THEORY OF EQUATIONS

CHAPTER 2

Algebra

- POLYNOMIAL EQUATION
- RELATION BETWEEN ROOTS AND COEFFICIENTS
- MULTIPLE ROOTS
- TRANSFORMATION OF EQUATIONS

2.0 INTRODUCTION

In preceding chapter we have studied quadratic expressions, equations and inequations. This chapter discusses the theory and solutions of polynomial equations of degree more than two.

Definition

An expression of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are complex or real $a_n \neq 0$ is called a polynomial in x of degree n. A complex number 'α' is said be a 'zero' of the polynomial $f(x)$ if $f(\alpha) = 0$.

Example :

1) $x^4 + 2x^3 + x^2 + x - 1$ is a polynomial of degree 4.
2) $x^5 - x^4 + x^3 - x + 1$ is a polynomial of degree 5.

2.1 SOME IMPORTANT RESULTS ON POLYNOMIALS

THEOREM-2.1 **Division Algorithm for polynomials**

If $f(x)$, $g(x)$ $(\neq 0)$ are two polynomials, then there exist polynomials $q(x)$, $r(x)$ uniquely, such that $f(x) = q(x)g(x) + r(x)$ where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$.

Note

i) *Here $q(x)$ is called quotient and $r(x)$ is called remainder of $f(x)$ when divided by $g(x)$.*

ii) *$f(x) = q(x)g(x) + r(x)$ is an identity in x.*

THEOREM-2.2 **If $f(x)$ is a polynomial, then the remainder of $f(x)$ when divided by $x - a$ is $f(a)$.**

Proof : Let $q(x)$, $r(x)$ be quotient and remainder when $x - a$ divides $f(x)$. Then by division algorithm $f(x) = (x - a)q(x) + r(x)$

Now $f(a) = r(a)$

$\therefore$ The remainder is $f(a)$

THEOREM-2.3 **(Factor theorem)**

If $f(x)$ is a polynomial then $f(a) = 0$ iff $(x - a)$ is a factor of $f(x)$.

2.2 DIVISION OF A POLYNOMIAL BY $x - a$ (SYNTHETIC DIVISION)

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is a polynomial,

$q(x) = q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + ... + q_{n-1}$ is the quotient and $r(x) = r$ be the remainder of $f(x)$ when divided by $x - a$ then $f(x) = (x - a)\, q(x) + r(x)$.

$$\Rightarrow a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = (x-a)(q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + ... + q_{n-1}) + r$$

$$\Rightarrow a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$$

$$= q_0x^n + (q_1 - aq_0)x^{n-1} + (q_2 - aq_1)x^{n-2} + ... + (q_{n-1} - aq_{n-2})x + r - aq_{n-1}$$

On comparing both sides, $a_0 = q_0$

$a_1 = q_1 - aq_0 \Rightarrow q_1 = a_1 + aq_0 = a_1 + aa_0$

$a_2 = q_2 - aq_1 \Rightarrow q_2 = a_2 + aq_1$

$a_{n-1} = q_{n-1} - aq_{n-2} \Rightarrow q_{n-1} = a_{n-1} - aq_{n-2}$

$a_n = r - aq_{n-1} \Rightarrow r = a_n + aq_{n-1}$

We can find $q_0, q_1, q_2, ..., q_{n-1}, r$ in the following proceedure which is known as Horner's method of synthetic division.

	a_0	a_1	a_2	a_3	.....	a_{n-1}	a_n
a	–	aq_0	aq_1	aq_2	.....	aq_{n-2}	aq_{n-1}
	$a_0 = q_0$	q_1	q_2	q_3		q_{n-1}	r

In the first row we write the coefficients of the given polynomial. We divide the polynomial by $x - a$ by writing a in the left corner as shown. We write the first term in third row by $a_0 = q_0$. Now multiply q_0 by a and write it below a_1. We add q_0a to a_1 write the sum as q_1 in third row. Now multiply q_1 by a and write below a_2. Then add q_1a and a_2 to write this q_2 in third row. Continue this process to get q_{n-1} in third row. Finally multiply q_{n-1} by a and write below a_n and add it to a_n to get r.

Note

If any term of $f(x)$ is missing then '0' will be taken as its coefficient.

Ex 1. Find the quotient and remainder when $3x^4 - x^3 + 2x^2 - 2x - 4$ is divided by $x + 2$.

Sol.

	3	–1	2	–2	–4
–2	–	–6	14	–32	68
	3	–7	16	–34	64

$\therefore$ Quotient is $3x^3 - 7x^2 + 16x - 34$, Remainder is 64

Ex 2. Divide $2x^5 - 3x^4 + 5x^3 - 7x^2 + 3x - 4$ by $x - 2$.

Sol.

	2	–3	5	–7	3	–4
2	–	4	2	14	14	34
	2	1	7	7	17	30

$\therefore$ Quotient is $2x^4 + x^3 + 7x^2 + 7x + 17$, Remainder is 30

2.3 DIVISION OF POLYNOMIAL BY $x^2 - ax - b$

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, be polynomial, $q(x) = q_0x^{n-2} + q_1x^{n-3} + ... + q_{n-2}$ and $r(x) = mx + n$ quotient, remainder respectivey when $f(x)$ is divided by $x^2 - ax - b$.

First write down the coefficient of x^n, x^{n-1}, x^{n-2}, ... respectively in a row. Draw a vertical line to the line left of a_0. Write down a, b as column figures to the left of the drawn vertical line in 2nd and 3rd row. Below a_0 write 0 in 2nd, 3rd row. Write this sum is 4th row as q_0. Now multiply q_0 by a write below a_1 in 2nd row and write 0 below it in third row. Now add these a_1, aq_0, 0 write this sum as q_1 in 4th row. Now multiply q_1 with a and q_0 with b write these two product aq_1, bq_0 below a_2 and add all these to write as q_2 continue this process until the terms under a_n are obtained. Now the quotient $q_0x^{n-2} + q_1x^{n-3} + ... + q_{n-2}$ $(q_0 \neq 0)$ and remainder is $mx + n$.

	a_0	a_1	a_2	a_3	.....	a_{n-2}	a_{n-1}	a_n
a	0	aq_0	aq_1	aq_2		aq_{n-3}	aq_{n-2}	0
b	0	0	bq_0	bq_1		bq_{n-4}	bq_{n-3}	bq_{n-2}
	$q_0=a_0$	q_1	q_2	q_3		q_{n-2}	m	n

Ex 1. Find quotient and the remainder when $2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ is divided by $x^2 - x - 3$.

Sol.

	2	–3	5	–3	7	–9
1	0	2	–1	10	4	0
3	0	0	6	–3	30	12
	2	–1	10	4	41	3

$\therefore$ Quotient is $2x^3 - x^2 + 10x + 4$

Remainder is $41x + 3$

Ex 2. Find the quotient and remainder when $x^4 - 11x^3 + 44x^2 - 76x + 48$ is divided by $x^2 - 7x + 12$.

Sol.

	1	–11	44	–76	48
7	0	7	–28	28	0
-12	0	0	–12	48	–48
	1	–4	4	0	0

$\therefore$ Quotient is $x^2 - 4x + 4$

Remainder is $0x + 0 = 0$

2.4 ALGEBRAIC EQUATIONS AND ITS ROOTS

Definition

An equation $f(x) = 0$ is said to be an algebraic equation or a polynomial equation.

A complex number 'α' is said to be a 'root' of the equation $f(x) = 0$ if $f(\alpha) = 0$

Example :

1. $2x^5 - x^4 + 3x^3 - x^2 + x + 1 = 0$
2. $x^5 - (2 + i)x^4 + 3ix^2 - 2 = 0$
3. $2x^4 - 5x^3 + 3x + 5 = 0$ is an equation with integer coefficients.
4. $17x^3 - \sqrt{3}\,x^2 + (2 + \sqrt{5})x - 5 = 0$ is an equation with real coefficients.

2.5 SOME IMPORTANT THEOREMS AND RESULTS REGARDING ALGEBRAIC EQUATIONS AND ROOTS

THEOREM-2.4 (Fundamental theorem of Algebra)

Every algebraic equation or polynomial equation of degree nonzero has a root.

THEOREM-2.5 **Every polynomial equation of degree n has n roots and cannot have more than n roots.**

THEOREM-2.6 **If $f(x)$ is a polynomial of degree n with leading coefficient a and $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of the equation $f(x) = 0$ then $f(x) = a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) .... (x - \alpha_n)$, which is an identity.**

2.6 RELATION BETWEEN THE ROOTS AND THE COEFFICIENS OF A POLYNOMIAL EQUATION

THEOREM-2.7 **If $\alpha_1, \alpha_2, .., \alpha_n$ be the roots of the equation say $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} +..+a_n=0$ then**

i) sum of roots $= \Sigma\alpha_1 = s_1 = -\dfrac{a_1}{a_0}$

ii) sum of the products of roots taken two at a time $= \Sigma\alpha_1\alpha_2 = s_2 = \dfrac{a_2}{a_0}$

iii) sum of the products of roots taken three at a time

$= \Sigma\alpha_1\alpha_2\alpha_3 = s_3 = \dfrac{-a_3}{a_0}$ ..... Product of (n) the roots

$= \alpha_1\alpha_2....\alpha_n = s_n = (-1)^n \dfrac{a_n}{a_0}$

Corollary - 1 :

1. If α, β, γ are roots of $ax^3 + bx^2 + cx + d = 0$ then

   i) $\alpha + \beta + \gamma = s_1 = -b/a$;

   ii) $\alpha\beta + \beta\gamma + \gamma\alpha = s_2 = c/a$;

   iii) $\alpha\beta\gamma = \dfrac{-d}{a}$

Corollary - 2 :

If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

i) $\alpha + \beta + \gamma + \delta = -\dfrac{b}{a}$

ii) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \dfrac{c}{a}$

iii) $\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -\dfrac{d}{a}$

iv) $\alpha\beta\gamma\delta = \dfrac{e}{a}$

THEOREM-2.8 **If the coefficients of a polynomial equation $f(x) = 0$ are real, and if α is any complex number and root of $f(x) = 0$ then the conjugate of α is also a root of $f(x) = 0$ i.e., in an equation with real coefficients, imaginary roots occur in conjugate pairs.**

Corollary : Every odd degree polynomial equation with real coefficients has atleast one real root.

THEOREM-2.9 If coefficients of a polynomial equation $f(x) = 0$ are rational, and if $\alpha = a + \sqrt{b}$ where a, b are rational, $b > 0$ then conjugate surd of α i.e., $a - \sqrt{b}$ is a root of $f(x) = 0$. i.e., in an equation with rational coefficeints, irrational roots occur in pairs of conjugate surds.

THEOREM-2.10 If coefficients are rational of a polynomial equation $f(x) = 0$ and if $\sqrt{a}, \sqrt{b}$ are irrational numbers such that one of the numbers $\sqrt{a}+\sqrt{b}, \sqrt{a}-\sqrt{b}, -\sqrt{a}+\sqrt{b}, -\sqrt{a}-\sqrt{b}$ is a root of the equation $f(x) = 0$ then all the four numbers are roots of $f(x) = 0$.

