COMPLEX NUMBERS

CHAPTER 4

Algebra

- COMPLEX NUMBER MODULUS AND ITS AMPLITUDE
- POLAR FORM, PROPERTIES & CONI'S THEOREM
- LOCUS OF A POINT, STANDARD LOCI
- EQUATION OF A STRAIGHT LINE AND CIRCLE

4.0 INTRODUCTION

Consider a simple algebraic equation $x^2 + 1 = 0$ which have real coefficients but do not possess a real number as its solution. i.e. no number in the set of real numbers acts as a solution of $x^2 + 1 = 0$. Thus if we restrict ourselves to the set of real numbers the solution set of the above equation is null set. Thus in order to have solutions of such equations, the concept of number was extended beyond that of real numbers. Such extension of set of real numbers is the set of complex numbers.

Leonard Euler (1707 – 83) was the first to use the symbol i for $\sqrt{-1}$ i. e. $i^2 = -1$.

Euler called the symbol 'i' imaginary. Once the symbol 'i' was introduced the symbol (or) structure $a + ib$, where a, b are real numbers, came into picture and was called a complex number.

The theory of complex numbers was later on developed by Gauss (1777 - 1855) and Hamilton (1805 - 1865).

4.1 COMPLEX NUMBER

Remember : *There is one to one correspondence between set of complex numbers and set of points in a two dimensional cortesian plane*

Definition

An ordered pair (a, b) of real numbers a and b is called a complex number. The set of all such pairs is called set of complex numbers denoted by C.

Thus $C = \{(a,b)/a,b \in R\}$

for example $(2, 3)$, $(-2, 1)$, $\left(\sqrt{2}, \frac{1}{2}\right)$, $(2, \pi)$ *are complex numbers.*

Note : *(a, b) is denoted by z i. e. z = (a, b)*

4.1.1 EQUALITY OF TWO COMPLEX NUMBERS

Definition

Two complex numbers (a_1, b_1) *and* (a_2, b_2) *are said to be equal iff* $a_1 = a_2$ *and* $b_1 = b_2$

i.e. $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$ *then* $z_1 = z_2 \Leftrightarrow (a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2, b_1 = b_2$

4.2 FUNDAMENTAL ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

i) **Addition of complex numbers :**

Definition

The sum of two complex numbers (a_1, b_1) *and* (a_2, b_2) *is defined as the complex number* (a_1+a_2, b_1+b_2).

i.e. for $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$

$$z_1 + z_2 = (a_1, b_1)+(a_2, b_2) = (a_1+a_2, b_1+b_2)$$

ii) **Multiplication of complex numbers :**

Definition

The product of two complex numbers (a_1, b_1) *and* (a_2, b_2) *is defined as the complex number* $(a_1a_2 - b_1b_2, a_1b_2+b_1a_2)$.

ie. Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$ *then* $z_1z_2 = (a_1, b_1).(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2+b_1a_2)$

4.3 PROPERTIES (Addition)

1) **Associative law :**

For $z_1, z_2, z_3 \in C$, $z_1+(z_2+z_3) = (z_1+z_2)+z_3$

2) **Commutative law :**

for $z_1, z_2 \in C$, $z_1+z_2 = z_2+z_1$

3) **Identity : (additive)**

complex number $0 = (0, 0)$ exists in C such that for any $z \in C$, $z+0 = 0+z = z$

4) **Inverse (additive) :**

for $z = (a,b)$ in C there exists '$-z$' $= (-a,-b)$ such that $z+(-z) = (-z)+z = 0$

$-z$ is the additive inverse of z.

i.e. $(-a, -b)$ is the additive inverse of $z = (a, b)$.

Example : i) The additive inverse of $(2, 3)$ is $(-2, -3)$

ii) The additive inverse of $(0,\sqrt{2})$ is $(0,-\sqrt{2})$

Note

The complex number $(-a, -b)$ ***is written as*** $-(a, b)$.

4.4 PROPERTIES (Multiplicatin)

1) **Association law :**

for $z_1, z_2, z_3 \in C$

$z_1(z_2.z_3) = (z_1.z_2)z_3$.

2) **Commutative law :**

for $z_1, z_2 \in C$

$z_1z_2 = z_2z_1$

3) **Identity for multiplication :**

for $z \in C$ these exists complex number $(1, 0)$ denoted by 1 such that $z.1 = 1.z = z$

i.e. $(a, b) . (1, 0) = (1, 0) . (a, b) = (a, b)$.

$(1, 0)$ is called multiplicative identity.

4) **Multiplicative inverse :**

Every complex number $z = (a, b)$ has an inverse with respect to multiplication except when $a = 0$, $b = 0$

i.e. Let $z = (a,b) \neq (0,0) \in C$ then there exists

$$z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) \in C \text{ such that } z.z^{-1} = z^{-1}.z = 1$$

Thus z^{-1} is the inverse of z.

i.e. $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the inverse of (a,b).

Example : i) The inverse of (2, 3) is $\left(\frac{2}{13}, -\frac{3}{13}\right)$

ii) The inverse of (0, 5) is $\left(0, -\frac{1}{5}\right)$

Note

z^{-1} is also denoted by $\frac{1}{z}$.

4.5 SUBTRACTION OF COMPLEX NUMBERS

Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$ Then $z_1 - z_2 = (a_1, b_1) - (a_2, b_2) = (a_1-a_2, b_1-b_2)$

4.6 DIVISION OF COMPLEX NUMBERS

Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$ such that

$z_2 = (a_2, b_2) \neq (0,0)$ then $\frac{z_1}{z_2} = z_1 . \frac{1}{z_2}$

$$= (a_1, b_1)\left(\frac{a_2}{a_2^2+b_2^2}, \frac{-b_2}{a_2^2+b_2^2}\right)$$

$$= \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}, \frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)$$

Example : $\frac{(2,3)}{(4,5)} = \left(\frac{(2)(4)+(3)(5)}{41}, \frac{(3)(4)-(2)(5)}{41}\right) = \left(\frac{23}{41}, \frac{2}{41}\right)$

Note

i) ***Multiplication over addition of complex number is distributive.***

Let $z_1, z_2, z_3 \in C$, $z_1.(z_2+z_3) = z_1z_2 + z_1z_3$

ii) ***The complex number (0, 0) is denoted by 0***

iii) ***The complex number (1, 0) is denoted by 1***

iv) ***Complex number (0,1) : The complex number (0, 1) is generally called imaginary unit and is denoted by i. i.e. i = (0, 1)***

Also $i^2 = i.i = (0, 1).(0, 1) = (-1, 0)$ i.e. $i^2 = -1$ i.e. $i = \sqrt{-1}$

4.7 DEFINITION

Remember :
$\sqrt{a}\sqrt{b} \neq \sqrt{ab}$
if a and b are both negative.

We define the square root of a negative real number as given below :

$$\sqrt{-a^2} = \sqrt{(a^2)(-1)} = \sqrt{a^2}\sqrt{-1} = |a|\, i.$$

For example : $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$

$\sqrt{-3} = \sqrt{3}\sqrt{-1} = \sqrt{3}\, i$

Ex : Simplify $\sqrt{5}\sqrt{-3}$.

Sol. $\sqrt{5}\sqrt{-3} = \sqrt{5}\left\{\sqrt{3}\sqrt{-1}\right\} = \sqrt{5}\sqrt{3}i = \sqrt{15}\, i$

Ex : Simplify $\sqrt{-7} \times \sqrt{-8}$?

Sol. $\sqrt{-7} \times \sqrt{-8} = \left\{\sqrt{7}\sqrt{-1}\right\} \times \left\{\sqrt{8}\sqrt{-1}\right\} = \left\{\sqrt{7}\, i\right\}\left\{\sqrt{8}\, i\right\}$

$= \sqrt{7} \times \sqrt{8} \times i^2 = \sqrt{56}\, i^2 = \sqrt{56}(-1) = -2\sqrt{14}$

4.8 DEFINITION

If $z(\neq 0) \in C$ and $n \in Z$ we define

i) $z^1 = z$

ii) $z^2 = z.z$

iii) $z^{n+1} = z^n.z \quad (n > 0)$

iv) $z^0 = 1$

v) $z^n = (z^{-n})^{-1} \quad (n < 0)$

4.9 RECTANGULAR FORM OF COMPLEX NUMBER

Since $(a, b) = (a + 0, 0 + b) = a(1, 0) + b(0, 1) = a(1) + b(i) = a + b\, i = a + ib$

This binomial form $a+bi$ (or) $a+ib$ of complex number (a, b) is very useful and is called rectangular form of the complex number (a, b).

Hence forth $a+ib$ means (a, b).

We denote this by single letter z.

Thus $z = a+ib \in C$ when $a, b \in R$.

'a' is called real part of z and 'b' called imaginary part of z and they are respectively denoted by Re(z) and Im (z).

Thus $z = a + ib \Rightarrow \text{Re}(z) = a$, $\text{Im}(z) = b$, which are real numbers.

Note

i) If b = 0 then it is equivalent to a which is purely real number.

ii) If a = 0 and b ≠ 0 then it is 'bi' called purely imaginary number.

iii) Every real number is a complex number but a complex number may not be a real number.

iv) Every imaginary quantity is a complex number but a complex number may not be imaginary.

4.10 PROPERTIES

1) $a+ib = 0 \Leftrightarrow a=0,\ b=0$
2) $a_1+ib_1 = a_2+ib_2 \Leftrightarrow a_1= a_2$ and $b_1= b_2$
3) $(a_1+ib_1) + (a_2+ib_2) = (a_1+ a_2) + i(b_1+ b_2)$
4) $(a_1+ib_1) - (a_2+ib_2) = (a_1-a_2) + i(b_1- b_2)$
5) $(a_1+ib_1)(a_2+ib_2) = (a_1a_2- b_1b_2) + i(a_1b_2+ a_2b_1)$
6) $\dfrac{(a_1+ib_1)}{(a_2+ib_2)} = \left(\dfrac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right) + i\left(\dfrac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)$ (where $(a_2,b_2)\neq(0,0)$)
7) $(a+ib)^2 = (a+ib)(a+ib) = (a^2-b^2) + i(2ab)$
8) $(a+ib)^3 = (a^3-3ab^2) + i(3a^2b-b^3)$

4.11 INTEGRAL POWERS OF THE IMAGINARY UNITY 'i'

1) $i^2 = i.i = (0,1)\ (0,1) = (0-1,\ 0+0)= (-1,\ 0) = -1$
2) $i^3 = i^2.i = (-1)i = -i$
3) $i^4 = (i^2)^2 = (-1)^2 = 1$
4) $i^5 = (i^4).i = (1)i = i$
5) $i^6 = -1$
6) $i^7 = -i$
7) $i^8 = 1$ and so on.

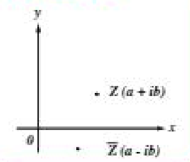
Thus i^n $(n \in Z)$ always gives one of the four values i, -1, $-i$, 1 periodically

Note

i) $i^{4n} = 1$
ii) $i^{4n+1} = i$
iii) $i^{4n+2} = -1$
iv) $i^{4n+3} = -i$ $(n \in Z)$

4.12 CONJUGATE OF A COMPLEX NUMBER

Remember :



Definition

The complex numbers $a + ib$ and $a - ib$ are called conjugate complex numbers to each other.

If z is a complex number its conjugate is denoted by $\bar{z}$

For example :

i) the conjugate of $-3 + 2i$ is $-3 - 2i$
ii) the conjugate of $5i$ is $-5i$
iii) the conjugate of 9 is 9
iv) the conjugate of -6 is -6
v) the conjugate of 0 is 0

Note

i) *The sum and product of two conjugate complex numbers are real numbers*
ii) *The difference of two conjugate complex numbers is purely imaginary or zero.*

4.13 PROPERTIES

Let $z_1, z_2 \in C$ then

1) $\overline{(\overline{z_1})} = z_1$

2) $\overline{(-z_1)} = -\overline{z_1}$

3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

4) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

5) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

6) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ $(z_2 \neq 0)$

7) $\overline{z_1 z_2 z_3 \ldots\ldots\ldots z_n} = \overline{z_1}\ \overline{z_2}\ \overline{z_3} \ldots\ldots\ \overline{z_n}$

8) $\overline{z_1 + z_2 + z_3 + \ldots\ldots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \ldots\ldots + \overline{z_n}$

9) $\overline{\left(\dfrac{z_1 + z_2}{z_3 z_4}\right)} = \dfrac{\overline{z_1} + \overline{z_2}}{\overline{z_3}\ \overline{z_4}}$

10) $\overline{i} = -i$ and $\overline{(-i)} = i$

11) $\overline{KZ_1} = K.\overline{Z_1}$ $(K \in R)$

12) If $z = a+ib$ then

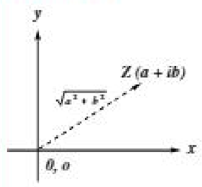
i) $z + \overline{z} = 2a = 2\,\mathrm{Re}(z)$

ii) $z - \overline{z} = 2ib = 2i\ \mathrm{Im}\,(z)$

iii) $z\overline{z} = a^2 + b^2$

4.14 MODULUS OF A COMPLEX NUMBER

Remember :



Definition

If $z = a + ib \in C$ is a complex number then the non - negative real number $\sqrt{a^2 + b^2}$ is called modulus of z.

It is denoted by $|z|$

Thus $z = a + ib \Rightarrow |z| = \sqrt{a^2 + b^2}$

for example if $z = 3+2i$ then $|z| = |3+2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$

similarly $|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$

Note

i) $|a + ib| = |a - ib| = |-a + ib| = |-a - ib| = \sqrt{a^2 + b^2}$

ii) $|z| = 0 \Leftrightarrow z = 0$

iii) ***The only complex number in C whose modulus is zero is the number 0***

iv) ***If the modulus of a complex number is one i.e. unity then it is called unimodular complex number. ie. z is unimodular*** $\Leftrightarrow |z| = 1$.

v) ***There are infinitely many unimodular complex numbers.***

For example 1, –1, i, –i, $\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}, \dfrac{1}{5} + \dfrac{2\sqrt{6}}{5}i$ ***etc., are unimodular.***

4.15 PROPERTIES

Let $z_1, z_2 \in C$ then

1) $|z_1 + z_2| \leq |z_1| + |z_2|$

2) $|z_1 - z_2| \leq |z_1| + |z_2|$

3) $|z_1 - z_2| \geq ||z_1| - |z_2||$

4) $|z_1 z_2| = |z_1||z_2|$

5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$

6) $\left|\frac{z_1 z_2}{z_3 z_4}\right| = \frac{|z_1||z_2|}{|z_3||z_4|} \quad (z_3 \neq 0,\ z_4 \neq 0)$

7) $|\overline{z_1}| = |z_1|$

8) $|\overline{z_1 + z_2 + z_3 + ... + z_n}| = |z_1 + z_2 + z_3 + ... + z_n|$

9) $|\overline{z_1 z_2 \,............\, z_n}| = |z_1||z_2||z_3|.........|z_n|$

10) $|z_1 + z_2 + z_3 + ... + z_n| \leq |z_1| + |z_2| + |z_3| + ... + |z_n|$

11) $z\bar{z} = |z|^2$ and its converse $|z|^2 = z\bar{z}$

12) If |z| = 1 i.e z is unimodular then $z\bar{z} = 1$ i.e., $\bar{z} = \frac{1}{z}, z = \frac{1}{\bar{z}}$

13) $|z^n| = |z|^n \quad (n \in N)$

14) $|-z| = |\bar{z}| = |-\bar{z}| = |z|$

15) $\text{Re}(z) \leq |z|$, $\text{Im}(z) \leq |z|$

16) $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$

17) $|a_1 z_1 + a_2 z_2 + ... + a_n z_n| \leq |a_1||z_1| + |a_2||z_2| + ... + |a_n||z_n|$

where $a_1, a_2 \,.........\, a_n \in R$

18) $|1 + z + z^2 + ... + z^n| < \frac{1}{1 - |z|} (|z| \neq 1)$

4.16 SQUARE ROOT OF A COMPLEX NUMBER

THEOREM-4.1 $\sqrt{a+ib}=\pm(x+iy)$ where $x=\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}$, $y=\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}$

Proof :

$\sqrt{a+ib}=\pm(x+iy)$

$\Rightarrow a+ib = (x+iy)^2 = x^2-y^2+2ixy$

$\Rightarrow a = x^2-y^2 \quad \rightarrow (1), \qquad b = 2xy \quad \rightarrow (2)$

$\therefore (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 = a^2+b^2$

$\Rightarrow x^2+y^2=\sqrt{a^2+b^2} \rightarrow (3)$

adding (1) and (3), $2x^2=\sqrt{a^2+b^2}+a$

$$x=\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}$$

Subtracting (1) from (3), $2y^2=\sqrt{a^2+b^2}-a$

$$\Rightarrow y=\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}$$

$\therefore \sqrt{a+ib}=\pm(x+iy)$, where $x=\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}$, $y=\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}$

Note

i) If $\sqrt{a+ib}=\pm(x+iy)$ then $\sqrt{a-ib}=\pm(x-iy)$

ii) $\sqrt{a+ib}+\sqrt{a-ib}=\pm\sqrt{(2\sqrt{a^2+b^2}+2a)}$ (or) $\pm i\sqrt{2\sqrt{a^2+b^2}-2a}$

iii) $\sqrt{i}=\pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\pm\left(\frac{1+i}{\sqrt{2}}\right)$

iv) $\sqrt{-i}=\pm\left(\frac{1-i}{\sqrt{2}}\right)$

v) $\sqrt{i}+\sqrt{-i}=\sqrt{2}$ (or) $-\sqrt{2}$ (or) $\sqrt{2}i$ (or) $-\sqrt{2}i$

vi) $\sqrt{z}=\pm(x+iy)$ where $z=a+ib$, $x=\sqrt{\frac{|z|+Rez}{2}}$, $y=\sqrt{\frac{|z|-Rez}{2}}$

4.17 n^{th} ROOTS OF A COMPLEX NUMBER

Definition

Let n be a positive integer and z be a complex number. Then the solutions of $x^n=z$ are called n^{th} roots of z i.e., the complex number w such that $w^n = z$ is called n^{th} root of z and is denoted as $z^{1/n}$.

