Introductory Actuarial Science

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Contents

1 Pooling and Insurance

1.1 Introduction

In the following sections we illustrate the effect of pooling, and then insurance, on the level of risk for an individual Steps

1. Define a situation, where risks are independent
2. Explore possible outcomes from simulation, comparing the cases without and with pooling
3. Analyze that situation mathematically
4. Extend this to situation where risks are dependent
5. Contract a pure pool with an insurance company, and explore how insurance adds to pooling

Central question: "what is the basic idea behind insurance"

1.2 Risk Pooling in case of independent risks

1.2.1 Assumptions

1.2.1.1 Independence

- Information about one variable which carries no information about the other F(x,y) = F(x)F(y)
- It is sometimes a reasonable assumption
- For instance: the risk of dying for most individuals is an independent risk since the chance that a person dies does not usually depend on whether or not someone else has died
- There are exceptions, usually related to extreme events/disasters.

1.2.1.2 Risks

- n independent risks X_i for i = 1 to n (eg. n individuals)
- Each risk–a loss of L with probability q or no loss with probability $1-q$ (q is often used to represent the probability of death)
- Each risk can incur only one loss

$$Pr[X_i = L] = q$$

$$Pr[X_i = 0] = (1 - q)$$

1.2.2 Example with formulas

"An individual has a 0.01 chance of dying and the insurer will pay $10,000 upon their death. Calculate the expected death benefit and variance of the death benefit for this individual"

- Expected benefit

$$0.01 \times 10,000 = 100$$

- Variance

$$0.01 \times 0.99 \times 10,000^2 = 990,000$$

- Standard Deviation

$$\sqrt{Variance} = \sqrt{990,000} = 995$$

This corresponds to the moments of this unique risk. Of course, insurers have more than one risk in their portfolio. **This leads to Pooling**

1.2.3 With Pooling

What is Risk Pooling?

1. Pool the n risks
2. Each individual agrees to pay the actual average of the total loses into the pool
3. The pool pays the losses of the members of the pool who incur a loss
4. This gives us an average loss:

$$A = \frac{\sum_{i=1}^{n}(X_i)}{n}$$

   With X_i = L if a loss occurs or $X_i = 0$ if no loss occurs

This is a random variable - we do not know what it will be until the losses occur

"The good luck of some compensates for the bad luck of others"

1.2.3.1 Expectation with Pooling

- Expected value of the average loss for the pool

$$E[A] = \mu_A = E[\frac{\sum_{i=1}^{n}(X_i)}{n}]$$

The Number of losses (in this case) has a Binomial (n,q) distribution, so the probability that j individuals incur a loss is

$$Pr(\sum_{i=1}^{n} X_i = jL) = \binom{n}{j} q^j (1-q)^{n-j}, j = 0, 1, 2 \ldots, n$$

Note:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$Pr(N = j)$$

N = total number of losses

- Expected Value of the Average Loss

$$\mu_A = E[\frac{\sum_{i=1}^{n}(X_i)}{n}] = E[\frac{N \cdot L}{n}] = \frac{L}{n}E[N]$$

$$= \frac{L}{n}\sum_{j=0}^{n}(j)\binom{n}{j}q^j(1-q)^{n-j}$$

Where X $\sim$ Bin(n,p), E(X) = np and Var(X) = np(1-p)

$$= \frac{L}{n} \cdot nq = qL$$

Expected Value of the average loss in the pool is the same as the expected value of the individual loss

- Expectation is a linear operator

We have

$$E[k] = k$$

where k is a constant, and

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

where X and Y are random variables that need not be independent, and a, b, c are constants

1.2.3.2 Properties of Variance

For any random variable X,

$$Var[X] = E[X^2] - E[X]^2$$

so that

$$E[X^2] = Var[X] + E[X]^2$$

and also

$$Var[kX] = k^2 Var[X]$$

Also,

$$Var(X+Y) = Var(X) + Var(Y)$$

If X and Y are independent

1.2.3.3 Variance with Pooling

$$\sigma_A^2 = Var[\frac{\sum_{i=1}^{n}(X_i)}{n}]$$

$$= \frac{1}{n^2}Var[\sum_{i=1}^{n}(X_i)]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)$$

(X_i's are independent)

$$= \frac{n}{n^2}Var(X_i)$$

(X_i's have the same distribution)

$$= \frac{n}{n^2} Var(L \cdot N)$$

Var(X_i) = Var(L $\cdot N^*$) = L^2 Var(N)

$$= \frac{n}{n^2} L^2 Var(N) = \frac{n}{n^2} L^2 Var(Y_i)$$

*Given that N $\sim$ Bin(1,q) = Bernoulli(q) = Y_i

$$= \frac{n}{n^2} L^2 \cdot [1 \cdot q(1-q)] = \frac{L^2}{n}[q(1-q)]$$

"The variance of the average of the pool losses is the variance of each individual loss *q(1-q)*L^2 divided by the number of individuals in the pool"

$$\sigma_A^2 = \frac{q(1-q)L^2}{n} = \frac{Var(X)}{n}$$

1.2.3.4 So why pool?

1. For risk averse individuals, the "happiness" of each individuals who participates in the risk pooling is higher (variability is lower)
2. A mathematical model for explaining this in microeconomics, would use so-called "utility functions"
3. (In the context of insurance) If they fully-insure, they exchange all their risk, otherwise they exchange only a portion of it

1.3 Risk Pooling in case of correlated risks

1.3.1 Assumptions

- Here we relax the assumption of independence, as this is a major assumption
- We consider the simplest dependence model possible - **Correlation**
- The basic formulas about correlation are:

  $$Var(X+Y) = Var(X) + Var(Y) + 2Cor(X,Y)$$

  And then the **coefficient of correlation** is defined as

  $$-1 \leq \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X}\sqrt{Var(Y)}} \leq 1$$

1.3.2 Pooling of Correlated Risks

The Simplest Case

- Two losses X and Y – each with the same expected value μ and the same variance σ^2
- Correlation ρ(X,Y) = ρ, equal variances with $\sigma_X = \sigma_Y = \sigma$. Hence Cov(XY) = $\rho\sigma^2$

Average expected loss is:

$$E[\frac{X+Y}{2}] = \frac{E[X+Y]}{2} = \frac{E[X]+E[Y]}{2} = \frac{\mu}{2} = \mu$$

which is what we expected

1.3.3 Effect of Pooling

Variance of the average loss is

$$Var[\frac{X+Y}{2}] = \frac{1}{4}[\sigma^2 + \sigma^2 + 2\rho\sigma^2]$$
$$= \frac{\sigma^2}{2}(1+\rho)$$

($\frac{\sigma^2}{2}$ represents the independent portion and $1+\rho$ represents the correlation)

Pooling of correlated risks multiplies the variance, relative to the variance if the risks were independent, by the factor $(1 + \rho)$ where -1 $\leq \rho \leq 1$

If risks are perfectly correlated, ($\rho = 1$), the variance is

$$Var[\frac{X+Y}{2}] = \sigma^2$$

There would be no apparent benefit in pooling

1.3.4 Diversification Benefits

AS soon as dependence is not perfect, there are **diversification benefits**

$$\text{Diversification Benefit} = \sigma^2 - \frac{\sigma^2}{2}(1+\rho) = \sigma^2\frac{1-\rho}{2}$$

The more the diversification benefits, the lower the risk, and the lower the capital requirements. Accurate calculation of diversification benefits is difficult but crucial

1.4 Basic Risk transfer arrangements

Essentially there are two possible risk transfer arrangements

- Mutual company/Friendly Society/ Simple pool:
  - Individuals exchange (some of) the the individual risk X_i for (some of) the less variable average risk of the total pool $\frac{\sum_{i=1}^n X_i}{n}$

  (Calculations in this lecture assume we are in this situation)
- Insurance
  - Individuals exchange (some of) the individual risk X_i for a known, deterministic premium.The shareholders cover the downside risk by making capital available, and expect remuneration for it. Hence, this is generally more expensive (but less risky)
  - In this case, the premium would be exactly ($qL = 100$), plus loadings (eg. cost of capital). In any case, it is fully **deterministic**, in contrast to the former arrangement

Historically, the former arrangement above appeared first, and "modern" insurance as we know it developed only later

2 Time Value of Money

2.1 Introduction

The time value of money (TVM) is the concept that money you have now is worth more than the identical sum in the future due to its potential earning capacity. A factor which plays a large role is Interest which corresponds to the time value of money

It is a mathematical model to reflect the fact that most people would rather have a dollar now than in the future
How much this time value of money is will depend on:

1. How impatient you are
2. Any perceived risk

2.2 Simple Interest

2.2.1 Interest Rates

Typically, interest is paid after a while (not continually). Let us focus on interest earnt in-between payment times. Note that this model is generally valid only for short term situations.

2.2.2 Examples

1. Suppose John lends \$10,000 to a bank and bank agrees to pay simple interest at 12% per annum.

Suppose further that John will require repayment of his lent funds, along with the simple interest earned, in three months' time
How much money will John be paid at maturity, ie. in three months' time?

Given that we are operating with simple interest, John earns interest on his initial investment at the rate of 1% each month. The cash flow at the end of each month is:

$$Interest = 10,000 \cdot 0.01 = 100$$

His closing balance at the end of month 3 is therefore equal to

$$10,000 + (100 \times 3) = 10,300$$

We can generalise this into a formula

$$I = P \times r \times t$$
$$A = P + I = P(1 + rt)$$

- P - Principal
- r - rate of simple interest (per term)
- t - number of terms of the investment
- I - interest earned
- A - accumulated value of the investment at the end of the term

2. Suppose I want to be paid $10,000 in two and a half years' time and interest is quoted as 8% per annum simple Find the amount that I need to invest today

$$A = P(1 + rt)$$
$$10,000 = P(1 + 0.08 \times 2.5) = 1.2P$$
$$P = 8,333.33$$

3. Suppose you invested $10,000 on 23 September 2019 Find the accumulated value on 29 November 2019 assuming 8% per annum simple interest applies

Days between 23 September and 29 November - (30-23)+31+29=67 days

$$A = P(1 + rt)$$
$$= 10,000(1 + (0.08)(\frac{67}{365})) = 10,146.85$$

2.2.3 Applications

Commercial Bills

- Examples of commercial bills include bank accepted bills and treasury notes
- These financial instruments require an amount to be paid at a specific Tim hint he future. The date of this future payment is called the maturity date of the bill
- The bill is sold at a date prior to the maturity date at a discount to the amount required to be paid at maturity (Face Value). The amount of this discount if often calculated using simple interest

Example
Consider a bank bill which will mature on 31 August this year and was purchased on 21 July this year. The maturity value (also called the face value) of the bill is $100,000

Find the price paid for the bill on 21 July so that the holder of the bill earns 6% per annum simple interest

$$A = P(1 + rt)$$
$$t(days) = (31 - 21) + 31 = 41$$
$$100,000 = P(1 + (0.06)(\frac{41}{365}))$$
$$P = 99,350.54$$

2.3 Simple Discount

We now turn our attention to simple discount. This is different to simple interest in one particular way

- Under simple interest we have seen that the amount of interest is calculated by applying the simple interest rate to the amount of money present **at the onset of the investment period**
- Under simple discount, the amount of the discount is calculated by applying the same discount rate to the amount of money present **at the end of the investment period**

We can generalise the differences into a formula

$$d = I/A$$

$$P(A) = A - D = A(1 - dt)$$

- d - rate of simple discount (per term)
- D - Discount Earned

$$D = A \times d \times t$$

$$A = \frac{P}{(1 - dt)}$$

Example

Consider we have a bank bill with a face value of $100,000 due to be paid on 31 August. The bill was purchased on 21 July in the same year as the maturity date.

Find the price paid on 21 July for the bill so that the holder earns 6% per annum simple discount

Days between 21 July and 31 August - (31-21)+31=41 days

$$P(A) = A(1 - dt)$$

$$= 100,000(1 - (0.06)(\frac{41}{365})) = 99,326.03$$

Note when earning 6% simple interest the price of the bill was 99,330.54 (simple interest < simple discount)

2.4 Simple Interest vs Simple Discount

From the Bank bill example it should be clear that a simple interest year of 6% is NOT equivalent to a simple discount rate of 6%. The bank bill has a different price when it was priced to earn 6% per annum simple interest compared to when I was priced to earn 6% per annum simple discount. The reason for this can be made clear by considering an item in a shop for which the price is discounted and then subsequently marked up

Suppose a suit costs $1000 ($A$). The price is reduce in a sale by 20% - this is like applying a simple discount rate of 20% per annum to the price over a one year period. The new price (P) is $800

Suppose now that the discounted price, P, is now increased by 20% - this is like applying a simple interest rate of 20% to the price of the suit for one year. The new price is $960

After application of simple discount and then simple interest both at 20% for a one year period, we do NOT get back to the same starting position, that is to say the price does not return to $1,000

2.4.1 The relationship between simple interest rates r and simple discount rates d

Case 1: the investment is one year (t=1)

Define the following notation and procedure

- Initial investment of X dollars
- Invest for one year at simple interest rate r per annum
- Let Y denote the accumulated value of X after one year
- Discount Y for one year at simple discount rate d per annum

Find d in terms or r such that the discounted value of Y is exactly X

From simple interest results, we have

$$Y = X(1 + r)$$

Now, we require

$$\frac{X}{1-d} = Y \Rightarrow \frac{1}{1-d} = 1 + r$$
$$\Rightarrow r = \frac{d}{1-d}$$
$$\Rightarrow d = \frac{r}{1+r}$$

Rearranging the terms,

$$\mathbf{d = r(1\text{-}d)}$$
$$\mathbf{r = d(1\text{+}r)}$$

Case 2: The investment term is t-years

Define the following notation and procedure

- Initial investment of X dollars
- Invest for t-year at simple interest rate r per annum
- Let Y denote the accumulated value of X after one year
- Discount Y for t-year at simple discount rate d per annum

Find d in terms or r such that the discounted value of Y is exactly X

If r is given,

$$Y = X(1 + rt) \Rightarrow X = \frac{Y}{1+rt}$$

on the other hand if d is given,

$$X = Y - Ydt = Y(1 - dt)$$

equating the two quantities gives the relationship

$$1 + rt = \frac{1}{1 - dt} \Rightarrow r = \frac{d}{1 - dt}$$

$$\Rightarrow d = \frac{r}{1 + rt}$$

Rearranging the terms,

$$\mathbf{d = r(1\text{-}dt)}$$

$$\mathbf{r = d(1\text{+}rt)}$$

2.4.2 Examples

1. Find the simple interest equivalent to a discount rate of 20% p.a

a) assuming a one-year investment

$$r = \frac{d}{1 - dt}$$

$$= \frac{0.2}{1 - (0.2)(1)} = 0.25$$

This again reaffirms our understanding that the simple discount rate is lower than the equivalent simple interest rate

b) assuming a three-month investment

$$r = \frac{d}{1 - dt}$$

$$= \frac{0.2}{1 - (0.2)(\frac{3}{12})} = 0.21$$

2. On 7 May 2020 an investor purchased a \$1,000 bill, maturing on 8 August 2020, at 6% per annum simple discount

a) Calculate the price paid to the nearest cent
Number of days from 7 May - 8 August: $(31 - 7) + 30 + 31 + 8 = 93$ days

$$P = A(1 - dt) = 1,000(1 - (0.06)[\frac{93}{365})] = 984.71$$

b) What is the rate of simple interest per annum implied by this price? Give your answer to 4 decimal places

$$r = \frac{d}{1 - dt} = \frac{0.06}{1 - (0.06)(\frac{93}{365})} = 0.0609$$

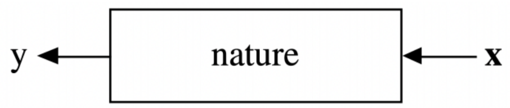
3 Application: Data and Predictive Analytics

Actuarial Science is more than just maths. Here we take a look into the increasing role data analytics and big data play in the jobs of actuaries.

3.1 Modelling Culture and AI

3.1.1 What are we trying to do?

Consider outputs (y) from inputs x in nature:



Statistical modelling tries to "model" *nature* in order to reach two types of conclusions from data(x)

- *Prediction*: predict y from future sets of x
- *Information*: extract information about the association of y and x - *understand nature*

3.1.2 Artificial Intelligence

Some vocabulary:

- **Artificial Intelligence**: Smart (intelligent) stuff performed by a computer
  - **Machine learning** is a subset: here the program "learns" to perform a given task better (according to some success criteria)as more data is fed to it
    - **Deep learning** is a subset: usually refers to "deep artificial neural networks." Such networks have a certain number of layers, leading to the ideas of depth

$$\text{Deep learning} \subset \text{Machine learning} \subset \text{AI}$$

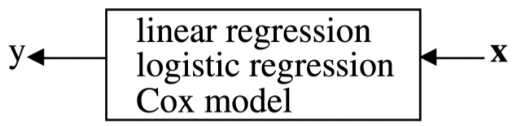
3.1.3 Modelling Culture

Breiman (2001) distinguishes two modelling cultures

- data modelling culture
- algorithmic modelling culture

This distinction helps work out the main differences between traditional statistical models, and the newer ones included in AI

Data Modelling Culture
Here we assume that data is generated from an underlying stochastic model which is explicit



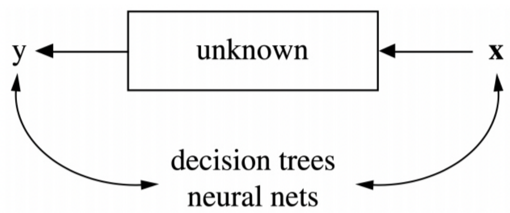
We have

$$y = f(x)$$

where f needs to be chosen and is relatively inflexible

- This is the older, traditional approach
- This model will be as good as the choice of f, and might be very "precisely wrong" (as opposed to "approximately right")

Algorithmic Modelling Culture
Here the mechanism by which data is generated is considered as unknown, and is modelled with algorithmic approaches



- This has no (or less) explicit structure and has perhaps better chances of modelling "nature's processes"
- It is the newer, "machine learning approach," which is more organic
- It thrives with more data and computer resources
- It is also harder to interpret (and communicate) as the middle is more of a "black box"

3.2 Impact of Big Data on the Australian Insurance Industry

Green paper on how Big Data is transforming the insurance industry and the implications for the cost and availability of insurance for all consumers

"Improved data will produce winners and losers amongst insurance customers"

"Insurers that choose not to use available data will end up in the unsustainable position of only insuring the higher risks. Hence big data usage is likely to be widely adopted"

"The goods news is that many consumers will benefit from this new technology. Premium pricing will more accurately reflect risk behaviour - good young drivers will pay less than risky young drivers"

3.2.1 Adverse Selection

With Big Data insurers have more information about consumers than before. This allows them to apply adverse selection to their consumer base. This results in Risk Factor Identification which forces competitors to follow:

Example:
If insurer A differentiates with respect to one (significant) risk factor, but not insurer B, all "good" risk will move from insurer B to A, and all "bad" risks will move from insurer A to insurer B

With more data on consumers insures are able to better able to price discriminate between customers. This is good news for some as premium prices will reward those who exhibit less risky behaviour - "good young drivers will pay less than risky young drivers"

From microeconomics we know that from a business's perspective it is always better to price discriminate. However as you discriminate you reduce the mutuality of your product, which may raise ethical questions.

3.2.2 Discussion: Precision vs Mutuality

The insurance business is the business of risk diversification
Precise pricing is when we segment the policyholder base by providing a lower premium to those with a lower risk of claim, and a larger premium (which may be very high) to those with a larger risk of claim.
Mutuality (when considered as opposed to precision) is when we set one price or a general price for both lower risk and larger risk policyholders, so that we pool the risks of both sets of policyholders, for a general affordable price for everyone.

More precise pricing leads to less mutuality

- What are the pros and cons?
- What if the factor is in or outside the control of the policyholder? (Smoking? Genetics? Address? Income?)
- What factors are unacceptable to consider? (illegal, unethical)

Including a risk factor that a policyholder has control on may reduce risk however if the factors are uncontrollable by an individual this might lead to the lack of access to insurance due to unaffordability. (Do people have a right to access insurance? What can be done if they can't)

3.2.3 Privacy and Discrimination

Privacy Issues

- Who owns the data?
- How can it be used?

Discrimination Issues

- Is this fair? How do we determine fair?
- What unintended consequences might there be?
- Direct vs Indirect discrimination

3.3 Insurtech

Loosely speaking, "insurtech" refers to the use of technology for insurance
Insurtech trends: using data to create dynamic insurance

- alot of data can be gathered, which can improve the risk assessment and pricing (underwriting)
- moreover, this could be done in a *dynamic* way
- we are back in the precision vs mutuality issue, and there are ethical issues too

4 Compound Interest

4.1 Compounds Interest

4.1.1 Definition

Compound interest differs from simple interest initiate the investor can earn interest no only on the original investment (called the principal) but also on interest paid in the past

Consider an investment of \$10,000 for a period of 3 years. Suppose that the interest rate offered is 6% per annum **simple**. The accumulated value of this investment after 3 year can easily be calculated

$$A = P(1 + rt) = 10,000[1 + (0.06 \times 3)] = 11,800$$

Here we assume that interest is not paid at the end of each year

Under compound interest, interest is paid periodically. If it is paid annually then we can see the increase in the principal as per the table below

Year	Starting Amount	Interest	Ending Amount	Ending Amount with $r = 0.06$ (simple)
1	\$10,000.00	\$600.00	\$10,600.00	\$10,600.00
2	\$10,600.00	\$636.00	\$11,236.00	\$11,200.00
3	\$11,236.00	\$674.16	\$11,910.16	\$11,800.00

Note that effect of earning interest on interest (the *compounding* effect giving its name to *compound* interest)

Example

Suppose that \$10,000 , in invested for a period of 10 years

Suppose also that the compound interest rate offered is 7% per annum effective

Find the accumulated value of this investment after

- One year

$$10,000(1 + 0.07) = \$10,700$$

- Two Years

$$10,000(1 + 0.07)^2 = \$11,449$$

- Ten years

$$10,000(1 + 0.07)^{10} = \$19,671.51$$

4.1.2 General Formula

- P = principal
- i = rate of compound interest per period
- n = term of investment (number of periods)
- A = accumulated value of original investment

$$A = P(1 + i)^n$$

Note that this formula also works for non-integer time periods n. If one period is one year, then i is the compound interest rate per annum

Example

Assume that compound interest is earned at the rate of 7% per annum effective

How much money do I need to invest today if I want to have \$10,000 in ten year's time

$$10,000 = P(1 + 0.07)^{10}$$

$$P = \frac{10,000}{1.07^{10}} = \$5,083.49$$

Note here that we have essentially *discounted* \$10,000 for 10 years

4.1.3 Comparison of Simple and Compound Interest

An important result

$$1 + it > (1 + i)^t, \text{ for } 0 < t < 1$$

$$1 + it = (1 + i)^t, \text{ for } t = 1$$

$$1 + it < (1 + i)^t, \text{ for } t > 1$$

As an illustration, consider two cases

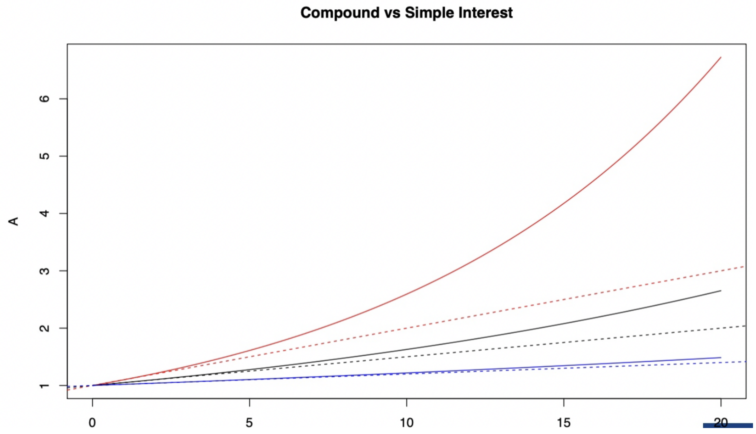
1. Let $i = 0.05$ and $t = 1/2$

$$1 + it = 1.025 \text{ and } (1 + i)^t = 1.0247$$

2. Let $i = 0.05$ and $t = 2$

$$1 + it = 1.1 \text{ and } (1 + i)^t = 1.105$$

In the long run (10% in red, 5% in black, 2% in blue)



What the illustration and graph show is that the accumulation with simple interest compared to the accumulation with compound interest will be *more* when $t < 1$, *equal* when $t = 1$ and *less* when $t > 1$

4.2 Discounting

4.2.1 Present Value and "Discounting"

"Discounting" is the inverse of "Accumulating". By discounting we generate what is called a "Present Value" (as opposed to an "accumulated value" or Future value)
The amount of money, denoted as P, is called the **present value (PV)** which is the amount of money needed **now** so that its accumulated value is A in n years' time

Rearranging our compound interest accumulation result, $A = P(1+i)^n$, we have

$$P = A(1+i)^{-1}$$

It follow therefore that we need to invest $A(1+i)^{-n}$ today in order to have A dollars after n years
We would refer to $A(1+i)^{-n}$ as the *(discounted) present value* of A due in n years' time

4.2.2 Notation

Let us define the **discounting factor v** as

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

It should be clear that v is the present value of \$1 due in one years' time when compound interest is levied at the rate of i per annum effective

This allows we rewrite our Present Value formula as

$$P = Av^n$$

Example
Find the present value of \$25,000 due in 15 years' time
Assume compound interest at 5% per annum effective

$$v = \frac{1}{1+0.05} = 1.05^{-1}$$

$$P = Av^n = 25,000(1.05)^{-1})^{15}$$

$$= 25,000(1.05^{-15} = \$12,025.43$$

4.3 Nominal and Effective Interest Rates

4.3.1 Effective Rate of Interest

"Effective" has meaning and is used when compounding is involved. It tell us that the compounding occurs once per effective time period. This means that there will be rates that are *not* effective These rates are called *"nominal" rates p.a.* which are typically effective for a different time period than annual

For example:
Consider an investment of \$10,000. Suppose that the compound rate of interest is 10% per annum. We consider two situations

1. If compounding occurs only once in a year, then the accumulated value of the investment at the end of the year will be
$$A = 10,000(1 + 0.10) = \$11,000$$
Here we say that the interest rate is 10% per annum effective
2. If instead compounding occurs twice in the year, then the accumulated value of the investment at the end of the year is calculated as
$$A = 10,000(1 + \frac{0.1}{2})^2 = \$11,025$$
Here we say that the interest rate is 10% per annum ***convertible half-yearly***
It can also be referred to as 5% effective per 6 months or 10.25% effective per annum

4.3.2 Connecting effective and nominal rates with formulas

Introductory example:
What is the annual effective interest rate equivalent to an interest rate of 10% per annum convertible half-yearly?

We can answer this question by noting that under an interest rate of 10% per annum convertible half-yearly, \$10,000 grows to \$11,025 at the end of the year. Hence we can solve for the unknown annual effective rate using
$$11,025 = 10,000(1 + i)$$
giving $i = 10.25\%$
Note: It makes sense that the annual effective rate is higher than the equivalent compound interest rate convertible half-yearly

In General,
We now define $i^{(m)}$ as the nominal rate of interest convertible *m-thly*, with
$$\frac{i^{(m)}}{m}$$
being the $\frac{1}{m}$*-year effective rate* of compound interest earned

4.3.3 Accumulation and Discounting under nominal $i^{(m)}$

If the nominal rate of interest is $i^{(m)}$ convertible *m-thly*, the accumulated value of principal P after n years (after mn compounding periods) is
$$A = P(1 + \frac{i^{(m)}}{m})^{mn}$$
Example
Suppose that \$5,000 is invested at a nominal rate of interest of 9% per annum convertible monthly The accumulated value of this investment after 2.5 years is
$$A = 5,000(1 + \frac{0.9}{12})^{12\times 2.5}$$
$$= 5,000(1.0075)^{30} = \$6256.36$$

4.3.4 Present Values and Discounting

We can also find present values when the interest rate is expressed as a nominal rate convertible *m-thly*. By re-arranging the formula

$$A = P(1 + \frac{i^{(m)}}{m})^{mn}$$

we get

$$P = A(1 + \frac{i^{(m)}}{m})^{-mn}$$

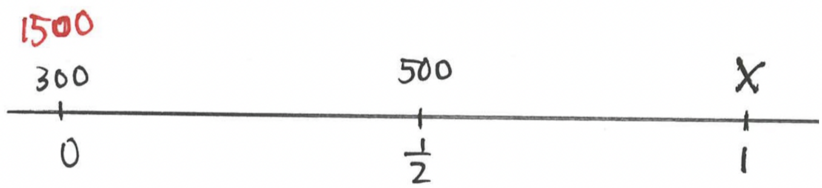
Example
What sum of money, due at the end of 5 years, is equivalent to \$1,800 due at the end of 12 years? Assume the nominal rate of interest is 11.75% per annum compounding half-yearly - that is

$$i^{(2)} = 11.75\%$$

$$P = 1,800(1 + \frac{0.1175}{2})^{-(2)(12-5)}$$

$$1,800(1.05875)^{-14} = \$809.40$$

Example
A consumer buys goods worth \$1,500. She pays \$300 deposit and will pay \$500 at the end of 6 months
If the store charges interest at $i^{(12)} = 18\%$ on the unpaid balance, what final payment will be necessary after one year?
- Let the unknown payment be X
- Let the present value of all payments be \$1,500, the purchase price of the goods



We have

$$1,500 = 300 + 500(1 + \frac{0.18}{12})^{-6} + X(1 + \frac{0.18}{12})^{-12}]$$

$$1,500 = 300 + 500(1.015^{-6}) + X(1.015^{-12})$$

$$X = \frac{1200 - 500(1.015^{-6})}{(1.015^{-12})} = \$888.02$$

4.3.5 The relationship between i and $i^{(m)}$

Let

- $i^{((m)}$ be the nominal rate of interest convertible *m-thly*, and
- i be the effective rate of compound interest per annum

Then we have

$$1 + i = (1 + \frac{i^{(m)}}{m})^m,$$

$$i^{(m)} = m\{(1 + i)^{1/m} - 1\}$$

4.4 The Force of Interest - Continuous Compounding

4.4.1 Concept

What we have is a nominal rate of interest with compounding occurring infinitely many times per year. This means that compounding must be occurring continuously
Continuous compounding has a very important role to play in theoretical actuarial and financial mathematics. The nominal rate of interest with continuous compounding is called the *force of interest*

Notation:

$$\lim_{m\to\infty} i^{(m)} = \delta$$

4.4.2 Main Formulas

We have that the accumulation factor of δ p.a is

$$e^{\delta} = 1 + i$$

If i is the equivalent effective rate of interest per annum. For t years this becomes

$$(1+i)^t = (e^{\delta})^t = e^{\delta t}$$

Example
- Calculate the present value of \$10,000 due in 6 years at a force of interest of 7% per annum

$$10,000 = P(e^{-0.07\times 6})$$

$$P = 10,000(e^{-0.42}) = \$6570.47$$

- Calculate the nominal rate of interest, convertible half-yearly, that is equivalent to a force of interest of 8% per annum

$$e^{0.08} = (1 + \frac{i^{(2)}}{2}))^2$$

$$i^{(2)} = [(e^{0.08})^{0.5} - 1] \times 2$$

$$= 0.0816$$

4.4.3 Properties: $i > \delta$

Mathematics Reminder: Taylor series expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

so that

$$e^{\delta} = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + ... = 1 + i$$

hence,

$$i > \delta$$

which makes sense, since I is compounded only once a year, and δ is compounding continuously

4.4.4 A Proof

$$\begin{aligned}
\delta &= \lim_{m\to\infty} i^{(m)} \\
&= \lim_{m\to\infty} m\{(1+i)^{1/m}-1\} \\
&= \lim_{m\to\infty} m\{e^{D/m}-1\} \text{ where } D = log(1+i) \\
&= \lim_{m\to\infty} m\{1+\frac{D}{m}+\frac{D^2}{2m^2}+\frac{D^3}{6m^3}+...-1\} \\
&= \lim_{m\to\infty} \{D+\frac{D^2}{2m}+\frac{D^3}{6m^2}+...\} \\
&= D = log(1+i)
\end{aligned}$$

and hence

$$1+i = e^{\delta}$$

4.5 Varying Interest Rates

4.5.1 Summary

We have the following expressions for accumulating \$1 for one year

$$1+i = (1+\frac{1^{(m)}}{m})^m = e^{\delta}$$

We also have the following expressions for discounting \$1 for one year

$$v = \frac{1}{1+i} = (1+\frac{i^{(m)}}{m})^{-m} = e^{-\delta}$$

4.5.2 Equivalent Interest Rates

We have already seen that, for example. 10% per annum convertible quarterly is equivalent to 2.5% per quarter effective. This equivalence in interest rates means that we can calculate present values and accumulations of payments in different ways and get the same result

We must be able to move from one definition to another with ease

Example
Find the PV of \$10,000 due in 10 years' time at 10% per annum convertible quarterly

1. Method 1: Use the nominal rate of 10% and annual time period

$$PV = 10,000v^{10} = 10,000[(1+\frac{0.1}{4})^{-4}]^{10} = \$3,724.31$$

2. Method 2: Use a time period of quarter years

$$PV = 10,000v^{40} = 10,000(1+\frac{0.1}{4})^{-40} = \$3724.31$$

Example
- An insurer invests $50,000 at a nominal rate of 9% per annum convertible monthly - This investment has to provide payments in 3 years' time and in 8 years' time - If the payments are of equal amount, what is the amount of each payment?
Approach to Problem: Let the unknown payment be X. Find the present value of the payments and equate this to 50,000. Measuring time in months, we get

$$50,000 = X(1+\frac{0.09}{12})^{-12\times 3} + X(1+\frac{0.09}{12})^{-12\times 8}$$

$$1.2522X = 50,000$$

$$X = \$39,929,38$$

4.5.3 Calculations with a mix of rates

We often hear in the media that the RBA is considering a change in the official level of interest rates. Such changes in official interest rates are generally reflected in the interest rates charged by lenders of funds and the interest rates offered to lenders of funds.
We must therefore include methods for determining accumulation and present values when interest rates vary with time

Example
Suppose that the annual effective rate of compound interest is 8% per annum for 3 years and then 6% per annum effective for 5 years
The accumulated value after 8 years is

$$P(1+0.08)^3(1+0.06)^5 = 1.68578P$$

Example
Bob invests $500 for 4 years. The nominal interest rate remains 8% each of the following 4 years, but

1. In the first year it is convertible half-yearly
2. In the second year it is convertible quarterly
3. In the third year it is convertible monthly, and
4. In the fourth year it is convertible daily

What is the accumulated value?

$$A = 800(1+\frac{0.08}{2})^2(1+\frac{0.08}{4})^4(1+\frac{0.08}{12})^12(1+\frac{0.08}{365})^365$$

$$A = \$686.76$$

How much greater is this value than the corresponding value assuming that the first year rate had remained unchanged for the 4 years?

