SIDDHARTH NILAKHE Home Work 3

Page No.
Date

Q2] Suppose $\{Y_t\}$ is an AR(1) process with $-1 < \phi < +1$

Then we get $Y_t = \phi Y_{t-1} + e_t$

$Y_t - Y_{t-1} = (\phi - 1) Y_{t-1} + e_t$

Now for k=0

$Cov(W_t, W_t) = Var(W_t)$

which gives us $Var(Y_t - Y_{t-1})$

$\Rightarrow (\phi-1)^2 Var(Y_{t-1}) + Var(e_t) + 2(\phi-1) Cov(Y_{t-1}, e_t)$

Solving further

$\Rightarrow (\phi-1)^2 \dfrac{\sigma_e^2}{(1-\phi^2)} + \sigma_e^2 + 0$

We then get, $(1-\phi)^2 \dfrac{\sigma_e^2}{(1-\phi)(1+\phi)} + \sigma_e^2$

$\Rightarrow \dfrac{(1-\phi)\sigma_e^2 + (1+\phi)\sigma_e^2}{1+\phi}$

giving us $\dfrac{2\sigma_e^2}{(1+\phi)}$

In a similar way solving for k=1

$Cov(W_t, W_{t-1}) =$

$Cov((\phi-1)Y_{t-1} + e_t, (\phi-1)Y_{t-2} + e_{t-1})$

giving us $\rightarrow (\phi-1)^2 Cov(Y_{t-1}, Y_{t-2})$

$\Rightarrow (\phi-1)^2 \dfrac{\phi \sigma_e^2}{1-\phi^2} = \dfrac{\phi(1-\phi)\sigma_e^2}{(1+\phi)}$

Now as we know for k>1, $\gamma_k = 0$ ~~then~~ — (1)

Autocovariance γ_k is given by

$\dfrac{2\sigma_e^2}{1+\phi}$ k=0

$\dfrac{\phi(1-\phi)\sigma_e^2}{1+\phi}$ k=1

0 k>0 — from (1)

Page No.

Date

Q6] $Y_t = e_t + 5/2\, e_{t-1} - 3/2\, e_{t-2}$

$\{e_t\} \sim WN(0, 1)$

$\gamma_k = cov(Y_t, Y_{t-k})$

$= cov\left(e_t + \frac{5 e_{t-1}}{2} - \frac{3}{2} e_{t-2},\ e_{t-k} + \frac{5}{2} e_{t-k-1} - \frac{3}{2} e_{t-k-2}\right)$

Then,

$\gamma_0 = cov(Y_t\ Y_{t-0})$

$\hookrightarrow Var(Y_t) = Var\left(e_t + \frac{5}{2} e_{t-1} - \frac{3}{2} e_{t-2}\right)$

$= 1 + \left(\frac{5}{2}\right)^2 (1) + \left(\frac{-3}{2}\right)^2 (1)$

$= 1 + \frac{25}{4} + \frac{9}{4} = \frac{19}{2}$

$\gamma_1 = cov(Y_t, Y_{t-1})$

$= cov\left(e_t + \frac{5}{2} e_{t-1} - \frac{3}{2} e_{t-2},\ e_{t-1} + \frac{5}{2} e_{t-2} - \frac{3}{2} e_{t-3}\right)$

$\Rightarrow \frac{5}{2} cov(e_{t-1}, e_{t-1}) + \left(\frac{-3}{2}\right)\left(\frac{5}{2}\right) cov(e_{t-2}, e_{t-2})$

$= \frac{5}{2} Var(e_{t-1}) - \frac{15}{4} Var(e_{t-2})$

$= -\frac{5}{4}$

Similarly for

$\gamma_2 = cov(Y_t, Y_{t-2})$

$= cov\left(e_t + \frac{5}{2} e_{t-1} - \frac{3}{2} e_{t-2},\ e_{t-2} + \frac{5}{2} e_{t-3} - \frac{3}{2} e_{t-4}\right)$

giving us nothing but

$= 0 + \dots + cov\left(\frac{-3}{2} e_{t-2}, e_{t-2}\right) + \dots$

$\frac{-3}{2} Var(e_{t-2}) = \frac{-3}{2}$

Autocovariance fn is given by the following

$$\gamma_k = \begin{cases} 19/2 & \text{when } k=0 \\ -5/4 & \text{when } k=1 \\ -3/2 & \text{when } k=2 \\ 0 & \text{when } k>2 \end{cases}$$