Note : If $n=3$ then n^{th} roots of z are called cube roots of z.

Definition

Let z be a complex number , m be an integer and n be a positive integer. Then $z^{m/n}$ is defined as $z^{m/n}=(z^m)^{1/n}$.

THEOREM-4.2 **(Cube Roots of unity)**

The cube roots of unity are 1, $\omega = \dfrac{-1+\sqrt{3}i}{2}$, $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$.

Proof :

Let $\sqrt[3]{1} = x$. Then $x^3 = 1 \Rightarrow x^3-1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow x = 1$ or $x^2+x+1 = 0$

If $x^2+x+1 = 0$ then $x = \dfrac{-1\pm\sqrt{1-4}}{2} = \dfrac{-1\pm\sqrt{3}i}{2} = \dfrac{-1+\sqrt{3}i}{2}$ or $\dfrac{-1-\sqrt{3}i}{2}$

Note

i) $\omega^3 = 1,\ \omega = \dfrac{1}{\omega^2}, \omega^2 = \dfrac{1}{\omega}$

ii) $1+\omega+\omega^2 = 0$

iii) $\omega+\omega^2 = -1$

iv) $1+\omega = -\omega^2$

v) $1+\omega^2 = -\omega$

vi) $\omega^{3n} = 1$

vii) $\omega^{3n+1} = \omega$

viii) $\omega^{3n+2} = \omega^2$

Remember :
$a,b,c \in R$ and
$a+b\omega+c\omega^2 = 0$
$\Rightarrow a = b = c$
Also $x, y \in R$ and
$x\omega+y\omega^2 = 0$
$\Rightarrow x = 0, y = 0$

Fourth roots of unity :

The equation $x^4 = 1$ is a 4^{th} degree equation and is satisfied by 4 values of x.

i.e. $x^4-1 = 0 \Rightarrow (x^2-1)(x^2+1) = 0$

$\Rightarrow x^2 = 1$ or $x^2 = -1$

$\Rightarrow x = 1, -1$ (or) $x = \pm\sqrt{-1}$ $\Rightarrow x = 1, -1, i, -i$

Thus the complex numbers $1, -1, i, -i$ are called fourth roots of unity

4.18 OREDER RELATION

If a and b are real numbers then we may have $a = b$ (or) $a < b$ (or) $a > b$ i.e., one of three order relations hold good. No such order relation exists for the imaginary quantities.

It is wrong to say $1 + 3i < 4 + 5i$

i.e. in complex numbers (imaginary quantities) the two relations "less than" and "greater than" are meaningless.

SOLVED EXAMPLES

1. Find the value of $i^{17} - 3i^2 + (1+i)^2(1+i^4)$.

Sol. $\text{G.E} = (i^4)^4 i - 3(-1) + (1 + i^2 + 2i)(1 + i^4)$

$= (1)i + 3 + (1 - 1 + 2i)(1 + 1)$

$= i + 3 + 4i \;\; = 3 + 5i$

2. Find the value of $i^2 + i^4 + i^6 + \ldots\ldots (2n+1)$ terms.

Sol. $\text{G.E} = (-1) + (1) + (-1) + (1) \ldots\ldots (2n+1)$ terms $= -1$

3. Show that the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is 4.

Sol. Consider $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{2}$

$= \frac{1+i^2+2i}{2} = \frac{1-1+2i}{2} = i$

$\therefore \left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1$

$\Rightarrow n$ can be .... $-8, -4, 0, 4, 8, \ldots\ldots$

Least positive integral value of 'n' is 4

4. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$ find x and y.

Sol. $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{2i}{2} = i$

$\therefore \frac{1-i}{1+i} = \frac{1}{i} = -i$

$\text{G.E} = \{i\}^3 - \{-i\}^3 = -2i = 0 - 2i$

$\therefore x = 0$ and $y = -2$

5. Show that $\sum_{n=1}^{25}(i^n + i^{n+1}) = i - 1$.

Sol. $\text{LHS} = \sum_{n=1}^{25} i^n (1+i) = (1+i)\sum_{n=1}^{25} i^n$

$= (1+i) \cdot \{i + i^2 + i^3 + \ldots\ldots\ldots + i^{25}\}$

$= (1+i)\, i \frac{(1-i^{25})}{1-i}$ [since G.P.]

$= (1+i) i \frac{(1-i)}{(1-i)}$ $[i^{25} = i]$

$= (1+i)\, i = i - 1 = \text{RHS}$

> ***Remember :***
> *The complex numbers z_1, z_2 are conjugate if $\overline{z_1} = z_2$*

***6. Show that $\dfrac{2-i}{(1-2i)^2}$ and $\dfrac{-2-11i}{25}$ are conjugate to each other**

Sol. $$\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i} = \frac{2-i}{-(3+4i)} = \frac{2-i}{-(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{-2+11i}{25}$$

Since this complex number is the conjugate of $\dfrac{-2-11i}{25}$

$\therefore$ the given complex numbers are conjugate to each other

7. If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$ then show that $|z_1 + z_2 + z_3| = 1$.

Sol. $|z_1| = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \dfrac{1}{z_1}$ etc...

consider $|z_1 + z_2 + z_3| = |\overline{z_1 + z_2 + z_3}| = |\overline{z_1} + \overline{z_2} + \overline{z_3}| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

$\therefore |z_1 + z_2 + z_3| = 1$

8. If $w\ (\neq 1)$ is a cube root of unity and $(1+w^2)^n = (1+w^4)^n$ then find the least positive integral value of n.

Sol. $(1+w^2)^n = (1+w^4)^n \Rightarrow (-w)^n = (1+w)^n$

$\Rightarrow (-w)^n = (-w^2)^n \Rightarrow w^n = 1$

$\Rightarrow n = \ldots\ldots -6, -3, 0, 3, 6, \ldots..$

Least positive integral value of n is 3

EXERCISE - 4.1

1. Simplify

   i) $i^{18} - 3i^7 + i^2(1+i^4)(-i)^{26}$ [Ans : $1 + 3i$]

   ii) $\sqrt{-9}\sqrt{-4} + \sqrt{2}\sqrt{-8}$ [Ans : $-6 + 4i$]

2. Express the following in the form $a + ib$

   i) $\dfrac{a+ib}{a-ib}$ [Ans : $\dfrac{a^2-b^2}{a^2+b^2} + i\dfrac{2ab}{a^2+b^2}$]

   ii) $\dfrac{4-2i}{1-2i}$ [Ans : $\dfrac{8}{5} + i\dfrac{6}{5}$]

   iii) $(5-3i)^3$ [Ans : $-10-198i$]

iv) $(1-i)^3(1+i)$ [Ans : $0 + i(-4)$]

v) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$ [Ans : $(-6+\sqrt{2})+i(\sqrt{3}+2\sqrt{6})$]

vi) $\dfrac{4+2i}{1-2i}+\dfrac{3+4i}{2+3i}$ [Ans : $\dfrac{18}{13}+\dfrac{25}{13}i$]

vii) $\dfrac{2+5i}{3-2i}+\dfrac{2-5i}{3+2i}$ [Ans : $\dfrac{-8}{13}+i(0)$]

*viii) $(1+2i)^3$ **(March-17, 18)** [Ans : $\dfrac{-1}{2i}$]

3. i) If $z_1 = (6, 3)$, $z_2 = (2, -1)$, find $\dfrac{z_1}{z_2}$. [Ans : $\left(\dfrac{9}{5}, \dfrac{12}{5}\right)$]

ii) If $4x + i(3x - y) = 3 - 6i$, where x and y are real numbers, then find the values of x and y.

[Ans : $x = \dfrac{3}{4}, y = \dfrac{33}{4}$]

4. Find the additive inverse of the following

i) $(\sqrt{3}, 5)$ [Ans : $(-\sqrt{3}, -5)$]

ii) $(-6, 5)+(10, -4)$ [Ans : $(-4, -1)$]

iii) $(2,1)(-4,6)$ [Ans : $(14, -8)$]

iv) $\dfrac{1+i}{1-i}$ [Ans : $-i$]

5. Find the multiplicative inverse of the following

i) $7+24i$ [Ans : $\dfrac{7-24i}{625}$]

ii) $(-2, 1)$ [Ans : $\left(\dfrac{-2}{5}, \dfrac{-1}{5}\right)$]

iii) $(3, 4)$ [Ans : $\left(\dfrac{3}{25}, \dfrac{-4}{25}\right)$]

iv) $(\sin\theta, \cos\theta)$ [Ans : $(\sin\theta, -\cos\theta)$]

v) $\dfrac{3+i}{1-i}$ [Ans : $\dfrac{1-2i}{5}$]

6. Find the modulus of the following

i) $(3-4i)(3+4i)$ [Ans : 25]

ii) $\dfrac{4+3i}{5-12i}$ [Ans : $\dfrac{5}{13}$]

7. If $z = 5 + (3\sqrt{2})i$, find $z\bar{z}$. [Ans : 43]

8. Find the complex conjugate of the following

i) $\dfrac{3+i}{2+3i}$ [Ans : $\dfrac{9}{13}+\dfrac{7}{13}i$]

ii) $(3+4i)(2-3i)$ [Ans : $18+i$]

iii) $(2+5i)(-4+6i)$ [Ans : $-38+8i$]

*iv) $\dfrac{5i}{7+i}$ **(May-19)** [Ans : $\dfrac{1}{10}-\dfrac{7i}{10}$]

*9. If $z = (\cos\theta, \sin\theta)$, find $\left(z - \dfrac{1}{z}\right)$ **(March-19** [Ans : $(0, 2\sin\theta)$]

*10. If $z = 2 - 3i$, then show that $z^2 - 4z + 13 = 0$. **(March-18, 19)**

11. If $\dfrac{z_1+z_2}{z_1-z_2} = i$, then show that $\dfrac{z_1}{z_2}$ is purely imaginary.

12. i) If x and y are real numbers such that $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$, determine the values of x and y. [Ans : $x = 3, y = -1$]

ii) Write $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$ in the form $x+iy$. [Ans : $\dfrac{8ab(a^2-b^2)i}{(a^2+b^2)^2}$]

iii) Find the least positive integer n satisfying $\left(\dfrac{1+i}{1-i}\right)^n = 1$ [Ans : $n = 4$]

iv) Find x and y if $\dfrac{x-1}{3+i}+\dfrac{y-1}{3-i}=i$ where x and y are real. [Ans : $x = -4, y = 6$]

13. Prove that $z_1 = \dfrac{2+11i}{25}, z_2 = \dfrac{-2+i}{(1-2i)^2}$ are conjugate to each other

*14. If $x+iy = \dfrac{1}{1+\cos\theta+i\sin\theta}$, show that $4x^2 - 1 = 0$ **(March-19)**

15. If $x+iy = \dfrac{3}{2+\cos\theta+i\sin\theta}$, show that $x^2+y^2 = 4x-3$ **(March-17, 18 & May-19)

16. If $u+iv = \dfrac{2+i}{z+3}$, where $z = x+iy$, find the values of u and v. [Ans : $\dfrac{2(x+3)+y}{(x+3)^2+y^2}, \dfrac{x-2y+3}{(x+3)^2+y^2}$]

17. If $x+iy = \text{cis}\,\alpha . \text{cis}\,\beta$ then find the value of x^2+y^2 [Ans : 1]

18. i) If $z = 3-5i$, show that $z^3 - 10z^2 + 58z - 136 = 0$.

ii) If $z = 2 - i\sqrt{7}$, show that $3z^3 - 4z^2 + z + 88 = 0$.

19. If $(1-i)(2-i)(3-i)\ldots\ldots(1-ni) = x - iy$, prove that $2.5.10\ldots\ldots(1+n^2) = x^2 + y^2$. [Ans : 2]

20. If $(x - iy)^{\frac{1}{3}} = a - ib$, then show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$.

21. Simplify the following numbers and find their modulus.

i) $\frac{(2+4i)(-1+2i)}{(-1-i)(3-i)}$ [Ans : $2-i$ and $\sqrt{5}$]

ii) $\frac{(1+i)^3}{(2+i)(1+2i)}$ [Ans : $\frac{2+2i}{5}$ and $\frac{2\sqrt{2}}{5}$]

iii) $-2i(3+i)(2+4i)(1+i)$ [Ans : $8(4+3i)$, 40]

iv) $(-5i)\left(\frac{i}{8}\right)$ [Ans : 5/8]

22. Find the square root of the following.

i) $3+4i$ **(June-03)** [Ans : $\pm(2+i)$]

ii) $-5+12i$ [Ans : $\pm(2+3i)$]

iii) $7+24i$ **(March-15)** [Ans : $\pm(4+3i)$]

iv) $-47+i8\sqrt{3}$ [Ans : $\pm(1+4\sqrt{3}i)$]

v) $-8-6i$ [Ans : $\pm(1-3i)$]

23. i) Find the general value of x if $\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+i\tan\frac{x}{2}}{1+2i\sin\frac{x}{2}}$ is real. [Ans : $2n\pi, n \in z$]

ii) Find real value of θ inorder that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is

a) a real number [Ans : $\theta = n\pi, n \in Z$]

b) purely imaginary number [Ans : $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$]

iii) Find the least positive interger n for which $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\left(\mathrm{Sec}^{-1}\frac{1}{p} + \mathrm{Sin}^{-1}p\right)$

where $-1 \le x \le 1, x \ne 0$. [Ans : 4]

4.19 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

We know that the complex number $z = x + iy$ is an ordered pair (x, y) of real numbers, as discussed earlier in the definition.

If we consider the ordered pair (x, y) to be the two dimensional rectangular cartesian co-ordinates of a point, then

i) Every complex number (x, y) corresponds uniquely to a point in the co-ordinate plane.

ii) Conversely every point (x, y) in the co-ordinate plane corresponds to some complex number $z = x + iy$.

iii) Thus a one - one correspondence is established between the set of points in the co- ordinate plane and the set of complex numbers. The co-ordinate plane whose points are represented by complex numbers is called "complex plane" (or) "Gaussion plane" (or) "Argand plane" (or) Argand diagram. The axes ox, oy are known as "real axis" and "imaginary axis" respectively. In the argand diagram, the real number $x = x + 0i = (x, 0)$ corresponding to the point A, say on the x-axis and the pure imaginary number $yi = 0 + yi = (0, y)$ corresponds to a point B, say on the y–axis.

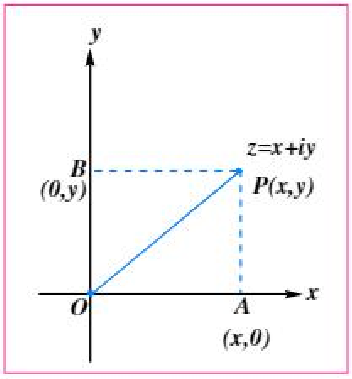


Fig. 4.1

The complex number z is called the "affix" of the point (x, y) which represents it.

4.20 VECTOR REPRESENTATION OF A COMPLEX NUMBER

Let $P(x, y)$ be the point corresponding to the complex number $z = x + iy$ in the complex plane (i.e., Argand diagram) with ox and oy as the co-ordinate axes.

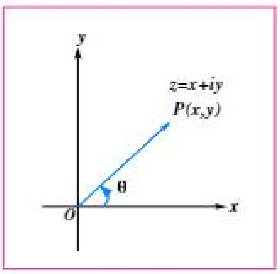


Fig. 4.2

Then the modulus and argument of the complex number z are given by the magnitude and direction of the vector $\overrightarrow{OP}$.

i.e. $\vec{OP} = x\bar{i} + y\bar{j}$

$$\left|\overline{OP}\right| = \sqrt{x^2 + y^2} = |z|$$

θ is argument of z ie. $\theta = \underline{|xOP}$

Observe the figures :

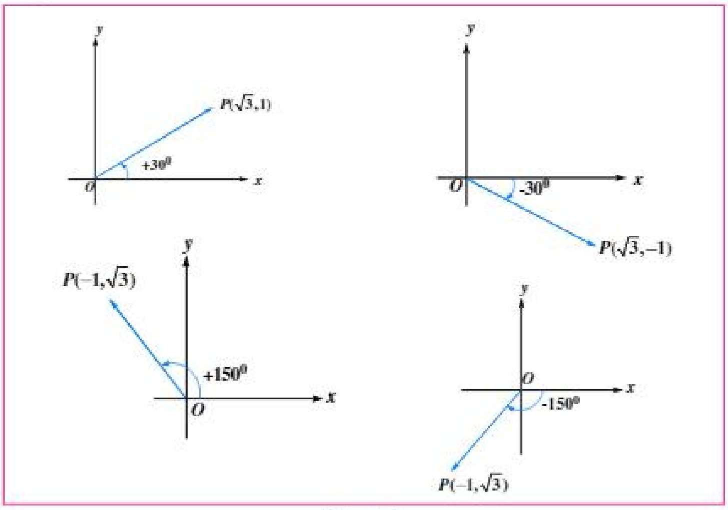


Fig. 4.3

4.21 POLAR FORM (OR) MODULUS ARGUMENT FORM (OR) TRIGONOMETRIC FORM OF A COMPLEX NUMBER

Definition

Let $z = a + ib$ be a complex number represented by $P(a, b)$ in the argand plane such that

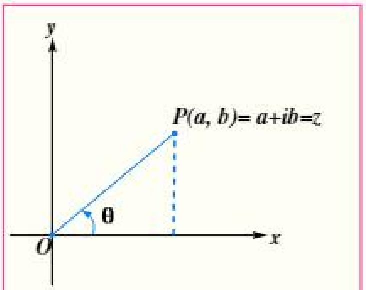


Fig. 4.4

$|z| = \sqrt{a^2 + b^2} = r$ *is called modulus of z and the real number θ satisfying simultaneously* $\cos\theta = \frac{a}{r}$ *and* $\sin\theta = \frac{b}{r}$ *is called "amplitude" or "argument" of z.*

We write $\theta = \arg(z)$. We write $z = a + ib = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$

Points to Remember on mod-amp form :

i) θ has infinitely many values in R.

ii) The value of θ satisfying $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$ is unique if θ is restricted in $(-\pi, \pi]$ (its least absolute value) and this value of θ is called "principal value of argument"

iii) If $\theta = \alpha$ is the principal argument of z then its general value is $2k\pi + \alpha$, $(k \in Z)$.

iv) The angle made by $\overrightarrow{OP}$ with the ray $\overrightarrow{OX}$ is the principal argument.

v) From the figures.

