CHAPTER 2

Algebra

PERMUTATIONS AND COMBINATIONS

• FUNDAMENTAL PRINCIPLE •
• LINEAR PERMUTATIONS WITH OR WITHOUT REPETITION •
• CIRCULAR PERMUTATIONS • PERMUTATIONS WITH LIKE OBJECTS •
• COMBINATIONS & DIVISION INTO GROUPS •
• DISTRIBUTION OF UNLIKE AND LIKE OBJECTS •

2.0 INTRODUCTION

Before going to discuss about permutations and combinations, let us learn about 'Fundamental Principle of counting' which plays a very crucial role in the theory of permutations and combinations.

2.1 FUNDAMENTAL PRINCIPLE (Product Rule)

If a work can be performed in 'm' ways and another different second work can be performed in 'n' ways, corresponding to each performance of the first work, then the two works one after another can be performed in $m \times n$ different ways. This is called fundamental principle or product rule.

Note

Alternately if a work has n parts and the work will be completed only when each part is completed and the first part can be completed in m_1 ways, the second part can be completed in m_2 ways and so on.... the n^{th} part can be completed in m_n ways, then the total number of ways of doing the work is $m_1m_2m_3....m_n$.

Note : *Counting without actually counting is the aim of this chapter*

Example :

1. In a shelf there are 3 different Hindi books and 2 different Sanskrit books. A student can select a Hindi Book and a Telugu book in $3 \times 2 = 6$ ways as explained below.

   Let the Hindi books be H_1, H_2, H_3 and sanskrit books be S_1, S_2.

   The six selections are H_1S_1, H_1S_2, H_2S_1, H_2S_2, H_3S_1, H_3S_2

2. If a man has 3 different coloured pants and 3 different coloured shirts. The number of ways that he can select a pair (i.e. one pant and one shirt) is $3 \times 3 = 9$ as explained below.

   Let the pants be P_1, P_2, P_3 and shirts be S_1, S_2, S_3. He can select a pair from the following

   P_1S_1, P_1S_2, P_1S_3, P_2S_1, P_2S_2, P_2S_3, P_3S_1, P_3S_2, P_3S_3 i.e., 9 ways

3. A two wheeler agent sells scooters, motorcycles. In each body pattern two capacities 100 C.C. and 150 C.C. available. In each capacity there are four colours. The number of choices a customer will have to buy a vehicle is $2 \times 2 \times 4 = 16$.

2.2 ADDITION PRINCIPLE

If a work can be performed in m different ways and another second work can be done in n different ways. Then either of these two works can be performed in $(m + n)$ ways.

Note

Alternately if a work can be done by n methods and by using the first method it can be done in a_1 ways or by second method in a_2 ways and so on... by the n^{th} method in a_n ways, then the number of ways to get the work done is $(a_1 + a_2 + ... + a_n)$.

Example :

1. A shelf contains 3 different Hindi books (H_1, H_2, H_3) and 2 different Sanskrit books (S_1, S_2).

   The number of ways to select one book of any language either Hindi or Sanskrit is (3 + 2) i.e., 5 ways.

   The 5 ways are H_1, H_2, H_3, S_1, S_2.

2. There are 16 two bed room flats in a building and 10 two bed room flats in another building and 8 two bedroom flats in a third building. The number of choices a customer will have for buying a flat is 16 + 10 + 8 = 24.

2.3 LINEAR PERMUTATION

Definition :

> ***Note :*** *Objects arranged along a row (line) is called linear permutation.*

Each of the different arrangements which can be made after selecting some or all of finite set of objects (elements) is called a permutation. If the objects are arranged along a row (line) then that arrangement is called linear permutation.

Note

i) ***If the objects are arranged around a circle then that arrangement is called circular permutation.***

ii) ***The permutation involves two steps.***

a) selection ***b) arrangement***

In first stage we select the objects and in second stage we arrange the selected objects according to different order.

iii) ***In permutation, the order of the objects (elements) in which they are arranged is considered (important).***

Example :

1. Consider 4 different objects *a, b, c, d*; suppose we first select 2 of them and arranging those two in order, we get the following 12 arrangements which are called permutations of 4 things taken 2 at a time.

   The 12 permutations are

   for the selection *ab*, the arrangements are *ab, ba*

   for the selection *ac*, the arrangements are *ac, ca*

   for the selection *ad*, the arrangements are *ad, da*

   for the selection *bc*, the arrangements are *bc, cb*

   for the selection *bd*, the arrangements are *bd, db*

   for the selection *cd*, the arrangements are *cd, dc*

2. Consider the letters of the word 'RAMA'.

   Here all the letters are not distinct. Two letters A, A are identical.

   Now let us select 2 of them and then arranging in order we get RA, AR, RM, MR, AM, MA, AA

   Observe that here we are not getting 12 permutations as in Example 1. Now we are getting only 7 arrangements. This is because all the 4 letters are not distinct.

> ***Note :*** *Objects arranged around a circle is called circular permutation.*

2.4 NOTATION

The number of permutations of n distinct things taken r at a time is denoted by ${}^nP_r$ or $P(n, r)$.

Here $1 \le r \le n$ and n, r are +ve integers.

We write ${}^nP_0 = 1$ by convention.

THEOREM-2.1 **If $1 \le r \le n$ and r, n are +ve integers, then ${}^nP_r = n(n-1)(n-2) \dots (n-r+1)$.**

Proof : Clearly, ${}^nP_r$ is equal to the number of ways of filling 'r' blank places in a row with n distinct things such that each blank place is filled with one thing only.

> ***Note :*** *Number of permutations of n things takes r at a time is equivalent to filling 'r' blanks with n things*

We prove this result by mathematical induction on r;

If $r = 1$, i.e., if there is one blank place, then it can be filled by any one of the n distinct things. So, It can be done in n ways.

$\therefore {}^nP_r = {}^nP_1 = n$

The result is true if $r = 1$.

Assume that the result is true for $r = k$

${}^nP_k = n(n-1)(n-2) \dots (n-k+1)$ i.e., we assumed that the number of ways of filling k blank places in a row with n distinct objects (things) is $n(n-1)(n-2)\dots(n-k+1)$.

We are going to prove that the result is true for $r = k + 1$.

Suppose there are $(k + 1)$ blank places.

By assumption, the first k blank places can be filled with n distinct things in $n(n-1)(n-2) \dots (n-k+1)$ ways.

After filling k blank places, there are $(n - k)$ things to fill $(k + 1)^{th}$ blank place.

So, $(k + 1)^{th}$ blank place can be filled in $(n - k)$ ways.

By fundamental principle, the number of ways to fill up $(k + 1)$ blank places with n distinct things is $n(n-1)(n-2) \dots (n-k+1)(n-k)$

$\therefore {}^nP_{k+1} = n(n-1)(n-2) \dots (n-k+1)(n-k)$

$\therefore$ The result is true for $n = k + 1$

By mathematical induction the result is true $\forall\, r \in N$ and $r \le n$;

$\therefore {}^nP_r = n(n-1)(n-2) \dots (n-r+1)$ if $1 \le r \le n$.

2.5 FACTORIAL NOTATION

If n is a non negative integer, then factorial n which is denoted by $\angle n$ or $n!$ is defined as

i) $\angle 0 = 1$

ii) If $n \ge 1$ then $\angle n = n(n-1)(n-2) \dots 3.2.1$

For example, $\angle 5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

THEOREM-2.2 **If n, r are positive integers and $n \ge r$ then ${}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$.**

Proof : By theorem 2.1, ${}^nP_r = n(n-1)(n-2) \dots (n-r+1)$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)\dots 3 \cdot 2 \cdot 1}{(n-r)(n-r-1)\dots\dots 3 \cdot 2 \cdot 1} = \frac{\lfloor n}{\lfloor n-r}$$

for example ${}^6P_2 = \dfrac{\lfloor 6}{\lfloor 6-2} = \dfrac{720}{24} = 30$

But the easy method is ${}^6P_2 = 6 \times 5 = 30$

${}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$

Note

i) ${}^{n}P_{n} = \dfrac{\underline{|n}}{\underline{|n-n}} = \dfrac{\underline{|n}}{\underline{|0}} = \underline{|n}$

ii) *If n and r are +ve integers and* $n \ge r$ *then* ${}^{n}P_{r} = n\,{}^{n-1}P_{r-1} = n(n-1)\,{}^{n-2}P_{r-2} = n(n-1)(n-2)\,{}^{n-3}P_{r-3} = \ldots$ *so*

on Since ${}^{n}P_{r} = \dfrac{\underline{|n}}{\underline{|n-r}} = n\dfrac{\underline{|n-1}}{\underline{|n-r}} = n\dfrac{\underline{|n-1}}{\underline{|(n-1)-(r-1)}} = n\,{}^{n-1}P_{r-1}$

We can proceed like this.

THEOREM-2.3 ${}^{n}P_{r} = {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1}$ **where** n, r **are positive integers such that** $n \ge r+1$.

Proof :

Method - 1

Note : ${}^{n}P_{r} = n.{}^{(n-1)}P_{(r-1)}$

$$\text{R.H.S.} = {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1} = \frac{\underline{|n-1}}{\underline{|n-1-r}} + r\frac{\underline{|n-1}}{\underline{|n-r}}$$

$$= \underline{|n-1}\left\{\frac{1}{\underline{|n-r-1}} + \frac{r}{\underline{|n-r}}\right\} = \underline{|n-1}\left\{\frac{n-r+r}{\underline{|n-r}}\right\}$$

$$= \underline{|n-1}\,\frac{n}{\underline{|n-r}} = \frac{\underline{|n}}{\underline{|n-r}} = {}^{n}P_{r}$$

Method - 2

We know that ${}^{n}P_{r}$ denotes the number of permutations of n different things taken r at a time.

Let us denote the n distinct things as $a_1, a_2, a_3, \ldots\ldots, a_n$;

These ${}^{n}P_{r}$ permutations can be divided into two categories which are exclusive.

1) The permutations which contain a_1

2) The permutations which do not contain a_1

Let us find the number of permutations of n different things taken r at a time which contain 'a_1'.

First fill up one of the r blank places with the particular thing a_1. It can be done in r ways.

Now the remaining $(r-1)$ blank places can be filled with the remaining $(n-1)$ distinct objects.

It can be done in ${}^{n-1}P_{r-1}$ ways.

By fundamental principle,

The number of permutations which contain a_1 is $r\,{}^{n-1}P_{r-1}$ .... (1)

Let us find the number of permutations of n things taken r at a time which do not contain a_1.

Here we have to fill up r blank places with $(n-1)$ distinct things only without using a_1. It can be done in ${}^{n-1}P_{r-1}$ ways.

$\therefore\ {}^{n}P_{r} = {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1}$ .... (2)

SOLVED EXAMPLES

1. If $^nP_4 = 1680$, find n.

Sol. We know that $^nP_4$ is the product of 4 consecutive integers of which n is the largest.

That is $^nP_4 = n(n-1)(n-2)(n-3)$

and $1680 = 8\times7\times6\times5$

on comparing the largest integers, we get $n = 8$.

2. If $^{12}P_r = 1320$, find r.

Sol. $1320 = 12\times11\times10 = {}^{12}P_3$; Thus $r = 3$.

3. If $^{(n+1)}P_5 : {}^nP_5 = 3:2$ find n.

Sol. $^{(n+1)}P_5 : {}^nP_5 = 3:2$

$$\Rightarrow \frac{(n+1)!}{(n-4)!}\times\frac{(n-5)!}{n!}=\frac{3}{2}\Rightarrow\frac{n+1}{n-4}=\frac{3}{2}$$

$$\Rightarrow 2n+2 = 3n-12 \Rightarrow n = 14$$

4. If $^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$, find r.

Sol. $^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$

$$\Rightarrow \frac{(56)!}{(56-(r+6))!}\times\frac{(54-(r+3))!}{(54)!}=\frac{30800}{1}$$

$$\Rightarrow \frac{(56)!}{(50-r)!}\times\frac{(51-r)!}{(54)!}=\frac{30800}{1}$$

$$\Rightarrow 56\times55\times(51-r)=30800$$

$$\Rightarrow (51-r)=\frac{30800}{56\times55}=10\Rightarrow r=41$$

5. Find the number of all 4 letter words that can be formed using the letters of the word EQUATION. How many of these words begin with E? How many end with N? How many begin with E and end with N?

Sol. The word EQUATION has 8 distinct letters. We have to fill up 4 places using these 8 letters

☐ ☐ ☐ ☐

This can be done in $^8P_4 = 8.7.6.5 = 1680$ ways. Hence the number of 4 letter words that can be formed using the letters of the word EQUATION is 1680.

Words beginning with E : Fill the first place with E as shown below

$\boxed{E}$ ☐ ☐ ☐

Now we are left with 7 letters and 3 places. They can be filled in

$^7P_3 = 7.6.5 = 210$ ways.

Thus the number of 4 letter words that begin with E is 210

Word ending with N : This can be done in the same way as above. First fill last place with N as shown below

☐ ☐ ☐ [N]

Then the remaining 3 places with the remaining 7 letters can be filled in

${}^7P_3 = 210$ ways. Hence the number of 4 letter words ending with N is 210.

Words beginning with E and ending with N : Fill the first place with E and the last place with N as shown below

[E] ☐ ☐ [N]

Now the remaining 2 places with the remaining 6 letters can be filled in

${}^6P_2 = 6 \times 5 = 30$ ways. Thus the number of 4 letter words that begin with E and ending with N is 30.

6. How many 4 letter words can be formed using the letters of the word 'ARTICLE' such that

a) each word begin with vowel b) the word contains A but not E

c) each word must contain atleast one vowel.

Sol. Word 'ARTICLE' contains 7 letters.

From these 7 letters we have to select 4 and then these 4 letters have to be arranged. We get ${}^7P_4$ words without any restriction.

a) Since each 4 - letter word must begin with a vowel, fill up the first blank place with any one of the 3 vowels i.e., A, I, E

This can be done in 3 ways.

The remaining 3 places can be filled with any one of the remaining six letters (2 vowels and 4 consonants) in ${}^6P_3$ ways.

By fundamental principle,

required number of 4 - letter words $= 3 \times {}^6P_3 = 3 \times 6 \times 5 \times 4 = 360$

b) Let us consider 4 blank places

☐ ☐ ☐ ☐

Since the 4 letter word must contain A, arrange A in any one of the 4 blank places. It can be done in 4 ways. Since the word must not contain E, the remaining 3 places can be filled from the remaining 5 letters R, T, I, C, L. This can be done in ${}^5P_3$ ways. By fundamental principle, the required number of arrangements $= 4 \times {}^5P_3 = 4 \times 5 \times 4 \times 3 = 240$

c) The number of 4 letter words which contain atleast one vowel = (the number of 4 letter words without any restriction) – (the number of 4 letter words which contain no vowel)

$= {}^7P_4$ – (The number of 4 - letter words which are formed using R, T, C, L only)

$= {}^7P_4 - {}^4P_4 = 7 \times 6 \times 5 \times 4 - 4 \times 3 \times 2 \times 1 = 24(35 - 1) = 24 \times 34 = 816$

7. Find the number of ways to arrange 5 boys and 5 girls in a row such that

a) all the girls must sit together b) no two girls sit together

c) all the boys sit together and all girls sit together

d) no two of the same sex sit together

Sol. a) Since all the girls must sit together, consider the 5 girls as a unit.

Now there are 5 boys + 1 unit of girls i.e., 6 different objects. $B_1B_2B_3B_4B_5$ $(G_1G_2G_3G_4G_5)$

Note :

When certain elements are to be together they can be considered as one unit.

> **Note :** *When n elements are arranged in a row (line) there are (n + 1) gaps available for further arrangements in that line*

They can be arranged along a row by taking all of them at a time in $\lfloor 6$ ways.
The 5 girls can be rearranged among themselves in $\lfloor 5$ ways.
By fundamental principle
Number of arrangements such that all girls sit together = $\lfloor 6 \times \lfloor 5$

b) First arrange the 5 boys in a row. It can be done in $\lfloor 5$ ways.
Then we get 6 gaps denoted by 'X' as shown below.
$X\ B_1\ X\ B_2\ X\ B_3\ X\ B_4\ X\ B_5\ X$
Since, no 2 girls come together, the 5 girls can be arranged by selecting 5 gaps from these six gaps
It can be done in ${}^6P_5$ ways
By fundamental principle,
The number of arrangements such that no two girls come together = $\lfloor 5 \times {}^6P_5$.

c) Since boys and girls must sit together,
Consider all the boys as a unit and all the girls as another unit. $(B_1B_2B_3B_4B_5)$ $(G_1G_2G_3G_4G_5)$
These two units can be arranged in ${}^2P_2$ ways i.e., $\lfloor 2$ ways.
Now the 5 boys can be arranged among themselves in $\lfloor 5$ ways. The 5 girls can be arranged among themselves in $\lfloor 5$ ways.
By fundamental principle, the required number of arrangements = $2 \times \lfloor 5 \times \lfloor 5$

d) Since no two of the same sex come together, we have to arrange the boys and girls alternatively. First arrange the 5 boys. It can be done in $\lfloor 5$ ways. We get 6 gaps denoted by $X_1, X_2, X_3, X_4, X_5, X_6$ as shown below.
$X_1\ B\ X_2\ B\ X_3\ B\ X_4\ B\ X_5\ B\ X_6$
We are going to arrange 5 girls in these 6 gaps. But the gaps named by X_2, X_3, X_4, X_5 must be filled with girls. If one of these gaps is not filled with a girl, then two boys on either side of that gap come side by side which is against to the restriction.
But these 4 gaps are not sufficient for 5 girls. So, we have to select one more gap from X_1, X_6.
The number of ways to arrange the 5 girls in 5 gaps using X_1 but not $X_6 = {}^5P_5 = \lfloor 5$
Similarly, using the gap X_6 but not X_1 we can arrange the 5 girls in $\lfloor 5$ ways
$\therefore$ The required number of arrangements $= \lfloor 5 \times (\lfloor 5 + \lfloor 5)$
(by using multiplication rule and addition rule) = $2 \times \lfloor 5 \times \lfloor 5$

Method - 2 :

1	2	3	4	5	6	7	8	9	10

Case(i) : odd places by boys and even places by girls

Case(ii) : odd places by girls and even places by boys

Required number of ways = $5! \times 5! + 5! \times 5! = 2(5!\,5!) = 28800$

8. In how many ways 9 mathematics papers can be arranged so that the best and the worst (i) may come together (ii) may not come together ?

Sol. i) If the best and worst papers are treated as one unit, then we have 9–2+1= 7+1= 8 papers. Now these can be arranged in (7+1)! ways and the best and worst papers between themselves can be permuted in 2! ways. Therefore the number of arrangements in which best and worst papers come together is 8!2!.

ii) Total number of ways of arranging 9 mathematics papers is 9!. The best and worst papers come together in 8!2! ways. Therefore the number of ways they may not come together is 9! –8!2! = 8!(9–2) = 8!×7.

9. Find the number of ways of arranging the letters of the word 'FATHER' so that

a) The relative positions of vowels and consonants are not disturbed.

b) no vowel occupies even place.