SOLVED EXAMPLES

***1. Find the roots of $x^3 - 6x^2 + 11x - 6 = 0$**

Sol. Let $f(x) = x^3 - 6x^2 + 11x - 6$

clearly $f(1) = 0$

$\therefore$ $x - 1$ is a factor. Now by dividing $f(x)$ by $x - 1$.

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\therefore$ $f(x) = (x - 1)(x^2 - 5x + 6)$

Now the factors $x^2 - 5x + 6$ are $x - 2$, $x - 3$

$\therefore$ $f(x) = (x - 1)(x - 2)(x - 3)$

$\therefore$ The roots of $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2, 3.

2. Solve the equation $6x^3 - 29x^2 - 17x + 60 = 0$, one root being 5.

Sol. Clearly $x - 5$ is a factor of $6x^3 - 29x^2 - 17x + 60$,

Now
$$\begin{array}{c|cccc} & 6 & -29 & -17 & 60 \\ 5 & 0 & 30 & 5 & -60 \\ \hline & 6 & 1 & -12 & 0 \end{array}$$

$\therefore$ $6x^3 - 29x^2 - 17x + 60 = (x - 5)(6x^2 + x - 12) = (x - 5)(2x + 3)(3x - 4)$

$\therefore$ The roots are 5, $-3/2$, $4/3$.

3. Solve $x^3 - 3x^2 - 6x + 8 = 0$, the roots being in A.P.

Sol. Let the roots be $a - d$, a, $a + d$

then $s_1 = a - d + a + a + d = 3a = 3$ or $a = 1$

and product of roots $s_3 = (a-d)a(a+d) = a(a^2-d^2) = -8$

$\Rightarrow 1 - d^2 = -8 \Rightarrow d^2 = 9$

$\therefore$ The roots are -2, 1, 4

4. Solve the equation $x^4 - 9x^3 + 27x^2 - 29x + 6 = 0$, one root being $2 - \sqrt{3}$.

Sol. Let $P(x) = x^4 - 9x^3 + 27x^2 - 29x + 6$, since the coefficients are rational and $2 - \sqrt{3}$ is one root hence $2 + \sqrt{3}$ is a root of $P(x) = 0$

Let the remaining two roots be α, β.

$\therefore$ $s_1 = \alpha + \beta + 2 - \sqrt{3} + 2 + \sqrt{3} = 9 \Rightarrow \alpha + \beta = 5$ and

$s_4 = \alpha\beta(2 - \sqrt{3})(2 + \sqrt{3}) = \alpha\beta = 6$

$\Rightarrow$ α, β are 2, 3 or 3, 2

$\therefore$ The roots of the given equation are 2, 3, $2 \pm \sqrt{3}$

5. Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$ if $-\sqrt{3}$, $1 - 2i$ are two of its roots.

Sol. Since $-\sqrt{3}, 1+2i$ are two if its roots.

$\sqrt{3}, 1-2i$ are also roots of the equation.

Let the remaining root be α

Since sum of roots equal to 1.

$\sqrt{3}+(-\sqrt{3})+(1+2i)+(1-2i)+\alpha = 1 \Rightarrow \alpha = -1$

$\therefore$ The roots are $-1, \sqrt{3}, 1-2i, -\sqrt{3}, 1+2i$.

***6. Solve $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$**

Sol. Let $P(x) = x^4 - 16x^3 + 86x^2 - 176x + 105$

$P(1) = 1 - 16 + 86 - 176 + 105 = 0$

$\therefore$ $x - 1$ is a factor of $P(x)$

Now

	1	–16	86	–176	105
1	0	1	–15	71	–105
	1	–15	71	–105	0

$\therefore$ $x^4 - 16x^3 + 86x^2 - 176x + 105 = (x - 1)(x^3 - 15x^2 + 71x - 105)$

Let $Q(x) = x^3 - 15x^2 + 71x - 105$, by trial and error $x = 3$

i.e., root of $Q(x) = 0$ $(\because Q(3) = 27 - 135 + 213 - 105 = 0)$

Now dividing $Q(x)$ by $x - 3$

	1	–15	71	–105
3	0	3	–36	105
	1	–12	35	0

$\therefore$ $Q(x) = (x - 3)(x^2 - 12x + 35)$

$\therefore$ $P(x) = (x - 1)(x - 3)(x^2 - 12x + 35) = (x - 1)(x - 3)(x - 5)(x - 7)$

$\therefore$ The roots of $P(x) = 0$ are 1, 3, 5, 7.

7. If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in H.P. show that $2q^3 = r(3pq - r)$

Sol. Given equation is $x^3 + 3px^2 + 3qx + r = 0$ ... (1)

Let α, β, γ be the roots of (1) and given that $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

i.e., $\alpha\beta + \beta\gamma = 2\alpha\gamma$

from the equation, $\alpha + \beta + \gamma = -3p$

$\alpha\beta + \beta\gamma + \gamma\alpha = 3q$

$\alpha\beta\gamma = -r$

$\alpha\beta + \beta\gamma + \gamma\alpha = 3q \Rightarrow 3\gamma\alpha = 3q \Rightarrow \gamma\alpha = q$

$\therefore$ $\beta = \frac{-r}{q}$

Since β is a root of (1)

$\left(-\frac{r}{q}\right)^3 + 3p\left(\frac{-r}{q}\right)^2 + 3q\left(\frac{-r}{q}\right) + r = 0$

$\Rightarrow$ $-r^3 + 3pqr^2 - 3q^3r + rq^3 = 0$

$\Rightarrow$ $2q^3 = 3pqr^2 - r^3 = r(3pq - r)$ $\therefore$ $2q^3 = r(3pq - r)$

2.7 MULTIPLE ROOTS

A root α of an algebraic equation $f(x) = 0$ is said to be a multiple root of order m if $f(x) = (x - \alpha)^m Q(x)$ for some $Q(x)$.

THEOREM-2.11 **If α is a multiple root of $f(x) = 0$ then $f(\alpha) = 0$ and $f'(\alpha) = 0$ where f' is the derivative of f.**

2.8 SOME IMPORTANT RESULTS TO REMEMBER

1. If α, β, γ are the roots of $ax^3+bx^2+cx+d = 0$ then

   i) $S_2 = \alpha^2 + \beta^2 + \gamma^2 = s_1^2 - 2s_2 = \dfrac{b^2 - 2ac}{a^2}$

   ii) $S_3 = \alpha^3 + \beta^3 + \gamma^3 = s_1^2 - 3s_1s_2 + 3s_3 = \dfrac{3abc - b^3 - 3a^2d}{a^3}$

   iii) $S_4 = \alpha^4 + \beta^4 + \gamma^4 = s_1^4 - 4s_1^2s_2 + 4s_1s_3 + 2s_2^2 = \dfrac{b^4 - 4ab^2c + 4a^2bd + 2a^2c^2}{a^4}$

2. If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

   i) $S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = s_1^2 - 2s_2 = \dfrac{b^2 - 2ac}{a^2}$

   ii) $S_3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 = s_1^2 - 3s_1s_2 + 3s_3 = \dfrac{3abc - b^3 - 3a^2d}{a^3}$

   iii) $S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 = s_1^4 - 4s_1^2s_2 + 4s_1s_3 + 2s_2^2 - 4s_4$

   $$= \frac{b^4 - 4ab^2c + 4a^2bd + 2a^2c^2 - 4a^3e}{a^4}$$

SOLVED EXAMPLES

1. Given that the sum of two roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero. Find the roots of the equation.

Sol. Let α, β, γ and δ be the roots of the given equation and $\alpha + \beta = 0$.

Now $s_1 = \alpha + \beta + \gamma + \delta = 2 \Rightarrow \gamma + \delta = 2$

Now $x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 - (\alpha+\beta)x + \alpha\beta)(x^2 - (\gamma+\delta)x + \gamma\delta)$

$\Rightarrow x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + \alpha\beta)(x^2 - 2x + \gamma\delta)$

Let $\alpha\beta = m$, $\gamma\delta = n$

Then $x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + m)(x^2 - 2x + n)$

On comparing both sides, $m + n = 4$; $-2m = 6$

$\Rightarrow m = -3$ and $n = 7$

$\therefore x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 - 3)(x^2 - 2x + 7)$

$\therefore$ The roots of the given equation are the roots of $x^2 - 3 = 0$ and $x^2 - 2x + 7 = 0$ which are $-\sqrt{3}, \sqrt{3}, 1 + i\sqrt{6}, 1 - i\sqrt{6}$.

2. Solve $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ given that the product of two of its roots is 3.

Sol. Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation given.

$s_1 = \alpha + \beta + \gamma + \delta = 5$

$s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 5$

$s_3 = \alpha\beta\gamma + \alpha\beta\delta + \gamma\delta\alpha + \gamma\delta\beta = -5$

$s_4 = \alpha\beta\gamma\delta = -6$

Let $\alpha\beta = 3$ then from s_4, $\gamma\delta = -2$

Now from s_3, we get $3(\gamma + \delta) - 2(\alpha + \beta) = -5$ and from s_1, $(\gamma + \delta) + (\alpha + \beta) = 5$

on solving these two equations on $\alpha + \beta, \gamma + \delta$ we get $\alpha + \beta = 4, \gamma + \delta = 1$

$\therefore \alpha + \beta = 4, \alpha\beta = 3$ and $\gamma + \delta = 1, \gamma\delta = -2$

$\Rightarrow \gamma = 2, \delta = -1$ or $\gamma = -1, \delta = 2$ and $\alpha = 3, \beta = 1$ or $\alpha = 1, \beta = 3$.

$\therefore$ The roots of the given equation are 1, 3, 2, –1.

