CHAPTER 3

Algebra

PROBABILITY

• CLASSICAL DEFINITION • ADDITION THEOREM •
• CONDITIONAL PROBABILITY • TOTAL PROBABILITY •
• BAYE'S THEOREM •

3.1 RANDOM EXPERIMENT

Note : *Probability is a measure of uncertainity*

An experiment is said to be a random experiment, if the outcome of the experiment can not be decided in advance even after knowing the results of all previous trials. (or) An experiment whose outcome can not be predicted with certainty is called random experiment.

Example :

1) Tossing a fair coin is a random experiment because when a coin is tossed we can not say with certainty whether a head or a tail will come up.

2) Throwing an unbiased die is a random experiment because when a die is rolled, we can not say which one of the numbers 1, 2, 3, 4, 5 and 6 will come up.

3.2 ELEMENTARY EVENT

Every possible out come in a single trial of a random experiment is called elementary event or simple event of that experiment.

Example :

1) If a coin is tossed, getting head is a simple event.

2) If 2 coins are tossed, getting both heads is a simple event.

Note : *Associated with a random experiment we can define more than one sample space*

Sample space :

The set of all possible outcomes of a random experiment is called the sample space of the experiment and every element of the sample space is called a sample point. The sample space of an experiment is denoted by S.

Example :

1) If a coin is tossed, the sample space S is given by $S = \{H, T\}$
Here H stands for Head and T stands for Tail.

2) If 2 coins are tossed, $S = \{HH, HT, TH, TT\}$

Note

i) Elementary event (simple event) of a random experiment is an event having only one sample point.

ii) If n coins are tossed, then the sample space contains 2^n sample points.

*iii) If a die is rolled, then $S = \{1, 2, 3, 4, 5, 6\}$
Here S contains 6 sample points*

*iv) If two dice are rolled, $S = \{(1, 1), (1, 2), (1, 3),..., (1, 6), (2, 1) .... (6, 6)\}$
Here S contains 36 sample points.*

v) If n dice are rolled, $S = \{(x_1, x_2, ..., x_n) / 1 \le x_i \le 6$ where $i = 1, 2,...n\}$. Here S contains 6^n sample points.

3.3 EVENT

A combination of two or more elementary events of a random experiment is called mixed event or simply an event.

For example, If 2 coins are tossed, getting atleast one head is a mixed event. It is the combination of 3 elementary events which are HT, TH and HH.

Note : *favourable cases are called 'successes' and remaining cases are called 'failures'.*

Note

i) Simple event is also an event.

ii) The favourable cases to a particular event of an experiment are called 'successes' and the remaining cases are called 'failures' with respect to the event.

iii) In a random experiment, let A be any event. The set of all outcomes of the experiment which are favourable to A is a subset of sample space S of the experiment.

Thus any event A of the experiment is a subset of the sample space S and elements of A are the favourable outcomes to the event A.

iv) A simple event is a singleton subset. We consider the subsets ϕ and S also as events.

v) When a random experiment is conducted if a sample point of an event A is the outcome then we say that A has happened.

vi) If the sample space S of a random experiment contains n elements then the number of possible events (including simple events) of the experiment is 2^n

vii) The set of all possible events of a random experiment associated with S is denoted by $\mathcal{P}(S)$.

3.4 CERTAIN EVENT AND IMPOSSIBLE EVENT

An event which is certain to happen is called certain event and it is denoted by S.

An event which can not happen at all is an impossible event and it is denoted by ϕ.

Note : *Any event A of an experiment is a subset of the sample space S.*

Note

Each sample point of S is a favourable point to the certain event and no sample point of S is a favourable point to impossible event.

Example :

If two dice are rolled, getting sum even or odd is a certain event where as getting sum 13 is an impossible event.

3.5 MUTUALLY EXCLUSIVE EVENTS

Two or more events are said to be mutually exclusive if the occurrence of one of the events prevents the occurrence of any of the remaining events.

We say the events $A_1, A_2, \ldots\ldots, A_n$ are mutually exclusive if $A_i \cap A_j = \phi$ for $i \neq j$ and $1 \le i, j \le n$.

For example, If two dice are rolled,

$S = \{(x_1, x_2) / 1 \le x_1, x_2 \le 6\}$

Let A : occurrence of both odd

B : occurrence of sum 12

Note : *A simple event is a singleton subset.*

Here $A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 2), (5, 3)\}$

$B = \{(6, 6)\}$

Here A and B are exclusive since $A \cap B = \phi$

Here it is evident that if A happens then B does not happen and vice-versa.

3.6 EXHAUSTIVE EVENTS

Two or more events are said to be exhaustive events if the outcome of the experiment always results in the occurrence of atleast one of them.

We say the events $A_1, A_2, \ldots, A_n$ are exhaustive if $A_1 \cup A_2 \cup \ldots \cup A_n = S$ where S is the sample space.

For example,

If a single die is rolled, then $S = \{1, 2, 3, 4, 5, 6\}$

Let A : occurrence of an even number; B : occurrence of an odd number. Here $A \cup B = S$

So, A, B are exhaustive. Here A, B are mutually exclusive also.

3.7 MUTUALLY EXCLUSIVE AND EXHAUSTIVE EVENTS

A set of events $A_1, A_2, \ldots, A_n$ is said to be mutually exclusive and exhaustive if

1) $A_i \cap A_j = \phi$ for $i \neq j$ and $1 \leq i, j \leq n$
2) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

For example,

If a card is drawn from pack of cards then the sample space contains 52 sample points.

Note : *Mutually exclusive events are disjoint subsets of a sample space*

Let A : getting hearts card

B : getting clubs card

C : getting diamond card

D : getting spade card

Here $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = C \cap D = \phi$ and $A \cup B \cup C \cup D = S$

So, A, B, C, D are exclusive and exhaustive.

3.8 EQUALLY LIKELY EVENTS

Two or more events of a random experiment are said to be equally likely events if there is no reason to expect one of them to happen in preference to others.

For example, if a fair coin is tossed, occurrence of head and occurrence of tail are equally likely events.

If a single die is thrown ocurence prime number, and occurense of even number are equally likely events. If a single fair die is thrown, all six simple events are equally likely simple events.

3.9 COMPLEMENTARY EVENT OF AN EVENT

Suppose A is any event of a random experiment associated with sample space S.

The complementary event of an event A denotedy by $\overline{A}$ or A^C is the event given by $A^C = (S - A)$ which is called the complementary event of A.

For example, if two coins are tossed, $S = \{HH, HT, TH, TT\}$

Let A : occurrence of atleast one head

Here A^c is the event of nonoccurrence of atleast one head.

i.e., A^C is the event of getting both tails.

Here $A = \{HH, HT, TH\}$

$A^c = \{TT\}$

Note

i) *An event A and its complementary event A^c are exclusive and exhaustive.*
Here $A \cap A^c = \phi$ and $A \cup A^c = S$;

ii) *A occurs $\Leftrightarrow$ A^c does not occur.*

iii) $S^c = \phi$ *and* $\phi^c = S$

3.10 DEMORGAN LAWS

If A, B, C are any three sets then

1) $A-(B \cup C)=(A-B) \cap (A-C)$

2) $A-(B \cap C)=(A-B) \cup (A-C)$

These are called Demorgan laws which are useful in solving some problems of this chapter.

3.11 CLASSICAL DEFINITION OF PROBABILITY

Note : *Probability of a certain event is '1'*

If a random experiment results n exhaustive, exclusive and equally likely elementary events and m of them are favourable to an event A then probability of occurrence of A denoted by $P(A)$ is given by m/n.

$$P(A) = \frac{\text{number of favourable cases to the event } A}{\text{number of possible outcomes}} = \frac{m}{n}$$

Note

i) *From the definition since* $0 \le m \le n$ *we have* $0 \le P(A) \le 1$.

ii) *If m outcomes are favourable to A then $(n-m)$ outcomes will be favourable to A^c.*

$$P(A^c) = \frac{n-m}{n}$$

Therefore $P(A) + P(A^c) = \frac{m}{n} + \frac{n-m}{n} = 1$

iii) *The classical defintion of probability has some limitations.*

a) *In some random experiments the sample space may be infinite. For example throwing a die until we get even number is such an experiment. In such random experiments, this defintion may not be applicable to find the probability of an event of the experiment.*

b) *In some random experiments, all the elementary events need not be equally likely. For example if a coin is biased (i.e., not fair) then getting head and getting a tail are not equally likely. In such experiments, probability of an event of the experiment can not be defined using classical definition.*

c) *In order to overcome these two inconveniences, we define relative frequency approach for the definition of probability.*

Note : *Probability of an impossible event is '0'.*

3.12 RELATIVE FREQUENCY DEFINITION OF PROBABILITY

Suppose a random experiment is repeated n times out of which an event A occurs m times then $\frac{m}{n}$ is called relative frequency of A and is denoted by $R_n(A)$.

If $\underset{n\to\infty}{Lt} R_n(A)$ exists then the limit is called probability of A.

So, $P(A) = \underset{n\to\infty}{Lt} R_n(A)$

Note

Relative frequency definition of probability has also an inconvenience. That is, $\underset{n\to\infty}{Lt} R_n(A)$ may not exist. In such a case probability of an event can not be decided. Keeping in mind all these facts, we are going to learn axiomatic approach to probability.

3.13 ODDS IN FAVOUR OF AN EVENT

Definition :

Suppose A is an event in a sample space S. Then the ratio $P(A) : P(A^c)$ is called the odds in favour of A and $P(A^c) : P(A)$ is called the odds against A.

> **Note :** $P(A) : P(A^c)$ is called the odds in favour of A. $P(A^c) : P(A)$ is called the odds against A.

Note

i) *If* $P(A) = \frac{m}{n}$ *then*

a) odds infavour of $A = m : (n-m)$ *b) odds against* $A = (n-m) : m$

ii) *Also, odds infavour of* $A = p : q \Rightarrow P(A) = \frac{p}{p+q},\ P(\bar{A}) = \frac{q}{p+q}$

odds against $A = l : m \Rightarrow P(A) = \frac{m}{l+m}, P(\bar{A}) = \frac{l}{l+m}$

3.14 AXIOMATIC APPROACH TO PROBABILITY

Definition :

Let S be the sample space of a random experiment. $\mathcal{P}(S)$ is the set of all possible events of the experiment.

Then a function $P : \mathcal{P}(S) \to R$ satisfying the following axioms is called a probability function.

1) **Axiom of non-negativity :** $P(A) \ge 0\ \forall\ A \in \mathcal{P}(S)$
2) **Axiom of certainity :** $P(S) = 1$
3) **Axiom of additivity :** If $A, B \in \mathcal{P}(S)$ and $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

Here for each $A \in \mathcal{P}(S)$, the non negative real number $P(A)$ is called the probability of the event A.

Note

i) *If* $A_1, A_2, A_3, \ldots, A_n$ *are finitely many pairwise mutually exclusive events then using mathematical induction we can prove* $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$

ii) *If* $A = \{a_1, a_2, \ldots, a_n\}$ *is an event then* $P(A) = \sum_{i=1}^{n} P(a_i)$

iii) *Let* $S = \{w_1, w_2, \ldots\ldots w_n\}$ *is sample space*

$\Rightarrow P(w_1) = P(w_2) = \ldots\ldots = P(w_n) \ldots\ldots (1)$

Also $P(w_1) + P(w_2) + \ldots\ldots + P(w_n) = 1 \ldots\ldots (2)$

From (1), (2), $P(w_i) = \frac{1}{n} \forall i$

Let $A = \{w_1, w_2, \ldots\ldots w_m\}$ $(m \le n)$ *be an event*

$$\Rightarrow P(A) = \sum_{i=1}^{m} w_i = \frac{1}{n} + \frac{1}{n} + \ldots\ldots + \frac{1}{n} = m\left(\frac{1}{n}\right) = \frac{m}{n} = \frac{n(A)}{n(S)}$$

THEOREM-3.1 **Let S be a sample space of a random experiment and $P : \mathcal{P}(S) \to R$ is a probaiblity function then**

1) $P(\phi) = 0$
2) If $A \in \mathcal{P}(S)$ then $P(A) + P(A^C) = 1$
3) $0 \le P(A) \le 1,\ \forall A \in \mathcal{P}(S)$
4) If $A \subseteq B$ then $P(B - A) = P(B) - P(A)$
5) If $A \subseteq B$ then $P(A) \le P(B)$

Proof : 1) We know that $(S \cap \phi) = \phi$ and $(S \cup \phi) = S$ So, $P(S) = P(S \cup \phi) = P(S) + P(\phi)$

$(\because S \cap \phi = \phi$ by applying Axiom of additivity)

$\therefore P(S) = P(S) + P(\phi)$ $\quad \therefore P(\phi) = 0$ (since $P(S) \in R$)

2) Let $A \in \mathcal{P}(S)$

since $A \cup A^C = S$ and $A \cap A^C = \phi$

we have $1 = P(S) = P(A \cup A^C) = P(A) + P(A^C)$ (Axiom of additivity)

$\therefore P(A) + P(A^C) = 1$

3) Let $A \in \mathcal{P}(S)$

By non-negativity Axiom,

we have $P(A) \geq 0$; $P(A^C) \geq 0$

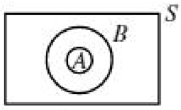
From the previous result we have $P(A) + P(A^C) = 1$

Therefore $P(A) = 1 - P(A^C) \leq 1$

So, $0 \leq P(A) \leq 1$

> ***Note :***
> $A \cap B \subseteq A$
> $\Rightarrow p(A \cap B) \leq p(A)$

4) Suppose $A \subseteq B$.



Since $A \subseteq B$ we have

$A \cup (B - A) = B$ and $A \cap (B - A) = \phi$

Therefore, $P(B) = P[A \cup (B - A)] = P(A) + P(B - A)$ (Axiom of additivity)

Therefore $P(B - A) = P(B) - P(A)$

> ***Note :***
> $A \cap B \subseteq A$
> $\Rightarrow p(B \cap B) \leq p(B)$

5) Suppose $A \subseteq B$.

we have to show that $P(A) \leq P(B)$

Since $A \subseteq B$, by above result we have $P(B - A) = P(B) - P(A)$

But by Axiom of non negativity,

$P(B - A) \geq 0$

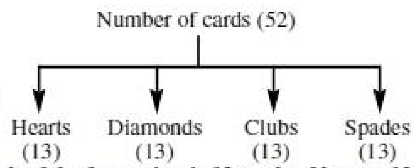
Therefore $P(B - A) = P(B) - P(A) \geq 0 \Rightarrow P(B) \geq P(A)$

> ***Note :***
> $p(A \cap B) \leq \min\{p(A), p(B)\}$

Note

i) ***In simple cases, number of elementary events favourable to an event and the total number of elementary events in the sample space can easily be counted, but in many cases direct counting is very difficult. In such cases we use permutations and combinations to know the number of elementary events favourable to an event and the total number of elementary events in the sample space and thus the calculation of probability of an event becomes easier.***

ii) ***Details of the cards of a pack of playing cards :***



a) ***In a deck of 52 cards there are 13 cards each of the four suits viz 13 cards of hearts, 13 cards of diamonds, 13 cards of clubs and 13 cards of spades. The cards of hearts and dimonds are of red colour where as the cards of clubs and spades are of black colur.***

b) ***In the thirteen cards of each suit there are four picture cards (honours) viz. ace, king, queen and jack. All the cards are known as court cards except the king.***

c) ***The king, the queen and the jack are known as face-cards.***

> ***Note :***
> ***Probability***
> ***The king, the queen and the jack are known as face-cards***

Note :

Equally likely events have the same number of favourable cases as subsets.

SOLVED EXAMPLES

1. In the experiment of throwing a die, consider the following events: A={1,3,5}, B={2,4,6},C={1,2,3}. Are these events equally likely ?

Sol. $n(A) = n(B) = n(C) \Rightarrow A, B, C$ are equally likely.

2. In the experiment of throwing a die, consider the following events : A={1,3,5}, B={2,4}, C={6}. Are these events mutually exclusive?

Sol. $A \cap B = \phi, B \cap C = \phi, A \cap C = \phi$ $\therefore$ A, B, C are mutually exclusive.

3. Suppose $S = \{0, 1, 2, 3\}$ be a sample space of a random experiment. If $P : \mathcal{P}(S) \to R$ defined by $P(0) = 0.1; P(1) = 0.3; P(2) = 0.3$ and $P(3) = 0.3$ and $P(A) = \sum_{a \in A} P(a)$ for any subset A of S, is P a probability function or not.

Sol. Clearly $P(A) \geq 0 \;\forall\; A \in \mathcal{P}(S)$

So, non negativity Axiom is satisfied.

Here $P(S) = P(0) + P(1) + P(2) + P(3) = 0.1 + 0.3 + 0.3 + 0.3 = 1$

So, axiom of certainity is satisfied.

Let $A, B \in \mathcal{P}(S)$ and $A \cap B = \phi$

Here $P(A \cup B) = \sum_{a \in A \cup B} P(a) = \sum_{a \in A} P(a) + \sum_{a \in B} P(a) = P(A) + P(B)$

Axiom of additivity is satisfied.

Therefore P is a probability function.

4. Find the probability of throwing a total score 8 with 2 uniform dice.

Sol. Since, two dice are rolled,

Sample space $S = \{(x_1, x_2) / 1 \leq x_1, x_2 \leq 6\}$

Here $n(S) = 6 \times 6 = 36$; Let A be the event of getting score 8

$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

Here $n(A) = 5;\; P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{36}$

5. A page is opened at random from a book containing 600 pages. What is the probability that the number on the page is a perfect square.

Sol. Since we are selecting one page from 600 pages

$S = \{1, 2, ..., 600\};\quad n(S) = 600;$

Let A be the event of selecting page whose number is perfect square.

Since 24 is the largest number such that $24^2 = 576 < 600$ we have,

$A = \{1^2, 2^2, ..., 24^2\}$ So, $n(A) = 24 \quad P(A) = \dfrac{24}{600} = \dfrac{3}{75}$

6. If two cards are drawn from from pack of 52 cards at random find the probability of getting both club cards.

Sol. Since 2 cards are drawn at random, $n(S) = {}^{52}C_2$

Let A be the event of selecting both club cards.

In a pack there are 13 club cards.

$n(A) = {}^{13}C_2$

$P(A) = \dfrac{{}^{13}C_2}{{}^{52}C_2} = \dfrac{13 \times 12}{52 \times 51} = \dfrac{1}{17}$

7. If 4 fair coins are tossed find the probability of getting 2 heads and 2 tails.

Sol. Since 4 coins are tossed,

we have, number of elementary events = $2^4 = 16$

Here $n(S) = 2^4 = 16$;

Let E be the event of getting 2 heads and 2 tails.

One of the favourable of E is HHTT;

If we rearrange these 4 letters in different orders we get all the favourables of E.

They can be arranged in $\frac{\underline{|4}}{\underline{|2}\,\underline{|2}}$ ways because these 4 letters are having 2 like letters of one kind and 2 like letters of another kind.

$$\text{So, } n(A) = \frac{\underline{|4}}{\underline{|2}\,\underline{|2}} = {}^4C_2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}^4C_2}{2^4} = \frac{6}{16} = \frac{3}{8}$$

Note :

i) If n fair coins are tossed, the total number of favourable outcomes to the event of getting exactly k heads and $(n-k)$ tails in ${}^nC_k$ or ${}^nC_{n-k}$

ii) If n fair coins are tossed, Probability of getting (k heads, $(n-k)$ tails)

= Probability of getting (k heads)

= Probability of gettig (k tails) = $\frac{{}^nC_k}{2^n}$

Note : *Probability of getting r heads in n tosses of a coin is* $\frac{{}^nC_r}{2^n}$

8. If three dice are rolled, find the probability of showing all different numbers.

Sol. Since three dice are rolled, $n(S) = 6 \times 6 \times 6 = 216$

Let A be the event that all the faces show different numbers.

