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MA - 641 HW 2

Q1] Let $Y_1, \dots, Y_n$ ~~the~~ be iid

Given - $E(Y_i) = \mu$ $\quad V(Y_i) = \sigma^2$

Lets Recall that $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$

To find - $E[(\bar{Y} - \mu)^2] = E[(\bar{Y} - E(\bar{Y}))^2]$

$$E(\bar{Y} - \mu)^2 = E((\bar{Y} - E(Y_i))^2)$$

$$= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu\right)^2\right]$$

$$= E\left[\frac{1}{n^2}\sum_{i=1}^{n} Y_i^2 + \mu^2 - \frac{2\mu}{n}\sum_{i=1}^{n} Y_i\right]$$

$$= \frac{1}{n^2} E\left(\sum_{i=1}^{n} Y_i^2\right) + E(\mu^2) - E\left(\frac{2\mu}{n}\sum_{i=1}^{n} Y_i\right)$$

Moving forward we get,

$$\{E[X^2] = Var[X] + E(X)^2\}$$

Then we solve for the first term, giving us $\frac{1}{n^2}\left(Var\left(\sum_{i=1}^{n} Y_i\right) + E\left(\sum_{i=1}^{n} Y_i\right)^2\right)$

which gives us $\frac{1}{n^2}\left(n\sigma^2 + (n\mu)^2\right)$

$$\Rightarrow \frac{\sigma^2}{n} + \mu^2$$

Similarly, solving for 2^{nd} and 3^{rd} term we get

$E(\mu^2) = \mu^2$ $\quad \frac{2\mu}{n} E\left(\sum_{i=1}^{n} Y_i\right) = \frac{2\mu}{n} n E\left(\sum_{i=1}^{n} Y_i\right)$

$$= 2\mu^2$$

Therefore we get, $E(\bar{Y} - \mu)^2 = \frac{\sigma^2}{n} + \mu^2 + \mu^2 - 2\mu^2$

$$\Rightarrow \frac{\sigma^2}{n}$$

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Q2] ~~It~~ Let $\{e_t\}$ be a zero mean white
$V(e_t) = \sigma^2$ for $t \geq 0$
$Y_t = \mu + e_t + \theta e_{t-1}$
If $\theta = 0$ Y_t are iid & $V(\bar{Y}) = \sigma^2/n$

a] $\theta = -1$
$Y_t = \mu + e_t - e_{t-1}$
$$\sum_{t=1}^{n} Y_t = n\mu + \sum_{i=1}^{n} (e_t - e_{t-1}) = n\mu + e_n - e_0$$
$$Var(\bar{Y}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} Var(n\mu + e_n - e_0)$$
$$\Rightarrow \frac{1}{n^2}(\sigma^2 + \sigma^2) = \frac{2\sigma^2}{n^2}$$

b] $\theta = 1$
$Y_t = \mu + e_t + e_{t-1}$
$$\sum_{t=1}^{n} Y_t = n\mu + e_n + e_0 + 2\sum_{t=1}^{n-1} e_t$$
$$\& \; Var(\bar{Y}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} Var\left(n\mu + e_n + e_0 + 2\sum_{t=1}^{n-1} e_t\right)$$
$$= \frac{1}{n^2}\left(0 + \sigma^2 + \sigma^2 + 4 Var\left(\sum_{t=1}^{n-1} e_t\right)\right)$$
$$4\sum_{t=1}^{n-1} Var(e_t) + \Sigma \sum_{t=1}^{n-1}\sum_{s=1}^{n-1} Cov(e_t, e_s)$$
$$= \frac{1}{n^2}(\sigma^2 + \sigma^2 + 4(n-1)\sigma^2)$$
$$= \frac{(4n - 4 + 2)\sigma^2}{n} = \frac{(4n-2)\sigma^2}{n}$$
$$\simeq \frac{4\sigma^2}{n}$$

c] Considering n is large, from what we got we can say that $\bar{Y}$ is best if $\theta = -1$ & next $\theta = 0$ & worst for $\theta = 1$