If the first year rate had remained unchanged for 4 years, our accumulation would have been

$$A = 500(1+\frac{0.08}{2})^{2\times 4} = \$684.28$$

So the difference is just $2.48

4.6 Describing Interest Rates Accurately

It is important to describe interest rates in an unambiguous manner. For example

- 10% per annum simple interest
- 8% per annum convertible half-yearly
- 8% per annum simple discount
- 5% per annum continuously compounded
- force of interest = 5% per annum

5 Application: Risk Management and Climate Risk

5.1 The World needs to wake up to long-term risks

Article: https://www.actuaries.digital/2021/04/06/the-world-needs-to-wake-up-to-long-term-risks/
Key Take-aways:

- Risks are categorised across five ares economic, environmental, geopolitical, societal, and technological
- Risks are classified by the Likelihood vs Impact
- "Climate action failure is the most impactful and second most likely long-term identified risk"
  - Top risks by likelihood include: "Extreme weather", "climate action failure and human-led environmental damage", "digital power concentration", "digital inequality", and "cybersecurity failure"
  - Top risks by impact include: "Infections diseases", "climate action failure and other environmental risks", "weapons of mass destruction", "livelihood crises", "debt crises", and "IT infrastructure breakdown".

5.2 Risk Management

Risk Management is the identification, evaluation, and prioritisation of risks followed by coordinated and economic application of resources to minimise, monitor, and control the probability or impact of unfortunate events or to maximise the realisation of opportunities

Note that this definition not only considers the downsides, but also the upsides of risk

However this definition could be seen as a little too broad for the actuarial focus

<u>"Insurance Risk Management" according to PwC</u>

> "The assessment and quantification of the likelihood and financial impact of events that may occur in the customer's world that require settlement by the insurer; and the ability to spread the risk of these events occurring across other insurance underwriter's in the market. Risk management work typically involves the application of mathematical and statistical modelling to determine appropriate premium cover and the value of insurance risk to 'hold' vs 'distribute' - the idea of reinsurance"

This is better tailored to the actuarial work - it essentially translates how the actuarial skill set is used to contribute to the broader risk management concept

5.3 The Expert Corner

Interview with Gloria Yu:
Key Takeaways:
- Actuaries and Risk Management

- "Risk management is a broad area ... we can actually bring to bear the actuarial skill set to areas which are potentially not seen as traditional actuarial space"
- "Risk management ... actually straddles across a number of different practice areas"
- "An actuary will need to comment on the suitability and adequacy of the risk management framework and there are prudential standards that govern these types of work, essentially saying that the statutory actuary or appointed actuary will need to comment on the risk management framework on an annual basis and to submit that to APRA as a regulator as well"

- Typical employers

- "Risk management essentially is important for every single company"
- "There are also other organisations even in non-financial services sector which actuaries are starting to get involved in, whether it's around the operational risk or whether it's about credit risk and strategic risk"

5.4 Australian Prudential Regulation Authority (APRA)

APRA's role is to protect the interests of policyholders (beneficiaries of the institutions under APRA's supervision in the broad sense). APRA supervises trillions of AUD. On the other hand, the Australian Securities and Investments Commission (ASIC) is the institution that is concerned with shareholders' interests (and market conduct in general)

APRA's mission:

- To establish and enforce prudential standards and practices designed to ensure that, **under all reasonable circumstances**, financial promises made by institutions that APRA supervises are met within a **stable, efficient and competitive** financial system
- There are no guarantees. APRA aims only to control the chance of failure to an acceptable small number

5.5 Enterprise Risk Management

Committee of Sponsoring Organisations of the Treadway Commission (COSO) defines:

"Enterprise risk management is a process, effected by an entity's board of director, management and other personnel, applied in strategy setting and across the enterprise, designed to identify potential events that may affect the entity, and manage risk to be within its risk appetite, to provide reasonable assurance regarding the achievement of entity objectives"

5.6 Climate Risk

- Climate change
There is evidence of climate change: by NASA, in Australia, etc.

- AD: "Ringing the bells of climate change"
How can climate change affect insurers and their financial stability?

1. **Physical risks** value of assets impacted by climate and weather related events. More events of bigger scale → impact on capitalisation
2. **Liability risks**: whose fault?
3. **Transition risk** assets revalued, changes in policies, changes in socio-economic political environment

- Actuaries Institute Climate Change Public Policy Statement
The AI's position is that climate change is a thing, and that it is "man-made"
Climate change has major consequences, and actuaries have skills that can help address those

- Identify and understand risks
- Develop policy and strategy to respond to those risks
- Develop and implement frameworks to manage those risks

Climate change will have a major impact on at least four key areas

1. Extreme weather events: frequency and severity
2. Population and health: pandemic and heat
3. Scarcity of resources: water food, ecosystem
4. Economic value of assets: shift away from fossil fuels

- Role of actuaries in climate risk (https://www.actuaries.digital/2021/06/23/the-role-of-actuaries-in-climate-risk/)

Tan Suee Chieh: "Ethical considerations that actuaries should take when weighing up the financial system, regulatory system and social system. Actuaries, said Tan Suee Chieh, should be part of the conversation or should we will just be technical experts waiting to be told what to do by the regulator"

Michael Eves: "Climate risk is not just about models, and actuaries need to be careful to not only project past trends but consider other factors such as people's behaviour and the impact of a changing climate. For example, Swiss Re observes that insurable hurricane risk has changed with climate; historically, the majority of insured damage from hurricanes arose from wind, but Swiss Re now sees flood risk associated with hurricanes is greatly increased as a percentage of event losses."

6 Compound Interest - Annuities and The Equation of Value

6.1 Valuing Annuities

Finding the present value of a series of equal payments = **"annuities"**
So far we have considered the valuation of the present value and the accumulated value of single cash flows. However it is quite common in practice for the same cash flows to be repeated many times. eg Mortgage Repayments, insurance payouts
In order to value (that is, to find the present value of) a series of payments, we could find the present value of each individual payment in the series of payments and sum the resulting series. This approach will very quickly become tedious for long series of cash flows. We therefore develop formulae for finding the present value of streams of equal payments

6.2 Annuities in arrears - Annuity - Immediate

Consider the figure below which contains n years. Consider the case where a payment of \$1 is made at the end of each of the n years marked in the diagram. Suppose we want to find the present value of these payments - that is, their value at time 0

We perform this calculation very often and so we give a symbol for the present value result. Define:

$$a_{\overline{n}|} = v + v^2 + ... + v^n$$

Note: angle symbol $\Rightarrow$ payment is certain
This type of annuity is called an **annuity in arrears** because the payments are made at the **end of each year**

Example:
1. Write down an expression in terms v for $(1+i)a_{\overline{n}|}$

$$a_{\overline{n}|} = v + v^2 + ... + v^n$$
$$(1+i)a_{\overline{n}|} = (1+i)(v + v^2 + ... + v^n)$$
$$v = \frac{1}{1+i} \Rightarrow (1+i) = v^{-1}$$
$$(1+i)a_{\overline{n}|} = v^{-1}(v + v^2 + ... + v^n) = 1 + v + v^2 + ... + v^{n-1}$$

2. Use part (1) and the definition of $a_{\overline{n}|}$ to derive and expression for $ia_{\overline{n}|}$.

$$(1+i)a_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1}$$
$$(1+i)a_{\overline{n}|} - a_{\overline{n}|} = [1 + v + v^2 + ... + v^{n-1}] - [v + v^2 + ... + v^n]$$
$$ia_{\overline{n}|} = 1 - v^n$$

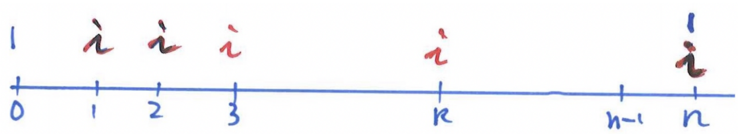
Hence show that

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

3. If we rearrange our result from (2), we get

$$1 = ia_{\overline{n}|} + v^n$$

Interpret this expression in terms of a loan of \$1 and the associated repayments



– For a loan of \$1 today, we can repay interest (only) on the loan at the end of each year for n years and then repay the amount borrowed (\$1) in n years time

or

– The present value of the borrowings (\$1) must equal the present value of the interest repayments ($ia_{\overline{n}|}$) plus the present value of the repayment of principal, v^n

$$1 = ia_{\overline{n}|} + v^n \text{ and hence } a_{\overline{n}|} = \frac{1 - v^n}{i}$$

6.2.1 A concrete example

Suppose I lend you \$100 and charge you interest at 5% per annum effective, $n = 10, \ i = 5\%$

- This means your debt in one year's time will be \$105
- Suppose you repay \$5 at the end of 1 year - you have repaid the interest only, and still owe me \$100
- At the end of the second year you again pay \$5 - your debt is still \$100
- Suppose this pattern continues for 10 years, each year you pay \$5 at the year end, and at the end of the 10th year you repay the \$100
- Your repayments can be represented as a 10 year annuity in arrears of amount 5, and a payment of \$100 at time 10 years
- As your loan is being repaid, the present value of my outgo equals that of my income which is:

$$100 = 5a_{\overline{10}|} + 100v^{10}$$

6.2.2 Alternative derivation of $a_{\overline{n}|}$

Another way of deriving $a_{\overline{n}|}$ is by using the sum of a geometric progression with n terms, first term v and common ration equal to v, we get

$$a_{\overline{n}|} = v + v^2 + v^3 + ... + v^n = v(v + v^2 + ... + v^{n-1})$$

$$= v\frac{1 - v^n}{1 - v}$$

$$= \frac{1 - v^n}{(1/v) - 1}$$

Since

$$\frac{1}{v} = 1 + i$$

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

6.3 Annuities in advance - Annuity-due

Definition:
– The annuity we considered above is called an annuity in arrears since the payments are made at the *end* of each of the n years
– Suppose instead that the payment are made at the *start* of each of the n years. The annuity is then called an *annuity in advance*, or *annuity-due*

The present value of an annuity in advance is written as $\ddot{a}_{\overline{n}|}$ and is pronounced "a due n"

Example
– Write down a sum in term of v for $\ddot{a}_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + ... + v^{n-1}$$

$$\ddot{a}_{\overline{n}|} = \frac{1}{v} a_{\overline{n}|}$$

– Explain in words why

$$\ddot{a}_{\overline{n}|} = (1+i) a_{\overline{n}|}$$

The number of payments and the valuation interest rates are the same. The only difference is that payments occur one year earlier in $\ddot{a}_{\overline{n}|}$ as compared with $a_{\overline{n}|}$, which explain why it is worth more, by a factor of (1+i)

– Show that

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

Using our definition of $a_{\overline{n}|}$, we can write

$$a_{\overline{n-1}|} = v + v^2 + v^3 + ... + v^{n-1}$$

This clearly includes every term in the summation for $\ddot{a}_{\overline{n}|}$ except the initial payment of 1. Therefore

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

Example:
– Find the PV of 20 annual payment of \$1,000 at 6% per annum effective with the first payment in 12 months time

$$\$1000 a_{\overline{20}|} = \frac{1 - (1.06)^{-20}}{0.06} = \$11,469.92$$

– Find the PV of 15 annual payments of \$700 at 5% per annum effective first payment due immediately

$$\$700 \ddot{a}_{\overline{15}|} = \$700(1+i) a_{\overline{15}|}$$

$$= \$700(1.05) \frac{1 - (1.05)^{-15}}{0.05} = \$7,629.05$$

6.4 Some reasonableness checks for annuities

Quite often in actuarial work, we are involved with very complex calculations. It is useful to be able to place a rough check (a reasonableness check) on our work at the end. This can be a useful way to remove any careless error that we may have made during the course of our work

6.4.1 First Check

First, (assuming $i \geq 0$

$$a_{\overline{n}|} = v + v^2 + v^3 + ... + v^n$$
$$\leq 1 + 1 + 1 + ... + 1 = n$$

Thus, **the value of $a_{\overline{n}|}$ cannot exceed** n

6.4.2 Second Check

Second, (assuming $i > 0$)

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \rightarrow \frac{1}{i} \text{ as } n \rightarrow \infty$$

Thus, **the value of $a_{\overline{n}|}$ cannot exceed** $\frac{1}{i}$

6.4.3 Third Check

Third, consider the amounts and the timing of the payments

- Each of the n payments is of amount 1, so the total amount is n
- The payment times are 1,2,. . .,n so the average payment time is:

$$\frac{1}{n}(1 + 2 + ... + n) = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

Hence, **an approximation of $a_{\overline{n}|}$** is

$$a_{\overline{n}|} \approx nv^{\frac{n+1}{2}}$$

Example
- Calculate the PV of a series of payments of \$100 at the beginning of each of the next 15 years at 8% per annum convertible quarterly

$$\ddot{a}_{\overline{n}|} = (1 + i)a_{\overline{n}|}$$

$$1 + i = (1 + \frac{i^{(4)}}{4})^4 = (1 + \frac{0.08}{4})^4$$

$$\$100a_{\overline{15}|} = \$100(1 + i)\frac{1 - (1 + i)^{-15}}{i}$$

$$= \$100(1.02^4)\frac{1 - (1.02)^{-15\times4}}{1.02^4 - 1} = \$912.90$$

- Apply a reasonableness check to you answer to Part (1)
The PV is approximately:

$$\$100a_{\overline{15}|} \approx \$100 \times 15(1 + \frac{0.08}{4})^{-1})^{\frac{15+1}{2}}$$

$$= \$1500 \times 1.02^{-28} = \$861$$

6.5 Perpetuity

In fact, this infinite term annuity seen before is called a *perpetuity*, and is written

$$a_{\overline{\infty}|i} = v + v^2 + v^3 + ...$$

$$= v(1 + v + v^2 + v^3 + ...)$$

$$= \frac{v}{1-v} = \frac{1}{i}$$

This can also be used as a building block to derive annuity formulas

Example:
Derive the formula for an annuity-immediate

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

Using perpetuity

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

Solution:
Consider the difference between two infinite series of payments of 1, one starting now with present value

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

and one starting in n years with present value

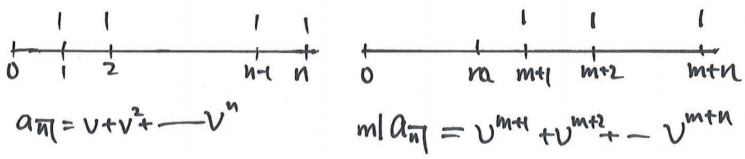
$$v^n a_{\overline{\infty}|i} = v^n \frac{1}{i}$$

The difference in cash flows corresponds exactly to that of an annuity-immediate over n years, with present value

$$\frac{1}{i} - v^n \frac{1}{i} = \frac{1-v^n}{i} = a_{\overline{n}|}$$

6.6 Deferred Annuities

We now develop develop formulae for annuities where the first payment is delayed by m years. We again consider the case where n payments are made in total
Consider first the annuity in arrears. In the non-deferred case, the first payment is made at the end of the first year, that is, at time 1. In the deferred annuity, the first payment is therefore made at time m+1, that is, at the end of the (m+1)st year



The notation used for the present value of an m year deferred, n year annuity with payments in arrears is $_{m|}a_{\overline{n}|}$

Example:
– Write down an expression for $_{m|}a_{\overline{n}|}$ in terms of v

$$_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + ... + v^{m+n}$$

– Write an expression for $_{m|}a_{\overline{n}|}$ in terms of v,m and $a_{\overline{n}|}$
From above, the expression for $_{m|}a_{\overline{n}|}$ involves m years further discounting for each term than is required under $a_{\overline{n}|}$. Therefore we have

$$_{m|}a_{\overline{n}|} = v^m(v + v^2 + ... + v^n) = v^m a_{\overline{n}|}$$

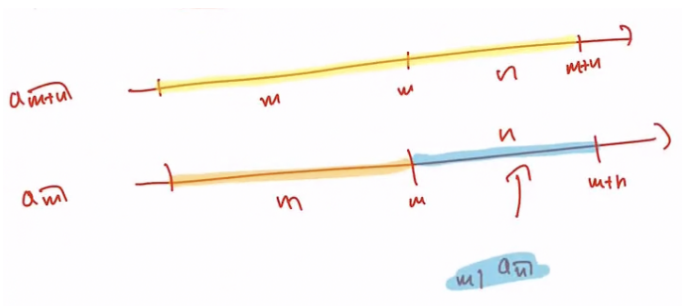
6.6.1 Formula $_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$

Proof:
Using the result in part (2) of the previous example, we have

$$_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} = v^m \frac{1 - v^n}{i} = \frac{v^m - v^{m+n}}{i}$$

$$= \frac{1 - v^{m+n} - (1 - v^m)}{i} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

Interpretation:



$$_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \Leftrightarrow a_{\overline{m+n}|} = a_{\overline{m}|} + {}_{m|}a_{\overline{n}|}$$

Example:
Find the PV of a series of a series of 15 payments of \$1 at yearly intervals beginning 11 years from now. Use an interest rate of 7% per annum effective
- Method 1: the PV is

$$_{10|}a_{\overline{15}|} = v^{10}\frac{1-v^{15}}{i} = (1+0.07)^{-10}\frac{1-1.07^{-15}}{0.07} = 4.63$$

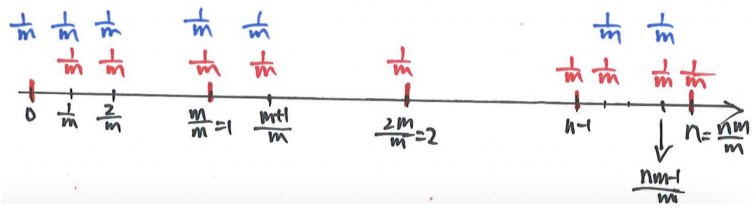
- Method 2: The PV is:

$$a_{\overline{25}|} - a_{\overline{10}|} = \frac{1-1.07^{-25}}{0.07} = \frac{1-1.07^{-10}}{0.07} = 4.63$$

6.7 Annuities payable in m partial payments

6.7.1 Annuities payable m-thly

We have already discussed annuities where payments are made for n years with one payment of \$1 each year
We now break up that single payment of \$1 into m payments of \$1/m



Note: Red = m-thly annuity in arrears, Blue = m-thly annuity due

6.7.2 m-thly annuity in arrears / annuity-immediate

The notation used for an m-thly annuity with payments at the end of each m-th of a year is

$$a_{\overline{n}|}^{(m)}$$

The derivation of a formula for this annuity is as follows:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}(v^{1/m} + v^{2/m} + ... + v^{nm/m})$$

and

$$(1+i)^{1/m}a_{\overline{n}|}^{(m)} = \frac{1}{m}(1 + v^{1/v} + ... + v^{(nm-1)/m})$$

Subtracting the first identity from the second gives

$$a_{\overline{n}|}^{(m)}((1+i)^{1/m} - 1) = \frac{1}{m}(1 - v^{nm/n})$$

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{m((1+i)^{1/m} - 1} = \frac{1-v^n}{i^{(m)}}$$

6.7.3 *m*-thly annuity due

The notation used for an m-thly annuity with payments at the beginning of each m-th of a year is

$$\ddot{a}_{\overline{n}|}^{(m)}$$

the derivation of a formula for this annuity is as follows:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}(1 + v^{1/m} + v^{2/m} + ... + v^{(nm-1)/m})$$

$$= v^{-1/m}\frac{1}{m}(v^{1/m} + v^{2/m} + ... + v^{nm/m})$$

$$\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{1/m} a_{\overline{n}|}^{(m)}$$

Examples
- Find the PV of \$100 per annum payable quarterly in arrears for 10 years at 7.5% per annum effective

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

$$i^{(4)} = 4[(1+0.07)^{1/4} - 1] = 0.072978...$$

$$100 a_{\overline{10}|}^{(4)} = 100\frac{1 - 1.075^{-10}}{i^{0.072978}} = 705.42$$

6.7.4 Alternative derivations

Another approach would be changing the time unit

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} = \frac{1}{m}\frac{1-(v^{1/m})^{nm}}{\frac{i^{(m)}}{m}}$$

$$= \frac{1}{m} a_{\overline{nm}|j}$$

In the formula above

- The time unit is 1/m of a year
- $j = \frac{i^{(m)}}{m}$ is the effective interest rate per time unit
- $v^{1/m} = 1/(1+j)$ is the discount per time unit ($1/m$ of a year)
- $1/m$ is the amount of payment per unit of time
- nm is the number of payments

Another approach is by using perpetuities
By analogy, we have

$$a_{\overline{\infty}|}^{(m)} i^{(m)} = \frac{1}{i^{(m)}}$$

the result immediately follows

$$a_{\overline{n}|}^{(m)} = a_{\overline{\infty}|}^{(m)} - v^n a_{\overline{\infty}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

6.8 Accumulated values of Annuities

Consider a superannuation fund into which payments are made during the working life of an individual. The payments will form a regular stream of payments and we are interested unknowing how much these payments will be worth at retirement, when interest is applied to each of the payments
We therefore are often interested in calculating the accumulated value of an annuity
$s_{\overline{n}|}$ represents the accumulation at time n of a series of payment of 1 at unit intervals in arrears (ie, at times 1,2,...,n)

We can also consider the accumulated value as the future value of sum of payments

6.8.1 Accumulated value of an annuity in arrears

We derive an expression for $s_{\overline{n}|}$ in a very similar way to how we derived formulae for present values of annuities

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i) + 1$$

gives

$$(1+i)s_{\overline{n}|} = (1+i)^{n} + (1+i)^{n-1} + ... + (1+i)$$

$$is_{\overline{n}|} = (1+i)^{n} - 1$$

Thus,

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

6.8.2 Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$

We now derive a relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$
Result

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$$

Proof:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} = (1+i)^n \frac{1-v^n}{i} = (1+i)^n a_{\overline{n}|}$$

6.8.3 Accumulated values of an annuity-due

Suppose now that the n payments considered above are made at the start of each of the years of the annuity. The payments are therefore made in advance. We aim to find the value of these payments at time n.

Illustration:
– Write down an expression in terms of i and n for $\ddot{s}_{\overline{n}|}$

$$\ddot{s}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + ... + (1+i)$$

– Show that :

$$\ddot{s}_{\overline{n}|} = (1+i)^{n+1} a_{\overline{n}|} = (1+i)^n \ddot{a}_{\overline{n}|}$$

Solution

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} = (1+i)\frac{(1+i)^n - 1}{i}$$
$$= (1+i)(1+i)^n a_{\overline{n}|}$$
$$= (1+i)^n \ddot{a}_{\overline{n}|}$$

– Find the PV of a series of 10 payments of \$100 at yearly intervals. The first payment is due in 3 months' time. The interest rate is 8% per annum effective
The PV is:

$$100(v^{\frac{1}{4}} + v^{\frac{5}{4}} + ... + v^{\frac{37}{4}}$$
$$= 100v^{\frac{1}{4}}(1 + v + ... + v^{\frac{36}{4}})$$
$$= 100v^{\frac{1}{4}} \ddot{a}_{\overline{10}|}$$
$$= 100(1.08^{-0.25})\frac{1 - 1.08^{-10}}{0.08}1.08$$
$$= \$710.88$$

- From first principles, find the accumulated value of the series above at time 12 years
The accumulated value is:

$$100((1+i)^{11\frac{3}{4}} + (1+i)^{10\frac{3}{4}} + ... + (1+i)^{2\frac{3}{4}}$$
$$= 100(1+i)^{2\frac{3}{4}}((1+i)^9 + (1+i)^8 + ... + 1)$$
$$= 100(1+i)^{2\frac{3}{4}} s_{\overline{10}|}$$
$$= 100(1.08)^{2\frac{3}{4}} \frac{1.08^{10} - 1}{0.08}$$
$$= \$1,790.11$$

- Check your answers are consistent
The valueation dates are 12 years apart. We can verify that

$$710.88(1.08^{12}) = 1,790.11$$

6.9 Further variations of Annuities

6.9.1 Annuities payable less frequently than once a year

Consider an annuity with regular payments of \$200 payable at two yearly intervals in arrears. The final payment is made after 20 years
Find the PV at 6% per annum effective

Method 1: Using first principles
Summing up the PVs of the individual payments, we get

$$PV = 200(1.06^{-2} + 1.06^{-4} + ... + 1.06^{-20}$$

$$= 200 \times 1.06^{-2}(1 + 1.06^{-2} + 1.06^{-4} + ... + 1.06^{-18})$$

$$= 200 \times 1.06^{-2} \frac{(1 - 1.06^{-20})}{1 - 1.06^{-2}} = \$1,113.58$$

Method 2: Change the time unit

New time unit is two-year
the effective 2-year interest is j such that

$$(1 + j) = (1 + i)^2 = 1.06^2 \Rightarrow j = 12.36\%$$

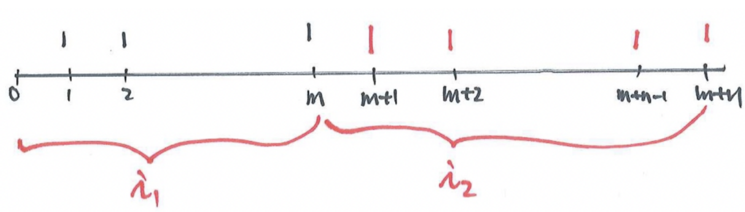
the 2-year discount factor is $v_i^2 = 1/(1 + j) = v_j$

then the PV for this annuity is

$$200 \times a_{\overline{10}|v_j} = 200\frac{1 - v_i^{20}}{j} = 200\frac{1 - v_j^{10}}{j} = \$1,113.58$$

6.9.2 Annuities with variable interest rate

Consider the following m+n year annuity in arrears
- If the effective annual interest rate is i throughout the term, then the PV of the annuity is $a_{\overline{n+m}|i}$, and the accumulated value is $s_{\overline{n+m}|i}$



If the interest rate is i_1 by time m years, and the interest rate is i_2 after time m years, then the PV of the annuity is:

$$a_{\overline{m}|i_1} + v_{i_1}^m a_{\overline{n}|i_2}$$

6.9.3 Annuities with variable payment amounts

Consider the following annuity in arrears with term $2n$ years with constant annual effective interest rate i

]

Define $j = (1+i)^2 - 1$ the PV of this annuity is

$$a_{\overline{2n}|i} + a_{\overline{n}|j} = (1+i)a_{\overline{n}|j} + 2a_{\overline{n}|j} = ...$$

7 Application: Professionalism and Actuaries

7.1 Professionalism

An Actuary is a professional - a member of a professional body

e.g. Institute of Actuaries ((IA), Institute of Actuaries of Australia (AI), Society of Actuaries (SoA), Casualty Actuarial Society (CAS), etc...

The Overarching body is the "international Actuarial Association". Professionals must protect the reputation of their profession, lest its value vanishes.Everyone must follow the law, but professionals (actuaries) MUST ALSO follow the rules set by the profession, and they are very wide-ranging

7.2 What are the additional rules?

Actuaries are governed by Code of Conduct, which saw major changes effective 31 March 2020

- Can be found on the Actuaries Institute Website

This governs general behaviour, but there are also rules about the actuarial work - Professional Standards and Guidance notes (eg. IAA ISAPs)

7.3 The Need for a Profession

- The individuals and society have a right to trust implicitly that the professional will adhere to standards
- The professional enjoys asymmetrical knowledge in relation to others. Knowledge is power, and where power is wielded, ethical concerns come into play
- Trust is underpinned where ethics and altruism direct professional skill

7.4 Benefits of a Profession

- The community
  consumers face a complex array of professional services choices, from medical and health to business and financial services. Professional play a vital role in providing trusted expertise founded on established standards that are policed to ensure community expectations of good practice and social purpose are met.
- The economy
  Professions improve consumers' access to services and support economic activity by encouraging confidence and trust in the services offered by professionals
- Regulators
  The burden of regulation and supervision by government can be reduced by improving the standards of practice of professionals and the regulatory capacity of professional communities
- Professionals
  Professionals enhance their reputations and skills by adhering to the professional standards and requirements of their professional bodies

7.5 The need for a Code of Conduct

- The Institute and its Members seek to serve the public interest.
- Professional actuarial services are typically provided to a Principal that offers a promise in exchange for money today.
- The actuary is uniquely placed to connect the interactions of all the moving parts involved in meeting the promise
- The public are reliant upon the actuary to join the dots.

The code of conduct (CC):

- Sets out the minimum standards of professional conduct.
- Members need to meet:
  - Legal requirements
  - AI's Constitution
  - Code of Conduct
  - Professional Standards.
- Disciplinary Scheme established to support CC

7.6 What is the role of a professional actuary?

According to Kevin:

- Most likely to be advising the ultimate decision makers rather than making the decision.
- The actuary needs to do all in his/her power by way of exposition to ensure that the people who have to make decisions know what they are doing.
- Incumbent upon the actuary to do his/her best in providing complete and appropriate advice to the decision maker; such that the decision maker can make a fully informed decision.

