0 & \left(a_2+a_3\right) c_2 & a_3 c_2 \end{pmatrix}.$$

It is easy to find a basis for each of the two requested subspaces:

$$\mathcal{N}(\boldsymbol{J}_s) = \begin{pmatrix} 0 \\ -a_3 \\ a_2 + a_3 \end{pmatrix} \qquad \mathcal{R}(\boldsymbol{J}_s) = \operatorname{span}\left\{ \begin{pmatrix} s_1 c_2 \\ c_1 c_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -c_1 s_2 \\ -s_1 s_2 \\ c_2 \end{pmatrix} \right\}.$$

As a result, in this singular configuration, the robot end effector will remain at rest when the first joint does not move while the second and third joints move in opposite directions with instantaneous velocities scaled by the relative factor $a_3/(a_2+a_3)$ (motion in the *null space*). If the two links have unit length ($a_1 = a_2 = 1$), then joint 3 will move instantaneously at twice the (opposite) velocity of joint 2.

On the other hand, to better visualize the *range space* motion at $\boldsymbol{\theta}_s$, consider the rotated Jacobian in the chosen singular configuration

$${}^1\boldsymbol{J}_s = {}^1\boldsymbol{J}(\boldsymbol{\theta}_s) = \begin{pmatrix} 0 & -\left(a_2+a_3\right) s_2 & -a_3 s_2 \\ 0 & \left(a_2+a_3\right) c_2 & a_3 c_2 \\ -\left(a_2+a_3\right) c_2 & 0 & 0 \end{pmatrix}.$$

Then

$$\mathcal{R}({}^1\boldsymbol{J}_s) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix}, \begin{pmatrix} -s_2 \\ c_2 \\ 0 \end{pmatrix} \right\}.$$

Therefore, when observing the end-effector motion in frame $(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1)$ with the robot in this singular configuration, an instantaneous velocity of the first joint produces only a velocity in the $\boldsymbol{z}_1$ (horizontal) direction; the second and third joint can move the end effector only along a single direction in the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$. If the two links have unit length and we set $\theta_2 = 0$, a unit velocity of joint 1 gives an end-effector velocity ${}^1\boldsymbol{v}_e = (0, 0, 2)$ [m/s]; a unit velocity of joint 2 (or of joint 3) gives ${}^1\boldsymbol{v}_e = (0, 2, 0)$ [m/s] (or ${}^1\boldsymbol{v}_e = (0, 1, 0)$ [m/s]).

Exercise 3

Let a point ${}^cP = (X, Y, Z)$ be expressed in the camera frame, which is placed on the physical image plane of the camera, i.e., behind the lens and at a focal distance f. The pinhole model maps point P into the image plane point $p = (u, v)$ (horizontal and vertical coordinates) as

$$u = u_0 - f\,\frac{X}{Z} \qquad v = v_0 - f\,\frac{Y}{Z}.$$

where (u_0, v_0) corresponds to the center point of the image on the lens axis (the origin of the camera coordinates is usually at the top-left of the image plane). Accordingly, a displacement $\boldsymbol{\Delta} = (\Delta X, \Delta Y)$ of point P on the plane at a distance $Z = L + f$ from the camera frame will result in variations of the coordinates of the point p on the image plane given by

$$\Delta u = -f\,\frac{\Delta X}{L+f} \qquad \Delta v = -f\,\frac{\Delta Y}{L+f}.$$

Therefore, being the physical size of the square pixels $\Delta u_{\min} = \Delta v_{\min} = 7\mu\text{m} = 7 \cdot 10^{-3}$ mm, the spatial resolution of a point P in the plane at $Z = L + f$ (namely, the minimum displacement detectable by the camera) is

$$\Delta X_{min} (= \Delta Y_{min}) = \Delta u_{\min}\,\frac{L+f}{f} = 7 \cdot 10^{-3} \cdot \frac{2008}{8} = 1.757 \text{ mm},$$

equal in the horizontal and vertical case.

The physical size of the sensor is $Wd \times Hd = 720 \cdot 7 \times 524 \cdot 7\,\mu\text{m} = 5.040 \times 3.668$ mm. The angular field of view (FOV) on the horizontal plane of this camera-lens system, expressed in degrees, is

$$\text{FOV}_H = 2 \arctan \frac{Wd/2}{f}\,[\text{rad}] = \frac{360°}{\pi} \cdot \arctan \frac{Wd/2}{f} = 114.59\,[°/\text{rad}] \cdot \arctan \frac{2.520}{8} \simeq 36°.$$

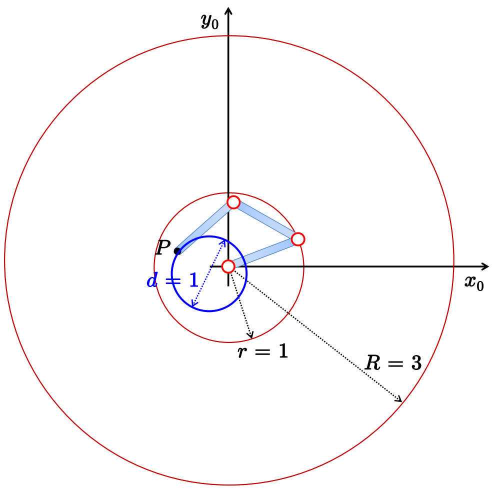


Figure 4: A circular path traced without singularities by a 3R planar robot with unit length links.

Exercise 4

The answer is yes. With reference to Fig. 4, a 3R planar robot is in a singular configuration for a task involving only its end-effector position (with a 2×3 Jacobian) when the three links are stretched or folded along a single radial direction from the robot base, i.e., when $\theta_2 = \{0, \pm\pi\}$ and $\theta_3 = \{0, \pm\pi\}$. When the links have unit length, these singular configurations correspond to the external boundary of the workspace (a circle of radius $R = 3$ m that the robot end effector can reach only in the fully stretched and singular configuration) and to a circle of radius $r = 1$ m inside the workspace (where the robot is in singularity only if folded along a radial direction). Therefore, a circular path of diameter $d = 1$ m (i.e., of radius $d/2 = 0.5$ m) for the end-effector can be defined safely (and in many ways!) inside the circle of unit radius, and certainly no singular configurations will be encountered.

Exercise 5

Being the given two Cartesian positions $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$ strictly inside the reachable workspace of the 2R planar robot (a circle of radius $r_{\max} = 2$, whereas $\|\boldsymbol{p}_i\| = 0.7211$ and $\|\boldsymbol{p}_f\| = 1.4142$), they do not correspond to singular configurations ($q_2 = 0$ or $\pm\pi$). Thus, there are two regular solutions to the inverse kinematics problem for these positions. We choose in both cases the *elbow-down* (viz. *right-arm*) solution, namely

$$\boldsymbol{q}_i = \begin{pmatrix} -1.7899 \\ 2.4039 \end{pmatrix} \qquad \boldsymbol{q}_f = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} \quad \text{[rad]}.$$

By doing so, we choose one out of the four possible combinations of boundary conditions for the interpolating joint path to be defined (and, correspondingly, four classes of trajectories). Moreover, choosing the initial and final inverse solutions in the same class will guarantee that interpolating joint paths that have no wandering (i.e., remain between the initial and final joint values) will not cross a singular configuration.[1] On the other hand, when choosing the two combinations with values of the second joint of opposite signs, a singular configuration will certainly be encountered.

In order to invert the Cartesian tangent data, we need to compute the 2×2 analytical Jacobian $\boldsymbol{J}$ of the robot. In fact, for the derivatives with respect to the path parameter s, one has

$$\boldsymbol{p}' = \frac{d\boldsymbol{p}}{ds} = \boldsymbol{J}(\boldsymbol{q})\,\frac{d\boldsymbol{q}}{ds} = \boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{q}',$$

just like for the transformation between joint velocity $\dot{\boldsymbol{q}}$ and end-effector velocity $\dot{\boldsymbol{p}}$ (which are derivatives of the position $\boldsymbol{p}$ with respect to time t in the two spaces). Being

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix},$$

we have

$$\boldsymbol{J}(\boldsymbol{q}_i) = \begin{pmatrix} 0.4000 & -0.5761 \\ 0.6000 & 0.8174 \end{pmatrix} \qquad \boldsymbol{J}(\boldsymbol{q}_f) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and thus

$$\boldsymbol{q}'_i = \boldsymbol{J}^{-1}(\boldsymbol{q}_i)\,\boldsymbol{p}'_i = \begin{pmatrix} -2.4305 \\ 1.7841 \end{pmatrix} \qquad \boldsymbol{q}'_f = \boldsymbol{J}^{-1}(\boldsymbol{q}_f)\,\boldsymbol{p}'_f = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad \text{[rad]}.$$

The interpolation problem for a geometric path $\boldsymbol{q}(s)$, with $s \in [0,1]$, has the four boundary conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_i \qquad \boldsymbol{q}'(0) = \boldsymbol{q}'_i \qquad \boldsymbol{q}(1) = \boldsymbol{q}_f \qquad \boldsymbol{q}'(1) = \boldsymbol{q}'_f,$$

and can be solved using the cubic polynomial

$$\boldsymbol{q}(s) = \boldsymbol{q}_i + \boldsymbol{q}'_i\,s + \left(3\,\Delta\boldsymbol{q} - (\boldsymbol{q}'_f + 2\boldsymbol{q}'_i)\right)s^2 + \left(-2\,\Delta\boldsymbol{q} + (\boldsymbol{q}'_f + \boldsymbol{q}'_i)\right)s^3,$$

where

$$\Delta\boldsymbol{q} = \boldsymbol{q}_f - \boldsymbol{q}_i = \begin{pmatrix} 1.7899 \\ -0.8331/2 \end{pmatrix} \quad \text{[rad]}.$$

[1]The situation is somewhat similar to Exercise 1.

Substituting the numerical data, the paths of the two joints for $s \in [0,1]$ are (in [rad]):

$$q_1(s) = -1.7899 - 2.4305\, s + 8.2308\, s^2 - 4.0104\, s^3$$
$$q_2(s) = 2.4039 + 1.7841\, s - 2.0674\, s^2 - -0.5498\, s^3.$$

The two components of the joint path $\boldsymbol{q}(s)$ and of its derivative (tangent) $\boldsymbol{q}'(s)$ are shown in Fig. 5. Since $q_2(t)$ never crosses 0 (or $\pm\pi$), the robot does not encounter a singular configuration. Figure 6 shows the Cartesian path corresponding to $\boldsymbol{q}(s)$.

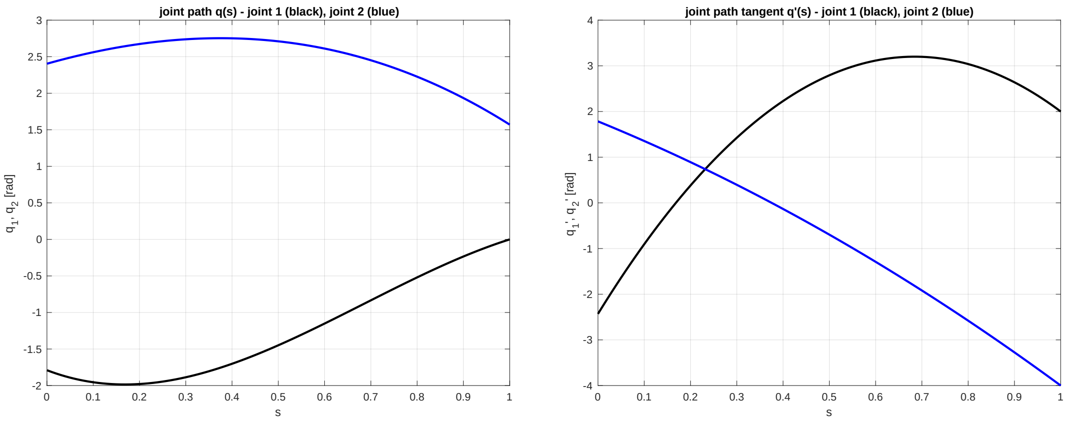


Figure 5: The components of the joint path $\boldsymbol{q}(s)$ and of its derivative $\boldsymbol{q}'(s)$.

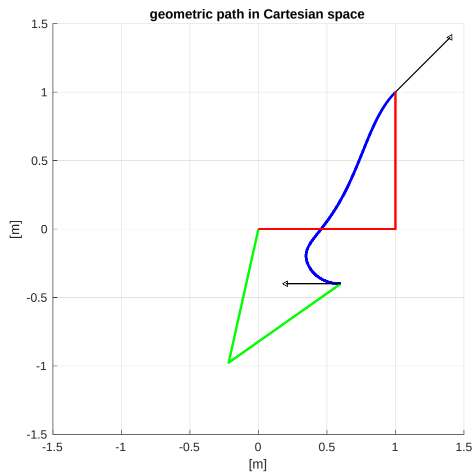


Figure 6: The Cartesian path $\boldsymbol{p}(s)$ (in blue) corresponding to $\boldsymbol{q}(s)$. Initial (green) and final (red) robot postures and initial and final tangent directions to the Cartesian path are also shown.

