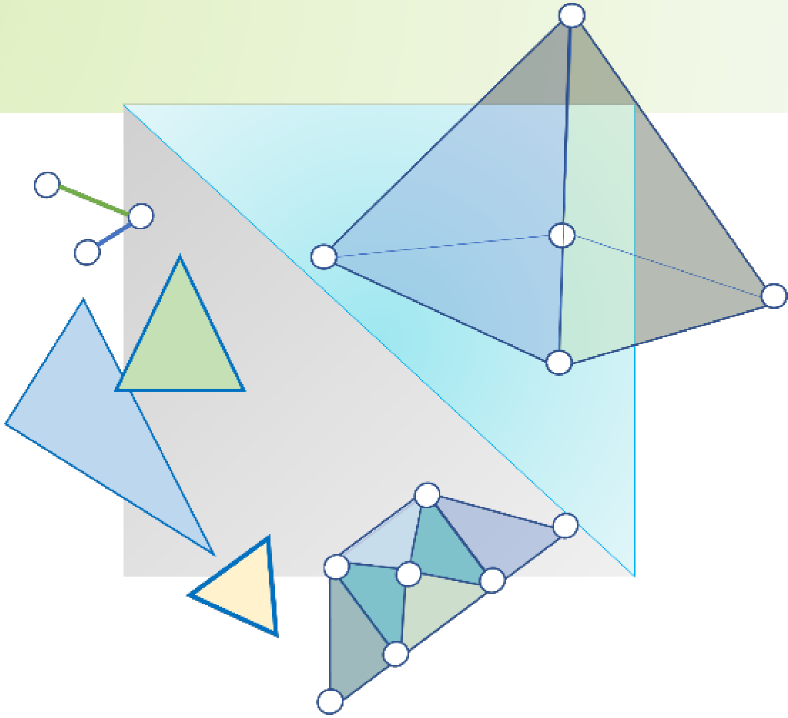
CITS3003 Graphics & Animation



Lecture 7:

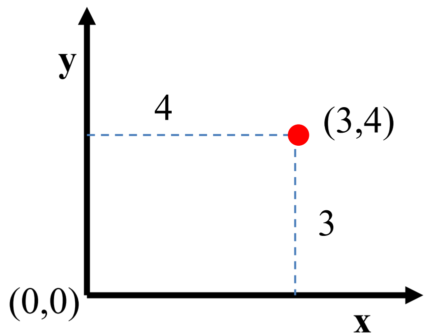
Representation and Coordinate Systems

Content

- Intro. to the elements of geometry:
    - points, scalars and vectors
- Dimensionality and linear independence
- Intro. to coordinate frame
- Learn how to define and change coordinate frames
- Derive homogeneous coordinate transformation matrices

