CHAPTER 1

Calculus

INDEFINITE INTEGRALS

• ANTIDERIVATIVE • INDEFINITE INTEGRAL •
• STANDARD INTEGRALS • METHOD OF SUBSTITUTION •
• INTEGRATION BY THE METHOD OF PARTIAL FRACTIONS •
• INTEGRATION OF ALGEBRAIC FUNCTIONS & TRIGONOMETIRIC FUNCTIONS •
• INTEGRATION BY PARTS • REDUCTION FORMULAE •

1.0 INTRODUCTION

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the ***area of the region bounded by the graph of the functions.***

If a function f is differentiable in an interval I, i.e.., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function ? The functions that could possibly have given function as a derivative are called ***anti derivatives (or primitive)*** of the function. Further, the formula that gives all these anti derivatives is called the ***indefinite integral*** of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant ? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types :

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

There is a connection, known as the ***Fundamental theorem of Calculus,*** between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

1.1 INTEGRATION AS AN INVERSE PROCESS OF DIFFERENTIATION

Integration is the inverse process of differentiation. Instead of differentiation a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called ***integration or anti differentiation.***

Let us consider the following examples :

We know that $\frac{d}{dx}(\sin x) = \cos x$ .... (1)

$$\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3 \quad .... (2)$$

and $\frac{d}{dx}(e^x) = e^x$ .... (3)

We observe that in (1), the function cos x is the derived function of sin x. We say that sin x is an anti derivative (or an integral) of cos x. Similarly, in (2) and (3), $\frac{x^4}{4}$ and e^x are the anti derivatives (or integrals) of x^3 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows:

$$\frac{d}{dx}(\sin x + C) = \cos x,\ \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3, \text{ and } \frac{d}{dx}(e^x + C) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant.*** In fact, C is the parameter by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx}F(x) = f(x),\ \forall x \in$ I (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}[F(x) + C] = f(x), x \in I$$

Thus, $\{F + C, C \in R\}$ denotes a family of anti derivatives of f.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x),\ \forall x \in I$

Then $\frac{df}{dx} = f' = g' - h'$ giving $f'(x) = g'(x) - h'(x)\ \ \forall x \in I$

or $f'(x) = 0,\ \forall x \in I$ by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in R\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x .

Symbolically, we write $\int f(x)dx = F(x) + C$.

For the sake of convenience, we mention below the following symbols/terms/ phrases with their meanings as given in the table

Symbols/Terms/Phrases	Meaning
$\int f(x)dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)dx$	Integrand
x in $\int f(x)dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Fundamental Integrals

Derivatives	Integrals
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
particularly, we note that $\frac{d}{dx}(x) = 1$	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\, dx = \sin x + C$
(iii) $\frac{d}{dx}(\cos x) = -\sin x$;	$\int \sin x\, dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\, dx = \tan x + C$
(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\, dx = \sec x + C$
(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$;	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

(viii) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$;	$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C$
(ix) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$;	$\int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+C$
(x) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$;	$\int\frac{dx}{1+x^2}=\tan^{-1}x+C$
(xi) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$;	$\int\frac{dx}{1+x^2}=-\cot^{-1}x+C$
(xii) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$;	$\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+C$
(xiii) $\frac{d}{dx}(\text{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$;	$\int\frac{dx}{x\sqrt{x^2-1}}=-\text{cosec}^{-1}x+C$
(xiv) $\frac{d}{dx}(e^x)=e^x$;	$\int e^x dx=e^x+C$
(xv) $\frac{d}{dx}(\log\lvert x\rvert)=\frac{1}{x}$;	$\int\frac{1}{x}dx=\log\lvert x\rvert+C$
(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x$;	$\int a^x dx=\frac{a^x}{\log a}+C$

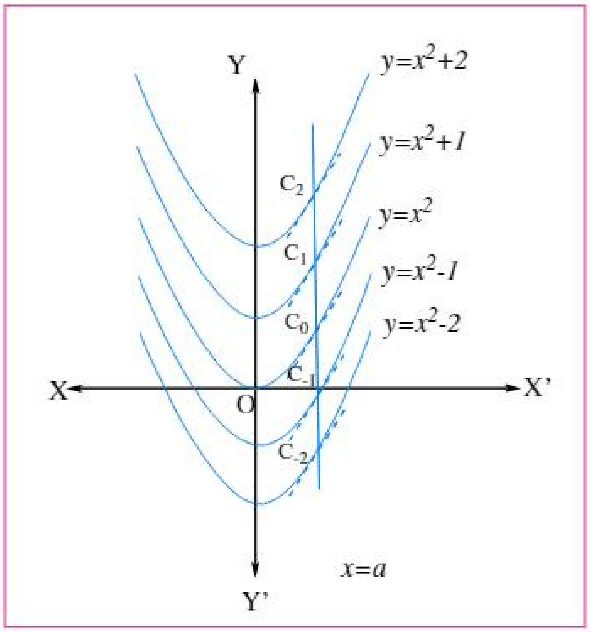
1.2 GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL

Let $f(x) = 2x$. Then $\int f(x)dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

Clearly, for $C = 0$, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for $C = 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in positive direction. For $C = -1$, $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in the negative direction. Thus, for each positive value of C, each parabola of the family has its vertex on the positive side of the y-axis and for negative values of C, each has its vertex along the negative side of the y-axis. Some of these have been shown in the figure.

Let us consider the intersection of all these parabolas by a line $x = a$. In the following Figure, we have taken $a > 0$. The same is true when $a < 0$. If the line $x = a$ intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at C_0, C_1, C_2, C_{-1}, C_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curve at these points are parallel. Thus, $\int 2x dx = x^2 + C = F_C(x)$ (say), implies that



the tangents to all the curves $y = F_C(x)$, $C \in \boldsymbol{R}$, at the points of intersection of the curves by the line $x = a$, ($a \in \boldsymbol{R}$), are parallel.

Further, the following equation (statement) $\int f(x)dx = F(x) + C = y$ (say), represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

1.3 SOME PROPERTIES OF INDEFINITE INTEGRAL

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results:

$$\frac{d}{dx}\int f(x)dx = f(x)$$

and $\int f'(x)dx = f(x) + C$, where C is any arbitrary constant.

Proof : Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}F(x) = f(x)$$

Then $\int f(x)dx = F(x)+C$

Therefore $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}(F(x)+C)$

$$\frac{d}{dx}F(x) = f(x)$$

Similarly, we note that $f'(x) = \frac{d}{dx}f(x)$

and hence $\int f'(x)dx = f(x)+C$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof : Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$$

or $\frac{d}{dx}[\int f(x)dx - \int g(x)dx] = 0$

Hence $\int f(x)dx - \int g(x)dx = C$, where C, is any real number (why?)

or $\int f(x)\,dx = \int g(x)dx + C$

So the families of curves $\left\{\int f(x)\,dx + C_1, C_1 \in R\right\}$

and $\left\{\int g(x)\,dx + C_2, C_2 \in R\right\}$ are identical.

Hence, in this sense, $\int f(x)dx$ and $\int g(x)dx$ are equivalent.

(III) $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Proof : By property (I), we have

$$\frac{d}{dx}\left[\int [f(x)+g(x)]dx\right] = f(x)+g(x) \qquad \text{..... (1)}$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx = f(x) + g(x) \qquad \text{.... (2)}$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

(IV) For any real number k, $\int k\ f(x)\ dx = k\int f(x)\ dx$

Proof : By the property (I), $\frac{d}{dx}\int k\,f(x)\ dx = k\int f(x)dx$

Also, $\frac{d}{dx}\left[k\int f(x)\ dx\right] = k\frac{d}{dx}\int f(x)dx = kf(x)$

Therefore, using the Property (II), we have $\int k\ f(x)\ dx = k\int f(x)dx$

(V) Properties (III) and (IV) can be generalised to a finite number of functions f_1, f_2 ..., f_n and the real numbers k_1, k_2, ... , k_n giving

$$\int[k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]dx$$

$$= k_1\int f_1(x)dx + k_2\int f_2(x)dx + ... + k_n\int f_n(x)dx$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

1.4 INDEFINITE INTEGRALS OF SOME ELEMENTARY REAL FUNCTONS (STANDARD INTEGRAL)

Since we already know the derivatives of some standard elementary real functions, we can now write the indefinite integrals of corresponding derivatives using the fundamental rule :

$\frac{d}{dx}(F(x)) = f(x) \Rightarrow \int f(x)dx = F(x) + c$ on an interval I.

1. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, (n \neq -1), \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$

2. $\frac{d}{dx}(x) = 1 \Rightarrow \int 1 dx = x + c$

3. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \Rightarrow \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

4. $\frac{d}{dx}(|x|) = \frac{|x|}{x}, x \neq 0 \Rightarrow \int \frac{|x|}{x} dx = |x| + c,\ x \neq 0$

5. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c$

6. $\frac{d}{dx}(a^x) = a^x.\log a \Rightarrow \int a^x dx = \frac{a^x}{\log a} + c \quad (a > 0,\ a \neq 1)$

7. $\frac{d}{dx}(\log|x|) = \frac{1}{x},\ x \neq 0 \Rightarrow \int \frac{1}{x} dx = \log|x| + c,\ x \neq 0$

In a similar way, the following indefinite integrals can be easily written using the corresponding differentiation formulae.

8. $\int \sin x \, dx = -\cos x + c$
9. $\int \cos x \, dx = \sin x + c$
10. $\int \sec^2 x \, dx = \tan x + c$
11. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
12. $\int \sec x \tan x \, dx = \sec x + c$
13. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
14. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$
15. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c = -\cot^{-1} x + c$
16. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$
17. $\int \sinh x \, dx = \cosh x + c$
18. $\int \cosh x \, dx = \sinh x + c$
19. $\int \operatorname{sech}^2 x \, dx = \tanh x + c$
20. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$
21. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$
22. $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c$
23. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c = \log\left|x + \sqrt{x^2+1}\right| + c$
24. $\int \frac{1}{\sqrt{x^2-1}} dx = \begin{cases} \cosh^{-1} x + c \text{ if } x > 1 \\ -\cosh^{-1}(-x) + c \text{ if } x < -1 \end{cases} = \log\left|x + \sqrt{x^2-1}\right| + c \quad (x < -1 \text{ or } x > 1)$
25. $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + c, \ |x| < 1$

The range of the values of the independent variable x for which $\int f(x)dx$ is valid, is to be considered as the set of all possible real values of x (the largest possible set) for which the integrand $f(x)$ is defined i.e., the domain of f, unless otherwise stipulated.

1.5 ALGEBRAIC PROPERTIES

THEOREM-1.1 **Let the functions f and g have integrals on I and k, l be real constants. Then the functions $f+g$, $f-g$, kf and $kf + \ell g$ have integrals on I and**

i) $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx + c$

ii) $\int (f-g)(x)dx = \int f(x)dx - \int g(x)dx + c$

iii) $\int (kf(x)dx = k\int f(x)dx + c$

iv) $\int (kf + \ell g)(x)dx = k\int f(x)dx + \ell \int g(x)dx + c$ **(Linearity property)**

where c is the constant of integration.

> ***Remark :*** *There is no need to introduce a constant of integration after calculating each integral. By combining all the arbitrary constants of integration we get a single arbitrary constant of integration denoted by c, which is added to the final answer.*

Example : i) $\int \frac{9x^2 - 6x\sqrt{x} + \sqrt{x} + 3x}{3x^2\sqrt{x}} dx = \int \left[3\left(\frac{1}{\sqrt{x}}\right) - 2\left(\frac{1}{x}\right) + \frac{1}{3}\left(\frac{1}{x^2}\right) + \frac{1}{x\sqrt{x}} \right] dx$

(dividing each term of the N^r with the D^r)

$= e\int \frac{1}{\sqrt{x}} dx - 2\int \frac{1}{x} dx + \int \frac{1}{x^{3/2}} dx$ (Linearity property)

$= 3(2\sqrt{x} + c_1) - 2(\log|x| + c_2) + \frac{1}{3}\left[\left(-\frac{1}{x}\right) + c_3\right] + \left[\frac{-2}{\sqrt{x}} + c_4\right] = 6\sqrt{x} - 2\log|x| - \frac{1}{3x} - \frac{2}{\sqrt{x}} + c$

where $c = 3c_1 - 2c_2 + \frac{1}{3}c_3 + c_4$

ii) $\int \left[\frac{2x^3 - 3x + 5}{2x^2} + \frac{4}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + (x^2 - 2)(x^2 + 3) \right] dx$

$= \int \left[x - \frac{3}{2}\left(\frac{1}{x}\right) + \frac{5}{2}\left(\frac{1}{x^2}\right) + 4\left(\frac{1}{1+x^2}\right) - 2\left(\frac{1}{\sqrt{1-x^2}}\right) + (x^4 + x^2 - 6) \right] dx$

$= \int x\, dx - \frac{3}{2}\int \frac{1}{x} dx + \frac{5}{2}\int \frac{1}{x^2} dx + 4\int \frac{1}{1+x^2} dx - 2\int \frac{1}{1-x^2} dx + \int x^4 dx + \int x^2 dx - 6\int 1\, dx$

$= \frac{x^2}{2} - \frac{3}{2}\log|x| + \frac{5}{2}\left(-\frac{1}{x}\right) + 4\tan^{-1} x - 2\sin^{-1}(x) + \frac{x^5}{5} + \frac{x^3}{3} - 6x + c$

