CHAPTER 4

Calculus

DIFFERENTIAL EQUATIONS

• ORDER AND DEGREE • FORMATION OF DIFFERENTIAL EQUATIONS •
• VARIABLES SEPARABLE TYPE •
• HOMOGENEOUS DIFFERENTIAL EUQATIONS •
• NON – HOMOGENEOUS DIFFERENTIAL EQUATIONS •
• LINEAR DIFFERENTIAL EQUATIONS •

4.0 INTRODUCTION

An equation involving derivatives or differentials is called a differential equation (abbreviated as D.E). The theory of differnetial equations is a vastly developed subject and it has applications in various fields of science like Physics (Dynamics, Thermodynamics, Heat, Fluid Mechanics, Electro Magnetism etc.,), Chemistry (rates of chemical reactions, Physical chemistry, radio active decay, etc.,). Biology (Bacteria growth rate, growth rates of plants and various organisms) and Economics (Economic growth rate, population growth rate, etc.,). In fact, almost all the modern scientific investigations involve differential equations.

In this chapter we study about the formation of ordinary differential equations, their order, degree and methods of solving various types of first order ordinary differential equations.

4.1 DIFFERENTIAL EQUATIONS

An equation involving one dependent variable, one or more independent variables and the derivatives of the dependent variable with respect to independent variables or differnetials of the dependent and independent variables, is called a differential equation

Example : i) $\frac{dy}{dx} = 6$ ii) $\frac{dy}{dx} = 3x + 4y$ iii) $(e^x + 1)ydy + (y+1)dx = 0$

iv) $\left(\frac{dy}{dx}\right)^2 + 3\frac{dy}{dx} + 2 = 0$ v) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = xe^{2x}$ vi) $\frac{dx}{dt} = ke^{\alpha t}$

vii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ viii) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

are all differential equations. Differential equations are mainly of two types.

i) **Ordinary differential equations (O.D.E's)**

ii) **Partial differential equations (P.D.E's)**

A differential equation is said to be an ordinary differential equation if the derivatives of the dependent variable are with respect to only one independent variable. If the derivatives of the dependent variable in the equation are with respect to two or more independent varaibles then it is said to be a partial differential equation.

In the above examples: Equations (i) to (vi) are ordinary differential equations and (vii) and (viii) are partial differntial equations.

In this chapter we study about ordinary differential equations only. Henceforth all the concepts, definitions and theoretical developments pertain to ordinary differential equations.

4.2 ORDER AND DEGREE

Definition

The order of an ordinary differential equation is defined to be the order of the highest order derivative occuring in the equation

Definition

The degree of an ordinary differential equation is defined to be the degree (exponent) of the highest order derivative occuring in the equation provided the equation is made free of radical signs and fractional powers as far as the derivatives are concerned (i.e., when the equation is expressed as a polynomial equation of the derivatives).

If it is possible to express a differential equation as a polynomial equation in the derivatives - an equation such that the number of terms in it is finite and all the derivatives involved in the equation have positive integral exponents - then the degree of the differential equation exists and is defined to be the positive integral exponent of the highest order derivative. Otherwise the degree of the differential equation is not defined.

Every differential equation has a certain order but need not have a degree

Example : i) $\frac{dy}{dx}=10$ is a D.E of 1st order and 1st degree

ii) $\left(\frac{dy}{dx}\right)^2+3\frac{dy}{dx}+2=0$ is a D.E of 1st order and2nd degree.

iii) $\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^3+y^4=e^x$ is a D.E of order 2 and degree 1

iv) $\frac{d^2y}{dx^2}=2\left(1+\frac{dy}{dx}\right)^{3/2}$ is a D.E of order 2 and degree2

($\because$ this equation can be expressed as $\left(\frac{d^2y}{dx^2}\right)^2=4\left(1+\frac{dy}{dx}\right)^3$)

v) $\left(\frac{d^3y}{dx^3}\right)^{2/3}=2\frac{d^2y}{dx^2}$ is a D.E of order 3 and degree 2.

($\because$ this equation can be expressed as $\left(\frac{d^3y}{dx^3}\right)^2=8\left(\frac{d^2y}{dx^2}\right)^3$)

vi) $\sqrt{x}\left(\frac{d^2y}{dx^2}\right)^{1/3}+x\frac{dy}{dx}+y=0$ is a D.E of order 2 and degree1.

($\because$ this equation can be expressed as $x^{3/2}\left(\frac{d^2y}{dx^2}\right)+\left(x\frac{dy}{dx}+y\right)^3=0$

vii) $\sin\left(\frac{dy}{dx}\right)=x$ is a D.E of order 1 and degree 1.

($\because$ this equation can be expressed as $\frac{dy}{dx}=\sin^{-1}x$)

viii) $\left(\frac{dy}{dx}\right)^{2/3}=x^2$ is a D.E of order 1 and degree 2.

($\because$ the equation can be written as $\left(\frac{dy}{dx}\right)^2=x^6$ but not as $\frac{dy}{dx}=x^3$ because the first form leads to $\left(\frac{dy}{dx}-x^3\right)\left(\frac{dy}{dx}+x^3\right)=0$ which gives two differential equations of Ist order and Ist degree).

ix) $\left(\frac{dy}{dx}\right)^{3/2} = x$ is a D.E of order 1 and degree 1. (This equation can be written as $\left(\frac{dy}{dx}\right)^{3} = x^2$ which leads to the conclusion that it is a D.E of order 1 and degree 3. There is nothing wrong in this conclusion by a beginner. How- ever, the factori-zation of the above form leads to only one differential equation with real coefficients viz., $\frac{dy}{dx} = x^{2/3}$. Hence the conclusion or-der 1 and degree1).

x) $x\frac{dy}{dx} = \log\left(1+\frac{dy}{dx}\right)$ is a D.E of order 1 and the degree is not defined.

($\because$ This equation can not be expressed as a polynomial equation in $\frac{dy}{dx}$)

4.3 FORMULATION OF DE'S

We discuss the formation of differential equations through some illustrations and proceed to the general case.

Consider a single parameter family of curves (straight lines) $y = mx + 6$ -- (1) where m is a parameter (or an orbitrary constant). The differential equation of this family of curves can be formed by eliminating the parameter m from equation (1) through the process of differentiation. Differentiating (1) w.r.t x,

we get, $\frac{dy}{dx} = m$ -- (2)

Eliminating m from (1) and (2) (by substituting the value of m from (2) in (1)

we get $y = x\frac{dy}{dx} + 6$ (or) $x\frac{dy}{dx} - y + 6 = 0$ -- (3)

This is the required differential equation of the family of curves (1).

It should be noted that both the equations (1) and (3) represent the same family of curves but the equation (1) is devoid of derivatives and the equation (3) is devoid of arbitrary constants.

As a second illustration, consider a two parameter family of curves

$$y = ae^x + be^{-x} \quad \text{-- (4)}$$

where a and b are parameters (or arbitrary constants). By eliminating these two parameters, we get the differential equations of this family of curves.

Since there are two parameters we need two more equations [in addition to equation (1)] for the elimination. These two equations, obtained by differentiating (4) two times successively, are

$$\frac{dy}{dx} = ae^x - be^{-x} \quad \text{-- (5)}$$

$$\frac{d^2y}{dx^2} = ae^x + be^{-x} \quad \text{-- (6)}$$

Eliminating a and b (using (1) in (3)), we find the D.E of the family of curves (4) as $\frac{d^2y}{dx^2} = y$

General Case:

Consider an n-parameter family of plane curves : $f(x, y, c_1, c_2, ..., c_n) = 0$ -- (1) where c_1, c_2...... c_n are n independent orbitrary constants (or parameters). By eliminating these n parameters, using differentiation, we find the differential equation of the family (1).

In order to eliminate the n parameters we need (n + 1) equations (i.e., n more equations in addition to equation (1)). The additional n equations are obtained by differentiating (1) n times successively. These equations are

$$f_1\left(x, y, \frac{dy}{dx}, c_1, c_2, ...c_n\right) = 0 \qquad \text{-- (2)}$$

$$f_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, c_1, c_2, ...c_n\right) = 0 \qquad \text{-- (3)}$$

$$f_n\left(x, y, \frac{dy}{dx}, ... \frac{d^ny}{dx^n}, c_1, ...c_n\right) = 0 \qquad \text{---(n+1)}$$

By eliminating the n orbitrary constants from the above (n + 1) simultaneous equations, we get an equation of the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \text{ -- (1) or equivalently } F(x, y, y_1, y_2, ..., y_n) = 0 \quad \text{--(1')}$$

This equation, which is devoid of any of the n arbitrary constants, is called the differential equation of the family of curves (1) and it is a D.E of order n. The above process (of finding equation (1') from equation (1) is called the formation of a differential equation.

It should be observed that if the equation of family of curves contains n independent arbitrary constants, then their elimination leads to an n^{th} order differential equation.

The general form of an n^{th} order differential equation in x and y can be taken as $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0$

A note on independent arbitrary constants:

In the equation of a given family of curves, if it is possible to combine two or more arbitrary constants so that the resulting combination can be considered as a single arbitrary constant then the arbitrary constants of the combination are said to be dependent, other wise they are said to be independent. For example, if c_1, c_2, c_3.......... are arbitrary constants then

i) in each of the combinations $c_1 + c_2 - c_3, 2c_1 + 3c_2c_3, c_1 \sin(c_2 - c_3), (c_1 + c_2) e^{c_3}$, $c_1 \log\left(\frac{c_2 + c_3}{c_1}\right)$...$c_1, c_2, c_3$ are dependent, since each combination can be treated as a single arbitrary constant

ii) In each of the combinations
$c_1 + c_2e^x + c_3\sin x$, $c_1 \cos(x + c_2) + e^{c_3x}$, $c_1e^{kx} + c_2e^{-kx} + c_3x^2$. c_1, c_2, c_3 are independent since it is not possible to combine any one of them with any of the others.

However, In the expression $c_1 \cos(x + c_2)$ even through c_1 and c_2 can be combined, the resulting combinations give rise to two independent arbitrary constants.

$$c_1 \cos(x + c_2) = c_1 \cos x \cos c_2 - c_1 \sin x \sin c_2 = (c_1 \cos c_2)\cos x + (-c_1 \sin c_2)\sin x$$

$= A\cos x + B\sin x$ where $A = c_1 \cos c_2$, $B = -c_1 \sin c_2$ are clearly two independent arbitrary constants.

The number of independent arbitrary constants in the equation of a family of curves decides the order of the differential equation of that family.

Therefore, one should ***not*** jump to the conclusion that the number of arbitrary constants (by direct count) is equal to the order of the differential equation. We have to first check whether all the arbitrary constants of the given equation are independent. For example, consider the family of curves

$$y=(c_1+c_2)\sin(x+c_3)+c_4e^{x+c_5} \quad \text{-- (1)}$$

where $c_1, c_2, \ldots\ldots c_5$ are arbitrary constants. Since there are 5 arbitrary constants (by direct count), we ***should not*** conclude that the order of the differential equation of this family is 5 as not all of them are independent. Rewriting the equation (1) as

$$y=(c_1+c_2)\sin(x+c_3)+(c_4.e^{c_5})\,e^x \quad \text{-- (2)}$$

we observe that the combinations c_1+c_2, $c_4.e^{c_5}$ can be considered as single arbitrary constants say A and B respectively. Then A, B and c_3 are independent and the equation (2) can, now be written as $y=A\sin(x+c_3)+Be^x$.

Hence the order of the differential equation of the family of curves (1) is only 3 and not 5.

4.4 SOLUTION OF A D.E

Definition

A relation between the dependent and independent variables without involving any derivatives is called a solution or integral of the differential equation if the relation and the derivatives obtained there from satisfy the differential equation.

Example : i) $y=e^x+e^{-x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2}=y$.

ii) $y=a\sin 2x+b\cos 2x$ where a, b are arbi-trary constants is a solution of $y_2+4y=0$.

Definition (General Solution)

A relation $\phi(x,y,c_1,c_2\ldots,c_n)=0$ where $c_1, c_2,\ldots,c_n$ are n independent arbitrary constants is called a general solution of an n^{th} order differential equations $F\left(x,y,\dfrac{dy}{dx},\ldots,\dfrac{d^ny}{dx^n}\right)=0$ if the relation and the derivatives obtained therefrom satisfy the differential equation.

Given the differential equation $F\left(x,y,\dfrac{dy}{dx},\ldots,\dfrac{d^ny}{dx^n}\right)=0$, the process of finding the general solution $\phi(x,y,c_1,c_2,\ldots\ldots,c_n)=0$ where $c_1, c_2,\ldots\ldots c_n$ are n arbitrary constants, through the process of integration, is called solving a differential equation.

It is important to note that *the number of independent arbitrary constants in the general solution of a differential equation is equal to the order of the differential equation*.

Example : i) $y=ae^{2x}+be^{2x}$ where a,b are arbitrary constants is the general solution of $y_2=4y$.

ii) $y=(a+bx)e^x$ where a,b are arbi-trary constants in the general solution of $y_2-2y_1+y=0$.

Definition (Particular Solution)

A solution of a differential equation is called a particular solution if it is obtained from the general solution of the equation by giving particular values to the arbitrary constants.

If $\phi(x, y, c_1, c_2, ..., c_n) = 0$ in the general solution of the differential equation $F\left(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}\right) = 0$ then $\phi(x, y, k_1, k_2, ...k_n) = 0$ where $k_1, k_2, ..., k_n$ are fixed values given to the arbitrary constants $c_1, c_2, ..., c_n$, is a particular solutions of the differential equation. Evidently there exist infinitely many particular solution for a differential equation having general solution.

Definition (Singular Solution)

A solution of the differential equation is called a singular solution if it does not involve any arbitrary constants and it can not be obtained from the general solution of the equation by giving any values to the arbitrary constants.

Example : For the differential equation $x\left(\frac{dy}{dx}\right)^3 - y\left(\frac{dy}{dx}\right)^2 = 1$, $y = cx - \frac{1}{c^2}$ where c is an arbitrary constant is the general solution and the relation $y^3 = -\frac{27}{4}x^2$ is the singular solution.

4.5 FORMALATION OF A D.E'S SOME STANDARD MODELS

1) The differential equation of $y = c_1 \cos mx + c_2 \sin mx$ where c_1 and c_2 are arbitrary constants, is $y_2 + m^2 y = 0$

Proof : The given family of curves is

$$y = c_1 \cos mx + c_2 \sin mx \qquad \text{-- (1)}$$

where c_1 and c_2 are arbitrary constants . Differentiating (1) w.r.t x, twice successively, we get

$$y_1 = m(-c_1 \sin mx + c_2 \cos mx) \qquad \text{-- (2)}$$

$$y_1 = m(-c_1 \sin mx + c_2 \cos mx) \qquad \text{-- (3)}$$

Eliminating c_1 and c_2 using (1) in equation (3) we get $y_2 = -m^2 y$

$\therefore$ The differential equation of the given family is $y_2 + m^2 y = 0$

For example, the differential equation of $y = A \cos 2x + B \sin 2x$ where A and B are arbitrary constants is $y_2 + 4y = 0$

2) The differential equation of $y = c_1 e^{\alpha x} + c_2 e^{\beta x}$ where c_1, c_2 are arbitrary constants is $y_2 - (\alpha + \beta) y_1 + \alpha\beta y = 0$

Proof : The given family of curves is

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} \qquad \text{-- (1)}$$

where c_1 and c_2 are arbitrary constants. Differential (1) w.r.t x, we get

$$y_1 = c_1\alpha e^{\alpha x} + c_2\beta e^{\beta x} \quad \text{-- (2)}$$

Differentiating (2) w.r.t x, we get

$$y_2 = c_1\alpha^2 e^{\alpha x} + c_2\beta^2 e^{\beta x} \quad \text{-- (3)}$$

Eliminating c_1, c_2 from the equations (1), (2) and (3) the eliminant is

$$\begin{vmatrix} e^{\alpha x} & e^{\beta x} & -y \\ \alpha e^{\alpha x} & \beta e^{\beta x} & -y_1 \\ \alpha^2 e^{\alpha x} & \beta^2 e^{\beta x} & -y_2 \end{vmatrix} = 0 \Rightarrow -e^{\alpha x} e^{\beta x} \begin{vmatrix} 1 & 1 & y \\ \alpha & \beta & y_1 \\ \alpha^2 & \beta^2 & y_2 \end{vmatrix} = 0$$

$$\Rightarrow y_2(\beta - \alpha) - y_1(\beta^2 - \alpha^2) + y(\alpha\beta^2 - \alpha^2\beta) = 0$$

$$\Rightarrow y_2 - (\alpha + \beta)y_1 + \alpha\beta.y = 0$$

For example, the differential equation of $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants is $y_2 + y_1 - 6y = 0$.

3) The differential equation of the family of curves $y = (c_1 + c_2 x)e^{\alpha x}$ where c_1 and c_2 are arbitrary constants is $y_2 - 2\alpha y_1 + \alpha^2 y = 0$.

Proof : The given family of curves is $y = (c_1 + c_2 x)e^{\alpha x}$ -- (1)

Differentiating (1) w.r.t x, $y_1 = (c_1 + c_2 x)\alpha e^{\alpha x} + c_2 e^{\alpha x} C$

$$\Rightarrow y_1 = \alpha y + c_2 e^{\alpha x} \quad \text{(using (1))}$$

$$\Rightarrow y_1 - \alpha y = c_2 e^{\alpha x} \quad \text{-- (2)}$$

Differentiating (2) w.r.t x,

$$y_2 - \alpha y_1 = c_2 \alpha e^{\alpha x} = \alpha(y_1 - \alpha y) \quad \text{(using (2))} \Rightarrow y_2 - 2\alpha y_1 + \alpha^2 y = 0$$

For example, the differential equation of $y = (A + Bx)e^{3x}$ where A and A are arbitrary constants is $y_2 - 6y_1 + 9y = 0$.