Points, Scalars and Vectors

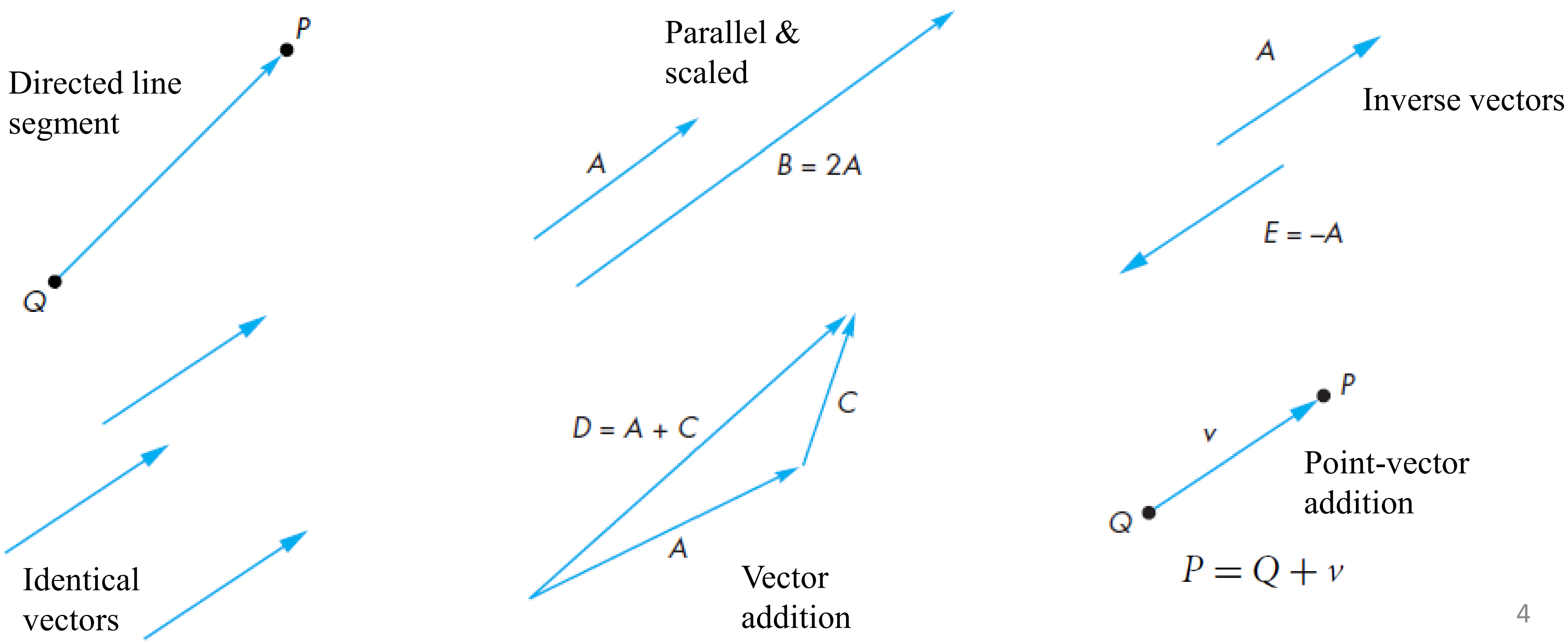
- **Point** (fundamental geometric object)
  - Location in space/coordinate system
    - Example: Point (3, 4)
  - Cannot add or scale points
  - mathematical point has neither a size nor a shape



- **Scalars**
  - Real /complex numbers
  - Used to specify quantities
  - Obey a set of rules
    - addition and multiplication
    - commutivity and associativity *// $a + b = b + a$; $(a + b) + c = a + (b + c)$*
    - multiplicative and additive inverses *// $a + (-a) = 0$; $a \,.\, a^{-1} = 1$*

Points, Scalars and Vectors

- **Vector**
  - Is any quantity with direction and magnitude
    - e.g., Force, velocity etc.
  - Can be added, scaled and rotated
  - A vector does not have a fixed location in space



Vector-Point Relationship

For computer graphics:

- scalars are the real numbers using ordinary addition and multiplication.
- geometric points are locations in space,
- and vectors are directed line segments.

These objects obey the rules of an affine space.

- Vector-vector addition
- Scalar-vector multiplication
- Point-vector addition
- Scalar-scalar operations
- No point-point addition

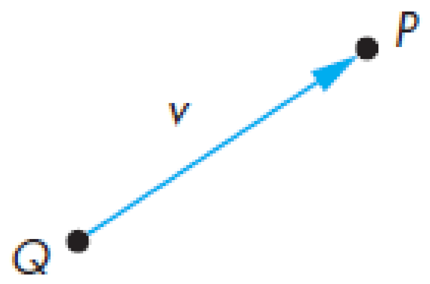
- For any point define
  - $1 \bullet P = P$
  - $0 \bullet P = 0$ (zero vector)

  No other point-scalar operations

Vector-Point Relationship

- **Vector**
  - Two points can be thought of defining a vector, i.e., *point-point-subtraction*

$$\boxed{\boldsymbol{v} = \boldsymbol{P} - \boldsymbol{Q}}$$



    - *Subtract 2 Points = vector*
    - *Point + vector =point*
  - Because vectors can be multiplied by scalars, expressions, below make sense

$$\boldsymbol{P} + \boldsymbol{3v}$$

$$\boldsymbol{2P} - \boldsymbol{Q} + \boldsymbol{3v}$$

  - But this does not $\boldsymbol{P} + \boldsymbol{3Q} - \boldsymbol{v}$

Magnitude of a Vector

The **magnitude** of a vector ***a*** is a real number denoted

- Magnitude of **a**

$$|\mathbf{a}| = \sqrt{a_1^{\ 2} + a_2^{\ 2} \ldots\ldots\ldots\ldots + a_n^{\ 2}}$$

