CHAPTER 2

Calculus

DEFINITE INTEGRALS

• INTEGRAL AS THE LIMIT OF A SUM •
• FUNDAMENTAL THEOREM OF CALCULUS •
• GEOMETRICAL INTERPRETATION •
• PROPERTIES OF DEFINITE INTEGRALS • REDUCTION FORMULAE •

2.0 INTRODUCTION

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. ***The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval [a,b], then its value is the difference between the values of F at the end points, i.e., F(b) – F(a).***

Here, we shall consider these two cases separately as discussed below :

2.1 DEFINITE INTEGRALS AS THE LIMIT OF A SUM

Let f be a continuous function defined on close interval $[a,b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y=f(x)$, the ordinates $x=a$, $x=b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x=a$ and $x=b$ (Fig 2.1).

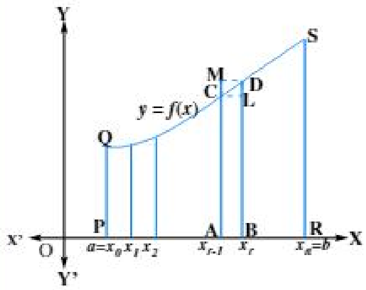


Fig 2.1

Divide the interval $[a,b]$ into n equal subintervals denoted by

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{r-1}, x_r], \ \ldots, \ [x_{n-1}, x_n],$$

where $x_0=a$, $x_1=a+h$, $x_2=a+2h$, ... , $x_r=a+rh$ and $x_n=b=a+nh$ or $n=\dfrac{b-a}{h}$.

We note that as $n \to \infty, h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r=1,2,3, \ldots, n$.

From Fig 2.1, we have area of the rectangle (ABLC)< area of the region (ABDCA)< area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, *i.e.*, $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h[f(x_0) + ... + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad ... (2)$$

and $$S_n = h[f(x_1) + f(x_2) + ... + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \quad ... (3)$$

Here s_n and S_n denote the sum of area of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for r=1,2,3, ..., n, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have s_n< area of the region PRSQP < S_n ... (4)

As $n \to \infty$ strips become narrower and narrower, it is assumed that limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} s_n$ =area of the region PRSQP= $\int_a^b f(x)dx$ ... (5)

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + ... + f(a+(n-1)h]$$

or $$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + ... + f(a+(n-1)h] \quad ... (6)$$

Where $h = \dfrac{b-a}{n} \to 0 \;\; as \;\; n \to \infty$

The above expression (6) is known as the definition of definite integral as the *limit of sum.*

SOLVED EXAMPLES

1. Evaluate $\int_a^b x\,dx$ as the limit of a sum.

Sol. Let as divide $[a, b]$ into n subintervals of equal length $h = \dfrac{b-a}{n}$. Then, by definition

$$\int_a^b f(x)dx = \underset{n\to\infty}{Lt}\, h\sum_{r=0}^{n-1} f(a+rh) \quad (\text{As } n\to\infty, h\to 0 \text{ such that } nh = b-a)$$

$$\Rightarrow \int_a^b xdx = \underset{n\to\infty}{Lt}\, h\begin{bmatrix} a+(a+h)+(a+2h)+.. \\ +[a+(n-1)h] \end{bmatrix} = \underset{n\to\infty}{Lt}\left(\frac{b-a}{n}\right)\left[na + h\frac{n(n-1)}{2}\right]$$

$$= (b-a)\underset{n\to\infty}{Lt}\left[a + (b-a)\left(\frac{n-1}{2n}\right)\right] = (b-a).\left[a + (b-a).\frac{1}{2}\right] = \frac{1}{2}(b^2 - a^2)$$

$$\therefore \int_a^b xdx = \frac{1}{2}(b^2 - a^2)$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \sin x\,dx$ as the limit of a sum.

Sol. Let us divide $\left[0, \frac{\pi}{2}\right]$ into n subintervals of equal length $h = \frac{\pi}{2n}$, then, by definition,

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \underset{n\to\infty}{Lt}\ h\sum_{r=0}^{n-1} f(a+rh) = \underset{n\to\infty}{Lt}\ h\sum_{r=0}^{n-1}\sin(rh) = \underset{n\to\infty}{Lt}\ h\left[\begin{matrix}\sin 0 + \sin h + \sin 2h + \ldots \\ + \sin(n-1)h\end{matrix}\right]$$

$$= \underset{n\to\infty}{Lt}\ h\left[\frac{\sin\left(\frac{(n-1)h}{2}\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right] = \underset{n\to\infty}{Lt}\ \frac{h}{2}\,\frac{\left[\cos\left(\frac{h}{2}\right) - \cos\left(nh - \frac{h}{2}\right)\right]}{\sin\left(\frac{h}{2}\right)}$$

$$= \underset{h\to 0}{Lt}\ \frac{\left(h/2\right)}{\sin\left(h/2\right)}\left[\cos\left(\frac{h}{2}\right) - \cos\left(\frac{\pi}{2} - \frac{h}{2}\right)\right] \quad \left(\because \text{As } n\to\infty,\ h\to 0 \text{ such that } nh = \frac{\pi}{2}\right)$$

$$= 1(1-0) = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x\,dx = 1$$

Note

i) $$\sin(\alpha) + \sin(\alpha+\beta) + \ldots + \sin(\alpha + (n-1)\beta) = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}.\sin\frac{[2\alpha + (n-1)\beta]}{2}$$

ii) $$\cos(\alpha) + \cos(\alpha+\beta) + \ldots + \cos[\alpha + (n-1)\beta] = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right).}\cos\frac{[2\alpha + (n-1)\beta]}{2}$$

3. Evaluate $\int_a^b e^x\,dx$ using the definition of a definite integral as the limit of a sum.

Sol. $\int_a^b f(x)dx = \underset{n\to\infty}{Lt}\ h\sum_{r=1}^{n} f(a+rh)$ when $n\to\infty, h\to 0$ such that $nh = b-a$

$$\therefore \int_a^b e^x dx = \underset{n\to\infty}{Lt}\ h\sum_{r=1}^{n} e^{a+rh} = e^a\ \underset{n\to\infty}{Lt}\ h\sum_{r=1}^{n} e^{rh} = e^a\ \underset{n\to\infty}{Lt}\ [he^h(1 + e^h + e^{2h} + \ldots + e^{(n-1)h})]$$

$$= e^a\ \underset{n\to\infty}{Lt}\ \frac{e^h(e^{nh}-1)}{\frac{(e^h-1)}{h}} = e^a\ \underset{h\to 0}{Lt}\left[\frac{e^h[e^{b-a}-1]}{\left(\frac{e^h-1}{h}\right)}\right] = e^a[e^{b-a}-1] = e^b - e^a$$

EXERCISE – 2.1

Evaluate the following integrals using the definition of a definite integral as the limit of a sum.

1. $\int_1^2 x\,dx$ *2. $\int_a^b x^2 dx$ 3. $\int_1^2 x^3 dx$

4. $\int_2^3 x^3 dx$ 5. $\int_0^{\frac{\pi}{2}} \cos x\,dx$ 6. $\int_0^1 e^x dx$

2.2 FUNDAMENTAL THEOREM OF (INTEGRAL)CALCULUS

A remarkable development in the evaluation of definite integrals was the introduction of the concept of anti - derivative by Newton and Leibnitz which led to the fund amental theorem of calculus. This theorem establishes the logical connection between Differentiation and Integration.

THEOREM-2.1 **(Fundamental Theorem of Integral Calculus)**

If f is integrable on $[a, b]$ and if there exists a differentiable function F such that $F' = f$ on $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$

(Proof of this theorem is beyond the scope of this book) by $[F(x)]_a^b$.

$$\therefore \int_a^b f(x)(x)dx = [F(x)]_a^b = F(b) - F(a)$$

This is called the ***Newton - Leibnitz formula***.

2.3 ANOTHER APPROACH TO DEFINITE INTEGRAL

Let $f : A \to R$ be a real function.

i) f is said to be a ***bounded function*** on A if there exist real numbers k_1 and k_2 such that $k_1 \le f(x) \le k_2 \ \forall x \in A$. k_1 is called a *lower bound* and k_2 is called an *upper bounds* of f.

ii) A real number ℓ is called the *greatest lower bound* or *infimum* of f if ℓ is a lower bound of f and k is a lower bound of $f \Rightarrow k \le \ell$. That is, the greatest of all lower bounds of f is called the infimum of f.

iii) A real number u is called the *least upper bound* or *supremum* of f if u is an upper bound of f and v is an upper bound of $f \Rightarrow u \le v$. That is, the least of all upper bounds of f is called the supremum of f.

iv) f is bounded on $A \Rightarrow f$ is bounded on every subset of A.

If f is continuous on $[a, b]$ then

Example :
$f(x)=\frac{1}{1+x^2}$ is a bounded function on R since $0<f(x)\le 1$ $\forall x\in R$. All negative numbers are lower bounds of f and all numbers greater than 1 are upper bounds of f. Any number greater than zero is not a lower bound and any number less than 1 is not an upper bound.
$\therefore$ inf f = 0, sup f = 1

i) f is bounded on $[a, b]$ i.e., there exists two numbers m and M such that $m \le f(x) \le M \;\; \forall x \in [a,b]$

ii) f attains its infimum and supremum on $[a, b]$

iii) inf f = the least value of f in $[a, b]$ sup f = the greatest value of f in $[a, b]$

iv) if $m \le f(x) \le M \;\; \forall x \in [a,b]$, $\exists$ points α and β in $[a, b]$ such that $f(\alpha)=m$ and $f(\beta)=M$

Definition : Let f a bounded function on $[a, b]$. Let $x_0, x_1, x_2, ..., x_n$ be $(n+1)$ points of $[a, b]$ such that $a=x_0<x_1<x_2<....<x_{n-1}<x_n=b$.

Let $\delta_r = x_r - x_{r-1}$ be the length of the r^{th} subinterval $[x_{r-1}, x_r]$.

Let m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$, $r = 1,2, .., n$. Then the sum $\sum_{r=1}^{n} m_r\delta_r$ is called the *lower Darboux sum* and $\sum_{r=1}^{n} M_r\delta_r$ is called the *upper Darboux sum* of the function f on $[a, b]$.

If the limits $\underset{n\to\infty}{Lt}\sum_{r=1}^{n} m_r\delta_r$ and $\underset{n\to\infty}{Lt}\sum_{r=1}^{n} M_r\delta_r$ exist finitely, and are equal then f is said to be Riemann integrable on $[a,b]$. The common limit is called the Riemanmm of integral of f on $[a,b]$. It is denoted by $\int_a^b f(x)dx$

$$\therefore \int_a^b f(x)dx = \underset{n\to\infty}{Lt}\sum_{n=1}^{n} m_r\delta_r = \underset{n\to\infty}{Lt}\sum_{r=1}^{n} M_r\delta_r$$

Note :
$m_r \le M_r$
$\Rightarrow m_r\delta_r \le M_r\delta_r$
$\Rightarrow \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r$

If f is continuous on $[a,b]$ then f is integrable on $[a, b]$. Even if f has a finite number of discontinuities in $[a, b]$ f is integrable on $[a,b]$ provided it is bounded on $[a,b]$.

If f is not bounded in $[a,b]$ then it need not be integrable on $[a, b]$.

2.4 GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

Remember :
A bounded function need not be Riemann integrable. A bounded function f is Reimann integrable on [a, b] iff f is integrable on [a,b].

Let f be integrable on $[a,b]$ and $f(x)\ge 0 \; \forall x\in[a,b]$. Let $P_0(x_0), P_1(x_1), ...P_n(x_n)$ be $(n+1)$ points such that $a=x_0<x_1<x_2<....<x_{r-1}<x_r<....<x_{n-1}<x_n=b$, dividing $[a,b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$. Let Q_0, Q_1, Q_2, Q_n be the points on the curve $y = f(x)$ such that P_0Q_0, P_1Q_1, P_2Q_2,...P_nQ_n be their respective ordinates.

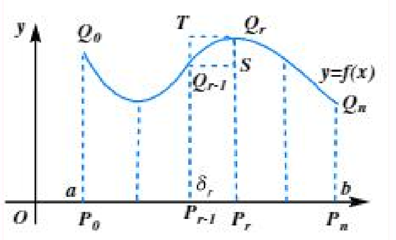


Fig 2.1

In the r^{th} subinterval $[x_{r-1}, x_r]$,

$P_{r-1}P_r = \delta_r,\ P_{r-1}Q_{r-1} = m_r,\ P_rQ_r = M_r$

Area of $P_{r-1}\ P_rQ_rQ_{r-1} = A$; Area of $P_{r-1}P_rSQ_{r-1} = m_r\delta_r$

Area of $P_{r-1}P_rQ_rT = M_r\delta_r$ Clearly $m_r\delta_r \le A_r \le M_r\delta_r$

For $r = 1, 2, 3.., n$, let $\delta_r = x_r - x_{r-1}$, $m_r =$ infimum f, $M_r =$ supremum of f in $[x_{r-1}, x_r]$.

Let A be the area of the region $P_0P_nQ_nQ_0$ enclosed by the curve $y = f(x)$ above the x-axis between $x = a$ and $x = b$, A_r be the area of the region $P_{r-1}, P_rQ_rQ_{r-1}$ enclosed by $y = f(x)$, x-axis, $x = x_{r-1}$ and $x = x_r$ $(r = 1, 2, ..., n)$.

Then $A = \sum_{r=1}^{n} A_r$ Now, $m_r\delta_r \le A_r \le M_r\delta_r$

$$\Rightarrow \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} A_r \le \sum_{r=1}^{n} M_r\delta_r \Rightarrow \sum_{r=1}^{n} m_r\delta_r \le A \le \sum_{r=1}^{n} M_r\delta_r$$

$$\Rightarrow \underset{n\to\infty}{Lt} \sum_{r=1}^{n} m_r\delta_r \le \underset{n\to\infty}{Lt}\ A \le \underset{n\to\infty}{Lt} \sum_{r=1}^{n} M_r\delta_r \ (n \to \infty \text{ such that each } \delta_r \to 0)$$

$$\Rightarrow \int_a^b f(x)dx \le A \le \int_a^b f(x)dx \quad (\because f \text{ is integrable on } [a, b]) \Rightarrow A = \int_a^b f(x)dx$$

2.5 GEOMETRICAL INTERPRETATON

Thus, if f is integrable on $[a, b]$ and $f(x) \ge 0\ \ \forall x \in [a,b]$ then $\int_a^b f(x)dx$ can be geometrically interpreted as the area of the region bounded by $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$.

2.6 EVALUATION OF DEFINITE INTEGRAL

In the previous chapter we have already discussed about various methods of evaluating indefinite integrals using the anti-derivative concept. Now we can make use of those methods and the fundamental theorem of calculus (Newton - Leibnitz formula) to evaluate definite integrals. Thus, for a function f defined on $[a, b]$ if $\int f(x)dx = F(x) + c$, then the value $F(b) - F(a)$ can be taken as the value of the definite integral of f on $[a, b]$. It is denoted by $\int_a^b f(x)dx$. Therefore, $\int f(x)dx = F(x) + c$ on $[a, b] \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$

For the sake of convenience, $F(b) - F(a)$ is denoted by the symbol $[F(x)]_a^b$

Thus $\int_a^b f(x)dx = [F(x) + c]_a^b = (F(b) + c) - (F(a) + c) = F(b) - F(c)$

Since the value of a definite integral does not depend on arbitrary constant c, we need not consider any arbitrary constant in its evaluation.