Sol. a) Since the relative positions of vowels and consonants are not disturbed, we have to arrange the 4 consonants F, T, H, R among themselves and 2 vowels A, E among themselves.

This can be done in ${}^4P_4 \times {}^2P_2$ i.e., $24 \times 2 = 48$ ways

b) Let us take 6 places

1	2	3	4	5	6

Among these 6 places, 3 are even places. Since no vowel occupies even place, we have to arrange the two vowels in 3 odd places. It can be done in ${}^3P_2$ ways. Now we shall be left with 4 places (3 even and one odd) in which the 4 consonants can be arranged. It can be done in ${}^4P_4$ ways.

Required number of arrangements = ${}^3P_2 \times {}^4P_4 = 6 \times 24 = 144$

10. Find the number of ways in which 5 boys and 4 girls can be arranged along a row such that atleast one of the first 3 places must be arranged with a girl.

Sol. The 5 boys and 4 girls can be arranged along a row without any restriction in $\underline{|9}$ ways.

Number of arrangements such that atleast one of the first 3 places must be arranged with a girl = (Number of arrangements without restriction) – (Number of arrangements in which the first 3 places are occupied by boys)

$= \underline{|9} - {}^5P_3 \times \underline{|6} = \underline{|6}\ \{7 \times 8 \times 9 - 5 \times 4 \times 3\} = 720 \times 444 = 319680$

Note : *If order among 'r' distinct elements is not to be considered then they can be taken as similar things.*

11. Find the number of ways of arranging 15 students $A_1, A_2, ..., A_{15}$ in a row such that

a) A_1, A_2, A_3 sit together in specified order

b) A_2 must be seated after A_1 and A_3 must come after A_2

c) neither A_2 nor A_3 be seated before A_1

Sol. a) Consider A_1, A_2, A_3 as a unit .

Now there are 12 persons + 1 unit

i.e., 13 things to be arranged in a row by taking all at a time. It can be done in ${}^{13}P_{13}$ i.e., $\underline{|13}$ ways.

Here we need not arrange the three persons A_1, A_2, A_3 among themselves because they must be seated together in specified order.

$\therefore$ Required number of ways = $\underline{|13}$

b) Consider 15 places to arrange 15 persons $A_1, A_2, A_3 ...., A_{15}$;

Keeping aside A_1, A_2, A_3 first arrange the remaining 12 persons in any 12 places of these 15 places. It can be done in ${}^{15}P_{12}$ ways. Now we shall be left with 3 places which need not be consecutive. Now arrange A_1, A_2, A_3 in these 3 places according to the desired order. It can be done in 1 way.

$\therefore$ Number of arrangements in which A_1 comes before A_2 and A_2 before $A_3 = {}^{15}P_{12} = \dfrac{\underline{|15}}{\underline{|3}}$

Note : *If 'r' distinct elements always occur in a particular order (not necessarily conse-cutive) then they can be takes as similar things*

c) First arrange the 12 persons $A_4, A_5, ..., A_{15}$ in any 12 places of these 15 places. It can be done in ${}^{15}P_{12}$ ways.

We shall be left with 3 places

Arrange A_1 in 1st place and arrange A_2, A_3 in the remaining two places in both orders because there is no restriction among A_2, A_3.

This can be done in $1 \times {}^2P_2$ i.e., 2 ways.

$\therefore$ Required number of arrangements $= {}^{15}P_{12} \times 1 \times {}^2P_2 = 2 \times \frac{\lfloor 15}{\lfloor 3}$

12. Find the number of 4 digit numbers that can be formed using the digits 2,3,5,6,8 (without repetition). How many of them are divisible by

i) 2 ii) 3 iii) 4 iv) 5 v) 25

Sol. The number of 4 digit numbers that can be formed using the 5 digits 2,3,5,6,8 is ${}^5P_4 = 120$.

i) Divisible by 2 : For a number to be divisible by 2, the units place should be filled with an even digits. This can be done in 3 ways (2 or 6 or 8).

			1

Now, the remaining 3 places can be filled with the remining 4 digits in ${}^4P_3 = 24$ ways. Hence the number of 4 digit numbers divisible by 2 is 3×24 = 72.

ii) Divisible by 3 : A number is divisible by 3 if the sum of digits in it is a multiple of 3. Since the sum of the given 5 digits is 24, we have to leave either 3 or 6 and use the digits 2,5,6,8 or 2,3,5,8. In each case, we can permute them in 4! ways. Thus the number of 4 digit numbers divisible by 3 is 2×4! = 48.

iii) Divisible by 4 : A number is divisible by 4 if the number formed by the digits in the last two places (tens and units places) is a multiple of 4.

		X	X

Thus we fill the last two places (as shown in the figure) with one of 28,32,36,52,56,68

That is done in 6 ways. After filling the last two places, we can fill the remaining two places with the remaining 3 digits in ${}^3P_2 = 6$ ways.

Thus, the number of 4 digit numbers divisible by 4 is 6×6 = 36.

iv) Divisible by 5 : After filling the units place with 5 (one way), the remaining 3 places can be filled with the remaining 4 digits in ${}^4P_3 = 24$ ways. Hence the number of 4 digit numbers divisible by 5 is 24.

v) Divisible by 25 : Here also we have to fill the last two places (that is, units and tens place) with 25 (one way) as shown below.

		2	5

Now the remaining 2 places can be filled with the remaining 3 digits in ${}^3P_2 = 6$ ways. Hence the number of 4-digit numbers divisible by 25 is 6.

13. Find the number of 4 digited numbers that can be formed by using the digits 0, 2, 3, 5, 7, 8 which are divisible by

a) 2 b) 3 c) 4

Sol. a) **Consider 4 blank places**

$\square \quad \square \quad \square \quad \square$

To get a number which is divisible by 2 we have to fillup the unit place either with 0, 2 or 8.

The number of 4 digited number ending with zero = $5 \times 4 \times 3 = 60$ .... (1)

[$\because$ 1000th place can be filled in 5 ways, 100th place in 4 ways and 10th place in 3 ways]

The number of 4 digited numbers ending with 2 or 8 $= 4 \times 4 \times 3 \times 2 = 96$ .... (2)

[Since the unit place can be filled in 2 ways and the 1000th place can be filled in 4 ways with the remaining digits except 0. 100th place can be filled in 4 ways. '0' also can be used here and 10th place can be filled in 3 ways]

$\therefore$ The number of 4 digited even numbers = 60 + 96 = 156

b) **Divisible by 3**

Note : *A number is divisible by 3 if the sum of the digits in that number is a multiple of 3.*

We know that a number is divisible by 3 if the sum of the digits in that number is a multiple of 3.

We always use only 4 numbers from 0, 2, 3, 5, 7, 8 to fill up 4 places.

The 4 digited numbers using 0, 3, 7, 8 or 0, 3, 5, 7 or 0, 2, 5, 8 or 2, 3, 5, 8 or 0, 2, 3, 7 are divisible by 3.

The number of 4 digited numbers

Using 0, 3, 7, 8 = ${}^4P_4 - {}^3P_3 = 18$

Using 0, 3, 5, 7 = ${}^4P_4 - {}^3P_3 = 18$

Using 0, 2, 5, 8 = ${}^4P_4 - {}^3P_3 = 18$

Using 0, 2, 3, 7 = ${}^4P_4 - {}^3P_3 = 18$

Using 2, 3, 5, 8 = ${}^4P_4 = 24$

$\therefore$ Total number of 4 - digited numbers which are divisible by

$$3 = 18 + 18 + 18 + 18 + 24 = 96$$

c) **Divisible by 4**

Note : *A number is divisible by 4 if the number in the last two places is a multiple of 4.*

A number is divisible by 4 if the number formed by the digits in the last two places is a multiple of 4.

The last two digits must be 08, 20, 28, 32, 52, 72, 80

The number of 4 - digited numbers ending with 08 or 20 or 80

$$= 3 \times {}^4P_2 = 3 \times 4 \times 3 = 36$$

The number of 4 digited numbers endingwith 28 or 32 or 52 or

$$72 = 4[{}^4P_2 - {}^3P_1] = 4[12 - 3] = 36$$

$\therefore$ Required number of 4-digited numbers divisible by 4 = 36 + 36 = 72

Note :

In decimal system the value of a digit 'n' in r^{th} place from the units place is $n.10^{r-1}$

***14. Find the sum of all 4 - digited numbers that can be formed using 1, 3, 4, 5, 7 without repetition.**

Sol. The number of 4 digited numbers that can be formed using 1, 3, 4, 5, 7 is ${}^5P_4 = 120$

In unit place each digit occur ${}^4P_3$ times.

Sum of unit place $= 1 \times {}^4P_3 + 3 \times {}^4P_3 + 4 \times {}^4P_3 + 5 \times {}^4P_3 + 7 \times {}^4P_3$

$= (1 + 3 + 4 + 5 + 7) \times {}^4P_3$

The value of sum of unit place $= (1 + 3 + 4 + 5 + 7) \times {}^4P_3 \times 1$

Similarly, the value of sum of 10^{th} place $= (1 + 3 + 5 + 7 + 4) \times {}^4P_3 \times 10$

The value of sum of 100^{th} place $= (1 + 3 + 5 + 7 + 4) \times {}^4P_3 \times 100$

The value of sum of 1000^{th} place $= (1 + 3 + 5 + 7 + 4) \times {}^4P_3 \times 1000$

$\therefore$ Sum of all 4 - digited numbers

$= (1 + 3 + 5 + 7 + 4) \times {}^4P_3 [1 + 10 + 100 + 1000]$

$= (1 + 3 + 5 + 7 + 4) \times {}^4P_3 [1111]$

$= 20 \times 24 \times 1111 = 480 \times 1111 = 533280$

Note :

The sum of k - digited numbers that can be formed using n nonzero distinct digits $(1 \le n \le 9)$ and $k \le n$ is (sum of n digits) $\times {}^{n-1}P_{k-1} \times 11111......1$ (k times).

***15. Find the sum of all 4 -digited numbers that can be formed using the digits 0, 2, 3, 4, 6 without repetition.**

Sol. The number of 4 digited numbers that can be formed using 0, 2, 3, 4, 6 is ${}^5P_4 - {}^4P_3 = 96$

In unit place 0 appears ${}^4P_3$ times and the remaining digits 2, 3, 4, 6 each appear in $({}^4P_3 - {}^3P_2)$ times.

$\therefore$ The value of sum of unit place

$= \{0 \times {}^4P_3 + ({}^4P_3 - {}^3P_2) [2 + 3 + 4 + 6] \times 1 = ({}^4P_3 - {}^3P_2) (2 + 3 + 4 + 6) \times 1$

Similarly value of sum of 10^{th} place $= ({}^4P_3 - {}^3P_2) (2 + 3 + 4 + 6) \times 10$

Value of sum of 100^{th} place $= ({}^4P_3 - {}^3P_2) (2 + 3 + 4 + 6) \times 100$

But in 1000^{th} place '0' does not occur and 2, 3, 4, 6 each occur ${}^4P_3$ times.

Sum of 1000^{th} place $= (2 + 3 + 4 + 6) \times {}^4P_3$

Value of sum of 1000^{th} place $= (2 + 3 + 4 + 6) \times {}^4P_3 \times 1000$

$\therefore$ Sum of all 4 - digited numbers

$= (2 + 3 + 4 + 6) ({}^4P_3 - {}^3P_2) \times (1 + 10 + 100) + (2 + 3 + 4 + 6) \times {}^4P_3 \times 1000$

$= (2 + 3 + 4 + 6) \times {}^4P_3 (1 + 10 + 100 + 1000) - (2 + 3 + 4 + 6) \times {}^3P_2 \times (1 + 10 + 100)$

$= 15 \times 24 \times 1111 - 15 \times 6 \times 111$

$= 15 \times 6 \{4444 - 111\} = 90 \times 4333 = 389970$

Note :

The sum of k digited numbers that can be formed using n distinct digits (zero is included) $0 \le n \le 9$ ***and*** $k \le n$ ***is***

(sum of n digits) $\times {}^{n-1}P_{k-1} \times 111.....1$ ***(k times) –***

(sum of n digits) $\times {}^{n-2}P_{k-2} \times 111.....1$ ***(k – 1 times)***

> **Note :**
> *Rank of a word is the positions at which the word occurs when the letters of the word are arranged as in a dictionary*

***16.** **If the letters of the word 'SIPRON' are arranged in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'PRISON'.**

Sol. The letters of the word 'SIPRON' in dictionary order is

I N O P R S

The number of words which begin with I = $^5P_5$ = 120

The number of words which begin with N = $^5P_5$= 120

The number of words which begin with O = $^5P_5$= 120

The number of words which begin with PI = $^4P_4$= 24

The number of words which begin with PN=$^4P_4$= 24

The number of words which begin with PO=$^4P_4$= 24

The number of words which begin with PRIN=$^2P_2$= 2

The number of words which begin with PRIO=$^2P_2$= 2

The number of words which begin with PRISNO =$^1P_1$=1

The next word is PRISON =1

$\therefore$ Rank of the word = 438

17. **If the letters of the word 'STREAM' are arranged in all possible ways and the words thus formed are arranged as in a dictionary. Find the word whose rank is 257.**

Sol. The order of the letters of the word 'STREAM' is A, E, M, R, S, T.

We know 257 = $\underline{|5} + \underline{|5} + \underline{|3} + \underline{|3} + \underline{|2} + \underline{|2} + \underline{|1}$

The first $\underline{|5}$ words begin with A

Next $\underline{|5}$ words begin with E

Next $\underline{|3}$ words begin with MAE

Next $\underline{|3}$ words begin with MAR

Next $\underline{|2}$ words begin with MASE

Next $\underline{|2}$ words begin with MASR

The next word is MASTER

T he word with rank 257 in 'MASTER'

2nd method : No.of words start with A = 5!

No.of words start with E = 5! ($\rightarrow$ 240)

No.of words start with MAE = 3! ($\rightarrow$ 246)

No.of words start with MAR = 3! ($\rightarrow$ 252)

No.of words start with MASE = 2! ($\rightarrow$ 254)

No.of words start with MASR = 2! ($\rightarrow$ 256)

Next word MASTER $\rightarrow$ (257)

18. **If there are 30 railway stations on a railway line, how many types of single second class tickets must be printed so as to enable a passenger to travel from one station to another.**

Sol. Let the stations be $S_1, S_2, S_3, \ldots\ldots, S_{30}$;

The Railway department has to print tickets from station S_i to another station S_j and from S_j to S_i (return journey) where $1 \le i \ne j \le 30$. i.e., we have to select 2 of these 30 and then we have to arrange them in order.

$\therefore$ Total number of tickets = number of permutations of 30 different things taken 2 at a time = $^{30}P_2 = 30 \times 29 = 870$

EXERCISE - 2.1

1. i) If $^{n}P_3 = 1320$ then find n. [Ans : 12]

   ii) If $^{n}P_7 = 42\,^{n}P_5$, then find n. **(March-17, 18) [Ans : 12]

   iii) If $^{18}P_{r-1} : {}^{17}P_{r-1} = 9 : 7$ then find r. [Ans : 5]

   iv) If $^{(n+1)}P_5 : {}^{n}P_6 = 2 : 7$, find n [Ans : 11]

   v) If $^{12}P_5 + 5 \cdot {}^{12}P_4 = {}^{13}P_r$, find r [Ans : 5]

   vi) If $^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$, find r [Ans : 41]

2. i) In a class there are 30 students on the new year day, every student posts a greeting card to all his/her classmates. Find the number of greeting cards posted by them. [Ans : 870]

   ii) If there are 25 railway stations on a railway line, how many types of single second class tickets must be printed, so as to enable a passenger to travel from one station to another. [Ans : $^{25}P_2$]

3. Find the number of 4-letter words that can be formed using the letters of the word MIRACLE. How many of them (i) Begin with an vowel (ii) Begin and end with vowels (iii) End with a consonant [Ans : (i) 360 (ii) 120 (iii) 480]

4. Find the number of 5 letter words that can be formed using the letters of the word 'CONSIDER'
   How many of them begin with 'C'
   How many of them end with 'R' and
   How many of them begin with 'C' and end with 'R' [Ans : 6720, 840, 840, 120]

5. Find the number of ways of arranging the letters of the word "TRIANGLE" so that the relative positions of the vowels and consonants are not disturbed [Ans : 720]

6. Find the number of ways of permuting the letters of the word PICTURE so that
   (i) All vowels come together (ii) No two vowels come together (iii) The relative positions of vowels and consonants are not disturbed [Ans : (i) 720 (ii) 1440 (iii) 144]

7. Find the number of ways arranging the letters of the word MONDAY so that no vowel occupies even place [Ans : 144]

8. In how many ways 6 boys and 6 girls can be arranged along a row so that (i) all the girls come together (ii) no two girls come together (iii) no two of the same sex come together
   [Ans : (i) $\underline{|7} \times \underline{|6}$ (ii) $7 \times \underline{|6} \times \underline{|6}$ (iii) $2 \times \underline{|6} \times \underline{|6}$]

9. Find the number of ways in which 5 red balls, 4 black balls of different sizes can be arranged in a row so that (i) no two balls of the same colour come together (ii) the balls of the same colour come together. [Ans : (i) 4!.5! (ii) 2!5!4!]

10. In how many ways 10 persons $A_1, A_2, A_3, A_4, \ldots\ldots, A_{10}$ can be seated along a row such that

    i) A_1, A_2, A_3 sit together [Ans : $\underline{|8} \times \underline{|3}$]

    ii) A_1, A_2, A_3 sit in a specified order need not be together [Ans : $^{10}P_7$]

    iii) A_1, A_2, A_3 sit together in a specified order [Ans : $\underline{|8}$]

    iv) A_2, A_3, A_4 sit always after A_1 [Ans : $^{10}P_6 \times \underline{|3}$]

11. There are 9 objects and 9 boxes. Out of 9 objects, 5 cannot fit in three small boxes. How many arrangements can be made such that each object can be put in the box only. [Ans : 17280]

12. i) How many numbers between 6000 and 10000 can be formed using the digits 2, 3, 4, 6, 7, 9 without repetition. **[Ans : 180]**

ii) How many numbers that can be prepared which are greater than 4000 using the digits 0, 2, 4, 6, 8 without repetition. **[Ans : 168]**

*13. The letters of the word 'MASTER' are permutated in all possible ways and the words thus formed are arranged as in dictionary. Find rank of words (i)'REMAST' '(ii) MASTER', (iii) EAMCET **(March-19, May-19)** **[Ans : 391, 257, 133]**

14. If the letters of the word 'BRING' are permutated in all possible ways and the words thus formed are arranged as in dictionary. Find the 59^{th} word. **[Ans : IGRBN]**

15. i) Find the sum of 4 - digited numbers that can be formed using the digits 1, 2, 4, 5, 6 without repetition. **[Ans : 479952]**

ii) Find the sum of 4 - digited numbers that can be formed using the digits 1, 3, 5, 7, 9 without repetition. **(May-18)**

16. Find the sum of 4 digited numbers that can be formed using the digits 0, 2, 4,7, 8 without repetition. **[Ans : 5459958]**

17. i) Find the number of 4 - digited numbers that can be formed using the digits 2, 4, 5, 6, 8. **[Ans : 120]**

ii) How many of them are divisible by

a) 2 **[Ans : 72]**

b) 3 **[Ans : 72]**

c) 4 **[Ans : 36]**

d) 5 **[Ans : 24]**

e) 25 **[Ans : 6]**

18. Find the number of 4 letter words that can be formed using the letters of the word MIXTURE which (i) Contain the letter X (ii) Do not contain the letter X **[Ans : (i) 480 (ii) 360]**

19. Find the number of 4 letter words that can be formed using the letters of the word 'EQUATION'. How many of them (i) begin with vowel (ii) begin and end with vowel. **[Ans : 1680, 1050, 600]**

20. Find the number of arrangements that can be made by using all the letters of the word 'EQUATION' so that the consonants may be in even places. **[Ans : 2880]**

21. Find the number of arrangements that can be formed by using all the letters of the word 'CONSIDER' so that relative positions of vowels and consonants remain unaltered. **[Ans : 720]**

22. Find the number of ways to arrange 10 students along a row by taking 4 at a time. In how many of these arrangements a specified student always occur. **[Ans : ${}^{10}P_4$, 4 . ${}^{9}P_3$]**

23. Find the number of 4-letter words that can be formed using the letters of the word 'ARTICLE' which
    a) contain the letter A [Ans : 480]
    b) do not contain E [Ans : 360]
    c) contain A but not E [Ans : 240]
    d) contain atleast one of A, E [Ans : 720]
24. In how many ways the 26 English letters can be arranged along a line so that all the 5 vowels must occur always after the letter 'B'. [Ans : ${}^{26}P_{20}\ \underline{|5}$]
25. If the letters of the word 'NORMAL' are arranged in all possible ways and the words thus formed are arranged as in dictionary. Then find the word whose rank is 455. [Ans : NORMAL]
26. If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the rank of the word PRISON. [Ans : 438]

2.6 PERMUTATIONS WHEN REPETITIONS ARE ALLOWED

In previous section, we have learnt the permutations of n distinct things taken $r(\leq n)$ at a time when repetition of things is not allowed.