To obtain a smooth rest-to-rest motion (strictly) inside the motion interval $[0,T]$, with $T = 3$ s, one can use a cubic polynomial as timing law — see Fig. 7:

$$s(\tau) = 3\,\tau^2 - 2\,\tau^3 \qquad \tau = \frac{t}{T} \qquad \Rightarrow \qquad s(t) = \frac{1}{3}\,t^2 - \frac{2}{27}\,t^3 \qquad t \in [0,3].$$

Combining the parametrized path $\boldsymbol{q}(s)$ with the timing law $s(t)$ yields the desired trajectory $\boldsymbol{q}(t)$ shown in Fig. 8. When comparing this with Fig. 5, one can appreciate the modulation of the joint path motion obtained through the timing law — in particular, the obtained zero initial and final velocities (horizontal tangents to $q_1(t)$ and $q_2(t)$ at $t = 0$ and $t = 3$ s) in face of nonzero tangents at the path boundaries $s = 0$ and $s = 1$.

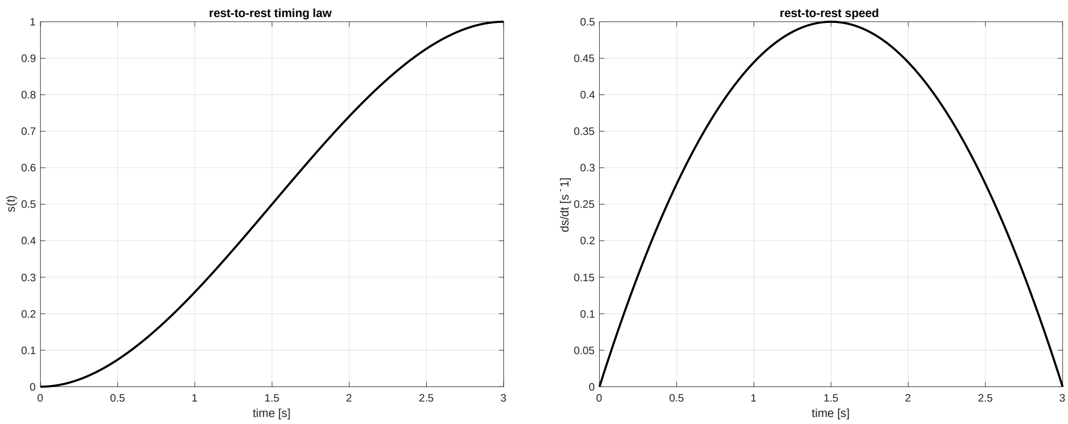


Figure 7: The rest-to-rest timing law $s(t)$ and its speed $\dot{s}(t)$.

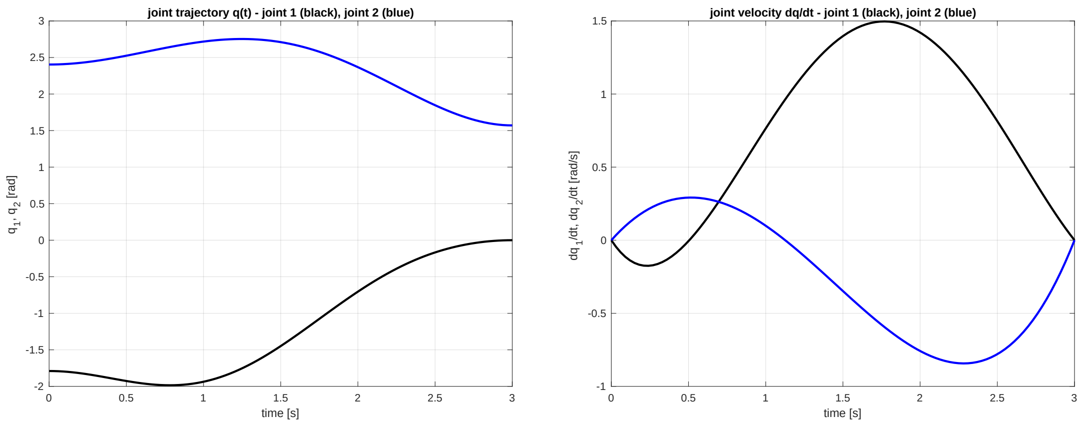


Figure 8: The two components of the resulting joint trajectory $\boldsymbol{q}(t)$ and velocity $\dot{\boldsymbol{q}}(t)$.

The requested values at the trajectory midtime $t = T/2 = 1.5$ s (or $\tau = 0.5$) are computed as follows. For the midtime configuration $\boldsymbol{q}_{\text{mid}}$:

$$s(t = 1.5) = s(\tau = 0.5) = 0.5 \quad \Rightarrow \quad \boldsymbol{q}_{\text{mid}} = \boldsymbol{q}(t = 1.5) = \boldsymbol{q}(s = 0.5) = \begin{pmatrix} -1.4488 \\ 2.7103 \end{pmatrix} \quad \text{[rad]};$$

for the midtime joint velocity $\dot{\boldsymbol{q}}_{\text{mid}}$:

$$\dot{\boldsymbol{q}}_{\text{mid}} = \dot{\boldsymbol{q}}(t = T/2) = \boldsymbol{q}'(s = 0.5)\,\dot{s}(t = T/2) = \begin{pmatrix} 2.7925 \\ -0.6956 \end{pmatrix} \cdot 0.5 = \begin{pmatrix} 1.3963 \\ -0.3478 \end{pmatrix} \quad \text{[rad/s]};$$

finally, for the midtime end-effector velocity $\dot{\boldsymbol{p}}_{\text{mid}}$:

$$\dot{\boldsymbol{p}}_{\text{mid}} = \boldsymbol{J}(\boldsymbol{q}_{\text{mid}})\,\dot{\boldsymbol{q}}_{\text{mid}} = \begin{pmatrix} 0.0400 & -0.9526 \\ 0.4260 & 0.3043 \end{pmatrix} \begin{pmatrix} 1.3963 \\ -0.3478 \end{pmatrix} = \begin{pmatrix} 0.3872 \\ 0.4890 \end{pmatrix} \quad \text{[m/s]}.$$

* * * * *

Robotics 1

July 8, 2024

Exercise 1

Based on the data sheet of the 7R robot shown in Fig. 1, assign the link frames and fill in the associated table of parameters according to the Denavit-Hartenberg (D-H) notation (use the attached sheet). The joint axes are labeled from A1 to A7. Frame 0 and frame 7 are already displayed; the green symbols $\otimes$ and $\odot$ denote here an axis *going in* or, respectively, *coming out* the sheet, so as to complete a right-handed frame. The assignment of the D-H frames should be such that all constant parameters are *non-negative*. Provide the numerical values of all parameters, including those of the joint variables $\theta_i \in (-\pi, \pi]$, for $i = 1, \ldots, 7$, in the configuration shown.

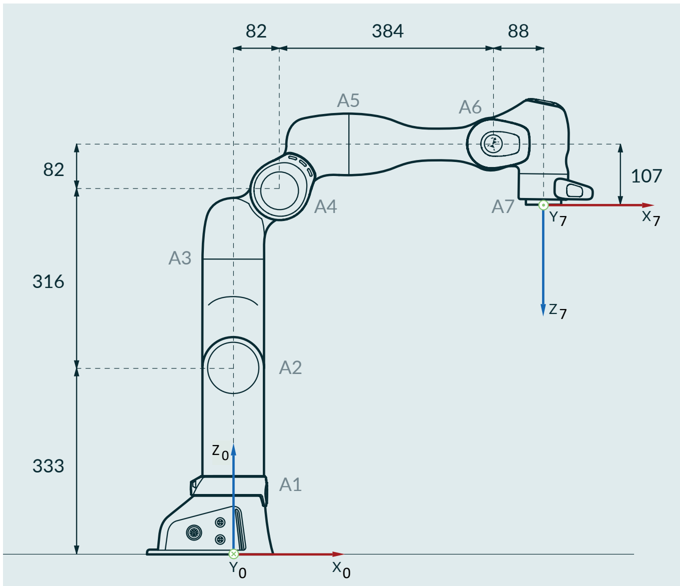


Figure 1: From the data sheet of a 7R robot. Lengths are in [mm].

Exercise 2

In the orientation specified by the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \end{pmatrix},$$

the end effector of a robot has the angular velocity

$$\boldsymbol{\omega} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{[rad/s]}.$$

Represent the orientation $\boldsymbol{R}$ with RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of fixed axes YZX and provide the value $\dot{\boldsymbol{\phi}}$ that produces the given angular velocity.

Exercise 3

Two equal 2R planar robots with unit length links share a collaborative task. With reference to Fig. 2, the base of robot A is placed at the origin of the world frame while robot B is mounted head down on the line Γ with a base that can slide on it. The line Γ is tilted by an angle $\gamma = 135°$ with respect to the x_w axis and intersects this axis at a distance $\Delta = 4$ m from the origin of the world frame. When robot A is in the configuration $\boldsymbol{q}_A = (\pi/4, -\pi/3)$ [rad], determine the position of the base of robot B on Γ and its configuration $\boldsymbol{q}_B$ such that the end effectors of the two robots are in the same position, aligned and facing each other. Is the solution found unique?

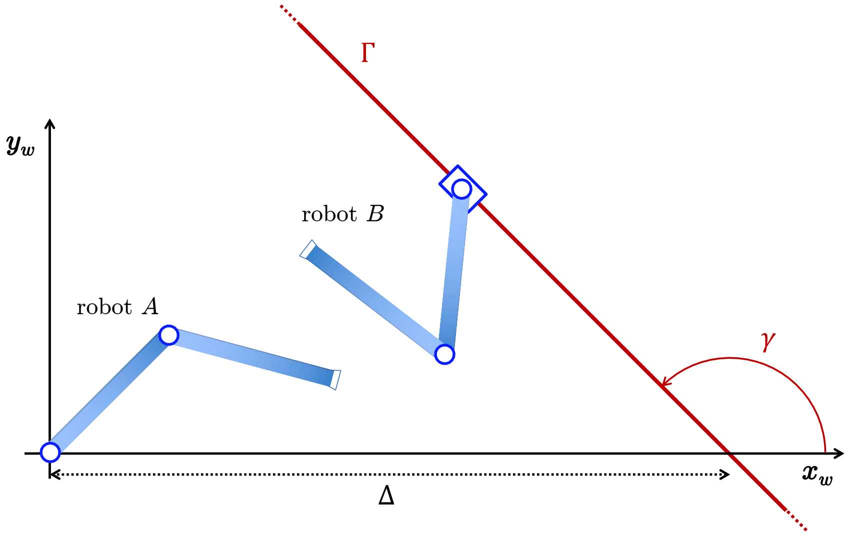


Figure 2: Set-up for the collaborative task.

Exercise 4

Consider a rest-to-rest trajectory planning problem for a RP planar robot. The robot should move its end effector along a linear path between the two Cartesian positions $\boldsymbol{p}_i = (0.6, -0.3)$ and $\boldsymbol{p}_f = (-0.3, 0.6)$ [m], using a trapezoidal speed profile. The velocities of the two joints are bounded by $|\dot{q}_1| \leq 2$ rad/s and $|\dot{q}_2| \leq 1$ m/s, while the acceleration along the path is bounded in norm as $\|\ddot{\boldsymbol{p}}\| \leq A = 0.5$ m/s^2. What is the minimum feasible motion time T for this task? Provide also the corresponding value of the joint velocity $\dot{\boldsymbol{q}}$ at the midpoint of the path.

[210 minutes (3,5 hours); open books]

Solution

July 8, 2024

Exercise 1

The 7R robot in Fig. 1 is a Franka Research 3. A correct assignment of D-H frames satisfying the requests is shown in Fig. 3, while Tab. 1 contains the corresponding (non-negative) constant parameters, as well as the values of the joint variables $\boldsymbol{\theta}$ in the configuration shown. The axes z_1, z_3 and z_5 are coming out the sheet (denoted with $\odot$).