3. Solve $x^3 - x^2 - 8x + 12 = 0$ if it has a multiple root.

Sol. Let $f(x) = x^3 - x^2 - 8x + 12 = 0$ then $f'(x) = 3x^2 - 2x - 8 = (x - 2)(3x + 4)$

Clearly $f(2) = 0$ and $f'(2) = 0$

$\therefore$ 2 is multiple root of $f(x) = 0$

Let the remaining root of $f(x) = 0$ be α then the sum of roots $= 2 + 2 + \alpha = 1$

$\Rightarrow \alpha = -3$

$\therefore$ The roots of the equation are 2, 2, –3

4. Find the multiple roots of $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$

Sol. Let $f(x) = x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12$

by trial and error $x = 1$ is a root of $f(x) = 0$ on dividing $f(x)$ by $x - 1$

$$\begin{array}{c|cccccc} & 1 & -3 & -5 & 27 & -32 & 12 \\ 1 & 0 & 1 & -2 & -7 & 20 & -12 \\ \hline & 1 & -2 & -7 & 20 & -12 & \boxed{0} \end{array}$$

$\therefore f(x) = (x - 1)\,Q(x)$

where $Q(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$

again by trial and error $x = 1$ is a root of $f(x) = 0$

Now dividing $Q(x)$ by $x - 1$ gives

$$\begin{array}{c|ccccc} & 1 & -2 & -7 & 20 & -12 \\ 1 & 0 & 1 & -1 & -8 & 12 \\ \hline & 1 & -1 & -8 & 12 & \boxed{0} \end{array}$$

Now $Q(x) = (x - 1)\,g(x)$ where $g(x) = x^3 - x^2 - 8x + 12$

Now $g'(x) = 3x^2 - 2x - 8 = (3x + 4)(x - 2)$

Clearly $g(2) = 0, g'(2) = 0$

$\therefore x = 2$ is also a multiple root of $f(x) = 0$

$\therefore$ The multiple roots of $f(x) = 0$ are $x = 1, x = 2$

5. Show that the condition that $ax^4+bx^3+cx^2+dx+e=0$ may have two pairs of equal roots is $ad^2=b^2e$

Sol. Let the roots of given equation be $\alpha, \alpha, \beta, \beta$

Then $s_1 = 2(\alpha+\beta) = -b/a$, $s_3 = \alpha^2\beta+\alpha^2\beta+\beta^2\alpha+\beta^2\alpha = -\frac{d}{a}$, $s_4 = \alpha^2\beta^2 = \frac{e}{a}$

from s_3, $2(\alpha\beta)(\alpha+\beta) = -\frac{d}{a}$

$\Rightarrow 4\alpha^2\beta^2(\alpha+\beta)^2 = \frac{d^2}{a^2}$

$\Rightarrow 4\frac{e}{a}\left(\frac{-b^2}{4a^2}\right) = \frac{d^2}{a^2} \Rightarrow b^2e = ad^2$

EXERCISE - 2.1

1. Find the quotient and the remainder when

*i) $x^4-6x^3+3x^2+26x-24$ is divided by $x-4$. [**Ans :** $q(x) = x^3-2x^2-5x+6$, $R=0$]

ii) $x^4-16x^3+86x^2-176x+105$ is divided by $x-1$. [**Ans :** $q(x) = x^3-15x^2+71x-105$, $R=0$]

iii) $x^4-16x^3+86x^2-176x+105$ is divided by x^2-8x+7. [**Ans :** $q(x) = x^2-8x+15$, $R=0$]

iv) $x^4+x^3-25x^2+41x+66$ is divided by $x^2-6x+11$. [**Ans :** $q(x) = x^2+7x+6$, $R=0$]

*v) $2x^5-3x^4+5x^3-3x^2+7x-9$ is divided by x^2-x-3.

[**Ans :** $q(x) = 2x^3-x^2+10x+4 \leftrightarrow R(x) = 41x+3$]

2. Form the equation of lowest degree whose roots are

i) $1\pm\sqrt{3}, 2, 5$ [**Ans :** $x^4-9x^3+22x^2-6x-20=0$]

ii) $1\pm i, -1\pm i$ [**Ans :** $x^4+4=0$]

iii) $2\pm 3i, 1\pm i$ [**Ans :** $x^4-6x^3+23x^2-34x+26=0$]

***iv) $0, 0, 2, 2, -2, -2$ **(March-17, 18)** [**Ans :** $x^6-8x^4+16x^2=0$]

v) $4\pm\sqrt{3}, 2\pm i$ [**Ans :** $x^4-12x^3+50x^2-92x+65=0$]

3. Solve the equation

i) $3x^3-4x^2+x+88=0$, one root being $2+\sqrt{7}i$ [**Ans :** $-\frac{8}{3}, 2+\sqrt{7}i$]

*ii) $9x^3-15x^2+7x-1=0$, two of the roots being equal. [**Ans :** $\frac{1}{3}, \frac{1}{3}, 1$]

iii) $4x^3-24x^2+23x+18=0$, the roots being in AP [**Ans :** $-\frac{1}{2}, 2, \frac{9}{2}$]

iv) $x^3-7x^2+14x-8=0$, the roots being in GP [**Ans :** $1, 2, 4$]

v) $15x^3-23x^2+9x-1=0$, the roots being in HP [**Ans :** $1, \frac{1}{3}, \frac{1}{5}$]

4. Solve the equation

i) $x^3 - 9x^2 + 14x + 24 = 0$, two of the roots are in the ratio 3 : 2. **[Ans : 6, 4, −1]**

ii) $2x^3 + 3x^2 - 8x + 3 = 0$, one root is double the another root. **[Ans :** $\frac{1}{2}, 1, -3$**]**

iii) $8x^3 - 20x^2 + 6x + 9 = 0$, given that there is a multiple root. **[Ans :** $\frac{3}{2}, \frac{3}{2}, \frac{-1}{2}$**]**

iv) Show that $x^5 - 5x^3 + 5x^2 - 1 = 0$, has three equal roots and find that root.

*v) $1, 1, \alpha$ are the roots of $x^3 - 6x^2 + 9x - 4 = 0$ then find α **(March-18)** **[Ans : 6]**

*vi) If $-1, 2, \alpha$ are the roots of $2x^3 + x^2 - 7x - 6 = 0$ then find α **(May-18)**

*vii) If the product of the roots of $4x^3 + 16x^2 - 9x - a = 0$ is 9. Find a **(March-19)** **[Ans : 36]**

5. Solve the equation

*i) $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal in magnitude but opposite in sign.
[Ans : $\pm\sqrt{3}, \frac{3}{4}, \frac{-1}{2}$**]**

*ii) $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$, if it has two pairs of equal roots **[Ans : −3, 1, −3, 1]**

*iii) $x^4 + x^3 - 16x^2 - 4x + 48 = 0$, the product of two of the roots being 6. **(March-19)**
[Ans : ±2, 3, −4]

*iv) $18x^3 + 81x^2 + 121x + 60 = 0$, given that one root is equal to half of the sum of remaining roots. **(March-19)**

6. Solve

i) $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that $1 - \sqrt{5}$ is a root. **[Ans :** $1 \pm \sqrt{5}, 1, 2$**]**

ii) $x^4 - 4x^2 + 8x + 35 = 0$, given that $2 + i\sqrt{3}$ is a root. **[Ans :** $2 \pm i\sqrt{3}, -2 \pm i$**]**

iii) $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$, one root being $\sqrt{2} + \sqrt{5}$. **[Ans :** $\pm\sqrt{2} \pm \sqrt{5}, \frac{4}{3}$**]**

iv) $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $-\sqrt{3}$ and the other $1 + 2i$. **[Ans :** $\pm\sqrt{3}, 1 \pm 2i, -1$**]**

7. Find the multiple roots of

i) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$ **[Ans :** $3, 3, \pm 2i$**]**

*ii) $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$ **[Ans : 2, 2, 1, 1, −3]**

*iii) $3x^4 + 16x^3 + 24x^2 - 16 = 0$ **[Ans : −2, −2, −2, 2/3]**

*8. Prove that the condition for the roots of equation $x^3 + 3lx^2 + 3mx + n = 0$ may be in

i) AP is $2l^3 - 3lm + n = 0$ ii) GP is $l^3 n = m^3$ iii) HP is $2m^3 = n(3lm - n)$

*9. Show that the condition that sum of two roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ is equal to the sum of the other two is $a^3 + 8c = 4ab$.

*10. Show that the equation $\frac{A_1^2}{x-\beta_1} + \frac{A_2^2}{x-\beta_2} + \frac{A_3^2}{x-\beta_3} + \ldots + \frac{A_k^2}{x-\beta_k} = x - a$ where $A_1, A_2, A_3, \ldots, \beta_1, \beta_2, \beta_3, \ldots, a$ are all real numbers, cannot have a non real (i.e., imaginary) root.

*11. If α, β, γ are the roots of equation $x^3 + px^2 + qx + r = 0$ then find the values of

i) $\alpha^2+\beta^2+\gamma^2$ [Ans : $p^2 - 2q$]

ii) $\alpha^3+\beta^3+\gamma^3$ [Ans : $-p^3 + 3pq - 3r$]

iii) $\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2$ [Ans : $q^2 - 2pr$]

iv) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$ [Ans : $-pq + r$]

v) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ [Ans : $-\frac{q}{r}$]

vi) $\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2}$ [Ans : $\frac{p^2-2q}{r^2}$]

vii) $\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$ [Ans : $\frac{3r-pq}{r}$]

viii) $\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3$ [Ans : $q^3 - 3pqr + 3r^2$]

*12. If α, β, γ are roots of $x^3 - 5x^2 + 6x - 7 = 0$ then find

i) $\sum\alpha^2\beta^2$ ii) $\sum(\alpha^2\beta+\alpha\beta^2)$ [Ans : (i) -34 (ii) 9]

2.9 — TRANSFORMATION OF EQUATIONS

THEOREM-2.12 **The equation whose roots are those of the equation $f(x) = 0$ with contrary signs is $f(-x) = 0$**

Proof : If α is a root of $f(x) = 0 \Leftrightarrow f(\alpha)=0$

$\Leftrightarrow f(-(-\alpha))=0 \Rightarrow -\alpha$ is a root of $f(-x) = 0$

THOEREM-2.13 **The equation whose roots are multiplied by $k(\neq 0)$ of those of the roots of equation $f(x) = 0$ is $f\left(\frac{x}{k}\right)=0$.**

Remember : *All the expressions are homogeneous expressions*

Proof : α is a root of $f(x) = 0 \Leftrightarrow f(\alpha)=0$

$\Leftrightarrow f\left(\frac{k\alpha}{k}\right)=0 \Rightarrow k\alpha$ is a root of $f\left(\frac{x}{k}\right)=0$

THEOREM-2.14 **The equation whose roots are reciprocals of the roots of $f(x) = 0$ is $f\left(\frac{1}{x}\right)=0$.**

Proof : α is a root of $f(x) = 0 \Leftrightarrow f(\alpha)=0$.

$\Leftrightarrow f\left(\frac{1}{1/\alpha}\right)=0 \Rightarrow \frac{1}{\alpha}$ is a root of $f\left(\frac{1}{x}\right)=0$.

THEOREM-2.15 **The equation whose roots exceed by h than those of $f(x) = 0$ is $f(x - h) = 0$.**

Proof : α is a root of $f(x) = 0 \Leftrightarrow f(\alpha)=0$

$\Leftrightarrow f(\alpha+h-h)=0 \Leftrightarrow \alpha+h$ is a root of $f(x - h) = 0$

Corollary :

If $\alpha_1, \alpha_2, \alpha_3, ...,\alpha_n$ are the roots of the polynomial equation $f(x) = 0$ then $\alpha_1 - h, \alpha_2 - h, ...,\alpha_n - h$ are the roots of the equation $f(x + h) = 0$.