b] $Y_t = e_t - \frac{1}{6} e_{t-1} - \frac{1}{6} e_{t-2}$

$$\gamma_0 = Cov(Y_t, Y_t) = Var(Y_t)$$
$$= Var(e_t) \left(-\frac{1}{6}\right)^2 Var(e_{t-1}) + \left(\frac{-1}{6}\right)^2 Var(e_{t-2})$$
$$= 9 + \frac{1}{3} \times 9 + \frac{9}{3} = 19/2$$

$$\gamma_1 = Cov(Y_t, Y_{t-1})$$
$$= Cov\left(e_t - \frac{1}{6}e_{t-1} - \frac{1}{6}e_{t-2},\ e_{t-1} - \frac{1}{6}e_{t-2} - \frac{1}{6}e_{t-3}\right)$$
$$= -\frac{1}{6} Cov(e_{t-1}, e_{t-1}) + \frac{1}{36} Cov(e_{t-2}, e_{t-2})$$
$$= \cancel{25} \ -\frac{3}{2} + \frac{1}{4} = -\frac{5}{4}$$

$$\gamma_2 = Cov(Y_t, Y_{t-2})$$
$$= Cov(e_t - \tfrac{1}{6}e_{t-1} - \tfrac{1}{6}e_{t-2},\ e_{t-2} - \tfrac{1}{6}e_{t-3} - \tfrac{1}{6}e_{t-4})$$
$$= 0 + \cdots + \left(\frac{-1}{6}\right) Cov(e_{t-2}, e_{t-2}) + 0$$
$$= -\frac{3}{2}$$

$\therefore$ From above we get autocovariance fn as

$$\gamma_k = \begin{cases} 19/2 & \text{where } k=0 \\ -3/4 & \text{where } k=1 \\ -3/2 & \text{where } k=2 \\ 0 & \text{where } k>2 \end{cases}$$

c] Equation $\Rightarrow 1+\theta_1 x+\theta_2 x^2$

$\theta_1 = 5/2 \quad \theta_2 = -3/2$

$1 + 5/2\,x - 3/2\,x^2 = 0$

Simplifying gives us $3x^2 - 5x - 2 = 0$

we need to check if it lies outside the circle

Iff sum of roots & product is greater than 1

for the above to be true

Sum $\rightarrow$ $-b/a$ Product $\rightarrow$ c/a

$5/3 > 1$ $\quad$ $\cancel{2/3 > 1}$ $-2/3 < 1$

Hence not inevertible

Now for (b) $\theta_1 = -1/6 \quad \theta_2 = -1/6$

$1 - \frac{x}{6} - \frac{x^2}{6} = 0 \rightarrow x^2 + x - 6 = 0$

Sum $\rightarrow$ $-b/a \rightarrow -1$ $\quad$ Product $\rightarrow$ $c/a \rightarrow -6$

Hence, not true

Therefore, both the models is not invertible.

Q5] For ARMA (1, 2)

$Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$

a) S.T $\rho_k = 0.8\rho_{k-1}$ for $k>2$

Now let us multiply by Y_{t-k} on either sides

$E(Y_t Y_{t-k}) = 0.8E(Y_{t-1}Y_{t-k}) + E(e_t Y_{t-k}) +$
$0.7E(e_{t-1}Y_{t-k}) + 0.6E(e_{t-2}Y_{t-k})$

Now for $k=0$

$\gamma_0 = 0.8\gamma_1 + E(Y_t e_t) + 0.7E(e_{t-1}Y_t) + 0.6E(e_{t-2}Y_t)$

$\gamma_0 = 0.8\gamma_1 + \sigma^2 + 0 + 0$

$k=1$

$\gamma_1 = 0.8E(Y_{t-1}Y_{t-1}) + E(Y_{t-1}e_t) + 0.7E(Y_{t-1}e_{t-1}) + 0.6E(Y_{t-1}e_{t-2})$

$= 0.8\gamma_0 + 0 + 0.7\sigma_e^2 + 0$

$\gamma_1 = 0.8\gamma_0 + 0.7\sigma_e^2$

$k=2$, $\gamma_2 = 0.8E(Y_{t-1}Y_{t-2}) + E(Y_{t-2}e_t) + 0.7E(Y_{t-2}e_{t-1})$
$+ 0.6E(Y_{t-2}e_{t-2}) \Rightarrow 0.8\gamma_1 + 0 + 0 + 0.6\sigma_e^2$