Consider 4 distinct things A, B, C, D.

We know that we get ${}^4P_2$ permutations by taking 2 at a time from the above 4 letters, if repetition is not allowed.

Those are $AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD$ and DC

Observe that in each of those 12 permutations, each letter occurs only once.

If repetition is allowed, in addition to the above we get 4 more permutations which are AA, BB, CC, DD.

So, the number of permutations of n distinct things taken r at a time is not equal to ${}^nP_r$ if repetition is allowed.

Now we are going to learn about the number of permutations of n distinct things taken r at a time when repetition of things is allowed.

THEOREM-2.4 **The number of permutations of n distinct things taken r at a time when repetition of things is allowed any number of times is n^r.**

Proof : We prove this theorem by using Mathematical induction on r.

Here the required number of permutations is equal to the number of ways of filling up r blank places with the n given distinct things such that each blank place can be used only once and the things are allowed to use any number of times.

If $r = 1$, the number of ways of filling up one blank place with 'n' distinct objects $= n = n^1 = n^r$

$\therefore$ The theorem is true if $r = 1$

Assume that the theorem is true if $r = k - 1$

i.e., the number of ways of filling up $(k-1)$ blank places with n distinct things is n^{k-1} when repetition of things is allowed.

We are going to prove that the theorem is true if $r = k$.

Consider k blank places. The first blank can be filled in n ways. Now we are left with $(k-1)$ blanks and n distinct things (because the thing used in 1st blank can also be used)

By induction hypothesis these $(k-1)$ blank places can be filled with n distinct objects when repetition of things is allowed in n^{k-1} ways.

By fundamental principle,

The number of ways of filling up r blank places with n distinct things when repetition is allowed is $n \times n^{k-1} = n^k$. So, the theorem is true by induction.

Note

i) ***In the above theorem, repetition of things is allowed but we are filling each blank exactly with one thing.***

ii) ***If a blank can be filled with any number of things i.e., 0, 1, 2, ..., n then the number of ways to fill up r blanks with n distinct things (each thing is used once) is r^n.***

iii) ***While using n^r formula, take care to correctly identity n and r in a problem.***

iv) ***Better use fundamental principle in such cases.***

Corollary :

The number of permutations of n distinct things taken not more than r at a time when repetition of things is allowed is $\dfrac{n(n^r-1)}{n-1}$.

We know that

The number of permutations of n distinct things taken r at a time is n^r when repetition is allowed.

$\therefore$ The number of permutations of n distinct things taken not more than r at a time

$$= n^1 + n^2 + n^3 + \dots + n^r = \frac{n(n^r-1)}{n-1}$$

Note

The number of permutations of n things (distinct) taken r at a time when atleast one thing is repeated $= n^r - {}^nP_r$

THEOREM-2.5 **The number of functions from a set A with m elements to another set B with n elements is n^m.**

Proof : Let $A = \{a_1, a_2, \dots, a_m\}$; $B = \{b_1, b_2, \dots, b_n\}$

To define the image of a_1, we have n choices. (i.e., $b_1, b_2, \dots\dots, $ or b_n)

Since distinct elements of A need not have distinct images, a_2 can also be mapped to any element of B i.e., in n ways.

Thus each element of A can be mapped to any element of B in n ways.

By fundamental principle,

The number of functions $= n \times n \times n \dots n$ (m times) $= n^m$.

THEOREM-2.6 **The number of one-one mappings (injections) from a set A with m elements to another set B with n elements where $n \geq m$ is ${}^nP_m$.**

Proof : Let $A = \{a_1, a_2, ..., a_m\}$; $B = \{b_1, b_2, ..., b_n\}$

A function is said to be one-one if distinct elements of A are mapped to distinct images of B.

To define the image of a_1, we have n choices. i.e., a_1 can be mapped to any element of B i.e., in n ways.

Since distinct elements of A must have distinct images, a_2 can be mapped in $(n-1)$ ways. Proceeding like this, a_m can be mapped in $n-(m-1)$ ways, i.e., $(n-m+1)$ ways.

According to fundamental principle

The number of injections = $n(n-1)(n-2).....(n-m+1) = {}^nP_m$

Note : ***The number of functions from a set A with m elements to another set B with n elements ($n \geq m$) which are not one-one is $n^m - {}^nP_m$. (i.e. number of many one functions)***

THEOREM-2.7 **The number of onto mappings from a set A with n elements onto another set B with 2 elements is $2^n - 2$.**

Proof : Let $B = \{x, y\}$

We know that the number of functions from A into B is 2^n.

A function is said to be onto

if range of function = $B = \{x, y\}$

The number of functions having range$\{x\}$=

The number of functions having Range $\{y\} = 1^n = 1$

$\therefore$ The number of on-to functions from A onto $B = 2^n - \{1 + 1\} = 2^n - 2$

Polindrome :

Definition

A number or a word which reads the same either from left to right or from right to left is called a Palindrome.

Note : ***A number or a word which reads the same either from left to right or from right to left is called a Palindrome.***

Ex: MADAM; ROTOR, 210012 etc...

Note : The number of Palindromes with r distinct letters that can be formed from n distinct letters is

i) $n^{\frac{r}{2}}$ if r is even ii) $n^{\frac{r+1}{2}}$ if r is odd

Ex: Find the number of seven digited palindromes that can be formed using 0, 1, 2, 3, 4.

Sol. First place can be filled in 4 was (using only non-zero digits). Remaining three places can be filled in 5 ways each.

$\therefore$ Number of palindromes = 4×5^3.

SOLVED EXAMPLES

1. **How many 4 digited numbers can be formed using 0, 1, 2, 3, 4 when repetition is allowed.**

Sol. Consider 4 blank places.

X X X X

The first place i.e., 1000th place can be filled in 4 ways

Since zero can not be used in first place.

Since repetition is allowed the remaining 3 places each can be filled in 5 ways.

Therefore, by product rule,

$\therefore$ The number of 4 - digited numbers = 4×5^3.

2. Find the number of 4 letter words that can be formed using the letters of the word 'ARTICLE' in which atleast one letter is repeated.

Sol. We know that the number of 4 letter words that can be formed using 7 letters = 7^4 (if repetition is allowed)

These 7^4 words can be divided into two exclusive categeries

i) The 4 letter words without repetition

ii) The 4 letter words with atleast one repetition of a letter

The 4 letter words without repetition will be ${}^7P_4$.

$\therefore$ The 4 letter words with atleast one repetition of a letter = $7^4 - {}^7P_4$

***3. Find the number of 5 letter words that can be formed using the letters of the word 'MIXTURE' which begin with an vowel when repetitions are allowed.**

Sol. X X X X X

First place can be filled in 3 ways with an vowel from I, U, E

The remaining 4 places each can be filled in 7 ways since repetition is allowed.

$\therefore$ Required number of arrangements = 3×7^4

4. Find the number of 4 digited numbers that can be formed using the digits 0, 1, 2, 3, 4, 5 which are divisible by 6 when repetition of digits is allowed

Sol. Consider 4 blank places

X X X X

The extreme left place i.e. 1000^{th} place can be filled in 5 ways (since 0 can not be used)

Since repetition is allowed, the remaining 3 places each can be filled in 6 ways (either 0, 1, 2, 3, 4 or 5)

So, by fundamental principle, we can prepare $5 \times 6 \times 6 \times 6$ i.e., 5×6^3 numbers having 4 digits which may or may not be divisible by 6.

By fixing first 3 places with a_1, a_2, a_3 where $a_1 \neq 0$ and varying the unit place with 0, 1, 2, 3, 4, 5 we get the following six consecutive integers.

$a_1 \; a_2 \; a_3 \; 0$

$a_1 \; a_2 \; a_3 \; 1$

$a_1 \; a_2 \; a_3 \; 2$

$a_1 \; a_2 \; a_3 \; 3$

$a_1 \; a_2 \; a_3 \; 4$

$a_1 \; a_2 \; a_3 \; 5$

Among these six consecutive integers exactly one is divisible by 6.

$\therefore$ The number of 4 digited numbers when repetition is allowed which are divisible by $6 = \frac{1}{6}[5 \times 6 \times 6 \times 6] = 5 \times 6 \times 6 = 180$

5. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 that are divisible by 3 when repetition is allowed.

Sol. Since we are going to prepare four digited numbers and since repetition is allowed, we can fill up each place in 6 ways.

X X X X

So, we get 6^4 numbers having 4 digits using 1, 2, 3, 4, 5, 6.

Among these 6^4 numbers, some of them may not be divisible by 3.

By fixing first 3 places with some fixed digits from 1, 2, 3, 4, 5, 6 we get the following six consecutive numbers.

$a_1\ a_2\ a_3\ 1$
$a_1\ a_2\ a_3\ 2$
$a_1\ a_2\ a_3\ 3$
$a_1\ a_2\ a_3\ 4$
$a_1\ a_2\ a_3\ 5$
$a_1\ a_2\ a_3\ 6$

We know that out of any six consecutive integers exactly two are divisible by 3. Therefore one third of these 6^4 numbers will be divisible by 3.

$\therefore$ The number of 4-digited numbers which are divisible by 3

$= \frac{1}{3}6^4 = 2\times 6^3 = 432$.

6. Find the number of Palindromes with 6 digits that can be formed using the digits 1). 0, 2, 4, 6, 8 2). 1, 3, 5, 7, 9

Sol. 1) $\bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{5}\ \bar{6}$ 4,5,6 Positions will have the same digits as 3,2,1 respectively

'1' can be filled in 4 ways {excluding 0}
'2' can be filled in 5 ways {excluding 0}
'3' can be filled in 5 ways {excluding 0}
number of Palindromes = $4.5.5 = 4.5^2$

2) $\bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{5}\ \bar{6}$

'1' can be filled in 5 ways
'2' can be filled in 5 ways
'3' can be filled in 5 ways
number of Palindromes = 5^3

EXERCISE - 2.2

1. i) Find the number of permutations of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 when repetition is allowed. [Ans : 1296]

   ii) Find the number of ways of arranging r things in a line using the given n different things in which atleast one letter is repeated. [Ans : $n^r - {}^nP_r$]

   iii) Find the number of 5 letter words that can be formed using the letters of the word NATURE that begin with N when repetition is allowed. [Ans : 1296]

   iv) Find the number of bijections from a set A containing 7 elements onto itself. [Ans : 7!]

   v) Find the number of functions from a set A containing 5 elements into a set B containg 4 elements. [Ans : ${}^5P_4$]

2. i) A number lock has 3 rings and each ring has 9 digits 1, 2, 3, ..., 9. Find the maximum number of unsuccessful attempts that can be made by a person who tries to open the lock without knowing the key code. [Ans : 728]

   ii) Find the number of 5 letters words that can be formed using the letters of the word "RHYME" if each letter can be used any number of times. [Ans : 3125]

   iii) Find the number of 5 letter words that can be formed using the letters of the word 'NATURE' that begin with N when repetition is allowed. [Ans : 1296]

3. i) 9 different letters of an alphabet are given. Find the number of 4 letter words that can be formed using these 9 letters which have (a) no letter is repeated (b) atleast one letter is repeated. **[Ans :** 3024, 3537]

ii) Find the number of 4-digit numbers which can be formed using the digits 0, 2, 5, 7, 8 that are divisible by (a) 2 (b) 4 when repetition is allowed **[Ans :** 300, 160]

iii) Find the number of 4-digit numbers that can be formed using the digits 0, 1, 2, 3, 4, 5 which are divisible by 6 when repetition of the digits is allowed. **[Ans :** 180]

4. i) How many numbers less than 2000 that can be formed using the digits 1,2 ,3, 4 if repetition is allowed. ii) Find the number of 5 digited numbers which are divisible by 5 that can be formed using the digits 0, 1,2, 3, 4, 5 when repetition is allowed. **[Ans :** (i) 148 (ii) 2160]

5. Find the number of (i) 6 (ii) 7 letter Palindromes that can be formed using the letters of the word EQUATION. **[Ans :** (i) 8^3 (ii) 8^4]

2.7 CIRCULAR PERMUTATIONS

Upto now we have learnt linear permutations of different things with or without repetition of objects.

Now we are going to learn the arrangement of objects around a circle which are called circular permutations.

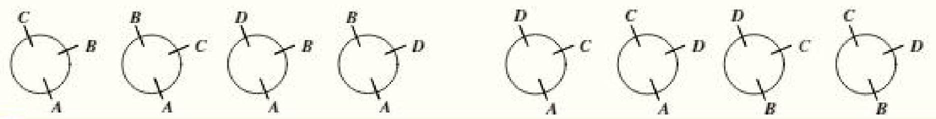
Note :

In a circular permutation where the first element is placed is not important but how the remaining elements are arranged relative to that element is important.

Definition

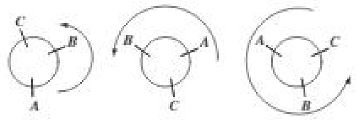
From a given finite set of things choose some or all of them and arrange them around a circle. Each such arrangement is called a circular permutation.

Example : Consider 4 distinct objects, *A, B, C, D*. By selecting 3 of them and arranging around a circle we get 8 circular permutations which are



Note

In circular permutations there is no beginning and there is no ending i.e., any element of the circular arrangement can be treated as 1st element. But the important thing is how the other elements are arranged relative to the first element.



Note that the above three circular arrangements are identical.

Finally we conclude that in circular permutations, which thing we are going to arrange at first and at which place the first thing is to be arranged is not important. But how we are going to arrange the remaining elements relative to the element which was already arranged at first whatever it may be is important.

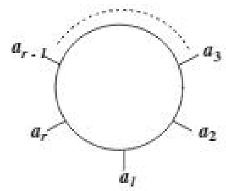
In the above example, we are getting 8 circular permutations, by arranging after selection of 3 at a time from 4 distinct things. But we know that the number of linear permutations of 4 distinct things taken 3 at a time is $^4P_3=24$. From above discussion, we can observe that the number of linear permutations of 4 things taken 3 at a time is 3 times to the number of circular permutations of 4 things taken 3 at a time.

So, we are interested to find the ratio between the number of arrangements of these two types.

THEOREM-2.8 **The number of circular permutations of n distinct things taken r at a time $= \frac{1}{r}$ (The number of linear permutations of n distinct things taken r at a time) $= \frac{^nP_r}{r}$ where $1 \le r \le n$.**

Proof : Let the n distinct things be $a_1, a_2, a_3, \ldots, a_n$

Consider one of the circular permutations of n things taken r at a time.



Starting with any one of these r elements, the above circular permutation gives rise to r linear permutations as shown below.

$a_1\ a_2\ a_3 \ldots a_{r-1}\ a_r$

$a_2\ a_3\ a_n \ldots a_{r-1}\ a_r\ a_1$

$a_3\ a_4\ a_5 \ldots a_r\ a_1\ a_2$

....

....

$a_r\ a_1\ a_2\ a_3 \ldots a_{r-1}$

Thus one circular permutation gives rise to r linear permutations.

Therefore, corresponding to each circular permutation, we can generate r linear permutations.

Therefore, the number of linear permutations of n things taken 'r' at a time $= r \times$ number of circular permutations of n things taken r at a time.

$\therefore$ Number of circular permutations of n distinct things taken r at a time $= \frac{^nP_r}{r}$

THEOREM-2.9 **The number of circular permutations of n different things taken all at a time is $\lfloor n-1$.**

Proof : **Method - 1**

In circular permutation, there is no beginning and there is no ending.

So, which thing at first we are going to arrange and at which place we are arranging the first thing is not important.

> ***Note :*** *In circular permutation, there is no beginning and there is no ending.*

So, put any one of the n given things in any one of the n places.

Now we have to arrange the remaining $(n-1)$ things relative to the first thing which was already arranged.

It can be done in $\lfloor n-1$ ways.

Therefore the number of circular permutations of n things taken all at a time $= \lfloor n-1$

Method - 2

By above result,

The number of circular arrangements of n distinct things taken r at a time $= \frac{^nP_r}{r}$ where $1 \le r \le n$.

By putting $r = n$,

The number of circular permutations of n distinct things taken all at a time $= \frac{^nP_n}{n} = \frac{\lfloor n}{n} = \lfloor n-1$

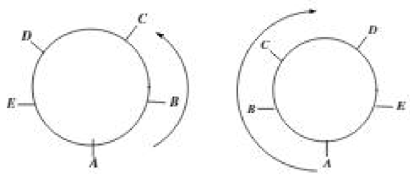
Note

Consider 5 different coloured beads A, B, C, D, E. Which we are going to arrange around a necklace.

These 5 beads can be arranged along a circle by taking all at a time in $\lfloor 4 = 24$ ways. i.e., we get 24 types of necklaces with the 5 different coloured beads.

Note that necklace arrangement is hanging type arrangement.

Let us consider,



The above two circular permutations are same but for the direction. In general, the direction is also important. So, we regard them as two different circular permutations. But here we are arranging beads. Since these arrangements are hanging type arrangements, if the first arrangement is turned over, we obtain the second arrangement and vice versa. So, we consider the above two circular arrangements as identical even though they are in different directions.

The number of circular permutations of n things taken r at a time in case of hanging type $= \frac{1}{2}\frac{^nP_r}{r}$

(Since we consider only one direction)

Note :

In hanging type arrangements clock-wise and anti-clockwise arrangements are similar.