In the symbol $\int_a^b f(x)dx, x$ is called the variable of integration, a is called the lower limit, b is called the upper limit and $[a, b]$ is called the range of the variable of integration x.

As the value of a definite integral $\int_a^b f(x)dx$ is independent of the variable of integration x (of course! Its value is dependent on f and the range of x, x can be treated as a dummy variable (or symbol) in the sense that it can be replaced by any other symbol having the same range. That is $\int_a^b f(x)dx \equiv \int_a^b f(t)dt$

For, if $\int f(x)dx = F(x)+c$ then $\int_a^b f(t)dt = F(t)+k$ and

$$\int_a^b f(x)dx = [F(x)+c]_a^b = F(b)-F(a)$$

$$\therefore \int_a^b f(t)dt = [F(t)+k]_a^b = F(b)-F(a) = \int_a^b f(x)dx$$

Thus $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy$ etc

Definition :

Let f be integrable on $[a, b]$. Then we define

i) $\int_b^a f(x)dx$ as the negative of $\int_a^b f(x)dx$ and ii) For $c \in [a,b]$, $\int_c^c f(x)dx$ as zero

$$\therefore \int_b^a f(x)dx = -\int_a^b f(x)dx \text{ and } \int_c^c f(x)dx = 0$$

2.7 ALGEBRAIC PROPERTIES

THEOREM-2.2 **If f and g are integrable on $[a,b]$ then**

i) $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

ii) For any constant $k \in R$, $k.f$ is integrable on $[a, b]$ and $\int_a^b (kf)(x)dx = k\int_a^b f(x)dx$

iii) for any $c \in (a,b)$, f is integrable on $[a, c]$ and $[c, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Remark : Proofs of the above statements using the general definition of integrability is beyond the scope of this book. Therefore we prove them using the anti derivative concept. This is the case with most of the theorems that follow.

Proof : Let $\int f(x)dx = F(x)$ and $\int g(x)dx = G(x)$.

Then $\int (f+g)(x) = F(x)+G(x)$, $\int_a^b f(x)dx = F(b)-F(a)$

and $\int_a^b g(x)dx = G(b)-G(a)$

i) Now $\int_a^b (f+g)(x)dx = [F(x)+G(x)]_a^b = [F(b)+G(b)]-[F(a)+G(a)]$

$= [F(b)-F(a)]+[G(b)-G(a)] = \int_a^b f(x)dx + \int_a^b g(x)dx$

ii) $\int (kf)(x)dx = kF(x) \Rightarrow \int_a^b (kf)(x)dx = [kF(x)]_a^b = kF(b)-kF(a)$

$= k[F(b)-F(a)] = k\int_a^b f(x)dx$

iii) $\int_a^c f(x)dx + \int_c^b f(x) = [F(x)]_a^c + [F(x)]_c^b dx$

$= [F(c)-F(a)]+[F(b)-F(c) = F(b)-F(a)] = \int_a^b f(x)dx$

$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Note

i) If f is inte grable on an interval I containing [a, b] and c is any point of I then

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ *(Note that the point c may lie out side [a, b].)*

ii) If $c_1, c_2, ..., c_n \in [a,b]$ then $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^{b} f(x)dx$

i.e., $\int_a^b f(x)dx = \left(\int_a^{c_1} + \int_{c_1}^{c_2} + ... + \int_{c_{n-1}}^{c_n} + \int_{c_n}^{b}\right) f(x)dx$

***Ex 1.** **Evaluate** $\int_0^\pi \sin x\, dx$

Sol. $\int_0^\pi \sin x\, dx = [-\cos x]_0^\pi = -[-1-1] = 2$

***Ex 2.** **Evaluate** $\int_0^a \frac{dx}{x^2+a^2}$ **(May-19)**

Sol. $\int_0^a \frac{1}{x^2+a^2}dx = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^a = \frac{1}{a}[\tan^{-1}(1)-\tan^{-1}(0)] = \frac{\pi}{4a}$

Ex 3. **Evaluate** $\int_1^2 \frac{(x+1)^3}{x^2}dx$

Sol. $\int_1^2 \frac{(x+1)^3}{x^2}dx = \int_1^2 \frac{x^3+3x^2+3x+1}{x^2}dx$

$= \int_1^2 \left(x+3+\frac{3}{x}+\frac{1}{x^2}\right)dx = \left[\frac{x^2}{2}+3x+3\log x+\left(\frac{-1}{x}\right)\right]_1^2$

$= \left(2+6+3\log 2-\frac{1}{2}\right)-\left(\frac{1}{2}+3-1\right) = 5+3\log 2$

Ex 4. **Evaluate i)** $\int_0^4 |2-x|dx$ **ii)** $\int_a^b \frac{|x|}{x}dx$

Sol. i) $|2-x| = \begin{cases} 2-x \text{ if } x<2 \\ x-2 \text{ if } x \ge 2 \end{cases}$ $\therefore \int_0^4 |2-x|dx = \int_0^2 |2-x|dx + \int_2^4 |2-x|dx$

$$= \int_0^2 (2-x)dx + \int_2^4 (x-2)dx = \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 = (2-0) + (0-(-2)) = 4$$

ii) We know that $\frac{d}{dx}|x| = \frac{|x|}{x}, x \ne 0$ $\therefore \int_a^b \frac{|x|}{x}dx = [|x|]_a^b = |b| - |a|$

Note

$$\int_a^b \frac{|x|}{x}dx = b-a \text{ if } 0<a<b = b\ a \text{ if } a<0<b = a-b \text{ if } a<b<0$$

THEOREM-2.3 **Let f be integrable on $[a,b]$. Let F be defined on $[a,b]$ as $F(x) = \int_a^x f(t)dt \quad \forall x \in [a,b]$. Then F is continuous on $[a,b]$ Further, if f is continuous on $[a,b]$, then F is differentiable on $[a,b]$ and $F'(x) = f(x)\ \forall x \in [a,b]$.**

THEOREM-2.4 **(Method of substitution or change of variable)**

Let $x = g(t)$ be a continuous function defined on $[\alpha,\beta]$ and have a continuous derivative $g'(t)$ on $[\alpha,\beta]$. Let f be a continuous function on $g([\alpha,\beta]) = [a,b]$, such that $g(\alpha) = a$ and $g(\beta) = b$

Then $(fog)g'$ is integrable on $[\alpha,\beta]$ and $\int_a^b f(x)dx = \int_\alpha^\beta f(g(t)).g'(t)dt$

That is, the subtitution $x = g(t)$ transforms $\int_a^b f(x)dx$ into $\int_\alpha^\beta f(g(t))g'(t)dt$

For, $x = g(t) \Rightarrow dx = g'(t)dt$

$x = a \Rightarrow g(t) = a \Rightarrow t = g^{-1}(a) = \alpha$

$x = b \Rightarrow g(t) = b \Rightarrow t = g^{-1}(b) = \beta$

$$\therefore \int_a^b f(x)dx = \int_\alpha^\beta f(g(t)).g'(t)dt$$

In practice, instead of the substitution $x = g(t)$ the inverse substitution $\phi(x) = t$ is used. Further the *substitution is usually performed with the help of monotonic, continuously differentiable functions.*

That is, to transform the integral $\int_a^b f(\phi(x)).\phi'(x)dx$ the substitution $\phi(x) = t$ is used so that $\phi'(x)dx = dt$, $x = a \Rightarrow t = \phi(a)$, $x = b \Rightarrow t = \phi(b)$

$\phi(a), \phi(b)$ are the limits of integration of the new variable t

$$\therefore \int_a^b f(\phi(x))\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)dt$$

***Ex 1.** **Evaluate** $\int_1^4 x\sqrt{x^2-1}\,dx$

Sol. To evaluate $\int_1^4 x\sqrt{x^2-1}\,dx$

Put $x^2-1=t$, then $2x\,dx=dt \Rightarrow x\,dx=\frac{1}{2}dt$

$x=1 \Rightarrow t=(1)^2-1=0 \qquad x=4 \Rightarrow t=16-1=15$

$$\therefore \int_1^4 x\sqrt{x^2-1}\,dx=\int_0^{15}\sqrt{t}\,\frac{1}{2}dt=\frac{1}{2}\left[\frac{2}{3}t^{3/2}\right]_0^{15}=\frac{1}{3}\left[(15)^{3/2}-0\right]=\frac{1}{3}(15)^{3/2}$$

Ex 2. **Evaluate** $\int_0^a \sqrt{a^2-x^2}\,dx$

Sol. To evaluate $\int_0^a \sqrt{a^2-x^2}\,dx$

> **Remember :**
> $\int_0^a \sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}$

Put $x=a\sin\theta$ so that $\theta=\sin^{-1}\left(\frac{x}{a}\right)$ then $dx=a\cos\theta\,d\theta$

$x=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$

$x=a \Rightarrow \sin\theta=1 \Rightarrow \theta=\frac{\pi}{2}$

$$\therefore \int_0^a \sqrt{a^2-x^2}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin\theta}\,a\cos\theta\,d\theta$$

$$=a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=a^2\int_0^{\frac{\pi}{2}}\frac{(1+\cos 2\theta)}{2}=\frac{a^2}{2}\left[x+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=a^2\,\frac{1}{2}\frac{\pi}{2}=\frac{\pi a^2}{4}$$

Ex 3. **Evaluate** $\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Sol. Put $\sin x-\cos x=t$. Then $(\cos x+\sin x)dx=dt$

$x=0 \Rightarrow t=-1, x=\frac{\pi}{4} \Rightarrow t=0$

Further, $(\sin x-\cos x)^2=t^2 \Rightarrow 1-\sin 2x=t^2 \quad \therefore \sin 2x=1-t^2$

$$\text{Now, } \int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx=\int_{-1}^{0}\frac{dt}{25-16t^2}=\frac{1}{4}\cdot\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\Bigg]_{-1}^{0}=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]=\frac{1}{20}\log 3$$

THEOREM-2.5 **Integration by parts**

i) If f and g are integrable on $[a,b]$ then their product $f.g$ is integrable on $[a,b]$

ii) If $f(x)$, $g(x)$ are integrable functions on $[a,b]$ such that $f'(x)$ and $g'(x)$ are integrable on $[a,b]$ then $f'(x)\,g(x)$ and $f(x)\,g'(x)$ are integrable on $[a,b]$ and

$$\int_a^b f(x)g(x)dx=[f(x)\int g(x)dx]_a^b-\int_a^b[f'(x)\int g(x)dx]dx$$

Ex. **Evaluate** $\int_0^{\frac{\pi}{2}} x^2\sin x\,dx$

Sol. $$\int_0^{\frac{\pi}{2}} x^2\sin x\,dx=x^2(-\cos x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(2x)(-\cos x)dx\;=0+2\int_0^{\frac{\pi}{2}}x\cos x\,dx$$

$$=2\left\{\left[x\sin x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}1.\sin x\,dx\right\}=2\left\{\left(\frac{\pi}{2}-0\right)-1\right\}=\pi-2$$

SOLVED EXAMPLES

1. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$

Sol. $\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx = \frac{1}{2}[x]_0^{\pi/2} - \frac{1}{4}[\sin 2x]_0^{\pi/2} = \frac{\pi}{4}$

2. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\, dx$

Sol. $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x} dx = \int_0^{\frac{\pi}{2}} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) dx = 2\left[\sin\frac{x}{2} - \cos\frac{x}{2}\right]_0^{\frac{\pi}{2}} = 2$

3. Evaluate $\int_0^1 \frac{1-x}{1+x} dx$

Sol. $\int_0^1 \frac{1-x}{1+x} dx = \int_0^1 \frac{2-(1+x)}{1+x} dx = [2\log(1+x) - x]_0^1 = 2\log 2 - 1$

4. Evaluate $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x} dx$

Sol. $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x} dx = \frac{1}{4}\int_0^{\frac{\pi}{3}} \frac{4\cos x}{3+4\sin x} dx = \frac{1}{4}[\log(3+4\sin x)]_0^{\frac{\pi}{3}}$

$= \frac{1}{4}[\log(3+2\sqrt{3}) - \log 3] = \frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$

5. Evaluate $\int_0^{\pi} \frac{1}{3+2\cos x} dx$

Sol. Let $I = \int_0^{\pi} \frac{1}{3+2\cos x} dx$

Put $\tan\frac{x}{2} = t$ then $dx = \frac{2}{1+t^2} dx$ and

$\cos x = \frac{1-t^2}{1+t^2}$, $x = 0 \Rightarrow t = 0$, $x \to \pi \Rightarrow t \to \infty$

Then $I = 2\int_0^{\infty} \frac{1}{t^2+5} dt = 2.\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)\Big]_0^{\infty} = \frac{2}{\sqrt{5}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{\sqrt{5}}$

6. Evaluate $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Sol. Put $e^x = t$

$\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_1^e \frac{dt}{1+t^2} = (\tan^{-1} t)_1^e = \tan^{-1} e - \frac{\pi}{4}$

7. Evaluate $\int_1^2 \log x\,dx$

Sol. $\int_1^2 \log x dx = [x\log x - x]_1^2 == 2\log 2 - 1$

8. Evaluate $\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}dx$

Remember :

$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}dx = \pi$

Sol. Put $x = a\cos^2\theta + b\sin^2\theta$

Then $dx = (b-a)\sin 2\theta\, d\theta$

$x - a = (b-a)\sin^2\theta,\ b - x = (b-a)\cos^2\theta$

$x = a \Rightarrow \theta = 0,\ x = b \Rightarrow \theta = \frac{\pi}{2}$

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}dx = \int_0^{\frac{\pi}{2}} \frac{(b-a)\sin 2\theta d\theta}{\sqrt{(b-a)\sin^2\theta(b-a)\cos^2\theta}} = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

9. Evaluate $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx\ (a < x < b)$

Remember :

$\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx = \frac{\pi}{2}(b-a)$

Sol. $$\int_a^b \sqrt{\frac{x-a}{b-x}}dx = \int_0^{\frac{\pi}{2}} \sqrt{\frac{(b-a)\sin^2\theta}{(b-a)\cos^2\theta}}(b-a)\sin 2\theta d\theta = 2(b-a)\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$$

$$= 2(b-a)\frac{1}{2}\frac{\pi}{2} = \frac{\pi}{2}(b-a)$$

Note : $\int_a^b \sqrt{\frac{b-x}{x-a}}dx \ = \frac{\pi}{2}(b-a)$

10. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}dx$

Remember :

$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}dx$

$= \sqrt{2}\log(\sqrt{2}+1)$

Sol. $$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)}dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\tan\frac{3\pi}{8}\right) - \log\left(\tan\frac{\pi}{8}\right)\right] = \frac{1}{\sqrt{2}}\log\left[\frac{\tan\left(\frac{3\pi}{8}\right)}{\tan\left(\frac{\pi}{8}\right)}\right] = \sqrt{2}\log\tan\left(\frac{3\pi}{8}\right)$$

$$= \sqrt{2}\log(\sqrt{2}+1)\left(\because \tan 67\frac{1}{2}^{\circ} = \sqrt{2}+1\right) \quad \therefore \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}dx = \sqrt{2}\log(\sqrt{2}+1)$$

