# PARABOLA

CHAPTER 3

Geometry

- CONIC SECTIONS • PARABOLA IN STANDARD FORM • TANGENT AND NORMAL • CHORD OF CONTACT • POLE AND POLAR • PARAMETRIC EQUATIONS

## 3.0 CONIC SECTIONS

The famous Greek mathematician Euclid, the father of creative geometry studied various plane sections of a right circular cone about 300 B.C, and discovered some remarkable curves which are known as conic sections or conics.

## 3.1 DEFINITION OF A CONIC

In fact circle, parabola, ellipse, hyperbola, a pair of a straight lines, a straight line and a point are called as conic sections because each is a section of a double napped right circular cone with a plane. These curves have a very wie range of applications in planetary motion, design of telescopes and antennas, reflectors in flash lights etc.

More generally, suppose the cutting plane makes an angle '$\beta$' with the axis of the cone and suppose the generating angle of the cone is $\alpha$. Then the section is

(i) a circle if $\beta = \frac{\pi}{2}$, (fig. 3.1 (a) (ii) an ellipse if $\alpha < \beta < \frac{\pi}{2}$, (fig. 3.1 (b)

(iii) a parabola if $\alpha = \beta$, (fig. 3.1 (c) (iv) a hyperbola if $0 < \leq \beta < \alpha$, (fig. 3.1 (d)

(v) **Degenerated conic sections**

We get the degenerated sections when the plane passes through the vertex of the cone

(a) a point when $\alpha < \beta \leq \frac{\pi}{2}$, (fig. 3.1 (2)

(b) a straight line when $\beta = \alpha$, (Fig. 3.1 (f) a generator of the cone.

(c) a pair of intersecting straight lines when $0 \leq \beta < \alpha$, (fig. 3.1(g), 3.1 (h))
it is the degenerated case of a hyperbola.

**Note:** ***A pair of parallel straight lines, however, is not a conic section as there is no plane which cuts the cone along two parallel lines.***

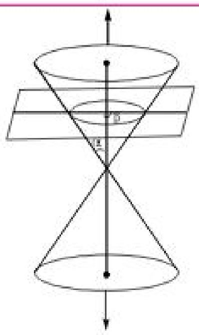


Fig.3.1(a)

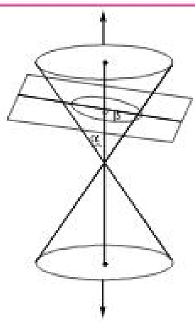


Fig.3.1(b)

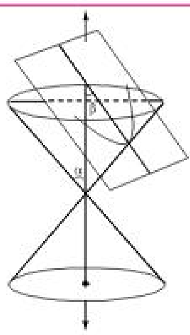


Fig.3.1(c)

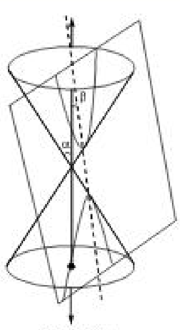


Fig.3.1(d)

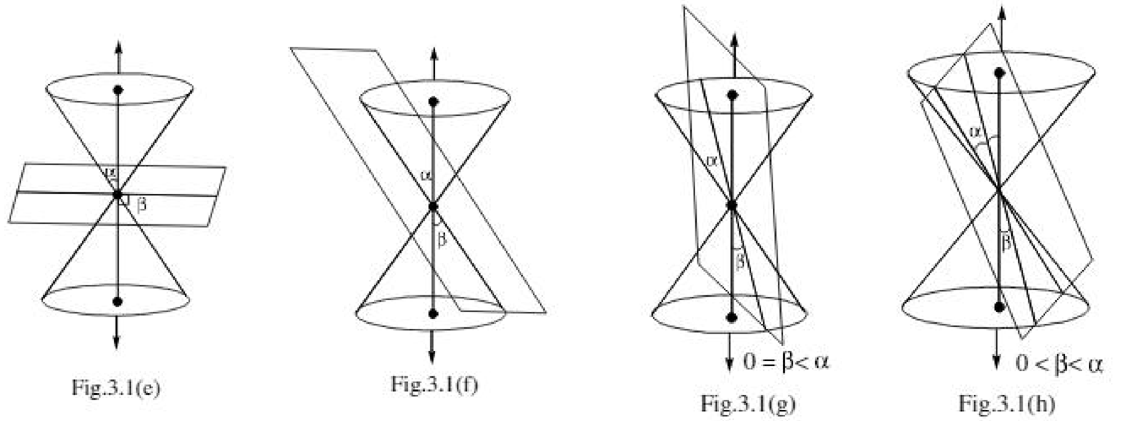



**Fig. 3.1**

***A conic section, can also be defined as the locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line are in constant ratio.***

## 3.2 DEFINITION OF A CONIC

**The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always a constant is called a *conic section* or a *conic*.**

Let $S$ be a fixed point and $L$ be a fixed line $P$ be any point in the plane. Let $M$ be the projection of $P$ on $L$. Then the locus of $P$ such that $\frac{SP}{PM} = a$ constant, is called a conic.

The fixed point $S$ is called the ***focus,*** the fixed line $L$ is called the ***directrix*** and the constant ratio, denoted by $e$, is called the ***eccentricity*** of the conic.

***If*** $e = 1$, ***the conic is a parabola***

***If*** $0 < e < 1$, ***the conic is an ellipse***

***If*** $e > 1$, ***the conic is a hyperbola***

***If*** $e = 0$, ***the conic is a circle***

***If*** $e \to \infty$, ***the conic is a pair of straight lines***

## 3.3 EQUATION OF A CONIC

Let $S(\alpha,\beta)$ be the focus and $L \equiv ax+by+c=0$ be the directrix and $e$ be the eccentricity of a conic. Let $P(x,y)$ be any point on the conic. Let $M$ be the projection of $P$ on $L$ .

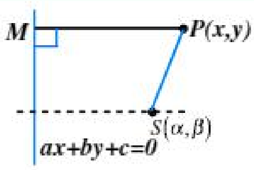



*Fig 3.2*

Then, by the definition of the conic

$$\frac{SP}{PM} = e \Rightarrow SP^2 = e^2.PM^2$$

$$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(ax+by+c)^2}{(a^2+b^2)}$$

$\therefore$ The locus of $P$ is $(a^2+b^2)[(x-\alpha)^2+(y-\beta)^2] = e^2(ax+by+c)^2$

On simplification, we get the equation of a conic as a second degree equation in $x$ and $y$. Therefore, the equation of a conic can be taken as $ax^2+2hxy+by^2+2gx+2fy+c=0$

We shall now define some important terms that frequently occur in the discussion of various properties of a conic.

***Definitions :***

***Principal Axis (Axis) :*** The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

***Vertex (Vertices) :*** The point(s) of intersection of the conic section and the axis is (are) called the vertex (vertices) of the conic.

***Chord :*** The line segment joining any two points on the conic is called a chord.

***Focal chord :*** Any chord passing through the focus is called a focal chord of the conic section.

***Double ordinate :*** A chord passing through a point $P$ on the conic and perpendicular to the axis of the conic is called the double ordinate of the point $P$.

***Latus rectum :*** The double ordinate passing through the focus is called the latus rectum of the conic.

***Centre :*** The point which bisects every chord of the conic passing through it is called the centre of the conic.

We know that the equation of a conic is a second degree equation in $x$ and $y$. But a second degree equation in $x$ and $y$ need not always represent a conic. Given that a general second degree equaion in $x$ and $y$ represents a conic, the following conditions help us to identify the conic.

**To Identify a conic :** Let the general second degree equation in $x$ and $y$, *viz.,*

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ --(1) represent a conic. Based on the following conditions satisfied by the coefficients of (1) we can identify the conic.

Let $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$.

If $\Delta = 0$ the curve represented by (1) is called a ***degenerate conic*** and if $\Delta \neq 0$, it is called a ***non - degenerate conic.***

| Condition | | Name of the conic |
|---|---|---|
| i) | $\Delta = 0$ and $h^2 = ab$ | a pair of parallel lines |
| ii) | $\Delta = 0$ and $h^2 > ab$ | a pair of intersecting lines |
| iii) | $\Delta = 0$ and $h^2 < ab$ | a pair imaginary lines intersecting at a real point |
| iv) | $\Delta \neq 0$, $h = 0$, $a = b$ | a circle |
| v) | $\Delta \neq 0$, $h^2 = ab$ | a parabola |
| vi) | $\Delta \neq 0$, $h^2 < ab$ | an ellipse (or an empty set) |
| vii) | $\Delta \neq 0$, $h^2 > ab$ | a hyperbola |
| viii) | $\Delta \neq 0$, $h^2 > ab$, | a rectangular hyperbola and $a + b = 0$ |

***To find the centre of a conic :***

The centre of the conic represented by (1) can be found by solving the equations $\frac{\partial S}{\partial x} = ax + hy + g = 0$, $\frac{\partial S}{\partial y} = hx + by + f = 0$ and is given by $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

## SOLVED EXAMPLES

**1. What conic does the equation $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent?**

**Sol.** Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

we have $a = 13$, $h = -9$, $b = 37$, $g = 1$, $f = 7$, $c = -2$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162 = 1600 \neq 0$

$h^2 = (-9)^2 = 81$ and $ab = 13 \times 37 = 481$

It is clear that $\Delta \neq 0$ and $h^2 < ab$

Hence the given equation represents an ellipse.

**2. Find the conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$ ?**

**Sol.** The given equation is $\sqrt{ax} + \sqrt{by} = 1$

Squaring both sides, we get $ax + by + 2\sqrt{abxy} = 1 \Rightarrow ax + by - 1 = -2\sqrt{abxy}$

Again squaring both sides, we get $(ax + by - 1)^2 = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 + 1 + 2abxy - 2by - 2ax = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 - 2abxy - 2ax - 2by + 1 = 0$

$\Rightarrow a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0$ --(1)

Comparing the equation (1) with the equation

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$,

$A = a^2, H = -ab, B = b^2, G = -a, F = -b, C = 1$

$\therefore \Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2$

$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2$

$= -4a^2b^2 \neq 0$ and $H^2 = a^2b^2 = AB$

So we have $\Delta \neq 0$ and $H^2 = AB$.

Hence the given equation represents a parabola.

**3. If the equation $x^2 - y^2 - 2x + 2y + \lambda = 0$ represents a degenerate conic then find the value of $\lambda$.**

**Sol.** For degenrate conic $\Delta = 0$

Comparing the given equation of conic with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore a = 1, b = -1, h = 0, g = -1, f = 1, c = \lambda$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or

$(1)(-1)(\lambda) + 0 - 1\times(1)^2 + 1\times(-1)^2 - \lambda(0)^2 = 0$ or $-\lambda - 1 + 1 = 0$

$\therefore \lambda = 0$

**4. If the equation $x^2 + y^2 - 2x - 2y + c = 0$ represents an empty set then find the value of $c$.**

**Sol.** For emtpy set $\Delta \neq 0$ and $h^2 < ab$

Now comparing the given equation of conic with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c' = 0$

then $a = 1, h = 0, b = 1, g = -1, f = -1, c' = c$

$h^2 = 0,\ ab = 1 \Rightarrow h^2 < ab$ is satisfied

$\Delta = abc' + 2fgh - af^2 - bg^2 - c'h^2 \neq 0$

$\Rightarrow (1)(1)(c) + 0 - 1\times(-1)^2 - 1\times(-1)^2 - 0 \neq 0$

$\Rightarrow c - 2 \neq 0 \quad \therefore c \neq 2$, Hence $c \in R - \{2\}$

**5. If the equation of conic $2x^2 + xy + 3y^2 - 3x + 5y + \lambda = 0$ represents a single point, then find the value of $\lambda$.**

**Sol.** For the equation to represent a single point $h^2 < ab$ and $\Delta = 0$. Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

$$a = 2,\ h = \frac{1}{2},\ b = 3,\ g = \frac{-3}{2},\ f = \frac{5}{2},\ c = \lambda$$

$$\therefore ab - h^2 = 6 - \frac{1}{4} = \frac{23}{4} > 0 \text{ and } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= (2)(3)(\lambda) + 2\times\frac{5}{2}\times -\frac{3}{2}\times\frac{1}{2} - 2\times\frac{25}{4} - 3\times\frac{9}{4} - \lambda\times\frac{1}{4}$$

$$= 6\lambda - \frac{15}{4} - \frac{25}{2} - \frac{27}{4} - \frac{\lambda}{4} = \frac{23\lambda}{4} - 23 = 0 \qquad \therefore \lambda = 4$$

**6. For what value of $\lambda$ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two intersecting straight lines ? If $\lambda = 17$ then what does this equation represent ?**

**Sol.** Comparing the given equation of conic with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore\ a = 0,\ b = 0,\ h = 1,\ g = 2,\ f = -3,\ c = \lambda$

For two intersecting lines $\Delta = 0, h^2 \neq ab$

$\therefore ab = 0, h = 1 \Rightarrow h^2 \neq ab \quad \therefore ab - h^2 = -1 \neq 0$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2\times -3\times 2\times 1 - 0 - 0 - \lambda(1)^2 = -12 - \lambda$

$\Delta = 0 \Rightarrow \lambda = -12$

If $\lambda = 17$, then the given equation of conic becomes $2xy + 4x - 6y + 17 = 0$

It is clear that $\Delta \neq 0$, $h^2 = 1, ab = 0 \Rightarrow h^2 > ab$

Hence the given equation represents a hyperbola for $\lambda = 17$

**7. Find the centre of the conic $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$**

*Sol.* Let $f(x, y) \equiv 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

Differentiating partially w.r.t to $x$ and $y$ we get,

$\frac{\partial f}{\partial x} = 28x - 4y - 44$ and $\frac{\partial f}{\partial y} = -4x + 22y - 58$

For the centre of the conic $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 28x - 4y - 44 = 0$ i.e., $7x - y - 11 = 0$ --(1)

and $-4x + 22y - 58 = 0$ i.e., $-2x + 11y = 29$ --(2)

Solving (1) and (2) we get $x = 2$ and $y = 3$

$\therefore$ Centre is (2,3)

**2nd Method :**

Comparing the given conic with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 14, h = -2, b = 11, g = -22, f = -29, c = 71$

$\therefore$ Centre $= \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

$= \left(\frac{(-2)(-29) - (11)(-22)}{(14)(11) - (-2)^2}, \frac{(-22)(-2) - (14)(-29)}{(14)(11) - (-2)^2}\right) = (2, 3)$

## 3.4 PARABOLA

*Definition*

**The locus of a point in a plane such that its distance from a fixed point (focus) is always equal to its distance from a fixed straight line (directrix) is called a *Parabola.***

Let $S$ be the focus $L$ be the directrix and $P$ be a point in the plane of $S$ and $L$. Let $M$ be the projection of $P$ on $L$. Then the locus of $P$ such that $SP = PM$ is a parabola. If $Z$ is the projection of $S$ on $L$. Then $\overleftrightarrow{SZ}$ is the principal axis of the parabola.

