CHAPTER 3

Calculus

AREAS

• DETERMINATION OF AREAS •

• AREAS BOUNDED BY SOME STANDARD CURVES •

3.0 INTRODUCTION

If $y = f(x)$ is a non -negative continuous function in $[a, b]$ then the definite integral $\int_a^b f(x)dx$ geometrically represents the area bounded by the curve $y = f(x)$ above the X-axis between the lines $x = a$ and $x = b$.

Based on the nature of the region bounded by the given curves we discuss the possible ways of calculating the areas of these regions using definite integrals. The process of finding areas using definite integrals is known as ***quadrature***.

3.1 DETERMINATION OF AREAS

1) If $f(x)$ is continuous on $[a,b]$ and $f(x) \geq 0\ \forall x \in [a,b]$ then the area A of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)dx$ (or) $A = \int_a^b y dx$

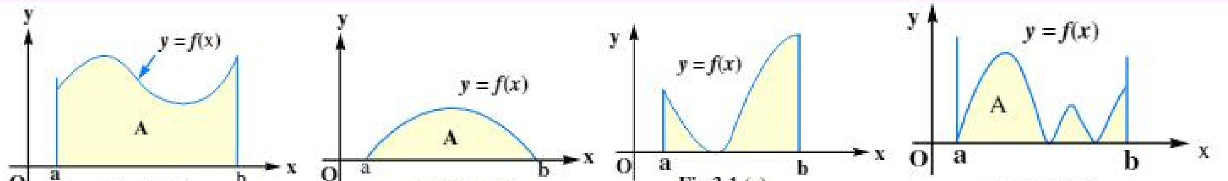


Fig 3.1 (a) Fig 3.1 (b) Fig 3.1 (c) Fig 3.1 (d)

2) If $f(x)$ is continuous on $[a, b]$ and $f(x) \leq 0\ \forall x \in [a,b]$ then the area A of the region bounded by the curve $y = f(x)$ the x-axis and the lines $x = a$, $x = b$ is given by $A = \int_a^b -f(x)dx$ (or) $A = -\int_a^b f(x)dx$ (or) $A = \left|\int_a^b f(x)dx\right|$.

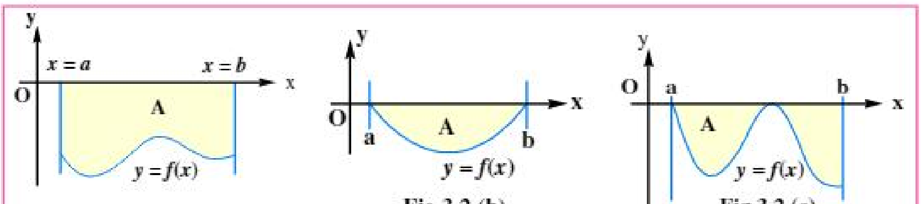


Fig 3.2 (a) Fig 3.2 (b) Fig 3.2 (c)

3) Let $f(x)$ be continuous on $[a, b]$ such that $f(x) \geq 0\ \forall x \in [a,c]$ and $f(x) \leq 0\ \forall x \in [c,b]$ where $a < c < b$. Then the area A of the region bounded by the curve $y = f(x)$ the x-axis and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^c f(x)dx + \int_c^b -f(x)dx \quad \text{i.e., } A = \int_a^c f(x)dx + \left|\int_c^b f(x)dx\right|$$

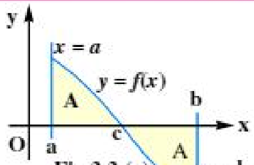


Fig 3.3 (a)

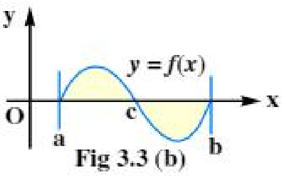


Fig 3.3 (b)

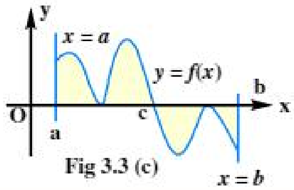


Fig 3.3 (c)

4) The area A of the region bounded by the curve $y = f(x)$, the x-axis between the lines $x = a$ and $x = b$ (as shown in the fig 3.4) is given by $A = \int_a^b |f(x)|\,dx$.

i.e., $A = \int_a^{c_1} f(x)\,dx + \left|\int_{c_1}^{c_2} f(x)\,dx\right| + \int_{c_2}^{c_3} f(x)dx \; + \left|\int_{c_3}^{b} f(x)dx\right|$

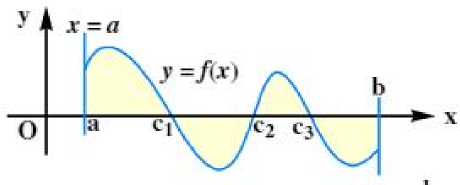


Fig 3.4

5) If f and g are two continuous functions on $[a, b]$ such that $f(x) \geq g(x)\ \forall x \in [a,b]$ then the area A of the region bounded by $y = f(x)$, $y = g(x)$ between the lines $x = a$ and $x = b$ (fig 3.5 (a), (b), (c)) is given by $A = \int_a^b f(x)dx - \int_a^b g(x)dx$ i.e., $A = \int_a^b [f(x) - g(x)]dx$

i.e., $A = \int_a^b (y_{UC} - y_{LC})dx$

where UC and LC stand for upper curve and lower curve respectively.