SOLVED EXAMPLES

1. Find the number of ways to arrange 8 persons around a circle by taking 4 at a time.

Sol. We know, number of circular permutations of n distinct things taken r at a time $= {}^nP_r / r$

Therefore the required number of arrangements $= {}^8P_4 / 4$

2. Find the number of ways to arrange 8 persons around circular table if

i) two specified persons wish to sit together

ii) never sit together

Sol. i) Since, in circular permutations, the person whom we are going to arrange at first and at which place we are arranging him is not important.

Let the specified persons be A, B

So, arrange A in any place.

Since A, B must sit together, B can be arranged in 2 ways i.e., in the two adjacent places on either side of A

It can be done in 2 ways.

The remaining 6 persons can be arranged in $\lfloor 6$ ways.

Therefore by product rule,

The number of arrangements $= 2\lfloor 6$.

ii) First arrange A in any place. Since A, B do not come together, deleting the two places on either side of A, now B can be arranged in 5 ways. The remaining six persons can be arranged in $\lfloor 6$ ways.

The required number of arrangements $= 5\lfloor 6$

Note : *When 'n' elements are arranged in a circle there are 'n' gaps available for any subsequent arrange-ment in that circle.*

3. A family consists of father, mother, 2 daughters and 2 sons. In how many different ways can they sit at a round table if 2 daughters wish to sit on either side of the father.

Sol. First arrange father in any place.

Since the 2 daughters wish to sit on either side of the father, they can be arranged in 2 places in ${}^{2}P_{2} = 2$ ways.

The remaining 3 persons (mother and 2 sons) can be arranged in $\lfloor 3$ ways.

So, by product rule,

The required number of arrangements $= 2\lfloor 3 = 12$

4. Find the number of ways of arranging 6 red roses and 3 yellow roses of different sizes into a garland. In how many of them all the yellow roses come together.

Sol. The nine different roses can be arranged along a circle in $\lfloor 8$ ways.

Since, it is hanging type circular arrangement, we consider only one direction.

Therefore the number of garlands $= \lfloor 8 / 2$

Since all the 3 different yellow roses come together, consider them as a unit.

The number of roses are 6 + 1 unit. They can be arranged in $\lfloor 6$ ways. The yellow roses can be arranged among themselves in $\lfloor 3$ ways.

The number of garlands $= \dfrac{\lfloor 6 \times \lfloor 3}{2}$

Note : *When n people are arranged in a circle a particular person will have the same neighbourhood in clockwise and anti-clockwise arrange-ments.*

5. In how many ways 20 different coloured flowers can be arranged into a garland by taking 10 at a time so that 2 specified colours must occur in the garland but not come together.

Sol. Let the two specified flowers be A, B

Now fix A at any place of the 10 places along the circle. Since A, B do not come together, we can arrange B only in 7 places. It can be done in 7 ways. Now the remaining 8 places can be filled with remaining 18 flowers by selecting 8 flowers from 18 flowers.

This can be done in ${}^{18}P_{8}$ ways.

So, by product rule,

The number of hanging type garlands $= 7 \times {}^{18}P_{8} / 2$

EXERCISE - 2.3

1. i) Find the number of ways of arranging 7 persons around a circle. **[Ans : 720]**
   ii) Find the number of ways of arranging 5 boys and 5 girls around a circle. **[Ans : $\lfloor 9$]**
   iii) Find the number of necklaces that can be prepared using 6 different coloured beads. **[Ans : 60]**

2. i) Find the number of ways of arranging 4 boys and 3 girls around a circle so that all the girls sit together. **[Ans : 144]**
   ii) Find the number of ways of arranging 7 gents and 4 ladies around a circular table if no two ladies wish to sit together. **[Ans : $6! \times {}^7P_4$]**
   iii) Find the number of ways of arranging 7 guests and a host around a circle if 2 particular guests wish to sit on either side of the host. **[Ans : 240]**
   iv) Find the number of ways of preparing a garland with 3 yellow, 4 white and 2 red roses of different sizes such that the two red roses come together. **[Ans : 5040]**

3. i) Find the number of ways in which 8 men and 4 ladies can sit around a round table so that
      a) no two ladies come together **[Ans : $7! \times {}^8P_4$]**
      b) all the ladies come together **[Ans : $8! \times 4!$]**
   ii) Find the number of ways of arranging 6 boys and 6 girls around a circular table so that a) all the girls sit together b) no two girls sit together c) boys and girls sit alternately **[Ans : $6! \times 6!$; $5! \times 6!$; $5! \times 6!$]**
   iii) Find the number of ways in which 6 red roses and 3 white roses of different sizes can be made out to form a garland so that
      a) no two white roses come together **[Ans : 7200]**
      b) all the white roses come together **[Ans : 2160]**

4. Find the number of ways of arranging the chief minister and 10 cabinet ministers at a circular table so that the chief minister always sit in a particular seat.

5. In how many ways 10 persons can be arranged around a circle by taking 4 of them at a time. **[Ans : ${}^{10}P_4 / 4$]**

6. Find the number of garlands that can be made using 6 different coloured flowers taken 4 at a time. **[Ans : 45]**

7. Find the number of ways of sating 5 Indians, 4 Americans and 3 Russians at a round table so that
   i) All Indians sit together **[Ans : 7! 5!]**
   ii) No two russians sit together **[Ans : 8! 9!]**
   iii) Persons of same nationality sit together. **[Ans : $2! \times 3! \times 4! \times 5!$]**

8. A family consists of father, mother, 2 daughters and 2 sons. In how many different ways can they sit at a round table if the 2 daughters wish to sit on either side of the father ? **[Ans : 12]**

2.8 PERMUTATIONS OF THINGS OF WHICH SOME ARE ALIKE AND THE REST ARE DIFFERENT

Up to now, we have learnt the permutations of distinct things along a row or circle with or without repetition.

Now we are going to study the permutations of things which are not all distinct.

Consider the three letters *ABC*.

If we arrange all these 3 letters, by taking all at a time, we get 6 permutations.

Consider the three letters *AAB*.

Eventhough these are also 3 letters, If we arrange these 3 letters by taking all at a time, we do not get 6 permutations. We just get only 3 permutations. Those are *AAB*, *ABA*, *BAA*. This is because all the letters of the word *AAB* are not different. If *A*, *A* are different then for a fixed position of *B* we get 2 permutations by arranging *A*, *A* among themselves. But since *A*, *A* are similar, It does not happen so.

Now we are going to find the number of permutations of things when they are not all distinct.

THEOREM-2.10 **The number of permutations of n things of which p are alike and the rest are different by taking all at a time is $\dfrac{\lfloor \underline{n}}{\lfloor \underline{p}}$.**

Proof : Let N be the number of permutations of n things of which p are alike and the rest are different by taking all at a time.

Let us take one permutation from these N permutations. This permutation contains p alike things and $(n - p)$ distinct things. If we replace these p like things with p dissimilar things and arranging these p dissimilar things among themselves without disturbing the positions of other $(n - p)$ things, we get $\lfloor \underline{p}$ arrangements.

In otherwords, one permutation from the above N permutations gives rise to $\lfloor \underline{p}$ permutations if the p things are also different.

Thus to each permutation of N we get $\lfloor \underline{p}$ permutations if all the 'n' things are distinct.

But we know that the number of permutations of n distinct things taken all at a time is $\lfloor \underline{n}$

Therefore $N \times \lfloor \underline{p} = \lfloor \underline{n}$

$$\therefore N = \frac{\lfloor \underline{n}}{\lfloor \underline{p}}$$

We can extend the result if the given things have more than one set of like things.

THEOREM-2.11 **The number of permutations of n things taken all at a time of which p are alike of one kind, q are alike of second kind, l are alike of 3rd kind and the rest are different is $\dfrac{\lfloor \underline{n}}{\lfloor \underline{p}\,\lfloor \underline{q}\,\lfloor \underline{l}}$.**

SOLVED EXAMPLES

***1. Find the number of arrangements by arranging all the letters of the word 'BANANA'.**

Sol. BANANA contains 3A's, 2N's and one B.

By above theorem.

The number of arrangements by taking all at a time

$$= \frac{\lfloor 6}{\lfloor 3 \lfloor 2} = \frac{720}{12} = 60$$

2. Find the number of ways of arranging the letters of the word $a^4 b^3 c^5$ in expanded form.

Sol. $a^4 b^3 c^5$ contains 12 letters of which 4 are alike of one kind, 3 are alike of second kind and 5 are alike of 3rd kind.

So, they can be arranged in $\frac{\lfloor 12}{\lfloor 4 \lfloor 3 \lfloor 5}$ ways

***3. Find the number of ways of arranging the letters of the word 'SHIPPING' such that**

i) 2 P's will come together

ii) 2 I's do not come together

Sol. i) The word 'SHIPPING' contains 2 P's, 2I's and S, H, N, G one each.

Since P's must come together, consider 2 P's as a unit.

No it will have 7 letters of which there are 2I's.

They can be arranged in $\frac{\lfloor 7}{\lfloor 2}$ ways.

We need not rearrange the 2P's among them selves because they are alike.

$\therefore$ required number of arrangements $= \frac{\lfloor 7}{\lfloor 2}$

ii) Since the 2I's do not come together, arrange the remaining letters at first.

It can be done in $\frac{\lfloor 6}{\lfloor 2}$ ways. Because there are 2P's among the remaining six.

Now we get 7 gaps in which 2I's can be arranged.

It can be done ${}^7P_2/\lfloor 2$

Total number of arrangements $= \frac{\lfloor 6}{\lfloor 2} \times \frac{{}^7P_2}{\lfloor 2} = 180 \times 42 = 7560$

> **Note :** *When certain number of elements are to be together consider them as one unit.*

***4. Find the number of ways of arranging the letters of the word 'BRINGING' so that they begin and end with I.**

Sol. The word contains 2I's, 2G's, 2N's and B, R

Fix I at first and at last. Arrange the remaining 6 letters between these 2I's.

It can be done in $\frac{\lfloor 6}{\lfloor 2 \lfloor 2} = \frac{720}{4} = 180$ ways.

5. How many numbers can be formed using all the digits 1, 2, 3, 4, 3, 2, 1 such that even digits always occupy even places.

Sol. Let us consider 7 places

X $\boxed{X}$ X $\boxed{X}$ X $\boxed{X}$ X

Among these 7 places there are 3 even places in which we have to arrange even digits 2, 4, 2.

They can be arranged in $\frac{\lfloor 3}{\lfloor 2}$ ways i.e., 3 ways.

The 4 odd digits 1, 3, 3, 1 can be arranged in the remaining 4 places.

It can be done in $\frac{\lfloor 4}{\lfloor 2 \lfloor 2}$ ways i.e. 6 ways.

By product rule, required number of arrangements $= 3 \times 6 = 18$

6. A book store has *m* copies each of *n* different books. Find the number of ways of arranging these books in a shelf.

Sol. In the book store there are 'mn' books and n sets where each set contains m alike books.

They can be arranged in $\frac{\lfloor mn}{\lfloor m \lfloor m \lfloor m ..... \lfloor m (n \text{ times})} = \frac{\lfloor mn}{(\lfloor m)^n}$ ways

***7. Find the number of 4 letter words that can be formed using the letters of the word 'RAMANA'.**

Sol. The word has 6 letters of which 3 are alike and the rest are different

The letters are R, M, N, A, A, A

The number of 4 letter words which are all different $= \lfloor 4 = 24$

The number of 4 letter words of which 3 are alike and one different i.e., using

AAAR, AAAM, AAAN $= \frac{\lfloor 4}{\lfloor 3} \times 3 = 12$

The number of 4 letter words of which 2 are alike and the rest are different

i.e., using AARM, AARN, AAMN $= \frac{\lfloor 4}{\lfloor 2} \times 3 = 36$

$\therefore$ Required number of arrangements $= 24 + 12 + 36 = 72$

***8. If the letters of the word 'AJANTA' are permutated in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of 'JANATA'.**

Sol. The order of letters is A, A, A, J, N, T

The number of words which begin with A $= \frac{\lfloor 5}{\lfloor 2} = 60$

The number of words which begin with JAA $= \lfloor 3 = 6$

The number of words which begin with JANAA = 1

The next word is JANATA

$\therefore$ Rank $= 60 + 6 + 1 + 1 = 68$

EXERCISE - 2.4

1. Find the number of ways of arranging all the letters of the word

**i) MATHEMATICS (March-18 & May-18) [Ans : $\frac{\lfloor 11}{\lfloor 2\,\lfloor 2\,\lfloor 2}$]

ii) INDEPENDENCE [Ans : $\frac{\lfloor 12}{\lfloor 4\,\lfloor 3\,\lfloor 2}$]

iii) COMBINATION [Ans : $\frac{\lfloor 11}{\lfloor 2\,\lfloor 2\,\lfloor 2}$]

*iv) SINGING (May-19) [Ans : $\frac{7!}{2!\,2!\,2!}$]

v) PERMUTATION [Ans : $\frac{11!}{2!}$]

*vi) INTERMEDIATE (March-19) [Ans : $\frac{12!}{3!\times 2!\times 2!}$]

2. Find the number of ways of arranging all the letters of the word '$a^4\,b^5\,c^3$' in expanded form. [Ans : $\frac{\lfloor 12}{\lfloor 4\,\lfloor 5\,\lfloor 3}$]

3. i) There are 5 copies each of 4 different books. Find the number of ways of arranging these books in a shelf. [Ans : $\frac{\lfloor 20}{(\lfloor 5)^4}$]

ii) Find the number of 7 digited numbers that can be formed using 2, 2, 2, 3, 3, 4, 4. [Ans : $\frac{\lfloor 7}{\lfloor 3\,\lfloor 2\,\lfloor 2}$]

4. i) Find the number of words of arranging the letters of the word 'MISSING' which do not begin with 'S'. [Ans : 900]

ii) Find the number of 5 digited numbers that can be formed using the digits 0,1,1, 2, 3. [Ans : 48]

iii) Find the number of 5 digited even numbers using the digits 1, 1, 2, 2, 4. [Ans : 18]

5. In how many ways the letters of the word 'ASSOCIATIONS' can be arranged so that

i) all S's come together [Ans : $\frac{\lfloor 10}{\lfloor 2\,\lfloor 2\,\lfloor 2}$]

ii) the 2A's do not come together [Ans : $\frac{\lfloor 10}{\lfloor 3\,\lfloor 2\,\lfloor 2}\times\frac{{}^{11}P_2}{2}$]

6. Find the number of ways of arranging the letters of the word SPECIFIC. In how many of them, a) the two C's come together b) the two I's do not come together [Ans : 10080, 2520]

7. Find the number of ways of arranging the letters of the word SINGING
   a) they begin and end with I
   b) the two G's come together
   c) relative positions of vowels and consonants are not disturbed [Ans : 30 ; 180 ; 30]

*8 The letters of the word 'EAMCET' are arranged in all possible ways and the words thus obtained are arranged as in dictionary. Find rank of 'EAMCET. **(March-18)** **[Ans : 133]**

9. Find the number of 5 digited numbers that can be formed using the digits 2, 2, 3, 3, 4. How many of them are greater than 30000. **[Ans : 30, 18]**

10. How many ways can the letters of the word 'BANANA' be arranged so that
   i) all A's come together **[Ans : 12]**
   ii) no two A's come together **[Ans : 12]**

11. In how many ways all the letters of the word 'ARRANGE' can be arranged so that the 2A's are separated by exactly 2 letters. **[Ans : 240]**

2.9 COMBINATIONS

Definition :

A selection that can be made by taking some or all of a finite set of things (objects) is called a combination.

Example : If we are asked to select any 3 from the four distinct things A, B, C, D then we get the following selections.

ABC, ABD, ACD, BCD

These 4 selections are called combinations of 4 things taken 3 at a time.

Combination is only selection where as permutation involve two steps (i) selection (ii) arrangement.

In combinations order of the things has no importance. i.e., we do not consider ABC, ACB as different combinations.

Whenever there is importance to the order in which the objects are placed, then it is a permutation and if there is no importance to the order in which the objects are placed but only selection is required then it is a combination.

In other words, in permutations we select and arrange the objects where as in combination we select only but need not arrange the selected objects.

Note : ***Every combination of selecting r elements from a given set A of n elements (distinct) is a r-subset of A.***

2.10 NOTATION

The number of combinations of n dissimilar things taken r at a time is denoted by

${}^nC_r$ or $c(n, r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$ where $0 \le r \le n$

THEOREM-2.12 $\quad {}^nC_r = \dfrac{\lfloor n}{\lfloor r \, \lfloor n-r}$

Proof : ${}^nC_r$ denotes the number of combinations of n dissimilar things taken r at a time.

Note :

$$^nC_r \begin{cases} \in N & ; n \geq r \\ 0 & ; n < r \end{cases}$$

Let us consider one of these ^nC_r combinations. This combination consists of r things in some order because order of things has no importance.

If we arrange these r things according to different order we get $\lfloor r$ permutations. Thus each combination gives rise to $\lfloor r$ permutations.

So, ^nC_r combinations give rise to $^nC_r \lfloor r$ permutations. But we know that the number of permutations of n things taken r at a time is ^nP_r

$$\therefore\ ^nP_r = {}^nC_r \lfloor r$$

$$\therefore\ ^nC_r = \frac{^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor n-r \lfloor r}$$

Note

i) $\quad ^nC_r \times \lfloor r = {}^nP_r$

ii) $\quad ^nC_r = \dfrac{^nP_r}{\lfloor r} = \dfrac{n(n-1)(n-2)......(n-r+1)}{1 \cdot 2 \cdot 3 \cdot .... \cdot r}$

For eg., $^9C_2 = \dfrac{9 \times 8}{1 \times 2}$

$^7C_4 = \dfrac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4}$

iii) $\quad \dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$ (*since* $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r} \times \dfrac{\lfloor r-1 \lfloor n-r+1}{\lfloor n} = \dfrac{n-r+1}{r}$)

iv) $\quad ^nC_r = \dfrac{n}{r} \cdot {}^{(n-1)}C_{r-1}$

Ex : $^nC_r = \dfrac{n}{r} \cdot {}^{(n-1)}C_{r-1}$

THEOREM-2.13

If n, r are non negative integers such that $0 \leq r \leq n$ then $^nC_r = {}^nC_{n-r}$.

Note : *Selection of r things out of n things is equalent to rejection of '$n-r$' things*

Proof : $^nC_{n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r} = \dfrac{\lfloor n}{\lfloor r \lfloor n-r} = {}^nC_r$

Example : 1) $^{25}C_{23} = {}^{25}C_2$

2) $^nC_{n-2} = {}^nC_{n-(n-2)} = {}^nC_2$

THEOREM-2.14

If $1 \leq r \leq n$; $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

Proof : **Method - 1**

$$^nC_{r-1} + {}^nC_r = \frac{\lfloor n}{\lfloor n-(r-1) \lfloor r-1} + \frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{\lfloor n}{\lfloor n-r+1 \lfloor r-1} + \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor n-r \lfloor r-1}\left\{\frac{1}{n-r+1} + \frac{1}{r}\right\} = \frac{\lfloor n}{\lfloor n-r \lfloor r-1}\left\{\frac{r+n-r+1}{r(n-r+1)}\right\}$$

$$= \frac{\lfloor n}{\lfloor n-r \lfloor r-1}\left\{\frac{n+1}{r(n-r+1)}\right\} = \frac{\lfloor n+1}{\lfloor n-r+1 \lfloor r} = \frac{\lfloor n+1}{\lfloor n+1-r \lfloor r} = {}^{n+1}C_r$$

Method - 2

Suppose $A_1, A_2, A_3, ..., A_n, A_{n+1}$ are $(n + 1)$ distinct things.

