CHAPTER 1
Algebra

QUADRATIC EQUATIONS AND EXPRESSIONS

- NATURE OF ROOTS
- SIGNS OF EXPRESSION
- MAXIMUM AND MINIMUM VALUES
- INEQUATIONS

1.0 INTRODUCTION

Theory of quadratic equations is an important concept in mathematics. Even though it is a topic of algebra, it acts as a tool in topics like trignometry, geometry etc., In this chapter, we discuss quadratic expressions, quadratic equations, nature of quadratic equations, sign changes of quadratic expressions and extreme values of quadratic expression.

1.1 QUADRATIC EXPRESSION

Definition

An expression of the form ax^2+bx+c, where a, b, c are real or complex numbers, $a \neq 0$ is called a quadratic expression in variable x.

Example : $x^2+5x+6, 3x^2-5x+7, x^2-(1-2i)x+i$

Definition

A complex number 'α' is said to be a 'zero' of quadratic expression $ax^2 + bx + c$, if $a\alpha^2 + b\alpha + c = 0$.

Example : 3 is zero of $x^2 - 5x + 6$

i is zero of $x^2 + 1$

1.2 QUADRATIC EQUATION

Remember : Root is a word used for polynomial equation only. Solution is a word used for any equation

Definition

An equation of the form $ax^2+bx+c= 0$ where a, b, c are real or complex numbers and $a \neq 0$ is called a quadratic equation in x, with a, b, c as coefficients.

Example : $x^2-7x+12 = 0, x^2 + x + 1 = 0, 3x^2 - 5x + 6 = 0$

Definition

A complex number 'α' is said to be a 'root' or 'solution' of the quadratic equation $ax^2 + bx + c = 0$ if $a\alpha^2 + b\alpha + c = 0$.

Example : 2 is a root of $x^2 - 5x + 6 = 0$

$1 - i$ is a root of $x^2 - 2x + 2 = 0$

THEOREM-1.1 The roots of the quadratic equation $ax^2+bx+ c = 0$ $(a \neq 0)$are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Proof : Given quadratic equation $ax^2 + bx + c = 0$

$\Rightarrow 4a(ax^2 + bx + c) = 0$

Let α be a root of the equation.

Then $4a(a\alpha^2 + b\alpha + c) = 0 \Rightarrow (2a\alpha)^2 + 2(2a\alpha)b + 4ac = 0$

$$\Rightarrow (2a\alpha + b)^2 - b^2 + 4ac = 0 \Rightarrow 2a\alpha + b = \pm\sqrt{b^2 - 4ac}$$

$$\Rightarrow \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example :

1. The roots of $x^2 - 7x + 12 = 0$ are $\frac{7 \pm \sqrt{49 - 4(1)12}}{2.1} = \frac{7+1}{2}, \frac{7-1}{2}$ i.e., 4, 3

2. The roots of $x^2 - 6x + 25 = 0$ are $\frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6+8i}{2}, \frac{6-8i}{2}$ i.e., $3 + 4i, 3 - 4i$

Remember : *Difference of roots of a quadratic equation* $= \frac{\sqrt{Discriminant}}{|Coefficient\ of\ x^2|}$

Note

i) The roots of $ax^2 + bx + c = 0,\ a \neq 0$ may be real or complex and the equation cannot have more than two roots.

ii) The roots of $ax^2 + bx + c = 0$ may be equal or distinct.

iii) If α and β are the roots of $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

iv) If α, β are the roots of $ax^2 + bx + c = 0$ then the quadratic expression $ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$.

v) The quadratic with roots α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e., $(x - \alpha)(x - \beta) = 0$

vi) Difference of roots : If α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha \sim \beta = \frac{\sqrt{b^2 - 4ac}}{|a|} = \frac{\sqrt{\Delta}}{|a|}$ where $\Delta = b^2 - 4ac$.

Example :

1. The quadratic equation whose roots are $3 + \sqrt{2}, 3 - \sqrt{2}$ is
$x^2 - (3 + \sqrt{2} + 3 - \sqrt{2})x + (3 + \sqrt{2})(3 - \sqrt{2}) = 0$ i.e., $x^2 - 6x + 7 = 0$.

2. The quadratic equation whose roots are $2 + 3i, 2 - 3i$ is $x^2 - (2 + 3i + 2 - 3i)x + (2 + 3i)(2 - 3i) = 0$ is $x^2 - 4x + 13 = 0$.

3. The quadratic equation whose roots are $1 + \sqrt{2}, \sqrt{2}$ is $x^2 - \left(1 + 2\sqrt{2}\right)x + \left(2 + \sqrt{2}\right) = 0$

1.3 NATURE OF THE ROOTS OF $ax^2 + bx + c = 0$

Definition

If $ax^2 + bx + c = 0$ a,b,c is a real quadratic equation then the real number $b^2 - 4ac$ is denoted by Δ and is called discriminant.

Example : The discriminant of $3x^2 + 4x - 5 = 0$ i.e., $\Delta = 4^2 - 4(3)(-5) = 76$

THEOREM-1.2

If a,b,c are real then the nature of the roots of quadratic equation $ax^2 + bx + c = 0$ is as follows :

i) If $\Delta = 0$ then the roots are real and equal and each root is $\frac{-b}{2a}$.

ii) If $\Delta > 0$ then the roots are real and distinct.

iii) If $\Delta < 0$ then the roots are imaginary and conjugate to each other.

THEOREM-1.3 **If a, b, c are rational numbers then the nature of the roots of quadratic equation $ax^2 + bx + c = 0$ is as follows :**

i) If $\Delta = 0$ then the roots are rational and equal, and each root is $\frac{-b}{2a}$

ii) If $\Delta > 0$ and if Δ is a perfect square then the roots are rational and distinct.

iii) If $\Delta > 0$ and if Δ is not a perfect square then the roots are irrational and not equal. Also they are conjugate surds.

iv) If $\Delta < 0$ then the roots are complex and conjugate to each other.

Results to remember

For $ax^2 + bx + c = 0$

i) $a+b+c = 0 \Rightarrow 1$ is a root and $\frac{c}{a}$ is the other root.

ii) $a-b+c=0 \Rightarrow -1$ is a root and $-\frac{c}{a}$ is the other root.

iii) difference of roots $= 1 \Rightarrow \Delta = a^2$

iv) roots are in the ratio $m : n \Rightarrow (m+n)^2 ac = mnb^2$

v) one root is k times the other $\Rightarrow (1+k)^2 ac = kb^2$

vi) one root is negative of the other $\Rightarrow b=0$

vii) one root is reciprocal of the other $\Rightarrow a=c$

viii) one root is the square of the other $\Rightarrow ac(a + c) + b^3 = 3abc$.

ix) One root is the n$^{\text{th}}$ power of the other $\Rightarrow (ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$

Remember :

If the roots of $ax^2 + bx + c = 0$ are in the ratio $m : n$ then $\frac{b^2}{ac} = \frac{(m+n)^2}{mn}$

SOLVED EXAMPLES

1. Find m so that equation $x^2 + 2(m + 2)x + 9m = 0$ will have equal roots ?

Sol. To have equal roots $\Delta = b^2 - 4ac$ must be zero

$\therefore (2(m + 2))^2 - 4\cdot(1)\cdot 9m = 0 \Rightarrow m^2 - 5m + 4 = 0$

$$\Rightarrow m = \frac{5 \pm \sqrt{25 - 4\cdot(1)\cdot 4}}{2} \Rightarrow m = 1, 4$$

$\therefore$ The values of m are 1, 4

2. If a, b, c are rational then show that the equation $x^2-2ax+a^2-b^2+2bc-c^2 = 0$ has rational roots.

Sol. To show a quadratic equation $ax^2 + bx + c = 0$ to have rational roots it is enough to prove Δ is a perfect square and coeffiecients are rational. Given quadratic equation $x^2-2ax+a^2-b^2+2bc-c^2= 0$. For this

$\Delta = (-2a)^2 - 4(1)(a^2 - b^2 + 2bc - c^2) = 4b^2 - 8bc + 4c^2$

$= 4(b^2 - 2bc + c^2) = (2(b - c))^2$

Which is a square of a rational number $2(b - c)$

$\therefore$ The given equation has rational roots.

3. Find quadratic equation, the sum of whose roots is 1 and sum of the squares of the roots is 13.

Sol. Let α, β be the roots of required equation.