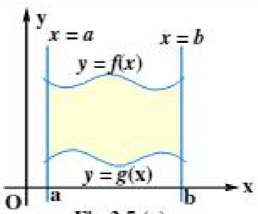


Fig 3.5 (a)

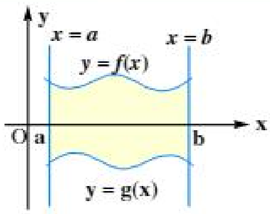


Fig 3.5 (b)

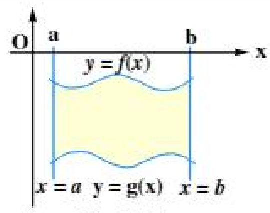


Fig 3.5 (c)

6) If f and g are two continuous functions on $[a, b]$ such that the curves $y = f(x)$ and $y = g(x)$ intersect at $c \in (a,b)$ and $f(x) > g(x)\ \forall x \in [a,c)$ and $f(x) < g(x)\ \forall x \in [a,c)$ then the area A bounded by the curves between the lines $x = a$ and $x = b$ is given by

$A = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$ (or) $A = \int_a^c (y_{UC} - y_{LC})dx + \int_c^b (y_{UC} - y_{LC})dx$

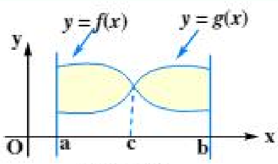


Fig 3.6 (a)

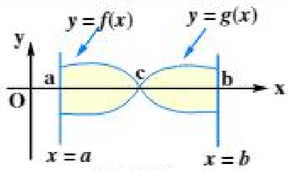


Fig 3.6 (b)

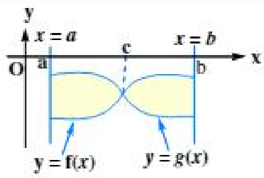


Fig 3.6 (c)

7) Let f and g be two continuous functions on $[a, b]$ such that the curves $y = f(x)$ and $y = g(x)$ intersect at $x_1, x_2 \in (a,b)$ and $x_1 < x_2$. If $f(x) > g(x)\ \forall x \in (x_1, x_2)$ then the area A of the region enclosed by the curves between their points of intersection is given by $A = \int_{x_1}^{x_2} [f(x) - g(x)]\,dx$ (or) $A = \int_{x_1}^{x_2} (y_{UC} - y_{LC})\,dx$

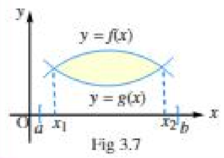


Fig 3.7

In the case of some regions it is more convenient to find the areas about y- axis (instead of x- axis as is done in the above cases 1 to 7). For this, we have to consider x as a function of y and determine the limits of y accordingly.

8) If $g(y) \geq 0\ \forall y \in [c,d]$, then the area bounded by the curve $x = g(y)$ the y-axis and the lines $y=c$ and $y=d$ is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy$$

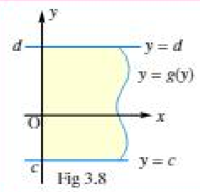


Fig 3.8

9) If $g(y) \leq 0\ \forall y \in [c,d]$ then the area A bounded by $x = g(y)$, y-axis, $y = c$ and $y = d$ is given by $A = -\int_c^d g(y)\,dy = \left|\int_c^d g(y)\,dy\right|$

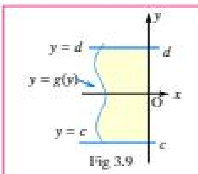


Fig 3.9

10) The areas of the regions shown in the following figures are given by

a) $A = \left|\int_c^{c_1} g(y)\,dy\right| + \int_{c_1}^{d} g(y)\,dy$ *(fig 3.10 (a))*

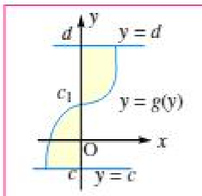


Fig 3.10 (a)

b) $A = \int_c^{d_1} g(y)\,dy + \left|\int_{d_1}^{d} g(y)\,dy\right|$ *(fig 3.10 (b))*

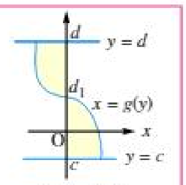


Fig 3.10 (b)

c) $A = \int_c^d [f(y) - g(y)]\,dy$

i.e., $A = \int_c^d [x_{RC} - x_{LC}]\,dy$ *(fig 3.10 (c))*

where RC = Right most curve;

LC = Left most curve.

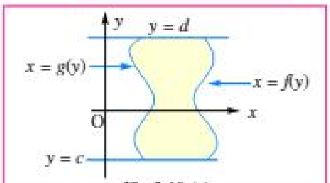


Fig 3.10 (c)

Remark :
If a region consists of n symmetric portions then the area A of the whole region is given by A = n (area of one symmetric portion).

3.2 AREAS BOUNDED BY SOME STANDARD CURVES

1. The area enclosed by one arch of $y = \sin x$ and the x-axis is 2 sq. units.

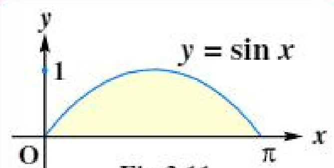


Fig 3.11

Note :

i) The area enclosed by $y = \sin x$ and x-axis between $x = 0$ and $x = n\pi$ is $2n$ sq. units ($\because$ there are n portions of 2 sq units each).

ii) The area enclosed by one arch of $y = \cos x$ and x-axis is 2 sq units.

iii) The area enclosed by $y = \cos x$ and x-axis between $x = 0$ and $x = n\pi$ is $2n$ sq units.

Sol. Area $= \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = 2$ sq units

2. The area enclosed by $y^2 = 4ax$ and its latus rectum is $\dfrac{8a^2}{3}$ sq units.

Sol. A rough sketch of the required region is shown in the fig

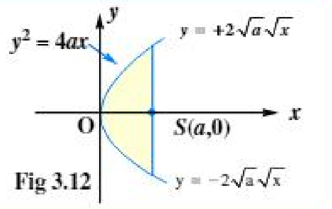


Fig 3.12

There are two symmetric portions in the region

$$\therefore Area = 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx = 4\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{8}{3}a^2 \text{ sq units}$$

3. The area enclosed between $y^2 = 4ax$ and $y = mx$ is $\dfrac{8a^2}{3m^3}$ sq units.

Sol. The curves $y^2 = 4ax$ and $y = mx (m > 0)$ intersect of $(0,0)$ and $\left(\dfrac{4a}{m^2}, \dfrac{4a}{m}\right)$

A rough sketch of the required region is shown in the figure.