Here $A = \{x, y, z) / 1 \le x, y, z \le 6$ and $x \ne y \ne z\}$

The number of favourable outcomes to A = the number of arrangements of 6 numbers taken 3 at a time = ${}^6P_3$

$$\text{So, } P(A) = \frac{{}^6P_3}{6^3} = \frac{5}{9}$$

9. Find the probability that a leap year will have 53 sundays.

Note : *A leap year contains 366 days*

Sol. A leap year contains 366 days.

$366 = 7 \times 52 + 2$

So, a leap year contains 52 complete weeks leaving two more days which are December 30^{th} and 31^{st}.

The last two consecutive days may be any one of the following 7 possibilities.

1) (sun, mon) 2) (mon, tue) 3) (tue, wed)

4) (wed, thu) 5) (thu, fri)

6) (fri, sat) 7) (sat, sun)

Let A be the event that the leap year will have 53 sundays.

Here $A = \{$(sat, sun) ; (sun, mon)$\}$

So, A has exactly two favourable outcomes

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$$

10. 4 boys and 4 girls are arranged in a row at random. Find the probability that the boys and girls sit alternatively.

Sol. Since 4 boys and 4 girls are arranged along a row,

$n(S) = \lfloor 8$

Let A be the event of arranging the boys and girls alternatively.

Consider 8 places to arrange these 4 boys and 4 girls.

X X X X X X X X

arrange the 4 boys in odd places and the 4 girls in even places.

It can be done in $\lfloor 4 \times \lfloor 4$ ways

But we can arrange 4 boys in even places and 4 girls in odd places also.

It can also be done in $\lfloor 4 \times \lfloor 4$ ways.

So, $n(A) = \lfloor 4 \times \lfloor 4 + \lfloor 4 \times \lfloor 4 = 2 \lfloor 4 \times \lfloor 4 \;\therefore\; P(A) = \dfrac{2\lfloor 4 \times \lfloor 4}{\lfloor 8}$

11. If 4 people are chosen at random, then find the probability that no two of them were born on the same day of the week.

Sol. Since a person can born on any day of a week, he can born in 7 ways.

By product rule,

The total number of ways that 4 people can born

$= 7 \times 7 \times 7 \times 7 = 7^4$

$n(S) = 7^4$;

Let A be the event that no two of the 4 people chosen were born on the same day of the week.

Here $n(A) =$ number of favourable cases to A

$=$ number of ways to arrange the 7 days of the week taken 4 at a time $= {}^7P_4$

$P(A) = \dfrac{{}^7P_4}{7^4}$

12. If the letters of the word 'QUESTION' are arranged at random. What is the probability that there are exactly two letters between Q and U.

Sol. Since the letters of the word 'QUESTION' are arranged at random,

$n(S) = \lfloor 8$;

Let A be the event that Q and U are separated by exactly two letters.

X X X X X X X X

Q U

The possible positions to arrange Q and U are (1^{st}, 4^{th}) (2^{nd}, 5^{th}) (3^{rd}, 6^{th}) (4^{th}, 7^{th}) and (5^{th}, 8^{th})

So, select one of these 5 possible positions and arrange Q and U in these two places in both different orders.

It can be done in ${}^5C_1 \times \lfloor 2$ i.e., in 10 ways.

The remaining six letters can be arranged in $\lfloor 6$ ways.

So, $n(A) = {}^5C_1 \times \lfloor 2 \times \lfloor 6 = 10 \times 720 = 7200$

So, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{7200}{\lfloor 8} = \dfrac{5}{28}$

> **Note :**
> *Among '6' consecutive integers exactly 2 are divisible by 3*

13. Two numbers are selected at random from 1, 2, 3, ..., 100 and multiplied. Find the probability that the product thus obtained is divisible by 3.

Sol. **Method - 1**

Among 1, 2, 3, ... 100 there are exactly 33 numbers which are divisible by 3.

Since we are selecting two numbers at random, number of possible out comes = $n(S) = {}^{100}C_2$.

Let A be the event of selecting the numbers whose product is divisible by 3.

We know product of 2 numbers is divisible by 3 if atleast one of them is divisible by 3 i.e., both are divisible by 3 or exactly one of them is divisible by 3.

Therefore $n(A) = {}^{33}C_2 + {}^{33}C_1 \times {}^{67}C_1$.

$$\therefore\ P(A) = \frac{{}^{33}C_2 + {}^{33}C_1 \times {}^{67}C_1}{{}^{100}C_2} = \frac{\frac{33\times 32}{2} + 33\times 67}{\frac{100\times 99}{1\times 2}} = \frac{33(83)}{50\times 99} = \frac{83}{150};$$

Method - 2

Here A^C is the event of selecting 2 numbers whose product is not divisible by 3.

$n(A^C) = {}^{67}C_2$

($\because$ product of 2 numbers is not divisible by 3 if both of them are not divisible by 3)

$$\therefore\ P(A^C) = \frac{{}^{67}C_2}{{}^{100}C_2} = \frac{66\times 67}{100\times 99} = \frac{67}{150}$$

$$\therefore\ P(A) = 1 - P(A^C) = 1 - \frac{67}{150} = \frac{83}{150}$$

> **Note :**
> *Two squares on a chessboard have exactly one common corner when they are on a diagonal.*

14. Two squares are chosen at random from the small squares on a chess board. What is the chance that the two squares have exactly one corner in common.

Sol. A chess board will have 64 small squares.

Since we are selecting 2 small squares from 64 small squares at random,

$n(S) = {}^{64}C_2$

Consider a diagonal of the chess board containing k small squares where $2 \le k \le 8$

A chess board will have 4 diagonals containing k squares where $2 \le k \le 7$ and exactly 2 diagonals containing 8 squares.

Along the diagonal containing k squares there are $(k - 1)$ pairs of consecutive squares having exactly one common corner.

So, total number of pairs of squares having exactly one common corner.

$$= 4\sum_{k=2}^{7}(k-1) + 2(8-1) = 4\{1+2+....+6\} + 2(7) = 4[21] + 14 = 98$$

Therefore the probability of selecting 2 small squares having exactly one common corner is $= \frac{98}{{}^{64}C_2}$

$$= \frac{98}{64\times 63}\times 2 = \frac{7}{144}$$

15. A and B are among 20 persons who sit at random along a round table. Find the probability that there are any six persons between A and B.

Sol. In circular permutations, there is no first place. At the same time, which thing we use first and in which place we fill first has no importance.

So, arrange A in any one of the 20 places. Now there are 19 places at which B can be arranged. But since 6 persons are to be seated between A and B, B has only two favourable positions to sit in both directions.

The required probability $= \frac{2}{19}$;

16. If 10 coins are tossed, find the odds against the event of getting atleast 2 heads.

Sol. Since 10 coins are tossed, $n(S) = 2^{10}$

Let E be the event of getting atleast 2 heads.

Then, E^C is the event of getting 1 head or no head.

$n(E^C) = {}^{10}C_0 + {}^{10}C_1 = 11$

$\therefore\ n(E) = 2^{10} - 11 = 1013$;

Therefore the odds against the event of getting atleast 2 heads

$= n(E^C) : n(E) = 11 : 1013$

Note :
Odds against E are $n(\bar{E}) : n(E)$
Odds infavour of E are $n(E) : n(\bar{E})$

17. Five coins are tossed whose faces are marked 2 and 3. Find the probability of getting sum 12.

Sol. Since 5 coins are tossed, $n(S) = 2^5 = 32$

Let A be the event of getting sum 12.

One of the favourable case to A is (2, 2, 2, 3, 3)

By rearranging these 5 numbers we get all the different favourables to A.

$\therefore\ n(A) = \frac{\lfloor 5}{\lfloor 3 \lfloor 2}$

This is because, among these 5 digits 3 are alike of one kind and 2 are alike of another kind.

$$P(A) = \frac{\frac{\lfloor 5}{\lfloor 3 \lfloor 2}}{2^5} = \frac{10}{32} = \frac{5}{16}$$

18. If p and q are chosen at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement. Find the probability that the roots of $x^2 + px + q = 0$ are imaginary.

Sol. Since p and q are selected from {1, 2, ..., 10} with replacement at random, $n(S) = 10 \times 10 = 100$;

Let A be the event of selecting p and q such that $x^2 + px + q = 0$ has imaginary roots

we know that $x^2 + px + q = 0$ has imaginary roots if $p^2 - 4q < 0$

Here $A = \{(p, q) / 1 \leq p, q \leq 10 \text{ and } p^2 - 4q < 0\}$

Let us find $n(A)$ as follows. Here $p^2 < 4q$

q	p	number of pairs (p, q)
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
10	1, 2, 3, 4, 5, 6	6
		38

$\therefore n(A) = 38$

$n(S) = 100;\ P(A) = \dfrac{38}{100} = \dfrac{19}{50}$

19. A natural number x is chosen at random from the first 100 natural numbers. Find the probability that $\dfrac{(x-20)(x-40)}{(x-30)} < 0$.

Sol. Since we are selecting x from $\{1, 2, ..., 100\}$, we have

$n(S) = {}^{100}C_1 = 100;$

Let A be the event of selecting 'x' for which $\dfrac{(x-20)(x-40)}{(x-30)} < 0$

Here $\dfrac{(x-20)(x-40)}{(x-30)} < 0 \quad \Leftrightarrow \dfrac{(x-20)(x-40)(x-30)}{(x-30)^2} < 0$

$\Leftrightarrow (x-20)(x-30)(x-40) < 0 \quad \Leftrightarrow x < 20 \text{ or } 30 < x < 40$

$\Leftrightarrow 1 \le x \le 19 \text{ or } 31 \le x \le 39$

$\therefore n(A) = 19 + (39 - 31) + 1 = 19 + 9 = 28$

$\therefore P(A) = \dfrac{28}{100} = \dfrac{7}{25}$

20. Two fair dice are rolled. Find the probability that the difference between the numbers is atleast 2.

Sol. **Method - 1**

Since 2 dice are rolled, $n(S) = 36$;

Let A be the event that the difference between the numbers on the dice is atleast 2.

The difference on the dice can not be more than 5.

Suppose $(x, x+k)$ is a favourable outcome with difference k between the numbers where $2 \le k \le 5$;

Now $1 \le x < x + k \le 6;\ \Rightarrow 1 \le x \le 6 - k$

So, there are $6 - k$ favourable ordered pairs of the form $(x, x + k)$

Again there are $6 - k$ favourable ordered pairs of the form $(x + k, x)$

The total number of favourables to $A = \sum_{k=2}^{5} 2(6-k) = 2\{4 + 3 + 2 + 1\} = 20$

$P(A) = \dfrac{20}{36} = \dfrac{5}{9}$

Note : *Probabilities of events calculated using classical definition are known as "a priori" probabilities.*

Method - 2

Let A be the event that the difference between the numbers on the dice is atleast 2.

A^C is the event that the difference between the events is either 0 or 1.

$A^C = \{(1, 1)(2, 2) .... (6, 6)(1, 2)(2, 3)(3, 4)(4, 5)(5, 6)(2, 1) .... (6, 5)\}$

$P(A^C) = \dfrac{16}{36} \quad \therefore P(A) = 1 - P(A^C) = \dfrac{20}{36} = \dfrac{5}{9}$

21. Two numbers are selected at random from 1, 2, 3, ... 100 without replacement. Find the probability that the minimum of the two numbers is less than 70.

Sol. Since two numbers are selected at random from 1, 2, ..., 100 we have $n(S) = {}^{100}C_2$

Let A be the event that the minimum of the two numbers is less than 70 ;

Then the complement of A i.e., A^C is the event of selecting both the numbers from $\{70, 71, ..., 100\}$

$n(A^C) = {}^{31}C_2$

$P(A^C) = \dfrac{{}^{31}C_2}{{}^{100}C_2} \Rightarrow P(A) = 1 - \dfrac{{}^{31}C_2}{{}^{100}C_2}$

22. Three fair dice are rolled. Find the probability that the greatest number on the dice must exceed 3.

Sol. Since 3 dice are rolled, $n(S) = 216$;

Let E be the event of getting the result in such a way the greatest number on the dice must exceed 3.

So, the complementary event E^C of E is the event that all the 3 dice show numbers which are ≤ 3;

$E^C = \{(x_1, x_2, x_3) / 1 \leq x_i \leq 3\}$

$n(E^C) = 3^3 = 27 \therefore P(E^C) = \dfrac{27}{216} = \dfrac{1}{8} \quad \therefore P(E) = \dfrac{7}{8}$

Note :

$P(E) = 1 - P(\bar{E})$

23. There are 100 stations between two stations A and B. A train is to stop at ten of these 100 stations. What is the probability that no two of these ten stations are consecutive.

Sol. Here $n(S) = {}^{100}C_{10}$;

Let E be the event that no two of the ten halting stations are consecutive.

Here $n(E)$ = number of ways to select 10 stations from the 100 stations such that no two of the selected 10 are consecutive.

= the number of binary sequences of ninety '0's and ten '1's of the form, for example

1 0 0 0 0 1 0 0 ..... 0 1 0 0 1

where no two '1's are consecutive

Here 1 at k^{th} place means k^{th} station is selected, and 0 at k^{th} place means k^{th} station is not selected

Arrange ninety '0's along a row. It can be done in one way.

We get 91 gaps as shown below marked with X symbols.

X 0 X 0 X 0 X 0 ..... X 0 X 0 X

Now arrange the ten '1's in these 91 places marked with X symbols. It can be done in ${}^{91}C_{10}$

So, the required number of binary sequences = ${}^{91}C_{10}$

Therefore, $n(E) = {}^{91}C_{10}$

Therefore, $P(E) = \dfrac{{}^{91}C_{10}}{{}^{100}C_{10}}$

24. Let F be the set of all 4 digited numbers whose sum is 34. If a number is selected from F then find the probability that the selected number is even.

Sol. Since sum of 4 digited number is 34.

It must contain atleast two '9's since $4 \times 8 = 32$

So, the 4 digits of the number will be as follows.

(i) 9, 9, 8, 8 (ii) 9, 9, 9, 7

The total number of 4 -digited numbers

whose sum is 34 = $\dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} + \dfrac{\lfloor 4}{\lfloor 3} = 6 + 4 = 10$

Among these 10 numbers, 4-digited numbers ended with 8 = $\dfrac{\lfloor 3}{\lfloor 2} = 3$

$\therefore$ The number of 4 digited even numbers with sum of digits 34 = 3

The required probability = $\dfrac{3}{10}$

25. From a heap containing 10 pairs of shoes, 6 shoes are selected at random. Find the probability that

i) There is no complete pair in the selected shoes

ii) 2 correct pairs in the selected shoes.

iii) atleast one correct pair in the selected shoes

Sol. There are 10 pairs of shoes i.e., 20 shoes since we are selecting 6 shoes at random, $n(S) = {}^{20}C_6$;

i) Let A be the event that there is no correct pair in the selected shoes.

First select 6 pairs from these given 10 pairs. It can be done in ${}^{10}C_6$ ways.

From these selected six pairs, let us select exactly one shoe from each pair which is either right or left.

It can be done in $2 \times 2 \times 2 \times 2 \times 2 \times 2$ i.e., in 2^6 ways.

Now we are having 6 shoes in which there is no correct pair.

The number of favourables for $A = {}^{10}C_6 \times 2^6$;

$$P(A) = \frac{{}^{10}C_6 \times 2^6}{{}^{20}C_6}$$

ii) Let A be the event of selecting exactly 2 correct pairs in the selected 6 shoes.

Here $n(S) = {}^{20}C_6$;

Let us select 2 pairs from 10 given pairs.

It can be done in ${}^{10}C_2$ ways.

Still we have to select 2 shoes which is not correct pair.

From the remaining 8 pairs, select 2 pairs It can be done in ${}^{8}C_2$ ways.

After selecting 2 pairs let us select one shoe (either right or left) from each pair. It can be done in 2^2 ways. These 2 selected shoes does not serve as a correct pair.

The number of favourables to $A = {}^{10}C_2 \times {}^{8}C_2 \times 2^2$

$$P(A) = \frac{{}^{10}C_2 \times {}^{8}C_2 \times 2^2}{{}^{20}C_6} = \frac{42}{323}$$

iii) P(atleast one correct pair in the selected 6 shoes)

$= 1 - p$(no correct pair in the selected 6 shoes) $= 1 - \dfrac{{}^{10}C_6 \times 2^6}{{}^{20}C_6} = \dfrac{211}{323}$

3.15 SOME IMPORTANT SET THEORETIC DESCRIPTION SYMBOLS

If A, B are any two sets then

i) $(A \cup B)^C = A^C \cap B^C$

ii) $(A \cap B)^C = A^C \cup B^C$

These are called demorgan's laws in set theory.

Suppose A, B are any two events in a random experiment, then

i) $A \cup B$ denotes the event of occurrence of either A or B.

i.e., $A \cup B$ denotes the event of occurrence of atleast one of the events A or B.

ii) $A \cap B$ denotes the occurrence of both A and B.

iii) $A - B = A \cap B^C$, denotes the occurrence of A and the non-occurrence of B.

iv) $(A \cup B)^C = (A^C \cap B^C)$ denotes the occurrence of neither A nor B. i.e., denotes the non occurrence of A and non occurrence of B.

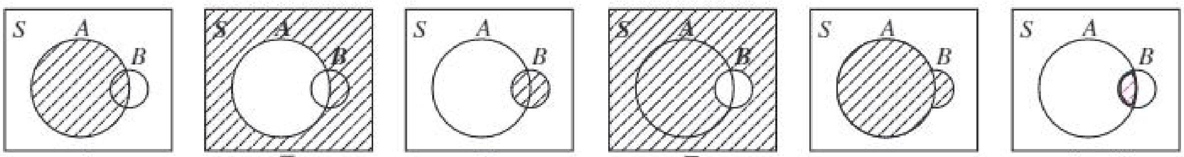
v) $(A \cap B)^C = (A^C \cup B^C)$ denotes the non-occurence of atleast one of A, B. i.e., both A and B does not occur simultaneously at a time.

vi) $(A \cap B^C) \cup (A^C \cap B) = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ denotes the occurrence of exactly one of A, B.

vii) $A \subseteq B$ denote that event B occurs whenever A occurs.

The above symbols can be listed as follows.

Event	Description
1) $A \cup B$	Event A or Event B occur (or) atleast one of A, B occur
2) $A \cap B$	both the events A and B occur
3) $(A \cup B)^C = A^C \cap B^C$	neither A nor B occur
4) $(A \cap B)^C = (A^C \cup B^C)$	either A does not happen (or) B does not happen. i.e., atleast one of A, B does not occur.
5) $(A \cap B^C) \cup (A^C \cap B)$ $= (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$	Exactly one of A, B occur
6) $A \subseteq B$	Event B occurs whenever A occur.