Given $\alpha+\beta=1$ and $\alpha^2+\beta^2=13$

Since $\alpha\beta=\dfrac{(\alpha+\beta)^2-(\alpha^2+\beta^2)}{2}$, $\alpha\beta=-6$

$\therefore$ The required equation i.e., $x^2-x-6=0$

4. Solve $3^{1+x}+3^{1-x}=0$.

Sol. Given equation $3\cdot 3^x+\dfrac{3}{3^x}=10 \Rightarrow 3(3^x)^2-10(3^x)+3=0$

Put $3^x=t$ then given equation will be reduced to quadratic equation $3t^2-10t+3=0$

Now $t=\dfrac{10\pm\sqrt{10^2-4\cdot 3\cdot 3}}{2\cdot 3}=3, \dfrac{1}{3}$

$\therefore 3^x=3; 3^x=\dfrac{1}{3} \Rightarrow x=1,-1$

5. Solve $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=\dfrac{13}{6}$.

Sol. Put $\sqrt{\dfrac{x}{1-x}}=t$

$\therefore$ Given equation becomes $t+\dfrac{1}{t}=\dfrac{13}{6}$

$\Rightarrow 6t^2-13t+6=0 \Rightarrow (2t-3)(3t-2)=0 \Rightarrow t=3/2$ or $2/3$

$\Rightarrow \dfrac{x}{1-x}=\dfrac{9}{4}$ or $\dfrac{x}{1-x}=\dfrac{4}{9} \Rightarrow 13x=9$ or $13x=4$

$\Rightarrow x=\dfrac{9}{13}$ or $\dfrac{4}{13}$

6. Solve $2\left(x^2+\dfrac{1}{x^2}\right)-3\left(x+\dfrac{1}{x}\right)=1$.

Sol. Given equation $2\left(x^2+\dfrac{1}{x^2}\right)-3\left(x+\dfrac{1}{x}\right)-1=0$

$\Rightarrow 2\left(\left(x+\dfrac{1}{x}\right)^2-2\right)-3\left(x+\dfrac{1}{x}\right)-1=0 \Rightarrow 2\left(x+\dfrac{1}{x}\right)^2-3\left(x+\dfrac{1}{x}\right)-5=0$

Put $x+\dfrac{1}{x}=t$ then $2t^2-3t-5=0$

$\Rightarrow (t+1)(2t-5)=0 \Rightarrow t=-1$ or $5/2 \Rightarrow x+\dfrac{1}{x}=-1$ or $x+\dfrac{1}{x}=\dfrac{5}{2}$

$\Rightarrow x^2+x+1=0$ or $2x^2-5x+2=0$

$\Rightarrow x=\dfrac{-1\pm i\sqrt{3}}{2}$ or $2, \dfrac{1}{2}$

Remember :
While solving equations of the type $f\left(x+\dfrac{1}{x}, x^2+\dfrac{1}{x^2}\right)=0$.
Put $x+\dfrac{1}{x}=t$ *so that it becomes* $f(t, t^2-2)=0$

***7.** **Solve $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}$.**

> **Remember :**
> *In real algebra, while solving irraional equations, we restrict ourselves to real solutions only.*

Sol. On squaring both sides of the equation,

$2x+1+3x+2+2\sqrt{(2x+1)(3x+2)}=5x+3$

$\Rightarrow (2x+1)(3x+2)=0 \Rightarrow x=-1/2$ or $-2/3$

For $x=-2/3$, $2x+1$ and $5x+3$ will be negative.

$\therefore x=-1/2$

8. **Solve $(x+1)(x+2)(x+3)(x+4)=120$**

> **Remember :**
> *The equation is of the type $(x+a)(x+b)(x+c)(x+d)=k$ such that sum of the two numbers of a, b, c, d is equal to the sum of the remaining. For if $a+b=c+d$ then seperate $(x+a)(x+b)$ and $(x+c)(x+d)$ and then put $x^2+(a+b)x$ as t*

Sol. The given equation can be written as $((x+1)(x+4))((x+2)(x+3))=120$

$(x^2+5x+4)(x^2+5x+6)=120$

Now put $x^2+5x=t$ then $(t+4)(t+6)=120$

$\Rightarrow t^2+10t-96=0 \Rightarrow (t+16)(t-6)=0$

$\Rightarrow x^2+5x=-16$ or $x^2+5x=6$

$\Rightarrow x^2+5x+16=0$ or $x^2+5x-6=0$

$$\Rightarrow x=\frac{-5\pm\sqrt{25-4\cdot16}}{2} \text{ or } x=\frac{-5\pm\sqrt{25-4\cdot1\cdot(-6)}}{2}$$

$$\Rightarrow x=\frac{-5\pm i\sqrt{39}}{2} \text{ or } x=-6,1$$

$$\therefore \text{ The solution set of } x=\left\{-6,1,\frac{-5\pm i\sqrt{39}}{2}\right\}$$

9. **Solve $2x^4+x^3-11x^2+x+2=0$**

> **Remember :**
> *For, $ax^4+bx^3+cx^2+bx+a=0$, in which coefficients equidistant from both ends are equal. Divide with x^2 and put $x+\frac{1}{x}=t$*

Sol. Divide the given equation by x^2 both sides then it becomes

$$2x^2+x-11+\frac{1}{x}+\frac{2}{x^2}=0$$

$$\text{i.e., } 2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0$$

$$\Rightarrow 2\left(\left(x+\frac{1}{x}\right)^2-2\right)+\left(x+\frac{1}{x}\right)-11=0$$

$$\Rightarrow 2\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-15=0$$

Now put $x+\frac{1}{x}=t$ then $2t^2+t-15=0$

$$\Rightarrow t=\frac{-1\pm\sqrt{1+120}}{4} \Rightarrow t=-3 \text{ or } \frac{5}{2}$$

$$x+\frac{1}{x}=-3 \text{ or } x+\frac{1}{x}=\frac{5}{2}$$

$\Rightarrow x^2+3x+1=0$ or $2x^2-5x+2=0$

$$\Rightarrow x=\frac{-3\pm\sqrt{9-4}}{2} \text{ or } x=\frac{5\pm\sqrt{25-16}}{4} \quad x=\frac{-3+\sqrt{5}}{2},\frac{-3-\sqrt{5}}{2},2,\frac{1}{2}$$

10. **If the roots of the equation $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$ are equal in magnitude but opposite in sign then show that $p+q=2r$ and that the product of the roots is equal to $\frac{-1}{2}(p^2+q^2)$**

Sol. The given equation can be written as $x^2+(p+q)x+pq=r(2x+p+q)$

i.e., $x^2+(p+q-2r)x+pq-r(p+q)=0$

According to the given condition, we have sum of the roots = zero

i.e., $p+q-2r=0$ gives $p+q=2r$ .... (1)

Now, we have product of the roots $= pq-r(p+q)$

$= pq-\frac{(p+q)^2}{2}$ (using result (1))

$= -\left[\frac{(p+q)^2-2pq}{2}\right]=-\frac{1}{2}(p^2+q^2)$

which is the desired result.

Remember : *For a quadratic $ax^2+bx+c=0$ if the roots are of opposite signs then* $\boxed{b=0}$

11. **If α,β are the roots of $ax^2+bx+c=0$ ($\alpha\neq\beta$) and α^4,β^4 are the roots of $lx^2+mx+n=0$, then prove that the roots of the equation $a^2lx^2-4aclx+2c^2l+a^2m=0$ are always real and opposite in sign.**

Sol. We have $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$ $(a\neq 0)$ and $\alpha^4+\beta^4=\frac{-m}{l}$, $\alpha^4\beta^4=\frac{n}{l}$ $(l\neq 0)$

The given equation $a^2lx^2-4aclx+2c^2l+a^2m=0$ has discriminant

$D=16a^2c^2l^2-4a^2l(2c^2l+a^2m)$

$=8a^2c^2l^2-4a^4lm$

$=4a^4l^2\left(\frac{2c^2}{a^2}-\frac{m}{l}\right)$

$=4a^4l^2\,(\alpha^2+\beta^2)>0$ $\left[\because \frac{-m}{l}=\alpha^4+\beta^4>0 \text{ and } \frac{2c^2}{a^2}=2\alpha^2\beta^2\right]$

Hence, the roots are real.

Also, we have product of the roots $=\frac{2c^2l+a^2m}{a^2l}=\frac{2c^2}{a^2}+\frac{m}{l}$

$=2\alpha^2\beta^2-(\alpha^4+\beta^4)=-(\alpha^2-\beta^2)^2<0$

which proves that the roots are of opposite signs.

Remember : *For a quadratic equation $ax^2+bx+c=0$ roots are real and of opposite sign, then* $\Delta>0$, $\frac{c}{a}<0$

THEOREM-1.4 **A necessary and sufficient condition for two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have common root.**

Proof : **Necessary condition :** Let α be the common root of the given equations

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ .....(1)

$a_2\alpha^2 + b_2\alpha + c_2 = 0$ .....(2)

Solve (1) & (2)
$$\begin{array}{cccc} & \alpha^2 & \alpha & 1 \\ b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

We get $\alpha^2 = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

eliminate 'α' we get $(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$

> ***Remember :***
> *Identify the scheme*
> $$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$
> *i)* $\Delta_2^2 = \Delta_1 \cdot \Delta_3$
> *ii) common root* $\dfrac{\Delta_1}{\Delta_2}$ $\left(or\ \dfrac{\Delta_2}{\Delta_3}\right)$

Sufficiency condition :

Suppose $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$

Case - I : $a_1b_2 - a_2b_1 = 0$ then $c_1a_2 - c_2a_1 = 0$

$\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ and $\dfrac{a_1}{a_2} = \dfrac{c_1}{c_2}$,

Hence $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$\therefore$ $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have same roots.