11. Evaluate $\int_0^1 \sin^{-1}(\sqrt{x})\,dx$

Sol. Put $\sqrt{x} = t$ then $x = t^2$ $dx = 2t\,dt$

$x = 0 \Rightarrow t = 0,\ x = 1 \Rightarrow t = 1$

$$\therefore \int_0^1 \sin^{-1}(\sqrt{x})dx = \int_0^1 \sin^{-1} t\, 2t\ dt = 2\int_0^1 t\ \sin^{-1} t dt = 2\left\{\left[\frac{t^2}{2}\sin^{-1} t\right]_0^1 - \int_0^1 \frac{t^2}{2}\frac{1}{\sqrt{1-t^2}}dt\right\}$$

$$= 2\left\{\frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{1-(1-t^2)}{\sqrt{1-t^2}}dt\right\} = \frac{\pi}{2} - \int_0^1\left(\frac{1}{\sqrt{1-t^2}} - \sqrt{1-t^2}\right)dt$$

$$= \frac{\pi}{2} - \left[\sin^{-1} t - \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t\right]_0^1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

12. Find the value of $\int_0^a \sqrt{\frac{a+x}{a-x}}dx$

Remember :

$$\int_0^a \sqrt{\frac{a+x}{a-x}}dx = \frac{a}{2}(\pi+2)$$

Sol. $$\int_0^a \sqrt{\frac{a+x}{a-x}}dx = \int_0^a \frac{a+x}{\sqrt{a^2-x^2}}dx = a\int_0^a \frac{1}{\sqrt{a^2-x^2}}dx + \int_0^a \frac{x}{\sqrt{a^2-x^2}}dx$$

$$= a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a - [\sqrt{a^2-x^2}]_0^a = a\left(\frac{\pi}{2} - 0\right) - (0-a) = a\left(\frac{\pi+2}{2}\right)$$

13. Evaluate $\int_{-1}^{3/2} |x \sin \pi x|\,dx$

Note :

$$|x\sin(\pi x)| = \begin{cases} x\sin \pi x; & -1 \le x \le 1 \\ -x\sin \pi x; & 1 < x \le \frac{3}{2} \end{cases}$$

Sol. $|x\sin(\pi x)| = x\sin(\pi x)$ if $-1 \le x \le 1 = -x\sin(\pi x)$ if $1 < x \le \frac{3}{2}$

$$\therefore \int_{-1}^{3/2} |x\sin \pi x|\,dx = \int_{-1}^{1} x\sin(\pi x)dx + \int_{1}^{3/2} -x\sin(\pi x)dx$$

$$= \left[x\left(\frac{-\cos \pi x}{\pi}\right)\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos \pi x}{\pi}dx - \left\{\left[x\left(\frac{-\cos \pi}{\pi}\right)\right]_1^{3/2} + \int_1^{3/2}\frac{\cos \pi x}{\pi}dx\right\}$$

$$= \left(\frac{1}{\pi} + \frac{1}{\pi}\right) + \left[\frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left\{\left(0 - \frac{1}{\pi}\right) + \left[\frac{\sin \pi x}{\pi^2}\right]_1^{3/2}\right\}$$

$$= \left[\frac{2}{\pi} + 0\right] - \left\{-\frac{1}{\pi} - \frac{1}{\pi^2}\right\} = \frac{3}{\pi} + \frac{1}{\pi^2}$$

EXERCISE – 2.2

Evaluate the following integrals

1. *a) $\int_1^2 x^5 dx$ *b) $\int_0^a (a^2x - x^3)dx$ *c) $\int_2^3 \frac{2x}{1+x^2}dx$

   *d) $\int_0^4 \frac{x^2}{1+x}dx$ *e) $\int_0^a (\sqrt{a}-\sqrt{x})^2 dx$ (March-19)   *f) $\int_0^1 \frac{1}{\sqrt{x+1}+\sqrt{x}}dx$

   *g) $\int_0^1 \frac{x^2}{1+x^2}dx$ *h) $\int_0^3 \frac{x\,dx}{\sqrt{x^2+16}}$ (March-17)

   *i) $\int_0^2 \sqrt{4-x^2}\,dx$ *j) $\int_{\pi/6}^{\pi/3} \cot^2 x\,dx$

2. *a) $\int_0^{\pi} \sqrt{2+2\cos\theta}\,d\theta$ (March-18)   **b) $\int_0^2 |1-x|dx$ (May-18)   *c) $\int_0^4 |2-x|dx$

   *d) $\int_0^{\pi/4} \frac{1+\sin 2x}{\cos x+\sin x}dx$ e) $\int_1^4 x\sqrt{x^2-1}\,dx$ **f) $\int_0^1 \frac{dx}{\sqrt{3-2x}}$ (March-19)

   **g) $\int_1^5 \frac{dx}{\sqrt{2x-1}}$ *h) $\int_0^{16} \frac{x^{1/4}}{1+x^{1/2}}dx$ *i) $\int_0^1 (2x+3)\sqrt{3-2x}\,dx$

   *j) $\int_0^1 xe^{-x^2}dx$ *k) $\int_1^2 \frac{x^2}{x^2+2}dx$

3. a) $\int_0^{\pi/2} x\cos x\,dx$ *b) $\int_0^{\sqrt{e}} x\log x\,dx$ *c) $\int_1^2 \frac{\log x}{x^2}dx$

   *d) $\int_0^1 \frac{xe^x}{(1+x)^2}dx$ *e) $\int_0^{\pi/2} x^2\sin x\,dx$

4. a) $\int_1^2 x^2\log x\,dx$ b) $\int_0^1 \sin^{-1}x\,dx$ c) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

   d) $\int_0^1 x^2\sin^{-1}x\,dx$

5. **a) $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}dx$ *b) $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$ *c) $\int_a^b \sqrt{(x-a)(b-x)}dx$ (May-18)

d) $\int_0^1 x\tan^{-1}x dx$ *e) $\int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ *f) $\int_{-\pi}^{\pi} \frac{\sin^2 x}{1+a^x}dx$

2.8 PROPERTIES OF DEFINITE INTEGRALS

Property 1: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: consider $\int_0^a f(a-x)dx$

Put $a-x=t$ then $dx=-dt$

$x=0 \Rightarrow t=a,\ x=a \Rightarrow t=0$

$$\therefore \int_0^a f(a-x)dx = \int_a^0 f(t)(-dt) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

As a consequence of this property we have

i) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

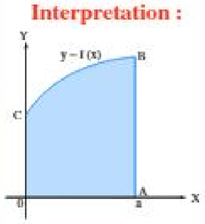
ii) $\int_0^{\frac{\pi}{2}} f(\tan x)dx = \int_0^{\frac{\pi}{2}} f(\cot x)dx$

iii) $\int_0^{\frac{\pi}{2}} f(\sec x)dx = \int_0^{\frac{\pi}{2}} f(\text{cosec } x)dx$

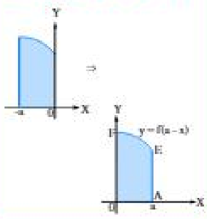
iv) $\int_0^{\frac{\pi}{2}} f(\sin 2x)\cos x dx = \int_0^{\frac{\pi}{2}} f(\sin 2x)\sin x dx$

v) $\int_0^a x^m(a-x)^m dx = \int_0^a x^n(a-x)^m dx$

Geometrical Interpretation :



Graph of $f(a-x)$:



Property 2: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof: consider $\int_a^b f(a+b-x)dx$

Put $a + b - x = t$ then $dx = -dt$

$x = a \Rightarrow t = b,\ x = b \Rightarrow t = a$

$$\therefore \int_a^b f(a+b-x)dx = \int_b^a f(t)(-dt) = -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx,$$

Deduction :

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}dx = \frac{1}{2}(b-a)$$

Property 3: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even $= 0$ if $f(x)$ is odd

Proof : $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$ --(1)

In the first integral on the RHS of (1)

Put $x = -t$, then $dx = -dt$

$x = -a \Rightarrow t = a, \quad x = 0 \Rightarrow t = 0$

$$\therefore \int_{-a}^{0} f(x)dx = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)dx = \int_0^a f(-x)dx$$

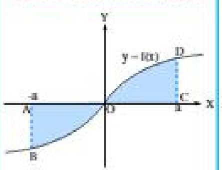
$$\therefore \text{ From (1), } \int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx = \int_0^a [f(x)+f(-x)]dx$$

$$= \begin{cases} \int_0^a [f(x)+f(x)]dx \text{ if } f(x) \text{ is even} \\ \int_0^a [f(x)-f(x)]dx \text{ if } f(x) \text{ is odd} \end{cases}$$

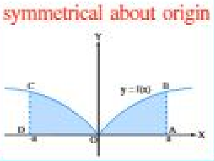
$$\Rightarrow \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(x) \text{ is even} \\ 0 \quad \text{ if } f(x) \text{ is odd} \end{cases}$$

Geometrical Interpretation :

Since odd function is



symmetrical about origin



Property 4 : $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x) = f(x) = 0$ if $f(2a-x) = -f(x)$

Proof : $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ -- (1)

In the 2nd integral on the RHS of (1)

Put $2a - x = t$, then $dx = -dt$,

$x = a \Rightarrow t = a;\ x = 2a \Rightarrow t = 0$

$$\therefore \int_a^{2a} f(x)dx = \int_a^0 f(2a-t)(-dt) = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

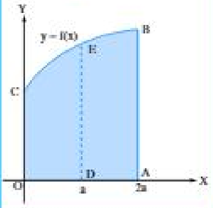
Now, from (1)

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx = \int_0^{a} [f(x)+f(2a-x)]dx$$

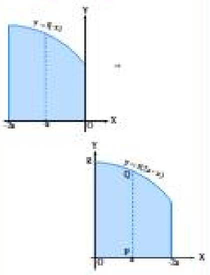
$$= \begin{cases} \int_0^{a} [f(x)+f(x)]dx \text{ if } f(2a-x)=f(x) \\ \int_0^{a} [f(x)-f(x)]dx \text{ if } f(2a-x)=-f(x) \end{cases}$$

$$\Rightarrow \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^{a} f(x)dx \text{ if } f(2a-x)=f(x) \\ 0 \quad\quad \text{if } f(2a-x)=-f(x) \end{cases}$$

Geometrical Interpretation :



Graph of $f(2a-x)$:



Deductions

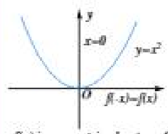
1) If $f(2a - x) = f(x)$ then $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}dx = a$

2) $\int_0^{a} f(x)dx = \int_0^{\frac{a}{2}} f(x)dx + \int_0^{\frac{a}{2}} f(a-x)dx$

Note

i) *If $f(x)$ is even then $f(-x)=f(x)$.*

In this case the graph of $y = f(x)$ is symmetric about the line $x = 0$ (y-axis)

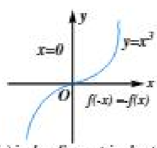


f(x) is symmetric about x = 0

Fig 2.2

ii) *If $f(x)$ is odd then $f(-x)=-f(x)$.*

In this case the graph of $y = f(x)$ is skew symmetric or anti symmetric about the line $x = 0$

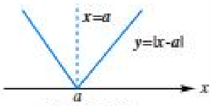


f(x) is skew Symmetric about x = 0

Fig 2.3

iii) *If $f(2a - x) = f(x)$ then $f(a-x) = f(a+x)$.*

In this case the graph of $y=f(x)$ is symmetric about the line $x = a$



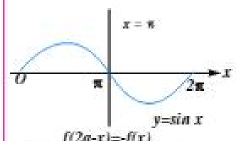
f(2a-x)=f(x)

f(x) is symmetric about x = a

Fig 2.4

iv) *If $f(2a - x) = -f(x)$ then $f(a-x)=-f(a+x)$.*

In this case the graph of $y = f(x)$ is skew symmetric about the line $x = a$.



f(2a-x)=-f(x)

f(x) is skew symmetric about x=π

Fig 2.5

Some Useful Results

A) i) $$\int_0^{\frac{\pi}{2}} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx = \int_0^{\frac{\pi}{2}} \frac{f(\cos x)}{f(\cos x)+f(\sin x)} = \frac{\pi}{4}$$

ii) $$\int_0^{\frac{\pi}{2}} \frac{f(\tan x)}{f(\tan x)+f(\cot x)}dx = \int_0^{\frac{\pi}{2}} \frac{f(\cot x)}{f(\cot x)+f(\tan x)}dx = \frac{\pi}{4}$$

iii) $$\int_0^{\frac{\pi}{2}} \frac{f(\sec x)}{f(\sec x)+f(\mathrm{cosec} x)}dx = \int_0^{\frac{\pi}{2}} \frac{f(\mathrm{cosec} x)}{f(\mathrm{cosec} x)+f(\sec x)}dx = \frac{\pi}{4}$$

Proof: (A) i) Let $I = \int_0^{\frac{\pi}{2}} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx$ -- (1)

$$= \int_0^{\frac{\pi}{2}} \frac{f\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{f\left(\sin\left(\frac{\pi}{2}-x\right)\right)+f\left(\cos\left(\frac{\pi}{2}-x\right)\right)}dx \Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{f(\cos x)}{f(\cos x)+f(\sin x)}dx \quad \text{-- (2)}$$

Adding (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{f(\sin x)+f(\cos x)}{f(\sin x)+f(\cos x)}dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2} \qquad \therefore I = \frac{\pi}{4}$$

The other two results can be proved in a similar way.

(B) i) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\cos x)}{f(\cos x)+f(\sin x)}dx = \frac{\pi}{12}$$

ii) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\tan x)}{f(\tan x)+f(\cot x)}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\cot x)}{f(\cot x)+f(\tan x)}dx = \frac{\pi}{12}$$

iii) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\sec x)}{f(\mathrm{cosec}\ x)+f(\mathrm{cosec}\ x)}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\mathrm{cosec}\ x)}{f(\mathrm{cosec}\ x)+f(\cot x)}dx = \frac{\pi}{12}$$

Proof: (B) ii): Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(\tan x)}{f(\tan x)+f(\cot x)}dx$ -- (1)

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f\left(\tan\left(\frac{\pi}{2}-x\right)\right)}{f\left(\tan\left(\frac{\pi}{2}-x\right)\right)+f\left(\cot\left(\frac{\pi}{2}-x\right)\right)}dx \quad \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{f(\cot x)}{f(\cot x) + f(\tan x)} dx \qquad \text{-- (2)}$$

Adding (1) & (2) we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{f(\tan x) + f(\cot x)}{f(\cot x) + f(\tan x)} dx = \int_{\pi/6}^{\pi/3} 1 dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \quad \therefore I = \frac{\pi}{12}$$

The other two results can be proved in a similar way.