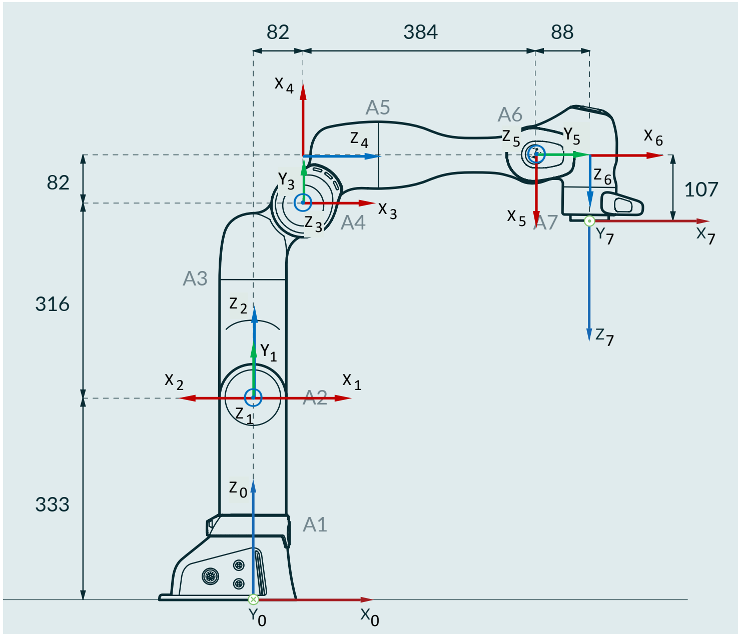


Figure 3: D-H frames for the 7R Franka Research 3 robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	333	0
2	$\pi/2$	0	0	π
3	$\pi/2$	82	316	π
4	$\pi/2$	82	0	$\pi/2$
5	$\pi/2$	0	384	π
6	$\pi/2$	88	0	$\pi/2$
7	0	0	107	0

Table 1: D-H parameters tor the frame assignment in Fig. 3 (units in [rad] or [mm]).

Exercise 2

The rotation matrix associated to the RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of fixed axes YZX is given by

$$\boldsymbol{R}_{YZX} = \boldsymbol{R}_X(\gamma)\boldsymbol{R}_Z(\beta)\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} c_\alpha c_\beta & -s_\beta & s_\alpha c_\beta \\ c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma & c_\beta c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\beta s_\gamma & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma \end{pmatrix}$$

The inverse relations from an orientation matrix $\boldsymbol{R} = \{R_{ij}\}$ to (α, β, γ) are given by

$$\beta = \text{atan2}\left\{-R_{12}, \pm\sqrt{R_{11}^2 + R_{13}^2}\right\} \qquad \alpha = \text{atan2}\left\{\frac{R_{13}}{c_\beta}, \frac{R_{11}}{c_\beta}\right\} \qquad \gamma = \text{atan2}\left\{\frac{R_{32}}{c_\beta}, \frac{R_{22}}{c_\beta}\right\},$$

out of the representation singularity $c_\beta = \sqrt{R_{11}^2 + R_{13}^2} = 0$.

For the given rotation matrix $\boldsymbol{R}$, this gives the two regular solutions

$$\boldsymbol{\phi}_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \pi/4 \\ 0 \\ \pi/2 \end{pmatrix} \qquad \boldsymbol{\phi}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -3\pi/4 \\ \pi \\ -\pi/2 \end{pmatrix}.$$

The contributions of the three time derivatives $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ to $\boldsymbol{\omega}$ when the orientation is $\boldsymbol{\phi}$ is computed as[1]

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\gamma}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\alpha}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\dot{\gamma} + \boldsymbol{R}_X(\gamma)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\dot{\beta} + \boldsymbol{R}_X(\gamma)\boldsymbol{R}_Z(\beta)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\alpha}$$

and thus $\boldsymbol{\omega} = \boldsymbol{T}(\beta, \gamma)\dot{\boldsymbol{\phi}}$ with

$$\boldsymbol{T}(\beta, \gamma) = \begin{pmatrix} -s_\beta & 0 & 1 \\ c_\beta c_\gamma & -s_\gamma & 0 \\ c_\beta s_\gamma & c_\gamma & 0 \end{pmatrix}.$$

Note that $\det \boldsymbol{T}(\beta, \gamma) = c_\beta$ vanishes exactly at the singularity of the YZX RPY-type representation. Evaluating $\boldsymbol{T}$ for the two solution triples $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ gives

$$\boldsymbol{T}_1 = \boldsymbol{T}(\beta_1, \gamma_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{T}_2 = \boldsymbol{T}(\beta_2, \gamma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Accordingly, we have the two solutions

$$\dot{\boldsymbol{\phi}}_1 = \boldsymbol{T}_1^{-1}\boldsymbol{\omega} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad \dot{\boldsymbol{\phi}}_2 = \boldsymbol{T}_2^{-1}\boldsymbol{\omega} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

[1] An alternative but more lengthy way would be to use the relationship $\dot{\boldsymbol{R}}(\boldsymbol{\phi})\boldsymbol{R}^T(\boldsymbol{\phi}) = \boldsymbol{S}(\boldsymbol{\omega})$, and then extracting from the off-diagonal elements of the skew-symmetric matrix $\boldsymbol{S}$ the components ω_x, ω_y and ω_z. Also, the elements of matrix $\dot{\boldsymbol{R}}$ have a linear dependence on the components $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ of $\dot{\boldsymbol{\phi}}$.

Exercise 3

The problem involves the use of a suitable homogeneous transformation between the base frames of the two robots and can be solved in different ways. The following is a simple one.

Through its direct kinematics, the end-effector position of robot A for $\boldsymbol{q}_A = (\pi/4, -\pi/3)$ [rad] is

$$ {}^w\boldsymbol{p}_A = \boldsymbol{f}_A(\boldsymbol{q}_A) = \begin{pmatrix} \cos q_{A1} + \cos(q_{A1} + q_{A2}) \\ \sin q_{A1} + \sin(q_{A1} + q_{A2}) \end{pmatrix} = \begin{pmatrix} 1.6730 \\ 0.4483 \end{pmatrix} \text{ [m].} $$

as expressed in the world frame, which coincides with the base frame of robot A. Robot B should place its end effector in this same position, facing the end effector of robot A and with an orientation that is aligned with the second link of this robot. Thus, it is convenient to *extend* the robot A by adding to its second link also the length of the second link of robot B (all links have unit length), namely with the modified direct kinematics

$$ {}^w\boldsymbol{p}_E = \begin{pmatrix} \cos q_{A1} + 2\cos(q_{A1} + q_{A2}) \\ \sin q_{A1} + 2\sin(q_{A1} + q_{A2}) \end{pmatrix} = \begin{pmatrix} 2.6390 \\ 0.1895 \end{pmatrix} \text{ [m].} $$

The position $\boldsymbol{p}_E$ is shown in Fig. 4. This point should be the target for the tip of the first link of robot B, without further conditions on the orientation part of the collaborative task (already satisfied by the 'trick' of extending the second link of robot A). Accordingly, the robot B mounted on a sliding base and taken up to the tip of the first link can be seen as an equivalent PR robot with a fixed base placed at the intersection between the line Γ and the world axis x_w.

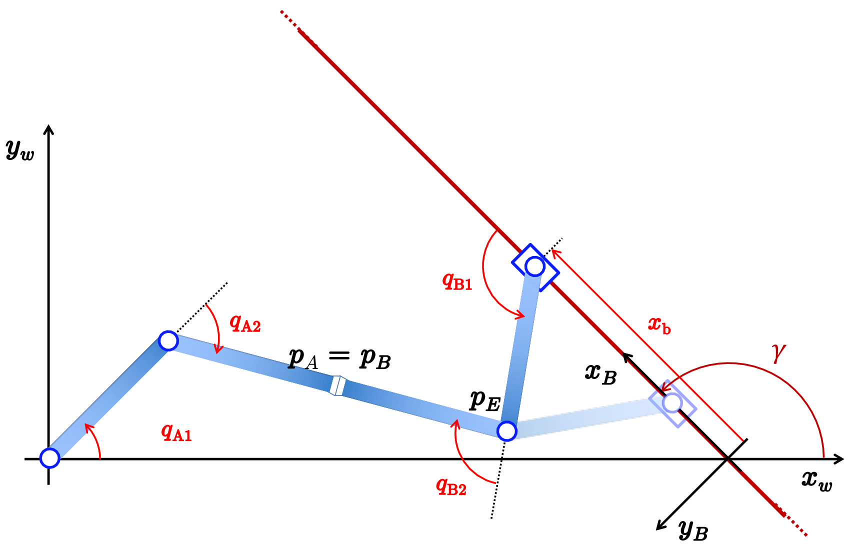


Figure 4: Graphical illustration of the two solutions for the collaborative task.

Place then the base frame of robot B as in Fig. 4. The homogeneous transformation[2] between the

[2]Since the problem is planar, we will use here a 3×3 homogeneous matrix, with a 2×2 rotation matrix $\boldsymbol{R} \in SO(2)$ and a position vector $\boldsymbol{p} \in \mathbb{R}^2$.

base frames $A\,(\equiv w)$ and B is then

$$
{}^w\boldsymbol{T}_B = {}^A\boldsymbol{T}_B = \begin{pmatrix} \cos\gamma & -\sin\gamma & \Delta \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 4 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

As a result, we compute

$$
\begin{aligned}
{}^B\boldsymbol{p}_{E,hom} &= {}^B\boldsymbol{T}_w\, {}^w\boldsymbol{p}_{E,hom} = {}^w\boldsymbol{T}_B^{-1} \begin{pmatrix} {}^w\boldsymbol{p}_E \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 2.8284 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 2.8284 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2.6390 \\ 0.1895 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.0964 \\ 0.8284 \\ 1 \end{pmatrix} = \begin{pmatrix} {}^B\boldsymbol{p}_E \\ 1 \end{pmatrix}.
\end{aligned}
$$

The direct kinematics of the equivalent PR robot is simply

$$
{}^B\boldsymbol{p}_{Beq} = \boldsymbol{f}_{\mathrm{PR}}(x_b, q_{B1}) = \begin{pmatrix} x_b + \cos q_{B1} \\ \sin q_{B1} \end{pmatrix}.
$$

The inverse kinematics is given by

$$
x_b = {}^B\boldsymbol{p}_{Beq,x} \pm \sqrt{1 - {}^B\boldsymbol{p}_{Beq,x}} \qquad q_{B1} = \mathrm{atan2}\left\{{}^B\boldsymbol{p}_{Beq,y}, {}^B\boldsymbol{p}_{Beq,x} - x_b\right\}.
$$

The joint angle q_{B2} of robot B is found by setting the difference between the absolute orientations of the two end effectors so that they face each other; i.e., the difference should be equal to π:

$$
(\gamma + q_{B1} + q_{B2}) - (q_{A1} + q_{A2}) = \pi \qquad \Rightarrow \qquad q_{B2} = q_{A1} + q_{A2} + \pi - (\gamma + q_{B1}).
$$

Setting now ${}^B\boldsymbol{p}_{Beq} = {}^B\boldsymbol{p}_E = (1.0964, 0.8284)$ [m], two solutions are found, as sketched graphically in Fig. 4; one solution is closer to the base frame of robot B

$$
x_b = 0.6037 \text{ m} \qquad \boldsymbol{q}_B = \begin{pmatrix} 1.0343 \\ -0.5107 \end{pmatrix} \text{ [rad]},
$$

while the other is further away

$$
x_b = 1.8647 \text{ m} \qquad \boldsymbol{q}_B = \begin{pmatrix} 2.3186 \\ -1.7950 \end{pmatrix} \text{ [rad]}.
$$

Exercise 4

The RP planar robot[3] and the desired motion task are shown in Fig. 5. The linear path has length $L = \left\|\boldsymbol{p}_f - \boldsymbol{p}_i\right\| = 1.2728$ m and is traced by

$$
\boldsymbol{p}(s) = \boldsymbol{p}_i + s\,\frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{L} \qquad s \in [0, L],
$$

where s is the path parameter (here, the arc length). The timing law $s(t)$, for $t \in [0, T]$, should have a rest-to-rest (symmetric) trapezoidal profile for the speed $\dot{s}$, which is fully described by the

[3]This is by default the most common structure of a PR planar robot. Moreover, we shall assume that $q_2 > 0$.