Case - II : $a_1b_2 - a_2b_1 \neq 0$

Let $\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

from (1)

$$a_1\alpha^2 + b_1\alpha + c_1 = a\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2 + b\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_1 = 0$$

similarly we can prove that $a_2\alpha^2 + b_2\alpha + c_2 = 0$

Thus 'α' is a common root of the given equations.

THEOREM-1.5 **If $f(x) \equiv ax^2 + bx + c = 0$ is quadratic equation then the quadratic equation whose roots are**

i) the reciprocals of the roots of $f(x) = 0$ is $f\left(\dfrac{1}{x}\right) = 0$

ii) greater by m than those of $f(x) = 0$ is $f(x-m) = 0$

iii) smaller by m than those of $f(x) = 0$ is $f(x+m) = 0$

iv) multiples by m of those of $f(x) = 0$ is $f\left(\dfrac{x}{m}\right) = 0$.

v) submultiples by m of those of $f(x) = 0$ is $f(mx) = 0$

vi) numerically equal but of opposite in sign of $f(x) = 0$ is $f(-x) = 0$.

vii) squares of the roots of $f(x) = 0$ is $f(\sqrt{x}) = 0$

viii) cubes of the roots of $f(x) = 0$ is $f(\sqrt[3]{x}) = 0$

Example :

1. The equation whose roots are reciprocals of the roots of $5x^2 + 6x + 7 = 0$ is $5\left(\frac{1}{x}\right)^2 + 6\left(\frac{1}{x}\right) + 7 = 0$ i.e., $7x^2 + 6x + 5 = 0$
2. The equation whose roots are equal but opposite in sign to the roots of $2x^2 + 3x + 4 = 0$ is $2(-x)^2 + 3(-x) + 4 = 0$ i.e., $2x^2 - 3x + 4 = 0$
3. The equation whose roots, are 3 times the roots of the equation $x^2 - 5x + 6 = 0$ is $\left(\frac{x}{3}\right)^2 - 5\left(\frac{x}{3}\right) + 6 = 0$ i.e., $x^2 - 15x + 54 = 0$.
4. The equation whose roots are $\frac{1}{4}$ times of the roots of the equation $x^2 - 3x + 2 = 0$ is $(4x)^2 - 3(4x) + 2 = 0$ i.e., $16x^2 - 12x + 2 = 0$
5. The equation whose roots are greater by 2 of the roots of $x^2 - 7x + 12 = 0$ is $(x-2)^2 - 7(x-2) + 12 = 0$ i.e., $x^2 - 11x + 30 = 0$.
6. If α, β are the roots of $2x^2 + x + 3 = 0$, then the equation whose roots are $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$ is $2(1-x)^2 + (1-x)(1+x) + 3(1+x)^2 = 0$, i.e. $2x^2 + x + 3 = 0$.

EXERCISE - 1.1

1. Find the nature of the roots of the following equations without finding roots.
   i) $9x^2 - 30x + 25 = 0$ [Ans : rational and equal]
   ii) $2x^2 - 7x + 10 = 0$ [Ans : imaginary]
   iii) $x^2 + 2\sqrt{3}x - 1 = 0$ [Ans : real and unequal]
   iv) $2x^2 - 6x + 7 = 0$ [Ans : imaginary]
2. Find the quadratic equations whose roots are
   i) $\frac{a}{b}, \frac{b}{a}$ $(a \neq 0, b \neq 0)$ [Ans : $abx^2 - (a^2+b^2)x + ab = 0$]
   ii) $2 \pm 3i$ [Ans : $x^2 - 4x + 13 = 0$]
   *iii) $\frac{a-b}{a+b}, \frac{a+b}{a-b}$ $(a \neq \pm b)$ [Ans : $(a^2 - b^2)x^2 - 2(a^2+b^2)x + a^2 - b^2 = 0$]
   *iv) $7 \pm 2\sqrt{5}$ (May-18)
3. Find the values of the following in terms of a, b, c if α, β are roots of quadratic equation $ax^2 + bx + c = 0, c \neq 0$.
   *i) a) $\frac{1}{\alpha} + \frac{1}{\beta}$ [Ans : $\frac{-b}{c}$]
   *b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ [Ans : $\frac{b^2 - 2ac}{c^2}$]
   ii) $\alpha^3 + \beta^3$ [Ans : $\frac{3abc - b^3}{a^3}$]
   *iii) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ [Ans : $\frac{b^2(b^2 - 4ac)}{c^2a^2}$]
   iv) $\alpha^4\beta^7 + \alpha^7\beta^4$ [Ans : $\frac{bc^4(3ac - b^2)}{a^7}$]
   v) $a(\alpha^3 + \beta^3) + b(\alpha^2 + \beta^2) + c(\alpha + \beta)$ [Ans : 0]

4. Find the value of a if the following quadratic equations have equal roots. (in each case)

*i) $x^2 - 15 - a(2x - 8) = 0$ **(March-19)** **[Ans : 3 or 5]**

*ii) $(a + 1)x^2 + 2(a + 3)x + (a + 8) = 0$ **[Ans : $\frac{1}{3}$]**

iii) $x^2 - 2(1 + 3a)x + 7(3 + 2a) = 0$ **[Ans : 2 or $-\frac{10}{9}$]**

5. If α, β are roots of the equation $ax^2 + bx + c = 0$ then find quadratic equations whose roots are

i) $m\alpha, m\beta$ **[Ans : $ax^2 + mbx + m^2c = 0$]**

*ii) $\alpha^2 + \beta^2, \frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **[Ans : $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$]**

iii) $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$ **[Ans : $cx^2 + (b + 2c)x + (a + b + c) = 0$]**

*6. Find the quadratic equation for which the sum of the roots is 7 and the sum of the squares of the roots is 25. **[Ans : $x^2 - 7x + 12 = 0$]**

*7. One fourth of a herd of goats was seen in the forest. Twice the square root of the number of the herd had gone up the hill and the remaining 15 goats were on the bank of a river. Find total number of goats. **[Ans : 36]**

*8. i) Prove that there is a unique pair of consecutive positive odd integers such that the sum of their squares is 290. Find them. **[Ans : (11, 13)]**

ii) If the harmonic mean between roots of $(2+\sqrt{5})x^2 - 3bx + 6 + 2\sqrt{3} = 0$ is 12 then show that $b = \frac{1}{3} + \frac{1}{3\sqrt{3}}$

9. Solve the following equations

*i) $4^{x-1} - 3.2^{x-1} + 2 = 0$ **[Ans : {1, 2}]**

*ii) $\sqrt{\frac{3x}{x+1}} + \sqrt{\frac{x+1}{3x}} = 2 \quad x \neq 0, x \neq 3$ **[Ans : $\left\{\frac{1}{2}\right\}$]**

*iii) $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0, \ x \neq 0$ **[Ans : $\left\{\frac{1 \pm i\sqrt{3}}{2}, 2 \pm \sqrt{3}\right\}$]**

*iv) $(x - 1)(x - 3)(x - 5)(x - 7) = 9$ **[Ans : $\{4, 4 \pm \sqrt{10}\}$]**

v) $x^4 - 2x^3 - x^2 - 2x + 1 = 0$ **[Ans : $\left\{\frac{3 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{3}i}{2}\right\}$]**

*vi) $\sqrt{3x+1} - \sqrt{x-1} = 2$ **[Ans : {1, 5}]**

10. i) Find the condition for the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root ? **[Ans : $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$]**

*ii) If x_1, x_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$ and $c \neq 0$, find the value of $(ax_1 + b)^{-2} + (ax_2 + b)^{-2}$ in terms of a, b, c **(March-17)** **[Ans : $\frac{b^2 - 2ac}{a^2c^2}$]**

iii) Find k if

a) $x^2-6x+5=0$ and $x^2-12x+k=0$ have a common root. **(March-17, 18)[Ans : 11 or 35]

b) $x^2+4kx+3=0$ and $2x^2-3kx-9=0$ have a common root. [Ans : $\pm 1, \pm 3$]

*c) If the equation $ax^2+2bx+c=0$ and $ax^2+2cx+b=0$ $(b \neq c)$ have a common root then show that $a+4b+4c=0$.

*d) If the equation $x^2+ax+b=0$ and $x^2+cx+d=0$ have a common root and if the first equation has equal roots then prove that $2(b+d)=ac$.

*e) If $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root then show that $a^3+b^3+c^3=3abc$.

*f) Show that the value of 'a' for which the equations $x^3+ax+1=0$ and $x^4+ax^2+1=0$ have a common root is -2.

*g) If $x^2+3x+5=0$ and $ax^2+bx+c=0$ have a common root, $a, b, c \in N$, find the minimum value of $a+b+c$. [Ans : 9]

1.4 SIGN OF THE QUADRATIC EXPRESSION ax^2+bx+c

THEOREM-1.6 **If the roots of $ax^2+bx+c=0$ are imaginary, then for $x \in R$, ax^2+bx+c and a have the same sign.**

> *Remember :* Roots are imaginary $\Rightarrow$ curve doesn't meet x-axis

Proof : The roots are imaginary

$\Rightarrow b^2-4ac<0 \Rightarrow 4ac-b^2>0$

$$\frac{ax^2+bx+c}{a}=x^2+\frac{b}{a}x+\frac{c}{a}=\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}=\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}>0$$

$\therefore$ For $x \in R$, ax^2+bx+c and a have the same sign.

