Chapter 3

Action and Conjugation

The notion of an *action* plays an important role in the theory of finite groups. The first section of this chapter introduces the basic ideas and results concerning group actions. In the other two sections the action on cosets is used to prove important theorems of Sylow, Schur-Zassenhaus and Gaschütz.

3.1 Action

Let $\Omega = \{\alpha, \beta, \ldots\}$ be a nonempty finite set. The set S_Ω of all permutations of Ω is a group with respect to the product

$$\alpha^{xy} := (\alpha^x)^y, \quad \alpha \in \Omega \quad \text{and} \quad x, y \in S_\Omega,$$

is the **symmetric group** on Ω. We denote by S_n the symmetric group on $\{1, \ldots, n\}$, which is the **symmetric group of degree** n. Evidently $S_n \cong S_\Omega$ if and only if $|\Omega| = n$.

A group G **acts** on Ω, if to every pair $(\alpha, g) \in \Omega \times G$ an element $\alpha^g \in \Omega$ is assigned[1] such that

$\mathcal{O}_1$ $\quad \alpha^1 = \alpha \quad$ for $1 = 1_G$ and all $\alpha \in \Omega$,

$\mathcal{O}_2$ $\quad (\alpha^x)^y = \alpha^{xy} \quad$ for all $x, y \in G$ and all $\alpha \in \Omega$.

[1] As in the definition of a group we are forming a *product*, but we write α^g instead of $\alpha\, g$.

The mapping

$$g^\pi : \Omega \to \Omega \quad \text{with} \quad \alpha \mapsto \alpha^g$$

describes the action of $g \in G$ on Ω. Because of

$$(\alpha^g)^{g^{-1}} \overset{\mathcal{O}_2}{=} \alpha^{gg^{-1}} = \alpha^1 \overset{\mathcal{O}_1}{=} \alpha,$$

$(g^{-1})^\pi$ is the inverse of g^π. In particular g^π is a bijection and thus a permutation on Ω. Now $\mathcal{O}_2$ implies that

$$\pi : G \to S_\Omega \quad \text{with} \quad g \mapsto g^\pi$$

is a homomorphism. The homomorphism theorem shows that $G/\operatorname{Ker}\pi$ is isomorphic to a subgroup of S_Ω and thus also to one of S_n, $n := |\Omega|$.

Conversely, every homomorphism $\pi : G \to S_\Omega$ gives rise to an action of G on Ω, if one defines $\alpha^g := \alpha^{g^\pi}$. A homomorphism $\pi : G \to S_\Omega$ is said to be an **action** of G on Ω.

If $\operatorname{Ker}\pi = 1$, then G acts **faithfully** on Ω; and if $\operatorname{Ker}\pi = G$, then G acts **trivially** on Ω.

Every action π of G on Ω gives rise to a faithful action of $G/\operatorname{Ker}\varphi$ on Ω, if we set

$$\alpha^{(\operatorname{Ker}\varphi)g} := \alpha^g.$$

Next we introduce some important actions, which we will frequently meet in the following chapters.

3.1.1 *The group G acts on*

(a) *the set of all nonempty subsets A of G by conjugation:*

$$A \overset{x}{\mapsto} x^{-1}Ax = A^x,$$

(b) *the set of all elements g of G by conjugation:*

$$g \overset{x}{\mapsto} x^{-1}gx = g^x,$$

(c) *the set of right cosets Ug of a fixed subgroup U of G by right multiplication:*

$$Ug \overset{x}{\mapsto} Ugx.$$

Proof. In all cases $1 = 1_G$ acts trivially; this is $\mathcal{O}_1$. Associativity gives $\mathcal{O}_2$. □

In (a) and (b) the permutation x^π is the inner automorphism induced by x (see 1.3 on page 15).

Also *left* multiplication on the set Ω of all left cosets of a fixed subgroup U leads to an action $\pi\colon G \to S_\Omega$. But here one has to define

$$x^\pi\colon G \to S_\Omega \quad \text{with} \quad gU \mapsto x^{-1}gU$$

since $gU \mapsto xgU$ is not a homomorphism (but an *anti*-homomorphism).[2]

Using (c) we obtain:

3.1.2 *Let U be a subgroup of index n of the group G. Then G/U_G is isomorphic to a subgroup of S_n.*[3]

Proof. As in 3.1.1 (c) let Ω be the set of all right cosets of U in G and $\pi\colon G \to S_\Omega$ the action by right multiplication. Then for $x, g \in G$

$$Ugx = Ug \iff gxg^{-1} \in U \iff x \in U^g,$$

and thus

$$x^\pi = 1_{S_\Omega} \iff x \in U_G,$$

i.e., $U_G = \mathrm{Ker}\ \pi$. □

In order to work with the actions given in 3.1.1 we first set some notation and collect some elementary properties of actions which follow more or less directly from the definition.