8 Loans and Bonds

8.1 Loans

We now turn our attention to applying our knowledge of compound interest and annuities to analysing loans
- Suppose you wish to borrow and amount of of money, say \$100,000, from a bank. This amount, lent by the bank, is called the **principal**. Of course, the bank will charge you interest on the balance of your loan

The interest charged by the bank is generally quoted as a nominal (compound) interest rate For instance it could be charged at 12% per annum, convertible monthly, ie. $i^{(12)} = 12\%$

8.1.1 Main Components

The main components of a loan are:

- Principal P: the amount of money that is borrowed/lent
- Interest rate i, usually an $i^{(m)}$p.a
- Repayments X (which usually include an interest component, and a reimbursement component), usually paid every 1/m years
- Maturity T years
- Number of payments n, usually $T \cdot m$. Typically one has

$$P = X a_{\overline{T|}i}$$

for annual repayments X in arrears or

$$P = (mX) a^{(m)}_{\overline{T|}i^{(m)}} = X a_{\overline{m \cdot T|}i^{(m)}/m}$$

for $1/m$-thly repayments X in arrears

For instance, a 30 year housing loan at interest $i^{(12)}$ becomes

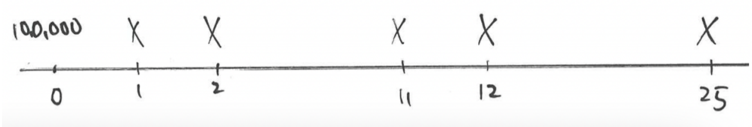
$$P = X a_{\overline{360|}i^{(12)}/12}$$

If P = 100,000, and $i^{(12)}$ = 12% then X = 1,028.61

$$P = X \frac{1 - v^n}{i}$$

$$100,000 = X \frac{1 - (1.1)^{-360}}{0.1} \Rightarrow X = 1,028.61$$

Example:
Consider a \$100,000 loan repayable by 25 equal annual instalments. i = 8%, per annum effective. Calculate X, the instalment each month.



$$P = X a_{\overline{25|}i}$$

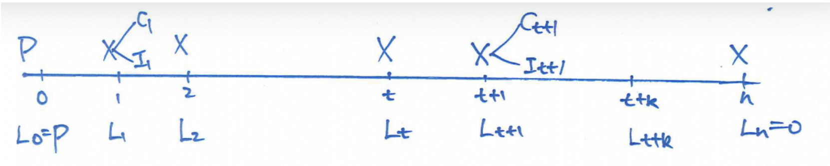
$$100,000 = X \frac{1 - (1 + 0.08)^{-25}}{0.08}$$

$$X = 9367.88$$

8.1.2 Loan Repayment Schedule

A loan repayment schedule is stable which sets out the amount to fate loan outstanding, the instalments paid, the interest charged and the capital repaid under a loan throughout its duration We define the following terms:

- L_t = principal (or loan) outstanding at time t after the instalment then due has been paid
- X = instalment
- i = effective rate of interest in time period t to t+1
- C_t = capital repaid in instalment number t
- I_t = interest component of instalment number t



The following relationships hold between the above variables

$$I_{t+1} = L_t \times i$$

$$C_{t+1} = X - I_{t+1} = X - L_t \times i$$

$$L_{t+1} = L_t - C_{t+1} = L_t(1 + i) - X$$

Example

Consider a loan of \$200,000 with a term of three years. Interest is charged at 10% per annum effective and instalments are paid annually in arrears for three years

- Calculate the annual instalment X

$$200,000 = X a_{\overline{3}|} = X \frac{1 - (1 + 0.1)^{-3}}{0.1}$$

$$X = 80,422.96$$

- Complete a loan repayment schedule for this loan, showing the loan outstanding at times 0,1,2 and 3 years as well as the repayments, interest charged and capital repaid at times 1,2, and 3 years

Time, t (years)	Payment	Interest due	Principal repaid	Outstanding balance
0	-	-	-	200,000.00
1	80,422.96	20,000.00	60,422.96	139,577.04
2	80,422.96	13,957.70	66,465.26	73,111.78
3	80,422.96	7,311.18	73,111.78	0.00

8.1.3 Outstanding Loan Calculations

There are three formulae for $L_t, t = 1.2, ..., n$

1. Recursive formula:
$$L_{t+1} = L_t(1+i) - X$$
with starting value $L_0 = P$

2. Prospective Formula (Present value of remaining payments)
$$L_t = Xa_{\overline{n-t}|}$$

3. Retrospective Formula (Accumulative value of the previous payments)
$$L_t = P(1+i)^t - xs_{\overline{t}|}$$

Example

Consider again the loan from the previous example. How would you calculate the amount that the person who originally borrowed the \$100,000 still owes the bank after 11 years?

After 11 years:

- 11 payments have been made under the loan
- There are 14 payments (each equal to \$9,367.88) remaining that still need to be paid

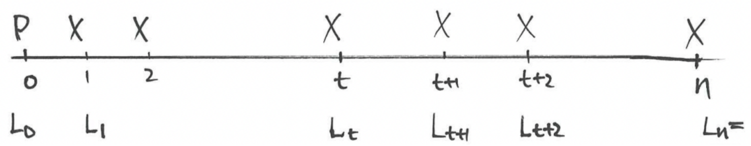
Note that when we calculate the loan outstanding at the time of an instalment we assume that the instalment payable on the date when the outstanding loan is being calculated has already been paid.

Hence in the example given above, the loan outstanding after 11 year is (using the prospective formula)
$$9367.88a_{\overline{14}|} = 9367.88\frac{1-(1_0.08)^{-14}}{0.08} = 77,231.02$$

8.1.4 Equivalence of prospective and retrospective formulas

Here we derive the prospective and retrospective formulas from the recursive one, and show that these are all indeed equivalent

- We consider a loan of P with a repayment period of n time periods
- Define X to be the amount of the instalment paid at the end of each time period
- Define L_t, to be the amount of the loan outstanding after t time periods. Note that of course
$$L_0 = P, \text{ and } L_n = 0$$



Clearly we have,
$$L_{t+1} = L_t(1+i) - X$$
$$L_{t+2} = L_{t+1}(1+i) - X$$
$$= L_t(1+i)^2 - X(1+i) - X = L_t(1+i)^2 - Xs_{\overline{2}|}$$

Similarly,

$$L_{t+3} = L_{t+2}(1+i) - X$$
$$= L_t(1+i)^3 - S(1+i)^2 - X(1+i) - X$$
$$= L_t(1+i)^3 3 - X s_{\overline{3}|}$$

In general,

$$L_{t+k} = L_t(1+i)^k - X s_{\overline{k}|},\ t+k \leq n$$

Also noticing the emerging pattern, we have

$$L_n = L_t(1+i)^{n-t} - X s_{\overline{n-t}|}$$

or equivalently, as $L_n = 0$

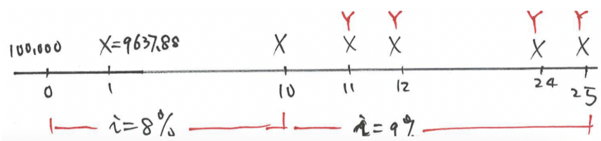
$$X s_{\overline{n-t}|} = L_t(1+i)^{n-t}$$
$$L_t = X s_{\overline{n-t}|}(1+i)^{-(n-t)} = X a_{\overline{n-t}|}$$

in particular,

$$L_0 = X a_{\overline{n}|}$$

Example:
– A loan of \$100,000 is repayable by equal annual instalments over 25 years at 8% per annum effective
– The annual repayment required is \$9,367.88
–Suppose that immediately after the 10th instalment is paid, the annual effective interest rate increases to 9%

Calculate the revised instalment so that the loan will still be repaid exactly 25 years after the fund were originally lent



The strategy is to find the loan outstanding at the time when the interest rate changes and then use this loan outstanding gas the amount owing that need to be paid within 15 years
- After 10 years, the loan outstanding is:

$$L_t = X a_{\overline{n-t}|}$$

$$L_{10} = 9,367.88 a_{\overline{25-10}|8\%}$$

$$= 9367.88 \frac{1-(1+0.08)^{-15}}{0.08} = 80,184.15$$

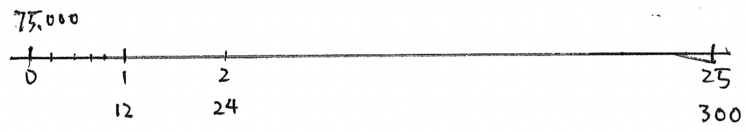
Now, if the annual effective rate of interest is 9%, we let the new repayable be Y and calculate:

$$Y a_{\overline{15}|9\%} = 80,184.15$$

$$Y = 80,184.15 \frac{0.09}{1-(1+0.09)^{-15}} = 9,947.56$$

Example:

A loan of \$75,000 is repaid by equal monthly instalments over 25 years with interest charged at 9% per annum convertible monthly



– Calculate the monthly repayment amount X

$$L_0 = X a_{\overline{n|}i^{(12)}}$$

$$X = 75,000/a_{\overline{25\times12|}0.75\%}$$

$$= 75,000\frac{0.0075}{1-(1+0.0075)^{-300}} = 629.40$$

– How much capital is repaid in the second year

To find the amount of the loan repaid (the capital repaid) in the second year, we need to calculate the difference between the loan outstanding at the start of the second year and the end of the second year.

Using L_t To denote the loan outstanding after t months, at $i = 0.0075$ we have:

$$L_{12} = X a_{\overline{300-12|}} = 629.40\frac{1-(1+0.0075)^{-288}}{0.0075} = 74,163.28$$

Similarly we have

$$L_{24} = X a_{\overline{300-24|}}$$

$$= 629.40\frac{1-1.0075^{-276}}{0.0075} = 73,248.06$$

Hence the capital repaid is

$$73,163.28 - 73,248.06 = 915.22$$

– How much interest is paid in the second year

During the second year we know that 12 instalments are paid. The total instalments paid in the second year are

$$12 \times X = 12 \times 629.40 = 7552.80$$

Therefore interest paid in the second year is

$$7,552.80 - 915.22 = 6,637.58$$

8.2 Bonds

Bonds are very large *securitised* loans, generally *interest only* until maturity. Borrowers are usually government (treasury notes) or companies.

Because of the securitisation, some features are standardised

- The loan is "cut" into small chunks (eg. \$10,000)
- The actual interest payments are fixed and don't change over the life of the bond
- The principal, or face value is reimbursed at maturity

Bonds are securities that can be traded. Because everything is set ("printed") their price will depend on how much interest/yield the market wants to obtain from this borrower
– The higher the risk, the high the expected yield
– The higher the inflation, the higher the expected yield

We will then need to learn **how to calculate a price from a yield , and yield from a price**

8.2.1 Prices and Yield to Maturity (YTM)

Yield to maturity (YTM) is the total rate of return that will have been earned by a bond when it makes all interest payments and repays the original principal.

Usually bond coupon payments occur every half yearly so let j = effective half year YTM, this means:

$$YTM = 2j = 2\frac{i^{(2)}}{2}$$

In general if bond payments occur m times annually

$$YTM = j^{(m)}$$

In general:

$$PV = Ca_{\overline{n}|j^{(m)}} + FVv^n$$

There are three possible cases for the price P

1. If $P = F$, the bond is sold at par
2. If $P < F$, the bond is sold at a discount
3. If $P > F$, the bond is sold at a premium

The "yield to maturity" (YTM) goes hand in hand with the price. It is the interest rate that makes the present value of all remaining cash flows equal to the price

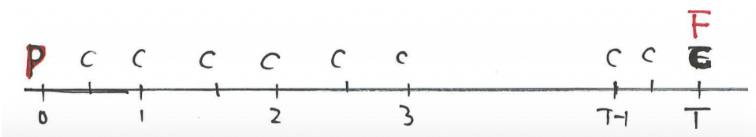
8.2.2 Some Typical Securities

Review of Treasury notes (T-Notes)

- Short term government debt security
- Terms are generally less than one year
- T-notes are sold at a discount to the face value
- The discount amount is calculated using simple interest rate r or simple discount rate d

Fixed Coupon Treasury Bonds

- A bond is a type of financial instrument issued by an organisation that wishes to borrow money
- A treasury bond (often called a t-bond) is a financial instrument issued by the government
- A T-bond is a medium to long-term government security
- A T-bond has a pre-specified face value (par value, nominal value), say F, at the maturity date T
- A T-bond is sold to a buyer at time zero at price P
- The government makes fixed interest payments of amount C (coupons, dividends) semi-annually



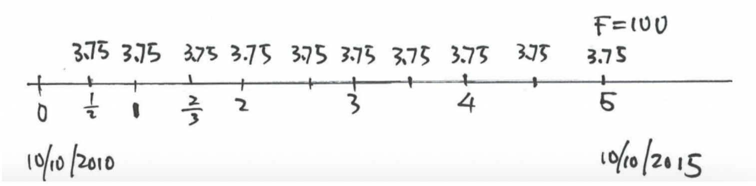
Note: The coupon rate would be $2C/F$ for semi-annual payments

8.2.3 Illustration

Lets say that today's date is the 10th October 2010

- Suppose that the government wishes to borrow \$100 today and the they will repay the \$100 in five years' time - that is, the government will repay the loan on the 10th October 2015 (We say that the government issues a T-bond with face value/par value/nominal value of \$100)
- The lender or buyer of the T-bond will receive coupons from the government during the 5 years

Suppose that the coupons are paid half-yearly with amount of \$3.75 at the end of each half year until the maturity date (10th October 2015)



The coupon rate is the annual coupon amount/face value. Here the coupon rate $= \frac{7.50}{100} = 7.5\%$ per annum

– Let $j^{(2)}$ be the yield per annum convertible half-yearly. The we have

$$100 = 3.75a_{\overline{10}|} + 100v^{10}$$

$$= 3.75\frac{1-(1+j^{(2)}/2)^{-10}}{j^{(2)}/2} + 100(1+j^{(2)}/2)^{-10}$$

This gives

$$100[1-(1+j^{(2)}/2)^{-10}] = \frac{7.5}{j^{(2)}}[1-(1+j^{(2)}/2)^{-10}]$$

if and only if $j^{(2)} = 7.5\%$

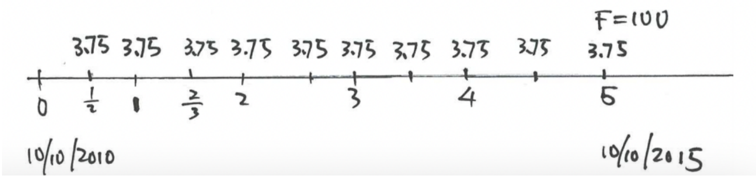
Remarks:

- If the purchasing price is equal to the face value, then the YTM is equal to the coupon rate
- Hence annual interest payable will be \$7.50 in our example. The semi-annual coupon will be \$3.75

Example:
Following the information from the illustration

- You lend the government \$100 today. They give you the \$100 face value T-bond
- Suppose now it is 10th October 2011 and you have just received you coupon of \$3.75. You decide to sell the bond at this time
- Let P be the price that the buyer wants to pay such that the yield on bond is $j^{(2)} = 7.2\%$ per annum convertible half-yearly (from 10th October 2011 to 10th October 2015)

Calculate P



It is said that you are selling the bond *ex-interest* (without interest).
When you sell the bond you are giving the buyer the right to receive future payments, including:

- Coupon paymetns of \$3.75 payable on 10 April 2012, 10 October 2012,...,10 October 2015
- Face value of \$100 from the government on 10 October 2015

The bond price P is the PV of all these future payments at the date of sale of the bond.
We therefore caculate the price, P, using

$$P = 3.75a_{\overline{8}|} + 100v^{8}$$

$$= 3.75\frac{1-1.036^{-8}}{0.036} + 100(1.036^{-8})$$
$$= 101.03$$

8.2.4 Determining the YTM (j) from a price

From the illustration above,

- Suppose the above bond was to be sold on 10th October 2011, however the price of the bond on that day is \$101.50
- Let $j^{(2)}$ be the (nominal) yield per annum convertible half-yearly (coupon rate)
- Then $j = j^{(2)}$ is the yield per half year implied by this bond. How would you calculate j?

From the illustration, it should be clear that we can set up the following equation:

$$101.50 = 3.75a_{\overline{8}|j} + 100v^8$$

evaluated at rate j per half year.
Our task is to calculate j. Looking at the above equation, we can write

$$101.50 = 3.75\frac{1-(1+j)^{-8}}{j} + 100(1+j)^{-8}$$

or equivalently, we can write the above equation as

$$P(j) = 3.75\frac{1-(1+j)^{-8}}{j} + 100(1+j)^{-8} - 101.50 = 0$$

The above equation is a non-linear in j and cannot be readily solved for j. We therefore adopt the following approach to solving the above equation - the following is a method that you should know for **calculating bond yields**

1. Estimate the solution to the bond price equation using general reasoning
2. Find tow estimated values of j, say j_1 and j_2, such that $P(j_1) > 0$ and $P(j_2) < 0$
3. Use linear interpolation to refine your estimate of j

Step 1:
Estimate j
There are two contributions to j - the coupon payments (3.75) and the capital gain (a loss of 1.5 here) at the end of the term, which both need to be compared to the initial price 101.5

$$j \approx \frac{3.75}{101.5} + \frac{(100-101.5)/8}{101.5} = 0.0351$$

Step 2:
Calculate both

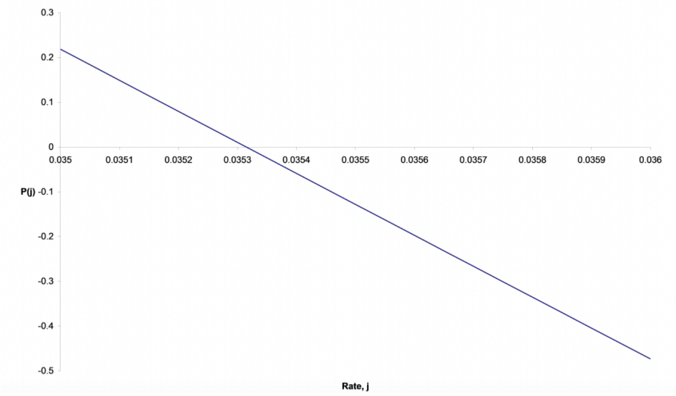
$$P(0.035) = 3.75\frac{1-(1+0.035)^{-8}}{0.035} + 100(1.035)^{-8} - 101.50 = 0.2185$$

and

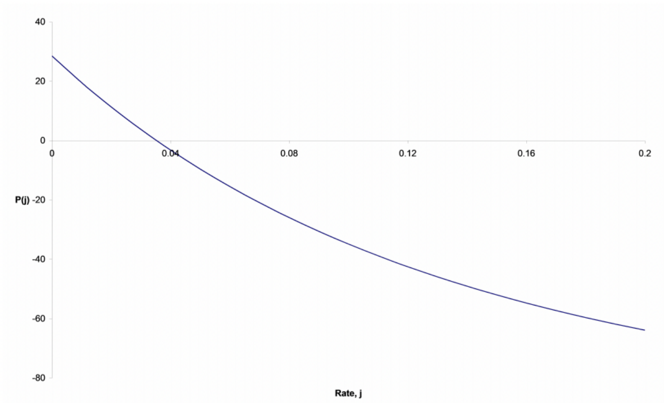
$$P(0.036) = 3.75\frac{1-(1+0.036)^{-8}}{0.036} + 100(1.036)^{-8} - 101.50 = -0.4732$$

Hence the true solution, the true yield per half year, j, lies between 3.5% and 3.6%

Step 3:
We assume that the function $P(j)$ is linear between $j = 0.035$ and j=0.036. the plot below shows $P(j)$ - it is not linear, but assuming linearity is reasonable for small interest spreads



The plot below shows $P(j)$ over a much wider range of values for j
It is clear that the assumption of linearity may be reasonable over narrow intervals, but not wide ones



Hence we assume

$$P(j) = \alpha + \beta j, j \in [3.5\%, 3.6\%]$$

The solution to the equation $P(j) = 0$ is therefore $j = -\alpha/\beta$
We use the results in Step 2 to find the values of α and β. We get two simultaneous equations, as follows

$$\alpha + 0.035\beta = 0.2185$$

$$\alpha + 0.036\beta = -0.4732$$

Solving these equations gives $\beta = -691.7$ and $\alpha = 24.428$ Therefore out approximation to the semi-annual yield is

$$j = \frac{-\alpha}{\beta} = \frac{-24.428}{-691.7} = 3.532\%$$

Summary
Wrapping up:

1. Given that our yield per half year is 3.532%, the nominal per annum yield convertible half-yearly is 7.064%

2. If the purchasing price is larger than the face value (eg. P =101.50 ¿ 100), then the yield is lower than the coupon rate (7.064% ¡ 7.5%)

3. If the purchasing price is smaller than the face value (eg. $P = 99 < 100$), then the yield ($3.90\% \times 2 = 7.8\%$) is higher than the coupon rate 7.5%

4. If Price is equal to the Face Value (100), the yield = 7.5% = coupon rate

8.3 The Effect of Tax

All individual and for-profit organisations are subject to tax on income and capital gains - In the context of bonds coupons are income
The difference between the price paid and the capital payments at the maturity date is a capital gain (or loss)

– Suppose a bond pays an annual coupon of \$5 per \$100 nominal
– Suppose that an investor is subject to income tax at 25%
– Assuming that coupons are paid net of tax, the investor would receive \$6 per \$100 nominal. Ie. 75% of \$8

To complicate matters, individuals face a different tax rate, depending on their level of income. This may be uncertain when making decisions

Example

- A government is about to issue a bond which has a term of 25 years and pay coupon half-yearly at 6% per \$100 nominal
- An investor is subject to income tax at 30%
- What price should this investor pay per \$100 nominal at issue to secure a yield of 5% per annum effective after tax?

Solution:
- The price is the present value at 5% per annum effective of the net of tax payments that the investor will receive
- The investor receives an annuity of $0.7 \times 6 = 4.2$ dollars per annum (payable half-yearly)
That is:

$$i^{(2)} = [(1+0.05)^{1/2} - 1] \times 2 = 0.04939...$$

$$a^{(m)}_{\overline{n}|} = \frac{1 - v^n}{i^{(m)}}$$

$$4.2a^{(2)}_{\overline{25}|,i=0.05} + 100v^{25} = 4.2\frac{1-(1+0.05)^{-25}}{0.04939...} + 100(1.05)^{-25} = 89.46$$

9 Demography

9.1 Introduction to Demography

Demography is the study of population, this includes populations of all sorts - animate or inanimate. A population is a group of individuals that is homogeneous *in some way* such that there is a method of joining the group and a method of leaving the group

We will discuss populations themselves, and also the invidious within the populations, and mostly we will be talking about people. Even when discussing human populations, there are many kinds of individuals who may be involved, for example: female people, Australian people

9.1.1 Importance

Actuaries are interested in the study of populations because much of an actuaries work is involved in predicting how a population group evolves.
– For example, when actuaries set the premiums for:

- **Life insurance**: they need to know what the probability is that the insurance benefit will have to be paid at any time - what the likelihood of the policy holder dying at a particular time is
  * Here we want to build a model for *exiting the population of the live people*
- **Superannuation**: they may need to model when an employee will retire
  * Here we want to build a model for *exiting the population of the employees by entering the population of the retirees* (excluding the exit cause *death*)
  * Here we have what we call "multiple decrements" - reasons for 'leaving'

9.1.2 Outline

Here we focus on analysis at the **population** (as opposed to individual) level

- What are human population characteristics?
  - Sex ratio
  - Child-women ratio
  - Dependency ratio (age dependency ratio, youth dependency ratio)
- How do populations evolve?
  * Rate of change

We will look at probabilities for **invidious** to

- Leave - mortality rates
- Enter - fertility rates

The (human) population later - in week 7
Later (week 7), we will also briefly discuss models used for 'population projections' - which enable us to form a view on what a population will look like int he future, based on the behaviour of the individuals within it.

9.2 Characteristics of a population

9.2.1 General Populations

Any population has a number of features which are characteristic. For our purposes, a population is a group of individuals that is homogeneous *in some way*, such there is

- (At least) one method of joining the group, and
- (At least) one method of leaving it

We also need to know what we mean by 'homogeneous' - at least one way of recognising why an individual belongs to this group

- For example, you are enrolled in ACTL10001, so that could be a criterion
- You can join and leave by enrolling / unenrolling

9.2.2 National Populations

When we consider the population of a country we can group the individuals in the population by:

- Age, or age range
- Gender
- Health conditions
- Incomes
- Smoking status, marital status, etc

Individuals join the group by - **being born or immigrating**
And they leave by - **dying or emigrating**

We can construct a profile of the country using these characteristics

9.3 Sources of Information

Census data

- In Australia, conducted every 5 year
- Comprehensive, but expensive

Population surveys

- Sample surveys
- Limited amount of information, but quick and cheap

Registration data (eg. Marriage)

- Quality is high, but scope is limited

Company / organisation data

- examples: social security register, insurance companies, state schools, pension funds, ...
- Insurance companies would collect data about their policy holders: age, gender, marital status, postcode, credit rating,...
- Specifically covers collect specific data
  - Eg. smoking status and health history for life insurance
  - Eg. Accident history, driving habits, value of car for car insurance

9.4 Summary Statistics

There are a number of summary statistics commonly calculated and used to describe populations. Some statistics focus on a characteristic of the population at one point in time: (snapshot ratios)

- Sex ratio
- Child-woman ratio
- Dependency ratio (age dependency ratio, youth dependency ratio)
- Labour force participation rate

Some statistic focus on rates of *change* (discussed in the next section)

- Crude rates (for whole population)
- Specific rates (for a specific part of the population)

9.4.1 Sex-Ratio

We can calculate a Sex ratio:

- For the total population
- For any particular age
- For an age group

It is calculated as

$$100 \times \frac{\#\text{ males}}{\#\text{ females}}$$

Sex ratio for the total population in 2020 (estimated)

- The Sex ratio for the entire world population is 101.8
- The overall Sex ratio for Australia is 99
- Furthermore, in Australia it is:
  - 106 at birth and for the 0-14 age bracket
  - 109 for the 15-24 bracket
  - 99 for the 25-54 bracket,
  - 93 for the 55-65 bracket

- The sex ratios vary a lot from country to country. According to UN 2015 data
  - The lowest tow ratios are 83 (Djibouti) and 86 (Latvia, Hong Kong, Lithuania, Russia, Ukraine)
  - The highest two are 256 (UAE), and 339(Qatar)

Australia's Population in 2016

Age	Males	Females	Persons
0-4 years	807,893	765,733	1,573,626
5-14 years	1,539,667	1,459,304	2,998,971
15-49 years	5,815,521	5,801,256	11,616,777
50-64 years	2,124,112	2,205,170	4,329,282
65 years +	1,715,846	1,956,405	3,672,251
Total	**12,003,039**	**12,187,868**	**24,190,907**

Sex ratio for Australian population in 2016 is:

$$100 \times \frac{12,003,039}{12,187,868} = 98.48$$

9.4.2 Child-Woman Ratio (CWR)

The child-woman ratio is calculated as the number of children (both males and females) in a population under a given age, typically 5, per 100 females in the population who are of reproductive age (typically taken as 15 to 49)

$$100 \times \frac{\#\text{ number of children ages 0 to 4}}{\#\text{ number of females aged 15 to 49}}$$

From our Australian population data we can calculate this per 100 women as:

$$100 \times \frac{1,573,626}{11,616,777} = 27.13$$

This is a crude measure for fertility - we will review more refined measures later

9.4.3 Dependency Ratios

9.4.3.1 Total Dependency ratio

This is the number of economically dependent individuals (children (0-14) and seniors (65+)) per 100 individuals of working age (15-64):

$$100 \times \frac{\#\text{ of children aged (0,14] + \$ of seniors aged 65+}}{\#\text{ of individuals of age [15,64]}}$$

"Economically dependent" may not be the best description for the 65+. More so, these are people who likely don't have (tax paying) salaries any more, and are living off pensions or savings

From our Australian population data owe can calculate this per 100 of economically viable persons as

$$100 \times \frac{(1,573,626 + 2,998,971) + 3,672,251}{11,616,777 + 4,329,282} = 51.70$$

For comparison this number was 49.58 in 2006

9.4.3.2 Age Dependency Ratio

This is the number of seniors per 100 individuals of working age (economically viable persons)

$$100 \times \frac{\#\text{ of seniors aged 65+}}{\#\text{ of individuals of age [15,64]}}$$

From our Australian population data, we can calculate this per 100 of economically viable person as

$$100 \times \frac{3,672,251}{11,515,777 + 4,329,282} = 23.03$$

For comparison the age dependency ratio for 2006 was 19.92

9.4.3.3 Youth dependency ratio

This is the number of children per 100 economically viable persons (ie. of working age)

$$100 \times \frac{\#\text{ of children aged (0,14]}}{\#\text{ of individuals of age [15,64]}}$$

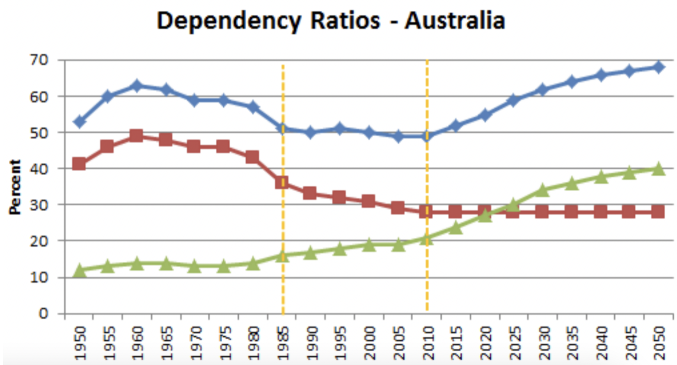
From our Australian population data we can calculate this per 100 of economically viable persons as

$$100 \times \frac{1,573,525 + 2,098,971}{11,616,777 + 4,329,282} = 28.68$$

For reference the youth dependency ratio for 2006 was 29.66

*Note that the sum of this and the Age Dependency Ratio gives the Dependency Ratio

Trends of dependency ratios of Australia (including future projections)



9.5 Rates of change

9.5.1 Crude vs Specific Rates

- A single figure statistic, based upon the number of events per 1,000 of population, is called a **crude rates**
- A figured based upon the number of events per 1,000 of a specific section of the population and relating only to that section, is called a **specific rate**
- Note that in the case the population size changes is the average population between time t and $t + 1$

9.5.2 Crude Rates of Migration

Simply, we can describe the change in a population from time t to $t+1$ in this way

$$P(t+1) = P(t) + Births(t, t+1) - Deaths(t, t+t) + Immigrants(t, t+1) - Emigrants(t, t+1)$$

Some definitions

- The **natural increase** in a population is $Birth - Deaths$
- The **net migration** in a population is $Immigrants - emigrants$
- The **population growth** is *Natural increase + net migration*

The Australian Bureau of Statistics (ABS) website has a population clock showing estimated of how Australia's population is changing

From the ABS population clock:
On 20 August 2021 at 10:37:03AM (Canberra time), the resident population of Australia is projected to be:

$$25,801,609$$

This population is based on the estimated resident population at 31 December 2019 and assumes growth since then of:

- one birth every 1 minute and 43 seconds
- one death every 3 minutes and 13 seconds
- one person arriving to live in Australia every 1 minutes and 30 seconds
- one Australian resident leaving Australia to live overseas every 1 minute and 26 seconds

Leading got an overall total population increase of one person every 3 minutes and 12 seconds. These assumptions are consistent with figures released in National, State and Territory population

27 August 2020 at 04:56:06PM (Canberra time), the resident population of Australia is projected to be:

25.653.866

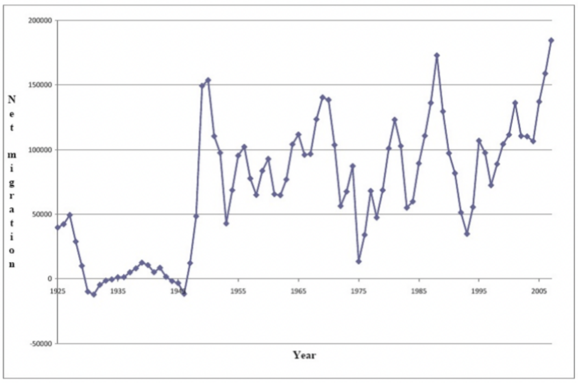
This projection is based on the estimated resident population at 31 December 2019 and assumes growth sine then of

- one birth every 1 minute and 44 seconds
- one death every 3 minutes and 15 seconds
- one person arriving to live in Australia every 3 minutes and 58 seconds
- one Australian resident leaving Australia to live overseas every 7 minutes and 4 seconds

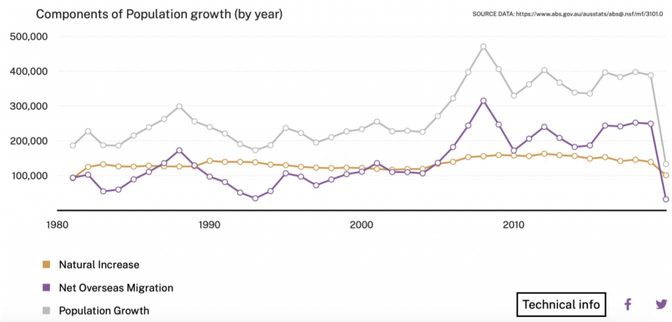
Leading to an overall total population increase of one person every 2 mixtures and 37 seconds. These assumptions are consistent with figures released in Australian Demographic Statistics, December Quarter 2019

Migration in Australia

Australia has had high net migration, especially in the post World War II period



Controlled migration is a way of stopping g a population dying out, or building numbers quickly, and may support economic growth - it is a way of controlling the age structure of the population reducing the age



Based on the trends there is a strong positive correlation between net overseas migration and population growth.

