CHAPTER 3

Algebra

PARTIAL FRACTIONS

- PROPER AND IMPROPER FRACTIONS
- DIVISION ALGORITHM
- RESOLVING INTO PARTIAL FRACTIONS

3.0 INTRODUCTION

In lower classes, a student learnt numbers particularly integers, rational numbers. In rational numbers, we have proper fractions, improper fractions and addition of several fractions into a single fraction. Present topic deals with splitting (or) expressing large fractions as sum of simpler fractions especially, for polynomial fractions, instead of number fractions.

The concept of decomposition of a given fraction into partial proper fractions is useful in the chapters of integration, differentiation, summing up infinite series.The method of partial fractions was introduced by Johann Bernoulli (1667-1748)

3.1 BASIC DEFINITIONS

In this section we shall learn some basic definitions.

Polynomial :

An expression $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, where $a_0, a_1, ..., a_n$ are real or complex constants, is called a polynomial in x $(n \in N)$. n is the degree of the polynomial where $a_0 \neq 0$.

Example :

1) $f(x) = 2x^3 - 3x^2 + 1$ is a third degree polynomial (cubic)
2) $g(x) = 3x^2 - 4x + 5$ is a second degree polynomial (quadratic)
3) $h(x) = 3x^{10} + x^8 + 3x^2 - 1$ is a 10^{th} degree polynomial.

Note

i) $\phi(x) = 3x^2 + \sqrt{x} + 1$ *is not a polynomial.*

ii) $\psi(x) = 2x^2 - x + \sin x$ *is not a polynomial.*

iii) $\mu(x) = \log_e (x^2 + x + 3)$ *is not a polynomial.*

iv) In this chapter unless otherwise specified we consider polynomials in which coefficients are real only.

v) Two polynomials $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, $g(x) = b_0x^m + b_1x^{m-1} + b_2x^{m-2} + ... + b_m$ *of* n^{th}, m^{th} *degree respectively are equal if* $a_i = b_i$ $\forall$ i *and* $n = m$

vi) A polynomial is a zero degree polynomial if all coefficients vanish except a_0. *That is every non-zero constant is a zero degree polynomial.*

vii) If all the coefficients $a_0, a_1, a_2, ..., a_n$ *are zero then it is called zero polynomial denoted by 0, for which degree is not defined.*

viii) A polynomial is reducible if it can be expressed as product of 2 polynomials, otherwise it is irreducible.

3.2 RATIONAL FRACTION

Definition

If f(x), g(x) are two polynomials (g(x) is non-zero polynomial) then $\frac{f(x)}{g(x)}$ is a Rational Fraction (or) Rational Function (or) fraction.

Example :

1) $\frac{2x+3}{x^2-2x+1}$ is a rational fraction.

2) $\frac{3x^2-2x+1}{x^4-2x+3}$ is a rational fraction.

3) $\frac{x^2+1}{x+1}$ is a rational fraction.

3.3 PROPER FRACTION, IMPROPER FRACTION

Definition

A rational fraction $\frac{f(x)}{g(x)}$ is called a proper rational fraction if degree of f(x) is less than degree of g(x). Otherwise it is called an improper fraction.

Example :

1) $\frac{2x+1}{x^2+3x+2}$ is a proper fraction.

2) $\frac{1}{2x-1}$ is a proper fraction.

3) $\frac{x^2-2x+1}{x^2+x+1}$ is an improper fraction.

4) $\frac{x^3+2x^2+x+1}{2x-1}$ is an improper fraction.

3.4 DIVISION ALGORITHM

The division algorithm which is a property of natural numbers is also applicable for polynomials.

Definition

If f(x) and g(x) are two polynomials where $g(x) \neq 0$, then there exists two polynomials q(x) and r(x) such that f(x) = g(x).q(x) + r(x)

Note

If f(x) is of degree n and g(x) is of degree m $(n \geq m)$ then q(x) is of $(n-m)^{th}$ degree and r(x) is of degree less than m.

SOLVED EXAMPLES

1. If $f(x)$ and $g(x)$ are of degrees 7 and 4 respectively such that $f(x) = g(x)\ q(x) + r(x)$ then find possible degrees of $q(x)$ and $r(x)$.