> *Remember :* Roots are real and equal $\Rightarrow$ curve touches x-axis

Corollary

If the roots of $ax^2+bx+c=0$ are real and equal then ax^2+bx+c and a will have same sign.

THEOREM-1.7 **Let α, β be the real roots of $ax^2+bx+c=0$ and $\alpha<\beta$. Then**

i) $x \in R, \alpha<x<\beta \Rightarrow ax^2+bx+c$ and a have the opposite signs

ii) $x \in R, x<\alpha$ or $x>\beta \Rightarrow ax^2+bx+c$ and a have the same sign.

> *Remember :* Roots are real and distinct $\Rightarrow$ curve intersects x-axis

Proof : α, β are the roots of $ax^2+bx+c=0$

$\Rightarrow ax^2+bx+c=a(x-\alpha)(x-\beta)$

$$\Rightarrow \frac{ax^2+bx+c}{a}=(x-\alpha)(x-\beta)$$

i) If $\alpha<x<\beta$ then $\dfrac{ax^2+bx+c}{a}<0$

$\Rightarrow ax^2+bx+c$ and a will have opposite signs.

ii) If $x<\alpha$ or $x>\beta$, then $(x-\alpha)(x-\beta)>0$

$\Rightarrow ax^2+bx+c$ and a will have same sign.

Ex: **Determine the sign of (i) x^2+x+1; (ii) $x^2-7x+12$ for $x \in R$.**

Sol. i) the roots of $x^2+x+1=0$ are imaginary as $\Delta = -3 < 0$ and coefficient is $x^2 = 1$ (i.e > 0)

$\Rightarrow x^2+x+1>0$

ii) $\therefore x^2-7x+12>0$ for $x<3$ or $x>4$

$x^2-7x+12<0$ for $3<x<4$

1.5 MAXIMUM AND MINIMUM VALUES OF ax^2+bx+c

The maximum and minimum values of a quadratic expression with real coefficients depend on the sign of the coefficient of x^2.

THEOREM-1.8 **Suppose that $a, b, c \in R$, $a \neq 0$ and $f(x) = ax^2+bx+c$;**

i) If $a > 0$, then $f(x)$ has absolute minimum at $x = -\dfrac{b}{2a}$ and the minimum value is $\dfrac{4ac-b^2}{4a}$.

ii) If $a < 0$ then $f(x)$ has absolute maximum at $x = -\dfrac{b}{2a}$ and the maximum value is $\dfrac{4ac-b^2}{4a}$.

Remember : *Either maximum or minimum of ax^2+bx+c is given by $\dfrac{4ac-b^2}{4a}$*

Proof : Since $f(x) = ax^2+bx+c$

$$= a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

i) If $a>0$ then $f(x) \geq \dfrac{4ac-b^2}{4a}$ for all $x \in R$ and when $x = \dfrac{-b}{2a}$ $f(x) = \dfrac{4ac-b^2}{4a}$.

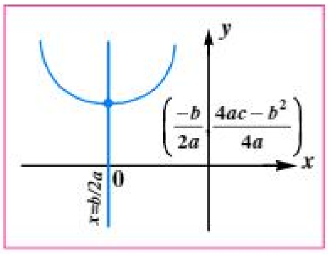
$\therefore$ $f(x)$ has minimum at $x = -\dfrac{b}{2a}$

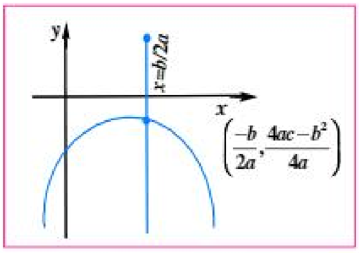
ii) If $a<0$ then $f(x) \leq \dfrac{4ac-b^2}{4a}$ for all $x \in R$ and when $x = \dfrac{-b}{2a}$ $f(x) = \dfrac{4ac-b^2}{4a}$.

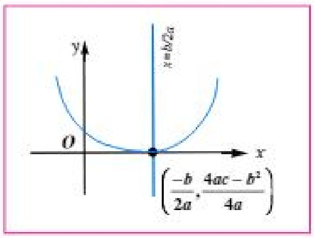
$\therefore$ $f(x)$ has miximum value $\dfrac{4ac-b^2}{4a}$ at $x = \dfrac{-b}{2a}$

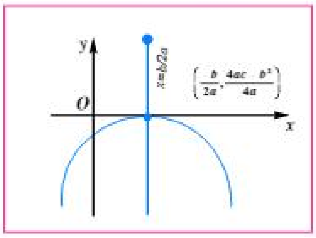
The graph of $f(x) = ax^2+bx+c = a\left(x+\dfrac{b}{2a}\right)^2 + \dfrac{4ac-b^2}{4a}$ is one of the following.

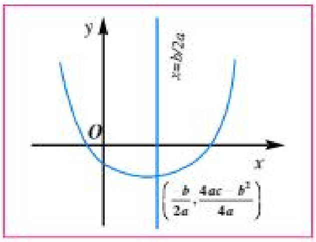
i) $a > 0$: **ii) $a < 0$**

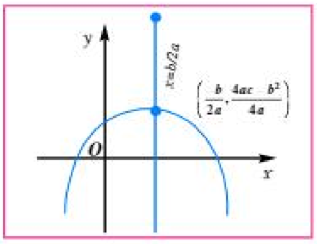












Note

i) The extreme value of f(x) is the ordinate of vertex of the parabola $\left(x+\frac{b}{2a}\right)^2=\frac{1}{a}\left(y-\frac{4ac-b^2}{4a}\right)$ *where* $y = f(x)$.

ii) Using calculus theorem, note that the extremum value of $y = f(x)$ *can be identified by the roots of* $f'(x)=0$. *Here* $y = ax^2 + bx + c$, $\frac{dy}{dx}=0$ *at* $x=-\frac{b}{2a}$; $\frac{d^2y}{dx^2}=2a$

$\therefore$ *If* $a > 0$ *; y has minimum value and if* $a < 0$ *; y has maximum value.*

Remember : *Every polynomial expression in x is continuous and differentiable everywhere*

Example :

Find maximum or minimum to the expressions (i) $x^2 - x + 2$; **(ii)** $4x - x^2 - 10$.

i) $a = 1, b = -1, c = 2$

Since $a > 0$ the expression has only minimum value at $x=\frac{1}{2}$ and the value is $\frac{4ac-b^2}{4a}=\frac{4(1)(2)-(1)}{4(1)}=\frac{7}{4}$.

ii) $a = -1, b = 4, c = -10$

Since $a < 0$. The expression has only maximum value at $x = 2$ and the maximum value is $\frac{4(-10)(-1)-(16)}{4(-1)}=-6$.

1.6 QUADRATIC INEQUALITIES

Definition

If $ax^2 + bx + c$, *a, b, c real, is a quadratic expression then* $ax^2 + bx + c > 0$ *or* $ax^2 + bx + c \geq 0$ *or* $ax^2 + bx + c < 0$ *or* $ax^2 + bx + c \leq 0$ *is called quadratic inequation or inequality.*

Example : $x^2 - 7x + 12 > 0$, $2x^2 - 3x - 4 \leq 0$, $3x^2 + 4x + 5 \geq 0$.

1.7 SOLVING QUADRATIC INEQUALITIES

There are two methods of solving inequations.

i) Algebraic method :

In this method, finding solution by observing the sign changes of quadratic expression.

ii) Graphical method :

In this method, finding solution by observing the graph of the quadratic expression.

SOLVED EXAMPLES

1. Solve $x^2 - 7x + 12 < 0$

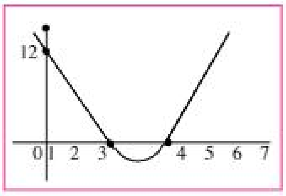
Sol. i) **Algebraic method :**

$x^2 - 7x + 12 = (x-3)(x-4) = x^2 - 7x + 12 < 0$ if $x \in (3, 4)$

ii) **Graphical method :**

Plot the graph of $y = x^2 - 7x + 12$ by taking values for $x = 0, 1, 2, 3, \ldots, 7$

x	0	1	2	3	4	5	6	7
y	12	6	2	0	0	2	6	12



Clearly the graph of $y = f(x)$ is below x-axis in the interval (3, 4).

$\therefore\ x^2 - 7x + 12 < 0 \Rightarrow x \in (3, 4)$

2. Solve (i) $x^2 - 5x + 6 < 0$; (ii) $2x^2 + 3x - 2 \geq 0$; (iii) $x^2 - 4x + 5 \leq 0$.

Sol. i) $x^2 - 5x + 6 = (x-2)(x-3) < 0$

$\Rightarrow x \in (2, 3)$

ii) $2x^2 + 3x - 2 = (x+2)(2x-1) = 2\left(x - \frac{1}{2}\right)(x+2)$

$2x^2 + 3x - 2 \geq 0 \Rightarrow 2\left(x - \frac{1}{2}\right)(x+2) \geq 0$

$\Rightarrow x \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

iii) $x^2 - 4x + 5 = (x-2)^2 + 1$ which is always positive.