Note

i) *Let $f(x)$ be a polynomial of degree $n > 0$. Then $f(x + h)$ is evidently a polynomial of degree n. Hence there exist unique constants $b_0, b_1, ..., b_n$ such that $f(x + h) = b_0x^n + b_1x^{n-1} + ... + b_{n-1}x + b_n$. on replacing x by $x - h$, we obtain $f(x) = b_0(x - h)^n + b_1(x - h)^{n-1} + ... + b_{n-1}(x - h) + b_n$*

For $k = 0, 1, 2, ... n$, let $Q_k(x) = b_0(x - h)^{n-k} + b_1(x - h)^{n-k-1} + ... + b_{n-k}$

Then $Q_0(x) = f(x)$, $Q_n(x) = b_0$ and $Q_k(x) = (x - h)Q_{k+1}(x) + b_{n-k}$, $k = 0, 1, ..., n$

Thus b_{n-k} is the remiander and $Q_{k+1}(x)$ is the quotient that we obtain when we divide $Q_k(x)$ with $x - h$.

Since $Q_0(x) = f(x)$, beginning $k = 0$, by dividng $Q_k(x)$ with $x - h$, we can determine $Q_{k+1}(x)$ and b_{n-k} in a successive manner by using synthetic division. Thus we can determine the constants $b_n, b_{n-1}, ..., b_0$ in that order. This is explained below with an example.

Suppose that $f(x) = x^5 + 4x^3 - x^2 + 11$ and $h = 3$

Let $f(x + 3) = b_0x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5$

3	1	0	4	–1	0	11
	0	3	9	39	114	342
	1	3	13	38	114	$353 = b_5$
	0	3	18	93	393	
	1	6	31	131	$507 = b_4$	
	0	3	27	174		
	1	9	58	$305 = b_3$		
	0	3	36			
	1	12	$94 = b_2$			
	0	3				
	1	$15 = b_1$				
	0					
	$1 = b_0$					

$\therefore$ $f(x + 3) = x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353$

This is known as Horner's process.

ii) *$f(x + h)$ can also be calculated using an important thoerem from real analysis i.e., Taylor's theorem given below*

TAYLOR'S THEOREM

If $f(x)$ is a polynomial function of degree n, then

$$f(x + h) = f(h) + \frac{x}{\lfloor 1} f^{I}(h) + \frac{x^2}{\lfloor 2} f^{II}(h) + ... + \frac{x^n}{\lfloor n} f^{(n)}(h).$$

Example : Let $f(x) = x^5 + 4x^3 - x^2 + 11$ and $h = 3$ then

$$f(x + 3) = f(3) + \frac{x}{\lfloor 1} f^{I}(3) + \frac{x^2}{\lfloor 2} f^{II}(3) + \frac{x^3}{\lfloor 3} f^{III}(3) + \frac{x^4}{\lfloor 4} f^{iv}(3) + \frac{x^5}{\lfloor 5} f^{v}(3)$$

$f(3) = 3^5 + 4.3^3 - 3^2 + 11 = 353$

$f^{i}(3) = 5.3^4 + 12.3^2 - 2.3 = 507$

$f^{ii}(3) = 20.3^3 + 24.3 - 2 = 305 \times 2$

$f^{iii}(3) = 60.3^3 + 24 = 94 \times 6$

$f^{iv}(3) = 120 \times 3 = 15 \times 24$

$f^{v}(3) = 120$

$\therefore$ $f(x + 3) = 353 + 507x + 305x^2 + 94x^3 + 15x^4 + x^5$

THEOREM-2.16 **If $f(x) = 0$ is an equation of degree n then to eliminate r^{th} term, $f(x) = 0$ can be transformed to $f(x + h) = 0$ where h is a constant such that $f^{(n-r+1)}(h) = 0$.**

Proof : Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n = 0$ be an equation of degree n and by Taylor's theorem. we have

$f(x + h) = f(h) + x f^1(h) + \frac{x^2}{\lfloor 2} f^{11}(h) + ...+ \frac{x^n}{\lfloor n} f^{(n)}(h)$ For r^{th} term to be eliminated in

$$f^{(n)}(h)\frac{x^n}{\lfloor n} + f^{(n-1)}(h)\frac{x^{n-1}}{\lfloor n-1} + ... + f^{(n-r+1)}(h)\frac{x^{n-(r-1)}}{\lfloor n-r-1} + ... + f^1(h)x +$$

$$f(h)f^{(n-r+1)}(h) = 0$$

Example :

Remove 2^{nd} term in $x^4 + 8x^3 + x - 5 = 0$

We have to translate the equation to $f(x + h) = 0$

$\because f^{(3)}(h) = 0 \quad (n = 4, r = 2)$

$f^1(x) = 4x^3 + 24x^2 + 1$

$f^{11}(x) = 12x^2 + 48x$

$f^{111}(x) = 24x + 48$

$f^{111}(h) = 0 \Rightarrow h = -2$

$\therefore f(x - 2) = 0$ can be obtained by Horners process.

–2	1	8	0	1	–5
	0	–2	–12	24	–50
	1	6	–12	25	–55
	0	–2	–8	40	
	1	4	–20	65	
	0	–2	–4		
	1	2	–24		
	0	–2			
	1	0			

$f(x - 2) = x^4 - 24x^2 + 65x - 55$ in which x^3 term is not present. (i.e. 2^{nd} term absent)

THEOREM-2.17 **The equation whose roots are the squares of the roots of $f(x) = 0$ is obtained by eliminating radical sign from $f(\sqrt{x}) = 0$**

Ex : **Find the equation whose roots are squares of the roots of $x^3 + x^2 - 2x + 5 = 0$**

Sol. The required equation is $(\sqrt{x})^3 + (\sqrt{x})^2 - 2\sqrt{x} + 5 = 0$

i.e., $x\sqrt{x} + x - 2\sqrt{x} + 5 = 0 \Rightarrow (x - 2)\sqrt{x} = -(x + 5)$

on squaring we get $(x^2 - 4x + 4)x = x^2 + 10x + 25$

$\Rightarrow x^3 - 5x^2 - 6x - 25 = 0$

SOLVED EXAMPLES

Remember :
If $\alpha_1, \alpha_2, \alpha_3$ are roots of $f(x) = 0$ then $-\alpha_1, -\alpha_2, -\alpha_3$, ... are roots of $f(-x) = 0$

1. Find the equation whose roots are those of the equation $x^7 + 3x^5 + x^3 - x^2 + 7x + 2 = 0$ with contrary signs.

Sol. The required equation is $(-x)^7 + 3(-x)^5 + (-x)^3 - (-x)^2 + 7(-x) + 2 = 0$

$\Rightarrow -x^7 - 3x^5 - x^3 - x^2 - 7x + 2 = 0$

i.e., $x^7 + 3x^5 + x^3 + x^2 + 7x - 2 = 0$

Remember :
If $\alpha_1, \alpha_2, \alpha_3$ .... are roots of $f(x) = 0$ then $k\alpha_1, k\alpha_2$ ..... are roots of $f\left(\frac{x}{k}\right) = 0$

2. Find the equation whose roots are multiplied by 2 of those of $x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$

Sol. For the required equation replace x by $\frac{x}{2}$ in the given equation.

$\therefore$ The required equation is $\left(\frac{x}{2}\right)^5 - 2\left(\frac{x}{2}\right)^4 + 3\left(\frac{x}{2}\right)^3 - 2\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 3 = 0$

$\therefore$ $x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96 = 0$

3. If α, β, γ are the roots of $x^3 + 4x^2 - 2x - 3 = 0$ then find the equation whose roots are $\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}$.

Sol. The required equation is $(3x)^3 + 4(3x)^2 - 2(3x) - 3 = 0$

i.e., $27x^3 - 36x^2 - 6x - 3 = 0$

$\Rightarrow$ $9x^3 + 12x^2 - 2x - 1 = 0$

Remember :
If α_1, α_2 ...... are roots of $f(x) = 0$ then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}$ ..... are roots of $f\left(\frac{1}{x}\right) = 0$

***4. Find the equation whose roots are the reciprocals of the roots of $x^5 + 11x^4 + x^3 + 4x^2 - 13x + 6 = 0$**

Sol. Replace x by $\frac{1}{x}$ for the required equation.

$\therefore$ The required equation is $\left(\frac{1}{x}\right)^5 + 11\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^3 + 4\left(\frac{1}{x}\right)^2 - 13\frac{1}{x} + 6 = 0$

i.e., $6x^5 - 13x^4 + 4x^3 + x^2 + 11x + 1 = 0$

Remember :
If α_1, α_2 ...... are roots of $f(x) = 0$ then α_1^k, α_2^k ..... are roots of $f(x^{1/k}) = 0$

***5. Find the equation whose roots are squares of the roots of $x^4 + x^3 + 2x^2 + x + 1 = 0$**

Sol. Replace x by $\sqrt{x}$ in the given equation, which gives

$(\sqrt{x})^4 + (\sqrt{x})^3 + 2(\sqrt{x})^2 + \sqrt{x} + 1 = 0$

i.e., $x^2 + 2x + 1 + \sqrt{x}\,(x + 1) = 0 \Rightarrow (x^2 + 2x + 1) = -(x + 1)\sqrt{x}$

On squaring both sides, we obtain $x^4 + 4x^3 + 6x^2 + 4x + 1 = (x^2 + 2x + 1)x$

$\Rightarrow x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$

$\therefore$ The required equation is $x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$

***6. Find the equation whose roots are the cubes of the roots $x^3 + 3x^2 + 2 = 0$.**

Sol. Replace x by $x^{1/3}$ in the given equation $x^3 + 3x^2 + 2 = 0$

Which gives $x + 3x^{2/3} + 2 = 0 \Rightarrow 3x^{2/3} = -(x+2)$

On cubing both sides, we get $27x^2 = (-(x + 2))^3$

$\Rightarrow$ $27x^2 = -(x^3 + 6x^2 + 12x + 8)$

$\Rightarrow x^3 + 33x^2 + 12x + 8 = 0$

$\therefore$ The required equation is $x^3 + 33x^2 + 12x + 8 = 0$

***7. Find the equation whose roots are the translates of the roots of $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$ by -2** **(May-19)**

Sol. Let $f(x) = x^4 - 5x^3 + 7x^2 - 17x + 11$

The required equation is $f(x + 2) = 0$

$\therefore\ f(x+2)=(x+2)^4-5(x+2)^3+7(x+2)^2-17(x+2)+11=0$

$\Rightarrow\ x^4 + 3x^3 + x^2 - 17x - 19 = 0$

$f(x + 2)$ can also be obtained by synthetic division (Horners process)

2	1	–5	7	–17	11
	0	2	–6	2	–20
	1	–3	1	–15	–19
	0	2	–2	–2	
	1	–1	–1	–17	
	0	2	2		
	1	1	1		
	0	2			
	1	3			
	0				
	1				

$\therefore$ The required equation is $x^4+3x^3+x^2-17x-19 = 0$

***8. Find the equation whose roots are the translates of the roots of $x^4 - x^3 + 10x^2 + 4x + 24 = 0$ by 2.**

Sol. Let $f(x) = x^4 - x^3 + 10x^2 + 4x + 24$

The required equation is $f(x - 2) = 0$

Now by Horners process

–2	1	–1	10	4	24
	0	–2	6	–32	56
	1	–3	16	–28	80
	0	–2	10	–52	
	1	–5	26	–80	
	0	–2	14		
	1	–7	40		
	0	–2			
	1	–9			
	0				
	1				