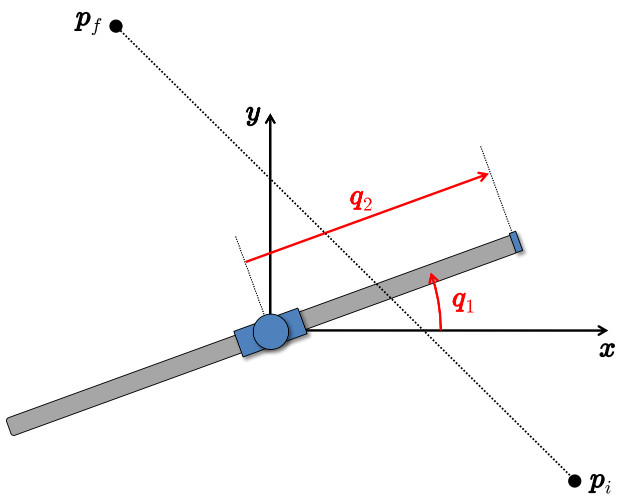


Figure 5: Trajectory planning task for a RP robot.

cruising speed V along the path and by the acceleration A in the first phase (with a rise time $T_r = V/A$). The Cartesian velocity and acceleration of the robot end effector are, respectively,

$$\dot{\boldsymbol{p}} = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{L}\,\dot{s} \qquad \ddot{\boldsymbol{p}} = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{L}\,\ddot{s}.$$

While we have for the acceleration norm $\|\ddot{\boldsymbol{p}}\| = |\ddot{s}| \leq A = 0.5$ m/s^2, there is instead no explicit bound specified in the Cartesian space for the velocity norm $\|\dot{\boldsymbol{p}}\| = |\dot{s}| \leq V$. This should be derived from the available velocity limits in the joint space.

Note first that, using the inverse kinematics of the RP robot

$$q_1 = \text{atan2}\{p_y, p_x\} \qquad q_2 = \|\boldsymbol{p}\| = \sqrt{p_x^2 + p_y^2},$$

we obtain from the initial and final Cartesian points $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$

$$\boldsymbol{q}_i = \begin{pmatrix} -0.4636 \\ 0.6708 \end{pmatrix} \text{[rad, m]} \qquad \boldsymbol{q}_f = \begin{pmatrix} 2.0344 \\ 0.6708 \end{pmatrix} \text{[rad, m]}.$$

The revolute (first) joint has to travel by $\Delta q_1 = q_{f,1} - q_{i,1} = 2.4981$ rad. Therefore, its motion time is lower bounded by $|\Delta q_1|/V_1 = 1.2490$ s (assuming an infinite joint acceleration). Moreover, the joint value at the midpoint is $q_{m,1} = (q_{i,1} + q_{f,1})/2 = 0.7854$ rad. On the other hand, since $q_{f,2} = q_{i,2} = 0.6708$, the prismatic (second) joint needs first to reduce its length in order to remain on the linear Cartesian path, and then to reverse motion increasing the length back to the initial value in a symmetric way with respect to the path midpoint; the minimum extension will be at $\boldsymbol{p}_m = (\boldsymbol{p}_i + \boldsymbol{p}_f)/2 = (0.15, 0.15)$ m, corresponding to $q_{m,2} = 0.2121$ m. Therefore, the motion time of the second joint is lower bounded by $(|q_{m,2} - q_{i,2}| + |q_{f,2} - q_{m,2}|)/V_2 = 0.9174$ s (assuming again an infinite joint acceleration).

The above analysis shows that the limiting velocity factor is due to the revolute joint. As a result, we can take as upper bound for the Cartesian speed along the path the worst case situation, namely when the distance to the path is minimum, i.e., at $q_{m,2} = 0.2121$ m, and evaluate

$$|\dot{s}| \leq V = q_{m,2} \cdot V_1 = 0.4243 \text{ m/s}.$$

With L, V, and A, we compute the minimum feasible motion time along the linear path when using a trapezoidal profile[4] as

$$T = \frac{LA + V^2}{VA} = \frac{L}{V} + \frac{V}{A} = 3.8485 \text{ s}.$$

To evaluate the joint velocity $\dot{\boldsymbol{q}}$ at the path midpoint (corresponding to $\boldsymbol{q}_m = (0.7854, 0.2121)$ [rad,m]), we need the task Jacobian for this robot:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}_m = \boldsymbol{J}(\boldsymbol{q}_m) = \begin{pmatrix} -0.15 & \sqrt{2}/2 \\ 0.15 & \sqrt{2}/2 \end{pmatrix}.$$

Being $\dot{\boldsymbol{p}}_m = V(\boldsymbol{p}_f - \boldsymbol{p}_i)/L = (-0.3, 0.3)$ [m/s] (the speed at the path midpoint is certainly at the cruise value), we have as expected

$$\dot{\boldsymbol{q}}_m = \boldsymbol{J}_m^{-1} \dot{\boldsymbol{p}}_m = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ [rad, m]}.$$

Figure 6 shows the components of the planned trajectory in the Cartesian space and of the corresponding trajectory in the joint space, together with their velocity and acceleration.

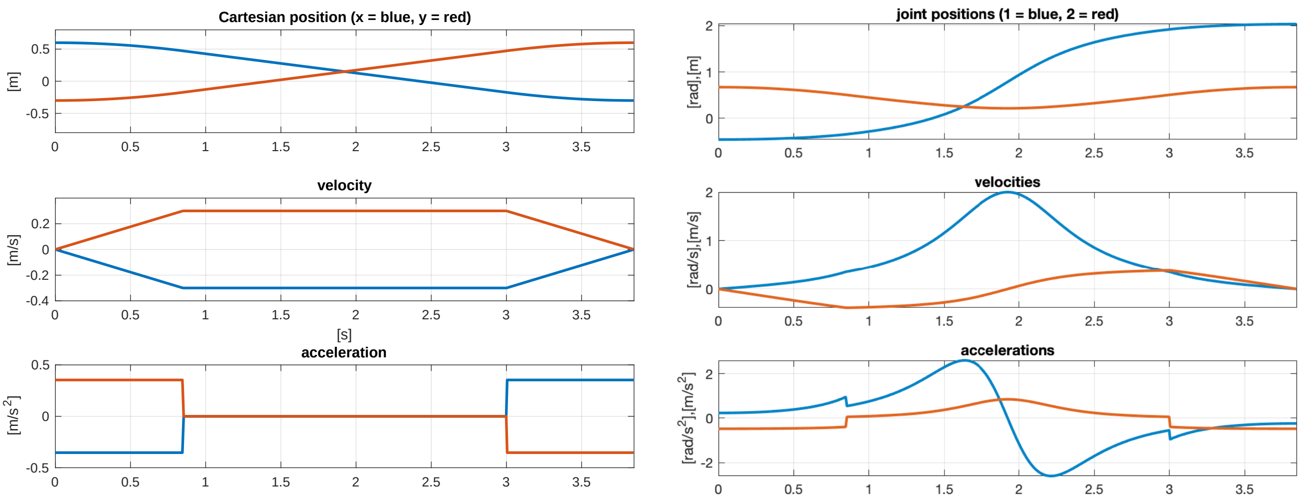


Figure 6: *Left:* The components of the minimum-time Cartesian trajectory using a trapezoidal speed profile. *Right:* The components of the corresponding jont trajectory.

* * * * *

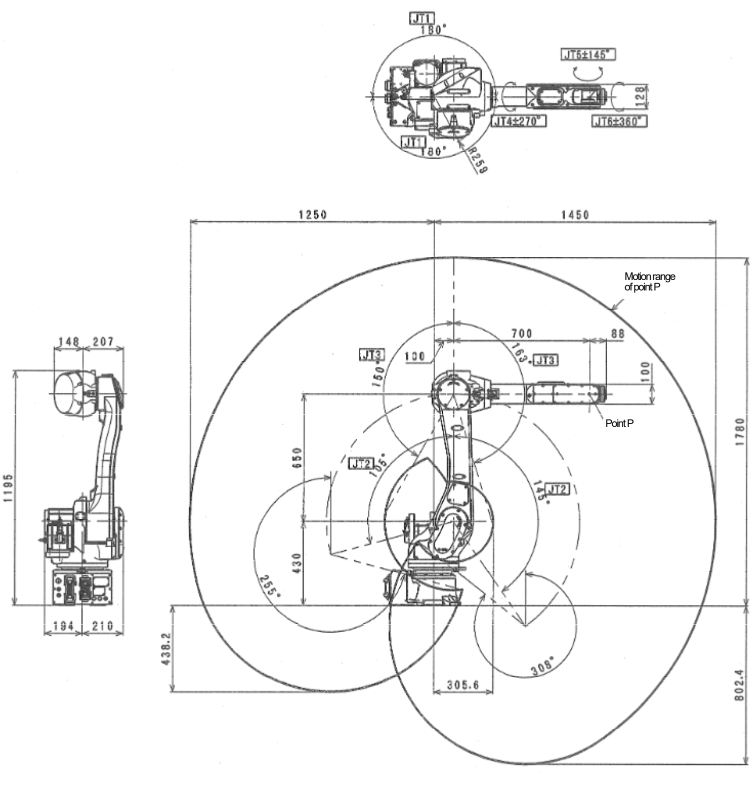
[4] Since $L > V^2/A$, the existence of a motion phase at cruise speed V is guaranteed.

Robotics 1

September 19, 2024

Exercise 1

Figure 1 is taken from the data sheet of the Kawasaki RS010N, a robot with 6 revolute joints and a spherical wrist. Assign the link frames to this robot and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) convention (use the attached sheet). Frame 0 is placed on the ground and frame 6 at the end of the final flange. Provide the numerical values of all parameters, including those of the joint variables θ_i, $i = 1, \dots, 6$, in the configuration shown.[1]



Model	Vertically articulated robot		
Degree of Freedom of Motion	6		
Motion Range and Maximum Speed	JT	Motion Range	Max. Speed
	1	±180°	250°/s
	2	+145° to -105°	250°/s
	3	+150° to -163°	215°/s
	4	±270°	365°/s
	5	±145°	380°/s
	6	±360°	700°/s

Figure 1: From the data sheet of the Kawasaki RS010N robot. Lengths are in [mm].

[1] Motion ranges in the data sheet do not necessarily correspond to the values assumed by the DH joint variables.

Exercise 2

With reference to the configuration shown in Fig. 1 for the Kawasaki robot, taking into account the (symmetric) limits of joint velocities given in the data sheet, compute numerically the following quantities (expressed in the base frame of the robot).

[a] The velocity $\boldsymbol{v}_P$ of Point P (the center of the spherical wrist) when $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ assume their maximum absolute values and their signs are chosen so to have the largest possible norm $\|\boldsymbol{v}_P\|$.

[b] The angular velocity $\boldsymbol{\omega}_6^{[b]}$ of the last DH frame when choosing the joint velocities as in **[a]** and with the wrist joints frozen.

[c] The angular velocity $\boldsymbol{\omega}_6^{[c]}$ of the last DH frame when the first three joints are frozen and $\dot{q}_4$, $\dot{q}_5$, and $\dot{q}_6$ assume their maximum *positive* values, according to the direction of the $\boldsymbol{z}_i$ axes $(i = 3, 4, 5)$ in the chosen DH assignment.

Exercise 3

Joints 2 and 3 of the robot in Fig. 1 should move from rest to rest, in minimum time, and in a coordinated way, starting from the lower limit and reaching the upper limit of their respective motion ranges. Assuming that the maximum absolute accelerations of the two joints are

$$A_{max,2} = 5.5 \qquad A_{max,3} = 7 \qquad [\text{rad/s}^2],$$

determine the minimum feasible time T_{min} for the coordinated joint motion. Draw the profiles of the planned velocity and acceleration of the two joints (using radians and not degrees!).

Exercise 4

Assume that the robot in Fig. 1 mounts two optical encoders of the incremental type on the motor axes of joint 2 and 3, respectively with $N_2 = 4000$ and $N_3 = 2600$ pulses per turn (after electronic quadrature), while the reduction ratios of the corresponding transmission gears are $n_{r2} = 40$ and $n_{r3} = 20$. When the robot is in the shown configuration, an instantaneous displacement is commanded to Point P in the upward vertical direction $\boldsymbol{z}_0$. What is the minimum displacement Δz of Point P in this direction that can be measured by the encoders?

[180 minutes (3 hours); open books]

Solution

September 19, 2024

Exercise 1

A possible assignment of DH frames for the Kawasaki RS010N robot is shown in Fig. 2, with Tab. 1 containing the corresponding constant parameters, as well as the values of the joint variables $\boldsymbol{\theta}$ in the configuration shown. In the figure, the x_i axes are shown in blue, the y_i axes in green, and the z_i axes in red. Off-plane axes are not indicated, but they complete as usual a right-handed frame.