Sol. Clearly degree of $q(x) = 7 - 4 = 3$

degree of $r(x) < 4$

i.e., possible degrees of $r(x)$ are 0 (or) 1 (or) 2 (or) 3

i.e., $r(x)$ is of the form $px^3 + qx^2 + rx + s$

2. If $f(x) = 2x^3 + x^2 - 5x + 1$ is divided with $x + 1$ and $x - 1$ and the respective remainders are 5 and –1 then find the remainder when $f(x)$ is divided with $x^2 - 1$?

Sol. Given $f(x) = 2x^3 + x^2 - 5x + 1$

Also by division algorithm,

$f(x) = (x+1)\,q_1(x) + 5$ .... (1)

$f(x) = (x-1)\,q_2(x) + (-1)$ .... (2)

$\therefore f(-1) = 0 + 5 = 5 \quad f(1) = 0 - 1 = -1$

Also, $f(x) = (x^2 - 1)\,q_3(x) + (ax + b)$ .... (3)

Put $x = -1 \Rightarrow 5 = 0 + a(-1) + b \Rightarrow -a + b = 5$

Put $x = 1 \Rightarrow -1 = 0 + a(1) + b \Rightarrow a + b = -1$

Solving we get $a = -3, b = 2$

$\therefore$ Remainder $= ax + b = -3x + 2$

> ***Remember :***
> *When $f(x)$ is divided with $(x-a)\ (x-b)$ then the remainder,*
> $$r(x) = \left(\frac{f(a) - f(b)}{a-b}\right)x + \frac{af(b) - bf(a)}{a-b}$$

3.5 PARTIAL FRACTIONS

The chapter of partial fractions is the reverse process of the following process.

Consider two proper fractions say $\frac{3}{2x-1}$ and $\frac{-1}{x-2}$

Their sum $= \frac{3}{2x-1} + \frac{-1}{x-2} = \frac{3(x-2) - 1(2x-1)}{(2x-1)(x-2)} = \frac{x-5}{2x^2 - 5x + 2}$ which is also a proper fraction.

Thus "Sum of two or more proper fractions is a proper fraction"

$\frac{x-5}{2x^2 - 5x + 2} = \frac{3}{2x-1} + \frac{-1}{x-2}$ is called resolving into partial fractions.

In the above, the two fractions $\frac{3}{2x-1}, \frac{-1}{x-2}$ are said to be partial proper fractions of $\frac{x-5}{2x^2 - 5x + 2}$.

In this chapter we learn methods of resolving the given proper fraction into two or more simpler partial proper fractions.

> ***Note***
> *If $\frac{f(x)}{g(x)}$ is an improper fraction then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ where $\frac{r(x)}{g(x)}$ is a proper fraction which may be further resolved.*

3.6 TYPE - I

Let $\frac{f(x)}{g(x)}$ be a proper fraction and $g(x)$ possesses only non-reducible linear factors. If $ax + b$ is a linear factor of $g(x)$ then $\frac{A}{ax+b}$ is a corresponding partial fraction and A has to be determined.

SOLVED EXAMPLES

1. Resolve $\frac{2x+3}{(x+3)(x+1)}$ into partial fractions.

Sol. Let $\frac{2x+3}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$ Where A and B are non-zero constants.

Clearly, $\frac{2x+3}{(x+3)(x+1)} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$

$\Rightarrow 2x+3 = A(x+1) + B(x+3)$ .... (1)

This is an identity

Put $x = -1$ in (1) we get, $2(-1)+3 = A(0) + B(-1+3) \Rightarrow 1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$

Put $x = -3$ in (1), we get, $2(-3)+3 = A(-3+1) + B(0) \Rightarrow -3 = -2A \Rightarrow \boxed{A = \frac{3}{2}}$

Now, $\frac{2x+3}{(x+3)(x+1)} = \frac{3/2}{x+3} + \frac{1/2}{x+1}$ i.e., $\frac{2x+3}{(x+3)(x+1)} = \frac{3}{2(x+3)} + \frac{1}{2(x+1)}$

which is the resolution of the given proper fraction into partial proper fractions.