$\gamma_2 = 0.8\gamma_1 + 0.6\sigma_e^2$

From above as we move ahead all terms will be 0

Now for $k>2$, $\gamma_k = 0.8\gamma_{k-1}$

b) $\dfrac{\gamma_2}{\gamma_0} = 0.8\dfrac{\gamma_1}{\gamma_0} + 0.6\dfrac{\sigma_e^2}{\gamma_0}$

$\rho_2 = 0.8\rho_1 + 0.6\dfrac{\sigma_e^2}{\gamma_0}$

MA_641_HW3

Siddharth Nilakhe

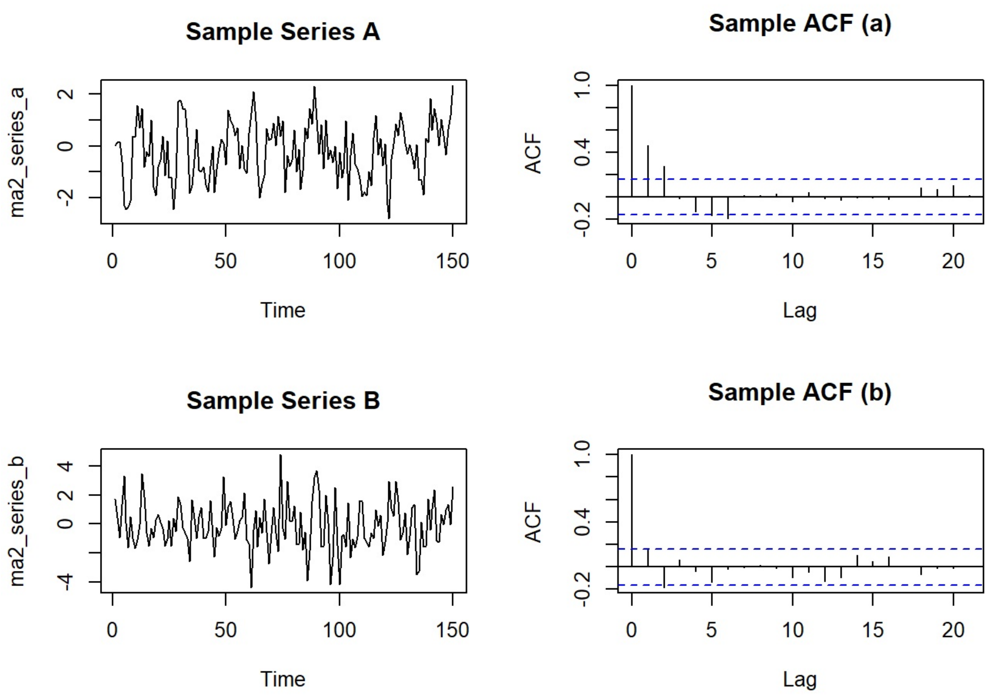
2023-10-19

Question 1

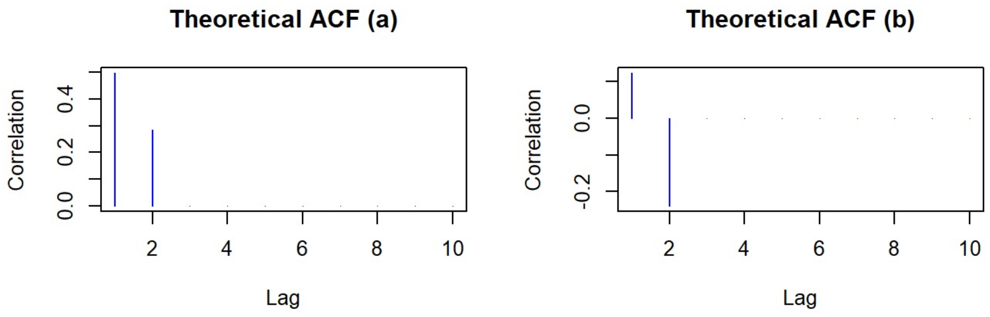
```
library(forecast)
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
set.seed(999)
ma2_coefficient1_a <- 0.5
ma2_coefficient1_b <- 1.2
ma2_coefficient2_a <- 0.4
ma2_coefficient2_b <- -0.7

ma2_series_a <- arima.sim(model = list(ma = c(ma2_coefficient1_a, ma2_coefficient2_a)), n = 150)
ma2_series_b <- arima.sim(model = list(ma = c(ma2_coefficient1_b, ma2_coefficient2_b)), n = 150)

par(mfrow = c(2, 2))
main_title <- "Question 1"

plot(ma2_series_a, type = "l", main = "Sample Series A")
acf(ma2_series_a, main = "Sample ACF (a)")

plot(ma2_series_b, type = "l", main = "Sample Series B")
acf(ma2_series_b, main = "Sample ACF (b)")
```