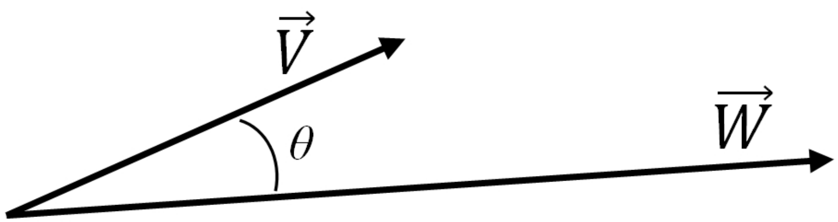
  – If a =(2, 5, 6) $|\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}$

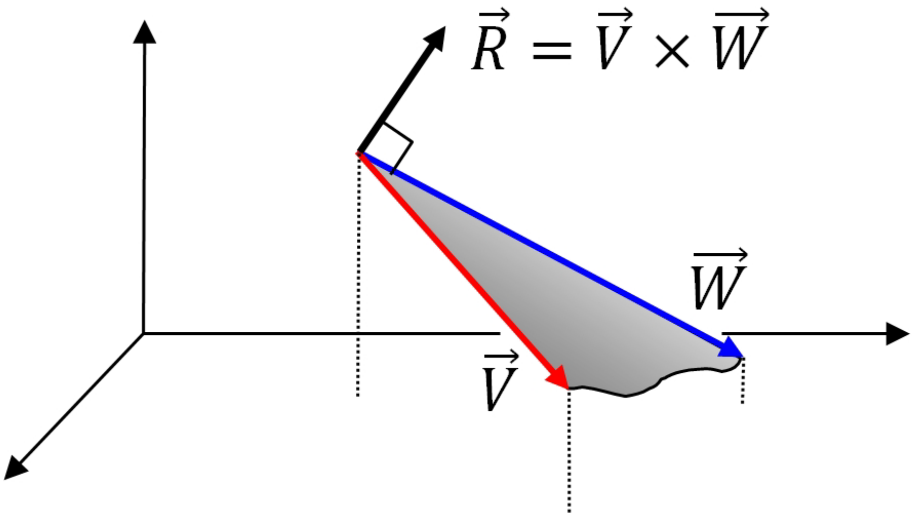
- Normalizing a vector $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{vector}{magnitude}$

$$\hat{\mathbf{a}} = \left(\frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)$$

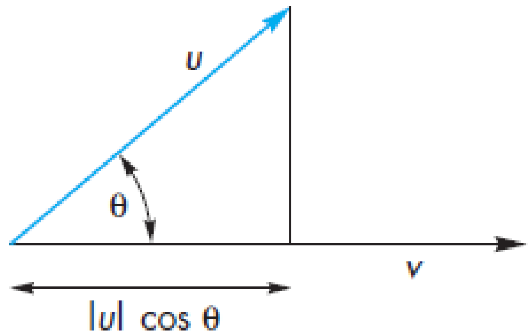
Dot and Cross Products

- Many of the geometric concepts relating the orientation between two vectors are in terms of the *dot (inner)* and *cross (outer)* products of two vectors.