2. Resolve $\frac{3x^2+1}{(x^2-3x+2)(2x+1)}$ into partial fractions.

Sol. Let $\frac{3x^2+1}{(x^2-3x+2)(2x+1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{2x+1}$

Clearly, $3x^2+1 = A(x-2)(2x+1) + B(x-1)(2x+1) + C(x-1)(x-2)$

Put $x = 1$, we get $4 = A(-1)(3) + B(0) + C(0)$

$\Rightarrow 4 = -3A \Rightarrow \boxed{A = -\frac{4}{3}}$

Put $x = 2$, we get $13 = A(0) + B(1)(5) + C(0)$

$\Rightarrow 13 = 5B \Rightarrow \boxed{B = \frac{13}{5}}$

Put $x = -\frac{1}{2}$, we get $\frac{7}{4} = A(0) + B(0) + C\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)$

$\Rightarrow \frac{7}{4} = \frac{15}{4}C \Rightarrow \boxed{C = \frac{7}{15}}$

Thus we have

$$\frac{3x^2+1}{(x^2-3x+2)(2x+1)} = \frac{-4}{3(x-1)} + \frac{13}{5(x-2)} + \frac{7}{15(2x+1)}$$

3. **If $\dfrac{3x-1}{(2x+1)(x+k)} = \dfrac{-5}{3(2x+1)} + \dfrac{7}{3(x+k)}$. Find k?**

Sol. Clearly, $3(3x-1) = -5(x+k) + 7(2x+1)$

This is an identity

Put $x = 0 \Rightarrow -3 = -5k + 7 \Rightarrow k = 2$

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4. **Show that**

$$\frac{n!}{x(x+1)(x+2)....(x+n)} = \frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \frac{{}^nC_3}{x+3} + ... + \frac{(-1)^n\ {}^nC_n}{x+n}.$$

Sol. Let $\dfrac{n!}{x(x+1)(x+2)....(x+n)} = \dfrac{A_0}{x} + \dfrac{A_1}{x+1} + \dfrac{A_2}{x+2} + ... + \dfrac{A_n}{x+n}$

$\Rightarrow n! = A_0(x+1)(x+2)..................(x+n) + A_1(x)(x+2)............(x+n) +$
$................ + A_n(x)(x+1)....(x+n-1)$

Put $x = 0$, we get, $n! = A_0.n! \Rightarrow A_0 = 1 = {}^nC_0$

Put $x = -1$, we get, $n! = A_1(-1)(n-1)! \Rightarrow A_1 = -{}^nC_1$

Put $x = -2$, we get, $n! = A_2(-2)(-1).(1.2.3.4.....n-2)$

$\Rightarrow n! = A_2(2!)(n-2)! \Rightarrow A_2 = {}^nC_2$

Clearly by symmetry, $A_3 = {}^{-n}C_3$

$A_4 = {}^nC_4....$

$A_n = (-1)n\ .\ {}^nC_n$

Thus $\dfrac{n!}{x(x+1).....(x+n)} = \dfrac{{}^nC_0}{x} - \dfrac{{}^nC_1}{x+1} + \dfrac{{}^nC_2}{x+2} + ... + (-1)^n \dfrac{{}^nC_n}{x+n}$

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5. **Resolve into partial fractions $\dfrac{(1+2x)(1+3x)(1+4x)}{(1-2x)(1-3x)(1-4x)}$.**

Sol. Given fraction is an improper fraction because polynomials in numerator and denominator are of same degree 3.

By actual division, we have

$$\begin{array}{l} -24x^3+....)\ 24x^3+....\ (-1 \\ \qquad\qquad\ \ 24x^3+.... \\ \hline \qquad\qquad\quad r(x) \end{array}$$

Here the quotient is '–1' and $r(x)$ is a polynomial of second degree, and thus

$$\text{G.E.} = -1 + \frac{r(x)}{(1-2x)(1-3x)(1-4x)}$$

Therefore let,

$$\frac{(1+2x)(1+3x)(1+4x)}{(1-2x)(1-3x)(1-4x)} = -1 + \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-4x}$$

$$\text{Clearly } A = \left\{\frac{(1+2x)(1+3x)(1+4x)}{(1-3x)(1-4x)}\right\}_{x=\frac{1}{2}} = \frac{2\times\frac{5}{2}\times 3}{\left(-\frac{1}{2}\right)(-1)} = 30$$

$$B = \left\{\frac{(1+2x)(1+3x)(1+4x)}{(1-2x)(1-4x)}\right\}_{x=\frac{1}{3}} = \frac{\frac{5}{3}\times 2\times\frac{7}{3}}{\frac{1}{3}\times-\frac{1}{3}} = -70$$

$$C = \left\{\frac{(1+2x)(1+3x)(1+4x)}{(1-2x)(1-3x)}\right\}_{x=\frac{1}{4}} = \frac{\frac{6}{4}\times\frac{7}{4}\times 2}{\frac{2}{4}\times\frac{1}{4}} = 42$$