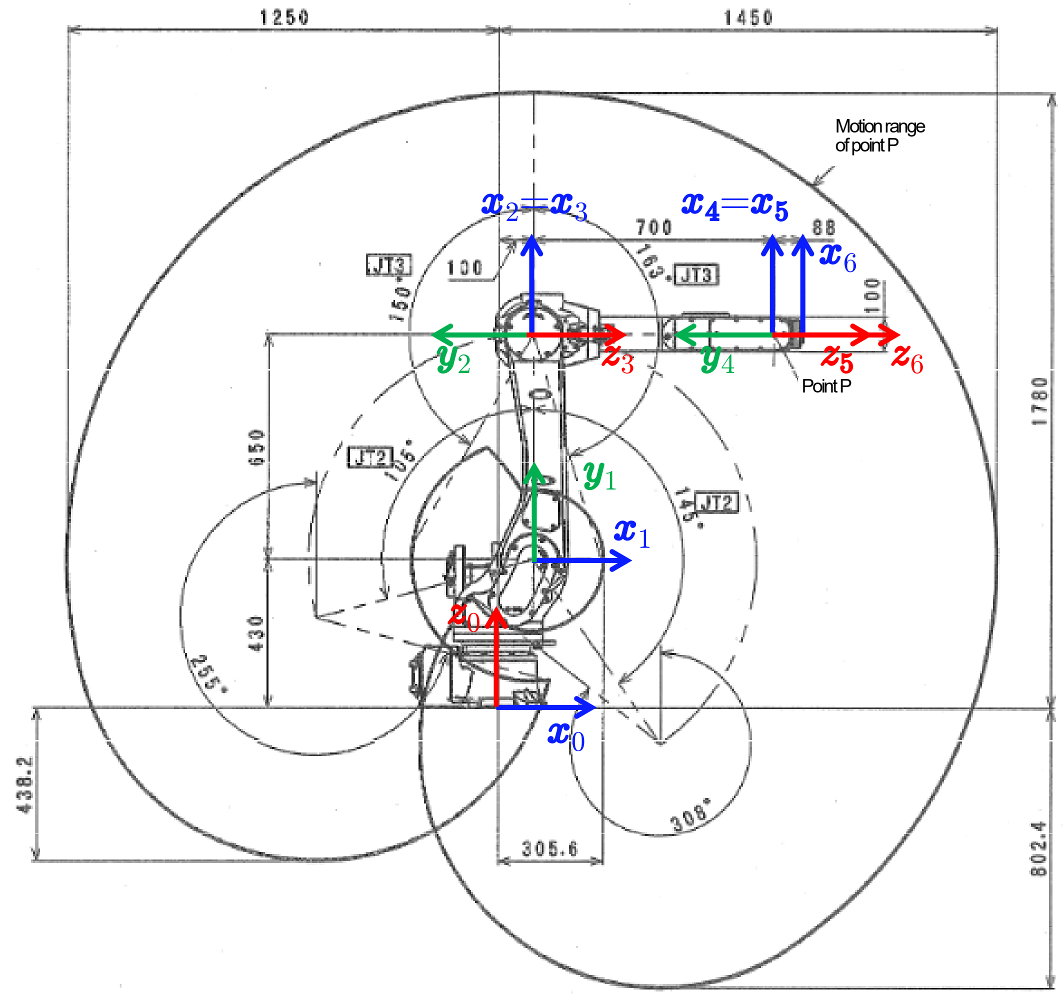


Figure 2: A possible assignment of DH frames for the Kawasaki RS010N robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	100	430	0
2	0	650	0	$\pi/2$
3	$\pi/2$	0	0	0
4	$-\pi/2$	0	700	0
5	$\pi/2$	0	0	0
6	0	0	88	0

Table 1: DH parameters tor the frame assignment in Fig. 2 (units in [rad] or [mm]).

Exercise 2

For all three cases, we need to compute the 6×6 geometric Jacobian of the robot which, in view of the presence of a spherical wrist, takes the form

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{L,P}(\boldsymbol{q}_b) & \boldsymbol{O} \\ \boldsymbol{J}_{A,b}(\boldsymbol{q}_b) & \boldsymbol{J}_{A,w}(\boldsymbol{q}) \end{pmatrix},$$

where we $\boldsymbol{q}_b = (q_1, q_2, q_3)$ is the vector of the first three (base) joints. Accordingly, $\boldsymbol{q}_w = (q_4, q_5, q_6)$ will be the vector of the last three (wrist) joints.

[**a**] For the velocity of Point P, only the first three joints matter. Moreover, to compute $\boldsymbol{v}_P$ one can equivalently use the 3×3 analytic Jacobian obtained by differentiation of the direct kinematics of Point P, which coincides with the origin of frame 4 in the DH convention. Using the parameters in Tab. 1, one has

$$\begin{aligned}\boldsymbol{p}_{P,hom}(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{p}_P(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)\,{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \\ &= {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(a_1 + a_2 c_2 + d_4 s_{23}) \\ s_1(a_1 + a_2 c_2 + d_4 s_{23}) \\ d_1 + a_2 s_2 - d_4 c_{23} \\ 1 \end{pmatrix}.\end{aligned}$$

Thus, we obtain

$$\boldsymbol{J}_{L,P}(\boldsymbol{q}_b) = \frac{\partial \boldsymbol{p}_P}{\partial \boldsymbol{q}_b} = \begin{pmatrix} -s_1(a_1 + a_2 c_2 + d_4 s_{23}) & c_1(d_4 c_{23} - a_2 s_2) & d_4 c_1 c_{23} \\ c_1(a_1 + a_2 c_2 + d_4 s_{23}) & s_1(d_4 c_{23} - a_2 s_2) & d_4 s_1 c_{23} \\ 0 & a_2 c_2 + d_4 s_{23} & d_4 s_{23} \end{pmatrix}. \tag{1}$$

When evaluated using the numerical DH parameters (with length expressed in [m]) and in the configuration shown in Fig. 2, we have

$$\boldsymbol{J}_{L,P} = \begin{pmatrix} 0 & -0.65 & 0 \\ 0.8 & 0 & 0 \\ 0 & 0.7 & 0.7 \end{pmatrix}.$$

The maximum velocity in norm of Point P is obtained when taking the velocities of joints 2 and 3 at their limit value and with the *same* (positive or negative) sign, independently of the velocity of joint

1 (which can be as well positive or negative in the solution). Taking into account the conversion from [°/s] to [rad/s], one has for example with all positive (and maximum) joint velocities

$$\boldsymbol{v}_P = \boldsymbol{J}_{L,P} \begin{pmatrix} \dot{q}_{max,1} \\ \dot{q}_{max,2} \\ \dot{q}_{max,3} \end{pmatrix} = \boldsymbol{J}_{L,P} \begin{pmatrix} 250\,°/\mathrm{s} \\ 250\,°/\mathrm{s} \\ 215\,°/\mathrm{s} \end{pmatrix} \cdot \frac{\pi}{180°} = \begin{pmatrix} -2.8362 \\ 3.4907 \\ 5.6810 \end{pmatrix} \text{[m/s]} \quad \Rightarrow \quad \|\boldsymbol{v}_P\| = 7.2459,$$

while flipping for instance the sign of the second joint velocity one obtains

$$\boldsymbol{v}_P = \boldsymbol{J}_{L,P} \begin{pmatrix} \dot{q}_{max,1} \\ -\dot{q}_{max,2} \\ \dot{q}_{max,3} \end{pmatrix} = \begin{pmatrix} 2.8362 \\ 3.4907 \\ -0.4276 \end{pmatrix} \text{[m/s]} \quad \Rightarrow \quad \|\boldsymbol{v}_P\| = 4.5179,$$

namely, a lower norm for the velocity of point P.

[b] The 3×6 angular part $\boldsymbol{J}_A$ of the geometric Jacobian is computed using the $\boldsymbol{z}_i$ axes of the DH frames, for $i = 0, 1, \ldots, 5$:

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 \qquad \ldots \qquad \boldsymbol{z}_5 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\ldots{}^4\boldsymbol{R}_5(q_5)\boldsymbol{z}_0,$$

where the rotation matrices are extracted from the DH homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i$, for $i = 1, \ldots, 5$. We obtain

$$\boldsymbol{J}_{A,b}(\boldsymbol{q}_b) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{J}_{A,w}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{pmatrix} = \begin{pmatrix} c_1 s_{23} & s_1 c_4 - c_1 s_4 c_{23} & s_5(s_1 s_4 + c_1 c_4 c_{23}) + c_5 c_1 s_{23} \\ s_1 s_{23} & -c_1 c_4 - s_1 s_4 c_{23} & c_5 s_1 s_{23} - s_5(c_1 s_4 - c_4 s_1 c_{23}) \\ -c_{23} & -s_{23} s_4 & s_{23} c_4 s_5 - c_{23} c_5 \end{pmatrix}.$$

Evaluating these matrices as before in the configuration shown in Fig. 2, we have

$$\boldsymbol{J}_{A,b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{J}_{A,w} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a result, when $\dot{\boldsymbol{q}}_b$ is chosen as in the solution used in **[a]** and $\dot{\boldsymbol{q}}_w = \boldsymbol{0}$, one has

$$\boldsymbol{\omega}_6^{[b]} = \boldsymbol{J}_{A,b} \begin{pmatrix} \dot{q}_{max,1} \\ \dot{q}_{max,2} \\ \dot{q}_{max,3} \end{pmatrix} = \begin{pmatrix} 0 \\ -8.1158 \\ 4.3633 \end{pmatrix} \text{[rad/s]}.$$

[c] Conversely, when $\dot{\boldsymbol{q}}_b = \boldsymbol{0}$ and $\dot{\boldsymbol{q}}_w$ has its components at their maximum positive values (i.e., rotating counterclockwise around the joint axes $\boldsymbol{z}_i$ of the wrist), one obtains

$$\boldsymbol{\omega}_6^{[c]} = \boldsymbol{J}_{A,w} \begin{pmatrix} \dot{q}_{max,4} \\ \dot{q}_{max,5} \\ \dot{q}_{max,6} \end{pmatrix} = \begin{pmatrix} 18.5878 \\ -6.6323 \\ 0 \end{pmatrix} \text{[rad/s]}.$$

Exercise 3

This trajectory planning problem is tackled first separately for each of the two joints. The rest-to-rest minimum time problem under bounds on velocity and acceleration is solved using a trapezoidal velocity profile (bang-coast-bang in acceleration), which may also collapse into a triangular velocity profile (without the coast phase) if the required displacement L is too short with respect to the given bounds (i.e., if $L \leq V_{max}^2/A_{max}$). However, this is not the case here. Once expressed in radians, the required displacements (from the lower to the upper limit of the motion range) are

$$L_2 = Q_{max,2} - Q_{min,2} = (145° - (-105°)) \cdot \frac{\pi}{180°} = 4.36 \text{ rad}$$
$$L_3 = Q_{max,3} - Q_{min,3} = (150° - (-163°)) \cdot \frac{\pi}{180°} = 5.46 \text{ rad},$$

and the check for the existence of a coast phase with constant (maximum) cruising velocity, with $V_{max,2} = 4.36$ and $V_{max,3} = 3.75$ [rad/s] (converted from the values in [°/s] of the data sheet)

$$L_2 = 4.36 > 3.46 = \frac{V_{max,2}^2}{A_{max,2}} \qquad L_3 = 5.46 > 2.01 = \frac{V_{max,3}^2}{A_{max,3}}$$

are satisfied in both cases. Therefore, the minimum time for the desired motion of the two joints when considered separately are

$$T_{min,2} = \frac{L_2}{V_{max,2}} + \frac{V_{max,2}}{A_{max,2}} = 1.79 \text{ s} \qquad T_{min,3} = \frac{L_3}{V_{max,3}} + \frac{V_{max,3}}{A_{max,3}} = 1.99 \text{ s},$$

with associated rising times $T_{s,2} = V_{max,2}/A_{max,2} = 0.79$, $T_{s,3} = V_{max,3}/A_{max,3} = 0.54$ [s].