We shall now derive the equation of a parabola in standard form. A conic is said to be in standard form if the principal axis of the conic is along the $X$-axis and the centre of the conic is at the origin.

## 3.5 EQUATION OF A PARABOLA

**THEOREM-3.1** ** The equation of a parabola in the standard form is $y^2=4ax$ (March-15,17& May-18)

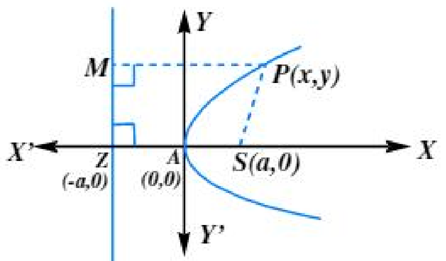



*Fig 3.3*

***Proof :***

Let $S$ be the focus and $L = 0$ be the directrix of the parabola. Let $S$ be on the right side of $L = 0$.

Let $P$ be a point on the parabola. Let $M$ and $Z$ be the projections (feet of the perpendiculars) of $P$ and $S$ respectively on the directrix.

Let $N$ be the projection of $P$ on $\overleftrightarrow{SZ}$. Let $A$ be the mid point of $\overline{SZ}$. Then $SA = AZ$ and hence, by definition, $A$ lies on the parabola ($A$ is called the vertex of the parabola).

Let us consider $\overrightarrow{AS}$ the principal axis as the positive $X$-axis and $\overrightarrow{Ay}$ perpendicular to $\overrightarrow{AS}$, as the positive $y$-axis. Then $A = (0, 0)$, the origin.

Let $AS=a$ then $S = (a, 0)$, $Z = (-a, 0)$ and the equation of the directrix is $x + a = 0$

Let $P = (x_1, y_1)$ $P$ lies on the parabola

$$\Rightarrow \frac{SP}{PM} = e = 1 \quad \Rightarrow SP = PM$$

But $PM = NZ = NA + AZ = x_1 + a$

Now, $SP = PM \Rightarrow \sqrt{(x_1 - a)^2 + y_1^2} = (x_1 + a)$

$$SP = PM \Rightarrow \sqrt{(x_1 - a)^2 + y_1^2} = (x_1 + a) \Rightarrow y_1^2 = 4ax_1$$

$\therefore$ The locus of $P$ is $y^2 = 4ax$.

This is the equation of the parabola in standard from.

**Note :**

*If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the fixed line. We call this straight line as degenerate case of the parabola.*

***Observation*** : For the parabola $y^2 = 4ax$

i) vertex $A = (0, 0)$

ii) focus $S = (a, 0)$

iii) equation of the directrix : $x + a = 0$

iv) equation of the principal axis : $y = 0$

The equation $y^2 = 4ax$ is called the *simplest form* of the equation of a parabola.

***Remarks:***

1. If the focus $S$ is on the left side of the directrix $L = 0$ then the equation of the parabola with vertex at the origin and axis along the $x$-axis is $y^2 = -4ax$ $(a > 0)$.

   In this case $S = (-a, 0)$, $z = (a, 0)$, Directrix is $x - a = 0$

2. The vertex being the origin, if the axis of the parabola is taken as the $y$-axis then the equation of the parabola is $x^2 = 4ay$ or $x^2 = -4ay$ according as the focus $S$ is above or below the $x$-axis.

## 3.6 NATURE OF THE CURVE $y^2 = 4ax$

Let $C$ be the curve represented by $y^2 = 4ax$ where $a > 0$. Then

i) $(x, y) \in C \Leftrightarrow (x, -y) \in C$. Thus the curve is symmetric about the $x$-axis. The $x$-axis is the principal axis of the parabola.

ii) $(x, y) \in C$ and $y = 0 \Rightarrow x = 0$. Thus the curve meets the $x$-axis at only one point (0,0). Hence the parabola has only one vertex.

iii) If $x < 0$ then there exists no $y \in R \ni (x, y) \in C$. Thus the parabola doesnot lie on the left of $y$-axis (i.e., in the second and third quadrants).

iv) If $x > 0$ then $y^2 = 4ax \Rightarrow$ $y$ has two real values equal in magnitude but opposite in sign. Hence the parabola lies in the first and fourth quadrants.

v) $(x, y) \in C,\ x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0, 0$. Thus $y$-axis meets the parabola in two coincident points and hence $y$-axis touches the parabola at (0, 0).

$\therefore x = 0$ is the tangent to the parabola at the vertex $A(0, 0)$.

vi) $x \to \infty \Rightarrow y^2 \to \infty \Rightarrow y \to \pm\infty$. Therefore the curve is not bounded (not closed) on the right side of the $y$-axis.

***Definitions :***

***Chord :*** The line segment joining two distinct points on a parabola is called a Chord.

***Focal Chord :*** A chord of the parabola passing through the focus is called a focal chord of the parabola.

***Double ordinate :*** A chord passing through a point $Q$ on the parabola and perpendicular to the axis of the parabola is called double ordinate of the point $Q$.

***Latus rectum :*** A focal chord of the parabola perpendicular to the principal axis of the parabola is called its latus rectum. If the latus rectum meets the parabola in $L$ and $L'$ then $LL'$ is called the *length of the latus rectum.*

***Focal Distance :*** The distance of a point $P$ on the parabola from its focus $S$ is called the *focal distance* of the point $P$.

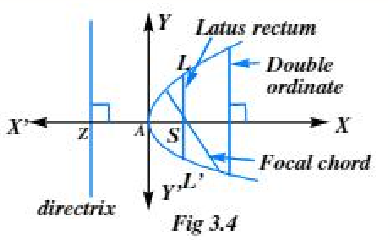


Fig 3.4

**THEOREM-3.2** **The length of the latus rectum of the parabola is $y^2 = 4ax$ is $4a$.**

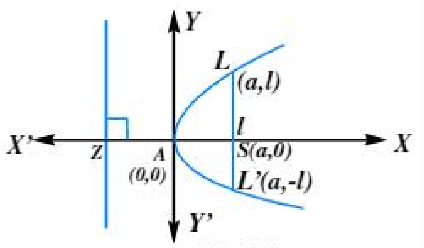


Fig 3.5

***Proof:*** Let $LL'$ be the length of the latusrectum of the parabola $y^2 = 4ax$.

If $SL = l$, then $l > 0$, $L = (a, l)$ and $L'(a, -\ell)$

$L$ lies on the parabola $y^2 = 4ax$

$\Rightarrow \ell^2 = 4a(a) \Rightarrow \ell = 2a \Rightarrow SL = 2a$

$\therefore LL' = 2SL = 4a$

Further, $L = (a, 2a), L' = (a, -2a)$

**Note :** *Latusrectum is the shortest focal chord.*

**THEOREM-3.3** **The focal distance of $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $|x_1 + a|$**

***Proof :*** $P(x_1, y_1)$ is a point on the parabola $y^2 = 4ax \Rightarrow y_1^2 = 4ax_1$, Focus $S = (a, 0)$

Now the focal distance of $P$ is $SP = \sqrt{(x_1 - a)^2 + (y_1 - 0)^2} = \sqrt{(x_1 - a)^2 + y_1^2}$

$= \sqrt{(x_1 - a)^2 + 4ax_1} = \sqrt{(x_1 + a)^2} = |a + x_1|$

Hence the focal distance of the point $P(x_1, y_1)$ is $|x_1 + a_1|$.

**Various forms of the parabola**

**I) Parabola $y^2 = 4ax$ $(a > 0)$**

i) Vertex $A = (0,0)$

ii) Focus $S = (a, 0)$

iii) Equation of the directrix is $x + a = 0$

iv) Equation of the axis is $y = 0$

v) Equation of the tangant at the vertex is $x = 0$

vi) Length of the latusrectum $LL' = 4a$

vii) Extremities of latusrectum are $L(a, 2a)$ and $L'(a, -2a)$

viii) Equation of latusrectum is $x = a$

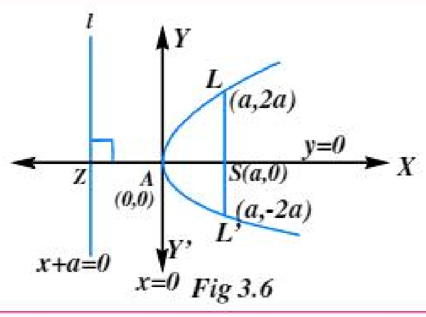



Fig 3.6

**II) Parabola $y^2 = -4ax$ $(a > 0)$**

i) Focus $S = (-a, 0)$

ii) Vertex $A = (0, 0)$

iii) Equation of the directrix is $x - a = 0$

iv) Equation of the axis is $y = 0$

v) Equation of the tangant at the vertex is $x = 0$

vi) Length of the latusrectum is $LL' = 4a$

vii) Extremities of latusrectum are $L(-a, 2a)$ and $L'(-a, -2a)$

viii) Equation of latusrectum is $x = -a$

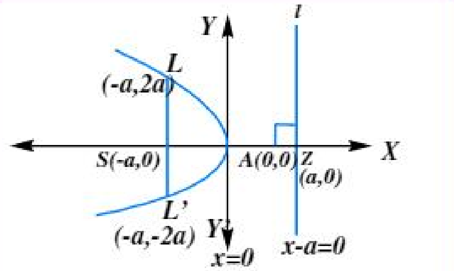


Fig 3.7

### III) Parabola $x^2 = 4ay$ $(a > 0)$

i) Vertex $A = (0, 0)$,

ii) Focus $S = (0, a)$

iii) Equation of the directrix is $y + a = 0$

iv) Equation of the axis is $x = 0$

v) Equation of the tangant at the vertex is $y = 0$

vi) Length of the latusrectum is $LL' = 4a$

vii) Extremities of latusrectum are $L(2a, a)$ and $L'(-2a, a)$

viii) Equation of latusrectum is $y = a$

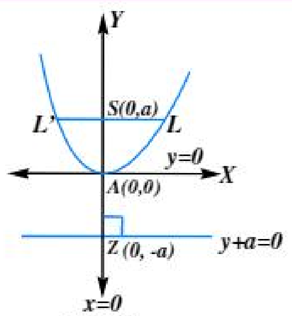


Fig 38

### IV) Parabola $x^2 = -4ay$ $(a > 0)$

i) Vertex $A = (0, 0)$,

ii) Focus $S$ $(0, -a)$

iii) Equation of the directrix is $y - a = 0$

iv) Equation of the axis is $x = 0$

v) Equation of the tangant at the vertex is $y = 0$

vi) Length of the latusrectum is $LL' = 4a$

vii) Extremities of latusrectum are $L$ $(2a, -a)$ and $L'(-2a, -a)$

viii) Equation of latusrectum is $y = -a$

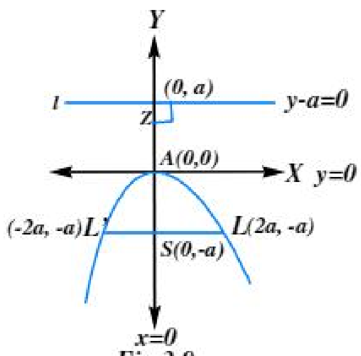


Fig 3.9

**Note :**
*Equation of a parabola with axis parallel to x-axis is* $x = ay^2 + by + c$

### V) Parabola $(y - k)^2 = 4a\,(x - h)\ (a > 0)$

i) Vertex $A = (h,\ k)$

ii) Focus $S = (h+a,k)$

iii) Equation of the directrix is $x = h - a$

iv) Equation of the axis is $y = k$

v) Equation of the tangant at the vertex is $x = h$

vi) Length of the latusrectum is $LL' = 4a$

vii) Extremities of latusrectum are $L'(a+h, k-2a)$

viii) Equation of latusrectum is $x = h + a$

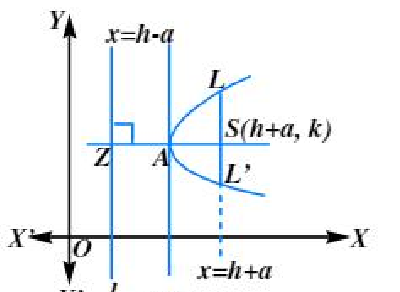


Fig 3.10

**Note :**
*Equation of a parabola with axis parallel to y-axis is* $y = ax^2 + bx + c$

### VI) Parabola $(x - h)^2 = 4a\,(y - k)\ (a > 0)$

i) Vertex $A = (h,\ k)$

ii) Focus $S = (h,\ a + k)$

iii) Equation of the directrix is $y = k - a$

iv) Equation of the axis is $x = h$

v) Equation of the tangant at the vertex is $y = k$

vi) Length of the latusrectum is $LL' = 4a$

vii) Extremities of latusrectum are $L(h+2a,\ k+a)$ and $L'(h-2a, k+a)$

viii) Equation of latusrectum is $y = k + a$

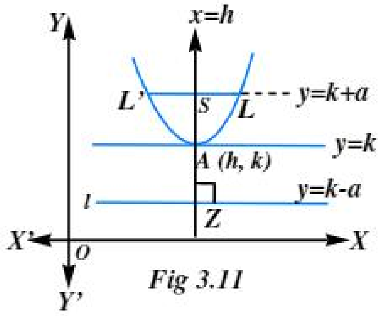


Fig 3.11

### VII) Parabola

$$(x-\alpha)^2+(y-\beta)^2=\frac{(\ell x+my+n)^2}{\ell^2+m^2}$$

i) Focus $(\alpha,\beta)$

ii) Equation of the directrix $lx+my+n = 0$

iii) Axis of the parabola $m(x-\alpha)-\ell(y-\beta)=0$

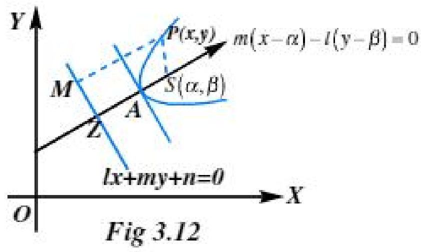


Fig 3.12

**Note :**

*Latusrectum of* $y=ax^2+bx+c$ *(or)* $x=ay^2+by+c$ *is* $\frac{1}{|a|}$

**Note**

*i) The equation (1) can be reduced to the form* $(mx-\ell y)^2+2gx+2fy+c=0$ *which is a second degree equation in x and y. From this we can conclude that if the second degree terms in the general equation forms a perfect square then that equation represents a parabola.*

*ii) Length of latusrectum of* $ax^2+bx+my+n=0$ *is* $\left|\frac{m}{a}\right|$

*iii) Length of latusrectum of* $ay^2+by+mx+n=0$ *is* $\left|\frac{m}{a}\right|$

*iv) Let P be a point ont he parabola and M,N be its projections on the principal axis and tangent at the vertex. The equation of parabola with latusrectum 4a is* $PM^2=(4a)PN$.