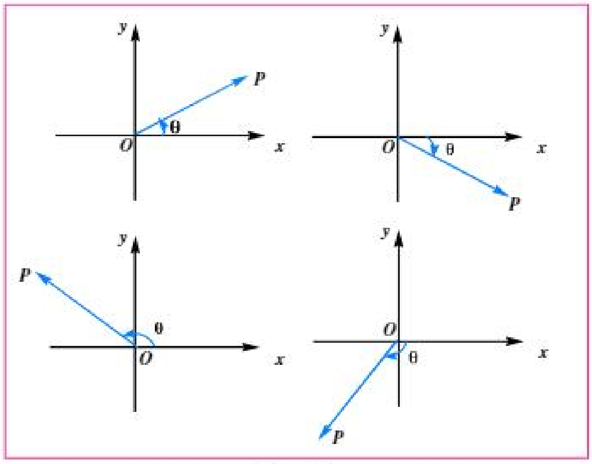


Fig. 4.5

If z lies in I (or) II quadrants Arg(z) will be positive and if z lies in III (or) IV quadrants Arg(z) will be negative.

z lies	value of argument (θ)
on + ve x –axis	0
on + ve y –axis	$\pi/2$
on – ve x–axis	π
on – ve y –axis	$-\pi/2$
in first quadrant	$0 < \theta < \pi/2$
in second quadrant	$\frac{\pi}{2} < \theta < \pi$
in third quadrant	$-\pi < \theta < -\frac{\pi}{2}$
in fourth quadrant	$\frac{-\pi}{2} < \theta < 0$
$z = 0$	undefined

vi) If $z = a + ib = (a, b)$ and $a > 0, b > 0$ and if principal argument of z is α ie. $Arg(z) = \alpha$.

Affix	Complex number	argument
$\bar{z}$	$(a, -b)$	$-\alpha$
$-\bar{z}$	$(-a, b)$	$\pi - \alpha$
$-z$	$(-a, -b)$	$-(\pi - \alpha)$

vii) *If a non principal value of the argument of z is* α *then its principal value is* $\alpha + 2k\pi$, $k = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$, *for some k.*

viii) *Modulus of a complex number* $z = a + ib$ *denoted by* $|z| = \sqrt{a^2+b^2} = \sqrt{(a-0)^2+(b-0)^2}$

= *distance of the point P(a, b) from origin*

$= |(a-0) + i(b-0)|$

$= |(a+bi) - (0+0i)| = |z-0|$

ix) $z_1 = a_1 + ib_1 = (a_1, b_1)$, $z_2 = a_2 + ib_2 = (a_2, b_2)$

$|z_1 - z_2| = |(a_1 + ib_1) - (a_2 + ib_2)|$

$= |(a_1 - a_2) + i(b_1 - b_2)| = \sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}$

= *Distance between points* $(a_1, b_1), (a_2, b_2)$

= *Distance between points whose affixes are* z_1, z_2

Thus distance between complex numbers z_1 *and* z_2 *is* $|z_1 - z_2|$.

x) $\cos\theta + i\sin\theta$ *is denoted by "cis* θ *"*

xi) $\cos\theta + i\sin\theta = e^{i\theta}$ *is called "Euler's formula"*

xii) $z = a + ib = r(\cos\theta + i\sin\theta) = r \cdot \text{cis}\,\theta = r.\,e^{i\theta}$

xiii) $\arg(z) = \alpha \Rightarrow$ i) $\arg(\bar{z}) = -\alpha$

a) $\arg(-z) = \begin{cases} \alpha - \pi & \text{if } \alpha > 0 \\ \alpha + \pi & \text{if } \alpha < 0 \end{cases}$

b) $\arg\left(\dfrac{1}{z}\right) = -\alpha$

c) $\arg\left(\dfrac{1}{\bar{z}}\right) = \alpha$

xiv) $\arg(z) = 0 \Leftrightarrow$ *z is real and positive*

$\arg(z) = \pi \Leftrightarrow$ *z is real and negative*

$\arg(z) = \dfrac{\pi}{2} \Leftrightarrow$ *z is purely imaginary,* $Im(z) > 0$

$\arg(z) = -\dfrac{\pi}{2} \Leftrightarrow$ *z is purely imaginary,* $Im(z) < 0$.

xv)

complex number	*principal argument*
1	0
-1	π
i	$\pi/2$
$-i$	$-\pi/2$
$1+i$	$\pi/4$
$-1+i$	$3\pi/4$
$-1-i$	$-3\pi/4$
$1-i$	$-\pi/4$
$\omega = \dfrac{-1+\sqrt{3}i}{2}$	$\dfrac{2\pi}{3}$
$\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$	$-\dfrac{2\pi}{3}$

THEOREM-4.3 If $z_1 = r_1 \operatorname{cis}\theta_1$, $z_2 = r_2 \operatorname{cis}\theta_2$ then

i) $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

ii) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

Proof : i) $z_1 z_2 = \{r_1 \operatorname{cis}\theta_1\}\{r_2 \operatorname{cis}\theta_2\}$

$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}$

$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$= r_1 r_2 . \operatorname{cis}(\theta_1 + \theta_2)$

ii) $\dfrac{z_1}{z_2} = \dfrac{r_1 cis\theta_1}{r_2 cis\theta_2} = \dfrac{r_1}{r_2} . \dfrac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \dfrac{r_1}{r_2} \cdot \dfrac{(\cos\theta_1, \sin\theta_1)}{(\cos\theta_2, \sin\theta_2)}$

$$= \frac{r_1}{r_2} . \left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}, \ \frac{\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2} \right)$$

$$= \frac{r_1}{r_2} . (\cos(\theta_1 - \theta_2), \sin(\theta_1 - \theta_2))$$

$$= \frac{r_1}{r_2} . (\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$= \frac{r_1}{r_2} . cis(\theta_1 - \theta_2)$$

Note

i) $|z_1 z_2| = r_1 r_2 = |z_1| |z_2|$

ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

iii) $z = \cos\theta + i\sin\theta \Rightarrow \dfrac{1}{z} = \cos\theta - i\sin\theta = \bar{z}$

iv) $r_1 cis\,\theta_1 = r_2\, cis\,\theta_2 \Rightarrow r_1 = r_2$ *and* $\theta_1 = 2k\pi + \theta_2\ (k \in z)$

v) If $Arg\, z_1 = \theta_1$, $Arg\, z_2 = \theta_2$ *where* θ_1, θ_2 *are principal values then*

a) $Arg(z_1 z_2) = \theta_1 + \theta_2 = Arg(z_1) + Arg(z_2)$

b) $Arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = Arg(z_1) - Arg(z_2)$

where $\theta_1 + \theta_2$ *and* $\theta_1 - \theta_2$ *are not necessarily principal values of arguments of* $z_1 z_2$ *and* $\dfrac{z_1}{z_2}$.

4.22 GEOMETRICAL REPRESENTATION OF ALGEBRAIC PROPERTIES OF COMPLEX NUMBERS

i) Addition :

$z_1 = x_1 + iy_1 = (x_1, y_1)$, $z_2 = x_2 + iy_2 = (x_2, y_2)$

Let $O(0, 0)$, $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

clearly from the figure (geometry) the complex number $z_1 + z_2$ i.e. $(x_1 + x_2, y_1 + y_2)$ is the fourth vertex opposite to the origin (O) of the parallelogram for which O, z_1, z_2 are three vertices.

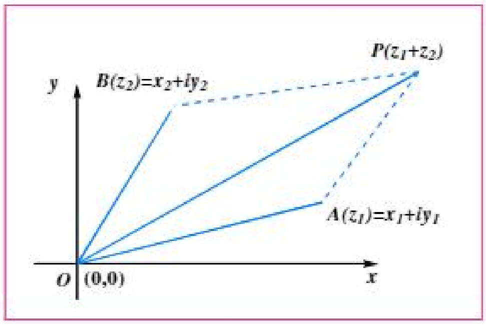


Fig. 4.6

Note

i) When $O(0, 0)$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are collinear then $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$ lies on the line, i.e. O, A, B are collinear $\Rightarrow$ O, A, B, P are collinear.

ii) $\overrightarrow{OP}$ is the resultant of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, vectorially speaking.

iii) $P = z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_1 + x_2 - 0, y_1 + y_2 - 0) = (x_1 + y_1) + (x_2 + y_2) - (0, 0) = z_1 + z_2 - 0.$

Important Inequality :

Clearly from the figure, $OA = BP = |z_1|$, $OB = AP = |z_2|$, $OP = |z_1 + z_2|$

In ΔOAP, using triangular inequality property

$OA + AP > OP \Rightarrow |z_1| + |z_2| > |z_1 + z_2| \Rightarrow |z_1 + z_2| < |z_1| + |z_2|$

applicable when O, z_1, z_2 form a triangle

(i.e. not collinear)

If O, z_1, z_2 are collinear and z_1, z_2 lie on same side of 'O' we have $|z_1 + z_2| = |z_1| + |z_2|$

Thus in general we have for two complex numbers z_1, z_2 ; $|z_1 + z_2| \leq |z_1| + |z_2|$.

Note

$$|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \arg z_1 - \arg z_2 = O \text{ and } \frac{z_1}{z_2} > O.$$

ii) Subtraction :

$A(z_1)$, $B(z_2)$ are complex numbers and O is origin $B^1(-z_2)$ such that O is the mid point of BB^1. Completing parallelogram $OAQB^1$, the point Q represents $z_1 + (-z_2)$ i.e. $z_1 - z_2$ complex number.

In vectors, $\overline{OQ} = \overline{BA} = \overline{OA} - \overline{OB} \Rightarrow z_1 - z_2 = \overline{OA} - \overline{OB}$

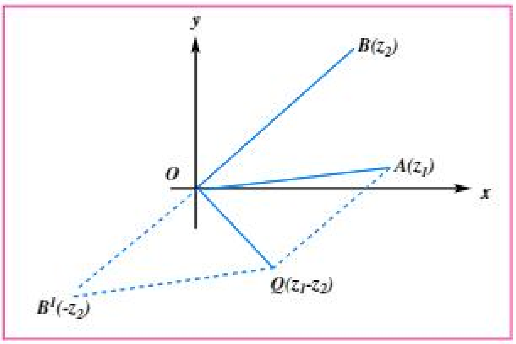


Fig. 4.7

Inclination of the line joining $B(z_2)$ to $A(z_1)$ with positive x-axis.

= Inclination of $\overline{OQ}$ with x-axis = $\arg(z_1 - z_2)$

Another Important Inequality :

From the figure, $OA = |z_1|$

$OB^1 = AQ = |-z_2| = |z_2|$

$OQ = |z_1 - z_2|$

Clearly in ΔOAQ, the diffrence of any two sides, is less than the third side,

i.e. $OA \sim AQ \leq OQ$

i.e. $|z_1| \sim |z_2| \leq |z_1 - z_2|$

i.e. $|z_1 - z_2| \geq \big| |z_1| - |z_2| \big|$

Note

i) ***Inequality holds when O, z_1, z_2 are non collinear.***

ii) ***Equality holds when O, z_1, z_2 are collinear z_1, z_2 lie on same side of O.***

iii) ***$|z_1 - z_2|$ denotes the distance between the points z_1, z_2.***

iii) Product and Quotient :

Let us first find the product of two complex numbers z_1, z_2. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be the points P_1 and P_2 in the complex plane. For convenience, take $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then it follows that $OP_1 = r_1$, $OP_2 = r_2$.

Now take $A(1, 0)$ on the real axis. Since $\angle AOP_1 = \theta_1$, $\angle AOP_2 = \theta_2$, construct ΔOP_2R similar to ΔOAP_1 as shown in Fig 4.8

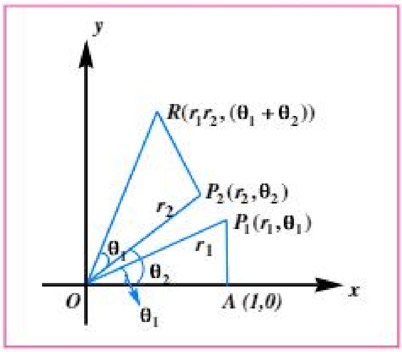


Fig. 4.8

Since ΔOAP_1, ΔORP_2 are similar, $\dfrac{OA}{OP_2} = \dfrac{OP_1}{OR}$

ie., $OP_1 \,.\, OP_2 = OA.OR$ ie., $r_1 \,.\, r_2 = 1.OR$ or $OR = r_1r_2$. Also $\angle AOR = \angle AOP_2 + \angle ROP_2 = \theta_2 + \theta_1$. Hence the polar coordinates of R are $(r_1r_2, (\theta_1+\theta_2))$. This means $r_1\, r_2\ cis\,(\theta_1+\theta_2)$ denotes the point R. Hence R denotes the complex number $z_1\, z_2$

We shall now find the quotient (z_1/z_2) of two complex numbers z_1 and z_2 when $z_2 \neq 0$. Following the discussion made on the product, take $z_1 = x_1 + iy_1 = r_1\, cis\,\theta_1$, $z_2 = x_2 + iy_2 = r_2\, cis\,\theta_2$ and denote z_1, z_2 with p_1, p_2 respectively.

From Fig. 4.9 $OP_1 = r_1$; $OP_2 = r_2$; $\angle XOP_1 = \theta_1$, $\angle XOP_2 = \theta_2$. Denote the point (1, 0) on OX by A. Construct ΔORP_1 similar to ΔOAP_2, as shown in the Fig 4.9. Then $\dfrac{OA}{OR} = \dfrac{OP_2}{OP_1}$,

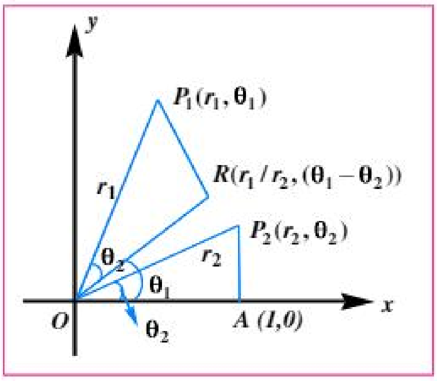


Fig. 4.9

i.e., $\dfrac{1}{OR} = \dfrac{r_2}{r_1}$ or $OR = \dfrac{r_1}{r_2}$; $\angle AOR = \angle AOP_1 - \angle P_1OR = \theta_1 - \theta_2$

$\therefore \dfrac{r_1}{r_2}\, cis(\theta_1-\theta_2)$ denotes the point R. Hence $R = \dfrac{z_1}{z_2}$

Important points :

1) $\left| |z_1| - |z_2| \right| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

2) $|z_1 + z_2| = |z_1| + |z_2|$ gives

   i) $\arg z_1 - \arg z_2 = 0$

   ii) $\frac{z_1}{z_2} > 0 \left(\because \frac{z_1}{z_2} = \frac{r_1}{r_2} > 0 \right)$

   iii) O, z_1, z_2 are collinear such that O, z_1, z_2 lie on same side of O.

3) $|z_1 - z_2| = |z_1| + |z_2|$ gives

   i) $\arg z_1 - \arg z_2 = \pi$

   ii) $\frac{z_1}{z_2} > 0 \left(\because \frac{z_1}{z_2} = \frac{r_1}{r_2}(-1) = -\frac{r_1}{r_2} < 0 \right)$

   iii) O, z_1, z_2 are collinear such that O lies between z_1 and z_2.

4) $|z_1 - z_2| = |z_1 + z_2|$ gives

   i) $\arg z_1 - \arg z_2 = \pm\frac{\pi}{2}$

   ii) $\bar{z}_1 . z_2$ is purely imaginary

   iii) $\frac{z_1}{\bar{z}_2}$ is purely imaginary

5) Parallelogram Law : $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

6) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$

   $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2)$

4.23 ROTATION (coni method)

CB is obtained by rotating CA through an angle α and multiplying with the number $\frac{CB}{CA}$.