$\therefore\ f(x - 2) = x^4 - 9x^3 + 40x^2 - 80x + 80$

$\therefore$ The required equation is $x^4 - 9x^3 + 40x^2 - 80x + 80 = 0$

Remember :
The equation whose roots are the translates of the roots of $f(x)=0$ by h is $f(x-h) = 0$

***9. Remove the second term from the equation $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$**

Sol. 2^{nd} term means the term with x^3 in $f(x) = x^4 + 4x^3 + 2x^2 - 4x - 2$

For this $f^{(4-2+1)}(h) = 0$ i.e., $f^{111}(h) = 0$

Here $f^1(x) = 4x^3 + 12x^2 + 4x - 4$

$f^{11}(x) = 12x^2 + 24x + 4$

$f^{111}(x) = 24x + 24$

$\therefore f^{111}(x) = 0 \Rightarrow f^{111}(h) = 0 \Rightarrow h = -1$

$\therefore$ The required equation is $f(x-1) = 0$

Now for $f(x-1)$, using Horners process

$$\begin{array}{r|rrrrr}
-1 & 1 & 4 & 2 & -4 & -2 \\
 & 0 & -1 & -3 & 1 & 3 \\
\hline
 & 1 & 3 & -1 & -3 & \boxed{1} \\
 & 0 & -1 & -2 & 3 & \\
\hline
 & 1 & 2 & -3 & \boxed{0} & \\
 & 0 & -1 & -1 & & \\
\hline
 & 1 & 1 & \boxed{-4} & & \\
 & 0 & -1 & & & \\
\hline
 & 1 & \boxed{0} & & & \\
 & 0 & & & & \\
\hline
 & 1 & & & &
\end{array}$$

$\therefore f(x-1) = x^4 + 0x^3 - 4x^2 + 0x + 1$

$\therefore$ The required equation is $x^4 - 4x^2 + 1 = 0$

Remember :
Remove 2^{nd} term means, translate the roots of the given equation $f(x) = 0$ by certain number 'h' so that in the tranformed equation $f(x+h)=0$. 2^{nd} largest power term is missing.

***10. Remove the third term from the equation $x^4 + 2x^3 - 12x^2 + 2x - 1 = 0$.**

Sol. Third term means term with x^2, in $f(x) = 0$

for this $f^{(4-3+1)}(h) = 0$ i.e., $f^{11}(h) = 0$

$f^1(x) = 4x^3 + 6x^2 - 24x + 2$

$f^{11}(x) = 12x^2 + 12x - 24$

$\therefore f^{11}(h) = \quad h^2 + h - 2 = 0 \Rightarrow h = -2, h = 1$

$\therefore$ The required equation is $f(x-2) = 0$, $f(x+1) = 0$

$f(x-2)$

$$\begin{array}{r|rrrrr}
-2 & 1 & 2 & -12 & 2 & -1 \\
 & 0 & -2 & 0 & 24 & -52 \\
\hline
 & 1 & 0 & -12 & 26 & \boxed{-53} \\
 & 0 & -2 & 4 & 16 & \\
\hline
 & 1 & -2 & -8 & \boxed{42} & \\
 & 0 & -2 & 8 & & \\
\hline
 & 1 & -4 & \boxed{0} & & \\
 & 0 & -2 & & & \\
\hline
 & 1 & \boxed{-6} & & & \\
 & 0 & & & & \\
\hline
 & 1 & & & &
\end{array}$$

$f(x-2) = x^4 - 6x^3 + 0x^2 + 42x - 53$

$f(x+1)$

$$
\begin{array}{r|rrrrr}
1 & 1 & 2 & -12 & 2 & -1 \\
 & 0 & 1 & 3 & -9 & -7 \\
\hline
 & 1 & 3 & -9 & -7 & \boxed{-8} \\
 & 0 & 1 & 4 & -5 & \\
\hline
 & 1 & 4 & -5 & \boxed{-12} & \\
 & 0 & 1 & 5 & & \\
\hline
 & 1 & 5 & \boxed{0} & & \\
 & 0 & 1 & & & \\
\hline
 & 1 & \boxed{6} & & & \\
 & 0 & & & & \\
\hline
 & 1 & & & &
\end{array}
$$

$f(x+1) = x^4 + 6x^3 + 0x^2 - 12x - 8$

$\therefore$ The required equations are $x^4 - 6x^3 + 0x^2 + 42x - 53 = 0$ and

$x^4 + 6x^3 + 0x^2 - 12x - 8 = 0$

2.10 RECIPROCAL EQUATIONS

Definition

An equation $f(x) = 0$ is said to be reciprocal equation if $\frac{1}{\alpha}$ is a root of $f(x) = 0$ of multiplicity m wherever α is a root of $f(x) = 0$ of multiplicity m.

THEOREM-2.18 **An equation $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ is a reciprocal equation $\Leftrightarrow$ either $a_i = a_{n-i}$ for every i, or $a_i = -a_{n-i}$ for every i.**

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$

The equation whose roots are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ is $f\left(\frac{1}{x}\right) = 0$

$$\Rightarrow a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \ldots + a_n = 0$$

$$\Rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

Since $f(x) = 0$ is a reciprocal equation, $f(x) = kg(x)$ for some k. where $g(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0$

$\Rightarrow a_0 = ka_n,\ a_1 = ka_{n-1}, \ldots\ldots, a_n = ka_0$

$\Rightarrow a_i = ka_{n-i}\ \forall i$

Now $a_0 = ka_n$ and $a_n = ka_0$

$\Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

$\therefore\ a_i = a_{n-i}$ or $a_i = -a_{n-i}$, $1 \le i \le n$ then $f(x) = 0$, $g(x) = 0$ have the same roots.

$\therefore\ f(x) = 0$ is a reciprocal equation.

THEOREM-2.19 **If $f(x) = 0$ is a reciprocal equation of degree n, then $x^n f\left(\frac{1}{x}\right) = \pm f(x)$.**

Proof : Let $f(x) = a_0x^n + a_1x^{n-1} + ...+ a_n = 0$ be a reciprocal equation.

If $a_i = a_{n-i}$, then $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2}........+ a_n = 0$

$$= x^n\left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a^n}{x^n}\right) = x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + ... + \frac{a_0}{x^n}\right) = x^n f\left(\frac{1}{x}\right)$$

If $a_i = -a_{n-i}$, then $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} .... + a_n$

$$= -x^n\left(-a_0 - \frac{a_1}{x} - \frac{a_2}{x^2} - .... - \frac{a^n}{x^n}\right) = -x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + ... + \frac{a_0}{x^n}\right) = -x^n f\left(\frac{1}{x}\right)$$

$\therefore$ $f(x) = 0$ is a reciprocal equation then $f(x) = \pm x^n f\left(\frac{1}{x}\right)$.

Definition

A reciprocal equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} .... + a_n = 0$ is said to be a reciprocal equation of class one if $a_i = a_{n-1}$ $\forall\, i$ and a reciprocal equation of class two if $a_i = -a_{n-i}$ $\forall\, i$.

Note

i) ***For an odd degree reciprocal equation of class one, -1 is a root and for an odd degree reciprocal equation of class two 1 is a root.***

ii) ***For an even degree reciprocal equation of class two 1 and -1 are roots.***

2.11 SOLVING RECIPROCAL EQUATIONS

Every reciprocal equation can be solved by transforming it into a reciprocal equation of class one and of even degree.

To solve a reciprocal equation of degree $2m$ divide the equation by x^m and put $x + \frac{1}{x} = y$ or $x - \frac{1}{x} = y$ according as the equation is of class one or class two. The degree of the transformed equation is m.

If a reciprocal equation of degree $2m + 1$ is given then divide it by $x + 1$ or $x - 1$ according as the equation is of class one or class two. Then the quotient $Q(x)$ is a reciprocal equation of degree $2m$ for which previous method (as explained above) will be applied.

SOLVED EXAMPLES

1. Show that $2x^3 + x^2 + 5x + 2 = 0$ is a reciprocal equation of class one.

Sol. $f(x) = 2x^3 + 5x^2 + 5x + 2 = x^3\left(2 + \frac{5}{x} + \frac{5}{x^2} + \frac{2}{x^3}\right) = x^3 f\left(\frac{1}{x}\right)$

$\therefore x^3 f\left(\frac{1}{x}\right) = f(x)$

$\therefore f(x) = 0$ is a reciprocal equation of class two.

2. Show that $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$ is reciprocal equation of class two.

Sol. $f(x) = 6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$

$= x^6\left(6 - \frac{35}{x} + \frac{56}{x^2} - \frac{56}{x^4} + \frac{35}{x^5} - \frac{6}{x^6}\right)$

$= -x^6\left(\frac{6}{x^6} - \frac{35}{x^5} + \frac{56}{x^4} - \frac{56}{x^2} + \frac{35}{x} - 6\right) = -x^6 f\left(\frac{1}{x}\right)$

$f(x) = 0$ is a reciprocal equation of class two.

3. Solve $x^4 + 3x^3 - 3x - 1 = 0$

Sol. Given equation is a reciprocal equation of even degree divide the given equation by x^2, we get $x^2 + 3x - \frac{3}{x} - \frac{1}{x^2} = 0$

$\Rightarrow \left(x^2 - \frac{1}{x^2}\right) + 3\left(x - \frac{1}{x}\right) = 0 \Rightarrow \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x} + 3\right) = 0$

$\Rightarrow x^2 - 1 = 0, x^2 + 3x + 1 = 0 \Rightarrow x = 1, -1, \frac{-3 \pm \sqrt{5}}{2}$

4. Solve $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$

Sol. Given equation is a reciprocal equation of class two and of odd degree. divide the equation by $x - 1$.

1	6	–1	–43	43	1	–6
	0	6	5	–38	5	6
	6	5	–38	–5	6	$\boxed{0}$

$\therefore 6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = (x - 1)(6x^4 + 5x^3 - 38x^2 + 5x + 6)$

Let $Q(x) = 6x^4 + 5x^3 - 38x^2 + 5x + 6$ which is a reciprocal equation of even degree.

Divide $Q(x) = 0$ by x^2,

we get $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0 \Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

Put $x + \frac{1}{x} = y$ then the equation will be transformed to

$6(y^2 - 2) + 5y - 38 = 0 \Rightarrow 6y^2 + 5y - 50 = 0$

$(3y + 10)(2y - 5) = 0 \Rightarrow y = -\frac{10}{3}, y = \frac{5}{2} \Rightarrow x + \frac{1}{x} = -\frac{10}{3}; x + \frac{1}{x} = \frac{5}{2}$

$3x^2 + 10x + 3 = 0$; $2x^2 - 5x + 2 = 0$

$x = -3, -1/3; x = 2, 1/2$

$\therefore$ The roots of the given equations are $1, -3, -1/3, 2, -1/2$

> ***Remember :***
> *Reciprocal equation of class two and of odd degree will have '1' as a root*

5. * Show that the equation $\frac{a^2}{x-a'}+\frac{b^2}{x-b'}+\frac{c^2}{x-c'}+.....+\frac{k^2}{x-k'}=x-m$ where a's and a's are all real numbers, cannot have non real root.