Note : *The above method is known as Eliminant Method.*

Corollary :

The differential equations of $y = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x}$ where c_1, c_2, c_3 are arbitrary constants is $y_3 - (\alpha + \beta + \gamma)y_2 + (\alpha\beta + \beta\gamma + \gamma\alpha)y_1 - \alpha\beta\gamma.y = 0$

For example, the differential equation of $y = Ae^{x} + Be^{2x} + Ce^{3x}$ where A, B, C are arbitrary constants is $y_3 - 6y_2 + 11y_1 - 6y = 0$.

4) The differential equation of the family of curves $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$ Where c_1 and c_2 are arbitrary constants, is $y_2 - 2\alpha y_1 + (\alpha^2 + \beta^2)y = 0$

Proof : The given family of curves is $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$ -- (1)

Differentiating (1) w.r.t x, $y_2 = \alpha e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + \beta e^{\alpha x}(-c_1 \sin\beta x + c_2 \cos\beta x)$

$\Rightarrow y_1 = \alpha y + \beta e^{\alpha x}(-c_1 \sin\beta x + c_2 \cos\beta x)$ (using (1))

$\Rightarrow y_1 - \alpha y = -\beta e^{\alpha x}(c_1 \sin\beta x - c_2 \cos\beta x)$ -- (2)

Differentiate (2) w.r.t x,

$y_2 - \alpha y_1 = -\beta\alpha e^{\alpha x}(c_1 \sin\beta x - c_2 \cos\beta x) \ -\beta^2 e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$\Rightarrow y_2 - \alpha y_1 = +\alpha(y_1 - \alpha y) - \beta^2 y$ (using (2) and (1))

$\Rightarrow y_2 - 2\alpha y_1 + (\alpha^2 + \beta^2)y = 0$

5) The differential equation of the family of curves $y = c_1 x^m + c_2 x^n$ where c_1, c_2 are parameters and $m \neq n$, is $x^2 y_2 - (m+n-1)xy_1 + mny = 0$

Proof : The given family of curves is $y = c_1 x^m + c_2 x^n$ -- (1)

Differentiating (1) constant x, $y_1 = c_1(mx^{m-1}) + c_2(nx^{n-1})$ -- (2)

Differentiating (2) w.r.t x, $y_2 = c_1[m(m-1)x^{m-2}] + c_2[n(n-1)x^{n-2}]$ -- (3)

Eliminating c_1 & c_2 from the equations (1), (2) and (3) the eliminant is

$$\begin{vmatrix} x^m & x^n & -y \\ mx^{m-1} & nx^{n-1} & -y_1 \\ m(m-1)x^{m-2} & n(n-1)x^{n-2} & -y_2 \end{vmatrix} = 0 \Rightarrow -x^{m-2}x^{n-2}\begin{vmatrix} x^2 & x^2 & y \\ mx & nx & y_1 \\ m(m-1) & n(n-1) & y_2 \end{vmatrix} = 0$$

$\Rightarrow y_2(nx^3 - mx^3) - y_1[n(n-1)x^2 - m(m-1)x^2] + y[mn(n-1)x - mn(m-1)x] = 0$

$\Rightarrow \{x^2 y_2 - xy_1[n+m-1] + mny\}(n-m)x = 0 \Rightarrow x^2 y_2 - (m+n-1)xy_1 + mny = 0$

For example, the differential equation of the family of curves $y = Ax^2 + Bx^3$ where A, B are arbitrary constants is $x^2 y_2 - 4xy_1 + 6y = 0$.

6) The differential equation of the family of circles of constant radius r given by $(x-a)^2 + (y-b)^2 = r^2$ where a and b are parameters is $r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$.

Proof : The given family of circles is $(x-a)^2+(y-b)^2 = r^2$ -- (1)

where a,b are parameters and r is a fixed constant.

Differentiating (1) w.r.t x, we get $(x-a) + (y-b)\frac{dy}{dx} = 0$ -- (2)

Differentiating (2) w.r.t x $\quad 1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\left(\frac{d^2y}{dx^2}\right) = 0$

$$\Rightarrow (y-b) = -\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\left(\frac{d^2y}{dx^2}\right)} \quad \text{-- (3)}$$

Eliminating a from (1) & (2) we have $(y-b)^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=r^2$ -- (4)

Eliminating b from (3) and (4) we get $r^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

This is the differential equation of the family (1)

7) The differential equation of the family of parabolas each of which has constant latus rectum $4a$ and whose axes are parallel to the X-axis, is $2ay_2+y_1^3=0$

Proof : The given family of parabola is $(y-\beta)^2=4a(x-\alpha)$ -- (1)

where $\alpha \,\&\, \beta$ are parameters

Differentiating (1) w.r.t x, we get $(y-\beta)\frac{dy}{dx}=2a$ -- (2)

Differentiating (2) w.r.t x, we get $(y-\beta)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$ -- (3)

Eliminating β from (2) & (3) $2a\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^3=0$

This is the required differential equation.

8) The differential equations of the central conics whose axes coincide with the axes of coordinates is $xy.y_2+x(y_1)^2-yy_1=0$.

Proof : The equation of the central conic whose axes coincde with the coordinate axes is given by $Ax^2+By^2=1$ -- (1)

where A & B are parameters

Differentiating (1) w.r.t x, we get $Ax+By.y_1=0$ -- (2)

Differentiating (2) w.r.t x, we get $A+B(yy_2+y_1^2)=0$ -- (3)

Eliminating A and B from equations (1), (2) & (3) we get $\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy_1 & 0 \\ 1 & yy_2+y_1^2 & 0 \end{vmatrix}=0$

$\Rightarrow xy.y_2+xy_1^2-yy_1=0$ which is the required differential equation.

Note

The differential equation of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the family of hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where a & b are arbitrary constants, is $xy.y_2+xy_1^2-yy_1=0$.

EXERCISE - 4.1

1. Find the order and degree of the following D.E's

*a) $x\left(\frac{dy}{dx}\right)^2+7y=0$ b) $\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+5y=0$ c) $\left[\left(\frac{dy}{dx}\right)^{1/2}+\left(\frac{d^2y}{dx^2}\right)^{1/3}\right]^{1/4}=0$

d) $2\frac{d^2y}{dx^2}=\left(5+\frac{dy}{dx}\right)^{5/3}$ e) $1+\left(\frac{d^2y}{dx^2}\right)^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ **f) $\left(\frac{d^3y}{dx^3}\right)^2-3\left(\frac{dy}{dx}\right)^5-e^x=4$

g) $\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right]^{6/5}=6y$ h) $\left[\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}\right]^{7/3}=\frac{d^3y}{dx^3}$ i) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\log\left(\frac{dy}{dx}\right)$

j) $\sin\left(\frac{dy}{dx}\right)=x+\frac{dy}{dx}$ k) $(y_2)^{3/2}-(y_1)^{1/2}-4=0$

*l) $x^{1/2}\left[\frac{d^2y}{dx^2}\right]^{1/3}+x\frac{dy}{dx}+y=0$ **(March-18)**

2. Find the order of the differential equation corresponding to

*a) $y=c(x-c)^2$ where c is an arbitrary constant.

b) $y=Ae^x+Be^{-x}+Ce^{5x}$ where A, B, C are arbitrary constants.

*c) $xy=ce^x+be^{-x}+x^2$ where b, c are arbitrary constants.

d) The family of all circles in the xy - plane with centre at the origin. **(March-2017)

e) $y=(c_1+c_2)e^{x+c_3}+c_4c_5\cos(x+c_6)$ where $c_1, c_2, \ldots, c_6$ are arbitrary constants.

3. Form the differential equations corresponding to the family of curves.

* a) $y=c(x-2c)$ where c is a parameter. **(March-19)**

b) $y=cx+c-c^2$ where c is a parameter.

**c) $y = a\cos 3x + b\sin 3x$ where a, b are parameters.

d) $y = a\cos x + b\sin x$ where a, b are parameters.

e) $y = ae^{x}+be^{-x}$ where a, b are parameters.

4. Form the differential equations of the following families of curves by eliminating the parameters (arbitrary constants) given against them in the brackets.

*a) $y=c(x-c)^2, (c)$ b) $xy=ae^x+be^{-x}, (a,b)$ c) $y=(a+bx)e^{kx}, (a,b)$

**d) $y=a\cos(nx+b), (a,b)$ (May-2017) *e) $y=ae^{3x}+be^{4x}, (a,b)$

*f) $y=ax^2+bx, (a,b)$ *g) $xy=ax^2+\frac{b}{x}, (a,b)$ h) $ax^2+by^2=1, (a,b)$

*5 Find the differential equation of the family of circles

a) touching the y-axis at the origin

b) having centres on the y-axis and passing through the origin.

*6. Obtain the differential equation of the family of parabolas having their focus at the origin and the axis along the x-axis.

*7. Obtain the differential equation of the family of parabolas each of which has latus rectum $4a$ and whose axes are parallel to x-axis.

*8. Find the differential equation of the family of ellipse having centres at the origin and axes along the coordinate axes.

*9. Find the D.E of the family of rectangular hyperbolas which have the coordinate axes as asymptotes.

10. Find the differential equation of the family of circles of fixed radius r and having their centres on the x-axis.

4.6 D.E'S OF FIRST ORDER AND FIRST DEGREE

The general form of a first order (and any degree) differential equation in x and y can be taken as $F\left(x, y, \frac{dy}{dx}\right) = 0.$

The general form of a first order and first degree differential equations in x and y can be taken as $\frac{dy}{dx} = \frac{x}{\sin y + y\cos y}$.

A first order first degree differential equation may or may not have a solution. Not all the first order differential equations are solvable. Based on the nature of the equations and the method of solution, the first order first degree differential equations are classified mainly into the following five types.

1) D.E's of variables separable type
2) Homogeneous differential equations
3) Non - homogeneous differential equations
4) Linear differential equations
5) Exact differential equations (Not in the syllabus)

4.7 D.E'S OF VARIABLE SEPERABLE TYPE

A first order first degree differential equation $\frac{dy}{dx} = f(x, y)$ -- (1)

is said to be of variables seperable type if f(x, y) can be expressed as a product (or quotient) of two explicit functions $f_1(x)$ ***and*** $f_2(y)$ ***in the variables x and y respectively. i.e., if equation (1) can be expressed as*** $\frac{dy}{dx} = f_1(x)f_2(y)$ ***or*** $\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)}$ ***or*** $\frac{dy}{dx} = \frac{f_2(y)}{f_1(x)}$ ***or*** $f_1(x)dx + f_2(y)dy = 0.$

To Find the General Solution

Let us suppose that the equation (1) can be expressed as $\frac{dy}{dx}=\frac{f_1(x)}{f_2(y)}$. *Seperating the variables, we get* $f_1(x)dx = f_2(y)dy$.

Integrating both sides, we get $\int f_1(x)dx = \int f_2(y)dy + c$, *c is an arbitrary constant.*

In evaluating the integrals we get the general solution of (1) in the form $F_1(x) = F_2(y) + c$

Example : i) $\frac{dy}{dx}=\frac{x}{y}$ is a differential equation of variables seperable type. Seperating the variables we get $ydy = xdx$

On integration,we get $\int ydy = \int xdx + c \Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c$

$\therefore$ The general solutions of the given differential equations in $y^2 = x^2 + 2c$ (or) $y^2 = x^2 + c$ where $2c$ is replaced with a new arbitrary constant C.

ii) $(1+x^2)\,dy + (1+y^2)\,dx = 0$ is a D.E of variables seperable type, since it can be written as

$\frac{1}{1+y^2}dy+\frac{1}{1+x^2}dx=0$ (variables are seperated)

On integration, we get $\int\frac{1}{1+y^2}dy+\int\frac{1}{1+x^2}dx=c \Rightarrow \tan^{-1}y+\tan^{-1}x=c$

Then is the general solution of the given D.E

SOLVED EXAMPLES

***1. Solve the following differential equations**

***(i)** $y^2 - x\frac{dy}{dx}=a\left(y+\frac{dy}{dx}\right)$ ***(ii)** $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$

Sol. i) $y^2 - x\frac{dy}{dx}=a\left(y+\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx}=\frac{y^2-ay}{x+a}$

Seperating the variables and integrating, we get $\int\frac{1}{y^2-ay}dy=\int\frac{1}{x+a}dx$

$\Rightarrow \frac{1}{a}\left[\log(y-a)-\log y\right]=\log(x+a)+\log c$

$\Rightarrow \left(\frac{y-a}{y}\right)^{\frac{1}{a}} = c(x+a)$ (or) $y-a=c.y(x+a)^a$

ii) $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$

$\Rightarrow \frac{\sqrt{1+x^2}}{x}dx+\int\frac{y}{\sqrt{1+y^2}}dy=0 \Rightarrow \int\frac{\sqrt{1+x^2}}{x}dx+\int\frac{y}{\sqrt{1+y^2}}dy=c$ --- (1)

To evaluate $\int\frac{\sqrt{1+x^2}}{x}dx$, Put $\sqrt{1+x^2}=t$ so that $x=\sqrt{t^2-1}$

then $dx=\frac{tdt}{\sqrt{t^2-1}}$ and $\int\frac{\sqrt{1+x^2}}{x}dx=\int\frac{t^2}{t^2-1}dt=\int\left(1+\frac{1}{t^2-1}\right)dt = t+\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$

$= \sqrt{1^2+x^2}+\frac{1}{2}\log\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$ -- (2)

From (1) and (2) the general solution of the D.E is

$$\sqrt{1+y^2}+\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|=c$$

(or) $\sqrt{1+y^2}+\sqrt{1+x^2}-\log|x|+\log\left|\sqrt{x^2+1}-1\right|=c$ (rationalising the D^r)

(or) $\sqrt{1+y^2}+\sqrt{1+x^2}+\log|x|-\log\left|1+\sqrt{1+x^2}\right|=c$ (rationalising the N^r)

***2. Find the equation of the curve whose slope at any point (x, y) is $\frac{y}{x^2}$ and which satisfies the condition $y=1$ when $x=3$**

Sol. Let the required curve be $y=f(x)$ so that the slope of the tangent at any point

$P(x, y)$ is $\frac{y}{x^2} \Rightarrow \frac{dy}{dx}=\frac{y}{x^2}$ -- (1)

Separating the variables and integrating, we get

$\int\frac{dy}{y}=\int\frac{1}{x^2}dx+c \Rightarrow \log y=-\frac{1}{x}+c$. Where $x=3, y=1 \Rightarrow c=\frac{1}{3}$

$\therefore$ The equation of the curve satisfying the given conditions is $\log y=\frac{1}{3}-\frac{1}{x}$

i.e., $y=e^{\frac{x-3}{3x}}$

3. Solve $\frac{dx}{dt}=\frac{t(2\log t+1)}{\sin x+x\cos x}$

Sol. The given differential equation can be written as

$$(\sin x+x\cos x)\,dx=(2t\log t+t)\,dt$$

Integrating both sides, we get $\int(\sin x+x\cos x)dx=\int(t+2t\log t)dt+c$

$\Rightarrow \int\sin x dx+\int x\cos x dx=\int t dt+\int 2t\log t dt+c$

$\Rightarrow \int\sin x dx+x\sin x-\int\sin x dx=\int t dt+t^2\log t-\int t^2\frac{1}{t}dt+c$

$\Rightarrow x\sin x=t^2\log t+c$

This is the general solution of the given D.E.

Aliter : $(\sin x+x\cos x)dx=(2t\log t+t)dt \Rightarrow d(x\sin x)=d(t^2\log t)$

$\Rightarrow \int d(x\sin x)=d(t^2\log t) \Rightarrow x\sin x=t^2\log t+c$

****4. Solve $ydx-xdy+4x^3y^2e^{x^4}dx=0$**

Sol. The given differential equation can be written as

$$\frac{ydx-xdy}{y^2}+4x^3e^{x^4}dx=0 \Rightarrow d\left(\frac{x}{y}\right)+d(e^{x^4})=0$$

On integration, $\int d\left(\frac{x}{y}\right)+\int d(e^{x^4})=c \quad \Rightarrow \frac{x}{y}+e^{x^4}=c$

This is in the general solution of the given D.E

Note : *The above method finding the general solution of a D.E is called the method of Inspection, which will be discussed later.*

EXERCISE - 4.2

1. *a) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ *b) $y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$

   *c) $\frac{dy}{dx}=e^{x-y}+x^2.e^{-y}C$ *d) $\frac{dy}{dx}+x^2=x^2e^{3y}$

2. *a) $\frac{dy}{dx}=\frac{2y}{x}$ (March-19) *b) $x+y\frac{dy}{dx}=0$ *c) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ (May-19)

   d) $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ *e) $\frac{dy}{dx}=2y\tanh x$ *f) $\frac{dy}{dx}=e^{x+y}$ (May-18)

   *g) $\frac{dy}{dx}=e^{y-x}$ *h) $\frac{dy}{dx}=\frac{xy+y}{yx+x}$ *i) $y(1+x)dx+x(1+y)dy=0$

**3. $(xy^2+x)dx+(yx^2+y)dy=0$ *4. $\frac{dy}{dx}=\frac{y^2+2y}{x-1}$ *5. $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

*6. $(e^x+1)ydy+(y+1)dx=0$ *7. $\tan y.\sec^2 xdx+\tan x.\sec^2 ydy=0$

*8. $\tan ydx+\tan xdy=0$ *9. $\sqrt{1+x^2}\,dx+\sqrt{1+y^2}\,dy=0$

*10. $\frac{dy}{dx}=\frac{1+y^2}{xy(1+x^2)}$ *11. $y-x\frac{dy}{dx}=5\left(y^2+\frac{dy}{dx}\right)$ 12. $\frac{dy}{dx}=x^2e^{-y}+e^{x-y}$

*13. $\frac{dy}{dx}+x^2=x^2e^{3y}$ *14. $\frac{dy}{dx}=\frac{x(1+2\log x)}{\sin y+y\cos y}$ 15. $xdy-ydx+3x^2y^2e^{x^3}dx=0$

4.8 DIFFERENTIAL EQUATIONS REDUCIBLE TO VARIABLES SEPERABLE TYPE

A differential equation of the form $\frac{dy}{dx}=f(ax+by+c)$ -- (1)

can be reduced to variables separable form by introducing a new dependent variable and hence can be solved. Put $ax+by+c=u$. *Then* $\frac{dy}{dx}=\frac{du}{dx}-a$

Now, equation (1) transform to $\frac{du}{dx}=a+f(u)$ -- (2)

The equation (2) is of variables separable type in u and x and hence can be solved to get a solution of the form.

