CHAPTER 1

Geometry

CIRCLES

• EQUATION OF A CIRCLE • TANGENT AND NORMAL •
• CHORD OF CONTACT • POLE AND POLAR •
• CONJUGATE POINTS • INVERSE POINT •
• COMMON TANGENTS • CENTRES OF SIMILITUDE •

1.0 INTRODUCTION

In this chapter we begin with the definition of a circle and discuss about its general equation, its tangents and normals and various other properties like the power of a point, pole and polar with respect to a circle, relative positions of two circles and their common tangents.

Definition

The locus of a point in a plane which moves in such a way that its distance from a fixed point in the plane is always a constant, is called a circle.

Let C be a given fixed point in a plane and r be a non - negative real number. Then the set of points P in the plane such that $CP = r$, is a circle. The fixed point C is called the ***centre*** and r is called the ***radius*** of the circle.

Definition

A circle of radius one unit is called a unit circle. A circle of radius zero is called a point circle.

A point circle contains only one point, the centre of the circle.

1.1 EQUATION OF A CIRCLE

In this section we discuss about different forms of the equation of a circle and the conditions for a general second degree equation in x and y to represent a circle.

THEOREM-1.1 **The equation of the circle with centre at (x_1,y_1) and radius r is $(x-x_1)^2+(y-y_1)^2=r^2$.**

Proof : Let $P(x, y)$ be any point on the circle with centre $C(x_1, y_1)$ and radius r. Then, by definition,

$$CP=r \Leftrightarrow CP^2 = r^2 \Leftrightarrow (x-x_1)^2 + (y-y_1)^2 = r^2$$

This equation is the locus of P and represents the equation of the required circle.

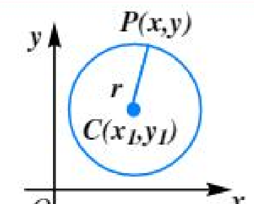


Fig1.1

Note

The equation of the circle with centre at the origin and radius r is $x^2 + y^2 = r^2$.
This is called the standard form of the equation of a circle.

Observation :

The equation of the circle with centre $C(x_1, y_1)$ and radius r is $(x-x_1)^2+(y-y_1)^2=r^2$.
On expanding, we get $x^2+y^2-2x_1x+2y_1y+(x_1^2+y_1^2-r^2) = 0$
This equation is of the form $x^2+y^2+2gx+2fy+c= 0$ and it can be observed that

a) **It is a second degree equation in x and y**

b) **The coefficients of x^2 and y^2 are equal and**

c) **The coefficient of xy is zero**

Note

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is called the general form of the equation of a circle.

Example :

i) The equation of the circle with centre (2, 3) and radius 5 is $(x-2)^2 + (y-3)^2 = 25$
$\Rightarrow x^2 + y^2 - 4x - 6y - 12 = 0$

ii) The equation of the unit circle with centre (1, 2) is $(x-1)^2 + (y-2)^2 = 1$
$\Rightarrow x^2 + y^2 - 2x - 4y + 4 = 0$

iii) The equation of the point circle with centre (1,3) is $(x-1)^2 + (y-3)^2 = 0$
$\Rightarrow x^2 + y^2 - 2x - 6y + 10 = 0$

iv) The equation of the circle with centre at origin and radius 3 is $x^2 + y^2 = 9$.

THEOREM-1.2 The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a real circle, if $g^2 + f^2 - c \geq 0$.

Proof : The given equation $x^2 + y^2 + 2gx + 2fy + c = 0$ -- (1)

can be written as $x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$

$$\Rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

$$\Rightarrow (x-(-g))^2 + (y-(-f))^2 = (\sqrt{g^2+f^2-c})^2 \quad ..(2)$$

Clearly this equation represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$. This circle is a real circle (i.e., a circle in the real plane) only if the radius is a non-negative real number i.e., if $g^2 + f^2 - c \geq 0$.

Note

i) *If $g^2 + f^2 - c > 0$ then equation (1) represents a circle with centre $(-g, -f)$ and non-zero radius $\sqrt{g^2+f^2-c}$.*

ii) *If $g^2 + f^2 - c = 0$ then equation (2) becomes $(x + g)^2 + (y + f)^2 = 0$, which is a point circle representing the point $(-g, -f)$.*

iii) *For the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$*

$$\text{Centre} = (-g, -f) = \left(\frac{-\text{coefficient of } x}{2}, \frac{-\text{coefficient of } y}{2}\right)$$

$$\text{Radius} = \sqrt{g^2+f^2-c} = \sqrt{\left(\frac{\text{coefficient of } x}{2}\right)^2 + \left(\frac{\text{coefficient of } y}{2}\right)^2 - (\text{constant})}$$

iv) *Equation of any circle passing through origin (0,0) can be taken as $x^2 + y^2 + 2gx + 2fy = 0$*
($\because$ (0,0) is a point on the circle $\Rightarrow c = 0$).

v) *Equation of a circle having its centre on the X-axis can be taken as $x^2 + y^2 + 2gx + c = 0$*
($\because$ Y-coordinate of centre is zero).

vi) *Equation of a circle having its centre on the Y-axis can be taken as $x^2 + y^2 + 2fy + c = 0$*
($\because$ X-coordinate of centre is zero).

Remark : Since there are three independent constants g, f, c in the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, three independent geometrical conditions are sufficient to determine the circle uniquely. For example, when three points on a circle or three tangents to circle or two tangents to a circle and a point on it are given, the circle can be determined.

THEOREM-1.3 **The conditions for the general second degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a circle are**

i) $a = b \neq 0$ ii) $h = 0$ and iii) $g^2 + f^2 - ac \geq 0$

Proof : The given equation can be written as

$$x^2 + \frac{2h}{a}xy + \frac{b}{a}y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0 \quad (a \neq 0)$$

Taking $g' = \frac{g}{a}$, $f' = \frac{f}{a}$ and $c' = \frac{c}{a}$, the above equation becomes

$$x^2 + \frac{2h}{a}xy + \frac{b}{a}y^2 + 2g'x + 2f'y + c' = 0$$

This represents a (real) circle if

i) coefficient of x^2 = coefficient of y^2 $\Rightarrow 1 = \frac{b}{a} \Rightarrow a = b$

ii) coefficient of $xy = 0$ $\Rightarrow \frac{2h}{a} = 0 \Rightarrow h = 0$

iii) radius $\geq 0 \Rightarrow g'^2 + f'^2 - c' \geq 0 \Rightarrow \frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a} \geq 0 \Rightarrow g^2 + f^2 - ac \geq 0$

Corollary :

If $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle, then its centre is $\left(\frac{-g}{a}, \frac{-f}{a}\right)$ and its radius is $\frac{\sqrt{g^2 + f^2 - ac}}{|a|}$.

Definition

Two circles are said to be concentric if they have the same centre.

Fig1.2

Note

i) *Equations of two concentric circles differ by a constant only.*

ii) *The equation of the circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is of the form $x^2 + y^2 + 2gx + 2fy + k = 0$, k is an unknown constant.*

SOLVED EXAMPLES

***1. Find the centre and radius of the circle $x^2+y^2+2x-4y-4=0$.**

Sol. The given circle is $x^2+y^2+2x-4y-4=0$

Here $2g=2 \Rightarrow g=1$,

$2f=-4 \Rightarrow f=-2, c=-4$

$\therefore$ Centre $=(-g,-f)=(-1,2)$ and

Radius $=\sqrt{g^2+f^2-c}=\sqrt{1+4+4}=\sqrt{9}=3$

***2. Find the centre and radius of the circle $3x^2+3y^2-6x+4y-4=0$**

Sol. The given circle is $x^2+y^2-2x+\frac{4}{3}y-\frac{4}{3}=0$

$$\text{Centre}=\left(\frac{-\text{coefficient of } x}{2},\frac{-\text{coefficient of } y}{2}\right)=\left(\frac{-(-2)}{2},\frac{-4/3}{2}\right)=\left(1,-2/3\right)$$

$$\text{Radius}=\sqrt{g^2+f^2-c}=\sqrt{(-1)^2+\left(\frac{2}{3}\right)^2+\frac{4}{3}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$$

> ***Remember :***
> *If (x_1, y_1) is centre and 'r' is radius, then the equation of the circle is*
> $(x-x_1)^2+(y-y_1)^2=r^2$.

3. Find the equation of the circle with centre (2, 3) and passing through the point (2, –1).

Sol. Let $C=(2,3)$ and $P=(2,-1)$; Radius of the circle $=CP=4$

$\therefore$ The equation of the required circle is $(x-2)^2+(y-3)^2=16$

$\Rightarrow x^2+y^2-4x-6y-3=0$

***4. Find the equation of the circle passing through (2, 3) and concentric with the circle $x^2+y^2+8x+12y+15=0$**

Sol. The equation of the circle concentric with the given circle is

$x^2+y^2+8x+12y+k=0$. where k is a constant

If this passes through (2,3), then $4+9+16+36+k=0 \Rightarrow k=-65$

$\therefore$ The equation of the required circle is $x^2+y^2+8x+12y-65=0$

> ***Remember :***
> *If $x^2+y^2+2gx+2fy+c=0$ represents a circle, then its centre is $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$*

***5. If $x^2+y^2+2gx+2fy-12=0$ is a circle with centre (2, 3), find g, f and the radius of this circle.**

Sol. Centre of the given circle $=(-g,-f)=(2,3) \Rightarrow g=-2$ and $f=-3$

radius $=\sqrt{g^2+f^2-c}=\sqrt{4+9+12}=5$

***6. If $x^2+y^2-4x+6y+c=0$ is a circle of radius 6, find the value of C.**

Sol. Radius $=\sqrt{4+9-c}=6 \Rightarrow 13-c=36 \Rightarrow c=-23$

***7. If $x^2+y^2+2gx+2fy=0$ represent a circle with centre (–4, –3) then find g, f and readius of the circle.** **(March-17)**

Sol. Centre $(-g,-f)=(-4,-3) \Rightarrow g=4, f=3$, radius $=\sqrt{g^2+f^2}=5$

Definition :

Let P and Q be any two points on a circle. Then

a) the line $\overleftrightarrow{PQ}$ passing through P and Q is called a secant.

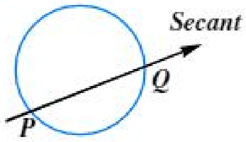
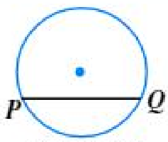


Fig1.3



Chord PQ
Fig1.4

> **Note :** *Diameter is the locus of mid points of a system of parallel chords.*

b) the line segment $\overline{PQ}$ joining P and Q is called a chord.

c) PQ is called the length of the chord $\overline{PQ}$.

d) Largest chord of the circle is called a diameter of the circle.

Note

i) A diameter of a circle is a chord but a chord need not be a diameter.

ii) Every diameter of a circle is bisected by the centre of the circle.

iii) Every diameter of a circle cuts the circle into two equal parts each of which is called a semi - circle.

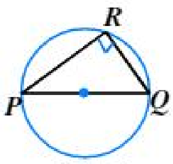


Fig1.5

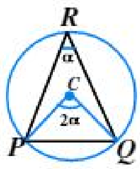


Fig1.6

iv) The angle subtendend by a diameter in a semi circle is 90^0. If PQ is a diameter of a circle then any point R other than P and Q lies on the circle iff $\angle PRQ = 90^0$.

v) Every chord subtends an angle at any point on the major arc of the circle formed by it which is exactly half of the angle that the chord subtends at the centre of the circle.

vi) Perpendicular bisector of any chord of a circle passes through the centre of the circle.

> **Note :** *Centre is the point of concurrence of diameters*

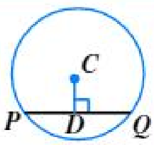


Fig1.7

It may be recalled that the equation of the perpendicular bisector of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $2(x_1 - x_2)x + 2(y_1 - y_2)y = (x_1^2 + y_1^2) - (x_2^2 + y_2^2)$

THEOREM-1.4

The equation of the circle having (x_1, y_1) and (x_2, y_2) as the extremities of a diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

Proof: Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and C be the centre of the circle.

> **Remember :** *Angle in Semi circle is right angle*

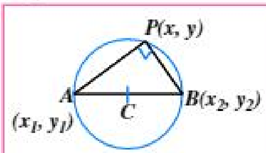


Fig1.8

Let $P(x, y)$ be any point on the circle other than A and B.

Then, $\angle APB = 90^0 \Rightarrow$ The lines AP and BP are perpendicular to each other.

$\therefore$ (slope of AP) (slope of BP) = -1

$$\Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \qquad \text{--(1)}$$

Also A and B clearly satsify (1)

$\therefore$ Any point $P(x, y)$ on the circle satisfies (1)

Conversely, if a point $P(x, y)$ satisfies (1), then $\angle APB = 90^0$ and hence P lies on the circle.

$\therefore$ (1) is the equation of the required circle

Note

i) ***For the above circle (1), Centre = Mid point of*** $\overline{AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Radius $= \frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{2} = \frac{1}{2}AB$

ii) ***One and only one circle passes through three given non - collinear points.***

THEOREM-1.5 **The equation of the circle passing through three non - collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is**

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2)+\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}x+\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}y+\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}=0$$

where $c_i = -(x_i^2+y_i^2), (i = 1,2,3)$.

Proof : Let the equation of the circle passing through the points P,Q and R be

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{-- (1)}$$

Since the points P,Q and R lie on (1), we have $x_1^2+y_1^2+2gx_1+2fy_1+c=0$ -- (2)

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \text{-- (3)}$$

$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \qquad \text{-- (4)}$$

Let $2g = a$, $2g = b$ and $c_i = -(x_i^2+y_i^2)$, $i = 1,2,3$ -- (5)

The equations (2) (3) and (4) can be written as

$$ax_1+by_1+c=c_1 \qquad \text{-- (6)}$$

$$ax_2+by_2+c=c_2 \qquad \text{-- (7)}$$

$$ax_3+by_3+c=c_3 \qquad \text{-- (8)}$$

Let $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ then $\Delta \neq 0$, (since P, Q, R are non - collinear)

Consider $\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} ax_1+by_1+c & y_1 & 1 \\ ax_2+by_2+c & y_2 & 1 \\ ax_3+by_3+c & y_3 & 1 \end{vmatrix} = \begin{vmatrix} ax_1 & y_1 & 1 \\ ax_2 & y_2 & 1 \\ ax_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} by_1 & y_1 & 1 \\ by_2 & y_2 & 1 \\ by_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} c & y_1 & 1 \\ c & y_2 & 1 \\ c & y_3 & 1 \end{vmatrix}$

$= a\Delta + 0 + 0$ ($\because$ the elements of two columns are proportional).

$$\therefore 2g = a = \frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{\Delta} \qquad \text{-- (9)}$$

$$\text{Similarly } 2f = b = \frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{\Delta} \qquad \text{-- (10)}$$

$$\text{and } c = \frac{\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}}{\Delta} \qquad \text{-- (11)}$$

Substituting the values of g, f and c in (1), we get the equation of the circle passing through the points P, Q, R as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix} = 0$$

Note

The centre of the circle passing through three non - collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\left(\frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{-2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{-2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}} \right) \quad (\text{where } c_i = -(x_i^2 + y_i^2),\ i = 1, 2, 3).$$

An Important Result :

If the circle $S = 0$ and the line $L = 0$ intersect, then the equation of the circle passing through the points of intersection of the circle and the line is $S + \lambda L = 0$ where λ is a parameter (For proof, see "System of Circles")

SOLVED EXAMPLES

Remember :

The equation of the circle passing through three non - collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}x + \begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}y + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix} = 0$$

*** 1. Find the equation of the circle passing through the points (1, 2) (3, –4) and (5, –6) .**

Sol. *1st Method :* Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$ -- (1)

This passes through (1, 2)

$\Leftrightarrow 1+4+2g+4f+c=0$

$\Leftrightarrow 2g+4f+c=-5$ -- (2)

(1) passes through (3, –4)

$\Leftrightarrow 9+16+6g-8f+c=0$

$\Leftrightarrow 6g-8f+c=-25$ -- (3)

(1) passes through (5, –6)

$\Leftrightarrow 25+36+10g-12f+c=0$

$\Leftrightarrow 10g-12f+c=-61$ -- (4)

solving (2), (3) and (4) we get $g=-11,\ f=-2,\ c=25$

$\therefore$ Equation of the required circle is $x^2+y^2-22x-4y+25=0$

2nd Method : Let $A=(1,2), B=(3,-4)$ & $C=(5,6)$

Equation of the circle with $\overline{AB}$ as diameter is $(x-1)(x-3)+(y-2)(y+4)=0$

$\Rightarrow x^2+y^2-4x+2y-5=0$ -- (1)

Slope of $\overleftrightarrow{AB} = \dfrac{2+4}{1-3} = \dfrac{6}{-2} = -3$

Equation of $\overleftrightarrow{AB}$ is $y-2=-3(x-1) \Rightarrow 3x+y-5=0$ -- (2)

Equation of the circle passing through A, B is

$(x^2+y^2-4x+2y-5)+\lambda(3x+y-5)=0$

If this passes through $C(5,-6)$, then $(25+36-20-12-5)+\lambda(15-6-5)=0$

$\Rightarrow 24+\lambda(4)=0 \Rightarrow \lambda=-6$

$\therefore$ The equation of the required circle is

$$(x^2+y^2-4x+2y-5)-6(3x+y-5)=0$$

$\Rightarrow x^2+y^2-22x-4y+25=0$

***2. Find the equation of the circle which passes through (6, 5) (4, 1) and whose centre lies on $4x+y-16=0$**

Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$ -- (1)

(1) passes through (6, 5) $\Leftrightarrow 12g+10f+c=-61$ -- (2)

(2) passes through (4, 1) $\Leftrightarrow 8g+2f+c=17$ -- (3)

The centre $(-g,-f)$ lies on $4x+y-16=0 \Leftrightarrow 4g+3f=-16$ -- (4)

Solving (2), (3) and (4) we get $g=-3, f=-4, c=15$

$\therefore$ The equation of the required circle is $x^2+y^2-6x-8y+15=0$

*** 3. Find the equation of the circumcircle of the triangle formed by the straight lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$**

Sol. The given lines are $x + y = 6$ --(1)

$2x + y = 4$ --(2)

$x + 2y = 5$ --(3)

Solving (1), (2) the point of intersection is $A = (-2, 8)$

Solving (2), (3) the point of intersection is $B = (1, 2)$

Solving (3), (1) the point of intersection is C= (7, –1)

Let the required circumcircle be $x^2 + y^2 + 2gx + 2fy + c = 0$ --(4)

A lies on (4) $\Leftrightarrow 68 - 4g + 16f + c = 0$ --(5)

B lies on (4) $\Leftrightarrow 5 + 2g + 4f + c = 0$ --(6)

C lies on (4) $\Leftrightarrow 50 + 14g - 2f + c = 0$ --(7)

Solving (5), (6) and (7) we get $2g = -17$, $2f = -19$, $c = 50$

$\therefore$ The equation of the required circle is $x^2 + y^2 - 17x - 19y + 50 = 0$

*** 4. Find the equation of circle passing through intersection points of line $ax + by + c = 0$ with coordinate axes and through origin.**

Sol. The straight line cuts the axes at $A\left(\frac{-c}{a}, 0\right)$ and $B\left(0, \frac{-c}{b}\right)$

AB subtends a right angle at O

The circle with A, B as the ends of diameter is

$$\left(x + \frac{c}{a}\right)(x - 0) + (y - 0)\left(y + \frac{c}{b}\right) = 0$$

$$\Rightarrow ab(x^2 + y^2) + c(bx + ay) = 0$$

Remember :

The equation of the circle passing through origin and through the points of intersection of the line $ax + by + c = 0$ with co-ordinate axes is $ab(x^2+y^2)+c(bx+ay)=0$.