9.5.3 Crude Rates of birth and death

These are usually quoted as rates per 1,000 of population

$$\text{Crude Death Rate} = 1,000 \times \frac{\text{total number of deaths}}{\text{Population size}}$$

$$\text{Crude Birth Rate} = 1,000 \times \frac{\text{total number of births}}{\text{Population Size}}$$

9.5.4 Specific Rates

'specific' than crude rates: they are rates which apply to a particular class of individual, for example

- Gender specific (rates for females or males)

$$\text{death rate for female} = 1,000 \times \frac{\text{total number of female deaths}}{\text{total women population}}$$

- Age specific rates (rates for a particular age range)

$$\text{death rate for seniors} = 1,000 \times \frac{\text{total number of deaths of age 65+}}{\text{total senior population}}$$

- Class specific (eg. Smoker, married, etc)

$$\text{death rate of smokers} = 1000 \times \frac{\text{total number of death from smokers}}{\text{total population of smokers}}$$

Illustration: Comparing Populations (crude rates)
Here are some statistics for two human populations.

Age group	Pop A	Deaths	Death rate	Pop B	Deaths	Death rate
1	2,000	4	2.00	5,000	11	2.20
2	3,000	3	1.00	4,000	4	1.00
3	4,000	5	1.25	3,000	4	1.33
4	5,000	12	2.40	2,000	5	2.50
Total	**14,000**	**24**		**14,000**	**24**	

The crude death rate of population A and B is the same:

$$1000 \times \frac{24}{14,000} = 1.7143$$

per thousand of population

9.5.5 Standardised Rates

In the illustration above, can we then say that mortality rates are similar for populations A and B?
Crude rates tell us nothing about the effect of the age distribution of the population we are comparing
We can remove this effect if we use **standardised death rates**

- Following the illustration from above
- We will use the statistics for populations A and B to calculate the crude death rate for population B, standardised to population (structure) A
- Procedure
  - Calculate age specific rates for both populations
  - Apply age specific rates to the reference population
  - Calculate standardised crude rate
  - Compare mortality experience
- As a sample calculation, for Age group 1, we apply the death rate for population B to the number of people in Age group 1 in Population A (multiply 2,000 by 2.20)

Age group	Pop A	Deaths	Death rate A	Death rate B	Exp deaths
1	2,000	4	2.00	2.20	4.4
2	3,000	3	1.00	1.00	3.0
3	4,000	5	1.25	1.33	5.3
4	5,000	12	2.40	2.50	12.5
Total	**14,000**	**24**			**25.2**

By comparing a standardised rate between Population A and B we can now tell that the death rate in Population B is higher than population A.

9.6 Demographic transition and population pyramids

9.6.1 Theory of Demographic Transition

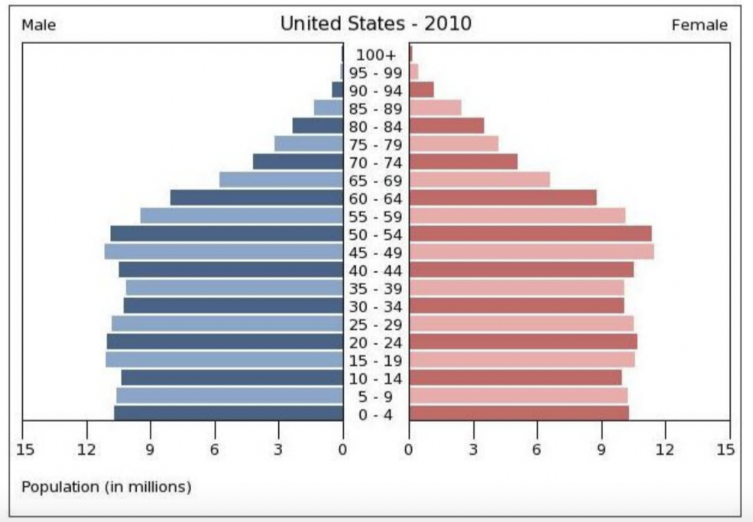
The theory of the Demographic Transition is a theory about the way human populations change under the effects of economic development, as a society moves from 'third world' conditions of tribal nomadic agrarian culture, into an industrial and urban society

- Increased and improved access to health services
- Improved general nutritional and sanitary standards (eg. Clean water)
  - Immediate reduction in infant mortality rate, leading to sharp increase of population
  - Slower reduction in overall mortality rates
- More wealth then leads to a drop of fertility (less family support, cost of having children, birth control...)
- After some time, increased longevity, leading to an older population

These effects can be observed in the structure of population ages in a given society. This is what a **population pyramid** depicts.

9.6.2 Population Pyramids

A population pyramid (age-gender pyramid) is a diagram showing the gender and age distribution of a country or a region in a year



<u>Common Features of Population Pyramids</u>
For most populations

- Higher number of male births than female births (Sex ratio at birth is approximately 1.05)
- Higher mortality rates for males
- Females are living longer than males

In the following slides, we review examples of the main types of pyramids:

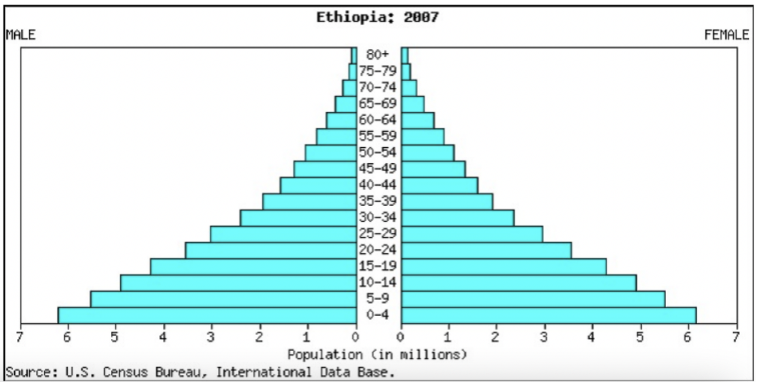
- Triangular-shaped (undeveloped countries, rapid population growth)
- Beehive-shaped (developed countries, slow population growth)
- Rectangular -shaped (aging population, stationary population)
- Upside-down triangular-shaped (shrinking population)

Shapes of the pyramids can be controlled by births, deaths and migrations

Social and economic changes result in changes in the distribution of populations. Over time, as a country develops, the shape changes from triangular shares to beehive shape, to barrel-like shape, to upside-down pyramid

9.6.3 Economically Undeveloped Countries

An example of such a country is Ethiopia in 2007



The population is characterised by

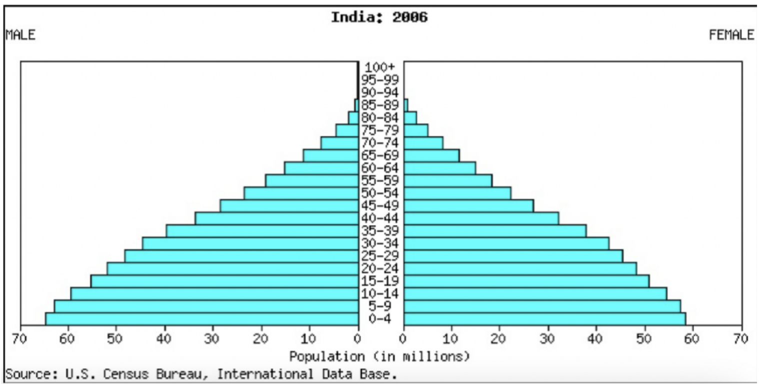
- High proportion of the population in young age groups
- High fertility rates
- High mortality rates, especially infant and child mortality
- Low life expectancy, age dependency ratio is low

The pyramid has a wide base and a narrow top. The majority of the population is below age 15

Population pyramids for other third world countries (low GDP, low quality of life, little medical care, inadequate food supply, poor water quality) generally have this shape

9.6.4 Emerging economies

An example of such a country is India in 2006



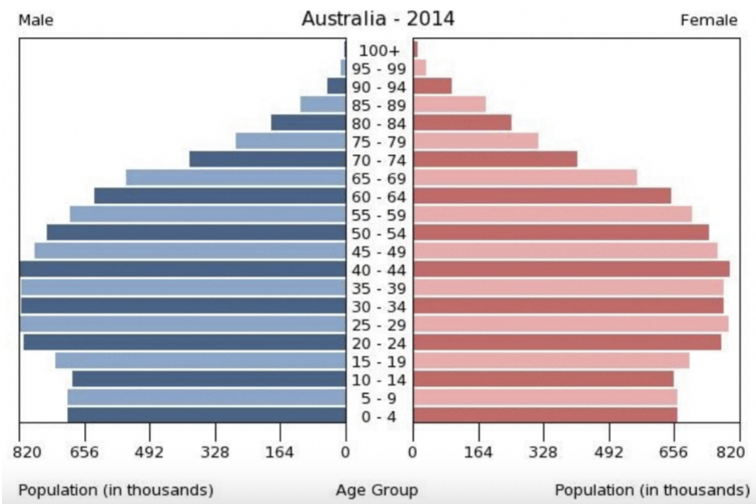
The population is characterised by:

- Lower fertility and child mortality than in under-developed countries
- Improving mortality
- Increasing maximum age
- High life expectancy than that for Ethiopia
- Rapid population growth

The pyramid has a fatter middle than in the case of Ethiopia - a higher proportion of the population is striving to reproductive ages

9.6.5 Established economies

Examples are Australia, USA, UK, Canada, HK, Singapore, etc



The population is characterised by

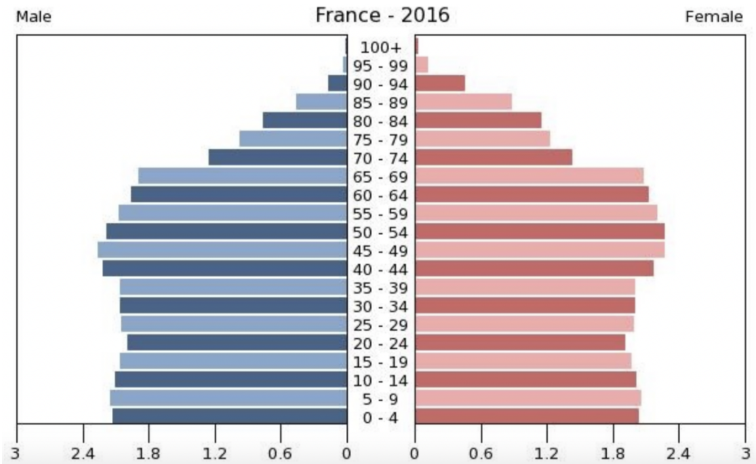
- High population in middle and higher age groups
- Lower mortality at all ages

- Lower fertility rates
- Lower youth dependency ratio
- Higher life expectancy
- Population is growing slowly

High quality of life, adequate health care, plenty of good supply and good water quality

9.6.6 Countries with an ageing population

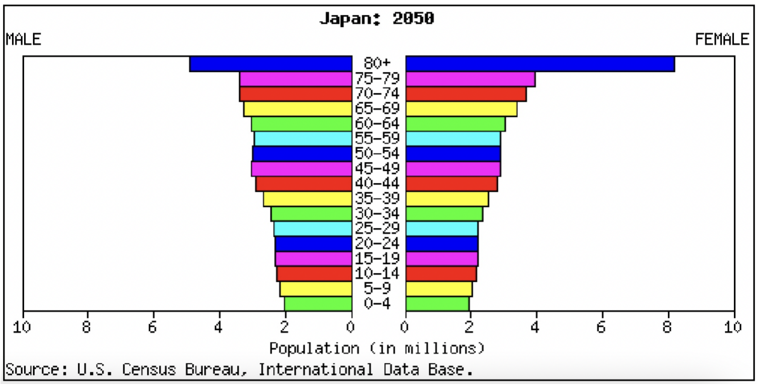
An example is France



The population is characterised by

- Low fertility rates, small base
- Low mortality rates
- Age distribution becoming even
- Stationary population
- Nearly zero population growth

Upside-down pyramid
Projections for Japan in 2050



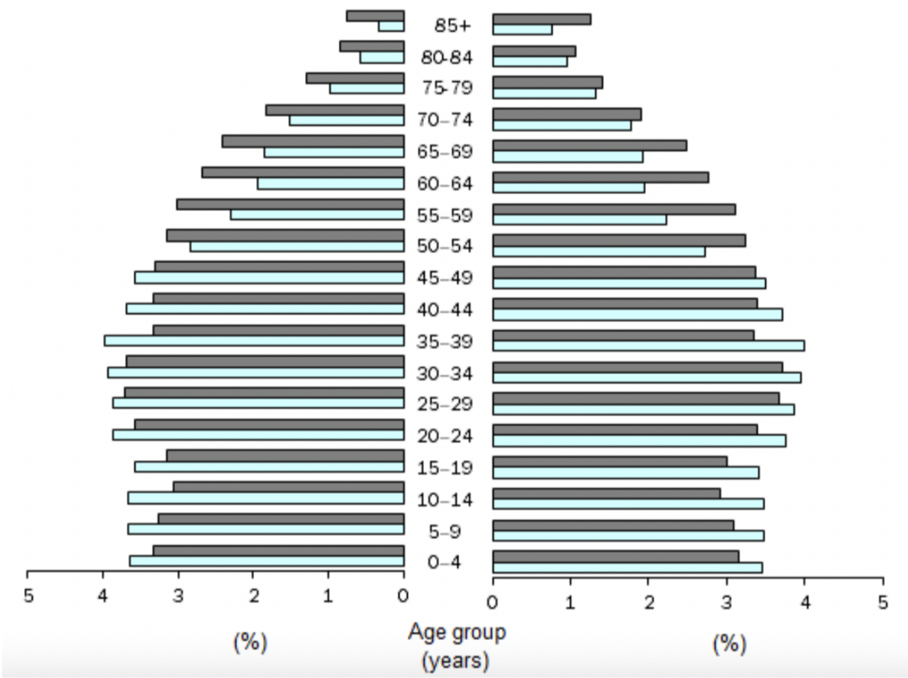
Aging is happening all round the world in developed countries
This means that the age dependency ratio is increasing: tax payers are having to support a higher number of aged people per capita

- Old people are expensive to support- they usually require subsidised income support (aged pensions) and their health costs are much higher than even middle aged people
- Improving mortality (ie. lower mortality rates) means that people love longer, and are more likely to become chronically ill and require long term care
- Health costs in the last year of life (in Australia) are about the same as for the WHOLE of the rest of their life

There has been world wide debate on how to provide for old age because of the expected strain of increasingly aged population. (Hence the introduction of compulsory superannuation saving in Australia)

9.6.7 Australian Pyramid

Population pyramids of Australia 1996 vs 2016



Questions

- Which size represents males?
    - In the older ages groups females live longer than males - Left side = males, right side = females
- Which color represents the 2016 pyramid?
    - There is larger percentage of older people in the dark grey columns = 2016 (aging populations)
    - 1996 = light blue
- Which year has longer life expectancy?
    - Can't really tell... pyramid only gives a snapshot of the population
- Which year has higher age dependency ratio?
    - 2016- larger proportion of population that are 65+ in 2016 compared to 1996
- Which year has higher youth dependency ratio?
    - 1996 - larger proportion of the population that are between 0-14 in 1996

10 Mortality, Fertility and Projections

10.1 The Survival Function

10.1.1 Review of Probability Discrete Variables

A **random variable**, denoted by capital letters such as $X, Y, or Z$, is a quantity whose value is subject to variations due to chance. It takes values from a set of possible different numbers

Examples

- The number (X) obtained when a die is thrown once
- Tomorrow's temperature at 12.00pm
- Tomorrow's exchange rate from AUD to USD
- The future life time of a life aged 30

10.1.2 Definitions

Mortality:

- There are many different ways of modelling mortality
  - Analytical models: such as the survival function - which are designed to 'fit' to statistical data which have been collected about a population
  - Life tables (see next section)
- Notation: (x) denotes an individual or 'life' aged exactly x

There are also different definitions of age

- Exact age: 30, 5, 19, or one month to 20
- Age last birthday
  - If exact age is in $(x, x+1)$ then aged x last birthday
- Age next birthday
  - If exact age is in $(x, x+1)$ then aged $x+1$ next birthday
- Age nearest birthday
  - If exact age is in $(x-0.5, x+0.5)$ then aged x nearest birthday

10.1.3 Survival Function

$s(x)$ is the probability of a life surviving from birth (0) to (at least) age x. We call s the survival function. This function can be of any analytical form provided it is consistent with the individual whose mortality experience it represents

However it must satisfy the following conditions

$$s(0) = 1$$

$$\lim_{x \to \infty} s(x) = 0$$

$$s(x) \text{ is a non-increasing function of } x$$

Any function fulfilling these three conditions can be used as a survival function, but may not be suitable for a human population

Example:
Let

$$h(x) = 1 - \frac{x}{20}, \text{ for } 0 \leq x \leq 20$$

Is this function suitable for use a survival function?
- We have

$$h(0) = 1$$

$$\lim_{x \to 20} h(x) = 0$$

$$h'(x) = \frac{-1}{20} < 0 \Rightarrow \text{decreasing function}$$

Therefore, it is suitable over [0,20]

However is the function h suitable as the survival function for a human population?

Probably not. Humans usually live the past the age of 20 and the mortality changes over time (not linear).

10.1.4 Lifetime Distribution

We define $F(x)$ to the probability that a newborn will die before age x, that is

$$F(x) = 1 - s(x)$$

This is what is called a distribution function, and it must fulfil conditions analogously to the survival function s

$$F(0) = 0$$

$$\lim_{x \to \infty} F(x) = 1$$

$$F(x) \text{ is a non-decreasing function of } x$$

To think of it another way

$$s(x) + F(x) = 1$$

Either you are alive or you are dead by age x: you must be one or the other but not both

10.1.5 Future Lifetime

Notation: $T(x)$ denotes the future lifetime of (x)
Important

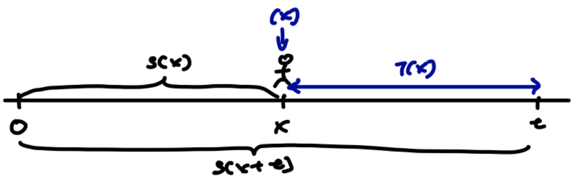
$$s(t) = Pr(T(0) > t)$$

Is the probability that a newborn life survives for (at least) t years.

What about $Pr(T(x) > t)$
An Important Result

$$Pr(T(x) > t) = \frac{s(x+t)}{s(x)}$$

$$Pr(T(x) \leq t) = 1 - \frac{s(x+t)}{s(x)}$$



$$Pr(T(x) > t) = Pr[\text{Survival}(x+t) \mid \text{Survival}(x)]$$

$$= \frac{s(x+t)}{s(x)}$$

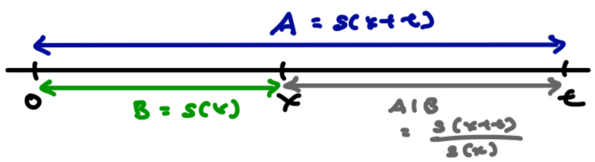
This stems from Bayes' rule: for two events A and B,

$$Pr(A|B) = \frac{Pr(A \text{ and } B)}{Pr(B)}$$

10.1.5.1 Proof that $Pr(T(x) > t) = \frac{s(x+t)}{s(x)}$

- Let A be the event that a newborn survives beyond age $x + t$, so that $Pr(A) = s(x + t)$
- Let B be the event that a newborn survives bygone age x, so that $Pr(B) = s(x)$
- Then $A|B$ is the event that a newborn survives beyond age $x + t$ given that the newborn survives beyond x
- Then $Pr(A|B)$ is the probability that a life who has survived to age x survives beyond $x + t$ ie. $Pr(T(x) > t)$
- As the events {A and B} and A are equivalent, $Pr(AandB) = Pr(A)$, and so

$$Pr(A|B) = Pr(T(x) > t) = \frac{s(x+t)}{s(x)}$$



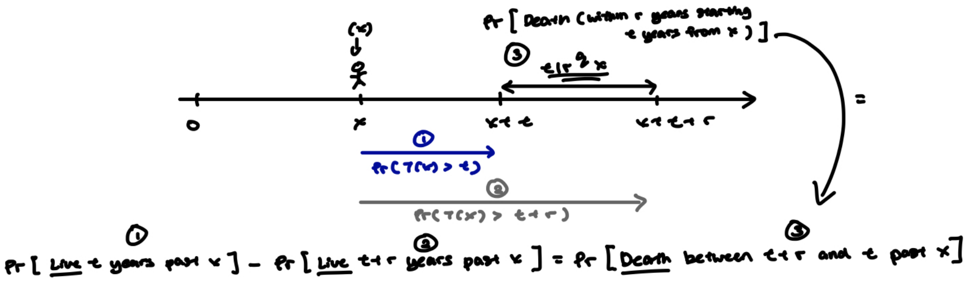
10.1.6 Probability of dying over the next r years, in t years

The probability (x) will die between ages $x + t$ and $x + t + r$ is

$$_{t|r}q_x = Pr(t < T(x) \le t + r) = \frac{s(x+t) - s(x+t+r)}{s(x)}$$

Read as: Pr(death of a person within r years starting t years from x)
This can be proven using the fact that

$$Pr(t < T(x) < t + r) = Pr(T(x) > t) - Pr(T(x) > t + r)$$



10.1.7 Assumption of Independence

Warning:
In actuarial problems it is usual to assume that lives are independent with respect to mortality. However in practice, this assumption may not be true. For example, events such as car crashes often result in the death of more than one member of a family. We might expect members of a family to be dependent mortality risks
Recall from probability theory that for two independent events A and B

$$Pr(A \text{ and } B) = Pr(A) \times Pr(B)$$

This is no longer always true if A and B are not independent. The mathematical modelling of such situations is beyond the scope of this course

10.1.8 Examples

1. There are three siblings, ages 10, 12 and 15, all born on the same (calendar) day. Today is their birthday. Assume their lives are independent

- What is the probability that they will all survive to attend each others' 21st birthday parties? They must all survive to the youngest's 21st:

$$\text{Pr(youngest surviving to 21)} \times \text{Pr(middle surviving to 23)} \times \text{Pr(eldest surviving to 26)}$$

$$\frac{s(21)}{s(10)}\frac{s(23)}{s(12)}\frac{s(26)}{s(15)}$$

- What is the probability that the youngest survives the next 5 years, and the eldest does not?

$$\text{Pr(youngest surviving next 5 years)} \times \text{Pr(eldest not surviving next 5 years)}$$

$$\frac{s(15)}{s(10)}(1 - \frac{s(20)}{s(15)})$$

2. Let

$$s(x) = 1 - 0.01x, \text{ for} 0 \leq x \leq 100$$

- What is the probability that (30) lives to age 60

$$\frac{s(60)}{s(30)} = \frac{1 - 0.01(60)}{1 - 0.01(30)} = \frac{4}{7}$$

- What is the probability that (30) will survive to age 60, but not to age 70?

$$Pr(30 < T(30) \leq 40) = \frac{s(60) - s(70)}{s(30)} = \frac{1}{7}$$

10.1.9 The force of mortality μ_x

10.1.9.1 Definition

Recall:

$${}_{s|t}q_x = Pr(s < T(x) \leq s+t) = \frac{s(x+s) - s(x+s+t)}{s(x)}$$

Now let $s = 0$ and t be infinitesimal (dt)

- ${}_{dt}q_x$ is the probability that (x) will die in the next instant
- This is ($\approx$) equal to the pdf of $T(x)$ at $s = 0$, time dt

This allows use to rearrange for

$${}_{dt}q_x = \frac{s(x) - s(x+dt)}{s(x)} = -\frac{s(x+dt) - s(x)}{dt}\frac{dt}{s(x)} = -\frac{s'(x)}{s(x)}dt$$

The coefficient of dt ($\frac{s'(x)}{s(x)}$) is called the **force of mortality**. It hence represents the 'likelihood' for the individual (x) to die in the next instant dt
We have

$$\mu_x = -\frac{s'(x)}{s(x)} = -[log[s(x)]]' = \frac{F'(x)}{s(x)}$$
$$= \frac{F'(x)}{1-F(x)}$$

This is because

$$s(x) = 1 - F(x)$$

and hence

$$s'(x) = -F'(x)$$

We now show that

$$s(t) = Pr(T(0) > t) = exp\{-\int_0^t \mu_x dx\}$$

The key is to recall that for a function h

$$\frac{d}{dx}log[h(x)] = \frac{h'(x)}{h(x)}$$

So

$$\mu_x = \frac{-s'(x)}{s(x)} = -\frac{d}{dx}log[s(x)]$$

We can integrate this with respect to x

$$\int_0^t \mu_x dx = -log[s(x)]|_0^t = -log[s(t)]$$

This gives the result. Note that $s(0) = 1$ and $log(1) = 0$

Example
Suppose that

$$\mu_{60+t} = 0.002 + 0.0001t$$

for $0 \leq t \leq 10$. The probability that a life aged 60 survives to 65 is

$$Pr(T(60) > 5) = exp\{-\int_0^5 \mu_{60+t} dt\}$$

$$= exp\{-\int_0^5 (0.002 + 0.0001t)dt\}$$

$$= 0.98881$$

One more step to Obtain the Probability Density Function (PDF) of $T(x)$
What if $s > 0$?

We have to account for the probability of surviving s years, before dying between s and $s + dt$

We have then

$$_{s|dt}q_x = -\frac{S(x+s)}{S(x)} \frac{S(x+s+dt) - S(x+s)}{S(x+s)dt} dt = {}_sp_x \mu_{x+s} dt$$

This integrates to 1, and is the probability density function of $T(x)$ (times dt)

We can now calculate the expected future lifetime of (x)

$$\mathring{e}_x = E[T(x)] = \int_0^\infty t \cdot {}_tp_x \cdot \mu_{x+t} dt$$

Where

$$_tp_x = e^{-\int_0^t \mu_{x+s} ds}$$

(note the additional of x as compared to the previous version)

10.1.9.2 Examples

Example:
The lifetime of a light bulb is sometimes modelled using an exponential distribution:

$$F'(z) = f(z) = \beta e^{-\beta z}$$

Find:
– The probability that the bulb will fail before x hours of usage

$$F(x) = \int_0^x \beta e^{-\beta x} dx$$

$$= [-e^{-\beta x}]_0^x$$

$$= 1 - e^{-\beta x}$$

– The probability that it will function more than x hours

$$s(x) = 1 - F(x)$$

$$= 1 - [1 - e^{-\beta x}] = e^{-\beta x}$$

– Its failure rate (that is, its "mortality" rate)

$$\frac{f(x)}{s(x)} = \frac{\beta e^{-\beta x}}{e^{-\beta x}} = \beta$$

– Its expected lifetime:

$$\int_0^\infty e^{-\beta x} dx = \frac{1}{\beta}$$

– The probability that it will function for another t years if it has already functioned for x hours

$$\frac{s(x+t)}{s(x)} = \frac{e^{-\beta(x+t)}}{e^{-\beta x}} = e^{-\beta t} = s(t)$$

This leads to the so-called "memory-less" property of the exponential distribution. The fact that you've already survived x years has no impact on the probability that you'll survive another t years

Example:
For a certain population:

$$s(x) = 1 - 0.01x, \text{ for} 0 \leq x \leq 100$$

– What is the force of mortality μ_x, for this population?

$$\mu_x = \frac{-s'(x)}{s(x)} = \frac{0.01}{1 - 0.01x}$$

– Calculate μ_{50}

$$\mu_{50} = \frac{0.01}{1 - (0.01)(50)} = 0.02$$

– What is the (exact) probability that (50) will die on his 50th birthday

$$Probability = 1 - \frac{s(50 + \frac{1}{365})}{s(50)} = 5.48 \times 10^{-5}$$

– Using μ_{50} for an approximation, what is the (approximate) probability that (50) dies on his 50th birthday?

$$Approximately \approx \mu_x dt = \mu_{50} \cdot \frac{1}{365} = 5.48 \times 10^{-5}$$

10.2 The Life Table

A life table

- Represents the experience of a particular group
- Is usually names after that group, and
- Is constructed based on data collected from the experience of that group

It is an historical alternative to using an analytical survival function, but it retains some advantages. The process of constructing life tables is called "graduation"
For example, Australia Life Tables 2000-02 show national population experience around 2001, based on the national census

There are life tables for a special group of lives, including:

- Annuitants, (often lower mortality)
- Pensioners
- Insured lives
- Particular groups of insured (eg. Members of a large industry super fund)
- Men or women, etc, based on data gathered from appropriate groups

Insurance companies have gathered much information from their policyholders over long periods of time, and the life tables they construct are widely used to calculate premium rates, probabilities of death and survival, etc...

10.2.1 Australian Life Tables 2000-02

Here is an extract from Australian Life Tables 2000-02 for Females, published by the Australian Government Actuary

x	l_x	d_x	q_x	p_x
0	100,000	466	0.00466	0.99534
1	99,534	42	0.00043	0.99957
2	99,492	19	0.00019	0.99981
3	99,473	16	0.00016	0.99984

What do these column represent?