$\therefore$ There is no real number x for which $x^2 - 4x + 5 \leq 0$

3. Solve $\dfrac{(x+1)(x-3)}{(x-2)} \geq 0$.

Sol. $f(x) = \dfrac{(x+1)(x-3)}{(x-2)} \geq 0$

$\Rightarrow \dfrac{(x+1)(x-3)(x-2)}{(x-2)^2} \geq 0$

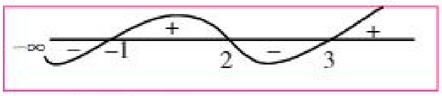
$\Rightarrow (x+1)(x-2)(x-3) \geq 0$

If $x < -1$ then $x + 1 < 0, x - 2 < 0, x - 3 < 0 \Rightarrow f(x) < 0$

If $-1 < x < 2$ then $x + 1 > 0, x - 2 < 0, x - 3 < 0 \Rightarrow f(x) > 0$

If $2 < x < 3$ then $x + 1 > 0, x - 2 > 0, x - 3 < 0 \Rightarrow f(x) < 0$

If $x > 3$ then $x + 1 > 0, x - 2 > 0, x - 3 < 0 \Rightarrow f(x) > 0$



$\therefore f(x) \geq 0$ for $x \in [-1, 2) \cup [3, \infty)$

4. Find the range of $\frac{x^2+x+1}{x^2-x+1}$, for $x \in R$

Sol. Let $y = \frac{x^2+x+1}{x^2-x+1}$ then $y(x^2-x+1)-(x^2+x+1)=0$

i.e., $(y-1)x^2-(y+1)x+(y-1)=0$

since x is real, discriminant ≥ 0

$\Rightarrow (-(y+1))^2-4(y-1)^2 \geq 0$

$\Rightarrow y^2+2y+1-4y^2+8y-4 \geq 0$

$\Rightarrow -3y^2+10y-3=0$

$\Rightarrow 3y^2-10y+3 \leq 0$

$\Rightarrow (y-3)(3y-1) \leq 0 \Rightarrow y \in \left[\frac{1}{3}, 3\right]$

$\therefore \frac{1}{3} \leq \frac{x^2+x+1}{x^2-x+1} \leq 3$

****5. Find the maximum value of the function $\frac{x^2+14x+9}{x^2+2x+3}$ for $x \in R$. (March-18)**

Sol. Let $y = \frac{x^2+14x+9}{x^2+2x+3}$

$y(x^2+2x+3)=x^2+14x+9$

$\Rightarrow (y-1)x^2+(2y-14)x+3y-9=0$

since x is real, discriminant ≥ 0

$\Rightarrow (2(y-7))^2-4(y-1)(3y-9) \geq 0$

$\Rightarrow -8y^2-8y+160 \geq 0 \Rightarrow y^2+y-20 \leq 0$

$\Rightarrow (y+5)(y-4) \leq 0 \Rightarrow -5 \leq y \leq 4$

$\therefore$ The maximum value of y is 4.

***6. If $\frac{x-c}{x^2-3x+2}$ takes all real values for $x \in R$ then show that $1<c<2$.**

Sol. Let $y = \frac{x-c}{x^2-3x+2}$

$\Rightarrow yx^2-(3y+1)x+(2y+c)=0$

Since x is real, $(-(3y+1))^2-4.y.(2y+c) \geq 0$

$\Rightarrow y^2+(6-4c)y+1 \geq 0$

But y is real and coefficient of $y^2=1>0$ the discriminant of $y^2+(6-4c)y+1=0$ should be negative.

$(6-4c)^2-4 \leq 0 \Rightarrow c^2-3c+2 \leq 0$

$\Rightarrow (c-1)(c-2) \leq 0 \Rightarrow 1 \leq c \leq 2$

but for $c=1$ or $c=2$

$\frac{x-c}{x^2-3x+2}$ can not take all real values.

$\therefore 1<c<2$

***7. Solve $\sqrt{(x-3)(2-x)} < \sqrt{4x^2+12x+11}$.**

Sol. $(x-3)(2-x) \geq 0;\ 4x^2+12x+11 \geq 0$

i.e., $(x-2)(x-3) \leq 0$; $4x^2+12x+11 \geq 0$ $x \in [2, 3]$;

$\because\ \Delta < 0, x \in R$

On squaring both sides

$-x^2+5x-6 < 4x^2+12x+11$

$\Rightarrow 5x^2+7x+17 > 0$

$\Delta = 49 - 4.5.17 < 0$

$\therefore\ 5x^2+7x+17 > 0$ for $x \in R$

$\therefore\ (x-3)(2-x) \geq 0; 4x^2+12x+11 \geq 0$ and $5x^2+7x+17 > 0$ simultaneously hold for $x \in [2, 3]$.

Remember :

Irrational inequation, and we search for real solutions only.

8. Solve $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+2}$.

Sol. $\dfrac{2x}{2x^2+5x+2} - \dfrac{1}{x+2} > 0$

$\Rightarrow \dfrac{2x}{(2x+1)(x+2)} - \dfrac{1}{x+2} > 0$

$\Rightarrow \dfrac{2x-(2x+1)}{(2x+1)(x+2)} > 0$

$\Rightarrow \dfrac{1}{(2x+1)(x+2)} < 0$

$\Rightarrow (x+2)(2x+1) < 0$

$\Rightarrow x \in \left(-2, -\dfrac{1}{2}\right)$

9. If x, a, b are real, then prove that the maximum value of $4(a-x)(x-a+\sqrt{a^2+b^2})$ is (a^2+b^2).

Sol. Let $y = 4(a-x)((x-a)+\sqrt{a^2+b^2})$

$y = -4(a-x)^2 + 4(a-x)\sqrt{a^2+b^2}$

$\therefore 4x^2 + 4x[\sqrt{a^2+b^2} - 2a] + 4a^2 + y - 4a\sqrt{a^2+b^2} = 0$

$\because\ x$ is real, $\Delta \geq 0$

$\therefore 16(\sqrt{a^2+b^2} - 2a)^2 - 4.4[4a^2 + y - 4a\sqrt{a^2+b^2}\,] \geq 0$

$\therefore (\sqrt{a^2+b^2} - 2a)^2 - [4a^2 + y - 4a\sqrt{a^2+b^2}\,] \geq 0$

$a^2+b^2+4a^2-4a^2-y \geq 0 \Rightarrow y \leq a^2+b^2$

$\therefore$ Maximum value of $4(a-x)(x-a+\sqrt{a^2+b^2})$ is (a^2+b^2)

EXERCISE - 1.2

1. For what values of x the following expressions are positive ?

*i) $x^2 - 5x + 14$ **(March-18)** [Ans : R]

ii) $x^2 - 8x + 12$ [Ans : $(-\infty, 2) \cup (6, \infty)$]

iii) $x^2 + 2x + 3$ [Ans : R]

iv) $x^2 + x + 1$ [Ans : R]

2. For what values of x the following expressions are negative ?

*i) $x^2 + x - 12$ **(March-19)** [Ans : $(3, 4)$]

ii) $4x^2 - 12x + 9$ [Ans : ϕ]

iii) $-6x^2 + 2x - 3$ [Ans : R]

iv) $-3x^2 + 6x - 7$ [Ans : R]

3. Find the maximum or minimum value of the expression

i) $12x - x^2 - 32$ [Ans : max. value = 4]

ii) $x^2 + 7x - 5$ [Ans : minimum value = $-\dfrac{69}{4}$]

iii) $-2x^2 + 3x + 4$ [Ans : max. value = $\dfrac{41}{8}$]

iv) $4 - \dfrac{3}{x^2 - 4x + 6}$ [Ans : min value = $\dfrac{5}{2}$]

*4. If $c^2 \neq ab$ and the roots of $(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$ are equal, then show that $a^3 + b^3 + c^3 = 3abc$ (or) $a = 0$. **(March-19)**

*5. If x is real show that $\dfrac{x}{x^2 - 5x + 9}$ lies between $\dfrac{-1}{11}$ and 1 **(March-19)**

6. Prove that $\dfrac{1}{3x+1} + \dfrac{1}{x+1} - \dfrac{1}{(3x+1)(x+1)}$ does not lie between 1 and 4 if x is real. **(May-18)

7. Find the least and greatest values of

*i) $\dfrac{x+2}{2x^2 + 3x + 6}$ **(May-19)** [Ans : $\dfrac{-1}{13}$ and $\dfrac{1}{3}$]

ii) $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$ [Ans : 5 and 9]

iii) $\dfrac{x^2 + 2x + 1}{x^2 + 2x + 7}$ [Ans : 0 and 1]

iv) $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ [Ans : $\dfrac{1}{7}$ and 7]

8. Solve

i) $\sqrt{6x-x^2-5} > 8-2x$ [Ans : (3, 5]]

*ii) $\sqrt{x+2} > \sqrt{8-x^2}$ [Ans : (2, $2\sqrt{2}$]]