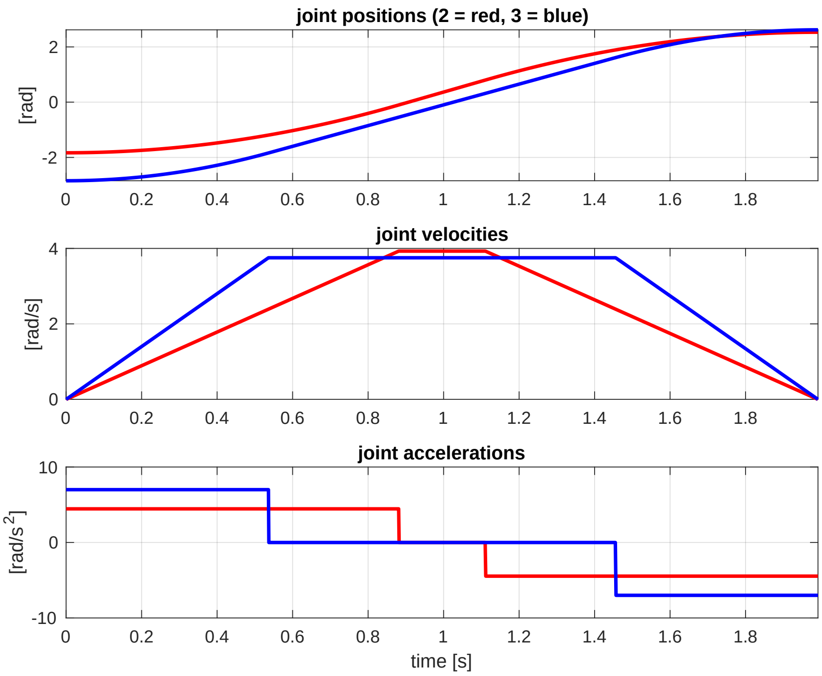


Figure 3: The coordinated minimum-time trajectory for the two joints 2 and 3.

At this stage, since a coordinated joint motion is required, the minimum coordinated motion time will be

$$T_{min} = \max\{T_{min,2}, T_{min,3}\} = 1.99 \text{ s},$$

and the fastest joint, namely joint 2, must be slowed down. While joint 3 will keep the same trapezoidal profile computed independently, with cruise speed $V_{max,3} = 3.75$ rad/s, acceleration $A_{max,3} = 7$ rad/s^2 and rising time $T_{s,3} = 0.53$ s, the trajectory of joint 2 will be uniformly scaled in time by a factor

$$k = \frac{T_{min}}{T_{min,2}} = \frac{1.99}{1.79} = 1.11 > 1,$$

thus specifying scaled rising time $T_2 > T_{s,2}$, cruise speed $V_2 < V_{max,2}$ and acceleration $A_2 < A_{max,2}$ as

$$T_2 = k\,T_{s,2} = 0.88 \text{ s} \qquad V_2 = \frac{V_{max,2}}{k} = 3.92 \text{ rad/s} \qquad A_2 = \frac{A_{max.2}}{k^2} = 4,46 \text{ rad/s}^2.$$

The resulting coordinated motion trajectories of the two joints are shown in Fig. 3, together with their trapezoidal velocity and bang-coast-bang acceleration profiles.

Exercise 4

The resolutions of the two encoders on the motor side are

$$r_{m2} = \frac{2\pi}{N_2} = 157.08 \cdot 10^{-5} \text{ rad} \qquad r_{m3} = \frac{2\pi}{N_3} = 241.66 \cdot 10^{-5} \text{ rad},$$

while on the link side of the gears we have

$$r_2 = \frac{r_{m2}}{n_{r2}} = 3.93 \cdot 10^{-5} \text{ rad} \qquad r_3 = \frac{r_{m3}}{n_{r3}} = 12.08 \cdot 10^{-5} \text{ rad}.$$

The part of the Jacobian $\boldsymbol{J}_{L,P}(\boldsymbol{q}_b)$ in eq. (1) that is involved in the assignment of a vertical velocity to Point P is given by the 2×2 matrix made by the first and third rows (respectively, along the $\boldsymbol{x}_0$ and $\boldsymbol{z}_0$ directions) and the second and third columns (corresponding to $\dot{q}_2$ and $\dot{q}_3$), namely

$$\bar{\boldsymbol{J}}_P(\boldsymbol{q}_b) = \begin{pmatrix} c_1(d_4 c_{23} - a_2 s_2) & d_4 c_1 c_{23} \\ a_2 c_2 + d_4 s_{23} & d_4 s_{23} \end{pmatrix}.$$

Evaluating this matrix in the configuration shown in Fig. 1 gives

$$\bar{\boldsymbol{J}}_P = \begin{pmatrix} -0.64 & 0 \\ 0.7 & 0.7 \end{pmatrix}.$$

Thus, a desired instantaneous displacement $\boldsymbol{\Delta p}_P = \begin{pmatrix} 0 & \Delta z \end{pmatrix}^T$ m of Point P along the upward vertical direction will be realized by the joint displacement

$$\boldsymbol{\Delta \bar{q}} = \begin{pmatrix} \Delta q_2 \\ \Delta q_3 \end{pmatrix} = \bar{\boldsymbol{J}}_P^{-1} \begin{pmatrix} 0 \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 10\Delta z/7 \end{pmatrix} \text{ [rad]}.$$

From this relation, it follows that the desired displacement $\boldsymbol{\Delta p}_P$ will not require any motion of joint 2 (so, no motion will be measured instantaneously by the encoder at this joint). On the other hand, the encoder at joint 3 will detect a Cartesian displacement of Point P along the $\boldsymbol{z}_0$ direction as long as this is larger or equal than its resolution beyond the transmission gear, or

$$\frac{10\Delta z}{7} \geq r_3 \qquad \Rightarrow \qquad \Delta z \geq 0.7 \cdot 12.08 \cdot 10^{-5} \text{ m} = 8.45 \cdot 10^{-5} \text{ m} = 0.0845 \text{ mm}.$$

* * * * *

Robotics 1

November 7, 2024

Exercise 1

Consider the PPR planar robot with a 2-jaw gripper in Fig. 1, shown together with the world frame RF_w. Assign the link frames and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) standard convention. Determine also the homogeneous transformation matrices ${}^w\boldsymbol{T}_0$ and ${}^3\boldsymbol{T}_e$, respectively between the world frame RF_w and the zero-th DH frame RF_0 and between the last DH frame RF_3 and the end-effector frame RF_e placed at the gripper with the usual convention ($\boldsymbol{z}_e$ in the approach direction and $\boldsymbol{y}_e$ in the open/close slide direction of the jaws). Provide the direct kinematics for the end-effector position ${}^w\boldsymbol{p}_e = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^3$.

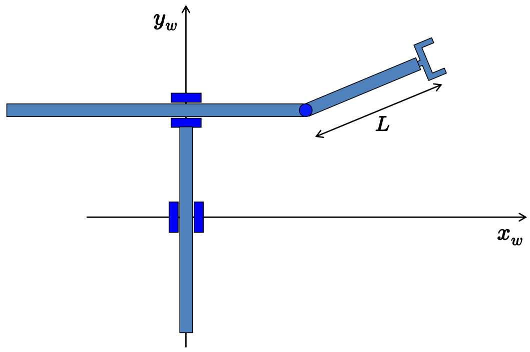


Figure 1: A PPR planar robot with the last link of length L.

Exercise 2

Based on the definition of joint variables in Exercise 1, solve the inverse kinematics problem in closed form for the PPR planar robot in terms of the components of the end-effector task vector $\boldsymbol{r} = ({}^w p_{e,x}, {}^w p_{e,y}, {}^w \alpha_e) \in \mathbb{R}^3$, where ${}^w\alpha_e = \text{atan2}\left\{\boldsymbol{z}_e^T \boldsymbol{y}_w, \boldsymbol{z}_e^T \boldsymbol{x}_w\right\}$ is the planar orientation angle of the gripper. For $\boldsymbol{r} = (1, 0, \pi/2)$ [m,m,rad], give the numerical solution(s) when $L = 0.5$ m.

Exercise 3

With reference to the DH assignment in Exercise 1, determine the 6×3 geometric Jacobian ${}^w\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the end-effector velocities ${}^w\boldsymbol{v}_e$ and ${}^w\boldsymbol{\omega}_e$, all expressed in the world frame. Find then all possible configurations for which this matrix loses rank. Next, with the robot in the configuration $\bar{\boldsymbol{q}} = (1, 1, 0)$ [m,m,rad] and with $L = 0.5$ m:

a) compute the joint velocity $\dot{\boldsymbol{q}}_a$ producing the end-effector velocity ${}^w\boldsymbol{v}_e = (-2, 1, 0)$ [m/s];

b) compute the joint velocity $\dot{\boldsymbol{q}}_b$ producing the angular velocity ${}^w\boldsymbol{\omega}_e = (0, 0, -3)$ [rad/s] of the end-effector frame;

c) determine whether the end-effector (twist) velocity $\left({}^w\boldsymbol{v}_e^T \;{}^w\boldsymbol{\omega}_e^T\right)^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T$ is admissible for this robot and, if so, compute the joint velocity $\dot{\boldsymbol{q}}_c$ that realizes it.

Exercise 4

For the given PPR robot. with a generic length L of the third link, consider the joint variables as in Exercise 1 and the task variables as in Exercise 2. Plan a path $\boldsymbol{r}_d(s)$, parametrized by $s \in [0, 1]$, made by a circle of radius $R = 3L/4$ traced counterclockwise, with the gripper always oriented along the path normal and pointing outside the circle. At the start ($s = 0$), the path should be matched with the initial robot configuration $\boldsymbol{q}_0 = (1, 2, -\pi/4)$ [m,m,rad].

[180 minutes (3 hours); open books]

Robotics 1

Midterm Test — November 22, 2024

Exercise 1

The end-effector of a robot manipulator has an initial orientation specified by the ZXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, \pi/4, -\pi/4)$ [rad] and should reach a final orientation specified by an axis-angle pair $(\boldsymbol{r}, \theta)$, with $\boldsymbol{r} = (0, -\sqrt{2}/2, \sqrt{2}/2)$ and $\theta = \pi/6$ rad. What is the required rotation matrix $\boldsymbol{R}_{if}$ between these two orientations? Represent $\boldsymbol{R}_{if}$ by the RPY-type angles (ϕ, χ, ψ) around the fixed-axes sequence YXY.

Exercise 2

A cylinder of height h and radius r lies on the plane (x_w, y_w) in the initial pose shown in Fig. 1, with a frame $RF_c = (x_c, y_c, z_c)$ attached to the geometric center of its body. The cylinder rolls without slipping by a ground distance $d > 0$ in the y_w-direction, and rotates then by an angle ϑ around the original z_w-axis. Finally, a rotation φ is performed around the current direction of the z_c-axis. Determine the expression of the elements of the homogeneous transformation matrix ${}^w\boldsymbol{T}_c(h, r, d, \vartheta, \varphi)$ that characterizes the final pose of the cylinder. Evaluate then ${}^w\boldsymbol{T}_c$ for $h = 0.5$, $r = 0.1$, $d = 1.5$ [m] and $\vartheta = \pi/3$, $\varphi = -\pi/2$ [rad]. *Hint: Check your intermediate results with simpler data.*

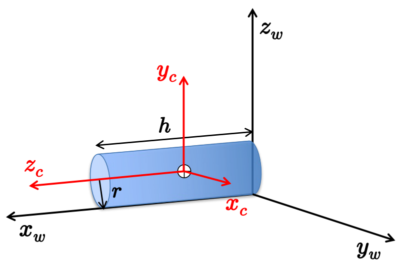


Figure 1: The initial set-up of a cylinder in the world frame.

Exercise 3

Consider the PPR planar robot with a 2-jaw gripper in Fig. 2, shown together with the world frame RF_w.

- Assign the link frames and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) convention (use the extra sheet). The origin of the last DH frame should be placed at the gripper's center (point P). Choose the frames so that there is **no** axis pointing inside the sheet.
- Determine the homogeneous transformation matrices ${}^w\boldsymbol{T}_0$ and ${}^3\boldsymbol{T}_e$, respectively between the world frame RF_w and the zero-th DH frame RF_0 and between the last DH frame RF_3 and the end-effector frame RF_e placed at the gripper, with the usual convention ($\boldsymbol{z}_e$ in the approach direction and $\boldsymbol{y}_e$ in the open/close slide direction of the jaws).
- Provide the direct kinematics for the end-effector position ${}^w\boldsymbol{p}_e \in \mathbb{R}^3$.
- When the two prismatic joints are limited as $q_i \in [q_{i,m}, q_{i,M}]$, under the assumption that $q_{i,M} - q_{i,m} > 2L$, for $i = 1, 2$, and the revolute joint is in the range $q_3 \in [-3\pi/4, 0]$, sketch the primary workspace of this robot and locate the relevant points on its boundary.

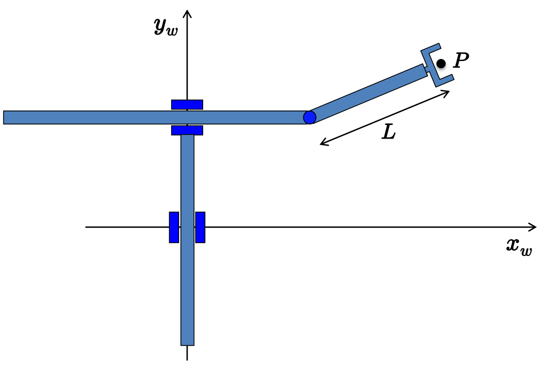


Figure 2: A PPR planar robot with last link of length L.