Example : i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

ii) $\int_0^{\pi/2} \frac{1}{1 + \tan x} dx = \int_0^{\pi/2} \frac{1}{1 + \cot x} dx = \frac{\pi}{4}$

iii) $\int_0^{\pi/2} \frac{\sqrt{\sec x}}{\sqrt{\sec x} + \sqrt{\operatorname{cosec} x}} dx = \int_0^{\pi/2} \frac{\sqrt{\operatorname{cosec} x}}{\sqrt{\operatorname{cosec} x} + \sqrt{\sec x}} dx = \frac{\pi}{4}$

Remember :

i) $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx = (a+b)\frac{\pi}{4}$

ii) $\int_0^{\pi/2} \frac{a \tan x + b \cot x}{\tan x + \cot x} dx = (a+b)\frac{\pi}{4}$

iii) $\int_0^{\pi/2} \frac{a \sec x + b \operatorname{cosec} x}{\sec x + \operatorname{cosec} x} dx = (a+b)\frac{\pi}{4}$ *(proof is left as an exercise)*

Property 5: Removal of x property

i) If $f(a-x) = f(x)$ $\forall x \in [0, a]$ then $\int_0^a x f(x) dx = \frac{a}{2}\int_0^a f(x) dx$

Proof: Let $I = \int_0^a x f(x) dx = \int_0^a (a-x) f(a-x) dx$

$$= \int_0^a (a-x) f(x) dx \, (\because f(a-x) = f(x))$$

$$= a\int_0^a f(x) dx - \int_0^a x f(x) dx$$

$$= a\int_0^a f(x) dx - I \Rightarrow I = \frac{a}{2}\int_0^a f(x) dx$$

Deduction

$$\int_a^b x\phi(x)dx = \left(\frac{a+b}{2}\right)\int_a^b \phi(x)dx \quad \textit{if} \quad \phi(a+b-x)=\phi(x)$$

ii) $\displaystyle\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

Let $\displaystyle I=\int_0^{\pi} xf(\sin x)dx = \int_0^{\pi}(\pi-x)f(\sin(\pi-x))dx$

$$=\int_0^{\pi}(\pi-x)f(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx - I$$

$$\therefore 2I = \pi\int_0^{\pi} f(\sin x)dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

Example : $\displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

$$=\frac{\pi}{2}2\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx \quad (\because f(2a-x)=f(x))$$

$$=-\pi\left[\tan^{-1}(\cos x)\right]_0^{\frac{\pi}{2}} \quad =-\pi\left[0-\frac{\pi}{4}\right]=\frac{\pi^2}{4}$$

Other Useful Properties (Practice and Remember)

i) $\displaystyle\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx$

ii) $\displaystyle\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$

iii) $\displaystyle\int_a^b f(x)dx = k\int_{\frac{a}{k}}^{\frac{b}{k}} f(kx)dx$

iv) $\displaystyle\int_a^b f(x)dx = \frac{1}{k}\int_{ak}^{bk} f\left(\frac{x}{k}\right)dx$

v) $\displaystyle\int_a^b f(x)dx = \int_{-a}^{-b} f(-x)dx$

vi) $\displaystyle\int_a^b f(x)dx = (b-a)\int_0^1 f(a+(b-a)x)dx$

The above properties can be easily proved by proceeding from RHS substituting t for the argument of each integrand.

SOLVED EXAMPLES

**1. Find $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx$

Sol. Let $I=\int_0^{\pi/2}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx=\int_0^{\pi/2}\frac{\sin^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx$

$$=\int_0^{\pi/2}\frac{\cos^5 x}{\cos^5 x+\cos^5 x}dx$$

$$\therefore 2I=\int_0^{\pi/2}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx=\int_0^{\pi/2}1\,dx=\frac{\pi}{2}\qquad \therefore I=\frac{\pi}{4}$$

*2. Find the value of $\int_0^{\pi}\frac{x}{1+\sin x}dx$

Sol. Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{\pi}{1+\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin x}dx$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{1}{1+\sin x}dx\quad\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$=\frac{\pi}{2}2\int_0^{\pi/2}\frac{1}{1+\sin x}dx\left[\because f\left(2.\frac{\pi}{2}-x\right)=f(x)\right]$$

$$=\pi\int_0^{\pi/2}\frac{1-\sin x}{\cos^2 x}dx\left[\because 1-\sin x\neq 0\, in\left(0,\frac{\pi}{2}\right)\right]$$

$$=\pi[\tan x-\sec x]_0^{\frac{\pi}{2}}=\pi\left[\frac{\sin x-1}{\cos x}\right]_0^{\pi/2}$$

$$\left[\because \underset{x\to\frac{\pi}{2}}{Lt}\frac{(\sin x-1)}{\cos x}=\underset{x\to\frac{\pi}{2}}{Lt}\left(\frac{\cos x}{-\sin x}\right)=0\right]=\pi(0-(-1))=\pi\quad\therefore\int_0^{\pi}\frac{x}{1+\sin x}dx=\pi$$

3. Find $\int_0^{\pi}\frac{x}{a^2-\cos^2 x}dx$

Sol. Let $I=\int_0^{\pi}\frac{x}{a^2-\cos^2 x}dx=\frac{\pi}{2}\int_0^{\pi}\frac{1}{a^2-\cos^2 x}$ (Removal of x property)

$$=\frac{\pi}{2}2\int_0^{\pi/2}\frac{1}{a^2-\cos^2 x}dx\left(\because f\left(2.\frac{\pi}{2}-x\right)=f(x)\right)=\pi\int_0^{\pi/2}\frac{\sec^2 x}{a^2\tan^2 x+(a^2-1)}dx$$

$$=\pi\frac{1}{a\sqrt{a^2-1}}\left[\tan^{-1}\left(\frac{a\tan x}{\sqrt{a^2-1}}\right)\right]_0^{\pi/2}(\therefore a>1)=\frac{\pi}{a\sqrt{a^2-1}}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{2a\sqrt{a^2-1}}$$

$$\therefore\int_0^{\pi}\frac{x}{a^2-\cos^2 x}dx=\frac{\pi^2}{2a\sqrt{a^2-1}}$$

Remember :

$$\int_0^{\pi}\frac{x}{a^2-\cos^2 x}dx=\frac{\pi^2}{2a\sqrt{a^2-1}}\ (\therefore a>1)$$

Remember :

$$\int_0^{\pi/2} \log(\sin x)dx = -\frac{\pi}{2}\log 2$$

4. **Evaluate $\int_0^{\pi/2} \log(\sin x)dx$**

Sol. $I = \int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx = \int_0^{\pi/2} \log\cos x\, dx$

$\therefore 2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$

$= \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} (\log 2)dx = \int_0^{\pi/2} \log(\sin 2x)dx - \frac{\pi}{2}\log 2$ -- (1)

Now, $\int_0^{\pi/2} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt$ where $t = 2x$

$= \frac{1}{2}2\int_0^{\pi/2} \log(\sin t)\left(\because f\left(2\frac{\pi}{2}-t\right) = f(t)\right)dt = \int_0^{\pi/2} \log(\sin t)dt = I$ -- (2)

From (1) and (2) we have

$2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2$

Note

i) $\int_0^{\pi/2} \log(\cos x)dx = \frac{-\pi}{2}\log 2$

ii) $\int_0^{\pi/2} \log(\tan x)dx = 0$

iii) $\int_0^{\pi/2} \log(\cot x)dx = 0$

iv) $\int_0^{\pi/2} \log(\sec x)dx = \frac{\pi}{2}\log 2$

v) $\int_0^{\pi/2} \log(\operatorname{cosec} x)dx = \frac{\pi}{2}\log 2$

vi) $\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\sin 2x)dx$

5. **Evaluate $\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}dx$**

Sol. Let $I = \int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}dx$

Put $x = \tan\theta$ then $dx = \sec^2\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0,\ x \to \infty \Rightarrow \theta \to \frac{\pi}{2}$

$\therefore I = \int_0^{\pi/2} \frac{\log(\tan\theta + \cot\theta)}{\sec^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta$

$= -\left[\int_0^{\pi/2} \log\sin\theta\, d\theta + \int_0^{\pi/2} \log(\cos\theta)d\theta\right] = \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2 = \pi\log 2$

EXERCISE - 2.3

Evaluate the following integrals

1. *a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x}\,dx$ *b) $\displaystyle\int_0^{\pi} \sin^3 x \cos^3 x\,dx$ c) $\displaystyle\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx$

   d) $\displaystyle\int_0^{2\pi} \sin^{11} x\,dx$ e) $\displaystyle\int_0^{2\pi} \cos x\,dx$

2. *a) $\displaystyle\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$ b) $\displaystyle\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{1-x^2}}\,dx$

   c) $\displaystyle\int_{-a}^{a} \left(\sqrt{k+x+x^2} - \sqrt{k-x+x^2}\right)dx$ d) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^{2009} x + x^{1009})\,dx$

   e) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f(x)+f(-x))(g(x)-g(-x))\,dx$ where $f(x)$ and $g(x)$ are continuous functions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

3. *a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^{5/2} x}{\sin^{5/2} x + \cos^{5/2} x}\,dx$ b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx$

   c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sec x}}{\sqrt{\sec x} + \sqrt{\operatorname{cosec} x}}\,dx$ d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt[4]{\cot x}}{\sqrt[4]{\tan x} + \sqrt[4]{\cot x}}\,dx$

4. a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2\sin x + 3\cos x}{\sin x + \cos x}\,dx$ b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3\tan x + 4\cot x}{\tan x + \cot x}\,dx$ c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4\sec x + 5\operatorname{cosec} x}{\sec x + \operatorname{cosec} x}\,dx$

   d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2\cos x - \sin x}{\cos x + \sin x}\,dx$ *e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$ **(May-17)**

5. **a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt[3]{\tan x}}\,dx$

   c) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^4 x}{\operatorname{cosec}^4 x + \sec^4 x}\,dx$ d) $\displaystyle\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\,dx$

6\. *a) $\int_{-a}^{a} \left(x^2 + \sqrt{a^2 - x^2}\right) dx$

**b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin|x|\, dx$ (March-17)

*c) $\int_{-1}^{1} |x|\, dx$ d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin|2x|\, dx$ e) $\int_{-1}^{1} (x^4 + \sin^3 x)\, dx$

7\. *a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx$ b) $\int_{0}^{2\pi} \frac{dx}{1+e^{\sin x}}$ c) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+e^{\tan x}}$

d) $\int_{-1}^{1} \frac{\cosh x}{1+e^{2x}} dx$

*8. Show that $\int_0^a x(a-x)^n\, dx = \frac{a^{n+2}}{(n+1)(n+2)}$. Hence or other wise find $\int_0^2 x\sqrt{2-x}\, dx$

9\. Evaluate

*a) $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$ (March-19) *b) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$ (May-19)

10\. Evaluate

*a) $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$ **b) $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x + \cos x}$ (March-17, 18)

11 Evaluate

**a) $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$ b) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

**c) $\int_0^{\pi} \frac{x \sin^3 x}{1+\cos^2 x} dx$ d) $\int_0^{\pi} x \sin^3 x\, dx$

*12. Evaluate $\int_0^{\pi} \sin^3\theta (1+2\cos\theta)(1+\cos\theta)^2\, d\theta$

2.9 REDUCTION FORMULAE

THEOREM-2.6 **If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ then $I_n = \frac{n-1}{n} I_{n-2} (n \in N, n \geq 2)$ and $\int_0^{\frac{\pi}{2}} \sin^n x dx$

$$= \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}\frac{\pi}{2} \text{ if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3}.1 \text{ if } n \text{ is odd} \end{cases}$$

Proof : $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x dx$

$$= [\sin^{n-1} x(-\cos x)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos^2 x dx = 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx = (n-1)I_{n-2} - (n-1)I_n \quad \Rightarrow nI_n = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} \qquad \text{-- (1)}$$

Applying the reduction formula (1) successively on I_n, we get

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}.\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} I_{n-6} = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....I_0 \text{ or } I_1$$

according as n is even or odd

But $I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$; $I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = 1 \quad \therefore I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$

$$= \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{1}{2}.\frac{\pi}{2} \text{ if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3}.1 \text{ if } n \text{ is odd} \end{cases}$$

Note

The above formulae can be directly used to evaluate $\int_0^{\frac{\pi}{2}} \sin^n x dx, n \in N$

Example : i) $\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ ($\because n = 6$ is even) ii) $\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1 = \frac{16}{35}$

THEOREM-2.7 If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ then $I_n = \frac{n-1}{n} I_{n-2} (n \in N, n \geq 2)$ and $\int_0^{\frac{\pi}{2}} \cos^n x dx$

$$\begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}\frac{\pi}{2} \text{ if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{2}{3}.1 \text{ if } n \text{ is odd} \end{cases}$$

Proof: $I_n=\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^{n-1}x\cos x\,dx$

$$=\cos^{n-1}x\sin x\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\cos^{n-2}x\sin^2 x\,dx$$

$$=0+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

$$=(n-1)I_{n-2}-(n-1)I_n \quad \Rightarrow nI_n=(n-1)I_{n-2} \;\therefore I_n=\frac{n-1}{n}I_{n-2}(n\geq 2)$$

The remaining part can be proved as in Theorem (2.6)

Example : i) $\int_0^{\frac{\pi}{2}}\cos^8 dx=\frac{7.5.3.1}{8.6.4.2}.\frac{\pi}{2}=\frac{35\pi}{256}$ ii) $\int_0^{\frac{\pi}{2}}\cos^9 dx=\frac{8.6.4.2}{9.7.5.3}.1=\frac{128}{315}$

Observation

$$\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\sin^n x\,dx \text{ For, } \int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x dx=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

Example : $\int_0^{\frac{\pi}{2}}\cos^5 x\,dx=\int_0^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^5 dx=\int_0^{\frac{\pi}{2}}\sin^5 x\,dx=\frac{4}{5}.\frac{2}{3}.1=\frac{8}{15}$

THEOREM-2.8 If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ then $I_n+I_{n-2}=\frac{1}{n-1}(n\in N, n\geq 2)$

Proof: $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx=\int_0^{\frac{\pi}{4}}\tan^{n-2}x\tan^2 x\,dx$

$$=\int_0^{\frac{\pi}{4}}\tan^{n-2}x\sec^2 x\,dx-\int_0^{\frac{\pi}{4}}\tan^{n-2}x\,dx=\frac{\tan^{n-1}x}{n-1}\Bigg]_0^{\frac{\pi}{4}}-I_{n-2}=\frac{1}{n-1}-I_{n-2}$$

$$\therefore I_n+I_{n-2}=\frac{1}{n-1}(n\geq 2)$$

Deduction: $I_n=\frac{1}{n-1}-\frac{1}{n-3}+\frac{1}{n-5}-\frac{1}{n-7}+....+I_0$ or I_1

according as n is even or odd, where $I_0=\int_0^{\frac{\pi}{4}}\tan^0 x\,dx=\frac{\pi}{4}$, $I_1=\int_0^{\frac{\pi}{4}}\tan x\,dx=\frac{1}{2}\log 2$

Example : i) $\int_0^{\frac{\pi}{4}} \tan^4 x dx = \frac{1}{3} - \frac{1}{1} + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}$

ii) $\int_0^{\frac{\pi}{4}} \tan^5 x dx = \frac{1}{4} - \frac{1}{2} + I_1 = -\frac{1}{4} + \frac{1}{2}\log 2$

THEOREM-2.9 If $I_n = \int_0^{\pi/4} \sec^n x dx$ then $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2} (n \in N, n \geq 2)$