10.2.2 Life Table Functions

l_x - is the expected number of survivors to age x out of l_0 births. We call l_0 the **radix** of the table, and it is common to set this to a large round number (l_x is nothing else than a scaled version of $s(x)$)

d_x - is the expected number of deaths at age x out of l_x lives aged x. It is calculated as $l_x - l_{x+1}$

q_x - is called the mortality rate at age x and is the probability that a life aged (exactly) x dies before age (exactly) $x + 1$

$p_x = 1 - q_x$ - is the probability that a life aged x survives to age $x + q$

We have

$$q_x = \frac{d_x}{l_x}$$

and

$$p_x = \frac{l_{x+1}}{l_x}$$

10.2.2.1 Mechanics of the Table

Given a survival function, s, or a set of mortality rates, the entire life table can be constructed Suppose we know mortality rates q_x for $x = 0, 1, 2, \ldots$

- First set the vales of l_0, say $l_0 = 100,000$
- Next, calculate $d_0 = l_0 q_0$, and $l_1 = l_0 - d_0$
- Repeat these steps at age 1: calculate $d_1 = l_1 q_1$, and $l_2 = l_1 - d_1$
- Continue in this manner

However, while this helps understanding what the tables mean, this is often besides the point. The point of having a table is to fit mortality rates to a given data set. The "shape" and "turns" of the function may not follow a simple parametric function

10.2.2.2 Two Identities

$$l_0 = d_0 + d_1 + d_2 + \ldots = \sum_{s=0}^{\infty} dx$$

Number of people alive at t=0 is the number of number of people who will die between t=0 and infinity

$$l_x = d_x + d_{x+1} + \ldots = \sum_{y=x}^{\infty} dy$$

Number of people alive at t=x is the number of people who will die between t=x and infinity

10.2.2.3 Basic Calculations

${}_tp_x$ is the probability that (x) survives to (at east) age $x + t$
${}_tq_x$ is the probability that (x) dies before age $x + t$

Note that:

$$ {}_tp_x = \frac{s(x+t)}{s(x)}$$

$$= \frac{s(x+1)}{s(x)} \frac{s(x+2)}{s(x+1)} \cdots \frac{s(x+t)}{s(x+t-1)}$$

$$p_x p_{x+1} \cdots p_{x+t-1}$$

$$\frac{l_{x+1}}{l_x} \frac{l_{x+2}}{l_{x+1}} \cdots \frac{l_{x+t}}{l_{x+t-1}}$$

$$= \frac{l_{x+t}}{l_x}$$

Example:
Based on the extract from Australian Life tables 2000-02 for Males (in tutorial booklet)
- What is the probability of (60) surviving for at least 5 years?
- What is the probability of (60) dying before age 65?
- What is the probability of (60) dying aged 62?

10.2.3 Aggregate Life Table Functions

10.2.3.1 L_x: Average number of survivors between age x and $x+1$

L_x: Average number of survivors between age x and $x+1$

Let $l_x = l_0 \cdot {}_xp_0$. $x > 0$, be the number of survivors reaching exact age x out of a group of newborn of size l_0. Define

$$L_x = \int_x^{x+1} l_y dy = \int_0^1 l_{x+t} dt, x = 0, 1, 2, \ldots$$

Then L_x is the average number of survivors between age x and $x+1$

Properties:
- Because there are no entries (and only deaths)

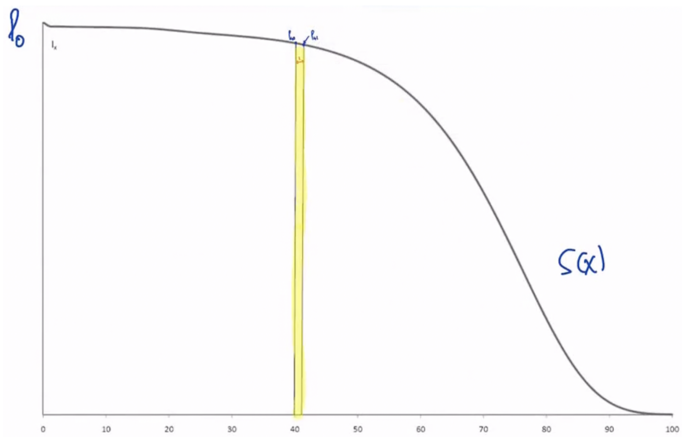
$$L_x < l_x, x = 0, 1, 2 \ldots$$

- Because deaths are roughly uniformly distributed in time (but more perfectly since there are less people around to die at the end at the end than at the beginning, and they are generally more likely to die)

$$L_x \approx \frac{l_x + l_{x+1}}{2} = l_x - \frac{1}{2} dx$$

which can also be written as

$$L_x = l_x \int_0^1 {}_tp_x dy$$



The curve is l_x and the area under the curve between 40 and 41 is L_{40}

10.2.3.2 T_x: Aggregate L_x

Define

$$T_x = \int_x^{\infty} l_y dy = \int_0^{\infty} l_{x+t} dt, x = 0, 1, 2, \ldots$$

Properties:
– $T_x = L_x + L_{x+1} + \ldots = \sum_{y=x}^{\infty} L_y$
– $T_x = L_x + T_{x+1}$

It can be shown that:

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt = \int s(x)dx = E[x]$$

which means that

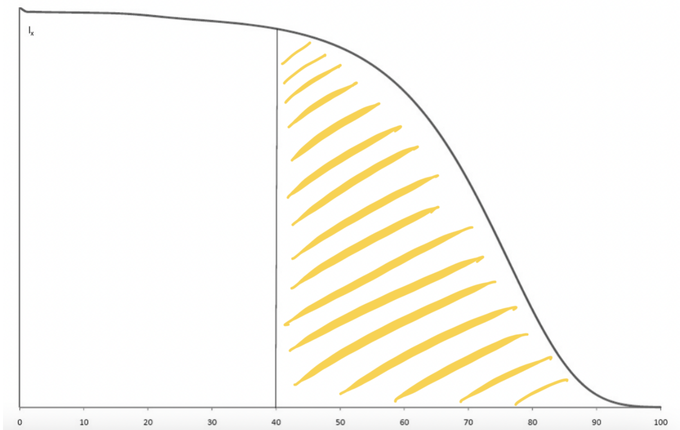
$$\mathring{e}_x = \frac{T_x}{l_x}$$

where $\mathring{e}_x$ represents the expected future life time of (x)

Additionally

$$T_x = l_x \mathring{e}_x$$

represents the expected total future life time of group of l_x lives all aged x



The curve is l_x and the area on the RHS of the vertical like at 40 is T_{40}

10.2.4 Stationary Populations

Let us conduct the following thought experiment: Let us assume the table represents a **stationary population**. This means that the number entering the population each year is the same as the number leaving it:

- The number of births is equal to the number of deaths if immigration and emigrations are not considered ($\sum d_x = l_0$)
- The population size is unchanged with time (zero growth rate)
- The size of a certain age group is unchanged with time
- The mortality rate q_x for each age remains the same from year to year
- We assume births are uniformly distributed throughout the year

We can interpret that:

- l_x - is the number of people in the population who have their x-th birthday each year
- d_x - is the number of deaths for each year between age x and $x+1$
  Note d_x does not change from one year to another
- $\sum_{x=0}^{\infty} d_x = l_0$ says that the number of deaths is the same as the number of births
- $L_x = l_x \int_0^1 {}_tp_x dt$ is the population size between ages x and $x+1$ (aged x last birthday) at any time point
- $T_x = \sum_{y=x}^{\infty} T_y$ is the population size above age x at any time point
- $T_0 = \sum_{y=0} T_y$ is the population at any time during the stationary period
- $T_0 - T_x$ is the population aged under age x, aged 0,1,2,... $x-1$, last birthday

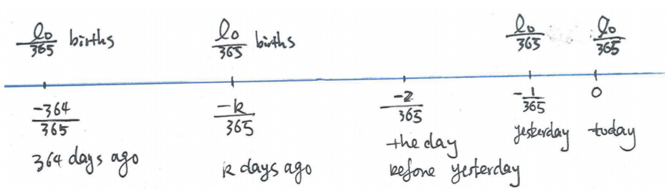
Why does L_0 represent the population aged 0 last birthday in a stationary population?

- Recall that l_0 is the number of births into a stationary population act year; l_x is the number of lives having their $x-th$ birthday in each year

$$L_0 = \int_0^1 l_t dt = l_0 \int_0^1 {}_tp_0 dt = lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{l_0}{n} {}_{\frac{k}{n}}p_0$$

$$\approx \frac{l_0}{365}[{}_0p_0 + {}_{\frac{1}{365}}p_0 + {}_{\frac{2}{365}}p_0 + \ldots + {}_{\frac{164}{365}}p_0]$$

- We assumed that births are uniformly distributed over the year, then $l_0/365$ is the number of births for each days over a year



- The number of births today (aged 0 exactly today) is

$$\frac{l_0}{365} {}_0p_0 = \frac{l_0}{165}$$

- The number of lives born yesterday surviving to today, aged 1 day or 1/365 year at today, is

$$\frac{l_0}{165} {}_{\frac{1}{165}}p_0$$

- Similarly,

$$\frac{l_0}{165} {}_{\frac{k}{165}}p_0$$

  is the number of lives born k days ago surviving to today, aged k days or $k/365$ year on today
- Finally, $\frac{l_0}{165} {}_{\frac{364}{365}}p_0$ is the number of lives born 364 days ago surviving to today, aged 364 days or 364/365 year at today

Wrapping up,

$$L_0 = \frac{l_0}{365} {}_0p_0 + \frac{l_0}{365} {}_{\frac{1}{365}}p_0 + \ldots \frac{l_0}{365} {}_{\frac{k}{365}}p_0 + \ldots + \frac{l_0}{365} {}_{\frac{364}{365}}p_0$$

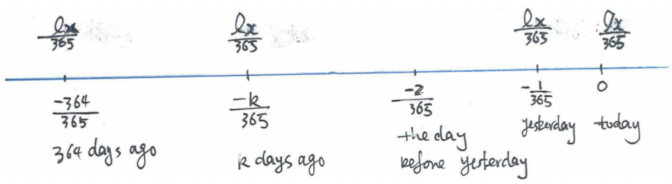
is the number of lives aged under one year old

Now generalising, why does L_x represent the population aged x last birthday in a stationary population?

- We assumed that birthdays are uniformly distributed over the year, then $l_x/365$ is the number of lives having their x-th birthday for each day over the year

$$L_x = \int_0^1 l_{x+t}dt = l_x \int {}_tp_x dt = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{l_x}{n} {}_{\frac{k}{x}}p_x$$

$$\approx \frac{l_x}{365} {}_0p_x + \frac{l_x}{365} {}_{\frac{1}{365}}p_x + \ldots + \frac{l_x}{365} {}_{\frac{k}{365}}p_x + \ldots + \frac{l_x}{365} {}_{\frac{364}{365}}p_x$$



10.2.4.1 Example 1

x	l_x	d_x	p_x	μ_x	L_x	T_x
0	100,000	986	0.99014	0.01520	99,424	6,909,573
1	99,014	66	0.99933	0.00529	98,942	6,810,149
2	98,948	40	0.99960	0.00054	98,927	6,711,208
3	98,908	30	0.99970	0.00035	98,892	6,612,281
4	98,878	26	0.99974	0.00028	98,865	6,513,389
5	98,852	24	0.99976	0.00025	98,840	6,414,524

- Calculate the probability that a newborn life survives to age 5. Give your answer to 5 decimal places

$$5p_0 = \frac{l_5}{l_0} = \frac{98,852}{100,000} = 0.98852$$

- Calculate the probability that of two lives aged 2, exactly one of them survives to age 5. Gives your answer to 5 decimal places. You may assume that these two lives are independent with respect to mortality

$$Pr(\text{a dies, b lives}) + Pr(\text{a lives, b dies})$$

$$= {}_3q_{2\,3}p_2 + {}_3p_{2\,3}q_2 = 2({}_3p_{2\,3}q_2)$$

$$= 2 \times \frac{l_5}{2}(1 - \frac{l_5}{l_2}) = 0.00194$$

- Calculate $\mathring{e}_5 = E[T(5)]$, the expected future life time of a life aged 5, from the table above. Give your answer to 2 decimal place

$$\mathring{e}_5 = \frac{T_5}{l_5} = \frac{6,414,524}{98,852} = 64.89$$

- Estimate the probability that a baby aged exactly 1 will die in two days. Give your answer to 5 decimals places.

The estimated probability is $\mu_1 \times (2/365) = 0.01520 \times 2/365 = 0.00003$

10.2.4.2 Example 2

The population of a country experiences the mortality of the life table in the previous example. The population of this country has reached a stationary condition with 100,000 births for each year during the stationary period. Calculate the following quantities to 2 decimal places
- The crude birth rate (CBR) for this country

$$CDR = 1000\frac{\text{number of births}}{\text{population size}} = 1000 \times \frac{l_0}{T_0} = 1000 \times \frac{100,000}{6,909,573} = 14.4$$

- The crude death rate (CDR) for this country

$$CDR = CBR, \text{in a stationary population number of births} = \text{number of deaths}$$

- The population size aged under 5

$$T_0 - T_5 = 6,909,573 - 6,414,524 = 495,049$$

10.2.4.3 Tutorial 8.1.1

A nursing home accepts patients on their 90th birthday and has now reached a stationary population. Their intake is 100 patients per year. Assuming that 3 times as many women as men are accepted and that the patients suffer mortality in accordance with the Male and Female mortality tables in AITAS2E, calculate:
1. the total population of the nursing home,

$$Pop = T_{90}^M + T_{90}^F$$

However Life Tables are normally scaled to a radix of 100,000. In order to scale the figures in the life table to our population we introduce K_M and K_F

$$Pop = K_M \cdot T_{90}^M + K_F \cdot T_{90}^F,\ K_M, K_F \Rightarrow \text{Unknown constants}$$

We know that of the 100 patients that enter each year, 25 are males and 75 are female. This also tell us that:

$$K_M \cdot l_{90}^M = 25$$

$$K_F \cdot l_{90}^F = 75$$

This allows us to rewrite out population equation:

$$K_M \cdot T_{90}^M + K_F \cdot T_{90}^F = K_M l_{90}^M \cdot \frac{T_{90}^M}{l_{90}^M} + K_F l_{90}^F \cdot \frac{T_{90}^F}{l_{90}^F}$$

Using the figures from the life tables

$$= 25\frac{8,488}{3,152} + 75\frac{72,266}{16,578}$$

$$= 67.322... + 326.936... = 394.2... \approx 394$$

2. The number of individuals reaching their 100th birthday each year
Number of people entering age 100:

$$l_{100}^M + l_{100}^F$$

However we again need to scale this to the fit our population

$$\begin{aligned}
&K_M l_{100}^M + K_F l_{100}^F \\
&= K_M l_{90}^M \cdot \frac{l_{100}^M}{l_{90}^M} + K_F l_{90}^F \cdot \frac{l_{100}^F}{l_{90}^F} \\
&= 25\frac{30}{3152} + 75\frac{1258}{16578} \\
&= 5.69 + 0.24 \approx 6
\end{aligned}$$

3. The number of deaths per year at ages 95 and above.
Since we have a stationary population, # deaths (0+) = # entries (0). Therefore we can also say:

$$\text{\# deaths } 95+ = \text{\# entries } 95$$

10.2.5 Curtate Future Lifetime

10.2.5.1 Revision: Expected value of a discrete random variable

Suppose a random variable, X, can take n values labelled $x_1, x_2, \ldots, x_n$
The expected value of the random variable X (denoted $E[x]$) is calculated using

$$E[X] = \sum_{I=1}^{n} x_i Pr[X = x_i]$$

10.2.5.2 Curtate future lifetime

Recall: $T(x)$ is the future lifetime of (x)
We define a random variable

$$K(x) = [T(x)]$$

to be the number of whole years of future lifetime from age x

We call $K(x)$ the **curtate future lifetime** at age x
For example:

- If (40) dies at age 75.6, $T(40) = 35.6$, and $K(40) = 35$
- If (30) dies at age 82.3, $T(30) = 52.3$, and $K(30) = 52$

10.2.5.3 Probability Mass Function of $K(x)$

How do we find $Pr(K(x) = 10)$?

$$K(x) = 10 \text{ if } 10 \leq T(x) < 11$$

Thus,

$$\begin{aligned}
Pr(K(x) = 10) &= Pr(10 \leq T(x) < 11) \\
&= \frac{s(x+10) - s(x+11)}{s(x)} \\
&= {}_{10}p_x - {}_{11}p_x
\end{aligned}$$

To generalise:

$$Pr(K(x) = k) = {}_{k}p_x - {}_{k+1}p_x$$

10.2.5.4 Curtate Expectation of Life

The expected value of $K(x)$ is called the curtate expectation of life at age x, and is denoted e_x. That is

$$E[K(x)] = e_x$$

The quantity $\mathring{e}_x = E[T(x)]$ is called the complete expectation of life, and

$$\mathring{e}_x \approx e_x + \frac{1}{2}$$

Using the pmf obtained earlier we have:

$$e_x = E[K(x)] = \sum_{k=0}^{\infty} k \times Pr[K(x) = k] = \sum_{k=1}^{\infty} k({}_kp_x - {}_{k+1}p_x)$$

Developing yields

$$e_x = \sum_{k=1}^{\infty} k({}_kp_x - {}_{k+1}p_x))$$

$$= {}_1p_x - {}_2p_x + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \dots$$

$$= {}_1p_x + {}_2p_x + {}_3p_x + \dots$$

Thus we get the important formula

$$e_x = \sum_{k=1}^{\infty} {}_kp_x$$

Example

Consider the survival function

$$s(x) = exp\{-\lambda x\}, \lambda > 0$$

- Show that e_x is independent of x

We have:

$${}_kp_x = \frac{s(x+k)}{s(x)} = \frac{exp\{-\lambda(x+k)\}}{exp\{-\lambda x\}} = exp\{-\lambda k\}$$

Thus

$$e_x = \sum_{k=1}^{\infty} {}_kp_x$$

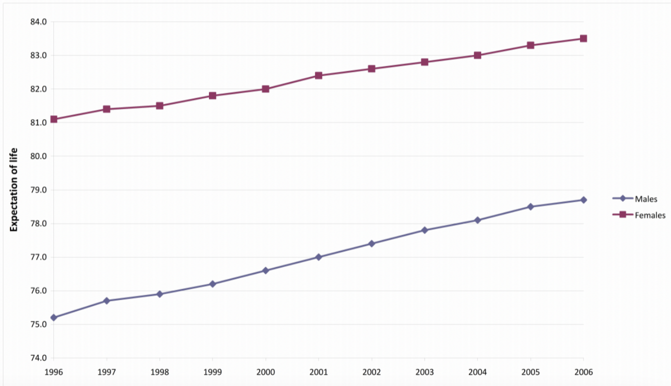
$$= \sum_{k=1}^{\infty} exp\{-\lambda k\}$$

$$= \frac{exp\{-\lambda\}}{1 - exp\{-\lambda\}}$$

which is independent of x

10.2.5.5 Expectation of Life in Australia

The plot shows expectation of life at birth in Australia



The table shows expectations of life from birth in 2005 for some countries

Country	Expectation of Male Life	Expectation of Female Life
Belgium	75.44	81.94
Canada	78.55	83.54
Kenya	53.35	53.10
Nicaragua	68.27	72.49
Pakistan	62.04	64.01
Singapore	79.05	84.39

These are life expectations calculated using a period table. They are often misinterpreted

10.2.6 Period vs Cohort Tables

- Period tables describe mortality of a population in a given year (so different individuals of different age, all in the same calendar year). So there are potentially different period tables for different calendar years of death
- Cohort tables look at the mortality of people born in the same calendar year - they tract mortality of a given cohort. So there are potentially different cohort tables for different calendar years of birth

10.3 Characteristics, Causes and Trends of Mortality Experience

Life tables are very important tin traditional actuarial work. They are constructed from past experience, but may also include an allowance for 'improving mortality' - changes in mortality rates since the data were collected

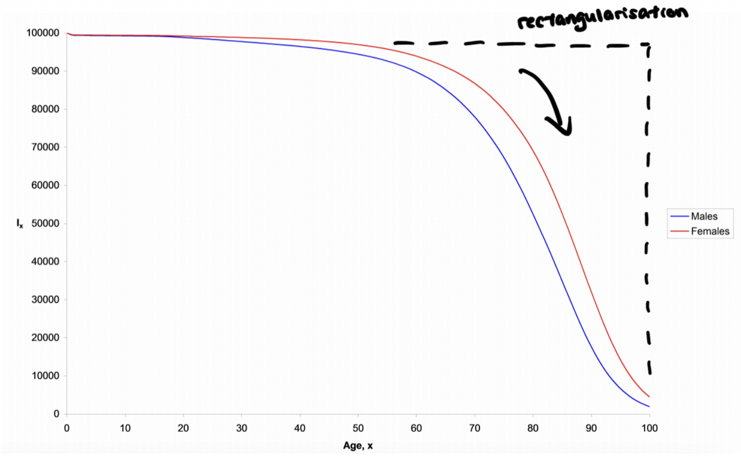
Different tables are constructed based on differed 'exposed to risk'. Calculations of quantities such as premiums are based on the assumption of a **particular** mortality experience, perhaps adjusted in some way - using the rate for higher or lower age, or making a percentage loading on the tabulated rate
For example, there are tables on annuitants' data, which reflect the mortality experience of people who have purchased annuities. These people expect to have low mortality, otherwise they would not buy an annuity - they are 'self selecting' on this basis

Australian Life Tables are based on the whole Australia population, without any special features to affect their experience. We would expect the mortality rates to be higher than for annuitants On the other hand, those who but life insurance are insuring their lives and so are self-selecting on that basis - so we would expect, and we find, that their morality rates are higher than for annuitants So, choosing the appropriate table, and whether and how to adjust the rates if of primary importance in actuarial calculations

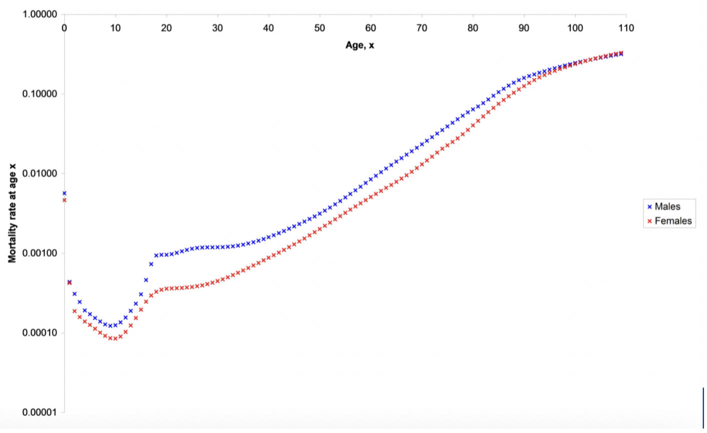
In what follows, we illustrate some of the main features of human mortality in Australia

10.3.1 Australian Life Tables: 2000-02

This graph shows l_x.



This graph shows q_x on a logarithmic scale.



Australia Life Tables 2000-02: Mortality Rates

Some of the key features include:

- Life in the first year is risky
- There is a 'bump' around the late twenties, where accidental death from risky behaviour becomes noticeable for males. This has tended to reduce for males, and increase for females
- Mortality rates start to increase more steeply around ages 40-45

10.4 Mathematical Models of Mortality

10.4.1 Two Theoretical Model

Various ‘laws of mortality’ have been proposed over the years. The two old and gold famous ones are:

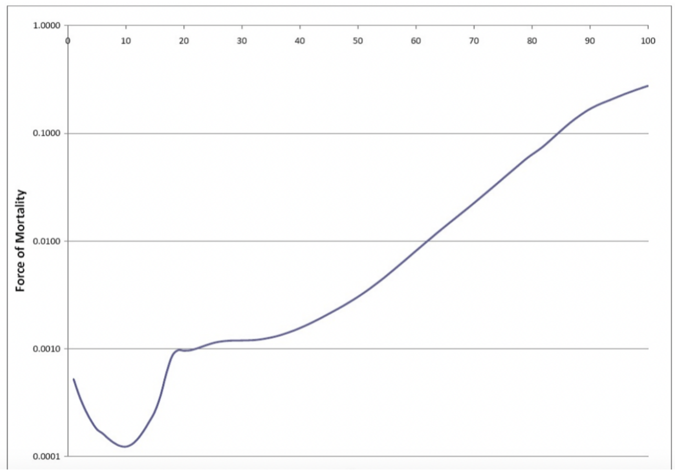
- Gompertz Law (1825)
$$\mu_x = Bc^x$$
Where $B > 0, c > 1$. This suggests that the force of mortality increases exponentially with age

- Makeham’s Law (1860)
$$\mu_x = A + Bc^x$$
Where $A, B > 0, c > 1$. Additionally, mortality is subject to a constant force, A, that is independent of age - eg. Death by accidental causes

How does these compare to mortality nowadays?

10.4.2 The force of mortality in Australia

The plot shows μ_x for males on a logarithmic scale - from Australian Life Tables Males 2000-02



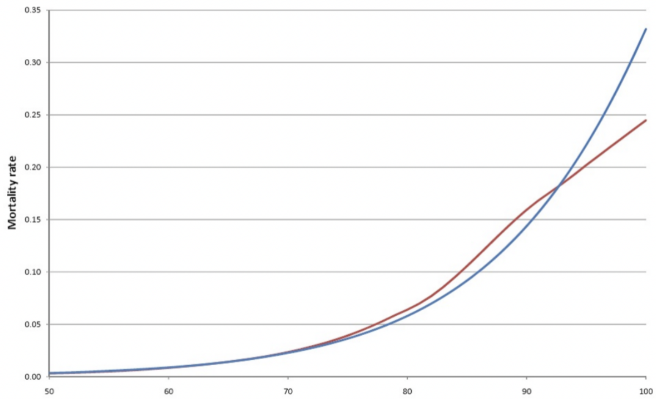
We observe that the force of mortality graph looks similar to a plot of q_x, namely

- Decreasing at young ages
- Increasing in late teen/early twenties, and
- Increasing as age increases

This is typical in the developed world
How does that compare with a formula like $\mu_x = Bc^x$ which says that μ_x increases with x?

10.4.3 Gompertz law on a portion of Australian Data

The following plot shows Gompertz law fitted to Australian Life Tables Males 2000-02, for ages 50-100; q_x is plotted in red



How was that fitted?
In Australian Life Tables 2000-02 Females, $\mu_{50} = 0.00193$ and $\mu_{70} = 0.01246$

We set:

$$\mu_{50} = 0.00193 = Bc^{50}$$
$$\mu_{70} = 0.01246 = Bc^{70}$$

Then

$$c^{20} = 6.45596$$

and so

$$B = 1.8224 \times 10^{-5}$$

Hence our estimate becomes

$$\mu_{60} = Bc^{60} = 0.00490$$

(The published value is 0.00488)

10.5 Measures of Fertility

10.5.1 Definition

Births are a natural source of increase for national populations. Another source of increase is immigration. Here we focus on births only.

There are number of summary statistics commonly used to described features of fertility experience, such as:

- General Fertility Rate (GFR)
- Age Specific Fertility Rate ($ASFR_x$)
- Total Fertility Rate (TFR)
- Gross Reproduction Rate (GRR)
- Net Reproduction Rate (NRR)

10.5.2 General Fertility Rate (GFR)

$$GFR = 1000 \times \frac{\text{\#of live births}}{\text{\#women aged } [15, 49]}$$

This is calculated as the number of live births per 1,000 women of reproductive age in a population The calculation is based on births in a fixed period of time, usually a calendar year. In Australia, there were 265,949 live births in 2006. Thus using the corresponding population data, the GFR is

$$1000 \times \frac{265,949}{4,918,284} = 54.07$$

Where we have taken women aged 15-49 to be of reproductive age

10.5.2.1 Comments on GFR

- Crude Birth Rate (CBR)

  $$CBR = 1000 \times \frac{\text{\# of live births}}{\text{Population size}}$$

  And hence we have

  $$GFR > CBR$$

- While GFR is better than CBR, a disadvantage of GFR is that it will be affected by the age distribution of women aged 15-49, which reduces its comparability

In what follows we define the age specific fertility rates (ASFR) which are independent of the age distribution of the female population

10.5.3 Age Specific Fertility Rate (ASFR) and Total Fertility Rate (TFR)

The **Age Specific Fertility Rate (ASFR)** is defined as the number of live births per 1,000 women of a specific age or an age group in a (calendar) year

$$ASFR_x = 1000 \times \frac{\text{\# of live births by women aged } [x, x+1)}{\text{\# of women aged } [x, x+1)}$$

$ASFR_x$ is the number of live births per 1,000 women aged x

The **Total Fertility Rate (TFR)** is

$$TFR = \sum_{x=15}^{49} ASFR_x$$

The ASFR can also be defined as fertility rates of women aged over an age group, 15-19, 20-24, 25-29, etc. For example

$$ASFR_{[15,19]} = 1000 \times \frac{\text{\# of live births by women aged } [15, 19]}{\text{\# of women aged } [15, 19]}$$

Thus, we could use the ASFR to compare, eg. fertility rates of women aged 30-34 in Victoria and NSW without concerning ourselves with the population structure in these states

10.5.3.1 Comparison of TFR and GFR

Consider the following data in a given table

age	15	16	$\cdots$	x	$\cdots$	49
# of women	w_{15}	w_{16}	$\cdots$	w_x	$\cdots$	w_{49}
# of live births	b_{15}	b_{16}	$\cdots$	b_x	$\cdots$	b_{49}

$$ASFR_x = 1000 \times \frac{b_x}{w_x}, x = 15, 16, \ldots, 49$$

$$TFR = \sum_{x=15}^{49} 1000 \times \frac{b_x}{w_x} = \sum_{x=15}^{49} ASFR_x$$

Now the GFR is calculated as

$$GFR = 1000 \frac{\sum_{x=15}^{49} b_x}{\sum_{x=15}^{49} w_x} \text{ which can be rewritten as}$$

$$1000 \sum_{x=15}^{49} \frac{b_x}{\sum_{x=15}^{49} w_x} = 1000 \sum_{x=15}^{49} \frac{w_x}{\sum_{x=15}^{49} w_x} \frac{b_x}{w_x} = \sum_{x=15}^{49} \theta_x ASFR_x$$

Here

$$\theta_x = \frac{w_x}{\sum_{x=15}^{49} w_x}$$

Is the percentage of women aged x last birthday to the total number of women at reproductive ages, with $\sum_{x=15}^{49} \theta_x = 1$

This is what changes from one population not another in the GFR, and which is somewhat standardised in the TFR (equal weights). In other words *if the population of women changes from one population to another, everything else being equal (same $ASFR_x$), this will impact the GFR but not the TFR*

Similarly, the age specific fertility rate over an age interval, say [15,19], is

$$ASFR_{[15,19]} = 1000 \frac{\sum_{x=15}^{19} b_x}{\sum_{x=15}^{19} w_x} = 1000 \sum_{x=15}^{19} \frac{b_x}{\sum_{x=15}^{19} w_x}$$

Note $ASFR_{[x,y]}$ is a weighted averages of $ASFR_x$

$$= 1000 \sum_{x=15}^{19} \frac{w_x}{\sum_{x=15}^{19} w_x} \frac{b_x}{w_x} = \sum_{x=15}^{19} \tau_x ASFR_x$$

where

$$\tau_x = \frac{w_x}{\sum_{x=15}^{19} w_x}, x = 15, 16, 17, 18, 19$$

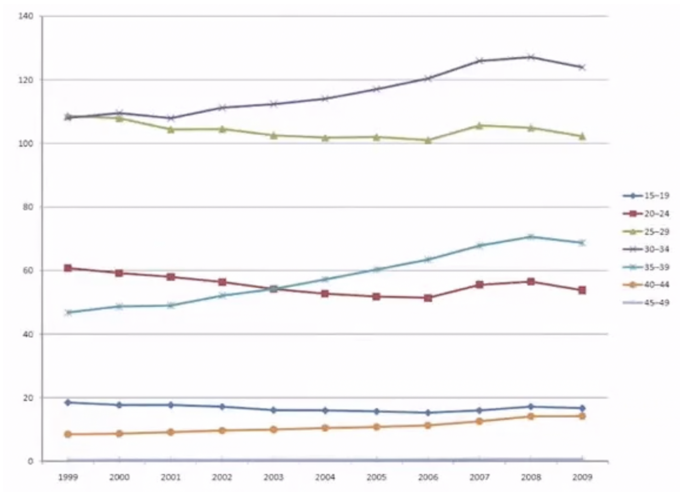
Is the percentage of women aged x last birthday to the total number of women aged [15,19], with $\sum_{x=15}^{19} \tau_x = 1$

10.5.3.2 Summary/some Comments

- $ASFR_{[15,19]}$ is the weighted average of $ASFR_x$ over the whole reproductive age interval [15,49]
- GFR is the weighted average of $ASFR_x$ over the whole reproductive age interval [15,49]
- TFR is the sum of $ASFR_x$ over all reproductive ages
- TFR is the number of the live births per 1000 women over their reproductive ages, assuming no mortality
- To sustain population, $TFR/1000 \geq 2.1$ (roughly, but that depends on mortality etc...)
- 2.1 is called replacement level fertility
- In Australia
  - TFR/1000 = 1.88 in 2013;
  - TFR/1000 = 1.83 in 2014;
  - TFR/1000 = 1.81 in 2015;
  - TFR/1000 = 1.77 in 2016;

10.5.4 Age Specific Fertility Rates in Australia

The chart below shows ASFRs for Australia over the period 1999-2009



Observations:

- Fertility rates haven fallen for women in their 20s
- Fertility rates have risen for women in their 30s
- There is a downward trend at ages 15-19 and an upward trend at 40-44, but these are small changes compared with the age groups
- No real change at ages 45-19

10.5.5 Gross Reproduction Rate (GRR)

Fertility rates tell us about how many babies are being born.