Dot and Cross Products



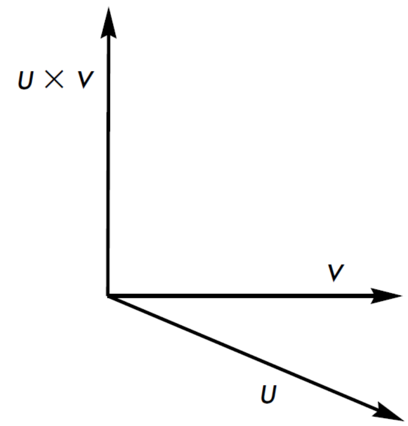
- Dot (inner) product:

$$u \cdot v = |u||v|\cos(\theta)$$

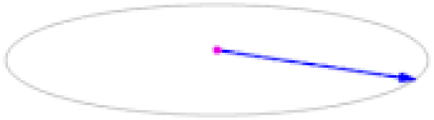
  - *angle between two vectors* $\cos\theta = \dfrac{u \cdot v}{|u||v|}$
  - Finding a vector's magnitude
    - Square of magnitude $|u|^2 = u \cdot u$
  - Finding whether two vectors are perpendicular,
    - If $u.v = 0$, u and v are orthogonal
  - Finding whether two vectors are parallel but pointing in opposite directions.

Dot and Cross Products

- Cross (outer) product
  - Given by $u \text{ x } v = |u||v| \, sin(\theta)$
  - Normal $n = u \times v$



An important property of the cross product of two vectors, is that it produces a vector that is normal (perpendicular) to the plane defined by the original two vectors.



Linear Independence

- A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is *linearly independent* when

$$a_1\mathbf{v}_1+a_2\mathbf{v}_2+\ldots a_n\mathbf{v}_n = \mathbf{0}, \qquad \text{only if } a_1=a_2=\ldots= 0$$

- If a set of vectors is *linearly independent*, we cannot represent any vector (in the set) in terms of the other vectors.
- If a set of vectors is *linearly dependent*, at least one can be written in terms of the others

Examples

- Example#1:
  - $v1=[1,2]^T$, $v2=[-5,3]^T$ Independent
- Example#2:
  - $v1=[2,-1,1]^T$, $v2=[3,-4,2]^T$, $v3=[5,-5,3]^T$ Dependent

Linear Independence (cont.)

- For example:

Let

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of linearly independent vectors.

- What are the values of $\alpha_1, \alpha_2,$ and α_3 if we want $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$?

Linear Independence (cont.)

- What are the values of $\alpha_1, \alpha_2,$ and α_3 if we want $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$?

- That is, we want

$$\alpha_1\begin{pmatrix}5\\0\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\3\\0\end{pmatrix} + \alpha_3\begin{pmatrix}0\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Dimension

- In a vector space, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space

- In an n-dimensional space, any set of n linearly independent vectors form a *basis* for the space

- Given a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, any vector $\mathbf{w}$ can be written as

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

where the coefficients $\{a_i\}$ are unique and are called representations of $\mathbf{w}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

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A 3-dimensional space can have 4 linearly independent vectors?

ⓘ Start presenting to display the poll results on this slide.

Dimension (cont.)

Let us define a basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

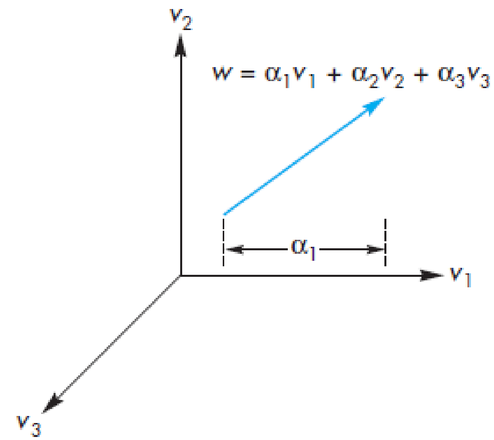
Orthonormal basis vectors

- The vector **w** can be written as:

$$\mathbf{w} = 10.5\,\mathbf{v}_1 + 21.3\,\mathbf{v}_2 + 0.9\,\mathbf{v}_3$$

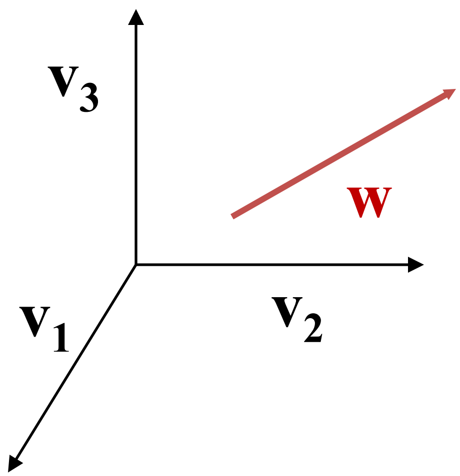
and the coefficients $\alpha_1 = 10.5, \alpha_2 = 21.3,$ and $\alpha_3 = 0.9$ are unique