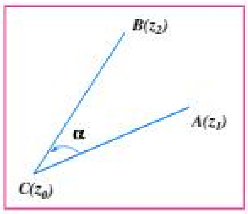


Fig. 4.10

$$\vec{CB} = \left\{\left(\frac{CB}{CA}\right)\vec{CA}\right\}(\cos\alpha + i\sin\alpha) \text{ i.e } \vec{CB} = \frac{|\vec{CB}|}{|\vec{CA}|}\vec{CA}\cdot e^{i\alpha}$$

$$\text{i.e } (z_2 - z_0) = \frac{|z_2 - z_0|}{|z_1 - z_0|}(z_1 - z_0)e^{i\alpha}$$

$$\text{i.e } \frac{(z_2 - z_0)}{|z_2 - z_0|} = \frac{(z_1 - z_0)}{|z_1 - z_0|}e^{i\alpha}.$$

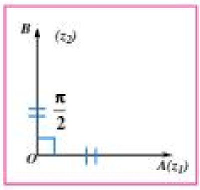
Note

i) *If CA = CB i.e* $|z_2 - z_0| = |z_1 - z_0|$ *then* $(z_2 - z_0) = (z_1 - z_0)\, e^{i\alpha}$.

ii) *If 'z_0' is origin then* $z_2 = \dfrac{|z_2|}{|z_1|} \cdot e^{i\alpha}$

iii) *If the rotation is* $\dfrac{\pi}{2}$ *i.e* $\alpha = \dfrac{\pi}{2}$. *i.e CB is obtained by rotating CA through* $\dfrac{\pi}{2}$

then $z_2 - z_0 = \dfrac{|z_2 - z_0|}{|z_1 - z_0|}(z_1 - z_0)\, i$

iv) 

From fig. $z_2 = iz_1$

Fig. 4.11

4.24 SECTION FOMRULA (Division formula)

Let z_1 and z_2 represent the points A and B. Let $C(z_3)$ divide AB in the ratio $l : m$

then $z_3 = \dfrac{lz_2 + mz_1}{l + m}$

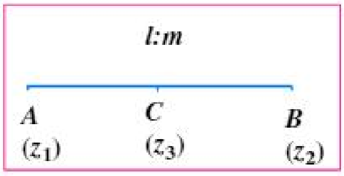


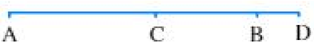
Fig. 4.12

Note

i) $z_3(l + m) = lz_2 + mz_1 \Rightarrow mz_1 + lz_2 + (-l - m)z_3 = 0 \Rightarrow pz_1 + qz_2 + rz_3 = 0$, *where* $p = m$, $q = l$, $r = -l - m$ *and* $p + q + r = 0$

Thus we have, z_1, z_2, z_3 *are collinear* $\Leftrightarrow$ *there exists nonzero scalars p, q, r such that* $pz_1 + qz_2 + rz_3 = 0$ *and* $p + q + r = 0$

ii) *If* $C(z_3), D(z_4)$ *divide AB in* $l : m$ *and* $-l : m$ *ratio then C, D are called harmonic conjugates of A and B. Thus AC, AB, AD are in H.P*



ie $|z_3 - z_1|, |z_2 - z_1|, |z_4 - z_1|$ *are in H.P.*

iii) *If* $A(z_1), B(z_2), C(z_3)$ *are vertices of* ΔABC *then centroid of* ΔABC : $\dfrac{z_1 + z_2 + z_3}{3}$ *Incentre* : $\dfrac{az_1 + bz_2 + cz_3}{a + b + c}$, *where* $a = |z_3 - z_2|$, $b = |z_1 - z_3|$, $c = |z_2 - z_1|$

Circumcentre $= \dfrac{\Sigma |z_1|^2 (z_2 - z_3)}{\Sigma \bar{z}_1 (z_2 - z_3)}$

THEOREM-4.4 Area of triangle with vertices $z_1, z_2, z_3 = \frac{1}{4}\left\| \begin{matrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{matrix} \right\|$

Proof :

Let $A(z_1) = x_1 + iy_1;\ B(z_2) = x_2 + iy_2;\ C(z_3) = x_3 + iy_3$

We know that Area (from geometry) of $\Delta ABC = |\Delta|$

$$\text{Where } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 & iy_1 & 1 \\ x_2 & iy_2 & 1 \\ x_3 & iy_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} \frac{z_1+\overline{z_1}}{2} & \frac{z_1-\overline{z_1}}{2} & 1 \\ \frac{z_2+\overline{z_2}}{2} & \frac{z_2-\overline{z_2}}{2} & 1 \\ \frac{z_3+\overline{z_3}}{2} & \frac{z_3-\overline{z_3}}{2} & 1 \end{vmatrix}$$

$$C_1 \to C_1 + C_2$$

$$= \frac{1}{2i}\begin{vmatrix} z_1 & \frac{z_1-\overline{z_1}}{2} & 1 \\ z_2 & \frac{z_2-\overline{z_2}}{2} & 1 \\ z_3 & \frac{z_3-\overline{z_3}}{2} & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\times\frac{1}{2}\begin{vmatrix} z_1 & z_1-\overline{z_1} & 1 \\ z_2 & z_2-\overline{z_2} & 1 \\ z_3 & z_3-\overline{z_3} & 1 \end{vmatrix} = \frac{1}{4i}\begin{vmatrix} z_1 & -\overline{z_1} & 1 \\ z_2 & -\overline{z_2} & 1 \\ z_3 & -\overline{z_3} & 1 \end{vmatrix}$$

$$\therefore \text{Area} = |\Delta| = \frac{1}{4}\left\| \begin{matrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{matrix} \right\|$$

Note

i) *Area* $= \frac{1}{4}\left|\sum \overline{z_1}(z_2 - z_3)\right|$

ii) z_1, z_2, z_3 *are collinear* $\sum \overline{z_1}(z_2 - z_3) = 0$

4.25 LOCUS

A point z which moves in the argand plane, according to a given geometrical condition, which is in the form of an equation involving complex numbers or real numbers, gives a real algebraic equation. This equation represents a geometrical path (or) sometimes a set of discrete points. Also Conditions with inequalities may represent regions.

4.26 SOME STANDARD LOCI

1) $|z| = k$, where k is non-negative real numbers represents a circle with centre at origin and radius k units

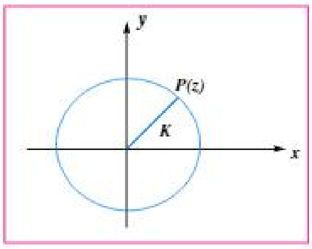


Fig. 4.13

2) $|z - z_0| = k$, where k is non-negative real number represents a circle with centre at z_0 and radius k units.

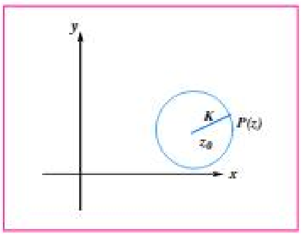


Fig. 4.14

3) $|z - z_1| = |z - z_2|$ where $z_1 \neq z_2$ represents the perpendicular bisector of the line segment joining the points z_1 and z_2

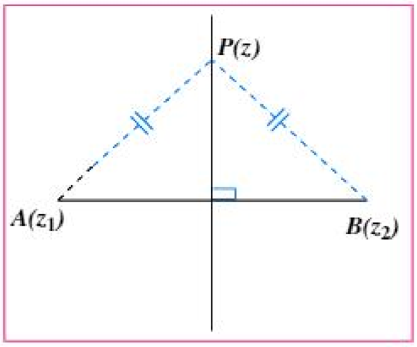


Fig. 4.15

4) $\left|\dfrac{z-z_1}{z-z_2}\right| = k$ where $k(\neq 1)$ is non-negative real number represents a circle. For this circle one pair of extremities of diameter are $\dfrac{kz_2+z_1}{k+1}$ and $\dfrac{kz_2-z_1}{k-1}$ and centre is at $\dfrac{k^2z_2-z_1}{k^2-1}$ which is the mid point of the above two points.

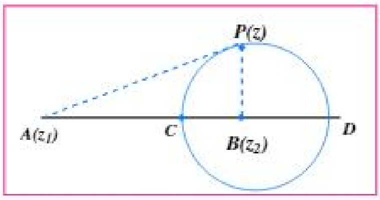


Fig. 4.16

Radius of circle $= \dfrac{1}{2}\left|\left(\dfrac{kz_2+z_1}{k+1}\right)-\left(\dfrac{kz_2-z_1}{k-1}\right)\right| = \dfrac{k}{(k^2-1)}|z_1-z_2|$

5) Arg $(z) = \alpha$, where α is fixed real number, represents a half line $(-\pi < \alpha \leq \pi)$.

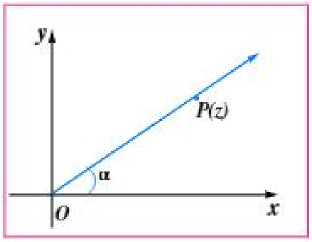


Fig. 4.17

6) $\text{Arg}(z-z_0) = \alpha,\ \alpha \in R, A(z_0)$, represents a ray starting from A.

7) Arg $\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha,\ \alpha \in R-\{O\}$ represents a circular arc.

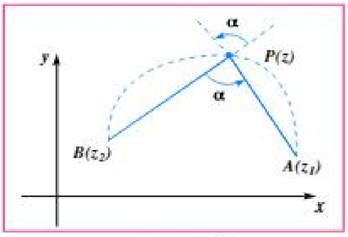


Fig. 4.18

For $\alpha = \pm\dfrac{\pi}{2}$, it represents a semicircle.

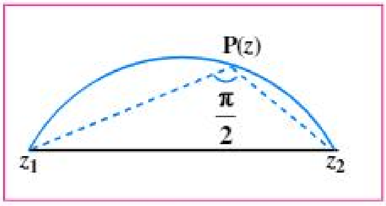


Fig. 4.19

8) z_1, z_2 are fixed points k is real number such that $|z - z_1| + |z - z_2| = k$ where $k > |z_1 - z_2|$ then locus of z is an ellipse.

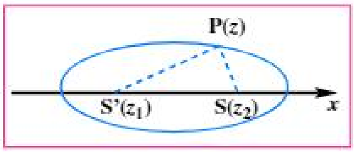


Fig. 4.20

9) z_1, z_2 are fixed points such that $|z - z_2| + |z - z_2| = k$ where $k = |z_1 - z_2|$ then locus of z is a line segment joining z_1 and z_2.

10) z_1, z_2 are fixed points such that $|z - z_1| - |z - z_2| = k$, where $k < |z_1 - z_2|$ then locus of z is a branch of hyperbola.

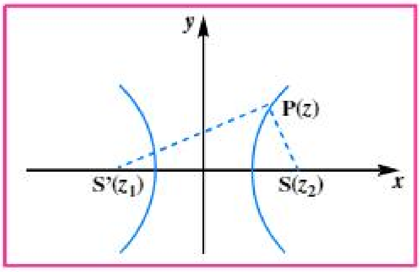


Fig. 4.21

4.27 STRAIGHT LINE

THEOREM-4.5 **Equation of a straight line passing through two given points $A\ (z_1)$ and $B\ (z_2)$ is**

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

Proof :

Given points $A(z_1)$, $B(z_2)$

Let $P(z)$ a variable point on the line through A and B clearly for all positions of P on the line P, A, B are collinear i.e area of $\Delta PAB = 0$

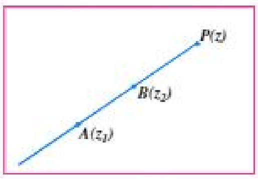


Fig. 4.22

$$\Rightarrow \frac{1}{4}\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

which is the required equation of the line.

Corollary :

On simplification the above equation takes the form

$$z(\overline{z}_1 - \overline{z}_2) - \overline{z}(z_1 - z_2) + (z_1\overline{z}_2 - z_2\overline{z}) = 0$$

This is in the form, $z(\overline{a}) + \overline{z}(a) + b = 0$, where $a = -i(z_1 - z_2)$, $\overline{a} = i(\overline{z}_1 - \overline{z}_2)$, $b = i(z_1\overline{z}_2 - z_1\overline{z}_2)$

which is the general equation of a straight line where 'a' is complex constant and b is real constant.

General Equation of a straight line :

Definition

The general equation of a straight line in the argand plane is of the form $\overline{a}z + a\overline{z} + b = 0$ where $a(\neq 0)$ is a complex constant and b is real constant.

For example,

i) $(1+i)z + (1-i)\overline{z} + 3 = 0$ is a straight line.

ii) $4z + 4\overline{z} + 7 = 0$ is a straight line.

iii) $2z - 2\overline{z} + 5i = 0$ is also a straight line since it simplifies to $(2i)z + (-2i)\overline{z} - 5 = 0$.

iv) $7z + 7\overline{z} + 2i = 0$ doesn't represent a straight line.

Slope of a straight line :

Let the equation of a straight line in the Argand plane be $\overline{a}z + a\overline{z} + b = 0$, $(b \in R)$

Put $z = x + iy$, $\overline{a}(x+iy) + a(x-iy) + b = 0$

$$\Rightarrow (a+\overline{a})x + i(\overline{a} - a)y + b = 0$$

clearly $a + \overline{a}$ and $i(\overline{a} - a)$ are real numbers and thus slope of the line is given by $\dfrac{-(a+\overline{a})}{i(\overline{a} - a)}$

$\therefore$ slope of line $= \dfrac{a+\overline{a}}{i(a-\overline{a})}$ which is a real quantity.

Parallality of two lines :

Let $\overline{\alpha}z + \alpha\overline{z} + k_1 = 0, \overline{\beta}z + \beta\overline{z} + k_2 = 0\ (k_1, k_2 \in R)$ be two given lines and if they are parallel i.e. their slopes are equal

$$\therefore \frac{\alpha+\overline{\alpha}}{i(\alpha-\overline{\alpha})} = \frac{\beta+\overline{\beta}}{i(\beta-\overline{\beta})}$$

$$\Rightarrow \frac{\alpha+\overline{\alpha}}{(\alpha-\overline{\alpha})} = \frac{\beta+\overline{\beta}}{(\beta-\overline{\beta})}$$

$$\Rightarrow \frac{\alpha}{\overline{\alpha}} = \frac{\beta}{\overline{\beta}} \Rightarrow \alpha\overline{\beta} - \overline{\alpha}\beta = 0$$

Remember :
The lines $\overline{\alpha}z + \alpha\overline{z} + k_1 = 0$, $\overline{\beta}z + \beta\overline{z} + k_2 = 0$ *are parallel if* $\dfrac{\overline{\alpha}}{\beta} = \dfrac{\alpha}{\beta}$

Perpendicularity of two lines :

> ***Remember :***
> *The lines*
> $\bar{\alpha}z + \alpha\bar{z} + k_1 = 0,$
> $\bar{\beta}z + \beta\bar{z} + k_2 = 0$ *are perpendicular if*
> $\alpha\bar{\beta} + \bar{\alpha}\beta = 0$

If the above two lines are perpandicular i.e product of slopes is –1

$$\Rightarrow \left\{\frac{\alpha+\bar{\alpha}}{i(\alpha-\bar{\alpha})}\right\}\left\{\frac{\beta+\bar{\beta}}{i(\beta-\bar{\beta})}\right\} = -1$$

$$\Rightarrow (\alpha+\bar{\alpha})(\beta+\bar{\beta}) = (\alpha-\bar{\alpha})(\beta-\bar{\beta})$$

$$\Rightarrow 2(\alpha\bar{\beta}+\bar{\alpha}\beta) = 0 \Rightarrow \alpha\bar{\beta}+\bar{\alpha}\beta = 0$$

Complex slope of line :

Definition

A line passing through two points A and B in the argand plane and if $A(z_1)$ and $B(z_2)$ then the complex slope of the line is defined as $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

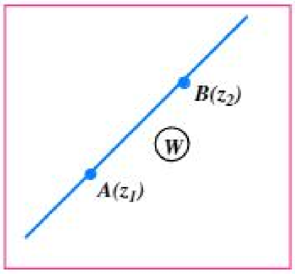


Fig 4.23

To find complex slope of the line $\bar{a}z + a\bar{z} + b = 0$

> ***Remember :***
> *Complex slope of the line* $\bar{a}z + a\bar{z} + b = 0$ *is* $-\dfrac{a}{\bar{a}}$

Let $A(z_1)$, $B(z_2)$ be two points on the line $\bar{a}z + a\bar{z} + b = 0$

Then we have $\bar{a}z_1 + a\bar{z}_1 + b = 0$ ..... (1)

$\bar{a}z_2 + a\bar{z}_2 + b = 0$ ..... (2)

(1) - (2) $\Rightarrow \bar{a}(z_1 - z_2) + a(\bar{z}_1 - \bar{z}_2) = 0$

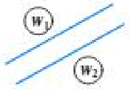
$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = -\frac{a}{\bar{a}}$$

$\Rightarrow$ Complex slope of the line $= -\dfrac{a}{\bar{a}} = -\dfrac{(coeff\ of\ \bar{z})}{(coeff\ of\ z)}$

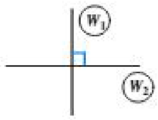
$\Rightarrow$ The complex slope of $\bar{a}z + a\bar{z} + b = 0$ is "$-\dfrac{a}{\bar{a}}$" and if this is denoted by 'W'

then $W = \dfrac{-a}{\bar{a}}$

Remember :



Parallel lines
$\Leftrightarrow W_1 = W_2$



Perpendicular lines
$\Leftrightarrow W_1 + W_2 = 0$

Note

i) *If W_1 and W_2 are complex slopes of the lines* $\bar{\alpha} z + \alpha \bar{z} + k_1 = 0,\ \bar{\beta} z + \beta \bar{z} + k_2 = 0$ *then*

a) *Lines are parallel*

$$\Rightarrow \alpha\bar{\beta} - \bar{\alpha}\beta = 0 \Rightarrow \frac{\alpha}{\bar{\alpha}} = \frac{\beta}{\bar{\beta}}$$

$$\Rightarrow -\frac{\alpha}{\bar{\alpha}} = -\frac{\beta}{\bar{\beta}} \Rightarrow W_1 = W_2$$

$\Rightarrow$ *complex slopes are equal.*

b) *Lines are perpendicular* $\Rightarrow \alpha\bar{\beta} + \bar{\alpha}\beta = 0$

$$\Rightarrow \frac{\alpha}{\bar{\alpha}} + \frac{\beta}{\bar{\beta}} = 0 \Rightarrow \left(\frac{-\alpha}{\bar{\alpha}}\right) + \left(\frac{-\beta}{\bar{\beta}}\right) = 0$$

$\Rightarrow W_1 + W_2 = 0$

$\Rightarrow$ *sum of complex slopes is zero.*

c) *Make* $\frac{\pi}{4}$ *with each other* $\Rightarrow W_1 = \pm\, i W_2$

ii) *Equation of a line parallel to the line* $\bar{\alpha} z + \alpha \bar{z} + b = 0$ *will be of the form* $\bar{\alpha} z + \alpha \bar{z} + \lambda = 0,\ \lambda \in R$.

iii) *Equation of a line perpendicular to the line* $\bar{a} z + a\bar{z} + b = 0$ *will be in the form* $\bar{a} z - a\bar{z} + i\mu = 0,\ \mu \in R$

iv) *The length of the perpendicular from a point $P(z_0)$ to the line* $\bar{a} z + a\bar{z} + b = 0$ *is given by*

$$\left|\frac{\bar{a} z_0 + a\bar{z}_0 + b}{2\bar{a}}\right|.$$

v) *The complex number corresponding to the foot of perpendicular from $P(z_0)$ to the line* $\bar{a} z + a\bar{z} + b = 0$ *is* $\frac{\bar{a} z_0 - a\bar{z}_0 - b}{2\bar{a}}$.