SOLVED EXAMPLES

***1. Evaluate $\int (1-x)(4-3x)(2x+3) dx$**

Sol. $\int (1-x)(4-3x)(2x+3) dx = \int (6x^3 - 5x^2 - 13x + 12) dx$

$= 6\int x^3 dx - 5\int x^2 dx - 13\int x dx + 12\int 1 dx$

$= 6\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) - 13\left(\frac{x^2}{2}\right) + 12x + c$

$= \frac{3}{2}x^4 - \frac{5}{3}x^3 - \frac{13}{2}x^2 + 12x + c$

***2. Evaluate $\int \left(x + \frac{1}{x}\right)^3 dx, x > 0$**

Sol. $\int \left(x + \frac{1}{x}\right)^3 dx, = \int \left(x^3 + 3x^2 \cdot \frac{1}{x} + 3x \cdot \frac{1}{x^2} + \frac{1}{x^3}\right) dx$

$= \int x^3 dx + 3\int x\, dx + 3\int \frac{1}{x} dx + \int x^{-3} dx$

$= \frac{x^4}{4} + 3 \cdot \frac{x^2}{2} + 3\log|x| - \frac{1}{2x^2} + c$

***3. Evaluate $\int \frac{x^6 - 1}{x^2 + 1} dx$**

Sol. $\int \frac{x^6 - 1}{x^2 + 1} dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2 + 1}\right) dx$ (by actual division)

$= \int x^4 dx - \int x^2 dx + \int 1 dx - 2\int \frac{1}{x^2 + 1} dx = \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + c$

Remember :
$\log\frac{a}{b} = \log a - \log b$

*4. **Find $\int \frac{(a^x - b^x)^2}{a^x b^x} dx \quad (a > 0, a \neq 1, b > 0, b \neq 1)$**

Sol. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx = \int \frac{a^{2x} + b^{2x} - 2a^x b^x}{a^x b^x} dx$

$= \int \left[\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x - 2\right] dx \quad = \int \left(\frac{a}{b}\right)^x dx + \int \left(\frac{b}{a}\right)^x dx - 2\int 1 dx$

$= \frac{1}{\log\left(\frac{a}{b}\right)}\left(\frac{a}{b}\right)^x + \frac{1}{\log\left(\frac{b}{a}\right)}\left(\frac{b}{a}\right)^x - 2x + c$

$= \frac{1}{\log a - \log b}\left[\left(\frac{a}{b}\right)^x - \left(\frac{b}{a}\right)^x\right] - 2x + c$

*5. **Find $\int \sec^2 x \operatorname{cosec}^2 x dx$** **(May-17)**

Sol. $\int \sec^2 x \operatorname{cosec}^2 x dx = \int \frac{1}{\sin^2 x \cos^2 x} dx$

$\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int (\sec^2 x + \operatorname{cosec}^2 x) dx = \tan x - \cot x + c$

6. **Evaluate $\int (a \tan x - b \cot x)^2 dx$**

Sol. $\int (a \tan x - b \cot x)^2 dx = \int (a^2 \tan^2 x + b^2 \cot^2 x - 2ab) dx$

$= \int \left[a^2(\sec^2 x - 1) + b^2(\operatorname{cosec}^2 x - 1) - 2ab\right] dx$

$= a^2 \int \sec^2 x dx + b^2 \int \operatorname{cosec}^2 x dx - (a+b)^2 \int 1 dx$

$= a^2 \tan x + b^2(-\cot x) - (a+b)^2 x + c$

7. **Evaluate $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$**

Sol. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx = \int \frac{2(\cos^2 x - \cos^2\alpha)}{\cos x - \cos\alpha} dx$

$= 2\int (\cos x + \cos\alpha) dx = 2(\sin x + x.\cos\alpha) + c$

Remember :
$\sqrt{1 + \sin 2x} = |\sin x + \cos x|$

$= \begin{cases} \sin x + \cos x \text{ if } \\ \left[2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right] \\ -(\sin x + \cos x) \text{ if } \\ \left[2n\pi + \frac{3\pi}{4}, 2n\pi + \frac{7\pi}{4}\right] \end{cases}$

where $n \in Z$

*8. **Evaluate $\int \sqrt{1 + \sin 2x} dx \quad (x \in R)$**

Sol. $\sqrt{1 + \sin 2x} = \sqrt{(\sin x + \cos x)^2} = |\sin x + \cos x|$

$= \begin{cases} \sin x + \cos x \text{ if } 2n\pi - \frac{\pi}{4} \leq x \leq 2n\pi + \frac{3\pi}{4}, n \in Z \\ -(\sin x + \cos x) \text{ if } 2n\pi + \frac{3\pi}{4} \leq x \leq 2n\pi + \frac{7\pi}{4}, n \in Z \end{cases}$

$\therefore \int \sqrt{1 + \sin 2x} dx$

$$=\begin{cases}\int(\sin x+\cos x)dx \text{ if } x\in\left[2n\pi-\frac{\pi}{4},2n\pi+\frac{3\pi}{4}\right]\\ \int-(\sin x+\cos x)\,dx \text{ if } x\in\left[2n\pi+\frac{3\pi}{4},2n\pi+\frac{7\pi}{4}\right]\end{cases}$$

$$=\begin{cases}\sin x+\cos x+c \text{ if } x\in\left[2n\pi-\frac{\pi}{4},2n\pi+\frac{3\pi}{4}\right]\\ \cos x-\sin x+c \text{ if } x\in\left[2n\pi+\frac{3\pi}{4},2n\pi+\frac{7\pi}{4}\right], n\in Z\end{cases}$$

Note :

$$\int\frac{1dx}{\sqrt{1+x^2}}=\sinh^{-1}x$$
$$=\log(x+\sqrt{1+x^2})$$

9. Evaluate $\int\frac{x^2}{\sqrt{1+x^2}(\sqrt{1+x^2}-1)}dx\ (x\neq 0)$

Sol. $\int\frac{x^2}{\sqrt{1+x^2}(\sqrt{1+x^2}-1)}dx=\int\frac{x^2(\sqrt{1+x^2}+1)}{\sqrt{1+x^2}[(\sqrt{1+x^2})-1]}dx$

$$=\int\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}}dx=\int\left(1+\frac{1}{\sqrt{1+x^2}}\right)dx=x+\sinh^{-1}x+c$$

Remember :

$a^{\log_a f(x)}=f(x)$

10. Evaluate $\int e^{\log(1+\tan^2 x)}dx$

Sol. $\int e^{\log(1+\tan^2 x)}dx=\int e^{\log\sec^2 x}dx=\int\sec^2 x\,dx=\tan x+c$

11. Evaluate $\int\frac{\sin^2 x}{1+\cos 2x}dx$

Sol. $\int\frac{\sin^2 x}{1+\cos 2x}dx=\int\frac{\sin^2 x}{2\cos^2 x}dx=\frac{1}{2}\int\frac{1-\cos^2 x}{\cos^2 x}dx$

$$=\frac{1}{2}(\tan x-x)+C=\frac{1}{2}(\tan x-x)+C$$

12. Evaluate $\int\left(\sqrt{x}-\frac{2}{1-x^2}\right)dx$

Sol. $\int\left(\sqrt{x}-\frac{2}{1-x^2}\right)dx=\int\sqrt{x}dx-2\int\frac{dx}{1-x^2}$

$$=\frac{x^{3/2}}{\left(\frac{3}{2}\right)}-2\tanh^{-1}x+C=\frac{2}{3}x\sqrt{x}-2\tanh^{-1}x+C$$

13. Evaluate $\int\sin^2 x\,dx$

Sol. $\int\sin^2 x\,dx=\int\left(\frac{1-\cos 2x}{2}\right)dx=\frac{1}{2}\int dx-\frac{1}{2}\int\cos 2x\,dx$

$$=\frac{1}{2}x-\frac{1}{4}\sin 2x+c$$

EXERCISE - 1.1

Find the values of the following integrals

1. *a) $\int 2x^7 dx$ *b) $\int 2x\sqrt{x}\,dx$ *c) $\int (2x^2)^{1/3} dx$ d) $\int e^{2\log x} dx$

e) $\int e^{-\log \operatorname{cosec} x} dx$ f) $\int \frac{e^{\log x}}{x} dx$ *g) $\int \cot^2 x\, dx$ h) $\int 3^{-\log_3 x^2} dx$

i) $\int (\tan^2 x + 1) dx$

2. *a) $\int (x^3 - 2x^2 + 3) dx$ b) $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right) dx$ *c) $\int \frac{x^2+3x-1}{2x} dx$

*d) $\int \frac{2x^3 - 3x + 5}{2x^2} dx$ *e) $\int \frac{1-\sqrt{x}}{x} dx$ *f) $\int \left(1 + \frac{2}{x} - \frac{3}{x^2}\right) dx$ *g) $\int \left(x + \frac{4}{1+x^2}\right) dx$

*h) $\int \left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}\right) dx, |x|>1$ *i) $\int \left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right) dx$

*j) $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1+x^2}}\right) dx, |x|<1$

3. *a) $\int (1-x^2)^3 dx, |x|<1$ *b) $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx$ on $(0,\infty)$

*c) $\int \frac{(3x+1)^2}{2x} dx, (x \neq 0)$ *d) $\int \left[\frac{2x-1}{3\sqrt{x}}\right]^2 dx, (x>0)$

*e) $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right) dx$ on $(1,\infty)$

4. *a) $\int (\sec^2 x - \cos x + x^2) dx$ *b) $\int \left(\sec x \tan x + \frac{3}{x} - 4\right) dx$

*c) $\int \left(x^3 - \cos x + \frac{4}{\sqrt{x^2+1}}\right) dx$ *d) $\int \left(\sinh x + \frac{1}{\sqrt{x^2-1}}\right) dx, |x|>1$

*e) $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}}\right) dx$ **f) $\int \frac{1}{\cosh x + \sinh x} dx$ (March-17)

5. *a) $\int \frac{1+\cos^2 x}{1-\cos 2x} dx$ (March-19) *b) $\int \frac{1}{\sin^2 x \cos^2 x} dx$

*c) $\int \sqrt{1-\cos 2x}\, dx \;\; x \in [0,\pi]$ **d) $\int \frac{1}{1+\cos x} dx$

e) $\int \frac{1}{1+\sin x} dx$ f) $\int \frac{1}{1-\cos x} dx$ g) $\int \frac{\sin x}{1+\sin x} dx$

h) $\int \frac{2\cos^3 x + 3\sin^3 x}{\sin^2 x \cos^2 x} dx$ i) $\int \sec^2 x \operatorname{cosec}^2 x\, dx$

Integrate the following functions with respect to x.

6. a) $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x}$ b) $\sqrt{1-\sin 2x},\ x \in \left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$

   c) $\sqrt{1-\sin 2x},\ x \in \left(\dfrac{5\pi}{4}, \dfrac{9\pi}{4}\right)$ d) $\sqrt{1+\sin 2x},\ x \in \left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$

7. a) $(a\tan x - b\cot x)^2$ b) $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$ c) $\dfrac{\sin x + \cos x}{\sqrt{1+\sin 2x}}$

   d) $\tan^{-1}\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$ e) $\dfrac{x^2}{\sqrt{1+x^2}\,(1+\sqrt{1+x^2})}$

1.6 METHODS OF INTEGRATION

There are mainly four methods of evaluating integrals by reducing them into integrals of standard form:

1) **Integration by the Method of substitution (change of variable)**
2) **Integration by the Method of partial fractions.**
3) **Integration by parts**
4) **Integration by successive reduction**

A combination of two or more of the above methods might be necessary to evaluate certain integrals

1.7 METHOD OF SUBSTITUTION

The method of substitution (or the method of change of variable) to evaluate $\int f(x)\,dx$ consists of substituting $\phi(t)$ for x where $\phi(t)$ is a ***continuously differentiable function*** of a new variable t. By this substitution, we have

$$\int f(x)\,dx = \int f(\phi(t))\,\phi'(t)\,dt \qquad \text{-- (1)}$$

After evaluating the integral on the R.H.S of (1) t is to be replaced by $\phi^{-1}(x)$ to get the result in terms of the original variable x.

But in practice, the indicated formula (1) is used in the reverse direction. That is, to evaluate a given integral of the form $\int f(\phi(t))\phi'(t)\,dt$ we substitute $\phi(t) = x$ so that $\int f(\phi(t))\,\phi'(t)dt = \int f(x)\,dx$ where $x = \phi(t)$.

Example : To evaluate $\int e^{t^2}\,2t\,dt$ we substitute $t^2 = x$ so that $2t\,dt = dx$ and therefore,

$$\int e^{t^2}\,2t\,dt = \int e^x dx = e^x + c \quad \text{where } x = t^2 = e^{t^2} + c$$

Thus, to evaluate $\int f(g(x)).g'(x)dx$, we substitute $g(x) = t$ so that $g'(x)dx = dt$ and hence $\int f(g(x))g'(x)dx = \int f(t)\,dt$.

After evaluating the integral on the R.H.S, we substitute $t = g(x)$ to get the *"value"* of the integral in terms of the original variable x.