Sol. Let us suppose that the equation has only imaginary roots

$\therefore$ Let $\alpha \pm i\beta$ be the pair of conjugate roots to the equation.

$$\therefore \frac{a^2}{(\alpha-a')+i\beta}+\frac{b^2}{(\alpha-b')+i\beta}+\frac{c^2}{(\alpha-c')+i\beta}+....\frac{k^2}{(\alpha-k')+i\beta}=(\alpha-m)+i\beta \quad ..(1)$$

$$\text{Again}\frac{a^2}{(\alpha-a')-i\beta}+\frac{b^2}{(\alpha-b')-i\beta}+\frac{c^2}{(\alpha-c')-i\beta}+....$$

$$\frac{k^2}{(\alpha-k')-i\beta}=(\alpha-m)-i\beta \quad ....(2)$$

$(2)-(1) \Rightarrow$

$$a^2\left[\frac{\alpha i\beta}{(\alpha-a')^2+\beta^2}\right]+b^2\left[\frac{\alpha i\beta}{(\alpha-b')^2+\beta^2}\right]+...+k^2\left[\frac{2i\beta}{(\alpha-k')^2+\beta^2}\right]=-2i\beta$$

$$\therefore 2i\beta\left[\begin{array}{l}1+\frac{a^2}{(\alpha-a')^2+\beta^2}+\frac{b^2}{(\alpha-b')^2+\beta^2}+...\\ \qquad +\frac{k^2}{(\alpha-k')^2+\beta^2}\end{array}\right]=0 \Rightarrow \beta=0$$

But this is a contradiction for our assumption.

$\therefore$ The equation cannot have non real roots.

Hence the result.

Remove the fractional coefficients from the following equations such that the coefficient of the leading term remains unity.

6. $x^3-\frac{3}{2}x^2-\frac{1}{16}x+\frac{1}{32}=0$

Sol. The transformed equation of the above equation is $f\left(\frac{y}{m}\right)=0$.

$$\Rightarrow \left(\frac{y}{m}\right)^3-\frac{3}{2}\left(\frac{y}{m}\right)^2-\frac{1}{16}\left(\frac{y}{m}\right)+\frac{1}{32}=0$$

$$y^3-\frac{3}{2}y^2m-\frac{1}{16}ym^2+\frac{m^3}{32}=0;\ y^3-\frac{3m}{2^1}y^2-\frac{m^2}{2^4}y+\frac{m^3}{2^5}=0$$

The exponents of 2 in order are 1, 4, 5.

Dividing them with the correspnding powers of m, we get 1, 2, $\frac{5}{3}$.

Now, the least integer not less than any quotient is '2'.

$\therefore m=2^2=4$

$\therefore$ Transformed equation is $x^3-6x^2-x+2=0$

7. $x^5-\frac{1}{3}x^4+\frac{25}{27}x^2+\frac{14}{81}x-\frac{8}{81}=0$

Sol. Put $x=\frac{y}{m}$ $\therefore$ $y^5-\frac{m}{3}y^4+\frac{25m^3}{3^3}y^2+\frac{14m^4}{3^4}y-\frac{8m^5}{3^4}=0$

The exponents of 3 are 1, 3, 4, 4.

Dividing with corresponding powers of m, we get 1, 1, 1, $\frac{4}{5}$.

$\therefore m=3^1$.

$\therefore$ Transformed equation is $x^5-x^4+25x^2+14x-24 = 0$

8. $x^4 + \frac{3}{10}x^2 + \frac{13}{25}x + \frac{77}{1000} = 0$

Sol. Put $x = \frac{y}{m}$

$\therefore\ y^4 + \frac{3m^2}{2.5}y^2 + \frac{13m^3}{2^0.5^2}y + \frac{77m^4}{5^3.2^3} = 0$

$1, 0, 3 \Rightarrow \frac{1}{2}, \frac{0}{3}, \frac{3}{4}$

$1, 2, 3 \Rightarrow \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

$\therefore\ m = 2^1 \cdot 5^1 \qquad \therefore\ m = 10$

$\therefore$ equation is $x^4 + 30x^2 + 20x + 770 = 0$

EXERCISE - 2.2

1. Find the equation whose roots are 2 times the roots of $x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$
   [**Ans :** $x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96 = 0$]

*2. Form the equation whose roots are m times the roots of the equation $x^3 + \frac{x^2}{4} - \frac{x}{16} + \frac{1}{72} = 0$ and deduce the case when $m = 4$ [**Ans :** $x^3 + \frac{m}{4}x^2 - \frac{m^2}{16}x + \frac{m^3}{72}x$, $9(x^3 + x^2 - x) + 8 = 0$]

3. Find the equation whose roots are the reciprocals of the roots of
   i) $x^4 + 3x^2 - 6x^2 + 2x - 4 = 0$ [**Ans :** $4x^4 - 2x^3 + 6x^2 - 3x - 1 = 0$]
   ii) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ [**Ans :** $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$]

4. Find the equation whose roots are
   i) squares of the roots of
      a) $x^3 + 2x^2 - x + 5 = 0$ [**Ans :** $x^3 - 6x^2 - 11x - 9 = 0$]
      b) $x^3 + 3x^2 - 7x + 6 = 0$ [**Ans :** $x^3 - 23x^2 + 13x - 36 = 0$]
   ii) cubes of the roots of
      a) $x^3 - 2x^2 + x + 2 = 0$ [**Ans :** $x^3 + 4x^2 + 25x + 8 = 0$]
      b) $x^3 + 3x^2 + 2 = 0$ [**Ans :** $x^3 + 33x^2 + 12x + 8 = 0$]

5. Find the equation whose roots are the translates of the roots of
   i) $2x^3 + 3x^2 - 4x + 5 = 0$ by 2 [**Ans :** $2x^3 - 9x^2 + 8x + 9 = 0$]
   ii) $x^4 - x^3 + 10x^2 + 4x + 24 = 0$ by 2 [**Ans :** $x^4 - 9x^3 + 40x^2 - 80x + 80 = 0$]
   *iii) $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$ by 2 [**Ans :** $4x^4 - 13x^2 + 9 = 0$]
   iv) $x^5 - 4x^4 + 3x^2 - 4x + 6$ by -3 [**Ans :** $x^5 + 11x^4 + 45x^3 + 81x^2 + 50x - 6 = 0$]
   *v) $3x^5 - 5x^3 + 7 = 0$ by 4 [**Ans :** $3x^5 - 60x^4 + 475x^3 - 1860x^2 + 3600x - 2745 = 0$]

6. Remove the second term from the equation

   i) $x^3 - 6x^2 + 4x - 7 = 0$ [Ans : $x^3 - 8x - 15 = 0$]

   ii) $x^4 + 8x^3 + x - 5 = 0$ [Ans : $x^4 - 24x^2 + 65x - 55 = 0$]

7. Remove the third term from the equation

   i) $x^3 + 2x^2 + x + 1 = 0$ [Ans : $x^3 - x^2 + 1 = 0$ (or) $27x^3 + 27x^2 + 27 = 0$]

   *ii) $x^4 + 2x^3 - 12x^2 + 2x - 1 = 0$ [Ans : $x^4 + 6x^3 - 12x - 8 = 0$ or $x^4 - 6x^3 + 42x - 53 = 0$]

8. Solve the following reciprocal equations.

   i) $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ (May-18**) [Ans : $2 \pm \sqrt{3}, 3 \pm 2\sqrt{2}$]

   *ii) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ [Ans : $\frac{1}{3}, \frac{1}{2}, 2, 3$]

   iii) $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ (March-18**) [Ans : $\pm 1, \frac{1}{2}, 2, \frac{5 \pm i\sqrt{11}}{6}$]

   *iv) $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ (**March-15**) [Ans : $1, \frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{5}}{2}$]

   *v) $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$ [Ans : $-1, -2, \frac{-1}{2}, \frac{3 \pm \sqrt{5}}{2}$]

*9. Show that $x^5 - 5x^3 + 5x^2 - 1 = 0$, has three equal roots and find the roots. (**March-18**)

SOLVED EXAMPLES

1. The roots of the equation $x^3 - x^2 + ax + b = 0$ are real and are in A.P. find the intervals in which a and b lie.

Sol. If x_1, x_2, x_3 be the roots of the given equation, then we have

$x_1 + x_2 + x_3 = 1$ [$\because$ sum of the roots = 1 .... (1)

and $x_1 + x_3 = 2x_2$ [$\because$ roots are in A.P.] .... (2)

Solving equations (1) and (2), we have $x_2 = \frac{1}{3}$

Hence, one of the roots of the given equation is 1/3 and therefore putting $x = 1/3$ in the given equation, we have

$$\frac{1}{27} - \frac{1}{9} + a\left(\frac{1}{3}\right) + b = 0 \text{ i.e., } a + 3b = \frac{2}{9} \quad .... (3)$$

The given equation can now be written as $\left(x - \frac{1}{3}\right)\left\{x^2 - \frac{2}{3}x + \left(a - \frac{2}{9}\right)\right\} = 0$ ....(4)

Since the roots of equation (4) are given to be real, therefore

$$D = \frac{4}{9} - 4\left(a - \frac{2}{9}\right) \geq 0 \qquad \therefore \boxed{a \leq \frac{1}{3}} \quad .... (5)$$

From results (3) and (5), we have

$$\frac{2}{9} - 3b \leq \frac{1}{3} \qquad \therefore \boxed{b \geq -\frac{1}{27}}$$

2. Find the condition for the biquadratic equation $px^4 + 4qx^3 + 6rx^2 + 4sx + t = 0$ may have two pairs of equal roots.

Sol. **Method - 1**

Let $\alpha, \alpha, \beta, \beta$ be the roots. Then

$$S_1 = 2(\alpha+\beta) = -\frac{4q}{p} \quad (\alpha+\beta) = -\frac{2q}{p} \qquad \text{... (1)}$$

$$S_2 = \alpha^2 4\alpha\beta^2 = \frac{6r}{p}$$

$$i.e., \alpha^2 + 2\alpha\beta + \beta^2 + 2\alpha\beta = \frac{6r}{p}$$

which, on using (1) gives $\frac{4q^2}{p^2} + 2\alpha\beta = \frac{6r}{p}$ or $2\alpha\beta = \frac{6r}{p} - \frac{4q^2}{p^2}$ ... (2)

$$S_3 = \alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = -\frac{4s}{p} \text{ or } 2\alpha\beta(\alpha+\beta) = -\frac{4s}{p},$$

which, on using (1) gives $2\alpha\beta\left(\frac{-2q}{p}\right) = -\frac{4s}{p}$ *or* $\alpha\beta = \frac{s}{q}$ ... (3)

$$S_4 = \alpha^2\beta^2 = \frac{t}{p} \qquad \text{... (4)}$$

(1) has already been used and by eliminating α and β between (2), (3) and (4) we get the required conditions.