Proof : $I_n = \int_0^{\frac{\pi}{4}} \sec^n x dx = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x dx$

$$= [\sec^{n-2} x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)\sec^{n-3} x \sec x . \tan^2 x dx$$

$$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x(\sec^2 x - 1)dx = (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2})$$

$$\Rightarrow (n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2} \Rightarrow I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Example : i) $\int_0^{\pi/4} \sec^4 x dx = \frac{(\sqrt{2})^2}{3} + \frac{2}{3}.\frac{7}{2} + \frac{2}{3}.\frac{7}{2} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

ii) $\int_0^{\pi/4} \sec^5 x dx = \frac{(\sqrt{2})^3}{4} + \frac{3}{9}.\frac{7}{3} = \frac{2\sqrt{2}}{4} + \frac{3}{4}\left[\frac{(\sqrt{2})'}{2} + \frac{1}{2} I_1\right]$

$$= \frac{2\sqrt{2}}{4} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) = \frac{1}{8}[7\sqrt{2} + 3\log(\sqrt{2}+1)]$$

THEOREM-2.10 If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ then $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

(or) $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$ $(m, n \in N, m \geq 2, n \geq 2)$

Proof: $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \int_0^{\frac{\pi}{2}} (\sin^{m-1} x)(\cos^n x \sin x) dx$

$$= -\sin^{m-1} x \frac{\cos^{n+1} x}{n+1}\Bigg]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (m-1)\sin^{m-2} x \frac{\cos^{n+2} x}{n+1} dx$$

$$= 0 + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x(1-\sin^2 x)dx$$

$$= \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} \cos^n xdx - \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^m x\cos^n xdx$$

$$\Rightarrow I_{m,n} = \frac{m-1}{m+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n} \Rightarrow \left(\frac{m+n}{n+1}\right) I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

Remarks :

1) The other result can be proved in a similar way by writing

$I_{m,n} = \int_0^{\pi/2} \sin^m x\cos^n xdx$ *as* $\int_0^{\pi/2} (\sin^m x.\cos x)\cos^{n-1} xdx$ *and appling integration by parts.*

2) If m = 1 then $\int_0^{\frac{\pi}{2}} \sin x\cos^n xdx = \frac{1}{n+1}$

3) If n = 1 then $\int_0^{\frac{\pi}{2}} \sin^m x\cos xdx = \frac{1}{m+1}$

Deduction : $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n},\ (m \geq 2, n \geq 2,\ m,n \in N)$

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}.I_{m-4,n} = \frac{m-1}{m+n}\frac{m-3}{m+n-2}\frac{m-5}{m+n-4}.I_{m-6,n}$$

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}\frac{m-5}{m+n-4}...I_{0,n} \text{ or } I_{1,n}$$

according as m is even or odd. But $I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n xdx$

$$I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x\cos^n xdx = -\frac{\cos^{n+1} x}{n+1}\Bigg]_0^{\frac{\pi}{2}} = \frac{1}{n+1}$$

$$\therefore I_{m,n} = \begin{cases} \dfrac{m-1}{m+n}.\dfrac{m-3}{m+n-2}.\dfrac{m-5}{m+n-4}....\displaystyle\int_0^{\frac{\pi}{2}} \cos^m xdx \\ \qquad\qquad\qquad \textit{if } m \textit{ is even} \\ \dfrac{m-1}{m+n}.\dfrac{m-3}{m+n-2}.\dfrac{m-5}{m+n-4}...\dfrac{1}{n+1} \\ \qquad\qquad\qquad \textit{if } m \textit{ is odd} \end{cases}$$

$$\Rightarrow I_{m,n} = \begin{cases} \frac{m-1}{m+n}.\frac{m-3}{m+n-2}.\frac{m-5}{m+n-4}....\frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{1}{2}.\frac{\pi}{2} \\ \qquad if\ m\,\&\,n\,are\ even \\ \frac{m-1}{m+n}.\frac{m-3}{m+n-2}.\frac{m-5}{m+n-4}...\frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{2}{3}.1 \\ \qquad if\ m\,is\ even\,\&\,n\,is\,odd \\ \frac{m-1}{m+n}.\frac{m-3}{m+n-2}.\frac{m-5}{m+n-4}....\frac{1}{n+1} \\ \qquad if\ m\,is\,odd\,(n\,is\,even\,or\,odd) \end{cases}$$

The above formula can be remembered as

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x dx \quad (m-1)(m-3)(m-5)....(2\text{ or }1)$$

$$= \frac{(n-1)(n-3)(n-5)......(2\text{ or }1)}{(m+n)(m+n-2)(m+n-4)(m+n-6).....(2\text{ or }1)(m,n \in N, m \le 2, n \ge 2)} k$$

where $k = \frac{\pi}{2}$ if both m and n are even; =1

otherwise (i.e., if atleast one of m,n is odd)

This formula is often referred to as *Wallis formula.*

Example : i) $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^4 xdx = \frac{(5.3.1)(3.1)}{10.8.6.4.2}.\frac{\pi}{2} = \frac{3\pi}{512}$

ii) $\int_0^{\frac{\pi}{2}} \sin^5 x\cos^4 xdx = \frac{(4.2)(3.1)}{9.7.5.3.1}.1 = \frac{8}{315}$

iii) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^5 xdx = \frac{(3.1)(4.2)}{9.7.5.3.1} = \frac{8}{3.15}$

iv) $\int_0^{\frac{\pi}{2}} \sin^5 x\cos^5 xdx = \frac{(4.2)(4.2)}{10.8.6.4.2} = \frac{1}{60}$

Working Rule : To find the value of $\int_0^{\frac{\pi}{2}} \sin^m x\cos^n xdx (m,n \in N), m \ge 2, n \ge 2$ using the above formula we proceed in the following way.

In the Numerator, first we begin with $(m-1)$ and go on writing successive factors each reduced by 2 until we get 2 or 1, next we begin with $(n-1)$ and go on writing successive factors each reduced by 2 until we get 2 or 1; and in the Denominator we begin with $(m + n)$ and go on writing successive factors each reduced by 2 until we get 2 or 1 and finally multiply the resulting fraction with $\frac{\pi}{2}$ only if both m and n are even (otherwise multiply with 1) (see the above examples (i) to (iv))

Observations: $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^m x \sin^m x\,dx$

For, $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \left[\sin\left(\frac{\pi}{2}-x\right)\right]^m \left[\cos\left(\frac{\pi}{2}-x\right)\right]^m dx = \int_0^{\frac{\pi}{2}} \cos^m x \sin^n x\,dx$

Properties : For $m,n \in N$

1) $$\int_0^{\pi} \sin^m x \cos^n x\,dx = \begin{cases} 2\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

For, $f(2a-x) = f(\pi - x) = [\sin(\pi - x)]^m \cos(\pi - x)^n$

$$= \sin^m x(-\cos x) = \begin{cases} \sin^m x \cos^n x \text{ if } n \text{ is even} \\ -\sin^m x \cos^n x \text{ if } n \text{ is odd} \end{cases}^n = \begin{cases} f(x) \text{ if } n \text{ is even} \\ -f(x) \text{ if } n \text{ is odd} \end{cases}$$

2) $$\int_0^{2\pi} \sin^m x \cos^n x\,dx = \begin{cases} 4\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx \text{ if } m \text{ and } n \text{ are even} \\ 0 \text{ otherwise} \end{cases}$$

For, $f(2\pi - x) = [\sin(2\pi - x)]^m [\cos(2\pi - x)]^n = (-\sin x)^m (\cos x)^n$

$$= \begin{cases} \sin^m x \cos^n x \text{ if } m \text{ is even} \\ -\sin^m x \cos^n x \text{ if } m \text{ is odd} \end{cases} = \begin{cases} f(x) \text{ if } m \text{ is even} \\ -f(x) \text{ if } m \text{ is odd} \end{cases}$$

$$\therefore \int_0^{2\pi} \sin^m x \cos^n x\,dx = \begin{cases} 2\int_0^{\pi} \sin^m x \cos^n x\,dx \text{ if } m \text{ is even} \\ 0 \text{ if } m \text{ is odd} \end{cases}$$

Now, applying property (1), we get the property (2)

For example,

1) $\int_0^{\pi} \sin^3 x \cos^6 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^3 x \cos^6 x\,dx$ $(\because n = 6 \text{ is even})$ $= 2.\frac{(2).(5.3.1)}{9.7.5.3.1}.1 = \frac{4}{63}$

2) $\int_0^{2\pi} \sin^2 x \cos^4 x\,dx = 4\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x\,dx = 4.\frac{(1)(3.1)}{6.4.2}.\frac{\pi}{2} = \frac{\pi}{8}$ $(\because m = 2, n = 4 \text{ are even})$

From the above properties it follows that for $m,n \in N$

3) $\int_0^{\pi} \cos^{2n-1} x\,dx = 0,\ \int_0^{\pi} \sin^m \cos^{2n-1} x\,dx = 0$

4) $\int_0^{2\pi} \sin^{2n-1} x \cos^n x \, dx = 0$

5) $\int_0^{2\pi} \sin^m x \cos^{2n-1} x \, dx = 0$

6) $\int_0^{2\pi} \sin^{2m-1} x \cos^{2n-1} x \, dx = 0$

SOLVED EXAMPLES

1. Find $\int_0^1 x^{\frac{3}{2}}\sqrt{1-x}\,dx$

Sol. Put $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta d\theta$

When $\theta = 0, x = 0$; when $\theta = \frac{\pi}{2}, x = 1$. When $\theta \in \left[0, \frac{\pi}{2}\right], x \in [0,1]$.

$$\int_0^1 x^{\frac{3}{2}}\sqrt{1-x}\,dx = \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta.2\sin\theta\cos\theta d\theta = 2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta = 2.\frac{\pi}{32} = \frac{\pi}{16}$$

2. Find $\int_{-3}^{3} (9-x^2)^{3/2} x\,dx$

Sol. Let $f(x) = (9-x^2)^{3/2}_x$

$f(x) = (9-(-x^2))^{3/2}(-x) = (9-x^2)^{3/2}x = -f(x)$

$\therefore f$ is odd function $\quad \therefore \int_{-3}^{3} (9-x^2)^{3/2}.x\,dx = 0$

3. Find $\int_0^1 x^{7/2}(1-x)^{5/2}dx$

Sol. Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta d\theta$

U.L. $x = \sin^2\theta \quad \Rightarrow 1 = \sin^2\theta \quad \Rightarrow \theta = \frac{\pi}{2}$

L.L. $x = \sin^2\theta \quad \Rightarrow 0 = \sin^2\theta \quad \Rightarrow \theta = 0$

Let $I = \int_0^1 x^{7/2}(1-x)^{5/2}dx = \int_0^{\pi/2} (\sin^2\theta)^{7/2}(1-\sin^2\theta)^{5/2} 2\sin\theta\cos\theta d\theta$

$$= 2\int_0^{\pi/2} \sin^7\theta\cos^5\theta.\sin\theta\cos\theta.d\theta = 2\int_0^{\pi/2} \sin^8\theta\cos^6\theta d\theta = 2I_{8,6}$$

$$= 2\left(\frac{5}{14}\right)I_{8,4} = 2\left(\frac{5}{14}\right)\left(\frac{3}{12}\right)\left(\frac{1}{10}\right)I_{8,0}$$

$$= 2\left(\frac{5}{14}\right)\left(\frac{3}{12}\right)\left(\frac{1}{10}\right)\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2} = \frac{5\pi}{2048}$$

4. Find $\int_0^5 x^2(\sqrt{5-x})^7 dx$

Sol. Put $x = 5\sin^2\theta$

$dx = 10\sin\theta\cos\theta\, d\theta$

U.L. $x = 5\sin^2\theta \Rightarrow 5 = 5\sin^2\theta \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

L.L. $x = 5\sin^2\theta \Rightarrow 0 = 5\sin^2\theta \Rightarrow \sin^2\theta = 0 \Rightarrow \theta = 0$

Let $I = \int_0^5 x^2(\sqrt{5-x})^7 dx$; $I = \int_0^{\pi/2} 25\sin^4\theta(\sqrt{5-5\sin^2\theta})^7$

$$10\sin\theta\cos\theta d\theta = 5^2.5^{7/2}.5.2\int_0^{\pi/2}\sin^4\theta\cos^7\theta.\sin\theta\cos\theta d\theta$$

$$= 5^{13/2}.2I_{5,8} = 2.5^{13/2}\left(\frac{7}{13}\right)I_{5,6}$$

$$= 2.5^{13/2}\left(\frac{7}{13}\right)\left(\frac{5}{11}\right)\left(\frac{3}{9}\right)\left(\frac{1}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right) = \frac{16.5^{13/2}}{1287}$$

5. Evaluate $\int_{-\pi}^{\pi} \sin^8 x\cos^7 x\, dx$

Sol. Let $f(x) = \sin^8 x.\cos^7$

$f(-x) = \sin^8(-x).\cos^7(-x) = \sin^8 x.\cos^7 x$

$\therefore$ f is even function

$$\therefore \int_{-\pi}^{\pi}\sin^8 x\cos^7 x\, dx = 2\int_0^{\pi}\sin^8 x\cos^7 x\, dx$$

Let $g(x) = \sin^8 x.\cos^7 x$

$g(\pi - x) = \sin^8(\pi - x).\cos^7(\pi - x) = -\sin^8 x.\cos^7 x = -g(x)$

$$\therefore \int_{-\pi}^{\pi}\sin^8 x\cos^7 x\, dx = 2\int_0^{\pi}\sin^8 x\cos^7 x = 0$$

6. Evaluate $\int_0^{\frac{\pi}{2}} \tan^5 x\cos^8 x\, dx$

Sol. Let $I = \int_0^{\pi/2}\tan^5 x\cos^8 x\, dx = \int_0^{\pi/2}\frac{\sin^5 x}{\cos^5 x}\cos^8 x\, dx$

$$= \int_0^{\pi/2}\sin^5 x\cos^3 x\, dx = I_{5,3} = \left(\frac{3-1}{5+3}\right)I_{5,1} = \frac{2}{8}.\frac{1}{6} = \frac{1}{24}$$

7. Evaluate $\int_3^7 \sqrt{\frac{7-x}{x-3}}dx$

Sol. Put $x = 3\cos^2\theta + 7\sin^2\theta$

$dx = (7-3)\sin 2\theta\, d\theta \Rightarrow dx = 4\sin 2\theta\, d\theta$

U.L. $x = 3\cos^2\theta + 7\sin^2\theta \Rightarrow 7 = 3\cos^2\theta + 7\sin^2\theta \Rightarrow 4\cos^2\theta = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

L.L. $x = 3\cos^2\theta + 7\sin^2\theta \Rightarrow 3 = 3\sin^2\theta + 7\sin^2\theta$

$4\sin^2\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$7 - x = 7 - (3\cos^2\theta + 7\sin^2\theta) = (7-3)\cos^2\theta = 4\cos^2\theta$

$x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3 = (7-3)\sin^2\theta = 4\sin^2\theta$

Let $I = \int_3^7 \sqrt{\frac{7-x}{x-3}}dx$; $I = \int_0^{\pi/2} \sqrt{\frac{4\cos^2\theta}{4\sin^2\theta}}4(2)\sin\theta\cos\theta\, d\theta$

$= 8\int_0^{\pi/2} \cos^2\theta d\theta = 8\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}d\theta = \frac{8}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]^{\pi/2}$