For example, to evaluate $\int \cos(x^3)\,x^2 dx$, we put $x^3 = t$ so that $3x^2\,dx = dt$.

$$\int \cos(x^3)x^2 dx = \int \cos t\left(\frac{1}{3}\right)dt = \frac{1}{3}\sin t + c \quad \text{where } t = x^3 = \frac{1}{3}\sin(x^3) + c$$

THEOREM-1.2 **Let I and J be two intervals in R. If $\int f(x)dx = F(x) + c$ on I and $g : J \to I$ is a continuously differentiable function on J then $(fog)g'$ has an integral on J and $\int f(g(x))g'(x)dx = \mathrm{F}(g(x)) + c$**

THEOREM-1.3 If $\int f(x)dx = F(x)+C$ then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C'$ where C and C' are constants of integration.

Note : $\int f'(g(x))g'(x)dx = f(g(x))+c$

Proof : To evaluate $\int f(ax+b)\,dx$, put $ax+b=t$

then $dx = \frac{1}{a}dt$ and $\int f(ax+b)dx = \int f(t)\frac{1}{a}dt = \frac{1}{a}(F(t)+c) = \frac{1}{a}F(ax+b)+C'$

1.8 — SOME STANDARD RESULTS

For $a \neq 0$

1) $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c \quad (n \neq -1)$ where $a \neq 0$

2) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+c \quad \left(x \neq {}^{-b}/_{a}\right)$ where $a \neq 0$

Note : $\int f'(ax+b)dx = \frac{1}{a}f(ax+b)+c$

3) $\int \frac{1}{\sqrt{ax+b}}dx = \frac{2}{a}\sqrt{ax+b}+c$ where $a \neq 0$

4) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$ where $a \neq 0$

5) $\int k^{ax+b}dx = \frac{1}{a}\frac{k^{ax+b}}{\log k}+c$ where $a \neq 0$

6) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$

7) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$

8) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$

9) $\int \mathrm{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

10) $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c$

11) $\int \mathrm{cosec}\,(ax+b)\,\cot(ax+b)dx = -\frac{1}{a}\mathrm{cosec}(ax+b)+c$

12) $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1}+c \quad (n \neq -1)$

13) $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$

14) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c$

15) $\int e^{f(x)}f'(x)dx = e^{f(x)}+c$

THEOREM-1.4 If $f(x) \neq 0$ for any x in an interval I then $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$ on I.

Proof : Let $f(x) = t$ then $f'(x)\,dx = dt$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{1}{t} dt = \log|t| + c \text{ where } t = f(x) = \log|f(x)| + c$$

1.9 STANDARD INTEGRALS

1) $\int \tan x\, dx = \log|\sec x| + c$

2) $\int \cot x\, dx = \log|\sin x| + c$

3) $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

4) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$

Proof :

1) $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{f'(x)}{f(x)} dx$ where $f(x) = \cos x$

$= -\log|f(x)| + c = -\log|\cos x| + c = \log|\sec x| + c$

Note : $\frac{d}{dx}(\sec x + \tan x) = \sec x\,(\sec x + \tan x)$

(or) $\int \tan x\, dx = \int \frac{\sec x \tan x}{\sec x} dx = \int \frac{(\sec x)'\, dx}{\sec x} = \log|\sec x| + c$

2) $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + c$ where $f(x) = \sin x$

Remember : $\sec x + \tan x = \frac{1 + \sin x}{\cos x} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

3) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$

$= \int \frac{f'(x)}{f(x)} dx$ where $f(x) = \sec x + \tan x$

$= \log|f(x)| + c = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

Note : $\frac{d}{dx}(\operatorname{cosec} x - \cot x) = \operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)$

4) $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx = \int \frac{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x} dx$

$= \int \frac{f'(x)}{f(x)} dx$ where $f(x) = \operatorname{cosec} x - \cot x$

$= \log|f(x)| + c = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$

Remember : $\operatorname{cosec} x - \cot x = \frac{1 - \cos x}{\sin x} = \tan\frac{x}{2}$

1.10 SOME STANDARD SUBSTITUTIONS

Depending on the form of the integrand, the following substitutions are made to transform the given integral into a standard form.

Form of Integrand	Substitution
i) $\sqrt{a^2-x^2}, \frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$ or $x = a\cos\theta$
ii) $\frac{1}{a^2+x^2}, \sqrt{a^2+x^2}, \frac{1}{\sqrt{a^2+x^2}}$	$x = a\tan\theta$ or $x = a\cot\theta$ or $x = a\sinh\theta$
iii) $\frac{1}{\sqrt{x^2-a^2}}, \sqrt{x^2-a^2}$	$x = a\sec\theta$ (or) $x = a$ cosecθ (or) $x = a\cosh\theta$
iv) $\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
v) $\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ (or) $x = a\cos^2\theta$
vi) $\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\sec^2\theta$
vii) $\sqrt{\frac{x-a}{b-x}}, \sqrt{(x-a)(b-x)}, \frac{1}{\sqrt{(x-a)(b-x)}}$	$x = a\cos^2\theta + b\sin^2\theta$
viii) $\sqrt{\frac{x-a}{x-b}}, \sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
ix) $\frac{1}{\sqrt{(x-a)(x-b)}}$	$x-a = t^2$ or $x-b = t^2$

1.11 NINE STANDARD INTEGRALS

1) $\int \frac{1}{x^2+a^2}dx$

Proof : Put $x = a\tan\theta$ then $dx = a\sec^2\theta\, d\theta$

so that $\theta = \tan^{-1}\left(\frac{x}{a}\right)$, $x^2+a^2 = a^2\sec^2\theta$

$$\therefore \int \frac{1}{x^2+a^2}dx = \int \frac{1}{a^2\sec^2\theta} a\sec^2\theta\, d\theta = \int \frac{1}{a}d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Remember :
$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Remember :

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c,\quad x \neq \pm a$$

2) $\int \frac{1}{a^2-x^2}dx$

Proof : $\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)} = \frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right]$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx = \frac{1}{2a}\left[\int\frac{1}{a+x}dx+\int\frac{1}{a-x}dx\right]$$

$$= \frac{1}{2a}[\log|a+x|-\log|a-x|]+c = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

3) $\int \frac{1}{x^2-a^2}dx$

Remember :

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c,\quad x \neq \pm a$$

Proof : $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$

$$= \frac{1}{2a}[\log|x-a|-\log|x+a|]+c = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

4) $\int \frac{1}{\sqrt{a^2-x^2}}dx$

Remember :

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c,\quad |x|<a$$

Proof : Put $x = a\sin\theta$ so that $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

Then $dx = a\cos\theta d\theta$ and $\int\frac{1}{\sqrt{a^2-x^2}}dx = \int\frac{1}{a\cos\theta}a\cos\theta\, d\theta = \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right)+c$

5) $\int \frac{1}{\sqrt{a^2+x^2}}dx$

Remember :

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)+c = \log(x+\sqrt{x^2+a^2})+c,\quad x \in R$$

Proof : Put $x = a\sinh\theta$ so that $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

then $dx = a\cosh\theta\, d\theta$ and $\int\frac{1}{\sqrt{a^2+x^2}}dx = \int\frac{1}{a\cosh\theta}a\cosh\theta\, d\theta$

$$= \int d\theta = \theta + c = \sinh^{-1}\left(\frac{x}{a}\right)+c = \log(x+\sqrt{x^2+a^2})+c$$

Note : The substitution $x = a\tan\theta$ gives the result interms of *logarithms*.

6) $\int \frac{1}{\sqrt{x^2-a^2}}dx$

Remember :

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)+c = \log(x+\sqrt{x^2-a^2})+c,\quad x>a$$

Proof : Put $x = a\cosh\theta$ so that $\theta = \cosh^{-1}\left(\frac{x}{a}\right)$

Then $dx = a\sinh\theta$ and $\int\frac{1}{\sqrt{x^2-a^2}}dx = \int\frac{1}{a\sinh\theta}a\sinh\theta\, d\theta = \int d\theta$

$$= \theta + c = \cosh^{-1}\left(\frac{x}{a}\right)+c = \log(x+\sqrt{x^2-a^2})+c$$

Note : The substitution $x = a\sec\theta$ gives the result in terms of *log*.

Remember :

$\int\sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c,\ |x|<a$

7) $\int\sqrt{a^2-x^2}dx$

Proof : Put $x=a\sin\theta$ so that $\theta=\sin^{-1}\left(\frac{x}{a}\right)$ then $dx=a\cos\theta\,d\theta$ and

$$\int\sqrt{a^2-x^2}=\int(a\cos\theta)\,a\cos\theta\,d\theta=\frac{a^2}{2}\int(1+\cos2\theta)\,d\theta=\frac{a^2}{2}\left(\theta+\frac{\sin2\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+c=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

8) $\int\sqrt{a^2+x^2}dx$

Remember :

$\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+c$

Proof : Put $x=a\sinh\theta$ so that $\theta=\sinh^{-1}\left(\frac{x}{a}\right)$

then $dx=a\cosh\theta$ and $\int\sqrt{a^2+x^2}dx=\int(a\cosh\theta)\,a\cosh\theta\,d\theta$

$$=a^2\int\cosh^2\theta\,d\theta=\frac{a^2}{2}\int(1+\cosh2\theta)\,d\theta=\frac{a^2}{2}\left[\theta+\frac{\sinh2\theta}{2}\right]d\theta$$

$$=\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}2\sinh\theta\cosh\theta\right]+c=\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}\right]+c$$

$$=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+c$$

Remember :

$\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c$

9) $\int\sqrt{x^2-a^2}dx$

Sol : Put $x=a\cosh\theta$ so that $\theta=\cosh^{-1}\left(\frac{x}{a}\right)$

then $dx=a\sinh\theta\,d\theta$ and $\int\sqrt{x^2-a^2}dx=\int a\sinh\theta\,a\sinh\theta\,d\theta$

$$=a^2\int\sinh^2\theta\,d\theta=\frac{a^2}{2}\int(\cosh2\theta-1)\,d\theta=\frac{a^2}{2}\left[\frac{\sinh2\theta}{2}-\theta\right]+c$$

$$=\frac{a^2}{2}\left[\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}-\cosh^{-1}\left(\frac{x}{a}\right)\right]+c=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c$$

Remark :

In the integral formulae 7,8 and 9 it may be observed that in the RHS the first term is x/2. (integrand) and the 2nd term is $a^2/2$. (integral of the reciprocal of the integrand) and the sign between the two terms is same as the sign before a^2 in the integrand.

The above Nine formulae can be directly used to evaluate the integrals in the corresponding forms.

Example : i) $\int \frac{1}{x^2+4}dx = \int \frac{1}{x^2+2^2}dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$

ii) $\int \frac{1}{9-x^2}dx = \frac{1}{6}\log\left|\frac{3+x}{3-x}\right|+c$

iii) $\int \frac{1}{x^2-16}dx = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right|+c$

*iv) $\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\left(\frac{x}{2}\right)+c$

*v) $\int \frac{1}{\sqrt{x^2+25}}dx = \sinh^{-1}\left(\frac{x}{5}\right)+c$

vi) $\int \frac{1}{\sqrt{x^2-25}} = \cosh^{-1}\left(\frac{x}{5}\right)+c$

vii) $\int \sqrt{25-x^2}\,dx = \frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)+c$

viii) $\int \sqrt{25+x^2}\,dx = \frac{x}{2}\sqrt{25+x^2} + \frac{25}{2}\sinh^{-1}\left(\frac{x}{5}\right)+c$

ix) $\int \sqrt{x^2-25}\,dx = \frac{x}{2}\sqrt{x^2-25} - \frac{25}{2}\cosh^{-1}\left(\frac{x}{5}\right)+c$

Remember :

i) $\int \frac{1}{(a^2+x^2)^{3/2}}dx = \frac{1}{a^2}\frac{x}{\sqrt{a^2+x^2}}+c$

ii) $\int \frac{1}{(a^2-x^2)^{3/2}}dx = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}}+c$

iii) $\int \frac{1}{(x^2-a^2)^{3/2}}dx = -\frac{1}{a^2}\frac{x}{\sqrt{x^2-a^2}}+c$

Note : i) $\frac{d}{dx}(\sqrt{x^2+a^2}) = \frac{x}{\sqrt{x^2+a^2}} \Rightarrow \int \frac{x}{\sqrt{x^2+a^2}}dx = \sqrt{x^2+a^2}+c$

ii) $\frac{d}{dx}(\sqrt{a^2-x^2}) = \frac{-x}{\sqrt{a^2-x^2}} \Rightarrow \int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}+c$

iii) $\frac{d}{dx}(\sqrt{x^2-a^2}) = \frac{x}{\sqrt{x^2-a^2}} \Rightarrow \int \frac{x}{\sqrt{x^2-a^2}}dx = \sqrt{x^2-a^2}+c$

Special Rational and Irrational integrands

To evaluate the integrals of the type $\int \frac{1}{x(x^n+1)}dx, \int \frac{1}{x^n(x^n+1)^{1/n}}dx, \int \frac{1}{x^2(x^n+1)^{\frac{n-1}{n}}}dx,$ ***take*** x^n ***common in Dr. and put*** $1 + x^{-n} = t$.