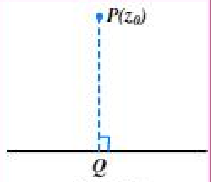


Fig 4.24

vi) *The image of $P(z_0)$ in the line* $\bar{\alpha} z + \alpha \bar{z} + b = 0$ *is* $-\frac{(a\bar{z}_0 + b)}{\bar{a}}$

4.28 CIRCLE

Equation of circle when centre, radius are given :

Let z_0 be centre and $r \in R$ be radius then for any $P(z)$ on the circle, we have $|z - z_0| = r \Rightarrow |z - z_0|^2 = r^2 \Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$$\Rightarrow z\bar{z} - \bar{z}_0 z - z_0\bar{z} + |z_0|^2 - r^2 = 0$$

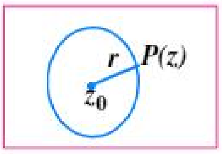


Fig 4.25

General Equation of a circle :

The equation above be simplified to $z\bar{z}+\bar{a}z+a\bar{z}+b=0$ where b is real, is the general equation of a circle in complex numbers and (i) Its centre is at "$-a$" (ii) Its radius is $\sqrt{|a|^2-b}$

Ex. **Find the centre and radius of the circle $iz\bar{z}+(3+4i)z-(3-4i)\bar{z}+9i=0$**

Sol. Given equation is $z\bar{z}+(4-3i)\bar{z}+(4+3i)\bar{z}+9=0$ which is in general form $z\bar{z}+\bar{a}z+a\bar{z}+b=0$

centre $=-a=-(4+3i)$

radius $=\sqrt{|a|^2-b}=\sqrt{25-9}=4$

Circle whose diametric ends are given :

Let a circle on $A(z_1)$ and $B(z_2)$ be described as its diameter.

For any point $P(z)$ on the circle we have $\vec{AB}\perp\vec{BP}$

$$\Rightarrow \arg\left(\frac{z-z_2}{z-z_1}\right)=\pm\frac{\pi}{2}\Rightarrow\frac{z-z_2}{z-z_1}+\frac{\bar{z}-\bar{z}_2}{\bar{z}-\bar{z}_1}=0$$

$$\Rightarrow (z-z_2)(\bar{z}-\bar{z}_1)+(z-z_1)(\bar{z}-\bar{z}_2)=0$$

which is the required equation of the circle on $\overline{AB}$ as diameter.

Note: Its another form is $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$

Condition for four points to be concyclic :

Four points z_1, z_2, z_3, z_4 will lie on the same circle if $\dfrac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely real.

SOLVED EXAMPLES

***1.** **Find the real values of θ, such that $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely**

(i) Real **(ii) Imaginary**

Sol.
$$\frac{3+2i\sin\theta}{1-2i\sin\theta}=\frac{(3-2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}=\frac{3+4i^2\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$$

$$=\frac{3-4\sin^2\theta}{1+4\sin^2\theta}+\frac{8\sin\theta}{1+4\sin^2\theta}i$$

i) If the given expression is purely real, then

$$\frac{8\sin\theta}{1+4\sin^2\theta}=0\Rightarrow\sin\theta=0\Rightarrow\theta=n\pi,n\in Z$$

ii) If the given expression is purely imaginary then

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0\Rightarrow 3-4\sin^2\theta=0$$

$$\Rightarrow\sin^2\theta=\frac{3}{4}\Rightarrow\sin^2\theta=\left(\frac{\sqrt{3}}{2}\right)^2=\sin^2\left(\frac{\pi}{3}\right)\Rightarrow\theta=n\pi\pm\frac{\pi}{3},n\in Z.$$

Remember :
$z=x+iy$ is purely real if $y=0$, purely imaginary if $x=0$

***2.** **If $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$, show that $a^2+b^2=4$.** **(March-19)**

Sol. $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$

$\Rightarrow \left|\sqrt{3}+i\right|^{100} = \left|2^{99}(a+ib)\right|$

$\Rightarrow 2^{100} = 2^{99}\sqrt{(a^2+b^2)} \Rightarrow a^2+b^2 = 4.$

3. **Find the polar form of $(2+\sqrt{3})+i$.**

Sol. $z = (2+\sqrt{3}) + i = a + ib$

$\Rightarrow |z| = \sqrt{(2+\sqrt{3})^2+1^2} = \sqrt{8+4\sqrt{3}}$

$\left.\begin{array}{l} a = 2+\sqrt{3} > 0 \\ b = 1 \quad\quad > 0 \end{array}\right\} \text{Arg}(z) = \theta$

$= \tan^{-1}\left(\dfrac{1}{2+\sqrt{3}}\right) = \tan^{-1}(2-\sqrt{3}) = \dfrac{\pi}{12}$

$\therefore (2+\sqrt{3}) + i = \sqrt{8+4\sqrt{3}}\left\{\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right\}$

4. **Find the polar form of complex number $1+\cos 2\alpha + i\sin 2\alpha$ where $\dfrac{\pi}{2} < \alpha < \pi$.**

Remember :
$z = f(\cos\theta + \sin\theta)$ *where*
$r > 0,\ \theta \in (-\pi, \pi]$

Sol. $z = 1 + \cos 2\alpha + i\sin 2\alpha$

$= 2\cos^2\alpha + i.(2\sin\alpha\cos\alpha) = 2\cos\alpha\{\cos\alpha + i\sin\alpha\}$

$= (-2\cos\alpha)\{-\cos\alpha + i(-\sin\alpha)\}$ [since $\cos\alpha < 0 \Rightarrow -\cos\alpha > 0$]

$= (-2\cos\alpha)\{\cos(\alpha-\pi) + i\sin(\alpha-\pi)\}$

$= (-2\cos\alpha).\ \text{cis}(\alpha-\pi)$ $\left[\because \dfrac{\pi}{2} < \alpha < \pi \Rightarrow -\dfrac{\pi}{2} < \alpha-\pi < 0\right]$ Which is the required polar form

5. **Show that the area of the triangle on the argand diagram formed by the complex numbers z, iz and $z+iz$ is equal to $\dfrac{1}{2}|z|^2$.**

Sol. Let $z = x + iy$, then co-ordinates in ordered pair of z and iz are (x, y) and $(-y, x)$.

Here origin 0, z, $z + iz$ and iz form a square of side $|z|$.

Hence area of the required triangle is $\dfrac{1}{2}|z|^2$. (Half the area of the square)

Note :

The area of triangle formed by the vertices z, $z\,\text{cis}\,\alpha$ and $z + z\,\text{cis}\,\alpha$ is $\left|\dfrac{1}{2}|z|^2\sin\alpha\right|$.

***6. If z_1 and z_2 are two non-zero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$ then show that Arg z_1 = Arg z_2.**

Sol. Let $z_1 = a_1 + ib_1,\ z_2 = a_2 + ib_2$

Then $|z_1+z_2| = |(a_1+a_2)+i(b_1+b_2)| = \sqrt{(a_1+a_2)^2+(b_1+b_2)^2}$

and $|z_1+z_2| = \sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+y_2^2}$

Given $|z_1+z_2|=|z_1|+|z_2| \Rightarrow \sqrt{(a_1+a_2)^2+(b_1+b_2)^2} = \sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}$

$\Rightarrow (a_1+a_2)^2+(b_1+b_2)^2 = (a_1^2+b_1^2)+(a_2^2+b_2^2)+2\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}$

$\Rightarrow 2a_1a_2+2b_1b_2 = \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)} \Rightarrow (a_1a_2+b_1b_2)^2 = (a_1^2+b_1^2)(a_2^2+b_2^2)$

$\Rightarrow 2a_1a_2b_1b_2 = a_1^2b_2^2+a_2^2b_1^2 \Rightarrow a_1^2b_2^2+a_2^2b_1^2-2(a_1b_2)(a_2b_1)=0$

$\Rightarrow (a_1b_2-a_2b_1)^2 = 0 \Rightarrow a_1b_2-a_2b_1=0 \Rightarrow \dfrac{b_1}{a_1}=\dfrac{b_2}{a_2}$

$\Rightarrow \tan^{-1}\left(\dfrac{y_1}{x_1}\right) = \tan^{-1}\left(\dfrac{b_2}{a_2}\right) \Rightarrow Arg z_1 = Arg\, z_2$

Aliter :

Given $|z_1+z_2| = |z_1|+|z_2|$ squaring

$\Rightarrow |z_1|^2+|z_2|^2+2\,\text{Re}(z_1\bar{z}_2) = |z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow \text{Re}(z_1\bar{z}_2) = |z_1\bar{z}_2|$

$\Rightarrow Arg(z_1\bar{z}_2) = 0$ $\qquad (\because \text{Re}\, z = |z| \Rightarrow z \text{ is real})$

$\Rightarrow$ Arg z_1 – Arg $z_2 = 0 \Rightarrow$ Amp z = 0)

$\Rightarrow$ Arg z_1 = Arg z_2

***7. Show that the locus of $z = x+iy$ such that the amplitude of $\dfrac{z-1}{z+1}$ is equal to $\dfrac{\pi}{4}$ is $x^2+y^2-2y+10$.**

Sol. $\dfrac{z-1}{z+1} = \dfrac{x+iy-1}{x+iy+1} = \dfrac{[(x+1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$

$\because Arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\dfrac{2y}{x^2+y^2-1} = \dfrac{\pi}{4} \Rightarrow \dfrac{2y}{x^2+y^2-1} = 1$

$\Rightarrow x^2+y^2-1 = 2y$

$\Rightarrow x^2+y^2-2y-1 = 0$ (Also $y>0$, $x^2+y^2>1$)

$\therefore$ Locus consists of set of points given by

$S = \{(x,y)/x^2+y^2-2y-1=0,\ y>0,\ x^2+y^2>1\}$

Remember :

$Arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha, \alpha \neq 0$

$\Rightarrow$ Locus of z is segment of a circle.

***8. Reduce the complex numbers $3+4i, \frac{3}{2}(7+i)(1+i), \frac{2(i-18)}{(1+i)^2}, \frac{5(i-3)}{1+i}$ to $a+ib$ form. Show that the four points represented by these complex numbers form a square.** **(March - 04, 08)**

Sol. Let the points are $A = 3+4i,\ B = \frac{3}{2}(7+i)(1+i),\ C = \frac{2(i-18)}{(1+i)^2},\ D = \frac{5(i-3)}{1+i}$

$A = 3+4i = (3, 4)$

$$B = \frac{3}{2}(7+7i+i+i^2) = \frac{3}{2}(6+8i) = 3(3+4i) = 9+12i = (9,12)$$

$$C = \frac{2(i-18)}{(1+i^2)} = \frac{2(i-18)}{(1+i^2+2i)} = \frac{i-18}{i} = \frac{i^2-18i}{-1} = 1+18i = (1,18)$$

$$D = \frac{5(i-3)}{1+i} = \frac{5(i-3)(1-i)}{(1+i)(1-i)} = \frac{5(i+1-3+3i)}{1+1} = \frac{5(-2+4i)}{2} = -5+10i = (-5,10)$$

$$AB = \sqrt{(3-9)^2+(4-12)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$BC = \sqrt{(9-1)^2+(12-18)^2} = \sqrt{64+36} = \sqrt{100} = 10$$

$$CD = \sqrt{(1+5)^2+(18-10)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$DA = \sqrt{(-5-3)^2+(10-4)^2} = \sqrt{64+36} = \sqrt{100} = 10$$

$$AC = \sqrt{(3-1)^2+(4-18)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$$

$$BD = \sqrt{(9+5)^2+(4-18)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$$

$\therefore AB = BC = CD = DA$ and $AC = BD$.

Hence the given points form a square

EXERCISE - 4.2

1. Express the following in polar form (mod- Amplitude form)

**i) $1+i\sqrt{3}$ (March-17,18) [Ans : $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$]

ii) $-1-i\sqrt{3}$ [Ans : $2\left[\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right]$]

iii) $-\sqrt{3}+i$ [Ans : $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$]

iv) $1+\cos\theta-i\sin\theta$ [Ans : $2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right]$]

v) $-1+i$ [Ans : $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$]

vi) $-2+2i\sqrt{3}$ [Ans : $4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$]

2. If $z_1 = -1$ and $z_2 = i$ then find

*i) Arg $z_1 z_2$ **(March-15, 19)** **[Ans : $3\pi/2$]**

*ii) $Arg\left(\frac{z_1}{z_2}\right)$ **(March-18)** **[Ans : $-\pi/2$]**

iii) If $Arg\,\bar{z}_1 = \frac{\pi}{5}$ and $Arg\, z_2 = \frac{\pi}{3}$ then find $Arg\, z_1 + Arg\, z_2$ **[Ans : $\frac{2\pi}{15}$]**

iv) If $z \neq 0$ find Arg z + Arg $\bar{z}$ **[Ans : 0 if Arg $z \neq \pi$, 2π if Arg $z = p$]**

3. If $z = x + iy$ then find the locus of z of the following

*i) $|z| = 1$ **(March-19)** **[Ans : $x^2 + y^2 = 1$]**

ii) $|z - 2 + 3i| = 4$ **[Ans : $x^2 + y^2 - 4x + 6y - 3 = 0$]**

iii) $|2z - 3| = 7$ **[Ans : $x^2 + y^2 - 3x - 10 = 0$]**

iv) Im $(z^2)=4$ **[Ans : $xy = 2$]**

v) $|z - 2| = 2|z - 1|$ **[Ans : $3x^2 + 3y^2 - 4x = 0$]**

4. i) Show that the points in the argand diagram represented by the complex numbers $5 - 5i$, $1 + 3i$, $4 - 3i$ lie on same line (collinear).

ii) Show that the equation of the straight line joining the points $2 - 3i$, $-4 + 3i$ is $x + y + 1 = 0$.

iii) Find the equation of the straight line joining the points $(-9 + 6i)$, $(11 - 4i)$ in the argand plane. **[Ans : $x + 2y - 3 = 0$]**

iv) Find Equation of perpendicular bisector of line segment joining the points $7 + 7i$, $7 - 7i$. **[Ans : $y = 0$]**

5. If $\frac{z_2}{z_1}$ $(z_1 \neq 0)$ is an imaginary number, find the value of $\left|\frac{2z_1 + z_2}{2z_1 - z_2}\right|$. **[Ans : 1]**

6. If $z_1 = 1 + \sqrt{3}i$, $z_2 = \sqrt{3} + i$, $z_3 = -1 + i$ then find the argument of $\frac{z_2 z_3}{z_1}$. **[Ans : $\frac{7\pi}{12}$]**

7. If $(1 - \sqrt{2}) + i = r(\cos\theta + i\sin\theta)$ then find θ. **[Ans : $\frac{5\pi}{8}$]**

8. Find the principal value of the argument of z if $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ **[Ans : $\frac{3\pi}{5}$]**

9. If $\frac{5z_2}{11z_1}$ is purely imaginary then show that $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = 1$

10. If $z + \frac{1}{z} = 1$ show that $z^4 + \frac{1}{z^4} = -1$

*11. Show that the points in argand diagram represented by $-2-2i, 2+2i, -2\sqrt{3}+2\sqrt{3}i$ is an equilateral triangle. **(May-18)**

12. Show that four points represented by complex numbers $\frac{6+8i}{2}, \left(\frac{21+3i}{2}\right)(1+i), \frac{-36+2i}{(1+i)^2}, \frac{-15+5i}{1+i}$ form a square.

13. Show that four points represented the complex numbers $\frac{-3}{i}, \frac{-1+2i}{i}, 4+3i, 2+5i$ form a square.

14. Show that the points in the argand diagram represented by $-2+7i, -\frac{3}{2}+\frac{1}{2}i, 4-3i, \frac{7}{2}(1+i)$ are the vertices of a rhombus.