F(u, c) = 0 where C is an arbitrary constant

Substituting u = ax+by+c, we get the general solution of equation (1) in the form F(ax+by+c;C) = 0.

SOLVED EXAMPLES

***1. Solve $\frac{dy}{dx} = (3x + y + 4)^2$**

Sol. i) $y^2 - x\frac{dy}{dx} = a\left(y + \frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = \frac{y^2 - ay}{x + a}$

Put $3x + y + 4 = u$ So that $\frac{dy}{dx} = \frac{du}{dx} - 3$

The equation (1) transforms to $\frac{du}{dx} - 3 = u^2 \Rightarrow \frac{du}{dx} = u^2 + 3$

Separating the varaibles and integrating, we get

$$\int \frac{1}{u^2 + 3} du = \int dx + c \Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) = x + c$$

This is the general solution of equation (2)

$\therefore$ The general solution of equation (1) is $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3x + y + 4}{\sqrt{3}}\right) = x + c$

***2. Solve $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$**

Sol. Put $x+y = u$ then $\frac{dy}{dx} = \frac{du}{dx} - 1$ $\therefore$ The given equation (1) transforms to

$\frac{du}{dx} - 1 = \sin u + \cos u \Rightarrow \frac{du}{dx} = 1 + \sin u + \cos u$

Separating the variable and integrating we get

$\int \frac{1}{1 + \sin u + \cos u} du = \int dx + c \Rightarrow \int \frac{1}{1 + \sin u + \cos u} du = x + c$

To evaluate the integral on the L.H.S,

Put $\tan\left(\frac{u}{2}\right) = t$ then $du = \frac{2dt}{1 + t^2}, \sin u = \frac{2t}{1 + t^2}, \cos u = \frac{1 - t^2}{1 + t^2}$

$\therefore \int \frac{1}{1 + \sin u + \cos u} du = \int \frac{dt}{1 + t} = \log|1 + t| = \log\left|1 + \tan\frac{u}{2}\right|$

$\therefore$ The general solution of equation (2) is $\log\left|1 + \tan\left(\frac{u}{2}\right)\right| = x + c$

Substituting $u = x + y$, the general solution of equation (1) is $1 + \tan\left(\frac{x + y}{2}\right) = ce^x$

***3. Solve $(x - y)^2 \frac{dy}{dx} = a^2$**

Sol. Put $x-y = u$ then $\frac{dy}{dx} = 1 - \frac{du}{dx}$

$\therefore$ Equation (1) transform to $u^2\left(1 - \frac{du}{dx}\right) = a^2 \Rightarrow \frac{du}{dx} = \frac{u^2 - a^2}{u^2}$ -- (2)

Separating the variables and integrating, we get

$\int \frac{u^2}{u^2 - a^2} du = \int dx + c \Rightarrow \int\left(1 + \frac{a^2}{u^2 - a^2}\right) du = x + c$

$\Rightarrow u + a^2 \frac{1}{2a} \log\left|\frac{u - a}{u + a}\right| = x + c \Rightarrow 2u + a\log\left|\frac{u - a}{u + a}\right| = 2x + c$

This is the general solution of (2).

$\therefore$ The general solution of equation (1) is $a\log\left|\frac{x - y - a}{x - y + a}\right| = 2y + c$

***4.** **Solve $\dfrac{dy}{dx} - x\tan(y - x) = 1$**

Sol. Put $y - x = t$

$$\frac{dy}{dx} - 1 = \frac{dt}{dx}\,;\quad \frac{dt}{dx} = x\tan t + 1 - 1 = x\tan t\,;\quad \frac{dt}{\tan t} = x\,dx$$

$$\int \cot t\; dt = \int x\,dx\,;\ \log\left|\sin t\right| = \frac{x^2}{2} + c$$

Solution is $\log\left|\sin(y - x)\right| = \dfrac{x^2}{2} + c$

EXERCISE - 4.3

*1. $\dfrac{dy}{dx} + 1 = e^{x+y}$ (May-18)

*2. $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$

**3. $\dfrac{dy}{dx} = \tan^2(x + y)$

*4. $\dfrac{dy}{dx} = \tan^2(x + y)$

*5. $\dfrac{dy}{dx} = 1 + x\tan(y - x)$

6. $\dfrac{dy}{dx} = (x - y)^2$

7. $\dfrac{dy}{dx} = (4x + 9y + 1)^2$

*8. $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{2x - 4y}$

4.9 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation $\dfrac{dy}{dx} = f(x, y)$ is said to be a homogeneous differential equation if f(x, y) is a homogeneous function of degree zero in x and y.

For example, the differential equations:

i) $\dfrac{dy}{dx} = \sin\left(\dfrac{y}{x}\right)$ ii) $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$ iii) $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 + y^2}$

are homogeneous differential equations because the function $f(x,x)$ on the R.H.S of each equation is a homogeneous function zero in x and x

Note

A differentiable equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$ is said to a homogeneous differential equation in x and y if f(x,y) and g(x,y) are homogeneous functions of the same degree in x and y.

Homogeneous Functions

A function f(x, y) is said to be a homogeneous function of degree n is the variables x and y if $f(kx, ky) = k^n f(x, y)$ for any $k \neq 0$ or if f(x, y) can be expressed as

$$f(x, y) = x^n f\left(\frac{y}{x}\right) \quad \text{or} \quad f(x, y) = y^n f\left(\frac{x}{y}\right)$$

where we have written $f\left(\dfrac{y}{x}\right)$ for $f\left(1, \dfrac{y}{x}\right)$ and $f\left(\dfrac{x}{y}\right)$ for $f\left(\dfrac{x}{y}, 1\right)$.

$f(x, y)$ is a homogeneous function of degree zero in x and y if $f(kx, ky) = f(x, y)$ for any $k \neq 0$, or if $f(x, y)$ can be expressed as $f\left(\frac{y}{x}\right)$ or $f\left(\frac{x}{y}\right)$.

Ex 1: $f(x,y) = ax^2 + 2hxy + by^2$ is a homogeneous function degree 2 in x and y since $f(kx, ky) = k^2 f(x, y)$

Ex 2 : $g(x,y) = \sin^{-1}\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)$ is a homogeneous function of degree zero is x and y, since $f(kx, ky) = k^0 f(x, y)$

Ex 3 : The function $f(x,y) = e^{x/y}\left(1 - \frac{x}{y}\right)$, $\cos^2\left(\frac{y}{x}\right)$, $1 + e^{x/y}$, $\frac{x-y}{x+y}$, $\frac{2xy}{x^2+y^2}$, $\frac{y^2 - x\sqrt{x^2+y^2}}{xy}$, $\frac{x^2 - xy + y^2}{y^2}$ are all homogeneous functions of degree zero in x and y.

Ex 4 : Show that $f(x, y) = 4x^2y + 2xy^2$ is homogeneous function of degree 3

Ex 5 : Show that $g(x, y) = xy^{1/2} + yx^{1/2}$ is a homogeneous function of degree $\frac{3}{2}$

Ex 6 : Show that $h(x, y) = \frac{x^2 + y^2}{x^3 + y^3}$ is a homogeneous function of degree -1

Ex 7 : Show that $f(x, y) = 1 + e^{x/y}$ is a homogeneous function of x and y of degree 0.

Ex 8 : Show that $f(x, y) = x\sqrt{x^2 + y^2} - y^2$ is a homogeneous function of x and y.

Ex 9 : Show that $f(x, y) = x - y \log y + y \log x$ is a homogeneous function of x and y.

To find the General Solution

Let $\frac{dy}{dx} = f(x,y)$ -- *(1) be a homogeneous differential equation*

Then $f(x, y)$ is a homogeneous function of degree zero and hence it can be written as $f\left(\frac{y}{x}\right)$.

Now, the equation (1) is equivalent to $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ -- *(2)*

To solve equation (2) put $\frac{y}{x} = v$ i.e., $y = vx$ where v is the new dependent variable.

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *and the equation (2) transforms to* $v + x\frac{dv}{dx} = f(v) \Rightarrow x\frac{dv}{dx} = f(v) - v$ -- *(3)*

This is a variables seperable equation in x and v.

Separating the variables and integrating we get $\int \frac{1}{f(v) - v}\, dv = \int \frac{1}{x}\, dx + c.$

On evaluation of these integrals,

we get the general solution of equation (3) in the form $F(v) = \ell n|x| + c$

Substituting $v = \frac{y}{x}$ *we get the general solution of equation (1) as* $F\left(\frac{y}{x}\right) = \ell n|x| + c$.

Remark :

If the given differential equa-tion can be expressed as $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$, it is convenient to substitute $\frac{x}{y} = v$ i.e., $x = vy$ so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$. This transforms the given equation into an equation of varaibles seperable type in v and y and hence can be solved (see Ex: 5)

SOLVED EXAMPLES

***1. Solve $(x^2 + y^2)dx = 2xydy$**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ --(1)

$$\text{R.H.S} = \frac{x^2 + y^2}{2xy} = \frac{1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

a homogeneous function of degree zero

$\therefore$ Equation (1) is a homogeneous differential equation

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore (1) \Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{2v} \Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v} \qquad \text{--(2)}$$

Seperating the variables and integrating, we get

$$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx + c \Rightarrow -\log(1-v^2) = \log x + c$$

This can be written as $\log x + \log(1-v^2) = \log c \Rightarrow x(1-v^2) = c$

This is the general solution of (2)

Substituting $v = \frac{y}{x}$, the general solution of (1) is $x^2 - y^2 = cx$, where c is an arbitrary constant.

***2. Solve $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{3xy^2 - x^3}{3x^2y - y^3}$ --(1)

R.H.S of (1) is a homogeneous function of degree zero since both the N^r and D^r are homogeneous function of the same degree 3.

$\therefore$ Equation (1) is a homogeneous D.E Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\text{Then equation (1)} \Rightarrow v + x\frac{dv}{dx} = \frac{3v^2 - 1}{3v - v^3} \Rightarrow x\frac{dv}{dx} = \frac{v^4 - 1}{3v - v^3} \qquad \text{--(2)}$$

Seperating the varaibles and integrating we get

$$\int \frac{3v - v^3}{v^4 - 1}dv = \int \frac{1}{x}dx + c \Rightarrow \frac{3}{2}\int \frac{2v}{(v^2)^2 - 1}dv - \frac{1}{4}\int \frac{4v^3}{v^4 - 1}dv = \log x + \log c$$

$$\Rightarrow \frac{3}{4}\log\left(\frac{v^2 - 1}{v^2 + 1}\right) - \frac{1}{4}\log(v^4 - 1) = \log(cx) \Rightarrow \frac{1}{4}\log\left[\frac{(v^2 - 1)^3}{(v^2 + 1)^3(v^4 - 1)}\right] = \log(cx)$$

$$\Rightarrow \frac{1}{2}\log\left[\frac{v^2 - 1}{(v^2 + 1)^2}\right] = \log(cx) \Rightarrow \frac{v^2 - 1}{(v^2 + 1)^2} = c^2x^2$$

This is the general solution of (2) Substituting $v = \frac{y}{x}$, the general solution of (1) is $y^2 - x^2 = c^2(y^2 + x^2)^2$

***3. Solve $(x - y\log y + y\log x)dx + x(\log y - \log x)dy = 0$**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{y\log\left(\frac{y}{x}\right) - x}{x\log\left(\frac{y}{x}\right)} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{\log\left(\frac{y}{x}\right)}$ --(1)

This is a homogeneous differential equation

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then equation (1) can be written as $u + x\frac{dv}{dx} = v - \frac{1}{\log v} \Rightarrow x\frac{dv}{dx} = -\frac{1}{\log v}$ -- (2)

Seperating the varaibles and integrating we get

$$\int \log v dv + \int \frac{1}{x} dx = c \Rightarrow v[\log v - 1] + \log x = c$$

Substituting $v = \frac{y}{x}$, the general solution of (1) is $\frac{y}{x}\left[\log\left(\frac{y}{x}\right) - 1\right] + \log x = c$

$\Rightarrow y\log y + (x - y)\log x = y + cx$

***4. Solve $x\sec\left(\frac{y}{x}\right)(ydx + xdy) = y\operatorname{cosec}\left(\frac{y}{x}\right)(xdy - ydx)$**

Sol. The given equation can be written as $\left[xy\cos ec\left(\frac{y}{x}\right) - x^2\sec\left(\frac{y}{x}\right)\right]dy$

$$= \left[xy\sec\left(\frac{y}{x}\right) + y^2\cos ec\left(\frac{y}{x}\right)\right]dx \Rightarrow \frac{dy}{dx} = \frac{xy\sec\left(\frac{y}{x}\right) + y^2\cos ec\left(\frac{y}{x}\right)}{xy\cos ec\left(\frac{y}{x}\right) - x^2\sec\left(\frac{y}{x}\right)} \quad \text{-- (1)}$$

This is clearly a homogeneous differential equation because both N^r and D^r on the R.H.S of (1) of are homogeneous functions of the same degree 2.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then equation (1) becomes $v + x\frac{dv}{dx} = \frac{2v\sec v}{v\cos ecv - \sec v}$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\sin v}{v\cos v - \sin v} \quad \text{-- (2)}$$

Seperating the variables and integrating, we get

$$\int \frac{v\cos v - \sin v}{v\sin v} dv = 2\int \frac{1}{x}dx + c \Rightarrow \int\left(\cot v - \frac{1}{v}\right)dv = 2\log x + \log c$$

$\Rightarrow \log\sin v - \log v = 2\log x + \log c \Rightarrow \frac{\sin v}{v} = cx^2$

This is the general solution of (2)

$\therefore$ The general solution of (1) is $\sin\left(\frac{y}{x}\right) = cxy$

Aliter: The given equation by inspection can be written as $\frac{ydx + xdy}{xy} = \cot\left(\frac{y}{x}\right)\frac{(xdy - ydx)}{x^2}$

$\Rightarrow \frac{1}{xy}d(xy) = \cot\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) \Rightarrow d[\log(xy)] = d\left[\log\sin\left(\frac{y}{x}\right)\right]$

On integration,. we get $\log(xy) = \log\sin\left(\frac{y}{x}\right) + \log c \Rightarrow xy = c\sin\left(\frac{y}{x}\right)$

*** 5. Solve $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$**

Sol. The given D.E can be written as $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{\left(e^{x/y}+1\right)}$ -- (1)

Since the RHS is a homogeneous function of degree zero the given equation is a homogeneous D.E and it is of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$

$\therefore$ Put $\frac{x}{y} = v$ so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$

Then equation (1) becomes $v + y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v+1} \Rightarrow y\frac{dv}{dy} = \frac{-(e^v+v)}{e^v+1}$ -- (2)

Seperating the varaibles and integrating, we get $\int \frac{e^v+1}{e^v+v}dv = -\int \frac{1}{y}dy + c$

$\Rightarrow \log(e^v+v) + \log y = \log c \Rightarrow y(e^v+v) = c$

Substituting $v = \frac{y}{x}$, the general solution of (1) is $ye^{x/y} + x = c$

Aliter : The given equation by inspection can be written as

$$dx + e^{x/y}\left[\frac{ydx - xdy}{y}\right] + e^{x/y}dy = 0$$

$\Rightarrow dx + ye^{x/y}d\left(\frac{x}{y}\right) + e^{x/y}dy = 0 \Rightarrow dx + yd(e^{x/y}) + e^{x/y}dy = 0$

$\Rightarrow dx + d(y.e^{x/y}) = 0$ on integration, we get $x + ye^{x/y} = c$

****6. Given the solution of $x\sin^2\left(\frac{y}{x}\right)dx = ydx - xdy$ which passes through the point $\left(1, \frac{\pi}{4}\right)$.** **(March-2014)**

Sol. The D.E of a family of curves is

$x\sin^2\left(\frac{y}{x}\right)dx = ydx - xdy \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$ -- (1)

This is clearly a homogeneous D.E

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ Then equation (1) becomes $x\frac{dv}{dx} = -\sin^2(v)$