SOLVED EXAMPLES

1. Evaluate $\int 2x\sin(x^2+1)dx$

Sol. Let $I = \int 2x\sin(x^2+1)dx$

Put $x^2+1 = t$, then $2xdx = dt$

$\therefore I = \int \sin t\, dt = -\cos t + c = -\cos(x^2+1)+c$

Note : $\int f(x)f'(x)dx = \dfrac{f^2(x)}{2}+c$

2. Evaluate $\int \dfrac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$

Sol. Let $I = \int \dfrac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$

Put $\sin^{-1}x = t$, then $\dfrac{1}{\sqrt{1-x^2}}dx = dt$

$\therefore I = \int t^2 dt = \dfrac{t^3}{3}+c = \dfrac{1}{3}(\sin^{-1}x)^3+c$

3. Evaluate $\int \dfrac{x^5}{1+x^{12}}dx$

Sol. Let $I = \int \dfrac{x^5}{1+x^{12}}dx$

Put $x^6 = t$, then $6x^5dx = dt$

$\therefore I = \dfrac{1}{6}\int \dfrac{dt}{1+t^2} = \dfrac{1}{6}\tan^{-1}(t)+c \;= \dfrac{1}{6}\tan^{-1}(x^6)+c$

Note : $\int e^{f(x)}f'(x)dx = e^{f(x)}+c$

4. Evaluate $\int e^{x+\frac{1}{x}}\left(1-\dfrac{1}{x^2}\right)dx$

Sol. Let $I = \int e^{x+\frac{1}{x}}\left(1-\dfrac{1}{x^2}\right)dx$

Put $x+\dfrac{1}{x} = t$, then $\left(1-\dfrac{1}{x^2}\right)dx = dt$

$\therefore I = \int e^t dt = e^t + c = e^{x+\frac{1}{x}}+c$

Note : *Technique involved in the solution is very very important*

5. Evaluate $\int \dfrac{dx}{x^2\sqrt{4+x^2}}$

Sol. $\int \dfrac{dx}{x^2\sqrt{4+x^2}} = \int \dfrac{dx}{x^3\sqrt{\dfrac{4}{x^2}+1}}$; Put $\dfrac{4}{x^2}+1 = t$; $\dfrac{-8}{x^3}dx = dt$

$\therefore \dfrac{-1}{8}\int \dfrac{dt}{\sqrt{t}} = \dfrac{-1}{4}\sqrt{t}+c \;= \dfrac{-1}{4x}\sqrt{4+x^2}+C$

6. Evaluate $\int \frac{dx}{\sqrt{x-x^2}} =$

Sol. $\int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{x}\sqrt{1-x}}$

Put $\sqrt{x} = \sin\theta \Rightarrow \frac{1}{2\sqrt{x}}dx = \cos\theta\, d\theta$

$\therefore = \int \frac{2\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = 2\theta + c = 2\sin^{-1}\sqrt{x} + c$

7. Evaluate $\int \frac{\sin^4 x}{\cos^6 x} dx$

Sol. $\int \frac{\sin^4 x}{\cos^6 x} dx = \int \tan^4 x . \sec^2 x\, dx$

Put $\tan x = t$, then $\sec^2 x\, dx = dt$ $= \int t^4 dt = \frac{t^5}{5} + c = \frac{1}{5}\tan^5 x + c$

8. Evaluate $\int \frac{x^2}{\sqrt{x+5}} dx$

Sol. Put $\sqrt{x+5} = t$, then $x = t^2 - 5$ and $dx = 2t\, dt$

$\therefore \int \frac{x^2}{\sqrt{x+5}} dx = \int \frac{1}{t}(t^2-5)^2 2t.dt$

$= 2\int (t^4 - 10t^2 + 25)dt = 2\left[\frac{t^5}{5} - 10t^2 + 25\right]dt$

$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50\sqrt{x+5} + c$

9. Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Sol. Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\therefore \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int \cos t\, 2dt = 2\sin t + c = 2\sin\sqrt{x} + c$

***10. Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$** **(March-17, 19 & May-17)**

Sol. Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Put $xe^x = t$, then $(x+1)e^x dx = dt$

$\therefore I = \int \sec^2 t\, dt = \tan t + c = \tan(xe^x) + c$

11. Evaluate $\int\sqrt{e^x-4}\,dx$

Sol. Put $e^x-4=t^2 \Rightarrow e^x dx = 2t\,dt \;\Rightarrow dx=\frac{2t}{t^4+4}dt$

Now, $\int\sqrt{e^x-4}\;dx=\int\sqrt{t^2}.\frac{(2t)}{t^2+4}dt\;=2\int\frac{t^2}{t^2+4}dt$

$=2\int\left(\frac{t^2+4-4}{t^2+4}\right)dt\;=2\int\left\{1-\frac{4}{t^2+4}\right\}dt$

$=2\left[t-4.\frac{1}{2}.\tan^{-1}\left(\frac{t}{2}\right)\right]+C\;=2\left[\sqrt{e^x-4}-2\tan^{-1}\left(\frac{1}{2}\sqrt{e^x-4}\right)\right]+C$

Note :

$\int\frac{dx}{\sqrt{a-bx^2}}$

$=\frac{1}{\sqrt{b}}\sin^{-1}\left(\sqrt{\frac{b}{a}}x\right)+c$

12. Evaluate $\int\frac{1}{\sqrt{4-9x^2}}dx$

Sol. (i) $\int\frac{1}{\sqrt{4-9x^2}}dx=\int\frac{1}{\sqrt{2x^2-(3x)^2}}dx$

Put $3x=t$, then $dx=\frac{1}{3}dt$

$=\frac{1}{3}\int\frac{dt}{\sqrt{2^2-t^2}}=\frac{1}{3}\sin^{-1}\left(\frac{t}{2}\right)+c\;=\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+c$

Note :

$\int\frac{dx}{\sqrt{a+bx^2}}$

$=\frac{1}{\sqrt{b}}\sinh^{-1}\left(\sqrt{\frac{b}{a}}x\right)+c$

13. Evaluate $\int\frac{2}{\sqrt{9x^2+25}}dx$

Sol. $\int\frac{2}{\sqrt{9x^2+25}}dx=\int\frac{2}{\sqrt{(3x)^2+5^2}}dx$

$=\frac{2}{3}\int\frac{dt}{\sqrt{t^2+5^2}}=\frac{2}{3}\sinh^{-1}\left(\frac{t}{5}\right)\;=\frac{2}{3}\sinh^{-1}\left(\frac{3x}{5}\right)+c$

Note :

$\int\frac{dx}{\sqrt{ax^2-b}}$

$=\frac{1}{\sqrt{a}}\cosh^{-1}\left(\sqrt{\frac{a}{b}}x\right)+c$

14. Evaluate $\int\frac{1}{\sqrt{3x^2-5}}dx$

Sol. $\int\frac{1}{\sqrt{3x^2-5}}dx=\int\frac{1}{\sqrt{(\sqrt{3}x^2)-5}}dx$

Put $\sqrt{3}x=t$, then $dx=\frac{1}{\sqrt{3}}dt$

$=\frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{t^2-(\sqrt{5})^2}}=\frac{1}{\sqrt{3}}\cosh^{-1}\left(\frac{t}{\sqrt{5}}\right)+c\;=\frac{1}{\sqrt{3}}\cosh^{-1}\left(\frac{\sqrt{3}x}{\sqrt{5}}\right)+c$

15. **Evaluate** $\int \frac{dx}{\sqrt{25+x^2}}$

Sol. $\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{dx}{\sqrt{5^2+x^2}} = \sinh^{-1}\left(\frac{x}{5}\right)+C$

16. **Evaluate** $\int \sqrt{9-4x^2}\,dx$

Sol. $\int \sqrt{9-4x^2}\,dx = \int \sqrt{3^2-(2x)^2}\,dx$

$= \frac{1}{2}\left[\frac{2x}{2}\sqrt{9-4x^2}+\frac{9}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right]+c$

$= \frac{x}{2}\sqrt{9-4x^2}+\frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right)+c$

17. **Evaluate** $\int \sqrt{25+16x^2}\,dx$

Sol. $\int \sqrt{25+16x^2}\,dx = \int \sqrt{5^2+(4x)^2}\,dx$

$= \frac{1}{4}\left[\frac{4x}{2}\sqrt{25+16x^2}+\frac{25}{2}\sinh^{-1}\left(\frac{4x}{5}\right)\right]+c$

$= \frac{x}{2}\sqrt{25+16x^2}+\frac{25}{8}\sinh^{-1}\left(\frac{4x}{5}\right)+c$

18. **Evaluate** $\int \sqrt{16x^2-9}\,dx$

Sol. $\int \sqrt{16x^2-9}\,dx = \int \sqrt{(4x)^2-3^2}\,dx$

$= \frac{1}{4}\left[\frac{4x}{2}\sqrt{16x^2-9}-\frac{9}{2}\cosh^{-1}\left(\frac{4x}{3}\right)\right]+c$

$= \frac{x}{2}\sqrt{16x^2-9}-\frac{9}{8}\cosh^{-1}\left(\frac{4x}{3}\right)+c$

***19.** **Evaluate** $\int \frac{1}{x\log x[\log(\log x)]}dx$ **(March-19)**

Sol. Let $I = \int \frac{1}{x\log x[\log(\log x)]}dx$

Put log (log$x$) = t, then $\frac{1}{\log x}.\frac{1}{x}dx = dt$

$\therefore I = \int \frac{1}{t}dt = \log|t|+c = \log|\log(\log x)|+c$

20. Evaluate $\int \sec x \log|sex + \tan x|\, dx$

Sol. Put $\log|\sec x + \tan x| = t$, then

$$\frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)dx = dt$$

$$\Rightarrow \sec x\, dx = dt$$

$$\therefore \text{ Given Integral } = \int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}(\log|\sec x + \tan x|)^2 + c$$

21. Evaluate $\int \frac{x^2}{(a+bx)^2} dx$

Sol. Put $a + bx = t$, then $x = \frac{t-a}{b}$ and $dx = \frac{1}{b}dt$

$$\therefore \int \frac{x^2}{(a+bx)^2} dx = \frac{1}{b}\int \frac{1}{t^2}\left(\frac{t-a}{b}\right)^2 dt$$

$$= \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2} dt \quad = \frac{1}{b^3}\int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) dt$$

$$= \frac{1}{b^3}\left[t - 2a\log t - \frac{a^2}{t}\right] + c = \frac{1}{b^3}\left[(a+bx) - 2a\log(a+bx) - \frac{a^2}{a+bx}\right] + c$$

22. Evaluate $\int \cos^4 x\, dx$

Sol. $$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}[1 + 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right] = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\therefore \int \cos^4 x\, dx = \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

23. Evaluate $\int \sqrt{1 + \sec x}\, dx$

Sol. $$\int \sqrt{\sec x + 1}\, dx = \int \sqrt{\frac{\sec^2 x - 1}{\sec x - 1}} dx$$

$$= \int \frac{\tan x}{\sqrt{\sec x - 1}} dx = \int \frac{\frac{\sin x}{\cos x}}{\sqrt{\frac{1-\cos x}{\cos x}}} dx = \int \frac{\sin x}{\sqrt{\cos x}\sqrt{1-\cos x}} dx$$

Put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$$= \int \frac{-dt}{\sqrt{t}\sqrt{1-t}} = -\int \frac{1}{\sqrt{t-t^2}}dt$$

$$= -\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}} = -\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right) + C \quad = -\left[t^2 - t + \frac{1}{4} - \frac{1}{4}\right]$$

$$= -\sin^{-1}(2t-1) + C = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 = -\sin^{-1}[2\cos x - 1] + C$$

Note :

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

24. **Evaluate** $\int \frac{\sin x}{\sin(a+x)}dx$

Sol. $\sin x = \sin(a + x - a) = \sin(a + x)\cos a - \cos(a + x)\sin a$

$$\int \frac{\sin x}{\sin(a+x)}dx = \cos a \int dx - \sin a \int \frac{\cos(a+x)}{\sin(a+x)}dx$$

$= x \,.\, \cos a - \sin a \,.\, \log \sin(a + x) + c$

25. **Evaluate** $\int \frac{e^{\log x}}{x}dx$

Sol. $\int \frac{e^{\log x}}{x}dx$

$t = \log x \Rightarrow dt = \frac{1}{x}.dx$

$$\int \frac{e^{\log x}}{x}dx = \int e^t dt = e^t + C = e^{\log x} + C = x + C$$

26. **Evaluate** $\int \cot h\, x\, dx$

Sol. $t = \sinh x \Rightarrow dt = \cos h\, x\, dx$

$$\int \cot h\, x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin h\, x| + C$$

27. **Evaluate** $\int \frac{1}{1+\sin 2x}dx$

Sol. $$\int \frac{1}{1+\sin 2x}dx = \int \frac{dx}{1+\frac{2\tan x}{1+\tan^2 x}}$$

$$= \int \frac{(1+\tan^2 x)dx}{1+\tan^2 x + 2\tan x} = \int \frac{\sec^2 x\, dx}{(1+\tan x)^2}$$

$t = 1 + \tan x \Rightarrow dt = \sec^2 x\, dx$

$$\int \frac{1}{1+\sin 2x}dx = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{1+\tan x} + C$$