Thus,

$$\frac{(1+2x)(1+3x)(1+4x)}{(1-2x)(1-3x)(1-4x)} = -1 + \frac{30}{1-2x} - \frac{70}{1-3x} + \frac{42}{1-4x}$$

3.7 TYPE - II

Let $\dfrac{f(x)}{g(x)}$ be a proper fraction and $g(x)$ contains a repeated factor $(ax+b)^n$ i.e., $ax+b$ is called a linear factor repeated n times, and the corresponding partial fractions will be of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + ... + \frac{A_n}{(ax+b)^n}$$

The method is illustrated below by examples.

SOLVED EXAMPLES

1. Resolve $\dfrac{x^2-3x+5}{(x-2)^3}$ into partial fractions

Sol. **Method - 1 :**

Clearly given fraction is a proper fraction

Let $\dfrac{x^2-3x+5}{(x-2)^3} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3}$

$\Rightarrow x^2-3x+5 = A(x-2)^2 + B(x-2) + C$ .... (1)

Put $x=2$, we get, $3 = 0+0+C \Rightarrow \boxed{C=3}$

Equating coefficient of x^2 on both sides of (1)

$1 = A \Rightarrow \boxed{A=1}$

Equating coefficient of x on both sides of (1)

$-3 = -4A + B \Rightarrow -3 = -4(1) + \text{B} \Rightarrow \boxed{B=1}$

$\therefore \dfrac{x^2-3x+5}{(x-2)^3} = \dfrac{1}{x-2} + \dfrac{1}{(x-2)^2} + \dfrac{3}{(x-2)^3}$

Method - 2

This method is applicable when the denominator contains only a repeated linear factor, $(ax+b)^n$ by substituting $ax+b=t$.

Given fraction $\dfrac{x^2-3x+5}{(x-2)^3}$

Let $x-2=t \Rightarrow x=t+2$

$$\therefore \frac{x^2-3x+5}{(x-2)^3}=\frac{(t+2)^2-3(t+2)+5}{t^3}=\frac{t^2+t+3}{t^3}$$

$$=\frac{t^2}{t^3}+\frac{t}{t^3}+\frac{3}{t^3}=\frac{1}{t}+\frac{1}{t^2}+\frac{3}{t^3}=\frac{1}{x-2}+\frac{1}{(x-2)^2}+\frac{3}{(x-2)^3}$$

2. Resolve $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ into partial fractions.

Sol. Let $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{1-2x}$

$\therefore\ 3x^2+x-2 = A(x-2)(1-2x)+B(1-2x)+C(x-2)^2$ .... (1)

Put $x=2$, $\therefore\ 12=-3B \Rightarrow \boxed{B=-4}$

Put $x=\dfrac{1}{2}$, $\therefore\ 3\left(\dfrac{1}{4}\right)+\dfrac{1}{2}-2=C\left(\dfrac{1}{2}-2\right)^2 \Rightarrow \boxed{C=\dfrac{-1}{3}}$

Equating the coefficient of x^2 on both sides of (1),

$\therefore\ 3=-2A+C \Rightarrow 3=-2A-\dfrac{1}{3} \Rightarrow \boxed{A=-\dfrac{5}{3}}$

$$\therefore \frac{3x^2+x-2}{(x-2)^2(1-2x)}=\frac{-5}{3(x-2)}-\frac{4}{(x-2)^2}-\frac{1}{3(1-2x)}$$

3. Resolve into partial fractions $\dfrac{x^4+3x+1}{x^3(x+1)}$.

Sol. It is an improper fraction and by actual division quotient = 1

Let $\dfrac{x^4+3x+1}{x^3(x+1)}=1+\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x+1}$

$\Rightarrow x^4+3x+1 = x^3(x+1)+Ax^2(x+1)+Bx(x+1)+C(x+1)+Dx^3$

Put $x=0 \Rightarrow 1=C(1) \Rightarrow \boxed{C=1}$

Put $x=-1 \Rightarrow -1=D(-1) \Rightarrow \boxed{D=1}$

Equating coefficient of x^3, $0=1+A+D \Rightarrow \boxed{A=-2}$

Equating coefficient of x^2, $0=A+B \Rightarrow \boxed{B=2}$

$$\therefore \frac{x^4+3x+1}{x^3(x+1)}=\frac{-2}{x}+\frac{2}{x^2}+\frac{1}{x^3}+\frac{1}{x+1}+1$$