*v) Let* $lx+my+n=0$ *and* $mx-ly+k=0$ *be the tangent at vertex and axis of the parabola with latusrectum 4a. The equation of parabola is* $(mx-ly+k)^2=(4a)\sqrt{\ell^2+m^2}(\ell x+my+n)$

*vi) The equation of the parabola whose axis is parallel to*

*a) x-axis is of the form* $x=ay^2+by+c$

*b) y-axis is of the form* $y=ax^2+bx+c$

***Notation :***

Here after the following notation will be adopted throughout this chapter

i) $S\equiv y^2-4ax$ ii) $S_1\equiv yy_1-2a(x+x_1)$

iii) $S_{12}\equiv y_1y_2-2a(x_1+x_2)$ iv) $S_{11}\equiv y_1^2-4ax_1$

In this notation, the equation of a parabola in the standard form is $S=0$.

## 3.7 POSITION OF A POINT WITH RESPECT TO PARABOLA

A parabola divides the plane in to three regions.

i) The region containing the focus, called interior of the parabola.

ii) The region consisting of the curve itself.

iii) The region containing the remaining portion of the plane, called the exterior of the parabola.

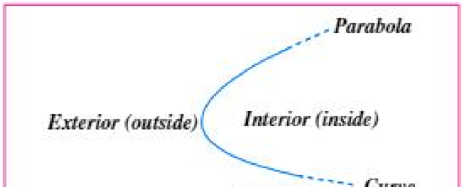



Fig 3.13

Let $P(x_1, y_1)$ be a point in the plane of the parabola $y^2 = 4ax$. Let $PN$ be the ordinate of $P$ meeting the curve in $Q$ (*Fig.3.18*). Then $N = (x_1, 0)$, $Q = (x_1, \sqrt{4ax_1})$ and $P(x_1, y_1)$.

$\therefore NP = y_1$ and $NQ = \sqrt{4ax_1}$

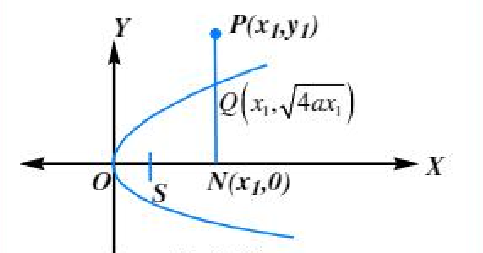



Fig 3.14

i) The point $P(x_1, y_1)$ lies outside the parabola i.e., $P$ is an external point of the parabola if $NP > NQ \Rightarrow NP^2 > NQ^2$

ie., if $y_1^2 > 4ax_1 \Rightarrow y_1^2 - 4ax_1 > 0 \Rightarrow S_{11} > 0$

ii) The point $(x_1, y_1)$ lies on the parabola if $NP = NQ \Rightarrow NP^2 = NQ^2$

i.e., if $y_1^2 = 4ax_1 \Rightarrow y_1^2 - 4ax_1 = 0 \Rightarrow S_{11} = 0$

iii) The point $P(x_1, y_1)$ lies inside the parabola i.e., $P$ is an internal point of the parabola if $NP < NQ \Rightarrow NP^2 < NQ^2$

i.e., if $y_1^2 < 4ax_1 \Rightarrow y_1^2 - 4ax_1 < 0 \Rightarrow S_{11} < 0$

Thus $P(x_1, y_1)$ is a point in the plane of the parabola $S = 0$ then

i) $P$ lies outside the parabola $\Leftrightarrow$ $S_{11} > 0$

ii) $P$ lies on the parabola $\Leftrightarrow$ $S_{11} = 0$

iii) $P$ lies inside the parabola $\Leftrightarrow$ $S_{11} < 0$

## SOLVED EXAMPLES

**1. Find the coordinates of the vertex and focus and the equations of the directrix and axis of the parabolas**

**i) $y^2 = 16x$ ii) $x^2 = -4y$**

**iii) $3x^2 - 9x + 5y - 2 = 0$ iv) $y^2 - x + 4y + 5 = 0$**

**Sol.** i) The given parabola is $y^2 = 16x$ --(1)

Comparing (1) with $y^2 = 4ax$, we get $4a = 16 \Rightarrow a = 4$

$\therefore$ The coordinates of Vertex = (0, 0), Focus $S = (a, 0) = (4, 0)$

Equation of directrix : $x + a = 0 \Rightarrow x + 4 = 0$

Equation of axis : $y = 0$

ii) $x^2 = -4y$

Comparing with $x^2 = -4ay$, we get $4a = 4 \Rightarrow a = 1$

The coordinates of vertex = (0, 0)

The coordinates of focus = $(0, -a) = (0, -1)$

The equation of directrix : $y - a = 0 \Rightarrow y - 1 = 0$

The equation of axis : $x = 0$

iii) $3x^2 - 9x + 5y - 2 = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 = -\frac{5}{3}\left(y - \frac{7}{4}\right)$

Comparing with $(x-h)^2 = -4a(y-k)$, we get $a = \frac{5}{12}, h = \frac{3}{2}, k = \frac{7}{4}$

Coordinates of the vertex $(h,k) = \left(\frac{3}{2}, \frac{7}{4}\right)$

Coordinates of the focus $S = (h, k - a) = \left(\frac{3}{2}, \frac{7}{4} - \frac{5}{12}\right) = \left(\frac{3}{2}, \frac{4}{3}\right)$

Equations of the directrix : $y - k - a = 0 \Rightarrow 6y = 13$

Equations of the axis : $x - h = 0 \Rightarrow 2x - 3 = 0$

iv) $y^2 - x + 4y + 5 = 0 \Rightarrow (y+2)^2 = (x-1)$

Comparing with $(y-k)^2 = 4a(x-h)$

$$a = \frac{1}{4}, h = 1, k = -2$$

Vertex $A = (h,k) = (1, -2)$

Focus $S = (h + a, k) = \left(1 + \frac{1}{4}, -2\right) = \left(\frac{5}{4}, -2\right)$

Equation of directrix is $x - h + a = 0 \Rightarrow 4x - 3 = 0$

Equation of axis is $y - k = 0 \Rightarrow y + 2 = 0$

***2. Find the equation of parabola whose focus is (1, –7) and the vertex is (1, –2).**

**Sol.** Given $S = (1, -7)$ and $A = (1, -2)$

The $x$ coordinates of $A$ and $S$ are equal

$\Rightarrow$ the axis $AS$ parallel to $y$-axis.

$\therefore$ The equation of the parabola can be taken in the form $(x - \alpha)^2 = \pm 4a(y - \beta)$ whose vertex is $A = (\alpha, \beta) = (1, -2)$

$\therefore AS = \sqrt{(-2+7)^2} = 5 \Rightarrow a = 5 \Rightarrow 4a = 20$

The focus $S$ lies below the vertex $A$

$\therefore$ The equation of the required parabola is $(x-1)^2 = -20(y+2)$

***3. Find the equation of parabola whose focus is (–2, 3) and the directrix is $2x + 3y = 4$**

**Sol.** Given $S = (-2, 3)$ and the directrix $L = 2x + 3y - 4 = 0$

Let $P(x_1, y_1) \in$ Parabola and $PM$ perpendicular to the directrix $L$

$$SP^2 = PM^2 \Rightarrow (x_1+2)^2 + (y_1-3)^2 = \left[\frac{2x_1+3y_1-4}{\sqrt{4+9}}\right]^2$$

$$\Rightarrow 13(x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9) = (2x_1 + 3y_1 - 4)^2$$

$$\Rightarrow 9{x_1}^2 - 12x_1y_1 + 4y_1^2 + 68x_1 - 54y_1 + 153 = 0$$

$\therefore$ Required parabola is $9x^2 - 12xy + 4y^2 + 68x - 54y + 153 = 0$

*****4. Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is $\frac{5}{2}$.**

*Remember :*

*The focal distance of the point $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $SP = |x_1 + a|$*

**Sol.** $y^2 = 2x \Rightarrow y^2 = 4\left(\frac{1}{2}\right)x \Rightarrow a = \frac{1}{2}$

Let $P(x_1, y_1)$ represent the required points.

The focal distance of $P = \frac{5}{2} \Rightarrow x_1 + \frac{1}{2} = \frac{5}{2} \Rightarrow x_1 = 2$

$\therefore y_1^2 = 2x_1 \Rightarrow y_1^2 = 4 \Rightarrow y_1 = \pm 2$

$\Rightarrow$ The required points are (2,–2) and (2,2)

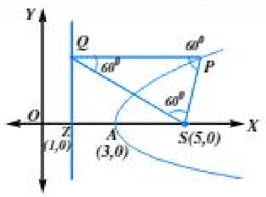



*Fig 3.15*

***5. If $Q$ is the foot of perpendicular from a point $P$ on the parabola $y^2 = 8(x-3)$ to its directrix. $S$ is the focus of the parabola and if $SPQ$ is an equilateral triangle then find the length of the side of the triangle.**

**Sol.** Given parabola is $y^2 = 8(x-3)$ Vertex $A = (3, 0)$ Focus $S = (5, 0)$

$SPQ$ is an equilateral triangle $\Rightarrow \angle PSQ = 60^0 = \angle PQS \Rightarrow \angle SQZ = 30^0$

From the $\Delta SQZ$, $\sin 30^0 = \frac{SZ}{SQ} = \frac{4}{SQ} \Rightarrow SQ = 8$ which is the length of the side

***6. Find the equation of the parabola whose focus is (3, 5) vertex is (1, 3).**

**Sol.** Given $S = (3, 5)$ and $A = (1, 3)$ **(March-19)**

Let $\overrightarrow{SA}$ meet the directrix in $Z(h, k)$

Mid point of $SZ = A(1,3) \Rightarrow \left(\frac{h+3}{2}, \frac{k+5}{2}\right) = (1,3) \Rightarrow Z(h,k) = (-1,1)$

Slope of $SZ = \frac{5-1}{3+1} = 1 \Rightarrow$ Slope of directri $x = -1$

$\therefore$ Equation to the directrix passing through $Z(-1, 1)$ is

$$y - 1 = -1(x+1) \Rightarrow x + y = 0$$

Let $P(x_1, y_1) \in$ parabola and $PM$ perpendicular to the directrix.

Then $SP^2 = PM^2 \Rightarrow (x_1-3)^2 + (y_1-5)^2 = \left[\frac{x_1+y_1}{\sqrt{1+1}}\right]^2$

$$\Rightarrow 2(x_1^2 - 6x_1 + 9 + y_1^2 - 10y_1 + 25) = x_1^2 + 2x_1y_1 + y_1^2$$

$$\Rightarrow x_1^2 - 2x_1y_1 + y_1^2 - 12x_1 - 20y_1 + 68 = 0$$

$\therefore$ Equation of the locus of $P$ is the parabola $x^2 - 2xy + y^2 - 12x - 20y + 68 = 0$

***7. If $L$ and $L'$ are the ends of the latusrectum of the parabola $x^2 = 6y$, find the equations of $OL$ & $OL'$ where $O$ is the origin. Find also angle between them.**

**Sol.** For the given parabola $x^2 = 6y$, vertex = (0, 0) and $S = \left(0, \frac{3}{2}\right)$

If the end of the latus rectum is $\left(x, \frac{3}{2}\right)$ then

$$x_1^2 = 6\left(\frac{3}{2}\right) \Rightarrow x_1 = \pm 3 \Rightarrow L = \left(3, \frac{3}{2}\right), L' = \left(-3, \frac{3}{2}\right)$$

$\therefore$ Equation to $OL$ is $y = \dfrac{\left(\frac{3}{2}\right)}{3}x \Rightarrow y = \frac{1}{2}x \Rightarrow x - 2y = 0$

Equation to $OL'$ is $y = \frac{-1}{2}x \Rightarrow x + 2y = 0$

If $\theta$ is the angle between $OL$ and $OL'$ then

$$\cos\theta = \left|\frac{(1)(1)+(2)(-2)}{\sqrt{1+4}\sqrt{1+4}}\right| = \frac{3}{5} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{5}\right)$$

**8. Find the equation of the parabola whose axis is parallel to $y$-axis and which passes through points (4, 5), (–2, 11) and (–4, 21).**

**Sol.** The equation of the parabola with axis parallel to the $y$-axis is of the form $y = lx^2+mx+n$

Since the parabola (1) passes through the points (4, 5), (–2, 11) and (–4, 21).

We have $5 = 16l + 4m + n$, $11 = 4l - 2m + n$, $21 = 16l - 4m + n$

Solving the above three equations simultaneously, we get $\ell = \frac{1}{2}, m = -2, n = 5$.