15. If $z = x + iy$ then find the locus of z of the following

i) $z(z+2) = 3$ **[Ans :** $x^2 - y^2 + 2x = 3,\ 2xy + 2y = 0$**]**

ii) Amplitude of $\frac{z-2}{z-6i} = \frac{\pi}{2}$

[Ans : $3x + y - 6 \geq 0, x^2 + y^2 - 2x - 6y = 0$ and $(x, y) \neq (0, 6)$**]**

iii) Amplitude of $(z - 1)$ is $\frac{\pi}{2}$ **[Ans :** $x = 1, y > 0$**]**

iv) $\frac{z-i}{z-1}$ is purely imaginary **[Ans :** $x^2 + y^2 - x - y = 0$ and $(x, y) \neq (1, 0)$**]**

v) $|z + ai| = |z - ai|$ **[Ans :** $y = 0$**]**

vi) $\left|\frac{z-a}{z+a}\right| = 1$ (real part of a is non-zero) **[Ans :** y-axis **]**

vii) $|z|^2 = 4\,\mathrm{Re}(z+2)$ **[Ans :** $x^2 + y^2 - 4x - 8 = 0$**]**

viii) $|z+i|^2 - |z-i|^2 = 2$ **[Ans :** $2y - 1 = 0$**]**

ix) $|z + 4i| + |z - 4i| = 10$ **[Ans :** $25x^2 + 9y^2 = 225$**]**

x) $z^2 + \bar{z}^2 = 2$ **[Ans :** $x^2 - y^2 = 1$**]**

xi) $\mathrm{Re}\left(\frac{z+1}{z+i}\right) = 1$ **[Ans :** $x - y = 1$**]**

xii) $|z + i| \leq 3$ **[Ans :** $x^2 + (y+1)^2 \leq 9$**]**

xiii) $w = \frac{z}{z-2i}, |w| = 1$ **[Ans :** $y - 1 = 0$**]**

16. i) Determine the locus of z, $z \neq 2i$, such that $\text{Re}\left(\dfrac{z-4}{z-2i}\right)=0$.

[**Ans** : $(x-2)^2+(y-1)^2=5,(x,y)\neq(0,2)$]

ii) If $\log_{\sqrt{3}}\left(\dfrac{|z|^2-|z|+1}{|z|+2}\right)<2$ then show that the locus of z is $|z|<5$

17. If $(1+x)^n=c_0+c_1x+c_2x^2+......+c_nx^n$, find the value of $c_0-c_2+c_4-.....$ [**Ans** : $2^{\frac{n}{2}}\cos\dfrac{n\pi}{4}$]

18. The points P, Q denote the complex numbers z_1, z_2 in the argand diagram. O is the origin. If $z_1\bar{z}_2+\bar{z}_1z_2=0$, show that $\angle POQ=90^0$.

19. Show that the points in the argand diagram represented by the complex numbers z_1, z_2, z_3 are collinear if and only if there exists three real numbers p,q,r not all zero satisfying $pz_1+qz_2+rz_3=0$ and $p+q+r=0$.

20. i) If $\dfrac{z_3-z_1}{z_2-z_1}$ is a real number, show that the points represented by complex numbers z_1, z_2 and z_3 are collinear.

ii) Let O, $A(z_1)$, $B(z_2)$ be vertices of isoceles right angled triangle with right angle at O then show that $(z_1-z_2)^2=-2z_1z_2$

21. If z_1, z_2 are complex numbers satisfying $\left|\dfrac{z_1-3z_2}{3-z_1\bar{z}_2}\right|=1$ and $|z_1|\neq 3$ then show that $|z_2|=1$

22. i) If $|z-1|=1$ then show that $\dfrac{z-2}{z}=i\tan(\arg z)$

ii) If $z=2(\cos\theta+i\sin\theta)$ and if $x+iy=z+\dfrac{1}{z}$ then prove that $\dfrac{x^2}{25}+\dfrac{y^2}{9}=\dfrac{1}{4}$

iii) If $(1-i)(2-i)(3-i).....(1-ni)=x-iy$ then prove that $2.5.10.....(1+n^2)=x^2+y^2$

iv) If z in any non-zero complex number show that $\left|\dfrac{z}{\bar{z}}-1\right|\leq|\arg z|$

23. If $|z_1|=2$, $|z_2|=3$, $|z_3|=4$ and $|2z_1+3z_2+4z_3|=9$ then show that $|8z_2z_3+27z_3z_1+64z_1z_2|=216$

24. If z has argument θ where $0<\theta<\dfrac{\pi}{2}$ such that $|z-3i|=3$, then prove that $\cot\theta-\dfrac{6}{z}=i$

4.29 MAX AND MIN VALUE OF |Z| IF $\left|z+\frac{1}{z}\right|=a$

Given $\left|z+\frac{1}{z}\right|=a$ --- (1)

let $z=r(\cos\theta+i\sin\theta)$

$$\therefore (1)\Rightarrow\left|\left(r+\frac{1}{r}\right)\cos\theta+i\left(r-\frac{1}{r}\right)\sin\theta\right|=a\Rightarrow\left(r+\frac{1}{r}\right)^2\cos^2\theta+\left(r-\frac{1}{r}\right)\sin^2\theta=a^2$$

$$\Rightarrow r^2+\frac{1}{r^2}+2\cos 2\theta=a^2$$

Differentiating w.r.t. θ

$$-4\sin 2\theta=0\ \Rightarrow\theta=0,\frac{\pi}{2}$$

Case (i) :

$$\theta=\frac{\pi}{2}\Rightarrow r^2+\frac{1}{r^2}-2=a^2\Rightarrow r-\frac{1}{r}=a\ \text{(or)}\ r-\frac{1}{r}=-a$$

$$\Rightarrow r=\frac{a+\sqrt{a^2+4}}{2}\ \text{(or)}\ r=\frac{\sqrt{a^2+4}-a}{2}$$

Case (ii) :

$$\theta=0\Rightarrow r^2+\frac{1}{r^2}-2=a^2$$

$$\Rightarrow r+\frac{1}{r}=a\ \text{only}$$

$$\Rightarrow r=\frac{a+\sqrt{a^2-4}}{2},\ \frac{a-\sqrt{a^2-4}}{2}$$

These two values lie between the two values of case(i)

$$\therefore r_{\max}=\frac{a+\sqrt{a^2+4}}{2};\ r_{\min}=\frac{\sqrt{a^2+4}-a}{2}$$

$$\therefore \text{Max value of } |z|=\frac{a+\sqrt{a^2+4}}{2}$$

$$\text{Min. value of } |z|=\frac{\sqrt{a^2+4}-a}{2}$$

Note

If $\left|z+\frac{c}{z}\right|=a$ is given then Max value of $|z|=\frac{a+\sqrt{a^2+4c}}{2}$ and Min value of $|z|=\frac{\sqrt{a^2+4c}-a}{2}$

Example :

If $\left|z+\frac{4}{z}\right|=3$ then Max. value of $|z|=\frac{3+\sqrt{3^2+4(4)}}{2}=4$ and Min. value of $|z|=\frac{\sqrt{3^2+4(4)}-3}{2}=1$

SOLVED EXAMPLES

1. Show that $\sum_{n=0}^{\infty}\left(\frac{2i}{3}\right)^n = \frac{9+6i}{13}$

Sol. LHS $= \underset{m\to\infty}{Lt}\sum_{n=0}^{m}\left(\frac{2i}{3}\right)^n = \underset{m\to\infty}{Lt}\left\{1+\left(\frac{2i}{3}\right)+\left(\frac{2i}{3}\right)^2+.........+\left(\frac{2i}{3}\right)^m\right\}$

$$= \underset{m\to\infty}{Lt}\left\{\frac{1\left(1-\left(\frac{2i}{3}\right)^{m+1}\right)}{1-\frac{2i}{3}}\right\} = \frac{1-0}{1-\left(\frac{2i}{3}\right)} = \frac{3}{3-2i} = 3\frac{(3+2i)}{13} = \frac{9+6i}{13}$$

2. If $\omega(\neq 1)$ be a cube root of unity and $(1+\omega^2)^n = (1+\omega^4)^n$, then find the least positive value of n

Sol. We have $(1+\omega^2)^n = (1+\omega^4)^n$

$\Rightarrow \quad (1+\omega^2)^n = (1+\omega)^n$

$\Rightarrow \quad \left(1+\frac{1}{\omega}\right)^n = (1+\omega)^n \Rightarrow \quad (1+\omega)^n = (1+\omega)^n\omega^n$

$\Rightarrow \quad \omega^n = 1 \quad \Rightarrow \quad n = 3, 6, 9,.........$

3. If a, b, c are integers not all equal and w is a cube root of unity $(w \neq 1)$ then find the minimum value of $|a + bw + cw^2|$.

Sol. consider $|a + bw + cw^2|^2 = (a+bw+cw^2)\,\overline{(a+bw+cw^2)}$

$= (a + bw + cw^2)(a+b\overline{w}+c\overline{w}^2) = (a+bw+cw^2)(a+bw^2+cw)$

$= a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\{2a^2+2b^2+2c^2-2ab-2bc-2ca\}$

$= \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$ when $a = b = 1, c = 2$ it gives minimum value

(since a, b, c not all equal)

$\therefore$ Minimum value of $|a + bw + cw^2| = \sqrt{\frac{1}{2}(0+1+1)} = \sqrt{1} = 1$

Note : The minimum value of $\left|a+b\omega+c\omega^2\right|+\left|a+b\omega^2+c\omega\right|$ is 2.

4. If z is a complex number such that $|z| = 2$, find the maximum and minimum value of $|z-2+3i|$

Sol. $|z-2+3i| = |z+(-2+3i)| \le |z|+|-2+3i| \quad [\because |z_1+z_2| \le |z_1|+|z_2|]$

$\le |z|+\sqrt{13} \qquad \le 2+\sqrt{13}$ (given $|z| = 2$)

$\therefore$ Max.value of $|z-2+3i|$ is $2+\sqrt{13}$

Also $|z-2+3i| = |z-(2-3i)|$

$\ge \left||z|-|2-3i|\right| \qquad \ge \left|2-\sqrt{13}\right| \qquad \ge \sqrt{13}-2$

$\therefore$ Minimum value of $|z-2+3i|$ is $\sqrt{13}-2$.

5. If complex number $A(2+\sqrt{3}\,i)$ is rotated about O through an angle 45° in anticlockwise sense and if A goes to B then find the complex number of the point B.

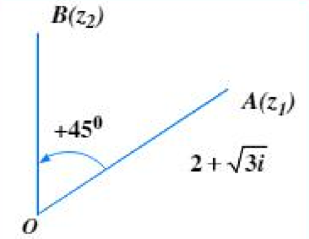


Fig. 4.26

Sol. We know that $A(z_1)$, $B(z_2)$ then

$$z_2 = z_1\, e^{i\frac{\pi}{4}} \text{ [since } OA = OB \text{ and } \angle AOB = +45^\circ\text{]} = (2+\sqrt{3}i)\, e^{i\frac{\pi}{4}}$$

$$(2+\sqrt{3}i)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = (2+\sqrt{3}i)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left\{\left(2-\sqrt{3}\right)+(2+\sqrt{3})i\right\}$$

$$= \frac{(2-\sqrt{3})+(2+\sqrt{3})i}{\sqrt{2}}$$

6. If the centre of a square $ABCD$ oriented in anticlockwise sense is at origin and vertex $A(z_1)$ then find the remaining vertices in terms of z_1.

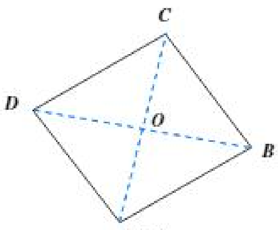


Fig. 4.27

Sol. Given $A(z_1)$

Let $B(z_2), C(z_3), D(z_4)$

To get B rotate OA by $+\frac{\pi}{2}$

To get C rotate OA by $+\pi$

To get D rotate OA by $-\frac{\pi}{2}$

Thus $z_2 = z_1 (e^{i\frac{\pi}{2}}) \quad \Rightarrow z_2 = iz_1$

Now $z_3 = z_1\, e^{i\pi} \quad \Rightarrow z_3 = -z_1$

Also $z_4 = z_1\, e^{i(-\frac{\pi}{2})} \quad \Rightarrow z_4 = -iz_1$

7. Let ΔABC is equlilateral with $A(z_1)$ and circum centre is $S(z_2)$. If the triangle is oriented in anticlockwise sense find vertex B.

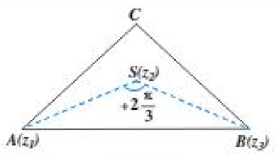


Fig. 4.28

Sol. Let $B(z_3)$

Rotate $\vec{SA}$ through $+\frac{2\pi}{3}$ to get $\vec{SB}$ (i.e. circum radius = SA = SB)

by the rotation formula.

$$\therefore\ \vec{SB} = \vec{SA}\quad e^{i\frac{2\pi}{3}}$$

$$\Rightarrow z_3 - z_2 = (z_1 - z_2)\ e^{i\frac{2\pi}{3}}$$

$$\Rightarrow z_3 - z_2 = (z_1 - z_2)\,w \quad [\because w \text{ is complex cube root of unity}]$$

$$\Rightarrow z_3 = z_2 + (z_1 - z_2)\,w$$

$$\Rightarrow z_3 = (1-w)\,z_2 + z_1 w$$

Thus the vertex B is given by, $z_3 = (1-w)\,z_2 + z_1 w$

8. **z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if $(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$.**

Sol. **Part - I :** z_1, z_2, z_3 are vertices of equilateral Δ, $(\vec{AC}, \vec{AB})=\frac{\pi}{3}$, $AB = AC$

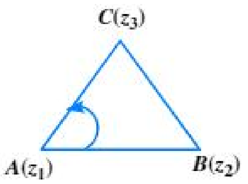


Fig. 4.29

$$\therefore\ z_3 - z_1 = (z_2 - z_1)\ e^{i\pi/3}$$

$$(z_3-z_1)^2=(z_2-z_1)^2\ e^{i2\pi/3}$$

$$\Sigma(z_3-z_1)^2=e^{i\frac{2\pi}{3}}\ \Sigma(z_3-z_1)^2$$

$$\Rightarrow \Sigma(z_2-z_1)^2 = w\Sigma(z_2-z_1)^2$$

$$\Rightarrow (1-w)\Sigma(z_2-z_1)^2=0$$

$$\Rightarrow \Sigma(z_2-z_1)^2=0$$

$$\Rightarrow\ (z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$$

Part - II :

$$(z_1-z_2)^2+(z_2-z_3)^2=-(z_3-z_1)^2$$

$$\Rightarrow\ (z_1-z_2)^2+(z_2-z_3)^2=-(z_3-z_1+z_2-z_1)^2$$

$$\Rightarrow\ (z_1-z_2)^2+(z_2-z_3)^2=-(z_2-z_3+z_1-z_2)^2$$

$$\Rightarrow\ p^2+q^2=-(p+q)^2$$

$$\Rightarrow\ p^2+q^2=-(p+q)^2$$

$$\Rightarrow\ p^2+q^2+pq=0$$

$$\Rightarrow\ \left(\frac{p}{q}\right)^2+\left(\frac{p}{q}\right)+1=0$$

$$p=z_1-z_2,\ q=z_2-z_3$$

$$\Rightarrow\ \frac{p}{q}=w\ \text{(or)}\ w^2$$

$$\Rightarrow\ \frac{z_1-z_2}{z_2-z_3}=w\ \text{(or)}\ w^2$$

$$\Rightarrow\ \frac{z_1-z_2}{z_3-z_2}-w\ \text{(or)}\ -w^2$$

$$\Rightarrow\ \frac{\vec{AB}}{\vec{AC}}=-w\ \text{(or)}\ -w^2$$

$$\Rightarrow\ \angle A=60^0$$

Similarly, $\angle B=60^0$, $\angle C=60^0$

$\therefore\ \Delta ABC$ is equilateral.

9. **Show that the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the argand diagram is equilateral if and only if**

i) $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$

ii) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

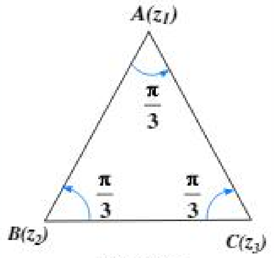


Fig. 4.30

Sol. **The triangle is equilateral.**

$\Rightarrow$ From Coni Method $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_2}\right) = \pi/3$

$\therefore \dfrac{z_1 - z_2}{z_3 - z_2} = \dfrac{AB}{BC} e^{i\pi/3}$

$\Rightarrow \dfrac{z_1 - z_2}{z_3 - z_2} = \left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) \quad (\because\ AB = BC)$

$\left(\dfrac{z_1 - z_2}{z_3 - z_2} - \dfrac{1}{2}\right) = \dfrac{i\sqrt{3}}{2}$

$(2z_1 - z_2 - z_3) = i\sqrt{3}\,(z_3 - z_2)$

squaring $(2z_1 - z_2 - z_3)^2 = -3(z_3 - z_2)^2$

$\Rightarrow 4z_1^2 + z_2^2 + z_3^2 - 4z_1z_2 + 2z_2z_3 - 4z_3z_1 = 3z_3^2 - 3z_2^2 + 6z_2z_3$

$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_1 - z_2)(z_2 - z_3) = 0$

Dividing by $(z_1 - z_2)\,(z_2 - z_3)\,(z_3 - z_1)$ then

we get $\dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} = 0$

After simplification

$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$

Conversely : $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$

Case I : $z_1 + \omega z_2 + \omega^2 z_3 = 0$

$z_1 + (-1 - \omega^2)z_2 + \omega^2 z_3 = 0$

$(z_1 - z_2) = \omega^2 (z_2 - z_3)$

$\Rightarrow |z_1 - z_2| = |\omega^2|\,|z_2 - z_3|$ and $\dfrac{z_1 - z_2}{z_3 - z_2} = -\omega^2$

$\Rightarrow |z_1 - z_2| = |z_2 - z_3|$ and $Arg\left(\dfrac{z_1 - z_2}{z_3 - z_2}\right) = \dfrac{\pi}{3}$

$\Rightarrow AB = BC$ and $\underline{|ABC} = 60°$

Case II : $z_1 + \omega^2 z_2 + \omega z_3 = 0$

$$z_1 + \omega^2 z_2 + (-1 - \omega^2) z_3 = 0$$

$$\omega^2 (z_2 - z_3) = (z_3 - z_1)$$

$$\Rightarrow |\omega^2| \, |z_2 - z_3| = |z_3 - z_1| \text{ and } \frac{z_2 - z_3}{z_1 - z_3} = -\omega$$

$$\Rightarrow |z_2 - z_3| = |z_3 - z_1| \text{ and } Arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = -\frac{\pi}{3}$$

$BC = CA$ and $\underline{|ACB} = 60^\circ$

Hence the triangle is equilateral, in both cases

10. If complex numbers z_1, z_2, z_3 are the vertices of an isosceles triangle with right angle at C show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Proof :

Method - 1

Since $\angle BCA = 90^\circ$, we have

$\arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{2}$, so that $\frac{z_2 - z_3}{z_1 - z_3}$ is purely imaginary.