$\Rightarrow -\text{cosec}^2 v dv = \frac{1}{x}dx$ (Separating the variables)

On integration, we get $\int -\text{cosec}^2 v dv = \int \frac{1}{x}dx + c = \cot v = \log x + c$

The general solution of the given equation (1) in $\cot\left(\frac{y}{x}\right) = \log x + c$ -- (2)

The curve (2) passes through $\left(1, \frac{\pi}{4}\right) \Rightarrow \cot\left(\frac{\pi}{4}\right) = a + c \Rightarrow c = 1$

$\therefore$ The required solution of (1) is $\cot\left(\frac{y}{x}\right) = 1 + \log x$

EXERCISE - 4.4

1. Express the following differential equaions in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

   *a) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$ *b) $\left[x - y\text{Tan}^{-1}\left(\frac{y}{x}\right)\right]dx + x\text{Tan}^{-1}\left(\frac{y}{x}\right)dy = 0$

   *c) $xdy = y(\log y - \log x + 1)dx$

2. Express the following differential equations in the form $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$

   *a) $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

   *b) $xdy - ydx + ye^{-\frac{2x}{y}}dy = 0$ c) $xydx + x^2dy - y\sqrt{x^2 + y^2}\,dy = 0$

3. Solve the following differential equations

   *a) $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$ *b) $y^2dx + (x^2 - xy)dy = 0$ *c) $\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$

   *d) $y^2dx + (x^2 - xy + y^2)dy = 0$ *e) $\frac{dy}{dx} = \frac{x - y}{x + y}$ *f) $(x^2 + y^2)dy = 2xydx$

   **g) $(x^2 - y^2)dx - xydy = 0$ (March-2017)

   Find the general solution the following differential equations

*4. $xdy = \left[y + x\cos^2\left(\frac{y}{x}\right)\right]dx$ *5. $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y\sin\left(\frac{y}{x}\right) - x$

*6. $xdy - ydx = \sqrt{x^2 + y^2}\,dx$ *7. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ *8. $xy^2dy = (x^3 + y^3)dx$

*9. $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - xy}$ (March-19)   *10. $(y^2 - 2xy)dx + (2xy - x^2)dy = 0$

*11. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$ (May-18)   *12. $(3x^2 + y^2)dy + (x^2 + 3y^2)dx = 0$

*13. $(2x - y)dy = (2y - x)dx$ *14. $(x^2 - y^2)\frac{dy}{dx} = xy$

*15. $2\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$ *16. $(ydx + xdy)x\cos\left(\frac{y}{x}\right) = (xdy - ydx)y\sin\left(\frac{y}{x}\right)$

*17. Find the equation of the curve whose gradient is $\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$ where $x>0$, $y>0$ and which passes through the point $\left(1, \frac{\pi}{4}\right)$

4.10 NON HOMOGENEOUS DIFFERENTIAL EQUATIONS

A first order first degree differential equation of the form $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ *where* $a_1, b_1, c_1, a_2, b_2, c_2$ *are real constants such that* $(c_1,c_2)\neq(0,0)$, *is called a non - homogeneous differential equation in x and y.*

For example, $\frac{dy}{dx}=\frac{x-y+1}{x+y+1}$; $\frac{dy}{dx}=\frac{2x+3y}{x+y+2}$, $(3x+4y+6)dx+(3x+4y+5)dy=0$ are non - homogeneous differential equations.

To find the general solution

Consider the D.E $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ --(1)

Where $(c_1,c_2)\neq(0,0)$ *the following cases arise.*

Case (i) : $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, *Let* $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=k$, *then equation (1) reduces to* $\frac{dy}{dx}=k$ *and its general solution is* $y = kx+c$

Case (ii) : $b_1=-a_2$, *i.e., coefficient of y in the Nr + coefficient of x in the Dr = 0.*

In this case, by cross-multiplication and regrouping the terms, equation (1) can be written as $a_1xdx+b_1(ydx+xdy)-b_2ydy+c_1dx-c_2dy=0$

$$\Rightarrow a_1d\left(\frac{x^2}{2}\right)+b_1d(xy)-b_2d\left(\frac{y^2}{2}\right)+c_1dx-c_2dy=0$$

On integration, we get the general solution is $a_1\left(\frac{x^2}{2}\right)+b_1(xy)-b_2\left(\frac{y^2}{2}\right)+c_1x-c_2y=0$

i.e., $a_1x^2+2b_1xy-b_2y^2+2c_1x-2c_2y=2c$

Case (iii) : $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ *i.e.,* $a_1b_2-a_2b_1\neq0$.

In this case the equation (1) can be transformed to a homogeneous D.E by introducing new variables x and y, as follows

Put $x = X+h$, $y = Y+h$ *where the constants h and k an to be choosen*

Such that $a_1h+b_1k+c_1=0$, $a_2h+b_2k+c_2=0$

Then $\frac{dy}{dx}=\frac{dY}{dX}$ *and the equation (1) transforms to* $\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$ -- (3)

Which is a homogeneous D.E in X and Y.

Substituting $Y = vX$, *equation (3) transforms to variables seperable form and hence can be solved to get the general solution in the form* $\phi(X,v,c)=0$

By substituting $v=\frac{Y}{X}$, *we get the general solution of equation (3) in the form* $\psi(X,Y,C)=0$. *Hence the general solution of equation (1), in this case, is given by* $\psi(x+h,y+k,c)=0$.

The method of finding the general solution in the case (ii), (iii) and (iv) is illustrated in the following examples.

SOLVED EXAMPLES

***1. Solve** $\dfrac{dy}{dx}=\dfrac{x-y+2}{x+y-1}$

Sol. The given D.E is $\dfrac{dy}{dx}=\dfrac{x-y+2}{x+y-1}$ --(1)

In the equation (1) $a_1 = 1, b_1 = -1, c_1 = 2$; $a_2 = 1, b_2 = 1, c_1 = -1$

Clearly $b_1 = -a_2$

$\therefore$ Equation (1) can be written as $(x-y+2)dx-(x+y-1)dy=0$

$\Rightarrow xdx-ydx+2dx-xdy-ydy+dy=0$

$\Rightarrow xdx-(ydx+xdy)-ydy+2dx+dy=0$ (Regrouping the terms)

$\Rightarrow d\left(\dfrac{x^2}{2}\right)-d(xy)-d\left(\dfrac{y^2}{2}\right)+2dx+dy=0$

On integration we get the general solution $\dfrac{x^2}{2}-xy-\dfrac{y^2}{2}+2x+y=c$

i.e., $x^2-2xy-y^2+4x+2y=2c$

***2 Solve** $\dfrac{dy}{dx}=\dfrac{4x+6y+5}{3y+2x+4}$ **(May-19)**

Sol. The given D.E is $\dfrac{dy}{dx}=\dfrac{4x+6y+5}{3y+2x+4}$ --(1)

Here, $a_1 = 4, b_1 = 6, c_1 = 5, \quad a_2 = 2, b_2 = 3, c_2 = 4,$

Clearly $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=2\neq\dfrac{c_1}{c_2}$

Put $2x+3y+4=u$ then $\dfrac{dy}{dx}=\dfrac{1}{3}\left(\dfrac{du}{dx}-2\right)$

Equation (1) transforms to $\dfrac{1}{3}\left(\dfrac{du}{dx}-2\right)=\dfrac{2u-3}{u} \Rightarrow \dfrac{du}{dx}=\dfrac{8u-9}{u}$ -- (2)

Seperating the varaibles and integrating, we get $\int\dfrac{u}{8u-9}du=\int dx+c$

$\Rightarrow \dfrac{1}{8}\int\left(1+\dfrac{9}{8u-9}\right)du=x+c \Rightarrow \dfrac{1}{8}\left[u+\dfrac{9}{8}\log(8u-9)\right]=x+c$

$\Rightarrow 8u+9\log(8u-9)=64x+c$

This is the general solution of (2), substituting $u = 2x+3y+4$ the general solution of (1) is $8(2x+3y+4)+9\log\left[8(2x+3y+4)-9\right]=64x+c$

$\Rightarrow 9\log\left[8(2x+3y+4)-9\right]=48x-24y-32+c$

$\Rightarrow \dfrac{3}{8}\log\left[16x+24y+23\right]=2x-y+k$

*3. Solve $\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}$

Sol. The given D.E is $\frac{dy}{dx}=\frac{-7x+3y+7}{3x-7y-3}$ --(1)

Hence $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ and $b_1+a_2\neq 0$

$\therefore$ Put $x = X+h$ and $y = Y+k$ where h and k are constants to be choosen such that

$$-7h+3k+7=0,\ 3h-7k-3=0$$

Solving for h and k, we get $h = 1, k = 0$

Then $x = X+1$, $y = Y$ and $\frac{dy}{dx}=\frac{dY}{dX}$

The equation (1) transformes to $\frac{dY}{dX}=\frac{-7X+3Y}{3X-7Y}$ -- (2)

This is a homogeneous D.E put $Y = vX$

so that $u+X\frac{dv}{dX}=\frac{dY}{dX}$. Then (2)

$$\Rightarrow v+X\frac{dv}{dX}=\frac{-7+3v}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{7(v^2-1)}{3-7v} \quad \text{-- (3)}$$

Seperating the varaibles and integrating, we get

$$\int\frac{3-7v}{v^2-1}dv=7\int\frac{1}{X}dX+c$$

$$\Rightarrow 3\int\frac{1}{v^2-1}dv-\frac{7}{2}\int\frac{2v}{v^2-1}=7\log X+\log c\ \text{s}$$

$$\Rightarrow 3\int\frac{1}{v^2-1}dv-\frac{7}{2}\int\frac{2v}{v^2-1}=7\log X+\log c$$

$$\Rightarrow \log\left[\left(\frac{v-1}{v+1}\right)^3\frac{1}{(v^2-1)^7}\right]=\log(cX^{14})$$

$$\Rightarrow \frac{1}{(v-1)^4(v+1)^{10}}=c.X^{14}$$

$$\Rightarrow (v-1)^4(v+1)^{10}X^{14}=\frac{1}{c}$$

This is the general solution of (3), putting $v=\frac{Y}{X}$, the general solution of (2) is $(Y-X)^4(Y+X)^{10}=k^2$, where $k^2=\frac{1}{c}$

Substituting $X = x-1$, $Y = y$ the general solution of (1) is given by

$$(y-x+1)^4(y+x-1)^{10}=k^2 \Rightarrow (y-x+1)^2(y+x-1)^5=k$$

EXERCISE - 4.5

1. Solve the following D.E's

*a) $\dfrac{dy}{dx}=\dfrac{3x-y+7}{x-7y-3}$ *b) $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$ *c) $\dfrac{dy}{dx}=\dfrac{-3x-2y+5}{2x+3y+5}$

*d) $\dfrac{dy}{dx}=\dfrac{-3x-2y+5}{2x+3y-5}$ *e) $\dfrac{dy}{dx}=\dfrac{(12x+5y-9)}{5x+2y-4}$

*f) $2(x-3y+1)dy=(4x-2y+1)dx$

*2. Find the general solution of the following D.E's

a) $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x-2y+5}$ *b) $(x-y)\,dy=(x+y+1)\,dx$ **(March-17)**

*3. $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}$

**4. $(2x+y+1)dx+(4x+2y-1)dy=0$

*5. $\dfrac{dy}{dx}=\dfrac{2y+x+1}{2x+4y+3}$ **(March-19)**

*6. $(x+y-1)dy=(x+y+1)dx$

*7. a) $\dfrac{dy}{dx}=\dfrac{6x+5y-7}{2x+18y-14}$ b) $(x-y-2)\,dx+(x-2y-3)\,dy=0$.

c) $(2x+3y-8)dx=(x+y-3)dy$ Where $X=x-1,\ Y=y-2]$

*8. $\dfrac{dy}{dx}+\dfrac{10x+8y-12}{7x+5y-9}=0$

*9. $\dfrac{dy}{dx}=\dfrac{2x+9y-20}{6x+2y-10}$

10. $\dfrac{dy}{dx}=\dfrac{2x+y+3}{2y+x+1}$

4.11 LINEAR DIFFERENTIAL EQUATIONS

Definition

A differential equation is said to be a linear differential equation if the dependent varaible and its derivatives occur only in first degree and are not multiplied together.

Definition

A differential equations of the form $\dfrac{dy}{dx}+P(x)y=Q(x)$ where P(x) and Q(x) are continuous functions of x only, is called a first order linear differential equation in y.

Example : i) $\dfrac{dy}{dx}+\dfrac{1}{x}y=\sin x$ ii) $\dfrac{dy}{dx}+\dfrac{x}{1+x^2}y=x^3e^{x^2}$

are linear differ-ential equations (L.D.E's) of first order in y.

Note

The equation $\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_{n-1}G_1\frac{dy}{dx} + P_n(x)y = Q(x)$ *is an* n^{th} *order L.D.E in y.*

Remark :

The equation $y\frac{dy}{dx} + xy^2 = (x+1)^3$ *is not a linear differential equation in y because of the presence of the product term* $y\frac{dy}{dx}$ *and the second degree term* y^2*. However, some differ-ential equations which are not linear can be made linear, by introducing a new dependent varaible through a suitable substitution (see sec 4.12)*

An equation of the form $a(x)\frac{dy}{dx} + b(x)y = c(x)$ where $a(x)$, $b(x)$ y and $c(x)$ are continuous functions of x only, is called the most general form of a first order L.D.E in y. This equation can be transformed to the standard form by dividing it with $a(x)$.

To Find the General Solution

Consider a linear differential equatin of first order in y $\frac{dy}{dx} + P(x)y = Q(x)$ -- (1)

Multiplying both sides of (1) with $e^{\int P(x)dx}$,

we get $\frac{dy}{dx}e^{\int P(x)dx} + ye^{\int P(x)dx}.P(x) = Q(x)e^{\int P(x)dx} \Rightarrow \frac{d}{dx}\left[ye^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$

Integrating both sides, we get $\int\frac{d}{dx}\left(ye^{\int P(x)dx}\right)dx = \int Q(x)e^{\int P(x)dx}dx + c \Rightarrow y.e^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + c$

This is the general solution of equation (1).

Integrating Factor

The factor $e^{\int P(x)dx}$, *which on multiplication, makes the L.H.S of (1) as the differential coefficient of* $\left(y.e^{\int P(x)dx}\right)$ *is called the integrating factor (usually written as I.F) of the differential equation (1).*

$\therefore$ *The I.F of the differential equation* $\frac{dy}{dx} + P(x).y = Q(x)$ -- (1) *is*

$I.F = e^{\int P(x)dx}$ *and the general solution of (1) can be written as* $y.(I.F) = \int Q(x)(I.F)dx + c$

Remark :

A function (fa-ctor) in x or y, which on multi-plication makes a differential exp-ression in x and y an exact differential, is called an integrating factor of that expression. For a given differential expression there may exist many integrating factors.

Linear differential equation in x

A differential equation of the form $\frac{dx}{dy} + P(y)x = Q(y)$ *where P(y) and Q(y) are continuous functions of y only, is called a linear differential equation in x.*

The I.F of this equation is $I.F = e^{\int P(y)dy}$ *and the general solution is given by* $x(I.F) = \int Q(y)(I.F)dy$

Example : i) $\frac{1}{x^2}$ is an I.F of the expression $(xdy-ydx)$ since $\frac{1}{x^2}(xdy-ydx)$ can be written as an exact differential $d\left(\frac{y}{x}\right)$

ii) $\frac{1}{y^2}$ is also an I.F of the expression $(xdy-ydx)$, since $\frac{1}{y^2}(xdy-ydx)$ can be written as the exact differential $d\left(-\frac{x}{y}\right)$.

iii) For any $x(\neq 0), k.e^{\int P(x)dx}$ is an integrating factor of the L.H.S of (1)

Note : i) To find the general solution of $Q(x)\frac{dy}{dx}+b(x)y=c(x)$, we have to transform the equation into the standard form (by dividing with $a(x)$).

$\frac{dy}{dx}+P(x)y=Q(x)$ Where $P(x)=\frac{b(x)}{a(x)}$ and $Q(x)=\frac{c(x)}{a(x)}$

ii) The general solution of $\frac{dy}{dx}+P(x)y=Q(x)$ is $y(I.F)=\int Q(x)(I.F)dx+c$ where $I.F=e^{\int P(x)dx}$.

iii) The general solution of $\frac{dx}{dy}+P(y)x=Q(y)$ is $x(I.F)=\int Q(y)(I.F)dy+c$ where $I.F=e^{\int P(y)dy}$

SOLVED EXAMPLES

1. Find the I.F's of the following differnetial equations by transforming them into linear form.

***i)** $\cos x\frac{dy}{dx}+y\sin x=\tan x$ ***ii)** $(2x-10y^3)\frac{dy}{dx}+y=0$

Sol. i) The given equation can be written in standard form as $\frac{dy}{dx}+\tan x.y=\tan x\sec x$

Here $P(x)=\tan x,\ Q(x)=\sec x.\tan x$

$\text{I.F}=e^{\int P|x|dx}=e^{\int \tan x dx}=e^{\log\sec x}=\sec x$

ii) This is not a linear equation in y because of the presence of $y^3\frac{dy}{dx}$. But it can be written as $y\frac{dx}{dy}+2x-10y^3=0\Rightarrow\frac{dx}{dy}+\frac{2}{y}x=10y^2$.

This is clearly a linear equation in x.