$= \frac{8}{2}\left[\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right] - 0 = \frac{8}{2}\left[\frac{\pi}{2}\right] = 2\pi$

8. Evaluate $\int_2^6 \sqrt{(6-x)(x-2)}dx$

Sol. Put $x = 2\cos^2\theta + 6\sin^2\theta$

$dx = (6-2)\sin 2\theta\, d\theta \Rightarrow dx = 4\sin 2\theta\, d\theta$

U.L. $x = 2\cos^2\theta + 6\sin^2\theta \Rightarrow 6 = 2\cos^2\theta + 6\sin^2\theta$

$4\cos^2\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

L.L. $x = 2\cos^2\theta + 6\sin^2\theta \Rightarrow 2 = 2\cos^2\theta + 6\sin^2\theta$

$4\sin^2\theta = 0 \Rightarrow \theta = 0$

$6 - x = 6 - (2\cos^2\theta + 6\sin^2\theta) = (6-2)\cos^2\theta = 4\cos^2\theta$

$x - 2 = 2\cos^2\theta + 6\sin^2\theta - 2 = (6-2)\sin^2\theta = 4\sin^2\theta$

Let $I = \int_2^6 \sqrt{(6-x)(x-2)}dx = \int_0^{\pi/2} \sqrt{4\cos^2\theta\, 4\sin^2\theta}\, 4.2\sin\theta\cos\theta\, d\theta$

$= 32\int_0^{\pi} \sin^2\theta.\cos^2\theta\, d\theta = 32\, I_{22} = 32\left(\frac{1}{4}\right)I_{20} = 32\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2} = 2\pi$

9. Evaluate $\int_0^{\pi}(1+\cos x)^3 dx$

Sol. $\int_0^{\pi}(1+\cos x)^3 dx = \int_0^{\pi}\left(2\cos^2\frac{x}{2}\right)^3 dx = \int_0^{\pi} 2^3.\cos^6\frac{x}{2}dx$

Put $\frac{x}{2}=t \Rightarrow dx = 2dt$

U.L. $\Rightarrow x=\pi \Rightarrow \frac{\pi}{2}=t$

L.L. $\Rightarrow x=0 \Rightarrow 0=t$ $=8.\int_0^{\pi/2}\cos^6 6(2dt) = 16\int_0^{\pi/2}\cos^6 t.dt$

$=\left[\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{n},\frac{n-3}{n-2}....\frac{1}{2}.\frac{\pi}{2}\text{ if } n \text{ is even}\right] = 16.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{2}$

10. Evaluate $\int_0^{2\pi}(1+\cos x)^5(1-\cos x)^3 dx$

Sol. $\int_0^{2\pi}(1+\cos x)^5(1-\cos x)^3 dx$

$(1+\cos x)^3(1+\cos x)^2(1-\cos x)^3 = \int_0^{2\pi}(1-\cos^2 x)^3(1+\cos x)^2 dx$

$=\int_0^{2\pi}(\sin^2 x)^3(1+\cos x)^2 dx = \int_0^{2\pi}\sin^6 x(1+\cos)^2 dx = \cos x = t$

$-\text{six } x\, dx = dt \Rightarrow \sin x\, dx = -dt$

U.L. $\Rightarrow \cos 2\pi = t \Rightarrow 1 = t$

L.L. $\Rightarrow \cos 0 = t \Rightarrow 1 = t$ $=\int_1^1(1+t)^2.dt = \left(\frac{(1+t)^3}{3}\right)_1^1 = \frac{8}{3}-\frac{8}{3}=0$

EXERCISE - 2.4

Find the values of the following integrals

1\. a) $\int_0^{\frac{\pi}{2}}\sin^8 x\,dx$ b) $\int_0^{\frac{\pi}{2}}\sin^9 x\,dx$ c) $\int_0^{\frac{\pi}{2}}\cos^6 x\,dx$

d) $\int_0^{\frac{\pi}{2}}\cos^7 x\,dx$ *e) $\int_0^{\frac{\pi}{2}}\sin^{10} x\,dx$ *f) $\int_0^{\frac{\pi}{2}}\cos^{11} x\,dx$ (March-19)

g) $\int_0^{\frac{\pi}{4}}\sec^3 x\,dx$ h) $\int_0^{\frac{\pi}{4}}\sec^6 x\,dx$

2. *a) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^4 x\,dx$ *b) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^6 x\,dx$ c) $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^8 x\,dx$

*d) $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^7 x\,dx$ **e) $\int_0^{\frac{\pi}{2}} \sin^5 x\cos^4 x\,dx$

3. a) $\int_0^{\pi} \sin^3 x\cos^3 x\,dx$ *b) $\int_0^{\pi} \cos^3 x\sin^4 x\,dx$ *c) $\int_0^{\pi} \sin^3 x\cos^6 x\,dx$

d) $\int_0^{\pi} \sin^7 x\cos^7 x\,dx$ e) $\int_0^{\pi} \sin^5 x\cos^4 x\,dx$

4. ***a) $\int_0^{2\pi} \sin^2 x\cos^4 x\,dx$ (March-14) *b) $\int_0^{2\pi} \sin^4 x\cos^6 x\,dx$ (May-17, 19)

c) $\int_0^{2\pi} \sin^7 x\cos^5 x\,dx$ d) $\int_0^{2\pi} \sin^{11} x\cos^{10} x\,dx$

5. a) $\int_{2\pi}^{2\pi} \cos^6 x\,dx$ *b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,dx$ (March-18)

*c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\cos^7\theta\,d\theta$ *d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta\cos^3\theta\,d\theta$ *e) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\cos^2 x(\sin x+\cos x)\,dx$

6. Evaluate the following integrals

*a) $\int_0^{2\pi} x\sin^6 x\cos^5 x\,dx$ *b) $\int_0^{\pi} x\sin^5 x\cos^6 x\,dx$ *c) $\int_0^{\pi} x\sin^7 x\cos^6 x\,dx$ (March-19)

7. Evaluate

a) $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\,dx$ b) $\int_0^{\frac{1}{2}} \frac{x^7}{\sqrt{1-x^4}}\,dx$ *c) $\int_{-a}^{a} x^2(a^2-x^2)^{3/2}\,dx$

*d) $\int_0^4 (16-x^2)^{5/2}\,dx$ *e) $\int_0^1 x^5(1-x)^{5/2}\,dx$ *f) $\int_0^a x(a^2-x^2)^{7/2}\,dx$

*g) $\int_0^2 x^{3/2}\sqrt{2-x}\,dx$ *h) $\int_0^1 x^6\sin^{-1}x\,dx$

8. If $I_n=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\,dx$, prove that $I_n+I_{n-2}=\frac{1}{n-1}$

9. If $I_n=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \mathrm{cosec}^n x\,dx$, show that $(n-1)I_n=2^{n-2}\sqrt{3}+(n-2)I_{n-2}$

2.10 DEFINITE INTEGRALS OF PERIODIC FUNCTIONS

Let f be a continuous periodic functions on R having fundamental period T. Then we have the following properties.

1) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in N$

Proof : f is a periodic functions of fundamental period T

$\Rightarrow f(x+T) = f(x)\ \forall x \in R \Rightarrow f(x) = f(x+T) = f(x+2T) = ...$

$= f(x+nT), n \in N$ -- (1)

Now, $\int_0^{nT} f(x)dx = \int_0^T f(x)dx + \int_T^{2T} f(x)dx + ... + \int_{(r-1)T}^{rT} f(x)dx + ... + \int_{(n-1)T}^{nT} f(x)dx$ -- (2)

In the r^{th} integral on RHS

Put $x = t + (r-1)T; r = 1, 2, ..., n$.

Then $dx = dt$; $x = (r-1)T \Rightarrow t = 0$, $x = rT \Rightarrow t = T$

$$\therefore \int_{(r-1)T}^{rT} f(x)dx = \int_0^T f(t+(r-1)T)dt = \int_0^T f(t)dt \text{ by (1)}$$

$$= \int_0^T f(x)dx, \text{ for } r = 1, 2, 3, ..., n$$

$$\therefore (2) \Rightarrow \int_0^{nT} f(x)dx = \int_0^T f(x) + \int_0^T f(x)dx + ... + \int_0^T f(x)dx \ (n \text{ times})$$

$$\Rightarrow \int_0^{nT} f(x)dx = n\int_0^T f(x)dx$$

Note

i) ***The above result is also valid for any integer.***

ii) ***The continuity of f on R can be replaced by integrability of f on [0, T].***

2) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$

Proof: $\int_{mT}^{nT} f(x)dx = \int_{mT-mT}^{nT-mT} f(x+mT)dt = \int_0^{(n-m)T} f(x)dx = (n-m)\int_0^T f(x)dx$

Hence the result

3) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$ for any in teger n

Proof : $\int_{a+nT}^{b+nT} f(x)dx = \int_{(a+nT)-nT}^{(b+nT)-nT} f(x+nT)dx = \int_a^b f(x+nT)dx = \int_a^b f(x)dx$

4) $\int\limits_{a}^{a+T} f(x)dx$ is independent of a and $\int\limits_{a}^{a+T} f(x)dx = \int\limits_{0}^{T} f(x)dx$

Proof : $\int\limits_{a}^{a+T} f(x)dx = \int\limits_{0}^{T} f(x)dx + \int\limits_{T}^{T+a} f(x)dx$

$$= \int\limits_{a}^{T} f(x)dx + \int\limits_{T-T}^{(T+a)-T} f(x+T)dx = \int\limits_{a}^{T} f(x)dx + \int\limits_{0}^{a} f(x)dx$$

$$= \int\limits_{a}^{T} f(x)dx \text{ which independent of } a$$

5) $\int\limits_{a}^{a+nT} f(x)dx = n.\int\limits_{a}^{a+T} f(x)dx = n.\int\limits_{0}^{T} f(x)dx,\ n \in N$

Proof : $\int\limits_{a}^{a+nT} f(x)dx = \int\limits_{a}^{T} f(x)dx + \int\limits_{T}^{2T} f(x)dx + \ldots.. + \int\limits_{(n-1)T}^{nT} f(x)dx + \int\limits_{nT}^{nT+a} f(x)dx$

$$= \int\limits_{a}^{T} f(x)dx + \int\limits_{0}^{T} f(x)dx + \ldots.. \int\limits_{0}^{T} f(x)dx + \int\limits_{0}^{a} f(x)dx \ (n-1) \text{ times}$$

$$= \int\limits_{0}^{T} f(x)dx + (n-1)\int\limits_{0}^{T} f(x)dx = n\int\limits_{0}^{T} f(x)dx$$

SOLVED EXAMPLES

1. Evaluate $\int\limits_{0}^{100} e^{x-[x]}dx$ where [ ] denotes the greatest integer function.

Sol. $x-[x]$ is a periodic function of period 1 so, is $e^{x-[x]}$

$$\therefore \int\limits_{0}^{100} e^{x-[x]}dx = 100\int\limits_{0}^{1} e^{x-[x]}dx = 100\int\limits_{0}^{1} e^{x}dx = 100(e-1)$$

2. Evaluate $\int\limits_{0}^{100\pi} \sqrt{1+\cos 2x}dx$

Sol. $\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}|\cos x|$

$|\cos x|$ is a periodic function of fundamental period π

$$\therefore \int\limits_{0}^{100\pi} \sqrt{1+\cos 2x}dx = \sqrt{2}\int\limits_{0}^{100\pi} |\cos x|dx = \sqrt{2}100\int\limits_{0}^{\pi} |\cos x|dx$$

$$= \sqrt{2}\,100\left[\int\limits_{0}^{\frac{\pi}{2}} \cos x dx + \int\limits_{\frac{\pi}{2}}^{\pi} (-\cos x)dx\right] = \sqrt{2}100[1+1] = 200\sqrt{2}$$

3. Evaluate $\int\limits_{-\pi}^{199\pi} \sqrt{\frac{1 \quad \cos 2x}{2}}dx$

Sol. $\sqrt{\frac{1-\cos 2x}{2}} = |\sin x|$, is a periodic function of period π

$$\therefore \int\limits_{-\pi}^{199\pi} \sqrt{\frac{1-\cos 2x}{2}}dx = \int\limits_{-\pi}^{199\pi} |\sin x|dx = [199-(-1)]\int\limits_{0}^{\pi} |\sin x|dx = 200\int\limits_{0}^{\pi} \sin x dx$$

$$= 200(2) = 400$$

4. **Show that $\int_0^{n\pi+\alpha} |\sin x|\, dx = (2n+1) - \cos\alpha$ where $n \in N$ and $0 \le \alpha \le \pi$**

Sol. $\int_0^{n\pi+\alpha} |\sin x|\, dx = \int_0^{n\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+\alpha} |\sin x|\, dx = n\int_0^{\pi} |\sin x|\, dx + \int_0^{\alpha} |\sin(x+n\pi)|dx$

($\because |\sin x|$ is periodic of period π)

$$= n(2) + \int_0^{\alpha} |\sin x|\, dx = 2n + \int_0^{\alpha} \sin x dx \quad (0 \le \alpha \le \pi)$$

$$= 2n + [-\cos x]_0^{\alpha} = 2n + [-\cos\alpha + 1] = (2n+1) - \cos\alpha$$

EXERCISE - 2.5

Evaluate the following integrals

1. $\int_0^{50} e^{\{x\}} dx$ where $\{x\}$ denotes the fractional part of x
2. $\int_0^{\frac{41}{2}} e^{2x-[2x]} dx$ where [ ] denotes the GIF
3. $\int_0^{100\pi} \sqrt{\frac{1-\cos 2x}{2}}\, dx$
4. $\int_0^{\frac{16\pi}{3}} |\sin x|\, dx$
5. Suppose $f : R \to R$ is a continuous periodic function and T is the period of it. Let $a \in R$. Then prove that, for any positive integer n

$$\int_a^{a+nT} f(x)dx = n\int_a^{a+T} f(x)dx$$

6. Let $f(x)$ be a periodic function with period 1 and integrable over any finite interval. Also for two real numbers a and b and for any two positive integers m and $n(m \ne n)$,

$$\int_a^{a+m} f(x)dx = \int_b^{b+n} f(x)dx.$$ Then calculate the value of $\int_m^n f(x)dx$

2.11 DEFINITE INTEGRALS INVOLVING THE GREATEST INTEGER FUNCTION

We know that, even if a function f has a finite number of discontinuities on $[a, b]$, it is integrable on $[a, b]$. Therefore $f(x)=[x]$, $x \in R$ the greatest integer function (abbreviated as GIF) is integrable on any finite interval $[a, b]$. We shall, now, evaluate some definite integrals involving the G.I.F.

SOLVED EXAMPLES

Remember :

$\int_0^n [x]dx = \frac{n(n-1)}{2}; n \in N$

where [ ] is G.I.F.