4. Resolve $\frac{1}{x^6(x+1)}$ into partial fractions.

Sol. Shortcut approach

Put $x = \frac{1}{y}$

$$\therefore \frac{1}{x^6(x+1)} = \frac{1}{\left(\frac{1}{y^6}\right)\left(\frac{1}{y}+1\right)} = \frac{y^7}{y+1}$$

$$= \frac{(y^7+1)-1}{y+1} = \frac{(y^7+1)}{y+1} - \frac{1}{y+1}$$

$$= y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 - \frac{1}{y+1}$$

$$= \frac{1}{x^6} - \frac{1}{x^5} + \frac{1}{x^4} - \frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$$

5. If $\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$ then show that

$$\frac{1}{(ax+b)^2(cx+d)} = \frac{A}{(ax+b)^2} + \frac{AB}{ax+b} + \frac{B^2}{cx+d}.$$

Sol. Given $\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$ .... (1)

Consider, $\frac{1}{(ax+b)^2(cx+d)} = \frac{1}{(ax+b)}\left\{\frac{A}{ax+b} + \frac{B}{cx+d}\right\}$

$$= \frac{A}{(ax+b)^2} + B\times\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)^2} + B\left\{\frac{A}{ax+b} + \frac{B}{cx+d}\right\}$$

$$= \frac{A}{(ax+b)^2} + \frac{AB}{ax+b} + \frac{B^2}{cx+d}$$

3.8 TYPE - III

If $\frac{f(x)}{g(x)}$ is a proper fraction and $g(x)$ contains a second degree non-reducible factor like $ax^2 + bx + c$ then the corresponding partial fraction will be of the form $\frac{Ax+B}{ax^2+bx+c}$ where A, B are constants.

The method is illustrated below.

SOLVED EXAMPLES

1. Resolve $\dfrac{2x+3}{(x-1)(x^2+x+1)}$ into partial fractions.

Sol. Since x^2+x+1 is a non reducible second degree factor.

Let $\dfrac{2x+3}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

$\Rightarrow 2x+3 = A(x^2+x+1)+(Bx+C)(x-1)$

Put $x = 1 \Rightarrow 5 = 3A \Rightarrow \boxed{A = \frac{5}{3}}$

Equating coefficient of x^2, $0 = A + B \Rightarrow \boxed{B = -\frac{5}{3}}$

Equating coefficient of x, $2 = A - B + C \Rightarrow \boxed{C = -\frac{4}{3}}$

$\therefore \dfrac{2x+3}{(x-1)(x^2+x+1)} = \dfrac{5}{3(x-1)} - \dfrac{(5x+4)}{3(x^2+x+1)}$

2. Resolve $\dfrac{x^3+x-1}{(x^2+1)(x^2+2x+3)}$ into partial fractions.

Sol. x^2+1, x^2+2x+3 are non-reducible second degree factors

Let $\dfrac{x^3+x-1}{(x^2+1)(x^2+2x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2x+3}$

$\Rightarrow x^3+x-1 = (Ax+B)(x^2+2x+3)+(Cx+D)(x^2+1)$

Equating coeff. of x^3 : $1 = A + C$ .... (1)

Equating coeff. of x^2 : $0 = 2A + B + D$ .... (2)

Equating coeff. of x : $1 = 3A + 2B + C$ .... (3)

Equating constant : $-1 = 3B + D$ .... (4)

(4) $\Rightarrow D = -1 - 3B$

(1) $\Rightarrow C = 1 - A$

(2) $\Rightarrow 0 = 2A + B - 1 - 3B \Rightarrow 2A - 2B = 1$ .... (5)

(3) $\Rightarrow 1 = 3A + 2B + 1 - A \Rightarrow 2A + 2B = 0$ .... (6)

Solving (5) & (6) : $\boxed{A = \frac{1}{4}}$, $\boxed{B = -\frac{1}{4}}$

Also, $\boxed{C = \frac{3}{4}}$, $\boxed{D = -\frac{1}{4}}$

$\therefore$ Partial fractions are, $\dfrac{x-1}{4(x^2+1)} + \dfrac{3x-1}{4(x^2+2x+3)}$.