28. Evaluate $\int \frac{1}{\cos^2 x+\sin 2x}dx$

Sol. $\int \frac{1}{\cos^2 x+\sin 2x}dx$

Dividing Nr and Dr by $\cos^2 x$

$$=\int \frac{\sec^2 x}{1+\frac{\sin 2x}{\cos^2 x}}dx = \int \frac{\sec^2 x}{1+2\tan x}dx$$

$$=\frac{1}{2}\int \frac{2\sec^2 x}{1+2\tan x}dx = \frac{1}{2}\log|1+2\tan x|+C$$

29. Evaluate $\int \frac{\sin 2x}{(a+b\cos x)^2}dx$

Sol. Put $a+b\cos x=t \Rightarrow \cos x=\frac{t-a}{b}$

Then $b(-\sin x)\,dx=dt \Rightarrow \sin x\,dx=\frac{-1}{b}dt$

Now, $\int \frac{\sin 2x}{(a+b\cos x)^2}dx=\int \frac{2\sin x\cos x}{(a+b\cos x)^2}dx$

$$=2\int \frac{\left(\frac{t-a}{b}\right)\left(\frac{-1}{b}\right)}{t^2}dt = \frac{-2}{b^2}\int\left(\frac{t}{t^2}-\frac{a}{t^2}\right)dt$$

$$=\frac{-2}{b^2}\left[\log|t|+\frac{a}{t}\right]+C = \frac{-2}{b^2}\left[\log|a+b\cos x|+\frac{a}{a+b\cos x}\right]+C$$

30. Evaluate $\int \frac{\sec x}{(\sec x+\tan x)^2}dx$

Sol. Put $\sec x+\tan x = t$

Then $(\sec x\tan x+\sec^2 x)\,dx=dt$

$\sec x(\sec x+\tan x)\,dx=dt$

Now, $\int \frac{\sec x}{(\sec x+\tan x)^2}dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^3}dx$

$$\int \frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-3+1}}{-3+1}+C=-\frac{1}{2}\cdot\frac{1}{t^2}+C = -\frac{1}{2}\cdot\frac{1}{(\sec x+\tan x)^2}+C$$

31. Evaluate $\int \frac{1-\tan x}{1+\tan x}\,dx$

Sol. $\int \frac{(1-\tan x)}{1+\tan x}\,dx = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\,dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

$t = \cos x + \sin x$

$dt = (-\sin x + \cos x)\,dx$

$\int \frac{1-\tan x}{1+\tan x}\,dx = \int \frac{dt}{t} = \log|t| + C = \log|\cos x + \sin x| + C$

Note : *For this kind of problem, mentioning the interval is very very important.*

32. Evaluate $\int \sqrt{1+\cos 2x}\,dx$

Sol. $\int \sqrt{1+\cos 2x}\,dx = \int \sqrt{2\cos^2 x}\,dx = \sqrt{2}\int \cos x\,dx = \sqrt{2}\sin x + C$

EXERCISE - 1.2

Evaluate the following integrals.

1. *a) $\int e^{2x}dx$ *b) $\int \sin 7x\,dx$ c) $\int 2x\cos(x^2+1)dx$

*d) $\int \frac{x}{x^2+1}dx$ *e) $\int \frac{1}{x}(\log x)^2 dx$ *f) $\int \frac{1}{1+x^2}e^{\tan^{-1}x}dx$

*g) $\int \frac{e^x}{e^x+1}dx$ **(May-18)**   *h) $\int \frac{6x}{3x^2-2}dx$ *i) $\int \frac{1}{1+(2x+1)^2}dx$

*j) $\int \cos^3 x.\sin x\,dx$ **(May-18)**   **k) $\int \sin(\tan^{-1}x)\frac{1}{1+x^2}dx$ **(March-18)**

*l) $\int \frac{3x^2}{1+x^6}dx$ *m) $\int \frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx$ *n) $\int \frac{2x+1}{x^2+x+1}dx$

*o) $\int \frac{1-\tan x}{1+\tan x}dx$ *p) $\int \frac{1}{7x+3}dx$ *q) $\int \frac{1}{\sqrt{1+5x}}dx$

*r) $\int (\sqrt{3x-2})dx$ *s) $\int (3x^2-4)x\,dx$ *t) $\int (1-2x^3)x^2dx$

2. **a) $\int \frac{\log(1+x)}{1+x}dx$ *b) $\int \frac{\sec^2 x}{(1+\tan x)^3}dx$ *c) $\int x^3\sin(x^4)\,dx$

*d) $\int \frac{\cos x}{(1+\sin x)^2 dx}$ *e) $\int \sqrt[3]{\sin x}\cos x\,dx$ *f) $\int 2xe^{x^2}dx$

*g) $\int \frac{e^{\log x}}{x}dx$ *h) $\int \frac{x^2}{\sqrt{1-x^6}}dx$ *i) $\int \frac{2x^3}{1+x^8}dx$

*j) $\int \frac{x^8}{1+x^{18}}dx$ (May-19) *k) $\int \frac{\operatorname{cosec}^2 x}{(a+b\cot x)^5}dx$ *l) $\int e^x \sin e^x dx$

*m) $\int \frac{\sin(\log x)}{x}dx$ *n) $\int \frac{1}{x\log x}dx$ *o) $\int \frac{(1+\log x)^n}{x}dx$

*p) $\int \frac{\cos(\log x)}{x}dx$ *q) $\int \frac{ax^{n-1}}{bx^n+c}dx$ *r) $\int \coth x\,dx$

**s) $\int \frac{1}{(x+3)\sqrt{x+2}}dx$ on $x \in I \subset (-2,\infty)$ *t) $\int \frac{dx}{(x+5)\sqrt{x+4}}$

3. *a) $\int \frac{1}{a\sin x+b\cos x}dx$ where $\cos\alpha=\frac{a}{r}$, $\sin\alpha=\frac{b}{r}$, $r=\sqrt{a^2+b^2}$]

b) $\int \frac{1}{\sin x+\sqrt{3}\cos x}dx$ *c) $\int \frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx$ *d) $\int \frac{\cot(\log x)}{x}dx$

*e) $\int e^x \cot e^x dx$ *f) $\int \sec(\tan x)\sec^2 x\,dx$ *g) $\int \sqrt{\sin x}\cos x\,dx$

*h) $\int \tan^4 x\sec^2 x\,dx$ *i) $\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx$ *j) $\int \operatorname{cosec}^2 x\sqrt{\cot x}\,dx$

*k) $\int \sin^3 x\,dx$ *l) $\int \cos^3 x\,dx$ *m) $\int \cos x\cos 2x\,dx$

*n) $\int \cos x\cos 3x\,dx$ *o) $\int x\sqrt{4x+3}\,dx$ *p) $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}}$

*q) $\int \frac{dx}{a^2+(b+cx)^2}$ *r) $\int \frac{dx}{1+e^x}$ *s) $\int \frac{x^2}{\sqrt{1-x}}dx$ t) $\int \frac{x^3}{\sqrt{x+1}}dx$

4. a) $\int \cos x\cos 2x\cos 3x\,dx$ b) $\int \sin x\sin 2x\sin 3x\,dx$ c) $\int \frac{1}{\sin(x-a)\sin(x-b)}dx$

d) $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ e) $\int \frac{\sec x}{(\sec x+\tan x)^n}dx$

5. *a) $\int \frac{1}{1+4x^2}dx$ *b) $\int \frac{1}{8+2x^2}dx$ c) $\int \sqrt{4x^2+9}\,dx$

*d) $\int \sqrt{9x^2-25}\,dx$ *e) $\int \sqrt{16-25x^2}\,dx$ f) $\int \frac{3}{\sqrt{9x^2-1}}dx$

g) $\int \frac{1}{\sqrt{1-4x^2}}dx$

1.12 INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS (METHOD OF PARTIAL FRACTIONS)

If the integrand *f(x)* is a rational algebraic function which cannot be integrated by direct methods, we have to look for the possibility of resolving it into a sum of simple partial fractions which can be directly integrated.

Example : $\int \frac{x+1}{x^2+x-2}dx = \int\left(\frac{A}{x+2}+\frac{B}{x-1}\right)dx = \frac{1}{3}\int\frac{1}{x+2}dx + \frac{2}{3}\int\frac{1}{x-1}dx$

$$= \frac{1}{3}\log|x+2| + \frac{2}{3}\log|x-1| + c = \log|(x+2)^{1/3}.(x-1)^{2/3}|+c$$

A Note on Partial Fractions :

Let $f(x)=\frac{P(x)}{Q(x)}$ be a proper fraction (i.e., *P(x)* and *Q(x)* are polynomial functions such that deg. $P(x) <$ deg. $Q(x)$. Based on the type of factors of the denominator *Q(x)*, the method of expressing *f(x)* as a sum of partial fractions, is illustrated below.

Type (i) : *Q(x)* contains non - repeated linear factors only

Example : $\frac{P(x)}{(ax+b)(cx-d)(x-\alpha)(x+\beta)} = \frac{A}{ax+b}+\frac{B}{cx-d}+\frac{C}{x-\alpha}+\frac{D}{x+\beta}$ -- (1)

In the equation (1) the L.H.S is the given fraction and R.H.S is the sum of partial fractions corresponding to the linear factors of *Q(x)*. The unknown constants *A,B,C,D* are to be found by solving the equations obtained by comparing the coefficients of different powers of *x* in the identity (which is obtained after multiplying both sides of equation (1) with the denominator (D^r) of *P*(*x*)).

Note : In the case of non - repeated linear factors, the values of *A,B,C,D* can also be found directly, using Horner's cover - up method :

$A = \frac{P\left(-b/a\right)}{\left[c\left(\frac{-b}{a}\right)-d\right]\left[-\frac{b}{a}-\alpha\right]\left[\frac{-b}{a}+\beta\right]}$ which is obtained by substituting $x=-\frac{b}{a}$ in the L.H.S of (1) after covering-up (or removing) the factor $ax+b$.

Similarly $B=\frac{P\left(\frac{d}{c}\right)}{\left[a\left(\frac{d}{c}\right)+b\right]\left[\frac{d}{c}-\alpha\right]\left[\frac{d}{c}+\beta\right]}$; $C=\frac{P(\alpha)}{[a\alpha+b][c\alpha-\alpha][\alpha+\beta]}$;

$$D=\frac{P(\beta)}{[a(-\beta)+b][c(-\beta)-d][-\beta-\alpha]}$$

Type (ii) : *Q(x)* contains repeated linear factors only

Example : $\frac{P(x)}{(ax+b)^2(x-\alpha)^3} = \frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{B_1}{x-\alpha}+\frac{B_2}{(x-\alpha)^2}+\frac{B_3}{(x-\alpha)^3}$ --(2)

The values of unknown constants A_1, A_2, B_1, B_2, B_3 are to be found by solving the equations obtained by comparing the coefficients of different powers of *x* in the identity (which is obtained after multiplying eq(2) with the D^r of *P*(*x*)).

Note : In this case, the values of A_2 and B_3 - the denominators of which have largest powers - can also be found by cover - up method, but not the other unknown constants A_1, B_1 and B_2

Type (iii) : $Q(x)$ contains non-repeated irreducible quadratic factors only

Example : $\dfrac{P(x)}{(x^2+a^2)(x^2+x+1)(x^2-x+1)} = \dfrac{Ax+B}{x^2+a^2} + \dfrac{Cx+D}{x^2+x+1} + \dfrac{Ex+F}{x^2-x+1}$ -- (3)

The values of the unknown constants A,B,C,D,E and F are to found by solving the equations obtained by equating the coefficients of like powers of x in the identity (obtained after multiplying the equation (3) with the denominator of $P(x)$ in the equation).

Type (iv) : $Q(x)$ contains repeated irreducible quadratic factors only

Example : $\dfrac{P(x)}{(x^2+a^2)^2(x^2+x+1)^3} = \dfrac{A_1x+B_1}{x^2+a^2} + \dfrac{A_2x+B_2}{(x^2+a^2)^2} + \dfrac{C_1x+D_1}{x^2+x+1} + \dfrac{C_2x+D_2}{(x^2+x+1)^2} + \dfrac{C_3x+D_3}{(x^2+x+1)^3}$ --(4)

The values of the unknown constants $A_1,B_1,A_2,B_2,C_1,D_1,C_2,D_2,C_3$ and D_3 are to be found by solving the equations obtained by equating the coefficients of like powers of x in the identity (obtained after multiplying the equation (4) with the D^r. of $P(x)$ in the equation).

Mixed type : $Q(x)$ contains all the above types of factors

where the unknown constants $A, B,...F$ are to be found by solving the equations obtained by equating the coefficients of like powers of x in the identity that results when the equation (5) is multiplical with the D^r of $P(x)$.

Note

If $f(x)=\dfrac{P(x)}{Q(x)}$ is not a proper fraction (i.e., if the deg $P(x) \ge \deg.Q(x)$ then f(x) should be expressed as the sum of a polynomial q(x) and a proper fraction $\dfrac{R(x)}{Q(x)}$ by dividing P(x) with Q(x). This proper fraction can be resolved into a sum of partial fractions.