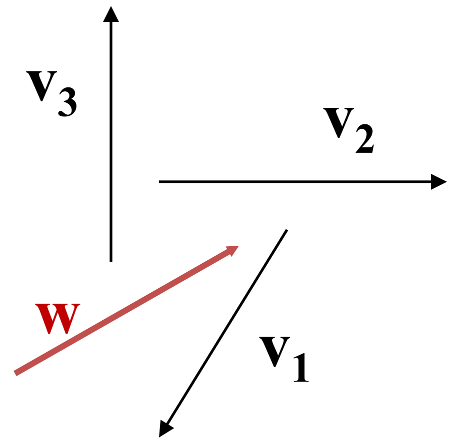
In a three-dimensional vector space, we can represent any vector *w* uniquely in terms of any three linearly independent vectors, *v*1, *v*2, and *v*3



Coordinate Systems

- Which one is correct?





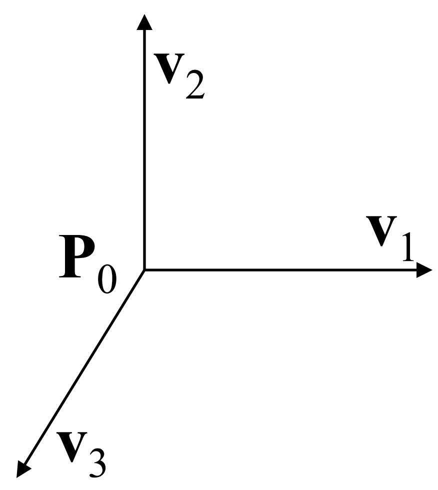
- Both are correct, because **vectors have no fixed location**

Coordinate Systems

- We need a frame of reference to relate points and objects to our physical world.
  - For example, where is a point? We can’t answer this without a reference system

Coordinate Frame

- Basis vectors alone cannot represent points
- We can add a single point, the *origin*, to the basis vectors to form a *coordinate frame*



Coordinate Frame

- A coordinate system (or coordinate frame) is determined by the origin and the basis vectors $(\mathbf{P}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

- Within this coordinate frame, every vector $\mathbf{w}$ can be written as

$$\boxed{\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3}$$

Every point can be written as

$$\boxed{\mathbf{P} = \mathbf{P}_0 + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3}$$

for some $\alpha_1, \alpha_2, \alpha_3$, and $\beta_1, \beta_2, \beta_3$

Representation in a Coordinate Frame

- Consider a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$
- A vector $\mathbf{w}$ is written $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$
- The list of scalars $\{a_1, a_2, \ldots\, a_n\}$ is the *representation* of $\mathbf{w}$ with respect to the given basis
- We can write the representation as a row or column matrix

$$\boldsymbol{\alpha} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^{\mathrm{T}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Coordinate systems (cont.)

For example, let $\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$.

If $\mathbf{v}_1 = [1 \quad 0 \quad 0]^{\mathrm{T}}$,
$\mathbf{v}_2 = [0 \quad 1 \quad 0]^{\mathrm{T}}$,
and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^{\mathrm{T}}$,

then $\boldsymbol{\alpha} = [2 \quad 3 \quad -4]^{\mathrm{T}}$

Note that this representation is with respect to a particular basis

Homogeneous Coordinates

- Consider the point **P** and the vector **v**, where

$$\mathbf{P} = \mathbf{P}_0 + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3$$