Remember :

A biquadratic having two pairs of equal roots can be expressed as the square of a quadratic expression

From (2) and (3) $\frac{6r}{p} - \frac{4q^2}{p^2} = \frac{2s}{q}$ *or*

$$p^2s = 3qpr - 2q^3 \qquad \text{... (5)}$$

From (3) and (4) $\frac{s^2}{q^2} = \frac{t}{p}$ *or* $ps^2 = tq^2$ ... (6)

$\therefore$ Conditions (5) and (6) are the required ones.

Method - 2

$$2(\alpha+\beta) = s_1 = \frac{-4q}{p}$$

Equation with $\alpha, \alpha, \beta, \beta$ as roots is

$(x - \alpha)^2(x - \beta)^2 = 0$ or $[x^2 - (\alpha+\beta)x + (\alpha\beta)]^2 = 0$ or $[x^2 + \frac{2q}{p}x + b]^2 = 0$

where $b = \alpha\beta$

$\therefore$ We may write $px^4 + 4qx^3 + 6rx^2 + 4sx + t \equiv p\left\{x^2 + \frac{2q}{p}x + b\right\}^2$ ... (1)

$$\equiv p\left\{x^4 + \frac{4q}{p}x^3 + \left(\frac{4q^2}{p^2}\right) + 2bx^2 + \frac{4qb}{p}x + b^2\right\}$$

(Note that the factor p in the right side of (1) is due to coefficient of x^4 on the left side being p and not 1)

$$\equiv px^4 + 4qx^3 + \left(\frac{4q^2}{p} + 2bp\right)x^2 + 4qbx + pb^2$$

Comparing x^2, x coefficients and constants on both sides, we have

$$\frac{4q^2}{p} + 2pb = 6r \quad or \quad b = \frac{3r}{p} - \frac{2q^2}{p^2} \qquad \dots (1)$$

$$4qb = 4s \text{ or } b = \frac{s}{q} \qquad \dots (2)$$

$$pb^2 = t \quad \text{or} \quad b^2 = \frac{t}{p} \qquad \dots (3)$$

Eliminating b between (1), (2) and (3) we get two conditions as follows :

From (1) and (2)

$$\frac{s}{q} = \frac{3r}{p} - \frac{2q^2}{p^2} \quad \text{or} \quad p^2 s = 3pqr - 2q^3 \qquad \dots. (4)$$

From (2) and (3) $\frac{s^2}{q^2} = \frac{t}{p} \qquad or \qquad ps^2 = tq^2 \qquad \dots (5)$

3. Solve $x^3 - 13x^2 + 15x + 189 = 0$ given that two of its roots differ by 2.

Sol. Taking the roots as $\alpha, \alpha+2, \beta$ we have

$s_1 = 2\alpha + 2 + \beta = 13 \quad \text{or} \quad \beta = 13 - 2(\alpha + 1) \quad \dots (1)$

$s_2 = \alpha(\alpha+2) + (\alpha+2)\beta + \alpha\beta = 15$

or $\alpha^2 + 2\alpha + 2\beta + 2\alpha\beta = 15 \qquad \dots (2)$

Substituting for β from (1) in (2)

$\alpha^2 + 2\alpha + 26 - 4(\alpha+1) + 2\alpha(13 - 2\alpha - 2) = 15$ or $3\alpha^2 - 20\alpha - 7 = 0$

Solving for α, we get

$\therefore \alpha = 7, \quad -1/3$

$\alpha = 7$ satisfies the equation and so the roots are $7, 7+2, \beta = 13 - 2(7+1) = -3$

Roots of the given equation are 7, 9, –3

Note : $\alpha = -\frac{1}{3}$ does not satisfy the equation

4. If $x^4 - px^2 + qx - r = 0$ has three equal roots, prove that $p^2 = 12r$ and $9q^2 = 32pr$ and the repeated root is $\frac{3q}{4p}$.

Sol. Roots be $\alpha, \alpha, \alpha, \beta$

Then $s_1 = 3\alpha + \beta = 0$

$\therefore \beta = -3\alpha$, so roots are $\alpha, \alpha, \alpha, -3\alpha$

$\therefore x^4 - px^2 + qx - r \equiv (x-\alpha)^3(x+3\alpha) \equiv x^4 - 6\alpha^2x^2 + 8\alpha^3x - 3\alpha^4$

$\therefore 6\alpha^2 = p; 8\alpha^3 = q, 3\alpha^4 = r$

Square the first and divide by the third. $p^2 = 12r$. Again multiply the first and the third square the second and divide one by the other. $9q^2 = 32pr$.

Dividing the second by the first $\alpha = \frac{3q}{4p}$

5. If α, β, γ be the roots of the equation $x^3 - x - 1 = 0$ form the equation whose roots are $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$ and hence show that

Sol. If y is the root of the required equation and corresponds to the root x of the given equation, then

$y = \frac{1+x}{1-x}$, solving for x, $x = \frac{y-1}{y+1}$

Substituting for x in the given equation $x^3 - x - 1 = 0$, $\frac{(y-1)^3}{(y+1)^3} - \frac{y-1}{y+1} - 1 = 0$ or

$(y-1)^3 - (y-1)(y+1)^2 - (y+1)^3 = 0$

Simplify, $y^3 + 7y^2 - y + 1 = 0$ .... (1) is got as the required equation

$\Sigma \frac{1+\alpha}{1-\alpha}$ = sum of the roots of (1) = -7

6. If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + \frac{1}{\alpha}$, $\gamma\alpha + \frac{1}{\beta}$ and $\alpha\beta + \frac{1}{\gamma}$.

Sol. Given α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$

$\Rightarrow s_1 = \alpha + \beta + \gamma = p$; $s_2 = \Sigma\alpha\beta = q$; $s_3 = \alpha\beta\gamma = r$

Let $y = \beta\gamma + \frac{1}{\alpha} = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha} \Rightarrow \frac{r+1}{y} = \alpha$ which is a root of the equation.

$\therefore$ The transformed equation is $\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$

$\Rightarrow ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0$

$\therefore$ The equation whose roots $\beta\gamma + \frac{1}{\alpha}$, $\gamma\alpha + \frac{1}{\beta}$, $\alpha\beta + \frac{1}{\gamma}$ is

$rx^3 - q(r+1)x^2 + p(r+1)^2 x - (r+1)^3 = 0$

7. If α, β, γ are the roots of equation $x^3 - 6x^2 + 11x - 6 = 0$ then find the equation whose roots are $\alpha^2 + \beta^2, \beta^2 + \gamma^2, \gamma^2 + \alpha^2$.

Sol. Given $\Sigma\alpha = 6; \Sigma\alpha\beta = 11; \alpha\beta\gamma = 6$

Let $y = \alpha^2 + \beta^2 = \alpha^2 + \beta^2 + \gamma^2 - \gamma^2$

$= (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta) - \gamma^2 \Rightarrow y = 36 - 2(11) - \gamma^2$

$\Rightarrow \gamma^2 = 14 - y \Rightarrow \gamma = \sqrt{14-y}$

Which is a root of given equation.

$\therefore$ The required equation for which y is a root is

$(\sqrt{14-y})^2 - 6(\sqrt{14-y})^2 + 11\sqrt{14-y} - 6 = 0$

$\Rightarrow (\sqrt{14-y})(11 + 14 - y) = 6(1 + 14 - y)$

On squaring both sides, we get $(625 + y^2 - 50y)(14 - y) = 36(225 + y^2 - 30y)$

$\Rightarrow y^3 - 28y^2 + 245y - 650 = 0$

$\therefore$ The required equation is $x^3 - 28x^2 + 245x - 650 = 0$

8. If α, β, γ are the roots of $x^3+x^2+2x+3=0$, find the equation whose roots are $\beta+\gamma-\alpha, \gamma+\alpha-\beta$ and $\alpha+\beta-\gamma$.

Sol. Given $\Sigma\alpha=-1, \Sigma\alpha\beta=2, \alpha\beta\gamma=-3$

If y is a root of transformed equation, then

$y=(\Sigma\alpha)-2\alpha=-1-2\alpha \Rightarrow \alpha=\frac{-(1+y)}{2}$ which is a root of given equation

$$\therefore \left(\frac{-(1+y)}{2}\right)^3+\left(\frac{-(1+y)}{2}\right)^2+2\left(\frac{-(1+y)}{2}\right)+3=0$$

$$\Rightarrow -\frac{(1+y)^3}{8}+\frac{(1+y)^2}{4}-(1+y)+3=0$$

$$\Rightarrow y^3+y^2+7y-17=0$$

$\therefore$ The required equation is $x^3+x^2+7x-17=0$

9. If α, β, γ are the roots of $x^3+px^2+qx+r=0$ then find the equation whose roots are $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$.

Sol. Given α, β, γ are the roots of the given equation $x^3+px^2+qx+r=0$

$$\Rightarrow \Sigma\alpha=-p, \Sigma\alpha\beta=q, \alpha\beta\gamma=-r$$

Let $y=\alpha(\beta+\gamma)=\Sigma\alpha\beta-\beta\gamma=q-\frac{\alpha\beta\gamma}{\alpha}=q-\frac{(-r)}{\alpha}$

$\Rightarrow y-q=\frac{r}{\alpha} \Rightarrow \alpha=\frac{r}{y-q}$ which is a root of given equation.

$\therefore$ The required equation is $\frac{r^3}{(y-q)^3}+p\left(\frac{r}{y-q}\right)^2+q\left(\frac{r}{y-q}\right)+r=0$

$$\Rightarrow r^3+pr^2(y-q)+qr(y-q)^2+r(y-q)^3=0$$

$$\Rightarrow y^3-2qy^2(pr+q^2)y+(r^2-pqr)=0$$

$\therefore$ The transformed equation is $x^3-2qx^2+(pr+q^2)x+(r^2-pqr)=0$

10. If α, β, γ are the roots of $x^3+3x^2+2=0$ then find the equation whose roots are $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\alpha+\beta}$.