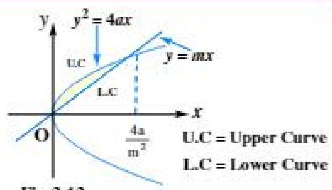


Fig 3.13

Note :

If $m<0$ then the required area is $\dfrac{8}{3}\dfrac{a^2}{|m|^3}$ sq units.

$$\text{Area} = \int_0^{4a/m^2} (y_{UC} - y_{LC})\,dx = \int_0^{4a/m^2} \left[2\sqrt{a}\sqrt{x} - mx\right]dx = 2\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{4a/m^2} - \frac{m}{2}\left[x^2\right]_0^{4a/m^2}$$

$$= \frac{32}{3}\frac{a^2}{m^3} - \frac{8a^2}{m^3} = \frac{8}{3}\frac{a^2}{m^3} \text{ sq units.}$$

4. The area enclosed by the curves $x^2 = 4by$ and $y = mx$ is $\dfrac{8}{3}b^2m^3$ sq units.

Sol. The curves $x^2 = 4by$ and $y = mx (m > 0)$ intersect at $O(0, 0)$ and $P(4bm, 4bm^2)$

A rough sketch of the required region is shown in the fig

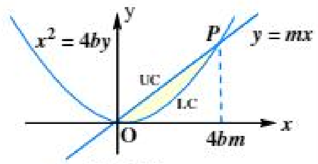


Fig 3.14

> **Note :**
> *If $m < 0$ then the area is $\frac{8}{3}b^2|m|^3$ sq units.*

$$\text{Area} = \int_0^{4bm}(y_{UC} - y_{LC})\,dx = \int_0^{4bm}\left(mx - \frac{x^2}{4b}\right)dx = \frac{m}{2}\left[x^2\right]_0^{4bm} - \frac{1}{12b}\left[x^3\right]_0^{4bm}$$

$$= 8b^2m^3 - \frac{16}{3}b^2m^3 = \frac{8}{3}b^2m^3 \text{ sq units}$$

***5. The area enclosed between two parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$ sq units.**

Sol. The two curves $y^2 = 4ax$ and $x^2 = 4by$ intersect at $O(0, 0)$ and $P\left(4a^{1/3}.b^{2/3}, 4a^{2/3}.b^{1/3}\right)$

A rough sketch of the region is shown in the figure.

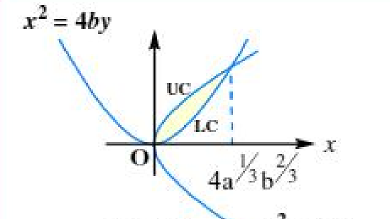


Fig 3.15

> **Dedcution :**
> *The area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$ sq units.*

$$\text{Area} = \int_0^{4a^{1/3}b^{2/3}}(y_{UC} - y_{LC})\,dx = \int_0^{4a^{1/3}b^{2/3}}\left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right)dx$$

$$= 2\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{4a^{1/3}b^{2/3}} - \frac{1}{12b}\left[x^3\right]_0^{4a^{1/3}b^{2/3}} = \frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab \text{ sq units}$$

6. The area enclosed between the parabolas $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$ where $a > 0, b > 0$ is $\frac{8}{3}(a+b)\sqrt{ab}$ sq units.

Sol. The two curves are symmetric about the x-axis and intersect at $P(b-a, 2\sqrt{ab})$ and $Q(b-a, -2\sqrt{ab})$

The required region (shown in the fig) lies between $x = -a$ and $x = b$ and the boundary of the region changes at $x = b - a$.

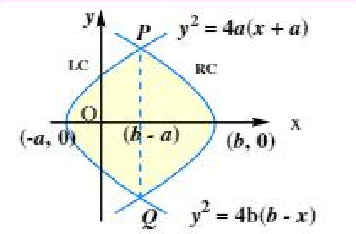


Fig 3.16

It contains 2 symmetric portions.

$$\text{Area} = 2\left\{\int_{-a}^{b-a} 2\sqrt{a}\sqrt{x+a}\,dx + \int_{b-a}^{b} 2\sqrt{b}\sqrt{b-x}\,dx\right\} = 2\left\{\left[\frac{4}{3}\sqrt{a}(x+a)^{3/2}\right]_{-a}^{b-a} - \left[\frac{4}{3}\sqrt{b}(b-x)^{3/2}\right]_{b-a}^{b}\right\}$$

$$= 2\left(\frac{4}{3}\sqrt{a}b^{3/2} + \frac{4}{3}\sqrt{b}a^{3/2}\right) = \frac{8}{3}\sqrt{ab}(a+b) \text{ sq units}$$

7. The area enclosed between two intersecting quadratic curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ is $\dfrac{\Delta^{3/2}}{6A^2}$ sq units where $\Delta = B^2 - 4AC$ and $A = a_1 - a_2$, $B = b_1 - b_2$ and $C = c_1 - c_2$.

Remarks :

1) If $\Delta = B^2 - 4AC < 0$ then the given curves doesnot intersect and hence no region is enclosed between them.

2) The above formula is also valid when one of the curves is a linear curve (straight line)

3) The area enclosed between two quadratic $x = a_1y^2 + b_1y + c_1$, $x = a_2y^2 + b_2y + c_2$ is $\dfrac{\Delta^{3/2}}{6A^2}$ sq units (using the same notation as above)

4) The area enclosed between $y = ax^2 + bx + c$ and $y = 0$ when they intersect, is $\dfrac{\Delta^{3/2}}{6a^2}$ sq units where $\Delta = b^2 - 4ac > 0$.