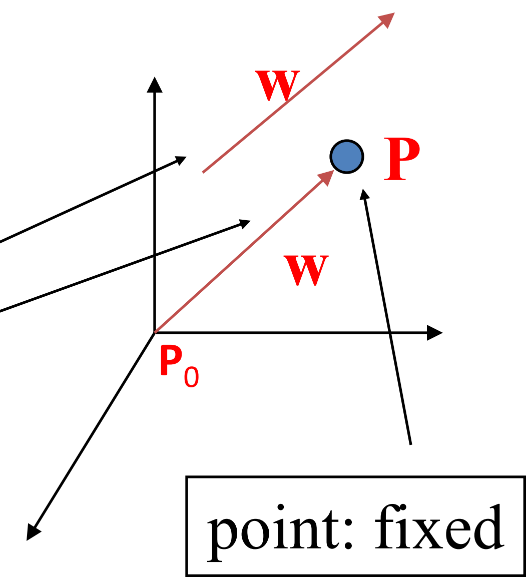
$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$$

- They appear to have similar representations:

$\mathbf{P} = [\beta_1, \beta_2, \beta_3]^T$, $\mathbf{w} = [\alpha_1, \alpha_2, \alpha_3]^T$ which confuses the point with the vector

A vector has no position

Vector can be placed anywhere



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Representation in a Coordinate Frame

$\mathbf{c} = [c_1, c_2, c_3]^T$

Representation of a vector or a point?

A Single Representation

- Assuming $0 \cdot \mathbf{P} = \mathbf{0}$ and $1 \cdot \mathbf{P} = \mathbf{P}$, we can write

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + 0 \cdot \mathbf{P}_0$$

$$\mathbf{P} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 + \mathbf{P}_0 = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 + 1 \cdot \mathbf{P}_0$$

- Thus, we obtain the four-dimensional ***homogeneous coordinate*** representation

$$\boxed{\mathbf{w} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0]^T}$$

$$\boxed{\mathbf{P} = [\beta_1 \quad \beta_2 \quad \beta_3 \quad 1]^T}$$

Homogeneous Coordinates

- The homogeneous coordinate form for a three-dimensional point $[x \quad y \quad z]^{\mathrm{T}}$ is given as

$$\mathbf{p} = [x \quad y \quad z \quad 1]^{\mathrm{T}} \rightarrow [wx \quad wy \quad wz \quad w]^{\mathrm{T}} = [x' \quad y' \quad z' \quad w]^{\mathrm{T}}$$

- We return to a three-dimensional point (for $w \neq 0$) by

$$x \leftarrow x'/w$$
$$y \leftarrow y'/w$$
$$z \leftarrow z'/w$$

- If $w = 0$, the representation is that of a vector

- Homogeneous coordinates replace points in three dimensions by lines through the origin in four dimensions

- For $w = 1$, the representation of a point is $[x \quad y \quad z \quad 1]^{\mathrm{T}}$

Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
  - Hardware pipeline works with 4 dimensional representations
  - For **orthographic viewing**, we can maintain $w = 0$ for vectors and $w = 1$ for points
  - For **perspective viewing** we need a *perspective division*

Change of Coordinate System

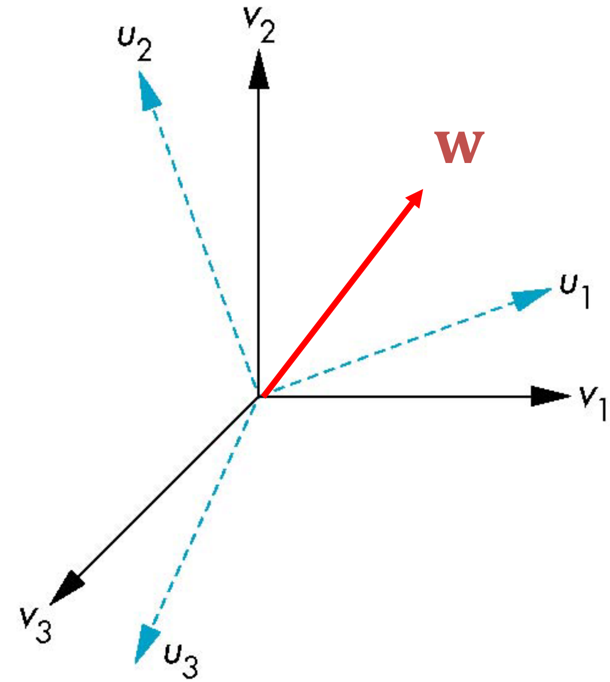
- Let's consider transformation of two bases
  - {v1, v2, v3} and {u1, u2, u3} are two bases.
  - Each basis vector in the second set can be represented in terms of the first basis