Substituting these values in (1), the equation of the parabola is

$$x^2 - 4x - 2y + 10 = 0.$$

**9. The points (–3, 2) and (–3, 1) are the end points of latusrectum of a parabola then find the equation of parabola.**

*Remember :*

*If Latus rectum of parabola is parallel ot x-axis then equation of parabola is* $(x-h)^2 = \pm 4a(y-k)$

**Sol.** Let $L = (-3, 2)$ and $L' = (-3, 1)$ $\quad LL' = 1 \Rightarrow 4a = 1$

Focus $S$ = Mid point of $LL' \Rightarrow S = \left(-3, \frac{3}{2}\right)$

Equation to line $LL'$ is $x = -3 \Rightarrow x + 3 = 0$

$\therefore$ Let the equation to the directrix which is parallel to the line $\overline{LL'}$ be $x + k = 0$

Now, the perpendicular distance from $S\left(-3, \frac{3}{2}\right)$ to (1) $= \frac{1}{2}(LL')$

$\Rightarrow |-3+k| = \frac{1}{2} \Rightarrow k = 3 \pm \frac{1}{2}$

$\therefore$ Equation to the directrix is $x + 3 \pm \frac{1}{2} = 0 \Rightarrow x + \frac{7}{2} = 0$ and $x + \frac{5}{2} = 0$

Let $P(x, y)$ be any point on the parabola

i) The equation to the parabola having focus $S = \left(-3, \frac{3}{2}\right)$ and directrix :

$x + \frac{7}{2} = 0$, is $(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(x + \frac{7}{2}\right)^2$

$\Rightarrow (2y-3)^2 = (2x+7)^2 - 4(x+3)^2 \Rightarrow (2y-3)^2 = 4x + 13$

ii) The equation to the parabola having focus $S = \left(-3, \frac{3}{2}\right)$ and directrix : $x + \frac{5}{2} = 0$, is $(x+3)^2 + \left(y - \frac{3}{2}\right) = \left(x + \frac{5}{2}\right)^2$

$\Rightarrow (2y-3)^2 = (2x+5)^2 - 4(x+3)^2$

$\Rightarrow (2y-3)^2 = (4x+11)$

**10. If $(a^2, a-2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, –4) then find the set of all possible real values of $a$.**

**Sol.** If $P(a^2, a-2)$ lies inside the parabola, then $(a-2)^2 - 2a^2 < 0$

$\Rightarrow a^2 - 4a + 4 - 2a^2 < 0$

$\Rightarrow -a^2 - 4a + 4 < 0$, $a^2 + 4a - 4 > 0$

$\Rightarrow (a+2)^2 - (2\sqrt{2})^2 > 0$

$\Rightarrow (a+2) < -2\sqrt{2}$ or $a + 2 > 2\sqrt{2}$

$\Rightarrow a < -2 - 2\sqrt{2}$ or $a > 2\sqrt{2} - 2$ -- (1)

The point $P(a^2, a-2)$ and the origin $O(0, 0)$ are on the same side of the chord joining (2, 2) & (8, –4) i.e., $x + y - 4 = 0$

$\therefore (0 + 0 - 4)(a^2 + a - 2 - 4) > 0$

$\Rightarrow a^2 + a - 6 < 0 \Rightarrow (a+3)(a-2) < 0 \Rightarrow -3 < a < -2$ -- (2)

Also, from the ranges of the abscissa and the ordinate, of the region we have

$0 < a^2 < 8$ and $-4 < a < -2 < 2$

$\Rightarrow -2\sqrt{2} < a < 2\sqrt{2}$ and $-2 < a < 4 \Rightarrow -2 < a < 2\sqrt{2}$ -- (3)

From (1), (2), (3) we obtain that $-2 < a < -2 + \sqrt{2}$,

Hence, $a \in (-2, -2 + \sqrt{2})$

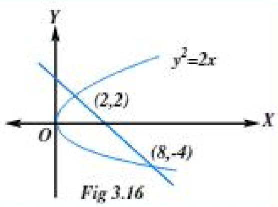


Fig 3.16

**11. Find the real values of a for which the point $(-2a, a+1)$ lies in the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$.**

**Sol.** If the point $(-2a, a+1)$ lies in the shaded region in fig,

Then $4a^2 + (a+1)^2 - 4 < 0$ and $(a+1)^2 - 4(-2a) < 0$

$\Rightarrow 5a^2 + 2a - 3 < 0$ and $a^2 + 10a + 1 < 0$

$\Rightarrow (a+1)(5a-3) < 0$ and $(a + 5 - 2\sqrt{6})(a + 5 + 2\sqrt{6}) < 0$

$\Rightarrow -1 < a < \frac{3}{5}$ and $-5 - 2\sqrt{6} < a < -5 + 2\sqrt{6}$

$\Rightarrow -1 < a < -5 + 2\sqrt{6}$

$\Rightarrow a \in (-1, -5 + 2\sqrt{6})$

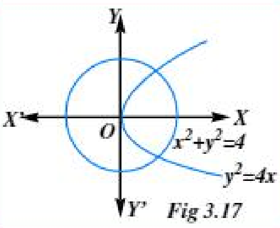


Fig 3.17

**12. Two parabolas $y^2 = 4a(x - \lambda_1)$ and $x^2 = 4a(y - \lambda_2)$ always touch each other, $\lambda_1$ and $\lambda_2$ being varaible parameters. Then show that the locus of their point of contact is a hyperbola.**

*Remember :*
*If angle between two Intersecting curves is $0^0$ then two curves are said to touch eachother.*

**Sol.** Let $P(x_1, y_1)$ be the point contact of the two parabolas.

Tangents as $P$ to the two parabolas are

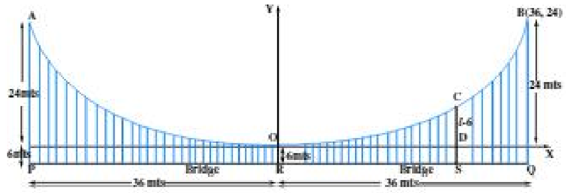
$$yy_1 = 2a(x + x_1) - 4a\lambda_1 \Rightarrow 2ax - yy_1 = 2a(2\lambda_1 - x_1) \quad \text{-- (1)}$$
$$xx_1 = 2a(y + y_1) - 4a\lambda_2 \Rightarrow xx_1 - 2ay = 2a(y_1 - 2\lambda_2) \quad \text{-- (2)}$$

Clearly (1) & (2) represent the same line

$$\therefore \frac{2a}{x_1} = \frac{y_1}{2a} \Rightarrow x_1 y_1 = 4a^2$$

Hence, the locus of $(x_1, y_1)$ is $xy = 4a^2$ which is a hyperbola.

**13. The cable of a uniformely loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 72 mt. long is supported by vertical wires attached to the cable, the longest being 30 mts. and the shortest being 6 mts. Find the length of the supporting wire attached to the road-way 18 mts. from the middle.**




**Sol.** Let $AOB$ be the cable [$O$ is its lowest point and $A$, $B$ are the highest points]. Let $PRQ$ be the bridge suspeded with $PR = RQ = 36$ mts .

$PA = QB = 30$ mts (longest vertical supporting wires)

$OR = 6$mts (shortest vertical supporting wire)[the lowest point of the cable is upright the mid-point R of the bridge]

Therefore, $PR$=$RQ$=36 mts. We take the origin of coordinates at $O$, $X$-axis along the tangent at $O$ to the cable and the $Y$-axis along $\overleftrightarrow{RO}$ . The equation of the cable would, therefore, be $x^2 = 4ay$ for some $a > 0$ passing through $B(36, 24)$

Therefore, $4a = \dfrac{36 \times 36}{24} = 54mts$

If $RS$ = 18mts. and $SC$ is the vertical through $S$ meeting the cable at $C$ and the $X$-axis at $D$, then $SC$ is the length of the supporting wire required .

If $SC$ = l mts, then $DC$ = (l-6) mts.

As such $C$ = (18, l-6).

Since $C$ is on the cable, $18^2 = 4a$ (l-6)

$$\Rightarrow l - 6 = \frac{18^2}{4a} = \frac{18 \times 18}{54} = 6 \quad \Rightarrow l = 12$$

## 3.8 EQUATION OF THE TANGENT AND NORMAL AT A POINT

***Tangent :*** Let $S = 0$ be a parabola and $P$ be a point on the parabola. Let $Q$ be another point on the parabola. If the secant line $\overrightarrow{PQ}$ approaches to the same limiting position as $Q$ moves along the curve and approaches to $P$ from either side, then the limiting position is called a tangent line (or) tangent to the parabola at $P$. The point $P$ is called the point of contact of the tangent to the parabola.

***THEOREM-3.4** **The equation of the chord joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is $S_1 + S_2 = S_{12}$.**

***Proof :*** Let $P(x_1, y_1)$and $Q(x_2, y_2)$ be the two points on the parabola $S = y^2 - 4ax = 0$, then $S_{11} = 0$ and $S_{22} = 0$. Consider the equation $S_1 + S_2 = S_{12}$.

This is a first degree equation in $x$ and $y$ representing a straight line. Substituting $(x_1, y_1)$ for $(x, y)$ we get $S_{11} + S_{12} = S_{12} \Rightarrow 0 + S_{12} = S_{12}$

$\therefore (x_1, y_1)$ satisfies the equation (1)

Similarly $(x_2, y_2)$ also satisfies (1). Therefore the equation of the chord $\overrightarrow{PQ}$ is given by $S_1 + S_2 = S_{12}$.

***Corollary :***

As $Q(x_2, y_2) \to P(x_1, y_1)$ the chord $PQ$ becomes the tangent at $P$ to the parabola.

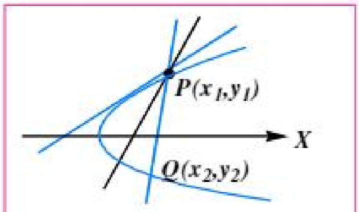



*Fig 3.18*

$\therefore$ Equation to the tangent at $P(x_1, y_1)$ is $\underset{(x_2,y_2)\to(x_1,y_1)}{Lt}[S_1 + S_2 = S_{12}]$

$\Rightarrow S_1 + S_1 = S_{11} \Rightarrow 2S_1 = 0$ [$\because (x_1, y_1)$ lies on $S = 0$, $S_{11} = 0$] i.e., $yy_1 - 2a(x + x_1) = 0$

***THEOREM-3.5** **The equation of tangent at $(x_1, y_1)$ to the parabola $S = y^2 - 4ax = 0$ is $S_1 = 0$.**

***Proof:*** Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4ax$ -- (1)

Differentiating (1) w.r.t $x$ we get $\dfrac{dy}{dx} = \dfrac{2a}{y}$

$\therefore$ The slope of the tangent at $P(x_1, y_1) = \dfrac{2a}{y_1}$ $(y_1 \neq 0)$

i) Now, the equation of the tangent at $P(x_1, y_1)$ is $y - y_1 = \dfrac{2a}{y_1}(x - x_1)$.

ie., $yy_1 - y_1^2 = 2ax - 2ax_1 \Rightarrow yy_1 - 4ax_1 = 2ax - 2ax_1 (\because y_1^2 - 4ax_1)$

$\Rightarrow yy_1 - 2a(x + x_1) = 0 \Rightarrow S_1 = 0$ -- (2)

If $y_1 = 0$, from the equation of parabola $x_1 = 0$.

Hence the tangent at the vertex $A(0, 0)$ is the line $x = 0$, the $y$-axis.

The equation (2) also holds good for the point (0, 0).

***Example :*** The equation to the tangent at (2, 4) on the parabola $y^2 = 8x$ is

$S_1 = 0 \Rightarrow yy_1 - 2a(x + x_1) = 0 \Rightarrow y(4) - 4(x + 2) = 0 \Rightarrow x - y + 2 = 0$

***THEOREM-3.6** **The condition that the line $y = mx + c$ may be a tangent to the parabola $y^2 = 4ax$ is $c = a/m$.**

***Proof :*** Let $(x_1, y_1)$ be the point where the line $y = mx+c$ touches the parabola $y^2 = 4ax$.

The equation of the tangent at $(x_1, y_1)$ on the parabola is $yy_1 - 2a(x + x_1)$ -- (1)

Given tangent is $y - mx - c = 0$ -- (2)

(1) and (2) represent the same line

$$\Rightarrow \frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c} \Rightarrow x_1 = \frac{c}{m} \text{ and } y_1 = \frac{2a}{m}$$

$(x_1, y_1)$ lies on the parabola

$$\Leftrightarrow y_1^2 = 4ax_1 \Leftrightarrow \frac{4a^2}{m^2} = 4a\left(\frac{c}{m}\right) \Leftrightarrow c = \frac{a}{m}$$

**Note :** *A line parallel to the axis of a parabola can never be a tangent to the parabola.*

**Note**

*i) The equation of the tangent to the parabola $y^2=4ax$ in the slope from can be taken as $y = mx + \frac{a}{m}$, for all values of $m(\neq 0)$. The point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.*

*ii) If $m = 0$ and $c \neq 0$ then the equation $y = mx + c$ reduces to $y = c$. This cuts the parabola at $\left(\frac{c^2}{4a}, c\right)$ and hence cannot be a tangent. 2 So we can conclude that any line parallel to the axis of the parabola meets the parabola at only one real point.*

*iii) If $m \neq 0, c = 0$ the line $y = mx + 0$ intersects the parabola at (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ and hence the line cannot be a tangent.*

*iv) If $m = 0$, $c = 0$ the equation reduces to $y = 0$(x-axis), which cuts the parabola at (0, 0) and is not a tangent to the parabola.*

*v) The line $x = c$, intersects the conic $y^2= 4ax$ in real points for $c > 0$, and touches the parabola for $c = 0$.*

*vi) The condition for the line $y = mx + c$ to be a tangent to the parabola $x^2 = 4ay$ is $c = -am^2$ and the point of contact is $(2am, am^2)$ (Prove this result)*

***THEOREM-3.7** **The condition that the line $lx + my + n = 0$ $(l \neq 0)$ may touch the parabola $y^2 = 4ax$ is $ln = am^2$.**

***Proof :*** Let the given line $lx + my + n = 0$ -- (1)

touch the parabola $y^2 = 4ax$ at $P(x_1, y_1)$

$\therefore$ Equation of the tangent at $P$ to the parabola is $yy_1 - 2a(x+x_1) = 0$ -- (2)

(1) & (2) represent the same line $\frac{\ell}{-2a} = \frac{m}{y_1} = \frac{n}{-2ax_1}$

$$\Rightarrow x_1 = \frac{n}{\ell} \text{ and } y_1 = \frac{-2am}{\ell} \Rightarrow P = \left(\frac{n}{\ell}, \frac{-2am}{\ell}\right)$$

But $P$ lies on $y^2 = 4ax \Leftrightarrow \frac{4a^2m^2}{\ell^2} = 4a\left(\frac{n}{\ell}\right) \Rightarrow \ell n = am^2$

**Note**

*The tangent line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ at $\left(\frac{n}{\ell}, \frac{-2am}{\ell}\right)$.*

***THEOREM-3.8** **Two tangents can be drawn from an external point $(x_1, y_1)$ to the parabola $y^2 = 4ax$.**

***Proof :*** Let $y = mx + \frac{a}{m}$ be the tangent to the parabola that passes through $(x_1, y_1)$.

Then $y_1 = mx_1 + \frac{a}{m} \Rightarrow m^2 x_1 - my_1 + a = 0 \quad \Rightarrow m = \frac{y_1 \pm \sqrt{(y_1^2 - 4ax_1)}}{2x_1}$

$(x_1, y_1)$ is an external point

$\Leftrightarrow y_1^2 - 4ax_1 > 0 \Rightarrow m$ has two distinct real values $m_1$, $m_2$.