Sol. Let the two curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ intersect at $x = x_1$ and $x = x_2$

Then x_1, x_2 are the roots of $(a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0$ i.e., $Ax^2 + Bx + C = 0$

where $A = a_1 - a_2$, $B = b_1 - b_2$, $C = c_1 - c_2$.

(Caution : Common factor, if any, among A, B, C should not be cancelled as the same symbols are being used in the area integral).

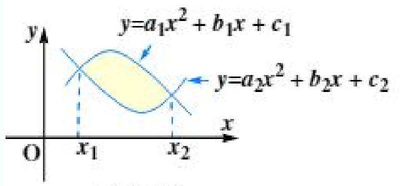


Fig 3.17

$\therefore x_1 + x_2 = -\dfrac{B}{A}, x_1x_2 = \dfrac{C}{A}$ Let $\Delta = B^2 - 4AC$.

Then $\Delta > 0$

The area enclosed between the curves is

$$\text{Area} = \left|\int_{x_1}^{x_2}\left[\left(a_1x^2 + b_1x + c_1\right) - \left(a_2x^2 + b_2x + c_2\right)dx\right]\right| = \left|\int_{x_1}^{x_2}\left(Ax^2 + Bx + C\right)dx\right|$$

$$= \left|\frac{A}{3}\left(x_2^3 - x_1^3\right) + \frac{B}{2}\left(x_2^2 - x_1^2\right) + C\left(x_2 - x_1\right)\right| = \left|\frac{\left(x_2 - x_1\right)}{6}\left\{2A\left(x_1^2 + x_1x_2 + x_2^2\right) + 3B\left(x_2 + x_1\right) + 6C\right\}\right|$$

$$= \left|\frac{\sqrt{\left(B^2 - 4AC\right)}}{6A}\left\{2A\left(\frac{B^2}{A^2} - \frac{C}{A}\right) + 3B\left(\frac{-B}{A}\right) + 6C\right\}\right| = \frac{\sqrt{B^2 - 4AC}}{6A^2}\left|B^2 - 4AC\right|$$

$$= \frac{\left(B^2 - 4AC\right)^{3/2}}{6A^2} = \frac{\Delta^{3/2}}{6A^2} \text{ sq units}$$

Example : The area enclosed between $y = 5x^2$ and $y = 2x^2 + 9$ is $\dfrac{\Delta^{3/2}}{6A^2} = \dfrac{(4.3.9)^{3/2}}{6.9} = 12\sqrt{3}$

****8. The area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab sq units. Also deduce the area of the circle $x^2 + y^2 = a^2$ (March-2014, 17)**

Sol. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is symmetric about both the axes and there are 4 symmetric portions in the region (see fig).

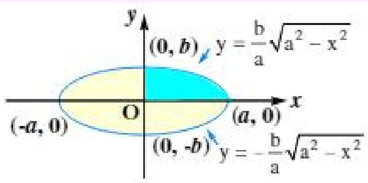


Fig 3.18

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Deduction : ***The area of the circle $x^2+y^2=a^2$ is πa^2 sq units.***

The curve meets the x-axis at $(a, 0)$ and $(-a, 0)$ and the y-axis at $(0, b)$ and $(0, -b)$

Area = 4 (area of one symmetric portion)

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a=\frac{4b}{a}\left[0+\frac{\pi a^2}{4}\right]=\pi ab$$

***9. The area enclosed by the line segments $|x|+|y|=1$ is 2 sq units.**

Sol. The given curves are $|x|+|y|=1 \Rightarrow \begin{cases} x+y=1 \text{ if } x\ge 0,\ y\ge 0 \\ -x+y=1 \text{ if } x<0, y\ge 0 \\ -x-y=1 \text{ if } x<0,\ y<0 \\ x-y=1 \text{ if } x\ge 0,\ y<0 \end{cases}$

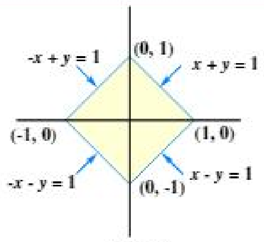


Fig 3.19

There are 4 symmetric portions in the region

Area = 4(area of one symmetric portion)

$$=4\int_0^1 (1-x)\,dx 4\left[x-\frac{x^2}{2}\right]_0^1=4\left(1-\frac{1}{2}\right)=2 \text{ sq units.}$$

10. The area enclosed between the curves $y=x$ and $y=x^3$ is $\frac{1}{2}$ sq unit.

Sol. The curves $y = x$ and $y = x^3$ intersect at $x=0,\pm 1$ the rough sketch of the region is shown in the figure.

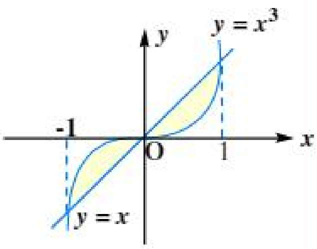


Fig 3.20

$$\text{Area}=\int_{-1}^{0}\left(x^3-x\right)dx+\int_0^1\left(x-x^3\right)dx=2\int_0^1\left(x-x^3\right)dx=2\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1=\frac{1}{2} \text{ sq unit.}$$

SOLVED EXAMPLES

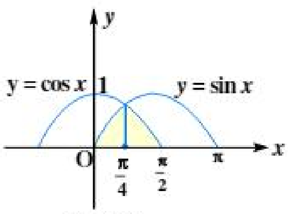


Fig 3.21

***1. Find the area of one of the curvili-near triangles formed by $y = \sin x$, $y = \cos x$ and x–axis.** **(March-19)**

Sol. The given curves are $y=\sin x, y=\cos x$ and $x-axis$

A rough sketch of the region is shown in the figure.

$$\text{Area} = \int_0^{\frac{\pi}{4}} \sin x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x dx = [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\left(\frac{1}{\sqrt{2}}-1\right)+\left(1-\frac{1}{\sqrt{2}}\right) = 2-\sqrt{2} \text{ sq units.}$$

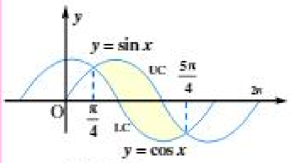


Fig 3.22

***2. Find the area bounded by $y = \sin x$ and $y = \cos x$ between any two consecutive points of their inter-section** **(March-18)**

Sol. Two of the consecutive points of intersection of $y=\sin x$ and $y=\cos x$ are $x=\dfrac{\pi}{4}$ and $x=\dfrac{5\pi}{4}$ A rough sketch of the region is shown in the figure.