*9. Solve $\dfrac{\sqrt{6+x-x^2}}{2x+5} \geq \dfrac{\sqrt{6+x-x^2}}{x+4}$ [Ans : [–2, –1] ∪ {3}]

10. Find the set of values of k for which $x^2+(4k-1)x+9>0\ \forall x \in R$ [Ans : $(\frac{-5}{4}, \frac{7}{4})$]

11. If $x \leq -3$, find the minimum value of the expression $\sqrt{9-6x+x^2}+\sqrt{9+6x+x^2}$ [Ans : 6]

1.8 IDENTITY PROPERTY

i) The equation $ax^2 + bx + c = 0$, $(a \neq 0)$ possesses exactly two roots. But if the equation $ax^2 + bx + c = 0$ is satisfied by more than two distinct values of x then it is called an identity, and we have $a = 0$, $b = 0$, $c = 0$.

since, if α, β, γ are roots & $\alpha \neq \beta \neq \gamma$ We have

$a\alpha^2 + b\alpha^2 + c = 0$ ----- (1)

$a\beta^2 + b\beta^2 + c = 0$ ----- (2)

$a\gamma^2 + b\gamma^2 + c = 0$ ----- (3)

(1) – (2) $\Rightarrow a(\alpha+\beta)+b=0$ ----- (4)

(2) – (3) $\Rightarrow a(\beta+\gamma)+b=0$ ----- (5)

Now (4) – (5) $\Rightarrow \boxed{a=0}$ and thus $\boxed{b=0}, \boxed{c=0}$

Thus

$$\left.\begin{array}{l} ax^2+bx+c=0 \text{ is an identity} \\ \text{i.e. possesses infinite solutions} \end{array}\right\} \Leftrightarrow a=o, b=0, c=0$$

Remember : *An equation is said to be identity, if it is true for all real or imaginary values of the variable in that equation*

Example :

The equation $a(a-1)x^2+(a^2-1)x+(a^2-3a+2)=0$ possesses infinite solutions, find a ?

$a(a-1)=0$ ---(1)

$a^2-1=0$ ---(2)

$(a-1)(a-2)$ ---(3)

commonly we have $\boxed{a=1}$

ii) Let α, β are the roots of quadratic $ax^2 + bx + c = 0$ then we have the identity in 'n' say $a(\alpha^n+\beta^n)+b(\alpha^{n-1}+\beta^{n-1})+c(\alpha^{n-2}+\beta^{n-2})=0$

Remember : *For $ax^2+bx+c=0$ if α, β are roots and if $S_n=\alpha^n+\beta^n$ then we have identity $aS_n+bS_{n-1}+cS_{n-2}=0$ for all $n \in Z$.*

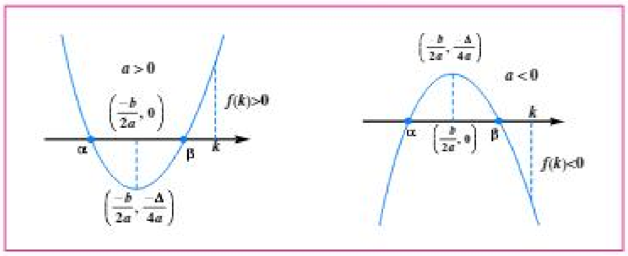
Example :

If α, β are the roots of $2x^2+3x+7=0$ then $\dfrac{2(\alpha^{10}+\beta^{10})+3(\alpha^9+\beta^9)}{(\alpha^8+\beta^8)}=-7$

1.9 CRITERIA FOR FINDING UNKNOWN FOR GIVEN CONDITION(S)

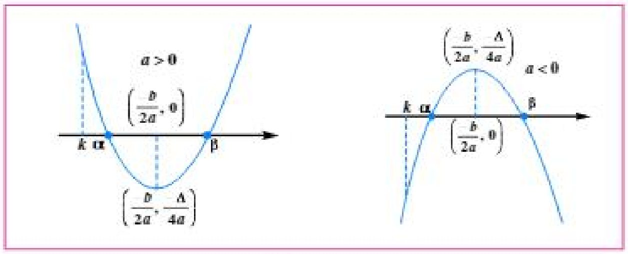
Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and $a, b, c \in R$, and α, β are roots of $f(x) = 0$. Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$, then remember the following.

a) Condition for a number k if both roots of $f(x) = 0$ are less than k.



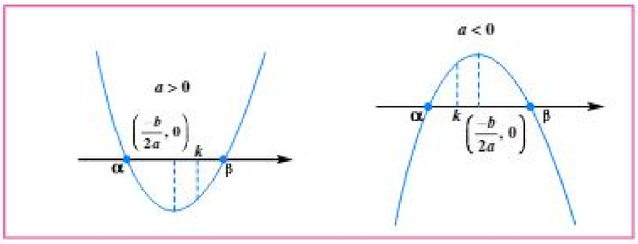
(i) $\Delta \geq 0$; (ii) $af(k) > 0$; (iii) $k > \dfrac{-b}{2a}$ where $\alpha \leq \beta$

b) Condition for a number k if both roots of $f(x) = 0$ are greater than k.



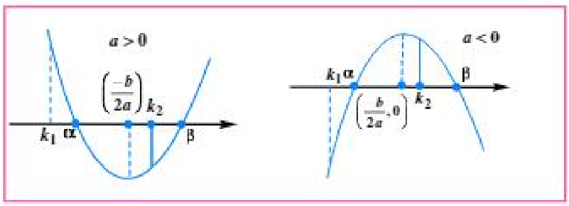
(i) $\Delta \geq 0$; (ii) $af(k) > 0$; (iii) $k < \dfrac{-b}{2a}$

c) Condition for k if k lies between the roots of $f(x) = 0$



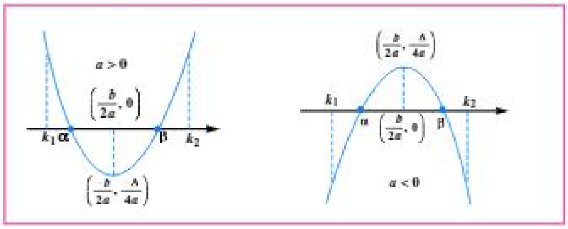
(i) $\Delta > 0$; (ii) $af(k) < 0$ where $\alpha < \beta$

d) Condition for the numbers k_1 and k_2 if exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2).



(i) $\Delta > 0$; (ii) $f(k_1)f(k_2) < 0$ where $\alpha < \beta$

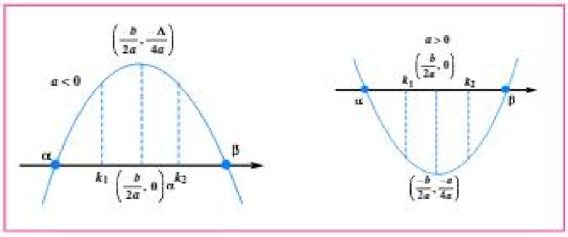
e) Conditions for numbers k_1 and k_2 if both roots of $f(x) = 0$ are confined between k_1 and k_2.



(i) $\Delta \geq 0$; (ii) $af(k_1) > 0$, $af(k_2) > 0$

(iii) $k_1 < \dfrac{-b}{2a} < k_2$ where $\alpha \leq \beta$ and $k_1 < k_2$

f) Condition for numbers k_1 and k_2 if k_1 and k_2 lie between the roots $f(x) = 0$



(i) $\Delta > 0$

(ii) $af(k_1) < 0$, $af(k_2) < 0$ where $k_1 < k_2$, $\alpha < \beta$.

1.10 FACTORIZATION OF SECOND DEGREE EXPRESSION IN x and y

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, in x and y may be resolved into two rational linear factors, by taking corresponding equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and putting it as $ax^2 + 2x(hy+g) + by^2 + 2fy + c = 0$ and solving for x.

we have,

$$x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2+2fy+c)}}{2a} \qquad (a \neq 0)$$

and the equation will have rational linear factors if the expression under the root is a perfect square i.e $4(hy+g)^2 - 4a(by^2+2fy+c) = 0$ has real and equal roots.

i.e its discrimant = 0 which gives

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example: If $2xy + y^2 + 2x + py - 3 = 0$ be capable of resolution into two rational linear factors, find p ?

Clearly $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & p/2 \\ 1 & p/2 & -3 \end{vmatrix} = 0 \quad \Rightarrow p = -2$

SOLVED EXAMPLES

1. Show that if p, q, r and s are real numbers and $pr = 2(q+s)$ then atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Sol. Let Δ_1 and Δ_2 be the discriminants of $x^2+px+q=0$ and $x^2+rx+s=0$ respectively,

then $\Delta_1+\Delta_2 = (p^2-4q)+(r^2-4s) = p^2+r^2-4(q+s)$

$= p^2+r^2-2pr = (p-r)^2 \geq 0$

$\therefore$ atleast one of $\Delta_1,\ \Delta_2$ is positive.

$\therefore$ atleast one of the two equations has real roots.

