Separation Axioms Summary

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Abstract

In this short essay I summarised the separation Axioms I have leared, in the topology course. Argued the subspace hereditary, productspace hereditary, and the Continuity-preserving property. And slightly discuss the relationships between these Axioms.

1 Motivation of Separation Axioms

The basic Axioms of topology only described the relationship of the opensets, the opensets captured the geometry structure in the point set. but it some of the coarse topology dose not have a much resonable behavior, such as the indiscrete topology, even the const sequence could converge to any point in the base set. this is because the topology is so coarse that we only have two opensets that is $\{\phi, X\}$ so all the structure we care is crushed together in this topology.

The Separation Axioms served as patched to the basic Axioms of topology to make sure we are deal with some topology space with some familiar behaviors. This is the motivation and mean of Introduction these Axioms patches. [1, script1]

2 All the Axioms

2.1 Definition of Separation Axioms

First lets see three basic separation concepts, that is Hausdorff, regular and normal. Figure1 has well illustrated the idea.

Definition 2.1 (regular)**.** *A topological space X is a regular space if, given any closed set F and any point x that does not belong to F, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint. Concisely put, it must be possible to separate x and F with disjoint neighborhoods. [4, Reglarspace] Figure1*

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Definition 2.2 (normal). *A topological space X is a normal space if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint. More intuitively, this condition says that E and F can be separated by neighbourhoods. [3, Normal]*

Definition 2.3 (closed neighborhoods separation). *We say that x and y can be separated by closed neighborhoods if there exists a closed neighborhood U of x and a closed neighborhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). (Note that a "closed neighborhood of x" is a closed set that contains an open set containing x.) [6, UandcHs]*

Another important concept is completeness. The prefix "completely" add ahead the Hausdorff, regular and normal neams the corresponding two object can be separate by functions.

Definition 2.4 (continuous separation). *We say that x and y can be separated by a function if there exists a continuous function $f : X \to [0, 1]$ (the unit interval) with $f(x) = 0$ and $f(y) = 1$. [6, UandcHs]*

Definition 2.5 (set separate by function). *We say that two sets X and Y can be separated by a function if there exists a continuous function $f : X \to [0, 1]$ (the unit interval) with $f|_X = 0$ and $|_Y = 1$. [6, UandcHs]*

Definition 2.6 (completely regular). *A topological space X is called completely regular if points can be separated from closed sets via (bounded) continuous real-valued functions. In technical terms this means: for any closed set $A \subseteq X$ and any point, $x \in X \setminus A$, there exists a real-valued continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f|_A = 0$. (Equivalently one can choose any two values instead of 0 and 1 and even require that f be a bounded function.) [5, Tychonoff]*

The following part is all the T Axioms.

Definition 2.7 (T_0). *A topological space $(X, \mathcal{T})$ is said to be T_0 (or much less commonly said to be a Kolmogorov space), if for any pair of distinct points $x, y \in X$ there is an open set U that contains one of them and not the other. [2, script5]*

Definition 2.8 (T_1). *A topological space $(X, \mathcal{T})$ is said to be T_1 (or much less commonly said to be a Fréchet space) if for any pair of distinct points $x, y \in X$, there exist open sets U and V such that U contains x but not y, and V contains y but not x. [2, script5]*

Definition 2.9 (T_2). *A topological space $(X, \mathcal{T})$ is said to be T_2, or more commonly said to be a Hausdorff space, if for every pair of distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. [2, script5]*

Theorem 2.1. *Let $(X, \mathcal{T})$ be a Hausdorff space. Then every sequence in X converges to at most one point. [2, script5]*

Proof. Suppose X is Hausdorff and let x_n be a sequence in X. Suppose $x_n \to x$ and $y \neq x$. Then there are disjoint open sets U and V such that $x \in U$ and $y \in V$. By definition of convergence, some tail of the sequence is in the set U. But then that tail (and therefore all tails of the sequence) is disjoint from V , meaning $x_n \not\to y$. [2, script5]

Definition 2.10 ($T_{2\frac{1}{2}}$)**.** *A Urysohn space, also called a $T_{2\frac{1}{2}}$ space, is a space in which any two distinct points can be separated by closed neighborhoods. [6, UandcHs]*

Definition 2.11 (completely T_2)**.** *A completely Hausdorff space, or completely T_2, or functionally Hausdorff space, is a space in which any two distinct points can be separated by a continuous function. [6, UandcHs]*

Definition 2.12 (T_3)**.** *A T_3 space or regular Hausdorff space is a topological space that is both regular and a Hausdorff space. [4, Reglarspace]*

Definition 2.13 ($T_{3\frac{1}{2}}$)**.** *A topological space is called a Tychonoff space (alternatively: $T_{3\frac{1}{2}}$ space, or T_π space, or completely T_3 space) if it is a completely regular Hausdorff space. [5, Tychonoff]*