*** 5. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y-3)^2 = 0$, a chord AB drawn and extended to a point M such that $AM = 2AB$. Find the equation of the locus of M.**

Sol. Let $M = (x_1, y_1)$

Given that $AM = 2AB \Rightarrow AB = BM$

$$B = \left(\frac{x_1}{2}, \frac{y_1 + 3}{2}\right)$$

B is a point on the given circle

$$\therefore \left(\frac{x_1}{2}\right)^2 + 4\left(\frac{x_1}{2}\right) + \left(\frac{y_1 + 3}{2} - 3\right)^2 = 0$$

$$\Rightarrow \frac{x_1^2}{4} + 2x_1 + \frac{y_1^2 - 6y_1 + 9}{4} = 0$$

$$\Rightarrow x_1^2 + y_1^2 + 8x_1 - 6y_1 + 9 = 0$$

$\therefore$ Locus of M is $x^2 + y^2 + 8x - 6y + 9 = 0$, which is a circle.

*** 6. Suppose a point (x_1, y_1) satisfies $x^2+y^2+2gx+2fy+c=0$ then show that it represents a circle whenever g, f and c are real.**

Sol. Given $x^2+y^2+2gx+2fy+c=0$ --(1)

Comparing the equation with the general equation of second degree, we have coefficient of x^2 = coefficient of y^2 and the coefficient of $xy=0$.

$\therefore$ The given equation represents a circle if $g^2+f^2-c \ge 0$

Since (x_1, y_1) is a point on (1), $x_1^2+y_1^2+2gx_1+2gy_1+c=0$ --(2)

Now, $g^2+f^2-c = g^2+f^2+x_1^2+y_1^2+2gx_1+2fy_1 = (x_1+g)^2+(y_1+f)^2 \ge 0$

Since g, f and c are real, (1) represents a circle.

EXERCISE - 1.1

1. Find the equation of the circle with centre C and radius r where

* a) $C=(a,-b), r=a+b$ [Ans : $x^2+y^2-2ax+2by-2ab=0$]

* b) $C=(-a,-b), r=\sqrt{a^2-b^2}\ (|a|>|b|)$ [Ans : $x^2+y^2+2ax+2by+2b^2=0$]

* c) $C=(\cos\alpha, \sin\alpha), r=1$ [Ans : $x^2+y^2-2x\cos\alpha-2y\sin\alpha=0$]

* d) $C=(-7,-3), r=4$ [Ans : $x^2+y^2+14x+6y+42=0$]

* e) $C=\left(-\frac{1}{2},-9\right), r=5$ [Ans : $4x^2+4y^2+4x+72y+225=0$]

* f) $C=\left(\frac{5}{2},\frac{-4}{3}\right), r=6$ [Ans : $36(x^2+y^2)-180x+96y-1007=0$]

* g) $C=(1,7), r=\frac{5}{2}$ [Ans : $4x^2+4y^2-8x-56y+175=0$]

* h) $C=(0,0), r=9$ [Ans : $x^2+y^2=81$]

2. Find the centre and radius of each of the circles whose equations are given below :

* a) $3x^2+3y^2-5x-6y+4=0$ [Ans : $\left(\frac{5}{6},1\right), \frac{\sqrt{13}}{6}$]

* b) $3x^2+3y^2+6x-12y-1=0$ [Ans : $(-1,2) \frac{4}{\sqrt{3}}$]

* c) $x^2+y^2+6x+8y-96=0$ [Ans : $(-3,-4), 11$]

* d) $2x^2+2y^2-4x+6y-3=0$ [Ans : $\left(1,\frac{-3}{2}\right), \frac{\sqrt{19}}{2}$]

* e) $2x^2+2y^2-3x+2y-1=0$ [Ans : $\left(\frac{3}{4},\frac{-1}{2}\right), \frac{\sqrt{21}}{4}$]

* f) $\sqrt{1+m^2}(x^2+y^2)-2cx-2mcy=0$ [Ans : $\left(\frac{c}{\sqrt{1+m^2}},\frac{mc}{\sqrt{1+m^2}}\right), |c|$]

3. * a) Find the equation of the circle passing through the origin and having the centre at $(-4, -3)$.
[**Ans :** $x^2 + y^2 + 8x + 6y = 0$]

* b) Find the equation of the circle passing through $(-2, 3)$ and having centre at $(0, 0)$.
[**Ans :** $x^2 + y^2 - 13 = 0$]

* c) Find the equation of the circle passing through $(3, 4)$ and having the centre at $(-3, 4)$.
[**Ans :** $x^2 + y^2 + 6x - 8y - 11 = 0$]

** d) Find the equation of the circle whose centre is $(-1, 2)$ and which passes through $(5, 6)$. **(May-18)**
[**Ans :** $x^2 + y^2 + 2x - 4y - 47 = 0$]

4. ***a) Find the value of a if $2x^2 + ay^2 - 3x + 2y - 1 = 0$ represents a circle and also find its radius.
[**Ans :** $a = 2$, radius $= \dfrac{\sqrt{21}}{4}$]

* b) Find the values of a, b if $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle. Also find the radius and centre of the circle.
[**Ans :** $a = 3$, $b = 0$ radius $= \dfrac{\sqrt{65}}{6}$, $\left(\dfrac{5}{6}, \dfrac{-1}{3}\right)$]

* c) If $x^2 + y^2 + 2gx + 2fy = 0$ represents a circle with centre $(-4, -3)$, then find g, f and the radius of the circle.
[**Ans :** $g = 4$, $f = 3$, radius $= 5$]

* d) If the circle $x^2 + y^2 + ax + by - 12 = 0$ has the centre at $(2, 3)$, then find a, b and the radius of the circle.
[**Ans :** $a = -4$, $b = -6$, $r = 5$]

* e) If the circle $x^2 + y^2 - 4x + 6y + a = 0$ has radius 4, find a.
[**Ans :** $a = -3$]

5. Find the equations of the circle for which the points given below are the end points of a diameter.

* a) $(1, 2)$ $(4, 6)$ [**Ans :** $x^2 + y^2 - 5x - 8y + 16 = 0$]

* b) $(-4, 3)$, $(3, -4)$ **(May 19)** [**Ans :** $x^2 + y^2 + x + y - 24 = 0$]

* c) $(7, -3)$, $(3, 5)$ [**Ans :** $x^2 + y^2 - 10x - 2y + 6 = 0$]

* d) $(1, 1)$, $(2, -1)$ [**Ans :** $x^2 + y^2 - 3x + 1 = 0$]

* e) $(0, 0)$, $(8, 5)$ [**Ans :** $x^2 + y^2 - 8x - 5y = 0$]

* f) $(3, 1)$, $(2, 7)$ [**Ans :** $x^2 + y^2 - 5x - 8y + 13 = 0$]

* g) $(4, 2)$, $(1, 5)$ **(March-19)** [**Ans :** $x^2 + y^2 - 5x - 7y + 14 = 0$]

6. * a) Find the other end of the diameter of the circle $x^2 + y^2 - 8x - 8y + 27 = 0$ if one end of it is $(2, 3)$.
[**Ans :** $(6, 5)$]

* b) Show that $A(3, -1)$ lies on the circle $x^2 + y^2 - 2x + 4y = 0$. Also find the other end of the diameter through A.
[**Ans :** $(-1, -3)$]

* c) Show that $A(-3, 0)$ lies on $x^2 + y^2 + 8x + 12y + 15 = 0$ and find the other end of the diameter through A.
[**Ans :** $(-5, -12)$]

* 7. Find the equation of the circle which is concentric with $x^2 + y^2 - 6x - 4y - 12 = 0$ and passing through $(-2, 14)$.
[**Ans :** $x^2 + y^2 - 6x - 4y - 156 = 0$]

*8. If the abscissae of points A, B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and ordinates of A, B are roots of $y^2 + 2py - q^2 = 0$, then find the equation of the circle for which $\overline{AB}$ is a diameter. [Ans : $x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$]

9. *a) Find the equation of the circle passing through $(2, -3)$, $(-4, 5)$ and having the centre on $4x + 3y + 1 = 0$. **(May-19)** [Ans : $x^2 + y^2 + 2x - 2y - 23 = 0$]

*b) Find the equation of a circle which passes through $(4,1)$, $(6,5)$ and having the centre on $4x + 3y - 24 = 0$. **(May-18)** [Ans : $x^2 + y^2 - 6x - 8y + 15 = 0$]

*c) Find the equation of the circle passing through $(-2, 3)$ $(4, 5)$ and whose centre lies on x-axis. [Ans : $3(x^2 + y^2) - 14x - 67 = 0$]

*10. If $ABCD$ is a square, then show that the points, A, B, C and D are concyclic.

*11. Find the equation of the circle passing through the points $(0, 0)$ $(2, 0)$ and $(0, 2)$. [Ans : $x^2 + y^2 - 2x - 2y = 0$]

*12. Find the equation of the circle passing through the points

a) $(3, 4)$ $(3, 2)$ $(1, 4)$ **(March-18) [Ans : $x^2 + y^2 - 4x - 6y + 11 = 0$]

*b) $(2, 1)$ $(5, 5)$ $(-6, 7)$ [Ans : $x^2 + y^2 + x - 12y + 5 = 0$]

*c) $(5, 7)$ $(8, 1)$ $(1, 3)$ [Ans : $3x^2 + 3y^2 - 29x - 19y + 56 = 0$]

13. Show that the following four points in each of the following are concyclic and find the equation of the circle on which they lie

a) $(1, 1)$, $(-6, 0)$, $(-2, 2)$, $(-2, -8)$. **(March-17, 19) [Ans : $x^2 + y^2 + 4x + 6y - 12 = 0$]

*b) $(1, 2)$, $(3, -4)$, $(5, -6)$, $(19, 8)$ [Ans : $x^2 + y^2 - 22x - 4y + 25 = 0$]

*c) $(1, -6)$, $(5, 2)$, $(7, 0)$, $(-1, -4)$ [Ans : $x^2 + y^2 - 6x + 4y - 7 = 0$]

** d) $(9, 1)$, $(7, 9)$, $(-2, 12)$ $(6, 10)$ **(March-19)** [Ans : $x^2 + y^2 - 6y - 76 = 0$]

***14. If $(2, 0)$, $(0, 1)$, $(4, 5)$ and $(0, c)$ are concyclic, then find c **(March-17)** [Ans : $\frac{14}{3}$]

15. Find the equation of the circum circle of the triangle formed by the lines

*a) $x + 3y = 1$, $x + y + 1 = 0$, $2x + 3y + 4 = 0$ [Ans : $x^2 + y^2 + 12x + 12y + 7 = 0$]

*b) $x + y + 1 = 0$, $3x + y - 5 = 0$, $2x + y - 5 = 0$ [Ans : $x^2 + y^2 - 30x - 10y + 25 = 0$]

*c) $5x - 3y + 4 = 0$, $2x + 3y - 5 = 0$, $x + y = 0$ [Ans : $49(x^2 + y^2) + 280x - 259y + 245 = 0$]

*d) $x - y - 2 = 0$, $2x - 3y + 4 = 0$, $3x - y + 6 = 0$ [Ans : $x^2 + y^2 - 24x + 16y - 52 = 0$]

*16. Show that the locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$, $x\sin\alpha - y\cos\alpha = b$, α is a parameter, is a circle.

*17. Show that the locus of a point such that the ratio of its distances from two given points is a constant $k(\neq 1)$, is a circle.

***18. Find the equation of circle whose centre lies on the x-axis and passing through $(-2, 3)$ and $(4, 5)$ [Ans : $x^2 + y^2 - 8x - 29 = 0$]

Notations : The following notations will be adopted throughout this chapter and the System of Circles.

$S \equiv x^2 + y^2 + 2gx + 2fy + c$

$S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$S_2 \equiv xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c$

$S_{11} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$S_{12} \equiv x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$

$S_{ij} \equiv x_ix_j + y_iy_j + g(x_i + x_j) + f(y_i + y_j) + c$

In this notation, the equation of a circle in general form is given by $S = 0$

1.2 POSITION OF A POINT W.R.T. A CIRCLE

A circle in a plane divides the plane into three regions, namely,

i) the interior of the circle (the region which contains the centre of the circle)

ii) the circular curve i.e., the circle it self and

iii) the exterior of the circle (the region which contains the remaining portion of the plane)

THEOREM-1.6 **Let $S = 0$ be a circle and $P(x_1, y_1)$ be any point in the plane of the circle. Then**

i) P lies in the interior of the circle $\Leftrightarrow S_{11} < 0$

ii) P lies on the circle $\Leftrightarrow S_{11} = 0$

iii) P lies in the exterior of the circle $\Leftrightarrow S_{11} > 0$

Proof : We have $S = x^2 + y^2 + 2gx + 2fy + c = 0$ -- (1)

Centre of (1), $C = (-g, -f)$

Radius of (1), $r = \sqrt{g^2 + f^2 - c}$

$P(x_1, y_1)$

i) P lies in the interior of the circle $\Leftrightarrow CP < r \Rightarrow CP^2 < r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 < g^2 + f^2 - c$

$\Leftrightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0 \Leftrightarrow S_{11} < 0$

ii) P lies on the circle $\Leftrightarrow CP = r \Leftrightarrow CP^2 = r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 = g^2 + f^2 - c$

$\Leftrightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \Leftrightarrow S_{11} = 0$

iii) P lies in the exterior of the circle

$\Leftrightarrow CP > r \Leftrightarrow CP^2 > r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 > g^2 + f^2 - c$

$\Leftrightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0 \Leftrightarrow S_{11} > 0$

Note : $CP^2 - r^2 = S_{11}$

*** *Ex.* Locate the position of the point (2, 4) w.r.t. the circle $x^2 + y^2 - 4x - 6y + 11 = 0$**

Sol. $S_{11} = 2^2 + 4^2 - 4(2) - 6(4) + 11 = 4 + 16 - 8 - 24 + 11 = -1 < 0(-ve)$

Since $S_{11} < 0$, the point (2, 4) lies inside the given circle.

1.3 POWER OF A POINT W.R.T. A CIRCLE

Definition :

Let C be the centre and r be the radius of a circle and let P be any point in its plane. Then the power of the point P w.r.t the circle is defined as $CP^2 - r^2$. i.e., S_{11}.

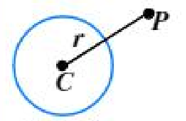


Fig1.9

Note

Let $S = 0$ be a circle and P be a point. Then

i) P lies out side the circle $\Leftrightarrow$ the power of P w.r.t. the circle is positive.

ii) P lies on the circle $\Leftrightarrow$ the power of P w.r.t. the circle is zero.

iii) P lies inside the circle $\Leftrightarrow$ the power of P w.r.t. the circle is negative.

* ***Example :*** The power of the point $P(-1, 1)$ w.r.t the circle $S = x^2+y^2-6x+4y-12=0$ is $S_{11}=1+1+6+4-12=0$

THEOREM-1.7 **If a secant through a point $P(x_1, y_1)$ meets the circle $S = 0$ in A and B, then the power of the point P is given by $PA.PB$ and $PA.PB = S_{11}$, where PA & PB are algebraic distances.**

Proof : Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle.

The equation of any line through $P(x_1, y_1)$ in the distance form is $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r$ where $|r|$ is the distance between (x, y) and (x_1, y_1).

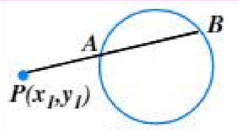


Fig1.10

Any point on the line is $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If this point lies on the circle $S = 0$, then

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0$$

$$\Rightarrow r^2 + 2r[(x_1 + g)\cos\theta + (y_1 + f)\sin\theta] + S_{11} = 0 \qquad \text{-- (1)}$$

Let r_1, r_2 be the roots of (1). Corresponding to these two roots there exist two points A, B on the circle such that $\Rightarrow PA = r_1$ and $\Rightarrow PB = r_2$

From (1), $r_1 r_2 = S_{11} \Rightarrow PA.PB = S_{11}$ = Power of P w.r.t $S = 0$.

Even if the line is parallel to x-axis or y-axis, the result can be proved to be true.

Note

i) P lies outside the circle $\Rightarrow S_{11} > 0 \Rightarrow PA.PB > 0$;

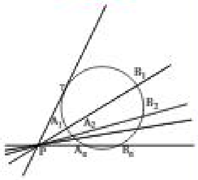
ii) P lies on the circle $\Rightarrow S_{11} = 0 \Rightarrow PA.PB = 0$

iii) P lies inside the circle $\Rightarrow S_{11} < 0 \Rightarrow PA.PB < 0$

1.4 CHORD JOINING TWO POINTS

THEOREM-1.8 **Equation to the chord joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the circle $S = 0$ is $S_1 + S_2 = S_{12}$**

Note :



$\Rightarrow PA_1.PB_1 = PA_2.PB_2 = PA_3.PB_3 = \ldots = PA_n.PB_n = (PT)^2$

1st Proof : Let the equation of the circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

Centre of the circle is $C(-g, -f)$

The middle point of the chord joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

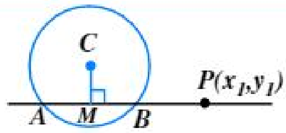


Fig1.11

If $P(x, y)$ is any point on chord AB (not passing through C), then

$\overleftrightarrow{CM} \perp \overleftrightarrow{AP} \Rightarrow$ (slope of $\overleftrightarrow{PA}$) (slope of $\overleftrightarrow{CM}$) $= -1$

$$\Rightarrow (x - x_1)\left(\frac{x_1 + x_2}{2} + g\right) + (y - y_1)\left(\frac{y_1 + y_2}{2} + f\right) = 0$$

$\therefore$ The equation of the chord $\overleftrightarrow{AB}$ is

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0$$

$$\Rightarrow x(x_1 + x_2) + y(y_1 + y_2) + 2gx + 2fy = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + x_1x_2 + y_1y_2$$

$$\Rightarrow (xx_1 + yy_1 + gx + fy + c) + (xx_2 + yy_2 + gx + fy + c)$$

$$= (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) + x_1x_2 + y_1y_2 + c$$

Adding $g(x_1 + x_2) + f(y_1 + y_2)$ to both sides,

$$(xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c) + (xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c)$$

$$= (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) + (x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c)$$

$$\Rightarrow S_1 + S_2 = S_{11} + S_{12} \Rightarrow S_1 + S_2 = S_{12} \quad (\because (x_1, y_1) \text{ lies on } S = 0, S_{11} = 0)$$

Even if $\overleftrightarrow{AB}$ passes through $C(-g, -f)$ the equation $S_1 + S_2 = S_{12}$ is satisfied.

$\therefore$ Equation to chord $\overleftrightarrow{AB}$ is $S_1 + S_2 = S_{12}$

2nd Proof :

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be the given circle.

Since $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on $S = 0$, $S_{11} = 0$, $S_{22} = 0$.

Consider the equation $S_1 + S_2 = S_{12}$ -- (1)

$$\Rightarrow (xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c) + (xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c)$$

$$= x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$$

It is first degree equation in x and y and hence it represents a line.

Substituting A in (1), we get $S_{11} + S_{12} = S_{12}$ which is true, since $S_{11} = 0$.