Hence $\text{Re}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = 0$ or $\frac{1}{2}\left[\frac{z_2 - z_3}{z_1 - z_3} + \frac{\overline{z}_2 - \overline{z}_3}{\overline{z}_1 - \overline{z}_3}\right] = 0$

$$\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = -\frac{\overline{z}_2 - \overline{z}_3}{\overline{z}_1 - \overline{z}_3} \qquad \ldots (1)$$

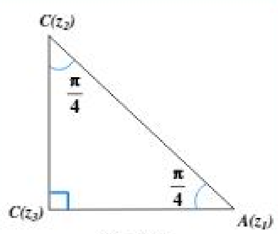


Fig. 4.31

Again $AC = BC$, so that $|z_2 - z_3| = |z_1 - z_3|$

$\therefore$ $|z_2 - z_3|^2 = |z_1 - z_3|^2$ or $(z_2 - z_3)(\overline{z}_2 - \overline{z}_3) = (z_1 - z_3)(\overline{z}_1 - \overline{z}_3)$

$\Rightarrow$ $(z_2 - z_3)^2 = -(z_1 - z_3)(z_1 - z_3)$ $\quad$ ($\because$ using [1])

$$\Rightarrow (z_2 - z_3)^2 + (z_1 - z_3)^2 = 0$$

$$\Rightarrow z_1^2 + z_2^2 - 2z_3^2 - 2z_1 z_3 - 2z_2 z_3 = 0$$

$$\Rightarrow z_1^2 + z_2^2 - 2z_1 z_2 = 2z_1 z_3 + 2z_2 z_3 - 2z_1 z_2 - 2z_3^2$$

$$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

Method - 2

By Coni Method

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{AC} e^{i\pi/2} \Rightarrow \frac{(z_2 - z_3)}{(z_1 - z_3)} = i \qquad (\therefore BC = AC)$$

$$\Rightarrow (z_2 - z_3) = i(z_1 - z_3)$$

squaring $(z_2 - z_3)^2 = -(z_1 - z_3)^2$

$$\Rightarrow z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 + 2z_1 z_3$$

$$\Rightarrow z_1^2 + z_2^2 = 2z_2 z_3 + 2z_1 z_3 - 2z_3^2$$

$$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

Note :

The above result also simplifies into $(z_1 - z_3)^2 + (z_2 - z_3)^2 = 0$, *i.e.*, $z_1^2 + z_2^2 = 2z_3(z_1 + z_2 - z_3)$

11. If z_1, z_2, z_3 are the vertices of an equilateral triangle with centroid at z_0 show that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Sol. z_1, z_2, z_3 are vertices of an equilateral triangle then $\Sigma z_1^2 = \Sigma z_1 z$

i.e. $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ ..... (1)

But centroid $= z_0 = \dfrac{z_1 + z_2 + z_3}{3}$ ..... (2)

Now $z_1 + z_2 + z_3 = 3z_0$

$\Rightarrow (z_1 + z_2 + z_3)^2 = (3z_0)^2 \Rightarrow \Sigma z_1^2 + 2\Sigma z_1z_2 = 9z_0^2$

$\Rightarrow \Sigma z_1^2 + 2\{\Sigma z_1^2\} = 9z_0^2 \Rightarrow 3\{\Sigma z_1^2\} = 9z_0^2$

$\Rightarrow \Sigma z_1^2 = 3z_0^2 \Rightarrow z_1^2 + z_2^2 + z_3^2 = 3.z_0^2$

12. Prove that the complex numbers z_1 and z_2 and the origin form an isosceles triangle with vertical angle $\frac{2\pi}{3}$ iff $z_1^2 + z_2^2 + z_1z_2 = 0$

Sol. **Part - 1**

Let $A(z_1), B(z_2)$

Given $z_1^2 + z_2^2 + z_1z_2 = 0$

$\Rightarrow (z_1 - \omega z_2)(z_1 - \omega^2 z_2) = 0 \Rightarrow z_1 = \omega z_2$ (or) $z_1 = \omega^2 z_2$

In the first case $|z_1| = |\omega z_2|$ and $\dfrac{z_1}{z_2} = \omega$

$\Rightarrow |z_1| = |z_2|$ and $\text{Arg}z_1 - \text{Arg}z_2 = \text{Arg}\,\omega$

$\Rightarrow |z_1| = |z_2|$ and $\text{Arg}z_1 - \text{Arg}z_2 = \dfrac{2\pi}{3}$

$\Rightarrow OA = OB$ and $\underline{|AOB} = \dfrac{2\pi}{3} \Rightarrow \Delta OAB$ is isoceles

Similarly, the other case.

Part - 2

Here $OA = OB$ ...(1)

$\therefore$ from Coni method $\dfrac{z_1 - 0}{z_2 - 0} = \dfrac{OA}{OB} e^{\frac{2\pi}{3}i}$

$\Rightarrow \dfrac{z_1}{z_2} = \left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ {from (1) }

$\Rightarrow \dfrac{z_1}{z_2} = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2} \Rightarrow \left(\dfrac{z_1}{z_2} + \dfrac{1}{2}\right) = \dfrac{i\sqrt{3}}{2}$

squaring both sides,

$\Rightarrow \dfrac{z_1^2}{z_2^2} + \dfrac{1}{4} + \dfrac{z_1}{z_2} = -\dfrac{3}{4} \Rightarrow \dfrac{z_1^2}{z_2^2} + \dfrac{z_1}{z_2} + 1 = 0$

$\Rightarrow z_1^2 + z_1z_2 + z_2^2 = 0$

$\therefore z_1^2 + z_1z_2 + z_2^2 = 0$

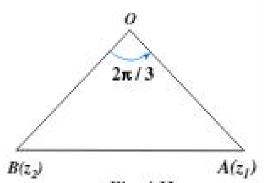


Fig. 4.32

13. Let two lines $z i - \bar{z} i + 2 = 0$ and $z(1+i) + \bar{z}(1-i) + 2 = 0$ intersect at a point P. Find the complex number of a point on the second line which is at a distance of 2 units from the point P.

Sol. Given lines $z i - \bar{z} i + 2 = 0$ .... (1)

$z(1+i) + \bar{z}(1-i) + 2 = 0$ .... (2)

Solving (1) & (2), we get P

$(2) \Rightarrow z + \bar{z} + (zi - \bar{z}i) + 2 = 0$

$\Rightarrow \bar{z} + \bar{z} + 0 = 0 \quad \Rightarrow z + \bar{z} = 0 \Rightarrow \bar{z} = -z$

$(1) \Rightarrow zi - (-z)i + 2 = 0 \Rightarrow z = i$

$\therefore P = (0,1)$

Let a point on second line be z_1 so that $|z_1 - i| = 2$

$\Rightarrow z_1 - i = 2e^{i\theta} \Rightarrow z_1 = 2e^{i\theta} + i$

substituting in (2), $\left\{2e^{i\theta} + i\right\}(1+i) + \left\{2e^{-i\theta} - i\right\}(1-i) + 2 = 0$

$\Rightarrow \cos\theta - \sin\theta = 0 \quad \Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ Required point $= 2e^{i\frac{\pi}{4}} + i = \sqrt{2} + (\sqrt{2}+1)i$

Note :

$\theta = \frac{3\pi}{4}$ is also a solution

Another point $= 2.e^{i\frac{3\pi}{4}} + i = 2\frac{(-1-i)}{\sqrt{2}} + i = -\sqrt{2} + (1-\sqrt{2})i$

14. Let $\bar{b}z + b\bar{z} = c$ be the equation of a straight line. If z_1 and z_2 be mirror images of each other in this line, then prove that $\bar{b}z_2 + b\bar{z}_1 = c$.

Sol. Given line $\bar{b}z + b\bar{z} = c$ ..... (1)

$A(z_1)$ and $B(z_2)$ are mirror images w.r.t. (1)

mid point of AB i.e. $\frac{z_1 + z_2}{2}$ lies on it.

$\therefore \bar{b}\left(\frac{z_1 + z_2}{2}\right) + b\left(\frac{\bar{z}_1 + \bar{z}_2}{2}\right) = c$

$\Rightarrow (\bar{b}z_1 + b\bar{z}_2) + (\bar{b}z_2 + b\bar{z}_1) = 2c$ ...... (2)

Also $AB \perp$ given line (1)

$\Rightarrow$ sum of complex slopes $= 0 \Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \left(\frac{-b}{\bar{b}}\right) = 0$

$\Rightarrow (\bar{b}z_1 + b\bar{z}_2) = (b\bar{z}_1 + \bar{b}z_2)$ ...... (3)

clearly from (2) & (3), we have $\bar{b}z_2 + b\bar{z}_1 = c$

[since $\because p + q = 2c$ and $p = q \Rightarrow p = q = c$]

15. Find the least value of p for which the two curves arg $(z) = \pi/6$ and $\left|z - 2\sqrt{3}i\right| = p$ intersect

Sol. $\left|z - 2\sqrt{3}i\right| = p$ represents a circle of radius p having centre at $(0, 2\sqrt{3})$ and $\arg(z) = \pi/6$ is a line making an angle of 30° with OX and lying in first quadrant.

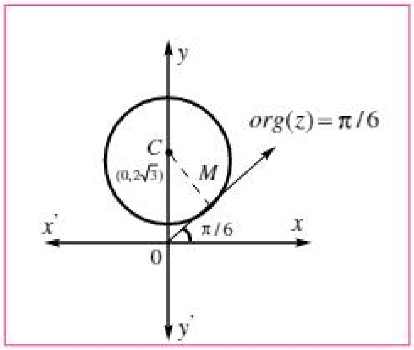


Fig. 4.33

Let CM be perpendicular from C on OA. Then,

$CM = OC \sin \pi/3 = 2\sqrt{3} \times \sqrt{3}/2 = 3$

Now, the two curves will intersect if

$CM \le p \Rightarrow 3 \le p \Rightarrow p \ge 3$

Hence, the least value of p is 3

16. If z_1 and z_2 are lying on $|z - 3| \le 4$ and $|z - 1| + |z + 1| = 3$ respectively, then show that $A = |z_1 - z_2| \in \left[0, \frac{17}{2}\right]$

Sol. We observe that $|z - 3| \le 4$ represents the set of all interior and boundary points of the circle having centre at (3, 0) and radius 4.

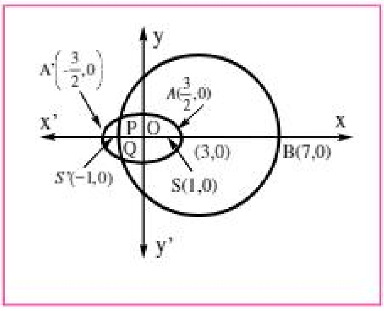


Fig. 4.34

The equation $|z - 1| + |z + 1| = 3$ represents the set of all points lying on the boundary of the ellipse having its two foci at (–1,0) and (1,0) and major axis of length 3.

Clearly $A = |z_1 - z_2| = 0$ when $z_1 = z_2$ (= affix of P or Q)

$\therefore$ min. $|z_1 - z_2| = 0$

Also, $A = |z_1 - z_2|$ is greatest when

z_1 = affix of B and z_2 = affix of A'

$\therefore$ max. $|z_1 - z_2| = A'B = A'O + OB = \frac{3}{2} + 7 = \frac{17}{2}$

Hence, $0 \le A \le \frac{17}{2}$

17. Find the complex numbers satisfying $|z+\bar{z}| + |z-\bar{z}| = 2$ and $|z+i| + |z-i| = 2$

Sol. Let $z = x + iy$

First equation gives $|x|+|y| = 2$ ...... (1) which is a square.

Second equation gives line segment joining the points (0, –1) and (0, 1)

$\therefore$ common points are (0, –1) and (0, 1) only

$\therefore$ Required numbers are $i, -i$

18. *ABCDEF* is a regular hexagon in anticlockwise sense with verties *A, B* respectively at $1 + 2i$, $1 + 3i$ in argand plane. Find complex numbers of vertex *F* and centre *P*.

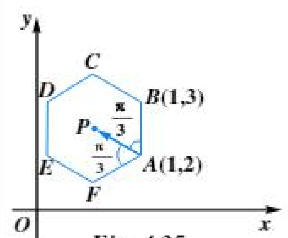


Fig. 4.35

Sol. Let $A(z_1) = 1 + 2i$, $B(z_2) = 1 + 3i$

$\because \angle PAB = \frac{\pi}{3}$, $AP = AB$, Let $P(z_3)$

$\vec{AP} = (\vec{AB})\ e^{i\pi/3}$

$\Rightarrow z_3 - z_1 = (z_2 - z_1)\ e^{i\pi/3} \Rightarrow z_3 = z_1 + (z_2 - z_1)\ e^{i\pi/3}$

$\Rightarrow z_3 = (1+2i) + \{(1+3i) - (1+2i)\}\ e^{i\frac{\pi}{3}}$

$\Rightarrow z_3 = (1+2i) + i\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \Rightarrow z_3 = \left(1 - \frac{\sqrt{3}}{2}\right) + i\,(2 + 1/2)$

$\Rightarrow z_3 = \left(\frac{2-\sqrt{3}}{2}\right) + (5/2)\,i$ which is P.

Let $F(z_4)$ $\therefore \angle BAF = \frac{2\pi}{3}$, $AF = AB$

$\vec{AF} = (\vec{AB})\ e^{i2\pi/3} \Rightarrow z_4 - z_1 = (z_2 - z_1)\,e^{i2\pi/3}$

$\Rightarrow z_4 = z_1 + (z_2 - z_1)\ e^{i2\pi/3}$

$\Rightarrow z_4 = (1+2i) + \{(1+3i) - (1+2i)\}\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$

$\Rightarrow z_4 = \left(\frac{2-\sqrt{3}}{2}\right) + i\left(2 - \frac{1}{2}\right) \Rightarrow z_4 = \left(\frac{2-\sqrt{3}}{2}\right) + \frac{3}{2}i$ which is F.

Note :

In the above problem since $e^{i2\pi/3} = w$

$F = z_1 + (z_2 - z_1)\,w \Rightarrow F = (1-w)\,z_1 + z_2 w$

19. **In the argand plane a man starts moving from origin in north east (NE) direction by 5 units then north west (NW) direction by 2 units again NE by 3 units and NW by one unit. Find the complex number corresponding to the final position of the person.**

Sol.

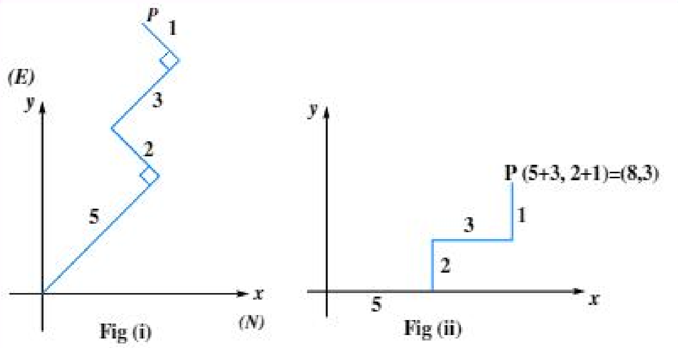


Fig. 4.36

Assming that the path as a bent string, if we rotate the entire string "clockwise $\frac{\pi}{4}$"

clearly

$P = (5 + 3, 2 + 1) = (8, 3) = 8 + 3i$ (in new position)

Now rotating $\overrightarrow{OP}$ of new position in anticlockwise direction by $\frac{\pi}{4}$ we get required point

Thus required point $= (8 + 3i)\ e^{i\pi/4} = (8 + 3i)\left(\frac{1+i}{\sqrt{2}}\right) = \left(\frac{5+11i}{\sqrt{2}}\right)$

20. **If z is a complex number and $z \neq 1$ then show that $\left|\frac{z}{|z|} - 1\right| \leq |\arg z|$**

Sol. Let $z = r(\cos\theta + i\sin\theta) \Rightarrow |z| = r,\ \arg z = \theta$

$$|\cos\theta + i\sin\theta - 1| = \left|-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right| = 2\left|\sin\frac{\theta}{2}\right| \quad (1)$$

$$\leq 2\left|\frac{\theta}{2}\right| \leq |\theta| \leq |\arg z|$$

21. **If z_1, z_2 are complex numbers show that $|z_1| \leq 1,\ |z_2| \leq 1$, then $|z_1 - z_2|^2 \leq \{|z_1| - |z_2|^2\} + \{\arg z_1 - \arg z_2\}^2$**

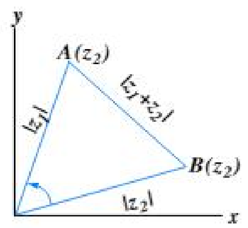


Fig. 4.37

Sol. From the figure, $AB^2 = (OA)^2 = (OB)^2 - 2(OA)(OB)\cos\alpha$

$$\Rightarrow |z_1 - z_2| = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\alpha$$

$$= \left\{|z_1|^2 - |z_2|^2 - 2|z_1||z_2| + 2|z_1||z_2| - 2|z_1||z_2|\cos\alpha\right\}$$

$$= \left\{|z_1|^2 - |z_2|^2\right\} + 2|z_1||z_2|(1 - \cos\alpha)$$

$$= \left\{|z_1|^2 - |z_2|^2\right\} + 4|z_1||z_2|\sin^2\frac{\alpha}{2}$$

$$\leq \left\{|z_1|^2 - |z_2|^2\right\} + 4\left(\frac{\alpha}{2}\right)^2 \quad \left[\begin{array}{l}\text{since } |z_1| \leq 1 |z_2| \leq 1 \\ \text{and } \sin\left(\frac{\alpha}{2}\right) \leq \left(\frac{\alpha}{2}\right)\end{array}\right]$$

$$\leq \left\{|z_1|^2 - |z_2|^2\right\} + \alpha^2$$

$$\leq \left\{|z_1|^2 - |z_2|^2\right\} + (\arg z_1 - \arg z_2)^2$$

22. The equation $|z - z_1|^2 + |z - z_2|^2 = k$, where k is a real number, will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$