In the following, G is a group that acts on the set Ω. For $\alpha \in \Omega$

$$G_\alpha := \{x \in G \mid \alpha^x = \alpha\}.$$

The set G_α is the **stabilizer** of α in G; and $x \in G$ **stabilizes** (**fixes**) α if $x \in G_\alpha$.

Notice that G_α is a subgroup of G because of $\mathcal{O}_2$.

[2] We will use the action by left multiplication only in 3.3.

[3] $U_G = \bigcap_{g \in G} U^g$

3.1.3 $G_\alpha{}^g \;=\; G_{\alpha^g}$ *for* $g \in G,\ \alpha \in \Omega$.

Proof. $(\alpha^g)^x = \alpha^g \iff \alpha^{gxg^{-1}} = \alpha \iff gxg^{-1} \in G_\alpha \iff x \in (G_\alpha)^g$. □

Two elements $\alpha, \beta \in \Omega$ are said to be **equivalent**, if there exists $x \in G$ such that $\alpha^x = \beta$. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ show that this notion of equivalence does indeed define an equivalence relation on Ω. The corresponding equivalence classes are called the **orbits** of G (or G-orbits) on Ω. For $\alpha \in \Omega$

$$\alpha^G := \{\alpha^x \mid x \in G\}$$

is the orbit that contains α. G acts **transitively** on Ω, if Ω itself is an orbit of G, i.e., for all $\alpha, \beta \in \Omega$ there exists $x \in G$ such that $\alpha^x = \beta$.

3.1.4 Frattini Argument. *Suppose that* G *contains a normal subgroup, which acts transitively on* Ω.[4] *Then* $G = G_\alpha N$ *for every* $\alpha \in \Omega$. *In particular,* G_α *is a complement of* N *in* G *if* $N_\alpha = 1$.

Proof. Let $\alpha \in \Omega$ and $y \in G$. The transitivity of N on Ω gives an element $x \in N$ such that $\alpha^y = \alpha^x$. Hence $\alpha^{yx^{-1}} = \alpha$ and thus $yx^{-1} \in G_\alpha$. This shows that $y \in G_\alpha x \subseteq G_\alpha N$. □

The following elementary result is similar to Lagrange's theorem:

3.1.5 $|\alpha^G| = |G : G_\alpha|$ *for* $\alpha \in \Omega$. *In particular, the* **length** $|\alpha^G|$ *of the orbit* α^G *is a divisor of* $|G|$.

Proof. For $y, x \in G$

$$\alpha^y = \alpha^x \iff \alpha^{yx^{-1}} = \alpha \iff yx^{-1} \in G_\alpha \iff y \in G_\alpha x.$$ □

Since Ω is the disjoint union of orbits of G we obtain:

3.1.6 *If* n *is an integer that divides* $|G : G_\alpha|$ *for all* $\alpha \in \Omega$, *then* n *also divides* $|\Omega|$. □

[4] Of course, here we mean the action of N as a subgroup of G.

For $U \subseteq G$

$$C_\Omega(U) := \{\alpha \in \Omega \mid U \subseteq G_\alpha\}$$

is the set of **fixed points** of U in Ω. Obviously, $\Omega \setminus C_\Omega(G)$ is the union of all G-orbits of length > 1.

3.1.7 *Let G be a p-group. Then*

$$|\Omega| \equiv |C_\Omega(G)| \pmod p.$$

Proof. For $\alpha \in \Omega' := \Omega \setminus C_\Omega(G)$ the stabilizer G_α is a proper subgroup of G. Hence, p is a divisor of $|G : G_\alpha|$ (Lagrange's Theorem), and 3.1.6 implies

$$|\Omega'| \equiv 0 \pmod p. \qquad \square$$

We now apply 3.1.3 and 3.1.5 using the actions given in 3.1.1.

Let Ω be the set of all nonempty subsets of G and $H \leq G$. Then H acts by conjugation on Ω. For $A \in \Omega$ the set consisting of the subsets

$$A^x = x^{-1}Ax \quad (x \in H)$$

is an orbit of H. The stabilizer

$$N_H(A) := \{x \in H \mid A^x = A\}$$

of A in H is the **normalizer** of A in H.