Reproduction rates focus on the birth of *female* babies, because they are the only ones who will eventually be able to have babies themselves, later on. As the Sex ratio at birth changes from on population to another (and also potentially across ages) then this makes a differences.
Let $ASFR_x^f$ denote the $ASFR_x$ for *female* births
Thus, $ASFR_x^f$ is defined as the number of live female births per 1,000 women at age x (per year)

The **Gross Reproductive Rate (GRR)** is then defined as

$$GRR = \sum_{x=15}^{49} ASFR_x^f$$

10.5.5.1 Comparing TFR and GRR

The Sex ratio at birth is 1.05 (males vs females) on average, therefore we can estimate

$$ASFR_x^f = \frac{1}{2.05} ASFR_x, x = 15, 16, \ldots, 49$$

and hence

$$GRR = \frac{1}{2.05} TFR$$

If we assumed we need

$$\frac{TFR}{1000} \geq 2.1$$

Then this translates into requiring (dividing by 2.05)

$$\frac{GRR}{1000} \geq \frac{2.1}{2.05 = 1.024}$$

Illustration:
Following the data from ABS relate to Australia in 2006. Estimate the GRR

Age Group	ASFR
15–19	15.4
20–24	51.6
25–29	100.8
30–34	120.1
35–39	63.3
40–44	11.3
45–49	0.6

The rates are per 5 year age group. When data is presented like this it is standard practice to assume that the given ASFR applies at each age in the group
Thus, eg. We assume $ASFR_{15} = ASFR_{16} = \ldots = ASFR_{19} = 15.4$

The same assumption applies to each age group so

$$\sum_{x=15}^{49} ASFR_x = \sum_{x=15}^{19} ASFR_x + \sum_{x=20}^{24} ... + \sum_{x=45}^{49} ASFR_x$$

$$= 5(15.4 + 51.6 + ... + 0.6) = 1,815.5 = TFR$$

Our data is not split by gender, so if we assume a Sex ratio at birth of 1.05, we get

$$GRR \approx \frac{1}{2.05}(1,815.5) = 885.6$$

per 1,000 women

Note that several layers of approximation were used here

We used grouped $ASFR$ to approximate the TFR

$$TFR \approx 5 \times (ASFR_{[15,19]} + ASFR_{[20,24]} + ... + ASFR_{[40,44]} + ASFR_{[45,49]})$$

–We estimated the female $ASFR$ using sex at birth
–We used a single sex ratio for all ages

10.5.6 Net Reproduction Rate (NRR)

This is defined as the number of live *female* births surviving to the ages their mothers were at their births per 1,000 woman in an age group.

It is a measure of the rate at which a women of age x is reproducing herself - with another woman who has survived to age x - so the probability of surviving to the age of the mother is taken into account
Thus:

$$NRR = \sum_{x=15}^{49} ASFR_x^f \frac{l_{x+1/2}}{l_0} \approx \frac{1}{2.05} \sum_{x=15}^{49} ASFR_x \frac{l_{x+1/2}}{l_0}$$

Note that we assume that women giving birth at age x are on average age $x + 1/2$, therefore we use the probability of survival to age $x + 1/2$
*Frequently the age range is 5 years, or 10. If so, remember that the average age is halfway through this range

Example:
Calculate the NRR from the following data

Age Group	ASFR	Survival Factor
15–19	15.4	0.99279
20–24	51.6	0.99107
25–29	100.8	0.98921
30–34	120.1	0.98703
35–39	63.3	0.98410
40–44	11.3	0.97991
45–49	0.6	0.97377

Assume that the survival factor applies from birth to the age in the middle of each age group

$$NRR = \frac{1}{2.05}[\sum_{x=15}^{19} ASFR_x \frac{l_{x+1/2}}{l_0} + \sum_{x=20}^{24} ASFR_x \frac{l_{x+1/2}}{l_0} + ... + \sum_{x=45}^{49} ASFR_x \frac{l_{x+1/2}}{l_0}]$$

$$= \frac{1}{2.05}[\sum_{x=15}^{19} 15.4 \frac{l_{x+1/2}}{l_0} + \sum_{x=20}^{24} 51.6 \frac{l_{x+1/2}}{l_0} + ... + \sum_{x=45}^{49} 0.6 \frac{l_{x+1/2}}{l_0}]$$

$$= \frac{5}{2.05}[\frac{15.4}{l_0} \sum_{x=15}^{19} \frac{l_{x+1/2}}{5} + \frac{5.16}{l_0} \sum_{x=20}^{24} \frac{l_{x+1/2}}{5} + ... + \frac{0.6}{l_0} \sum_{x=45}^{49} \frac{l_{x+1/2}}{5}]$$

*Note: ($\frac{5}{2.05}$ and $\frac{l_{x+1/2}}{5}$) because the age range for each $ASFR$ is 5)

$$= \frac{5}{2.05}[\frac{15.4}{l_0}l_{17.5} + \frac{5.16}{l_0}l_{22.5} + ... + \frac{0.6}{l_0}l_{47.5}]$$

Age Group	ASFR	Survival Factor	
(1)	(2)	(3)	(2) × (3)
15–19	15.4	0.99279	15.2889
20–24	51.6	0.99107	51.1392
25–29	100.8	0.98921	99.7119
30–34	120.1	0.98703	118.5417
35–39	63.3	0.98410	62.2932
40–44	11.3	0.97991	11.0730
45–49	0.6	0.97377	0.5843

$$NRR = \frac{5}{2.05}[15.2889 + 51.1392 + ... + 0.5843] = 874.7, \text{ per 1000 women}$$

10.5.7 Factors affecting fertility

Standards of nutrition, sanitation, and health more broadly mainly have an impact on child survival, but they do have some impact on the success of pregnancies (and the wish of parents to have children)
Then how many babies to have is mostly a matter of choice, and population circumstances have an impact of those choices:

- In general, increased prosperity leads to lower fertility rates
- Individuals delay parenthood, and limit the number of children
- Philosophical and religious convictions may have an impact, too
- Some countries may decide to implement strict birth policies (eg, China)

Importantly, developing countries may have high fertility rates, but low reproduction rates, due to poor health. These highlight how different both types of indicators are

10.5.8 Trends in fertility rates, and consequences

Trends in fertility rates for developing countries are typically as follows

- Women are having fewer children over their reproductive lifetime
- Women are choosing to have their first child later in life
- Completed family size is falling

A consequence is an increase in **generation gap,** whereby parents are typically 30-40 years older than their children (as opposed to 20-30 years older)
This, coupled with reduced mortality at all ages, leads to an **ageing of the population**, which has serious consequences

- Recall the increasing age dependency ratio
- Elderly people may not have large families available to care from the, (they had no children, or they moved overseas), and there are less active people to fund their care

10.6 Population Projections

A population projection gives a profile of a population in the future, based on current (and past) data, and our view as to how the population will change over time
There are many reasons why we might be interested in knowing what a population will look like in the future. For example, to estimate the cost of providing a particular service (education, health care, social security payments, etc) to a particular group in the population

There are a number of different models and method commonly used to estimate population size based on different types of growth

10.6.1 Polynomial Model

Let P_t be the size of the population at time t. Then

$$P_t = P_0 + \sum_{i=1}^{n} a_i t^I$$

for constants $a_1, a_2, \ldots, a_n$
As a special case, the linear model is

$$P_t = P_0 + at = P_0(1 + rt)$$

where $r = a/P_0$ is called simple annual growth rate

10.6.2 Geometric Population Model

This method assumes a compound rate of growth at rate r per unit time so that

$$P_t = P_0(1 + r)^t$$

For example, we may use the information from past years and extrapolate to the future
This method has the advantage that it is very simple to apply. However it has the disadvantage that it is very simplistic

Example:
A country's population in 1990 was 12 million. In 2000 it was 14 million
Assuming geometric population growth, project the country's population in 2015

We have

$$P_{2000} = (1 + r)^{10} P_{1990}$$
$$14(M) = (1 + r)^{10} 12(M)$$
$$(1 + r)^{10} = \frac{7}{6}$$

Then

$$P_{2015} = (1 + r)^{15} P_{2000}$$
$$P_{2015} = (\frac{7}{6})^{1.5} 14(M) = 17.642(M)$$

10.6.3 Logistic Model

Populating growth under the logistic model is represented as

$$P_t = \frac{1}{A + Be^{-rt}}$$

Thus, at $t = 0$

$$P_0 = \frac{1}{A + B}$$

And there is a limiting population size as $t \to \infty$

$$\lim_{t \to \infty} P_t = \frac{1}{A}$$

Example
The population of a certain country is currently 20 million
The population is estimated to grow at 1.5% over the next year
Assume that the population will grow to a limiting value of 25 million
Estimate the population size in 50 years' time under a logistic model of population growth

We know

$$\lim_{t \to \infty} P_t = \frac{1}{A} = 25(M)$$

$$\Rightarrow A = \frac{1}{25(M)} = 4 \times 10^{-8}$$

Also

$$P_0 = \frac{1}{A + B} = 20(M)$$

$$A + B = \frac{1}{20(M)} = 5 \times 10^{-8}$$

$$B = \frac{1}{20(M) - \frac{1}{25(M)}} = 10^{-8}$$

Lastly we know

$$\frac{P_1}{P_0} = 1.015 = \frac{A + B}{A + Be^{-rt}}$$

$$= \frac{4 \times 10^{-8} + 10^{-8}}{4 \times 10^{-8} + 10^{-8}e^{-r}}$$

$$1.015(4 + e^{-r}) = 5$$

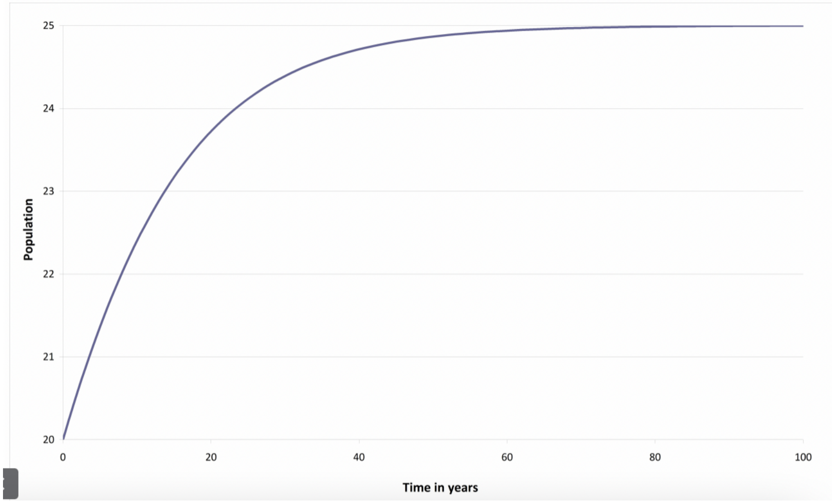
$$e^{-r} = 0.9251...$$

$$r = -log(0.9251...) = 0.07676...$$

Thus

$$P_{50} = \frac{1}{4 \times 10^{-8} + 10^{-8}e^{-50r}} = 24.87 \times 10^6$$

The population here grows as in the graph below.



10.6.4 Component Model

Finally, there is the component method. This is detailed and models the 'components' which contribute to growth or decline of the population,
Simply put,

$$P_t = P_0 + [births - deaths + immigration - emigration], from(0, t)$$

The Australian Bureau of Statistics (ABS) website has a population clock doing estimates of how Australia's population is projected into the future it uses such a component model

This method enables assumptions about the mortality, fertility, immigration and emigration rates in as detailed a way as we wish
It also enables us to investigate the effects of different rate/assumptions have on future population growth and distribution ("what if" analysis) which is crucial for decision making

11 Life Insurance and Wealth Management

11.1 Life Insurance

11.1.1 Life Insurance Products - Traditional policies

In case of death

- Whole of life
  - Sum insured on death of the life insured regardless of when the life dies, can be limited term premium
  - **Non-participating** - fixed sum insured
  - **Participating** - bonus (profit shares) is usually added as a percentage of the sum insured (*simple bonus*) or as a percentage of the sum insured plus previous bonuses (*compound bonus*)
- Term insurance
  - Sum insured paid on death within a specified term
  - Pays nothing on survival to death within a specified term
  - **Reducing term insurance** - to cover a housing loan with reducing balance
  - **Renewable term insurance policy** - increasing premium rates, guaranteed renewable

In case of life

- Life annuities
  - Lump sum (*consideration* or the *single premium*) paid to purchase the annuity
  - Income stream in return (may be indexed)
  - Variants: immediate; Deferred; Indexed; Joint
- Pure endowment
  - Sum insured benefit paid on survival to the end of the term of the policy
  - Earlier death - usually a return of premiums with interest

As a combination:

- Endowment assurance
  - An endowment assurance pays the sum insured death within a specified term or survival to maturity
  - Combines life insurance cover payable on death (*death cover*) with *savings* payable on maturity of the contract

11.1.2 Related Products

Disability

- Additional benefits - **disability benefits rider** or **Total and Permanent Disability (TPD) benefits**
- Separate **income protection policies** provide a percentage of the insured's income in the event of disability (here defined as inability to earn a salary - so loss of earnings) for a fixed period or to a fixed age

Critical Illness (also called Trauma)

- Critical illness policies pay a benefit if the insured suffers specifies illnesses or surgical procedures (heart. Attack, cancer)

Reverse mortgages

- Equity release product of the "asset rich, income poor" people
- Big unknown is whether there will be equity left at time of death of the borrower(s) - a guarantee may be involved (which is very complicated to calculate)

11.1.3 Lapse and Surrender

11.1.3.1 Definitions

Lapse:

- Policyholder does not respect their obligation (eg. Does not pay premium and there is no owed benefit to offset)

Surrender

- Policyholder decides to terminate the policy before maturity, and receives the *surrender value* of the policy
- Minimum surrender value required by life insurance law in many countries

These present distinct risks that need to be valued and monitored

11.1.3.2 Where does this surrender value come from?

Consider the sale of life insurance over a long period of time, with level premiums, and consider what happens at the start and at the end

- at the start, the insured life is young, and the (level) premium is more than enough to cover the likelihood of paying the benefit. In this case, the portion of premium that exceed the actual cost of coverage ($q_x \times Benefit$), is put aside in a reserve;
- at the end, the insured life is old, and the (level) premium is insufficient to cover the likelihood of paying the benefit. In this case, the shortfall of premium is funded from the reserve that accumulated in the early years of the contract.

  The principle of equivalence means that the reserve will, on average, accumulate and deplete exactly to a reserve of 0 at the end of the contract.

When the insured wants to leave the middle of that process, they have a claim to a portion of that reserve, because it is built on the excess premiums that they paid. The portion of that reserve that they receive is called the surrender value

11.1.3.3 Risk of Lapse and Surrender

Ideally one should make sure that policies cannot **lapse** with a negative value (by commission structure, minimum policy size, product design,...). Insurance legislation does not usually allow life insurance companies to treat negative liability values as an asset, because there is no guarantee that the policy holder will pay the later premiums so that the asset is not guaranteed

Surrender values are typically less than 100% of the reserve (because the insurance company is forfeiting some future profits it was entitled to) - this is of course regulated

11.1.4 Role of the Actuary in Life Insurance

11.1.4.1 Underwriting

Insurance policies are mostly sold at what are called **normal rates** or **standard rates**
Insurers subdivide lives into groups and charge premiums according to the grouping. Lives are often divided by:

- Age (when a policy was issued)
- Policyholder's sex

Further subdivision may take place by eg:

- Smoking or non-smoking
- Driving or not
- Marital status
- Health status

The purpose of underwriting is to assess whether an individual should be:

- Sold a policy at standard rates
- Sold a policy at higher than standard rates
- Or not sold a policy at all

Selection is an issue we touched on earlier, this is very important for underwriting. There are two types:

- Self-selection (eg. An individual buying a life annuity)
- Adverse selection
    - risks higher than expected (and this being unbeknownst of the insurer) take up the cover

The so-called "**Duty of disclosure**" (see eg. The ICA's explanation) is a powerful protection for the insurer, but often misunderstood by the insured

11.1.4.2 Pricing

This involves calculating the premium for a policy
The standard approach is through the 'principle of equivalence':

The Expected Present Value (EPV), at the time of issuing the policy, of all premium incomes must be equal to the EPVs of future benefits and expenses

In calculating premiums, actuaries must also set assumptions on interest

Types of expenses:

- Initial and renewable expenses:
  - Policy linked expenses include things like the cost of a medical examination, commission to agents for selling and policy, and the cost of regular correspondence with policyholders (eg. Annual statements about their policies)
- General expenses (overhead)
  - Business expenses have to be paid for - these include things like salaries, rent, investment expenses. Generally, it is a difficult task to allocate such expenses in precise manner to policies

11.1.4.3 Reserving

This is a statutory requirement in most countries. In Australia, a life insurance company has an **Appointed Actuary**

- An insurance company must hold **reserves**
- The reserve for a life insurance policy at time t (if the policy is still in force), denoted as ${}_tV$, is calculated as follows:

$$
\begin{aligned}
&{}_tV + \text{EPV at time } t \text{ of future premium income from the policy} \\
&= \text{EPV at time } t \text{ of future costs under this policy}
\end{aligned}
$$

Maintaining the solvency of an insurance company is a very important task for an actuary

11.1.4.4 Invest and Analysis of Surplus

Actuaries are typically involved in the investment function of insurance companies, so much so that it is a whole area of practice in itself => Analysis of Surpluses

Profit typically arises from three main reasons
When an insurer sets the premium for a policy, it is making assumptions about future levels of interest, mortality and expenses. It is unlikely that these assumptions will become reality exactly, and so the insurer make a profit or a loss

1. If the assumed interest rate is lower than the real interest rate, then the insurer will make a profit
2. If the expected number of deaths in a year is more than the actual number of deaths, then the insurer makes a profit (in a death benefit context)

3. If the assumed expenses in a year is more than the actual expenses occurred, the insurer makes a profit

Example:
Suppose an insurer sells one hundred one-year term insurance policies to individuals aged 30 with sum insured $100,000 for a pure premium of

$$\$952.38 = 100,000 v q_{30}$$

The following assumptions were made:

- Investment income : 5% per annum effective
- Mortality: it is expected that one of the policyholder will die, ie: $100 q_{30} = 1$

If the actual **experience** is as expected, at the end of the year the insurer's fund will be:

$$100 \times 952.38 \times 1.05 - 100,000 = 0$$

Consider the following deviations from the assumptions:

- Suppose that the insurer actually earns interest at **6% per annum effective** . Then the insurer's surplus is

$$100 \times 952.38 \times 1.06 - 100,000 = 952.38$$

And we would say that the profit has arisen through a favourable investment experience

- Similarly, if no policyholder dies and interest was earned at 5% per annum effective, the insurer would make a profit of $100,000

$$(100 \times 952.38 \times 1.05 - (0 \times 100,000) = 100,000$$

- If two policyholders died and interest was earned at 5% per annum effective, the insurer would have a loss of amount

$$100 \times 952.38 \times 1.05 - (2 \times 100,000) = -100,000$$

11.1.5 Interview: Edward Tan

Takeaways
Actuaries and Life Insurance

- Direct insurer: traditional actuarial work such as pricing and reporting
- With consulting, the role changes alot: actuarial **skill set** valued, for **any** job (also data analytics, mergers and acquisitions, risk management, claims assessment, industry surveys, climate change, government work,. . .)
- Also wealth management (and superannuation) space
- Secondment roles, any work to support the client (and where actuarial skill set is useful)

Typical employers

- Consulting and direct life insurers hire the build of life insurance graduates (but also super funds, risk management, but hire more experienced - not a lot of graduates)

Issues and Challenges

- Bad publicity for life insurance
- No focus on the benefits
- Technology: changes the landscape for clients

Advice

- What you learn at university gives useful context (understanding concepts in a simplified world before you can apply them in businesses context)
- The edge: good results necessary but not sufficient, relevant work experience, communication skills, computing skills

11.1.6 Further Insights

11.1.6.1 What factors impact mortality?

Actuary Magazine - Game on: Utilizing games to better understand policyholders in the life insurance realm

- Future expected mortality is very heterogeneous - see example with HK data
- Other factors than just age, gender and smoking status have predictive power
- Again, discussion of how more data will lead to more granular pricing due to adverse selection. This leads to "vulnerability" due to price segmentation

How is this data obtained?

- Purchase from vendors?
- Ask policyholders (eg. Genetic tests)?
- Lure them into providing information with games? (Think Facebook quizzes, Tiktok scrolling data, etc. . .)

Sounds unpalatable, but insurers argue this may lead to better health outcomes as insured learn about good behaviours

11.1.6.2 Ethical issues

AD - "Thinking about life insurance through a genetic lens"

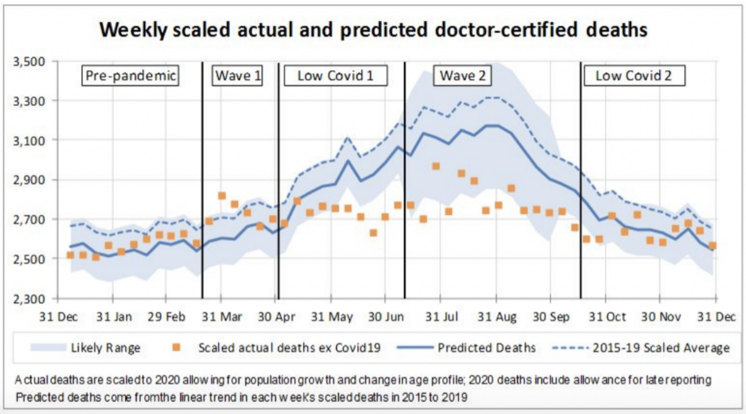
"Our ability to predict disease risk based on genetics is rapidly advancing. What does this mean for life insurance?"

- "Nature" (genetics) vs "Nurture" (environment and lifestyle) effects: likelihood of some diseases can be dramatically impacted by genetic profile (heritability)
- 'Polygenic risk score'
- Definitely more information
  - Becomes 'practical' (ratio of informative vs cost sky rockets)
  - Persistent
- Is it ok to require genetic tests?
  - If testing is conducted (and no necessarily disclosed): adverse selection, increased lapse
  - Tension between inclusivity and sustainability

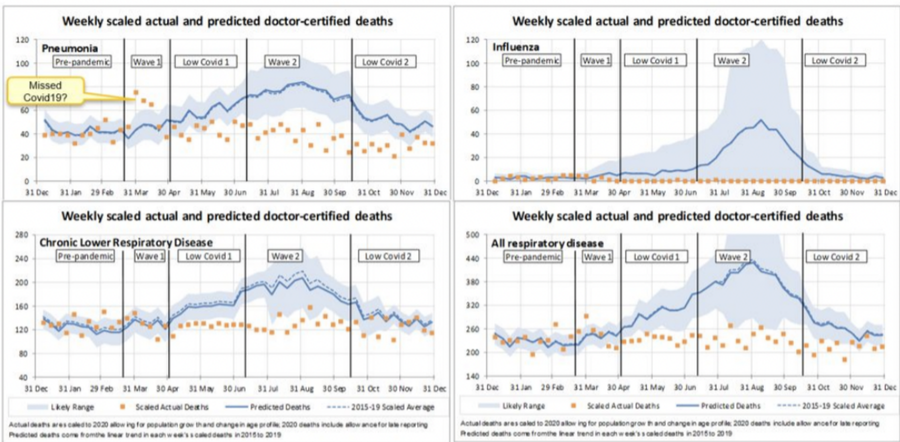
11.1.6.3 Impact of COVID 19

AD - impact of COVID-19 on Mortality and Morbidity in 2020

Overall there were around 3,900 (2.7%) fewer deaths in 2020 than predicted. Also, COVID measures almost totally cancelled the seasonal (winter) effects.



Drilling down into the data



11.2 Wealth Management

11.2.1 Importance for Insures

Asset management is a major issue for insurers

- There is a long **time gap** between receipt of premiums and payment of benefits (especially in life insurance, and in **long tail** general insurance lines, and in superannuation)
  - Premium money is invested in the meantime
  - Investment income is accounted for in the premium amount
- In some cases it is even a critical competitive factor (in life insurance in particle), almost becoming one of the core businesses of insurance
- That explain to some extent why banks sometimes do insurance as well(along with ready to use sales channels)
- It also explains why financial mathematics and economics is an important part of the training of actuaries

11.2.2 Asset-liability management

- One critical issues is to have assets that match well the type of liabilities that the company has (and which are best understood by actuaries)
- One has to balance risk and return, with matching liabilities well, all within the company's risk appetite
- Things to think about are duration, volatility, liquidity, correlation/diversification (with other assets, and between assets and liabilities)

Typical Asset Classes

- Cash
- Fixed interest (bonds) and inflation indexed
- Equities
- Property
- Derivatives

11.2.2.1 Asset Allocation

Strategic

- Defined by the board, at a high level
- Thought as an average to achieve in the long term
- An intention
- Communicates a risk appetite

Tactical

- Implemented by the fund manager
- An actual allocation
- Can change any minute
- Can differ from the strategic allocation, as long as it reverts back to it in the long run , and is true to its spirit

11.2.3 Investment Policy

Cash flow timing matters

- Investment policy determines the proportion of the total funds invested in the different asset classes such as shares, property, fixed interest and cash
- Investment policy also specifies the maturity term of any fixed interest investments and the currency of any investments
- Actuaries are concerned with the **matching** of asset cash flows and the liability expected cash flows
- Mismatching can cause insolvency or adverse profit results
- This has been a major problem with guarantees in life insurance products

Possible Solutions

- **Matching Investment Strategy -** When the cash flows on the assets from maturing investments and investment income is determined so that they occur at the same time and for the same amount as the expected future claims and expenses less premiums
- **Immunisation-** Matching averages (eg. Duration) rather than actual cash flows, an in theory investments are selected so that the change in the value of the assets for a small change of interest rates will equal the change in the value of the policy liability for the same small change in interest rates

12 Valuing Contingent Payments and Life Insurance

We now move on to consider the valuation of payments which are **not certain** - payments which have a probability of occurrence which is not necessarily 1

For example

- General insurance - claim size, number of claims per policy
- Life insurance - when will the policyholder die? How many premiums will be paid?
- Life annuity - how long the policyholder will survive?
- Coupon bonds - will the borrower default?

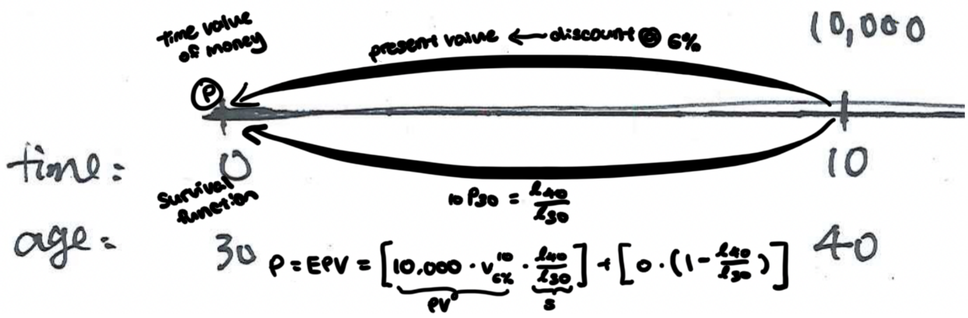
12.1 EPV of a single contingent payment

12.1.1 Contingent Payment

If a payment under an insurance policy, or any other financial transaction, depends on the occurrence of a certain event (eg. A person dying within a given period, or a person surviving a given period), we call such an event a **contingent event**
– The corresponding payment is referred to as a **contingent payments**

12.1.1.1 Illustration

In a pure endowment insurance, if the policy holder aged (30) survives 10 years, then the insurance benefit of \$10,000 payable at time 10. Assume an interest rate of 6% per annum effective
- Find an expression for the EPV of this survival benefit



$$EPV = [PV(10,000) \cdot Pr(\text{person aged 30 survives to 40})]$$
$$= 10,000 \cdot v^{10}_{6\%} \cdot \frac{l_{40}}{l_{30}}$$

12.1.2 Principle of Equivalence

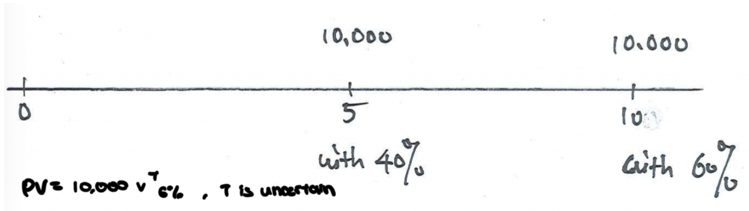
If a premium is calculated according to the **principle of equivalence**, then the expected present value of premium income equals the expected present value of the benefits under the policy

$$\text{EPV Premium} = \text{EPV Benefits}$$

The premium obtained using this principle is sometimes referred to as a **pure premium**. This is the pure expected value of the benefit, before nay adjustment is made (eg. Market adjustments, risk loading,. . .)

12.1.3 Case 1

Suppose that an insurance contract pays \$10,000 at two possible times in the future
In particular, there is a 40% chance that they payment will have to be made in 5 years' time and there is a 60% chance that the payment will have to be made in 10 years' time



Assuming an interest rate of 6% per annum effective, calculated the EPV of the required insurance benefit

Solution
The present value of the required payment is clearly uncertain. This present value, denoted PV, is a random variable that can take two distinct values depending on the timing of the \$10,00 payment

The values that the random variable PV can take are

$$10,000 \times 1.06^{-5} \text{ with probability } 0.4$$

$$10,000 \times 1.06^{-10} \text{ with probability } 0.6$$

The expected value of the PV is therefore

$$EPV = E[PV] = 0.4 \cdot [10,000 \times 1.06^{-5}] + 0.6 \cdot [10,000 \times 1.06^{-10}] = \$6,339.40$$

The EPV is the **pure premium** for this insurance contract

12.1.3.1 Analysis of Case 1

Suppose that the insurer sets aside the EPV = \$6,339.40 (pure premium income) to fund the future payment.

At 6% per annum effective, the accumulation of the EPV over 5 years is

$$6,339.40(1.06^5) = 8,4883.55$$

And the accumulation of the EPV over 10 years is

$$6,339.40(1.06^{10}) = 11,352.90$$

Thus, if the payment of \$10,000 is due is 10 years, we would have sufficient funds to make the payments, whereas if it were due in 5 years, we would not

Is this an issue?

Suppose now that there were 1,000 such contracts (all mutually independent)

- For each contract the probability that the payment is due in 5 years' time is 0.4, and the probability that the payment is due in 10 years' time is 0.6

- The total pure premium income for the 1000 contracts is

$$6,339.40 \times 1000 = 400 \times 10,000 \times 1.06^{-5} + 600 \times 10,000 \times 1.06^{-10}$$

- Among those 1,000 contracts, we expect 40% of the contracts to require a payment of $10,000 at time 5 years, and 60% to require a payment at time 10 years

After 5 years we expect to have

$$[1000 \times 6339.40 \times 1.06^5] - [400 \times 10,000] = 4,483,547.20$$

and After 10 years we expect to have

$$([1000 \times 6339.40 \times 1.06^5] - [400 \times 10,000]) \times 1.06^5 - (600 \times 10,000) = 0$$

Note
– Of course the 40% and 60% are probabilities, and the actual *realised* proportions may no be exactly the same. This is why some margin may have to be added to the pure premium, to guarantee all payments to a certain level of confidence

– If one focused exclusively on this randomness, how much would the insurance company need to be 100% certain of having enough money?