$\therefore$ A lies in (1)

Substituting B in (1), we get $S_{12} + S_{22} = S_{12}$ which is true, since $S_{22} = 0$

$\therefore B$ lies on (1)

$\therefore$ Equation of $\overleftrightarrow{AB}$ is $S_1 + S_2 = S_{12}$.

1.5 TANGENT LINE

Let P be any point on a given circle and Q be neighbouring point of P lying on the circle. If the secant $\overleftrightarrow{PQ}$ approaches the same limiting position, as Q moves along the curve and approaches P from either side, the limiting position of $\overleftrightarrow{PQ}$ is called the tangent line or tangent to the curve at P.

$\therefore \underset{Q\to P}{Lt}$ (secant through P,Q) = tangent at P.

P is called the point of contact of the tangent to the circle.

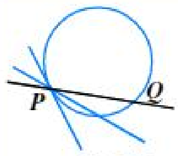


Fig1.12

Note

If a line meets a circle in two coincident points, then it is a tangent to the circle at the point of coincidence.

THEOREM-1.9 **The equation of the tangent at (x_1, y_1) on the circle $S = 0$ is $S_1 = 0$.**

Proof : 1st Method :

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points on the circle $S = 0$.

$\therefore$ Equation of the chord $\overleftrightarrow{PQ}$ is $S_1 + S_2 = S_{12}$.

As $Q \to P$, the chord $\overleftrightarrow{PQ}$ becomes the tangent at P to the circle.

$\therefore$ Equation to the tangent at $P(x_1, y_1)$ is $\underset{Q\to P}{Lt}(S_1 + S_2 = S_{12}) \Rightarrow S_1 + S_1 = S_{11}$

$\Rightarrow 2S_1 = 0 \Rightarrow S_1 = 0 \quad (\because (x_1, y_1)\text{lies on } S = 0 \Rightarrow S_{11} = 0)$

2nd Method :

Let the eqution of the circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ --(1)

Diff. (1) w.r.t x we get $2x + 2y\dfrac{dy}{dx} + 2g + 2f\dfrac{dy}{dx} = 0$

$\Rightarrow (y + f)\dfrac{dy}{dx} = -(x + g) \qquad \Rightarrow \dfrac{dy}{dx} = \dfrac{-(x + g)}{(y + f)} \ (y \neq -f)$

$\therefore$ The slope of the tangent at $P(x_1, y_1)$ to the circle $= \dfrac{-(x_1+g)}{(y_1+f)}, (y_1 \neq -f)$.

$\therefore$ The equation of the tangent at (x_1, y_1) to the circle is $y - y_1 = \dfrac{-(x_1+g)}{(y_1+f)}(x - x_1)$

$\Rightarrow (y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$

$\Rightarrow xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

Adding $gx_1 + fy_1 + c$ to both sides

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow S_1 = S_{11} \Rightarrow S_1 = 0 \quad (\because S_{11} = 0)$

Even if $y_1 = -f$, $S_1 = 0$ is satisfied .

$\therefore$ Equation of tangent at $P(x_1, y_1)$ to $S = 0$ is $S_1=0$.

Note

i) The 2nd method is useful in deriving the equation of the tangent to any curve at a point on the curve.

ii) The equation of the tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$.

iii) The equation of the tangent at $(r\cos\alpha, r\sin\alpha)$ on the circle $x^2 + y^2 = r^2$ is $x\cos\alpha + y\sin\alpha = r$

1.6 NORMAL

Let $S = 0$ be a circle and P be a point on the circle. The line passing through P and perpendicular to the tangent at P to the circle is called the normal at P to the circle.

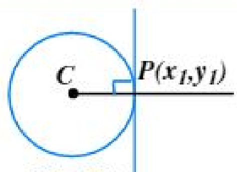


Fig1.13

THEREOM-1.10 **The equation of the normal to the circl $x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$ is $(y_1 + f)(x - x_1) - (x_1 + g)(y - y_1) = 0$.**

Proof : The equation of the circle is $S = x^2 + y^2 + 2gx + 2fy + c = 0$

The equation of the tangent at $P(x_1, y_1)$ is $S_1 = 0$.

$\Rightarrow (x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0 \quad$ -- (1)

The equation of any line perpendicular to (1) is $(y_1 + f)x - (x_1 + g)y + k = 0$

This passes through $P(x_1, y_1)$

$\Rightarrow k = -[(y_1 + f)x_1 - (x_1 + g)x_1]$

$\therefore$ The equation of the normal is $(y_1 + f)(x - x_1) - (x_1 + g)(y - y_1) = 0 \quad$ -- (2)

Note

i) It can be observed that the nomral (2) passes through the centre $(-g, -f)$ of the circle $S = 0$.

ii) The equation of the normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - x_1y = 0$.

SOLVED EXAMPLES

***1. Find the equation of the tangent and normal at (3,2) to the circle $x^2+y^2-x-3y-4=0$.**

Sol. Let $S=x^2+y^2-x-3y-4=0$

Let $P(x_1,y_1)=(3,2)$

$\therefore$ Equation of the tangent at P to (1) is $S_1=0$

$$\Rightarrow x(3)+y(2)-\frac{1}{2}(x+3)-\frac{3}{2}(y+2)-4=0$$

$$\Rightarrow \frac{5}{2}x+\frac{1}{2}y-\frac{17}{2}=0 \Rightarrow 5x+y-17=0$$

Slope of normal $=\frac{1}{5}$

Equation of the normal at P to (1) is $y-2=\frac{1}{5}(x-3)$

$\Rightarrow 5y-10=x-3 \quad \Rightarrow x-5y+7=0$

***2. Show that the line $lx+my+n=0$ is a normal to the circle $S=0$ iff $gl+mf=n$.**

Sol. The straight line $lx+my+n=0$ is a normal to the circle

$S=x^2+y^2+2gx+2fy+c=0$

$\Leftrightarrow$ Centre $(-g,-f)$ lies on $lx+my+n=0$

$\Leftrightarrow$ $l(-g)+m(-f)+n=0$

$\Leftrightarrow$ $lg+mf=n$

1.7 POSITION OF A STRAIGHT LINE W.R.T. A CIRCLE

Given a straight line $L=0$ and a circle $S=0$ we have the following three possibilities.

Case (i) :

The line intersects the circle in two distinct points.

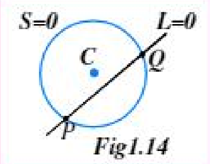


Fig1.14

Case (ii) :

The line meets the circle in one and only one point (touches the circle).

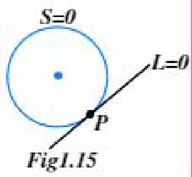


Fig1.15

Case (iii) :

The line neither touches nor intersects the circle.

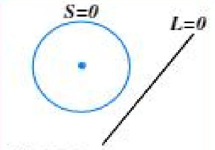


Fig1.16

THEOREM-1.11 **i) A straight line $y = mx + c$ (i) meets the circle $x^2 + y^2 = r^2$ in two distinct points if $\frac{c^2}{(1+m^2)} < r^2$**

ii) touches the circle $x^2 + y^2 = r^2$ if $\frac{c^2}{(1+m^2)} = r^2$

iii) has no points in common with the circle $x^2 + y^2 = r^2$ if $\frac{c^2}{(1+m^2)} > r^2$

Proof : The equation of the given circle is $x^2 + y^2 = r^2$ -- (1)

The equation of the given line is $y = mx + c$ -- (2)

Solving (1) & (2): $x^2 + (mx + c)^2 = r^2$

$$\Rightarrow (1+m^2)x^2 + 2mcx + (c^2 - r^2) = 0 \qquad \text{-- (3)}$$

The roots of (3) are real and distinct or real and equal or imaginary according as the discriminant of (3) $\gtreqless 0$.

$$\Rightarrow 4m^2c^2 - 4(1+m^2)(c^2 - r^2) \gtreqless 0$$

$$\Rightarrow 4m^2c^2 - 4(c^2 + m^2c^2 - r^2 - r^2m^2) \gtreqless 0$$

$$\Rightarrow c^2 - r^2(1+m^2) \lesseqgtr 0 \Rightarrow \frac{c^2}{1+m^2} \lesseqgtr r^2$$

Case (i) :

If $\frac{c^2}{1+m^2} < r^2$, then the line (2) intersects the circle in two distinct points.

Case (ii) :

If $\frac{c^2}{1+m^2} = r^2$, then the line (2) touches the circle. In this case, the line (2) is a tangent to the circle (1).

Case (iii) :

If $\frac{c^2}{1+m^2} > r^2$, then the line (2) neither touches nor intersects the circle. They do not have any common points.

Note

i) $\frac{|c|}{\sqrt{1+m^2}}$ *is the perpendicular distance of the line $y=mx+c$ from the centre (0, 0) of the circle $x^2+y^2=r^2$.*
From the above discussion it is clear that the condition for the line $y = mx + c$ to be a tangent to the circle $x^2+y^2=r^2$ is $c^2 = r^2\,(1+m^2)$. Hence for all values of m, the line $y = mx \pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=r^2$ and the point of contact is $\left(\frac{-r^2m}{c}, \frac{r^2}{c}\right)$.

ii) The equation of any tangent to the circle $x^2+y^2=r^2$ in the slope form can be taken as $y = mx \pm r\sqrt{1+m^2}$ where m is the slope of the tangent.

In general, we can conclude as follows :

Let $L = 0$ be a line and $S = 0$ be a circle of radius r. Let d be the perpendicular distance of the line from the centre of the circle. Then

i) **the line intersects the circle in two distinct points if $r > d$.**

ii) **the line touches the circle (i.e. meets the circle in two coincident points) if $r = d$.**

iii) **the line neither touches nor intersects the circle if $r < d$.**

Result : The equation of the tangent to the circle $(x - x_1)^2 + (y - y_1)^2 = r^2$ in the slope form is $y - y_1 = m(x - x_1) \pm r\sqrt{1+m^2}$ where m is the slope of the tangent.

Proof : Shift the origin to the point (x_1, y_1), so that (x, y) transforms to (X, Y) then

$x = X + x_1$ and $y = Y + y_1$ i.e., $x - x_1 = X$, $y - y_1 = Y$

The given circle transforms to $X^2 + Y^2 = r^2$ -- (1)

The equation of the tangent to (1) in the slope form is $Y = mX + r\sqrt{1+m^2}$ -- (2) where m is the slope of the tangent.

$\Rightarrow y - y_1 = m(x - x_1) + r\sqrt{1+m^2}$ is the equation of the tangent to the given circle in the slope form.

Note

The equation of the tangent to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ having slope m is $y + f = m(x+g) \pm r\sqrt{1+m^2}$

*** *1.* Find the equations of the tangents to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ which are inclined at 45^0 with X- axis.**

Sol. Slope of tangent, $m = \text{Tan } 45° = 1$

Centre of the circle $= (-g, -f) = (2, 3)$

Radius of the circle, $r = \sqrt{4+9-3} = \sqrt{10}$

$\therefore$ Required equations of the tangents are $y + f = m(x+g) \pm r\sqrt{1+m^2}$

$\Rightarrow y - 3 = 1(x-2) \pm \sqrt{10}\sqrt{1+1}$

$\Rightarrow x - y + 1 \pm 2\sqrt{5} = 0$

THEOREM-1.12 **The condition for the line $lx + my + n = 0$ to touch the circle $x^2 + y^2 = r^2$ is $n^2 = r^2(l^2 + m^2)$ and the point of contact is $\left(\frac{-r^2\ell}{n}, \frac{-r^2 m}{n}\right)$**

Proof : The line $lx + my + n = 0$ -- (1)

touches the circle $x^2 + y^2 = r^2$ -- (2)

if $r = \frac{|n|}{\sqrt{\ell^2 + m^2}} \Rightarrow n^2 = r^2(\ell^2 + m^2)$

Let $P(x_1, y_1)$ be the point of contact of the line (1) and circle (2).

The equation of the tangent at $P(x_1, y_1)$ to (2) is $S_1 = 0 \Rightarrow xx_1 + yy_1 = r^2$ -- (3)

Since (1) & (3) represent the same tangent line at (x_1, y_1)

$$\Rightarrow \frac{x_1}{\ell} = \frac{y_1}{m} = \frac{-r^2}{n} \Rightarrow x_1 = \frac{-r^2\ell}{n}; y_1 = \frac{-r^2 m}{n}$$

$\therefore$ The required point of contact is $(x_1, y_1) = \left(\frac{-r^2\ell}{n}, \frac{-r^2 m}{n}\right)$

1.8 POSITION OF THE CIRCLE $S = 0$ W.R.T. AXES

A) Position of the circle $S = 0$ w.r.t. the X-axis

Let the circle $S = x^2+y^2+2gx+2fy+c=0$ intersect the X-axis at $A(x_1, 0)$ and $B(x_2, 0)$.

Then $x_1^2 + 2gx_1 + c = 0$ and $x_2^2 + 2gx_2 + c = 0$

$\Rightarrow x_1, x_2$ are the roots of $x^2 + 2gx + c = 0$

$\therefore x^2 + 2gx + c \equiv (x - x_1)(x - x_2)$ and $x_1 + x_2 = -2g$, $x_1x_2 = c$

Now, $AB = |x_1 - x_2| = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = 2\sqrt{g^2 - c}$, which is real only if $g^2 \geq c$

$\therefore$ The condition for the circle $S = 0$ to meet the x-axis in real points is $g^2 \geq c$.

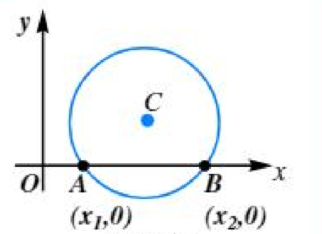


Fig1.17

i) If $g^2 > c$, the circle intersects the x-axis in two distinct points and in this case, the length of the intercept made by the circle on the x-axis is $AB = 2\sqrt{g^2 - c}$

ii) If $g^2 = c$, then the circle $S = 0$ meets the X-axis in two coincident points i.e., the circle touches the x-axis. If the point of contact is $A(x_1, 0)$ then $x^2 + 2gx + c \equiv (x - x_1)^2$, using which the point of contact can be easily found to be $(-g, 0)$. Further the equation of the circle touching the x-axis at $(x_1, 0)$ can be written as $(x - x_1)^2 + y^2 + 2fy = 0$

i.e., $(x + g)^2 + y^2 + 2fy = 0$.

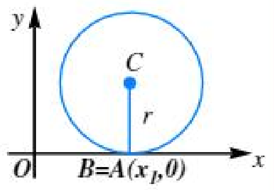


Fig1.18

Thus the condition for the circle $S = 0$ to touch the X-axis is $g^2 = c$ and the equation of any circle touching the X-axis can be written as $x^2 + y^2 + 2gx + 2fy + g^2 = 0$ or $x^2 + y^2 \pm 2\sqrt{c}x + 2fy + c = 0$ $(c \geq 0)$

This circle touches the X-axis from the above if $f < 0$ and from below if $f > 0$.

iii) If $g^2 < c$ then the circle $S = 0$ does not meet the X-axis.

B) Position of $S = 0$ w.r.t. Y axis

Let the circle $S = 0$ intersect the Y-axis at $P(0, y_1)$ and $Q(0, y_2)$.

Then $y^2 + 2fy + c \equiv (y - y_1)(y - y_2)$ and $PQ = 2\sqrt{f^2 - c}$

This is real only if $f^2 \geq c$

$\therefore$ The condition for the circle $S = 0$ to meet the y-axis is $f^2 \geq c$.

i) If $f^2 > c$ then the circle intersects the Y-axis in two distinct points and in this case the *length of the intercept made by the circle on the y-axis is* $PQ = |y_1 - y_2| = 2\sqrt{f^2 - c}$.

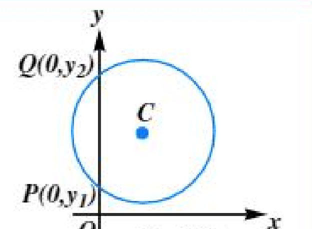


Fig1.19

ii) The conditions for the circle to touch the y-axis is $f^2 = c$. In this case $y^2 + 2fy + c \equiv (y - y_1)^2$ so that the point of contact is $(0, y_1) = (0, -f)$

$\therefore$ The equation of the circle touching the y-axis at $(0, y_1)$ can be written as

$$x^2 + (y - y_1)^2 + 2gx = 0$$

i.e., $x^2 + (y + f)^2 + 2gx = 0$

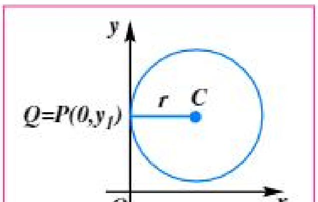


Fig1.20

Further, the equation of any circle touching the y-axis can be written as $x^2 + y^2 + 2gx + 2fy + f^2 = 0$ (or) $x^2 + y^2 + 2gx \pm 2\sqrt{c}y + c = 0$ $(c \geq 0)$

This circle touches the y-axis from the right if $g < 0$ and from the left if $g > 0$.

iii) The circle $S = 0$ does not meet the y-axis if $f^2 < c$.

C) **Position of $S = 0$ w.r.t. both the axes :**

i) The circle $S = 0$ meets both the axes if $g^2 \geq c$ and $f^2 \geq c$. The intercepts made by the circle on the axes are

X-intercept $= 2\sqrt{g^2 - c}$, Y-intercept $= 2\sqrt{f^2 - c}$.

ii) The conditions for the circle $S = 0$ to touch both the axes are $g^2 = f^2 = c$.

iii) The equation of any circle touching both the axes can be written as $x^2 + y^2 + 2gx + 2gy + g^2 = 0$ or $x^2 + y^2 \pm 2\sqrt{c}x \pm 2\sqrt{c}y + c = 0 \quad (c \geq 0)$ -- (1)

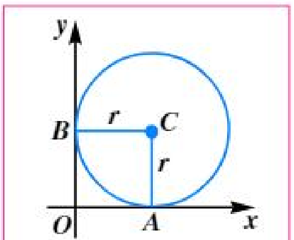


Fig1.21

The equation (1) represents four circles belonging to the four quadrants.

Thus, **the equation of any circle touching both the axes,**

I) **in the Ist quadrant can be taken as $x^2 + y^2 - 2kx - 2ky + k^2$ $(k > 0)$ centre $C = (k, k)$, radius $r = k$**

II) **in the 2nd quadrant can be taken as $x^2 + y^2 + 2kx - 2ky + k^2$ $(k > 0)$**

centre $C = (-k, k)$, radius $r = k$

III) **in the 3rd quadrant can be taken as $x^2 + y^2 + 2kx + 2ky + k^2$ $(k > 0)$**

centre $C = (-k, -k)$, radius $r = k$

IV) **in the 4th quadrant can be taken as $x^2 + y^2 - 2kx + 2ky + k^2$ $(k > 0)$**

centre $C = (k, -k)$, radius $r = k$

The centres of the circles II and III lie on $y = x$, III and IV lie on $y = -x$, which are bisectors of the angles between the coordinate axes.

1.9 LENGTH OF THE TANGENT

Definition :

Let a tangent be drawn from an external point P to the circle $S = 0$, touching the circle at Q. Then PQ is called the length of the tangent from P to the circle $S = 0$.