3. Resolve $\dfrac{2x^2+3}{(x^2+1)(x^2+2)(x^2+3)}$ into partial fractions.

Sol. The given functions is a proper fraction and denominator contains second degree non-reducible factors. Clearly it is an even function and by a simplified approach, Put $x^2 = y$

$$\therefore \text{ GE} = \frac{2y+3}{(y+1)(y+2)(y+3)} = \frac{A}{y+1} + \frac{B}{y+2} + \frac{C}{y+3}$$

Clearly $A = \left\{\dfrac{2y+3}{(y+2)(y+3)}\right\}_{y=-1} = \dfrac{1}{2}$

Similarly $B = \left\{\dfrac{2(-2)+3}{(-2+1)(-2+3)}\right\}_{y=-2} = \dfrac{-1}{-1} = 1$

$C = \left\{\dfrac{2(-3)+3}{(-3+1)(-3+2)}\right\}_{y=-3} = \dfrac{-3}{2}$

$\therefore$ Partial fractions are, $\dfrac{1}{2(x^2+1)} + \dfrac{1}{x^2+2} - \dfrac{3}{2(x^2+3)}$

4. Resolve $\dfrac{3x-1}{x^3+1}$ into partial fractions.

Sol. Let $\dfrac{3x-1}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$

$\Rightarrow 3x - 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$

Put $x = -1$, $-4 = A(3) \Rightarrow \boxed{A = -4/3}$

Equating coeff. of $x^2 : 0 = A + B \Rightarrow \boxed{B = \dfrac{4}{3}}$

Equating coeff. of $x : 3 = -A + B + C \Rightarrow \boxed{C = \dfrac{1}{3}}$

$\therefore$ Partial fractions are, $\dfrac{-4}{3(x+1)} + \dfrac{4x+1}{3(x^2-x+1)}$

3.9 TYPE - IV

If $\dfrac{f(x)}{g(x)}$ is a proper fraction and $g(x)$ contains a repeated second degree non-reducible factor $(ax^2 + bx + c)^n$ then the corresponding partial fractions are,

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$ where $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$ are constants.

The method is illustrated by the following examples.

SOLVED EXAMPLES

1. Resolve $\dfrac{3x-2}{(x^2+4)^2(x-1)}$ into partial fractions.

Sol. $(x^2+4)^2$ is a repeated non-reducible factor

$$\therefore \text{ Let } \frac{3x-2}{(x^2+4)^2(x-1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{x-1}$$

$\Rightarrow 3x-2 = (Ax+B)(x^2+4)(x-1) + (Cx+D)(x-1) + E(x^2+4)^2$

Put $x = 1$,

$\therefore 1 = E(25) \Rightarrow \boxed{E = \frac{1}{25}}$

Equating coeff. of $x^4 : 0 = A + E \Rightarrow \boxed{A = \frac{-1}{25}}$

Equating coeff. of $x^3 : 0 = -A + B \Rightarrow \boxed{B = \frac{-1}{25}}$

Equating coeff. of $x^2 : 0 = 4A - B + C + 8E \Rightarrow \boxed{C = \frac{-1}{5}}$

Equating constant $: -2 = -4B - D + 16E \Rightarrow \boxed{D = \frac{70}{25}}$

$$\therefore \frac{3x-2}{(x^2+4)^2(x-1)} = \frac{-(x+1)}{25(x^2+4)} - \frac{(5x-70)}{25(x^2+4)^2} + \frac{1}{25(x-1)}$$

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2. Resolve $\dfrac{x^4}{(x^2+1)^2}$ into partial fractions.

Sol. Given fraction is improper. By actual division quotient is 1

$$\therefore \text{ Let } \frac{x^4}{(x^2+1)^2} = 1 + \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$\Rightarrow x^4 = (x^2+1)^2 + (Ax+B)(x^2+1) + Cx + D$

Equating coeff. of $x^4 : 1 = 1$

Equating coeff. of $x^3 : 0 = A \Rightarrow \boxed{A = 0}$

Equating coeff. of $x^2 : 0 = 2 + B \Rightarrow \boxed{B = -2}$

Equating coeff. of $x : 0 = A + C \Rightarrow \boxed{C = 0}$

Equating constant $: 0 = 1 + B + D \Rightarrow \boxed{D = 1}$

$$\therefore \frac{x^4}{(x^2+1)^2} = 1 + \frac{-2}{x^2+1} + \frac{1}{(x^2+1)^2}$$

Note : It can also be solved by putting $x^2 = t$.