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (y_{UC}-y_{LC})dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx$$

$$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\left[\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]$$

$$= \sqrt{2}+\sqrt{2} = 2\sqrt{2} \text{ sq units}$$

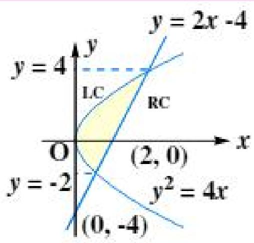


Fig 3.23

3. Find the area bouned by $y^2 = 4x$ and $y = 2x-4$

Sol. The given curves are $y^2 = 4x$ -- (1)

$y = 2x-4$ --(2)

F or points of intersection of (1) & (2)

$$y^2 = 4\left(\frac{y+4}{2}\right) \Rightarrow y^2-2y-8=0 \Rightarrow (y+2)(y-4)=0$$

A rough sketch of the region is showin in the figure.

It is bounded by the curves (1) & (2) between $y=-2$ and $y=4$.

It is convenient to find the area about the y-axis than the x-axis (1)

$$\Rightarrow x=\frac{y^2}{4} \ (2) \Rightarrow x=\frac{y+4}{2} \ \therefore \text{Area} = \int_{-2}^{4}(x_{RC}-x_{LC})dy$$

(RC : Right most curve LC : Left most curve)

$$= \int_{-2}^{4}\left(\frac{y+4}{2}-\frac{y^2}{4}\right)dy = \frac{1}{2}\left[\left(\frac{y^2}{2}+4y\right)\right]_{-2}^{4} - \frac{1}{4}\left[\frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}[(8+16)-(2-8)] - \frac{1}{4}\left(\frac{64}{3}+\frac{8}{3}\right) = 15-6 = 9 \text{ sq units.}$$

Aliter: The given curves are $x=\frac{y^2}{4}, x=\frac{y}{2}+2$ -- (2)

$$(1)-(2) \Rightarrow Ay^2+By+C \equiv \frac{y^2}{4}-\frac{y}{2}-2$$

$$A=\frac{1}{4}, B=-\frac{1}{2}, C=-2$$

$$\text{Area}=\frac{\Delta^{3/2}}{6A^2}=9 \text{ sq units.}$$

***4. Find the area enclosed between $y=\sin 2x, y=\sqrt{3}\sin x, x=0$ and $x=\frac{\pi}{6}$**

Sol. The given curves are $y=\sin 2x$ --(1)

$y=\sqrt{3}\sin x$ --(2)

and the lines are $x=0$ and $x=\frac{\pi}{6}$

For points of intersection of (1) & (2) $\sin 2x=\sqrt{3}\sin x$

$$\Rightarrow \sin x=0 \text{ or } \cos x=\frac{\sqrt{3}}{2}$$

$\therefore x=0, x=\frac{\pi}{6}$ (since x lies between 0 and $\frac{\pi}{6}$)

A rough sketch of the region is shown in the figure

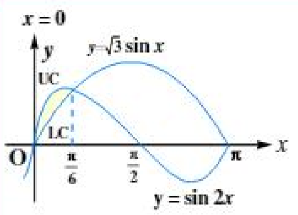


Fig 3.24

$$\text{Area}=\int_0^{\frac{\pi}{6}}(y_{UC}-y_{LC})dx=\int_0^{\frac{\pi}{6}}\left(\sin 2x-\sqrt{3}\sin x\right)dx$$

$$=\left[-\frac{1}{2}\cos 2x+\sqrt{3}\cos x\right]_0^{\frac{\pi}{6}}=\left(-\frac{1}{4}+\frac{3}{2}\right)-\left(-\frac{1}{2}+\sqrt{3}\right)=\left(\frac{7}{4}-\sqrt{3}\right) \text{ sq units}$$

5. Find the area bounded by the curves $y=x^3-x$ and $y=x^2+x$

Sol. The given curves are $y=x^3-x$ --(1)

$y=x^2+x$ --(2)

For points of intersection $x^3-x=x^2+x \Rightarrow x=0,2,-1$

curve (1) cuts the x - axis at $x=0, x=1, x=-1$ and the curve (2) at $x=-1$ and $x=0$

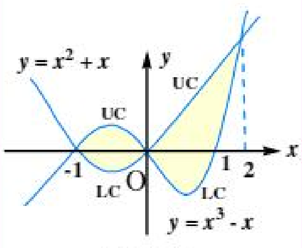
$$(1) \Rightarrow y=x(x-1)(x+1)$$

y is +ve for $x\in(-1,0)\cup(1,\infty)$ –ve for $x\in(-\infty,-1)\cup(0,1)$

(2) $\Rightarrow y=x(x+1)$ this is +ve for $x\in(-\infty,-1)\cup(0,\infty)$ and –ve for $x\in(-1,0)$

Sign scheme for $y=x^3-x$ Sign scheme for $y=x^2+x$

A rough sketch of the required region is shown in the figure.