SOLVED EXAMPLES

****1.*** **Evaluate** $\int \dfrac{2x^2-5x+1}{x^2(x^2-1)}dx$

Sol. Let $\dfrac{2x^2-5x+1}{x^2(x^2-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$

$\Rightarrow 2x^2-5x+1 \equiv A x(x^2-1)+B(x^2-1)+Cx(x+1)+Dx^2(x-1)$

Comparing the coefficients of different powers of x, we get $A=5, B=-1, C=-1, D=-4$

$\therefore \int \dfrac{2x^2-5x+1}{x^2(x^2-1)}dx = 5\int \dfrac{1}{x}dx - \int \dfrac{1}{x^2}dx - \int \dfrac{1}{x-1}dx - 4\int \dfrac{1}{x+1}dx$

$= 5\log|x| + \dfrac{1}{x} - \log|x-1| - 4\log|x+1| + c = \dfrac{1}{x} + \log\left|\dfrac{x^3}{(x^2-1)(x+1)^3}\right| + c$

Remember :

$\displaystyle\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\log\left|\frac{x+a}{x+b}\right| + c$

****2.*** **Evaluate** $\int \dfrac{x^3-2x+3}{x^2+x-2}dx$

Sol. $\dfrac{x^3-2x+3}{x^2+x-2} = (x-1) + \dfrac{x+1}{x^2+x-2}$ (by actual division)

Let $\dfrac{x+1}{x^2+x-2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-1} \Rightarrow x+1 \equiv A(x-1)+B(x+2) \Rightarrow A=\dfrac{1}{3}, B=\dfrac{2}{3}$

$\therefore \int \dfrac{x^3-2x+3}{x^2+x-2}dx = \int (x-1)dx + \dfrac{1}{3}\int \dfrac{1}{x+2}dx + \dfrac{2}{3}\int \dfrac{1}{x-1}dx$

$= \dfrac{x^2}{2} - x + \dfrac{1}{3}\log|x+2| + \dfrac{2}{3}\log|x-1| + c = \dfrac{x^2}{2} - x + \log\left|(x+3)^{1/3}(x-1)^{2/3}\right| + c$

EXERCISE - 1.3

Evaluate the following integrals.

*1. $\int \frac{x-1}{(x-2)(x-3)}dx$

*2. $\int \frac{dx}{(x+1)(x+2)}$ (March-19)

*3. $\int \frac{x+3}{(x-1)(x^2+1)}dx$

*4. $\int \frac{1}{(x^2+a^2)(x^2+b^2)}dx$

*5. $\int \frac{dx}{e^x+e^{2x}}$

*6. $\int \frac{1}{e^x-1}dx$

*7. $\int \frac{x^2}{(x+1)(x+2)^2}dx$

*8. $\int \frac{1}{(1-x)(x^2+4)}dx$

*9. $\int \frac{2x+3}{x^3+x^2-2x}dx$

*10. $\int \frac{dx}{6x^2-5x+1}$

*11. $\int \frac{dx}{x(x+1)(x+2)}$

*12. $\int \frac{3x-2}{(x-1)(x+2)(x-3)}dx$

*13. $\int \frac{7x-4}{(x-1)^2(x+2)}dx$

*14. $\int \frac{1}{(x-a)(x-b)(x-c)}dx$

*15. $\int \frac{2x+3}{(x+3)(x^2+4)}dx$

*16. $\int \frac{2x^2+x+1}{(x+3)(x-2)^2}dx$

*17. $\int \frac{\sin x \cos x}{\cos^2 x+3\cos x+2}dx$

*18. $\int \left(\sqrt{x}-\frac{2}{1-x^2}\right)dx$

*19. $\int \frac{1}{x^3+1}dx$

1.13 SOME STANDARD TYPES OF INTEGRALS

In the following, we discuss about some standard types of integrals involving rational and irrational functions with suitable examples.

Type 1 : **To evaluate** $\int \frac{1}{(ax^2+bx+c)}dx$

If ax^2+bx+c can be factorised then the integral can be resolved into a sum of partial fractions which can be integrated directly. If ax^2+bx+c can not be factorised (or if the factors involve irrational numbers) then the integral, can be expressed in one of the forms $\int \frac{1}{X^2+A^2}dX, \int \frac{1}{X^2-A^2}dX$ or $\int \frac{1}{A^2-X^2}dX$ and hence can be integrated. It may be noted that $ax^2+bx+c \equiv a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$

$$\Rightarrow \int \frac{1}{ax^2+bx+c}dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)}dx$$

which is of the form $\frac{1}{a}\int \frac{1}{X^2 \pm A^2}dx$ and hence can be evaluated.

Remember :

$\int \frac{1}{(ax+b)(cx+d)}dx$

$= \frac{1}{ad-bc}\log\left|\frac{ax+b}{cx+d}\right|+K$

Ex 1: $\int \frac{1}{2x^2+x-1}dx = \int \frac{1}{(2x-1)(x+1)}dx = \frac{1}{3}\int\left(\frac{2}{2x-1}-\frac{1}{x+1}\right)dx = \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right|+c$

Ex 2: $\int \frac{1}{2x^2+3x+4}dx = \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}dx = \frac{1}{2}\int \frac{1}{X^2+A^2}dx$

where $X = x+\frac{3}{4},\ A = \frac{\sqrt{23}}{4} = \frac{1}{2}\frac{4}{\sqrt{23}}\tan^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{23}}{4}}\right)+c = \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+c$

Ex 3: $\int \frac{1}{4+3x-2x^2}dx = \frac{1}{2}\int \frac{1}{\frac{41}{16}-\left(x-\frac{3}{4}\right)^2}dx = \frac{1}{2}\int \frac{1}{A^2-X^2}dX$ where $X = x-\frac{3}{4}, A = \frac{\sqrt{41}}{2}$

$= \frac{1}{2}\frac{1}{2A}\log\left|\frac{A-X}{A+X}\right|+c = \frac{1}{\sqrt{41}}\log\left|\frac{\sqrt{41}-3+4x}{\sqrt{41}+3-4x}\right|+c$

Ex 4 : $\int \frac{1}{1-x-x^2}dx = \int \frac{1}{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}dx$

$= \int \frac{1}{A^2-X^2}dX$ where $X = x+\frac{1}{2}, A = \frac{\sqrt{5}}{2} = \frac{1}{2A}\log\left|\frac{A+x}{A-x}\right|+c = \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}+1+2x}{\sqrt{5}-1-2x}\right|+c$

Note

i) *If $b^2-4ac<0$ and $a>0$ then* $\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+k$

ii) *If $b^2-4ac>0$ and $a>0$ then* $\int \frac{1}{ax^2+bx+c}dx = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+k$

iii) *If $b^2-4ac>0$ and $a<0$ then* $\int \frac{1}{ax^2+bx+c}dx = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{\sqrt{b^2-4ac}-(2ax+b)}{\sqrt{b^2-4ac}+(2ax+b)}\right|+k$

Note that $2ax+b = \frac{d}{dx}(ax^2+bx+c)$

Type 2 : **To evaluate** $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

This integral can be transformed into one of the following types:

$\int \frac{1}{\sqrt{X^2+A^2}}dx,\ \int \frac{1}{\sqrt{X^2-A^2}}dx$ or $\int \frac{1}{\sqrt{A^2-X^2}}dx$

depending on the sign of a and (b^2-4ac).

Let $I = \int \frac{1}{\sqrt{ax^2+bx+c}}dx$

i) If $b^2-4ac<0$ and $a>0$ then $I = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}}dx$

ii) If $b^2-4ac>0$ and $a>0$

then $I = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}}dx$

iii) If $b^2 - 4ac > 0$ and $a < 0$ then

$$I = \frac{1}{\sqrt{\lambda}} \int \frac{1}{\sqrt{\left(\frac{b^2 + 4\lambda c}{4\lambda^2}\right) - \left(x - \frac{b}{2\lambda}\right)^2}}\, dx \quad \text{where } \lambda = -a \, (\lambda > 0)$$

Ex 1: $\int \frac{1}{\sqrt{x^2 + 2x + 26}}\, dx = \int \frac{1}{\sqrt{(x+1)^2 + 5^2}}\, dx$

$$= \int \frac{1}{\sqrt{X^2 + A^2}}\, dX,\ X = x + 1, A = 5 = \sinh^{-1}\left(\frac{X}{A}\right) + c = \sinh^{-1}\left(\frac{x+1}{5}\right) + c$$

Ex 2: $\int \frac{1}{\sqrt{3 + 2x - x^2}}\, dx = \int \frac{1}{\sqrt{4 - (x-1)^2}}\, dx = \int \frac{1}{\sqrt{A^2 - X^2}}\, dX,\ X = x - 1,\ A = 2$

$$= \sin^{-1}\left(\frac{X}{A}\right) + c = \sin^{-1}\left(\frac{x-1}{2}\right) + c$$

Ex 3: $\int \frac{1}{\sqrt{x^2 - 4x - 12}}\, dx = \int \frac{1}{\sqrt{(x-2)^2 - 16}}\, dx$

$$= \int \frac{1}{\sqrt{X^2 - A^2}}\, dx,\ X = x - 2,\ A = 4 \ = \cosh^{-1}\left(\frac{X}{A}\right) + c = \cosh^{-1}\left(\frac{x-2}{4}\right) + c$$

Note

i) *If $b^2 - 4ac < 0$ and $a > 0$ then* $\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx = \frac{1}{\sqrt{a}} \sinh^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + k$

ii) *If $b^2 - 4ac > 0$ and $a > 0$ then* $\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx = \frac{1}{\sqrt{a}} \cosh^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right) + k$

iii) *If $b^2 - 4ac > 0$ and $a < 0$ then* $\int \frac{1}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{(-a)}} \sin^{-1}\left[\frac{-(2ax + b)}{\sqrt{b^2 - 4ac}}\right] + k$

Type 3: **To evaluate** $\int \sqrt{ax^2 + bx + c}\ dx$

The integrand can be transformed into one of the forms $\sqrt{X^2 + A^2}, \sqrt{X^2 - A^2}$ or $\sqrt{A^2 - X^2}$ depending on the sign of a and hence can be integrated.

Writing $ax^2 + bx + c = \frac{1}{4a}[(2ax + b)^2 + (4ac - b^2)]$, we can show that

Remember :

$$\int \sqrt{ax^2 + bx + c}\, dx = \frac{(2ax + b)}{4a} \sqrt{ax^2 + bx + c} - \frac{(b^2 - 4ac)}{8a} \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

where the last integral can be evaluated as in

Note

$$\int \sqrt{ax^2 + c}\ dx = \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2} \int \frac{1}{\sqrt{ax^2 + c}}\, dx$$

Ex 1. **Evaluate** $\int \sqrt{2x^2+3x+4}\,dx$

Sol. $ax^2+bx+c \equiv 2x^2+3x+4 \;\therefore a=2, b=3, c=4, b^2-4ac=-23$

derivative $=4x+3 \;\therefore \int\sqrt{2x^2+3x+4} = \dfrac{(4x+3)}{8}\sqrt{2x^2+3x+4}+\dfrac{23}{16}\int\dfrac{1}{\sqrt{2x^2+3x+4}}\,dx$ --(1)

$$\text{Now, } \int\frac{1}{\sqrt{2x^2+3x+4}}\,dx=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}}\,dx$$

$$=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{23}}{4}}\right)+c=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+c \quad \text{-- (2)}$$

$\therefore$ From (1) & (2)

$$\int\sqrt{2x^2+3x+4}\,dx=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+\frac{23}{16\sqrt{2}}\sinh^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+c$$

Note : $\sinh^{-1}\left(\dfrac{4x+3}{\sqrt{23}}\right)=\log[(4x+3)+2\sqrt{2}\sqrt{2x^2+3x+4}]$

Ex 2. **Evaluate** $\int\sqrt{1-x-x^2}\,dx$

Sol. Here $a=-1, b=-1, c=1, b^2-4ac=5$, derivative $=-2x-1$

$$\therefore\int\sqrt{1-x-x^2}\,dx=\frac{-(2x+1)}{4(-1)}\sqrt{1-x-x^2}-\frac{5}{8(-1)}\int\frac{1}{\sqrt{1-x-x^2}}\,dx$$

$$=\frac{(2x+1)}{4}\sqrt{1-x-x^2}+\frac{5}{8}\int\frac{1}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}}\,dx$$

$$=\frac{2x+1}{4}\sqrt{1-x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{5}/2}\right)+c=\frac{2x+1}{4}\sqrt{1-x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+c$$

Type 4 : To evaluate $\int\dfrac{px+q}{ax^2+bx+c}\,dx$

To evaluate this integral, we take

$$px+q\equiv A\frac{d}{dx}(ax^2+bx+c)+B$$

so that $px+q=(2aA)x+bA+B$

$$\Rightarrow 2aA=p \text{ and } bA+B=q \Rightarrow A=\frac{p}{2a} \text{ and } B=\frac{2aq-bp}{2a}$$

$$\therefore\int\frac{px+q}{ax^2+bx+c}\,dx=\int\frac{\left[\frac{p}{2a}(2ax+b)+\frac{2aq-bp}{2a}\right]}{ax^2+bx+c}\,dx$$

$$=\frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}\,dx+\frac{(2aq-bp)}{2a}\int\frac{1}{ax^2+bx+c}\,dx$$

Remember :

$$\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\log\left|ax^2+bx+c\right| + \frac{2aq-bp}{2a}\int \frac{1}{ax^2+bx+c}dx$$

where the last integral can be evaluated as in Type 1.

Ex 1. **Evaluate** $\int \frac{2x-5}{3x^2+4x+5}dx$.