Definition 2.14 (T_4)**.** *A T4 space is a T_1 space X that is normal; this is equivalent to X being normal and Hausdorff. [5, Tychonoff]*

Definition 2.15 (T_5)**.** *A T_5 space, or completely T_4 space, is a completely normal T_1 space X, which implies that X is Hausdorff; equivalently, every subspace of X must be a T_4 space. [5, Tychonoff]*

Definition 2.16 (T_6)**.** *A T_6 space, or perfectly T_4 space, is a perfectly normal Hausdorff space. [5, Tychonoff]*

2.2 Some properties

Theorem 2.2 (T_0 hereditary)**.** *If a space is T_0 then its subspace is also T_0*

Proof. The subspace presere the open sets of the original space. The definition of the T_0 space only contains the definition of the open sets.

Suppose $x, y \in E \subset X$ then $\exists U \in \mathcal{T}$ that $x \in U, y \notin U$ then $U_E = E \cap U \in \mathcal{T}_E$ that $x \in U_E, y \notin U_E$

Theorem 2.3 (T_0 productspace)**.** *If a space is T_0 then its productspace is also T_0*

Proof. The productspace also can presere the open sets of the original space.

Suppose $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are T_0 space. then the product space $(P, \mathcal{T}_P) = (X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$ if two points $(x, y) \neq (x', y') \in P$ then there at least one coordinate are distinct Suppose $x \neq x'$ then $\exists U \in \mathcal{T}_\mathcal{X}$ that $x \in U, x' \notin U$ then $\exists U_P = U \times Y \in \mathcal{T}_\mathcal{P}$ that that $x \in U_P, x' \notin U_P$

For the continuous image $X \to Y$. We can just let $(Y, \mathcal{T}_{trivial})$ then any Separation Axioms would not presere in the image $f(X) \subset Y$

2.3 Examples and counterExamples

Spaces that are not T_0

A set with more than one element, with the trivial topology. No points are distinguishable.

The set R2 where the open sets are the Cartesian product of an open set in R and R itself, i.e., the product topology of R with the usual topology and R with the trivial topology; points (a,b) and (a,c) are not distinguishable.

The space of all measurable functions f from the real line R to the complex plane C such that the Lebesgue integral $\left(\int_{\mathbb{R}} |f(x)|^2 \, dx\right)^{\frac{1}{2}} < \infty$. Two functions which are equal almost everywhere are indistinguishable. See also below.

Spaces that are T_0 but not T_1

The Zariski topology on Spec(R), the prime spectrum of a commutative ring R, is always T_0 but generally not T_1. The non-closed points correspond to prime ideals which are not maximal. They are important to the understanding of schemes.

The particular point topology on any set with at least two elements is T_0 but not T_1 since the particular point is not closed (its closure is the whole space). An important special case is the Sierpiński space which is the particular point topology on the set 0,1.

The excluded point topology on any set with at least two elements is T_0 but not T_1. The only closed point is the excluded point.

The Alexandrov topology on a partially ordered set is T_0 but will not be T_1 unless the order is discrete (agrees with equality). Every finite T_0 space is of this type. This also includes the particular point and excluded point topologies as special cases.

The right order topology on a totally ordered set is a related example.

The overlapping interval topology is similar to the particular point topology since every non-empty open set includes 0.

Quite generally, a topological space X will be T_0 if and only if the specialization preorder on X is a partial order. However, X will be T_1 if and only if the order is discrete (i.e. agrees with equality). So a space will be T_0 but not T_1 if and only if the specialization preorder on X is a non-discrete partial order.

2.4 Relationship between Separation Axioms

$T_6 \Rightarrow T_5 \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_{2\frac{1}{2}} \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

Each implication is strict

References

[1] Ivan Khatchatourian. 1. introduction. `https://www.math.utoronto.ca/ivan/mat327/docs/notes/01-intro.pdf`. Accessed: 2024-11-20.

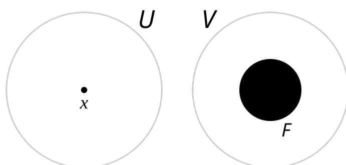

Figure 1: regular

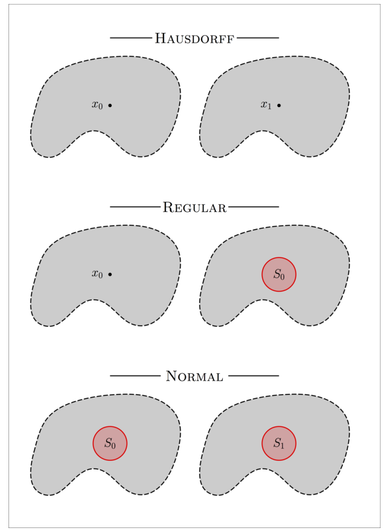

Figure 2: regular, normal and Hausdorff

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