THEOREM-1.13 **The length of the tangent drawn from an external point $P(x_1, y_1)$ to the circle $S = 0$ is $\sqrt{S_{11}}$.**

Proof : Let the equation of the circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $C = (-g, -f)$; radius, $r = \sqrt{g^2 + f^2 - c}$

Let Q be the point of contact of the tangent from $P(x_1, y_1)$ to the circle.

Then PQ is the length of the tangent from P to the circle $S=0$ Now $\angle PQC = 90^0$

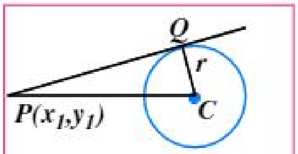


Fig1.22

$$\Rightarrow PQ^2 = CP^2 - r^2 = (x_1+g)^2 + (y+f)^2 - g^2 - f^2 + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_{11}$$

$\therefore$ Length of tangent from P to the circle $\sqrt{S_{11}}$

Note

If P lies outside the circle S = 0, then the square of the length of the tangent from P to the circle is the power of P w.r.t S = 0.

THEOREM-1.14 **In general, two tangents can be drawn to a circle from an external point $P(x_1, y_1)$.**

Proof : Let the equation of the circle be $x^2 + y^2 = a^2$

The equation of any tangent to the circle $x^2 + y^2 = a^2$ in the slope form is $y = mx + a\sqrt{1+m^2}$ where m is the slope of the tangent.

If the tangent passes through given point $P(x_1, y_1)$ then $y_1 = mx_1 + a\sqrt{1+m^2}$

$$\Rightarrow (y_1 - mx_1)^2 = a^2(1+m^2)$$

$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$$

This is a quadratic in m and hence m has two values. Discriminant of (1) is $4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$

$$= 4x_1^2y_1^2 - 4x_1^2y_1^2 + 4x_1^2a^2 + 4a^2y_1^2 - 4a^4$$

$$= 4a^2(x_1^2 + y_1^2 - a^2) > 0 \quad (\because S_{11} > 0)$$

$\therefore$ The two values of m are real and distinct.

Thus there exist two tangents corresponding to the two values of m through P to the circle.

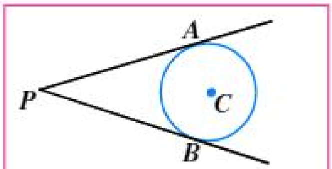


Fig1.23

Note

i) ***If m_1 and m_2 are the slopes of the tangents drawn from an external point (x_1, y_1) to the circle $x^2+y^2=a^2$, then $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$ and $m_1m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$.***

ii) If $S_{11} = 0$, the point $P(x_1, y_1)$ lies on the circle and hence the two tangents are coincident touching the circle at (x_1, y_1).

iii) If $S_{11} < 0$, the point $P(x_1, y_1)$ lies inside the circle and hence no tangent can be drawn to the circle.

iv) The number of tangets from a point $P(x_1, y_1)$ to the circle $S = 0$ is

a) 0 if P lies inside the circle ($S_{11} < 0$)

b) 1 if P lies on the circle ($S_{11} = 0$)

c) 2 if P lies outside the circle ($S_{11} > 0$)

THEOREM-1.15 **If θ is the angle between the tangents through a point P to the circle $S = 0$, then $\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}}$ where r is the radius of the circle.**

Proof : Let the two tangents from P to the circle $S = 0$ touch the circle at Q, R and θ be the angle between them. Let C be the centre of the circle.

We have $CQ = r$, $PQ = \sqrt{S_{11}}$, $\angle RPC = \angle CPQ = \frac{\theta}{2}$ and $\angle CQP = 90^0$

$$\therefore \text{Tan}\left(\frac{\theta}{2}\right) = \frac{CQ}{PQ} = \frac{r}{\sqrt{S_{11}}}$$

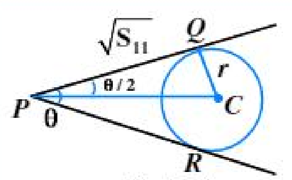


Fig1.24

*** *Ex.*** **Find the angle between the pair of tangents drawn from origin to the circle $x^2+y^2-14x+2y+25=0$**

Sol. Here $P(x_1, y_1)\,0\,(0, 0)$

radius $r = \sqrt{49+1-25} = \sqrt{25} = 5$

Length of tangent = $\sqrt{S_{11}} = \sqrt{0+0-0+0+25} = 5$

If θ is the angle between the tangents, then

$$Tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}} = \frac{5}{5} = 1 \Rightarrow \frac{\theta}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$$

1.10 PARAMETRIC EQUATIONS

THEOREM-1.16 **The parametric equations of the circle with centre (x_1, y_1) and radius r are $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$, where $0 \le \theta < 2\pi$.**

Proof : Let $P(x, y)$ be any point on the circle with centre $C(x_1, y_1)$ and radius r.

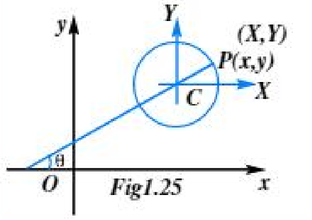


Fig1.25

Shift the origin to (x_1, y_1) without changing the directions of the axes.

Let the new axes be $\overleftrightarrow{CX}$ and $\overleftrightarrow{CY}$

Let CP have an inclination θ so that $\angle XCP = \theta$ in the standard position.

Let $P(X, Y)$ so that $x = X + x_1$ and $y = Y + y_1$

$\therefore$ By definition $\cos\theta = \dfrac{X}{r} = \dfrac{x - x_1}{r}$ and $\sin\theta = \dfrac{Y}{r} = \dfrac{y - y_1}{r}$.

$\therefore x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

$\therefore$ Any point on the circle is given by $(x_1 + r\cos\theta, y_1 + r\sin\theta),\ 0 \le \theta < 2\pi$

Hence the parametric equations of the circle with centre (x_1, y_1) and radius r are $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta,\ 0 \le \theta < 2\pi$

Note

i) *The parametric equations of the circle $x^2+y^2=r^2$ are $x = r\cos\theta$, $y = r\sin\theta$ where θ is a parameter and $0 \le \theta < 2\pi$. Any point 'θ' on the circle is given by $P(x, y) = P(\theta) = (r\cos\theta, r\sin\theta)$.*

ii) *The parametric equations of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are given by $x = -g + r\cos$, $y = -f + r\sin\theta$ where $r = \sqrt{g^2 + f^2 - c}$ and θ is a parameter $(0 \le \theta < 2\pi)$. Any point "θ" on the circle is $(-g + r\cos\theta, -f + r\sin\theta)$.*

THEOREM-1.17 **The equation of the chord joining the two points $P(\alpha)$ and $Q(\beta)$ on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x+g)\cos\left(\dfrac{\alpha+\beta}{2}\right) + (y+f)\sin\left(\dfrac{\alpha+\beta}{2}\right) = r\cos\left(\dfrac{\alpha-\beta}{2}\right)$**

Proof : Let r be the radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Then $r = \sqrt{g^2 + f^2 - c}$ and

$P(\alpha) = P(-g + r\cos\alpha, -f + r\sin\alpha)$, $Q(\beta) = Q(-g + r\cos\beta, -f + r\sin\beta)$.

$\therefore$ The equation of the line joining these two points is

$$\frac{y + f - r\sin\alpha}{r\sin\alpha - r\sin\beta} = \frac{x + g - r\cos\alpha}{r\cos\alpha - r\cos\beta}$$

$$\Rightarrow (x + g - r\cos\alpha)(\sin\alpha - \sin\beta) = (y + f - r\sin\alpha)(\cos\alpha - \cos\beta)$$

$$(x + g - r\cos\alpha)\cos\left(\frac{\alpha+\beta}{2}\right) = (y + f - r\sin\alpha)\left(-\sin\left(\frac{\alpha+\beta}{2}\right)\right)$$

$$\Rightarrow (x+g)\cos\left(\frac{\alpha+\beta}{2}\right) + (y+f)\sin\left(\frac{\alpha+\beta}{2}\right) = r\cos\left(\frac{\alpha-\beta}{2}\right)$$

Note

i) *The equation of the chord joining the points α and β on the circle $x^2 + y^2 = r^2$ is*
$x\cos\left(\dfrac{\alpha+\beta}{2}\right) + y\sin\left(\dfrac{\alpha+\beta}{2}\right) = r\cos\left(\dfrac{\alpha-\beta}{2}\right)$

ii) *The equation of the tangent at θ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $(x+g)\cos\theta + (y+f)\sin\theta = r$*

iii) *The equation of the tangent at θ to the circle $x^2 + y^2 = r^2$ is given by $x\cos\theta + y\sin\theta = r$*

***Ex.** **Write the parametric equation of the circle represented by $x^2+y^2+6x+8y-96=0$**

Sol. Centre of the circle, $C=(x_1, y_1)=(-3,-4)$

Radius of the circle, $r=\sqrt{9+16+96}=\sqrt{121}=11$

$\therefore$ The parametric equation of the given circle are $x=-3+11\cos\theta$ and $y=-4+11\sin\theta$, θ is a parameter, $0\le\theta<2\pi$.

SOLVED EXAMPLES

If $S=0$ is a circle $P(x_1, y_1)$ is a point, then
i) $S_{11}>0 \Leftrightarrow P$ lies outside the circle
ii) $S_{11}=0 \Leftrightarrow P$ lies on the circle
iii) $S_{11}<0 \Leftrightarrow P$ lies inside of the circle

*** 1. Locate the position of the point $P(4, 2)$ with respect to the circle $2x^2+2y^2-5x-4y-3=0$**

Sol. Given circle is $S=x^2+y^2-\frac{5}{2}x-2y-\frac{3}{2}=0$

Let $P=(x_1, y_1)=(4,2)$

$S_{11}=16+4-\frac{5}{2}(4)-2(2)-\frac{3}{2}$

$\therefore P$ lies outside $S=0$

*** 2. Find the power of the point $P(2, 3)$ with respect to the circle $S=x^2+y^2-2x+8y-23=0$**

Sol. The power of $P(2,3)$ w.r.to $S=0$ is $S_{11}=4+9-4+24-23=10$

*** 3. If the length of the tangent from (2, 5) to the circle $x^2+y^2-5x+4y+k=0$ is $\sqrt{37}$, then find k.** **(March-17, May-18)**

Sol. Given circle is $S=x^2+y^2-5x+4y=0$ --(1)

Let $P=(x_1, y_1)=(2,5)$

Length of the tangent from P to (1) $=\sqrt{S_{11}}=\sqrt{37}$

$\Rightarrow 4+25-10+20+k=37 \Rightarrow 39+k=37 \Rightarrow k=-2$

*** 4. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ are in the ratio 2:3, then find the equation to the locus of P.** **(March-17, 19)**

Sol. Given circles are

$S=x^2+y^2-4x-6y-12=0$ --(1)

$x^2+y^2+46+26=018y-12=0$ $S'=x^2+y^2+6x+18y+26=0$ --(2)

Let $P(x_1, y_1)$ be a point on the locus

Given $\frac{\sqrt{S_{11}}}{\sqrt{S'_{11}}}=\frac{2}{3} \Rightarrow 9S_{11}=4S'_{11}$

cus of P is $5(x^2+y^2)-60x-126y-212=0$

*** 5. Find the length of the chord intercepted by the circle $x^2+y^2-x+3y-22=0$ on the line $y=x-3$.** **(March-18)**

Sol. Given circle is $x^2+y^2-x+3y-22=0$ --(1)

Centre of (1), $C=\left(\frac{1}{2},\frac{-3}{2}\right)$

Remember :

If 'r' is radius of a circle then a line which is at a distance 'd' from centre of the circle, cuts (a chord) an intercept of length $2\sqrt{r^2-d^2}$

Radius of (1), $r=\sqrt{\frac{1}{4}+\frac{9}{4}+22}=\sqrt{\frac{49}{2}}=\frac{7}{\sqrt{2}}$

Given line AB is $x-y+3=0$ --(2)

Perpendicular distance from C to (2), $CD=d=\frac{\left|\frac{1}{2}+\frac{3}{2}+3\right|}{\sqrt{2}}=\frac{5}{\sqrt{2}}$

Length of chord, $AB=2\sqrt{r^2-d^2}\quad=2\sqrt{\frac{49}{2}-\frac{25}{2}}=2\sqrt{12}=4\sqrt{3}$

*** 6. Find the equation of the circle with centre (2, 3) and touching the line $3x-4y+1=0$.**

Sol. Centre of the circle, $C=(2,3)$

Given line is $3x-4y+1=0$ --(1)

Since (1) is a tangent to the circle,

radius of the circle = perpendicular distance from to (1) $=\frac{|6-12+1|}{5}=1$

$\therefore$ Equation of the required circle is $(x-2)^2+(y-3)^2=1$

$\Rightarrow x^2+y^2-4x-6y+12=0$

***7. Find the equations of tangents of the circle $x^2+y^2-8x-2y-12=0$ at the point whose ordinates are 1.**

Sol. Given circle is $S=x^2+y^2-8x-2y+12=0$ --(1)

If y=1, then $x^2+1-8x-2+12=0$

$\Rightarrow x^2-8x+11=0 \Rightarrow x=\frac{8\pm\sqrt{64-44}}{2}=4\pm\sqrt{5}$

$\therefore$ The points on (1) are $P(4+\sqrt{5},1)$ and $Q(4-\sqrt{5},1)$

The equation of the tangent at P to (1) is $S_1=0$

$\Rightarrow x(4+\sqrt{5})+y(1)-4(x+(4+\sqrt{5}))-1(y+1)+12=0$

$\Rightarrow \sqrt{5}x=5+4\sqrt{5} \Rightarrow x=4+\sqrt{5}$

The equation of the tangent at Q to (1) is $S_1=0$

$\Rightarrow x(4-\sqrt{5})+y(1)-4(x+4-\sqrt{5})-1(y+1)+12=0 \Rightarrow x=4-\sqrt{5}$

*** 8. Find the equation of the tangent to $x^2+y^2-2x+4y=0$ at (3, –1). Also find the equation of tangent parallel to it.**

Remember :

The equation of the tangent at $P(x_1, y_1)$ of the circle $S=0$ is $S_1=0$.

Sol. Given circle is $S=x^2+y^2-2x+4y=0$ --(1)

Centre of (1), $C=(1,-2)$

Let $P=(x_1,y_1)=(3,-1)$

$\therefore$ Equation of the tangent at P to (1) is $S_1=0$

$\Rightarrow x(3)+y(-1)-1(x+3)+2(y-1)=0$

$\Rightarrow 2x+y-5=0$ --(2)

The other end of the diameter passing through P is $Q=2C-P(-1,-3)$

$\therefore$ The equation of the tangent at Q to (1) is

$x(-1)+y(-3)-1(x-1)+2(y-3)=0$

$\Rightarrow -2x-y-5=0 \Rightarrow 2x+y+5=0$ is the equation of the required tangent to (1) parallel to (2).

***9. If $x^2+y^2=c^2$ and $\frac{x}{a}+\frac{y}{b}=1$ intersect at A and B, then find AB. Hence deduce the condition that the line touches the circle.**

Sol. Given circle is $x^2+y^2=c^2$ --(1)

Centre of (1), $C=(0,0)$

Radius of (1), $r=c$

Given line is $\frac{x}{a}+\frac{y}{b}=1$ --(2)

Perpendicular distance from C to (2) $= CD = d = \frac{|0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

From ΔACD, $AD^2 = AC^2 - CD^2 = c^2 - \frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$

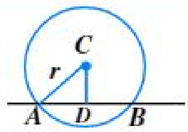


Fig1.26

$AB = 2AD = 2\sqrt{c^2 - \frac{a^2b^2}{a^2+b^2}}$

The line AB touches (1)

$\Rightarrow AB = 0 \Rightarrow c^2 = \frac{a^2b^2}{a^2+b^2} \Rightarrow \frac{1}{c^2} = \frac{1}{a^2}+\frac{1}{b^2}$

This is the required condition.

Remember :
Condition for $lx+my+n=0$ to touch $x^2+y^2+2gx+2fy+c=0$ is $(l^2+m^2)(g^2+f^2-c)=(lg+mf-n)^2$.

***10. Show that the tangent at $(-1, 2)$ of the circle $x^2+y^2-4x-8y+7=0$ touches the circle $x^2+y^2+4x+6y=0$ and also find its point of contact.**

Sol. Given circles are

$x^2+y^2-4x-8y+7=0$ --(1)

$x^2+y^2+4x+6y=0$ --(2)

Let $P=(x_1,y_1)=(-1,2)$

The equation of the tangent at P to (1) is $x(-1)+y(2)-2(x-1)-4(y+2)+7=0$

$\Rightarrow -3x-2y+1=0 \Rightarrow 3x+2y-1=0$ --(3)

Centre of (2), $C=(-2,-3)$

radius of (2), $r=\sqrt{4+9}=\sqrt{13}$

Perpendicular distance from C to (3) $= \left|\frac{3(-2)+2(-3)-1}{\sqrt{13}}\right| = \frac{13}{\sqrt{13}} = \sqrt{13} = r$

$\therefore$ (3) touches (2)

Let $Q(h, k)$ be the point of contact of (2) & (3)

$\therefore Q$ is the foot of the perpendicular from C to (3)

$\therefore \frac{h+2}{3} = \frac{k+3}{2} = \frac{-(-13)}{13} \Rightarrow h=1, k=-1$

$\therefore Q=(1,-1)$

Remember :
Point of contact of tangent is the foot of perpendicular from centre of circle on the Tangent

*** 11. Find the equations of the tangents to the circle $x^2+y^2+2x-2y-3=0$ which are perpendicular to $3x-y+4=0$**

Sol. Given circle is $x^2+y^2+2x-2y-3=0$ --(1)

Centre of (1), $C(x_1, y_1) = (-1, 1)$

Radius of (1), $r=\sqrt{1+1+3}=\sqrt{5}$

Given line is $3x-y+4=0$ ---(2)

The slope of the line perpendicular to (2) is, $m=\frac{-1}{3}$.