$\therefore I.F=e^{\int P(y)dy}=e^{\int\frac{2}{y}dy}=e^{2\log y}=y^2$

2. Solve $(1+x)\frac{dy}{dx}+4xy=\frac{1}{1+x^2}$

Sol. The given equation can be written as $\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(1+x^2)^2}$ --(1)

This is a linear differential equation in y

$\text{I.F}=e^{\int P(x)dx}=e^{\int\frac{4x}{1+x^2}dx}=e^{2\log(1+x^2)}=(1+x^2)^2$

$\therefore$ The general solution of (1) is given by $y(I.F)=\int Q(x)(I.F)dx+c$

$\Rightarrow y(1+x^2)^2=\int\frac{1}{(1+x^2)^2}(1+x^2)^2dx+c\Rightarrow y(1+x^2)^2=x+c$

***3. Solve** $\frac{1}{x}\frac{dy}{dx}+ye^{x}=e^{(1-x)e^{x}}$

Sol. The given equation can be written as $\frac{dy}{dx}+(xe^{x})y=xe^{(1-x)}e^{x}$ --(1)

This is a linear differnetial equation in y

$I.F=e^{\int xe^{x}dx}=e^{e^{x}(x-1)}$

$\therefore$ The general solution of equation (1) is given by $y(I.F)=\int Q(x)(I.F)dx+c$

$\Rightarrow y.e^{e^{x}(x-1)}=\int xe^{e^{x}(1-x)}.e^{e^{x}(x-1)}dx+c$

$\Rightarrow y.e^{e^{x}(x-1)}=\int x\,dx+c \Rightarrow y.e^{e^{x}(x-1)}=\frac{x^{2}}{2}+c$

****4. Solve** $(x+y+1)\frac{dy}{dx}=1$ **(March-2017)**

Sol. The given equation is not a linear equation in y. It can be written as

$\frac{dx}{dy}=x+y+1 \Rightarrow \frac{dx}{dy}-x=y+1$ --(1)

Which is clearly a linear equation in x

I.F $=e^{\int -1dy}=e^{-y}$ $\therefore$ The general solution of equation (1) is

$x(I.F)=\int Q(y)(I.F)dy+c \Rightarrow xe^{-y}=\int e^{-y}(y+1)dy+c$

$\Rightarrow xe^{-y}=-e^{-y}(y+2)+c \Rightarrow x=c.e^{y}-(y+2)$

This is the required solution.

4.12 — D.E'S REDUCIBLE TO LINEAR EQUATIONS

This is not in the syllabus

Bernoulli's Differential Equation

A differnetial equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^{n}$

where P(x) and Q(x) are continuous functions of x only and $n\in R$ *is called a Bernoulli's equation.*

To find the general solution

Consider a Bernoulli's differential equation $\frac{dy}{dx}+P(x)y=Q(x)y^{n},\ (n\neq 1)$ -- (1)

This equation can be written as $\frac{1}{y^{n}}\frac{dy}{dx}+P(x)\frac{1}{y^{n-1}}=Q(x)$ -- (2)

Put $\frac{1}{y^{n-1}}=u$ *then* $(1-n)\frac{1}{y^{n}}\frac{dy}{dx}=\frac{du}{dx}$ *and the equation (2) transforms to*

$\frac{1}{1-n}\frac{du}{dx}+P(x)u=Q(x) \Rightarrow \frac{du}{dx}+(1-n)P(x)u=(1-n)Q(x)$ -- (3)

This is a linear differential equation in u and hence can be solved.

(If n = 1 the equation (1) reduces to variables seperable type)

A General form

A differential equation of the form $f'(y)\dfrac{dy}{dx}+P(x)f(y)=Q(x)$ -- *(4)*

can be transformed into a linear differential equation by substituting f(y)=u.

Then $f'(y)\dfrac{dy}{dx}=\dfrac{du}{dx}$ *and equation (4) reduces to* $\dfrac{du}{dx}+P(x)u=Q(x)$

Which is a linear equation in u and hence can be solved.

SOLVED EXAMPLES

1. Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2y^6$

Sol. The given equation is $\dfrac{dy}{dx}+\left(\dfrac{1}{x}\right)y=x^2y^6$ -- (1)

This is a Bernoulli's equation.

This equation can be written as $\dfrac{1}{y^6}\dfrac{dy}{dx}+\dfrac{1}{x}\cdot\dfrac{1}{y^5}=x^2$ -- (2)

Put $\dfrac{1}{y^5}=u$ then $-5\dfrac{1}{y^6}\dfrac{dy}{dx}=\dfrac{dy}{dx}$

Equation (2), now, reduces to $-\dfrac{1}{5}\dfrac{du}{dx}+\dfrac{1}{x}u=x^2\Rightarrow\dfrac{du}{dx}+\left(\dfrac{-5}{x}\right)u=-5x^2$ -- (3)

Equation (3) is a L.D.E in $u=I.F=e^{\int-\frac{5}{x}dx}=e^{-5\log x}=\dfrac{1}{x^5}$

$\therefore$ The general solution of (3) is $u(I.F)=\int d(x)(I.F)\,dx+c$

$\Rightarrow u\dfrac{1}{x^5}=\int-5x^2\dfrac{1}{x^5}dx+c=-5\int\dfrac{1}{x^3}dx+c=-5\left(\dfrac{1}{-2x^2}\right)+c\Rightarrow x\dfrac{1}{x^5}=\dfrac{5}{2x^2}+c$

Substituting $u=\dfrac{1}{y^5}$, the general solution of equation (1) is $\dfrac{1}{x^5y^5}=\dfrac{5}{2x^2}+c$

***2. Solve** $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

Sol. The given D.E can be written as $\sec^2\dfrac{dy}{dx}+2x\tan y=x^3$ --(1)

Put $\tan y=u$ then $\sec^2 y\dfrac{dy}{dx}=\dfrac{du}{dx}$

The equation (1) transforms to $\dfrac{du}{dx}+2xu=x^3$ -- (2)

This is a linear equation in u. $I.F=e^{\int 2x\,dx}=e^{x^2}$

The general solution of (2) is $ue^{x^2}=\int x^3e^{x^2}dx+c=\dfrac{1}{2}\int te^t\,dt+c$ where $t=x^2$

$=\dfrac{1}{2}e^t(t-1)+c$ where $t=x^2\Rightarrow ue^{x^2}=\dfrac{1}{2}e^{x^2}(x^2-1)+c$

$\therefore$ The general solution of (1) is

$$\tan y e^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+c\Rightarrow\tan y=\frac{1}{2}(x^2-1)+c.e^{-x^2}$$

EXERCISE - 4.6

1. Find the I.F. of the following differential equations by transforming them into linear form.

*a) $x\frac{dy}{dx}-y=2x^2\sec^2 2x$ *b) $y\frac{dx}{dy}-x=2y^2$ *c) $(1+x^2)\frac{dy}{dx}+xy=\frac{x^4}{(1+x^5)}(\sqrt{1-x^2})^3$

2. Solve the following differential equation $\frac{dy}{dx}+\frac{2}{x}y=2x^2$

*3. Solve $\frac{dy}{dx}-y\tan x=e^x\sec x$ (May-19)

*4. Solve $\frac{dy}{dx}+y\tan x=\sin x$

*5. Solve $\frac{dy}{dx}-y=-2e^{-x}$

*6. Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$

*7. $\frac{dy}{dx}+y\sec x=\tan x$

8. $\frac{dy}{dx}+y\tan x=\cos^3 x$

9. $x\frac{dy}{dx}+2y=\log x$

10. $x\frac{dy}{dx}+y=(1+x)e^x$

*11. $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$ (March-18)

*12. $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$

*13. $x\log x\frac{dy}{dx}+y=2\log x$

*14. $\sec x\frac{dy}{dx}-y=\sin x$

**15. $\cos x\frac{dy}{dx}+y\sin x=\sec^2 x$ (March-19)

*16. $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$

**17. $\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(1+x^2)^2}$ (May-17)

*18. $\sin^2 x\frac{dy}{dx}+y=\cot x$

*19. $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3$ (March-19)

*20. $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

*21. $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

*22. $x(x-2)\frac{dy}{dx}-2(x-1)y=x^3(x-2)$ given that $y=9$ when $x=3$.

*23. $(x+2y^3)\frac{dy}{dx}=y$

*24. $\left(x-\frac{1}{y}\right)\frac{dy}{dx}+y^2=0$

*25. $\frac{dy}{dx}(x^2y^3+xy)=1$

26. $\frac{dy}{dx}=x^3y^3-xy$

***27. $(1+y^2)dx=(\tan^{-1}y-x)dy$ (May-18)

28. $(x^3y^2+xy)dx=dy$

4.13 METHOD OF INSPECTION

Some of the first order first degree differential equations can be easily solved (or integrated) by expressing them as a sum of exact differentials by a proper regrouping of the terms after making suitable algebraic mantipulations. This method of solving a differential equation is called the ***method of inspection.***

To make an effective use of this method we need to remember certain differential expressions which occur frequently and which can be written as exact differentials. A list of such expressions and the corresponding exact differentiatials are given below.

Standard inspections techequnies :

i) $dx \pm dy = d(x \pm y)$

ii) $xdx \pm ydy = \frac{1}{2}d(x^2 \pm y^2)$

iii) $xdy + ydx = (xy)$

iv) $\frac{xdy + ydx}{xy} d[\ell n(xy)]$

v) $\frac{xdy + ydx}{x^2y^2} = d\left[\frac{-1}{xy}\right]$

vi) $\frac{xdy + ydx}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$

vii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

viii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

ix) $\frac{xdy - ydx}{xy} = d\left[\ell n\left(\frac{y}{x}\right)\right]$

x) $\frac{xdy - ydx}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$

xi) $\frac{xdy - ydx}{x^2 - y^2} = d\left[\frac{1}{2}\ell n\left(\frac{x+y}{x-y}\right)\right]$

xii) $\frac{xdx + ydy}{x^2 + y^2} = d\left[\frac{1}{2}\ell n(x^2 + y^2)\right]$

xiii) $\frac{2xydy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

xiv) $\frac{2xydx - x^2dx}{y^2} = d\left(\frac{x^2}{y}\right)$

xv) $\frac{xydy - y^2dx}{x^3} = \frac{1}{2}d\left(\frac{y^2}{x^2}\right)$

xvi) $\frac{xydy - x^2dx}{y^3} = \frac{1}{2}d\left(\frac{x^2}{y^2}\right)$

xvii) $\frac{e^x(ydx - dy)}{y^2} = d\left(\frac{e^x}{y}\right)$

xviii) $\int\frac{e^x(ydx - dy)}{y^2} = d\left(\frac{e^x}{y}\right)$

xix) $e^{y/x}\left(\frac{xdy - ydx}{x^2}\right) = d(e^{y/x})$

xx) $e^{y/x}\left(\frac{ydx - xdy}{y^2}\right) = d(e^{x/y})$

xxi) $\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d(\sqrt{x^2 + y^2})$

xxii) $\frac{xdy - ydx}{x\sqrt{x^2 - y^2}} = d\left[\sin^{-1}\left(\frac{y}{x}\right)\right]$

xxiii) $\frac{dx + dy}{x + y} = d[\ell n(x + y)]$

xxiv) $x^{m-1}y^{n-1}(mydx + nxdy) = d(x^m y^n)$

In some cases a change of the pair of varaibles (x, y) to the pair (r,θ) through a suitable transformation facilitates the integration of a differential equation. Two such transformations are

1) $x = r\cos\theta, y = r\sin\theta$ *so that* $x^2 + y^2 = r^2$ *and* $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ *then*

   i) $xdx + ydy = rdr$

   ii) $xdy - ydx = r^2 d\theta$

   iii) $(dx)^2 - (dy)^2 = (dr)^2 + r^2 (d\theta)^2$

2) $x = r\sec\theta, y = r\tan\theta$ *so that* $x^2 - y^2 = r^2$ *and* $\theta = \sin^{-1}\left(\frac{y}{x}\right)$ *that*

   i) $xdx - ydy = rdr$

   ii) $xdy - ydx = r^2 \sec\theta\, d\theta$

SOLVED EXAMPLES

1. Solve $(xy^4 + y)dx - xdy = 0$

Sol. The given equation can be written as $xy^4dx + ydx - xdy = 0$

$$\Rightarrow xdx + \frac{ydx - xdy}{y^4} = 0 \Rightarrow x^3dx + \left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = 0 \Rightarrow d\left(\frac{x^4}{4}\right) + d\left[\frac{1}{3}\left(\frac{x}{y}\right)^3\right] = 0$$

On integration, we get $\int d\left(\frac{x^4}{4}\right) + \int d\left[\frac{1}{3}\left(\frac{x}{y}\right)^3\right] = c \Rightarrow \frac{x^4}{4} + \frac{1}{3}\frac{x^3}{3} = c$

This is the required general solution.

2. Solve $y\log y dx + (x - \log y)dy = 0$

Sol. The given equation can be written as $y\log y dx + xdy - \log y\, dy$

$$\Rightarrow \log y dx + x\left(\frac{1}{y}dy\right) - \frac{1}{y}\log y dy = 0 \Rightarrow d(x\log y) - d\left[\frac{1}{2}(\log y)^2\right] = 0$$

On integration, we get the general solution as $x\log y - \frac{1}{2}(\log y)^2 = c$

3. Solve $xdy + ydx = a^2\frac{(xdy - ydx)}{x^2 + y^2}$

Sol. The given D.E is $xdy + ydx = a^2\left(\frac{xdy - ydx}{x^2 + y^2}\right)$

$$\Rightarrow d(xy) = a^2\frac{1}{\left(1 + \frac{y^2}{x^2}\right)}\left(\frac{xdy - ydx}{x^2}\right) = a^2\frac{1}{\left(1 + \frac{y^2}{x^2}\right)}d\left(\frac{y}{x}\right) = a^2 d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

On integration, we get $\int d(xy) = a^2\int d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] + c \Rightarrow xy = a^2\tan^{-1}\left(\frac{y}{x}\right) + c$

This is the general solution of the given D.E

4. Solve $(1 - xy + x^2y^2)dx + (x^3y - x^2)dy = 0$

Sol. The given equation can be written as $dx - xydx + x^2y^2dx + x^3ydy - x^2dy = 0$

$\Rightarrow dx - x(xdy + ydx) + x^2y(ydx + xdy) = 0$

$\Rightarrow dx - xd(xy) + x^2yd(xy) = 0$

$\Rightarrow \frac{1}{x}dx - d(xy) + (xy)d(xy) = 0 \Rightarrow d[\log x] - d(xy) + d\left[\frac{1}{2}(xy)^2\right] = 0$

On integration we get the general solution as $\log x - xy + \frac{1}{2}x^2y^2 = c$

5. Solve $y(xy+1)dx + x(1 + xy + x^2y^2)dy = 0$

Sol. The given equation can be written as $xy^2dx + ydx + xdy + x^2y\,dy + x^3y^2dy = 0$

$\Rightarrow (xdy + ydx) + xy(ydx + xdy) + x^3y^2dy = 0$

$\Rightarrow d(xy) + xy(dxy) + x^3y^3\left(\frac{1}{y}dy\right) = 0$

$\Rightarrow \frac{1}{x^3y^3}d(xy) + \frac{1}{x^2y^2}d(xy) + (\log y) = 0$

$\Rightarrow d\left(\frac{-1}{2x^2y^2}\right) + d\left(\frac{-1}{xy}\right) + d(\log y) = 0$

On integration we get the general solution as $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$

6. Solve $(1 + 2e^{x/y})dx + 2e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Sol. The given D.E can be written as $dx + 2e^{x/y}dx + 2e^{x/y}dy - 2\frac{x}{y}e^{x/y}dy = 0$

$\Rightarrow dx + 2e^{x/y}dy + 2e^{x/y}\frac{(ydx - xdy)}{y} = 0 \Rightarrow dx + 2\left[e^{x/y}dy + ye^{x/y}\frac{(ydx - xdy)}{y^2}\right] = 0$

$\Rightarrow dx + 2[e^{x/y}dy + yd(e^{x/y})] = 0 \Rightarrow dx + 2d(ye^{x/y}) = 0 \Rightarrow \int dx + 2\int d(ye^{x/y}) = c$

On integration, we get $x + 2ye^{x/y} = c$ as the general solution.

7. Solve $\frac{xdx - ydy}{xdy - ydx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 - y^2}}$

Sol. Put $x = r\sec\theta$, $y = \tan\theta$ so that $x^2 - y^2 = r^2$ and $\theta = \sin^{-1}\left(\frac{y}{x}\right)$

then $xdx - ydy = rdr$, $xdy - ydx = r^2\sec\theta\,d\theta$

The given equation transforms to $\frac{rdr}{r^2\sec\theta d\theta} = \sqrt{\frac{1+r^2}{r^2}}$

$\Rightarrow \frac{dr}{\sqrt{1+r^2}} = \sec\theta d\theta \Rightarrow \int\frac{dr}{\sqrt{1+r^2}} = \int\sec\theta d\theta + c$

$\Rightarrow \log(r + \sqrt{1+r^2}) = \log(\sec\theta + \tan\theta) + \log c$

$\Rightarrow r + \sqrt{1+r^2} = c(\sec\theta + \tan\theta) \Rightarrow \sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = c\left(\frac{x+y}{\sqrt{x^2 - y^2}}\right)$

This is the general solution of the given D.E

EXERCISE - 4.7

Solve the following differential equations using inspection method or otherwise.