> **Note :**
> ${}^{n}C_r = \frac{n}{r}\cdot{}^{(n-1)}C_{(r-1)}$

If we select r things from these $(n + 1)$ distinct things we get ${}^{n+1}C_r$ combinations.

These ${}^{n+1}C_r$ combinations can be divided into two exclusive categories.

1) The combinations in which a particular thing say A_1 is included.
2) The combinations in which A_1 is excluded.

Let us count these two categories

a) The number of combinations in which A_1 is included : Select first A_1 and then we have to select $(r - 1)$ things from the remaining 'n' things. This can be done in ${}^{n}C_{r-1}$ ways.

$\therefore$ Number of combinations of $(n + 1)$ things taken r at a time in which A_1 is included. $= {}^{n}C_{r-1}$

b) The number of combinations in which A_1 is excluded : Delete A_1 from the given $(n + 1)$ things. Now we have to select r things from the remaining n things. It can be done in ${}^{n}C_r$ ways. These ${}^{n}C_r$ combinations do not contain A_1.

$\therefore$ Number of combinations of $(n + 1)$ things taken r at a time in which A_1 is excluded $= {}^{n}C_r$

$\therefore\ {}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$

THEOREM-2.15 **If $0 \le r, s \le n$ and ${}^{n}C_r = {}^{n}C_s$ then either $r = s$ or $r + s = n$.**

Proof : Before going to prove the theorem, let us prove the following lemma.

Lemma : If a, b are +ve real numbers and k is any +ve integer such that $a(a + 1)(a + 2) \ldots\ldots (a + k) = b(b + 1)(b + 2) \ldots\ldots (b + k)$ then $a = b$.

Proof : Given $a(a + 1)(a + 2) \ldots\ldots (a + k) = b(b + 1) \ldots\ldots (b + k)$ we have to prove $a = b$.

suppose $a \neq b$

Without loss of generality,

let us assume $a < b$

Since $a < b$ we have $(a + i) < b + i$

This is true for all i $(0 \le i \le k)$

Therefore $a(a + 1)(a + 2) \ldots\, a + k) < b(b + 1) \ldots (b + k)$

Which is a contradiction.

$\therefore\ a \neq b$ is wrong assumption. $\quad \therefore\ a = b$

Proof of Main Theorem :

Given ${}^{n}C_r = {}^{n}C_s$

We have to show $r = s$ or $r + s = n$.

If $r = s$ then there is nothing to prove. Suppose $r \neq s$

Without loss of generality,

Let us assume $r < s$

since $r < s \Rightarrow -r > -s \Rightarrow n - r > n - s$

$${}^{n}C_r = {}^{n}C_s \Rightarrow \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}\,\lfloor\underline{r}} = \frac{\lfloor\underline{n}}{\lfloor\underline{n-s}\,\lfloor\underline{s}}$$

$$\Rightarrow \lfloor\underline{n-r}\,\lfloor\underline{r} = \lfloor\underline{n-s}\,\lfloor\underline{s}$$

$\Rightarrow (n-s+1)(n-s+2) \ldots (n-r) = (r+1)(r+2) \ldots s$ $[\because r<s \text{ and } n-r > n-s]$

By above lemma,

Since LHS and RHS of above are both products of $(s-r)$ consecutive integers,

we have $n-s+1 = r+1 \Rightarrow n-s = r \Rightarrow n = r+s$

Hence the theorem is proved

2.11 DISTRIBUTION OF DISSIMILAR THINGS INTO GROUPS

THEOREM-2.16 **The number of ways in which $(m+n)$ dissimilar things $(m \neq n)$ can be divided into two different groups of m and n things respectively is $\dfrac{\lfloor m+n}{\lfloor m \lfloor n}$.**

Proof : Whenever we select m things from $(m+n)$ distinct things, automatically we are left with n things and hence two groups containing m things in one group and n things in another group will be formed.

Therefore,

The number of ways to divide $(m+n)$ things into two groups containing m things and n things.

$= $ The number of ways to select m things from $(m+n)$ things $= {}^{m+n}C_m = \dfrac{\lfloor m+n}{\lfloor m \lfloor n}$

Corollary :

If m, n, p are +ve integers, The number of ways to divide $(m+n+p)$ things into 3 groups containing m things, n things and p things is $\dfrac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}$ $(m \neq n \neq p)$.

Proof : First we select m things from $(m+n+p)$ things to form one group containing m things.

It can be done in ${}^{(m+n+p)}C_m$ ways.

From the remaining $(n+p)$ things we select n things to form another group of n things. It can be done in ${}^{n+p}C_n$ ways.

Now we are left with p things from which we can form another group.

By product rule, the required number of ways

$$= {}^{m+n+p}C_m \times {}^{(n+p)}C_n \times 1 = \frac{\lfloor m+n+p}{\lfloor m \lfloor n+p} \times \frac{\lfloor n+p}{\lfloor n \lfloor p} \times 1 = \frac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}$$

Corollary :

The number of ways in which $2n$ dissimilar things can be divided into two equal groups of n things in each is $\dfrac{\lfloor 2n}{\lfloor n \lfloor n \lfloor 2}$.

Proof : We can divide '$2n$' dissimilar things into two groups containing n things in each in ${}^{2n}C_n$ ways.

Consider one division of these ${}^{2n}C_n$ divisions. This division contains two distinct groups containing n things in each.

Since the number of elements in these two groups are equal by interchanging these two groups among themselves we get another division which is also one among the ${}^{2n}C_n$ divisions.

The difference between these two divisions is only the order of the groups.

Since the order of the groups has no importance, we consider these two divisions as identical divisions and hence we count these divisions as a single.

So the actual number of ways to divide $2n$ distinct things into two groups of n things each $= \dfrac{1}{2}\ {}^{2n}C_n = \dfrac{1}{2}\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$

Corollary :

The number of ways of dividing 'mn' dissimilar things into m equal groups containing n things each is $\dfrac{\lfloor mn}{(\lfloor n)^m \lfloor m}$.

Corollary :

The number of ways to divide $2n$ distinct things among 2 persons equally is $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.

Proof : First we divide $2n$ things into 2 equal groups without giving importance to the order.

> ***Note :*** *At least one of 'n' dissimilar things can be selected in $2^n - 1$ ways*

It can be done in $\dfrac{\lfloor 2n}{\lfloor n \lfloor n \lfloor 2}$ ways.

These 2 different groups can be distributed among 2 persons in $\lfloor 2$ ways.

By fundamental principle,

The number of ways to divide $2n$ distinct things among 2 persons equally is $\dfrac{\lfloor 2n}{\lfloor n \lfloor n \lfloor 2} \times \lfloor 2$ is $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.

Corollary :

The number of ways in which mn distinct things can be divided among m persons equally is $\dfrac{\lfloor mn}{(\lfloor n)^m \lfloor m} \times \lfloor m$ i.e, $\dfrac{\lfloor mn}{(\lfloor n)^m}$.

> ***Note***
>
> *In Binomial Theorem, we shall learn the following results which are very useful in this chapter also.*
>
> *i)* *If n is +ve integer* $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$
>
> *ii)* *If n is any +ve integer, then* $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$ *and* $(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r\ {}^{n+r-1}C_r x^r$
>
> *iii)* ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

2.12 DISTRIBUTION OF IDENTICAL OBJECTS INTO SOME GROUPS

In previous section we have studied about the distribution of distinct objects into some groups.

Now we are going to learn about the distribution of n identical objects among r groups putting zero or more in a group.

Let us discuss different methods to divide identical objects into some groups.

For example, let us suppose the number of identical objects to be distributed is 10 and the number of groups be 4.

Suppose the number of objects placed in k^{th} group where $1 \le k \le 4$ is a_k; Here each a_k is a non - negative integer and $a_1 + a_2 + a_3 + a_4 = 10$.

Let us define a set A as $A = \left\{(a_1, a_2, a_3, a_4) \Big/ \begin{array}{l} each\ a_k \ge 0\ \forall\ 1 \le k \le 4 \\ and\ a_1 + a_2 + a_3 + a_4 = 10 \end{array}\right\}$

Here each element of A is a 4-tuple which represents a distribution of 10 identical objects into 4 groups putting zero or more in a group.

For example, the 4-tuple (6, 4, 0, 0) represents the distribution in which 6 objects are placed in 1st group, 4 objects are placed in 2^{nd} group and 0 objects are placed in the remaining 2 groups.

Therefore, the number of ways to distribute 10 identical objects among 4 groups putting 0 or more in a group = the number of distinct 4-tuples in set $A = n(A)$.

The concept of use of binomial theorem to find $n(A)$:

Now we are interested to find $n(A)$; for that let us define another set B as

$$B = \left\{ x^{a_1}\ x^{a_2}\ x^{a_3}\ x^{a_4} \Big/ \begin{array}{l} 0 \le a_1, a_2, a_3, a_4 \le 10 \\ \text{and } a_1 + a_2 + a_3 + a_4 = 10 \\ \text{where each } a_i \text{ is an integer} \end{array} \right\}$$

Note that each element of B appears exactly once in the expansion of $(x^0 + x^1 + x^2 + ... + x^{10})^4$ and number of elements in B is nothing but the coefficient of x^{10} in $(x^0 + x^1 + x^2 + ... + x^{10})^4$

(Since $x^{a_1}\ x^{a_2}\ x^{a_3}\ x^{a_4} = x^{a_1+a_2+a_3+a_4} = x^{10}$)

Clearly, there is one-to-one correspondence between the sets A and B. This is because corresponding to a 4-tuple (a_1, a_2, a_3, a_4) of A there is a term $x^{a_1}\ x^{a_2}\ x^{a_3}\ x^{a_4}$ in B and vice-versa.

So $n(A) = n(B)$ = the coefficient of x^{10} in$(x^0 + x^1 + ... + x^{10})^4$.

So, in general we conclude that the number of ways to divide n identical objects among r groups putting zero or more in a group = coefficient of x^n in $(x^0+x^1+....+ x^n)^r$.

The concept of use of binary sequences to find $n(A)$:

Suppose the number of identical objects placed in k^{th} group is a_k for $k = 1, 2, 3, 4$.

Here a_k is non negative integer and $a_1 + a_2 + a_3 + a_4 = 10$

This distribution can be associated with a binary sequence with two symbols 0, 1 which is of the form

$$\underbrace{\overbrace{0\ 0 ..... 0}^{1^{st}\text{ group}}}_{a_1 \text{ times}}\ 1\ \underbrace{\overbrace{0\ 0 ..... 0}^{2^{nd}\text{ group}}}_{a_2 \text{ times}}\ 1\ \underbrace{\overbrace{0\ 0 ..... 0}^{3^{rd}\text{ group}}}_{a_3 \text{ times}}\ 1\ \underbrace{\overbrace{0\ 0 ..... 0}^{4^{th}\text{ group}}}_{a_4 \text{ times}}$$

Where each $a_k \geq 0$ for $k = 1, 2, 3, 4$ and $a_1 + a_2 + a_3 + a_4 = 10$

In the above sequence the 10 identical objects are marked as ten '0's and the '0's are seperated into 4 groups with the help of 3 seperators which are marked as '1's. Note that to divide into 4 groups exactly 4 – 1 i.e., 3 seperators are needed.

For example, consider a distribution in which 9 objects are placed in 2nd group and 1 object is placed in 4th group and no object is placed in the remaining groups. This distribution is associated with

$$1 \quad \underbrace{0\ 0\ 0 \ldots\ldots 0}_{9 \text{ times}} \quad 1 \quad 1 \quad \underbrace{0}_{1 \text{ time}}$$

Suppose, C is the set of all binary sequences of ten '0's and three '1's.

Clearly, there is one-to-one correspondance between the sets A and C also.

So, $n(A) = n(C)$ = number of binary sequences with ten '0's and three '1's.

So, in general we conclude that the number of ways to divide n identical objects into r groups putting zero or more in a group = the number of binary sequences of n zeros and $(r - 1)$ ones (separators).

THEOREM-2.17 **The number of ways to distribute 'n' identical objects among 'r' groups putting zero or more in a group is ${}^{n+r-1}C_{r-1}$ where r is a positive integer and n is non-negative integer. (or)**

The number of non-negative integral solutions of $x_1 + x_2 + \ldots\ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$ where r is a positive integer and n is non-negative integer.

Proof : **Method - 1**

The number of ways to divide 'n' identical objects among 'r' groups putting zero or more in a group = the coefficient of x^n in $(x^0 + x^1 + \ldots\ldots + x^n)^r$

$$= \text{the coefficient of } x^n \text{ in } \left[\frac{x^0[1-x^{n+1}]}{1-x}\right]^r$$

$$= \text{the coefficient of } x^n \text{ in } (1 - x^{n+1})^r (1 - x)^{-r} = \text{the coefficient of } x^n \text{ in } (1 - x)^{-r}$$

$$= \text{the coefficient of } x^n \text{ in } \sum_{k=0}^{\infty} {}^{r+k-1}C_k\, x^k = {}^{r+n-1}C_n \text{ (by putting } k=n \text{ in } {}^{r+k-1}C_k)$$

$$= {}^{n+r-1}C_{r-1}$$

Method - 2

The number of ways to divide 'n' identical objects among 'r' groups putting zero or more in a group

= The number of binary sequences of n zeros and $(r - 1)$ ones (seperators) which are of the form

$$\overbrace{\underbrace{0\ 0 \ldots\ldots 0}_{a_1 \text{ times}}}^{1^{st} \text{ group}} \; 1 \; \overbrace{\underbrace{0\ 0 \ldots\ldots 0}_{a_2 \text{ times}}}^{2^{nd} \text{ group}} \; 1 \; \ldots\ldots 1 \; \overbrace{\underbrace{0\ 0 \ldots\ldots 0}_{a_r \text{ times}}}^{r^{th} \text{ group}}$$

Where $a_1 + a_2 + ... + a_r = n$ and each $a_i \geq 0 \ \forall\ 1 \leq i \leq r = \dfrac{\lfloor n+r-1}{\lfloor n \lfloor r-1} = {}^{n+r-1}C_{r-1}$

(since we know that the number of permutations of n things taken all at a time of which p are alike of one kind and q are alike of second kind and the rest are all different is $\dfrac{\lfloor n}{\lfloor p \lfloor q}$).

It is easy to observe that the number of ways to distribute n identical objects among r groups putting zero or more in a group is equal to the number of non-negative integral solutions of $x_1 + x_2 + ... + x_r = n$.

Therefore, the number of non-negative integral solutions of $x_1 + x_2 + ... + x_r = n$ is ${}^{n+r-1}C_{r-1}$ where r is a positive integer and n is non-negative integer.

THEOREM-2.18 **The number of ways to distribute 'n' identical objects among r groups putting atleast one object in each group is ${}^{n-1}C_{r-1}$ where $1 \leq r \leq n$. (or)**

The number of positive integral solutions of $x_1 + x_2 + ... + x_r = n$ is ${}^{n-1}C_{r-1}$ where n and r are positive integers such that $n \geq r$.

Proof : **Method - 1**

The number of ways to distribute 'n' identical objects among r groups putting atleast one object in each group

= the coefficeint of x^n in $(x^1 + x^2 + ...+x^n)^r$

= the coefficeint of x^n in $\left[\dfrac{x(1-x^n)}{1-x}\right]^r$ = the coefficeint of x^n in $\dfrac{x^r(1-x^n)^r}{(1-x)^r}$

= the coefficeint of x^{n-r} in $(1-x^n)^r\,(1-x)^{-r}$

= the coefficeint of x^{n-r} in $(1-x)^{-r}$

= the coefficeint of x^{n-r} in $\sum\limits_{k=0}^{\infty} {}^{r+k-1}C_k\, x^k$

$= {}^{r+(n-r)-1}C_{n-r} = {}^{n-1}C_{n-r} = {}^{n-1}C_{r-1}$

Method - 2

We have to distribute n identical objects among r groups putting atleast one in each group.

Here every distribution can be identified with a binary sequence of n zeros and $(r-1)$ ones (seperators) which are of the form

$$\overbrace{\underbrace{0\ 0 .....0}_{a_1 \text{ times}}}^{1^{st}\text{ group}}\ 1\ \overbrace{\underbrace{0\ 0 .....0}_{a_2 \text{ times}}}^{2^{nd}\text{ group}}\ 1\ ....1\ \overbrace{\underbrace{0\ 0 .....0}_{a_r \text{ times}}}^{r^{th}\text{ group}}$$

where $a_1 + a_2 + ... + a_r = n$ and each $a_i \geq 1 \ \forall\ 1 \leq i \leq r$

Since each group must contain atleast one object we have each $a_i \geq 1 \ \forall\ 1 \leq i \leq r$.

Since $a_1 \geq 1, a_r \geq 1$ the binary sequence must begin and must be ended with zeros only.

Since each $a_i \geq 1$, for $i = 2, 3, ... (r - 1)$, no two ones of the binary sequence occupy consecutive positions.

Further corresponding to every such binary sequence there exists a distribution of identical objects into r groups putting atleast one object in each group and vice-versa.

So, the required number of distributions

= the number of binary sequences of n zeros and $(r - 1)$ ones such that no two '1's are consecutive and the sequence must begin and end with zero only.

Let us count the number of such binary sequences. We first arrange n zeros as shown below.

0 X 0 X 0 X .... X 0 X 0

This can be done in one way. Now there are $n - 1$ places marked with a cross as shown above at which the $(r-1)$ ones can be arranged.

It can be done in ${}^{n-1}C_{r-1}$

Therefore, the number of ways to distribute 'n' identical objects among r groups putting atleast one object in each group = ${}^{n-1}C_{r-1}$.

But we know the number of ways to distribute 'n' identical objects among r groups putting atleast one object in each group = number of positive integral solutions of $x_1 + x_2 + ... + x_r = n$.

= ${}^{n-1}C_{r-1}$. Hence theorem proved.

2.13 SELECTION OF THINGS (ONE OR MORE) FROM GIVEN THINGS WHICH ARE NOT ALL DISTINCT

We know that the number of ways to select 2 things from 3 different given things A, B, C is ${}^3C_2$ i.e, 3.

Those selections are AB, AC, BC.

If we are asked the number of ways to select 2 things from 3 similar things $A\ A\ A$ then also saying ${}^3C_2$ is wrong. We get only one selection which is $A\ A$.

Note

i) The number of ways to select r things from n similar things (where $0 \leq r \leq n$) at a time is only one.

ii) The number of ways of selecting none or more from n similar things is (n + 1) for example, the number of ways to select none or more from 3 similar rupee coins is 4 which are

a) no coin b) one coin c) two coins d) 3 coins

THEOREM-2.19 **The total number of combinations of $(p + q)$ things taken any number at a time i.e, none or more when p things are alike of one kind and q things are alike of second kind is $(p + 1)(q + 1)$.**

Proof : From first kind of p things (alike) we can select 0 or 1 or 2..... or p things.