Sol. To evaluate the given integral, let us take $2x-5 \equiv A\frac{d}{dx}(3x^2+4x+5)+B$

$$\Rightarrow 2x-5 = 6Ax+(4A+B) \;\Rightarrow A = \frac{1}{3} \text{ and } B = \frac{-19}{3}$$

$$\therefore I = \int \frac{2x-5}{3x^2+4x+5}dx = \int \frac{\frac{1}{3}(6x+4)-\frac{19}{3}}{3x^2+4x+5}dx = \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}dx - \frac{19}{3}\int \frac{1}{3x^2+4x+5}dx$$

$$= \frac{1}{3}\log\left|3x^2+4x+5\right| - \frac{19}{9}\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}dx = \frac{1}{3}\log\left|3x^2+4x+5\right| - \frac{19}{3\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)+c$$

****Ex 2.*** **Evaluate** $\int \frac{(3e^x-4)}{e^{2x}-2e^x+10}e^x dx$

Sol. Put $e^x = t$. Then $e^x dx = dt$

$$\therefore I = \int \frac{(3e^x-4)}{e^{2x}-2e^x+10}e^x dx = \int \frac{3t-4}{t^2-2t+10}dt$$

Now, let us take $3t-4 = A\frac{d}{dt}(t^2-2t+10)+B \Rightarrow A = \frac{3}{2}, B = -1$

$$\int \frac{3t-4}{t^2-2t+10}dt = \frac{3}{2}\int \frac{2t-2}{t^2-2t+10}dt - 1\int \frac{1}{t^2-2t+10}dt = \frac{3}{2}\log\left|t^2-2t+10\right| - \int \frac{1}{(t-1)^2+3^2}dt$$

$$= \frac{3}{2}\log\left|t^2-2t+10\right| - \frac{1}{3}\tan^{-1}\left(\frac{t-1}{3}\right)+c = \frac{3}{2}\log\left|e^{2x}-2e^x+10\right| - \frac{1}{3}\tan^{-1}\left(\frac{e^x-1}{3}\right)+c$$

Type 5: **To evaluate** $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ **To evaluate this type of integrals, we take**

$px+q \equiv A\frac{d}{dx}(ax^2+bx+c)+B$. So that $A = \frac{p}{2a}$ and $B = \frac{2aq-bp}{2a}$

Then $I = \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \frac{2aq-bp}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

$$\Rightarrow I = \left(\frac{p}{2a}\right)2\sqrt{ax^2+bx+c} + \frac{2aq-bp}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

The last integral can be evaluated as in **Type 2**.

Ex 1. Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx$ (March-2017)

Sol. Let $2x+5 \equiv A\frac{d}{dx}(x^2-2x+10)+B \Rightarrow A=1, B=7$

$$\therefore I = \int \frac{2x+5}{\sqrt{x^2-2x+10}}dx = \int \frac{2x-2}{\sqrt{x^2-2x+10}}dx + 7\int \frac{1}{\sqrt{x^2-2x+10}}dx$$

$$= 2\sqrt{x^2-2x+10} + 7\int \frac{1}{\sqrt{(x-1)^2+3^2}}dx = 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right)+c$$

Ex 2. Evaluate $\int \frac{x}{\sqrt{x^2+x+1}}dx$

Sol. $\int \frac{x}{\sqrt{x^2+x+1}}dx = \frac{1}{2}\int \frac{(2x+1)-1}{\sqrt{x^2+x+1}}dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}}dx$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}dx = \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Type 5(a): **To evaluate $\int \sqrt{\frac{x-a}{b-x}}\,dx$ and $\int \sqrt{\frac{b-x}{x-a}}\,dx$ $(a<x<b)$**

The above two integrals can be written as $\int \frac{x-a}{\sqrt{(x-a)(b-x)}}dx$ and $\int \frac{b-x}{\sqrt{(x-a)(b-x)}}dx$ which are of type (5) and hence can be evaluated.

Ex 1. Evaluate $\int \sqrt{\frac{5-x}{x-2}}dx$

Sol. $\int \sqrt{\frac{5-x}{x-2}}dx = \int \frac{5-x}{\sqrt{(x-2)(5-x)}}dx = \int \frac{-x+5}{\sqrt{-x^2+7x-10}}dx$

$$= \frac{1}{2}\int \frac{-2x+7}{\sqrt{-x^2+7x-10}}dx + \frac{3}{2}\int \frac{1}{\sqrt{-x^2+7x-10}}dx$$

$$= \sqrt{-x^2+7x-10} + \frac{3}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}}dx = \sqrt{(x-2)(5-x)} + \frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)+c$$

Note

i) The integrals $\int \sqrt{\frac{b-x}{x-a}}\,dx, \int \sqrt{\frac{b-x}{x-a}}\,dx, \int \frac{1}{\sqrt{(x-a)(b-x)}}dx$ *and* $\int \sqrt{(x-a)(b-x)}\,dx$ *can be evaluated using the substitution* $x = a\cos^2\theta + b\sin^2\theta$ $(a<b)$

ii) To evaluate $\int \sqrt{\frac{x-a}{x-b}}dx, \int \sqrt{(x-a)(x-b)}$, *put* $x = a\sec^2\theta - b\tan^2\theta$

iii) To evaluate $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$, *put* $x-a=t^2$ *or* $x-b=t^2$

Remember :

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\left[\frac{2}{3}(ax^2+bx+c)^{3/2}\right] + \frac{2aq-bp}{2a}\int\sqrt{ax^2+bx+c}$$

Type 6: **To evaluate** $\int (px+q)\sqrt{ax^2+bx+c}\;dx$

To evaluate this integral, we take $px+q \equiv A\frac{d}{dx}(ax^2+bx+c)+B$

so that $A=\frac{p}{2a}, B=\frac{2aq-bp}{2a}$ $\quad \therefore I=\int (px+q)\sqrt{ax^2+bx+c}\;dx$

$$=\frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\;dx+\left(\frac{2aq-bp}{2a}\right)\int\sqrt{ax^2+bx+c}\;dx$$

$$\Rightarrow I=\frac{p}{2a}\left[\frac{2}{3}(ax^2+bx+c)^{3/2}\right]+\frac{2aq-bp}{2a}\int\sqrt{ax^2+bx+c}\;dx$$

where the last integral can be evaluated as in **Type 3**.

Ex 1. **Evaluate** $\int (3x+2)\sqrt{x^2+2x+5}\,dx$

Sol. Let $3x+2 \equiv A\frac{d}{dx}(x^2+2x+5)+B$

Then $A=\frac{3}{2}$ and $B=-1$

$$\therefore I=\int (3x+2)\sqrt{x^2+2x+5}\;dx$$

$$=\frac{3}{2}\int (2x+2)\sqrt{x^2+2x+5}\;dx-1\int\sqrt{x^2+2x+5}\;dx$$

$$=(x^2+2x+5)^{3/2}-\int\sqrt{(x+1)^2+2^2}\,dx$$

$$=(x^2+2x+5)^{3/2}-\left(\frac{x+1}{2}\right)\sqrt{x^2+2x+5}-2\sinh^{-1}\left(\frac{x+1}{2}\right)+c$$

****Ex 2.*** **Evaluate** $\int x\sqrt{1+x-x^2}\,dx$

Sol. Let $x \equiv A\frac{d}{dx}(1+x-x^2)+B$ then

$$A=-\frac{1}{2}, B=\frac{1}{2} \quad \therefore I=\int x\sqrt{1+x-x^2}\,dx$$

$$=-\frac{1}{2}\int(-2x+1)\sqrt{1+x-x^2}\,dx+\frac{1}{2}\int\sqrt{1+x-x^2}\,dx$$

$$=-\frac{1}{2}\cdot\frac{2}{3}(1+x-x^2)^{3/2}+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$=-\frac{1}{3}(1-x-x^2)^{3/2}+\frac{1}{2}\left[\frac{2x-1}{4}\sqrt{1+x-x^2}+\frac{5}{8}\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\sqrt{5}/2}\right]+c$$

$$=-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{(2x-1)}{8}\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\frac{(2x-1)}{\sqrt{5}}+c$$

Type 7: **To evaluate** $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}dx$

To evaluate this integral, we use to substitution $px+q=\frac{1}{t}$. The method is explained through the following examples.

Ex 1. Evaluate $\int \frac{1}{(1+x)\sqrt{3+2x-x^2}}dx$

Sol. Put $1+x=\frac{1}{t}$ then $dx=-\frac{1}{t^2}dt$ $=3+2x-x^2=\frac{1}{t^2}(4t-1)$

$$\therefore \int \frac{1}{(1+x)\sqrt{3+2x-x^2}}dx = -\int \frac{1}{\sqrt{4t-1}}dt$$

$$= -\frac{1}{4}.2\sqrt{4t-1}+c = -\frac{1}{2}.\sqrt{\frac{4}{1+x}-1}+c \quad = -\frac{1}{2}.\sqrt{\frac{3-x}{1+x}}+c$$

Type 8 : **To evaluate** $\int \frac{1}{(px+q)\sqrt{ax+b}}dx$, $\int \frac{px+q}{\sqrt{ax+b}}dx$

To evaluate the integrals of the type $\int \frac{1}{(px+q)\sqrt{ax+b}}dx$, $\int \frac{px+q}{\sqrt{ax+b}}dx$,

$\int \frac{\sqrt{ax+b}}{px+q}dx$ or $\int (px+q)\sqrt{ax+b}\,dx$,

put $\sqrt{ax+b}=t$.

Ex 1. Evaluate $\int \frac{1}{(2x+3)\sqrt{4x+5}}dx$

Sol. Put $\sqrt{4x+5}=t$. Then $4x+5=t^2$ and $dx=\frac{1}{2}tdt$

$$2x+3=\frac{1}{2}[(4x+5)+1]=\frac{1}{2}(t^2+1)$$

$$\therefore \int \frac{1}{(2x+3)\sqrt{4x+5}}dx = \int \frac{1}{t^2+1}dt,\ t=\sqrt{4x+5}$$

$$= \tan^{-1}(t)+c = \tan^{-1}(\sqrt{4x+5})+c$$

Ex 2. Evaluate $\int \frac{1}{(x+1)\sqrt{x+2}}dx$

Sol. Put $\sqrt{x+2}=t$, then $x+2=t^2$

$dx=2tdt$ $\qquad x+1=t^2-1$

$$\therefore \int \frac{1}{(x+1)\sqrt{x+2}}dx = 2\int \frac{1}{t^2-1}dt = \log\left|\frac{t-1}{t+1}\right|+c = \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right|+c$$

Type 9: **To evaluate** $\int \frac{1}{(px^2+q)\sqrt{ax^2+b}}dx$

To evaluate this type of integrals, put $x=\frac{1}{t}$

Ex 1. **Evaluate** $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx$

Sol. Put $x=\frac{1}{t}$, then $dx=-\frac{1}{t^2}dt$

$$\therefore \int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx = -\int \frac{tdt}{(t^2+1)\sqrt{t^2-1}} \quad \text{--- (1)}$$

Now, Put $\sqrt{t^2-1}=y$ so that $t^2=1+y^2$ and $tdt=ydy$ $=-\int \frac{1}{y^2+2}dy = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+c$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{t^2-1}{2}}\right)+c \quad = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right)+c$$

Type 10: **A special type of Integrals :**

To evaluate the integrals of the type $\int \frac{x^2 \pm 1}{x^4+\lambda x^2+1}dx$, $\int \frac{x^2}{x^4+\lambda x^2+1}dx$,

$\int \frac{1}{x^4+\lambda x^2+1}dx$, **where** λ **is a constant, we make use of suitable substitutions:**

$x+\frac{1}{x}=t$ **or** $x-\frac{1}{x}=t$

Ex 1: **Evaluate** $\int \frac{x^2+1}{x^4+1}dx$

Sol. Let the above twelve integrals be denoted by $I_1, I_2, \ldots, I_{12}$ respectively.