Flg 3.25

$$\text{Area} = \int_{-1}^{0}\left(y_{UC} - y_{LC}\right)dx + \int_{0}^{2}\left(y_{UC} - y_{LC}\right)dx$$

$$= \int_{-1}^{0}\left[\left(x^3 - 3\right) - \left(x^2 + x\right)\right]dx + \int_{0}^{2}\left(x^2 + x\right) - \left(x^3 - x\right)dx$$

$$= \int_{-1}^{0}\left(x^3 - x^2 - 2x\right)dx + \int_{0}^{2}\left(2x + x^2 - x^3\right)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_{0}^{2} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \text{ sq units.}$$

***6. Find the area of the region $\{(x, y) : x^2 - x - 1 \le y \le -1\}$**

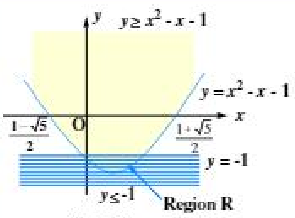


Fig 3.26

Sol. Let the given region be $R = \{(x, y) : x^2 - x - 1 \le y \le -1\}$

Consider the curves $y = x^2 - x - 1$ --(1)

and $y = -1$ --(2)

The curves (1) & (2) intersect at the points x for which $x^2 - x - 1 = -1$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

$y = x^2 - x - 1 \Rightarrow \left(x - \frac{1}{2}\right)^2 = y + \frac{5}{4}$

This is a parabola with vertex at $\left(\frac{1}{2}, -\frac{5}{4}\right)$ and cuts the

x- axis at $x = \frac{1-\sqrt{5}}{2}$ and $x = \frac{1+\sqrt{5}}{2}$

Further $y < 0$ for $x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ and $y > 0$ for $x < \frac{1-\sqrt{5}}{2}$ or $x > \frac{1+\sqrt{5}}{2}$

$\therefore$ The parabola is oriented upwards with a vertical axis.

The origin $O(0, 0)$ lies inside the parabola and also satisifies the inequality

$$y \ge x^2 - x - 1$$

$y \ge x^2 - x - 1$ represents the region inside the parabola (1) including the curve.

$y \le -1$ represents the region below the line $y = -1$ including the line.

A rough sketch of the region R is shown in the figure.

It is bounded between $x = 0$ and $x = 1$ $\therefore$ Area $= \int_{0}^{1}\left(y_{UC} - y_{LC}\right)dx$

$$= \int_{0}^{1}\left[(-1) - \left(x^2 - x - 1\right)\right]dx = \int_{0}^{1}\left(x - x^2\right)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{1} = \frac{1}{6} \text{ sq units.}$$

***7. Find the area of the region enclosed by $y^2 = 4(4-x)$ and $y^2 = 4x$ (May-19)**

Sol. The given curves $y^2 = 4(4 - x)$ --(1)

$y^2 = 4x$ --(2)

are both parabolas symmetric about x- axis

For the points of intersection $4(4 - x) = 4x \Rightarrow x = 2,\ y = \pm\sqrt{8}$

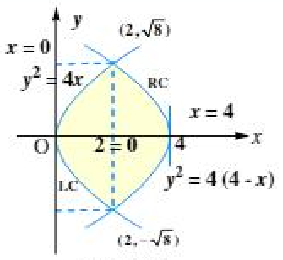


Fig 3.27

(1) and (2) intersect at $(2, \pm 8)$.

A rough sketch is shown in the figure.

The region has two symmetric portions and it is bounded between $x = 0$ and $x = 4$ and the boundary of the region changes at $x = 2$

$$\therefore \text{Area} = 2\left[\int_0^2 2\sqrt{x}dx - \int_2^4 2\sqrt{4-x}dx\right] = 4\left\{\left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}(4-x)^{3/2}\right]_2^4\right\}$$

$$= \frac{8}{3}\left(2\sqrt{2} + 2\sqrt{2}\right) = \frac{32\sqrt{2}}{3} \text{sq units (or) Area} = \int_{-\sqrt{8}}^{\sqrt{8}}\left(x_{RC} - x_{LC}\right)dy = \int_{\sqrt{8}}^{\sqrt{8}}\left[\left(4 - \frac{y^2}{4}\right) - \frac{y^2}{4}\right]dy$$

$$= 2\int_0^{\sqrt{8}}\left(4 - \frac{y^2}{2}\right)dy = 2\left[4y - \frac{y^3}{6}\right]_0^{\sqrt{8}} = 2\left(8\sqrt{2} - \frac{16}{6}\sqrt{2}\right) = \frac{32\sqrt{2}}{3} \text{ sq units.}$$

***8. The parabola $y = \frac{1}{2}x^2$ divides the circle $x^2 + y^2 = 8$ into two parts find the area of each part.**

Sol. The given curves $y = \frac{1}{2}x^2$ -- (1)

$x^2 + y^2 = 8$ -- (2)

are both symmetric about y - axis for the points of intersection of (1) & (2)

$= 2y + y^2 = 8$

$\Rightarrow (y-2)(y+4) = 0 \Rightarrow y = 2, y = -4$

$y = -4$ does not satisfy equation (1) $\therefore y = 2 \Rightarrow x = \pm 2$

The points of intersection are $(-2, 2)$, $(2, 2)$.

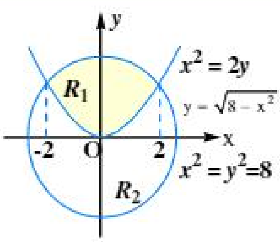


Fig 3.28

The smaller region R_1 is shown in the figure as shaded portion.

It is bounded between $x = -2$ and $x = 2$ and it has 2 symmetric portions.

$$\therefore \text{Area of } R_1 = 2\int_0^2\left(y_{UC} - y_{LC}\right)dx = 2\int_0^2\left(\sqrt{8 - x^2} - \frac{x^2}{2}\right)dx$$

$$= 2\left\{\frac{x}{2}\sqrt{8 - x^2} + \frac{8}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) - \frac{x^3}{6}\right\}_0^2 = 2\left\{\left(2 + \pi - \frac{4}{3}\right) - 0\right\} = 2\pi + \frac{4}{3} \text{ sq units}$$

Since the area of the circle is 8π, the area of the larger portion R_2 is

$$8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3} \text{ sq units}$$

$\therefore$ The areas of the two parts are $2\pi + \frac{4}{3}$ sq units and $6\pi - \frac{4}{3}$ sq units.