By 3.1.5 $|H : N_H(A)|$ is the number of H-conjugates of A.

Let $B \in \Omega$. Then B **normalizes** A if $B \subseteq N_G(A)$.

By 3.1.1 (b) H acts by conjugation on the elements of G. For this action the stabilizer

$$C_H(g) := \{x \in H \mid g^x = g\}$$

of $g \in G$ is the **centralizer** of g in H. It is evident that this subgroup consists of those elements $x \in H$ that satisfy $xg = gx$.

Because of 3.1.5 $|H : C_H(g)|$ is the number of H-conjugates of g.

For a nonempty subset A of G

$$C_H(A) := \bigcap_{g \in A} C_H(g)$$

is the **centralizer** of A in H. Thus, $C_H(A)$ contains exactly those elements of H that commute with every element of A. For example, $C_G(A) = G$ if and only if A is a subset of $Z(G)$. A subset $B \subseteq G$ **centralizes** A, if $B \subseteq C_G(A)$ (or equivalently $[A,B] = 1$; see page 25).

3.1.3 implies for $x \in G$

$$N_G(A)^x = N_G(A^x) \quad \text{and} \quad C_G(A)^x = C_G(A^x);$$

and more generally

$$N_H(A)^x = N_{H^x}(A^x) \quad \text{and} \quad C_H(A)^x = C_{H^x}(A^x).$$

In the case $H = G$ the G-orbit g^G of the elements conjugate to g is the **conjugacy class** of g in G, and

$$|g^G| = |G : C_G(g)|.$$

The center $Z(G)$ contains exactly those elements of G whose conjugacy class has length 1, i.e., those elements that are only conjugate to themselves.

G is the disjoint union of its conjugacy classes since these classes are the G-orbits with respect to the action by conjugation. This gives:

3.1.8 Class Equation. *Let $K_1, \dots, K_h$ be the conjugacy classes of G that have length larger than 1, and let $a_i \in K_i$ for $i = 1, \dots, h$. Then*

$$|G| = |Z(G)| + \sum_{i=1}^{h} |G : C_G(a_i)|. \qquad \square$$

We note:

3.1.9 *Let U be a subgroup of G. Then $N_G(U)$ is the largest subgroup of G in which U is normal. The mapping*

$$\varphi\colon N_G(U) \to \operatorname{Aut} U \quad \text{with} \quad x \mapsto (u \mapsto u^x)$$

is a homomorphism with $\operatorname{Ker} \varphi = C_G(U)$. *In particular, $N_G(U)/C_G(U)$ is isomorphic*[5] *to a subgroup of* $\operatorname{Aut} U$. $\square$

We close this section with two fundamental properties of p-groups and p-subgroups, which follow from 3.1.7.

[5] Homomorphism Theorem 1.2.5 on p. 13.

3.1.10 *Let P be a p-subgroup of G and p be a divisor of $|G:P|$. Then $P < N_G(P)$.*

Proof. By 3.1.1 (c) P acts on the set Ω of right cosets Pg, $g \in G$, by right multiplication, and

$$|\Omega| = |G:P| \equiv 0 \pmod p.$$

From 3.1.7 we get (with P in place of G):

$$|C_\Omega(P)| \equiv |\Omega| \equiv 0 \pmod p.$$

Moreover $C_\Omega(P) \neq \varnothing$ since $P \in C_\Omega(P)$. Hence there exists $Pg \in C_\Omega(P)$ such that $P \neq Pg$. This implies $g \notin P$ and $PgP = Pg$. Thus $gPg^{-1} = P$ and $g \in N_G(P) \setminus P$. □

3.1.11 *Let P be a p-group and $N \neq 1$ a normal subgroup of P. Then $Z(P) \cap N \neq 1$. In particular $Z(P) \neq 1$.*

Proof. P acts on $\Omega := N$ by conjugation, and

$$C_\Omega(P) = Z(P) \cap N.$$

Since N is a p-group we get from 3.1.7

$$|C_\Omega(P)| \equiv |\Omega| \equiv 0 \pmod p.$$

Now $1 \in C_\Omega(P)$ gives $|C_\Omega(P)| \geq p$. □

Exercises

Let G be a group.

1. Let G be the semidirect product of a subgroup K with the normal subgroup N, and let $\Omega := N$. Then

$$\omega^{kn} := \omega^k n \quad (\omega \in \Omega, \quad k \in K, \quad n \in N)$$

defines an action of G on Ω.