Since all these p things are alike, we can do this work in $(p + 1)$ ways.

Note : *'P' similar things can be dealt whithin $p + 1$ ways*

Similarly from second kind of q things (alike) we can select none or more in $(q+1)$ ways. By fundamental principle, the total number of selections is $(p + 1)(q + 1)$.

Corollary :

The total number of combinations (selections) of $(p + q)$ things taken one or more at a time when p things are alike of one kind and q things are alike of another kind is $(p + 1)(q + 1) - 1$.

Proof : By above theorem, the total number of selections from $(p + q)$ things taken none or more at a time where p things are alike of one kind and q things are alike of second kind is $(p + 1)(q + 1)$.

These combinations includes the selection in which nothing was selected.

$\therefore$ Required number of selections taken one or more at a time $= (p+1)(q+1)-1$.

Note : The above theorem can be extended to any finite number of operations.

THEOREM-2.20 **The number of ways of selecting one or more from n different things is $2^n - 1$.**

Proof : The n dissimilar things can be regarded as n different kinds where each kind is having one alike thing.

So, by above corollory,

The number of ways of selecting one or more from n different things =

$$\underbrace{(1+1)(1+1)(1+1)....(1+1)}_{\text{n times}}-1 = 2^n - 1$$

2.14 POSITIVE DIVISORS OF GIVEN POSITIVE INTEGER

We know that any positive integer can be expressed uniquely as product of primes.

With help of this fact we are going to find the number of positive divisors of a given +ve integer.

THEOREM-2.21 **If $p_1, p_2, p_3, ...\ p_k$ are distinct primes and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ....p_k^{\alpha_k}$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are positive integers then the number of positive divisors of n including one and n itself is $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) .....(\alpha_k + 1)$.**

Proof : Given $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ....p_k^{\alpha_k}$

where each $p_i\ (1 \le i \le k)$ is prime.

Any positive divisor of n is of the form $p_1^{\beta_1} p_2^{\beta_2} .....p_k^{\beta_k}$ where each β_i is an integer and $0 \le \beta_i \le \alpha_i$ for $1 \le i \le k$.

If any β_i exceeds α_i, then it can not be a divisor.

Here β_1 can be replaced either with 0 (or) 1 (or) 2, ... (or) α_1 to get a divisor. It can be done in $(\alpha_1 + 1)$ ways.

Similarly each β_i can be replaced either with 0 (or)1 (or) 2, ...(or) α_i for $1 \le i \le k$. It can be done in $(\alpha_i + 1)$ ways.

> *Note :* *1 and n are called trivial divisors or improper divisors.*

By fundamental principle

The number of positive divisors $= (\alpha_1 + 1)(\alpha_2 + 1).....(\alpha_k + 1)$

These divisors includes 1 and n itself. If we take $\beta_1 = \beta_2 = \beta_3 = ... = \beta_k = 0$ then we get 1 and if we take $\beta_i = \alpha_i$ for $1 \le i \le k$ then we get n.

Note

i) The positive divisors of n other than 1 and n itself are called proper divisors. Here 1 and n are called the trivial divisors or improper divisors.

ii) Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$ where each $p_i (1 \le i \le k)$ is a prime and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are +ve integers. The number of positive divisors of n which are divisible by p_1.

$= \alpha_1(1+\alpha_2)(1+\alpha_3) \ldots (1+\alpha_k)$ (by taking $\beta_1 = 1, 2, 3, \ldots, \alpha_1$ but not 0 in the proof of above theorem)

iii) The number of +ve divisors of n which are not divisible by p_1. $= (1+\alpha_2)(1+\alpha_3) \ldots\ldots (1+\alpha_k)$ (by taking $\beta_1 = 0$ only in the proof of above theorem)

2.15 SUM OF DIVISORS

If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ where p_1, p_2, p_3 are distinct primes and $\alpha_1, \alpha_2, \alpha_3$ and positive integers then sum of positive divisors of

$$n = (1 + p_1 + p_1^2 + \ldots.. + p_1^{\alpha_1}) \times (1 + p_2 + p_2^2 + \ldots.. + p_2^{\alpha_2}) \times (1 + p_3 + p_3^2 + \ldots\ldots + p_3^{\alpha_3})$$

$$= \frac{(p_1^{\alpha_1+1} - 1)(p_2^{\alpha_2+1} - 1)(p_3^{\alpha_3+1} - 1)}{(p_1 - 1)(p_2 - 1)(p_3 - 1)}$$

Explanation of the above formula with an example :

Consider positive integer 36.

Here $36 = 2^2 \times 3^2$.

So, it has (2 + 1) (2 + 1) i.e., 9 divisors.

Those are $2^0 3^0, 2^0 3^1, 2^0 3^2, 2^1 3^0, 2^1 3^1, 2^1 3^2, 2^2 3^0, 2^2 3^1, 2^2 3^2$

Sum of these divisors $= 2^0 3^0 + 2^0 3^1 + 2^0 3^2 + 2^1 3^0 + 2^1 3^1 + 2^1 3^2 + 2^2 3^0 + 2^2 3^1 + 2^2 3^2$

$= 2^0(3^0 + 3^1 + 3^2) + 2^1(3^0 + 3^1 + 3^2) + 2^2(3^0 + 3^1 + 3^2)$

$= (2^0 + 2^1 + 2^2)(3^0 + 3^1 + 3^2) = \left(\frac{2^3 - 1}{2 - 1}\right)\left(\frac{3^3 - 1}{3 - 1}\right) = 7 \times 13 = 91$

Note : *Number of positive divisors of a perfect square is always odd.*

Note

i) Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ where $\alpha_1, \alpha_2, \alpha_3$ are positive integers and p_1, p_2, p_3 are distinct primes.

If n is a perfect square then $\alpha_1, \alpha_2, \alpha_3$ must be even positive integers.

So, the number of positive divisors of $n = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) =$ odd integer.

So, the number positive divisors of a perfect square is always odd. This is necessary and sufficient condition.

ii) The number of ways in which $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ can be resolved as product of two positive factors (order of factors has no importance) is $\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) - 1] + 1$ if n is perfect square.

If n is not perfect square, it can be resolved as product of two factors (order of factors has no importance) is $\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)]$

Explanation :

Consider positive integer 36.

It is a perfect square. $36 = 2^2 \times 3^2$

It has nine positive divisors. It has odd number of positive divisors because 36 is a perfect square.

Note : *Number of positive divisors of a non-perfect square integer is always even.*

We have to find the number of different ways to express as $36 = a \times b$ (i.e., product of 2 factors).

First, let us find the number of ways in which $a \neq b$.

Since $a \neq b$ any divisor except 6 can be assigned to the variable a. It can be done in 8 ways. Now we can assign only one divisor $\frac{36}{k}$ (if $a = k$) to b. It can be done in one way.

By fundamental principle,

The number of ways to express $36 = a \times b$ is (8×1) i.e, 8. (where $a \neq b$)

But since the factors has no order, we can resolve 36 as product of 2 positive integers in 4 ways only.

In addition to these, $36 = 6 \times 6$ (i.e., $a = b$) is another way.

$\therefore$ The required number of ways $= 4 + 1 = 5$

By formula, the required number of divisors $= \frac{1}{2}[(p_1+1)(p_2+1)-1]+1$

$$= \frac{1}{2}[(2+1)(2+1)-1]+1 = 5$$

2.16 SELECTION OF SPECIFIED NUMBER OF THINGS FROM GIVEN THINGS WHICH ARE ALL NOT DISTINCT

Result : If there are l like objects of one kind and m like objects of second kind then the number of ways of selecting r objects out of these objects $(r \leq l+m)$ is the coefficient of x^r in the expansion of $(1 + x + x^2 + \ldots\ldots + x^l)(1 + x + x^2 + \ldots\ldots + x^m)$.

Result : If there are l like objects of one kind and m like of objects of second kind then the number of ways to select r objects from these $(l + m)$ objects such that atleast one object must be included from each kind $(2 \leq r \leq l+m)$ is the coefficient of x^r in $(x + x^2 + \ldots\ldots + x^l)(x + x^2 + \ldots\ldots + x^m)$.

Result : If there are l objects of one kind, m objects of second kind then the number of arrangmenets of r objects out of these $(l + m)$ objects is the coefficient of x^r in the expansion of $\underline{|r}\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots\ldots+\frac{x^l}{\underline{|l}}\right)\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots\ldots+\frac{x^m}{\underline{|m}}\right)$

Selection of committes

Result : If there are m distinct objects of one kind, l distinct objects of another second kind, n distinct objects of third kind then the number of ways to select r distinct objects from $(l + m + n)$ distinct objects where $r \leq (l+m+n)$ and including atleast one of each kind is the coefficient of x^r in the expansion of $({}^mC_1\, x + {}^mC_2\, x^2 + {}^mC_3\, x^3 + \ldots\ldots + {}^mC_m\, x^m)\ ({}^lC_1\, x + {}^lC_2\, x^2 + \ldots\ldots + {}^lC_l\, x^l)\ ({}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots\ldots + {}^nC_n\, x^n)$

Explanation :

Let us find the number of ways to select 11 members from 7 batsmen 5 bowlers and 3 wicket keepers with atleast 3 bowlers and 2 wicket keepers.

Method - 1

We can select 11 cricketers as explained below.

S. No.	No. of Batsmen	Bowlers	Wicket keepers	No. of teams
1	6	3	2	${}^7C_6 \times {}^5C_3 \times {}^3C_2 = 210$
2	5	4	2	${}^7C_5 \times {}^5C_4 \times {}^3C_2 = 315$
3	4	5	2	${}^7C_4 \times {}^5C_5 \times {}^3C_2 = 105$
4	5	3	3	${}^7C_5 \times {}^5C_3 \times {}^3C_3 = 210$
5	4	4	3	${}^7C_4 \times {}^5C_4 \times {}^3C_3 = 175$
6	3	5	3	${}^7C_3 \times {}^5C_5 \times {}^3C_3 = 35$
				1050

Total number of teams = 1050

Method - 2

We can do the same problem with help of formula.

The number of ways selecting cricket 11 with atleast 3 bowlers and 2 wicket keepers.

$= \text{coeff. of } x^{11} \text{ in } ({}^7C_0 + {}^7C_1 x + {}^7C_2 x^2 + ... + {}^7C_7 x^7)({}^5C_3 x^3 + {}^5C_4 x^4 + {}^5C_5 x^5)({}^3C_2 x^2 + {}^3C_3 x^3)$

$= \text{coeff. of } x^{11} \text{ in } (1 + {}^7C_1 x + ... + {}^7C_7 x^7)\, x^3 (10 + 5x + x^2) \cdot x^2 (3 + x)$

$= \text{coeff. of } x^6 \text{ in } (1 + {}^7C_1 x + {}^7C_2 x^2 + {}^7C_3 x^3 + {}^7C_4 x^4 + {}^7C_5 x^5 + {}^7C_6 x^6 + {}^7C_7 x^7).(10 + 5x + x^2)(3 + x)$

$= \text{coeff. of } x^6 \text{ in } (1 + {}^7C_1 x + {}^7C_2 x^2 + ... + {}^7C_6 x^6 + {}^7C_7 x^7).(30 + 25x + 8x^2 + x^3)$

$= 30\,{}^7C_6 + 25\,{}^7C_5 + 8\,{}^7C_4 + {}^7C_3 = 30(7) + 25(21) + 8(35) + 35$

$= 7\{30 + 75 + 40 + 5\} = 7 \times 150 = 1050$

2.17 NUMBER OF DIAGONALS OF 'n' SIDED POLYGON

Result : The number of diagonals of a regular polygon of n sides is $\dfrac{n(n-3)}{2}$ (Remember).

Proof : Since no three of the n vertices of polygon are collinear, we can get ${}^nC_2$ straight lines by selecting and joining these n vertices by taking 2 at a time.

But these ${}^nC_2$ straight lines includes both the sides of the polygon and diagonals.

$\therefore$ Number of sides of the polygon + number of diagonals of the polygon = ${}^nC_2$

But polygon has 'n' sides.

Therefore number of diagonals $= {}^nC_2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$

SOLVED EXAMPLES

***1. If ${}^{17}C_{2t+1} = {}^{17}C_{3t-5}$ then find t.**

Sol. Since ${}^{17}C_{2t+1} = {}^{17}C_{3t-5}$

we have $2t + 1 = 3t - 5$ (or) $2t + 1 + 3t - 5 = 17$

$\Rightarrow t = 6$ or $5t = 21 \Rightarrow t = 6$ or $t = 21/5$

But if $t = 21/5$ then $2t + 1 = \dfrac{47}{5}$ which is not an integer.

$\therefore t \neq \dfrac{21}{5} \quad \therefore t = 6$

Note :

${}^nC_r = {}^nC_s$ then either $r = s$ (or) $r + s = n$

2. Prove for $3 \le r \le n$, ${}^{n-3}C_r + 3\,{}^{n-3}C_{r-1} + 3\,{}^{n-3}C_{r-2} + {}^{n-3}C_{r-3} = {}^{n}C_r$

Sol. We know ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$.... (1)

L.H.S. $= {}^{n-3}C_r + 3\,{}^{n-3}C_{r-1} + 3\,{}^{n-3}C_{r-2} + {}^{n-3}C_{r-3}$

$= {}^{n-3}C_r + {}^{n-3}C_{r-1} + 2({}^{n-3}C_{r-1} + {}^{n-3}C_{r-2}) + {}^{n-3}C_{r-2} + {}^{n-3}C_{r-3}$

$= {}^{n-2}C_r + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$ [from (1)]

$= {}^{n-2}C_r + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$

$= {}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^{n}C_r$ [from (1)]

3. Show that ${}^{10}C_2 + {}^{11}C_2 + {}^{12}C_2 + {}^{13}C_2 + ... + {}^{20}C_2 = {}^{21}C_3 - {}^{10}C_3$.

Sol. We know ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

Therefore, ${}^{n}C_{r-1} = {}^{n+1}C_r - {}^{n}C_r$

Using this ${}^{10}C_2 = ({}^{11}C_3 - {}^{10}C_3)$

${}^{11}C_2 = ({}^{12}C_3 - {}^{11}C_3)$

${}^{12}C_2 = ({}^{13}C_3 - {}^{12}C_3)$

....

${}^{20}C_2 = ({}^{21}C_3 - {}^{10}C_3)$

Adding ${}^{10}C_2 + {}^{11}C_2 + {}^{12}C_2 + {}^{13}C_2 + ... + {}^{20}C_2 = {}^{21}C_3 - {}^{10}C_3$

***4. Prove ${}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3 = {}^{30}C_4$** **(March-18)**

Sol. L.H.S. $= {}^{25}C_4 + {}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{26}C_4 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{27}C_4 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{28}C_4 + {}^{28}C_3 + {}^{29}C_3 = {}^{29}C_4 + {}^{29}C_3 = {}^{30}C_4$

5. In a class there are 50 students. If each student plays a chess game with each other student, then find the total number of games played by them.

Sol. For every selection of 2 students from 50, a game will be conducted.

By interchanging these two selected students we are not going to conduct another game.

Therefore the number of games $= {}^{50}C_2 = \dfrac{50 \times 49}{1 \times 2} = 49 \times 25 = 1225$.

6. Find the number of ways of forming a committee of 5 members from 6 men and 3 ladies.

Sol. Since there is no restriction to form the committee, we have to select 5 members from 9 members.

This can be done in ${}^{9}C_5$ ways i.e., in ${}^{9}C_4$ ways.

7. Find the number of ways of selecting 5 objects from 9 dissimilar objects such that (i) a particular object is included (ii) a particular object is not included.

Note : *Number of ways of selecting 'r' things out of 'n' things always including s particular things is $^{(n-S)}C_{(r-S)}$*

Sol. i) The number of ways to select 5 objects from 9 distinct objects without any restriction in $^9C_5$ ways.

To know the number of selections in which a particular object to be included, let us select that particular object at first.

We have to select the remaining 4 objects from the remaining 8 objects.

It can be done in $^8C_4$ ways.

$\therefore$ Number of selections that a particular object included = $^8C_4$

ii) To know the number of selections in which a particular object is not included, select 5 objects from the remaining 8 objects after deleting the particular object.

It can be done in $^8C_5$ ways.

Number of selections such that a particular object is not included = $^8C_5$

Here note that $^8C_4 + {}^8C_5 = {}^9C_5$

8. Find the number of ways in which 3 numbers in A.P. can be selected from 1, 2, 3, ... 21.

Sol. If a, b, c are in A.P then $2b = a + c$

$\therefore$ $a + c$ is even. $\therefore$ both a and c are either even or odd.

From 1 to 21 there are 10 even and 11 odd natural numbers

$\therefore$ Therefore the number of ways to select 3 numbers a, b, c in A.P. from 1 to 21

= The number of ways to select 2 numbers a, c which are either both even or both odd

$= {}^{10}C_2 + {}^{11}C_2 = 45 + 55 = 100$

9. Out of 8 gentlemen and 5 ladies a committee of 5 is to be formed. Find the number of ways in which this can be done so as to include atleast 2 ladies.

Sol. The selection of committee can be done as follows :

S.No.	No. of gentle men	No. of ladies	No. of ways to select committee
1	3	2	$^8C_3 \times {}^5C_2 = 560$
2	2	3	$^8C_2 \times {}^5C_3 = 280$
3	1	4	$^8C_1 \times {}^5C_4 = 40$
4	0	5	$^8C_0 \times {}^5C_5 = 1$

Total number of committees = 560 + 280 + 40 + 1 = 881

***10. For a cricket team 6 people from one class and 8 people from another class have come for selection. In how many ways can we select a cricket team of 11 people taking atleast 2 from the first class and atleast one from another class.**

Sol. **Method - 1**

The required cricket team can be selected as follows.

S.No.	1st class	2nd class	no. of ways of selecting the team
1	3	8	$^6C_3 \times {}^8C_8 = 20$
2	4	7	$^6C_4 \times {}^8C_7 = 120$
3	5	6	$^6C_5 \times {}^8C_6 = 168$
4	6	5	$^6C_6 \times {}^8C_5 = 56$

Total number of cricket teams = 20 + 120 + 168 + 56 = 364

Note : *Number of ways of selecting r things from n things always rejecting 's' particular things is $^{(n-S)}C_r$*

Method - 2

The number of ways to select a cricket team of 11 people with the given restriction

= coefficient of x^{11} in the expansion

$({}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6\,x^6) \times ({}^8C_1\,x + {}^8C_2\,x^2 + ... + {}^8C_8\,x^8)$

= coefficient of x^{11} in

$x^2({}^6C_2 + {}^6C_3x + {}^6C_4x^2 + {}^6C_5x^3 + {}^6C_6\,x^4)\ x\,({}^8C_1 + {}^8C_2\,x + ... + {}^8C_8\,x^7)$

= coefficient of x^8 in

$({}^6C_2 + {}^6C_3x + {}^6C_4x^2 + {}^6C_5x^3 + {}^6C_6\,x^4)({}^8C_1 + {}^8C_2\,x + ... + {}^8C_7\,x^6 + {}^8C_8\,x^7)$

$= {}^6C_3 \times {}^8C_8 + {}^6C_4\,{}^8C_7 + {}^6C_5\,{}^8C_6 + {}^6C_6\,{}^8C_5$

$= 20 + (15)\,(8) + (6)\,(28) + 56 = 20 + 120 + 168 + 56 = 364$

***11. How many 4 letter words can be formed using the letters of the word 'PROPORTION'.**

Sol. Method - 1

The word contains 10 letters of which there are 2P's, 3O's, 2R's and I, T, N each one.