***1. Evaluate $\int_0^n [x]dx$, $(n \in N)$ where [ ] denotes the GIF**

Sol. We know that $k \le x < k-1 \Rightarrow [x] = k, k \in z$

$$\therefore \int_0^n [x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + ... + \int_{n-1}^n [x]dx$$

$$= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + ... + \int_{n-1}^n (n-1)dx$$

$$= 0 + 1(2-1) + 2(3-2) + ... + (n-1)[n-(n-1)]$$

$$= 1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2} \qquad \therefore \int_0^n [x]dx = \frac{n(n-1)}{2}, n \in N$$

Remember :

$\int_0^n \{x\}dx = \frac{n}{2}; n \in N$

where { } is a functional part

2. Evaluate $\int_0^n (x-[x])dx, n \in N$, where [ ] denotes the GIF

Sol. $\int_0^n (x-[x])dx = n.\int_0^1 (x-[x])dx$ ($\because x-[x]$ is periodic of period 1)

$$= n\int_0^1 xdx = n\left(\frac{1}{2}\right) = \frac{n}{2}$$

Note : $\dfrac{\int_0^n [x]dx}{\int_0^n (x-[x])dx} = n-1$

Remember :

$\int_0^{n^2} [\sqrt{x}]dx = \frac{n(n-1)(4n+1)}{6}; n \in N$

where [ ] is G.I.F.

3. Evaluate $\int_0^{n^2} [\sqrt{x}]dx$ $(n \in N)$ where [ ] denotes the G I F

Sol. $$\int_0^{n^2} [\sqrt{x}]dx = \int_0^{1^2} [\sqrt{x}]dx + \int_{1^2}^{2^2} [\sqrt{x}]dx + \int_{2^2}^{3^2} [\sqrt{x}]dx + ... + \int_{(n-1)^2}^{n^2} [\sqrt{x}]dx$$

$$= \int_0^{1^2} 0dx + \int_{1^2}^{2^2} 1dx + \int_{2^2}^{3^2} 2dx + ... + \int_{(n-1)^2}^{n^2} (n-1)dx$$

$$= 0 + 1(2^2 - 1^2) + 2(3^2 - 2^2) + ... + (n-1)[n^2 - (n-1)^2]$$

$$= \sum_{r=1}^{n-1} r(2r+1) = \sum_{r=1}^{n-1} (2r^2 + r) = 2\frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} = \frac{n(n-1)(4n+1)}{6}$$

$$\therefore \int_0^{n^2} [\sqrt{x}]dx = \frac{n(n-1)(4n+1)}{6}, \quad n \in N$$

4. Evaluate $\int_0^2 [x^2]dx$ where [ ] denotes the GIF

Sol. $$\int_0^2 [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^{\sqrt{4}} [x^2]dx$$

$$= \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^2 3dx$$

$$= 0 + 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}$$

5. Evaluate $\int_0^{100}[\tan^{-1}x]dx$ where [ ] denotes the GIF

Sol. The range of $\tan^{-1}x$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$\therefore$ when $x\in[0,100], 0\le x<\tan 1 \Rightarrow [\tan^{-1}x]=0$

$\tan 1\le x<\infty \Rightarrow [\tan^{-1}x]=1 \quad \therefore \int_0^{100}[\tan^{-1}x]dx=\int_0^{\tan 1}0dx+\int_{\tan 1}^{100}1dx=100\tan 1$

6. Evaluate $\int_0^{2\pi}[\sin x+\cos x]dx$

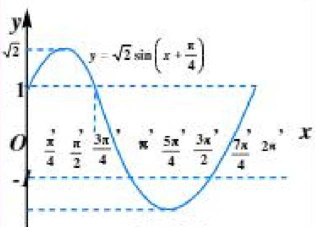


Fig 2.6

Sol. $[\sin x+\cos x]=\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]=\begin{cases}1 \text{ if } 0\le x\le\frac{\pi}{2}\\ 0 \text{ if } \frac{\pi}{2}<x\le\frac{3\pi}{4}\\ -1 \text{ if } \frac{3\pi}{4}<x\le\pi\\ -2 \text{ if } \pi<x<\frac{3\pi}{2}\\ -1 \text{ if } \frac{3\pi}{2}<<\frac{7\pi}{4}\\ 0 \text{ if } \frac{7\pi}{4}\le x<2\pi\end{cases}$

$$\therefore \int_0^{2\pi}[\sin x+\cos x]dx=\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}+\int_{\frac{3\pi}{4}}^{\pi}+\int_{\pi}^{\frac{3\pi}{2}}+\int_{\frac{3\pi}{2}}^{\frac{7\pi}{4}}+\int_{\frac{7\pi}{4}}^{2\pi}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]dx$$

$$=\int_0^{\frac{\pi}{2}}1dx+\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}0dx+\int_{\frac{3\pi}{4}}^{\pi}-1dx+\int_{\pi}^{\frac{3\pi}{2}}-2dx+\int_{\frac{3\pi}{2}}^{\frac{7\pi}{4}}-1dx+\int_{\frac{7\pi}{4}}^{2\pi}1dx$$

$$=\frac{\pi}{2}+0-1\left(\frac{\pi}{4}\right)-2\left(\frac{\pi}{2}\right)-1\left(\frac{\pi}{4}\right)+0 \quad =\frac{\pi}{2}-\frac{3\pi}{2}=-\pi$$

7. Evaluate $\int_0^2[x^2-x+1]dx$ where [ ] denotes the GIF

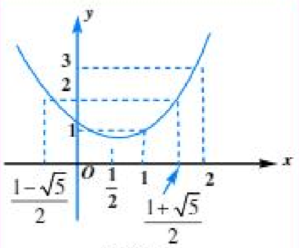


Fig 2.7

Sol. Consider the graph of $y=x^2-x+1,\ x\in[0,2]$

This is a parabola with axis parallel to y-axis and vertex at $\left(\frac{1}{2},\frac{3}{4}\right)$ and oriented upwards.

$y=1\Rightarrow x=0, x=1;\quad y=2\Rightarrow x=\frac{1\pm\sqrt{5}}{2};\quad y=3\Rightarrow x=-1, x=2$

$$\therefore \int_0^2[x^2-x+12]dx=\int_0^1+\int_1^{\frac{1+\sqrt{5}}{2}}+\int_{\frac{1+\sqrt{5}}{2}}^{2}[x^2-x+1]dx=\int_0^1 0dx+\int_1^{\frac{1+\sqrt{5}}{2}}1dx+\int_{\frac{1+\sqrt{5}}{2}}^{2}2dx$$

$$=0+\left(\frac{1+\sqrt{5}}{2}-1\right)+2\left(2-\frac{1+\sqrt{5}}{2}\right)=3-\frac{1+\sqrt{5}}{2}=\frac{5-\sqrt{5}}{2}$$

$[x^2-x+1]=0$ if $0<x<1=1$ if $1\le x<\frac{1+\sqrt{5}}{2}=2$ if $\frac{1+\sqrt{5}}{2}\le x<2$

Note :

$\int_0^{2n\pi}[\sin x+\cos x]dx=-2\pi$

EXERCISE – 2.6

Find the values of the following integrals (where [.] is gif and {.} is fractional part)

1. Evaluate the following

a) $\int_0^{30} [x]dx$ b) $\int_{-2}^{2} [x]dx$ c) $\int_0^{5} x[x]dx$ d) $\int_0^{1.5} [x^2]dx$

e) $\int_0^{5} [\sqrt{x}]dx$ f) $\int_0^{25} [\sqrt{x}]dx$ g) $\int_0^{30} (x-[x])dx$

2. Prove that $\int_0^{x} [t]dt = \frac{[x]([x]-1)}{2} + [x](x-[x])$

3. Evaluate $\int_0^{\pi} [2\sin x]dx$ 4. Evaluate $\int_0^{\infty} [2e^{-x}]dx$

5. Evaluate $\int_1^{15} sgn(x-[x])dx \int_0^{\infty} [2e^{-x}]dx$

6. Evaluate $\int_0^{\frac{\pi}{4}} \left[\sin x + \left[\cos x + [\tan x + [\sec x]]\right]\right]dx$

7. Evaluate $\int_{-n}^{n} (-1)^{[x]} dx, n \in N$ (Hint : Integrand is odd)

8. Evaluate $\int_0^{\frac{5\pi}{12}} [\tan x]dx$

2.12 LIMIT OF A SUM AS A DEFINITE INTEGRAL

In the definition of the definite integral, of $f(x)$ on $[a,b]$ in sec 2.1, we have arrived at the definition on [0,1] as

$$\int_0^1 f(x)dx = \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \text{ or } \int_0^1 f(x)dx = \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$$

These definitions can be used to find the sum of certain infinite series whose n^{th} term is of the form $\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ by expressing it as a definite integral as $n \to \infty$ i.e., $\underset{n\to\infty}{Lt} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$ which can be evaluated to get the value of the limit.

Working Rule

To transform the given sum as $n \to \infty$ into a definite integral, we proceed along the following steps.

Step 1: *Express the given sum in the form* $\sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$ *(or)* $\sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

[In some cases the final value of r may be in the form $pn \pm q$ where p and q are fixed positive integers]

Step 2: *Replace $\frac{r}{n}$ with x and $\frac{1}{n}$ with dx to get f(x)dx.*

Step 3: *Calculate the lower and upper limits of the integral as*

Lower limit $= \underset{n\to\infty}{Lt} \dfrac{\text{Initial value of } r}{n}$ *(Usually, this limit is zero)*

Upper Limit $= \underset{n\to\infty}{Lt} \dfrac{\text{Final value of } r}{n} \begin{pmatrix} = 1 \text{ if the final value is } n \text{ or } n \pm 1 \\ = p \text{ if the final value is } pn \pm q \end{pmatrix}$

Step 4: *Replace* $\underset{n\to\infty}{Lt} \sum_{r=1}^{n}$ with $\int_0^1$ *Thus we get* $\underset{n\to\infty}{Lt} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

SOLVED EXAMPLES

1. **Find** $\underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}}$

Sol. $\underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{n} \frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^n}} = \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

where $f\left(\frac{r}{n}\right) = \frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$

so that $f(x) = \frac{1}{\sqrt{1-x^2}} \quad = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1 = \frac{\pi}{2}$

2. **Evaluate** $\underset{n\to\infty}{Lt} \left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n} \right)$

Sol. $\underset{n\to\infty}{Lt} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n} \right] = \underset{n\to\infty}{Lt} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+2n} \right]$

$= \underset{n\to\infty}{Lt} \sum_{r=1}^{2n} \frac{1}{n+r} = \underset{n\to\infty}{Lt} \sum_{r=1}^{2n} \frac{1}{n} \left[\frac{1}{1+\left(\frac{r}{n}\right)} \right] = \underset{n\to\infty}{Lt} \sum_{r=1}^{2n} \frac{1}{n} f\left(\frac{r}{n}\right)$

Where $f\left(\frac{f}{n}\right) = \frac{1}{1+\left(\frac{r}{n}\right)}$ so that $f(x) = \frac{1}{1+x} = \int_0^2 \frac{1}{1+x} dx = \log(1+x)]_0^2 = \log_e 3$

Remember :

$\underset{n\to\infty}{Lt} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{kn} \right]$

$= \log_e k \,(k \in N)$

3. **Evaluate** $\underset{n\to\infty}{Lt}\frac{1}{n}\left[\sin^2\left(\frac{\pi}{2n}\right)+\sin^2\left(\frac{2\pi}{2n}\right)+\sin^2\left(\frac{3\pi}{2n}\right)+....\text{ to } n \text{ terms}\right]$

Sol. $\underset{n\to\infty}{Lt}\frac{1}{n}\left[\sin^2\left(\frac{\pi}{2n}\right)+\sin^2\left(\frac{2\pi}{2n}\right)+\sin^2\left(\frac{3\pi}{2n}\right)+..+\sin^2\left(\frac{n\pi}{2n}\right)\right]$

$$=\underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\sin^2\left(\frac{r\pi}{2n}\right)=\int_0^1\sin^2\left(\frac{\pi x}{2}\right)dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\,dt=\frac{2}{\pi}.\frac{1}{2}.\frac{\pi}{2}=\frac{1}{2}$$

4. **Find** $\underset{n\to\infty}{Lt}\left\{\frac{\sqrt{n}}{n^{3/2}}+\frac{\sqrt{n}}{(n+3)^{3/2}}+...+\frac{\sqrt{n}}{[n+3(n-1)]^{3/2}}\right\}$

Sol. Given expression $=\underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{(m+3r)^{3/2}}$

$$=\underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\frac{1}{n}\frac{1}{\left[1+3\left(\frac{r}{n}\right)\right]^{3/2}}=\int_0^1\frac{1}{(1+3x)^{3/2}}dx$$

$$=\frac{1}{3}\left[\frac{-1}{2(1+3x)^{1/2}}\right]_0^1=-\frac{1}{6}\left[\frac{1}{2}-1\right]=\frac{1}{12}$$

5. **Find** $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

Sol. Let $P=\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$$\Rightarrow \log p=\frac{1}{n}\left[\begin{array}{l}\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+...\\ \qquad\qquad +\log\left(1+\frac{n^2}{n^2}\right)\end{array}\right]=\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$\therefore \underset{n\to\infty}{Lt}(\log P)=\underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\log\left[1+\left(\frac{r}{n}\right)^2\right]$$

$$\Rightarrow \log(\underset{n\to\infty}{Lt}P)=\int_0^1\log(1+x^2)dx$$

$$=x\log(1+x^2)\Big]_0^1-\int_0^1\frac{2x^2}{1+x^2}dx=\log 2-2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\log 2-2[x-\tan^{-1}x]_0^1=\log 2-2\left(1-\frac{\pi}{4}\right)$$

$$\therefore \underset{n\to\infty}{Lt}P=e^{\log 2+\left(\frac{\pi-4}{2}\right)}=2e^{\frac{\pi-4}{2}}$$

6. Apply the definition of definite integral as the limit of a sum to evaluate

$$\underset{n\to\infty}{Lt}\left(\frac{n!}{n^n}\right)^{1/n}$$

Sol. Let $P=\left(\frac{n!}{n^n}\right)^{1/n}$. Then $P=\left[\frac{1}{n}.\frac{2}{n}.\frac{3}{n}....\frac{n}{n}\right]^{1/n}$

We have to find $\underset{n\to\infty}{Lt}P$ Now, $\log P=\frac{1}{n}\left[\begin{array}{r}\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+..\\+\log\left(\frac{n}{n}\right)\end{array}\right]=\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$

$\therefore \underset{n\to\infty}{Lt}(\log P)=\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$

$\Rightarrow \log(\underset{n\to\infty}{Lt}P)=\int_0^1\log x dx=[x\log x-x]_0^1$

$=-1-0=-1$ $(\because \underset{x\to 0+}{Lt}x\log x=0)$

$\therefore \underset{n\to\infty}{Lt}P=e^{-1}=\frac{1}{e}$

7. Evaluate $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{n-i}{n+1}\right]$ by using the method of finding definite integral as the limit of a sum

Sol. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{n-i}{n+1}\right]=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{1-\frac{i}{n}}{1+\frac{i}{n}}\right]=\int_0^1\left(\frac{1-x}{1+x}\right)dx$

$f(x)=\frac{1-x}{1+x}, x\in[0,1]$. Now, $\int_0^1\frac{1-x}{1+x}dx=-\int_0^1\frac{x+1-2}{1+x}dx=-\int_0^1 dx+2\int_0^1\frac{1}{1+x}dx$

$=-[x]_0^1+2\ln(1+x)\big|_0^1=-1+2\ln 2$

8. Evaluate $\lim\limits_{n\to\infty}\frac{2^k+4^k+6^k+....+(2n)^k}{n^{k+1}}$ by using the method of finding definite integral as the limit of a sum.