$\therefore$ Equation of the required tangent to (1) is $y-y_1=m(x-x_1)\pm r\sqrt{1+m^2}$

$\Rightarrow y-1=\frac{-1}{3}(x+1)\pm\sqrt{5}\sqrt{1+\frac{1}{9}}$

$\Rightarrow x+3y-2\pm5\sqrt{2}=0$

*** 12. Find the equations of the circles which touch $2x-3y+1=0$ at (1, 1) and having radius $\sqrt{13}$.**

Sol. The centres of the required circles lie on a line which is perpendicular to the line $2x-3y+1=0$ and passing through $(x_1, y_1)=(1, 1)$

i.e., on $3x+2y-5=0$

$\therefore$ The centres lie on $3x+2y-5=0$ at a distance $r=\sqrt{13}$ from (1, 1)

$\therefore$ The centres are given by $(x_1+r\cos\theta, y_1+r\sin\theta)$

$=\left(1+\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1+\sqrt{13}\frac{3}{\sqrt{13}}\right)=(-1, 4)$ and

$(x_1-r\cos\theta, y_1-r\sin\theta)=\left(1-\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1-\sqrt{13}\left(\frac{3}{13}\right)\right)=(3,-2)$

$\therefore$ The equations of the required circles are $(x+1)^2+(y-4)^2=13$

and $(x-3)^2+(y+2)=13$ i.e., $x^2+y^2+2x-8y+4=0$ and $x^2+y^2-6x+4y=0$

13. Find the equations of the circles passing through (1, –1), touching the lines $4x+3y+5=0$ and $3x-4y-10=0$

Sol. Since the circle touches the given lines, which are perpendicular, the centre of the circle lies on the bisectors of the angle between the given lines.
The bisectors of the angles between the given lines are

$$\frac{4x+3y+5}{5}\pm\frac{3x-4y-10}{5}=0$$

$\Rightarrow 7x-y-5=0$ or $x+7y+15=0$

$\Rightarrow y=7x-5$ or $x=-(7y+15)$

If the centre lies on $x=-(7y+15)$, then the centre is of the form $(-7k-15, k)$

$\therefore \frac{|4(-7k-15)+3k+5|}{5}=\sqrt{(-7k-15-1)^2+(k+1)^2}$

$\Rightarrow(-25k-55)^2=25(+(7k+16)^2+(k+1)^2)$

Remember :

Equations of Angle bisectors between the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

$\Rightarrow 25k^2+116k+136=0 \Rightarrow k$ is imaginary

If centre lies on $y=7x-5$, centre is of the form $(k,7k-5)$

$$\therefore \frac{|4k+3(7k-5)+5|}{5}=\sqrt{(k-1)^2+(7k-5+1)^2}$$

$\Rightarrow (5k-2)^2=(k-1)^2+(7k-4)^2$

$\Rightarrow 25k^2-38k+13=0$

$\Rightarrow (25k-13)(k-1)=0 \Rightarrow k=1$ or $k=\frac{13}{25}$

If $k=1$, then centre $=(1,2)$, and radius $=\sqrt{(1-1)^2+(2+1)^2}=3$

$\therefore$ The equation of one of the required circles is $(x-1)^2+(y-2)^2=3^2$

$\Rightarrow x^2+y^2-2x-4y-4=0$ --(1)

If $k=\frac{13}{25}$, centre $=\left(\frac{13}{25},\frac{-34}{25}\right)$

$$\text{radius}=\sqrt{\left(\frac{13}{25}-1\right)^2+\left(\frac{-34}{25}+1\right)^2}=\sqrt{\frac{225}{625}}=\frac{15}{25}=\frac{3}{5}$$

$\therefore$ The equation of the other circle is $\left(x-\frac{13}{25}\right)^2+\left(y+\frac{34}{25}\right)^2=\frac{9}{25}$

$\Rightarrow 25(x^2+y^2)-26x+68y+44=0$ --(2)

Thus, (1) & (2) are the required circles.

*** 14. Find the locus of the point of intersection of the tangents drawn to the circle $x^2+y^2=a^2$ which makes a constant angle α to each other.**

Sol. Let $P(x_1,y_1)$ be a point on the locus

The equation of the tangent to the circle $x^2+y^2=a^2$ in the slope form is $y=mx\pm a\sqrt{1+m^2}$, where m is the slope of the tangent

If this tangent passes through P, $y_1=mx_1\pm a\sqrt{1+m^2}$

$\Rightarrow (y_1-mx_1)^2=a^2(1+m^2) \Rightarrow (x_1^2-a^2)m^2-2x_1y_1m+(y_1^2-a^2)=0$ ---(1)

If m_1,m_2 are the slopes of the tangents through P, then m_1,m_2 are the roots of (1).

$$\therefore m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2},\ m_1m_2=\frac{y_1^2-a^2}{x_1^2-a^2}$$

Since α is the angle between the tangents,

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}=\frac{\sqrt{\left(\frac{2x_1y_1}{x_1^2-a^2}\right)^2-4\left(\frac{y_1^2-a^2}{x_1^2-a^2}\right)}}{1+\left(\frac{y_1^2-a^2}{x_1^2-a^2}\right)}$$

$$=\frac{\sqrt{4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)}}{x_1^2-a^2+y_1^2-a^2}\quad\therefore \tan^2\alpha=\frac{4a^2(x_1^2+y_1^2-a^2)}{(x_1^2+y_1^2-2a^2)^2}$$

$\therefore$ The equation of the locus of P is $(x^2+y^2-2a^2)^2=4a^2(x^2+y^2-a^2)\cot^2\alpha$

*** 15. If θ_1 and θ_2 are the angles of inclination of tangents through a point P to the circle $x^2+y^2=a^2$, then find the locus of P when $\cot\theta_1+\cot\theta_2=k$**

Sol. The equation of the tangent to $x^2+y^2=a^2$ having slope m is $y=mx\pm a\sqrt{1+m^2}$ --(1)

Let $P(x_1, y_1)$ be a point on the locus

If (1) passes through P, then $y_1=mx_1\pm a\sqrt{1+m^2}$ $\Rightarrow (y_1-mx_1)^2=a^2(1+m^2)$

$\Rightarrow (x_1^2-a^2)m^2-2x_1y_1m+(y_1^2-a^2)=0$

If m_1, m_2 are the roots of the above equation, then

$m_1+m_2=\tan\theta_1+\tan\theta_2=\dfrac{2x_1y_1}{x_1^2-a^2}$; $m_1m_2=\tan\theta_1\tan\theta_2=\dfrac{y_1^2-a^2}{x_1^2-a^2}$

Given $\cot\theta_1+\cot\theta_2=k$

$\Rightarrow \tan\theta_1+\tan\theta_2=k(\tan\theta_1.\tan\theta_2)\Rightarrow \dfrac{2x_1y_1}{x_1^2-a^2}=k\left(\dfrac{y_1^2-a^2}{x_1^2-a^2}\right)$

$\therefore$ The locus of $P(x_1, y_1)$ is $k(y^2-a^2)=2xy$

*** 16. Find the equation of the normal to the circle $x^2+y^2-4x-6y+11=0$ at (3,2). Also find the other point where the normal meets the circle.**

Sol. Let $C(2, 3)$ be the centre of given circle.

Let $A(3, 2)$.

Let the normal at A meets the circle at $B(a, b)$. Since the normal to the circle always passes through C, the equation of normal at $A(3, 2)$ is

$(x-3)(2-3)-(y-2)(3-2)=0$ i.e., $x+y-5=0$

The centre of the circle is the mid point of A and B

$\therefore \dfrac{a+3}{2}=2\Rightarrow a=1$ and $\dfrac{b+2}{2}=3\Rightarrow b=4$ $\therefore B=(1,4)$

Remember :

In a circle, normal at any point of the circle passes through its centre

*** 17. Find the equation of the circle which touhes X-axis at a distance of 3 units from the origin and making an intercept of length 6 on Y-axis.**

Sol. Let the equation of the required circle be $x^2+y^2+2gx+2fy+c=0$ --(1)

This meets the X - axis at (3,0), (3,0) is a point on (1)

$\therefore 9+0+6g+0+c=0$

i.e., $6g+c=-9$ --(2)

We have $g^2-c=0$ --(3)

Adding (2) and (3) we get

$g^2+6g+9=0$ --(4)

i.e., $(g+3)^2=0$ --(5)

i.e., $g=-3$

From (3) and (4), we get $c=9$

Given that the intercept on Y-axis made by (1) is 6.

Therefore we have $2\sqrt{f^2-c}=6$

$\Rightarrow 2\sqrt{f^2-9}=6 \Rightarrow \sqrt{f^2-9}=3 \Rightarrow f^2-9=9\Rightarrow f^2=18 \therefore f=\pm3\sqrt{2}$

Since $g=-3, f=\pm3\sqrt{2}$ and $c=9$, we have two circles satisfying the hypothesis, these circles are $x^2+y^2-6x+6\sqrt{2}y+9=0$ and $x^2+y^2-6x-6\sqrt{2}y+9=0$

Remember :

Length of y-Intercept made by a circle S = 0 is $2\sqrt{f^2-c}$

EXERCISE - 1.2

1. Locate the position of the point P with respect to the circle $S = 0$ when
   - *a) $P(1, 2)$ and $S \equiv x^2 + y^2 + 6x + 8y - 96$ **[Ans : interior]**
   - *b) $P(3, 4)$ and $S \equiv x^2 + y^2 - 4x - 6y - 12$ **[Ans : interior]**
   - *c) $P(2, -1)$ and $S \equiv x^2 + y^2 - 2x - 4y + 3$ **[Ans : exterior]**

2. Find the power of the point P w.r.t the circle $S = 0$ when
   - *a) $P(1, 2)$ and $S \equiv x^2 + y^2 + 6x + 8y - 96$ **[Ans : -69]**
   - *b) $P(5, -6)$ and $S \equiv x^2 + y^2 + 8x + 12y + 15$ **[Ans : 44]**
   - *c) $P(2, 4)$ and $S \equiv x^2 + y^2 - 4x - 6y - 12$ **[Ans : -24]**

3. Find the equation of the tangent at P of the circle $S = 0$ where P and S are given by
   - *a) $P(3, 4)$ and $S \equiv x^2 + y^2 - 4x - 6y + 11$ **[Ans : $x + y - 7 = 0$]**
   - *b) $P(-1, 1)$ and $S \equiv x^2 + y^2 - 6x + 4y - 12$ **[Ans : $4x - 3y + 7 = 0$]**
   - *c) $P(-6, -9)$ and $S \equiv x^2 + y^2 + 4x + 6y - 39$ **[Ans : $2x + 3y + 39 = 0$]**

4. Find the equation of the normal at P of the circle $S = 0$ where P and S are given by
   - *a) $P(3, -4)$, $S \equiv x^2 + y^2 + x + y - 24$ **[Ans : $x + y + 1 = 0$]**
   - *b) $P(1, 3)$, $S \equiv 3(x^2 + y^2) - 19x - 29y + 76$ **[Ans : $11x - 13y + 28 = 0$]**
   - *c) $P(3, 5)$, $S \equiv x^2 + y^2 - 10x - 2y + 6$ **(March-18)**

*5. Find the equation of tangent and normal at (1,1) to the circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$
[Ans : $2x - y - 1 = 0$, $x + 2y - 3 = 0$]

*6. Find the area of the triangle formed by the tangent at $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ with the coordinate axis, where $x_1 y_1 \neq 0$ **[Ans : $\frac{a^4}{2|x_1 y_1|}$]**

7. Find the area of the triangle formed by the normal at $(3, -4)$ to the circle $x^2+y^2-22x-4y+25 = 0$ with the coordinate axes. **(May-18) **[Ans : $\frac{625}{24}$ sq.units]**

8. *a) Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$ **[Ans : $2\sqrt{7}$]**

   *b) Find the length of the chord intercepted by the circle $x^2 + y^2 + 8x - 4y - 16 = 0$ on the line $3x - y + 4 = 0$ **[Ans : $2\sqrt{26}$]**

   *c) Find the length of the chord formed by $x^2+y^2 = a^2$ on the line $x\cos\alpha + y\sin\alpha = p$
   [Ans : $2\sqrt{a^2 - p^2}$]

*9. If $y = mx+c$ and $x^2+y^2=a^2$ intersect at A and B and if $AB = 2\lambda$, then show that $c^2 = (1 + m^2)(a^2 - \lambda^2)$. **(March-19)**

*10. Find the equation of the circle with centre $(-2, 3)$ cutting a chord of length 2 units on $3x + 4y + 4 = 0$. **[Ans : $x^2 + y^2 + 4x - 6y + 8 = 0$]**

*11. Find the equations of tangents of the circle $x^2+y^2=10$ at the points whose abscissae are 1. [Ans : $x+3y-10=0$, $x-3y-10=0$]

12. *a) Find the equation of the circle passing through (0, 0) and making intercepts 4, 3 on X-axis and Y-axis respectively. [Ans : $x^2+y^2\pm 4x\pm 3y=0$]

*b) Find the equation of the circle passing through (0, 0) and making intercept 6 units on X-axis and intercept 4 units on Y-axis. [Ans : $x^2+y^2\pm 6x\pm 4y=0$]

*13. If $S\equiv x^2+y^2+2gx+2fy+c=0$ represents a circle then show that the straight line $lx+my+n=0$

a) touches the circle $S=0$ if $g^2+f^2-c=\dfrac{(gl+mf-n)^2}{(l^2+m^2)}$

b) meets the circle $S=0$ in two points if $g^2+f^2-c>\dfrac{(gl+mf-n)^2}{(l^2+m^2)}$

c) will not meet the circle if $g^2+f^2-c<\dfrac{(gl+mf-n)^2}{(l^2+m^2)}$

*14. Find the equations of the tangents to the circle $x^2+y^2-4x+6y-12=0$ which are parallel to $x+y-8=0$. [Ans : $x+y+1\pm 5\sqrt{2}=0$]

*15. Show that the circle $S=x^2+y^2+2gx+2fy+c=0$ touches the

a) X-axis if $g^2=c$

b) Y-axis if $f^2=c$

*16. Find the equation of the circle with centre (–3,4) and touching Y-axis. [Ans : $x^2+y^2+6x-8y+16=0$]

17. *a) Show that the line $5x+12y-4=0$ touches the circle $x^2+y^2-6x+4y+12=0$.

b) If $4x-3y+7=0$ is a tangent to the circle represented by $x^2+y^2-6x+4y-12=0$, then find its points of contact. **(March-17) [Ans : (–1, 1)]

*c) Prove that the tangent at (3, –2) of the circle $x^2+y^2=13$ touches the circle $S\equiv x^2+y^2+2x-10y-26=0$ and find its point of contact. [Ans : (5, 1)]

*d) Show that $x+y+1=0$ touches the circle $x^2+y^2-3x+7y+14=0$ and find its point of contact. [Ans : (2, –3)]

*18. Find the equation of the circle passing through (–1, 0) and touching $x+y-7=0$ at (3, 4). [Ans : $x^2+y^2-2x-4y-3=0$]

19. Find the length of the tangent from P to the circle $S=0$ when

a) $P=(-2,5)$ and $S\equiv x^2+y^2-25$ [Ans : 2]

b) $P(0,0)$ and $S\equiv x^2+y^2-14x+2y+25=0$ [Ans : 5]

c) $P=(2,5)$ and $S\equiv x^2+y^2-5x+4y-5$ [Ans : $\sqrt{34}$]

***20. If the length of the tangent from (5, 4) to the circle $x^2+y^2+2ky=0$ is 1, then find k. [Ans : –5]

*21. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2 + y^2 + 8x + 12y + 15 = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are equal, then find the equation of the locus of P [Ans : $4x + 6y + 9 = 0$]

*22. If a point P is moving such that the lengths of tangents drawn from P to $x^2 + y^2 - 2x + 4y - 20 = 0$ and $x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio 2:1, then show that the equation of the locus of P is $x^2 + y^2 - 2x - 12y + 8 = 0$. [Ans : $x^2 + y^2 - 2x - 12y + 8 = 0$]

*23. Show that the locus of P where the tangents drawn from P to $x^2 + y^2 = a^2$ are prependicular to each other is $x^2 + y^2 = 2a^2$

*24. From a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, two tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0 \left(0 < \alpha < \frac{\pi}{2}\right)$. Prove that the angle between them is 2α

25. Find the parametric equations of the circles

a) $x^2 + y^2 - 4x + 6y - 12 = 0$ [Ans : $x = 2 + 5\cos\theta, y = -3 + 5\sin\theta$]

*b) $4(x^2 + y^2) = 9$ **(March-17)** [Ans : $x = \frac{3}{2}\cos\theta, y = \frac{3}{2}\sin\theta$]

*c) $(x - 3)^2 + (y - 4)^2 = 8^2$ **(March-18)** [Ans : $x = 3 + 8\cos\theta, y = 4 + 8\sin\theta$]

*d) $2x^2 - 2y^2 = 7$ **(March-19)** [Ans : $x = \sqrt{\frac{7}{2}}\cos\theta, y = \sqrt{\frac{7}{2}}\sin\theta$]

26. If $x = -1 + 5\cos\theta, y = 2 + 5\sin\theta$, show that the locus of the point (x, y) is a circle. Find its centre and radius. [Ans : $(-1, 2), 5$]

*27. If the parametric values of two points A and B lying on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are 30^0 and 60^0 respectively, then find the equation of the chord joining A and B. [Ans : $2x + 2y - (7 + 5\sqrt{3}) = 0$]

*28. Find the equation of the tangent at the point 30^0(parametric value θ) of the circle $x^2 + y^2 + 4x + 6y - 39 = 0$. **(May-2019)** [Ans : $\sqrt{3}x + y + 3 + 2(\sqrt{3} - 2\sqrt{13}) = 0$]

1.11 CHORD OF CONTACT

The line joining the points of contact of the tangents to a circle $S = 0$ drawn from an external point P is called the chord of contact of P w.r.t $S = 0$

THEOREM-1.18 **The equation to the chord of contact of $P(x_1, y_1)$ w.r.t the circle $S = 0$ is $S_1 = 0$.**

Proof : Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be the given circle. Let the tangents from P to the circle meet the circle at $A(x_2, y_2)$ and $B(x_3, y_3)$ as shown in fig.

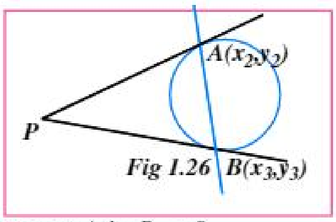


Fig 1.26

The equation of the tangent at A is $S_2 = 0$

$$\Rightarrow xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0$$

Since this tangent passes through P, we get

$$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \qquad \text{--(1)}$$

The equation of the tangent at B is $S_3 = 0$

$$\Rightarrow xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0$$

Since this tangent passes through P,

we get $x_1x_2 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0$ --(2)

From (1) and (2) it is clear that A and B satisfy the equation

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$ --(3)

Equation (3) is a first degree equation in x and y and hence it represent a straight line

$\therefore$ The equation (3) represents $\overleftrightarrow{AB}$

$\therefore$ Hence the equation of $\overleftrightarrow{AB}$ is $S_1 = 0$.

Note

i) *If the point $P(x_1, y_1)$ is on the circle $S = 0$, then the tangent itself can be taken as the chord of contact.*

ii) *If the point $P(x_1, y_1)$ is an interior point of the circle $S = 0$, then the chord of contact of P w.r.t the circle doesnot exist.*

Ex : **Find the chord of contact of (2, 5) w.r.t the circle $x^2 + y^2 - 5x + 4y - 2 = 0$**

Sol. The equation of the chord of contact of (2, 5) w.r.t the circle $S = x^2 + y^2 - 5x + 4y - 2 = 0$ is $S_1 = 0$

$$\Rightarrow x(2) + y(5) - \frac{5}{2}(x+2) + 2(y+5) - 2 = 0 \Rightarrow x - 14y - 6 = 0$$

1.12 POLE AND POLAR

Let $S = 0$ be a circle and P be any point in the plane other than the centre of $S = 0$. If any line drawn through the point P meets the circle in two points Q and R, then the points of intersection of tangents drawn at Q and R lie on a line called polar of P and P is called pole of that line.

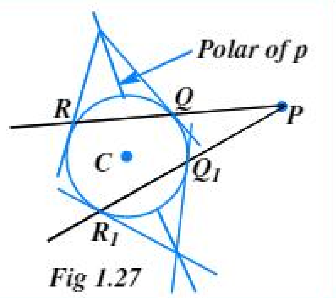


Fig 1.27

Note :

Locus of harmonic conjugate of a point P w.r.t. a chord of the circle S = 0 passing through P is polar of P.

Definition :

The locus of the point of intersection of the tangents to the circle $S = 0$ drawn at the extremities of the chord passing through a point P is a straight line (or a part of a straight line), called the polar of P w.r.t the circle $S = 0$. The point P is called the pole of the locus.