Exercise 4

With reference to the scheme in Fig. 3, assume that the three toothed gears of the transmission have radius, respectively, $r_m = 0.5$, $r_e = 40$, and $r_l = 10$ [cm]. The motor inertia is $J_m = 7.1 \cdot 10^{-4}$ kgm^2, while the inertia of the link around its rotation axis is denoted by J_l. An incremental encoder is mounted on the axis of the middle gear. Gravity is absent and inertia and friction of the gears are negligible.

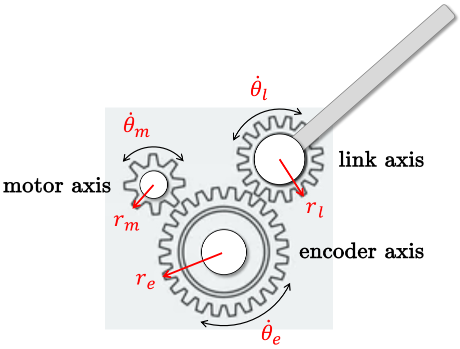


Figure 3: Transmission gears from motor to link, using an incremental encoder.

- What is the value of the link inertia J_l that optimizes torque transmission?
- With this J_l, what is the acceleration $\ddot{\theta}_l$ when the motor delivers on its axis a torque $\tau_m = 10$ [Nm]?
- For a link resolution of 0.01°, how many pulses per turn (with quadrature) should the encoder have?
- With this resolution, what is the average speed $\dot{\theta}_m$ when the encoder increments 100 pulses per second?

Exercise 5

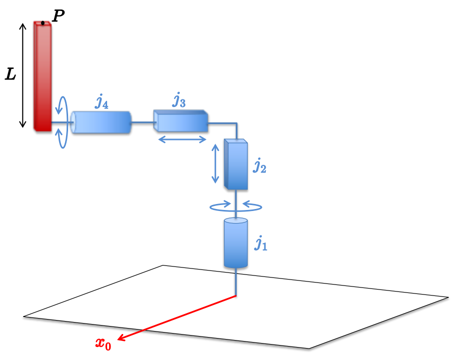


Figure 4: An RPPR spatial robot.

i	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$\pi/2$	0	q_2	0
3	0	0	q_3	0
4	0	L	0	q_4

Table 1: D-H parameters of the RPPR robot.

The RPPR spatial robot shown in Fig. 4 has the DH parameters given in Tab. 1.

- Draw the corresponding DH frames (use the extra sheet) and give the values, or at least the signs, of the components of $\boldsymbol{q}$ in the shown configuration.
- Consider the task vector

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \alpha \end{pmatrix} = \begin{pmatrix} \sin q_1 q_3 + L \cos q_1 \cos q_4 \\ -\cos q_1 q_3 + L \sin q_1 \cos q_4 \\ q_2 + L \sin q_4 \\ q_4 \end{pmatrix}. \tag{1}$$

  Solve the inverse kinematics problem in closed form for a given $\boldsymbol{r}_d \in \mathbb{R}^4$, determining also the possible singular situations. With $L = 1.5$ m, provide the numerical solutions for these data: $\boldsymbol{r}_{d1} = (2, 2, 4, -\pi/4)$, $\boldsymbol{r}_{d2} = (0, 0, 3, \pi/2)$, $\boldsymbol{r}_{d3} = (1, 1, 2, 0)$, and $\boldsymbol{r}_{d4} = (0, 1.5, 4, 0)$ [m,m,m,rad].

[180 minutes, open books]

Solution

November 22, 2024

Exercise 1

The initial orientation is specified by a ZXY Euler sequence (α,β,γ), which is associated to the rotation matrix

$$\boldsymbol{R}_{in}(\alpha,\beta,\gamma) \;=\; \boldsymbol{R_z}(\alpha)\boldsymbol{R_x}(\beta)\boldsymbol{R_y}(\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}.$$

When evaluated with the data, we obtain

$$\boldsymbol{R}_i = \boldsymbol{R}_{in}(\pi/2, \pi/4, -\pi/4) = \begin{pmatrix} 0.5000 & -0.7071 & 0.5000 \\ 0.7071 & 0 & -0.7071 \\ 0.5000 & 0.7071 & 0.5000 \end{pmatrix}.$$

On the other hand, the final orientation is given by the axis-angle method, with unit vector $\boldsymbol{r}$ and angle θ

$$\boldsymbol{R}_{fin}(\boldsymbol{r},\theta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta.$$

When evaluated with the data, we obtain

$$\boldsymbol{R}_f = \boldsymbol{R}_{fin}\left((0, -\sqrt{2}/2, \sqrt{2}/2), \pi/6\right) = \begin{pmatrix} 0.8660 & -0.3536 & -0.3536 \\ 0.3536 & 0.9330 & -0.0670 \\ 0.3536 & -0.0670 & 0.9330 \end{pmatrix}.$$

Therefore, the relative rotation to be realized is

$$\boldsymbol{R}_{if} = \boldsymbol{R}_i^T \boldsymbol{R}_f = \begin{pmatrix} 0.8598 & 0.4495 & 0.2424 \\ -0.3624 & 0.2026 & 0.9097 \\ 0.3598 & -0.8700 & 0.3371 \end{pmatrix}.$$

The RPY-type YXY sequence (ϕ,χ,ψ) is associated to the rotation matrix

$$\begin{aligned} \boldsymbol{R}_{if}(\phi,\chi,\psi) \;&=\; \boldsymbol{R_y}(\psi)\boldsymbol{R_x}(\chi)\boldsymbol{R_y}(\phi) \\ &= \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\chi & -\sin\chi \\ 0 & \sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\chi\sin\psi & \sin\chi\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\chi\sin\psi \\ \sin\phi\sin\chi & \cos\chi & -\cos\phi\sin\chi \\ -\cos\phi\sin\psi - \sin\phi\cos\chi\cos\psi & \sin\chi\cos\psi & \cos\phi\cos\chi\cos\psi - \sin\phi\sin\psi \end{pmatrix}. \end{aligned}$$

The solution of the inverse problem for this sequence of angles

$$\boldsymbol{R}_{if}(\phi,\chi,\psi) = \boldsymbol{R}_{if}$$

is obtained from the expressions in the second row and second column of the above matrix. Denoting the numerical elements of matrix $\boldsymbol{R}_{if}$ by R_{ij}, under the regularity assumption

$$\sigma = \sqrt{R_{21}^2 + R_{23}^2} = |\sin\chi| \neq 0,$$

one obtains

$$\chi^{+,-} = \mathrm{ATAN2}\left\{\pm\sqrt{R_{21}^2 + R_{23}^2}, R_{22}\right\},$$

and for each sign in this expression, the two pairs

$$\phi^+ = \mathrm{ATAN2}\left\{R_{21}, -R_{23}\right\} \qquad \psi^+ = \mathrm{ATAN2}\left\{R_{12}, R_{32}\right\}$$

and

$$\phi^- = \mathrm{ATAN2}\left\{-R_{21}, R_{23}\right\} \qquad \psi^- = \mathrm{ATAN2}\left\{-R_{12}, -R_{32}\right\}.$$

When substituting the numerical values, we find $\sigma = 0.9793$ and thus the two regular solutions

$$\begin{pmatrix} \phi^+ \\ \chi^+ \\ \psi^+ \end{pmatrix} = \begin{pmatrix} -2.7625 \\ 1.3668 \\ 2.6647 \end{pmatrix} \qquad \begin{pmatrix} \phi^- \\ \chi^- \\ \psi^- \end{pmatrix} = \begin{pmatrix} 0.3791 \\ -1.3668 \\ -0.4769 \end{pmatrix} \quad \text{[rad]}.$$

Finally, a convenient check of the correctness of the obtained results is to verify that

$$\boldsymbol{R}_i \boldsymbol{R}_{if}(\phi^+, \chi^+, \psi^+) \boldsymbol{R}_f^T = \boldsymbol{I},$$

and the same for the second solution (with the $-$ superscripts).

Exercise 2

The initial pose of the cylinder with respect to the world frame is

$${}^w\boldsymbol{T}_c^{in} = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When rolling the cylinder, a displacement $d > 0$ in the current x_c-direction corresponds to a clockwise rotation α around $\boldsymbol{z}_c$. Setting $\alpha = -d/r < 0$, the three elementary motions are described respectively by

$$\boldsymbol{T}_d = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & d \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{T}_\vartheta = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 & 0 \\ \sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{T}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which are combined as follows. The first rolling motion is defined with respect to the *current* pose ${}^w\boldsymbol{T}_c$ (namely, the initial one); thus

$$\boldsymbol{T}_1 = {}^w\boldsymbol{T}_c^{in}\, \boldsymbol{T}_d = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ \cos\alpha & -\sin\alpha & 0 & d \\ \sin\alpha & \cos\alpha & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ \cos(d/r) & \sin(d/r) & 0 & d \\ -\sin(d/r) & \cos(d/r) & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and one can see that the displacement $d > 0$ occurs in fact along the y_w-direction of the world frame. Moreover, if the distance travelled was $d = 2\pi r$ ($\alpha = -2\pi$, one full rotation), the orientation of the frame RF_c at the end would be again ${}^w\boldsymbol{R}_c$, as in the initial configuration, whereas for $d = \pi r/2$ ($\alpha = -\pi/2$, one fourth of a clockwise rotation), the axis x_c would be aligned with $-z_w$. The second rotation by ϑ occurs around the *fixed* axis z_w; thus, the order in the matrix product is

$$\boldsymbol{T}_2 = \boldsymbol{T}_\vartheta\, \boldsymbol{T}_1 = \boldsymbol{T}_\vartheta\, {}^w\boldsymbol{T}_c^{in}\, \boldsymbol{T}_d = \begin{pmatrix} -\sin\vartheta\cos\alpha & \sin\vartheta\sin\alpha & \cos\vartheta & (h/2)\cos\vartheta - d\sin\vartheta \\ \cos\vartheta\cos\alpha & -\cos\vartheta\sin\alpha & \sin\vartheta & d\cos\vartheta + (h/2)\sin\vartheta \\ \sin\alpha & \cos\alpha & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can easily check that a rotation by $\vartheta = \pi/2$ (counterclockwise around z_w) brings the position of the origin O_c to ${}^w\boldsymbol{p}_c = (-d, h/2, r)$, as expected. Finally, the third rotation by φ is defined again with respect to the *current* orientation of axis z_c; thus, we obtain the general symbolic expression

$$\boldsymbol{T}_3 = \boldsymbol{T}_2\,\boldsymbol{T}_\varphi = \boldsymbol{T}_\vartheta\,{}^w\boldsymbol{T}_c^{in}\,\boldsymbol{T}_d\,\boldsymbol{T}_\varphi$$

$$= \begin{pmatrix} -\cos\alpha\sin\vartheta\cos\varphi + \sin\alpha\sin\vartheta\sin\varphi & \cos\alpha\sin\vartheta\sin\varphi + \sin\alpha\sin\vartheta\cos\varphi & \cos\vartheta & (h/2)\cos\vartheta - d\sin\vartheta \\ \cos\alpha\cos\vartheta\cos\varphi - \sin\alpha\cos\vartheta\sin\varphi & -\sin\alpha\cos\vartheta\cos\varphi - \cos\alpha\cos\vartheta\sin\varphi & \sin\vartheta & (h/2)\sin\vartheta + d\cos\vartheta \\ \cos\alpha\sin\varphi + \sin\alpha\cos\varphi & \cos\alpha\cos\varphi - \sin\alpha\sin\varphi & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Substituting the given data, the final pose of the cylinder is

$${}^w\boldsymbol{T}_c = \boldsymbol{T}_{\vartheta=\pi/3}\,{}^w\boldsymbol{T}_c^{in}(h=0.5, r=0.1)\,\boldsymbol{T}_{d=1.5}\,\boldsymbol{T}_{\varphi=-\pi/2} = \begin{pmatrix} 0.5632 & 0.6579 & 0.5000 & -1.1740 \\ -0.3251 & -0.3798 & 0.8660 & 0.9665 \\ 0.7597 & -0.6503 & 0 & 0.1000 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3

The assignment of DH frames for the PPR planar robot is shown in Fig. 5, with Tab. 2 containing the corresponding parameters. By following the given specifications, this assignment is unique up to the choice of the direction of the axis x_3 (which may instead point toward the axis of joint 3). Note that the third link points along y_w when $q_3 = 0$.