1. $xdy + ydx + 2x^3dx = 0$
2. $ydx + (x + x^2y)dy = 0$
3. $ydx - (x - 2y^3)dy = 0$
4. $(x + y)dx - (x - y)dy = 0$
5. $(1 + xy)ydx + (1 - xy)xdy = 0$
6. $xdx + ydy = y(x^2 + y^2)dy$
7. $xdy - ydx = (x^2 + y^2)dx$
8. $y(axy + e^x)dx - e^xdy = 0$
9. $2xy^2dx = e^x(dy - ydx)$
10. $(y^2e^x + 2xy)dx = x^2dy$
11. $(ydx - xdy)\cos\left(\frac{x}{y}\right) = xy^3(xdy + ydx)$
12. $x\frac{dy}{dx} - y = \sin x^2\sqrt{x^2 - y^2}\,dx$
13. $\frac{xdx + ydy}{ydx - xdy} = x^2 + 2y^2 + \frac{y^4}{x^2}$
14. $\frac{xdx + ydy}{ydx - xdy} = \frac{x}{y^3}\cos^2(x^2 + y^2)$
15. $\frac{xdx + ydy}{ydx - xdy} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$

4.14 DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

An equation of the form $F\left(x, y, \frac{dy}{dx}\right) = 0$ is called a differential equation of first order in x and y. In the discussion of differential equations of 1st order but not of first degree, it is usual to denote $\frac{dy}{dx}$ by p. Therefore, an equation of the form $F(x, y, p) = 0$ where $p = \frac{dy}{dx}$ is considered as a first order and higher degree differential equation in x and y. We discuss the following four types of these equations.

A) Equations solvable for p

B) Equations solvable for y

C) Equations solvable for x

D) Clairaut's equations

Equations Solvable for p

Consider a differential equations of the form

$$p^n + f_1(x \cdot y)p^{n-1} + f_2(x, y)p^{n-2} + ... + f_{n-1}(x, y)p + f_n(x, y) = 0 \quad ....(1)$$

where $p = \frac{dy}{dx}$ *and n is a positive integer.*

If this polynomial equation in p can be resolved into a product of n linear factors in p then it is said to be an equation solvable for p.

To find the General Solutions :

Let the equation (1) be solvable for p. Then it can be expressed as

$$[p-F_1(x,y)][p-F_2(x,y)].....[p-F_n(x,y)]=0 \quad .....(2)$$

Equating each further to zero, we get n first order and first degree differential equations which can be solved. Let their general solutions be $\phi(x,y,c_1)=0, \phi_2(x,y,c_2)=0.....\phi_n(x,y,c_n)=0$ *.....(3)*

Where $c_1, c_2 ... c_n$ are n orbitrary constants. Since the general solutions of a first order D.E should certain only one arbitrary constant, we take $c_1 = c_2 = ... = c_n = c$

Now, each of $\phi_k(x,y,c)=0, k=1,2,.....,n$ *is a solution of equation (1) the combined general solution of (1) is given by* $\phi_1(x,y,c)\cdot\phi_2(x,y,c)......\phi_n(x,y,c)=0$

SOLVED EXAMPLES

1. Solve $p^3-6p^2+11p-6=0$

Sol. The given equation in solvable for p as it can be written as $(p-1)(p-2)(p-3)=0$

$\Rightarrow p=1$ or $p=2$ or $p=3$ i.e., $\frac{dy}{dx}=1, \frac{dy}{dx}=2, \frac{dy}{dx}=3$

The general solutions of these three equations are

$$y-x-c_1=0,\ y-2x-c_2=0,\ y-3x-c_3=0$$

Since the general solution of a first order differential equations should contain only one arbitrary constant, we take $c_1=c_2=c_3=c$

$\therefore$ The general solution of (1) is $(y-x-c)(y-2x-c)(y-3x-c)=0$

where c is an orbitrary constant.

2. Solve $x^2(p^2-y^2)+y^2=x^4+2xyp$

Sol. The given equation can be written as $x^2p^2-2xyp+y^2= x^4+x^2y^2$ --(1)

$$\Rightarrow (xp-y)^2=x^2(x^2+y^2) \Rightarrow xp-y=\pm x\sqrt{x^2+y^2}$$

$$\Rightarrow \left[xp-y-x\sqrt{x^2+y^2}\right]\left[xp-y+\sqrt{x^2+y^2}\right]=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}+x\sqrt{\left(1+\left(\frac{y}{x}\right)^2\right)},\ \frac{dy}{dx}=\frac{y}{x}-x\sqrt{\left(1+\left(\frac{y}{x}\right)^2\right)}$$

$$\Rightarrow x\frac{dv}{dx}=\pm x\sqrt{1+v^2}$$

where $y=vx$ Solving these equations we get $\int\frac{dv}{\sqrt{1+v^2}}=\pm\int dx+c$

$\Rightarrow \log(v+\sqrt{1+v^2})=\pm x+c$, where $v=\frac{y}{x}$.

The general solution the given equation is

$$\left\{\log\left[\frac{y+\sqrt{x^2+y^2}}{x}\right]-x-c\right\}\left\{\log\left[\frac{y+\sqrt{x^2+y^2}}{x}\right]+x-c\right\}=0$$

3. Solve $p^2 + 2py\cot x - y^2 = 0$

Sol. The given equation can be written as $p^2 + 2py\cot x + y^2\cot^2 x = y^2\text{cosec}^2 x$

$$\Rightarrow (p+y\cot x)^2 = (y\text{cosec}x)^2 \Rightarrow [p+y(\text{cosec}x+\cot x)]\,[p-y(\text{cosec}x-\cot x)]=0$$

$$\Rightarrow \frac{dy}{dx} = -(\text{cosec}x+\cot x)dx,\ \frac{dy}{y} = (\text{cosec}x-\cot x)dx$$

$$\Rightarrow \int\frac{dy}{y} = -\int\cot\left(\frac{x}{2}\right)dx+c,\ \int\frac{dy}{y} = \int\tan\left(\frac{x}{2}\right)dx+c$$

$$\Rightarrow \ell ny = -2\ell n\left(\sin\frac{x}{2}\right)+\ell nc,\ \ell ny = 2\ell n\left(\sec\frac{x}{2}\right)+\ell nc$$

$$\Rightarrow y = c.\text{cosec}^2\left(\frac{x}{2}\right), y = c\sec^2\frac{x}{2}$$

$\therefore$ The general solution of the given equation is given by

$$\left[y-\text{cosec}^2\left(\frac{x}{y}\right)\right]\left[y-c\text{cosec}^2\left(\frac{x}{2}\right)\right]=0 \text{ or } \left(y-\frac{c}{1+\cos x}\right)\left(y-\frac{c}{1-\cos x}\right)=0$$

Equations Solvable for y

An equation $f(x, y, p) = 0$ is (said to be) solvable for y if y occurs in first degree and if y can be expressed as an explicit function of x and p. In the case $f(x, y, p) = 0$ can be written as $y = F(x, p)$.

To find the General Solution

Let the given equation $f(x, y, p) = 0$ be solvable for y. Then it can be written as $y = F(x, p)$ --(1)

Differentiating (1) w.r.t x, we get an equation of the form $\frac{dy}{dx} = G\left(x, p, \frac{dp}{dx}\right)$

i.e., $p = G\left(x, p, \frac{dp}{dx}\right)$ *--(2) Equation (2) is a first order D.E in x and p.*

Let $\phi(x,p,c)=0$ --(3) be its general solution. Eliminating p from (1) & (3), we get the general solution of equation (1) in the form $\psi(x,y,c)=0$ If it is not possible to eliminate p from (1) & (3), we write (3) as $x = f_1(p, c)$ and using this, equation (1) as $y = f_2(p, c)$ so that the general solution of (1) is given by $x = f_1(p, c)$ and $y = f_2(p, c)$. Where p is treated as parametric and c is an arbitrary constant and this solution is called Parametric solution.

Note

If the equation (1) does not contain x, then $y = F(P)$. If this is solvable for p we get $p = G(y)$ which can be solved by separating the variables. If it is not solvable for p, we have to proceed as above.

Ex 1. Solve $y = 2px - p^2$

Sol. The given equation $y = 2px - p^2$ -- (1)

is clearly solvable of y. Differentiating (1) w.r.t x we get $p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} \Rightarrow \frac{dx}{dp} + \frac{2}{p}x = 2$ -- (2)

This is a linear equation in x

I.F. $= e^{\int\frac{2}{p}dp} = p^2$

$\therefore$ The general solution of (2) is $xp^2 = \frac{2}{3}p^3 + c \Rightarrow x = \frac{2}{3}p + cp^{2}$ -- (3)

Since p can not be eliminated from (1) and (3) the general solutions of (1) is given by $x = \frac{2}{3}p + cp^{2}, y = 2xp - p^2$ where p is a parameter and c is an orbitrary constant.

Ex 2. **Solve $y = 2xp + x^2p^4$**

Sol. The given equation $y = 2xp + x^2p^4$ -- (1) is clearly solvable for y. Differentiating (1) w.r.t x, and simplifying we get

$$(2xp^3+1)\left(2x\frac{dp}{dx}+p\right)=0 \Rightarrow 2xp^3+1=0 \text{ -- (2) (or) } 2\frac{dp}{dx}+\frac{p}{x}=0 \text{ -- (3)}$$

$$(2) \Rightarrow p = \frac{-1}{\sqrt[3]{2x}} \quad \text{-- (4)}$$

Eliminating p from (1) & (4) We get a singular solution $y=-3\left(\frac{x}{4}\right)^{2/3}$

(which doesnot contain an arbitrary constant).

Solving (3) by seperating the variables, we get $p^2x = c$ -- (5)

Eliminating p from (1) & (5) we get the general solution $y=\pm 2\sqrt{c}\sqrt{x}+c^2$

i.e., $(y-c^2)^2 = 4cx$ where c is an arbitrary constant.

Equations Solvable for x

An equation $f(x,y,p) = 0$ is said to be solvable for x if x occurs in first degree and if x can be expressed as an explicint function of y and p. In this case $f(x,y,p) = 0$ can be written as $x = F(y, p)$

To find the General Solution

Let the given equation be solvable for x. Then it can be written as $x = F(y, p)$ -- (1)

Differentiating w.r.t y, we get an equation of the form $\frac{1}{p}=G\left(y,p,\frac{dp}{dy}\right)$ -- (2)

$\left(\therefore \frac{dx}{dy}=\frac{1}{p}\right)$. This is a first order D.E is y and p. Let $\phi(y,p,c)=0$ -- (3) be its general solution. Eliminating p from (1) & (3), we get the general solution of (1) in the form $\psi(x,y,c)=0$. If it is not possible to eliminate p from (1) & (3), we write (3) as: $x = f_1(p, c)$ and using the equation (1) as $y = f_2(p, c)$, so that the general solution of (1) is given by $x = f_1(p, c)$ and $y = f_2(p, c)$ where p is treated as a parameter and c is an arbitrary constant and this solution is called Parametric solution.

Note

If the equation (1) does not contain y, then $x = F(p)$. If this equation is solvable of p, we get $P=\psi(x)$ and hence can be solved by seperating this variables. If $x = F(p)$ is not solvable for p, we have to proceed as above.

Ex 1. **Solve $y = 2px + y^2p^3$ to find the general solution.**

Sol. The given equation is solvable for x and can be written as $x=\frac{y}{2p}-\frac{1}{2}y^2p^2$ -- (1)

Diff (1), w.r.t y, we get $\frac{1}{p}=\frac{1}{2p}+\frac{y}{2}\left(-\frac{1}{p^2}\right)\frac{dp}{dy}-yp^2-y^2p\frac{dp}{dy}$

$$\Rightarrow \frac{dp}{dy}\left(\frac{y}{2p^2}+y^2p\right)+\left(yp^2+\frac{1}{2p}\right)=0 \Rightarrow \left(\frac{1}{2p}+y^2\right)\left(\frac{y}{p}\frac{dp}{dy}+1\right)=0$$

$$\Rightarrow y^2+\frac{1}{2p}=0 \text{ -- (2) } \frac{dp}{dy}+\frac{p}{y}=0 \text{ -- (3)}$$ (2) leads to a singular solution of equation (1).

$\therefore$ solving (3) $\int\frac{dp}{p}+\int\frac{dy}{y}=\log c \Rightarrow py=c$ i.e., $p=\frac{c}{y}$ -- (4)

Eliminating p from (1) & (4), the general solution of (1) is given by $y^2 = 2cx + c^3$ where c is an arbitrary constant.

Ex 2. **Solve $(x - \tan^{-1}p)(1 + p^2) = p$**

Sol. The given equation is solvable for x and can be written as $\frac{1}{p} = \left[\frac{1}{1+p^2} - \frac{2p^2}{(1+p^2)^2} + \frac{1}{1+p^2}\right]\frac{dp}{dy}$ -- (1)

This equation does not contain y and is not solvable for p.

Differential (1) w.r.t y, we get $\frac{1}{p} = \left[\frac{1}{1+p^2} - \frac{2p^2}{(1+p^2)^2} + \frac{1}{1+p^2}\right]\frac{dp}{dy}$

$$\Rightarrow \frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy} \Rightarrow \frac{2p}{(1+p^2)^2}dp = dy \quad \text{-- (2)}$$

Solving (2), we get $y = c - \frac{1}{1+p^2}$ -- (3)

Even though p can be eliminated from (1) & (3), we can consider the general solution of (!)

$$x = \frac{p}{1+p^2} - \tan^{-1}p \text{ and } y = c - \frac{1}{1+p^2}$$

Where p is treated as a parameter and c is an arbitrary constant.

Clairaut's Equation

A differential equation of the form y = xp + f(p) is called as Clairaut's equation.

Example : i) $y = xp + p - p^3$

ii) $y = px + \frac{p}{\sqrt{1+p^2}}$ are Clairaut's Equations A remarkable fact about Clairaut's equation is its general solution can be obtained simply by replacing p with C in the given equation, where C is an arbitrary constant for example, the general solution of $y = xp + \frac{p}{\sqrt{1+p^2}}$ is $y = cx + \frac{c}{\sqrt{1+c^2}}$ (c is an arbitrary constant)

To find the General Solution

Consider a Clairaut's Equation y = xp + f(p) --- (1) This equation is clearly solvable for y

Differentiable (1) w.r.t x, we get $p = p + [x + f'(p)]\frac{dp}{dx}$

$$\Rightarrow [x + f'(p)]\frac{dp}{dx} = 0 \Rightarrow x + f'(p) = 0 \text{ --- (2) or } \frac{dp}{dx} = 0 \text{ -- (3)}$$

Eliminating p from (1) and (2), we get a singular solution Solving (3), we get p = c -- (4)

Eliminating p from (1) & (4) we get the general solution of (1) as y = cx + f(c) --- (5)

where c is an arbitrary constant. (This justifies the above mentioned remarkable fact).

Ex 1. **Solve $y = xp + p - p^3$, to find the singular and general solution.**

Sol. The given equation $y = xp + (p - p^3)$ --- (1)

is clearly a chairant equation and is solvable for y.

Differential (1) w.r.t x

$$p = p + \left[x + (1 - 3p^2)\right]\frac{dp}{dx} \Rightarrow 3p^2 = x + 1 \text{ or } \frac{dp}{dx} = 0$$

i.e., $p = \pm\sqrt{\dfrac{x+1}{3}}$ --- (2)

or $p = c$ -- (3)

Eliminating p from (1) and (2), we get a singular solution $y^2 = \dfrac{4}{27}(x+1)^3$

Eliminating p from (1) and (3), we get the general solution of (1) as $y = cx + c - c^3$

where c is an arbitrary constant.

Remark :

Some differential equtions (of higher degree) can be transformed to Clairaut's form by a change of dependent and or indpendent variable using some simple substitutions (like $y^2 = Y$, $x^2 = X$, $\log y = Y$ etc).

Ex 2. Find the general solution of $y^2 \log y = pxy + p^2$

Sol. The given D.E is $y^2 \log y = pxy + p^2$ -- (1)

put $\log y = Y$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dY}{dX} \Rightarrow p = yP \text{ where } P = \frac{dY}{dX}$$

Then equation (1) becomes $y^2 Y = xy^2 P + y^2 p^2$

$\Rightarrow Y = xP + P^2$ -- (2)

Equation (2) is a clairant equation

$\therefore$ The general solution of (2) (obtained by subtstituting $P = c$) is $Y = xc + c^2$

$\therefore$ The general solution of (1) is $\log y = cx + c^2$ where c is an arbitrary constant.