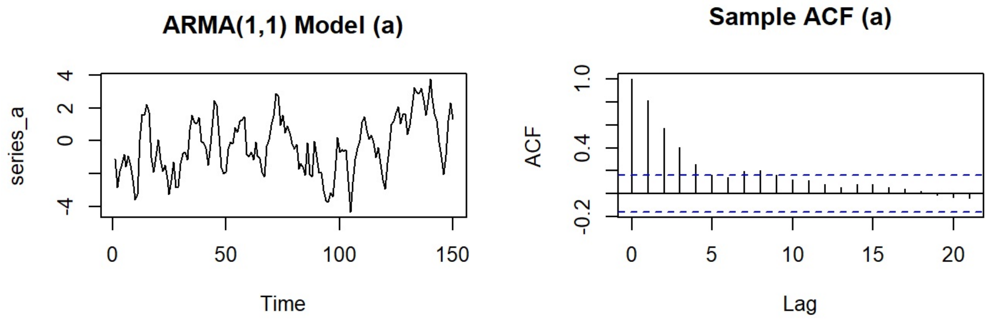


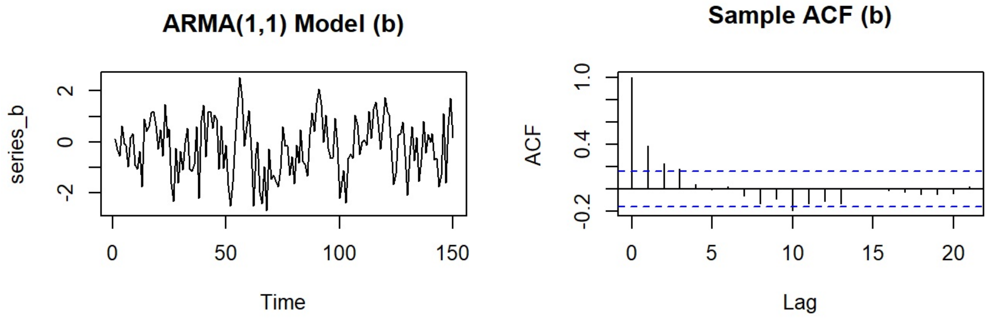

```
acf_theoretical_a <- ARMAacf(ma = c(ma2_coefficient1_a, ma2_coefficient2_a), lag.max = 10)
acf_theoretical_b <- ARMAacf(ma = c(ma2_coefficient1_b, ma2_coefficient2_b), lag.max = 10)

plot(acf_theoretical_a[2:length(acf_theoretical_a)], type = 'h', xlab = "Lag", ylab = "Correlation", col = "blue"
, main = "Theoretical ACF (a)")
plot(acf_theoretical_b[2:length(acf_theoretical_b)], type = 'h', xlab = "Lag", ylab = "Correlation", col = "blue"
, main = "Theoretical ACF (b)")
```



Question 4

```r
library(forecast)
set.seed(999)

phi_a <- 0.7
phi_b <- 0.7
theta_a <- 0.4
theta_b <- -0.4

series_a <- arima.sim(model = list(ar = phi_a, ma = theta_a), n = 150)
series_b <- arima.sim(model = list(ar = phi_b, ma = theta_b), n = 150)

par(mfrow = c(2, 2))
plot(series_a, type = "l", main = "ARMA(1,1) Model (a)")
acf(series_a, main = "Sample ACF (a)")

plot(series_b, type = "l", main = "ARMA(1,1) Model (b)")
acf(series_b, main = "Sample ACF (b)")
```





```r
theoretical_a <- ARMAacf(ar = phi_a, ma = theta_a, lag.max = 10)
theoretical_b <- ARMAacf(ar = phi_b, ma = theta_b, lag.max = 10)

plot(theoretical_a[2:length(theoretical_a)], type = 'h', xlab = "Lag", ylab = "Correlation", col = "red", main =
"Theoretical ACF (a)")
plot(theoretical_b[2:length(theoretical_b)], type = 'h', xlab = "Lag", ylab = "Correlation", col = "red", main =
"Theoretical ACF (b)")
```