Hence two distinct tangents can be drawn to the parabola from an external point $(x_1, y_1)$.

***Note***

*i) If the point $(x_1, y_1)$ lies on the parabola then $y_1^2 - 4ax_1 = 0$ and hence the two tangents are concident touching the conic at $(x_1, y_1)$.*

*ii) The condition that the line $lx + my + n = 0$ $(m \neq 0)$ may touch the parabola $x^2 = 4ay$ is $al^2 = mn$ and the point of contact is $\left(\frac{-2al}{m}, \frac{n}{m}\right)$*

## 3.9 NORMAL

***Definition :***

**Let $S = 0$ be a parabola and $P$ be a point on the parabola. The line passing through $P$ and perpendicular to the tangent of $S = 0$ at $P$ is called the normal to the parabola at $P$.**

**THEOREM-3.9** ***The equation of the normal to the parabola $y^2 = 4ax$ having the slope $m$, is $y = mx - 2am - am^3$***

***Proof:*** Differentiating the equation $y^2 = 4ax$ w.r.t $x$,

we get $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore$ The slope of the tangent at the point $P(x_1, y_1)$ is $\frac{2a}{y_1}$,

$\therefore$ The slope of the normal is $m = \frac{-y_1}{2a} \Rightarrow y_1 = -2am$

$P(x_1, y_1)$ lies on $y^2 = 4ax \Rightarrow y_1^2 = 4ax_1 \Rightarrow 4a^2m^2 = 4ax_1 \Rightarrow x_1 = am^2$

Hence the equation to the normal at $P(am^2, -2am)$ is $y + 2am = m(x - am^2)$

$\Rightarrow y = mx - 2am - am^3$

## 3.10 NUMBER OF NORMALS THROUGH A GIVEN POINT

(i) The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + xt = 2at + at^3$, if this line passes through $(x_1, y_1)$, then $y_1 + x_1 t =$

$2at + at^3$ is $+ t(2a - x_1) - y_1 = 0$. This is a cubic equation in '$t$' and has, at most three real roots.

Hence the number of normals through a given point $(x_1, y_1)$ to a parabola $y^2 = 4ax$ is either 1 or 2 or 3 accordingly as the number of distinct real roots of the cubic equation

$at^3 + (2a - x_1)\, t - y_1 = 0$ is 1 or 2 or 3.

**Criterion for the number of normals**

Write $H = \frac{2a - x_1}{3a}$, $G = -\frac{y_1}{a}$ and $\Delta = G^2 + 4GH^3$ if $x_1 = 2a$ and $y_1 = 0$ then the number of normals = 1

Assume either $x_1 \neq 2a$ or $y_1 \neq 0$

(i) If $\Delta > 0$ then the number of normals is 1

(ii) If $\Delta = 0$ then the number of normals is 2

(iii) If $\Delta < 0$ then the number of normals is 3

(ii) The equation of the tangent '$t$' is $yt = x + at^2$. Hence slope of the normal at '$t$' *is* $m = -t \Rightarrow t = -m$, substituting in the equation of the normal at $t$

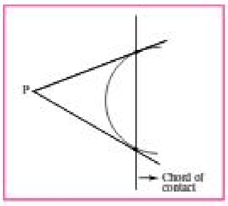
(i.e., $y + xt = 2at + at^3$) we get

$y - mx = -2am - am^3$ is $y = mx - 2am - am^3$

$\therefore$ The equation of the normal to the prabola $y^2 = 4ax$, having slope $m$, is

$y = mx - 2am - am^3 = m(x - 2a - am^2)$

## 3.11 CHORD OF CONTACT




*Definition :*

**If two tangents are drawn to a conic from a point, the straight line joining the point of contact is called chord of contact of that point with respect to the conic.**

*Note : The equation to the chord of contact of $P(x_1, y_1)$ with respect to the parabola $S = 0$ is $S_1 = 0$.*

*Remember :*
*The condition for the line $lx+my+n=0$ to be a tangent to parabola $y^2 = 4ax$ is $am^2 = ln$ and the point of contact is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$*

**SOLVED EXAMPLES**

**1. The line $2x - y + 2 = 0$ touches the parabola $y^2 = 4px$. Find $P$ and also find the point of contact.**

**Sol.** The line $lx + my + n = 0$ touches $y^2 = 4ax \Leftrightarrow \ell n = am^2$ -- (1)

The given line $2x - y + 2 = 0$, $l = 2$, $m = -1$, $n = 2$

$\therefore$ Condition (1) $\Rightarrow (2)(2) = P(-1)^2$

$\Rightarrow P = 4$

The coordinates of the point of contact $= \left(\frac{n}{\ell}, \frac{-2am}{\ell}\right)$

i.e., $\left(\frac{2}{2}, \frac{-2(4)(-1)}{2}\right) = (1, 4)$

**2. Find the equation of the tangent to the parabola $y^2 = 8x$ and which is parallel to the line $x - y + 3 = 0$**

**Sol.** Slope of the line $x - y + 3 = 0$ is 1.

For the parabola $y^2 = 8x$, $a = 2$

$\therefore$ Equation to the tangent parallel to $x - y + 3 = 0$ is

$$y = mx + \frac{a}{m} \Rightarrow y = x + \frac{2}{1} \Rightarrow x - y + 2 = 0$$

**3. Find the equation of normal to the parabola $y^2 = 4x$ and whose slope is 2.**

**Sol.** Given parabola $y^2 = 4x$, $\therefore a = 1$

Given slope of the normal $m = 2$

Equation to the normal is $y = mx - 2am - am^3$

$$\Rightarrow y = 2x - 2(1)(2) - (1)(8)$$

$$\Rightarrow 2x - y - 12 = 0$$

***4. Find the equation of the tangent and normal at $\left(4, \frac{3}{2}\right)$ on the parabola $x^2 - 4x - 8y + 12 = 0$.**

**Sol.** Let $S = x^2 - 4x - 8y + 12 = 0$ and $P(x_1, y_1) = \left(4, \frac{3}{2}\right)$

$\therefore$ Equation to the tangent $P$ is $S_1 = 0$

$$\Rightarrow x(4) - 2(x+4) - 4\left(y + \frac{3}{2}\right) + 12 = 0 \Rightarrow x - 2y - 1 = 0$$

$\Rightarrow$ Slope of the tangent $= \frac{1}{2}$ $\quad\Rightarrow$ Slope of the normal $= -2$

$\therefore$ Equation to the normal at $\left(4, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = -2(x-4) \Rightarrow 2x + y - \frac{19}{2} = 0 \quad \Rightarrow 4x + 2y - 19 = 0$$

**5. Find the equation of the tangents to the parabola $y^2 = 9x$ and which pass through (4, 10).**

**Sol.** Given parabola $y^2 = 9x \Rightarrow 4a = 9$

Equation to the tangent in slope form is $y = mx + \frac{a}{m} \Rightarrow y = mx + \frac{9}{4m}$

$\Rightarrow 16m^2 - 40m + 9 = 0 \Rightarrow (4m-1)(4m-9) = 0 \Rightarrow m = \frac{1}{4}$ (or) $m = \frac{9}{4}$

$\therefore$ The equations of tangents are

i) $y = \frac{x}{4} + \frac{\left(\frac{9}{4}\right)}{\left(\frac{1}{4}\right)} \Rightarrow x - 4y + 36 = 0$ $\quad$ ii) $y = \frac{9x}{4} + \frac{\left(\frac{9}{4}\right)}{\left(\frac{9}{4}\right)} \Rightarrow 9x - 4y + 4 = 0$

***6. $\alpha$ and $\beta$ are the angles made by two tangents to the parabola $y^2 = 4ax$, with its axis. Find the locus of their point of intersection, if $\cot\alpha + \cot\beta = p$**

**Sol.** Let $P(x_1, y_1)$ be the point of intersection of the two tangents to $y^2 = 4ax$ -- (1)

The tangent line $y = mx + \frac{a}{m}$ to (1) passes through $P(x_1, y_1) \Leftrightarrow y_1 = mx_1 + \frac{a}{m}$

$\Leftrightarrow m^2 x_1 - my_1 + a = 0$ -- (2)

Let $m_1 m_2$ be the roots of (2) then $\tan\alpha = m_1, \tan\beta = m_2$

Given $\cot\alpha + \cot\beta = p \Rightarrow \frac{1}{m_1} + \frac{1}{m_2} = p$

$$\Rightarrow m_1 + m_2 = pm_1 m_2 \Rightarrow \frac{y_1}{x_1} = p\left(\frac{a}{x_1}\right) \Rightarrow y_1 = pa$$

Locus of $P$ is the line $y = ap$.

***7. The normals from $(P, 0)$ are drawn to the parabola $y^2 = 8x$, one of them is the axis. If the remaining two normals are perpendicular find the value of $P$.**

**Sol.** Equation to the normal to $y^2 = 8x$ is $y = mx - 2am - am^3$

$\Rightarrow y = mx - 4m - 2m^3$ --(1)

(1) passes through $(p,0) \Rightarrow 0 = pm - 4m - 2m^3$

$\Rightarrow 2m^2 + 4 - p = 0$ --(2)

$\Rightarrow m^2 = \dfrac{p-4}{2}$

Let the roots of (2) be $m_1$, $m_2$

Given that the normals are perpendicular

$\Rightarrow m_1 m_2 = -1 \Rightarrow -\dfrac{p-4}{2} = -1 \Rightarrow p = 6$

*Remember :*

*If $m_1$ and $m_2$ are the slopes of tangents from an external point $(x_1, y_1)$ to the parabola $y^2=4ax$ then they are the roots of $m^2x_1 - my_1 + a = 0$ and hence $m_1 + m_2 = \dfrac{y_1}{x_1}$ and $m_1 m_2 = \dfrac{a}{x_1}$*

***8. Show that the locus of the point of intersection of perpendicular tangents to the parabola $y^2 = 4ax$ is the directrix, $x + a = 0$.**

**Sol.** Let the two tangents be drawn from $P(x_1, y_1)$ to the parabola $y^2 = 4ax$

$\therefore$ Equation to pair of tangents from $P$ is

$S_1^2 = S.S_{11} \Rightarrow [yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$ --(1)

The pair of tangents contain a right angle

$\Leftrightarrow$ coefficient of $x^2$ + coefficient of $y^2 = 0$

$\Rightarrow 4a^2 + y_1^2 - (y_1^2 - 4ax_1) = 0 \Rightarrow 4a^2 + 4ax_1 = 0$

$\therefore$ Locus of $P$ is $x + a = 0$, which is the directrix.

***9. Show that the equation of the common tangent to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ is $y = \pm(x + 2a)$** **(March-17, May-19)**

**Sol.** Given the equation of the parabola is $y^2 = 8ax$ --(1)

and the circle is $x^2 + y^2 = 2a^2$ --(2)

Any tangent of (1) is $y = mx + \dfrac{2a}{m}$

$\Rightarrow mx - y + \dfrac{2a}{m} = 0$ --(3)

line (3) is also a tangent to circle (2) $\Rightarrow \dfrac{\left|\dfrac{2a}{m}\right|}{\sqrt{(1+m^2)}} = a\sqrt{2}$

If $\dfrac{4}{m^2} = 2(1 + m^2)$ i.e., $m^4 + m^2 - 2 = 0$

If $(m^2 + 2)(m^2 - 1) = 0$ i.e., $\Rightarrow m^2 \neq -2$, $m^2 = 1 \Rightarrow m \pm 1$

$\therefore$ The equation of the common tangent (1) & (2) are

$y = \pm x \pm 2a \Rightarrow y = \pm(x + 2a)$

*Remember :*

*If $\theta$ is the acute angle between tangents drawn from $(x_1, y_1)$ to parabola $S \equiv y^2 - 4ax = 0$ then $Tan\ \theta = \dfrac{\sqrt{S_{11}}}{|x_1 + a|}$.*

**10. The locus of the foot of the perpendicular from the focus to the tangent of the parabola $y^2 = 4ax$ is $x = 0$, the tangent at the vertex.**

**Sol.** Given parabola $y^2 = 4ax$

Equation to any tangent to the parabola is

$$y = mx + \frac{a}{m} \Rightarrow m^2x - my + a = 0 \quad \text{--(1)}$$

Equation to any the line perpendicular to (1) and passing through $S(a, 0)$ is

$$m(x-a) + m^2y = 0 \Rightarrow x + my = a$$

Solving (1) & (2) we get $P\left(0, \frac{a}{m}\right)$

Which is the foot of the perpendicular from $S(a, 0)$ on the tangent.