A I; $\bar{A}$ II; B III; $\bar{B}$ IV; $A \cup B$ V; $A \cap B$ VI

Note

i) *Since* $(A \cup B)^C = A^C \cap B^C$

We have, $P(A \cup B) = 1 - P(A^C \cap B^C)$

i.e., *P(atleast one of A, B occur) = 1 − P(none of A, B occur)*

In general, $P(A_1 \cup A_2 \cup .... \cup A_n) = 1 - P(A_1^{\,C} \cap A_2^{\,C} .... \cap A_n^{\,C})$

i.e., P(atleast one) = 1 − P(none)

ii) *Since* $(A \cap B)^C = A^C \cup B^C$

We have, $P(A \cap B) = 1 - P(A^C \cup B^C)$

i.e., *P(both A and B occur)* = *1 – P(atleast one of A, B doesnot occur)*

In general, $P(A_1 \cap A_2 \cap \dots \cap A_n) = 1 - P(A_1^C \cup A_2^C \dots \cup A_n^C)$

i.e., *P(all the events must occur)* = *1 – P(atleast one of the events doesnot occur)*

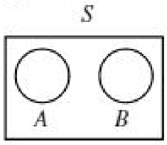
3.16 ADDITION THEOREM ON PROBABILITY

***If A, B are any two events of a random experiment and P is a probability function then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.** (March-18, May-18))

Proof : **Case - 1 :**

Suppose A and B are mutually exclusive.

i.e., $A \cap B = \phi$



From Axiom of additivity,

$P(A \cup B) = P(A) + P(B)$

$= P(A) + P(B) - 0$

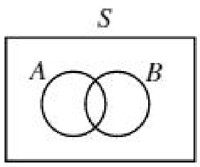
$= P(A) + P(B) - P(\phi)$

$= P(A) + P(B) - P(A \cap B)$

Case - 2 :

Suppose A and B are not mutually exclusive.

i.e., $A \cap B \neq \phi$



Here $A \cup B = A \cup (B - A)$

Therefore,

$P(A \cup B) = P[A \cup (B - A)]$

$= P(A) + P(B - A)$ (By axiom of additivity and A, $B - A$ are disjoint)

$= P(A) + P[B - (A \cap B)]$

$= P(A) + P(B) - P(A \cap B)$ $[\because A \cap B \subseteq B]$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A, B are exclusive, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$

Corollary :

If A, B, C are three events then $P(A \cup B \cup C)$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof : Let $B \cup C = D$

$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$ .... (1)

$P(A \cap D) = P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$ ...(2)

from (1) & (2)

$P(A \cup B \cup C) = P(A) + P(D) - P(A \cap D)$

$= P(A) + P(B \cup C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

3.17 GENERAL FORM OF ADDITION THEOREM ON PROBABILITY

If $A_1, A_2, ..., A_n$ are n events then,

$$P(A_1 \cup A_2 \cup .... \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- ...... + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \cap .... \cap A_n)$$

SOLVED EXAMPLES

1. If A and B are two events then show that

i) $P(A \cap B^C) = P(A) - P(A \cap B)$

ii) The probability that exactly one of them occurs is given by $P(A) + P(B) - 2P(A \cap B)$

Sol. i) $P(A \cap B^C) = P(A - B) = P(A - (A \cap B)) = P(A) - P(A \cap B)$ [$\because (A \cap B \subseteq A)$]

ii) P(exactly one of A, B occurs)

$= P[(A \cap B^C) \cup (A^C \cap B)] = P(A \cap B^C) + P(A^C \cap B)$

(by using axiom of additivity and $A \cap B^C, A^C \cap B$ are disjoint)

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$ (using (1)) $= P(A) + P(B) - 2P(A \cap B)$

Note :

$A \cap \bar{B} = A - (A \cap B)$

$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$

2. $P(A) = 0.5$; $P(B) = 0.3$ and A, B are exclusive. Then what is the probability that neither A nor B occurs.

Sol. Given $P(A) = 0.5$; $P(B) = 0.3$ and $A \cap B = \phi$

P(neither A nor B occurs) $= P(A^C \cap B^C)$

$= 1 - P(A \cup B)$ (by Demorgan law)

$= 1 - [P(A) + P(B)]$

$[\because A \cap B = \phi] = 1 - [0.5 + 0.3] = 0.2$

3. A contractor submitted tenders for 2 works. If 0.4, 0.6, 0.1 are the respective probabilities that his first tender, atleast one tender, both the tenders are accepted, what is the probability that his second tender is accepted.

Sol. Let A_k be the event of acceptance of k^{th} tender

where $k = 1, 2$

given, $P(A_1) = 0.4$; $P(A_1 \cup A_2) = 0.6$; $P(A_1 \cap A_2) = 0.1$

Now, by addition theorem, $P(A_2) = P(A_1 \cup A_2) - P(A_1) + P(A_1 \cap A_2)$

$= 0.6 - 0.4 + 0.1 = 0.3$

***4.** **If $P(A)=0.3, P(B)=0.4, P(C)=0.8, P(A\cap B)=0.08, P(A\cap C)=0.28, P(A\cap B\cap C)=0.09, P(A\cup B\cup C)\geq 0.75$ then show $P(B\cap C)$ lies in [0.23, 0.48].**

Sol. Given $P(A\cup B\cup C)\geq 0.75$

So, $0.75\leq P(A\cup B\cup C)\leq 1$

$\Rightarrow 0.75\leq P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)\leq 1$

$\Rightarrow 0.75\leq 0.3+0.4+0.8-0.08-0.28-P(B\cap C)+0.09\leq 1$

$\Rightarrow 0.75\leq 1.23-P(B\cap C)\leq 1$

$\Rightarrow -1\leq P(B\cap C)-1.23\leq -0.75$

$\Rightarrow 0.23\leq P(B\cap C)\leq 0.48$

5. **If three dice are rolled. Find the probability of getting sum 16 or getting 6 on first die.**

Sol. Since three dice are rolled,

$S=\{(x_1,x_2,x_3)/1\leq x_1,x_2,x_3\leq 6\}$

Here $n(S)=6^3=216$

Let A be the event of getting sum 16;

$A=\{(4,6,6),(5,6,5),(5,5,6),(6,5,5),(6,6,4),(6,4,6)\}$

Here $n(A)=6$; Let B be the event of getting 6 on first die.

$B=\{(6,x_2,x_3)/1\leq x_2\leq 6; 1\leq x_3\leq 6\}$ $n(B)=36$;

A, B are not exclusive $A\cap B=\{(6,5,5),(6,6,4),(6,4,6)\}$

Therefore $P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{6}{216}+\frac{36}{216}-\frac{3}{216}=\frac{39}{216}=\frac{13}{72}$

6. **If two cards are drawn from a pack of 52 cards then find the probability of getting both red or both kings.**

Sol. Since 2 cards are drawn from pack of cards, $n(S)={}^{52}C_2$

Let A be the event of getting both red

Here $n(A)={}^{26}C_2$

Let B be the event of getting both kings $n(B)={}^{4}C_2$

Here A, B are not exclusive

The simple event of getting both red kings is common to both A and B.

$n(A\cap B)=1$;

By addition theorem, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=\frac{{}^{26}C_2}{{}^{52}C_2}+\frac{{}^{4}C_2}{{}^{52}C_2}-\frac{1}{{}^{52}C_2}=\frac{325+6-1}{{}^{52}C_2}=\frac{330}{26\times 51}=\frac{55}{221}$

7. **A number is chosen from the first 100 natural numbers. Find the probability that it is a multiple of 4 or 6.**

Sol. Since one number is selected from first 100 natural numbers, $n(S)=100$,

Let A be the event of selecting a number which is multiple of 4

Here $A=\{4,8,12,......,100\}\Rightarrow n(A)=25$;

Let B be the event of selecting a number which is multiple of 6 ;

$B=\{6,12,18,...,96\}\Rightarrow n(B)=16$;

Note :

$\bar{A}\cap\bar{B}=\overline{A\cup B}$

$\Rightarrow P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})$

$=1-P(A\cup B)$

Here A, B are not exclusive.

$(A \cap B)$ is the event of selecting a number which is a multiple of 4 and 6 i.e., it is a multiple of 12.

$A \cap B = \{12, 24, ..., 96\} \Rightarrow n(A \cap B) = 8;$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{25}{100} + \frac{16}{100} - \frac{8}{100} = \frac{33}{100}$

8. A box contains 2 red, 3 blue and 4 black balls. Three balls are drawn at random. What is the probability that two balls are of the same colour and the third of a different colour.

Sol. Number of balls = 2 + 3 + 4 = 9

Since we are selecting 3 balls at random, $n(S) = {}^9C_3 = 84$;

Let A be the event of getting 2 red balls and one non red.

Let B be the event of getting 2 blue balls and one non blue.

Let C be the event of getting 2 black balls and one non black.

Here A, B, C are mutually exclusive.

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$= \frac{{}^2C_2 \times {}^7C_1}{{}^9C_3} + \frac{{}^3C_2 \times {}^6C_1}{{}^9C_3} + \frac{{}^4C_2 \times {}^5C_1}{{}^9C_3} = \frac{7+18+30}{84} = \frac{55}{84}$

9. Three electric bulb holders are fixed in a room. 3 bulbs are chosen at random from a set of 20 bulbs of which 16 are good and fitted to the holders. What is the probability that the room is lighted.

Sol. **Method - 1**

Since we are selecting 3 bulbs from 20 bulbs at random, $n(S) = {}^{20}C_3$

Let A_k be the event of selecting k good bulbs and $(3-k)$ defective bulbs where $k = 1, 2, 3$.

$n(A_1) = {}^{16}C_1 \times {}^4C_2$

$n(A_2) = {}^{16}C_2 \times {}^4C_1$ and $n(A_3) = {}^{16}C_3$

Here A_1, A_2, A_3 are mutually exclusive.

P(the room gets light) = P(choosing atleast one good bulb)

$= P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

$= \frac{{}^{16}C_1 \times {}^4C_2}{{}^{20}C_3} + \frac{{}^{16}C_2 \times {}^4C_1}{{}^{20}C_3} + \frac{{}^{16}C_3}{{}^{20}C_3} = \frac{(96+480+560)\times 6}{20 \times 19 \times 18} = \frac{1136 \times 6}{20 \times 19 \times 18} = \frac{284}{285}$

Note :

$P(x \geq 1) = 1 - P(x = 0)$

Probability of atleast one event occuring = 1- probability that none of the events occur.

Method - 2

P(room gets light) = P(choosing atleast one good bulb)

$= 1 - P$(choosing no good bulb) $= 1 - \frac{{}^4C_3}{{}^{20}C_3} = 1 - \frac{1}{285} = \frac{284}{285}$

***10. A and B are seeking admission into IIT. If the probability for A to be selected is 0.5 and that of both to be selected is 0.3. Is it possible that, the probability of B to be selected is 0.9 ?**

Sol. Given $P(A) = 0.5; P(A \cap B) = 0.3$

We know for any event A, $0 \leq P(A) \leq 1$

So, $0 \leq P(A \cup B) \leq 1$

Now by addition theorem, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(B) = P(A \cup B) - P(A) + P(A \cap B) = P(A \cup B) - 0.5 + 0.3 \leq 1 - 0.2 = 0.8$

$P(B) \leq 0.8$ So, $P(B)$ can not be equal to 0.9

***11. A, B, C are 3 horses in a race. The probability of A to win the race is twice to that of B and the probability of B is twice to that of C. What are the probabilities of A, B and C to win the race.**

Sol. Let A, B, C be the events that the horses named by A, B, C win the race respectively.

Given, $P(A) = 2P(B)$ and $P(B) = 2P(C)$

So, $P(A) = 2P(B) = 4P(C)$

But A, B, C are exclusive and exhaustive.

So, $P(A) + P(B) + P(C) = 1$ .... (1)

Here $P(A) = 2P(B) = 4P(C)$

$$\Rightarrow \frac{P(A)}{4} = \frac{P(B)}{2} = \frac{P(C)}{1} = \frac{P(A)+P(B)+P(C)}{4+2+1} = \frac{1}{7} \text{ (from (1))}$$

So, $P(A) = 4/7$; $P(B) = 2/7$; $P(C) = 1/7$

12. In a swimming competition, only 3 students A, B, C are taking part. The probability of A's winning or the probability of B's winning is three times the probability of C's winning. Find the probability of the event either B or C to win.

> **Note :**
> $A \cup B \supseteq A; A \cup B \supseteq B$
> $\Rightarrow P(A \cup B) \geq \max\{P(A), P(B)\}$

Sol. Suppose A, B, C denote the events that the 3 students A, B, C win the completition respectively.

By data, $P(A) = P(B) = 3P(C)$

But A, B, C are mutually exclusive and exhaustive because one and only one of A, B, C win the competition.

$\therefore P(A) + P(B) + P(C) = 1$

$3P(C) + 3P(C) + P(C) = 1 \therefore P(C) = 1/7$

$P(A) = P(B) = 3/7$

$P(B \cup C)) = P(B) + P(C)$ $[\because B \cap C = \phi] = \frac{3}{7} + \frac{1}{7} = \frac{4}{7}$

13. If A, B, C are mutually exclusive and exhaustive events such that $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{3}P(B)$. Find odds in favour of $(A \cup B)$.

Sol. Since A, B, C are mutually exclusive and exhaustive,

$P(A) + P(B) + P(C) = P(A \cup B \cup C) = P(S) = 1$

But given, $2P(B) = 3P(A)$ and $P(B) = 3P(C)$

So, $3P(A) = 2P(B) = 6P(C) \Rightarrow \frac{P(A)}{2} = \frac{P(B)}{3} = \frac{P(C)}{1} = \frac{P(A)+P(B)+P(C)}{6} = \frac{1}{6}$

$P(A) = \frac{1}{3}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{1}{6}$

Since A, B are exclusive, $P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

Odds in favour of $(A \cup B)$ are 5 : 1

14. If $\frac{1+3p}{3}, \frac{1-p}{4}, \frac{1-2p}{2}$ are the probabilities of 3 mutually exclusive events then find the set of all values of p.

Sol. Suppose A, B, C are the three events

such that $P(A) = \frac{1+3p}{3}$; $P(B) = \frac{1-p}{4}$; $P(C) = \frac{1-2p}{2}$

We know that $0 \le P(A), P(B), P(C) \le 1$

Therefore, $0 \le \frac{1+3p}{3} \le 1 \Rightarrow 0 \le \frac{1}{3} + p \le 1 \Rightarrow -\frac{1}{3} \le p \le \frac{2}{3}$ .... (1)

$0 \le \frac{1-p}{4} \le 1 \Rightarrow 0 \le 1 - p \le 4 \Rightarrow -1 \le -p \le 3 \Rightarrow -3 \le p \le 1$ .... (2)

$0 \le \frac{1-2p}{2} \le 1 \Rightarrow 0 \le 1 - 2p \le 2 \Rightarrow -1 \le -2p \le 1 \Rightarrow -\frac{1}{2} \le p \le \frac{1}{2}$ .... (3)

Since A, B, C are exclusive,

$0 \le P(A \cup B \cup C) = P(A) + P(B) + P(C) \le 1$

$\Rightarrow 0 \le \frac{1+3p}{3} + \frac{1-p}{4} + \frac{1-2p}{2} \le 1$

$\Rightarrow 0 \le \frac{13-3p}{12} \le 1 \Rightarrow 0 \le 13 - 3p \le 12$

$\Rightarrow -13 \le -3p \le -1 \Rightarrow \frac{1}{3} \le p \le \frac{13}{3}$

Now Max. of $\left\{-\frac{1}{3}, -3, -\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{3}$ and Minimum of $\left\{\frac{2}{3}, 1, \frac{1}{2}, \frac{13}{3}\right\} = \frac{1}{2}$

$\therefore \frac{1}{3} \le p \le \frac{1}{2}$

EXERCISE - 3.1

1. In the experiment of throwing a die, consider the following events: A= {1,3,5}, B= {2,4,6}, C = {1,2,3} are these events equally likely ? [Ans : Yes]
2. In the experiment of throwing a die, consider the following events: A= {1,3,5}, B= {2,4}, c={6} are these events mutually exclusive ? [Ans : Yes]
3. In the experiment of throwing a die, consider the events A={2,4,6}, B= {3,6}, C={1,5,6} are these events exhaustive ? [Ans : Yes]
4. Give two examples of mutually exclusive and exhaustive events.

   In the experiment of throwing a die, the event

   E_1: Occurence of an even number (on the face of the die) and

   E_2: Occurence of an odd number are mutually exclusive events. They are also exhaustive.

5. Give examples of two events that are neither mutually exclusive nor exhaustive ?
In the experiment of throwing a pair of dice, let us consider the following events
E_1: A sum 7(of the numbers that appear on the uppermost faces of the dice)
E_2: A sum 6(of the number that appear on the uppermost face of the dice)
E_3: A sum 5(of the number that appear on the uppermost face of the dice)
E_1, E_2, E_3 are mutually exclusive

6. Give two examples of events that are neither equally likely nor exhaustive.
If a coin is tossed, Occurence of a Head (H) or occurence of a Tail (T) are equally likely

7. If 4 fair coins are tossed simultaneously, then find the probability that 2 heads and 2 tails appear
[Ans : 3/8]

8. Find the probability that a non leap year contains (i) 53 sundays (ii) 52 sundays only
[Ans : (i) 1/7 (ii) 6/7]

9. Two dice are rolled. What is the probability that none of the dice shows the number 2?
[Ans : $(5/6)^2$]

10. In an experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probabilities of A, B, $A \cap B$ and $A \cup B$ [Ans : 1/4, 3/13, 3/52, 11/26]

11. In a class of 60 boys, and 20 girls, half of the boys and half of the girls know cricket. Find the probability of the event that a person selected from the class is either a boy, or a girl knowing cricket [Ans : 7/8]

12. For any two events A and B show that $p(A^c \cap B^c) = 1 + P(A \cap B) - P(A) - P(B)$

13. Two persons A and B are rolling a die on the condition that the person who gets 3 will win the game. if A starts the game, then find the probabilities of A and B respectively to win the game.
[Ans : 6/11, 5/11]

14. A, B, C are 3 news papers from a city. 20% of the population read A, 16% read B, 14%read C, 8% both A & B, 5% both A and C, 4% both B and C and 2% all the three. Find the percentage of the population who read atleast one news paper [Ans : 35%]

15. If one ticket is randomly selected from tickets numbered 1 to 30, then find the probability that the number on the ticket is (i) a multiple of 5 or 7 (ii) a multiple of 3 or 5 [Ans : (i) 1/3, (ii) 7/15]

16. If two numbers are selected randomly from 20 consecutive natural numbers find the probability that the sum of the two numbers is (i) an even number (ii) an odd number
[Ans : (i) 9/19 (ii) 10/19]

17. A game consist of tossing a coin 3times and noting its outcome. A boy wins if all tosses give the same outcome and loses otherwise. Find the probability that the boy loses the game.[Ans : 3/4]

18. If E_1, E_2 are two events with $E_1 \cap E_2 = \phi$, then show that $P(E_1^c \cap E_2^c) = P(E_1^c) - P(E_2^C)$

19. A pair of dice is rolled 24 times. A person wins by not getting a pair of 6s on any of the 24 rolls. What is the probability of his winning ? **[Ans : $(35/36)^{24}$]**

20. If P is probability function, then show that for any two events A and B
$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

21. In a box containing 15 bulbs, 5 are defective. If 5 bulbs are selected at random from the box, find the probability of the event, that
   i) none of them is defective **[Ans : 12/143]**
   ii) Only one of them is defective **[Ans : 50/143]**
   iii) atleast one of them is defective **[Ans : 131/143]**

22. A and B are seeking admission into IIT. If the probability for A to be selected is 0.5 and that both to be selected is 0.3, then is it possible that, the probability of B to be selected is 0.9 ? **[Ans : No]**

*23. The probability for a contractor to get a road contract is 2/3 and to get a building contract is 5/9 the probability to get atleast one contract is 4/5. Find the probability that he gets both the contracts **(May-19)** **[Ans : 19/45]**

24. In a committee of 25 members, each member is proficient either in mathematics or in statistics or in both. If 19 of these are proficient in mathematics, 16 in statistics, find the probability that a person selected from the committee is proficient in both. **[Ans : 2/5]**

*25. A,B,C are three horses in a race. The probability of A to win the race is twice that of B, and probability of B is twice that of C. What are the probabilities of A,B and C to win the race. **(May-18, 19)** **[Ans : 4/7, 2/7, 1/7]**

26. A bag contains 12 two rupee coins, 7 one rupee coins and 4 half a rupee coins. If three coins are selected at random, then find the probability that
   I) the sum of three coins is maximum **[Ans : $12_{C_3}/23_{C_3}$]**
   II) The sum of three coins is minimum **[Ans : $4_{C_3}/23_{C_3}$]**
   III) Each coin is of different value **[Ans : $\dfrac{12 \times 7 \times 4}{23_{C_3}}$]**

27. The probabilities of three events A, B, C are such that $P(A)=0.3$, $P(B)=0.4$, $P(C)=0.8$, $P(A \cap B) = 0.08, P(A \cap C) = 0.28, P(A \cap B \cap C) = 0.09)$ and
$P(A \cup B \cup C) \ge 0.75$ show that $P(B \cap C)$ lie in the interval [0.23, 0.48]

