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Theory

SML

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ASSIGNMENT-1

Q1

We have

mean cov.

$P(X/\omega_1) \sim N([0,0], I)$

$P(X/\omega_2) \sim N([1,1], I)$

$P(X/\omega_3) \sim \frac{1}{2} N([0.5, 0.5], I) + \frac{1}{2} N([-0.5, 0.5], I)$

$x = [0.3, 0.3]$

Normal Distribution

Gaussian $\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}$

$\Sigma = I, \ \mu = [0,0], \ x = [0.3 \ 0.3]$

$P(X/\omega_1) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[0.3-0, 0.3-0]^T \Sigma^{-1} [0.3-0, 0.3-0]\right\}$

$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[0.3, 0.3]^t [0.3, 0.3]\right\}$

$= 0.14$

$P(X/\omega_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[0.3-1, 0.3-1]^t \Sigma^{-1} [0.3-1, 0.3-1]\right\}$

$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[-0.7, -0.7]^t [-0.7, -0.7]\right\}$

$= 0.133$

$$P(x/\omega_3) = \frac{1}{2\cdot 2\pi} e \left\{ -\frac{1}{2} [0.3-0.5, 0.3-0.5]^t \Sigma^{-1} [0.3-0.5, 0.3-0.5] \right. +$$

$$\left. \frac{1}{4\pi} \exp\left\{ -\frac{1}{2} [0.3+0.5, 0.3-0.5]^t \Sigma^{-1} [0.3+0.5, 0.3-0.5] \right\} \right\}$$

$$= \frac{1}{4\pi} \exp\left\{ -\frac{1}{2} [-0.2, -0.2]^t [-0.2, -0.2] \right\} +$$

$$\frac{1}{4\pi} \exp\left\{ -\frac{1}{2} [0.8, -0.2]^t [0.8, -0.2] \right\}$$

$$= 0.1336$$
$$= 0.134 \, .$$

NOW For $x = [0.3 \;\; 0.3]$

Minimim prob. of error dme.

$\boxed{\text{x belongs to class 1}}$

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Q2 Given:- $p(x|\omega_i) = \frac{1}{2b} e^{-\frac{|x-a_i|}{b}}$

for $i = 1, 2$

$a_1 = 0;\ a_2 = 1;\ b = 1.$

(a) **Likelihood Ratio**

$$P(x|\omega_1)/P(x|\omega_2) = \frac{1}{2b} e^{-\frac{|x-a_1|}{b}} \Big/ \frac{1}{2b} e^{-\frac{|x-a_2|}{b}}$$

$$= e^{-|x-0|} / e^{-|x-1|}$$

$$= e^{(+|x-a_1| - |x|)}$$

OR

$$p(x|\omega_1)/p(x|\omega_2) = e^{(-|x| + |x-1|)}$$

//Plot appears to be somewhat this

(b) Zero-One Loss

Equating Posteriors

$P(1|x) = P(x|1) * P(1) / [P(x|1) * P(1) + P(x|2) * P(2)]$

$P(x|2) * P(2) / [P(x|1) * P(1) + P(x|2) * P_2]$

Equate $P(1|x) = P(2|x)$; we get decision boundary

say $D1 = (-\infty, x_0)$ //we need to find x_0

$D2 = (x_0, \infty)$

$$f1(x|i=1) = \frac{1}{2b} e^{\left(-\frac{|x-a_1|}{b}\right)}$$

$$= \frac{1}{2} e^{(-|x|)}$$

$$f2(x|i=2) = \frac{1}{2b} e^{\left(-\frac{|x-a_2|}{b}\right)}$$

$$= \frac{1}{2} e^{(-|x-1|)}$$

$$\frac{1}{2} e^{(-|x|)} = \frac{1}{2} e^{(-|x-1|)}$$

Expanding right side

$$\frac{1}{2} e^{(-|x|)} = \frac{1}{2} e^{(-(1-x))} \text{ if } x >= 1$$

OR

$$\frac{1}{2} e^{(-x+1)} \text{ if } x < 1$$

For $x >= 1$

$$\frac{1}{2} e^{(-|x|)} = \frac{1}{2} e^{(-(1-x))}$$

taking log and rearranging we get

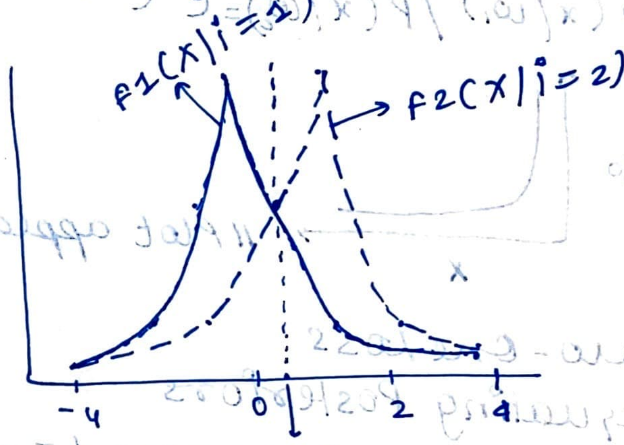
$$\boxed{x = \frac{1}{2}}$$

$x < 1/2$

$x > 1/2$

$D = (-\infty, 1/2)$

$D_2' = (1/2, \infty)$



ⓒ Average Probability of Error

$P(error|i=1) = \int_{D2} f1(X|i=1)\, dx$

$P(error\,|\,i=2) = \int_{D1} f2(X|i=2)\, dx$

// we have all the values

$$P_e = \frac{1}{2} \int f2(X|i=2),\ x < \frac{1}{2} + \frac{1}{2} * \int f1(X|i=1),\ x > 1/2$$