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Q3] $Y_t = \mu - \theta e_{t-1} + e_t$

a] $E(Y_t) = \mu - \theta E(e_{t-1}) + E(e_t)$
$= \mu$

b] $V(Y_t) = Cov(Y_t, Y_t)$

$\gamma_0 = Var[Y_t] = Var(\mu - \theta e_{t-1} + e_t)$
$= \theta^2 Var(e_{t-1}) + Var(e_t) + 2 Cov(e_{t-1}, e_t)$
$= \theta^2 \sigma^2 + \sigma^2$

$V(Y_t) = \sigma^2 (1 + \theta^2)$

c] $Cov(Y_t, Y_{t-1})$

$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(\mu - \theta e_{t-1} + e_t, \mu - \theta e_{t-2} + e_{t-1})$
$= -\theta Cov(e_{t-1}, e_{t-1})$

gives us $\Rightarrow -\theta \sigma^2$

d] ~~To show~~ $\rightarrow Cov(Y_t, Y_{t-k}) = 0$ for $k > 1$

Is $\{Y_t\}$ stationary?

As (c) we can show that for $k > 1$, $Cov(Y_t, Y_{t-k}) = 0$

Now coming to whether $\{Y_t\}$ is stationary:
$\{Y_t\}$ is stationary, simply because the mean, variance & Cov are not dependent on time.

Q4 $\{W_t\}$ and $\{Z_t\}$ are independent and each are iid

$\Rightarrow P(W_t=0) = P(W_t=1) = \frac{1}{2}$ & ~~$P(Z_t=-1)$~~

$\Rightarrow P(Z_t=-1) = P(Z_t=1) = \frac{1}{2}$

$\Rightarrow X_t = W_t(1-W_{t-1})Z_t$

Finding mean & covariance

$E[X_t] = E[W_t(1-W_{t-1})Z_t]$

$= E[W_t(1-W_{t-1})]E[Z_t]$

$= 0$

Note that $E[Z_t] = -1 \times \frac{1}{2} + 1 \times \frac{1}{2} = 0$

Moving on to the covariance

We get $Cov(X_t, X_s) = Cov(W_t(1-W_{t-1})Z_t, W_s(1-W_{s-1})Z_s)$

$\Rightarrow E[W_t(1-W_{t-1})W_s(1-W_{s-1})]E[Z_tZ_s]$

But here if $t \neq s$ then

$E[Z_tZ_s] = E[Z_t]E[Z_s] = 0$

Meaning $\{X_t\}$ is uncorrelated

If $t=s$

$E[Z_tZ_s] = E[Z_t^2] = (-1)^2\frac{1}{2} + 1^2\frac{1}{2} = 1$

$\therefore Var(X_t) \Rightarrow Cov(X_t, X_t) = E[W_t^2]E[(1-W_{t-1})^2]E[Z_t^2]$

$= \frac{1}{2} \times \frac{1}{2} \times 1 = \boxed{\frac{1}{4}}$

Telling us that $\{X_t\}$ is white noise

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Now we need to proove $\{X_t\}$ is not iid

X_t can have 0, -1 & 1 as its values

~~P(X_t)~~ We know, $P(X_t=1, X_{t-1}=1)=0$

$\therefore P(X_t=-1) = P(W_t=1, W_{t-1}=0, Z_t=-1) = 1/8$

$\therefore P(X_t=0) = P(W_t=0, W_{t-1}=0, Z_t=1) + P(W_t=0, W_{t-1}=0, Z_t=-1) + P(W_t=0, W_{t-1}=1, Z_t=1) + P(W_t=0, W_{t-1}=1, Z_t=-1) + P(W_t=1, W_{t-1}=1, Z_t=1) + P(W_t=1, W_{t-1}=1, Z_t=-1) \Rightarrow 6/8$