Hence the locus of $P$ is $x = 0$

which is the tangent at the vertex.

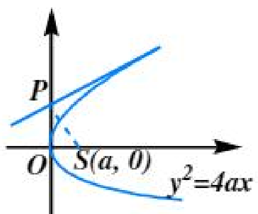



Fig 3.19

***11. Show that the straight line $7x + 6y = 13$ is a tangent to the parabola $y^2 - 7x + 8y + 14 = 0$ and find point of contact.**

**Sol.** Solving the two equation $7x + 6y = 13$

and $y^2 - 7x + 8y + 14 = 0$

$$\Rightarrow y^2 + 6y - 13 - 8y + 14 = 0$$

$$\Rightarrow y^2 - 2y + 1 = 0 \Rightarrow y = 1, 1$$

$\therefore$ Since roots are equal.

The line is a tangent to the parabola substituting in equation of line $7x + 6y = 13$.

$$\Rightarrow x = 1$$

The point of contact is (1, 1)

***12. Find the condition for the line $y = mx + c$ to be a tangent to the parabola $x^2 = 4ay$**

**Sol.** Let $(x_1, y_1)$ be the point of contact

The equation of tangent is $xx_1 = 2a(y + y_1)$

$$\Rightarrow xx_1 - 2ay - 2ay_1 = 0 \quad \text{--(1)}$$

Comparing the coefficients of (1) and $mx - y + c = 0$

$$\Rightarrow \frac{x_1}{m} = \frac{-2a}{-1} = \frac{-2ay_1}{c} \Rightarrow x_1 = 2am;\ y_1 = -c$$

Substituting $(2am, -c)$ on the parabola $x^2 = 4ay$

$$4a^2m^2 = 4a(-c) \Rightarrow c = -am^2$$

$\therefore$ The condition is $c = -am^2$

***13. If a chord of the parabola $y^2 = 4ax$ touches the parabola $y^2 = 4bx$, show that the tangents at its extremities meet on the parabola $by^2 = 4a^2x$**

**Sol.** Let $QR$ be a chord of the parabola $y^2 = 4ax$ -- (1)

Let $P(x_1, y_1)$ be the point of intersection of the tangent at $Q$ and $R$ to (1)

then $QR$ is the chord of contact of $P$ with respect to (1)

$\therefore$ The equation of $QR$ is $yy_1 - 2a(x + x_1) = 0$

$\Rightarrow 2ax - yy_1 + 2ax_1 = 0$ -- (2)

Given (2) touches the parabola $y^2 = 4bx$

$\Rightarrow (2a)(2ax_1) = by_1^2 \Rightarrow 4a^2x_1 = by_1^2$

$\therefore$ The locus of $P(x_1, y_1)$ is $by^2 = 4a^2x$

*Remember :* ***The condition for the line $lx+my+n=0$ to be a tangent to parabola $y^2=4ax$ is $am^2 = ln$.***

**14. Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect (other than the origin) at an angle of $\mathrm{Tan}^{-1}\left[\dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}\right]$**

**Sol.** Without loss of generality we assume $a > 0$ and $b > 0$.

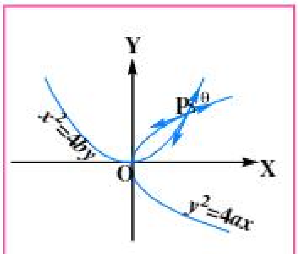


*Fig 3.20*

Let $P(x, y)$ be the point of intersection of the parabolas other than the origin.

Then $y^4 = 16a^2x^2 = 16a^2(4by) = 16a^2by$

$\therefore y[y^3 - 64a^2b] = 0$

$\Rightarrow y^3 - 64a^2b = 0 \Rightarrow y^3 = (64a^2b)^{1/3}$ $[\because y > 0]$ $= 4a^{2/3}b^{1/3}$

Also from $y^2 = 4ax$, $x = \dfrac{16a^{4/3}b^{2/3}}{4a} = 4a^{1/3}b^{2/3}$

$\therefore P = (4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Differentiating both sides of $y^2 = 4ax$ w.r.t. '$x$',

we get $\dfrac{dy}{dx} = \dfrac{2a}{y}$

$\therefore \left[\dfrac{dy}{dx}\right]_P = \dfrac{2a}{4a^{2/3}b^{1/3}} = \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{1/3}$

If $m_1$ be the slope of the tangent at $P$ to $y^2 = 4ax$, then $m_1 = \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{1/3}$

Similarly, we get $m_2 = 2\left(\dfrac{a}{b}\right)^{1/3}$ where $m_2$ is the slope of the tangent at $P$ to $x^2 = 4ay$.

If $\theta$ is the acute angle between the tangents to the curves at $P$,

then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1m_2}\right| = \dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$ so that $\theta = \mathrm{Tan}^{-1}\left[\dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}\right]$

## EXERCISE - 3.1

I. 1. Find the vertex and focus of $4y^2+12x-20y+67=0$ [Ans : Vertex $\left(-\frac{7}{2},\frac{5}{2}\right)$, Focus $\left(-\frac{17}{4},\frac{5}{2}\right)$]

2. Find the vertex and focus of $x^2-6x-6y+6=0$ [Ans : Vertex $\left(3,-\frac{1}{2}\right)$, Focus (3, 1)]

3. Find the equation sof axis and directrix of the parabola $y^2+6y-2x+5=0$
[Ans : Equationof the axis is $y+3=0$, equation of the directrix is $2x+5=0$]

4. Find the equations of axis and directrix of the parabola $4x^2+12x-20y+67=0$
[Ans : axis $2x+3=0$, directrix $20y-33=0$]

5. Find the equationof the parabola whose focus is $S(1,-7)$ and vertex is $A(1,-2)$
[Ans : $(x-1)^2=-20(y+2)$]

*6. Find the equation of the parabola whose focus is $S(3,-1)$ and vertex is $A(3,-2)$ **(May-19)**

7. Find the equation of the parabola whose latus rectumis the line segment joining the points $(-3,2)$ and $(-3,1)$ [Ans : $(2y-3)^2=(4x+13)$ or $(2y-3)^2=(4x+11)$]

8. Find the position (interior or exterior or on) of the following points with respect to the parabola $y^2=6x$. (i) $(6,-6)$ (ii) (0, 1) (iii) (2, 3)
[Ans : (i) on the parabola (ii) outside the parabola (iii) inside the parabola]

*9. Find the coordinates of the points on the parabola $y^2=8x$ whose focal distance is 10.
**(March-17)** [Ans : $(8,\pm 8)$]

10. If (1/2,2) is one extremity of a focal chord of the parabola $y^2=8x$. Find the coordinates of the other extremity. [Ans : (8,8)]

11. Prove that the point on the parabola $y^2=4ax$, $(a>0)$ nearest to the focus is its vertex.

12. A comet moves in a parabolic orbit with the sun as focus. When the comet is $2\times 10^7$ K.M. from the sun, the line from the sun to it makes an angle $\frac{\pi}{2}$ with the axis of the orbit. Find how near the comet come to the sun. [Ans : $10^7$ k.m.]

II. 1. Find the locus of the point of trisection of double ordinate of a parabola $y^2=4ax$, $(a>0)$
[Ans : $9y^2=4ax$]

2. Find the equation of the parabola whose vertex and focus are on the positive x-axis at a distance '$a$' and '$a^1$' from the origin respectively. [Ans : $y^2=4(a^1-a)(x-a)$]

3. If $L$ and $L'$ are the ends of the latus rectum of the parabola $x^2=6y$, find the equations of $OL$ and $OL'$ where '$O$' is the origin. Also find the angle between them.
[Ans : $x+2y=0$, $x-2y=0$, $\pi-\mathrm{Tan}^{-1}(4/3)$]

4. Find the equation of the parabola whose axis is parallel to x-axis and which passes through the points $(-2,1)$, (1, 2) and $(-1,3)$ [Ans : $5y^2+2x-21y+20=0$]

5. Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points (4, 5), $(-2,11)$, and $(-4,21)$ [Ans : $x^2-4x-2y+10=0$]

III. 1. Find the equation of the parabola whose focus is $(-2, 3)$ and directrix is the line $2x+3y-4=0$. Also find the length of the latus rectum and the equation of the axis of the parabola.

[Ans : $9x^2 - 12xy + 4y^2 + 68x - 34y + 153 = 0$, length of the latus rectum $= \frac{2}{\sqrt{13}}$

Equation of the axis of the parabola is $3x - 2y + 12 = 0$]

*2. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ sq. units where $y_1, y_2, y_3$ are the ordinates of its vertices. (March-18)

3. Find the coordinates of the vertex and focus, the equation of the directrix and axis of the following parabolas. (i) $y^2 + 4x + 4y - 3$ (ii) $x^2 - 2x + 4y - 3 = 0$

[Ans : (i) Vertex $\left(\frac{7}{4}, -2\right)$, Focus $\left(\frac{3}{4}, -2\right)$, Directrix $4x - 11 = 0$, axis $y + 2 = 0$

(ii) Vertex $(1, 1)$, Focus $(1, 0)$, Directrix $y = 2$, axis $x = 1$]

## 3.12 PARAMETRIC EQUATIONS OF THE PARABOLA

The point $(at^2, 2at)$ satisfies the equation of a parabola $y^2 = 4ax$, for all real values of "$t$". What $t$ is called a parameter.

$\therefore$ The parametric equations of the parabola $y^2 = 4ax$ are, $x = at^2$ and $y = 2at$. where $t$ is a parameter. The point $P(at^2, 2at)$ is denoted by $t$ (or) $P(t)$.

*Note*

***The following table gives the parametric coordinates of a parabola in different forms and their parametric equations.***

| Parabola | $y^2 = 4ax$ | $y^2 = -4ax$ | $x^2 = 4ay$ | $x^2 = -4ay$ | $(y-k)^2 = 4a(x-h)$ | $(x-h)^2 = 4a(y-k)$ |
|---|---|---|---|---|---|---|
| Parametric coordinates | $(at^2, 2at)$ | $(-at^2, 2at)$ | $(2at, at^2)$ | $(2at, -at^2)$ | $(h+at^2, k+2at)$ | $(h+2at, k+at^2)$ |
| Parametric equations | $x=at^2$<br>$y = 2at$ | $x = -at^2$<br>$y = 2at$ | $x = 2at$<br>$y = at^2$ | $x = 2at$<br>$y = -at^2$ | $x = h+at^2$<br>$y = k+2at$ | $x = h+2at$<br>$y = k + at^2$ |

**THEOREM-3.10** **The equation of chord joining the points $t_1$ and $t_2$ on the parabola, $y^2 = 4ax$ is $y(t_1 + t_2) = 2x + 2at_1t_2$**

***Proof :*** The chord joining two points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $\frac{y-2at_1}{2at_1 - 2at_2} = \frac{x - at_1^2}{at_1^2 - at_2^2}$

$$\Rightarrow \frac{y-2at_1}{2} = \frac{x - at_1^2}{t_1 + t_2} (\because t_1 \neq t_2 \Rightarrow y(t_1 + t_2) - 2x - 2at_1t_2 = 0)$$

***Corollary-1 :***

The chord joining the points $t_1$ & $t_2$ on the parabola $y^2 = 4ax$ is a focal chord then $t_1t_2 = -1$.

***Proof :*** The equation of the chord is $y(t_1+t_2) = 2x+2at_1t_2$.

If it is a focal chord then it passes through the focus $S(a, 0)$.

$\therefore O = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$

**Note**

*If $P(at^2, 2at)$ is one end of the focal chord $PQ$ of the parabola $y^2 = 4ax$, then the coordinates of the other end $Q$ are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$. Which are obtained by replacing $t$ with $-\frac{1}{t}$ in $(at^2, 2at)$.*

**Corollary-2 :** Length of the focal chord $PQ$ is $a\left(t+\frac{1}{t}\right)^2$

**Proof :** $P(at^2, 2at)$ is one end of the focal chord $PQ$ of a parabola $y^2 = 4ax$,

then $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

$$\therefore PQ = \sqrt{\left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2} \Rightarrow PQ = a\sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + 4\left(t+\frac{1}{t}\right)^2}$$

$$\Rightarrow PQ = a\left(t+\frac{1}{t}\right)\sqrt{\left(t-\frac{1}{t}\right)^2 + 4} \Rightarrow PQ = a\left(t+\frac{1}{t}\right)^2$$

**Remark :** The length of the smallest focal chord of the parabola is $4a$, which is the length of latus rectum.

(We know that $\left|t+\frac{1}{t}\right| \geq 2$ for all $t \neq 0 \Rightarrow a\left(t+\frac{1}{t}\right)^2 \geq 4a$)

## SOLVED EXAMPLES

**1. Show that the focal chord of the parabola $y^2 = 4ax$, which makes an angle $\alpha$ with the $x$-axis, is of length $4a\operatorname{cosec}^2\alpha$.**

**Sol.** Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord $PQ$ which makes an angle $\alpha$ with the axis of the parabola. Then

$\therefore PQ = a(t_2 - t_1)^2$ and $\tan\alpha = \text{slope of } PQ = \frac{2}{t_2 + t_1} \Rightarrow t_1 + t_2 = 2\cot\alpha$

$PQ = a(t_2 - t_1)^2 = a[(t_2 + t_1)^2 - 4t_1t_2]$

$= a\{4\cot^2\alpha + 4\} = 4a\operatorname{cosec}^2\alpha\ [\because t_2 + t_1 = 2\cot\alpha \text{ and } t_1t_2 = -1]$

**Note :** $P(t)$ is a point on the parabola $y^2 = 4ax$ then $SP$ = focal distance of $p = (1+t^2)$

**Remember :**

*The length of focal chord drawn at a point 't' on the parabola $y^2=4ax$ is $a\left(t+\frac{1}{t}\right)^2$*

**2. Show that semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord.**

**Sol.** Let $PQ$ be a focal chord of the parabola $y^2 = 4ax$ having focus at $S(a, 0)$.

Let the coordinates of $P$ & $Q$ be $(at_1^2, 2at_1)$ and $(at_2^2 + 2at_2)$ then $t_1t_2 = -1$.