3. **Resolve $\dfrac{x^2+1}{x^4+x^2+1}$ into partial fractions.**

Sol. Consider denominator $= x^4 + x^2 + 1$

$= (x^4 + 2x^2 + 1) - x^2 = (x^2 + 1)^2 - x^2$

$= (x^2 + 1 + x)(x^2 + 1 - x) = (x^2 + x + 1)(x^2 - x + 1)$

Which are non-reducible second degree factors.

Let $\dfrac{x^2+1}{x^4+x^2+1} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{x^2-x+1}$

$\Rightarrow x^2+1 = (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)$

Equating coeff. of $x^3 : 0 = A + C$ .... (1)

Equating coeff. of $x^2 : 1 = -A + B + C + D$ .... (2)

Equating coeff. of $x : 0 = A - B + C + D$ .... (3)

Equating constant : $1 = B + D$ .... (4)

(1) & (3) $\Rightarrow B = D$ .... (5)

(2) & (4) $\Rightarrow C - A = 1$ .... (6)

Solving (4) & (5) $\boxed{B = \dfrac{1}{2}}$, $\boxed{D = \dfrac{1}{2}}$

Solving (1) & (6) $\boxed{A = \dfrac{-1}{2}}$, $\boxed{C = \dfrac{1}{2}}$

$\therefore \dfrac{x^2+1}{x^4+x^2+1} = \dfrac{-x+1}{2(x^2+x+1)} + \dfrac{x+1}{2(x^2-x+1)}$

4. **If $\dfrac{x^2}{x^6-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + f(x)$ then find the value of $f(2)$?**

Sol. $\dfrac{x^2}{(x-1)(x+1)(x^4+x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + f(x)$

Clearly $A = \dfrac{1^2}{(1+1)(1^4+1^2+1)} = \dfrac{1}{6}$

$B = \dfrac{(-1)^2}{(-1-1)((-1)^4+(-1)^2+1)} = \dfrac{1}{-6}$

$\therefore \dfrac{x^2}{x^6-1} = \dfrac{1}{6(x-1)} - \dfrac{1}{6(x+1)} + f(x)$

Put $x = 2 \Rightarrow \dfrac{4}{63} = \dfrac{1}{6} - \dfrac{1}{18} + f(2) \Rightarrow \boxed{f(2) = -\dfrac{1}{21}}$

EXERCISE

1. Find the remainders when $x^4 + 2x^2 - 3x + 7$ is divided with $x + 2$ and $x - 1$? [Ans : 37 and 7]

2. If the remainders when $x^5 - 2x^4 + px^3 + 4x + 9$ is divided with $x - 2$ and $x + 2$ respectively are 2 and -1 them find (p, q). [Ans : $(p, q) = (-5, 32)$]

3. If the remainders when $x + 3$ and $x - 1$ divide the polynomial expression $f(x)$ are respectively 1 and 2 then find the remainder when it is divided with $x^2 + 2x - 3$? [Ans : $\left(\frac{x+7}{4}\right)$]

4. Find k in the following

i) $\frac{2x+5}{(x+1)(2x+k)} = \frac{3}{x+1} - \frac{4}{2x+k}$ [Ans : 3]

ii) $\frac{3x+1}{(x-2)(x+4)} = k\left\{\frac{7}{x-2} + \frac{11}{x+4}\right\}$ [Ans : 1/6]

iii) $\frac{x^3+1}{x^2-3x+2} = (kx-5) + \frac{7x-5}{(x-1)(x-2)}$ [Ans : 7]

5. Resolve the following Fractions into Partial fractions

i) $\frac{2x+3}{(x+1)(x-3)}$ ii) $\frac{5x+6}{(2+x)(1-x)}$ iii) $\frac{3x+7}{x^2-3x+2}$ iv) $\frac{x+4}{(x^2-4)(x+1)}$

v) $\frac{2x^2+2x+1}{x^3+x^2}$ **(March-17, 18) vi) $\frac{2x+3}{(x-1)^3}$ vii) $\frac{x^2-2x+6}{(x-2)^3}$