$\therefore P(X_t=1) = P(W_t=1, W_{t-1}=0, Z_t=1) = 1/8$

$\because P(X_t=1, X_{t-1}=1) \neq P(X_t=1)\, P(X_{t-1}=1)$

$$0 \neq \frac{1}{8} \times \frac{1}{8}$$

Telling us X_t & X_{t-1} are not iid & are white

641_HW2

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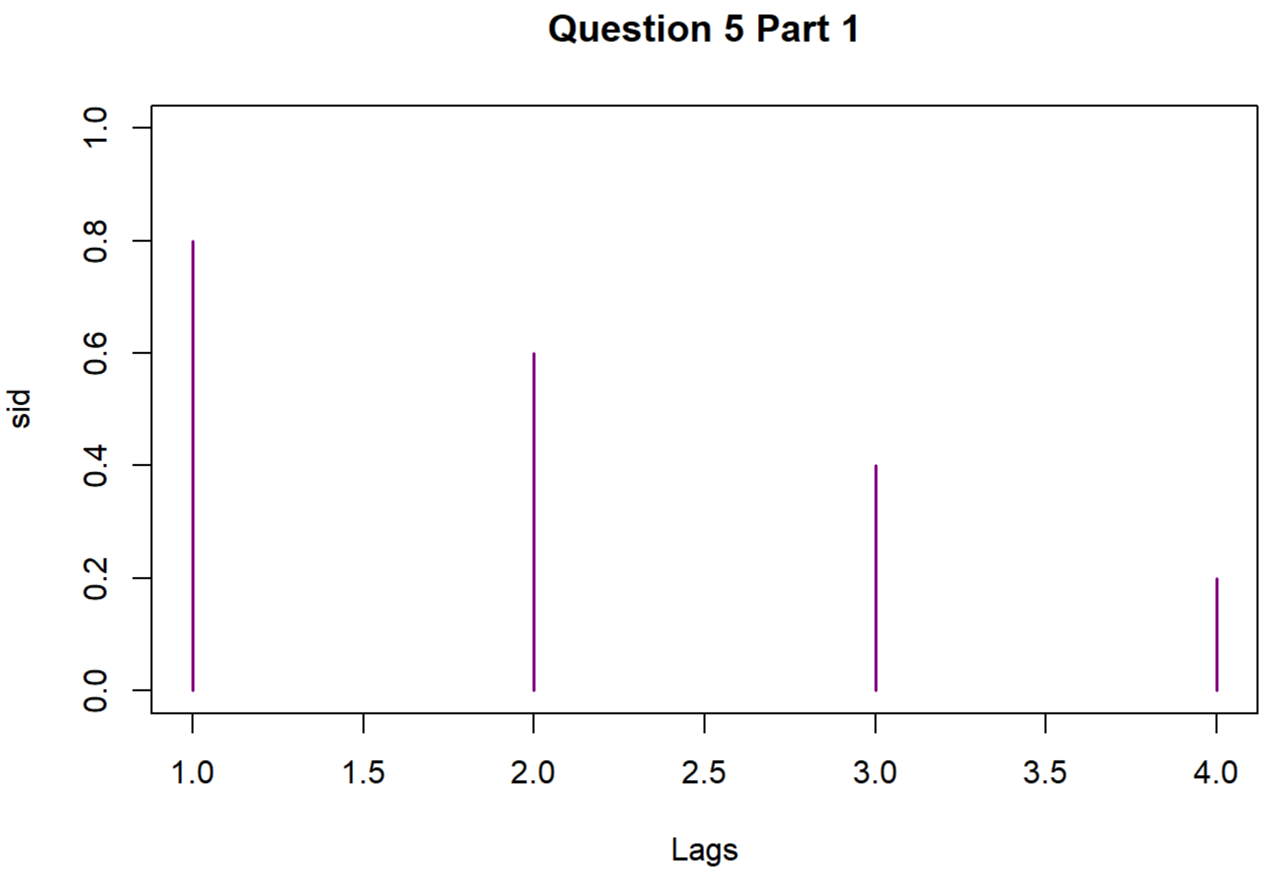
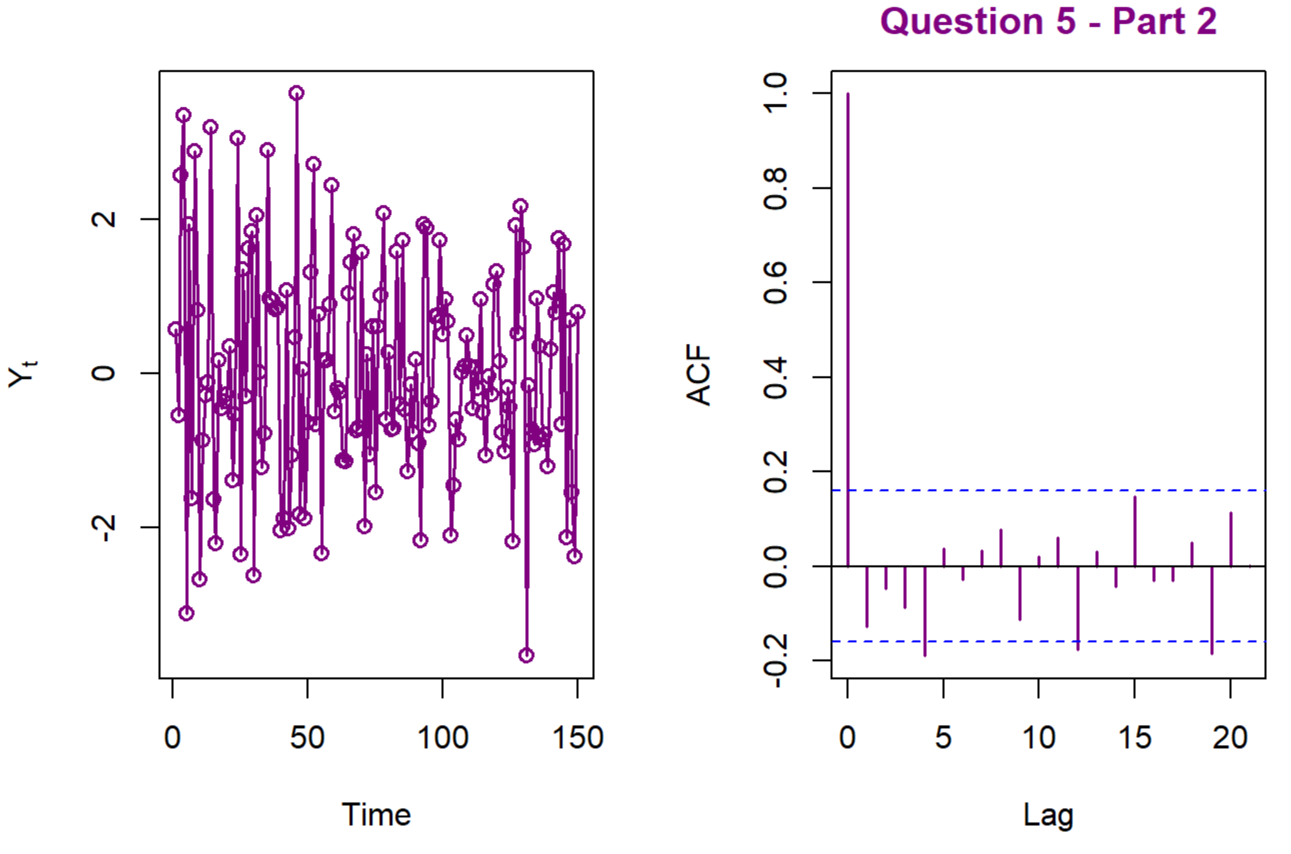
```
#Question 5 Part 1
library(stats)

#Defining the parameters
sid = c(0.8, 0.6, 0.4, 0.2)

#Plot the ACF
plot(sid, type='h', col='#800080', ylim=c(0, 1), xlab="Lags", ylab=expression(sid), lwd=1.5, main="Question 5 Par
t 1")
```




```
#Part 2

set.seed(100)
white_noise = rnorm(200, mean=0, sd=1)
n = 150
MA4 = numeric(n)
theta = c(0.8, 0.6, 0.4, 0.2)
time_vector = 1:n

#Generating the time series

for (i in 1:n) {
  MA4[i] = white_noise[50 + i] - sum(theta * white_noise[50 + i - 1:4])
}

#Plot for Part 2

par(mfrow=c(1,2))
plot(time_vector, MA4, ylab=expression(Y[t]), xlab='Time', type='o', col='#800080', lwd=1.5)
acf_result = acf(MA4, col='#800080', lwd=1.5, main="")
title("Question 5 - Part 2", line = 1, col.main = '#800080', font.main = 2)
```




```
par(mfrow=c(1,1))
```