28. The probability of three mutually exclusive events are respectively given as $\dfrac{1+3p}{3}, \dfrac{1-p}{4}, \dfrac{1-2p}{2}$, prove that $\dfrac{1}{3} \le p \le \dfrac{1}{2}$

29. On a festival day, a man plans to visit 4 holy temples, A, B, C, D in a random order. Find the probability that he visits

i) A before B [Ans :1/2]

ii) A before B and B before C. [Ans : 1/6]

30. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. The particulars of 5 persons are follows

S.No.	Name	Sex	Age in years
1	Harish	M	30
2	Rohan	M	33
3	Sheetala	F	46
4	Alis	F	28
5	Salim	M	41

A person is selected at random from this group to act as a spokesperson. Find the probability that the spokesperson will be either male or above 35 years. [Ans : 4/5]

31. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100students, find the probability that

i) you both enter the same section [Ans : 17/33]

ii) you both enter the different sections [Ans : 16/33]

32. Two dice are thrown. We now find the probability of getting the same number on both the faces. [Ans : 1/6]

33. An integer is picked from 1 to 20, both inclusive. Let us find the probability that it is a prime. [Ans : 2/5]

34. A bag contains 4 red, 5 black and 6 blue balls. Let us find the probability that two balls drawn at random simultaneously from the bag are a red and a black ball. [Ans : 4/21]

35. Ten dice are thrown. Find the probability that none of the dice shows the number 1.

[Ans : $P(A)=\left(\frac{5}{6}\right)^{10}$]

36. A number x is drawn arbitrarily from the set {1,2,3,......,100}. Find the probability that $\left(x+\frac{100}{x}\right)>29$ [Ans : 0.78]

37. Two squares are chosen at random on a chess board, show that the probability that they have a side in common is 1/18

38. A fair coin is tossed 200 times. Find the probability of getting a head an odd number of times. [Ans : 1/2]

39. A and B are among 20 persons who sit at random along a round table. Find the probability that there are any six persons between A and B [Ans : 2/19]

40. Out of 30 consecutive integers, two integers are drawn at randow. Find the probability that their sum is odd. [Ans : 15/29]

41. Out of 1,00,000 new born babies, 77,181 survived till the age of 20. Find the probability that a new born baby survives till 20 years of age. [Ans : 0.77181]

42. Suppose S={0,1,2,3,4} is the sample space of a random experiment P. Define $P(0)=0$, $P(1)=0$, $P(2)=1/2$, $P(3)=1/4$, and $P(4)=1/4$. Define $P(A)=\sum_{a\in A} P(a)$ for $A\subseteq S$

Then P defines a probability function. Here

i) $P(A)\geq 0\ \forall a\in S$

ii) $\sum_{a\in S} P(a)=P(0)+P(1)+P(2)+P(3)+P(4)=1$ if $A\subseteq S$ and $B\subseteq S, A\cap B=\phi$

then $P(A\cup B)=\sum_{a\in A\cup B} P(a)=\sum_{a\in A} P(a)+\sum_{a\in B} P(a)=P(A)+P(B)$

43. Find the probability of throwing a total square of 7 with 2 dice [Ans : 1/6]

44. Find the probability of obtaining two tails and one head when 3 coins are tossed [Ans : 3/8]

45. A page is opened at random from a book containing 200 pages. What is the probability that the number on the page is a perfect square. [Ans : 7/100]

46. Find the probability of drawing an Ace or a spade from a well shuffled pack of 52 playing cards A pack of cards contains a total of 52 cards. **(March-17, 18 & May-19)

(**Hint :** These 52 cards are divided into 4 sets namely Hearts, Diamonds, Spades and Clubs. Each set consists of 13 cards namely : A, 2,3,4,5,6,7,8,9,10,J,Q, Here A : Ace, K: King, Q : Queen, J: Jack) [Ans : 4/13]

47. If A and B are two events then show that

i) $P(A\cap B^C)=P(A)-P(A\cap B)$ and

ii) The probability that one of them occurs is given by $P(A)+P(B)-2P(A\cap B)$

**48. A and B are events with $P(A)=0.5, P(B)=0.4$ and $P(A\cap B)=0.3$. Find the probability that

i) A doesnot occur ii) neither A nor B occurs **(March-17, 18)** [Ans: (i) 0.5 (ii) 0.4]

49. If A,B,C are three events shows that

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

3.18 CONDITIONAL PROBABILITY

Suppose two dice are rolled.

We know that $S = \{(1, 1), (1, 2), \ldots\ldots, (6, 6)\}$

Here $n(S) = 36$; where S is the sample space

Let A, B be the two events of getting sum 10 and getting a doublet respectively.

$A = \{(4, 6)\ (6, 4)\ (5, 5)\}$

$B = \{(1, 1)\ (2, 2)\ (3, 3)\ (4, 4)\ (5, 5)\ (6, 6)\}$

Here $P(\text{A}) = 3/36 = 1/12$

Here the probability of A is measured with respect to sample space S.

So, the chance of happening the event A without any condition (restriction) is $\frac{1}{12}$.

Suppose we are interested to find the probability of A with a condition that the event B has already occurred.

The event "happening of A after happening of B" is called conditional event and is denoted by A/B (we read it as happening of A given that B has occurred).

Since the event B has already occurred, the sample space S reduces to B. This is because the outcome of the experiment is one of the favourables to B.

The set of favourables to A need not be the set of favourables to A/B.

For example (6, 4) and (4, 6) are favourables to A but they are not favourables to A/B;

This is because they do not belong to the reduced sample space B.

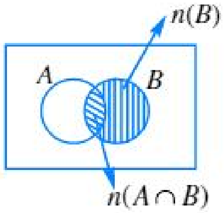
So, we conclude that, a favourable to A/B must be a favourable not only to A but also to B;

So, $(A \cap B)$ is the set of favourables to A/B; So, in the above example,

$$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{6}$$

Definition

If A and B are two events in a sample space S and P(B) $\neq$ 0;
The probability of A given the event B has occurred is called the conditional probability of
A given B and is denoted by P(A/B) we define,



$$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)}$$

3.19 MULTIPLICATION THEOREM ON PROBABILITY

If A and B are any two events of a random experiment where $P(A) \neq 0$ and $P(B) \neq 0$ then $P(A \cap B) = P(A)\, P(B/A) = P(B)\, P(A/B)$.

Proof : Let S be the sample space of the random experiment.

Here A, B are events of the experiment such that $P(A) \neq 0$; $P(B) \neq 0$.

According to the definition of conditional probability,

$$P(B/A) = \frac{n(B \cap A)}{n(A)} = \frac{\frac{n(B \cap A)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{P(B \cap A)}{P(A)}$$

Therfore, $P(B \cap A) = P(A \cap B) = P(A)\, P(B/A)$

similary we can prove, $P(A \cap B) = P(B) P(A/B)$

THEOREM-3.2 If $A_1, A_2, A_3, \ldots, A_n$ are n events of a random experiment with $P\left(\bigcap_{i=1}^{n-1} A_i\right) \neq 0$ then,

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_n / A_1 \cap A_2 \ldots \cap A_{n-1})$

This can be proved by applying mathematical induction.

3.20 INDEPENDENT EVENTS

Definition :

Two events A and B of a random experiment are called independent events if the occurrence or non occurrence of one of them does not influence the occurrence or non occurrence of the other.

A, B are independent if $P(B/A) = P(B/\overline{A}) = P(B)$

If $P(B/A) \neq P(B)$ then we say A, B are dependent events.

Note

i) *If A, B are independent events then* $P(A \cap B) = P(A)\, P(B/A) = P(A)\, P(B)$

ii) *So, A, B are independent events* $\Leftrightarrow$

a) $P(A/B) = P(A)$ *(or)*

b) $P(B/A) = P(B)$ *(or)*

c) $P(A \cap B) = P(A)\, P(B)$

3.21 THREE PAIRWISE INDEPENDENT EVENTS

Three events A, B, C are said to be pairwise independent if $P(A \cap B) = P(A)P(B)$

$P(B \cap C) = P(B)P(C)$

$P(C \cap A) = P(C)P(A)$

hold good simultaneously

3.22 THREE INDEPENDENT EVENTS

Three events A, B, C are said to be mutually independent (or) independent if

$P(A \cap B) = P(A)P(B)$

$P(B \cap C) = P(B)P(C)$

$P(C \cap A) = P(C)P(A)$

$P(A \cap B \cap C) = P(A)P(B)P(C)$

hold good simultaneously

THEOREM-3.3 **If A and B are independent, then**

a) A, B^C are independent

b) A^C, B^C are independent

Proof : Since A, B are independent $P(A \cap B) = P(A)\ P(B)$ ..... (1)

a) Consider $P(A \cap B^C) = P(A - B) = P(A - (A \cap B))$

$= P(A) - P(A \cap B)$ $[\because A \cap B \subseteq A]$

$= P(A) - P(A)P(B)$ [by (i)]

$= P(A)\ [1 - P(B)] = P(A)\ P(B^C)$

Therefore we proved, $P(A \cap B^C) = P(A)\ P(B^C)$

$\therefore$ A, B^C are independent.

Note : ***Mutually exclusive events are not independent events***

b) To show A^C, B^C are independent if A, B are independent.

Since A, B are independent, we can use the fact, $P(A \cap B) = P(A)\ P(B)$ ..... (1)

Consider, $P(A^C \cap B^C) = P[(S - A) \cap (S - B)]$

$= P[S - (A \cup B)]$ (Demorgan law)

$= P(S) - P(A \cup B)$ $[\because A \cup B \subseteq S]$

$= 1 - P(A \cup B)$ [by axiom of certainity]

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)\ P(B)$ [by using 1]

$= [1 - P(B)]\ [1 - P(A)]$

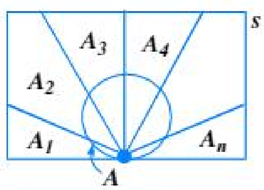
$= P(B^C)\ P(A^C) = P(A^C)\ P(B^C)$

Therefore A^C, B^C are also independent.

*Note : **If events A, B are mutually exclusive events of a random expermient then A, B are not independent events.***

3.23 THEOREM ON TOTAL PROBABILITY

THEOREM-3.4 **If $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive events of a random experiment associated with sample space S such that $P(A_i)>0$ for $i = 1,2,...., n$ and A is any event which takes place in conjuction with any one of A_i, then $P(A) = \sum_{i=1}^{n} P(A_i)P(A/A_i)$.**



Proof : Since $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive in sample space S, we have $A_i \cap A_j = \phi$ if $i \neq j$ for $1 \leq i, j \leq n$ and $A_1 \cup A_2 \cup .... \cup A_n = S$.

Since $A_i \cap A_j = \phi$ we have $(A \cap A_i) \cap (A \cap A_j) = \phi$

i.e., $A \cap A_i$, $A \cap A_j$ are mutually disjoint for $i \neq j$.

Now $P(A) = P(A \cap S) = P\left(A \cap \bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n}(A \cap A_i)\right)$

$= P(A \cap A_1) + P(A \cap A_2) + ... + P(A \cap A_n)$ (by axiom of additivity)

$= P(A_1)\ P(A/A_1) + P(A_2)\ P(A/A_2) + ... + P(A_n)\ P(A/A_n) = \sum_{i=1}^{n} P(A_i)\ P(A/A_i)$

3.24 *BAYE'S THEOREM

Suppose $A_1, A_2, ..., A_n$ are n mutually exclusive and exhaustive events of a random experiment associated with sample space S such that $P(A_i) \neq 0$, and A is any event which takes place in conjuction with any one of A_i. **(March-18, 19)**

$$P(A_k/A) = \frac{P(A_k)P(A/A_k)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)} \text{ for any } k = 1, 2, ..., n;$$

(or) State & Prove Baye's theorem (**March-17**)

Proof : Given that $P(A_i) \neq 0$ for $i = 1, 2, ..., n$;

Since, $A_1, A_2, ..., A_n$ are mutually exclusive.

$A_i \cap A_j = \phi$ for $i \neq j$ and $1 \leq i, j \leq n$

Since, $A_1, A_2, ..., A_n$ are exhaustive

$A_1 \cup A_2 \cup .... \cup A_n = S$

Here for any event A of S we have $(A \cap A_i)$, $(A \cap A_j)$ are also exclusive for $i \neq j$.

$P(A) = P(A \cap S) = P[A \cap (A_1 \cup A_2 \cup ..... \cup A_n)]$

$= P[(A \cap A_1) \cup (A \cap A_2) ...... \cup (A \cap A_n)]$

$= P(A \cap A_1) + P(A \cap A_2) + ...+ P(A \cap A_n)]$

(by axiom of additivity)

$= P(A_1)\ P(A/A_1) + P(A_2)\ P(A/A_2) + ... + P(A_n)\ P(A/A_n)$

$= \sum_{i=1}^{n} P(A_i)P(A/A_i)$ .... (1)

Consider for $1 \leq k \leq n$, $P(A_k/A) = \dfrac{P(A_k \cap A)}{P(A)}$

(by definition of conditional probability)

$= \dfrac{P(A_k)P(A/A_k)}{P(A)} = \dfrac{P(A_k)P(A/A_k)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$ (from (1))

SOLVED EXAMPLES

1. If $P(\overline{A}) = 0.7; P(B) = 0.7 ; P(B/A) = 0.5$ then find $P(A \cup B)$.

Sol. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.7 - P(A)\, P(B/A)$

$= 1 - 0.3 \times 0.5 = 1 - 0.15 = 0.85$

2. If A, B, C are any three events in an experiment then show that

i) $P(A/B^C) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$ if $P(B^C) > 0$

ii) $A \subseteq B \Rightarrow P(A/C) \le P(B/C)$ if $P(C) > 0$

iii) If A, B are mutually exclusive, then $P(A/B^C) = \dfrac{P(A)}{1 - P(B)}$ if $P(B) \neq 1$

iv) If A, B are mutually exclusive and

$P(A \cup B) \neq 0$ then $P(A/A \cup B) = \dfrac{P(A)}{P(A) + P(B)}$

Note : $A = (A \cap \overline{B}) \cup (A \cap C)$

Sol. i) Since $P(B^C) \neq 0$, we have

$P(A/B^C) = \dfrac{P(A \cap B^C)}{P(B^C)}$ (by def. of conditional probability)

$= \dfrac{P(A \cap B^C)}{1 - P(B)}$ .... (1)

$P(A \cap B^C) = P(A - B) = P(A - P(A \cap B)) = P(A) - P(A \cap B)$ $[\because A \cap B \subseteq A]$

from (1) $P(A/B^C) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

ii) Since $A \subseteq B$

We have $A \cap C \subseteq B \cap C$ for any event C

But we know, if A, B are any two events of a random experiment such that $A \subseteq B$ then $P(A) \le P(B)$

Therefore $P(A \cap C) \le P(B \cap C)$

$\Rightarrow \dfrac{P(A \cap C)}{P(C)} \le \dfrac{P(B \cap C)}{P(C)}$ $[\because P(C) > 0]$ $\Rightarrow P(A/C) \le P(B/C)$

iii) Since A and B are exclusive

$A \cap B = \phi$

Therefore $A \subseteq B^C \Rightarrow A \cap B^C = A;$

Now $P(A/B^C) = \dfrac{P(A \cap B^C)}{P(B^C)} = \dfrac{P(A)}{1 - P(B)}$

iv) Since A and B are exclusive,

$A \cap B = \phi$ $P(A \cap B) = 0$

$P(A/A \cup B) = \dfrac{P[A \cap (A \cup B)]}{P(A \cup B)} = \dfrac{P(A)}{P(A \cup B)}$ $[\because A \subseteq A \cup B]$

$= \dfrac{P(A)}{P(A) + P(B) - P(A \cap B)}$

$= \dfrac{P(A)}{P(A) + P(B)}$ $(\because P(A \cap B) = 0)$

3. If A and B are any two events of a random experiment then show that

i) $P(A^C \cap B^C) = P(A^C) - P(B)$ if $A \cap B = \phi$

ii) $P(A^C/B^C) = \dfrac{1 - P(A \cup B)}{1 - P(B)}$ with $P(A) \neq 0$ and $P(B) \neq 1$

Sol. Given A, B are any two events of a random experiment.

i) Given $A \cap B = \phi$

$P(A^C \cap B^C) = 1 - P(A \cup B)$ [By demorgan law]

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= 1 - P(A) - P(B) + 0$ $(\because A \cap B = \phi)$ $= P(A^C) - P(B)$

ii) Given $P(A) > 0$; $P(B) \neq 1$

By def. of conditional probability,

$$P(A^C/B^C) = \frac{P(A^C \cap B^C)}{P(B^C)} = \frac{1 - P(A \cup B)}{1 - P(B)} \text{ (by demorgan law)}$$

Note :

$A \subseteq B \Rightarrow P\left(\frac{A}{C}\right) \leq P\left(\frac{B}{C}\right)$

***4. For any two events A, B show that**

$P(A \cap B) - P(A)P(B) = P(A^C)P(B) - P(A^C \cap B)$

$= P(A)P(B^C) - P(A \cap B^C)$

Sol. $P(A^C)P(B) - P(A^C \cap B)$

$= (1 - P(A))P(B) - (P(B) - P(A \cap B))$

$= P(B) - P((A).P(B) - P(B) + P(A \cap B)$

$= P(A \cap B) - P(A).P(B)$

$P(A).P(B^C) - P(A \cap B^C) = P(A)(1 - P(B)) - [P(A) - P(A \cap B)]$

$= P(A) - P(A)P(B) - P(A) + P(A \cap B)$

$= P(A \cap B) - P(A).P(B)$

5. If A and B are two independent events, and $P(A) = 1/4$; $P(B) = 1/3$ then find $P((A - B) \cup (B - A))$

Sol. Given A and B are independent

$P(A) = 1/4$; $P(B) = 1/3$

$P[(A - B) \cup (B - A)]$

$= P(A - B) + (B - A)$ $(\because A - B$ and $B - A$ are disjoint)

$= P(A - (A \cap B)) + P(B - (A \cap B))$

$= \{P(A) - P(A \cap B)\} + \{P(B) - P(A \cap B)\}$ $(\because A \cap B \subseteq A$ and $A \cap B \subseteq B)$

$= P(A) + P(B) - 2\,P(A \cap B)$

$= P(A) + P(B) - 2\,P(A)\,P(B)$ $(\because A$ and B are independent)

$$= \frac{1}{4} + \frac{1}{3} - 2\frac{1}{4}\,\frac{1}{3} = \frac{3 + 4 - 2}{12} = \frac{5}{12}$$

Note :

Only one of the events A or B occurs

$= (A - B) \cup (B - A)$

$= (A \cap \bar{B}) \cup (\bar{A} \cap B)$

6. If A, B, C are three independent events of an experiment such that $P(A \cap B^C \cap C^C) = \frac{1}{4}$; $P(A^C \cap B \cap C^C) = \frac{1}{8}$; $P(A^C \cap B^C \cap C^C) = \frac{1}{4}$. Find $P(A)$, $P(B)$, $P(C)$

Sol. Since A, B, C are independent, A^C, B^C, C^C are also independent.

$$P(A \cap B^C \cap C^C) = \frac{1}{4} \Rightarrow P(A)\,P(B^C)\,P(C^C) = \frac{1}{4} \quad \ldots (1)$$

$$P(A^C \cap B \cap C^C) = \frac{1}{8} \Rightarrow P(A^C)\,P(B)\,P(C^C) = \frac{1}{8} \quad \ldots (2)$$

$$P(A^C \cap B^C \cap C^C) = \frac{1}{4} \Rightarrow P(A^C)\,P(B^C)\,P(C^C) = \frac{1}{4} \ldots (3)$$

(3)/(1) gives, $\frac{P(A^C)}{P(A)} = 1 \Rightarrow P(A^C) = P(A)$

$$\Rightarrow 1 - P(A) = P(A) \Rightarrow P(A) = \frac{1}{2}$$

(3)/(2) $\Rightarrow \frac{P(B^C)}{P(B)} = 2 \Rightarrow 1 - P(B) = 2P(B) \Rightarrow P(B) = \frac{1}{3}$

from (1) $\frac{1}{2} \times \frac{2}{3} P(C^C) = \frac{1}{4} \Rightarrow P(C^C) = \frac{3}{4} \Rightarrow P(C) = \frac{1}{4}$

***7. Two dice are rolled. Let A be the event of getting sum 10 and B be the event of getting even on both the dice. Find whether A, B are independent or not.**

> **Note :** $P\left(\frac{A_i}{A}\right)$ *are called posterior probabilities*

Sol. Here $n(S) = 36$ where S is sample space.