THEOREM-1.19 **The equation of the polar of $P(x_1, y_1)$ w.r.t $S = 0$ is $S_1 = 0$.**

Note : *The polar of centre of circle w.r.t the same circle does not exist.*

Proof : Let QR be any chord drawn through $P(x_1, y_1)$ and let the tangents at Q and R meet at the point $A(\alpha,\beta)$. Then $\overleftrightarrow{QR}$ is the chord of contact of $A(\alpha,\beta)$

$\therefore$ The equation of $\overleftrightarrow{QR}$ is $x\alpha + y\beta + g(x+\alpha) + f(y+\beta) + c = 0$

It passes through $P(x_1, y_1)$

$\therefore x_1\alpha + y_1\beta + g(x_1+\alpha) + f(y_1+\beta) + c = 0$

$\therefore A(\alpha,\beta)$ satisfies $S_1 = 0$

$\therefore$ The locus of the polar of $P(x_1, y_1)$ is $S_1 = 0$

Note

i) *If P lies outside the circle S = 0, then the polar of P meets the circle in two points and the polar becomes the chord of contact of P.*

ii) *If P lies on the circle, then the polar of P becomes the tangent at P to the circle S = 0.*

iii) *If P lies inside the circle S = 0, then the polar of P doesnot meet the circle.*

iv) *If C is the centre of the circle, then the polar of P has slope* $\dfrac{-(x_1+g)}{(y_1+f)}$ *and hence it is perpendicular to CP whose slope is* $\dfrac{y_1+f}{x_1+g}$

v) *If $P(x_1, y_1)$ = centre of the circle S = 0, then the polar of P w.r.t S = 0 doesnot exist i.e. the polar of P(−g, −f) of the circle* $S = x^2 + y^2 + 2gx + 2fy + c = 0$ *doesnot exist.*

vi)

$P(x_1, y_1)$	Tangent at P	Chord of contact at P	Polar at P
Interior of circle ($S_{11} < 0$)	Does not exist	Does not exist	$S_1 = 0$ (P is different from centre)
On the circle ($S_{11} = 0$)	$S_1 = 0$	$S_1 = 0$	$S_1 = 0$
Exterior of the circle	Does not exist	$S_1 = 0$	$S_1 = 0$

THEOREM-1.20 **The pole of $\ell x + my + n = 0$ $(n \neq 0)$ with respect to the circle $x^2 + y^2 = a^2$ is $\left(\dfrac{-a^2\ell}{n}, \dfrac{-a^2m}{n}\right)$**

Proof : Let $P(x_1, y_1)$ be the pole of the line $lx + my + n = 0$ -- (1)

w.r.t the circle $x^2 + y^2 = a^2$ -- (2)

But the polar of P w.r.t the circle (2) is $xx_1 + yy_1 - a^2 = 0$ -- (3)

Since (1) and (3) represent the same line, $\dfrac{x_1}{\ell} = \dfrac{y_1}{m} = \dfrac{-a^2}{n}$

$\Rightarrow x_1 = \dfrac{-a^2\ell}{n};\ y_1 = \dfrac{-a^2m}{n}$

$\therefore$ The pole of $\ell x + my + n = 0$ with respect to the circle (2) is $\left(\dfrac{-a^2\ell}{n}, \dfrac{-a^2m}{n}\right)$.

***Ex 1. Find the equation of the polar of (2, 3) w.r.t the circle $x^2+y^2+6x+8y-96=0$.**

Sol. The equation of the polar of $P(x_1, y_1) = (2, 3)$ w.r.t the circle $S = x^2 + y^2 + 6x + 8y - 96 = 0$ is $S_1 = 0$

$\Rightarrow x(2)+y(3)+3(x+2)+4(y+3)-96=0$

$\Rightarrow 5x+7y-78=0$

THEOREM-1.21 **If (x_1, y_1) is the pole of the line $lx + my + n = 0$ w.r.t the circle $x^2+y^2+2gx+2fy+c = 0$, then $\dfrac{x_1+g}{\ell}=\dfrac{y_1+f}{m}=\dfrac{r^2}{\ell g+mf-n}$ where $r^2 = g^2+f^2-c$**

> **Note :** *Pole of diameter of a circle w.r.t the same circle does not exist.*

Proof : Equation to polar of (x_1,y_1) w.r.t the circle $S = x^2+ y^2 + 2gx + 2fy+c = 0$ is $S_1= 0$

$\Rightarrow xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$\Rightarrow x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$ --(1)

Given equation of the polar is $lx + my + n = 0$ --(2)

$\therefore$ Equations (1) & (2) represents the same line

$\therefore \dfrac{x_1+g}{\ell}=\dfrac{y_1+f}{m}=\dfrac{gx_1+fy_1+c}{n}=\dfrac{g(x_1+g)+f(y_1+f)-(gx_1+fy_1+c)}{g\ell+fm-n}$ (by ratio and proportion)

$=\dfrac{g^2+f^2-c}{\ell g+mf-n}=\dfrac{r^2}{\ell g+mf-n}$

$\therefore \dfrac{x_1+g}{\ell}=\dfrac{y_1+f}{m}=\dfrac{r^2}{\ell g+mf-n}$

> **Note**
>
> *The pole of the line $lx+my+n=0$ w.r.t the circle $x^2+y^2+2gx+2fy+c=0$ is*
>
> $$\left(-g+\frac{\ell r^2}{\ell g+mf-n}, -f+\frac{mr^2}{\ell g+mf-n}\right)$$
>
> *i.e.,* $\left(-g-\dfrac{\ell r^2}{N}, -f-\dfrac{mr^2}{N}\right)$
>
> *where* $N \equiv \ell(-g)+m(-f)+n$ *and r is the radius of the circle.*

***Ex. Find the pole of $x+y+2=0$ with respect to the circle $x^2+y^2-4x+6y-12=0$**

Sol. Here $l=1, m=1, n=2, g=-2, f=3$

radius of the circle $r=\sqrt{4+9+12}=5$

Pole of $lx+my+n=0$ w.r.t $S=0$ is $\left(-g+\dfrac{\ell r^2}{\ell g+mf-n}, -f+\dfrac{mr^2}{\ell g+mf-n}\right)$

$\therefore$ The pole of $x+y+2=0$ w.r.t the circle $x^2+y^2-4x+6y-12=0$ is

$$\left(2+\frac{(1)(5)^2}{(1)(-2)+(1)(3)-2}, -3+\frac{(1)(5)^2}{(1)(-2)+(1)(3)-2}\right)=(-23,-28)$$

THEOREM-1.22 **The polar of $P(x_1, y_1)$ with respect to the circle $S = 0$ passes through $Q(x_2, y_2) \Leftrightarrow$ the polar of Q passes through P.**

Proof : Suppose that the polar of $P(x_1, y_1)$ with respect to the circle

$$S = x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{--(1)}$$

passes through $Q(x_2, y_2)$.

We shall prove that the polar of $Q(x_2, y_2)$ passes through P.

The polar of P w.r.t (1) is $S_1 = 0$. If it passes through $Q(x_2, y_2)$ then $S_{12} = 0$. -- (2)

Now, the polar of $Q(x_2, y_2)$ w.r.t (1) is $S_2 = 0$ --(3)

It passes through P if $S_{12} = 0$. In view of (2), the condition $S_{12} = 0$ is satisfied.

$\therefore$ The polar of P passes through Q and vice versa.

1.13 CONJUGATE POINTS

Definition :

Two points P and Q are said to be conjugate points with respect to a circle $S=0$ if Q lies on the polar of P. (If Q lies on the polar of P, then P lies on the polar of Q).

Result : The condition that the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to a circle $S = 0$ is $S_{12} = 0$.

1.14 CONJUGATE LINES

Definition :

If two lines are such that the pole of either of which with respect to a circle lies on the other, then the two lines are called the conjugate lines w.r.t the circle.

THEOREM-1.23 **If the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate with respect to the circle $x^2 + y^2 = r^2$ then $r^2 (l_1l_2+m_1m_2) = n_1n_2$**

> **Note :**
> *If three points ABC are such that polars of each point w.r.t. a circle passes through the other two then orthocentre of ΔABC is the centre of the circle*

Proof : Given circle is $x^2 + y^2 = r^2$ --(1)

Let (x_1, y_1) be the pole of the line $l_1x + m_1y + n_1 = 0$ -- (2) w.r.t (1)

$\therefore$ Polar of (x_1, y_1)w.r.t (1) is $xx_1 + yy_1 - r^2 = 0$ -- (3)

$\therefore$ (2) & (3) represent the same line

$$\Rightarrow \frac{x_1}{\ell_1} = \frac{y_1}{m_1} = \frac{-r^2}{n_1} \Rightarrow x_1 = \frac{-r^2\ell}{n_1}; \ y_1 = \frac{-r^2 m}{n_1}$$

$$\therefore \text{ Pole of (2) w.r.t (1) is } \left(\frac{-r^2\ell_1}{n_1}, \frac{-r^2 m_1}{n_1}\right).$$

But the two lines are conjugate.

$\therefore$ Pole of (2) lies on $l_2x + m_2y + n_2 = 0$

$$\Rightarrow \ell_2\left(\frac{-r^2\ell_1}{n_1}\right) + m_2\left(\frac{-r^2 m_1}{n_1}\right) + n_2 = 0 \Rightarrow -\ell_1\ell_2 r^2 - m_1m_2r^2 + n_1n_2 = 0$$

$$\Rightarrow r^2(\ell_1\ell_2 + m_1m_2) = n_1n_2$$

This is the required condition.

THEOREM-1.24 **The two lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then $r^2(l_1l_2 + m_1m_2) = (l_1g + m_1f - n_1)(l_2g + m_2f - n_2)$ where r is the radius of the circle.**

Proof : Let $L_1 = l_1x + m_1y + n_1 = 0$ and $L_2 = l_2x + m_2y + n_2 = 0$ and

$S = x^2 + y^2 + 2gx + 2fy + c = 0.$

Let $r^2 = g^2 + f^2 - c$

Pole of L_1 w.r.t $S = 0$ is $P\left(-g + \dfrac{r^2\ell_1}{\ell_1 g + m_1 f - n_1}, -f + \dfrac{r^2 m_1}{\ell_1 g + m_1 f - n_1}\right)$

Since lines L_1 and L_2 are conjugate w.r.t $S = 0 \Rightarrow P$ lies on $L_2 = 0$.

$$\therefore \ell_2\left(-g + \frac{r^2\ell_1}{\ell_1 g + m_1 f - n_1}\right) + m_2\left(-f + \frac{r^2 m_1}{\ell_1 g + m_1 f - n_1}\right) + n_2 = 0$$

$$\Rightarrow (-\ell_2 g - m_2 f + n_2) + \frac{r^2(\ell_1\ell_2 + m_1m_2)}{(\ell_1 g + m_1 f - n_1)} = 0$$

$$\Rightarrow r^2(\ell_1\ell_2 + m_1m_2) = (\ell_1 g + m_1 f - n_1)(\ell_2 g + m_2 f - n_2)$$

1.15 INVERSE POINTS

Definition :

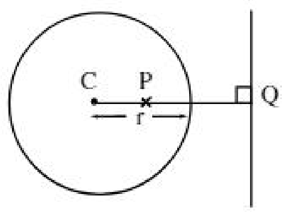
Let C be the centre and r be the radius of the circle $S = 0$. The points P and Q are said to be inverse points with respect to the circle $S = 0$ if C, P, Q are collinear such that P, Q are on the same side of C and $CP.CQ = r^2$.

THEOREM-1.25 **Let C be the centre and r be the radius of the circle $S \equiv x^2 + y^2 - r^2 = 0$. Two points P and Q are inverse points iff Q is the point of intersection of the polar of P w.r.t $S = 0$ and the line joining P and C.**

Proof : We have $C = (0, 0)$. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are inverse points. Then

i) $CP.CQ = r^2$

ii) C, P, Q are collinear.



From (i), $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = r^4$ --(1)

From (ii) we have area of $\Delta CPQ = 0$

$\Rightarrow x_1y_2 - x_2y_1 = 0$ --(2)

(1) $\Rightarrow x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 = r^4$

$\Rightarrow x_1^2x_2^2 + y_1^2y_2^2 + (x_1y_2 - x_2y_1)^2 + 2x_1x_2y_1y_2 = r^4$

$\Rightarrow (x_1x_2 + y_1y_2)^2 + 0 = r^4$ (using (2))

$\Rightarrow x_1x_2 + y_1y_2 = \pm\, r^2$

Since P and Q lie on the same side of C, we get $x_1x_2 > 0$, $y_1y_2 > 0$

$\therefore x_1x_2 + y_1y_2 = r^2$

$\Rightarrow$ P and Q are conjugate points

$\Rightarrow$ Q lies on the polar of P

$\therefore$ Q is the point of intersection of CP and the polar of P.

Conversely, suppose Q is the intersection of the polar of P and CP.

We shall prove that $CP.CQ = r^2$

Equation of the polar of P is $xx_1 + yy_1 - r^2 = 0$

$$CQ = \frac{\left|0+0-r^2\right|}{\sqrt{x_1^2+y_1^2}} = \frac{r^2}{\sqrt{x_1^2+y_1^2}} = \frac{r^2}{CP} \quad \therefore CP.CQ = r^2$$

$\therefore$ P and Q are inverse points.

Note :

If three points in a plane are collinear then their polars w.r.t. a circle are

i) Concurrent : when the centre is not on the line through the three points.

ii) Parallel : When the centre of the circle is on the line through the three points.

Note

i) *P lies on the circle $\Rightarrow CP = r$ and $CQ = \dfrac{r^2}{CP} = r \Rightarrow P = Q$*

$\therefore$ Any point on the circle is the inverse point to itself.

ii) *P lies inside the circle*

$\Rightarrow CP = r$ and $CQ = \dfrac{r^2}{CP} \Rightarrow CQ > r \Rightarrow$ Q lies outside the circle

iii) *P lies outside the circle $\Rightarrow$ Q lies inside the circle.*

iv) *If P,Q are inverse points w.r.t the circle $S = 0$, then P, Q are conjugate points w.r.t $S = 0$.*

v) *If P,Q are inverse points w.r.t the circle $S = 0$, then Q is the foot of the perpendicular from P on the polar of P w.r.t $S = 0$.*

Also, Q is the foot of the perpendicular from centre C to the polar of P w.r.t $S = 0$.

vi) *The inverse point of a point w.r.t a circle is unique.*

THEOREM-1.26

The inverse point of the point $P(x_1, y_1)$ with respect to the circle $x^2+y^2 = r^2$ is $\left(\dfrac{r^2x_1}{x_1^2+y_1^2}, \dfrac{r^2y_1}{x_1^2+y_1^2}\right)$

Proof : Let $S = x^2 + y^2 - r^2 = 0$

Equation of the polar of $P(x_1, y_1)$ w.r.t $S = 0$ is $S_1 = 0 \Rightarrow xx_1 + yy_1 - r^2 = 0$ -- (1)

Equation of the line joining (x_1, y_1) and the centre (0,0) of the circle is $xy_1 - x_1y = 0$ --(2)

Solving (1) & (2), we get the inverse point of P as $\left(\dfrac{r^2x_1}{x_1^2+y_1^2}, \dfrac{r^2y_1}{x_1^2+y_1^2}\right)$

Example : The inverse point of (2, 3) w.r.t $x^2 + y^2 = 10$ is $\left(\dfrac{2(10)}{4+9}, \dfrac{3(10)}{4+9}\right) = \left(\dfrac{20}{13}, \dfrac{30}{13}\right)$

THEOREM-1.27 **If $P(x_1, y_1)$ is the mid point of a chord AB (other than diameter) of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ then the equation of secant line $\overleftrightarrow{AB}$ is $S_1 = S_{11}$**

Note : *Polar is a straight line and it is perpendicular to the line joining centre and the point.*

Proof : Let C be the centre of the circle $S = 0$ then $C = (-g, -f) \neq (x_1, y_1)$

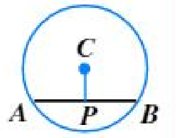


Fig 1.28

We know that $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CP}$. we may suppose that $y_1 \neq -f$.

$$\text{Slope of } AB = \frac{-1}{\text{slope of } CP} = \frac{-(x_1+g)}{(y_1+f)}$$

$$\therefore \text{ Equation of } \overleftrightarrow{AB} \text{ is } y - y_1 = \frac{-(x_1+g)}{(y_1+f)}(x - x_1)$$

$$\Rightarrow (y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

Adding $gx_1 + fy_1 + c$ on both sides of the above equation, we get $S_1 = S_{11}$.

THEOREM-1.28 **The length of the chord of the circle $S = 0$ having $P(x_1, y_1)$ as its mid-point is $2\sqrt{|S_{11}|}$.**

Proof : Let $\overline{AB}$ be the chord of the circle $S = 0$ having P as its mid - point

Now $PA.PB$ and $PA.PB = |S_{11}| \Rightarrow PA^2 = |S_{11}| \Rightarrow PA = \sqrt{|S_{11}|}$

Length of chord, $AB = 2PA = 2\sqrt{|S_{11}|}$

THEOREM-1.29 **If the line $lx + my + n = 0$ cuts the circle $x^2 + y^2 = a^2$ in A and B, then mid point of $\overline{AB}$ is $\left(\frac{-\ell n}{\ell^2+m^2}, \frac{-mn}{\ell^2+m^2}\right)$.**

Proof : For the circle $x^2 + y^2 = a^2$, centre $C = (0, 0)$ and radius, $r = a$.

If $P(x_1, y_1)$ is the mid point of the chord $\ell x + my + n = 0$, then P is the foot of the perpendicular from C on the chord.

$$\therefore \frac{x_1 - 0}{\ell} = \frac{y_1 - 0}{n} = \frac{-(\ell(0) + m(0) + n)}{\ell^2 + m^2} = \frac{-n}{(\ell^2 + m^2)}$$

$$\Rightarrow x_1 = \frac{-n\ell}{\ell^2 + m^2} \text{ and } y_1 = \frac{-mn}{\ell^2 + m^2}$$

$$\therefore \text{ Mid point } P = \left(\frac{-\ell n}{\ell^2 + m^2}, \frac{-mn}{\ell^2 + m^2}\right)$$

Ex. **Find the equation and length of the chord of the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ having (1, –2) as its mid point.**

Sol. Equation of the chord having (1,–2) as its mid point is $S_1 = S_{11}$

$\Rightarrow x(1) + y(-2) - 2(x+1) + 3(y-2) - 3 = (1)^2 + (-2)^2 - 4(1) + 6(-2) - 3$

$\Rightarrow x - 2y - 2x - 2 + 3y - 6 - 3 = 1 + 4 - 4 - 12 - 3 \Rightarrow x - y - 3 = 0$

Length of the chord = $2\sqrt{|S_{11}|} = 2\sqrt{14}$

1.16 PAIR OF TANGENTS

THEOREM-1.30 The combined equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the circle $S = 0$ is $S_1^2 = SS_{11}$.

Proof : Suppose that the tangents drawn from P to the circle $S = 0$ touch the circle at A and B.

The equation of AB is $S_1 = 0$

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ --(1)

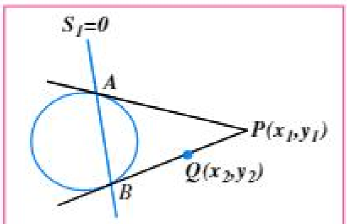


Fig 1.29

Let $Q(x_2, y_2)$ be any point on these tangents.

The locus of Q will be the equation of pair of tangents drawn from P.