Worse case scenario: 1000 at time 5

$$P : 10,000v^5 = 10,000 \cdot \frac{1}{1.06}^5 = 7,472.58$$

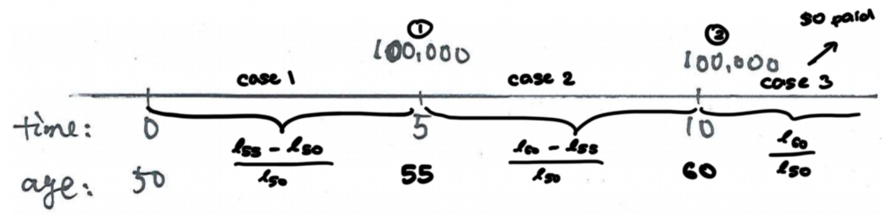
Additionally, typically, the interest rate would also be uncertain, and there would be additional uncertainties to take into account as well

12.1.4 Case 2

Consider an insurance policy which is sold to a female currently aged 50. The benefits under this insurance policy are:

- If she dies within five years, her estate is paid $100,000 after five years
- If she dies no earlier than five year but no later than ten years, then her estate is paid $100,00 after ten years
- Nothing is paid to her estate (or her) if she survives to age 60
- The effective rate of interest per annum is 6%. Assume the mortality of the female follows the Female Mortality Table in the tutorial book

Calculate the EPV of the benefits paid under this insurance policy



<u>Solution</u> Calculate present values as a random variable

We create a table of present values and their associated probability as follows

Present Value	Probability	
$100{,}000 \times 1.06^{-5}$	$(l_{50} - l_{55})/l_{50}$	$_5q_{50}$
$100{,}000 \times 1.06^{-10}$	$(l_{55} - l_{60})/l_{50}$	$_{5\|5}q_{50}$
0	l_{60}/l_{50}	$_{10}p_{50}$

$\Sigma = 1$

Thus the EPV is

$$\begin{aligned}
&100,000 \times 1.06^{-5} \times \frac{94,692 - 92,349}{94,692} \\
&+100,000 \times 1.06^{-10} \times \frac{92,3439 - 88,703}{94,692} \\
&= \$3,999
\end{aligned}$$

12.1.5 Tutorial 10.1.1.2

The probability that a life aged 20 survives for t years is 0.998^t for $0 \leq t \leq 20$. Using an interest rate of 6% per annum effective, calculate:
– The EPV of a payment of \$100,000 at the end of the year of (20)'s death should death occur before age 40

Present Value of the Payment

$$PV(t) = 100,000v^t$$

Probability of Payment being paid at time t

$$\text{Payment at } t \Rightarrow \text{(20) dies between } t \text{ and } t-1$$

$$\begin{aligned}
P_t &= {}_{t-1}p_{20} - {}_tp_{20} \\
&= 0.998^{t-1} - 0.998^t \\
&= (1 - 0.998)0.998^{t-1} = 0.002(0.998^{t-1})
\end{aligned}$$

Expected Present Value

$$\begin{aligned}
EPV &= \sum_{t=1}^{20} 100,000v^t \cdot 0.002(0.998^{t-1}) \\
&= \frac{200}{0.998} \sum_{t=1}^{20} v^t \cdot 0.998^t
\end{aligned}$$

IMPORTANT IDENTITY - From week 1

$$\sum_{i=1}^{n} x^i = x\frac{1-x^n}{1-x}$$

contd.

$$\begin{aligned}
EPV &= \frac{200}{0.998} \cdot 0.998v\frac{1-(0.998v)^{20}}{1-0.998v}, \; i = 0.06 \\
&= 200 \cdot v \cdot \frac{1-(0.998v)^{20}}{1-0.998v} \\
&= \$2259.46
\end{aligned}$$

12.2 EPV of a series of contingent payments

Previously we considered the calculation of the EPV of a single payment made (perhaps) at some (possibly random) time in the future. We now consider payments which are made if and only if a condition is realised at particular times.

It is clear that this could be considered as "being alive" and /or "being dead" leading to the context of life insurance products, but the formulas are general and apply to multiple possible (contingent) payments

We thus generalise results to multiple possible (continent) payments

12.2.1 A Life Insurance Example

- Consider a life aged 30
- Consider also a series of payments made at times 1,2 and 3
- These payments are made only if the life currently aged 30 is alive at those times

Determine an expression for the EPV of the payments made under this agreement
– Note this is a term life annuity which we will denote

$$a_{30:\overline{3}|}$$

(Annuity of a person aged 30 for 3 years)

Considering all possible outcomes (either/or) of present values PV (a random variable)

Event	Present Value (PV)	Probability
(30) dies during period [0,1)	0	q_{30}
(30) dies during period [1,2)	$a_{\overline{1}|}$	$p_{30} - {}_2p_{30}$
(30) dies during period [2,3)	$a_{\overline{2}|}$	${}_2p_{30} - {}_3p_{30}$
(30) survives to time 3	$a_{\overline{3}|}$	${}_3p_{30}$

Hence the exception of the PV, the "EPV" of the series of payments is

$$EPV = a_{\overline{1}|}(p_{30} - {}_2p_{30}) + a_{\overline{2}|}({}_2p_{30} - {}_3p_{30}) + a_{\overline{3}|} \cdot {}_3p_{30}$$

$$= v(p_{30} - {}_2p_{30}) + (v + v^2)({}_2p_{30} - {}_3p_{30}) + (v + v^2 + v^3){}_3p_{30}$$

$$= vp_{30} + v^2{}_2p_{30} + v^3{}_3p_{30}$$

Note:

- The EPV of three contingent payment is the sum of the EPV of each of the three individual contingent payments
- This is crucial, because this gives us two different ways of calculating the EPV
    1. Calculate a weighted average of the PV's, with weights being the associated probabilities. This is the *first line* in the equation above.
    2. Calculate a weighted average of the present value of each individual cash flows, weighted by their associated probabilities of occurrence. This is the *last line* in the equation above

12.2.2 Preliminary: changing the order of summation

Because of the alternative ways of calculating the EPV as discussed above, we may need to swap summations to move from one case to the other

Consider

$$\sum_{k=1}^{n}\sum_{j=1}^{k} a_{k,j}$$

This can be written as

$$\begin{aligned}
&a_{1,1}+\\
&a_{2,1}+a_{2,2}+\\
&a_{3,1}+a_{3,2}+a_{3,3}+\\
&...\\
&a_{n,1}+a_{n,2}+a_{n,3}+...+a_{n,n}
\end{aligned}$$

The sum of the first column is $\sum_{k=1}^{n} a_{k,1}$, the sum of the second column is $\sum_{k=2}^{n} a_{k,2}$, and so on until the sum of the last column is $\sum_{k=n}^{n} a_{k,n} = a_{n,n}$

Thus

$$\sum_{k=1}^{n}\sum_{j=1}^{k} a_{k,j} = \sum_{k=1}^{n} a_{k,1} + \sum_{k=2}^{n} a_{k,2} + \sum_{k=n}^{n} a_{k,n} = \sum_{j=1}^{n}\sum_{k=j}^{n} a_{k,j}$$

12.2.3 The EPV of a series of payments - general case

We now consider the more general case of the above example

We noted that *the EPV of a series of payments is equal to the sum of the EPVs of each of the payments in the series*

We will assume the following

- Suppose we have a series of payments made at times $1, 2, 3, \dots, n$
- These payments are denoted $X_1, X_2, \dots, X_n$
- The present value of the first k payments is denoted V_k such that

$$V_k = \sum_{j=1}^{k} v^j X_j$$

- The payment at time t can be made only if all previous payments have been made. In other word, what is random is when the payment will stop (the final k in the expression above)

Of course these are quite restrictive assumptions (especially the last one) but we focus on this for now as it simplifies calculation and it ties in well with the life insurance applications

12.2.3.1 Necessary Probabilities

- Let p_k be the probability that **at least** k payments are made
- We define π_k to be the probability that exactly k payments are made
- Then we have the following relationship between $\{p_k\}_{k=1}^{n}$ and $\{\pi_k\}_{k=1}^{n}$

$$\pi_1 = p_1 - p_2$$
$$\pi_2 = p_2 - p_3$$
$$\dots$$
$$\pi_k = p_k - p_{k+1}$$
$$\dots$$
$$\pi_{n-1} = p_{n-1} - P_n$$
$$\pi_n = p_n$$

The notation of p used here does not clash as it also has implications regarding probabilities of survival

12.2.3.2 The EPV of a series of payment contd

Then we have

$$p_k = \sum_{j=k}^{n} \pi_j$$

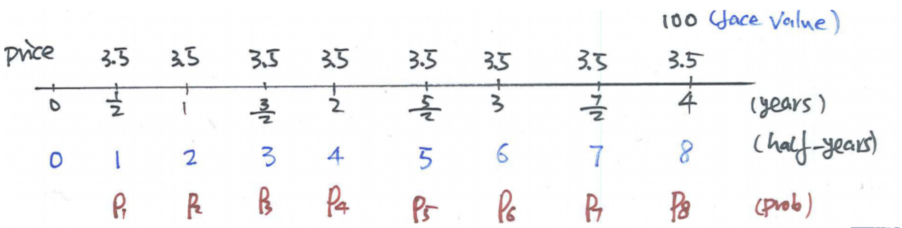
The EPV of the series of n payments is

$$\sum_{k=1}^{n} \pi_k V_k = \sum_{k=1}^{n} \pi_k \sum_{j=1}^{k} v^j X_j = \sum_{k=1}^{n} \sum_{j=1}^{k} \pi_k v^j X_j$$
$$= \sum_{j=1}^{n} \sum_{k=j}^{n} \pi_k v^j X_j$$
$$= (\sum_{j=1}^{n} v^j X_j)(\sum_{k=j}^{n} \pi_k) = (\sum_{j=1}^{n} v^j X_j) p_j$$

This proves that the EPV of a series of payments can here be calculated as the sum of the EPVs of each payment in the series

12.2.4 Example: Fixed Coupon Bond Subject to Default

- Suppose that corporation issues a bond to the public
- The bond's face value is \$100
- The bond has 4 years until maturity and will pay coupons of 7% per annum, payable half yearly
- Find the price of the bond assuming a yield of 8% per annum convertible half yearly
- Assume that the probability that the corporation will be able to make the payment in t half years is $p_t = 0.99^t$



The price of this defaultable bond is the EPV of the <u>coupon payments</u> and the <u>face value</u>

$$Price = \sum_{t=1}^{8}(3.5 \times 1.04^{-t} \times 0.99^{t}) + (100 \times 1.04^{-8} \times 0.99^{8})$$

$$= 3.5\sum_{t=1}^{8}(\frac{0.99}{1.04})^{t} + (100 \times \frac{0.99}{1.04}^{8}), \quad \frac{0.99}{1.04} = v = \frac{1}{1+j}$$

$$= 3.5\sum_{t=1}^{8}(\frac{1}{1+j})^{t} + (100 \times \frac{1}{1+j}^{8})$$

$$= 3.5a_{\overline{8}|j=0.0505} + (100 \times \frac{1}{1+j}^{8})$$

$$= 3.5 \times 6.450 + 67.4242 = 90$$

12.2.5 Tutorial 10.1.1.3

The probability that a life aged 20 survives for t years is 0.998^t for $0 \leq t \leq 20$. Using an interest rate of 6% per annum effective, calculate:
– the EPV of payments of \$10,000 annually in arrear for 20 years provided (20) is alive

Present Value of Payments:

$$PV(t) = 10,000v^{t}$$

Probability of payment being paid at time t

$$P_t = 0.998^{t}$$

Expected Present Value

$$EPV = \sum_{t=1}^{20} 10,000v^{t} \cdot 0.998^{t}$$

$$= 10,000\sum_{t=1}^{20}(0.998v)^{t}$$

$$= 10,000[0.998v + (0.998v)^{2} + ... + (0.998v)^{20}]$$

$$= 10,000 \cdot 0.998v \cdot \frac{1-(0.998v)^{20}}{1-0.998v}, \ i = 0.06$$

$$= \$112747.17$$

12.3 Life Insurance Premium Calculations

12.3.1 Introductory Example

Recall the insurance policy we considered before. From that exercise we had

- A sum insured of \$100,000
- The life insured is a 50 year old female
- There is payment:
  - After 5 years if (50) dies in the first five years (before 55)
  - After 10 years if (50) dies between times five years and ten years (between 55 and 60)
  - No payment otherwise
- Effective rate of interest was 6% per annum
- EPV of death benefit = \$3,999

12.3.1.1 Annual premiums and the equation of value

Consider now the case where premiums for this insurance contract are paid annually in advance while (50) is alive but for a maximum of three years
– What is the fair annual premium P?

To solve for annual premium P, we use the "equation of value"

$$\text{EPV of the premiums} = \text{EPV of the death benefit}$$

thus

$$3,999 = P(1 + vp_{50} + v^2{}_2p_{50})$$

with $p_{50} = l_{51}/l_{50} = 0.99590$ and ${}_2p_{50} = l_{52}/l_{50} = 0.99141$
Finally,

$$3,999 = 2.82188P$$

giving $P = \$1,417$ to the nearest dollar

12.3.2 General Assumptions/Notation

- We generally assume that the death benefit is payable at the end of the year of death
- We generally assume that the annual premium is payable in advance
- Let S be the sum insured
- Premiums are calculated using the principle of equivalence

12.3.3 EPV of Life Insurance Contracts

12.3.3.1 Whole Life Insurance

- Insurance contract that will pay an amount of money (the sum insured) on the death, whenever that might be, of a life currently aged x
- EPV of such a contract with sum insured \$1 is denoted

$$A_x$$

12.3.3.2 Term Life Insurance

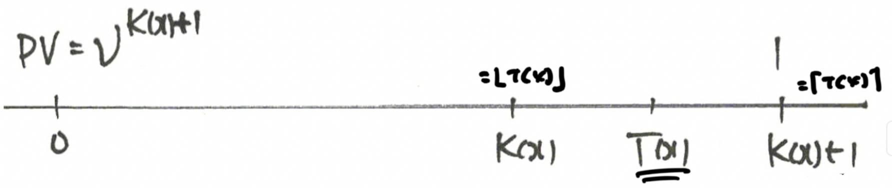
- Insurance contract that will pay an amount of money (the sum insured) on the death, provided this occurs before n years, of a life currently aged x
- EPV of such a contract with sum insured \$1 is denoted

$$A^{1}_{x:\overline{n}|}$$

(Payment of "1" is made within the x years)

12.3.3.3 EPV of A_x

A payment of \$1 is made at the end of the year of death



Recall:

$$\begin{aligned}
Pr[K(x)=t] = Pr[t \leq T(x) < t+1] &= {}_tp_x - {}_{t+1}p_x \\
&= {}_tp_x - [{}_tp_x \times {}_1p_{x+t}] \\
&= {}_tp_x[1 - p_{x+t}] \\
&= {}_tp_x \times q_{x+t} \\
&Or \\
&= \frac{l_{x+t}}{l_x} - \frac{l_{x+t+1}}{l_x} = \frac{d_{x+t}}{l_x}
\end{aligned}$$

The PV of the death benefit is v^{K_x+1} which follows the distribution

PV	v	v^2 - - - -	v^{t+1} - - - -
prob	$Pr(K(x)=0) = \frac{d_x}{l_x}$	$Pr(K(x)=1) = \frac{d_{x+1}}{l_x}$	$Pr(K(x)=t) = \frac{d_{x+t}}{l_x}$

The expected present value (EPV) of the death benefit then becomes

$$A_x = \sum_{t=0}^{\infty} v^{t+1} \frac{d_{x+t}}{l_x}$$

This can be easily calculated using a spreadsheet. For example, to compute A_{60} when $i = 0.05$ from the set of values $l_{60}, l_{61}, l_{62}, \dots$, we would set up columns like

t	l_{60+t}	v^{t+1}	d_{60+t}/l_{60}	Product
0	79,062	0.9524	0.0209	00199
1	77,411	0.9070	0.0224	0.0204
2	75,637	0.8638	0.0241	0.0208
...	...	...	...	...
$w-60$	l_w	v^{w-60+1}	d_w/l_{60}	

Where ω is the final age of a life table
The sum of the terms in the 'Product' column gives the value of A_{60}

12.3.3.4 EPV of $A^1_{x:\overline{n}|}$

For a term life insurance, simply stop the sum after n years rather than running the sum all the way to ω

Also

$$A^1_{x:\overline{n}|} = A_x - \frac{l_{x+n}}{l_x} v^n A_{x+n}$$

This is analogous to the formula

$$a_{\overline{n}|} = a_\infty - v^n a_\infty$$

but taking probability of death (contingencies) on top

Another method would be to take the sum of EPV of payments at each t for $n-1$ years

$$A^1_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^{t+1} \frac{d_{x+t}}{l_x}$$

12.3.3.5 Tutorial 11.1.4

Calculate A_{30}, the expected present value of 1 payable at the end of the year of death of a life now aged 30 assuming a rate of interest of 6% per annum effective and a force of mortality of 0.002 per annum for ages up to 50 and 0.03 for ages above 50. Show your answer correct to five decimal places.

$$A_{30} = \sum_{t=0}^{\infty} v^{t+1} \frac{d_{30+t}}{l_{30}} = \sum_{t=0}^{\infty} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t}$$

$$\begin{cases} \mu_x = 0.002 & x \le 50 \\ \mu_x = 0.03 & x > 50 \end{cases} = \begin{cases} \mu_{s+30} = 0.002 & s \le 20 \\ \mu_{s+30} = 0.03 & s > 20 \end{cases}$$

When $0 < t \le 20$

$$\begin{aligned} {}_tp_{30} &= exp\{-\int_0^t 0.002 \cdot ds\} \\ &= exp\{-0.002t\} \end{aligned}$$

When $t > 20$

$$ {}_tp_{30} = exp\{-\int_0^t \mu_{30+s} ds\} $$

$$ = exp\{-\int_0^{20} 0.002 \cdot ds - \int_{20}^t 0.03 \cdot ds\} $$

$$ = exp\{-0.04 - 0.03(t-20)\} $$

$$ = exp\{0.56 - 0.03t\} $$

For q_{30+t}

$$ q_{30+t} = 1 - p_{30+t} = \begin{cases} 1 - exp\{-\int_0^1 0.002ds\} & t < 20 \\ 1 - exp\{-\int_0^1 0.03ds\} & t \geq 20 \end{cases} $$

$$ = \begin{cases} 1 - exp\{-0.002\} & t < 20 \\ 1 - exp\{-0.03\} & t \geq 20 \end{cases} $$

So,

$$ A_{30} = \sum_{t=0}^{\infty} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t} = \sum_{t=0}^{19} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t} + \sum_{t=20}^{\infty} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t} $$

Firstly

$$ \sum_{t=0}^{19} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t} = \sum_{t=0}^{19} v^{t+1} \cdot e^{-0.002t} \cdot (1 - e^{-0.002}) $$

$$ = v(1 - e^{-0.002}) \sum_{t=0}^{19} (ve^{-0.002})^t $$

$$ = v(1 - e^{-0.002}) \frac{1 - (e^{-0.002}v)^{20}}{1 - e^{-0.002}v}, \text{ where } v = (1.06)^{-1} $$

$$ = 0.0225724 $$

Second

$$ \sum_{t=20}^{\infty} v^{t+1} \cdot {}_tp_{30} \cdot q_{30+t} = \sum_{t=20}^{\infty} v^{t+1} \cdot e^{0.56-0.03t} \cdot (1 - e^{-0.03}) $$

$$ = v(1 - e^{-0.03}) e^{0.56} \sum_{t=20}^{\infty} (ve^{-0.03})^t $$

$$ = v(1 - e^{-0.03}) e^{0.56} [\sum_{t=0}^{\infty} (ve^{-0.03})^t - \sum_{t=0}^{19} (ve^{-0.03})^t] $$

$$ = v(1 - e^{-0.03}) e^{0.56} [\frac{1}{1 - ve^{-0.03}} - \frac{1 - (ve^{-0.03})^{20}}{1 - ve^{-0.03}}] $$

$$ = (1 - e^{-0.03}) \frac{v^{21} e^{-0.04}}{1 - ve^{-0.03}} \text{ where } v = (1.06)^{-1} $$

$$ = 0.098866 $$

Therefore $A_{30} = 0.12144$

12.3.4 EPV of Endowments

12.3.4.1 Pure Endowment

- Insurance contract that will pay an amount of money (the sum insured) after n years, provided its insured life, currently aged x, has survived those n years
- EPV of such a contract with sum insured \$1 is denoted

$$A_{x:\overline{n}|}^{\;\;1}$$

(Payment of "1" is made at n)

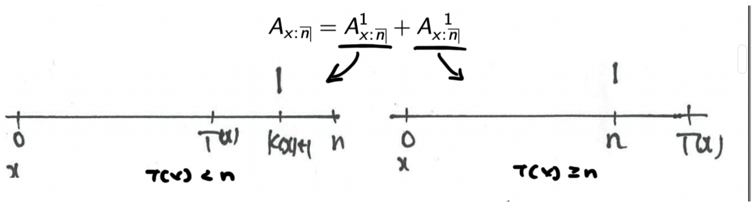
12.3.4.2 Endowment

- Insurance contract that will pay an amount of money (the sum insured) on the death of al life currently aged x if this happens before n years, or in exactly n years if the insured has survived that time
- EPV of such a contract with sum insured \$1 is denoted

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

12.3.4.3 EPV of $A_{x:\overline{n}|}^{\;\;1}$ and $A_{x:\overline{n}|}$

$A_{x:\overline{n}|}$ is the EPV of a benefit of 1 payable at the end of the year of death of a life now aged x, if this occurs within n years, or at time n if (x) survives for n years



The EPV of the benefit of a pure endowment is simply the RHS of the EPV of classical endowment which is

$$A_{x:\overline{n}|}^{\;\;1} = v^n {}_np_x$$

Hence, to calculate $A_{x:\overline{n}|}$, we use

$$\begin{aligned} A_{x:\overline{n}|} &= A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1} \\ &= [\sum_{t=0}^{n-1} v^{t+1} \frac{d_{x+t}}{l_x}] + [v^n {}_np_x] \end{aligned}$$

PV of the death benefit	v	v^2	....	v^{t+1}	....	v^n
prob	$\frac{d_x}{l_x}$	$\frac{d_{x+1}}{l_x}$		$\frac{d_{x+t}}{l_x}$		$\frac{d_{x+n-1}}{l_x}$

PV of the survival benefit	v^n	0
prob	${}_np_x = \frac{l_{x+n}}{l_x}$	${}_nq_x$

Note:

- if $n = 1$, then $A^1_{x:\overline{n}|} = vq_x$ and $A_{x:\overline{1}|} = v(q_x + p_x) = v$ (payment at time 1 is certain)
- if n goes to ∞, then

$$\lim_{n\to\infty} A^1_{x:\overline{n}|} = A_x, \; \lim_{n\to\infty} A_{x:\overline{n}|}^{\;\;1} = 0$$
$$\Rightarrow \lim_{n\to\infty} A_{x:\overline{n}|} = A_x$$

12.3.5 EPV of Life Annuities

12.3.5.1 Life Annuity

- Insurance contract that will pay an amount to money (the sum insured) every year for as long as the insured, currently aged x, survives
- If payments are made at the end of each year (that is, payments start in 1 year) is denoted

$$a_x$$

- If payments are made at the beginning each year (that is, payments start in 1 year), this is denoted

$$\ddot{a}_x$$

- These notations are for payments of \$1 p.a, as usual

Note that the subscript uses x (age) instead of n. This is because we do not the term of the product
$\Rightarrow$ That's why there's no angle symbol

12.3.5.2 Term life annuity

- Insurance contract that will pay an amount of money (the sum insured) every year for as long as the insured, currently aged x, survived, but no longer than n years
- If payments are made at the end of each year (that is, payments start in 1 years), this is denoted

$$a_{x:\overline{n}|}$$

- If payment are made at the beginning of each year (that is, payments start immediately), this is denoted

$$\ddot{a}_{x:\overline{n}|}$$

- These notations are for payments of \$1 p.a, as usual

12.3.5.3 EPV of Life Annuities

Using the result that the EPV of a series of payments is the sum of the EPVs of each payment in the series, we have

$$\ddot{a}_x = \sum_{t=0}^{\infty} v^t {}_tp_x$$

to calculate $\ddot{a}_{x:\overline{n}|}$. we use

$$\ddot{a}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t {}_tp_x$$

Such calculations can be easily be performed using a spreadsheet
or example, to compute $\ddot{a}_{60}$ at $i = 0.05$ from the set of values $l_{60}, l_{61}, l_{62}, ...$, we would set up columns like:

t	l_{60+t}	v^t	${}_tp_{60}$	Product
0	79,062	1.0000	1	1
1	77,411	0.9524	0.97911	0.93249
2	75,637	0.9070	0.95668	0.86773
...	...	...	...	...
$w-60$	l_w	v^{w-60}	${}_{w-60}p_{60}$	
				$\ddot{a}_{60}$ =Sum

Where ω is the final age of a life table. in ALT 2000-2002, $\omega = 110$
As for the life insurance table, the desired result ($\ddot{a}_{60}$) is the sum of the values in the "Product" column

12.3.5.4 Tutorial 11.1.1

Calculate $\ddot{a}_{30}$, the expected present value of a payment of 1 per annum at the beginning of every year while a life now aged 30 is alive using a force of interest of 6% per annum and assuming a force of mortality of 0.005 per annum for all ages.

$$\ddot{a}_{30} = \sum_{t=0}^{\infty} v^t \cdot {}_tp_{30}$$

$$= \sum_{t=0}^{\infty} v^t \frac{s(t+30)}{s(30)}$$

$$= \sum_{t=0}^{\infty} e^{-0.06t} \frac{exp\{-\int_0^{t+30} 0.005dt\}}{exp\{-\int_0^{30} 0.005dt\}}$$

$$= \sum_{t=0}^{\infty} e^{-0.06t} e^{-0.005t} = \sum_{t=0}^{\infty} e^{-0.065t}$$

IMPORTANT IDENTITY - From week 1

$$\sum_{i=0}^{\infty} x^i = \lim_{n\to\infty} \sum_{i=0}^{n} x^i = \frac{1}{1-x}$$

contd.

$$\sum_{t=0}^{\infty} e^{-0.065t} = \frac{1}{e^{-0.065}}$$

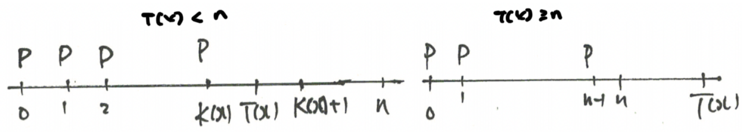
$$= 15.8900$$

12.3.6 Calculating Insurance Premiums

12.3.6.1 Typical Assumptions

Typically, premiums are paid

- In advance
- For lie or up to n years
- In the latter case, we can distinguish two scenarios



Hence, the EPV of premiums needs to be calculated with the help of life annuities, as the number of payments is uncertain and depends on survival of the insured.

12.3.6.2 Principle of Equivalence

Recall that the principle of equivalence requires

$$\text{EPV Premiums} = \text{EPV Benefits}$$

The EPV of premiums will be

- $P\ddot{a}_{x:\overline{n}|}$ in case of n year payment period
- $P\ddot{a}_x$ potentially, but only in the case of whole life payments (people would not pay premiums beyond their coverage period)

The EPV of benefit will be simply

$$S \cdot A_x \text{ in case of whole life insurance}$$
$$S \cdot A^1_{x:\overline{n}|} \text{ in case of term life insurance}$$
$$S \cdot A_{x:\overline{n}|}^{\ \ 1} \text{ in case of pure endowment}$$
$$S \cdot A_{x:\overline{n}|} \text{ in case of endowment}$$

12.3.6.3 Endowment Payments example

An endowment insurance with a term of 3 years if sold to a life currently aged 45
The sum insured is \$10,000. Premiums are payable annually in advance while the life is alive for a maximum of three years
Assume an interest rate of 6% per annum effective in your calculations
– Using the mortality rates from the Male Mortality Table in Atkinson and Dickson (2011), calculate the actuarially fair annual premium

Use the principle of equivalence to get

$$P\ddot{a}_{45:\overline{3}|} = 10,000A_{45:\overline{3}|}$$

We have

$$\ddot{a}_{45:\overline{3}|} = 1 + vp_{45} + v^2{}_2p_{45} = 2.8191$$

and

$$A_{45:\overline{3}|} = \frac{1}{l_{45}}(vd_{45}v^2d_{46} + v^3d_{47} + v^3l_{48}) = 0.840428$$

substituting, we finally obtain

$$2.8191 \cdot P = 8,404.28$$

$$\Rightarrow P = 2,981.19$$

12.4 Relationships between actuarial functions

Recall that we were able to derive identities and relationships in the context of Time Value of Money

For instance, we showed that

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

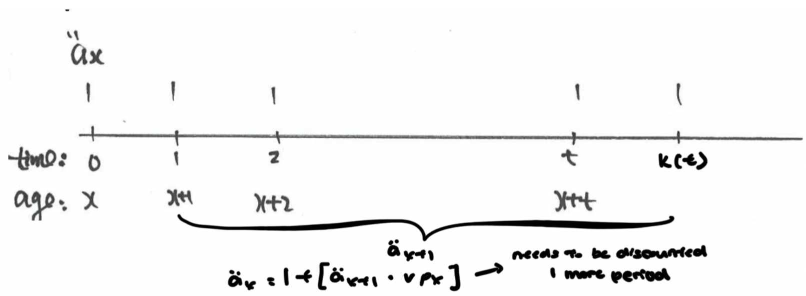
Can we derive similar relationships in the context of life annuities?