A : occurrence of sum 10

$A = \{(4,6)\,(6,4)\,(5,5)\}$

B : occurrence of both even

$B = \{(2,2)\,(2,4)\,(2,6)\,(4,2)\,(4,4)\,(4,6)\,(6,2)\,(6,4)\,(6,6)\}$

Here $(A \cap B) = \{(4,6)\,(6,4)\}$

$$P(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

$$P(A) = \frac{3}{36} = \frac{1}{12}\ ;\ P(B) = \frac{9}{36} = \frac{1}{4}$$

$P(A \cap B) \neq P(A)\,P(B)$

Therefore A, B are not independent.

Here further observe that

$$P(A/B) = \frac{2}{9}; P(A) = \frac{1}{12}$$

$P(A/B) \neq P(A)$

So, A, B are not independent.

***8. A card is selected at random from a pack of 52 cards. Let A be the event that the card is a face card and B be the event that the card is a heart card show that A and B are independent.**

Sol. Since a card is drawn from pack, $n(S) = 52$

Let A be the event that the selected card is face card $P(A) = \frac{16}{52} = \frac{4}{13}$

Let B be the event that the selected card is hearts $P(B) = \frac{13}{52} = \frac{1}{4}$

Here $A \cap B$ is the event that the card is both heart and face card.

$P(A \cap B) = \frac{4}{52} = \frac{1}{13}$ (since heart face cards are 4 in pack)

$P(A \cap B) = P(A)\,P(B)$ $\therefore$ A, B are independent.

Further, here $P(A/B) = 4/13$ $P(A) = 16/52 = 4/13$

Therefore $P(A/B) = P(A)$

So, A, B are independent.

9. The probabilities of a problem being solved by three students are $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ respectively. Find the probability that the problem is being solved.

Sol. Let A, B, C be the events that the problem being solved by students A, B, C respectively.

Clearly, A, B, C are independent.

P(Problem being solved by atleast one student)

$= P(A \cup B \cup C) = 1 - P[(A \cup B \cup C)^C]$

$= 1 - P[A^C \cap B^C \cap C^C]$ (Demorgan law)

$= 1 - P(A^C)\,P(B^C)\,P(C^C)$ [$\because A^C, B^C, C^C$ are also independent]

$= 1 - \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) = 1 - \frac{5}{12} = \frac{7}{12}$

10. A box contains 4 defective and 6 good bulbs. Two bulbs are drawn at random without replacement. Find the probability that both bulbs are good.

Sol. Let A and B be the events of drawing a good bulb in 1st draw and second draw respectively.

Here, $P(A \cap B) = P(A)\,P(B/A) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

11. A couple has two children. Find the probability that both are male if it is known that atleast one of them is a male child.

Sol. Since couple has two children,

$S = \{MM, MF, FM, FF\}$ where M stands for a male child, F stands for female child.

Let A be the event that atleast one of them is a male.

$A = \{MM, MF, FM\}$

Let B be the event that both are male.

Here $B = \{MM\}$ and $A \cap B = \{MM\}$

$$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$$

12. What is the probability that 6 is obtained on one of the dice in a throw of two dice, given that the sum is 7.

Sol. Since two dice are rolled. $S = \{(x_1, x_2) / 1 \le x_1, x_2 \le 6\}$

$n(S) = 36;$

Let A be the event of getting 6 on one of the dice.

$A = \{(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)(1,6)(2,6)(3,6)(4,6)(5,6)\}$

i.e., $n(A) = 11$; Let B be the event of getting sum 7.

$B = \{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\}$

$A \cap B = \{(1,6)(6,1)\}$

By def. of conditional probability,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{2}{36}\right)}{\frac{6}{36}} = \frac{1}{3}$$

Note : $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = 1$

***13. The probability that Australia wins a match against India in a cricket game is given to be 1/3. If India and Australia play 3 matches, what is the probability that (i) Australia will loose all the three matches (ii) Australia will win atleast one match ?**

Sol. $P(A)$ = probability that Australia wins a match $P(A) = \frac{1}{3}$

$P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$

i) $P(\bar{A} \cap \bar{A} \cap \bar{A}) = P(\bar{A}).P(\bar{A}).P(\bar{A}) = \frac{2}{3}.\frac{2}{3}.\frac{2}{3} = \frac{8}{9}$

ii) Probability that Australia wins atleast one match.

$=$ 1– Probability that Australia looses all matches $= 1 - \frac{8}{9} = \frac{1}{9}$

14. In a certain college, 25% of the boys and 10% of the girls are studying mathematics. The girls constitute 60% of the student strength. If a student is selected at random is found studying mathematics, find the probability that the student is a girl.

Sol. Let us suppose,

Total number of students = 100

The number of girls = 60

The number of boys = 40

The number of boys who are studying mathematics $= \frac{25}{100} \times 40 = 10$

The number of girl students who are studying mathematics $= \frac{10}{100} \times 60 = 6$;

Let M, G be the events of selecting mathematics student, girl student respectively.

$$P(G/M) = \frac{P(G \cap M)}{P(M)} = \frac{\frac{6}{100}}{\frac{10+6}{100}} = \frac{3}{8}$$

15. A die is thrown 3 times. Find the probability of the event of getting sum of the numbers thrown as 15 when it is known that the first throw was a five.

Sol. Since a die is rolled 3 times,

$S = \{(x_1, x_2, x_3) / 1 \le x_i \le 6 \text{ for } i = 1, 2, 3\}$

$n(S) = 216$

Let A be the event of getting 5 on the first die

$A = \{(5, x_2, x_3) / 1 \le x_2, x_3 \le 6\}$

So, $n(A) = 36$;

Here $P(A) = 36/216$

Let B be the event of getting sum 15;

$B=\{(x_1, x_2, x_3)/1 \le x_i \le 6 \text{ for } i=1, 2, 3 \text{ and } x_1+x_2+x_3=15\}$

Here $A \cap B = \{(x_1\, x_2\, x_3)/x_1= 5 \text{ and } 1 \le x_i \le 6 \text{ for } i = 2, 3 \text{ and } x_1+x_2+x_3=15\}$

$= \{(5, x_2, x_3) / 1 \le x_2, x_3 \le 6 \text{ and } x_2+x_3=10\} = \{(5, 5, 5), (5, 6, 4), (5, 4, 6)$

$$P(A \cap B) = \frac{3}{216}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{3}{216}}{\frac{36}{216}} = \frac{3}{36} = \frac{1}{12}$$

***16. A survey shows that in a certain village 2 out of every 100 men and 1 out of every 100 women have stomach ulcers. A person selected at random from the village is found to have stomach ulcer. Find the probability that the person is a male, given that the probability of selecting a male from the village is 0.55.**

Sol. Without loss,

We can assume number of male persons = 5500

Number of women = 4500

This supposition is acceptable because according to data, probability of selecting a male is 0.55

Number of male persons who are having stomach ulcer $= \frac{2}{100} \times 5500 = 110$;

Number of females who are having stomach ulcer $= \frac{1}{100} \times 4500 = 45$

Let S be the event of selecting a person having stomach ulcer and M be the event of selecting a male person respectively.

By def. of conditional probability, $P(M/S) = \frac{P(M \cap S)}{P(S)} = \frac{\frac{110}{10000}}{\frac{155}{10000}} = \frac{110}{155} = \frac{22}{31}$

17. Find the probability of drawing 2 red balls in succession from a bag containing 4 red balls and 5 black balls when the ball that is drawn first is (i) not replaced (ii) replaced.

Sol. Let R_i be the event of drawing red ball in i^{th} draw for $i = 1, 2$

i) **without replacement**

P(drawing both red balls) $= P(R_1 \cap R_2) = P(R_1)\, P(R_2/R_1) = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$

ii) **with replacement**

$P(R_1 \cap R_2) = P(R_1)\, P(R_2/R_1) = P(R_1)\, P(R_2) = \frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$

(Since the first ball is replaced, R_1, R_2 are independent)

***18. Three cards are drawn from pack of 52 cards one after another without replacement. Find the probability of getting king in 1st draw, queen in 2nd draw and ace in 3rd draw.**

Sol. Let A be the event of getting king in 1st draw.

Let B be the event of getting queen in 2nd draw.

Let C be the event of getting ace in 3rd draw.

$P(A \cap B \cap C) = P(A)\, P(B \cap C/A)$

$= P(A)\, P(B/A)\, P(C/A \cap B) = \frac{4}{52} \times \frac{4}{51} \times \frac{4}{50} = \frac{16}{33150}$

19. Bag A contains 4 white and 3 black balls. Bag B contains 3 white and 2 black balls. One ball is transfered from bag A to bag B. Now one ball is drawn from bag B. Find the probability that it is white .

Sol. Let T_W, T_B be the events of transfering white ball and black ball from 1st bag to second bag.

Here $P(T_W) = 4/7$; $P(T_B) = 3/7$

Here T_W, T_B are exclusive and exhaustive.

Let W be the event of selecting white ball from 2nd bag.

$P(W) = P(W \cap S) = P(W \cap (T_W \cup T_B) = P[(W \cap T_W) \cup (W \cap T_B)]$

$= P(W \cap T_W) + (W \cap T_B)$ (Additive axiom)

$= P(T_W)\, P(W/T_W) + P(T_B)\, P(W/T_B)$ ... (1)

$P(W/T_W) = P$(drawing white ball from 2nd bag if white ball is transfered from 1st bag to 2nd bag) $= \frac{4}{6} = \frac{2}{3}$

Similarly, $P(W/T_B) = \frac{3}{6} = \frac{1}{2}$

From (1) $P(W) = \frac{4}{7} \times \frac{2}{3} + \frac{3}{7} \times \frac{1}{2} = \frac{8}{21} + \frac{3}{14} = \frac{25}{42}$

***20. A bag contains 6 white balls and 4 black balls. A ball is drawn and is put back in the bag with 5 balls of the same colour as that of the ball drawn. A ball is drawn again at random. What is the probability that the ball drawn now is white.**

Sol. Let A, B be the events of drawing white ball and black ball from the bag in 1st draw respectively.

Here A, B are exclusive and exhaustive

$P(A) = \frac{6}{10} = \frac{3}{5}$; $P(B) = \frac{4}{10} = \frac{2}{5}$;

Let E be the event of getting white ball in 2nd draw.

Now $P(E) = P(E \cap S) = P[E \cap (A \cup B)] (\because A \cup B = S)$

$= P[(E \cap A) \cup (E \cap B)] = P(E \cap A) + (E \cap B)$

$= P(A)\, P(E/A) + P(B)\, P(E/B)$ .... (1)

$P(E/A) = P$(getting white ball when 1st ball drawn from the bag is white)

$= \frac{11}{15}$ [$\because$ white ball is put back with 5 white balls]

similarly $P(E/B) = \frac{6}{15}$

[since black ball is put back along with 5 black balls]

from (1), $P(E) = \frac{11}{15} \times \frac{3}{5} + \frac{6}{15} \times \frac{2}{5} = \frac{3}{5}$

***21. A person secures a job in a construction company in which the probability that the workers go on strike is 0.65 and the probability that the construction job will be completed on time if there is no strike is 0.80. If the probability that the construction job will be completed on time even if there is a strike is 0.32, determine the probability that the construction job will be completed on time.**

Sol. $P(S) = 0.65$; $E \rightarrow$ Job completed on time

$P\left(\frac{E}{S}\right) = 0.32$ $\frac{E}{S} \rightarrow$ Job completed on time given that there is a strike

$P\left(\frac{E}{\overline{S}}\right) = 0.80$ $\frac{E}{\overline{S}} \rightarrow$ Job completed on time given that there is no strike

$$P(E) = P(S).P\left(\frac{E}{S}\right) + P(\overline{S}).P\left(\frac{E}{\overline{S}}\right)$$

$= (0.65)\,(0.32) + (0.35)\,(0.80) = .2080 + .2800$

$P(E) = 0.488$

***22. Bag *A* contains 4 white and 7 black balls. Bag *B* contains 5 white and 6 black balls. A die is rolled. If 2 or 5 turns up then choose bag *A* otherwise choose *B*. If one ball is drawn at random from the selected bag, then find the probability that it is black.**

Sol. Let B be the event of selecting black ball from the selected bag.

Let E be the event of throwing 2 or 5 on the die.

P(drawing black ball)

$= P(B) = P(B \cap S) = P[B \cap (E \cup E^C)] = P[(B \cap E) \cup (B \cap E^C)]$

$= P(B \cap E) + P(B \cap E^C)$ (by axiom of additivity)

$= P(E)\, P(B/E) + P(E^C)\, P(B/E^C) = \frac{2}{6} \times \frac{7}{11} + \frac{4}{6} \times \frac{6}{11}$

(since $P(B/E) = P$(drawing black ball from 1st bag) $= \frac{7}{11}$) $= \frac{38}{66} = \frac{19}{33}$

***23. Three boxes numbered I, II, III contain the balls as follows.**

	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One box is randomly selected and a ball is drawn from it. If the ball is red, then find the probability that it is from box II.

Sol. Let B_i be the event of selecting i^{th} bag for $i = 1, 2, 3$

Here B_1, B_2, B_3 are mutually exclusive and exhaustive events.

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}.$$

Let R be the event of getting red ball.

By Baye's theorem, $$P(B_2/R) = \frac{P(B_2)P(R/B_2)}{\sum_{i=1}^{3} P(B_i)P(R/B_i)} \quad \text{.... (1)}$$

Now, $P(R/B_1) = \frac{3}{6}$; $P(R/B_2) = \frac{1}{4}$; $P(R/B_3) = \frac{3}{12}$

From (1), $$P(B_2/R) = \frac{\frac{1}{3}\frac{1}{4}}{\frac{1}{3}\frac{3}{6} + \frac{1}{3}\frac{1}{4} + \frac{1}{3}\frac{3}{12}} = \frac{\frac{1}{4}}{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = \frac{1}{4}$$

***24. A bag contains 5 balls. Two balls are drawn and found them to be red. Find the probability that all the balls are red.**

Sol. It is very clear that the bag contains atleast 2 red balls.

Let R_k be the event that the bag contains k red balls where $k = 2, 3, 4, 5$.

Here R_2, R_3, R_4, R_5 are exclusive and exhaustive

$$P(R_2) = P(R_3) = P(R_4) = (R_5) = \frac{1}{4} \text{ (Since } R_2, R_3, R_4, R_5 \text{ are equiprobable)}$$

Let A be the event that the two balls which are drawn are red.

By Baye's theorem, $$P(R_5/A) = \frac{P(R_5)P(A/R_5)}{\sum_{k=2}^{5} P(R_k)P(A/R_k)} = \frac{P(A/R_5)}{\sum_{k=2}^{5} P(A/R_k)} \quad .... (1)$$

($\because$ R_2, R_3, R_4, R_5 are equiprobable)

$$P(A/R_2) = \frac{{}^2C_2}{{}^5C_2} = \frac{1}{10}$$

$$P(A/R_3) = \frac{{}^3C_2}{{}^5C_2} = \frac{3}{10}; \; P(A/R_4) = \frac{{}^4C_2}{{}^5C_2} = \frac{6}{10}$$

$$P(A/R_5) = \frac{{}^5C_2}{{}^5C_2} = \frac{10}{10} = 1$$

From (1), $$P(R_5/A) = \frac{1}{\frac{1}{10}+\frac{3}{10}+\frac{6}{10}+\frac{10}{10}} = \frac{1}{2}$$

***25. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually six.**

Note : *Probabilites calculated using Baye's theorem are known as inverse probabilities or posterior probabilities*

Sol. Let A be the event that six occurs on the die. Let E be the event that the man speaks truth

$$P(E) = \frac{3}{4}; \; P(E^C) = \frac{1}{4}; \quad P(A/E) = \frac{1}{6}; P(A/E^C) = \frac{5}{6}$$

By Baye's theorem $$P(E/A) = \frac{P(E)P(A/E)}{P(E)P(A/E)+P(E^C)P(A/E^C)}$$

$$= \frac{\frac{3}{4}\times\frac{1}{6}}{\frac{3}{4}\times\frac{1}{6}+\frac{1}{4}\times\frac{5}{6}} = \frac{3}{8}$$

***26. A card from pack of 52 cards is lost. From the remaining cards of pack, two cards are drawn and are found to be spades. Find the probability of the missing card to be a spade.**

Sol. Let E_C, E_S, E_H, E_D be the events of losing a card from clubs, spades, hearts and diamonds respectively

Here E_C, E_S, E_H, E_D are exclusive and exhaustive and they are equiprobable.

Let A be the event of drawing 2 spades from the remaining cards.

Now by Baye's Theorem, $P(E_S/A)$

$$= \frac{P(E_S)P(A/E_S)}{P(E_C)P(A/E_C)+P(E_S)P(A/E_S)+P(E_H)P(A/E_H)+P(E_D)(P_A/E_D)}$$

$$= \frac{P(A/E_S)}{P(A/E_C)+P(A/E_S)+P(A/E_H)+P(A/E_D)}$$

$(\because P(E_S) = P(E_C) = P(E_H) = P(E_D) = 1/4)$

$P(A/E_S) = \frac{{}^{12}C_2}{{}^{51}C_2}$ (since the card which was lost is spade)

$P(A/E_C) = \frac{{}^{13}C_2}{{}^{51}C_2}$; $P(A/E_H) = \frac{{}^{13}C_2}{{}^{51}C_2}$; $P(A/E_D) = \frac{{}^{13}C_2}{{}^{51}C_2}$

from (1)

$$P(E_S/A) = \frac{{}^{12}C_2}{{}^{12}C_2 + 3({}^{13}C_2)} = \frac{12\times 11}{12\times 11 + 3(13)\times 12} = \frac{11}{50}$$

***27. A letter is known to have come from either 'MAHARASTRA' or 'MADRAS' on the post mark only consecutive letters 'RA' can be read clearly. What is the chance that the letter came from 'MAHARASTRA'.**

Sol. Let A, B be the events that the letter will come from 'MAHARASTRA' and 'MADRAS' respectively.

Let E be the event that the two consecutive letters 'RA' can be read clearly.

Here A, B are exclusive and exhaustive $P(A) = P(B) = \frac{1}{2}$

By Baye's theorem, $P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A)+P(B)P(E/B)}$

$= \frac{P(E/A)}{P(E/A)+P(E/B)}$ ....(1) $(\because P(A) = P(B))$

$P(E/A) = \frac{2}{9}$

($\because$ MAHARASTRA contains 9 possible consecutive letters MA, AH, HA, AR, RA, AS, ST, TR and RA among these 9 possibilites two are favour to E)

Similarly, $P(E/B) = \frac{1}{5}$

from (1) $P(A/E) = \frac{\frac{2}{9}}{\frac{2}{9}+\frac{1}{5}} = \frac{10}{19}$

28. Four persons *A, B, C, D* cut a pack of 52 cards successively in that order given. If the person who cuts a spade first wins, find their probability of winning.