2. Show that the roots of the equation $(a^4+b^4)x^2+4abcdx+c^4+d^4=0$ cannot be distinct, if real.

Sol. The discriminant Δ of the equation given is $\Delta = 16a^2b^2c^2d^2-4(a^4+b^4)(c^4+d^4)$

$= -4[(a^4c^4+b^4d^4-2a^2b^2c^2d^2)+a^4d^4+b^4c^4-2a^2b^2c^2d^2)]$

$= -4((a^2c^2-b^2d^2)^2+(a^2d^2-b^2c^2)^2) \leq 0 \quad ...(1)$

If the roots are real then $\Delta \geq 0 \quad ....(2)$

$\therefore\ \Delta = 0$ hence the roots are equal. [From (1) & (2)]

3. If $a<b<c<d$ then show that the roots of the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ are real and distinct.

Sol. Given $(x-a)(x-c)+2(x-b)(x-d)=0 \Rightarrow 3x^2-(a+c+2b+2d)x+(ac+2bd)$

Let D be the discriminant of this equation, then

$D=(a+c+2b+2d)^2-12(ac+2bd)=0=((a+2d)+(c+2b))^2-12(ac+2bd)$

$=((a+2d)-(c+2b))^2+4(a+2d)(c+2b)-12(ac+2bd)$

$=((a+2d)-(c+2b))^2+8(ab+cd-ac-bd)$

$=((a+2d)-(c+2b))^2+8(c-b)(d-a)>0$

$\therefore$ The roots are real and distinct.

Aliter :

Let $f(x)=(x-a)(x-c)+2(x-b)(x-d)$ which is continuous and $f(b)=(b-a)(b-c)<0$

$f(d)=(d-a)(d-c)>0$

So $f(x)=0$ has a root lies between b and d and $f(a)=2(a-b)(a-d)>0$

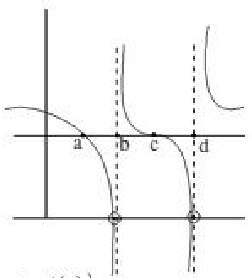
So, $f(x)=0$ has another root lie between a and b.

$\therefore\ f(x)=0$ has real roots

Remember :

$\frac{(x-a)(x-c)}{(x-b)(x-d)}=-2$

$\Rightarrow \phi(x)=\psi(x)$



$\left.\begin{array}{l} y=\phi(x) \\ y=\psi(x)\end{array}\right\}$ Intersect at 2 points

4. If a, b, c are positive integers and if the roots of the equation $ax^2-bx+c=0$ lie in the open interval (0,1) and distinct, find the minimum values of a, b, c.

Sol. Let α, β be the roots of the equation $ax^2-bx+c=0$.

Then $\alpha+\beta=\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. It is given that $\alpha, \beta \in (0,1)$.

On applying AM – GM inequality to $\alpha, 1-\alpha$ and also to $\beta, 1-\beta$

$\frac{\alpha+1-\alpha}{2} \geq \sqrt{\alpha(1-\alpha)}\ ;\ \frac{\beta+1-\beta}{2} \geq \sqrt{\beta(1-\beta)}$

Remember :

If both roots of a quadratic $f(x)$ lie in the interval (l, m) then $f(l).f(m)>0$

$\Rightarrow\ 0 < \alpha(1-\alpha) < \frac{1}{4}$ and $0 < \beta(1-\beta) < \frac{1}{4}$

$\Rightarrow\ \alpha\beta(1-\alpha)(1-\beta) < \frac{1}{16}$

Since a, b, c are positive integers $c(a - b + c)$ is an integer.

Let $f(x) = a(x-\alpha)(x-\beta)$

$f(0)$, $f(1)$ have same sign and $f(0)\,f(1) \geq 1$

$\Rightarrow 1 \leq f(0)f(1)$

$\Rightarrow 1 \leq a^2\alpha\beta(1-\alpha)(1-\beta) < a^2\left(\frac{1}{16}\right)$

$\Rightarrow a^2 > 16 \Rightarrow a \geq 5$

$\therefore$ Minimum value of a is 5

Since $ax^2 - bx + c = 0$ are real and distinct therefore $b^2 > 4ac$.

$\Rightarrow\ b^2 > 20c \Rightarrow b^2 > 20 \Rightarrow b \geq 5$

$\therefore$ Minimum value of b is 5 (Since $c \in N$ we have $c \geq 1$.)

$\therefore$ Minimum value of c is 1; b is 5; a is 5

5. If α, β are the roots of the equation $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $x^n + 1 + (x+1)^n = 0$ then prove that n is an even integer.

Sol. Since α, β are the roots of $x^2 + px + q = 0$

$\therefore\ \alpha + \beta = -p;\ \alpha\beta = q$

It is given that α, β are also the roots of $x^{2n} + p^n x^n + q^n = 0$

$\therefore\ \alpha^{2n} + p^n\alpha^n + q^n = 0,\ \beta^{2n} + p^n\beta^n + q^n = 0$

$\Rightarrow\ \alpha^{2n} - \beta^{2n} + p^n(\alpha^n - \beta^n) = 0$

$\Rightarrow\ \alpha^n + \beta^n + p^n = 0 \Rightarrow \alpha^n + \beta^n = -p^n \quad \ldots (1)$

Since $\left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$

$\Rightarrow\ \alpha^n + \beta^n + (\alpha + \beta)^n = 0$

$\Rightarrow\ \alpha^n + \beta^n + (-p)^n = 0 \quad \ldots. (2)$

(1) and (2) $\Rightarrow n$ is even integer

6. Solve $|x^2 - 3x - 4| = 9 - |x^2 - 1|$.

Sol. $x^2 - 3x - 4 = (x+1)(x-4)$

$\therefore\ x^2 - 3x - 4 \geq 0 \quad x \leq -1$ or $x \geq 4$

$x^2 - 3x - 4 < 0,\ -1 < x < 4$

Case (i) : $x \leq -1$

$x^2 - 3x - 4 \geq 0$ and $x^2 - 1 \geq 0$

$\therefore\ x^2 - 3x - 4 = 9 - (x^2 - 1)$

$\Rightarrow 2x^2 - 3x - 14 = 0 \Rightarrow x = 7/2, -2 \Rightarrow x = -2$

Case (ii) : $-1 < x < 1$

$x^2 - 1 < 0$ and $x^2 - 3x - 4 < 0$,$-(x^2 - 3x - 4) = 9 + x^2 - 1$

$2x^2 - 3x + 4 = 0$ which has no real roots

Remember :

While solving equations of the type $f(|\phi(x)|, |\psi(x)|) = 0$ key points are given by $\phi(x) = 0$ or $\psi(x) = 0$

Case (iii) : $1 \le x < 4$

$x^2 - 1 \ge 0$ and $x^2 - 3x - 4 < 0$

$-(x^2 - 3x - 4) = 9 - (x^2 - 1) \Rightarrow 3x - 6 = 0 \Rightarrow x = 2$

Case (iv) : $x \ge 4$

$x^2 - 3x - 4 \ge 0$ and $x^2 - 1 \ge 0$

$x^2 - 3x - 4 = 9 - (x^2 - 1)$

$\Rightarrow 2x^2 - 3x - 14 = 0 \Rightarrow x = 7/2, -2$

$\therefore$ No solution in this case

Hence, the solutions of the given equation are $x = -2, 2$

7. Show that the expression $\dfrac{\tan(x+\alpha)}{\tan(x-\alpha)}$ where $0 < \alpha < \dfrac{\pi}{4}$ cannot lie between the values $\tan^2\left(\dfrac{\pi}{4}-\alpha\right)$ and $\tan^2\left(\dfrac{\pi}{4}+\alpha\right)$.

Sol. Let $y = \dfrac{\tan(x+\alpha)}{\tan(x-\alpha)}$ $\alpha \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow y(1 - \tan x \tan\alpha)(\tan x - \tan\alpha) = (\tan x + \tan\alpha)(1 + \tan x \tan\alpha)$

$\Rightarrow (y+1)\tan\alpha \tan^2 x + (-y+1)(1+\tan^2\alpha)\tan x + (y+1)\tan\alpha = 0$

As $\tan x$ is real for $x \in R$, therefore

$(1-y)^2(1+\tan^2\alpha)^2 - 4(y+1)^2\tan^2\alpha \ge 0$

$((1-y)(1+\tan^2\alpha) - 2(y+1)\tan\alpha)\ ((1-y)(1+\tan^2\alpha) + 2(y+1)\tan\alpha) \ge 0$

$$\Rightarrow (1-\tan\alpha)^2(1+\tan\alpha)^2 \left(y - \left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2\right) \left(y - \left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)^2\right) \ge 0$$

$$\Rightarrow \left(y - \tan^2\left(\frac{\pi}{4}-\alpha\right)\right)\left(y - \tan^2\left(\frac{\pi}{4}+\alpha\right)\right) \ge 0$$

$\Rightarrow$ y cannot lie between $\tan^2\left(\dfrac{\pi}{4}-\alpha\right)$ and $\tan^2\left(\dfrac{\pi}{4}+\alpha\right)$

8. Find all real values of a for which $-3 < \dfrac{x^2+ax-2}{x^2-x+1} < 2$ holds for all $x \in R$.

Sol. Since $x^2 - x + 1 > 0$

$-3 < \dfrac{x^2+ax-2}{x^2-x+1} < 2 \Rightarrow -3(x^2 - x + 1) < x^2 + ax - 2 < 2(x^2 - x + 1)$

$\Rightarrow -3(x^2 - x + 1) < x^2 + ax - 2$ and $x^2 + ax - 2 < 2(x^2 - x + 1)$ for all $x \in R$.