The number of words with 3 like and one different = ${}^5C_1 \times \dfrac{\lfloor 4}{\lfloor 3} = 20$

The number of words with 2 like of one kind and 2 like of another kind

$= {}^3C_2 \times \dfrac{\lfloor 4}{\lfloor 2\,\lfloor 2} = 18$

The number of words with 2 like and 2 different $= {}^3C_1 \times {}^5C_2 \times \dfrac{\lfloor 4}{\lfloor 2} = 360$

The number of words with 4 different $= {}^6P_4 = 360$

$\therefore$ The number of 4 letter words = 20 + 18 + 360 + 360 = 758

Method - 2

The number 4 letter words = coeff. of x^4 in

$$\lfloor 4\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}\right)^2\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}\right)(1+x)^3$$

= coeff. of x^4 in $(2 + 2x + x^2)^2\,(6 + 6x + 3x^2 + x^3)\,(1 + 3x + 3x^2 + x^3)$

= coeff. of x^4 in $[4 + 4x^2 + x^4 + 8x + 4x^2 + 4x^3]\,[6 + 24x + 39x^2 + 34x^4 + 18x^4]$

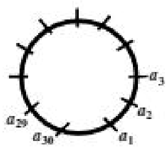
(neglecting x^5 and x^4)

= coeff. of x^4 in $[4 + 8x + 8x^2 + 4x^3 + x^4]\,[6 + 24x + 39x^2 + 34x^3 + 18x^4]$

= 72 + 272 + 312 + 96 + 6 = 758

12. If 30 persons are sitting around a circle in how many ways can 2 persons out of them be selected so that they are not adjacent.

Sol. Suppose the circular arrangement of 30 persons $a_1, a_2, a_3, ..., a_{29}, a_{30}$ is as shown below



Without any restriction we can select 2 persons in ${}^{30}C_2$ ways. Among these selections we do not accept the following 30 selections.

$a_1a_2,\ a_2a_3,\ ......\ a_{30}a_1$

$\therefore$ Required number of selections $= {}^{30}C_2 - 30$

Note :

Number of non-negative integral solution of $x_1 + x_2 + x_3 ... + x_r = n$ is ${}^{(n+r-1)}C_{(n)} = {}^{(n+r-1)}C_{(r-1)}$

13. There are 20 points in a plane out of which 7 points are collinear and no three of the points are colinear unless all the three are from these 7 points. Find the number of different straight lines.

Sol. From the given 20 points, by selecting 2 of them and joining we are supposed to get ${}^{20}C_2$ lines. But since 7 points are collinear, by selecting 2 points from these 7 points we get only one line instead of getting ${}^{7}C_2$ lines.

$\therefore$ Number of different lines $= {}^{20}C_2 - {}^{7}C_2 + 1$

14. How many selections of atleast one red ball can be made from 4 red balls and 3 green balls if balls of same colour are different in size.

Sol. Since 4 red balls are different, we can select one or more from red balls in $(2^4 - 1) = 15$ ways

Since 3 green balls are also different,

we can select none or more from green balls in 2^3 ways.

By Product rule,

Number of selections $= 15 \times 8 = 120$

15. There are 4 mangoes, 3 apples, 2 oranges in a bag, fruits of the same variety being identical. In how many different ways can a selection of fruits be made if

i) atleast one fruit is to be selected.

ii) atleast one mango is to be selected.

iii) atleast one of each kind is to be selected.

Sol. i) Number of ways to select none or more from 4 mangoes $= 4 + 1$ ways $= 5$ ways

Number of ways to select none or more from 3 apples $= 4$

Number of ways to select none or more from 2 oranges $= 3$

By fundamental principle

The total nubmer of ways to select $= 5 \times 4 \times 3 = 60$

But these 60 selections includes selection of none also

Since we have to select atleast one fruit,

the required numbers of selections $= 60 - 1 = 59$

ii) The number of ways to select one or more mangoes $= 4$

So, by fundamental principle

The number of ways to select $= 4 \times 4 \times 3 = 48$

These 48 selections does not include selection of none because one mango is already selected.

iii) The number of ways to select atleast from each variety $= 4 \times 3 \times 2 = 24$

16. Find the number of proper divisors of 2520.

i) How many of them are odd ? Find their sum

ii) How many of them are divisible by 10. Find their sum ?

Sol. The prime factorization of 2520 is $2^3 \times 3^2 \times 5^1 \times 7$

Any divisor will be of the form

$2^{\alpha}\, 3^{\beta}\, 5^{\gamma}\, 7^{\delta}$ where $0 \le \alpha \le 3$

$0 \le \beta \le 2$

$0 \le \gamma \le 1$

$0 \le \delta \le 1$

We can assign any one of 4 values to α which are 0, 1, 2, 3. Similarly we can assign a value to β, γ, δ in 3 ways, 2 ways, 2 ways respectively.
by product rule,
Number of divisors $= 4 \times 3 \times 2 \times 2 = 48$
Number of proper divisors $= 48 - 2 = 46$

i) To get odd divisor,
We have to take $\alpha = 0$
$\therefore$ Number of odd divisors $= 1 \times 3 \times 2 \times 2 = 12$
$\therefore$ Number of odd proper divisors $= 12 - 1 = 11$ ($\because$ 2560 is even improper)
Sum of these odd proper divisors =
$$2^0 \times (3^0 + 3^1 + 3^2)(5^0 + 5^1)(7^0 + 7^1) - 1 = 1 \times 13 \times 6 \times 8 - 1 = 623$$

ii) To get a divisor divisible by 10,
we have to take $\alpha = 1, 2$ or 3 and $\gamma = 1$ (i.e., $\alpha \neq 0$ and $\gamma \neq 0$)
$\therefore$ The number of divisors divisible by 10 $= 3 \times 3 \times 1 \times 2 = 18$
$\therefore$ Number of proper divisors divisible by 10 $= 18 - 1 = 17$
($\because$ 2520 is the only improper divisor which is divisible by 10)
Sum $= (2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(5^1)(7^0 + 7^1) - 2520$
$= 14 \times 13 \times 5 \times 8 - 2520 = 4760$

17. In how many ways 4 rupee coins can be distributed among 5 persons so that any person may have either 0, 1, 2, 3, 4.

Sol. We know that the number of ways to distribute 'n' identical objects among r groups putting zero or more in a group $= {}^{n+r-1}C_{r-1}$

Here we have to distribute 4 rupee coins (identical objects) among 5 persons (groups) so that any person may have either 0, 1, 2, 3, 4. The required number of ways to distribute $= {}^{n+r-1}C_{r-1}$ where $n = 4$ and $r = 5 = {}^{8}C_4 = 70$

18. Find the number of selections of 10 balls from unlimited number of red, black, white and green balls so that the each selection must contain atleast one ball of each colour.

Sol. Suppose x_R, x_B, x_W, x_G denote the numbers of red, black, white and green balls in the selected ten balls.

Here $x_R + x_B + x_W + x_G = 10$
Since we have to select atleast one ball from each colour we have each $x_R, x_B, x_W, x_G \geq 1$
So, the required number of selections = the number of positive integral solutions of $x_R+x_B+x_W+x_G=10$
$= {}^{n-1}C_{r-1}$ where $n = 10$ and $r = 4 = {}^{9}C_3 = 84$

19. There are 8 intermediate railway stations between Vijayawada and Hyderabad. In how many ways can a train be stopped at 3 of these stations such that no two of them are consecutive.

Sol. Method - 1

We have to select 3 stations from 8 so that no two are consecutive.
Suppose x_1 is the number of stations on left side of 1st selected station; x_2 is the number of stations between 1st selected and 2nd selected. x_3 is the number of stations between 2nd selected and 3rd selected. Suppose x_4 is the number of stations on the right side of 3rd selected if any

Note :
Number of ways of selecting r non consecutive objects from n objects arranged in a row (line) is ${}^{(n-r+1)}C_r$

$$\underbrace{X \quad X}_{x_1} \; (X) \; \underbrace{X}_{x_2} \; (X) \; \underbrace{X}_{x_3} \; (X) \; \underbrace{X}_{x_4}$$

Here $x_1 + x_2 + x_3 + x_4 = (8-3) = 5$ $\quad x_1 \geq 0; x_4 \geq 0$ and $x_2 \geq 1; x_3 \geq 1$

Put $x_1 = y_1$; $x_4 = y_4$ $\quad$ Put $y_2 = (x_2 - 1)$; $y_3 = (x_3 - 1)$ $\quad$ Now $y_2 \geq 0; y_3 \geq 0$

$y_1 + y_2 + y_3 + y_4 = 3$. where each $y_i \geq 0$ for $y = 1, 2, 3, 4$

The number of ways to stop at 3 stations of which no two are consecutive.

= The number of non-negative integral solution of $y_1 + y_2 + y_3 + y_4 = 3$

$= {}^{n+r-1}C_{r-1}$ where $n=3$ and $r = 4 = {}^6C_3 = \dfrac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$

Method - 2

We have to select 3 stations from 8 stations such that no two of the selected three are consecutive.

To each selection of 3 stations there exists a binary sequence of five '0's and three '1's of the form 10010001 (1 at k^{th} place means k^{th} station is selected and 0 at k^{th} place means k^{th} station is not selected) where no two '1's are consecutive.

Clearly, there exists a bijection (one-to-one correspondence) between set of selections of three stations from 8 stations so that no two of the selected three are consecutive and set of binary sequences of five '0's and three '1's in which no two '1's are consecutive.

Let us count the number of such binary sequences. First arrange five '0's as shown below.

X 0 X 0 X 0 X 0 X 0 X

It can be done in one way. Now three '1's can be arranged at any of the 6 places marked with 'X's in the above diagram.

This is because no two '1's come together.

The three '1's can be arranged in ${}^6C_3$ ways.

So, we conclude that the required number of selections = number of binary sequences of five '0's and three '1's in which no two '1's are consecutive.

$= {}^6C_3 = 20$

20. Two students A and B are having 8 different books and 5 different books respectively. In how many ways they can exchange the books.

Sol. Totally there are 13 books. From the heap of these 13 books A can select 8 books in ${}^{13}C_8$ ways i.e., ${}^{13}C_5$ ways.

These ${}^{13}C_5$ selections selected by A includes the selection of his own books, which can not be treated as an exchange.

$\therefore$ Number of exchanges $= {}^{13}C_8 - 1 = {}^{13}C_5 - 1$

This can also be done as,

Number of exchanges $= {}^8C_1 \times {}^5C_1 + {}^8C_2 \times {}^5C_2 + {}^8C_3 \times {}^5C_3 + {}^8C_4 \times {}^5C_4 + {}^8C_5 \times {}^5C_5$

Note :

Formula, ${}^8C_0{}^5C_0 + {}^8C_1{}^5C_1 + {}^8C_2{}^5C_2 + \ldots + {}^8C_5{}^5C_5 = {}^{13}C_5$

In general if m and n are +ve integers, $(m < n)$ then

${}^mC_0{}^nC_0 + {}^mC_1{}^nC_1 + {}^mC_2{}^nC_2 + \ldots + {}^mC_m{}^nC_m = {}^{m+n}C_m$

EXERCISE - 2.5

1. i) If ${}^nC_4 = 210$ then find n. **[Ans : 10]**

   ii) If ${}^{12}C_r = 495$ find possible values of r. **[Ans : (4, 8)]**

   *iii) If ${}^{12}C_{r+1} = {}^{12}C_{3r-5}$ then find r. **(May-19)** **[Ans : (3, 4)]**

   iv) If $10 \cdot {}^nC_2 = 3 \cdot {}^{n+1}C_3$ then find n **[Ans : 9]**

   v) If ${}^9C_3 + {}^9C_5 = {}^{10}C_r$ find r. **[Ans : 4 or 6]**

   vi) If ${}^nC_4 = {}^nC_6$, find n **[Ans : 10]**

   vii) If ${}^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r **[Ans : 3]**

   viii) If ${}^{17}C_{2t+1} = {}^{17}C_{3t-5}$, find t **[Ans : 6]**

   *ix) If ${}^nC_5 = {}^nC_6$, then find ${}^{13}C_n$ **(May-18 & March-19)**

2. *i) Show that ${}^{34}C_5 + \sum_{r=0}^{4} {}^{(38-r)}C_4 = {}^{39}C_5$ **(March-19)**

   ii) If ${}^nP_r = 5040$; ${}^nC_r = 210$ then find n, r. **[Ans : 10, 4]**

   ** iii) Show that $\dfrac{{}^{4n}C_{2n}}{{}^{2n}C_n} = \dfrac{1 \cdot 3 \cdot 5 \ldots\ldots (4n-1)}{\left[1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)\right]^2}$ **(March-17, 18 & May-19)**

3. i) Find the number of ways of selecting 6 members out of 12 members always including a specified member. **[Ans : ${}^{11}C_5$]**

   ii) Find the number of 4 letter words that can be formed using one vowel and 3 consonants from the letters of the word 'ARTICLE' **[Ans : 288]**

4. i) If n persons are sitting in a row, find the number of ways of selecting two persons, who are sitting adjacent to each other. **[Ans : $n-1$]**

   ii) Find the number of ways of giving away 4 similar coins to 5 boys if each boy can be given any number (less than or equal to 4) of coins. **[Ans : 70]**

5. If a set A has 12 elements. Find the number of subsets of A having

   (i) 4 elements (ii) atleast 3 elements (iii) atmost 3 elements **[Ans : (i) 495 (ii) 4017 (iii) 299]**

6. A class contains 4 boys and g girls. Every Sunday, five students with atleast 3 boys go for a pincnic. A different group is being sent every week. During the picnic, the class teacher gives each girls in the group a doll. If the total number of dolls distributed is 85, find g. **[Ans : 5]**

7. A teacher wants to take 20 students to a park. He can take exactly 5 students at a time and will not take the same group more than once. Find the number of times

   i) each student can go to the park **[Ans : ${}^{19}C_4$]**

   ii) the teacher can go to the park **[Ans : ${}^{20}C_5$]**

8. Find the number of ways in which 12 things be

i) Divided into 4 equal groups [**Ans** : $\frac{12!}{(3!)^4 4!}$]

ii) Distributed to 4 persons equally [**Ans** : $\frac{12!}{(3!)^4}$]

9. *i) Find the number of ways of selecting a cricket team of 11 players from 7 batsmen and 6 bowlers such that there will be atleast 5 bowlers in the team. (**May-18**) [**Ans** : 63]

ii) A question paper is divided into 3 sections *A, B, C* containing 3, 4, 5 questions respectively. Find the number of ways of attempting 6 questions choosing atleast one from each section. [**Ans** : 805]

iii) Find the number of ways of forming a committee of 5 members out of 6 Indians and 5 Americans so that always the Indians will be in majority in the committee. [**Ans** : 281]

iv) A double decker bus has 15 seats in the lower deck and 13 seats in the upper deck. In how many ways can a marriage party of 28 persons be arranged if 4 old people refuse to go to the upper deck and 4 children wish to travel in the upper deck only.[**Ans** : ${}^{20}C_9 \times \lfloor 15 \times \lfloor 13$]

10. Out of 3 different books on Economics, 4 different books on political science and 5 different books on Geography, how many collections can be made, if each collection consists of

i) Exactly one book of each subject [**Ans** : 60]

ii) Atleast one book of each subject [**Ans** : 3255]

11. i) Find the number of divisors of 1080. [**Ans** : 32]

ii) Find the number of proper divisors of 540. [**Ans** : 22]

iii) Find the number of even divisors of 720. [**Ans** : 24]

iv) Find the number of even proper divisors of $2^3 \cdot 3^2 \cdot 5^3$ [**Ans** : 35]

v) In how many ways 4900 can be expressed as product of 2 positive integers. [**Ans** : 14]

12. i) There are 20 points in a plane of which 5 are collinear and no three of the points are collinear unless all the three are from these 5 points. Find the number of different

a) straight lines passing through these points [**Ans** : 181]

b) triangles fromed by joining these points [**Ans** : 1130]

ii) Find the maximum number of points into which 4 circles and 4 straight lines intersect. [**Ans** : 50]

iii) If a set of *m* parallel lines intersect another set of *n* parallel lines (not parallel to the lines in 1[st] set) then find the number of parallelograms formed. [**Ans** : $\frac{m(m-1)n(n-1)}{4}$]

iv) Find the number of diagonals of a polygon having 20 sides. [**Ans** : 170]

v) If a polygon has 35 diagonals find the number of sides. [**Ans** : 10]

13. **i) Find the value of ${}^{10}C_5 + 2.{}^{10}C_4 + {}^{10}C_3$ (**March-17, 18**) [**Ans** : 792]

ii) If ${}^nC_2 + 2({}^nC_3) + {}^nC_4 > {}^{(n+2)}C_3$ then find *n*. [**Ans** : $n > 5$]

2.18 NUMBER OF RECTANGLES AND SQUARES

Consider a square of size $n \times n$. The square is divded into n^2 squares each of area 1 sq. unit with help of $(n + 1)$ horizontal lines and $(n + 1)$ vertical lines.

1) Number of rectangles :

To get a rectangle whatever its size may be we have to select 2 lines from $(n + 1)$ horizontal and 2 lines from $(n + 1)$ vertical.

By product rule, it can be done in ${}^{(n+1)}C_2 \times {}^{(n+1)}C_2$

Therefore total number of rectangles (including squares) = ${}^{(n+1)}C_2 \times {}^{(n+1)}C_2$

$$= \left[\frac{(n+1)n}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

= sum of cubes of first 'n' natural numbers = $\sum_{r=1}^{n} r^3$

2) Number of squares :

First we count the number of squares with size $k \times k$. where $k \le n$.

To get a square with size $k \times k$ we have to select two lines from horizontal and two lines from vertical lines in such a way they are at a distance of k units.

First we select 2 lines from horizontal lines.

Suppose r^{th} line is the first selected line and $(r + k)^{th}$ is the second selected line which are measured from top side of the square board.

Here $1 \le r < r + k \le n+1$

($\because$ There are $(n + 1)$ horizontal lines)

$\therefore$ $1 \le r \le (n+1-k)$

So, first line can be selected in $(n + 1 - k)$ ways where as the second line can be selected in only one way i.e., at a distance of k units from first selected.

Therefore, the number of ways to select 2 horizontal lines at a distance of k units $= (n + 1 - k) \times 1$

Similarly, we can select 2 vertical lines at a distance of k units $= (n + 1 - k) \times 1$

$\therefore$ Number of squares of size $k \times k = (n+1-k) \times 1 \times (n+1-k) \times 1 = (n+1-k)^2$

$\therefore$ Total number of squares $= \sum_{k=1}^{n} (n+1-k)^2$ Put $n + 1 - k = r$

$$= \sum_{r=1}^{n} r^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Note

i) ***The number of squares from $n \times n$ square board with size $k \times k = (n + 1 - k)^2$ $(1 \le k \le n)$.***

Here number of horizontals= number of verticals = $n + 1$.