Sol. Let E be the event of any one cut a spade in one cut.

$$P(E) = \frac{13}{52} = \frac{1}{4} = p \text{ (say)}$$

$$P(E^C) = 1 - \frac{1}{4} = \frac{3}{4} = q \text{ (say)}$$

Since A starts the game, A wins if he cuts spade in the first attempt or A, B, C, D all fail and A cuts spade card in his 2nd attempt and so on.

$$P(A \text{ wins in his first attempt}) = p = \frac{1}{4}$$

$$P(A \text{ wins in his second attempt}) = q\,q\,q\,q\,p = q^4 p = \left(\frac{3}{4}\right)^4 \frac{1}{4}$$

$$P(A \text{ wins in his third attempt}) = (q\,q\,q\,q)^2\,p = q^8 p = \left(\frac{3}{4}\right)^8 p \text{ and so on}$$

Since all these attempts are mutually exclusive,

$$P(A \text{ wins}) = p + q^4 p + q^8 p + \ldots\ldots \infty = \frac{p}{1-q^4}$$

(infinte G.P. with common ratio q^4 where $0 < q^4 < 1$)$= \dfrac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^4} = \dfrac{64}{175}$;

In a similar manner, $P(B \text{ wins}) = qp + q\,q\,q\,q\,q\,p + (q\,q\,q\,q)^2\,qp + \ldots \infty$

$$= qp + q^4 ap + q^8 qp + \ldots\ldots \infty$$

$$= \frac{qp}{1-q^4} = q\,.\,P(A \text{ wins}) = \frac{3}{4} \times \frac{64}{175} = \frac{48}{175}$$

$$P(C \text{ wins}) = q\,q\,p + (q\,q\,q\,q)\,q^2 p + (q\,q\,q\,q)^2\,q^2 p + \ldots \infty$$

$$= \frac{q^2 p}{1-q^4} = q^2\,P(A \text{ wins}) = q^2\,\frac{64}{175} = \frac{9}{16}\,\frac{64}{175} = \frac{36}{175}$$

Similarly, $P(D \text{ wins}) = q^3\,P(A \text{ wins}) = \dfrac{27}{64}\,\dfrac{64}{175} = \dfrac{27}{175}$

Note :

In the above problem, we can observe that the ratio of their winning chances is $1 : q : q^2 : q^3$ where $q = 1 - p$ and p is probability of happening the desired event in a single attempt.

***29. An urn contains 10 white balls and 5 black balls. Two players *Q* and *R* alternatively draw a ball with replacement from the urn. The player that draws a white ball first wins the game. If *Q* begins the game, find the probability of his winning the game.**

Sol. Let p be the probability of drawing a white ball from bag by any person in one trail.

Here $p=\frac{10}{15}=\frac{2}{3}$;

Let $q=1-p=1-\frac{2}{3}=\frac{1}{3}$;

Since Q starts the game, Q wins the game if he draws a white ball in 1st attempt or Q, R both fail and Q draws a white ball in second attempt and so on.

P(Q wins in 1st attempt) = p

P(Q wins in 2nd attempt) = $q\,q\,p = q^2 p$

P(Q wins in 3rd attempt) = $(q\,q)^2 p = q^4 p$ and so on

Since all the above cases are mutually exclusive,

$$P(Q \text{ wins}) = p + q^2p + q^4p + \ldots\ldots \infty = \frac{p}{1-q^2} = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{3}{4}$$

Here $P(R \text{ wins}) = 1 - P(Q \text{ wins}) = 1-\frac{3}{4}=\frac{1}{4}$

Note :

The ratio of their winning chances is $1 : q = 1 : \frac{1}{3} = 3 : 1$

So, $P(Q \text{ wins}) = \frac{3}{4}$; $P(R \text{ wins}) = \frac{1}{4}$

***30.** **A bag contains10 white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. What is the probability that this procedure will come to an end at the seventh draw.**

Sol. The procedure for drawing balls has to come to an end at the 7th draw if all but one black ball must be drawn in the first six draws and the last black ball in 7th draw.

$$\text{Probabilty of required event} = \frac{{}^{10}C_4 \times {}^{3}C_2}{{}^{13}C_6} \times \frac{1}{7}$$

***31.** **A consignment of 15 record players contains 4 defective. The record players are selected at random one by one and examined. The ones examined are not placed back. What is the probability that the 9th one examined is the last defective.**

Sol. The 9th examined record player will be the last defective only if all but one defective record player must be examined in first 8 draws and in last 9th draw the last defective is examined.

$$\text{So, the required probability} = \frac{{}^{11}C_5 \times {}^{4}C_3}{{}^{15}C_8} \times \frac{1}{7}$$

EXERCISE - 3.2

1. Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of the event that all 3 screws are non- defective, assuming that the drawing is

   a) with replacement [Ans : $\left(\frac{9}{10}\right)^3$]

   b) without replacement [Ans : $\frac{1419}{1960}$]

2. If A,B,C are three independent events of an experiment such that,

   $P(A \cap B^C \cap C^C) = \frac{1}{4}, P(A^C \cap B \cap C^C) = \frac{1}{8}, P(A^C \cap B^C \cap C^C) = \frac{1}{4}$

   Then find $P(A)$, $P(B)$ and $P(C)$. [Ans : 1/2, 1/3, 1/4]

3. There are 3 black and 4 white balls in one bag, 4 black and 3 white balls in the second bag. A die is rolled and the first bag is selected if the die shows up 1 or 3, and the second bag for the rest. Find the probability of drawing a black ball, from the bag thus selected. [Ans : 11/21]

4. A,B,C are aiming to shoot a balloon, A will succeed 4 times out of 5 attempts. The chance of B to shoot the Balloon is 3 out of 4 and that of C is 2 out of 3. If the three aim the balloon simulineously then find the probability that atleast two of them hit the balloon. [Ans : 5/6]

5. If A,B are two events, then show that $P(A/B)P(B) + P(A/B^C)P(B^C) = P(A)$

6. A pair of dice is rolled, what is the probability that they sum to 7 given that neither die shows a 2? [Ans : 4/25]

7. A pair of dice is rolled, what is the probability that neither die shows a 2 given that they sum to 7. [Ans : 2/3]

8. If A,B are any two events in an experiment and $P(B) \neq 1$. Show that $P(A/B^C) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

9. An urn contains 12 red balls and 12 green balls. Suppose two balls are drawn one after another without replacement. Find the probability that the second ball drawn is green, given that the first ball drawn is red [Ans : 12/23]

10. A single die is rolled twice in succession. what is the probability that the number on the second toss is greater than that on the first rolling? [Ans : 15/36]

11. If one card is drawn at random from a pack of cards, then show that the event of getting an ace and getting a heart are independent events.

12. The probability that a Boy A will get a scholarship is 0.9 and that another boy B will get is 0.8. What is the probability that atleast one of them will get the scholarship? [Ans : 0.98]

*13. If A,B are two events with $P(A \cup B) = 0.65, P(A \cap B) = 0.15$ then find the value of $P(A^C) + P(B^C)$. (May-19) [Ans : 1.2]

14. If A,B,C are independent events, then show that $A \cup B$ and C are also independent events.

15. A,B are two independent events, such that the probability of both the events to occur is 1/6 and the probability of both the events do not occurs is 1/3 find $P(A)$ [Ans : 1/2 or 1/3]

*16. A fair die is rolled consider the events $A=\{1,3,5\}$, $B=\{2,3\}$ and $C=\{2,3,4,5\}$. Find
i) $P(A \cap B), P(A \cup B)$ ii) P (A/B), P(B/A) iii) P(A/C), P(C/A) iv) P(B/C), P(C/B) **(May-19)**
[Ans : (i) 1/6,2/3 (ii) 1/2,1/3 (iii) 2/4, 2/3 (iv) 1/2,1]

17. If A,B,C are three events in a random experiment, prove the following
i) P(A/A) = 1
ii) $P(\phi / A) = 0$
iii) $A \subseteq B \Rightarrow P(A/C) \le P(B/C)$
iv) $P(A-B) = P(A) - P(A \cap B)$
v) If A,B are mutually exclusive and $P(B) > 0$ then $P(A/B) = 0$
vi) If A,B are mutually exclusive then $P(A/B^C) = \dfrac{P(A)}{1-P(B)}$ when $P(B) \neq 1$
vii) If A and B are mutually exclusive and $P(A \cup B) \neq 0$, Then $P(A/A \cup B) = \dfrac{P(A)}{P(A)+P(B)}$

18. Suppose that a coin is tossed three times. Let event A be "getting three heads" and B be the event of "getting a head on the first toss". Show that A and B are dependent events.

19. Suppose that an unbaised pair of dice is rolled. Let A denote the event that the same number shows on each die. Let B denote the event that the sum is greater than 7.
find i) $P(A/B)$ ii) $P(B/A)$ [Ans : (i) 1/5, (ii) 1/2]

20. Prove that A and B are independent events if and only if $P\left(\dfrac{A}{B}\right) = P\left(\dfrac{A}{B^C}\right)$

*21. Suppose A and B are independent events with $P(A) = 0.6$, $P(B) = 0.7$. Then compute (May-18)
i) $P(A \cap B)$ [Ans : 0.42]
ii) $P(A \cup B)$ [Ans : 0.88]
iii) $P(B/A)$ [Ans : 0.7]
iv) $P(A^C \cap B^C)$ [Ans : 0.12]

22. The probability that Australia wins a match against India in a cricket game is given to be 1/3. If India and Australia play 3 matches, what is the probability that
i) Australia will loose all the three matches ? [Ans : 8/27]
ii) Australia will atleast one match ? [Ans : 19/27]

*23. Three boxes numbered I, II, III contain the balls as follows:

	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One box is randomly selected and a ball is drawn from it. If the ball is red, then find the probability that it is from box II. **(May-19)** **[Ans : 1/4]**

24. A person secures a job in a construction company in which the probability that the workers go on strike is 0.65 and the probability that the construction job will be completed on time if there is no strike is 0.80. If the probability that the construction job will be completed on time even if there is a strike is 0.32, determine the probability that the construction job will be completed on time. **[Ans : 0.4880]**

25. For any two events A, B show that

$P(A \cap B) - P(A)P(B) = P(A^C)P(B) - P(A^C \cap B) = P(A)P(B^C) - P(A \cap B^C)$

26. Three urns have the following composition of balls

Urn I : 1 white, 2 black

Urn II: 2 white, 1black

Urn III: 2 white, 2 black

One of the urns is selected at random and a ball is drawn. It turns out to be white.Find the probability that it came from urn III **[Ans : 1/3]**

27. In a shooting test the probability of A,B,C hitting the targets are 1/2, 2/3 and 3/4 respectively. If all of them fire at the same target, find the probability that i) only one of them hits the target ii) atleast one of them hits the target **[Ans : 6/24, 23/24]**

28. In a certain college, 25% of the boys and 10% of the girls are studying mathamatics. The girls constitute 60% of the student strength. If a student selected at random is found studying mathematics, find the probability that the student is a girl. **[Ans : 3/8]**

29. a) A person is known to speak truth 2 out of 3 times. He throws a die and reports that it is 1. Find the probability that it is actually 1 **[Ans : 2/7]**

*b) A speaks truth in 75% cases and B in 80% cases. What is the probability that their statements about an incident donot match. **(March-19)** **[Ans : 35%]**

30. A box contains 4 defective and 6 good bulbs. Two bulbs are drawn at random without replacement. Let us find the probability that both bulbs drawn are good. Let A denote the event of drawing a good bulb in the first draw and B denote the event that the second bulb drawn is also good. **[Ans : 1/3]**

31. A bag contains 10 identical balls, of which 4 are blue and 6 are red. Three balls are taken out at random from the bag one after the other. Let us find the probability that all the three balls drawn are red. **[Ans : 1/6]**

32. An urn contains 7 red and 3 black balls. Two balls are drawn without replacement. What is the probability that the second ball is red if it is known that the first ball drawn is red [Ans : 2/3]

33. Let A and B be independent events with $P(A) = 0.2$, $P(B) = 0.5$. Let us find

i) $P(A/B)$ [Ans : 0.2]

ii) $P(B/A)$ [Ans : 0.5]

iii) $P(A \cap B)$ [Ans : 0.1]

iv) $P(A \cup B)$ [Ans : 0.6]

34. Bag B_1 contains 4 white and 2 black balls. Bag B_2 contains 3 white and 4 black balls. A bag is drawn at random and a ball is choosen at random from it. What is the probability that the ball drawn is white ? [Ans : 23/42]

35. A bag contains 10 balls ; 5 of which are red and the remaining blue. Two balls are drawn at random from the bag one after the other with replacement. Let A be the event that the first ball drawn is red and B be the event that the second ball is red. They are independent or dependent. [Ans : Independent]

36. A shop keeper buys a particular type of electric bulbs from three manufactures M_1, M_2 and M_3. He buys 25% of his requirement from M_1, 45% from M_2 and 30% from M_3. Based on the past experi-ence, he found that 2% of type M_3 bulbs are defective, where as only 1% of type $M_1$and type M_2 are defective. If a bulb chosen by him at random is found defective, let us find the probability that it was of type M_3 [Ans : 0.46]

37. Suppose that an urn B_1 contains 2 white and 3 black balls and another urn B_2 contains 3white and 4 black balls. One urn is selected at random and a ball is drawn from it. If the ball drawn is found black, Let us find the probability that the urn chosen was B_1 [Ans : 21/41]

38. Suppose there are 12 boys and 4 girls in a class. If we choose three children one after another in succession at random, find the probability that all the three are boys. [Ans : 11/28]

*39. A problem in Calculus is given to two students A and B whose chances of solving it are 1/3 and 1/4 respectively. Find the probability of the problem being solved if both of them try indepen-dently **(March-18)** [Ans : 1/2]

40. If A and B are independent events of a random experiment, show that A^C and B^C are also indepen-dent.

41. An urn contains w white balls and b black balls. Two players Q and R alternatively draw a ball with replacement from the urn. The player that draws a white ball first wins the game. if Q begins the game, find the probability of his winning the game. [Ans : $\frac{w+b}{w+2b}$]

42. Three boxes B_1, B_2 and B_3 contain balls with different colours as shown below

	White	Black	Red
B_1	2	1	2
B_2	3	2	4
B_3	4	3	2

A die is thrown. B_1 is chosen if either 1 or 2 turns up. B_2 is chosen if 3 or 4 turns up and B_3 is chosen if 5 or 6 turns up. Having chosen a box in this way, a ball is chosen at random from this box. If the ball drawn is found to be red, find the probability that it is drawn from box B_2.

[Ans : 5/12]

SOLVED EXAMPLES

1. Two integers x and y are chosen one by one with replacement at random from the set $\{x \,/\, 0 \le x \le 10$ and x is an integer$\}$. Find the probability that $|x - y| \le 5$.

Sol. Let $A = \{x/x$ is an integer and $0 \le x \le 10\}$

Since two integers x and y are chosen from A one by one with replacement,

$S = A \times A = \{(x, y) / x, y \in A\}$

$\therefore\ n(S) = 11 \times 11 = 121$

Let E be the event of selecting x and y

Such that $|x - y| \le 5$

Here $(x,y) = (0,0); (1, 1); (2,2); ....(10,10)$ are ordered pairs such that $|x - y| = 0 \le 5$

So, these 11 pairs are favourable to E;

Let us count the number of ordered pairs (x, y) such that $|x - y| = k$

where $k = 1, 2, 3, 4, 5$;

Consider a fixed integer k where $1 \le k \le 5$;

Suppose $(x, x + k)$ is a favour to E.

Here $0 \le x < x + k \le 10 \Rightarrow 0 \le x \le 10 - k$

So, the number of ordered pairs of the form $(x, x + k) = 11 - k$.

If $(x, x + k)$ is a favourable to E then $(x + k, x)$ is also a favourable to E.

(since $|(x + k) - x| = |x - (x + k)| = k$)

So, for a fixed $k (1 \le k \le 5)$

There are $2(11 - k)$ favourable outcomes to E such that $|x-y| = k$

Therefore total number of favourable outcomes to E

$$= 11 + \sum_{k=1}^{5} 2(11 - k) = 11 + 2\sum_{k=1}^{5}(11 - k) = 11 + 2\{55 - 15\} = 11 + 80 = 91$$

$P(A) = \dfrac{91}{121}$

Method - II : $\underbrace{00}_{x_1}0\underbrace{000}_{x_2}0\underbrace{0000}_{x_3}$

$x_1 + x_2 + x_3 = 9,\ 0 \leq x_2 \leq 5$

R.N.W. $= 11 + 2\left(\sum_{k=0}^{5}(9-k) + (2-1)_{C_{2-1}}\right) = 11 + 2\sum_{k=0}^{5}(10-k) = 11 + 2(40) = 91$

$P(E) = \dfrac{91}{121}$

2. A is a set containing n elements. A subset P of A is choosen at random. The set A is reconstructed by replacing the elements of the subset of P, a subset Q of A is again chosen at random. Find the probability that
(i) $P \cap Q = \phi$ (ii) $P \cup Q = A$ (iii) $P \cup Q = A$ and $P \cap Q = \phi$

Sol. Let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$

for each a_k $(1 \leq k \leq n)$ we have the following 4 choices

i) $a_k \in P$ and $a_k \in Q$

ii) $a_k \in P$ and $a_k \notin Q$

iii) $a_k \notin P$ and $a_k \in Q$

iv) $a_k \notin P$ and $a_k \notin Q$

The total number of ways choosing the subsets P and Q is 4^n.

i) For the event $P \cap Q = \phi$, out of the above 4 choices (i) choice is not favourable. So, the number of favourables for the event $P \cap Q = \phi$ is 3^n;

required probability $= \left(\dfrac{3}{4}\right)^n$

ii) Out of 4 choices (iv) is not favourable to the occurence of the event $P \cup Q = A$;

The number of favourable ways to the occurrence of $P \cup Q = A$ is 3^n;

Probability $= \left(\dfrac{3}{4}\right)^n$

iii) Out of the 4 choices (i) and (iv) are not favourables for the occurrence of the event $P \cup Q = A$ and $P \cap Q = \phi$

The number of favourables is 2^n.

Probability $= \left(\dfrac{2}{4}\right)^n = \left(\dfrac{1}{2}\right)^n$

3. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that (i) Q is subset of P (ii) The number of elements in P is more than the number of elements in Q.

Sol. i) Let $A = \{a_1, a_2, \ldots\ldots, a_n\}$

for each $a_k \in A$ $(1 \leq k \leq n)$

we have the following 4 choices

a) $a_k \in P$ and $a_k \in Q$

b) $a_k \notin P$ and $a_k \notin Q$

c) $a_k \in P$ and $a_k \notin Q$

d) $a_k \notin P$ and $a_k \in Q$

Among these 4 choices, (4) is unfavourable to the occurrence of the event $Q \subseteq P$.
i.e., $a_k \in Q$ and $a_k \notin P$ is not favour to the occurence of $Q \subseteq P$.

$\therefore$ Probability $= \left(\frac{3}{4}\right)^n$

ii) Let E be the event of selecting subsets P and Q such that number of elements in P is more than that of Q.

Here $P(E) = \dfrac{\sum\limits_{0 \le i < j \le n} {}^nC_i \, {}^nC_j}{4^n}$;

Consider $({}^nC_0 + {}^nC_1 + ... + {}^nC_n)^2 =$

$$({}^nC_0)^2 + ({}^nC_1)^2 + ... + ({}^nC_n)^2 + 2\left[\sum_{0 \le i < j \le n} {}^nC_i \, {}^nC_j\right]$$

Therefore, $(2^n)^2 = {}^{2n}C_n + 2 \sum\limits_{0 \le i < j \le n} {}^nC_i \, {}^nC_j$

So, $\sum\limits_{0 \le i < j \le n} {}^nC_i \, {}^nC_j = \frac{1}{2}(4^n - {}^{2n}C_n)$

So, $P(E) = \dfrac{\frac{1}{2}(4^n - {}^{2n}C_n)}{4^n} = \frac{1}{2} - \frac{1}{2} \cdot \frac{{}^{2n}C_n}{4^n} = \frac{1}{2} - \frac{{}^{2n}C_n}{2^{2n+1}}$

4. A pair of fair dice is rolled repeatedly. Find the probability of getting doublet 4th time in the 9th trail.

Sol. Probability of occuring a doublet when a pair of dice is rolled $= \frac{6}{36} = \frac{1}{6}$

Let E be the event that doublet occurs 4th time in 9th trail.

Here E happens only if in 1^{st} 8 trails exactly 3 times doublets must occur and in the next 9^{th} trail another doublet must occur.