For $x < 1/2$ $\quad \int f2(X|i=2),\ x < 1/2$

$$= \frac{1}{b} e^{(-1/2)} - \frac{1}{b} e^{(-1)}$$

For $x > 1/2$

$$\int f1(X|i=1),\ x > 1/2$$

$$= \frac{1}{b} e^{(-1/2)} - \frac{1}{b} e^{(-1)}$$

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Average Prob

$$P_{\text{avg.}} = \frac{1}{2}\left(e^{(-1/2)} - e^{(-1)}\right) + \frac{1}{2}\left(e^{(-1/2)} - e^{(-1)}\right)$$

$$\boxed{P_{\text{avg.}} = 1 * e^{(-1/2)} - e^{(-1)}}$$

Q3 Prove:- $\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is an unbiased estimator of variance.

If a simple random sample $x_1, x_2, \ldots$ has unknown finite variance, we have

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

variance mean

To find mean of S^2

$$x_i - \mu = (x_i - \bar{x}) + (\bar{x} - \mu)$$

$$\sum_{i=1}^{n}(x_i - \mu)^2 = \sum_{i=1}^{n}(x_i - \bar{x}) + (\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + 2\sum_{i=1}^{n}(x_i - \bar{x})(\bar{x} - \mu) + \underbrace{\sum_{i=1}^{n}}_{n}(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + 2(\bar{x} - \mu)\underbrace{\sum_{i=1}^{n}(x_i - \bar{x})}_{0} + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

subtract $(\bar{x} - \mu)^2$ from both sides

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}(x_i - \mu)^2 - (\bar{x} - \mu)^2$$

using linearity property of Expectation.

$$ES^2 = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$E = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_i)^2 - (\bar{x} - \mu)^2\right]$$

Now $Var(X) = E((X - E(X))^2)$

$$= \frac{1}{n} \sum_{i=1}^{n} E[(X_i - \mu)^2] - E[(\bar{X} - \mu)^2]$$

$$= \frac{1}{n} \sum_{i=1}^{n} Var(X_i) - Var(\bar{X})$$

$$S^2 = \frac{1}{n} \sigma^2 - \frac{1}{n} \sigma^2 = \frac{n-1}{n} \sigma^2 \neq \sigma^2 \quad \text{— (I)}$$

this is biased

using mean of difference

$$b_d(\theta) = E_\theta d(X) - h(\theta)$$

$$b(\sigma^2) = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{1}{n} \sigma^2$$

$$E\left[\frac{n}{n-1} S^2\right] = \frac{n}{n-1} E[S^2] \text{ — From (I)}$$

$$E\left[\frac{n}{n-1} S^2\right] = \frac{n}{n-1} \left(\frac{n-1}{n} \sigma^2\right)$$

$$= \sigma^2$$

$$S_u^2 = \frac{n}{n-1} S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is an unbiased estimator of σ^2

Q4 (a) $E[aX+b]$

Here a & b are constants and we know expected value of a constant is a constant

$$\boxed{E[aX+b] = aE[X] + b}$$

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(b) $Var(aX+b) = ?$

$Var(X) = E[X^2] - (E[X])^2$

$Var(aX+b) = E[(aX+b)^2] - (E[aX+b])^2$

$= E[a^2X^2 + 2abX + b^2] - (aE(X)+b)^2$

$= a^2E(X^2) + 2abE(X) + b^2 - a^2E^2(X) - 2abE(X) - b^2$

$= a^2E(X^2) - a^2E^2(X)$

$= a^2 [\underbrace{E(X^2) - (E(X))^2}_{Var(X)}]$

$= a^2 Var(X)$

$\boxed{Var(aX+b) = a^2 Var(X)}$

Q5. The prob. of error $P(error|x)$, for a cauchy distribution can't be computed analytically as it is not defined for mean or variance.
We will use cumulative distribution function (CDF) for cauchy evaluated at X. (// Referenced Book) & notes

CDF say is $F(x)$

$P(error|x) = 1 - F(x)$

Probability of correct $P(C)$, is given by the expected value of prob. of correct given x

$P(C) = E[P(C|x)] = E[1 - P(error|x)]$

$= E[1 - f(x)]$

// as given in Q

Now

PDF for cauchy $f(x) = \frac{1}{(\pi(1+x^{\wedge}2))}$

CDF $F(X) = \frac{1}{\pi} \arctan(x) + 1/2$

$$P(\text{error}/x) = 1 - F(X)$$
$$= 1 - \left(\frac{1}{\pi} * \arctan(x) + 1/2\right)$$

$$P(C) = E[P(C|X)]$$
$$= \int_{-\infty}^{\infty} (1 - P(\text{error}|x)) * f(x)\,dx$$
$$= \int_{-\infty}^{\infty} (1-(1-(1/\pi * \arctan(x) + 1/2))) * (1/\pi * (1+x^{\wedge}2))) \, dx.$$

It can't be solved in closed form. It has to be evaluated numerically.