12.4.1 Recursive formula for $\ddot{a}_x$

We have:

$$\ddot{a}_x = 1 + v \cdot p_x \ddot{a}_{x+1}$$

Interpretation



12.4.1.1 Proof

Starting with the LHS, we have

$$\ddot{a}_x = \sum_{t=0}^{\infty} v^t {}_tp_x = 1 + \sum_{t=1}^{\infty} v^t {}_tp_x$$

$$= 1 + \sum_{r=0}^{\infty} v^{r+1} \cdot {}_{r+1}p_x$$

$$= 1 + v \sum_{r=0}^{\infty} v^r \cdot {}_1p_x \cdot {}_rp_{x+1}$$

$$= 1 + v \cdot p_x \sum_{r=0}^{\infty} v^r \cdot {}_rp_{x+1}$$

$$= 1 + v \cdot p_x \cdot \ddot{a}_{x+1}$$

12.4.2 Relationship between $A_{x:\overline{n}|}$ and $\ddot{a}_{x:\overline{n}|}$

One can show that the following two identities hold

$$A_{x:\overline{n}|} + (1 - v)\ddot{a}_{x:\overline{n}|} = 1$$

$$A_x + (1 - v)\ddot{a}_x = 1$$

12.4.2.1 Proof

We have

$$A_{x:\overline{n}|} + (1-v)\ddot{a}_{x:\overline{n}|}$$

$$\begin{aligned}
&= [\sum_{t=0}^{n-1} v^{t+1}\frac{d_{x+t}}{l_x} + v^n\frac{l_{x+n}}{l_x}] + [(1-v)\sum_{t=0}^{n-1} v^t \cdot {}_tp_x] \\
&= [\sum_{t=0}^{n-1} v^{t+1}({}_tp_x - {}_{t+1}p_x) + v^n{}_np_x] + [\sum_{t=0}^{n-1} v^t \cdot {}_tp_x - \sum_{t=0}^{n-1} v^{t+1} \cdot {}_tp_x] \\
&= -\sum_{t=1}^{n} v^t \cdot {}_tp_x + \sum_{t=0}^{n} v^t \cdot {}_tp_x = 1
\end{aligned}$$

The second case (involving A_x) is shown in the same way

12.4.2.2 Alternative Proof using perpetuities

First we introduce the effective rate of discount interest d p.a. Note that

$$\ddot{a}_{\overline{t}|} = (1+i)a_{\overline{t}|} = (1+i)\frac{1-v^t}{i} = \frac{1-v^t}{d}$$

where

$$d = \frac{i}{1+i} = 1-v$$

is the discounted amount of i to time 0. in particular

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d} \;\; [= (1+i)a_{\overline{\infty}|}]$$

hence it makes sense that

$$\ddot{a}_x = \frac{1-A_x}{d}$$

This generalises to other cases

12.4.2.3 Corollary

From

$$P \cdot \ddot{a}_{x:\overline{n}|} = S \cdot A_{x:\overline{n}|}$$

we get

$$P = \frac{SA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

The equation

$$A_{x:\overline{n}|} + (1-v)\ddot{a}_{x:\overline{n}|} = 1$$

thus yields

$$P = \frac{S(1-(1-v)\ddot{a}_{x:\overline{n}|})}{\ddot{a}_{x:\overline{n}|}} = S[\frac{1}{\ddot{a}_{x:\overline{n}|}} - (1-v)]$$

$$S(\frac{1}{\frac{1-A_{x:\overline{n}|}}{(1-v)}} - \frac{(1-v)\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}) = S(1-v)[\frac{1-\ddot{a}_{x:\overline{n}|}(1-v)}{1-A_{x:\overline{n}|}}]$$

$$= (1-v)\frac{SA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}} = d\frac{SA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}}$$

12.4.3 Illustration

You are given that $A_{x:\overline{n|}} + (1-v)\ddot{a}_{x:\overline{n|}} = 1$
Suppose that $l_{40+t} = 95,000 - 300t$ for $t = 0, 1, 2..., 10$
- Compute the annual premium, P, for a 10 year endowment insurance for a life aged exactly 40 with sum insured \$100,000 payable at the end of the year of death or at maturity using an effective rate of interest 6% per annum

Solution - Brute Force Approach (Solving for the value of Payments)
Calculate $\ddot{a}_{40:\overline{10|}}$ first as

$$\ddot{a}_{40:\overline{10|}} = \sum_{t=0}^{9} v^t {}_tp_{40} = \sum_{t=0}^{9} v^t \frac{l_{40+t}}{l_{40}}$$

$$= \sum_{t=0}^{9} v^t \frac{95,000 - 300t}{95,000}$$

$$= \sum_{t=0}^{9} v^t (1 - \frac{3}{950}t)$$

$$= \ddot{a}_{\overline{10|}} - \frac{3}{950} \sum_{t=0}^{9} t \cdot v^t$$

Now let

$$S = \sum_{t=0}^{9} t \cdot v^t = v + 2v^2 + 3v^3 + ... + 9v^9$$

So that

$$(1+i)S = 1 + 2v + 3v^2 + 4v^3 + ... + 9v^8$$

giving

$$i(S) = 1 + v + v^2 + v^3 + ... + v^8 - 9v^9$$

as such

$$S = \frac{\ddot{a}_{\overline{9|}} - 9v^9}{i} = 31.37846$$

Hence $\ddot{a}_{40:\overline{10|}} = 7.70260$, and we can calculate P

Solution - Alternative Method (Solving for the value of Benefits)
We have $P\ddot{a}_{40:\overline{10|}} = 100,000 A_{40:\overline{10|}}$ where

$$A_{40:\overline{10|}} = \sum_{t=0}^{9} v^{t+1} \frac{d_{40+t}}{l_{40}} + v^{10} \frac{l_{50}}{l_{40}}$$

As $d_{40+t} = 300$,

$$\sum_{t=0}^{9} v^{t+1} \frac{d_{40+t}}{l_{40}} = \frac{300}{95,000} \sum_{t=0}^{9} v^{t+1} = \frac{3}{950} a_{\overline{10|}}$$

giving,

$$A_{40:\overline{10|}} = \frac{3}{950} a_{\overline{10|}} + v^{10} \frac{92}{95} = 0.564004$$

Hence $\ddot{a}_{40:\overline{40|}} = 7.70260$ and $P = \$7,322$

12.4.4 Sensitivity Analysis

Sensitivities that are generally true about premiums for endowments

- The premiums increase with age
- The premium decreases as
  - The term increases, or
  - The interest rate increases
- The premium increases as the sum insured increases

The table below demonstrates this. It uses the base case has age $x = 30$, sum insured $S = 100,000$, term 20 years, interest rate 5$ per annum effective, and the Female Mortality table in Atkinson and Dickson (2011)

Case	x	S	n	i	P
base	30	100,000	20	5%	2,951
1.	40	100,000	20	5%	3,058
2.1	30	100,000	25	5%	1,811
2.2	30	100,000	20	6%	2,637
3.	30	120,000	20	5%	3,541

12.5 Parameter variability - Calculation of A_x under variable interest

Consider a whole life insurance with sum insured $1 payable at the end of the year of death, issued to a life aged 25
Suppose that the interest rate is 10% per annum effective for 10 years, and 9% per annum thereafter

We seek an expression for the EPV of the benefit

If the interest rate was constant i, then the solution would be

$$A_{25} = \sum_{t=0}^{\infty} v^{t+1} \frac{d_{25+t}}{l_{25}}$$

But i is not constant

Let v(t) be the present value of a payment of 1 at time t, then v(t) =

$$\begin{cases} 1.1^{-t}, 0 \leq t \leq 10 \\ 1.1^{-10} \times 1.09^{-(t-10)}, t > 10 \end{cases}$$

Under this variable interest

$$EPV = \sum_{t=0}^{\infty} v(t+1) \frac{d_{25+t}}{l_{25}}$$

$$= [\sum_{t=0}^{9} (\frac{1}{1.1})^{t+1} \frac{d_{25+t}}{l_{25}}] + [\sum_{t=10}^{\infty} (\frac{1}{1.1})^{10} (\frac{1}{1.09})^{t-10+1} \frac{d_{25+t}}{l_{25}}]$$

$$= A^{1}_{25:\overline{10}|@10\%} + [(\frac{1}{1.1})^{10} \sum_{s=0}^{\infty} (\frac{1}{1.09})^{s+1} \frac{d_{35+s}}{l_{35}} \frac{l_{35}}{l_{25}}]$$

$$= A^{1}_{25:\overline{10}|@10\%} + (\frac{1}{1.1})^{10} \cdot {}_{10}p_{25} \cdot A_{35@9\%}$$

13 Superannuation

13.1 Ageing Society

- Reduced mortality - people live longer, including need for long term care → Longevity risk
- Improved medical research - people need health benefits for longer, and which get more expensive at a higher rate than CPI or GDP growth
- Lower fertility rates
- Fewer younger people paying taxes to support growing proportion of older people
- **Higher dependency ratio**

This leads to increasing costs - and increasing taxation

Bismarck's Pension Trap:

> "How to support the growing number of retirees without bankrupting the economy"

This comes about due to countries having a fixed aged for retirement, where individuals would be entitled to a pension.

13.2 Pay-as-you-go (PAYG) vs Funded Systems

– **PAYG**: benefits of year N are funded by contribution of year N
– **Funded system**s: benefits of year N (and potentially N+1, N+2,...) are funded by accumulated capital over years 1,...,N-1

- Usually, PAYG systems are set up at the country level. Funded systems are usually individual
- PAYG systems are good for immediate benefits (introduction of a new scheme), and they are more resilient to big affordability shocks (crisis, inflation, etc) (contributions would be made on the same basis are required benefits). However, you may be at a risk of contributing for many years, then get nothing
- Funded systems are better in that hey do not rely on the promise of someone else paying the benefit later (the money is there). But they are more sensitive to market shots (the money is invests, so If investment do not perform then the money may disappear altogether, leaving you with nothing)
- A combination of both is usually best, and is the solution in many developed countries. Often, systems have three pillars
  - A first pillar corresponding to a safety net, universal, PAYG
  - A second pillar complementing the first pillar, often mandatory for salary earning workforce, funded, with tax breaks
  - A third pillar that is voluntary funded, with ore without tax breaks

  The mix changes from one country to another, and may be more or less sustainable
- To assess the sustainability of such systems is the job of some actuaries. How is Australia doing?

13.3 The Australian System: Three Pillars

1st pillar: Government provided pension

- Means tests (assets and income)
- Benefits are small

2nd pillar: Superannuation Guarantee

- Minimum is 10% (used to be 9%, will continue to increase to 12% by 2025). Can be more (eg. Australian academic (to Unisuper) 17% from employers + up to 7% from the employee)
- Incentives include tax breaks: contributions taxed at flat 15% (with limits). Benefits generally not taxed

3rd Pillar: Private savings (voluntary)

Major asset class in Australia is real estate (due to a variety of reasons, including tax rules related to the above), which presents issues (lack of diversification, lack of liquidity, housing unaffordability). Own home down not attract capital gains tax

13.4 Employer Sponsored Superannuation Funds

13.4.1 The Accumulation and Decumulation Phases

Typically, superannuation funds (also called "occupational pension funds" internationally) are funded systems which are typically based on individuals - they belong to Pillar 2

This means the lifecycle of membership includes two major phases

1. The accumulation phase: when contributions are aggregated and invested, typically during the active years of the members
2. The decumulation phase: when funds are being drawn to fund the member's expenses, typically during retirement

13.4.1.1 Accumulation Phase

Members typically join a fund when they start employment and pay contributions to the fund whilst in employment. Employers usually contribute to the fund too

Unlike life insurance premiums, contributions are not usually level, but depend on a member's salary. The contributions are invested, so that the main income to a superannuation fund is from contributions and investment income

13.4.1.2 Decumulation Phase

Outgo from a superannuation fund is in the form of benefits

Benefits can be payable under a variety of circumstances, including:

- Retirement, due to reaching a certain age, eg. 60 or 65
- Retirement, due to failing health

- Death, whilst still in the accumulation phase
- Resignation from employment, which may cause a member to leave the superannuation fund (with their balance or not)

13.4.2 DC and DB Funds

There are two main types of superannuation funds:

- Defined Contribution (DC)
- Defined Benefit (DB)

Note:

- In Australia, DC is the dominant form, but this is not true everywhere
- Not all funds are purely DC or DB, as some may include DC and DB components
- This distinction is mostly about the *accumulation phase* in case of lumps sum benefits, but in DB plan benefits can be formulated as life annuity (pension)

13.4.2.1 Defined Consumption (DC) Funds

Main Principles

- A defined contribution fund acts like a bank account, with a variable rate of return on a member's contributions
- This rate of return depends on the investment performance of the fund (usually depends on investment choices made by the member)
- The final retirement benefits is the total balance of a member's Super account (minus fees and taxes) and is dependent on investment performance

Advantages

- It is relatively transparent/ understandable - most people understand the concept of a bank account
- Members have some control over their investments (they typically choose a strategy, not the actual investments)
- The fund won't be forced to adjust contributions, although changes can always be possible if everyone agree

Disadvantages

- An important drawback about DC super is that the **investment risk is borne by the member**. Furthermore, it pre-supposes that members know how to make choices about their investments, which is a strong assumption

What is the typical role of actuaries in DC funds?
– For the accumulation phase, not much, although actuaries may be involved in advising on investments
– For the decumulation phase, no more no less than its a DB fund. It really depends on how benefits are organised

13.4.2.2 Defined Benefit (DB) Funds

Main Principles

- In a Defined Benefit plan, the retirement benefit is calculated according to a deterministic formula (rather than by random investment returns)
- The fund is responsible for collecting sufficient contributions so as to find those benefits
- In other words: benefits are not defined by the contributions, but are defined in the rules of the planning itself (hence the "defined benefit" denomination)

For example:
In a given DB fund, the formula for calculating the annual amount of the annuity benefits is

$$\frac{n}{60} \times \text{Final Average Salary}$$

where n denotes years of service

In another fund, the lump sum benefit of retirement is the product of the following

- Final average salary
- Number of years of service
- Lump sum factor s(depends on the age of retirement)
- Average salary fraction (employment % age)
- Average contribution factor (depends on how much extra contributions the member paid)

Advantages

- Benefits are deterministically calculated, based on some factors (eg. Final salary). It is easier for members to understand (and product) ho much they will receive
- Related to the above, the investment fluctuations do not affect benefits. This means that most of the investment risk is is borne by the fund. Of course, if things go really wrong some measures may need to be taken, but this should only be in extreme cases

Disadvantages

- No exposure to the downside investment risk means that there is also less exposure to the upside investment risk: the fund will set aside above-than-average returns for the difficult years, rather than distribute them immediately to members
- Since benefits are fixed, contributions may have to be adjusted if assumptions change (eg. Investment returns, longevity, mortality,...)

What is the typical role of actuaries in DB funds?
– Because the timing of benefits doesn't match necessarily that of investments (contrary to DC funds), careful modelling required. Actuaries can help (remember the "Wealth Management" practice area)
Actuaries have an essential role in valuing the funds's liabilities, and setting contribution rates:

- Funds undergo regular valuations, the main purpose of which is to determine whether the fund's liability can be met by existing assets and future contributions

- The valuation is also used to determine the contribution rates; mostly it assesses whether the current contribution rate is adequate. This involves building models for population entry (new employees, transfers), exit (resignations, retirement, death and disability), and other major assumptions (interest rates, salary of members, family situation, health status,...

Valuations can occur at the individual level, too. When a member leaves before retirement, the value of his acquired rights needs to be valued so that they can leave with their savings

13.5 The Actuarial Expert Corner

– Interview: Anthony Saliba
Takeaways:

- Actuaries in Superannuation
  - Beyond traditional life insurance: asset liability modelling, modelling of customer outcomes
  - In particular, retirement income strategies
  - Policy risk (what is policy changes, what is the impact?)
  - financial advice
- Current and Future Challenges
  - Accumulation quite mature, but how do we keep up income on those balances?
  - Main focus now is on decumulation: how do you turn the accumulated lump sum into required income
  - This is difficult
    - No benchmark
    - Longevity risk, investment risk, health risk, policy risk
  - I would add: Circumstances and plan change from one individual to another, too. For instance
    - Family (do kids want money for their home deposit?)
    - Risk aversion (investment safety vs high return)
    - plans (eg. Do you want to travel a lot?)
    - Housing needs (do you have a house? Do you want to downsize? Move elsewhere?...)
- Advice
  - Broaden skill set
  - Actuaries studies is a great place to start
  - Don't specialise too soon
  - Opportunity: nexus pure actuarial and IT

13.6 Industry Insights

AD: "investing for the Different Phases of Retirement"

- "Matching strategies, products and advice to the specific demands of an ageing population is one of the greatest challenges facing the retirement sector in Australia today"
- " Encouraged funds to begin looking at better segmentation of their members along with revised investment strategies which utilise separate "buckets" of funds to meet different financial needs in retirements rather than one generic approach"
- Phases of Retirement
  - Active phase
  - Sedentary phase (perhaps 75/80+) - everything slows down (inducing expenditure)
  - Trial phase (85+) - mind and body decline

AD: "what is an investment-linked annuity?"

- **An investment-linked lifetime annuity** produces "**longevity protection** to ensure the retiree never runs out of money but can also offer a **choice of investments** used to support that income"
- "the main problem with allocating 100% of you superannuation to an **Accounts Based Pension** in retirement is you don't know how long you are going got live. This means you don't know how much income you can safely draw from you ABP year year." → **longevity risk**

14 Health Insurance

14.1 National Insurance

14.1.1 Universal Public Coverage

Some societies (mainly, developed societies) collectively decide that some risks should be covered for everyone, and decide to offer that coverage by the government. **National insurance** essentially refers to such coverage of certain risks by the government.

These include welfare benefits, covering for basic necessities such as an old age pension, a disability pension, basic health care, and unemployment benefits

Australia has a "government Actuary" who proves advice to the government on a range of issues. These would be heavily influenced by demographic considerations, and would involve estimating the current and future costs of benefits under a range of scenarios

National insurance differs from conventional insurance in a number of ways

- Benefits are usually financed via government levies/ taxes (specific or not) rather than premiums. Furthermore, these are often finance in a PAYG fashion
- The government is not subject to the same statutory requirements as private insurance companies. This is because hey have very strong ability to raise money and can force its population pay
- It can also change the law - the "terms" of the (social) "contract"

14.1.2 Medicare

Medicare is the heath cover that all Australians and Permanent residents enjoy (there are others under certain conditions). The scheme is a national scheme which is free for people with no income

It is funded

- Via a Medicare Levy, which is a fixed percentage of income (2021: 2% of taxable income)
- A Medicare Levy surcharge (for people without private health insurance), which is a percentage of income (2021: 1-1.5%) above a certain threshold (2021: 90k for singles, and 180l for couples/families)
- Income taxes (in particular for infrastructure such as hospitals, as well as research)

Medicare covers:

- Free or subsided treatment by health professionals, such as doctors, specialists, optometrists, and in specific circumstances, dentists and other allied health professionals
- Free treatment and accommodation for patients in a public hospital;
- 75% of the Medicare schedule fee for services and procedures if you're a private patient in a public of private hospital (but not including hospital accommodation, theatre fees, medicines and other items)

A safety net reduces gaps under certain conditions

14.2 Private Health Insurance

One of the largest risk insurance sector in Australia: private health insurance
- Heavily regulated, additional premiums
- National Health Act 1964 & Regulations
- Complements Medicare

Products

- Hospital cover (no surcharge levy)
  - Accommodation, theatre fees, medicare fees, private/public hospitals, excesses, exclusions, benefit level etc.
- Ancillary cover
  - Optical, chiropractic, dental, pharmacy, physiotherapy etc.

14.2.1 Principle of Community Rating

Community rating means that one cannot discriminate in private health insurance pricing in Australia by either

- Health status or similar
- Age (other than for Lifetime Health Cover)
- Sex, or
- Other demographic features

Also, one cannot refuse applicants

Community rating presents issues

- Does not rate according to risk
- Adverse selection (unless compulsory for all)
- Increasing costs as healthier lives opt out

14.2.2 Lifetime Health Cover

To deal with Community Rating issues, an age at entry community rating was introduced

- Proxies age as a rating factor
- More expensive as you enter at older ages

There are also tax incentives to encourage people to be covered by Private Health Insurance, such as the Medicare Levy surcharge, as well as premium rebates (2021: up to 32.812% with income thresholds. Nothing for incomes above 140k/280k for singles/families)

14.3 The Actuarial Export Corner

– Interview: Ignatius Li
Takeaways:

- Actuaries in Health
  - Focus on payers (insurers and government), and in particular health insurance companies
  - questions: risks involved in new products, premium setting, risk profile likely to attract, how much capital is required?
- Who hires Health Actuaries
  - Consultancies (bg 4 through to boutique consultants focusing on health)
  - Direct insurers with actuaries on staff (Medibank, BUPA,...)
  - New 'broader health' area: government (health policies), understand drivers of health (data analytics), new types of insurance (eg. NDIS)
- Current and Future Challenged
  - Knowledge and understanding of products (simply products? Educate the public?)
  - What is the minimum we should offer? (Everything, where do we stop?)
  - Sustainability (related to previous point)
  - Emerging technologies, big data (wearable technology incentivise health behaviour, detects early signs of health issues)
  - Problem with community rating is that insurers don't reap all the benefits of their successes in terms of risk mitigation
- Advice
  - Work out where your passion is
  - By curious, challenge the status quo, make a contribution to that area that passions you
  - Actuaries have a role of 'disinterested voice' in important area
  - Keep an open mind, and expose yourself to as many different issues as possible
  - Don't specialise too early

14.4 Industry Insights

AD: Future of Health Seminar 2021 - an overview
"How healthcare providers, governments and insurers are using data to create better patient outcomes and improve resource efficiency"

- "How healthcare providers, governments and insurers are using data to create better patient outcomes and improve resource efficiency"
  - Growing role of data science, analytics, and technology in healthcare
  - Exploding amount of data presents challenges
- Costs rising faster than economic growth across OECD
- Study of demand, supply, and sustainability
- Mental health as a majority and increasing issues

15 General Insurance

15.0.0.1 Equivalent Names

General insurance can be broadly defined as anything that is not life insurance

Several other names:

- General insurance (GI) - traditionally in Australia
- Non-life insurance - traditionally in Europe
- Property and Casualty Insurance (P&C) - traditionally in North America

There is political push within the ASTIN section of the IAA for generalising the term of "General Insurance," but this is ongoing

15.0.0.2 Two Main Types

Property insurance - covers the loss arising from damage to property such as buildings, contents, motor vehicles, aircraft and cargo

Liability Insurance - liability insurance (also sometimes called "Casualty Insurance") that covers the liability to provide compensation to another party when the insured is at fault (negligent acts) or where compensation is required by law

Liability insurance can include liability for damage to property and injury to persons, an example is Workers Compensation

15.1 Main Classes of General Insurance Business

Motor Insurance

- Compulsory Third Party (CTP) insurance
  - CTP is designed to provide cover for liability for bodily injury or death as a result of an accident
  - CTP is administered differently across the different Australian states
- Third Party Property Damage cover: protects the policy holder from the costs of damage you car causes to someone else's car or property
- Fire and Theft cover: this has all benefits from the Third Party Property Damage cover, as well as protection to your car against fire and theft
- Comprehensive insurance cover: cover your car against damage caused by a number of events including theft, accidents (including "at fault"), flood, storm, fire, earthquake, as well as the benefits from the Third Party Property Damage cover

Building and Contents Insurance

- Household contents insurance: this protects a policyholder from damage to personal possessions, and burglary. Essentially, goods are replaced by the insurance company
- Buildings Insurance: this protects a policyholder from damage to a home/factory/ office by an event such as fire. Such an Insurance policy would normally pay for rebuilding or repairing the property

Other Classes of Insurance include

- Public and Products Liability
- Workers Compensation
- Private Health Insurance

Emerging: related to sharing economy (Uber, AirBnB,...) as well as cyber risk

15.2 Main Features of General Insurance Products

Shorter coverage periods (PAYG coverage) than for life insurance contracts - usually one year coverage policies

Longer settlement:Payment of claims can extend over many years into the future for **long tail** classes such as liability or workers compensation (as opposed to **short-tail** classes such as motor and home). But how long is a very long settlement period?

Random frequencies and severities: Insured can claim more than once - amount of a claim is variable, and high variability of claims

No (or very high) upper bounds for outcomes: Risk of large claims arising from one event such as a cyclone, fire or earthquake

Particular Challenges

- Modelling of **catastrophe risk** ("modelled" risks, rather than fitted to past data)
- Controlling for **moral hazard and fraud**
- Presence of **experience rating**: Bonus/malus, no claim discounts,...
- Presence of **excess/ deductibles**
  - Reduction of moral hazard and administrative costs
  - Insured have a choice of how much risk to retain/transfer

15.2.1 Contrast with Life Insurance

Life Insurance

- Usually long term cover
- Frequency is binary (0 or 1)
- Severity is usually fixed or deterministic
- No moral hazard and no small claims so usually no excess (murder is an exclusion)
- Level premiums are usually paid over several years
- Renewal guaranteed
- Core difficulties: investment of premiums over long periods of time, and longevity risk (mortality risk)

General Insurance

- Usually short term cover
- Frequency is more complex (0,1,2,...)
- Severity is typically random (and different for each claim)
- Excess due to significant moral hazard and potentially small claims
- Single premiums, which can vary on renewal
- No guarantee of renewal
- Core difficulty: estimation of existing, outstanding liability (IBNR, RBNS)

15.3 Role of the Actuary in General Insurance

15.3.1 Modelling of Risks

Understanding risks and their interactions is of paramount importance

This typically goes through modelling of frequency and severity separately; for instance for a particular risk, one might assume:

- Data from the past k years include claim counts, say, $n_1, n_2, \ldots, n_j, \ldots, n_k$, from which we can estimate the distribution of the claim frequency (say N)
- Data from the past years include claim severities, say, $x_1, x_2, \ldots, x_j$ $(j = 1, \ldots, k)$ from which we can estimate the distribution of the claim severity (say X_i) Then the aggregate amount of claim from this risk is modelled as:

$$S = \sum_{i=1}^{N} X_i$$

  Where N is the claim frequency, $X_1, X_2, \ldots$ are claim severities. This is what we call a **random sum**

Dependencies between classes of business, between insurance and other risks (such as investment risk), as well as between severity and frequency, are all potentially material and need to be considered

15.3.2 Pricing

Just as in life insurance, one must understand what the main drivers of risk are, and when someone seeks coverage, an assessment of their level of risk will be carried out (underwriting)
The actuary will typically analyse data in order to determine statically significant **rating factors**

Once a risk is classified into group, the gross premium for S, the aggregate claim amount from the risk is given as follow

- Gross premium = pure premium + risk loading + expenses + profit loading - investment income
  It is usually at the "pure premium" level that relativities are applied, to reflect levels of risk

How to adjust premiums for future years based on past claim experience (the so-called "experience rating") is also part of the actuary's responsibilities

15.3.3 Reserving

Determining the amount of reserves is one of the most important jobs of the general insurance actuary

Reserves are of the utmost importance: they typically represent a significant proportion of the balance sheet of insurers

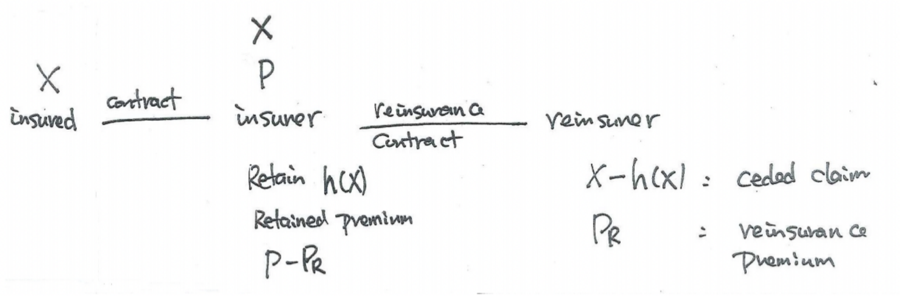
- 1% variation may represent tens of millions of $
- Reserves are usually a multiple of Equity (eg. 2-5 times equity)

Reserving involves a significant amount of past data analysis, and trying to infer

- How many insured events occurred (even through not all of the may be reported) ← IBNR ("incurred But Not Reported")
- And for those that are down, predicting how much they might eventually cost ← RBNS ("Reported But Not Settled")

15.3.4 Reinsurance

In simple terms, a reinsurance company act as an insurer to an insurance company



15.3.4.1 Two Commonly used Types of Reinsurance Arrangements

Proportion reinsurance: for each claim of amount X, where $0 < a < 1$

- The retained claim amount by the insurer is

$$h(X) = aX$$

- And the ceded amount to the reinsurer is

$$X - h(X) = (1 - a)X$$

Stop Loss Reinsurance: for each claim of amount X

- The retained claim amount by the insurer is

$$h(x) = min(X, M)$$

- And the ceded claim amount to the reinsurer is

$$X - h(x) = max(0, X - M)$$

15.3.4.2 The Choice of Reinsurance Level

The appointed actuary (AA) of a general insurance company typically advises the board when it determined its **risk appetite**, and then helped the company achieve its desired level of risk. This involves

- Assessing what type of reinurnace might be appropriate
- Assess the cost of purchasing such reinsurance
- Assess the impact of reinsurance on the risk level of the insurance company in view of its risk appetite

Of course, the pricing actuary of the reinsurer will be naturally pricing reinsurance covers

Optimal reinsurance design is a research field in itself. When making simple (even simplistic assumptions, some conclusions can be drawn.

15.4 The Actuarial Expert Corner

– Interview: Luke Cassar
Takeaways:

- Actuaries and general insurance
  - Bread and better "appointed actuary" role: (i) Insurance liability valuation, (ii) Financial condition report
  - Also: workers compensation schemes (also liabilities), advice on pricing (including marketing strategy), capital modelling (including DFA), reinsurance strategy, CTP, liability practice are (modelling of risk)
- Who hires GI actuaries?
  - Insurance companies ('corporates'): big ones such as IAG, QBE, Suncorp; but also boutique ones
  - Consulting companies: actuaries arm in big consultants, or boutique actuarial
  - Government agencies: state workers compensation, self insurance schemes, NDIA
- What's great about GI?
  - Digital disruption: both types of products AND the way insurance overstates
  - Eg. Drivel's cars, shared economy (Uber, Airbnb)
  - Other issues are product focussed: CTP in NSW, NDIS, climate change and property insurance
- Advice
  - Studies provides tool kit
  - Spend time on AI website + Actuaries Digital magazine
  - Keep in touch with people in the industry (what is happening?, what is their company like?)
  - Keep an open mid, and expose yourself to as many different issues as possible
  - Don't specialise too early

15.5 Industry Insights

15.5.1 (still) Emerging area: Cyber Risk

AD: What's covered in a cyber insurance policy?
AD: Insuring Cyber Risk in 2020

Two types of coverage

- First party: incident response, business interruption, data restoration, cyber extortion, etc
- Third party: private and confidential liability, network liability (network infected from your computer), media liability (copyright)

Cyber insurance market is 5 years old, with now approximately $130 mio of annual premium (tiny!- total premiums is about $bio) across more than 25 insurers

Some trends

- Standardisation of coverage
- Market growth (to smaller businesses)
- Increase in clay frequency (first party ransomware), but data is limited
- Combined ratio is approximately 75% in Australia
- Gets into governance (appointed of CISO - Chief Information Security Officer, Board involvement)

Covid-19 led to a five-fold increase of cyber attacks (phasing, vulnerability due to WFH)

Suggests 12 elements (relativities) to be considered in the underwriting process - but typically only 3 are being used

The risk is not global, which makes it special

15.5.2 Insurance Affordability

15.5.2.1 AD: Insurance Affordability in Northern Queensland

"General Insurance Affordability Working Group": looked at the measurement and scale oft he affordability problem, as well as possible solution

What is "insurance affordability?" How do you measure it?

- Option 1
  - "Housing Affordability pressure metric (essentially net disposable income after housing costs using ABS data) vs
  - Relative Insurance risk metric" (retail premiums per sum insured)
- Option 2
  - Average number of weeks needed to pay for the annual cost of home insurance

Possible Solutions

- Mitigation is key, but risk can't be eliminated - who picks up the tab?

- Non-pool options
    - Community rating
    - Reduce non-risk costs (taxes, brokerage,...)
    - Government subsidies
- Pool options
    - Spread losses across space (including overseas) and time (across generations) to smooth out fluctuations
    - Reinsurance pool vs insurance pool (where government act as a direct insurer)
- Any solution will involve spreading losses across time and space

The article also discusses the decision making process, without lobbying one way or another

15.5.2.2 AD: Virtual Summit Shorts: A grand proposal for uninsurable risks

- How can new make the risk more insurable (less risky)?
- *Some* coverage is better than none - why take an all or nothing approach?
- Idea: share the tail

15.5.2.3 AD: IBM: Insurers welcome government reinsurance pool for cyclone and flood risk

Final Solution:

- 10 billion reinsurance pool for cyclone and cyclone-related flood risk in northern Australia (the spreading of risk bit)
- 40 million Investment in making older strata buildings more resilient (the mitigation bit)

Insurers had set up a "Reinsurance Pool Working Group" which led to the solution

15.5.3 What about COVID-19?

Business Interruption (BI):
AD: Business interruption in a COVID-19 Australia

- Importance of working
- Was the pandemic an exclusion or not?
- What will happen in the future?

Workers Compensation
IN: Workers' Comp claims set to rise as economy reopens: Finity

- "Workers' compensation claims are likely to increase when the economy opens up, as states and territories transition to "living with COVID-19," actuarial firm Finity has warned in a new report"
- "The big question is, how many of those infections are actually going to be compensable under a workers' compensation policy."

But also travel insurance, and others...