The segment $\overline{PQ}$ is divided by the line AB whose equation is $S_1 = 0$ in the ratio $-S_{11} : S_{12}$

$\therefore PB : QB = -S_{11} : S_{12}$ or S_{11}:S_{12} according as $S_{11} : S_{12} < 0$ or $S_{11} : S_{12} > 0$

$$\Rightarrow \frac{PB}{QB} = \left|\frac{S_{11}}{S_{12}}\right| \text{ ------ (2)}$$

But $PB = \sqrt{S_{11}}$ and $QB = \sqrt{S_{22}}$ (lengths of the tangents from P and Q)

$$(2) \quad \Rightarrow \frac{S_{11}^2}{S_{12}^2} = \frac{S_{11}}{S_{22}} \Rightarrow S_{11}.S_{22} = S_{12}^2$$

Locus of $Q(x_2, y_2)$ is $SS_{11} = S_1^2$.

****Ex.*** **Find the equation of the pair of tangents from (10, 4) to the circle $x^2 + y^2 = 25$**

Sol. $S_1^2 = SS_{11} \Rightarrow (10x + 4y - 25)^2$

$= (x^2 + y^2 - 25)(100 + 16 - 25)$

$\Rightarrow 9x^2 + 80xy - 75y^2 - 500x - 200y + 2900 = 0$ is the equation of the required pair of tangents.

SOLVED EXAMPLES

Note : *The polar of a point w.r.t a circle if exists is unique.*

***1. If the chord of contact of a point P with respect to the circle $x^2+y^2=a^2$ cut the circle at A and B such that $\angle AOB = 90^0$. Find the locus of P.**

Sol. Given circle is $x^2+y^2=a^2$ --(1)

Let $P(x_1, y_1)$ be a point on the locus

$\therefore$ Equation of the chord of contact of P w.r.t (1) is $xx_1+yy_1 \Rightarrow \frac{xx_1+yy_1}{a^2}=1$ --(2)

The equation to the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is obtained by homogenising (1) with the help of (2)

$\therefore$ The equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is $x^2+y^2-a^2\left(\frac{xx_1+yy_1}{a^2}\right)^2=0$

$\Rightarrow a^2(x^2+y^2)-a^2(xx_1+yy_1)^2=0$

$\Rightarrow (a^2-x_1^2)x^2-2x_1y_1xy+(a^2-y_1^2)y^2=0$

Since $AOB=90^0$, coefficient of x^2+ coefficient of y^2 of (3) $=0$

$\therefore a^2-x_1^2+a^2-y_1^2=0 \Rightarrow x_1^2+y_1^2=2a^2$

$\therefore P(x_1, y_1)$ lies on the circle $x^2+y^2=2a^2$

***2. Show that the poles of tangents to the circle $x^2+y^2=a^2$ with respect to the circle $(x+a)^2+y^2=2a^2$ lie on $y^2+4ax=0$**

Sol. Given circles are $x+a^2+y^2=a^2$ --(1)

and $(x+a)^2+y^2=2a^2 \Rightarrow x^2+y^2+2ax-a^2=0$ --(2)

Let $P(x_1, y_1)$ be the pole of the tangent to the circle (1) with respect to the circle (2)

i.e., the polar of P w.r.t (2) is a tangent to the circle (1)

$\therefore$ Polar of P w.r.t (2) is $xx_1+yy_1+a(x+x_1)-a^2=0$

$\Rightarrow x(x_1+a)+yy_1+(ax_1-a^2)=0$ --(3)

(3) is a tangent to (1)

$$\Rightarrow a=\left|\frac{0+0+ax_1-a^2}{\sqrt{(x_1+a)^2+y_1^2}}\right| \Rightarrow Y_1^2+4ax_1=0$$

$\therefore$ The poles of tangents of (1) w.r.t (2) lie on the curve $y^2+4ax=0$

Remember : *If $(x_1\ y_1)$ and $(x_2\ y_2)$ are conjugate points w.r.t a circle $S=0$ then $S_{12}=0$*

***3. Find the value of k if the points (4, 2) and (k, –3) are conjugate points with respect to the circle $x^2+y^2-5x+8y+6=0$ (March-17, 19)**

Sol. Given circle is $S = x^2+y^2-5x+8y+6=0$ --(1)

Let $A(x_1, y_1)=(4, 2)$ and $B=(x_2, y_2)=(k, 3)$

A and B are conjugate w.r.t (1) $\Rightarrow S_{12}=0$

$$\Rightarrow (4)(k)+(2)(-3)-\frac{5}{2}(4+k)+4(2-3)+6=0$$

$$\Rightarrow k=\frac{28}{3}$$

Remember :

The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate w.r.t the circle $x^2+y^2+2gx+2fy+c=0$ is $(g^2+f^2-c)(l_1l_2+m_1m_2) = (l_1g+m_1f-n_1)(l_2g+m_2f-n_2)$

***4. Find the value of k if $x+y-5=0$ and $2x+ky-8=0$ are conjugte w.r.t the circle $x^2+y^2-2x-2y-1=0$**

Sol. Given circle is $x^2+y^2-2x-2y-1=0$ -- (1)

Centre of (1), $c(x_1, y_1) = (1, 1)$

radius of (1) $r = \sqrt{1+1+1} = \sqrt{3}$

Given lines are $\ell_1x + m_1y + n_1 \equiv x + y - 5 = 0$ -- (2)

and $\ell_2x + m_2y + n_2 \equiv 2x + ky - 8 = 0$ -- (3)

$N_1 = \ell_1x_1 + m_1y_1 + n_1 = 1+1-5 = -3$

$N_2 = \ell_2x_1 + m_2y_1 + n_2 = 2+k-8 = k-6$

(2) & (3) conjugate w.r.t (1)

$\Rightarrow r^2(\ell_1\ell_2 + m_1m_2) = N_1N_2 \Rightarrow 3(2+k) = (-3)(k-6)$

$\Rightarrow 2+k = 6-k \Rightarrow +2k = +4 \Rightarrow k = +2$

***5. Show that the area of the triangle formed by the two tangents through $P(x_1, y_1)$ to the circle $S = x^2+y^2+2gx+2fy+c = 0$ and the chord of contact of P w.r.t $S=0$ is $\dfrac{r(S_{11})^{3/2}}{S_{11}+r^2}$, where r is the radius of the circle.**

Sol. Let PA and PB be two tangents through P to the circle $S=0$ and θ be the angle between these two tangents

We know that $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{r}{\sqrt{S_{11}}}$

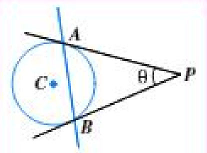


Fig 1.30

Required area = Area of $\Delta APB = \dfrac{1}{2}(PA)(PB)\sin\theta$

$$= \frac{1}{2}(\sqrt{S_{11}})(\sqrt{S_{11}})\left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right) = \frac{S_{11}\left(\frac{r}{\sqrt{S_{11}}}\right)}{1+\frac{r^2}{S_{11}}} = \frac{r(S_{11})^{3/2}}{(S_{11}+r^2)}$$

Remember :

Inverse point of 'P' w.r.t to circle is the foot of perpendicular from centre (or) P on polar of P w.r.t to circle

***6. Find the inverse point of $(-2, 3)$ with respect to the circle $x^2+y^2-4x-6y+9=0$**

Sol. Given circle is $x^2+y^2-4x-6y+9=0$ -- (1)

Centre of (1), $c = (2, 3)$

Let P = $(-2, 3)$

Polar of P w.r.t (1) is $S_1 = 0$

$\Rightarrow x(-2) + y(3) - 2(x-2) - 3(y+3) + 9 = 0 \quad \Rightarrow x = 1$ -- (2)

Equation of line $\overleftrightarrow{CP}$ is $y - 3 = \dfrac{3-3}{2+1}(x+2) \quad \Rightarrow y = 3$ -- (3)

Point of intersection of (2) & (3) i.e., $(1,3)$ is the inverse point of P$(-2, 3)$ w.r.t (1)

***7. Find the mid point of the chord intercepted by $x^2+y^2-2x-10y+1=0$ on the line $x-2y+7=0$.**

Remember :
Mid point of a chord is the foot of perpendicular from centre on the chord.

Sol. Given circle is $x^2+y^2-2x-10y+1=0$ -- (1)

Centre of (1), $c=(x_1, y_1)=(1, 5)$

Given line is $ax+by+c=x-2y+7=0$ -- (2)

Mid - point of chord (2) is the foot of the perpendicular $P(h, k)$ from c to (2)

$$\therefore \frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{(a^2+b^2)}$$

$$\Rightarrow \frac{h-1}{1}=\frac{k-5}{-2}=\frac{-(1-10+7)}{5}=\frac{2}{5} \Rightarrow h=\frac{7}{5} \text{ and } k=\frac{21}{5}$$

$\therefore P\left(\frac{7}{5}, \frac{21}{5}\right)$ is the mid point of given chord

***8. Find the locus of the mid points of the chord of contact of $x^2+y^2=a^2$ from the points lying on the line $\ell x+my+n=0$.**

Sol. Let $P(x_1, y_1)$ be a point on the locus. Thus P is the mid point of a chord of the circle $S=x^2+y^2-a^2=0$ --(1) and this chord is the chord of contact of P lying on $\ell x+my+n=0$ -- (2)

i.e., pole of this chord lies on (2)

Equation of the chord of (1) with $P(x_1, y_1)$ as its mid point is

$$S_1=S_{11} \Rightarrow xx_1+yy_1-(x_1^2+y_1^2)=0 \quad \text{-- (3)}$$

Pole of (3) w.r.t (1) is $\left(\frac{-a^2x_1}{-(x_1^2+y_1^2)}, \frac{-a^2y_1}{-(x_1^2+y_1^2)}\right)=\left(\frac{a^2x_1}{(x_1^2+y_1^2)}, \frac{a^2y_1}{(x_1^2+y_1^2)}\right)$

Since this lies on (2), $\ell\left(\frac{a^2x_1}{x_1^2+y_1^2}\right)+m\left(\frac{a^2y_1}{x_1^2+y_1^2}\right)+n=0$

$$\Rightarrow a^2(\ell x_1+my_1)+n(x_1^2+y_1^2)=0$$

Locus of P is $a^2(\ell x+my)+n(x^2+y^2)=0$

***9. Find the condition that the tangents drawn from the exteriorpoint (g, f) to $S \equiv x^2+y^2+2gx+2fy+c=0$ are perpendicular to each other.**

Remember :
Locus of point of intersection of perpendicular tangents is called the director circle.

Sol. Let $P(x_1, y_1)=(g, f)$

If θ is the angle between the tangents drawn from P to the circle $S=0$, then

$\tan\left(\frac{\theta}{2}\right)=\frac{r}{\sqrt{S_{11}}}$, r is the radius of the circle.

Since $\theta=90^0$, $1=\frac{\sqrt{g^2+f^2-c}}{\sqrt{g^2+f^2+2g^2+2f^2+c}}$

$$\Rightarrow 3g^2+3f^2+c=g^2+f^2-c$$

$\Rightarrow g^2+f^2+c=0$ is the required condition

***10.** **Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle $S = x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.**

Sol. Given circle is $S = x^2+y^2+2gx+2fy+c=0$ --(1)

Let $P(x_1, y_1)$ be the foot of the perpendicular drawn from the origin to the chord AB of (1)

Slope of $OP = \frac{y_1}{x_1}$ $\quad\therefore$ Slope of $AB = \frac{-x_1}{y_1}$

$\therefore$ Equation of chord AB is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$$\Rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2 \Rightarrow \frac{xx_1 + yy_1}{x_1^2 + y_1^2} = 1 \qquad \text{--(2)}$$

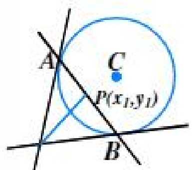


Fig 1.31

The combined equation of OA and OB is obtained by homogenising (1) with the help of (2)

$\therefore$ The combined equation of OA & OB is

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{xx_1 + yy_1}{x_1^2 + y_1^2}\right) + c\left(\frac{xx_1 + yy_1}{x_1^2 + y_1^2}\right)^2 = 0 \qquad \text{--(3)}$$

Since $\angle AOB = 90^0$, coefficent of x^2 + coefficient of y^2 of (3) = 0

$$\Rightarrow 1 + 1 + \frac{2gx_1 + 2fy_1}{(x_1^2 + y_1^2)} + c\left(\frac{x_1^2 + y_1^2}{(x_1^2 + y_1^2)^2}\right) = 0$$

$$\Rightarrow (2x_1^2 + y_1^2) + 2(gx_1 + fy_1) + c = 0$$

Locus of $P(x_1, y_1)$ is $2(x^2 + y^2 + gx + fy) + c = 0$

11. **Find the locus of midpoints of chords of the circle $x^2+y^2=r^2$, subtending a right angle at the point (a, b)**

Sol. Given circles $S \equiv x^2 + y^2 - r^2 = 0$

Let $P(x_1, y_1)$ be the midpoint of a chord AB of the circle $S = 0$

$P(x_1, y_1)$ lies inside $S = 0 \Rightarrow S_{11} < 0 \Rightarrow |S_{11}| = -S_{11}$

Length of the chord, $AB = 2\sqrt{|S_{11}|} \Rightarrow AP = \sqrt{|S_{11}|} \Rightarrow AP^2 = |S_{11}| = -S_{11}$

Let $C = (a, b)$. Given $\angle ACB = 90^0$

$\Rightarrow$ Circle on AB as diameter passes through C

$\Rightarrow PC = PA$ (radius)

$\Rightarrow PC^2 = PA^2 \Rightarrow (x_1 - a)^2 + (y_1 - b)^2 = -S_{11}$

$\Rightarrow x_1^2 - 2ax_1 + a^2 + y_1^2 - 2by_1 + b^2 = -(x_1^2 + y_1^2 - r^2)$

$\therefore$ Locus of P is $2x^2+2y^2-2ax-2by + a^2 + b^2-r^2=0$

Remember :

The locus of mid points of chords of circle $x^2+y^2 = r^2$ which subtend angle 'θ' at centre is $x^2+y^2 = r^2\cos^2\theta/2$

EXERCISE - 1.3

1. Find the chord of contact of
   *a) (1, 1) with respect to the circle $x^2+y^2=9$ **[Ans : Does not exist]**
   *b) (0, 5) with respect to the circle $x^2+y^2-5x+4y-2=0$ **[Ans : $-5x+14y+16=0$]**

2. Find the polar of
   *a) (1, 2) with respect to $x^2+y^2=7$ **[Ans : $x+2y=7$]**
   *b) (3, –1) with respect to $2x^2+2y^2=11$ **(May-19)** **[Ans : $6x-2y=11$]**
   ***c) (1, –2) with respect to $x^2+y^2-10x+10y+25=0$ **[Ans : $4x-3y-10=0$]**

3. Find the pole of
   *a) $3x+4y-45=0$ with respect to $x^2+y^2-6x-8y+5=0$ **[Ans : (6, 8)]**
   *b) $x-2y+22=0$ with respect to $x^2+y^2-5x+8y+6=0$ **[Ans : (2, –3)]**

*4. Find the slope of the polar of (1, 3) with respect to the circle $x^2+y^2-4x-4y-4=0$. Also find the distance from the centre to it. **[Ans : 1, $6\sqrt{2}$]**

*5. If $ax+by+c=0$ is the polar of (1, 1) with respect to the circle , $x^2+y^2-2x+2y+1=0$ and H.C.F of a, b, c is equal to one, then find $a^2+b^2+c^2$ **[Ans : 5]**

*6. Find the coordinates of the point of intersection of tangents at the points where $x+4y-14=0$ meets the circle $x^2+y^2-2x+3y-5=0$ **[Ans : $\left(\frac{109}{76},\frac{9}{38}\right)$]**

*7. If the polar of the points on the circle $x^2+y^2=a^2$ with respect to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$, then prove that a, b, c are in geometric progression.

*8. Tangents are drawn to the circle $x^2+y^2=16$ from the point $P(3, 5)$. Find the area of the triangle formed by these tangents and the chord of contact of P. **[Ans : $\frac{108\sqrt{2}}{17}$ sq.units]**

*9. Find the locus of the point whose polars with respect to the circles $x^2+y^2-4x-4y-8=0$ and $x^2+y^2-2x+6y-2=0$ are mutually perpendicular. **[Ans : $x^2+y^2-3x+y-4=0$]**

10. Show that the poles of tangents of the circle $(x-p)^2+y^2=b^2$ with respect to the circle $x^2+y^2=a^2$ lie on the curve $(px-a^2)^2=b^2(x^2+y^2)$

11. If the polars of the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) with respect to a circle are concurrent, prove that these points are collinear.

12. Show that
   *a) (–6, 1) and (2, 3) are conjugate points w.r.t the circle $x^2+y^2-2x+2y+1=0$
   *b) (4, 2) and (3, –5) are conjugate points w.r.t the circle $x^2+y^2-3x-5y+1=0$
   *c) (4, –2) and (3, –6) are conjugate points w.r.t the circle $x^2+y^2=24$

13. If (4, k) and (2, 3) are conjugate points with respect to the circle $x^2+y^2=17$, then find k. **[Ans : 3]**

*14. If (1, 3) and (2, k) are conjugate points with respect to the circle $x^2 + y^2 = 35$, then find k.
(March-19) [Ans : 11]

*15. Show that the lines $2x+3y+11 = 0$ and $2x-2y-1 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 4x + 6y - 12 = 0$

*16. Find the value of k if $kx+3y-1 = 0$, $2x+y+5 = 0$ are conjugate lines with respect to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ [Ans : 2]

*17. Find the inverse point of (1, 2) w.r.t the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ [Ans : (0, 1)]

18. Find the length and mid-point of the chord $2x + y - 5$ with respect to the circle $x^2 + y^2 = 9$.
[Ans : 4, (2, 1)]

19. Find the equation to the pair of tangents drawn from
*a) (0, 0) to the circle $x^2 + y^2 + 10x + 10y + 40 = 0$ [Ans : $3x^2-10xy+3y^2 = 0$]
*b) (3, 2) to the circle $x^2 + y^2 - 6x + 4y - 2 = 0$ [Ans : $x^2-15y^2-6x+60y-51 = 0$]
c) (1, 3) to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and also find the angle between them.
[Ans : $9x^2-16y^2-18x+96y-135 = 0$, $2\tan^{-1}\left(\frac{4}{3}\right)$]

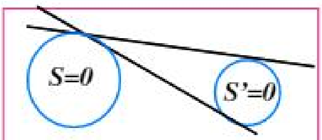
*20. Find the equation to the pair of tangents drawn from the origin to the circle $x^2+y^2+2gx+2fy+c = 0$ and hence deduce a condition for these tangents to be perpendicular.
[Ans : $(gx+fy)^2 = c(x^2+y^2)$, $g^2+f^2 = 2c$]

1.17 COMMON TANGENTS

The number of common tangents that can be drawn to two given circles depends on their relative positions. Any two intersecting common tangents of two circles and the line of centres of the circles are concurrent, and the point of intersection of two common tangents (if it exists) of two circles and their centres are collinear. In this section, we discuss the different possible relative positions of two circles and the number of common tangents that exist in each case.

Definition :

A line L is said to be a common tangent to the circles $S = 0$ and $S' = 0$ if it is a tangent to both the circles.



Common tangents

Fig 1.32

Definition : **(Touching Circles)**

If two circles meet at one and only one point P, then they are said to touch each other at P. The common point P is called the point of contact of the two circles.

Note

If two circles touch each other, then there exists only one tangent at their point of contact.