9. Show that the area enclosed between the curves $y^2 = 12(x+3)$ and $y^2 = 20(5-x)$ is $64\sqrt{\frac{5}{3}}$.

Sol. Equation of the curves are $y^2 = 12(x+3)$ ....(1)

$y^2 = 20(5-x)$ ....(2)

Eliminating $12(x+3) = 20(5-x)$

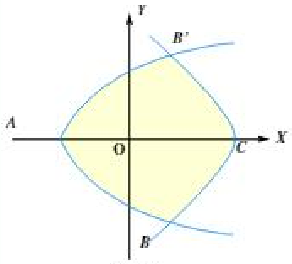


Fig 3.29

$3x + 9 = 25 - 5x$

$8x = 16$

$x = 2$

$y^2 = 12(2 + 3) = 60$

$y = \sqrt{60} = \pm 2\sqrt{15}$

Points of intersection are $B(2, 2\sqrt{15}); B'(+2, -2\sqrt{15})$

The required area is symmetrical about x- axis Area ABCB'

$$= 2\left[\int_{-3}^{2} 2\sqrt{3}\sqrt{x+3}\,dx + \int_{2}^{5} 2\sqrt{5}\sqrt{5-x}\,dx\right] = 4\sqrt{3}\left[\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-3}^{2} + 4\sqrt{5}\left[\frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{2}^{5}$$

$$= \frac{8\sqrt{3}}{3}\left(5^{\frac{3}{2}} - 0\right) - \frac{8\sqrt{5}}{3}\left[0 - 3^{\frac{3}{2}}\right] = \frac{8\sqrt{3}}{3}.5\sqrt{5} + \frac{8\sqrt{5}}{3}.3\sqrt{3}$$

$$= \frac{40.\sqrt{15}}{3} + \frac{24\sqrt{15}}{3} = \frac{64}{3}\sqrt{15} \text{ sq units}$$

$$= 64\sqrt{\frac{15}{9}} \text{ sq units} = 64\sqrt{\frac{5}{3}} \text{ sq units}$$

EXERCISE – 3.1

1. Find the area of the region bounded by

*a) $y = \sin x$, $y = 0$, $x = 0$ and $x = \pi$ [Ans : 2]

*b) $y = |\cos x|$, $y = 0$, $x = -\pi$ and $x = \pi$ [Ans : 4]

*c) $y = e^x$, $y = 0$, $x = 0$ and $x = \ln 2$ [Ans : 1]

*d) $y = |x|$, $y = 0$, $x = -1$ and $x = 1$ [Ans : 1]

*e) $y = x^3$, $y = 0$, $x = 0$ and $x = 2$ [Ans : 4]

f) $y = \ln x$, $y = 0$, $x = 1$ and $x = e$ [Ans : 1]

2. Find the area of the region bounded by

*a) $y = \sin x$ and the x-axis in the interval $[0, 2\pi]$ [Ans : 4]

b) $y = \cos x$ and the x-axis in the interval $[0, 2\pi]$ [Ans : 4]

*c) the parabola $y = x^2$, x-axis and the lines $x = -1$ and $x = 2$ [Ans : 3]

*d) $y = e^x$ and $y = x$ between $x = 0$ and $x = 1$. [Ans : $\left(e - \frac{3}{2}\right)$]

*e) $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ [Ans : $(2(\sqrt{2} - 1))$]

*f) $y = \cos x$ and $y = \sin 2x$ between $x=0$ and $x = \frac{\pi}{2}$ [Ans : $\left(\frac{1}{2}\right)$]

*g) $y^2 = 4 - x$ and the y-axis [Ans : $\frac{32}{3}$]

*h) $y = \cos x$ and $y = 1 - \frac{2x}{\pi}$ [Ans : $\left(2 - \frac{\pi}{2}\right)$]

*i) $y = x^3 + 3$ and $y = 0$ between $x = -1$ and $x = 2$ [Ans : $\left(\frac{51}{4}\right)$]

*j) the hyperbola $xy = 1$ and $x = 0$ between $y = 1$ and $y = 2$ [Ans : $(\ell n 2)$]

*k) $y = 1 - |x|$ and the x-axis [Ans : 1]

*l) $x^2 = 8y$, x-axis and the line $x = 4$ [Ans : $\frac{8}{3}$]

*m) $y = x^2$ and $y = 2$ [Ans : $\frac{32}{3}$]

*n) $y = 4x - x^2 - 3$ and the x-axis [Ans : $\frac{4}{3}$]

*o) $y = x^2$ and $y = x^3$ **(May-19)** [Ans : $\frac{1}{12}$]

*p) $x = 4 - y^2$, $x = 0$ [Ans : $\frac{32}{3}$]

3. Find the area of the region enclosed by the curves

*a) $y = x^2$ and $y = 2x$ [Ans : $\frac{4}{3}$]

*b) $y^2 = 3x$ and $x = 3$ [Ans : 12]

*c) $x = 2 - 5y - 3y^2$ and $x = 0$ [Ans : $\left(\frac{343}{54}\right)$]

*d) $x^2 = 4y$, $x = 2x$ and $y = 0$ [Ans : $\left(\frac{2}{3}\right)$]

*e) $y^2 - 1 = 2x$ and $x = 0$ [Ans : $\left(\frac{2}{3}\right)$]

*f) $y = 6x - x^2$ and $y = 3x$ [Ans : $\left(\frac{9}{2}\right)$]

*g) $y = x^2$ and $y = \sqrt{x}$ (May-18) [Ans : $\left(\frac{1}{3}\right)$]

*h) $y^2 = 8x$ and $y = 2x$ [Ans : $\left(\frac{4}{3}\right)$]

*i) $y = x^2$ and $y = 3x$ [Ans : $\left(\frac{9}{2}\right)$]