Ex 3. Find the general solution of $xy\,(y - px) = x + yp$

Sol. The given equation can be written as $y^2 = xyp + 1 + \dfrac{y}{x}p$ -- (1)

Put $x^2 = X$ and $y^2 = Y$

Then $\dfrac{y}{x} = \dfrac{dy}{dx} = \dfrac{dY}{dX} \Rightarrow \dfrac{y}{x}p = P$ where $P = \dfrac{dY}{dX}$

The equation (1), now becomes $y^2 = x^2 P + 1 + P \Rightarrow Y = XP + 1 + P$ -- (2)

This is a clairaut equation and its general solution is $Y = cX + c + 1$

$\therefore$ The general solution of the given equation is $y^2 = cx^2 + c + 1$

EXERCISE - 1

Solve the following differential equations

1. a) $p^2 - 7p + 12 = 0$ b) $x^2p^2 + xyp - 6y^2 = 0$

c) $xy^2(p^2 + 2) = 2py^3 + x^3$ d) $xyp^2 - (x^2 - y^2)p - xy = 0$

e) $xyp^2 + (x^2 + xy + y^2)p + (x^2 + xy) = 0$

2. a) $y + px = p^2x^4$ b) $y = xp^2 + p$

   c) $y = 3x + \log p$ d) $4yp^2 + 2px - y = 0$

3. a) $y = 2px + p^2y$ b) $x\sqrt{1+p^2} + p = a\sqrt{1+p^2}$

   c) $2px = 2\tan y + p^3\cos^2 y$ d) $p^2 - (y+3)p + x = 0$

4. Find the general solution of the following equations

   a) $y = px + \dfrac{p}{\sqrt{1+p^2}}$ b) $\sin px \cos y = \cos px \sin y + p$

   c) $\sqrt{1+p^2} = \tan(px - y)$ d) $p^3x - p^2y - 1 = 0$

   e) $y^2 = pxy + f\left(\dfrac{y}{x}p\right)$ f) $(px - y)(py + x) = 2p$

   g) $x^2(y - px) = yp^2$ h) $y = 2px + y^2p^3$

   i) $y - 2px + ayp^2 = 0$

   (*Hint:* Put $x^2 = X$ and $y^2 = Y$ in (v) (vi) and (vii), $y^2 = Y$ in (viii) & (ix))

4.15 APPLICATIONS OF D.E'S

If $y = f(x)$ is a non -negative continuous function in $[a, b]$ then the definite integral $\int_a^b f(x)\,dx$ geometrically represents the area bounded by the curve $y = f(x)$ above the X-axis between the lines $x = a$ and $x = b$.

A) Orthogonal Trajectories :

A curve which cuts every member of a given family of curves according to a given law, is called a *trajectory* of the given family.

Definition

A curve which cuts every member of a given family of curves at a right angle, is called an orthogonal trajectory of the given family.

If every member of one family of curves cuts every member of another family of curves at a right angle then the two families of curves are said to be ***orthogonal to each other***.

If every member of a family of curves cuts every other member of the same family at right angles then the family of curves is orthogonal to it self and hence it is called a self orthogonal family or ***self orthogonal system of curves***.

Working Rule

To find the orthogonal trajectories of a given family of curves, f(x, y, c) = 0, where c is an arbitrary constant, the following three steps are to be followed.

Step (i) : Form the D.E of the given family by eliminating c. Let this D.E be $F\left(x, y\frac{dy}{dx}\right) = 0$

Step (ii) : Replace $\frac{dy}{dx}$ *with* $-\frac{dx}{dy}$ *in the above D.E to get the D.E of the orthogonal trajectories as* $F\left(x, y, -\frac{dx}{dy}\right) = 0.$ *Let the new D.E be written as* $G\left(x, y, \frac{dy}{dx}\right) = 0.$

Step (iii) : Solve $G\left(x, y, \frac{dy}{dx}\right) = 0$ *to get the equation of the family of the orthogonal trajectories in the form g(x, y, c') = 0 where c' is an arbitrary consant.*

Note

Given a family of curves $f(r, \theta, c) = 0$ *in polar coordinates, c being a parameter, to get the differential equation of the orthogonal trajectories replace* $\frac{dr}{d\theta}$ *with* $-r^2\frac{d\theta}{dr}$ *in the differential of the given family.*

SOLVED EXAMPLES

1. Find the orthogonal trajectories of the family of parabolas through the origin and foci on the *x*-axis.

Sol. The equation of the given family of parabolas is $y^2 = 4ax$ -- (1)

Where a is a parameter, Eliminating a, the D.E of (1) is $\frac{dy}{dx} = \frac{y}{2x}$ -- (2)

The D.E of the orthogonal trajectories (obtained by replacing $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in (2) is $-\frac{dx}{dy} = \frac{y}{2x} \Rightarrow \frac{dx}{dy} = -\frac{2x}{y}$ -- (3)

Seperating the varaibles and integrating, we get $\int 2x dx + \int y dy = c$

$\Rightarrow x^2 + \frac{y^2}{2} = c \Rightarrow \frac{x^2}{c} + \frac{y^2}{2c} = 1,$ where c is an arbitrary constant.

This is the equation of the orthogonal trajectories of (1)

2. Find the orthogonal trajectories of the family of coaxal circles $x^2 + y^2 = 2\lambda x + c = 0$ where λ is a parameter.

Sol. The given family of circles is $x^2 + y^2 = 2\lambda x + c = 0$ -- (1)

where λ is a parameter and c is a given constant.

Diff (1) w.r.t x, we get $2\lambda = -\left(2x + 2y\frac{dy}{dx}\right)$

$\therefore$ The D.E of the given family (1) is given be $x^2 + y^2 + c = x\left(2x + 2y\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2 + c}{2xy}$ -- (2)

The D.E of the orthogonal trajectories is $-\frac{dx}{dy} = \frac{y^2 - x^2 + c}{2xy}$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2 - c} \qquad \text{-- (3)}$$

Solving by inspection method (3)

$$\Rightarrow x^2 dy - 2xy\,dx = (y^2 + c)dy \;\Rightarrow \frac{x^2 dy - 2xy\,dx}{y^2} = \left(1 + \frac{c}{y^2}\right)dy$$

$$\Rightarrow d\left(-\frac{x^2}{y}\right) = d\left(y - \frac{c}{y}\right) \;\Rightarrow -\frac{x^2}{y} = y - \frac{c}{y} + 2\mu \text{ (no integration)}$$

$\Rightarrow x^2 + y^2 + 2\mu y - c = 0$, where μ is a parameter.

This is the equation of the orthogonal trajectories of the family (!)

3. Show that the family of confocal curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is a parameter is self orthogonal.

Sol. The given family of curves is $\frac{x^2}{a^2+\lambda} + \frac{b^2}{b^2+\lambda} = 1$ -- (1)

$$\Rightarrow (b^2+\lambda)x^2 + (a^2+\lambda)y^2 = (a^2+\lambda)(b^2+\lambda)$$

Diff w.r.t x we get $x(b^2+\lambda) + y\frac{dy}{dx}(a^2+\lambda) = 0 \Rightarrow \lambda = -\left[\frac{b^2x + a^2yy_1}{x + yy_1}\right]$

$$\Rightarrow a^2 + \lambda = \frac{(a^2-b^2)x}{x+yy_1} \quad \text{-- (2)} \qquad b^2 + \lambda = \frac{-(a^2-b^2)yy_1}{x+yy_1} \quad \text{-- (3)}$$

Eliminating (λ) from (1) using (2) and (3) we get the differential equation of (1)

as $(x + yy_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2$ -- (4)

Replacing y_1 with $-\frac{1}{y_1}$ in (4), the D.E of the orthogonal trajectories is

$$\left(x - \frac{y}{y_1}\right)(x + yy_1) = a^2 - b^2 \qquad \text{-- (5)}$$

Since the D.E's (4) and (5) are identical, the given family of curves is a self orthogonal system.

Note :

Equation (3) may be transformed into a linear equation is u and y by substituting $x^2 = u$

4. Find the orthogonal trajectories of the family of cardioids $r = a\,(1 - \cos\theta)$ where a is a parameter.

Sol. The given family of curves is $r = a\,(1 - \cos\theta)$ -- (1)

Diff. w.r.t θ, we get $\frac{dr}{d\theta} = a\sin\theta$ -- (2)

Eliminating a from (1) and (2) we get the D.E. of (1) as $\frac{dr}{d\theta} = r\cot\frac{\theta}{2}$ -- (3)

The D.E of the orthogonal trajectories (obtained by replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$)

is $-r^2\frac{d\theta}{dr} = r\cot\frac{\theta}{2} \;\Rightarrow \tan\left(\frac{\theta}{2}\right)d\theta + \frac{1}{r}dr = 0$

On integration, we get $2\log\left(\sec\frac{\theta}{2}\right) + \log r = \log c \;\Rightarrow r\sec^2\frac{\theta}{2} = c$

$\Rightarrow r = c(1 + \cos\theta)$, where c is a parameter.

This is the equation of the orthogonal trajectories of (1).

EXERCISE - 2

*1. What is the area under the curve $y = \sin x$ between the ordinates $x = 0$ and $x = \pi$

2. Find the orthogonal trajectories of the following families of curves.

   a) $x^2 + y^2 = a^2$ b) $ay^2 = x^3$ c) $xy = a^2$ d) $x^{2/3} + y^{2/3} = a^{2/3}$

   where a is a parameter.

3. Find the orthogonal trajectories of the family of parabolas through the origin and foci on the y-axis.

4. Find the orthogonal trajectories of the circles passing through the origin and having centres on the x-axis.

5. Find the orthogonal trajectories of the family of circles $x^2 + y^2 + 2fy + 1 = 0$ where f is a parameter.

6. Show that the system of confocal and co-axial parabolas $y^2 = 4a(x+a)$ where a is a parameter, is self orthogonal.

7. Find the orthogonal trajectories of the following families of curves, a being the parameter.

   a) $r^n \sin n\theta = a^n$ b) $r\theta = a$ c) $r = a\theta$ d) $r^n = a^n \cos n\theta$ e) $r(1 + \cos\theta) = 2a$

B) Geometrical Applications

Let $y = f(x)$ be a differentiable curve and $P(x, y)$ be a fixed point on the curve. Then, at the point P,

i) the slope of the tangent to the curve is $m = \left(\dfrac{dy}{dx}\right)_P = \dfrac{dy}{dx}$

ii) the gradient of the curve is $\left(\dfrac{dy}{dx}\right)_P$

iii) the slope of the normal to the curve is $m^1 = -\left(\dfrac{dx}{dy}\right)_P$

iv) Length of the tangent, $PT = \left|\dfrac{y}{\left(\dfrac{dy}{dx}\right)}\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\right| = \left|y\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\right|$

v) Length of the normal, $PN = \left|y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\right|$

vi) Length of the subtangent $ST = \left|\dfrac{y}{\left(\dfrac{dy}{dx}\right)}\right| = \left|y\dfrac{dx}{dy}\right|$

vii) Length of the subnormal $SN = \left|y\dfrac{dy}{dx}\right|$

For the tangent at P, x intercept $= x - y\left(\frac{dx}{dy}\right)$, y intercept $= y - x\left(\frac{dy}{dx}\right)$

For the normal at P, x intercept $= x + y\left(\frac{dy}{dx}\right)$ y intercept $= y + x\left(\frac{dy}{dx}\right)$

The *radius vector* of the point P is $OP = \sqrt{x^2 + y^2}$

SOLVED EXAMPLES

1. Find the equation of the curve for which the cartesian subtangent is a constant

Sol. Let $y = f(x)$ be the required curve

The cartesian subtangent at any point is a constant

$\Rightarrow y\frac{dx}{dy} = k$ (k is a constant) $\Rightarrow \frac{k}{y}dy = dx \Rightarrow \int \frac{k}{y}dy = \int dx + c_1$

$\Rightarrow k \log y = x + c_1 \Rightarrow y = e^{\frac{x}{k}+c_1} \Rightarrow y = ce^{x/k}$

where c is an arbitrary constant. This is the required curve.

2. Find the equation of the curve for which the cartesian subtangent varies as the abscissa

Sol. Cartesian subtangent varies as absiccsa $\Rightarrow y\frac{dx}{dy} \propto x$

Let $y\frac{dx}{dy} = kx$ (k is the proportionality constant)

$\Rightarrow \frac{k}{y}dy = \frac{1}{x}dx \Rightarrow \int \frac{k}{y}dy = \int \frac{1}{x} + \log c \Rightarrow k \log y = \log x + \log c$

$\Rightarrow y^k = cx\, c$ is an arbitrary constant.

3. Find the equation of the curve for which the cartesian subnormal is a constant

Sol. The cartesian subnormal is constant

$\Rightarrow y\frac{dy}{dx} = k$ (k is a constant) $\Rightarrow ydy = kdx$

$\Rightarrow \int ydy = k\int dx + c \Rightarrow \frac{y^2}{2} = kx + c_1$

$\Rightarrow y^2 = 2kx + c, c$ is an arbitrary constant.

This is the required curve.

4. Find the equation of the curve for which the cartesian subnormal is equal to the abscissa

Sol. The equation subnormal is equal to absiccsa.

$y\frac{dy}{dx} = \pm x \Rightarrow ydy = \pm xdx \Rightarrow \int ydy = \int \pm xdx \Rightarrow y^2 = \pm x^2 + c$

$\Rightarrow y^2 \pm x^2 = c$; where c is an arbitrary

5. A curve is passing through the point (1, 1) and has the property that the perpendicular distance of the origin from the normal at any point *P* from the *x*-axis. Determine the equation of the curve.

Sol. Let $P(x_1, y_1)$ be a point on the curve $m = \left(\frac{dy}{dx}\right)_P \neq 0$

Then the equation of the normal at P is $x + my = x_1 + my_1$ --(1)

The perpendicular distance of the origin to the normal (1) is equal to the distance of P from the x-axis

$$\Rightarrow \frac{|x_1 + my_1|}{\sqrt{1+m^2}} = |y_1|$$

$$\Rightarrow (x_1 + my_1)^2 = y_1^2(1+m^2) \Rightarrow 2x_1y_1m = y_1^2 - x_1^2$$

Replacing (x_1, y_1) with (x, y) and m with $\frac{dy}{dx}$,

we get $2xy\frac{dy}{dx} = y^2 - x^2$ --(2)

This is the D.E of the family of curves satisfying the given condition. Solving (2)

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This is homogeneous D.E. Put $y = vx$.

Then $v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$ $\Rightarrow x\frac{dv}{dx} = \frac{-(v^2+1)}{2v}$

$$\Rightarrow \int \frac{2v}{v^2+1}dv + \int \frac{1}{x}dx = \log c \Rightarrow \log(v^2+1) + \log x = \log c$$

$$\Rightarrow (v^2+1)x = c \qquad \Rightarrow x^2 + y^2 = cx \quad \left(\because v = \frac{y}{x}\right)$$

This curve passes through $(1,1) \Rightarrow c = 2$

$\therefore$ The required curve is $x^2 + y^2 = 2x$

6. A curve is such that the segment of the normal, drawn at any point *P* on the curve, between *x*-axes and *y*-axes, is bisected at the point *P*. Find the curve, given that it passes through the point (1, 3).

Sol. $P(x, y)$ is the mid point of the normal at P, intercepted between the coordinate axes

$$\Rightarrow \frac{1}{2}(x \text{ intercept of the normal}) = x$$

$$\Rightarrow \frac{1}{2}\left(x + y\frac{dy}{dx}\right) = x \Rightarrow y\frac{dy}{dx} = x \Rightarrow ydy - xdx = c$$

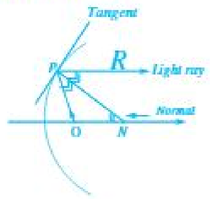
$$\Rightarrow y^2 - x^2 = 2c \text{ (no integration)}$$

This curve passes through $(1, 3) \Rightarrow 2c = 8$

$\therefore$ The required curve is $y^2 - x^2 = 8$

Note : The above problem can also be solved using $\frac{1}{2}$ (y intercept of the normal) = y

Ex 7. Consider a curved mirror $y = f(x)$ passing through (8, 6) having the property that all light rays emerging from the origin, after getting reflected from the mirror, travel in the direction parallel to the positive x-axis. Find the equation of the curved mirror.