$$
\begin{aligned}
u_1 &= \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3, \\
u_2 &= \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3, \\
u_3 &= \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3.
\end{aligned}
$$

$$\mathbf{u} = \mathbf{M}\mathbf{v}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$



The same vector $\mathbf{w}$ represented in two coordinate systems

- Consider the same vector $\mathbf{w}$ with respect to two different coordinate systems having basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Suppose that

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3$$
$$\mathbf{w} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3$$

- Then the representations are:

$$\mathbf{a} = [\alpha_1 \quad \alpha_2 \quad \alpha_3]^{\mathrm{T}}$$
$$\mathbf{b} = [\beta_1 \quad \beta_2 \quad \beta_3]^{\mathrm{T}}$$

- Equivalently,

$$\mathbf{w} = \mathbf{a}^{\mathrm{T}} \mathbf{v}$$

and $\mathbf{w} = \mathbf{b}^{\mathrm{T}} \mathbf{u}$

$$\mathbf{v} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]^{\mathrm{T}}$$
$$\mathbf{u} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]^{\mathrm{T}}$$

Representing the Second Basis in Terms of the First (cont.)

- In this example, we have $\mathbf{w} = \mathbf{a}^{\mathrm{T}}\mathbf{v}$ and $\mathbf{w} = \mathbf{b}^{\mathrm{T}}\mathbf{u}$

So

$$\mathbf{a}^{\mathrm{T}}\mathbf{v} = \mathbf{b}^{\mathrm{T}}\mathbf{u}$$

- With $\mathbf{u} = \mathbf{M}\mathbf{v}$, we have

$$\mathbf{a}^{\mathrm{T}}\mathbf{v} = \mathbf{b}^{\mathrm{T}}\mathbf{M}\mathbf{v}$$
$$\Rightarrow \mathbf{a} = \mathbf{M}^{\mathrm{T}}\mathbf{b}$$

- Thus, $\mathbf{a}$ and $\mathbf{b}$ are related by $\mathbf{M}^{\mathrm{T}}$

$\boldsymbol{b} = \boldsymbol{Ta}$
where,
$\boldsymbol{T} = (\boldsymbol{M}^{\boldsymbol{T}})^{-1}$

Representation w.r.t the second basis (u)

Representation w.r.t first basis (v)

Change of Coordinate System

Example:

Suppose **u** and **v** are two basis related to each other as follows:

$$u_1 = v_1,$$

$$u_2 = v_1 + v_2,$$

$$u_3 = v_1 + v_2 + v_3.$$

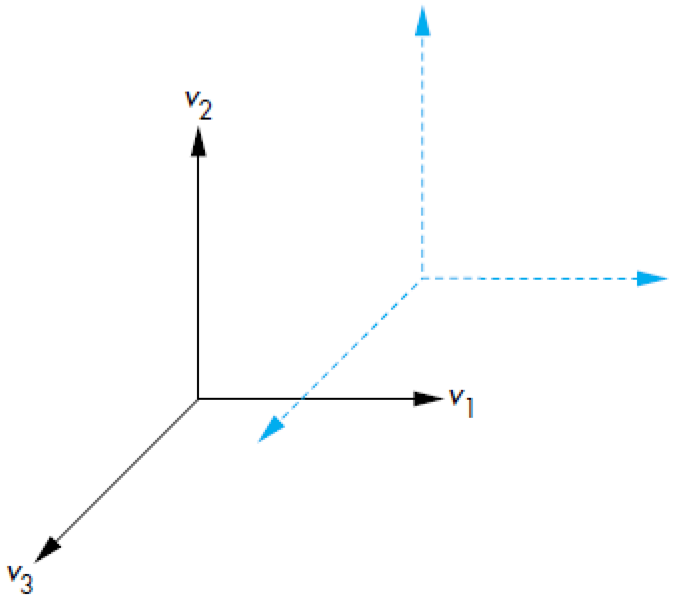
We have a vector **w** that is represented in basis **v** as:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

what will be its representation in **u** ?