The number of rectangles from $n \times n$ square board with size $k \times l$ $(1 \le k, l \le n) = (n+1-k)(n+1-l)$

ii) ***In a rectangle of $n \times p$ $(n < p)$***

The total number of rectangles (including squares) of any size $= {}^{n+1}C_2 \times {}^{p+1}C_2 = \frac{(n+1)n(p+1)p}{4}$

The number of squares having size $k \times k$ is $(n + 1 - k)(p + 1 - k)$

The total number of squares with any size from $n \times p$ rectangular board where $n < p$ is $\sum_{r=1}^{n} (n+1-r)(p+1-r)$

Ex : **Find the number of rectangles excluding squares from a rectangle of size 9×6.**

Sol. The total number of rectangles including squares with any size $= {}^{10}C_2 \times {}^{7}C_2$

$(\because$ It will have 10 horizonal and 7 vertical lines$)$ $= \dfrac{10 \times 9 \times 7 \times 6}{4} = 945$

Number of squares with size $k \times k$

Where $1 \le k \le 6 \quad = (9 + 1 - k)(6 + 1 - k) = (10 - k)(7 - k)$

Total number of squares $= \sum_{k=1}^{6} (10-k)(7-k) = \sum_{k=1}^{6} (70 - 17k + k^2)$

$$= 420 - 17 \times \frac{\overset{3}{\cancel{6}} \times 7}{2} + \frac{6 \times 7 \times 13}{6} = 420 - 21 \times 17 + 7 \times 13$$

$$= 7(60 - 51 + 13) = 7 \times 22 = 154$$

$\therefore$ The number of rectangles excluding squares $= 945 - 154 = 791$

2.19 THE INCLUSION AND EXCLUSION PRINCIPLE

Suppose $A_1, A_2, \ldots\ldots, A_n$ are finite sets and $A = A_1 \cup A_2 \cup \ldots \cup A_n$ then $n(A) =$

$$n(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1}\, n(A_1 \cap A_2 \cap \ldots\ldots \cap A_n)$$

2.20 DERANGEMENTS

Consider the word 'RAM'. In this word, R occupies 1^{st} place, A,M are in 2^{nd} and 3^{rd} places.

We know that the letters of the word 'RAM' can be arranged by taking all a time in $\underline{|3}$ ways.

The arrangements are RAM, RMA, AMR, ARM, MAR, MRA

In these six arrangements (except in AMR, MRA) atleast one letter occupies its original position for example in ARM, M occupies its original position. In MAR, A occupies its original position.

Consider AMR, MRA. Here no letter of these two arrangements is in its original position. Such arrangements are called derangements.

An arrangement of certain objects is said to be derangement if no object goes to its scheduled place.

Result : The number of derangements of n distinct objects is $\underline{|n}\left\{\dfrac{1}{\underline{|2}} - \dfrac{1}{\underline{|3}} + \dfrac{1}{\underline{|4}} - \dfrac{1}{\underline{|5}} + \ldots\ldots + (-1)^n \dfrac{1}{\underline{|n}}\right\}$

(or)

If n distinct letters are placed in n corresponding addressed envelopes. Then the number of ways in which they can be arranged such that no letter is placed in the right envelope. i.e., all the letters are placed in wrong envelopes is $\underline{|n}\left\{\dfrac{1}{\underline{|2}} - \dfrac{1}{\underline{|3}} + \dfrac{1}{\underline{|4}} - \dfrac{1}{\underline{|5}} + \ldots\ldots + (-1)^n \dfrac{1}{\underline{|n}}\right\}$.

> ***Note :***
> ***Rearrnging the elements of a given arrangement so that no element remains in its original position is a Derangement.***

Proof : Suppose μ is the set of all possible arrangements of placing n different letters in n addressed envelopes such that each envelope is filled with exactly one letter.

Here $n(\mu) = \lfloor n$.

Let $A_k (1 \le k \le n)$ be the set of all arrangements in which k^{th} letter is placed in its scheduled right envelope.

The complement set of A_k denoted by A_k^C $(= \mu - A_k)$ is the set of all arrangements of letters in which k^{th} letter is placed in wrong addressed envelope.

So, $A_1^C \cap A_2^C \cap A_3^C \cap ..... \cap A_n^C$ is the set of all arrangements in which all the letters are placed in wrong envelopes.

Here we know $n(A_1 \cup A_2 \cup .... \cup A_n) + n(A_1^C \cap A_2^C \cap .... \cap A_n^C)$

$= \text{total number of arrangements} = \lfloor n$ .... (1)

By inclusion and exclusion principle $n(A_1 \cup A_2 \cup ...... \cup A_n)$.

$$= \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) .... +$$

$$(-1)^{n+1} \; n(A_1 \cap A_2 \cap ...... \cap A_n) \quad .... (2)$$

Here $n(A_i) = \lfloor n-1$ (place ith letter in its scheduled envelope and arrange the remaining $(n-1)$ letters in the remaining $(n-1)$ envelopes with out any restriction)

$$\therefore \; \sum n(A_i) = n\lfloor n-1 = \lfloor n$$

Similarly, $n(A_i \cap A_j) = \lfloor n-2$

(After placing i^{th} and j^{th} letters in their scheduled envelopes and arrange the remaining$(n-2)$ letters in $(n-2)$ envelopes without any restriction)

$$\therefore \; \sum_{1 \le i < j \le n} n(A_i \cap A_j) \; {}^nC_2 \lfloor n-2 = \frac{\lfloor n}{\lfloor 2}$$

Similarly, $\displaystyle\sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) = {}^n C_3 \lfloor n-3 = \frac{n(n-1)(n-2)}{\lfloor 3} \lfloor n-3 = \frac{\lfloor n}{\lfloor 3}$

Proceeding like this $n(A_1 \cap A_2 \cap .... \cap A_n) = 1 = \dfrac{\lfloor n}{\lfloor n}$

from (1) and (2) the number of ways to place all the letters in wrongly addressed envelopes

$$= n(A_1^C \cap A_2^C \cap ..... \cap A_n^C) = \lfloor n - n(A_1 \cup A_2 \cup .... \cup A_n)$$

$$= \lfloor n - \left\{ \lfloor n - \frac{\lfloor n}{\lfloor 2} + \frac{\lfloor n}{\lfloor 3} - .... + (-1)^{n+1} \frac{\lfloor n}{\lfloor n} \right\} = \lfloor n - \lfloor n + \frac{\lfloor n}{\lfloor 2} - \frac{\lfloor n}{\lfloor 3} + ... + (-1)^n \frac{\lfloor n}{\lfloor n}$$

$$= \lfloor n \left\{ 1 - 1 + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + ... + (-1)^n \frac{1}{\lfloor n} \right\}$$

Example :

The number of derangements of 3 distinct objects $= \lfloor 3 \left\{ \dfrac{1}{\lfloor 2} - \dfrac{1}{\lfloor 3} \right\} = 3 - 1 = 2$

This fact is already discussed in above explanation.

Result: **The number of onto functions from a set A onto another set B where $n(A)=m$ and $n(B)=n$ is $^nC_0n^m - {^nC_1}(n-1)^m + {^nC_2}(n-2)^m - {^nC_3}(n-3)^m + \dots + (-1)^{n-1}\,{^nC_{n-1}}(1)^m$ where $m \geq n$.**

Proof : Suppose $A = \{a_1, a_2, ..., a_m\}$; $B = \{b_1, b_2, ..., b_n\}$

Let μ be the set of all functions from A to B either may be onto or not.

Here $n(\mu) = n^m$.

Let us suppose A_k $(1 \leq k \leq n)$ be the set of all functions from A to B in which no element of A is mapped to the element b_k of B.

The compliment set of A_k is denoted by A_k^C.

Here, A_k^C is the set of all functions from A to B in which atleast one element of A is mapped to b_k of B.

Clearly, $A_1^C \cap A_2^C \cap ..... \cap A_n^C$ is the set of all onto functions from A to B.

We know $n(A_1 \cup A_2 \cup .... \cup A_n) + n(A_1^C \cap A_2^C \cap ... \cap A_n^C) = n^m$ .... (1)

By inclusion and exclusion principle $n(A_1 \cup A_2 \cup ...... \cup A_n)$.

$$= \sum_{i=1}^{n} n(A_i) - \sum_{1\le i<j\le n} n(A_i \cap A_j) + \sum_{1\le i<j<k\le n} n(A_i \cap A_j \cap A_k) - .... + (-1)^{n+1}\, n(A_1 \cap A_2 \cap ...... \cap A_n) \quad .... (2)$$

Here $n(A_i)$ = The number of functions from A to $B - \{b_i\} = (n-1)^m$

$$\sum_{i=1}^{n} n(A_i) = {^nC_1}\,(n-1)^m$$

Again $n(A_i \cap A_j)$ = number of functions A to $B - \{b_i, b_j\} = (n-2)^m$

$$\sum_{1\le i<j\le n} n(A_i \cap A_j) = {^nC_2}(n-2)^m$$

Similarly, $\displaystyle\sum_{1\le i<j<k\le n} n(A_i \cap A_j \cap A_k) = {^nC_3}(n-3)^m$

Proceeding like this $n(A_1 \cap A_2 \cap .... \cap A_n) = {^nC_n}(n-n)^m = 0$

from (1) and (2), the number of onto functions from A to B =

$$n(A_1^C \cap A_2^C \cap ..... \cap A_n^C) = n^m - n(A_1 \cup A_2 \cup ...... \cup A_n)$$

$$= \{{^nC_1}(n-1)^m - {^nC_2}(n-2)^m + {^nC_3}(n-3)^m + ... + (-1)^{n-2}\,{^nC_{n-1}}(1)^m + (-1)^{n-1}0\}$$

$$= {^nC_0}n^m - {^nC_1}(n-1)^m + {^nC_2}(n-2)^m - {^nC_3}(n-3)^m + ... + (-1)^{n-1}\,{^nC_{n-1}}(1)^m$$

2.21 EXPONENT OF PRIME 'P' IN $\lfloor n$

Suppose P is a prime number and n is a +ve integer. The exponent of prime P in $\lfloor n$ is denoted by $E_P(\lfloor n)$.

Note that, the last integer among 1, 2, 3, ...$(n-1)$, n which is divisible by P is $\left[\dfrac{n}{P}\right]P$ where $[x]$ stands for the integral part of x.

Now $E_P(\lfloor n) = E_P(1 \cdot 2 \cdot 3 \cdot ..... \cdot (n-1) \cdot n) = E_P\,(P \cdot 2P \cdot 3P \cdot 4P \cdot ...... \cdot [n/P]P)$

(Here we are writing only the integers which are divisible by P)

$$= \left[\frac{n}{P}\right] + E_P(1 \cdot 2 \cdot 3 \cdot ..... \cdot [n/P]) \quad ....(1)$$

Among 1, 2, 3, ... $\left[\frac{n}{P}\right]$, the last integer divisible by 'P' is $\left[\frac{\left[\frac{n}{P}\right]}{P}\right]P = \left[\frac{n}{P^2}\right]P$

From (1) $E_P(\lfloor n) = \left[\frac{n}{P}\right] + E_P(1\cdot2\cdot3\cdot....[n/P]) = \left[\frac{n}{P}\right] + E_P\left(P\cdot 2P\cdot 3P\cdot .....\left[\frac{n}{P^2}\right]P\right)$

(we are not writing the integers which are not divisible by P)

$= \left[\frac{n}{P}\right] + \left[\frac{n}{P^2}\right] + E_P(1\cdot2\cdot3\cdot.....\cdot[n/P^2])$

Proceeding like this, $E_P(\lfloor n) = \left[\frac{n}{P}\right] + \left[\frac{n}{P^2}\right] + \left[\frac{n}{P^3}\right] + ... + \left[\frac{n}{P^s}\right]$ (remember)

Where $P^s \le n < P^{s+1}$

Note

i) $E_p({}^nC_r) = E_p(n!) - E_p(r!) - E_p(n-r)!$

ii) number of zeroes at the end of $n! = E_5(n!)$

iii) number of zeroes at the end of $N = 2^a.3^b,5^c$ *is equal to c.*

Ex : **Find $E_2(\lfloor 15)$.**

Sol. $E_2(\lfloor 15) = \left[\frac{15}{2}\right] + \left[\frac{15}{2^2}\right] + \left[\frac{15}{2^3}\right] = 7 + 3 + 1 = 11$

SOLVED EXAMPLES

1. Find the number of integral solutions of $x+y+z+t=29$ where $x \ge 1;\ y \ge 1;\ z \ge 3$ and $t \ge 0$.

Sol. **Method - 1 :** The number of integral solutions of $x+y+z+t=29$ where $x \ge 1;\ y \ge 1;\ z \ge 3$ and $t \ge 0$

= the number of ways to distribute 29 ruppee coins (identical things) among 4 persons A, B, C, D such that A, B must receive atleast one coin and C must receive at least 3 coins.

= The number of ways to distribute 24 coins among A, B, C, D without any restriction after giving one coin each to A, B and 3 coins to C.

$= {}^{n+r-1}C_{r-1}$ where $n = 24, r = 4 = {}^{27}C_3$

Method - 2

$x+y+z+t=29$

Where $x \ge 1$; $y \ge 1$; $z \ge 3$; $t \ge 0$

Put $x-1=x_1$; $y-1=y_1$; $z-3=z_1$

Now $x_1+y_1+z_1+t=24$

Where $x_1 \ge 0$; $y_1 \ge 0$; $z_1 \ge 0$; $t \ge 0$

$\therefore$ The number of non negative integral solutions of $x_1+y_1+z_1+t=24$

$= {}^{n+r-1}C_{r-1}$ where $n = 24, r = 4 = {}^{27}C_3$

2. Find the number of non negative integral solutions of $x_1+x_2+x_3+x_4 \le 20$.

Sol. **Method - 1 :** Let $x_1+x_2+x_3+x_4 = 20-k$

where $0 \le k \le 20$;

The number of non negative integral solutions of $x_1 + x_2 + x_3 + x_4 = 20 - k$ for some fixed k,

$= {}^{n+r-1}C_{r-1}$ where $n = 20 - k$; $r = 4 = {}^{23-k}C_3$

The number of non negative integral solutions of $x_1 + x_2 + x_3 + x_4 \le 20$

$$= \sum_{k=0}^{20} {}^{23-k}C_3 = {}^{23}C_3 + {}^{22}C_3 + {}^{21}C_3 + {}^{20}C_3 + ... + {}^{3}C_3$$

$$= ({}^{24}C_4 - {}^{23}C_4) + ({}^{23}C_4 - {}^{22}C_4) + ({}^{22}C_4 - {}^{21}C_4) + ... + ({}^{5}C_4 - {}^{4}C_4) + {}^{3}C_3$$

$$= {}^{24}C_4 \; (\because {}^{4}C_4 = {}^{3}C_3 = 1)$$

we applied the result, ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ in the above simplification.

Method - 2

Given $x_1 + x_2 + x_3 + x_4 \le 20$

Let us introduce a new variable 'x_5'

Such that $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where $0 \le x_5 \le 20$

Therefore the number of non negative integral solutions for $x_1 + x_2 + x_3 + x_4 \le 20$

= The number of non negative integral solutions for

$x_1 + x_2 + x_3 + x_4 + x_5 = 20 = {}^{n+r-1}C_{r-1}$

where $n = 20$; $r = 5 = {}^{24}C_4$

3. In how many ways the number 18900 can be split into two factors which are relative prime.

Sol. Here $18900 = 2^2 \times 3^3 \times 5^2 \times 7$

We have to express 18900 as $18900 = a \times b$

Where a, b are relative prime.

Here we have to assign entire 2^2 with out splitting to just one variable either to a or to b.

If you split 2^2, then a and b can not be relative prime.

So, each of 2^2, 3^3, 5^2, 7 can be assigned in two ways either to a or b.

By product rule it can be done in $2^4 = 16$ ways. But a, b has no order.

Therefore the number of ways to split 18900 into two factors which are relative prime = 16/2 = 8

4. Find the number of positive integral solutions of $x_1x_2x_3 = 30$.

Sol. $30 = 2 \times 3 \times 5 =$ (prime factors)

We can assign 2, 3, 5 to any one of the 3 variables

So, each can be assigned in 3 ways

By product rule, the number of integral solutions = $3^3 = 27$.

This is just like to post 3 different letters in 3 boxes.

5. Find the number of ways keeping 5 letters in 5 addressed envelopes.

Sol. Requried number of ways = $5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$

6. Find the exponent of 3 in $\underline{|100}$.

Sol. $E_3(\underline{|100}) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{81}\right] = 33 + 11 + 3 + 1 = 48$

7. What is the largest positive integer n such that $\underline{|33}$ is divisible by 2^n.

Sol. $E_2(\underline{|33}) = \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right] = 16 + 8 + 4 + 2 + 1 = 31$

$\therefore$ Exponent of 2 in $\underline{|33}$ in 31.

$\therefore$ 2^{31} divides $\underline{|33}$ and 31 is the largest value of n such that 2^n divides $\underline{|33}$.

8. Find the exponent of 2 in ${}^{20}C_{10}$.

Sol. We know ${}^{20}C_{10} = \frac{\underline{|20}}{\underline{|10}\,\underline{|10}}$

$$E_2(\underline{|20}) = \left[\frac{20}{2}\right] + \left[\frac{20}{2^2}\right] + \left[\frac{20}{2^3}\right] + \left[\frac{20}{2^4}\right] = 10 + 5 + 2 + 1 = 18$$

$$E_2(\underline{|10}) = \left[\frac{10}{2}\right] + \left[\frac{10}{2^2}\right] + \left[\frac{10}{2^3}\right] = 5 + 2 + 1 = 8$$

$\therefore E_2({}^{20}C_{10}) = 18 - 2(8) = 2$

9. Find the number of proper divisors of $\underline{|15}$.

Sol. Suppose $\underline{|15} = 2^a\, 3^b\, 5^c\, 7^d\, 11^e\, 13^f$

$$E_2(\underline{|15}) = \left[\frac{15}{2}\right] + \left[\frac{15}{2^2}\right] + \left[\frac{15}{2^3}\right] = 7 + 3 + 1 = 11$$

$$E_3(\underline{|15}) = \left[\frac{15}{3}\right] + \left[\frac{15}{3^2}\right] = 5 + 1 = 6$$

$$E_7(\underline{|15}) = \left[\frac{15}{7}\right] = 2$$

$E_{11}(\underline{|15}) = E_{13}(\underline{|15}) = 1$

$\therefore \underline{|15} = 2^{11}\, 3^6\, 5^3\, 7^2\, 11^1\, 3^1$

$\therefore$ The number of proper divisors $= (12 \times 7 \times 4 \times 3 \times 2 \times 2) - 2 = 4030$

10. Find the number of zeroes at the end of 15!

Sol. $E_5(15!) = \left[\frac{15}{5}\right] + \left[\frac{15}{5^2}\right] = 3 + 0 = 3$

$\therefore$ No.of zeroes at the end of 15! = 3.