$SP = a(1+t_1^2)$; $SQ = a(t_2^2 + 1) = a\left(\frac{1}{t_1^2} + 1\right) = \frac{a(1+t_1^2)}{t_1^2}$ $[\because t_1t_2 = -1]$

$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t_1^2+1)} + \frac{t_1^2}{a(t_1^2+1)} = \frac{1}{a} \Rightarrow \frac{SP+SQ}{SP.SQ} = \frac{1}{a} \Rightarrow 2a = \frac{2SP\,SQ}{SP+SQ}$

**Remark :** If $l_1$ and $l_2$ are the lengths of the segmets of a focal chord of a parabola, then its latus rectum is $\frac{4\ell_1\ell_2}{\ell_1 + \ell_2}$

**Note :**

*Tangents drawn at the ends of focal chord of a parabola are at right angles, they intersect on directrix.*

**THEOREM-3.11** **The equations of the tangent to the parabola $y^2 = 4ax$ at the point "$t$" is $yt = x + at^2$.**

***Proof :*** For parabola $y^2 = 4ax, \frac{dy}{dx} = \frac{2a}{y}$

$\therefore$ slope of the tangent at $(at^2, 2at) = \frac{2a}{2at} = \frac{1}{t}$

Equation of the tangent at $(at^2, 2at)$ is $y - 2at = \frac{1}{t}(x - at^2) \Rightarrow yt = x + at^2$

**THEOREM-3.12** **The point of intersection of two tangents to the parabola $y^2 = 4ax$ at the point $t_1$ and $t_2$ is $(at_1t_2, a(t_1+t_2))$.**

***Proof :*** The equation of tangent to the parabola at $t_1$ is $yt_1 = x + at_1^2$ -- (1)

The equation of the tangent to the parabola at $t_2$ is $yt_2 = x + at_2^2$ -- (2)

(1) - (2) $\Rightarrow y(t_1 - t_2) = a(t_1^2 - t_2^2)$

$\Rightarrow y = a(t_1 + t_2)$

(1) $\Rightarrow a(t_1 + t_2)t_1 = x + at_1^2$

$\Rightarrow at_1^2 + at_1t_2 = x + at_1^2 \Rightarrow x = at_1t_2$

$\therefore$ Point of intersection is $(at_1t_2, a(t_1 + t_2))$

***Example :*** The area of triangle formed by the tangents at $P(t_1)$ and $Q(t_2)$ and the chord $PQ$ is $\frac{1}{2}a^2|t_1 - t_2|^3$

***Proof :*** Let $R$ be the point of intersection of tangents at $P$ and $Q$.

Then the coordinates of the $R$ are $[at_1t_2, a(t_1 + t_2)]$

$\therefore$ Area of $\Delta PQR = \frac{1}{2}|\Delta|$, where

$$\Delta = \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix} = a^2\begin{vmatrix} t_1^2 & 2t_1 & 1 \\ t_2^2 & 2t_2 & 1 \\ t_1t_2 & t_1+t_2 & 1 \end{vmatrix} = a^2\begin{vmatrix} t_1^2 - t_1t_2 & t_1 - t_2 & 0 \\ t_2^2 - t_1t_2 & t_2 - t_1 & 0 \\ t_1t_2 & t_1 + t_2 & 1 \end{vmatrix}$$

$$= a^2(t_1 - t_2)^2\begin{vmatrix} t_1 & 1 & 0 \\ -t_2 & -1 & 0 \\ t_1t_2 & t_1t_2 & 1 \end{vmatrix} = a^2(t_1 - t_2)^2(-t_1 + t_2) = -a^2(t_1 - t_2)^3$$

Hence the area of $\Delta PQR = \frac{1}{2}a^2|t_1 - t_2|^3$

**THEOREM-3.13** **The tangents at extremities of a focal chord of a parabola intersect at right angles on the directrix.**

***Proof :*** Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$.

$R[at_1t_2, a(t_1 + t_2)]$ is the point of intersection of tangents at $P$ & $Q$.

Since $PQ$ is a focal chord of the parabola, therefore $t_1t_2 = -1$.

Thus the coordinates of $R$ are $(-a, a(t_1+t_2))$ clearly it lies on the directrix.

The equations of the tangents $PR$ & $PQ$ are $t_1 y = x + at_1^2$ and $t_2 y = x + at_2^2$

$\therefore m_1 = \text{slope of } PR = \dfrac{1}{t_1}$ ; $m_2 = \text{slope of } QR = \dfrac{1}{t_2}$ $\quad \therefore m_1 m_2 = \dfrac{1}{t_1 t_2} = -1 \quad (\because t_1 t_2 = -1)$

Hence the tangents $P$ & $Q$ intersect at right angles on the directrix.

**THEOREM-3.14** **The equation of the normal to the parabola $y^2 = 4ax$ at the point $t$ is $y + xt = 2at + at^3$**

***Proof :*** The equation of the tangent at $t$ is $yt = x + at^3$.

The equation of the normal at $(at_1^2\, 2at)$ is $t(x - at^2) + 1(y - 2at) = 0$

$\Rightarrow xt - at^3 + y - 2at = 0 \Rightarrow y + xt = 2at + at^3$

**THEOREM-3.15** **In general three normals can be drawn from a point $(x_1, y_1)$ to the parabola $y^2 = 4ax$.**

***Proof :*** The equation of the normal to the parabola $y^2 = 4ax$ is

$y + xt = 2at + at^3$ if this normal pass through $(x_1, y_1)$ then $y_1 + x_1 t = 2at + at^3$

$\Rightarrow at^3 + (2a - x_1)t - y_1 = 0 \qquad$ -- (1)

Equation (1) is a cubic equation in $t$ and hence $t$ has three values.

Corresponding to these three values there exists three normals to the parabola $y^2 = 4ax$ passing through $(x_1, y_1)$

***Note***

*If $t_1, t_2, t_3$ are the roots of the equation (1) then $t_1 + t_2 + t_3 = 0$, $t_1t_2 + t_2t_3 + t_3t_1 = \dfrac{2a - x_1}{a}$, $t_1t_2t_3 = \dfrac{y_1}{a}$*

***Corollary :*** All the three normals drawn to the parabola $y^2 = 4ax$ from the point $(x_1, y_1)$ are real and distrinct if $x_1 > 2a$ and $27ay_1^2 < 4[x_1 - 2a]^3$.

***Proof :***

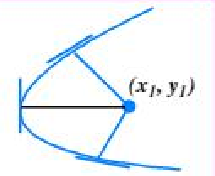



*Fig 3.21*

The equation of normal to the parabola $y^2 = 4ax$ is $y + xt = 2at + at^3$ -- (1)

(1) passes through $(x_1, y_1) \Rightarrow y_1 + x_1 t = 2at + at^3$

$\Rightarrow at^3 + (2a - x_1)t - y_1 = 0 \qquad$ -- (2)

Let $f(t) = at^3 + (2a - x_1)t - y_1$

This is a cubic equation in $t$. A cubic equation has three distinct real root if $f'(t) = 0$ has two distinct real root say $\alpha$ and $\beta$ such that $f(\alpha)f(\beta) < 0$

$f'(t) = 3at^2 + (2a - x_1) = 0$ have real roots if $\dfrac{x_1 - 2a}{3a} > 0$

$\Rightarrow x_1 - 2a > 0 \Rightarrow x_1 > 2a \qquad$ -- (3)

Two distinct roots of $f'(t)=0$ are $\alpha=\sqrt{\frac{x_1-2a}{3a}}$ and $\beta=-\sqrt{\frac{x_1-2a}{3a}}$

$f(\alpha)f(\beta)<0 \Rightarrow f(\alpha).f(-\alpha)<0$ $[\because \beta=-\alpha]$

$\Rightarrow (a\alpha^3+(2a-x_1)\alpha-y_1)(-a\alpha^3-(2a-x_1)\alpha-y_1)<0$

$\Rightarrow y_1^2-(a\alpha^2+(2a-x_1)^2)\alpha^2<0$

$\Rightarrow y_1^2-\left(\frac{x_1-2a}{3}+2a-x_1\right)^2\left(\frac{x_1-2a}{3a}\right)<0 \Rightarrow y_1^2-\left(\frac{4a-2x_1}{3}\right)^2\left(\frac{x_1-2a}{3a}\right)<0$

$\Rightarrow y_1^2-\frac{4}{27a}(x_1-2a)^3<0,\ 27ay_1^2<4(x_1-2a)^3$

$\therefore$ The required conditions are $x_1>2a$ and $27ay_1^2<4(x_1-2a)^3$

**THEOREM-3.16** **A tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.**

***Proof :*** Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2=4ax$.

$\therefore t_1t_2=-1$ -- (1)

Equation of the tangent at $P(at_1^2, 2at_1)$ is $yt_1-x-at_1^2=0$ -- (2)

Slope of (2) = $\frac{1}{t_1}$. Equation of normal at $Q(at_2^2, 2at_2)$ is $y+t_2x=2at_2+at_2^3$ -- (3)

Slope of (3) = $-t_2=-\left(\frac{-1}{t_1}\right)=\frac{1}{t_1}$ = Slope of (2)

$\therefore$ Tangent at $P$ is parallel to the normal at $Q$

**THEOREM-3.17** ***The normal at $t_1$ on the parabola $y^2=4ax$ meets the curve again at a point $t_2$ then $t_2=-t_1-\frac{2}{t_1}$.**

***Proof :*** The equation to the normal at $t_1$ on the parabola $y^2=4ax$ is

$y+xt_1=2at_1+at_1^3$ -- (1)

If (1) passes through $(at_2^2, 2at_2)$ then $2at_2+at_2^2t_1=2at_1+at_1^3$

$\Rightarrow 2(t_2-t_1)=t_1(t_1^2-t_2^2)-2=t_1(t_1+t_2) \Rightarrow -\frac{2}{t_1}=t_1+t_2 \Rightarrow t_2=-t_1-\frac{2}{t_1}$

***Corollary:***

If the normals at the points $t_1$ and $t_2$ on the parabola $y^2=4ax$ meet on the parabola, then $t_1t_2=2$.

***Proof :*** The equations of the normals at $t_1$ & $t_2$ are

$y+t_1x=2at_1+at_1^3$ -- (1) $\qquad y+t_2x=2at_2+at_2^3$ -- (2)

Let the normals $t_1$ & $t_2$ meet at $t_3$ on the parabola

$\therefore (at_3^2, 2at_3)$ lies on (1) & (2)

$\Rightarrow 2at_3+t_1(at_3^2)=2at_1+at_1^3$ -- (3) $\qquad 2at_3+t_2(at_3^2)=2at_2+at_3^2$ -- (4)

***Note :*** *If the normals at $t_1$ and $t_2$ on the parabola $y^2=4ax$ meet again on parabola at $t_3$ then $t_1t_2=2$ and $t_1+t_2+t_3=0$*

from (3) $2(t_3 - t_1) = t_1(t_1^2 - t_3^2)$ -- (5) $(\because t_1 \neq t_3)$

$$\Rightarrow 2 = -t_1(t_1 + t_3) \Rightarrow t_3 = -\frac{2}{t_1} - t_1$$

from (4) $t_3 = -\frac{2}{t_2} - t_2 \quad \therefore -\frac{2}{t_1} - t_1 = -\frac{2}{t_2} - t_2$

$$\Rightarrow \frac{2(t_1 - t_2)}{t_1 t_2} = t_1 - t_2 \Rightarrow t_1 t_2 = 2 \quad (\because t_1 \neq t_2)$$

*****Corollary :***

A normal chord drawn at the point $P(t)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex then, $t = \pm\sqrt{2}$.

***Proof :***

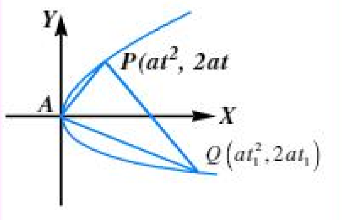


*Fig 3.22*

Given parabola $y^2 = 4ax$

Let the normal at $P(at^2, 2at)$ meet the conic again at $Q(at_1^2, 2at_1)$

$$\Rightarrow t_1 = -t - \frac{2}{t} \quad \text{-- (1)}$$

Slope of $AP = \frac{2at}{at^2} = \frac{2}{t}$, Slope of $AQ = \frac{2}{t_1}$

Given $AP \perp AQ \Rightarrow \frac{2}{t} \cdot \frac{2}{t_1} = -1 \Rightarrow t_1 t = -4$

from (1) $tt_1 + t^2 = -2 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

**Note :** *If the normal chord at 't' on $y^2 = 4ax$ subtends a right angle at the focus then $t^2 = 4$.*

**THEOREM-3.18**

**The condition for the straight line $lx + my + n = 0$ to be a normal to the parabola $y^2 = 4ax$ is $a\ell^3 + 2a\ell m^2 + m^2 n = 0$**

***Proof :*** $lx + my + n = 0$ -- (1)

is a normal to $y^2 = 4ax$ -- (2)

Let $P(t)$ be the foot of the normal equation of the normal is $y + xt = 2at + at^3$

$$\Rightarrow xt = y - (2at + at^3) = 0 \quad \text{-- (3)}$$

(1) & (3) represent the same line

$$\Rightarrow \frac{\ell}{t} = \frac{m}{1} = \frac{n}{-(2at + at^3)} \Rightarrow t = \frac{\ell}{m} \text{ and } 2at + at^3 = -\frac{n}{m}$$

$$\Rightarrow 2a\left(\frac{\ell}{m}\right) + a\frac{\ell^3}{m^3} = \frac{-n}{m} \Rightarrow a\ell^3 + 2a\ell m^2 + m^2 n = 0$$

## SOLVED EXAMPLES

**1. Prove that the locus of point of intersection of two perpendicular normals to the parabola $y^2 = 4ax$ is the parabola $y^2 = a(x-3a)$.**