P(occurrence of exactly 3 times doublet in first 8 trails) $= {}^8C_3\,(1/6)^3\,(5/6)^5$

$$P(E) = {}^8C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5 \times \frac{1}{6} = \frac{{}^8C_3 \times 5^5}{6^9}$$

***5. If a pair of dice is rolled until sum more than 10 appears first time. Find the probability of getting different numbers in last throw.**

Sol. **Method - 1**

We are interested only in last throw of the experiment.

The possbile outcomes in last throw are (6, 5), (5, 6) and (6, 6)

Out of these 3 outcomes, only (6, 5) and (5, 6) are favourable to the occurrence of different numbers on the dice.

So, required probability $= \frac{2}{3}$

Method - 2

Since 2 dice are rolled, $n(S) = 36$

Let A be the event of getting sum more than 10.

Here $P(A) = \frac{3}{36} = \frac{1}{12}$

$P(A^C) = \frac{11}{12}$

Let B be the event of getting different numbers on the dice and sum more than 10 i.e., B is the event of getting sum more than 10 with different numbers.

$P(B) = \frac{2}{36} = \frac{1}{18}$ (since $B = \{(6,5), (5,6)\}$

Let E be the event of getting different numbers in last throw.

Let X denote the number of the last throw.

$P(X = 1) = P(B) = \frac{1}{18}$

$P(X = 2) = P(A^C \cap B) = \frac{11}{12} \times \frac{1}{18}$

$P(X = 3) = P(A^C \cap A^C \cap B) = \left(\frac{11}{12}\right)^2 \frac{1}{18}$ and so on

$\therefore\ P(E) = P(X = 1 \text{ or } X = 2 \text{ or } X = 3 \text{ or } .....)$

$$= \frac{1}{18} + \frac{11}{12}\frac{1}{18} + \left(\frac{11}{12}\right)^2 \frac{1}{18} + ...\infty \quad = \frac{\frac{1}{18}}{1 - \frac{11}{12}} = \frac{\frac{1}{18}}{\frac{1}{12}} = \frac{2}{3}$$

6. A bag contains 'a' white and 'b' black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. If the probability of A winning the game is three times that of B; then find the ratio $a : b$.

Sol. Let W be the event of getting white ball at any draw.

$P(W) = \frac{a}{a+b} = p$ (say)

$P(W^C) = \frac{b}{a+b} = q$ (say)

$P(A$ wins in 1st attempt$) = \frac{a}{a+b} = p$

$P(A$ wins in his 2nd attempt$) = P(W^C \cap W^C \cap W) = q^2 p$ .... and so on.

Therefore $P(A$ wins the game$) = p + q^2p + q^4p + ... = \frac{p}{1-q^2}$

$P(B$ wins the game$) = 1 - P(A$ wins the game$) = 1 - \frac{p}{1-q^2}$

$$= \frac{1-q^2-p}{1-q^2} = \frac{q-q^2}{1-q^2} \ (\because 1-p=q) \ = \frac{q(1-q)}{1-q^2} = \frac{q \cdot p}{1-q^2}$$

By data, $P(A$ wins the game$) = 3P(B$ wins the game$)$

$\Rightarrow \frac{p}{1-q^2} = \frac{3qp}{1-q^2} \Rightarrow q = \frac{1}{3} \Rightarrow \frac{b}{a+b} = \frac{1}{3}$

$\Rightarrow 3b = a + b \Rightarrow 2b = a$

$\Rightarrow a : b = 2 : 1$

7. A biased coin with probability $p(0 < p < 1)$ of getting head is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5 then find p.

Sol. Let X denote the number of tosses required

$P(X=2) = (1-p)p$

$P(X=4) = (1-p)^3 p$ and so on

P(the number of tosses required is even) $= P(X=2) + P(X=4) + P(X=6) + ...$

$$= (1-p)p + (1-p)^3 p + (1-p)^5 p + ... = \frac{(1-p)p}{1-(1-p)^2}$$

By data, $\dfrac{(1-p)p}{1-(1-p)^2} = \dfrac{2}{5}$

$$\Rightarrow \frac{(1-p)p}{2p-p^2} = \frac{2}{5} \Rightarrow \frac{1-p}{2-p} = \frac{2}{5} \Rightarrow 5-5p = 4-2p$$

$\Rightarrow 3p = 1 \Rightarrow p = 1/3$

8. Let F be the set of all on-to functions from a set $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ to another set $B = \{b_1, b_2, b_3, b_4, b_5\}$. If a function f is selected from F at random, then find the probability that the selected function f is such that $f^{-1}(b_1) = \{a_1\}$

Sol. Given $A = \{a_1, a_2, ..., a_6\}$

$B = \{b_1, b_2, ..., b_5\}$

Here $F = \{f / f : A \to B \text{ is onto}\}$

Since we are selecting one function from F at random,

$n(S)$ = number of onto functions from A onto B.

= the number of ways to distribute six different things among 5 persons so that each person receives atleast one thing.

= the number of ways to distribute six different things among 5 persons so that one receives 2 things and the remaining 4 persons each receive one thing

$$= \frac{\lfloor 6}{\lfloor 2 \lfloor 1 \lfloor 1 \lfloor 1 \lfloor 1 \lfloor 4} \times \lfloor 5 = 360 \times 5 = 1800$$

Let E be the event of selecting the function 'f' from F for which $f^{-1}(b_1) = \{a_1\}$

Let us count $n(E)$

$n(E)$ = number of onto functions from $A - \{a_1\}$ onto the set $B - \{b_1\}$

= the number of ways to distribute 5 different things among 4 persons

so that each receives atleast one thing $= \dfrac{\lfloor 5}{\lfloor 2 \lfloor 1 \lfloor 1 \lfloor 1 \lfloor 3} \times \lfloor 4 = 60 \times 4 = 240$

$P(E) = \dfrac{240}{1800} = \dfrac{2}{15}$

9. Ten rupee coins are distributed among 5 children at random. Find the probability that the first child gets atmost 3 coins.

Sol. Let us suppose that the kth child receives 'x_k' rupees for $k = 1, 2, 3, 4, 5$

Here $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ where each $x_i \geq 0$

since 10 rupee coins are disbributed among 5 children at random,

$n(S)$ = number of non negative integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 10$

$= {}^{(n+r-1)}C_{r-1}$ where $n = 10$ and $r = 5 = {}^{14}C_4 = \dfrac{14\times13\times12\times11}{24} = 7\times 143 = 1001$

Let E be the event that first child gets atmost 3 coins

Here $n(E)$ = number of non negative integral solutions of

$x_1 + x_2 + x_3 + x_4 + x_5 = 10$ where $x_1 \le 3$;

For a fixed value k where $0 \le k \le 3$

if $x_1 = k$ then the equation becomes $x_2 + x_3 + x_4 + x_5 = 10 - k$

The number of non-ve integral solutions of the equation $x_2 + x_3 + x_4 + x_5 = 10 - k$

$= {}^{(n+r-1)}C_{r-1}$ (where $n = 10 - k$ and $r = 4$) $= {}^{13-k}C_3$ $\therefore$ $n(E) = \sum_{k=0}^{3} {}^{(13-k)}C_3$

$= {}^{13}C_3 + {}^{12}C_3 + {}^{11}C_3 + {}^{10}C_3$

$= {}^{14}C_4 - {}^{13}C_4 + {}^{13}C_4 - {}^{12}C_4 + {}^{12}C_4 - {}^{11}C_4 + {}^{11}C_4 - {}^{10}C_4$

$= {}^{14}C_4 - {}^{10}C_4 = 1001 - 210 = 791$ $\therefore$ $P(E) = \dfrac{791}{1001}$

10. A determinant is chosen at random from the set of all 2×2 determinants with elements –1, 0, 1 only. Find the probability that the determinant chosen is positive.

Sol. Here $S = \left\{ \begin{vmatrix} a & b \\ c & d \end{vmatrix} / a,b,c,d \text{ each can take values } -1,0,1 \text{ only} \right\}$

By product rule, $n(S) = 3^4 = 81$

Let E be the event of choosing a determinant A of 2×2 order from

S such that $|A| > 0$

Here $E = \left\{ \begin{vmatrix} a & b \\ c & d \end{vmatrix} / a,b,c,d \text{ each can take values } -1,0,1 \text{ only and } ad - bc > 0 \right\}$

Let us find $n(E)$

i.e., let us find number of solutions for $ad - bc > 0$ where $a, b, c, d = -1, 0$ or 1

Here $ad = 1, 0, -1$ and $bc = 1, 0$ or -1

Since $ad > bc$ we have $ad \ne -1$

Case - 1:

Suppose $ad = 1$

since $ad = 1$ and $ad > bc \Rightarrow bc = 0$ or -1

$ad = 1 \Rightarrow (a, d) = (1, 1)$ or $(-1, -1)$

$bc = 0 \Rightarrow (b, c) = (1, 0)\ (0, 0)\ (-1, 0)\ (0, 1)\ (0, -1)$

$bc = -1 \Rightarrow (b, c) = (1, -1)$ or $(-1, 1)$

Therefore the number of solutions for

$ad = 1$ and $bc = 0$ or -1 is $2 \times (5 + 2) = 14$

(using product rule and addition rule)

Case - 2 :

Suppose $ad = 0$

$ad = 0$; $ad > bc$

$\Rightarrow bc = -1$

$ad = 0 \Rightarrow (a, d) = (0, 1)\ (0, 0)\ (0, -1), (1, 0)$ or $(-1, 0)$

$bc = -1 \Rightarrow (b, c) = (1, -1)\ (-1, 1)$

By product rule,

The number of solutions for $ad = 0$, $bc = -1$ is $5 \times 2 = 10$

So, $n(E) = 14 + 10 = 24$

$\therefore P(E) = \frac{24}{81} = \frac{8}{27}$

11. Two numbers x and y are chosen at random from $\{1, 2, 3, .., 5n\}$ where $n \geq 2$; show that the probability $x^4 - y^4$ is divisible by 5 is $\frac{17n-5}{5(5n-1)}$.

Sol. Let $A = \{1, 2, 3, ..., 5n-1, 5n\}$ where $n \geq 2$

Since we are selecting two numbers x and y at random, $n(S) = {}^{5n}C_2$

Let us consider 5 subsets of A denoted by A_0, A_1, A_2, A_3, A_4 whose elements are integers of A which leave remainders 0, 1, 2, 3, 4 respectively when divided by 5.

$A_0 = \{5, 10, 15, ..., 5n\}$

$A_1 = \{1, 6, 11, ... 5n-4\}$

$A_2 = \{2, 7, 12, ... 5n-3\}$

$A_3 = \{3, 8, 13, ... 5n-2\}$

$A_4 = \{4, 9, 14, ... 5n-1\}$

Since $x, y \in A$ we can suppose

$x = 5k + r_1$, and $y = 5l + r_2$

where $0 \leq k, l \leq n$ and $0 \leq r_1 \leq 4$ and $0 \leq r_2 \leq 4$

$x^4 - y^4 = (5k + r_1)^4 - (5l + r_2)^4 = 5$ (some integer) $+ r_1^4 - r_2^4$

Therefore, $(x^4 - y^4)$ is divisible by 5

$\Rightarrow r_1^4 - r_2^4$ is divisible by $5 \Rightarrow r_1 = r_2$ (or)

r_1, r_2 are different non zero integers from 1, 2, 3, 4

(since $2^4 - 1^4 = 15$; $3^4 - 2^4 = 65$; $4^4 - 3^4 = 175$ etc.,)

So, both $x, y \in$ same A_i for $i = 0, 1, 2, 3, 4$ (or)

one of $x, y \in A_i$ and another belongs to A_j

where $1 \leq i \neq j \leq 4$

The number of favourable selections of x, y such that $x^4 - y^4$ is divisible by

$5 = 5 \times {}^nC_2 + {}^4C_2\,{}^nC_1 \times {}^nC_1 = \frac{5n(n-1)}{2} + 6n^2 = \frac{17n^2 - 5n}{2}$

Therefore required probability $= \frac{\frac{17n^2-5n}{2}}{{}^{5n}C_2} = \frac{17n^2 - 5n}{5n(5n-1)} = \frac{17n-5}{5(5n-1)}$

12. Two natural numbers a and b are selected at random, find the probability that $a^2 + b^2$ is divisible by 7.

Sol. Since we are selecting two natural numbers at random, the sample space is not finite. So, we can not apply classical definition to find the required probability.

So, let us do the following problem whose sample space is finite.

If two natural numbers a, b are selected from the set $A = \{1, 2, 3, ..., 7n\}$ where $n \geq 2$; then let us find the probability P_n that $a^2 + b^2$ is divisible by 7.

Here $n(S) = {}^{7n}C_2$

Consider the following subsets of A denoted by $A_0, A_1, A_2, A_3, A_4, A_5, A_6$ which leave remainders 0, 1, 2, 3, 4, 5, 6 respectively when divided by 7

$A_0 = \{7, 14, 21, ..., 7n\}$

$A_1 = \{1, 8, 15, ... 7n-6\}$

$A_2 = \{2, 9, 16, ... 7n-5\}$

....

....

$A_6 = \{6, 13, 20, ... 7n-1\}$

We can suppose $a = 7k + r_1$ and $b = 7l + r_2$

where $0 \leq k, l \leq n$ and $0 \leq r_1, r_2 \leq 6$;

$a^2 + b^2 = (7k + r_1)^2 + (7l + r_2)^2 = 49k^2 + 49l^2 + 14kr_1 + 14lr_2 + r_1^2 + r_2^2$

$= 7[\text{some integer}] + r_1^2 + r_2^2$

So, $a^2 + b^2$ is divisible by 7

$\Leftrightarrow r_1^2 + r_2^2$ is divisible 7 $\Leftrightarrow r_1 = r_2 = 0$ ($\because 0^2 + 0^2 = 0$ is divisible by 7)

except $r_1 = r_2 = 0$ for other values of r_1, r_2 we have $r_1^2 + r_2^2$ is not divisible by 7.

So, $a^2 + b^2$ is divisible by 7 only if $a, b \in A_0$ only

The number of favourable selections of a, b

such that $(a^2 + b^2)$ is divisible by 7 $= {}^nC_2$

$\therefore P_n =$ The probability of selecting a, b from the set $A = \{1, 2, 3, ..., 7n\}$ in such a way that $a^2 + b^2$ is divisible by 7.

$$= \frac{{}^nC_2}{{}^{7n}C_2} = \frac{n(n-1)}{7n(7n-1)} = \frac{n-1}{7(7n-1)}$$

$\therefore$ The probability of selecting a, b from set of natural numbers so that $a^2 + b^2$ is divisible by 7.

$$= \lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{n-1}{7(7n-1)} = \frac{1}{49}$$

13. Seven digits from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are written in random order. Find the probability that the seven digited number is divisible by 9.

Sol. Since we have to make seven different digited numbers using 1, 2, 3, 4, 5, 6, 7, 8, 9. We have to select and arrange 7 of these nine digits after omitting 2 digits.

These 2 numbers which we omit may be any two of these 9 numbers. It can be done in ${}^9C_2$ ways i.e., in 36 ways.

A seven digited number is divisible by 9 if sum of the seven digits is a multiple of 9.

We know $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$

The least sum of seven numbers from these nine is $1+2+3+4+5+6+7=28$

The maximum sum of seven digits from these nine

$=3+4+5+6+7+8+9=42$

So, $28 \leq$ sum of any 7 digits from the given nine digits ≤ 42.

So, we can conclude that a seven digited number using the numbers 1 to 9 which is divisible by 9 must have sum of their digits = 36

So, to make a seven digited number which is divisible by 9, we have to omit two numbers from 1 to 9 whose sum is 9 i.e., those two numbers must be

$\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$

So, among 36 possible ommissions, only 4 are favourable.

Required probability $= \dfrac{4}{36} = \dfrac{1}{9}$

14. If the sum of five natural numbers is 50. Find the probability that the five numbers are even.

Sol. Let the five natural numbers be x_1, x_2, x_3, x_4, x_5

Here $S = \{(x_1, x_2, x_3, x_4, x_5) / x_i \in N \text{ for } i = 1,2,3,4,5 \text{ and } x_1+x_2+x_3+x_4+x_5 = 50\}$

$n(S)$ = number of +ve integral solutions of $x_1+x_2+x_3+x_4+x_5 = 50$

We know that the number of +ve integral solutions of

$$x_1+x_2+x_3+x_4+x_5+.....+x_r = n \text{ is } {}^{n-1}C_{r-1}$$

Therefore $n(S) = {}^{n-1}C_{r-1}$

where $n = 50$ and $r = 5 = {}^{49}C_4$;

Let A be the event that the five numbers are even.

Here $A = \{(2n_1, 2n_2, 2n_3, 2n_4, 2n_5) / \text{each } n_i \in N \text{ and } 2n_1+2n_2+2n_3+2n_4+2n_5 = 50\}$

Here $n(A)$ = The number of +ve integral solutions of $2n_1+2n_2+2n_3+2n_4+2n_5=50$

= The number of +ve integral solutions of $n_1+n_2+n_3+n_4+n_5=25={}^{25-1}C_{5-1}={}^{24}C_4$

Therefore, $P(A) = \dfrac{{}^{24}C_4}{{}^{49}C_4} = \dfrac{24\times23\times22\times21}{49\times48\times47\times46} = \dfrac{33}{658}$

15. Out of $(4n+1)$ tickets numbered $m, m+1, m+2, ..., m+4n$, five tickets are chosen at random without replacement. Find the probability that these tickets are in A.P.

Sol. Since 5 tickets are drawn at random from $(4n+1)$ tickets.

$n(S) = {}^{4n+1}C_5$;

Let A be the event that these selected tickets are in AP. Consider an AP. of the given $(4n+1)$ numbers.

Let it be $a, a+k, a+2k, a+3k, a+4k$

Where $m \leq a < a+4k \leq m+4n \Rightarrow m \leq a \leq m+4n-4k$

So, the first term of an AP may take any value from m to $m+4n-4k$.

So there exists $(4n-4k+1)$ A.P's with common difference k.

Since the least value of a is m, the maximum value of k is n.

Therefore number of successes of A =

$$n(A) = \sum_{k=1}^{n}(4n-4k+1) = (4n+1)n - 4\left[\frac{n(n+1)}{2}\right]$$

$$= 4n^2+n-2n^2-2n = 2n^2-n$$

$P(A) = \dfrac{2n^2-n}{{}^{4n+1}C_5}$

16. **A special die with numbers 1, –1, 2, –2, 0 and 3 is thrown thrice. What is the probability that the total is 0.**

Sol. Since 3 dice are rolled,

$n(S) = 216$; (since the die has 6 different numbers)

Let A be the event of getting sum 0.

Number of successes to $A = n(A) =$ coefficeint of x^0 in $(x^1 + x^{-1} + x^2 + x^{-2} + x^0 + x^3)^3$

$= \text{coefficeint of } x^0 \text{ in } \left[x + \frac{1}{x} + x^2 + \frac{1}{x^2} + 1 + x^3\right]^3$

$= \text{coefficeint of } x^0 \text{ in } \left[\frac{x^3 + x + x^4 + 1 + x^2 + x^5}{x^2}\right]^3$

$= \text{coefficeint of } x^0 \text{ in } x^{-6}(1 + x + x^2 + x^3 + x^4 + x^5)^3$

$= \text{coefficeint of } x^6 \text{ in } \left(\frac{1-x^6}{1-x}\right)^3$

$= \text{coefficient of } x^6 \text{ in } (1-3x^6)\sum_{r=0}^{\infty} {}^{r+2}C_r x^r = {}^8C_6 - 3(1) = 28 - 3 = 25$

$P(A) = \frac{25}{216}$

17. **Five ordinary dice are rolled at random and sum of the numbers shown on them is 16. What is the probability that the numbers shown on each is any one from 2, 3, 4, 5.**

Sol. Since five dice are rolled, $n(S) = 6^5$.

Let A be the event of getting sum 16 ;

Let B be event of that the numbers

shown on each die is any one from 2, 3, 4, 5.

we have to find $P(B/A)$

number of favourables to A.