Sol. Given $\Sigma\alpha=-3, \Sigma\alpha\beta=0, \alpha\beta\gamma=-2$

Let $y=\frac{\alpha}{\beta+\gamma}=\frac{\alpha}{(\Sigma\alpha)-\alpha}=\frac{\alpha}{-3-\alpha}$

$\Rightarrow \alpha=\frac{-3y}{y+1}$ which is a root of the given equation.

$$\therefore \left(\frac{-3y}{y+1}\right)^3+3\left(\frac{-3y}{y+1}\right)^2+2=0$$

$$\Rightarrow -27y^3+27y^2(y+1)+2(y+1)^3=0$$

$$\Rightarrow -27y^3+27y^3+27y^2+2+6y^2+6y=0$$

$$\Rightarrow 2y^3+33y^2+6y+2=0$$

$\therefore$ The required equation is $2x^3+33x^2+6x+2=0$

11. **If α, β, γ are the roots of the equation $x^3 - 6x + 7 = 0$ form the equation whose roots are $\alpha^2 + 2\alpha + 3, \beta^2 + 2\beta + 3, \gamma^2 + 2\gamma + 3$ and hence find $(\alpha^2 + 2\alpha + 3)(\beta^2 + 2\beta + 3)(\gamma^2 + 2\gamma + 3)$.**

Sol. If x is a general root of the given equation and y is the corresponding root of the required equation, then $y = x^2 + 2x + 3$

$\therefore\ x^2 + 2x + (3 - y) = 0$ .... (1)

Given equation is $x^3 - 6x + 7 = 0$ .... (2)

We have to eliminate x between (1) and (2)

(1) – (2) × x gives

$-2x^2 - x(6 + 3 - y) + 7 = 0$ or $2x^2 + (9 - y)x - 7 = 0$ .... (3)

We now eliminate x from (1) and (3)

By the method of cross multiplication

$$\frac{x^2}{-14 - (3 - y)(9 - y)} = \frac{x}{2(3 - y) + 7} = \frac{1}{(9 - y) - 4}$$

i.e., $$\frac{x^2}{-(y^2 - 12y + 41)} = \frac{x}{13 - 2y} = \frac{1}{5 - y}$$

$$\therefore\ x^2 = -\frac{(y^2 - 12y + 41)}{5 - y},\ x = \frac{13 - 2y}{5 - y}$$

$$\therefore\ -\frac{(y^2 - 12y + 41)}{(5 - y)} = \frac{(13 - 2y)^2}{(5 - y)^2} \text{ or } y^3 - 21y^2 + 153y - 374 = 0 \quad .... (4)$$

$\therefore$ Product of the roots of (4) = 374 i.e., $\pi(\alpha^2 + 2\alpha + 3) = 374$

2.12 NATURE OF ROOTS OF A CUBIC POLYNOMIAL

$f(x) = x^3 + ax^2 + bx + c$ (where $a, b, c \in R$)

Let $f(x) = x^3 + ax^2 + bx + c$

Differentiating $f'(x) = 3x^2 + 2ax + b$

This is a quadratic expression whose discriminant $\Delta = 4(a^2 - 3b)$

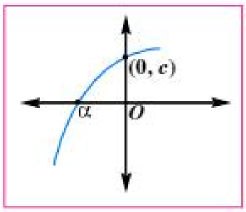
Depending on the nature of Δ we have the following cases.

Case (i) :

Let $\Delta < 0$

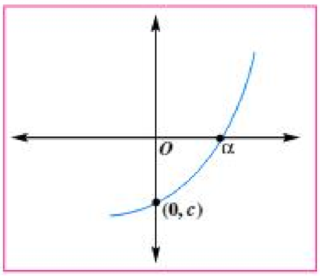
Clearly coefficient of x^2 in $f'(x)$ is 3 which is positive.

$\therefore$ Clearly $f'(x) > 0$



$\therefore\ f(x)$ increases on R and cuts y-axis at $(0, c)$

also, $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$



$\Rightarrow f(x) = 0$ has exactly 1 real root.

Let it be 'α'

Clearly, i) $\alpha > 0$ if $c < 0$ ii) $\alpha < 0$ if $c > 0$

Case (ii) : If $\Delta = a^2 - 3b > 0$, then $f'(x) = 0$ has 2 distinct real roots.

Let they be x_1, x_2 where $x_1 < x_2$. (say)

Now, $f'(x) = 3(x - x_1)(x - x_2)$

$f'(x) > 0 \ \forall x \in (-\infty, x_1) \cup (x_2, \infty)$

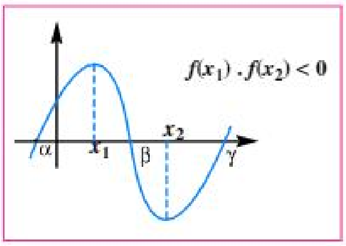
$f'(x) < 0 \ \forall x \in (x_1, x_2)$

$\therefore f$ increases on $(-\infty, x_1) \cup (x_2, \infty)$

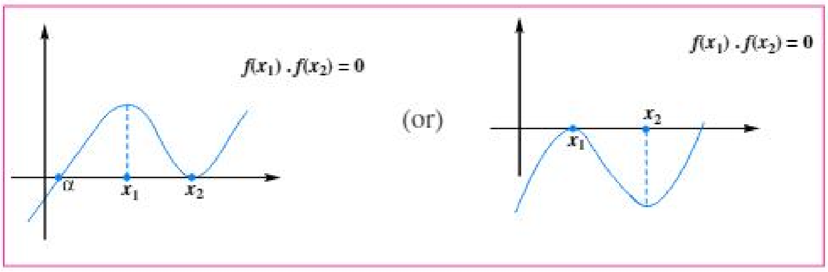
f decreases on (x_1, x_2)

Now observe the following cases.

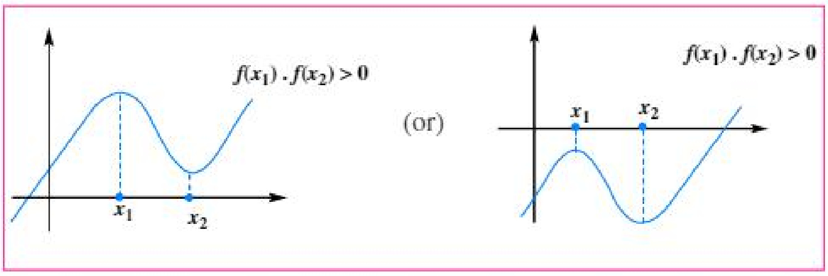
i)



ii)



iii)



From the above cases, it is clear that

i) If $f(x_1) \,.\, f(x)_2 < 0$,

$\Rightarrow f(x) = 0$ would have 3 real and distinct roots

ii) If $f(x_1) \,.\, f(x)_2 = 0$,

$\Rightarrow f(x) = 0$ would have 3 real roots but one of them is repeated. (1 is distinct and 2 equal roots)

iii) If $f(x_1) \,.\, f(x)_2 > 0$,

$\Rightarrow f(x) = 0$ would have just one real root where x_1, x_2 are the roots of $f'(x) = 0$

Case (iii) : If $\Delta = 0 \Rightarrow f'(x)=3(x-x_1)^2$ [$\because\ x_1 = x_2$]

$\Rightarrow f(x) = (x - x_1)^3 + k$

if i) $k = 0$, then $f(x) = 0$ has 3 equal roots and

if ii) $k \neq 0$, then $f(x) = 0$ has atleast 1 real root.

Ex : **$f(x) \equiv 2x^3 - 6x + p$, then find the interval in which p lies so that the equation $f(x) = 0$ has 3 real and distinct roots.**

Sol. $f(x) \equiv 2x^3 - 6x + p, \ \Rightarrow f(-\infty) = -\infty, f(\infty) = \infty$

$f'(x) = 6x^2 - 6$

$f'(x) = 0 \Rightarrow x = \pm 1$

$\because$ It has 3 real and distinct roots $f(1) \,.\, f(-1) < 0$

$\therefore (2 - 6 + p)(-2 + 6 + p) < 0$

$(p - 4)(p + 4) < 0$

$\therefore p \in (-4, 4)$

EXERCISE - 2.3

1. If the roots of the equation $x^3 - px^2 - r = 0$ are $\tan\alpha, \tan\beta, \tan\gamma$ then find the value of $\sec^2\alpha \sec^2\beta \sec^2\gamma$. [Ans : $p^2 + r^2 - 2rp + 1$]

2. If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ then show that $\dfrac{\alpha - 1}{\beta - 1} + \dfrac{\beta - 1}{\gamma - 1} + \dfrac{\gamma - 1}{\alpha - 1} = \omega^2$ where ω is complex cube root of unity.

   (Hint : Given equation $(x-1)^3 + 8 = 0 \Rightarrow x - 1 = -2, -2\omega, -2\omega^2$)

3. If α, β, γ are the roots of $x^3 + 2x + r = 0$ then find $\dfrac{1}{\alpha + \beta - \gamma} + \dfrac{1}{\beta + \gamma - \alpha} + \dfrac{1}{\gamma + \alpha - \beta}$ [Ans : $\dfrac{q}{2r}$]

4. If α, β, γ are roots of $x^3 - 7x + 6 = 0$ then find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$

   [Ans : $x^3 - 42x^2 + 441x - 400 = 0$]

5. If α, β, γ are roots of $x^3 + ax^2 + bx + ab = 0$ find the equation whose roots are $\alpha^3, \beta^3, \gamma^3$
[Ans : $x^3 + a^3x^2 + b^3x + a^3b^3 = 0$]

6. If α, β, γ are roots of $x^3 + qx + r = 0$ find the equation whose roots are $\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}, \frac{\alpha+\beta}{\gamma^2}$
[Ans : $rx^3 - qx^2 - 1 = 0$]

7. If α, β, γ are roots of $x^3 + qx + r = 0$ find the equation whose roots are $\beta^2 + \beta\gamma + \gamma^2$, $\gamma^2 + \gamma\alpha + \alpha^2, \alpha^2 + \alpha\beta + \beta^2$
[Ans : $(x + q)^3 = 0$]

8. If α, β, γ are the roots of $x^3 + x^2 - 5x - 1 = 0$ then find the value of $[\alpha] + [\beta] + [\gamma]$ where [ ] is G.I.F.
[Ans : -3]

9. If the roots of the equation $x^4 - 12x^3 + bx^2 + cx + 81 = 0$ are positive then find the values of b and c.
[Ans : 54, -108]

10. If x, y, z are real $x + y + z = 5$ and $xy + yz + zx = 8$ then find the minimum value of x, and maximum value of y
[Ans : 1, $\frac{7}{3}$]

11. If x and y are positive integers such that $xy + x + y = 71$ and $x^2y + xy^2 = 880$ then find the value of $x^2 + y^2$
[Ans : 146]

12. Find the sum of the roots of the equation $1 + 2^{22x+1} = 2^{33x-2} + 2^{11x+2}$ [Ans : $\frac{2}{11}$]

13. If $\frac{1}{3}, \frac{1}{3}\log_x y, \frac{1}{3}\log_y z, \frac{1}{7}\log_z x$ are in H.P. and $y = x^p$, $z = x^q$ then find the value of $p + q$. [Ans : $\frac{3}{2}$]

14. If $x^2 - 5x + 1 = 0$ then show that $\frac{x^{10}+1}{x^5} = 2525$

15. Find the number of non-real roots of $2x^{100} + 7x^{99} + 7x^{98} + ...+ 7x^3 + 7x^2 + 7x + 5 = 0$ [Ans : 98]

16. Let $a \in R$ and $f : R \to R$ be given by $f(x) = x^5 - 5x + a$ then show that $f(x)$ has only one real root if $a > 4$ and it has three real roots if $-4 < a < 4$.

17. Let $P(x)$ be a polynomial of fifth degree with leading Coefficient unity show that $P(1)=5$, $P(2)=6$, $P(3)7$, $P(4)=8$, $P(5)=9$ then find the value of $P(6)$
[Ans : 130]

18. Consider the equation $x^3 - ax^2 + bx - c = 0$, where a, b, c are rational numbers $a \neq 1$. Given that x_1, x_2 and x_1x_2 are the real roots of the equation then find the value of $x_1x_2\left(\frac{a+1}{b+c}\right)$ [Ans : 1]