**Sol.** Let $P(x_1, y_1)$ be the point of intersection of normals at $A(t_1)$, $B(t_1)$

Equation of the normal at $t$ is

$$y + tx = 2at + at^3 \text{ if it passes through } (x_1, y_1)$$

$$y_1 + tx_1 = 2at + at^3$$

$$\Rightarrow at^3 + (2a - x_1)t - y_1 = 0$$

if $t_1, t_2, t_3$ are the roots of (1), $t_1 t_2 t_3 = \frac{y_1}{a}$ -- (2)

Slope of normal at $t_1$ is $= -t_1$

Slope of normal at $t_2 = -t_2$

Given $PA$ is perpendicular to $PB$

$$\Rightarrow (-t_1)(-t_2) = -1 \Rightarrow t_1 t_2 = -1$$

from (2) & (3) $(-1)t_3 = \frac{y_1}{a} \Rightarrow t_3 = -\frac{y_1}{a}$

Since the normal at $t_3$ also passes through

$$P(x_1, y_1)\ y_1 + x_1 t_3 = 2at_3 + at_3^3$$

Substituting the value of $t_3$ from (4) we get

$$y_1 + x_1\left(\frac{-y_1}{a}\right) = 2a\left(\frac{-y_1}{a}\right) + a\left(\frac{-y_1^3}{a^3}\right)$$

$$\Rightarrow 1 - \frac{x_1}{a} = -2 - \frac{y_1^2}{a^2} \Rightarrow y_1^2 = a(x_1 - 3a)$$

Hence the locus of $P(x_1, y_1)$ is $y^2 = a(x-3a)$

> **Note :** *Slope of the tangent at 't' on $y^2 = 4ax$ is $\frac{1}{t}$.*

***2. If $\left(\frac{1}{2}, 2\right)$ is one extremity of a focal chord of a parabola $y^2 = 8x$, find the coordinate of the other extremity.**

**Sol.** Let one extremity be $(at^2, 2at)$

$$y^2 = 8x \Rightarrow a = 2$$

$$(2t^2, 4t) = \left(\frac{1}{2}, 2\right) \Rightarrow t = \frac{1}{2}$$

for a focal chord $t_1 t_2 = -1$

$\Rightarrow$ other extremity is $t_2 = -2t$

$\therefore$ the point is $[2(-2)^2, 2(2)(-2)] = (8, -8)$

> **Remember :** *If $(at^2, 2at)$ is one end of focal chord of the parabola $y^2 = 4ax$ then its other end is $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.*

***3. Prove that the point on the parabola $y^2 = 4ax(a > 0)$ nearest to the focus is its vertex.**

**Sol.** Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$, which is nearest to the focus $S(a, 0)$ then $SP^2 = (at^2-a)^2 + (2at-0)^2$

Let $f(t) = a^2(t^2-1)^2 + 4a^2t^2$

$f'(t) = a^2 2(t^2-1)(2t) + 4a^2(2t) = 4a^2t(t^2-1+2) = 4a^2t(t^2+1)$

for minimum value of $f(t)$, $f'(t) = 0 \Rightarrow t = 0$

$f''(t) = 4a^2[3t^2+1]$, $f''(0) = 4a^2 > 0$

$\therefore$ At $t = 0$, $f(t)$ is minimum, then $P = (0, 0)$

$\therefore$ The point on the parabola $y^2 = 4ax$,

which is nearest to the focus is vertex $A(0, 0)$.

***4. Two parabola have the same vertex and equal length of latus rectum such that their axis are right angles. Prove that the common tangent touches each at the end of a latus rectum.**

**Sol.** Equations of the parabola can be taken as $y^2 = 4ax$ and $x^2 = 4ay$

Equation of the tangent at $(2at, at^2)$ to $x^2 = 4ay$ is $2atx = 2a(y+at^2)$

$\Rightarrow y = tx - at^2$

This is a tangent to $y^2 = 4ax \Rightarrow c = \dfrac{a}{m} \Rightarrow -at^2 = \dfrac{a}{t} \Rightarrow t^3 = -1 \Rightarrow t = -1$

Equation of the tangent at $L'(a,-2a)$ is $y(-2a) = 2a(x+a) \Rightarrow x + y + a = 0$

$\therefore$ The common tangent to the parabola touches the parabola $x^2 = 4ay$ at $L'(a,-2a)$

Equation of the tangent at $L(a,-2a)$ to $x = 4ay$ is $x(-2a) = 2a(y+a) \Rightarrow x + y + a = 0$

Common tangent to the parabola touches the parabola cut $L' = (-2a, a)$

***5. The sum of ordinates of two points on $y^2 = 4ax$ is equal to the sum of the ordinates of the two other points on the same curve show that the chord joining the first two points is parallel to the chord joining the other two points.**

**Sol.** Equation of the parabola is $y^2 = 4ax$

Equation of the chord joining $P(t_1)$ and $Q(t_2)$ is $(t_1+t_2)y = 2x+2at_1t_2$

Slope of $PQ = \dfrac{2}{t_1+t_2}$ --(1)

Equation of the chord joining $R(t_3)$ and $S(t_4)$ is $(t_3+t_4)y = 2x+2at_3t_4$

Slope of $RS = \dfrac{2}{t_3+t_4}$ --(2)

Given $2at_1 + 2at_2 = 2at_3 + 2at_4$

$\Rightarrow 2a(t_1+t_2) = 2a(t_3+t_4) \Rightarrow t_1+t_2 = t_3+t_4$ --(3)

from (1) (2) (3) we get slope of $PQ$ = slope of $RS$

i.e., $PQ$ and $RS$ are parallel.

*Note :*

*If $t_1, t_2$ are the ends of the focal chord of $y^2 = 4ax$, then $t_1t_2 = -1$*

**6. Prove that the portion of the tangent intercepted between the point of contact and the directrix of the parabola $y^2 = 4ax$ subtends a right angle at its focus.**

**Sol.** Equation of the tangent at $P(at^2, 2at)$ to the parabola

$y^2 = 4ax$ is $x - ty + at^2 = 0$ -- (1)

Equation of the directrix of the parabola is

$x + a = 0$ -- (2)

Solving (1) & (2) the point of intersection is $\left[-a, \dfrac{a(t^2-1)}{t}\right]$

Now $PQ$ is the position of the tangent intercepted between the point of contact and the directrix focus $S(a, 0)$

Slope of $SP$ x Slope of $SQ = \left[\dfrac{2at-0}{at^2-a}\right]\left[\dfrac{\dfrac{a(t^2-1)}{t}-0}{-a-a}\right] = -1$

$\therefore$ $PQ$ subtends a right angle at the focus of the parabola.

**7. Prove that the radius of the circle whose centre is (–4, 0) and which cuts the parabola $y^2 = 8x$ at $A$ and $B$ such that its common chord $AB$ subtends a right angle at the vertex of the parabola is equal to "4".**

**Sol.** Let $r$ be the radius of the circle. Then its equation is $(x+4)^2 + y^2 = r^2$ -- (1)

This cuts the parabola $y^2 = 8x$ at points $A(x_1, y_1)$ and $B(x_2, y_2)$

The abscicsae of $A$ & $B$ are the roots of the equation

$(x+y)^2 + 8x = r^2$ (or) $x^2 + 16x + 16 - r^2 = 0$, $\therefore x_1x_2 = 16 - r^2$ -- (2)

The ordinates of $A$ and $B$ are given by $y_1^2 = 8x_1$ and $y_2^2 = 8x_2$ respectively.

$\therefore y_1 = \pm 2\sqrt{2x_1}$ and $\therefore y_2 = \pm 2\sqrt{2x_2}$

Since $AB$ subtends a right angle at the vertex of the parabola

$\therefore \dfrac{y_1}{x_1} \times \dfrac{y_2}{x_2} = -1 \Rightarrow x_1x_2 + y_1y_2 = 0 \Rightarrow x_1x_2 + 8x_1x_2 = 0 \Rightarrow x_1x_2 = 0$

$\Rightarrow 16 - r^2 = 0 \Rightarrow r = 4$

**8. A triangle $ABC$ of area $\Delta$ is inserted in the parabola $y^2 = 4ax$ such that $A$ is the vertex and $BC$ is a focal chord of the parabola then find the difference of the ordinates of $B$ and $C$.**

**Sol.** Let $B(at_1^2, 2at_1)$ and $C(at_2^2, 2at_2)$ be the coordinate of the end points of focal chord $BC$.

Then, $\Delta$ = area of the triangle

$$= \Delta = \frac{1}{2}\text{absolute value of}\begin{vmatrix} 0 & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix}$$

$\Rightarrow \Delta = \left|a^2 t_1 t_2 (t_1 - t_2)\right|$ $(\because t_1t_2 = -1)$

$\Rightarrow \Delta = a^2\left|t_1 - t_2\right| \Rightarrow \left|2at_1 - 2at_2\right| = \dfrac{2\Delta}{a}$

*Note :* *The orthocenter if the triangle formed by three tangents of a parabola lies on directrix.*

**9. Prove that the circle on a focal radius of a parabola, as diameter touches the tangent at the vertex.**

***Sol.*** Let the parabola be $y^2 = 4ax$

Let $P(at^2, 2at)$ be a point on the parabola and $S(a, 0)$

Equation to the circle on $SP$ as diameter

$(x-a)(x-at^2)y(y-2at)=0$

$\Rightarrow x^2+y^2-ax(1+t^2)-2aty+a^2t^2=0$

In this circle $f=-at$ and

$c=a^2t^2$ and $f^2-c=a^2t^2-a^2t^2=0$

$\Rightarrow$ circle touches the $y$-axis

Hence the circle touches the tangent at the vertex

**10. Prove that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix of the parabola.**

***Sol.*** Let $y^2 = 4ax$ be the parabola and

$A=(at_1^2, 2at_1)$, $B=(at_2^2, 2at_2)$, $C=(at_3^2, 2at_3)$ be any three points on it.

Now we consider the triangle $PQR$ formed by the tangents to the parabola at $A, B, C$ where $P=(at_1t_2, a(t_1+t_2))$, $Q=(at_2t_3, a(t_2+t_3))$ and $R=(at_3t_1, a(t_3+t_1))$.

Equation of $\overline{QR}$ (i.e., the tangent at $C$) is $x-t_3y+at_3^2=0$.

Therefore, the attitude through $P$ of triangle $PQR$ is

$t_3x+y=at_1t_2t_3+a(t_1+t_2)$ ...(1)

Similarly, the attitude through $Q$ is $t_1x+y=at_1t_2t_3+a(t_2+t_3)$ ....(2)

Solving (1) and (2), we get $(t_3-t_1)x=a(t_1-t_3)$ i.e., $x=-a$

Therefore, the orthocenter of the triangle $PQR$, with abscissa as $-a$, lies on the directrix of the parabola.

## EXERCISE - 3.2

1. 1. Find the equations of the tangent and normal to the parabola $y^2 = 6x$ at thepositive end of the latus rectum. **[Ans : Tangent $2x-2y+3=0$ Normal $2x+2y-9=0$]**

*2. Find the equation of the tangent and normal to the parabola $x^2-4x-8y+12=0$ at $\left(4, \frac{3}{2}\right)$ **(March-19)** **[Ans : Tangent $x-2y-1=0$, Normal $4x+2y-19=0$]**

*3. Find the value of $k$ if the line $2y=5x+k$ is a tangent to the parabola $y^2=6x$ **(March-18)** **[Ans : $k=\frac{6}{5}$]**

*4. Find the equation of the normal to the parabola $y^2=4x$ which is parallel to $y-2x+5=0$ **(March-19)** **[Ans : $2x-y-12=0$]**

5. Show that the line $2x-y+2=0$ is a tangent to the parabola $y^2=16x$. Findthe point of contact also. **[Ans : (1, 4)]**

6. Find the equation of tangent to the parabola $y^2=16x$ inclined at an angle $60^0$ with its axis and also find the point of contact. **[Ans : $3x-\sqrt{3}y+4=0\left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$]**

**II.** 1. Find the equation of tangents of the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$, also find the coordinates of their points of contact.

[**Ans :** $2x - y + 2 = 0$, point of contact $(1, 4)$
$x - 2y + 16 = 0$, point of contact $(16, -16)$]

2. If $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, then show that $al^3 + 2alm^2 + nm^2 = 0$

3. Show that the equation of common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are $y = \pm(x + 2a)$

4. Prove that tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix

5. Find the condition for the line $y = mx + c$ to be a tangent to the parabola $x^2 = 4ay$

[**Ans :** $am^2 + c = 0$]

6. Three normals are drawn from $(k, 0)$ to the parabola $y^2 = 8x$ one of the normal is the axis and the remaining two normals are perpendicular to each other, then find the value of $k$ [**Ans :** $k = 6$]

7. Show that the locus of point of intersection of perpendicular tangents to the parabola $y^2 = 4ax$ is the directrix $x + a = 0$

8. Two parabolas have the same vertex and equal length of latus rectum such that their axes are at right angle. Prove that the common tangents touch each at the end of latus rectum.

9. Show that the foot of the perpendicular from focus to the tangent of the parabola $y^2 = 4ax$ lies on the tangent at vertex.

10. Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

**III.** 1. The normal at a point $t_1$ on $y^2 = 4ax$ meets the parabola again in the point $t_2$. Then prove that $t_1t_2 + t_1^2 + 2 = 0$

*2. From an external point $P$ tangents are drawn to the parabola $y^2 = 4ax$ and these tangents make angles $\theta_1, \theta_2$ with its axis, such that $\cot\theta_1 + \cot\theta_2$ is a constant '$d$'. Then show that all such $P$ lie on a horizontal line. **(March-19)**

3. Show that the common tangents to the circle $2x^2 + 2y^2 = a^2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola $y^2 = -4ax$

4. The sum of the ordinates of two points on $y^2 = 4ax$ is equal to the sum of the ordinates of two other points on the same curve. Show that the chord joining the first two points is parallel to the chord joining the other two points.

5. If a normal chord a point '$t$' on the parabola $y^2 = 4ax$ subtends a right angle at vertex, then prove that $t = \pm\sqrt{2}$