Sol. $\lim\limits_{n\to\infty}\frac{2^k+4^k+6^k+....+(2n)^k}{n^{k+1}}=\lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}2^k\left(\frac{i}{n}\right)^k=\int_0^1 2^k x^k dx$

$f(x)=2^k x^k, x\in[0,1]$, k is a fixed real number not equal to -1.

Now, $\int_0^1 2^k x^k dx=2^k\left[\frac{x^{k+1}}{k+1}\right]_0^1=\frac{2^k}{k+1}$

Hence, $\lim\limits_{n\to\infty}\frac{2^k+4^k+6^k+....+(2n)^k}{n^{k+1}}=\frac{2^k}{k+1}$

9. Evaluate $\lim\limits_{n\to\infty}\frac{1}{n}\left[\tan\frac{\pi}{4n}+\tan\frac{2\pi}{4n}+....+\tan\frac{n\pi}{4n}\right]$

Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\left[\tan\frac{\pi}{4n}+\tan\frac{2\pi}{4n}+....+\tan\frac{n\pi}{4n}\right]$

$$=\underset{n\to\infty}{lt}\frac{1}{n}\sum_{i=1}^{n}\tan\left(\frac{i\pi}{4n}\right)=\underset{n\to\infty}{lt}\frac{1}{n}\sum_{i=1}^{n}\tan\left(\frac{\pi}{4}.\frac{i}{n}\right)=\int_0^1\tan\left(\frac{\pi}{4}x\right)dx$$

$$=\left[\frac{\log\left|\sec\frac{\pi}{4}x\right|}{\left(\frac{\pi}{4}\right)}\right]_0^1=\frac{4}{\pi}.\left[\log\sec\frac{\pi}{4}-\log\sec 0\right]=\frac{4}{\pi}[\log\sqrt{2}-\log 1]$$

$$=\frac{4}{\pi}[\log\sqrt{2}-0]=\frac{4}{\pi}.\log 2^{1/2}=\frac{4}{\pi}.\frac{1}{2}.\log 2=\frac{2}{\pi}\log 2$$

10. Evaluate $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i^3}{i^4+n^4}$

Sol. $$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{i^4+n^4}=\underset{n\to\infty}{lt}\sum_{i=1}^{n}\frac{n^3\frac{i^3}{n^3}}{1+\frac{i^4}{n^4}.n^4}=\underset{n\to\infty}{lt}\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\frac{i}{n}\right)^3}{1+\left(\frac{i}{n}\right)^4}=\int_0^1\frac{x^3}{1+x^4}dx$$

$$=\frac{1}{4}[\log(1+x^4)]_0^1=\frac{1}{4}\log 2$$

11. Evaluate $\lim\limits_{n\to\infty}\frac{1+2^4+3^4+....+n^4}{n^5}$

Sol. $$\underset{n\to\infty}{lt}\frac{1^4+2^4+3^4+....+n^4}{n^5}$$

$$=\underset{n\to\infty}{lt}\frac{1^4+2^4+3^4+....+n^4}{n^4n}=\underset{n\to\infty}{lt}\frac{1}{n}\sum_{r=1}^{n}\frac{r^4}{n^4}=\int_0^1 x^4dx=\left[\frac{x^5}{5}\right]_0^1=\frac{1}{5}$$

EXERCISE - 2.7

Evaluate the following

1. $\underset{n\to\infty}{Lt}\left[\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{2n}\right]$

2. $\underset{n\to\infty}{Lt}\left[\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{6n}\right]$

3. $\underset{n\to\infty}{Lt}\left[\frac{1}{2n+1}+\frac{1}{2n+2}+....+\frac{1}{3n}\right]$

4. $\underset{n\to\infty}{Lt}\left[\frac{1}{3n+1}+\frac{1}{3n+2}+\ldots+\frac{1}{4n}\right]$

5. $\underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\left(\frac{r}{n^2+r^2}\right)$

6. $\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left[\frac{1}{n}\sqrt{\frac{n+r}{n-r}}\right]$

7. $\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left[\frac{1}{\sqrt{4n^2-r^2}}\right]$

8. $\underset{n\to\infty}{Lt}\left[\frac{\sqrt{n+1}+\sqrt{n+2}+\ldots+\sqrt{n+n}}{n\sqrt{n}}\right]$

9. $\underset{n\to\infty}{Lt}\left[\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\frac{1}{\sqrt{n^2+3^2}}+\ldots\ldots\text{to } n \text{ terms}\right]$

10. $\underset{n\to\infty}{Lt}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\ldots\ldots\text{to } n \text{ terms}\right]$

11. $\underset{n\to\infty}{Lt}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\frac{n^2}{(n+3)}+\ldots\ldots\text{to } n \text{ terms}\right]$

12. $\underset{n\to\infty}{Lt}\left[\frac{\sqrt{n^2-1}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\frac{\sqrt{n^2-3^2}}{n^2}+\ldots\ldots\text{to } n \text{ terms}\right]$

13. $\underset{n\to\infty}{Lt}\frac{1}{n}\left[\sec^2\left(\frac{\pi}{4n}\right)+\sec^2\left(\frac{2\pi}{4n}\right)+\ldots\ldots+\sec^2\left(\frac{n\pi}{4n}\right)\right]$

14. $\underset{n\to\infty}{Lt}\left[\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n^2+2)^{3/2}}+\ldots\ldots+\frac{n^2}{\left[n^2+(n-1)^2\right]^{3/2}}\right]$

15. $\underset{n\to\infty}{Lt}\left[\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\ldots\ldots+\frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}\right]$

16. $\underset{n\to\infty}{Lt}\left[\frac{1}{n^2}\sec^2\left(\frac{1}{n^2}\right)+\frac{2}{n^2}\sec^2\left(\frac{4}{n^2}\right)+\frac{3}{n^2}\sec\left(\frac{9}{n^2}\right)+........+\frac{1}{n}\sec^2 1\right]$

17. $\underset{n\to\infty}{Lt}\left[\frac{1^3}{n^4+1^4}+\frac{2^3}{n^4+2^4}+....+\frac{n^3}{n^4+n^4}\right]$

18. $\underset{n\to\infty}{Lt}\frac{1}{n}\left[e^{\frac{1}{n}}+e^{\frac{2}{n}}+.....+e^{\frac{n}{n}}\right]$

19. $\underset{n\to\infty}{Lt}\left[\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}}\right]$

20. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)....\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

21. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n^2}\right)^{\frac{2}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}}\left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}}.......\left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right]$

EXERCISE

1. Evaluate $\int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$

   Hint : write Nr = $2x^3(x^2-1)+(x^2+1)^2$ and write I as $I_1+I_2; I_1=\log 2-\frac{1}{10}, I_2=\frac{1}{2}\log\left(\frac{3}{2}\right)$

2. Show that $\int_0^{\pi/4}\frac{x^2}{(x\sin x+\cos x)^2}dx=\frac{4-\pi}{4+\pi}$

3. Show that $\int_{\sqrt{3}}^{2\sqrt{3}} x^3(x^2-3)^{3/2}dx=\frac{16038}{35}$

4. Prove that $\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2ab(a+b)}$

5. Show that $\int_0^{\infty}\frac{dx}{\left[x+\sqrt{x^2+1}\right]^3}=\frac{3}{8}$

6. Show that $\int_0^{\pi/2} \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx = -\frac{1}{2}$

7. Show that $\int_0^{\pi/2} \frac{dx}{3+4\sin x} = \frac{1}{\sqrt{7}}\log\left(\frac{4+\sqrt{7}}{3}\right)$

8. a) If $\int_{\log 2}^{x} \frac{dy}{\sqrt{e^y-1}} = \frac{\pi}{6}$, prove that $x = \log 4$

b) If $\int_{\log 2}^{x} \frac{1}{e^x-1}dx = \log\frac{3}{2}$, show that $x = \log 4$

9. Show that $\int_0^{1/2} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = \frac{1}{2} - \frac{\sqrt{3}}{12}\pi$

10. Prove that $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) = \frac{\pi}{2} - 1$

11. Show that a) $\int_e^{e^2} \frac{1}{\log x}dx = \int_1^2 \frac{e^x}{x}dx$ b) $\int_x^1 \frac{dx}{1+x^2} = \int_1^{1/x} \frac{dx}{1+x^2}$

12. Prove that $\int_0^1 \log(\sqrt{1-x}+\sqrt{1+x})dx = \frac{1}{2}\log 2 + \frac{\pi}{4} - \frac{1}{2}$

13. Show that $\int_0^{\pi/4} x\cos x\cos 3x\, dx = \frac{\pi-3}{16}$

14. Show that $\int_1^{\sqrt[7]{5}} \frac{1}{x(2x^7+1)}dx = \frac{1}{7}\log\left(\frac{6}{5}\right)$

15. Show that $\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + 3\cos x + 2}dx = \log\left(\frac{9}{8}\right)$

ANSWERS

EXERCISE – 2.1

1. $\frac{3}{2}$ 2. $\frac{1}{3}(b^3-a^3)$ 3. $\frac{15}{4}$ 4. $\frac{65}{4}$ 5. 1 6. $e-1$

EXERCISE – 2.2

1. a) $\frac{21}{2}$ b) $\frac{a^4}{4}$ c) $\log 2$ d) $4+\log 5$ e) $\frac{a^2}{6}$ f) $\frac{4}{3}(\sqrt{2}-1)$
   g) $1-\frac{\pi}{4}$ h) 1 i) π j) $\frac{2}{\sqrt{3}}-\frac{\pi}{8}$
2. a) 4 b) 1 c) 4 d) 1 e) $5\sqrt{15}$ f) $\sqrt{3}-1$
   g) 2 h) $4\left(\frac{2}{3}+\mathrm{Tan}^{-1}2\right)$ i) $\frac{3}{\sqrt{3-2x}}$ j) $\frac{1}{2}\left(1-\frac{1}{e}\right)$
   k) $3-\sqrt{2}\left(\mathrm{Tan}^{-1}\sqrt{2}+\mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$
3. a) 1 b) $\frac{3e+1}{4}$ c) $\frac{1}{2}(1-\log 2)$ d) $\frac{e-2}{2}$ e) $\pi-2$
4. a) $\frac{8}{3}\log 2-\frac{7}{9}$ b) $\frac{\pi-2}{2}$ c) $\frac{\pi}{2}-\log 2$ d) $\frac{\pi}{6}-\frac{2}{9}$
5. a) $\frac{1}{3}\log 2$ b) $\frac{\pi}{2}-1]$ c) $\frac{\pi}{8}(b-a)^2$ d) $\frac{\pi-2}{4}$ e) $\frac{1}{2}-\frac{\pi\sqrt{3}}{12}$ f) π

EXERCISE - 2.3

1. a) 0 b) 0 c) 0 d) 0 e) 0
2. a) 0 b) 0 c) 0 d) 0 e) 0
3. a) $\frac{\pi}{4}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{4}$
4. a) $\frac{5\pi}{4}$ b) $\frac{7\pi}{4}$ c) $\frac{9\pi}{4}$ d) $\frac{\pi}{4}$ e) $(a+b)\frac{\pi}{4}$

5. a) $\frac{\pi}{12}$ b) $\frac{\pi}{12}$ c) $\frac{\pi}{12}$ d) $\frac{\pi}{20}$

6. a) $\frac{2}{3}a^3+\frac{\pi}{2}a^2$ b) 2 c) 1 d) 2 e) $\frac{2}{5}$

7. a) 1 b) π c) $\frac{\pi}{4}$ d) $\frac{1}{2}\left(e-\frac{1}{e}\right)$ 8. $\frac{16\sqrt{2}}{15}$

9. a) $\frac{\pi}{8}\log 2$ b) $\frac{\pi}{8}\log 2$ 10. a) $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ b) $\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

11. a) $\frac{\pi}{2}(\pi-2)$ b) $\frac{\pi^2}{4}$ c) $\frac{\pi}{2}(\pi-2)$ d) $\frac{2\pi}{3}$ 12. $\frac{8}{3}$

EXERCISE - 2.4

1. a) $\frac{35\pi}{256}$ b) $\frac{128}{315}$ c) $\frac{5\pi}{32}$ d) $\frac{16}{35}$ e) $\frac{63\pi}{512}$ f) $\frac{256}{693}$

g) $\frac{1}{\sqrt{2}}+\frac{1}{2}\log(\sqrt{2}+1)$ h) $\frac{28}{15}$

2. a) $\frac{3\pi}{256}$ b) $\frac{3\pi}{512}$ c) $\frac{5\pi}{2048}$ d) $\frac{16}{315}$ e) $\frac{8}{315}$

3. a) 0 b) 0 c) $\frac{4}{63}$ d) 0 e) $\frac{16}{315}$

4. a) $\frac{\pi}{8}$ b) $\frac{3\pi}{128}$ c) 0 d) 0

5. a) $\frac{5\pi}{4}$ b) $\frac{\pi}{16}$ c) $\frac{32}{315}$ d) 0 e) $\frac{4}{15}$

6. a) 0 b) $\frac{8\pi}{693}$ c) $\frac{16\pi}{3003}$

7. a) $\frac{8-5\sqrt{2}}{12}$ b) $\frac{1}{3}-\frac{11\sqrt{15}}{128}$ c) $\frac{\pi a^6}{16}$ d) 640π e) $\frac{512}{153153}$

f) $\frac{a^9}{9}$ g) $\frac{\pi}{2}$ h) $\frac{\pi}{14}-\frac{16}{245}$

EXERCISE - 2.5

1. $50(e-1)$ 2. $\frac{41}{2}(e-1)$ 3. 200 4. $\frac{21}{2}$ 6. 0

EXERCISE – 2.6

1. a) 435 b) -2 c) 35 d) $2-\sqrt{2}$ e) 5 f) 70 g) 25

3. $\frac{2\pi}{3}$ 4. $ln\,2$ 5. 16 6. $\frac{\pi}{4}$ 7. 0

8. $\frac{5\pi}{4}-\frac{\pi}{4}-\tan^{-1}3-\tan^{-1}2=\frac{\pi}{4}$

EXERCISE - 2.7

1. $\log 2$ 2. $\log 6$ 3. $\log\left(\frac{3}{2}\right)$ 4. $\log_e\left(\frac{4}{3}\right)$ 5. $\frac{1}{2}\log 2$ 6. $\frac{\pi}{2}+1$

7. $\frac{\pi}{6}$ 8. $\frac{2}{3}(2\sqrt{2}-1)$ 9. $\log(\sqrt{2}+1)$ 10. $\frac{\pi}{2}$ 11. $\frac{3}{8}$ 12. $\frac{\pi}{4}$

13. $\frac{4}{\pi}$ 14. $\frac{1}{\sqrt{2}}$ 15. $\frac{1}{10}(5-\sqrt{5})$ 16. $\frac{1}{2}\tan 1$ 17. $\frac{1}{4}\log 2$ 18. $e-1$

19. $\frac{1}{100}$ 20. $\frac{4}{e}$ 21. $\frac{4}{e}$

EXERCISE

1. $\left[\frac{1}{2}\left(\log 6-\frac{1}{5}\right)\right]$ 26. $\frac{1}{12}$ 27. $\frac{22}{7}-\pi$