$= n(A) = \text{coefficient of } x^{16} \text{ in } (x^1 + x^2 + ... + x^6)^5$

$= \text{coefficient of } x^{11} \text{ in } (1 + x + ... + x^5)^5 = \text{coefficient of } x^{11} \text{ in } \left(\frac{1-x^6}{1-x}\right)^5$

$= \text{coefficient of } x^{11} \text{ in } (1-x^6)^5(1-x)^{-5} = \text{coefficient of } x^{11} \text{ in } [1-5x^6]\sum_{r=0}^{\infty} {}^{r+4}C_r x^r$

$= {}^{15}C_{11} - 5 \,.\, {}^9C_5 = {}^{15}C_4 - 5.{}^9C_4 = 735$

Here $(A \cap B)$ is the event of getting sum 16 with the digits 2, 3,4 ,5 only

$n(A \cap B) = \text{coefficient of } x^{16} \text{ in } (x^2 + x^3 + x^4 + x^5)^5$

$= \text{coefficient of } x^{16} \text{ in } x^{10}\ (1 + x + x^2 + x^3)^5$

$= \text{coefficient of } x^{16} \text{ in } x^{10}\left(\frac{1-x^4}{1-x}\right)^5$

$= \text{coefficient of } x^{16} \text{ in } x^{10}(1-x^4)^5(1-x)^{-5}$

$= \text{coefficient of } x^6 \text{ in } (1-5x^4)\sum_{r=0}^{\infty} {}^{r+4}C_r x^r$

$= {}^{10}C_6 - 5\,{}^6C_2 = 210 - 75 = 135$

$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{135}{735} = \frac{9}{49}$

18. A coin is tossed 20 times. Find the probability of getting atleast 12 consecutive heads.

Sol. Since 20 coins are tossed, $n(S) = 2^{20}$;

Let A be the event of getting atleast 12 consecutive heads. i.e., 12 or more consecutive heads.

Suppose H_k, T_k, A_k are the events of getting head, tail, either head or tail in k^{th} toss of coin respectively for $k = 1, 2, 3, ..., 20$.

Let us count the number of successes to A as explained below.

S.No.	favourable outcome to A	number of favourable outcomes to A
1.	$H_1H_2.....H_{12}A_{13}A_{14}...A_{20}$	$1^{12} \times 2^8 = 2^8$
2.	$T_1H_2H_3....H_{13}A_{14}A_{15}....A_{20}$	$1 \times 1^{12} \times 2^7 = 2^7$
3.	$A_1T_2H_3H_4....H_{14}A_{15}....A_{20}$	$2 \times 1 \times 1^{12} \times 2^6 = 2^7$
4.	$A_1A_2T_3H_4H_5...H_{15}A_{16}....A_{20}$	$2^2 \times 1 \times 1^{12} \times 2^5 = 2^7$
	....	....
	....	....
9.	$A_1A_2A_3.......A_7T_8H_9H_{10}....A_{20}$	$2^7 \times 1 \times 1^{12} = 2^7$

Here all the above 9 cases are exclusive cases

Total number of favourable outcomes to A

$= 2^8 + 8 \times 2^7 = 2^7(2 + 8) = 10 \times 2^7$

$$P(A) = \frac{10 \times 2^7}{2^{20}} = \frac{10}{2^{13}} = \frac{5}{2^{12}}$$

19. If 10 coins are tossed, find the probability that no two or more consecuitve heads occur.

Sol. Since 10 coins are tossed, $n(S) = 2^{10}$.

Let A be the event that no two or more consecutive heads occur.

Let us count the number of favourables to A having k heads and $(10 - k)$ tails in which no two heads are consecuitve.

First arrange $(10 - k)$ 'T' symbols in a row. It can be done in one way.

✓ T ✓ T ✓ T ✓ .... ✓ T ✓

We get $(10 - k + 1)$ gaps as shown above in which we can arrange k 'H' symbols. It can be done in ${}^{11-k}C_k$ ways.

Thus the number of out comes having k heads and $(10 - k)$ tails such that no two heads are consecutive is ${}^{11-k}C_k$.

Here $11 - k \geq k \Leftrightarrow 2k \leq 11 \Leftrightarrow k \leq 5.5 \Leftrightarrow k \leq 5$ ($\because$ k is integer)

So, the maximum value of k is 5;

i.e., for $6 \leq k \leq 10$, there will be no favour to A;

Total number of favourables for A

$$= \sum_{k=0}^{5} {}^{11-k}C_k = {}^{11}C_0 + {}^{10}C_1 + {}^{9}C_2 + {}^{8}C_3 + {}^{7}C_4 + {}^{6}C_5$$

$= 1 + 10 + 36 + 56 + 35 + 6 = 144$

$$P(A) = \frac{144}{2^{10}} = \frac{9}{64}$$

20. **Suppose two persons A and B each toss 11 coins and 10 coins respectively. Show that the probability that A gets more heads than B is $\frac{1}{2}$.**

Sol. By data, A, B each toss 11 coins and 10 coins respectivley.

Suppose E is the event, that A gets more heads than B.

Consider one favourable outcome of E, in which the coins tossed by A are showing m heads and $(11-m)$ tails. Whereas the coins tossed by B are showing n heads and $(10-n)$ tails where $m > n$.

If we replace each head with a tail and tail with a head in the above outcome we get another outcome which is not favourable outcome to E.

This is because in this new outcome the coins tossed by A are showing m tails and $(11-m)$ heads. Whereas the coins tossed by B are showing n tails and $(10-n)$ heads. But $m > n \Rightarrow -m < -n \Rightarrow 11-m < 11-n$

$\Rightarrow\ 11-m \le 10-n \Rightarrow\ 11-m \not> 10-n$

$\Rightarrow$ The number of heads tossed by A is not more than the number of heads tossed by B.

So, corresponding to each favourable outcome there is an unfavourable outcome to E in sample space and viceversa.

So, there is one-to-one correspondance between the sets E and E^C.

So, the number favourable outcomes to E = number of unfavourable outcomes to E.

$\therefore\ P(E) = \frac{1}{2}$

***21.** **A bag contains 5 white and 4 black balls. 3 balls are drawn and laid aside. Without noting their colour. Then one more ball is drawn. Find the probability that it is white.**

Sol. Let W_i be the event of getting 'i' white balls and $(3-i)$ black balls when 3 balls are drawn from bag at random for $i = 0, 1, 2, 3$

Here W_0, W_1, W_2, W_3 are exclusive and exhaustive

$P(W_0) = \frac{{}^5C_0 \times {}^4C_3}{{}^9C_3} = \frac{4}{84} = \frac{2}{42}$; $\quad P(W_1) = \frac{{}^5C_1 \times {}^4C_2}{{}^9C_3} = \frac{30}{84} = \frac{15}{42}$

$P(W_2) = \frac{{}^5C_2 \times {}^4C_1}{{}^9C_3} = \frac{40}{84} = \frac{20}{42}$; $\quad P(W_3) = \frac{{}^5C_3 \times {}^4C_0}{{}^9C_3} = \frac{10}{84} = \frac{5}{42}$

Let E be the event of selecting white ball in second draw.

$P(E/W_0) = 5/6$; $P(E/W_1) = 4/6$;

$P(E/W_2) = 3/6$; $P(E/W_3) = 2/6$

$P(E) = P(E \cap S) = P\left(E \cap \bigcup_{i=0}^{3} W_i\right)$ [$\because$ W_0, W_1, W_2, W_3 are exhaustive]

$= P\left[\bigcup_{i=0}^{3} (E \cap W_i)\right] = \sum_{i=0}^{3} P(E \cap W_i) = \sum_{i=0}^{3} P(W_i)\,P(E/W_i)$

$= P(W_0)\,P(E/W_0) + P(W_1)\,P(E/W_1) + P(W_2)\,P(E/W_2) + P(W_3)\,P(E/W_3)$

$= \frac{2}{42} \times \frac{5}{6} + \frac{15}{42} \times \frac{4}{6} + \frac{20}{42} \times \frac{3}{6} + \frac{5}{42} \times \frac{2}{6} = \frac{10+60+60+10}{42 \times 6} = \frac{140}{42 \times 6} = \frac{5}{9}$

***22. A bag contains 10 white and 15 black balls. The balls are drawn one at a time until only those of the same colour are left. Show that the probability that they are all black is 3/5.**

Sol. Let A_k denote the event that k black balls are left in the bag when the last white ball is drawn for $k = 1, 2, ..., 15$

That is A_k is the event that when $(25 - k)$ balls are drawn from bag one at a time then in first $(24 - k)$ draws 9 white balls and $(15 - k)$ black balls are drawn and in last $(25 - k)$ th draw a white ball is drawn.

$$P(A_k) = \frac{{}^{10}C_9 \times {}^{15}C_{15-k}}{{}^{25}C_{24-k}} \times \frac{1}{k+1} = \frac{10 \times {}^{15}C_k}{{}^{25}C_{k+1}} \times \frac{1}{k+1} \quad [\text{ since } {}^nC_r = {}^nC_{n-r}]$$

$$= 10 \times \frac{\lfloor 15}{\lfloor 15-k \lfloor k} \times \frac{\lfloor k+1 \times \lfloor 24-k}{\lfloor 25} \times \frac{1}{k+1}$$

$$= 10 \times \frac{\lfloor 15 \ \lfloor 24-k}{\lfloor 15-k \ \lfloor 25} = \frac{\lfloor 24-k}{\lfloor 15-k} \times \frac{10 \times \lfloor 15}{\lfloor 25} = \frac{\lfloor 24-k}{\lfloor 15-k \lfloor 9} \times \frac{9 \times 10 \lfloor 15}{\lfloor 25}$$

$$= {}^{24-k}C_9 \times \frac{\lfloor 10 \times \lfloor 15}{\lfloor 25} = \frac{{}^{24-k}C_9}{{}^{25}C_{10}}$$

Let A be the event that only black balls remain in the bag.

$$P(A) = P\left(\bigcup_{k=1}^{15} A_k\right) = \sum_{k=1}^{15} P(A_k) \quad [\because A_1, A_2, ..., A_{15} \text{ are exclusive}]$$

$$= \sum_{k=1}^{15} \frac{{}^{(24-k)}C_9}{{}^{25}C_{10}} = \frac{{}^{23}C_9 + {}^{22}C_9 + ... + {}^{9}C_9}{{}^{25}C_{10}}$$

$$= \frac{({}^{24}C_{10} - {}^{23}C_{10}) + ({}^{23}C_{10} - {}^{22}C_{10}) + ... + ({}^{11}C_{10} - {}^{10}C_{10}) + {}^{9}C_9}{{}^{25}C_{10}}$$

$$= \frac{{}^{24}C_{10}}{{}^{25}C_{10}} = \frac{\frac{\lfloor 24}{\lfloor 10 \lfloor 14}}{\frac{\lfloor 25}{\lfloor 10 \lfloor 15}} = \frac{15}{25} = \frac{3}{5}$$

Note :

Suppose a bag contains 'a' white and 'b' black balls. The balls are drawn one at a time until only those of the same colour are left. The ratio between the probabilities that only white balls left and only black balls left is $a : b$.

23. Eighteen rupee coins are distributed among 6 children at random in such a way that each child receives atleast one coin. Find the probability that the total number of coins received by first five children is atleast 8 and atmost 12.

Sol. Suppose the number of coins received by k th child is x_k where $k = 1, 2, 3, 4, 5, 6$ and $1 \le x_k \le 13$.

Here $S = \{(x_1, x_2, x_3, x_4, x_5, x_6) / \text{each } x_i \ge 1 \text{ and } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18\}$

$\therefore n(S) = {}^{n-1}C_{r-1}$ where $n = 18; r = 6 = {}^{17}C_5$

Let A be the event that $8 \le x_1 + x_2 + x_3 + x_4 + x_5 \le 12$

Here $A = \{(x_1, x_2, x_3, x_4, x_5, x_6) \,/\, \text{each } x_i \ge 1 \text{ and } 8 \le x_1 + x_2 + x_3 + x_4 + x_5 \le 12$ and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18\}$

Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 = p$ where $8 \le p \le 12$

The number of +ve integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = p$ is ${}^{p-1}C_4$

$\therefore$ $n(A)$ = The number of +ve integral solutions of $8 \le x_1 + x_2 + x_3 + x_4 + x_5 \le 12$

$$= \sum_{p=8}^{12} {}^{p-1}C_4 = {}^7C_4 + {}^8C_4 + {}^9C_4 + {}^{10}C_4 + {}^{11}C_4$$

$$= ({}^8C_5 - {}^7C_5) + ({}^9C_5 - {}^8C_5) + ({}^{10}C_5 - {}^9C_5) + ({}^{11}C_5 - {}^{10}C_5) + ({}^{12}C_5 - {}^{11}C_5)$$

$$= {}^{12}C_5 - {}^7C_5$$

$$P(A) = \frac{{}^{12}C_5 - {}^7C_5}{{}^{17}C_5} = \frac{771}{6188}$$

24. Let A, B, C be three events. If the probability of exactly one event of A and B is $1 - x$, out of B and C is $1 - 2x$ and out of A and C is $1 - x$. The probability that the three events occur simultaneously is x^2 then prove that the probability that atleast one out of A, B, C will occur is greater than $\frac{1}{2}$.

Sol. P(happening of exactly one event of A and B)

$= P[(A - B) \cup (B - A)] = P(A - B) + P(B - A)$ (since $A - B$ and $B - A$ are disjoint)

$= P[A - (A \cap B)] + P[B - (A \cap B)]$

$= P(A) + P(B) - 2(P \cap B)$ $[\because A \cap B \subseteq A; A \cap B \subseteq B]$

So, $P(A) + P(B) - 2P(A \cap B) = 1 - x$ ...... (1)

Similarly, $P(B) + P(C) - 2P(B \cap C) = 1 - 2x$ ...... (2)

$P(C) + P(A) - 2P(A \cap C) = 1 - x$ ...... (3)

$P(A \cap B \cap C) = x^2$

Adding (1), (2) and (3) we have,

$2P(A) + 2P(B) + 2P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) = 3 - 4x$

$$\therefore P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3 - 4x}{2}$$

Consider, $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{3 - 4x}{2} + x^2 = \frac{2x^2 - 4x + 3}{2} = \frac{2(x-1)^2 + 1}{2} \ge \frac{0 + 1}{2} = \frac{1}{2}$$

25. There are two balls in an urn whose colours are not known (each ball can be either white or black). A white ball is put in the urn. A ball is drawn from the urn. Find the probability that it is white.

Sol. Suppose A_k is the event that the urn contains k white balls and $(2 - k)$ black balls where $k = 0, 1, 2$

Here A_0, A_1, A_2 are exclusive and exhaustive. They are equiprobable.

$P(A_0) = P(A_1) = P(A_2) = 1/3$

Let A be the event of drawing a white ball

$P(A) = P(A \cap S) = P(A \cap (A_0 \cap A_1 \cap A_2) = \sum_{k=0}^{2} P(A \cap A_k)$

$= \sum_{k=0}^{2} P(A_k) \times P(A / A_k)$ ..... (1)

Here $P(A/A_0) = \frac{1}{3}$ (even though the urn contains no white ball at first but we are putting a white ball)

$P(A/A_1) = \frac{2}{3}$ $\quad P(A/A_2) = 1$ $\quad$ from (1) $\frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times 1 = \frac{2}{3}$

26. In a bag there are six balls of unknown colours. Three balls are drawn at random and found to be all black. Find the probability that the bag contains exactly 3 black balls.

Sol. Since the 3 balls drawn at random are black, we can easily understand that the bag contains atleast 3 black balls.

Suppose A_3, A_4, A_5, A_6 be the events that the bag contains exactly 3,4,5,6 black balls.

Let A be the event of drawing 3 black balls.

Here A_3, A_4, A_5, A_6 are exclusive and exhaustive.

$P(A_3) = P(A_4) = P(A_5) = P(A_6) = 1/4$

By Baye's theorem, $P(A_3/A) = \frac{P(A_3)P(A/A_3)}{\sum_{k=3}^{6} P(A_k)P(A/A_k)} = \frac{P(A/A_3)}{\sum_{k=3}^{6} P(A/A_k)}$ .... (1)

Here, $P(A/A_3) = \frac{^3C_3}{^6C_3}$; $P(A/A_4) = \frac{^4C_3}{^6C_3}$; $P(A/A_5) = \frac{^5C_3}{^6C_3}$; $P(A/A_6) = \frac{^6C_3}{^6C_3}$

from (1) $\quad P(A_3/A) = \frac{^3C_3}{^3C_3 + {^4C_3} + {^5C_3} + {^6C_3}} = \frac{1}{1+4+10+20} = \frac{1}{35}$

27. A man has 3 coins *A, B, C*. The coin *A* is unbiased. The probability that a head will show when *B* is tossed is 2/3. While it is 1/3 in case of the coin *C*. A coin is chosen at random and tossed 3 times giving 2 heads and one tail. Find the probability that the coin *A* was chosen.

Sol. Suppose A, B, C are the events of choosing the coins A, B, C respectively.

Here A, B, C are exclusive and exhaustive.

$P(A) = P(B) = P(C) = 1/3$;

Let E be the event of getting 2 heads and one tail when the selected coin tossed 3 times.

According to Baye's theorem,

$$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B) + P(C)P(E/C)}$$

$$= \frac{P(E/A)}{P(E/A) + P(E/B) + P(E/C)} \quad .... (1) \quad (\because P(A) = P(B) = P(C) = 1/3)$$

$P(E/A) = P$(getting 2 heads and one tail when coin A is tossed 3 times)

$= \frac{^3C_2}{2^3} = \frac{3}{8}$ ($\because$ A is unbiased)

$P(E/B) = P$(getting 2 heads and one tail when coin B is tossed 3 times)

$${}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)=\frac{12}{27}$$

[$\because$ Probability of getting head when B is tossed in 2/3]

$P(E/C) = P$(getting 2 heads and one tail when coin C is tossed 3 times)

$${}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)=\frac{6}{27}$$

[$\because$ Probability of getting head when C is tossed in 1/3]

from (1), $P(A/E)=\dfrac{\frac{3}{8}}{\frac{3}{8}+\frac{12}{27}+\frac{6}{27}}=\dfrac{\frac{3}{8}}{\frac{3}{8}+\frac{2}{3}}=\dfrac{\frac{3}{8}}{\frac{25}{24}}=\dfrac{9}{25}$

28. *A* and *B* are two independent witnesses in a case. The probability that *A* will speak truth is 3/5 and the probability that *B* will speak truth is 1/4. *A* and *B* agree in a certain statement. Find the probability that the statement is true.

Sol. Let E_1, E_2, E_3, E_4 be the events that both A and B speak truth, both A and B tell a lie, only A speaks truth and only B speaks truth respectively.

Here E_1, E_2, E_3, E_4 are exclusive and exhaustive.

$$P(E_1)=\frac{3}{5}\times\frac{1}{4}=\frac{3}{20};\quad P(E_2)=\left(1-\frac{3}{5}\right)\left(1-\frac{1}{4}\right)=\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$$

$$P(E_3)=\frac{3}{5}\left(1-\frac{1}{4}\right)=\frac{9}{20};\quad P(E_4)=\left(1-\frac{3}{5}\right)\times\frac{1}{4}=\frac{2}{20}$$

Let E be the event that both agree a statement

By Baye's throem, $P(E_1/E)=\dfrac{P(E_1)P(E/E_1)}{\sum_{k=1}^{4}P(E_k)P(E/E_k)}$ .... (1)

Note that if both speak truth or if both tell lie then the event that both agree a statement becomes certain event. Otherwise it will be impossible event.

i.e., $P(E/E_1) = P(E/E_2) = 1 \qquad P(E/E_3) = P(E/E_4) = 0$

from (1)

$$P(E_1/E)=\frac{\frac{3}{20}\times 1}{\frac{3}{20}\times 1+\frac{6}{20}\times 1+\frac{9}{20}\times 0+\frac{2}{20}\times 0}=\frac{3}{9}=\frac{1}{3}$$