Direct Common Tangent :

Let l be a common tangent to two circles S and S'. If the circles S and S' lie on the same side of l, then l is called a direct common tangent to two circles.

Transverse Common Tangent :

Let l be a common tangent to two circles S and S'. If the circles S and S' lie on the either side of l, then l is called a transverse common tangent to two circles.

In general, there may be at most two direct and two transverse common tangents to a given pair of non-concentric circles.

1.18 CENTRES OF SIMILITUDE

1) External centre of similitude :

The point of intersection of the direct common tangents of two circles is called the *external centre of similitude.*

2) Internal centre of similitude :

The point of intersection of the transverse common tangents of two circles is called the *internal centre of similitude.*

Let S, S' be two circles with centres C_1, C_2 and radii r_1, r_2 respectively.

Case(i) : Let $\overleftrightarrow{Q_1Q_2}$ $(r_1 \neq r_2)$

Then the two circles donot intersect each other and each will be outside the other. In this case, there exist two direct and two transverse common tangents.

Let a direct common tangent $\overleftrightarrow{P_1P_2}$ meet the circles at P_1 and P_2. Let it cut the line of centres C_1C_2 at Q. Join C_1P_1 and C_2P_2.

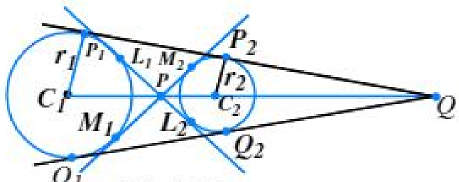


Fig 1.33

Since the right angled ΔC_1P_1Q /// the right angled ΔC_2P_2Q

$$\frac{C_1P_1}{C_2P_2} = \frac{C_1Q}{C_2Q} \Rightarrow \frac{C_1Q}{C_2Q} = \frac{r_1}{r_2}$$

Similarly, we can prove that the other direct common tangent $\overleftrightarrow{Q_1Q_2}$ also meets the line of centres $\overleftrightarrow{C_1C_2}$ at Q.

The two direct common tangents meet $\overleftrightarrow{C_1C_2}$ at Q and divide C_1C_2 in the ratio $r_1 : r_2$ externally.

Similarly, the pair of transverse common tangents $\overleftrightarrow{L_1L_2}$, $\overleftrightarrow{M_1M_2}$ intersect on $\overleftrightarrow{C_1C_2}$ at P and divide C_1C_2 in the ratio $r_1 : r_2$ internally.

If $r_1 = r_2$, then the direct common tangents will be parallel to $\overleftrightarrow{C_1C_2}$ and the external centre of similitude doesnot exist. Also, the internal centre of similitude P becomes the mid-point of C_1C_2.

Note

The pair of tangents from $Q(P)$ to any of the circles $S = 0$, $S' = 0$ is the pair of direct (transverse) common tangents to the two circles.

Case (ii) : Let $C_1C_2 = r_1 + r_2 (r_1 \neq r_2)$.

The two circles touch each other externally at P, the internal centre of similitude.

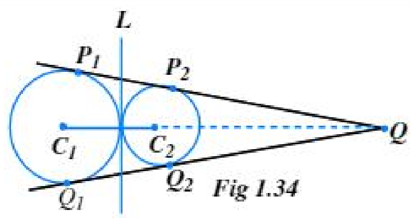


Fig 1.34

In this case, the two common tangents $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{Q_1Q_2}$ and only one transverse common tangent $\overleftrightarrow{LM}$ exist. Further, the transverse common tangent $\overleftrightarrow{LM}$ is perpendicular to C_1C_2.

Case (iii) : Let $C_1C_2 = r_1 \sim r_2 (r_1 > r_2)$. The two circles touch internally at P. Then only centre of similitude P exists. In this case, only one direct common tangent exists. Further it is perpendicular to $\overleftrightarrow{C_1C_2}$.

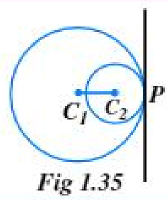


Fig 1.35

Note : *Centres of similitude are harmonic conjugates w.r.t. the linesegment joining centers.*

Case (iv) : Let $C_1C_2 < r_1 + r_2 (r_1 \neq r_2)$. The two circles intersect each other. In this case, only two direct common tangents exist and they intersect at Q, the external centre of similitude, on $\overleftrightarrow{C_1C_2}$.

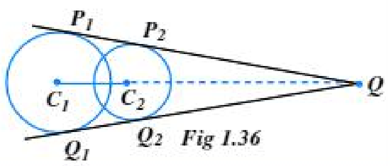


Fig 1.36

1.19 RELATIVE POSITIONS OF TWO CIRCLES

Let C_1 and C_2 be the centres and r_1, r_2 be the radii of two circles respectively. Let n be the number of common tangents to the two circles.

1) $C_1C_2 > r_1 + r_2$ (One circle lies completely outside the other) Number of common tangents, $n = 4$ *(See Fig 1.34)*

2) $C_1C_2 = r_1 + r_2$ (Two circles touch each other externally. The point of contact divides $\overline{C_1C_2}$ internally in the ratio $r_1 : r_2$) Number of common tangents, $n = 3$ *(See Fig (1.35)*

3) $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ (The two circles intersect each other) Number of common tangents, $n = 2$ *(See Fig 1.37)*

4) $C_1C_2 = |r_1 - r_2|$ (The two circles touch each other internally. Their point of contact divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ externally)

Number of common tangents, $n = 1$ *(See Fig 1.36)*

5) $C_1C_2 < |r_1 - r_2|$ (The circles do not intersect and one will be completely inside the other)

Number of common tangents, $n = 0$

S.No.	Condition	No. of common tangents	No. of D.C.T's	No. of T.C.T's
1.	$C_1C_2 < \|r_1 - r_2\|$	0	0	0
2.	$C_1C_2 = \|r_1 - r_2\|$	1	1	0
3.	$\|r_1 - r_2\| < \overline{C_1C_2} < r_1 + r_2$	2	2	0
4.	$r_1 + r_2 > \overline{C_1C_2}$	3	2	1
5.	$\overline{C_1C_2} > r_1 + r_2$	4	2	2

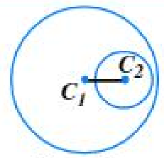


Fig 1.37

SOLVED EXAMPLES

***1. Discuss the relative positions of the following pairs of circles.**

$x^2 + y^2 - 2x + 4y - 4 = 0,\quad x^2 + y^2 + 4x - 6y - 3 = 0$

Sol. Given circles are

$$x^2 + y^2 - 2x + 4y - 4 = 0 \qquad \text{--(1)}$$
$$x^2 + y^2 + 4x - 6y - 3 = 0 \qquad \text{--(2)}$$

Centre of (1), $C_1 = (1, -2)$

Centre of (2) $C_2 = (-2, 3)$

Radius of (1) $r_1 = \sqrt{1+4+4} = 3$

Radius of (2) $r_2 = \sqrt{4+9+3} = 4$

$C_1C_2 = \sqrt{9+25} = \sqrt{34}$

$|r_1 - r_2| < C_1C_2 < r_1 + r_2 \Rightarrow$ The two circles intersect each other.

2. Find the number of possible common tangents that exist for the following pairs of circle.

***a) $x^2 + y^2 - 4x - 2y + 1 = 0, x^2 + y^2 - 6x - 4y + 4 = 0$**

***b) $x^2 + y^2 - 4x + 2y - 4 = 0, x^2 + y^2 + 2x - 6y + 6 = 0$**

Sol. a) Given circles are $x^2 + y^2 - 4x - 2y + 1 = 0$ --(1)

$x^2 + y^2 - 6x - 4y + 4 = 0$ --(2)

$C_1 = (2, 1), C_2 = (3, 2),\ r_1 = \sqrt{4+1-1} = 2,\ r_2 = \sqrt{9+4-4} = 3$

$\therefore C_1C_2 = \sqrt{1+1} = \sqrt{2}$

$|r_1 - r_2| < C_1C_2 < r_1 + r_2 \Rightarrow$ The two circles intersect each other

$\therefore$ Number of common tangents 2

b) Given circles are $x^2 + y^2 - 4x + 2y - 4 = 0$ --(1)

and $x^2 + y^2 + 2x - 6y + 6 = 0$ --(2)

$C_1 = (2, -1), C_2 = (-1, 3), r_1 = \sqrt{4+1+4} = 3, r_2 = \sqrt{1+9-6} = 2$

$C_1C_2 = \sqrt{9+16} = 5$

$C_1C_2 = r_1 + r_2$

$\Rightarrow$ The two circles touche each other externally

$\therefore$ Number of common tangents of (1) & (2) = 3

***3. Find the internal centre of similitude for the circles $x^2 + y^2 + 6x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 6y + 9 = 0$.**

Sol. Given circles are $x^2 + y^2 + 6x - 2y + 1 = 0$ -- (1)

and $x^2 + y^2 - 2x - 6y + 9 = 0$. -- (2)

$C_1 = (-3, 1), C_2 = (1, 3), r_1 = \sqrt{9+1-1} = 3, r_2 = \sqrt{1+9-9} = 1$

Let P be the internal centre of similitude of (1) & (2)

$\therefore P$ divides C_1C_2 in the ratio $r_1 : r_2 = 3 : 1$ internally $= \left(0, \frac{5}{2}\right)$

*** 4. Find the external centre of similitude for the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 = 4$**

Sol. Given circles are $x^2 + y^2 - 2x - 6y + 9 = 0$ -- (1)

$x^2 + y^2 = 4$ -- (2)

$C_1 = (1,3), C_2 = (0,0), r_1 = \sqrt{1+9-9} = 1, r_2 = 2$

Let Q be the external centre of similitude of (1) & (2)

$\therefore Q$ divides C_1C_2 in the ratio $r_1 : r_2 = 1 : 2$ externally

$$\therefore Q = \left(\frac{1(0)-2(1)}{1-2}, \frac{1(0)-2(3)}{1-2}\right) = (2,6)$$

Remember :

The point of intersection of direct common tangents of two circles is called as external centre of similitude.

***5. Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$, $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other find the point of contact and the equation of the common tangent at their point of contact.** **(March-2017)**

Sol. Given circles are $S = x^2 + y^2 - 6x - 2y + 1 = 0$ -- (1)

$S' = x^2 + y^2 + 2x - 8y + 13 = 0$ -- (2)

$C_1 = (3, 1), C_2 = (3, 1), r_1 = \sqrt{9+1-1} = 3, r_2 = \sqrt{1+16-13} = 2$

$C_1C_2 = \sqrt{16+9} = 5, C_1C_2 = r_1 + r_2$

$\Rightarrow$ The circles (1) & (2) touch each other externally.

Let P be the point of contact of (1) & (2)

$\therefore P$ divides C_1C_2 in the ratio $r_1 : r_2 = 1 : 2$ internally

$$\therefore P = \left(\frac{3(-1)+2(3)}{3+2}, \frac{3(4)+2(1)}{3+2}\right) = \left(\frac{3}{5}, \frac{14}{5}\right)$$

Equation of the common tangent at P to (1) & (2) is $S_1 = 0$

$$\Rightarrow x\left(\frac{3}{5}\right) + y\left(\frac{14}{5}\right) - 3\left(x + \frac{3}{5}\right) - 1\left(y + \frac{14}{5}\right) + 1 = 0 \Rightarrow 4x - 3y + 6 = 0$$

Remember :

If C_1, C_2 are centres r_1, r_2 are radii, then external centre of similitude divide the line joining C_1, C_2 in the ratio $r_1 : r_2$ externally.

***6. Find the equation of the circle which touches $x^2+y^2-4x+6y-1=0$ at $(-1, 1)$ internally with a radius of 2.**

Sol. Given circle is $x^2+y^2-4x+6y-1=0$ -- (1)

Centre of (1) $C_1=(2,-3)$, $r_1=\sqrt{4+9+12}=5$

Let C_2 the centre of the required with radius $r_2=2$ circle which touches (1) internally at $P(-1, 1)$

$C_1P=5, C_2P=2$

$\therefore C_1C_2=C_1P-C_2P=3$

$\therefore C_2$ divides C_1P in the ratio 3 : 2 internally

$$\therefore C_2=\left(\frac{3(-1)+2(2)}{3+2},\frac{3(1)+2(-3)}{3+2}\right)=\left(\frac{1}{5},\frac{-3}{5}\right)$$

$\therefore$ Equation of the required circle is $\left(x-\frac{1}{5}\right)^2+\left(y+\frac{3}{5}\right)^2=4$

$\Rightarrow 5x^2+5y^2-2x-6y-18=0$

7. Show that the circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2by+c=0$ to touch each other if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$

Sol. Given circles are $x^2+y^2+2ax+c=0$ -- (1)

$x^2+y^2+2by+c=0$ -- (2)

$C_1=(-a,0)$, $C_2=(0,-b)$ $r_1=\sqrt{a^2-c}, r_2=\sqrt{b^2-c}$, $C_1C_2=\sqrt{a^2+b^2}$

(1) & (2) touch each other if $C_1C_2=r_1\pm r_2$

i.e., if $\sqrt{a^2+b^2}=\sqrt{a^2-c}\pm\sqrt{b^2-c}$

i.e., if $a^2+b^2=a^2-c+b^2-c\pm 2\sqrt{a^2-c}\sqrt{b^2-c}$

i.e., if $c=\pm\sqrt{a^2-c}\sqrt{b^2-c}$

i.e., if $c^2=(a^2-c)(b^2-c)$

i.e., if $e^2=a^2b^2-c(a^2+b^2)+c^2$

i.e., if $a^2b^2=c(a^2+b^2)$

i.e., if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$ which is true

***8. Find the equation to all possible tangents of the circles $x^2+y^2-2x-6y+6=0$ and $x^2+y^2=1$**

Sol. Given circles are $x^2+y^2-2x-6y+6=0$ -- (1)

and $x^2+y^2=1$ -- (2)

Let C_1, C_2 be the centres and r_1, r_2 be the radii of the circles given by (1) & (2)

$\therefore C_1=(1,3), C_2=(0,0)$, $r_1=2, r_2=1$

Internal centre of similitude of (1)& (2), $P=\left(\frac{1}{3},1\right)$

External centre of similitude of (1) & (2), $Q=(-1,-3)$

Note : *Internal centre of similitude divides the line of centre in the ratio $r_1 : r_2$ internally.*

Note :

Internal centre of similitude and external centre of similitude are harmonic conjugate w.r.t. the line of centre.

Equation of the pair of transverse common tangents is given by $S_1^2 = SS_{11} \Rightarrow \left(x\left(\frac{1}{3}\right) + y(1) - 1\right)^2 = (x^2 + y^2 - 1)\left(\frac{1}{9} + 1 - 1\right)$

$\Rightarrow 4y^2 + 3xy - 3x - 9y + 5 = 0$

$\Rightarrow (y-1)(3x+4y-5) = 0$

Equation of the pair of direct common tangents is given by $S_1^2 = SS_{11}$

$\Rightarrow (x(1) + y(-3) - 1)^2 = (x^2 + y^2 - 1)(1+9-1)$

$\Rightarrow (x+3y+1)^2 = 9(x^2 + y^2 - 1)$

$\Rightarrow 4x^2 - 3xy - x - 3y - 5 = 0$

$\Rightarrow (x+1)(4x-3y-5) = 0$

$\therefore$ The equation of the common tangents of (1) & (2) are

$$y-1=0,\ 3x+4y-5=0,\ x+1=0 \text{ and } 4x-3y-5=0$$

EXERCISE - 1.4

1. Discuss the relative positions of the following pairs of circles

   *a) $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 6x + 18y + 26 = 0$ [Ans : Touch each other]

   *b) $x^2 + y^2 + 6x + 6y + 14 = 0$, $x^2 + y^2 - 2x - 4y - 4 = 0$ [Ans : One lies completely outside the other]

2. Find the number of possible common tangents that exist for the following pairs of circles

   *a) $x^2 + y^2 + 6x + 6y + 14 = 0$, $x^2 + y^2 - 2x - 4y - 4 = 0$ [Ans : 4]

   *b) $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 8y + 16 = 0$ [Ans : 3]

   *c) $x^2 + y^2 + 4x - 6y - 3 = 0$, $x^2 + y^2 + 4x - 2y + 4 = 0$ [Ans : 0]

3. *a) Show the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $5(x^2 + y^2) - 8x - 14y - 32 = 0$ touch each other and find their point of contact. [Ans : (−1, −1)]

   *b) Show that $x^2 + y^2 - 6x - 9y + 13 = 0$, $x^2 + y^2 - 2x - 16y = 0$ touch each other. Find the point of contact and the equation of common tangent at their point of contact. **(May-18)**

   [Ans : (3, 1), $4x - 7y - 13 = 0$]

   *c) Show that the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ touch each other. Also find the point of contact and common tangent at this point of contact.

   [Ans : $\left(\frac{1}{13}, \frac{-21}{13}\right)$, $5x + 12y + 19 = 0$]

4. *a) Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at (5, 5) with radius 5. [Ans : $x^2 + y^2 - 18x - 16y + 120 = 0$]

   **b) Show that the circles $S \equiv x^2 + y^2 - 2x - 4y - 20 = 0$, $S' \equiv x^2 + y^2 + 6x + 2y - 90 = 0$ touch each other internally. Find their point contact.

*5. Show that four common tangents can be drawn the circles given by $x^2+y^2-14x+6y+33=0$ and $x^2+y^2+30x-2y+1=0$ and find the internal and external centres of similitude. **(March-19)**

[**Ans :** $\left(\frac{3}{2},-2\right)$, $(18,-5)$]

*6. Prove that the circles $x^2+y^2-8x-6y+21=0$ and $x^2+y^2-2y-15=0$ have exactly two common tangents. Also find the intersection of those tangents. [**Ans :** $(8, 5)$]

***7. Find the direct common tangents of the circles $x^2+y^2+22x-4y-100=0$ and $x^2+y^2-22x+4y+100=0$ **(March-18)** [**Ans :** $3x+4y-50=0$, $7x-24y-250=0$]

***8. Find the transverse common tangents of the circles $x^2+y^2-4x-10y+28=0$ and $x^2+y^2+4x-6y+4=0$ **(March-17, 19 & May-19)** [**Ans :** $x-1=0$, $3x+4y-21=0$]

9. Find all common tangents of the pairs of circles

*a) $x^2+y^2=9$ and $x^2+y^2-16x-2y+49=0$

[**Ans :** $4x-3y-15=0$, $12x+5y-39=0$, $y-3=0$, $16x+63y+195=0$]

*b) $x^2+y^2+4x+2y-4=0$ and $x^2+y^2-4x-2y+4=0$

[**Ans :** $y-2=0$, $y-3=0$, $x-1=0$, $3x+4y-5=0$]

10. Let A be the centre of the circle $x^2+y^2-2x-4y-20=0$. Suppose that the tangents at $B(1, 7)$ and $D(4, -2)$ which are on the circle, meet at C. Find the area of the quadrilateral $ABCD$.

[**Ans :** 75 sq units]

11. Find the value of c for which $A(2,0)$, $B(0,14/3)$, $C(4, 5)$ and $D(0, c)$ are concyclic. [**Ans :** 1]

12. Let $x^2+y^2-4x-2y-11=0$ be a circle. A pair of tangents drawn from the point $(4,5)$. Find the area of the quadrilateral formed by these two tangents and a pair of radii of the circle which are line joining the centres and the point contact of those two tangents. [**Ans :** 8 sq. units.]