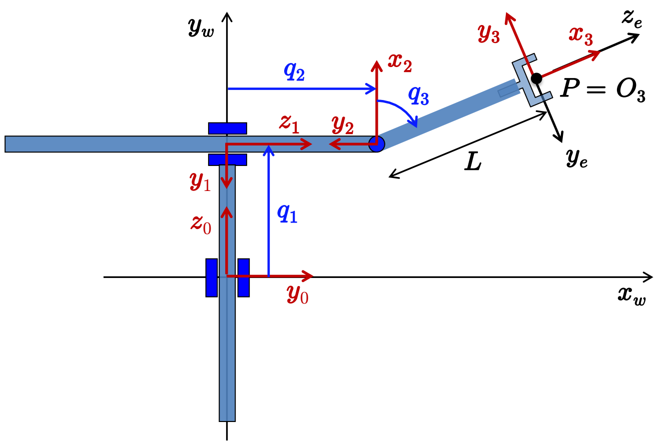


Figure 5: Assignment of DH frames for the PPR robot of Fig. 2.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	L	0	q_3

Table 2: DH parameters for the frame assignment in Fig. 5.

The two constant homogenous transformation matrices are

$$
{}^{w}\boldsymbol{T}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^{3}\boldsymbol{T}_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Being $P = O_3 = O_e$, the direct kinematics for the end-effector position is extracted from

$$
{}^{w}\boldsymbol{p}_{e,hom} = {}^{w}\boldsymbol{T}_0 \, {}^{0}\boldsymbol{A}_1(q_1) \, {}^{1}\boldsymbol{A}_2(q_2) \, {}^{2}\boldsymbol{A}_3(q_3) \, {}^{3}\boldsymbol{T}_e \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = \begin{pmatrix} q_2 - L\sin q_3 \\ q_1 + L\cos q_3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{w}\boldsymbol{p}_e \\ 1 \end{pmatrix}.
$$

Figure 6 shows the primary workspace of the PPR planar robot for the given joint ranges. The darker yellow region shows the excursion of the two prismatic joints; the bottom-left corner is cut away by the length L of the third link and specifically by the lower limit $q_3 = -3\pi/4$ of the revolute joint (i.e., the last link points at most 45° downwards).

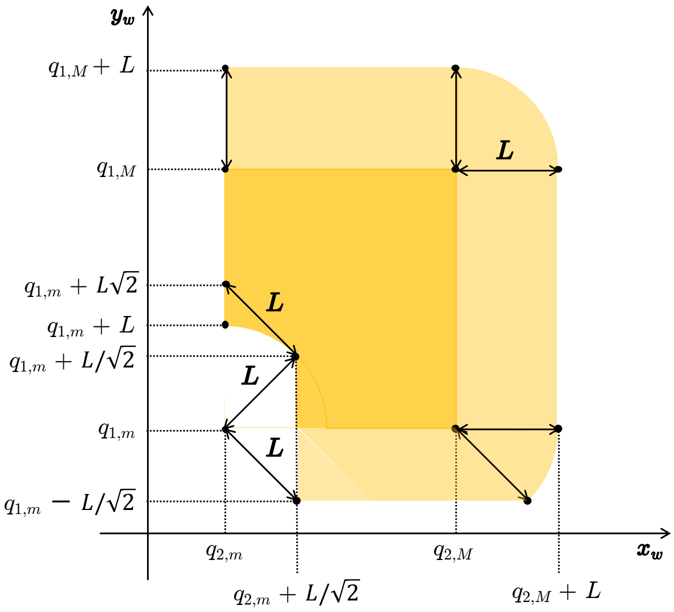


Figure 6: Primary workspace of the PPR robot, with the relevant points of interest.

Exercise 4

The transmission ratios of interest are

$$
n_{me} = \frac{r_e}{r_m} = \frac{40}{0.5} = 80 \qquad n_{el} = \frac{r_l}{r_e} = \frac{10}{40} = 0.25 \qquad \Rightarrow \qquad n = n_{me} \cdot n_{el} = \frac{r_l}{r_m} = 20.
$$

The matching condition for the link inertia that optimizes the transfer from motor torque to link acceleration is

$$
J_l = J_m n^2 = 7.1 \cdot 10^{-4}\, 400 = 0.284 \text{ kgm}^2.
$$

From the torque balance

$$
\tau_m = J_m \ddot{\theta}_m + \frac{1}{n} J_l \ddot{\theta}_l = J_m \left(n \ddot{\theta}_l\right) + \frac{1}{n}\left(J_m n^2\right)\ddot{\theta}_l = 2 J_m n \ddot{\theta}_l,
$$

we compute the link acceleration

$$\ddot{\theta}_l = \frac{\tau_m}{2nJ_m} = \frac{10}{40 \cdot 7.1 \cdot 10^{-4}} = 352 \text{ rad/s}^2,$$

which is indeed a very large value (because the delivered torque τ_m is already extremely high!). To obtain the desired resolution Δ_l on the link side, we have on the encoder axis

$$\Delta_e = n_{el}\,\Delta_l = n_{el}\,0.01^\circ = 0.25\,\frac{0.01}{360} = \frac{2.5 \cdot 10^{-3}}{360} \text{ fraction of a turn.}$$

Thus, the number of pulses per turn of the incremental encoder should be

$$N = \frac{1}{\Delta_e} = \frac{360}{2.5 \cdot 10^{-3}} = 144000.$$

Accordingly, the pulses per turn of the optical disc are at least $N_e = \lceil N/4 \rceil = 36000$ (i.e., before electronic quadrature). When counting an increment of 100 pulses per second on the encoder axis, the motor velocity will be

$$\dot{\theta}_m = -n_{me}\,\dot{\theta}_e = -80\,\frac{100}{144000} = -0.056 \text{ turns/s } (\cdot 2\pi = -0.349 \text{ rad/s},)$$

with the sign $-$ due to the inverse rotation between motor and encoder (whereas $\text{sign}(\dot{\theta}_l) = \text{sign}(\dot{\theta}_m)$).

Exercise 5

Using the set of DH parameters given in Tab. 1, the unique corresponding assignment of DH frames is given in Fig. 7. In the shown configuration, we have $\boldsymbol{q} = (\pi/4, q_2 > 0, q_3 > 0, \pi/2)$.

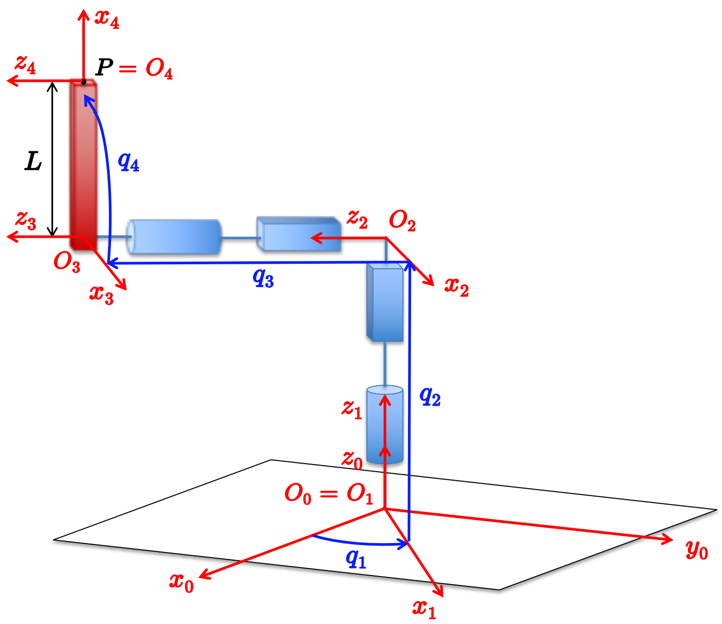


Figure 7: Assignment of DH frames for the RPPR robot of Fig. 4.

For completeness, the final pose of the frame RF_4 is expressed by the homogeneous transformation matrix

$$^0\boldsymbol{T}_4(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \cos q_4 & -\cos q_1 \sin q_4 & \sin q_1 & \sin q_1 q_3 + L \cos q_1 \cos q_4 \\ \sin q_1 \cos q_4 & -\sin q_1 \sin q_4 & -\cos q_1 & -\cos q_1 q_3 + L \sin q_1 \cos q_4 \\ \sin q_4 & \cos q_4 & 0 & q_2 + L \sin q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can easily recognize that the first three components of the task vector $\boldsymbol{r}$ in eq. (1) coincide with the position ${}^0\boldsymbol{p}_4$. For a given $\boldsymbol{r}_d = (p_{xd}, p_{yd}, p_{zd}, \alpha_d)$, the inverse kinematics (IK) problem is solved as follows. First set

$$q_4 = \alpha_d, \tag{2}$$

and from the third equation in (1)

$$q_2 = p_{zd} - L \sin \alpha_d. \tag{3}$$

Replace (2) in the first two equations in (1), which are rearranged as

$$\begin{aligned} p_{xd} &= q_3 \sin q_1 + L \cos \alpha_d \cos q_1 \\ p_{yd} &= -q_3 \cos q_1 + L \cos \alpha_d \sin q_1. \end{aligned} \tag{4}$$

Squaring and summing yields

$$p_{xd}^2 + p_{yd}^2 = q_3^2 + L^2 \cos^2 \alpha_d,$$

and thus

$$q_3^{\{+,-\}} = \pm\sqrt{p_{xd}^2 + p_{yd}^2 - L^2 \cos^2 \alpha_d}. \tag{5}$$

When the argument of the square root in (5) is negative, the result is imaginary and the desired positionis outside the reachable workspace of the robot.[1] On the other hand, when $q_3 = 0$, we have a singular configuration. For each solution in (5), we obtain from (4) a linear system in the two unknowns $(\cos q_1, \sin q_1)$

$$\begin{pmatrix} L \cos \alpha_d & q_3^{\{+,-\}} \\ -q_3^{\{+,-\}} & L \cos \alpha_d \end{pmatrix} \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} p_{xd} \\ p_{yd} \end{pmatrix},$$

which is solved by

$$q_1^{\{+,-\}} = \text{ATAN2}\left\{ p_{yd} L \cos \alpha_d + p_{xd}\, q_3^{\{+,-\}}, p_{xd} L \cos \alpha_d - p_{yd}\, q_3^{\{+,-\}} \right\} \tag{6}$$

under the assumption that the determinant of the coefficient matrix

$$L^2 \cos^2 \alpha_d + (q_3^{\{+,-\}})^2 = p_{xd}^2 + p_{yd}^2 > 0.$$

When $p_{xd}^2 + p_{yd}^2 = 0$, then in order to have a real value for q_3 from (5), it must be $\alpha_d = q_3 = \pm\pi/2$. In this case, q_1 is undefined and we are again in a singularity. The four numerical cases to be solved summarize all possible cases for the IK problem (with units of $\boldsymbol{q}$ being [rad,m,m,rad]):

$$\boldsymbol{r}_{d1} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ -\pi/4 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{\{+\}} = \begin{pmatrix} 1.9718 \\ 5.0607 \\ 2.6220 \\ -0.7854 \end{pmatrix} \quad \boldsymbol{q}^{\{-\}} = \begin{pmatrix} -0.4010 \\ 5.0607 \\ -2.6220 \\ -0.7854 \end{pmatrix} \quad \text{(two regular solutions)}$$

$$\boldsymbol{r}_{d2} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ \pi/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} \text{undefined} \\ 1,5 \\ 0 \\ 1.5708 \end{pmatrix} \quad \text{(singularity with infinite solutions)}$$

$$\boldsymbol{r}_{d3} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \text{no solution for } \boldsymbol{q} \quad \text{(the task data are out of the reachable workspace)}$$

$$\boldsymbol{r}_{d4} = \begin{pmatrix} 0 \\ 1.5 \\ 4 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} 1.5708 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad \text{(singularity with only one solution).}$$

[1] One can consider the reachable workspace of this robot as a collection of primary workspaces $WS_1(\alpha)$, parametrized by the angle α (the fourth component in $\boldsymbol{r}$). The primary workspace WS_1 will be the *union* of the sets $WS_1(\alpha)$ over all possible values of α. A point $\boldsymbol{p} \in \mathbb{R}^3$ belongs to the secondary workspace WS_2 if it belongs to $WS_1(\alpha)$ for all possible values of α (thus, to the intersection of all sets $WS_1(\alpha)$).

In the last case, the point $\boldsymbol{p} = (0, 1.5, 4) \in \mathbb{R}^3$ is on the boundary of $WS_1(0)$, i.e., the primary workspace obtained for $\alpha = 0$. This boundary is the surface of an infinite cylinder having its main axis coincident with z_0 and radius $R = L = 1.5$ m; the primary workspace is the (unlimited) part of $\mathbb{R}^3$ in the *outside* of this surface.

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