Sol. $|z - z_1|^2 + |z - z_2|^2 = k$

$$\Rightarrow |z|^2 + |z_1|^2 - 2\operatorname{Re}(z\overline{z_1}) + |z|^2 + |z_2|^2 - 2\operatorname{Re}(z\overline{z_2}) = k$$

$$\Rightarrow |z|^2 - 2\operatorname{Re}\left(z\frac{(\overline{z_1} + \overline{z_2})}{2}\right) = \frac{1}{2}[k - |z_1|^2 - |z_2|^2]$$

$$\Rightarrow |z|^2 - 2\operatorname{Re}\left(z\frac{(\overline{z_1} + \overline{z_2})}{2}\right) + \left|\frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}[k - |z_1|^2 - |z_2|^2] + \left|\frac{z_1 + z_2}{2}\right|^2$$

$$\Rightarrow \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{4}[2k - 2|z_1|^2 - 2|z_2|^2 + |z_1|^2 + |z_2|^2| + 2\operatorname{Re}(z_1\overline{z}_2)]$$

$$\Rightarrow \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{4}[2k - |z_1 - z_2|^2]$$

$$\Rightarrow \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$$

Clearly, it represents a circle having centre at $\frac{z_1 + z_2}{2}$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$. For the circle to be real, we must have

$$2k - |z_1 - z_2|^2 \geq 0 \Rightarrow k \geq \frac{1}{2}|z_1 - z_2|^2$$

23. If z is a complex number satisfying $|z^2 + 1| = 4|z|$, then find the minimum value of $|z|$ and maximum value of $|z|$

Sol. We have, $|z^2 + 1| = 4|z| \quad \Rightarrow \left|\frac{z^2 + 1}{z}\right| = 4$

$$\Rightarrow \left|z + \frac{1}{z}\right| = 4 \qquad \Rightarrow \left||z| - \left|\frac{1}{z}\right|\right| \leq 4 \qquad \left[\because \left||z| - \left|\frac{1}{2}\right|\right| \leq \left|z + \frac{1}{z}\right|\right]$$

$\Rightarrow \left||z|^2 - 1\right| \le 4|z| \qquad \Rightarrow -4|z| \le \{|z|^2 - 1\} \le 4|z|$

$\Rightarrow |z|^2 - 4|z| - 1 \le 0$ or, $|z|^2 + 4|z| - 1 \ge 0$

Case (i) : When, $|z|^2 + 4|z| - 1 \ge 0$

In this case, we have $|z|^2 + 4|z| - 1 \ge 0$

$\Rightarrow |z| \ge -2 + \sqrt{5}$

$\Rightarrow$ Minimum value of $|z|$ is $\sqrt{5} - 2$

Case (ii) : When, $|z|^2 - 4|z| - 1 \le 0$

In this case, we have $|z|^2 - 4|z| - 1 \le 0$

$\Rightarrow \; 2 - \sqrt{5} \le |z| \le 2 + \sqrt{5}$

$\Rightarrow$ Maximum value of $|z|$ is $2 + \sqrt{5}$

24. Let α, β are real numbers such that $\sin\alpha + \sin\beta = \frac{\sqrt{2}}{2}$ and $\cos\alpha + \cos\beta = \frac{\sqrt{6}}{2}$. Using complex numbers find the value of $\sin(\alpha + \beta)$.

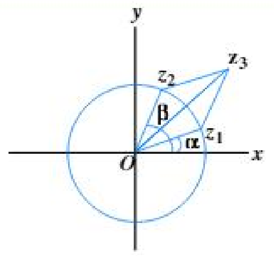


Fig. 4.38

Sol. Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$

Now, $z_3 = z_1 + z_2 = (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)$

$= (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)$

$$= 2\cos\left(\frac{\alpha-\beta}{2}\right)\left\{\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha-\beta}{2}\right) \cdot cis\left(\frac{\alpha+\beta}{2}\right) \qquad \text{...... (1)}$$

But $z_3 = z_1 + z_2 = (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta) = \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{2}cis\left(\frac{\pi}{6}\right) \qquad \text{..... (2)}$$

$\therefore$ From (1) & (2), $\frac{\alpha+\beta}{2} = \frac{\pi}{6} \Rightarrow \alpha + \beta = \frac{\pi}{3} \Rightarrow \sin(\alpha+\beta) = \frac{\sqrt{3}}{2}$

25. If $|a_r| \le 1$ for $1 \le r \le n$ show that all the roots of the equation $1 + a_1 z + a_2 z^2 + ... + a_n z^n = 0$ lie outside the circle $|z| = \frac{1}{2}$

Sol. $-1 = a_1 z + a_2 z^2 + ... + a_n z^n$

$\Rightarrow |-1| = \left|a_1 z + a_2 z^2 + ... + a_n z^n\right|$

$\Rightarrow 1 = \left|a_1 z + a_2 z^2 + ... + a_n z^n\right| \le |a_1||z| + |a_2||z^2| + ... + |a_n||z^n|$

$\leq |z|+|z|^2+....+|z|^n \qquad (\because |a_r \leq 1|) \qquad$ ---- (1)

$< |z|+|z|^2+....+......$ infinity (series extended)

$< \dfrac{|z|}{1-|z|} \qquad$ (let $|z|<1$)

Thus $1 < \dfrac{|z|}{1-|z|} \quad \Rightarrow 1-|z|<|z| \Rightarrow |z|>\dfrac{1}{2}$

$\Rightarrow |z| \in \left(\dfrac{1}{2},1\right) \qquad$ ----- (2)

Also (1) is true $\forall |z| \geq 1 \qquad$ ----- (3)

From (2), (3) $|z| \in \left(\dfrac{1}{2},1\right) \cup [1,\infty) \Rightarrow |z| \in \left(\dfrac{1}{2},\infty\right) \Rightarrow |z| > \dfrac{1}{2}$

$\Rightarrow z$ lies outside $|z| = \dfrac{1}{2}$

26. $z^2+\alpha z+\beta=0$ **(α,β are complex numbers) has a real root, then show that** $(\beta-\bar{\beta})^2=(\bar{\alpha}-\alpha)(\alpha\bar{\beta}-\bar{\alpha}\beta)$

Sol. Let $z=x+0i$ be a real root of the given equation. Then,

$x^2+\alpha x+\beta=0$

$\Rightarrow \quad x^2+(a+ib)x+(c+id)=0,$ where $\alpha=ib, \beta=c+id$

$\Rightarrow \quad (x^2+ax+c)+i(bx+d)=0$

$\Rightarrow \quad x^2+ax+c$ and $bx+d=0$

$\Rightarrow \quad x^2+ax+c=0$ and $x=-\dfrac{d}{b}$

$\Rightarrow \quad \dfrac{d^2}{b^2}-\dfrac{ad}{b}+c=0$

$\Rightarrow \quad a^2-abd+b^2c=0$

$\Rightarrow \quad \left(\dfrac{\beta-\bar{\beta}}{2i}\right)^2-\left(\dfrac{\alpha+\bar{\alpha}}{2}\right)\left(\dfrac{\alpha-\bar{\alpha}}{2i}\right)\left(\dfrac{\beta-\bar{\beta}}{2i}\right)+\left(\dfrac{\alpha-\bar{\alpha}}{2i}\right)^2\left(\dfrac{\beta+\bar{\beta}}{2}\right)=0$

$\Rightarrow \quad -2(\beta-\bar{\beta})^2+(\alpha+\bar{\alpha})(\alpha-\bar{\alpha})(\beta-\bar{\beta})-(\alpha-\bar{\alpha})^2(\beta+\bar{\beta})=0$

$\Rightarrow \quad 2(\beta-\bar{\beta})^2=(\alpha-\bar{\alpha})\left\{(\alpha+\bar{\alpha})(\beta-\bar{\beta})-(\alpha-\bar{\alpha})(\beta+\bar{\beta})\right\}=0$

$\Rightarrow \quad 2(\beta-\bar{\beta})^2=(\alpha-\bar{\alpha})(-2\alpha\bar{\beta}+2\bar{\alpha}\beta)$

$\Rightarrow \quad (\beta-\bar{\beta})^2=(\bar{\alpha}-\alpha)(\alpha\bar{\beta}-\bar{\alpha}\beta)$

Hence proved.

EXERCISE - 4.3

1. i) Show that $\frac{1}{1+2w} - \frac{1}{1+w} + \frac{1}{2+w} = 0$.

   ii) If $1, w, w^2$ are cube roots of unity show that the roots of the equation $(x-1)^3+8 = 0$ are $-1, 1-2w, 1-2w^2$.

   iii) If $x^2 + x + 1 = 0$ then show that $\sum_{r=1}^{25}\left(x^r + \frac{1}{x^r}\right)^2 = 49$

   iv) If w is a cube root of unity then show that $\frac{a+bw+cw^2}{c+aw+bw^2} + \frac{a+bw^2+cw}{bw+aw^2+c} = -1$

2. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\,\text{Sin}^{-1}\left(\frac{1+x^2}{2x}\right)$ where $x>0$

   [Ans : 4]

3. If w is a cube root of unity then find the value of $\cos\left\{\left\{\begin{array}{l}(1-w)(1-w^2)+(2-w)(2-w^2)\\+\ldots\ldots+(10-w)(10-w^2)\end{array}\right\}\frac{\pi}{1800}\right\}$

   [Ans : $\frac{1}{\sqrt{2}}$]

4. i) If w is a complex cube root of unity and a,b,c are such that $\frac{1}{a+w} + \frac{1}{b+w} + \frac{1}{c+w} = 2w^2$ and $\frac{1}{a+w^2} + \frac{1}{b+w^2} + \frac{1}{c+w^2} = 2w$ then show that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$

   ii) If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ and w is a complex cube root of unity then show that $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3w^2$.

5. i) If $\arg(z) < 0$ then show that the value of $\arg(-z) - \arg(z) = \pi$.

   ii) If z_1, z_2 are complex numbers and if $|z_1+z_2| = |z_1 - z_2|$ show that $\arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$.

   iii) If z_1, z_2 are complex numbers and if $|z_1 + z_2| = |z_1| - |z_2|$ show that $\arg(z_1) - \arg(z_2) = \pi$

   iv) Let z, w be complex numbers such that $z + iw = 0$ and $\arg(zw) = \pi$, then show that $\arg z = \frac{3\pi}{4}$.

   v) If z and w are non – zero complex numbers such that $|zw| = 1$ and $\arg z - \arg w = \frac{\pi}{2}$ then show that the value of $\bar{z}w$ is equal to $-i$.

6. i) If z_1, z_2 are two non- zero complex numbers then prove that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$.

ii) Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if $\dfrac{z_1}{z_2}$ is purely imaginary.

7. If $8iz^3 + 12z^2 - 18z + 27i = 0$ (where $i = \sqrt{-1}$) then show that $|z| = \dfrac{3}{2}$

8. Find the number of common roots of the equations $z^3 + (1+i)z^2 + (1+i)z + i = 0$ and $z^{1993}+z^{1994}+1=0$.

[Ans : 2]

9. i) If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$ prove that $\dfrac{1+iz}{1-iz} = \dfrac{a+ib}{1+c}$.

ii) Let P be a complex number and $|P| \neq 1$ and if $|z + P| = |\bar{p}z + 1|$ show that $|z| = 1$.

10. i) Find the locus of z if $|z - i| < |z + 2i|$. [Ans : $2y + 1 > 0$]

ii) If $w = \dfrac{z}{z - \left(\dfrac{i}{3}\right)}$ and $|w| = 1$ then find the locus of z. [Ans : $6y - 1 = 0$]

iii) Determine the locus of z such that $\dfrac{z^2}{z-1}$ is real. [Ans : $y = 0$, $(x-1)^2 + y^2 = 1$]

11. If $z = re^{i\theta}$ then find the value of $|e^{iz}|$. [Ans : $e^{-\sin\theta}$]

12. i) If z is such that $|z| = 2$ then find the greatest value of $|z - 4 + 3i|$. [Ans : 7]

ii) Find the minimum value of $|z - 2 + i| + |z + 3i|$. [Ans : $2\sqrt{2}$]

13. i) If $\left|z - \dfrac{2}{z}\right| = 1$ then show that the maximum value of $|z|$ is 2.

ii) Show that the area bounded by the curves $\arg(z) = \pi/3$ and $\arg(z) = 2\pi/3$ and $\arg(z - 2 - 2i\sqrt{3}) = \pi$ in the argand plane is $4\sqrt{3}$.

iii) Show that the greatest value of $|z|$ if $\log_{\tan(\pi/8)}\left(\dfrac{|z|^2 - |z| + 1}{2 + |z|}\right) > -2$ is $2 + \sqrt{2} + \sqrt{11 + 8\sqrt{2}}$

14. i) Find the value of least positive argument z where z satisfies $|z - 1 - 2i| \leq 1$.

[Ans : $\tan^{-1}\left(\dfrac{3}{4}\right)$]

ii) If $|z - i| \leq 2$ and $z_1 = 3 + 4i$ then find the maximum value of $|iz + z_1|$. [Ans : $2 + \sqrt{20}$]

15. i) If $|z| = 1$ show that $\tan\left(\frac{\arg(z)}{2}\right) = \left(\frac{1-z}{1+z}\right)$.

ii) Show that $e^{2im\cot^{-1}x}\left(\frac{ix+1}{ix-1}\right) = 1$

16. i) Find the number of solutions of the equation $z^2 + |z|^2 = 0$. **[Ans : Infinite]**

ii) Find the solutions of the equation $z^2 + |z| = 0$. **[Ans : O, i, $-i$]**

17. Find the value of z satisfying the equation $z + \sqrt{2}|z + 1| + i = 0$ where $i = \sqrt{-1}$. **[Ans : $-2-i$]**

18. i) Find the area of the triangle whose vertices are the roots of $z^3 + iz^2 + 2i = 0$.

[Ans : 2 SQ units]

ii) If a and b are real numbers between 0 and 1 such that $z_1 = a + i$ and $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then find a and b. **[Ans : $a = b = 2 - \sqrt{3}$]**

iii) If $z_1^2 + z_2^2 - 2z_1 z_2 \cos\theta = 0$ then show that origin, z_1, z_2 form vertices of an isoceles triangle with vertical angle 'θ'.

iv) Find the nature of the triangle with vertices z_1, z_2, z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

[Ans : Equilateral]

19. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular. Show that z_1 lies on a circle of radious 2.

20. If all the roots of $z^3 + pz^2 + qz + r = 0$ are of unit modulus then show that $|p| \leq 3$, $|q| \leq 3$, $|r| = 1$.

21. Prove that the condition for the quadratic equation $az^2 + bz + c = 0$ to possess both real roots where a, b, c are complex constants is $\frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{c}{\bar{c}}$.

22. If z_1, z_2 are non – real complex roots of the equation $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} + \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = 0$ then show that $\arg(z_1) + \arg(z_2) = 0$.

23. If z_1, z_2, z_3 are vertices A, B, C of ΔABC respectively such that $|z_1| = |z_2| = |z_3|$ and $|z_1 + z_2| = |z_1 - z_2|$ then find the value of $\tan C$. **[Ans : 1]**

24. If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1\bar{z}_2) = 0$ then prove that for the complex numbers $z_3 = a + ic$, $z_4 = b + id$, $\operatorname{Re}(z_3\bar{z}_4) = 0$.

25. If z_1, z_2, z_3, z_4 are the vertices of a square with centre at z_0 then prove that $\sum z_1^2 = 4z_0^2$.

26. i) Prove that $|1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1-|z_1|^2)(1-|z_2|^2)$

ii) Show that $\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right| = |z_1 + z_2| + |z_1 - z_2|$

27. For the complex numbers z and w if $|z|^2 w - |w|^2 z = z - w$ prove that $z = w$ or $z\bar{w} = 1$

28. If z_1, z_2, z_3 are vertices of an equilateral triangle then prove that $\dfrac{z_2 + z_2 - 2z_1}{z_2 - z_3}$ is purely imaginary.

29. Let A, B, C be an isosceles triangle inscribed in the circle $|z| = r$ with $AB = AC$. If z_1, z_2, z_3 represent the points A, B, C respectively show that $z_2 z_3 = z_1^2$.

30. If the complex number z satisfies the equation $|z - 3i| = 3$ then find the value locus of $i - \dfrac{6}{z}$.

[Ans : $y = 0$]

31. If $x_1, x_2, x_3, \ldots\ldots, x_n$ are the roots of $x^n + ax + b = 0$ then show that

$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots\ldots (x_1 - x_n) = n x_1^{n-1} + a$

32. Let $A(z_1), B(z_2), C(z_3)$ be three points such that $az_1 + bz_2 + cz_3 = 0$, $a+b+c = 0$ for some $a, b, c \in R$, and atleast one of a, b, c is non-zero. If $|z_1 - z_2| = |z_2 - z_3| = |z_2|$ then find area of triangle AOC.

[Ans : $\frac{1}{2}|z_1||z_3|$]

33. z_1 and z_2 are two distinct complex numbers such that $|z_1|^2 z_2 - |z_2|^2 z_1 = z_1 - z_2$ then show that $\bar{z}_1 z_2 = \bar{z}_2 z_1 = 1$.