Sol. Let $P(x, y)$ be any point on the curve

Slope of $OP = \frac{y}{x}$, Slope of $PN = -\frac{dx}{dy}$

Slope of $PR = 0$

$$\angle OPN = \angle NPR \Rightarrow \frac{\left(-\frac{dx}{dy}\right)-\left(\frac{y}{x}\right)}{1+\left(\frac{-dx}{dy}\right)\left(\frac{y}{x}\right)} = \frac{0-\left(-\frac{dx}{dy}\right)}{1+(0)\left(\frac{-dx}{dy}\right)}$$

$$\Rightarrow y\left(\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0 \quad \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

$$\Rightarrow yp^2 + 2xp - y = 0, p = \frac{dy}{dx}$$

This D.E is solvable for x and $x = \frac{y}{2}\left(\frac{1}{p} - p\right)$ -- (1)

Diff (1) w.r.t y

$$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - p\right) + \frac{y}{2}\left(-\frac{1}{p^2} - 1\right)\frac{dx}{dy} \Rightarrow \left(\frac{1}{2p} + \frac{p}{2}\right)\left(1 + \frac{y}{p}\frac{dp}{dx}\right) = 0 \quad \Rightarrow 1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\Rightarrow \int \frac{1}{p}dp + \int \frac{1}{y}dy = \log c \quad \Rightarrow yp = c \quad \Rightarrow p = \frac{c}{y}$$ -- (2)

Eliminating P from (1) & (2),, we get $y^2 = 2cx + c^2$

This curve passes through (8, 6)

$\Rightarrow c^2 + 16c - 36 = 0 \quad \Rightarrow (c+18)(c-2) = 0 \Rightarrow \; c = 2 \quad (\because c \neq -18)$

$\therefore$ The required equations of the curved mirror is $y^2 = 4x + 4$

Aliter : $\angle OPN = \angle NPR = \angle ONP$

$$\Rightarrow OP = ON \quad \Rightarrow \sqrt{x^2 + y^2} = \left|x + y\frac{dy}{dx}\right| \quad \Rightarrow \pm\left(\frac{xdx + ydy}{\sqrt{x^2 + y^2}}\right) = dx$$

$$\Rightarrow \pm d(\sqrt{x^2 + y^2}) = dx \qquad \Rightarrow \pm\sqrt{x^2 - y^2} = x + c \text{ (on integration)}$$

$$\Rightarrow x^2 + y^2 = x^2 + 2cx + c^2 \qquad \Rightarrow y^2 = 2cx + c^2$$

This passes through $(8,6) \Rightarrow c^2 + 16c - 36 = 0 \quad \Rightarrow c = -18, c = 2$

$\therefore$ The curve we get are $y^2 = 36(9 - x)$, $y^2 = 4x + 4$

The required curve is $y^2 = 4x + 4$

EXERCISE - 3

1. The tangent at any point P of a curve meets the x-axis in T. Find the equation of the curve for which $OP = PT$, "O" being the origin.
2. A curve is such that any point P on it is as four from the origin as prove the point in which the normal at P meets the x-axis. Show that it must be an equilateral hyperbola or a circle (or) Find the equation of the curve for which the length of the normal is equal to the radius vector.
3. The normal is drawn at a point $P(x,y)$ of a curve. It meets the x-axis at G. If PG is of constant length k, then show that the differential equation describing such curves is $y\dfrac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through $(0, k)$.
4. Find the curve for which the intercept cut off by a tangent on x-axis is equal to 4 times the ordinate of the point of contact.
5. Find the equation of the curve which is such that the position of the x-axis cut off between the origin and the tangent at any point is proportional to the ordinate of the point.
6. The tangent at a point P of a curve meets the y-axis is N and the line through P parallel to the y-axis meets the x-axis at M. O is the origin. If the area of ΔMON is a constant. Show that the curve is a hyperbola.
7. The normal at any point P of a curve cuts OX is G and N is the foot of the ordinate of P. If NG varies as the square of the radius vector from the origin "O", find the curve.
8. A curve is such that the length of the perpendicular from the origin on the tangent at any P of the curve is equal to the abscissa of P. Prove that the different equation of the curve is $y^2 - 2xy\dfrac{dy}{dx} - x^2 = 0$ and hence find the curve.
9. Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\dfrac{x^4 + 2xy - 1}{1 + x^2}$
10. Find the equation of the curve such that the distance between the tangent at any point and the origin is equal to the distance between the origin and the normal at that point.
11. A normal is drawn at a point $P(x,y)$ of a curve. If it meets the X and Y axes at A and B respectively, such that $\dfrac{1}{OA} + \dfrac{1}{OB} = 1$ where O is the origin. Find the equation of such a curve passing through the point $(5, 4)$.
12. Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point (i.e., y-intercept of the tangent) is a constant equal to a^2.
13. The population of a country doubles in 50 years. In how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants.

14. The rate of cooling of a substance is proportional to the difference of the temperature of the substance and the air. If the substance cools from 36°C to 34°C in 15 minutes find when the substane will have the temperature 32°C, it being known that the constant temperature of the air is 30°C.

15. A radio-active substance decays with time such that at any moment the rate of dacay of volume is proportional to the volume at that time. Calculate the half life of the substance (i.e., the time it taken for half two substance to disappear) if 20% of it disappears in 15 years.

ANSWERS

EXERCISE - 4.1

1. a) order 1, degree 2 b) order 2, degree 1 c) order 2, degree 2
   d) order 2, degree 3 e) order 2, degree 4 f) order 3, degree 2
   g) order 2, degree 6 (one the factors has order 2, degree 1)
   h) order 3, degree 3 i) order 2, degree is not defined
   j) order 1, degree is not defined k) order 2, degree 6
2. a) Since there is only one arbitrary constant, the order is 1
   b) Since there are 3 independent arbitrary constants, order 3
   c) Since there are 2 independent arbitrary constants, order 2
   d) $x^2 + y^2 = a^2$, a is a parameter has only one arbitrary constant, order 1
   e) Since these are only 3 independent arbitrary constants, order 3
3. a) $y = x\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right)^2$ b) $\left(\frac{dy}{dx}\right)^2 - (x+1)\frac{dy}{dx} + y = 0$
   c) $y_2 + 9y = 0$ d) $\frac{d^2y}{dx^2} + y = 0$ e) $y_2 = y$
4. a) $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$ b) $xy_2 + 2y_1 - xy = 0$
   c) $y_2 - 2Ky_1 + K^2y = 0$ d) $y_2 + n^2y = 0$ e) $y_2 - 7y_1 + 12y = 0$ f) $x^2y_2 - 2xy_1 + 2y = 0$
   g) $x^2y_2 + 2xy_1 - 2y = 0$ h) $xyy_2 + x(y_1)^2 - yy_1 = 0$
5. a) $y^2 - x^2 = 2xy\frac{dy}{dx}$ b) $(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$
6. $yy_1^2 + 2xy_1 - y = 0$ 7. $2ay_2 + y_1^3 = 0$ 8. $xyy_2 + x(y_1)^2 - yy_1 = 0$
9. $x\frac{dy}{dx} + y = 0$ 10. $y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = r^2$

EXERCISE - 4.2

1. a) $\frac{1}{1+x^2}dx+\frac{1}{1+y^2}dy=0$ b) $\frac{1}{x+a}dx+\frac{1}{ay^2-y}dy=0$

   c) $(e^x+x^2)dx-e^y dy=0$ d) $x^2dx+\frac{1}{(1-e^{3y})}dy=0$

2. a) $x^2=cy$ b) $x^2+y^2=2c$ c) $\tan^{-1}y=\tan^{-1}+c$ d) $\sin^{-1}x+\sin^{-1}y=c$

   e) $y=c\cosh^2 x$ f) $e^x+e^{-y}=c$ g) $e^{-x}-e^{-y}=c$ h) $y=cx.e^{x-y}$

   i) $x+y+\log(xy)=c$

3. $(x^2+1)(y^2+1)=c$ 4. $(x-1)^2(y+2)=cy$ 5. $\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)=c$

6. $e^{x+y}=c(e^x+1)(y+1)$ 7. $\tan x.\tan y=c$ 8. $\sin x.\sin y=c$

9. $x\sqrt{1+x^2}+y\sqrt{1+y^2}+\log[(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})]=c$

10. $(1+x^2)(1+y^2)=cx^2$ 11. $y=c(x+5)(1-5y)$ 12. $e^y=e^x+\frac{x^3}{3}+c$

13. $1-e^{-2y}=ce^{x^2}$ 14. $y\sin y=x^2\log x+c$ 15. $\frac{x}{y}+e^{x^2}=c$

EXERCISE - 4.3

1. $x+e^{-(x+y)}=c$ 2. $\tan(x+y)-\sec(x+y)=x+c$

3. $\sin(2x+2y)=2x-2y+c$ 4. $2\sqrt{y-x}+2\log(\sqrt{y-x}-1)=x+c$

5. $\sin(y-x)=ce^{x^2/2}$ 6. $(x-y+1)=c(1-x+y)e^{2x}$

7. $3(4x+9y+1)=2\tan(6x+c)$ 8. $(x-2y)^2+2x=c$

EXERCISE - 4.4

1. a) $\frac{dy}{dx}=\frac{y}{x}+\sqrt{1+\left(\frac{b}{x}\right)^2}$ b) $\frac{dy}{dx}=\frac{\frac{y}{x}\tan^{-1}\left(\frac{y}{x}\right)-1}{\tan^{-1}\left(\frac{y}{x}\right)}$ c) $\frac{dy}{dx}=\frac{y}{x}\left[\log\left(\frac{y}{x}\right)+1\right]$

2. a) $\frac{dx}{dy}=\frac{e^{x/y}\left(\frac{x}{y}-1\right)}{1+e^{x/y}}$ b) $\frac{dx}{dy}=\frac{x}{y}+e^{-2\left(\frac{x}{y}\right)}$ c) $\frac{dx}{dy}=\frac{\sqrt{\left(\frac{x}{y}\right)^2+1}-\left(\frac{x}{y}\right)^2}{\left(\frac{x}{y}\right)}$

3. a) $2x=(x-y)(\log x+c)$ b) $e^{y/x}=ky$ c) $2\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)=\log x+c$

   d) $2\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)=\log x+c$ e) $x^2-2xy+y^2=c$ f) $x^2-y^2=ky$ g) $x^2(x^2-2y^2)=c$

4. $\tan\left(\frac{y}{x}\right)=\log x+c$ 5. $kx=e$ 6. $cx^2=y+\sqrt{x^2+y^2}$ 7. $y-2x=kx^2y$

8. $y^3=3x^3\log(cx)$ 9. $xy\sqrt{2xy-3x^2}=c$ 10. $xy(y-x)=c$ 11. $y^3e^{x/y}=cx^2$

12. $\log\left(\frac{x+y}{e}\right)=\frac{-2xy}{(x+y)^2}$ 13. $(x+y)^3=c(x-y)$ 14. $x^2+2y^2(c+\log y)=0$ 15. $xy^2=c(x-y)^2$

16. $xy\cos\left(\frac{y}{x}\right)=c$ 17. $\tan\left(\frac{y}{x}\right)=1-\log x$

EXERCISE - 4.5

1. a) $3x^2-2xy+7y^2+14x+16y=2c$ b) $x^2-xy-y^2+x+3y=c$

c) $3x^2+4xy+3y^2-10x+10y=c$ d) $3x^2+4xy+3y^2-10x-10y=c$

e) $6x^2+5xy+y^2-9x-4y=c$ f) $2x^2-2xy+3y^2+x-2y=k$

2. a) $x-2y+\log(x-y+2)=c$ b) $\left[2\tan^{-1}\left(\frac{2y+1}{2x+1}\right)=\log\left|c^2\left(x^2+y^2+x+y+\frac{1}{2}\right)\right|\right]$

3. $\left[6y-3x+\log(3x+3y+4)\right]=0$ 4. $x+2y+\log(2x+y-1)=c$

5. $\log(4x+8y+5)=4x-8y+c$ 6. $(x-y)+\log(x+y)=c$

7. a) $(3y-2x-1)^2(2y+x-2)=c$ b) $\left[(x^2-2y^2-2x-4y-2)=c\left[\frac{x-y\sqrt{2}-\sqrt{2}-1}{x+y\sqrt{2}+\sqrt{2}-1}\right]^{1/\sqrt{2}}\right]$

c) $(\sqrt{3}\log(Y^2-2XY-2X^2)+2\log\left\{\frac{Y-(1+\sqrt{3})X}{Y-(1-\sqrt{3})X}\right\}=C$ Where $X=x-1$, $Y=y-2$

8. $(x+y-1)^2(y+2x-3)^3=c$ 9. $(2x-y)^2=c(x+2y-5)$

10. $\left(x+y+\frac{4}{3}\right)(x-y+2)^3=c$

EXERCISE - 4.6

1. a) $\frac{1}{x}$ b) $\frac{1}{y}$ c) $\frac{1}{\sqrt{1-x^2}}$

2. $yx^2=\frac{2}{5}x^5+c$ 3. $y\cos x=e^x+c$ 4. $y\sec x=\log(\sec x)+c$ 5. $y=ce^x+e^{-x}$

6. $y(1+x^3)=x+\frac{x^3}{3}+c$ 7. $y(\sec x+\tan x)=\sec x+\tan x-x+c$

8. $y\sec x=\frac{1}{4}[2x+\sin 2x]+c$ 9. $yx^2=\frac{x^2}{4}(2\log x-1)+c$

10. $xy=xe^x+c$ 11. $ye^{\tan^{-1}x}=\frac{1}{2}e^{2\tan^{-1}x}+c$

12. $y=\tan^{-1}x-1+ce^{-\tan^{-1}x}$ 13. $y\log x=(\log x)^2+c$ 14. $y=-(\sin x+1)+ce^{\sin x}$

15. $y\sec x=\tan x+\frac{1}{3}\tan^3 x+c$ 16. $y(1+x^2)=\frac{4}{3}x^3+c$

17. $y(x^2+1)^2 = x+c$ 18. $y = 1+y = 1+\cot x + ce^{\cot x}$ 19. $\frac{xy}{x-1} = \frac{x^5}{5} - \frac{x^4}{4} + c$

20. $y\left(\frac{x-1}{x^2}\right) = x^2 - x + c$ 21. $y = \sqrt{1-x^2} + c(1-x^2)$ 22. $y = x(x-2)[x+2\log(x-2)]$

23. $\frac{x}{y} = y^2 + c$ 24. $xy = 1 + y + cye^{1/y}$ 25. $\frac{1}{x}e^{\frac{y^2}{2}} = e^{\frac{y^2}{2}}((2-y^2)) + c$

26. $y^2(1+x^2+ce^{x^2}) = 1$ 27. $xdx \pm ydy = \frac{1}{2}d(x^2 \pm y^2)$ 28. $\frac{1}{y} = x^2 - 2 + ce^{-\frac{x^2}{2}}$

EXERCISE - 4.7

1. $2xy + x^4 = c$ 2. $-\frac{1}{xy} + \log y = c$ 3. $x + y^3 = cy$ 4. $x^2 + y^2 = ce^{2\tan^{-1}\left(\frac{y}{x}\right)}$

5. $\log\left(\frac{x}{y}\right) = c + \frac{1}{xy}$ 6. $\log(x^2+y^2) = y^2 + ca$ 7. $\tan^{-1}\left(\frac{y}{x}\right) = x + c$

8. $ax^2y + 2e^x = cy$ 9. $\frac{e^x}{y} + x^2 = c$ 10. $ye^x + x^2 = cy$ 11. $\sin\left(\frac{x}{y}\right) = \frac{1}{2}x^2y^2 + c$

12. $\sin^{-1}\left(\frac{y}{x}\right) = m\frac{x^2}{2} + c$ 13. $\frac{y}{x} - \frac{1}{2}\frac{1}{x^2+y^2} = c$ 14. $\tan(x^2+y^2) = \frac{x^2}{y^2} + c$

15. $\sin^{-1}\left(\frac{1}{a}\sqrt{x^2+y^2}\right) = \tan^{-1}\left(\frac{b}{x}\right) + c$

EXERCISE - 1

1. a) $(y-3x-c)(y-4x-c) = 0$ b) $(x^2y-c)(y-cx^2) = 0$
   c) $(y^2-x^2-c)(y^2-x^2) - cx^4 = 0$ d) $(xy-c)(y^2-x^2-c) = 0$
   e) $(x^2+y^2-c)(2xy+x^2-c) = 0$

2. a) $y + \frac{c}{x} = c^2$ b) $x = c - p + \log|p|(p-1)^{-2}$, $y = xp^2 + p$, p is a parameter
   c) $y = 3x + \log\left(\frac{3}{1-ce^{3x}}\right)$ d) $y^2 = 2cx + 4c^2$

3. a) $y^2 = 2cx + c^2$ b) $(x-a)^2 + (y+c)^2 = 1$ c) $2cx = 2\sin y + c^2$
   d) $x = 2p + \frac{c}{\sqrt{p^2-1}}$, $y = p^2 - 1 + \frac{c}{\sqrt{p^2-1}}$ where p is a parametric, c is an arbitrary constant.

4. a) $y = cx + \frac{c}{\sqrt{1+c^2}}$ b) $y = cx - \sin^{-1}(c)$ c) $y = cx - \tan^{-1}\sqrt{1+c^2}$
   d) $y = cx - \frac{1}{c^2}$ (singular solution: $4y^3 = -27x^2$) e) $y^2 = cx^2 + f(c)$
   f) $y^2 = cx^2 - \frac{c}{c+1}$ g) $y^2 = cx^2 + c^2$ h) $y^2 + 2cx + c^3 = 0$
   i) $y^2 = cx - \frac{ac^2}{4}$ (*Hint:* Put $x^2 = X$ and $y^2 = Y$ in (v) (vi) and (vii), $y^2 = Y$ in (viii) & (ix))

EXERCISE - 2

1. 2

2. a) $y = cx$ b) $2x^2 + 3y^2 = c$ c) $x^2 - y^2 = c$ d) $y^{4/3} - x^{4/3} = c^{4/3}$

where a is a parameter.

3. $\dfrac{x^2}{c} + \dfrac{y^2}{2c} = 1$ 4. $x^2 + y^2 - cy = 0$ 5. $x^2 + y^2 + 2gx - 1 = 0$]

7. a) $r^n \cos n\theta = c$ b) $r^2 = ce^{\theta^2}$ c) $r^2 = ce^{-\theta a^2}$

d) $r^n = c^n \sin n\theta$ e) $r(1-\cos\theta) = 2c$

EXER CISE - 3

1. $xy = c^2$ (or) $y = cx$ 2. $y^2 \pm x^2 = c^2$ 3. $x^2 + y^2 = k^2$ 4. $cy^4 = e^{4x}$

5. $y = c \cdot e^{-\frac{x}{ky}}$ 6. $2cx^2 - 2xy + k = 0$ 7. $y^2 + x^2 + \dfrac{x}{k} + \dfrac{1}{2k^2} = 2c(e^{2kx})$

8. $x^2 + y^2 = cx$ 9. $y = (x^2 + 1)(x - 2\tan^{-1}x)$

10. $\sqrt{x^2 + y^2} = ce^{-\tan^{-1}(y/x)}, c\sqrt{x^2 + y^2} = e^{\tan^{-1}\left(\frac{y}{x}\right)}$

11. $(x - 1)^2 + (y - 1)^2 = 25$

12. $y = \pm\dfrac{a^2}{2x} + cx$

13. 79 years (approximately)

14. 40.6 minutes (approximately)

15. 46.6 years (approximately)