*j) $y = x^3 - 6x^2 + 8x$ and the x-axis. [Ans : 0]

4. Find the area enclosed between the curves

*a) $y = 4x - x^2$ and $y = 5 - 2x$ [Ans : $\left(\frac{32}{3}\right)$]

*b) $y = x^2 - 5x$ and $y = 4 - 2x$ [Ans : $\left(\frac{125}{6}\right)$]

*c) $y = 2 - x^2$ and $y = x^2$ [Ans : $\left(\frac{8}{3}\right)$]

*d) $y = x^2 + 1$, $y = 2x - 2$ and the ordinates $x = -1$ and $x = 2$ [Ans : 9]

*e) $y^2 = 2x$, $y = 4x - 1$ [Ans : $\left(\frac{9}{32}\right)$]

*f) $y^2 = 2x + 6$ and $y = x - 1$ [Ans : 18]

*g) $y = 4x^2$ and $y = x^2 + 3$ [Ans : 4]

*h) $y = x^2$, $y = 2x - x^2$ [Ans : $\left(\frac{1}{3}\right)$]

*i) $y = |x|$ and $y = x^2 - 2$ [Ans : $\left(\frac{20}{3}\right)$]

*j) $y = \sin \pi x$, $y = x^2 - x$ and the line $x = 2$ [Ans : $\frac{5}{6} + \frac{2}{\pi}$]

*5. Find the area between x-axis and the curve $y = (x - 1)^2 - 25$. [Ans : $\frac{500}{3}$]

*6. Find the area of the region formed by the segment cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$ and the line. [Ans : 36]

*7. Find the area of the region bounded by $y^2 = 4ax$ between the lines $x = a$ and $x = 9a$.
[Ans : $\left(\frac{208}{3}a^2\right)$]

*8. Find the ara of the region bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}\,(x \geq 0, y \geq 0)$ and the coordinate axes. [Ans : $\left(\frac{a^2}{6}\right)$]

*9. Find the area bounded by the x-axis part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates $x=2$ and $x=4$
[Ans : 4]

*10. Let AOB be the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $OA = a$ and $OB = b$. Show that the area bounded between the chord AB and the arc AB of the ellipse is $\left(\frac{\pi - 2}{4}\right)ab$ sq units.

*11. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. [Ans : 4π]

12. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by the lines $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts. **(March-17)

*13. Find the area of the right angled Δle with base b and altitude h using the fundamental theorem of integral calculus. [Ans : $\left(\frac{1}{2}bh\right)$]

*14. Find the area of the Δle formed by the straight line $2x + y = 2$ and the coordinate axes using integration. [Ans : 1]

15. If the regions A and B are given by $A = \{(x,y) : y \geq x\}, B = \{(x,y) : y \leq 2 - x^2\}$ find the area of $A \cap B$. [Ans : $\left(\frac{9}{2}\right)$]

EXERCISE - 3.2

1. Find the area of the region bounded by $y = x(x-1)(x-2)$, the x-axis and $x = 0$ and $x = 4$. [Ans : $\frac{33}{2}$]

2. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$ and the line $x = 1$ [Ans : $e + \frac{1}{e} - 2$]

3. Find the area of the region bounded by $y = 3^x$ and the lines $y = 3$ and $x = 0$. [Ans : $3 - \left(\frac{2}{ln\,3}\right)$]

4. Find the area of the region enclosed by the parabola $y = x^2 + 2$, the lines $y = -x$, $x = 0$ and $x = 1$. [Ans : $\frac{17}{6}$]

5. Find the area enclosed by the curves $y = ln\,x$, $y = 2^x$ and the lines $x = \frac{1}{2}$ and $x = 2$ [Ans : $\frac{3}{2} - \frac{5}{2}ln\,2 + \frac{4-\sqrt{2}}{ln 2}$]

6. Find area enclosed by $|y| = 1 - x^2$. [Ans : $\frac{8}{3}$]

7. Find the area enclosed between $y = \sqrt{5 - x^2}$ and the lines $y = |x - 1|$. [Ans : $\frac{5\pi - 2}{4}$]

8. Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$. [Ans : 15 : 49]

9. Find the ratio of the curves into which the circle $x^2 + y^2 = 64a^2$ is divided by the curve $y^2 = 12ax$. [Ans : $4\pi + \sqrt{3} : 8\pi - \sqrt{3}$]

10. Compute the area of the region bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$ and $y = 2x - x^2$. [Ans : $\frac{3}{ln\,2} - \frac{4}{3}$]

11. Find the area of the region bounded by the curves $y = x^2$ and $y = \frac{2}{1 + x^2}$. [Ans : $\pi - \frac{2}{3}$]

12. Find the area of the region bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$. [Ans : $\frac{2}{e}$]

13. Find the area of the region bounded by the lines $y = |x-1|$ and $y = 3-|x|$. [Ans : 4]

14. Find the area of the region bounded by the curves $x = |y^2 - 1|$ and $y = x - 5$. [Ans : $\frac{109}{6}$]

15. Find the area of the region bounded by $y = \log_e x$ and $y = \sin^4(\pi x)$ [Ans : $\frac{11}{8}$]

16. Find the area of the region bounded by $4y = |4 - x^2|$ and $y = 7 - |x|$ [Ans : 32]

17. Find the area enclosed by $y = \log_e(x + e)$ and $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis. [Ans : 2]

18. Let $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$. Determine the area of the region bounded by $y = f(x)$, x-axis and $x = 2\pi$ [Ans : $\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}$]

19. Let $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$. Determine the area of the region bounded by $y = f(x)$, x-axis, $x = 0$ and $x = 1$. [Ans : 7/27]

20. Let A_n between area bounded by the curve $y = (\tan x)^n, n \in N$ and the lines $y = 0$ and $x = \frac{\pi}{4}$. For $n > 2$, prove that $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce that $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

21. Find the area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$. [Ans : $4\sqrt{2} - 2$]