[Ans : (i) $\frac{9}{4(x-3)} - \frac{1}{4(x+1)}$ (ii) $\frac{11}{3(1-x)} - \frac{4}{3(x+2)}$ (iii) $\frac{-10}{(x-1)} + \frac{13}{(x-2)}$

(iv) $\frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{(x+1)}$ (v) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$ (vi) $\frac{2}{(x-1)^2} + \frac{5}{(x-1)^3}$ (vii) $\frac{1}{(x-2)} + \frac{2}{(x-2)^3}$]

6. Resolve the following Fractions into Partial fractions

i) $\frac{x^2-x+1}{(x+1)(x-1)^2}$ ii) $\frac{9}{(x-1)(x+2)^2}$ iii) $\frac{1}{(1-2x)^2(1-3x)}$

iv) $\frac{1}{x^3(x+a)}$ *v) $\frac{x^2+5x+7}{(x-3)^3}$ **(March-18)** vi) $\frac{3x^3-8x^2+10}{(x-1)^4}$

[Ans : (i) $\frac{3}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}$ (ii) $\frac{1}{(x-1)} - \frac{1}{(x+2)} - \frac{3}{(x+2)^2}$

(iii) $\frac{9}{1-3x} - \frac{6}{1-2x} - \frac{2}{(1-2x)^2}$ (iv) $\frac{1}{a^3x} - \frac{1}{a^2x^2} + \frac{1}{ax^3} - \frac{1}{a^3(x+a)}$

(v) $\frac{1}{(x-3)} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}$ (vi) $\frac{3}{(x-1)} + \frac{1}{(x-1)^2} - \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}$]

7. Resolve the following Fractions into Partial fractions

*i) $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}$ (May-18, 19) ii) $\dfrac{3x-1}{(x+2)(1-x+x^2)}$ iii) $\dfrac{x^2-3}{(x+2)(x^2+1)}$

*iv) $\dfrac{x^2+1}{(x^2+x+1)^2}$ (March-19) v) $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}$

[Ans : (i) $\dfrac{3}{(x-1)}+\dfrac{2-x}{x^2+2}$ (ii) $\dfrac{x}{(1-x+x^2)}-\dfrac{1}{x+2}$ (iii) $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$

(iv) $\dfrac{1}{(1+x+x^2)}-\dfrac{x}{(x^2+x+1)^2}$ (v) $\dfrac{2}{3(x-1)}+\dfrac{1}{(x-1)^2}+\dfrac{x+2}{3(x^2+x+1)}$]

8. Resolve the following Fractions into Partial fractions

i) $\dfrac{x^2}{(x-1)(x-2)}$ *ii) $\dfrac{x^3}{(x-1)(x+2)}$ (March-19)

iii) $\dfrac{x^3}{(2x-1)(x-1)^2}$ iv) $\dfrac{x^3}{(x-a)(x-b)(x-c)}$

[Ans : (i) $1-\dfrac{1}{x-1}+\dfrac{4}{x-2}$ (ii) $x-1+\dfrac{1}{3(x-1)}+\dfrac{8}{3(x+2)}$ (iii) $\dfrac{1}{2}+\dfrac{1}{2(2x-1)}+\dfrac{1}{(x-1)}+\dfrac{1}{(x-1)^2}$

(iv) $1+\dfrac{a^3}{(a-b)(a-c)(x-a)}+\dfrac{b^3}{(b-c)(b-a)(x-b)}+\dfrac{c^3}{(c-a)(c-b)(x-c)}$]

9. Find the coefficient of x^3 in the power series expansion of $\dfrac{5x+6}{(x+2)(1-x)}$ specifying the region in which the expansion is valid. [Ans : $\dfrac{15}{4}$]

10. Find the coefficient of x^4 in the power series expansion of $\dfrac{3x^2+2x}{(x^2+2)(x-3)}$ specifying the interval in which the expansion is valid. [Ans : $\dfrac{77}{324}$]

11. Find the coefficient of x^n in the power series expansion of $\dfrac{x-4}{x^2-5x+6}$ specifying the region in which the expansion is valid. [Ans : $\dfrac{1}{3^{n+1}}-\dfrac{1}{2^n}$]

12. Find the coefficient of x^n in the power series expansion of $\dfrac{3x}{(x-1)(x-2)^2}$ [Ans : $-3+\dfrac{3}{2^{n+1}}+\dfrac{3(n+1)}{2^{n+1}}$]