Now $-3(x^2 - x + 1) < x^2 + ax - 2$ for all $x \in R$.

$\Rightarrow 4x^2 + (a-3)x + 1 > 0$ for all $x \in R$

$\Rightarrow (a-3)^2 - 4.4.1 < 0 \Rightarrow (a-7)(a+1) < 0 \Rightarrow -1 < a < 7$

and for $x^2 + ax - 2 < 2(x^2 - x + 1)$ for all $x \in R$

$x^2 - (a+2)x + 4 > 0 \Rightarrow (a+2)^2 - 16 < 0$

$\Rightarrow (a-2)(a+6) < 0 \Rightarrow -6 < a < 2$

$\therefore$ Common set is $-1 < a < 2$

i.e., $a \in (-1, 2)$

9. For what values of 'a' one of the roots of the equation $(2a+1)x^2 - ax + a - 2 = 0$ greater and the other smaller than unity ?

Sol. As per the condition, the given quadratic should have real roots and 1 lies between the roots.

$\therefore\ \Delta > 0$ and $(2a+1)f(1) < 0$

$\Rightarrow a^2 - 4(2a+1)(a-2) > 0 \Rightarrow 7a^2 - 12a - 8 < 0$

$$\Rightarrow \frac{6-2\sqrt{23}}{7} < a < \frac{6+2\sqrt{23}}{7} \quad \text{.... (1)}$$

$(2a+1)f(1) < 0$

$\Rightarrow (2a+1)(2a+1-a+a-2) < 0$

$\Rightarrow (2a+1)(2a-1) < 0$

$$\Rightarrow -\frac{1}{2} < a < \frac{1}{2} \quad \text{.... (2)}$$

Since $\dfrac{6-2\sqrt{23}}{7} < -\dfrac{1}{2}$ and $\dfrac{6+2\sqrt{23}}{7} < \dfrac{1}{2}$

$\therefore$ Required values of a lie in the interval $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

> ***Remember :***
> *For $ax^2 + bx + c = 0$ if k lie between the roots then*
> *i) $\Delta > 0$*
> *ii) $a\,f(k) < 0$*

10. For what real values of 'a' do the roots of the equation $x^2 - 2x - a^2 + 1 = 0$ lie between the roots of the equation $x^2 - 2(a+1)x + a(a-1) = 0$

Sol. The roots of $x^2 - 2x - a^2 + 1 = 0$ are $1+a,\ 1-a$

Let $f(x) = x^2 - 2(a+1)x + a(a-1)$

For the roots of first quadratic equation lie between the roots of $f(x) = 0$ then the following must hold.

i) discriminant D of $f(x) = 0$ should be greater than 0.

ii) $f(1 \pm a) < 0$

$\therefore$ For $D > 0$

$4(a+1)^2 - 4a(a-1) > 0 \Rightarrow 3a + 1 > 0 \Rightarrow a > -\dfrac{1}{3}$

$f(1-a) < 0$

$(1-a)^2 - 2(a+1)(1-a) + a(a-1) < 0$

$\Rightarrow (1-a)^2 - 2(1-a^2) + a(a-1) < 0$

$\Rightarrow -1 - 2a + 3a^2 + a^2 - a < 0$

$\Rightarrow 4a^2 - 3a - 1 < 0 \Rightarrow -\dfrac{1}{4} < a < 1$ and now $f(1+a) < 0$

$\Rightarrow (1+a)^2 - 2(a+1)^2 + a(a-1) < 0$

$\Rightarrow -(1+a)^2 + a(a-1) < 0$

$\Rightarrow -3a - 1 < 0$

$\Rightarrow a > -\dfrac{1}{3}$

Thus for $D > 0,\ f(1-a) < 0,\ f(1+a) < 0$

we have $a \in \left(-\dfrac{1}{4}, 1\right)$

> ***Remember :***
> *If $ax^2+bx+c=0$ is such that k_1, k_2 lie between the roots, then*
> *i) $\Delta > 0$*
> *ii) $a\,f(k_1) < 0$*
> *iii) $a\,f(k_2) < 0$*

11. If $a(p+q)^2 + 2bpq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$, then show that $qr = p^2 + \frac{c}{a}$.

Sol. We have $a(p+q)^2 + 2bpq + c = 0$

i.e., $aq^2 + 2(a+b)pq + (c+ap^2) = 0$ .... (1)

and $a(p+r)^2 + 2bpr + c = 0$

i.e., $ar^2 + 2(a+b)pr + (c+ap^2) = 0$ .... (2)

From equations (1) and (2), we can see that q and r satisfy the quadratic equation

$ax^2 + 2(a+b)px + (c+ap^2) = 0$

Hence, we have product of the roots = $\frac{c+ap^2}{a}$

i.e., $qr = p^2 + \frac{c}{a}$

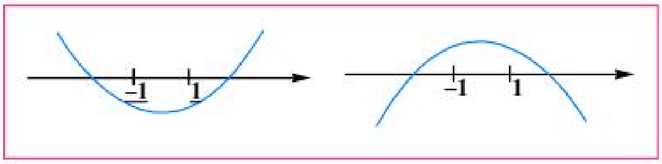
which is the desired result.

Remember :
If $f(\alpha)=0, f(\beta)=0$ then, clearly α, β are solutions of $f(x)=0$

12. The equation $ax^2 + bx + c = 0$ has two real roots $\alpha < -1, \beta > 1$. Show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Sol. Let $f(x) = ax^2 + bx + c = 0$

According to the given conditions, the graph of $f(x)$ is as shown below.



From the figure the necessary and sufficient conditions are $f(-1)\,f(1) > 0$

i.e., $(a-b+c)(a+b+c) > 0$ i.e., $(a+c)^2 - b^2 > 0$

i.e., $\left(1+\frac{c}{a}\right)^2 - \frac{b^2}{a^2} > 0$ i.e., $\frac{b^2}{a^2} < \left(1+\frac{c}{a}\right)^2$

i.e., $\left|\frac{b}{a}\right| < \left|1+\frac{c}{a}\right|$ i.e., $\left|\frac{b}{a}\right| < -\left(1+\frac{c}{a}\right)$

$\left[\because \text{Products of the roots} = \frac{c}{a} < -1 \therefore 1+\frac{c}{a} < 0\right]$

i.e., $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$

14. Find the number of real solutions of the equation $e^x(e^x - 2) = 1 + |e^x - 1|$. [Ans : 1]

15. For a given θ, show that $\frac{2a(x-1)\sin^2\theta}{x^2 - \sin^2\theta}$ doesn't lie between $2a\cos^2\frac{\theta}{2}$ and $2a\sin^2\frac{\theta}{2}$

16. Show that $\frac{(x+a)(1+ax)}{(x-a)(1-ax)}$ where $x \in R, x \neq a, \frac{1}{a}$ cannot lie between $\left(\frac{1-a}{1+a}\right)^2$ and $\left(\frac{1+a}{1-a}\right)^2$

17. Find the set of real values of parameter c so that $\frac{x^2+2x+c}{x^2+4x+3c}$ can take all real values for $x \in R$.

[Ans : [0, 1]]

*18. Solve $\sqrt{(x-3)(2-x)} < \sqrt{4x^2+12x+11}$ [Ans : [2, 3]]

*19. Solve $\sqrt{x^2-3x-10} > (8-x)$ [Ans : $\left(\frac{74}{13}, \infty\right)$]

20. Solve the equation $\left|\frac{x^2-3x+2}{x^2-7x+12}\right| + \left(\frac{x^2-3x+2}{x^2-7x+12}\right) = 0$ [Ans : $[1, 2) \cup (3, 4]$]

21. i) Find k if $x^2 + 4xy + 4y^2 + 2kx + 4y + 1$ can be represented as product of two linear factors.

[Ans : $k = 1$]

ii) If x and y are real, find the range of x and y for the equation $x^2 + 2y^2 + 3xy - x - 2y = 0$

[Ans : $x \in [-\sqrt{5}, \sqrt{5}]$, $y \in [-\sqrt{2}, \sqrt{2}]$]

22. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is $b + 2$, then find the value of $\frac{a^2+a-14}{a+1}$ [Ans : 4]

23. The quadratic equation $f(x) = 0$ with real coefficients has purely imaginary roots that prove that the equation $f(f(x)) = 0$ has neither real nor purely imaginary roots.

24. Find the number of polynomials $f(x)$ both non-negative integer coefficients, of degree=2 satisfying $f(0)=0$, $\int_0^1 f(x)dx = 1$ [Ans : 2]

25. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$ then find the value of $\frac{a_{10} - 2a_8}{2a_9}$.

[Ans : 3]