Coordinate Frame

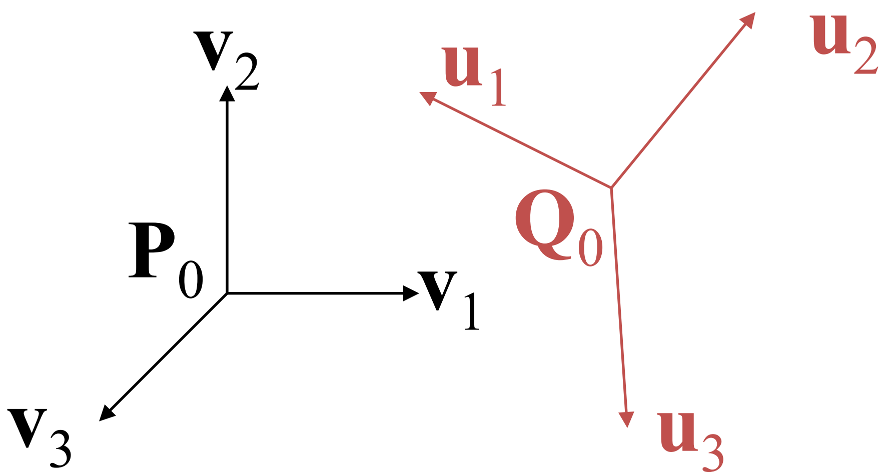
- We can also do all this in coordinate systems:



Change of Coordinate Frames

- We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two coordinate frames:
$(\mathbf{P}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$
$(\mathbf{Q}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$



- Any point or vector can be represented in either coordinate frame.
- We can represent $(\mathbf{Q}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ in terms of $(\mathbf{P}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Representing One Coordinate Frame in Terms of the Other

- We can extend what we did with the change of basis vectors:

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$
$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$
$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$
$$\mathbf{Q}_0 = \gamma_{41}\mathbf{v}_1 + \gamma_{42}\mathbf{v}_2 + \gamma_{43}\mathbf{v}_3 + \mathbf{P}_0$$

by replacing the 3×3 matrix $\mathbf{M}$ by a 4×4 matrix as follows:

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

Working with Representations

- Within the two coordinate frames any point or vector has a representation of the same form:

$\mathbf{a} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4]$ in the first frame

$\mathbf{b} = [\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors and

$$\mathbf{a} = \mathbf{M}^{\mathrm{T}}\mathbf{b}$$

or

$\boldsymbol{b} = \boldsymbol{Ta}$
where,
$\boldsymbol{T} = (\boldsymbol{M}^{\boldsymbol{T}})^{-1}$

- The matrix $\mathbf{M}^{\mathrm{T}}$ is 4×4 and specifies an affine transformation in homogeneous coordinates

Further Reading

“Interactive Computer Graphics – A Top-Down Approach with Shader-Based OpenGL” by Edward Angel and Dave Shreiner, 6th Ed, 2012

- Sec 3.3 *Coordinate Systems and Frames* (all subsections)
- Sec 3.4 *Frames in OpenGL*

Vector and Affine Spaces

- **Scalar field**
  - A pair of scalars can be combined to form another scalar
    - two operations: *addition* and *multiplication*
  - obey the closure, associativity, commutivity, and inverse properties
- **Vector space**
  - Contains vectors and scalars
  - Vector-scalar and vector-vector interactions
  - *Euclidean vector space*
    - is an extension of a vector space that adds a measure of size or distance
    - e.g., length of a line segment
  - *Affine vector space*
    - Extension of vector space and includes "point"
    - No point serves as origin, we have displacement vectors and points
    - Vector-point addition and point-point subtraction are possible
    - No point-point addition and point-scalar operation are possible