Remember :

$$\int \frac{1}{(x^2+a^2)(x^2+b^2)}dx = \frac{1}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right]+c$$

$$I_1 = \int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx \text{ (Dividing both } N^r \text{ and } D^r \text{ by } x^2) = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}$$

Put $x-\frac{1}{x}=t \Rightarrow d\left(x-\frac{1}{x}\right)=\left(1+\frac{1}{x^2}\right)dx=dt$ and $x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2=t^2+2$

$$\therefore I_1 = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c \quad = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right)+c$$

$$\Rightarrow I_1 = \int \frac{x^2+1}{x^4+1} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+c$$

Similarly

Ex 2: **Evaluate** $\int \frac{x^2+1}{x^4+x^2+1}dx$

Sol. $I_2 = \int \frac{x^2+1}{x^4+x^2+1}dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)+c$

Ex 3: **Evaluate** $\int \frac{x^2+1}{x^4-x^2+1}dx$

Sol. $I_2 = \int \frac{x^2+1}{x^4-x^2+1}dx = \tan^{-1}\left(\frac{x^2-1}{x}\right)+c$

Ex 4: **Evaluate** $\int \frac{x^2-1}{x^4+1}dx$

Sol. $I_4 = \int \frac{x^2-1}{x^4+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}$ Put $x+\frac{1}{x}=t$ then $I_4 = \int \frac{dt}{t^2-2} = \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{\left(x+\frac{1}{x}\right)-\sqrt{2}}{\left(x+\frac{1}{x}\right)+\sqrt{2}}\right|+c \quad \therefore I_4 = \int \frac{x^2-1}{x^4+1}dx = \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+c,$

Similarly

Ex 5: **Evaluate** $\int \frac{x^2-1}{x^4+x^2+1}dx$

Sol. $I_5 = \int \frac{x^2-1}{x^4+x^2+1}dx = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+c$

Ex 6: **Evaluate** $\int \frac{x^2-1}{x^4-x^2+1}dx$

Sol. $I_6 = \int \frac{x^2-1}{x^4-x^2+1}dx = \frac{1}{2\sqrt{3}}\log\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right|+c$

Ex 7: **Evaluate** $\int \frac{x^2}{x^4+1}dx$

Sol. Now, $I_7 = \int \frac{x^2}{x^4+1}dx = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}dx$

$= \frac{1}{2}\left[\int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx\right] = \frac{1}{2}(I_1+I_4) \quad \therefore I_7 = \frac{1}{2}(I_1+I_4)$

Ex 8: **Evaluate** $\int \frac{x^2}{x^4+x^2+1}dx$

Sol. $\therefore I_7 = \frac{1}{2}(I_1+I_4)$

Ex 9: **Evaluate** $\int \frac{x^2}{x^4 - x^2 + 1} dx$

Sol. $I_9 = \frac{1}{2}(I_3 + I_6)$

Ex 10: **Evaluate** $\int \frac{1}{x^4 + 1} dx$

Sol. $I_{10} = \frac{1}{2}(I_1 - I_4)$

Ex 11: **Evaluate** $\int \frac{1}{x^4 + x^2 + 1} dx$

Sol. $I_{11} = \frac{1}{2}(I_2 - I_5)$

Ex 12: **Evaluate** $\int \frac{1}{x^4 - x^2 + 1} dx$

Sol. $I_{12} = \frac{1}{2}(I_3 - I_6)$

Note

To evaluate integrals of the type

$\int f\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) dx$, *put* $x + \frac{1}{x} = t$

$\int f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right) dx$, *put* $x - \frac{1}{x} = t$

$\int f\left(x^2 + \frac{1}{x^2}\right)\left(x - \frac{1}{x^3}\right) dx$, *put* $x^2 + \frac{1}{x^2} = t$

$\int f\left(x^2 - \frac{1}{x^2}\right)\left(x + \frac{1}{x^3}\right) dx$, *put* $x^2 - \frac{1}{x^2} = t$

EXERCISE - 1.4

Evaluate the following integrals.

Type I :

1. $\int \frac{1}{x^2 - 3x + 2} dx$
2. $\int \frac{1}{6x^2 - 5x + 1} dx$
3. $\int \frac{1}{x^2 + 2x + 5} dx$
4. $\int \frac{1}{x^2 + 6x + 10} dx$
5. $\int \frac{1}{3x^2 + 2x + 1} dx$

*6. $\int \frac{1}{x^2 + x + 1} dx$

*7. $\int \frac{1}{3x^2 + x + 1} dx$

*8. $\int \frac{1}{4x^2 - 4x - 7} dx$

*9. $\int \frac{1}{5 + 4x - 2x^2} dx$

10. $\int \frac{1}{2x^2 + 3x - \frac{11}{4}} dx$

1.14 INTEGRATION OF RATIONAL TRIGONOMETRIC FUNCTIONS

In this section we discuss about integration of some standard types of functions which are rational in sin x and cos x. The classification of types is mainly based on the substitution that we use.

Type 1: **To evaluate integrals of the type**

$$\int \frac{1}{a+b\cos^2 x}dx,\ \int \frac{1}{a+b\sin^2 x}dx,\ \int \frac{1}{a\cos^2 x+b\sin^2 x}dx,\ \int \frac{1}{a\cos^2 x+b\sin x\cos x+c\sin^2 x}dx$$

To evaluate the above type of integrals, we ***multiply both numerator and denominator with*** **$\sec^2 x$** ***and put*** **$\tan x = t$.**

***Ex 1. Evaluate $\int \frac{1}{2\sin^2 x+3\cos^2 x}dx$**

Sol. Let $I = \int \frac{1}{2\sin^2 x+3\cos^2 x}dx$

Multiplying both N^r and D^r with $\sec^2 x$, we get $I = \int \frac{\sec^2 x}{2\tan^2 x+3}dx$

Put $\tan x = t$. Then $\sec^2 x dx = dt$

$$\therefore I = \int \frac{dt}{2t^2+3} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{3}}\right)+c = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{3}}\right)+c$$

Ex 2. Evaluate $\int \frac{1}{2\sin^2 x+3\sin x\cos x-2\cos^2 x}dx$

Sol. Let $I = \int \frac{1}{2\sin^2 x+3\sin x\cos x-2\cos^2 x}dx = \int \frac{\sec^2 x}{2\tan^2 x+3\tan x-2}dx$

$$= \int \frac{dt}{2t^2+3t-2} \text{ where } t=\tan x = \frac{1}{5}\int\left(\frac{2}{2t-1}-\frac{1}{t+2}\right)dt = \frac{1}{5}\log\left|\frac{2t-1}{t+2}\right|+c = \frac{1}{5}\log\left|\frac{2\tan x-1}{\tan x+2}\right|+c$$

Type 2: **To evaluate the integrals of the type**

$$\int \frac{1}{a+b\cos x}dx,\ \int \frac{1}{a+b\sin x}dx,\ \int \frac{1}{a\cos x+b\sin x+c}dx$$

To evaluate the above type of integrals we take the substitution $\tan\frac{x}{2}=t$ and transform the integral into a rational algebraic function in t, which can be evaluated using the methods discussed in Sec 1.13.

$$\boxed{\text{If } \tan\frac{x}{2}=t, \text{ then } dx=\frac{2dt}{1+t^2},\quad \sin x=\frac{2t}{1+t^2} \text{ and } \cos x=\frac{1-t^2}{1+t^2}}$$

***Ex 1. Evaluate $\int \frac{1}{5+4\cos x}dx$**

Sol. Let $I = \int \frac{1}{5+4\cos x}dx$

Put $\tan\left(\frac{x}{2}\right)=t$ then $dx=\frac{2dt}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

$$\therefore I = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)}\frac{2dt}{1+t^2} = \int \frac{2dt}{t^2+9} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+c = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right)+c$$

***Ex 2. Evaluate $\int \frac{1}{4+5\sin x}dx$**

Sol. Let $I=\int \frac{1}{4+5\sin x}dx$. Put $\tan\left(\frac{x}{2}\right)=t$.

Then $dx=\frac{2dt}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$ $\quad\therefore I=\int \frac{1}{4+5\left(\frac{2t}{1+t^2}\right)}\frac{2dt}{1+t^2}$

$$=\int \frac{dt}{2t^2+5t+2}=\frac{1}{3}\int\left(\frac{2}{2t+1}-\frac{1}{t+2}\right)dt=\frac{1}{3}\log\left|\frac{2t+1}{t+2}\right|+c=\frac{1}{3}\log\left|\frac{2\tan\left(x/2\right)+1}{\tan\left(x/2\right)+2}\right|+c$$

****Ex 3. Evaluate $\int \frac{1}{1+\sin x+\cos x}dx$**

Sol. Let $I=\int \frac{1}{1+\sin x+\cos x}dx$

Put $\tan\left(\frac{x}{2}\right)=t$, then $dx=\frac{2dt}{1+t^2}$

$\sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$

$$\therefore I=\int \frac{2dt}{(1+t^2)+2t+(1-t^2)}=\int\frac{dt}{1+t}=\log|1+t|+c=\log\left|1+\tan\frac{x}{2}\right|+c$$

***Ex 4. $\int \frac{1}{\sin x+\sqrt{3}\cos x}dx$**

Sol. Let $t=\tan\frac{x}{2}$ so that $dx=\frac{2\,dt}{1+t^2}$

$\sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$

$$I=\int\frac{2\frac{dt}{1+t^2}}{\frac{2t}{1+t^2}+\frac{\sqrt{3}(1-t^2)}{1+t^2}}=2\int\frac{dt}{\sqrt{3}(1-t^2)+2t}=\frac{2}{\sqrt{3}}\int\frac{dt}{1-t^2+\frac{2}{\sqrt{3}}t}=\frac{2}{\sqrt{3}}\int\frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2-\left(t-\frac{1}{\sqrt{3}}\right)^2}$$

$$=\frac{1}{2}\log\left|\frac{t+\frac{1}{\sqrt{3}}}{\sqrt{3}-t}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{3}t+1}{\sqrt{3}(\sqrt{3}-t)}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\left(\sqrt{3}-\tan\frac{x}{2}\right)}\right|+C$$

***Ex 5. Evaluate $\int \frac{1}{5+4\cos 2\theta}d\theta$**

Sol. Let $I=\int\frac{1}{5+4\cos 2\theta}d\theta$

Put $\tan\theta=t$. Then $\sec^2\theta d\theta=dt\Rightarrow d\theta=\frac{dt}{1+t^2}$

$\sin 2\theta=\frac{2t}{1+t^2}$ and $\cos 2\theta=\frac{1-t^2}{1+t^2}$

$$\therefore I=\int\frac{dt}{5(1+t^2)+4(1-t^2)}=\int\frac{dt}{t^2+9}=\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+c=\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)+c$$

***Ex 6.** **Evaluate** $\int \frac{1}{2-3\cos 2x}dx$

Sol. $t=\tan x \Rightarrow dt=\sec^2 x dx = (1+\tan^2 x)dx = (1+t^2)\,dx$

$$dx=\frac{dt}{1+t^2}$$

$$\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{1-t^2}{1+t^2}$$

$$I=\int \frac{\frac{dt}{1+t^2}}{2-3\left(\frac{1-t^2}{1+t^2}\right)}=\int \frac{dt}{2+2t^2-3+3t^2}=\int \frac{dt}{5t^2-1}=\frac{1}{5}\int \frac{dt}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2}$$

$$=\frac{1}{5}\frac{\frac{1}{2}}{\sqrt{5}}\log\left|\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right|+c=\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right|+C=\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right|+C$$

Note

To evaluate the integrals of the form

i) $\int \frac{a\sin x+b}{(a+b\sin x)^2}dx$, *divide both N^r and D^r by $\cos^2 x$ and put $a\sec x+b\tan x=t$*

ii) $\int \frac{a\cos x+b}{(a+b\cos x)^2}dx$ *divide both N^r and D^r by $\sin^2 x$ and put $a\operatorname{cosec}x+b\cot x=t$*

iii) $\int f(\sin 2x)(\sin x+\cos x)dx$, *put $\sin x-\cot x=t$*

iv) $\int f(\sin 2x)(\sin x-\cos x)dx$, *put $\sin x+\cot x=t$*

v) $\int \sin^m x\cos^n x\,dx$

a) Put $\cos x=t$ if m is even and n is odd

b) Put $\sin x=t$ if m is odd and n is even

c) Put $\tan x=t$ if m+n is a negative even integer.

Type 3: **To evaluate the integrals of the type** $\int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx$

To evaluate the above type of integrals we take $a\cos x+b\sin x$

$\equiv A(c\cos x+d\sin x)+B\frac{d}{dx}(c\cos x+d\sin x)$ (i.e., Nr. $\equiv A(\text{Dr.})+B\frac{d}{dx}(\text{Dr.})$)

$\Rightarrow a\cos x+b\sin x=A(c\cos x+d\sin x)+B(-c\sin x+d\cos x)$

$=(Ac+Bd)\cos x+(Ad-Bc)\sin x$

$\Rightarrow Ac+Bd=a$ and $Ad-Bc=b \Rightarrow A=\frac{ac+bd}{c^2+d^2}$, $B=\frac{ad-bc}{c^2+d^2}$

$$\therefore \int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=A\int \frac{c\cos x+d\sin x}{c\cos x+d\sin x}dx+B\int \frac{\frac{d}{dx}(c\cos x+d\sin x)}{c\cos x+d\sin x}dx$$

$$=Ax+B\log|c\cos x+d\sin x|+k$$

$$\Rightarrow \int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=\left(\frac{ac+bd}{c^2+d^2}\right)x+\left(\frac{ad-bc}{c^2+d^2}\right)\log|c\cos x+d\sin x|+k$$

Remember :

$\int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx$
$=\left(\frac{ac+bd}{c^2+d^2}\right)x+\left(\frac{ad-bc}{c^2+d^2}\right)$
$\log|c\cos x+d\sin x|+k$

Ex 1. **Evaluate** $\int \frac{3\cos x + 2\sin x}{4\cos x + 3\sin x} dx$

Sol. Let $I = \int \frac{3\cos x + 2\sin x}{4\cos x + 3\sin x} dx$

Let us take $3\cos x + 2\sin x \equiv A(4\cos x + 3\sin x) + B\frac{d}{dx}(4\cos x + 3\sin x)$

$\Rightarrow 3\cos x + 2\sin x = A(4\cos x + 3\sin x) + B(-4\sin x + 3\cos x)$

$\Rightarrow 4A + 3B = 3$ and $3A - 4B = 2$ $\Rightarrow A = \frac{18}{25}$ and $B = \frac{1}{25}$

$\therefore I = \int \frac{\frac{18}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx + \int \frac{\frac{1}{25}(-4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx = \frac{18}{25}\int 1 dx + \frac{1}{25}\int \frac{-4\sin x + 3\cos x}{4\cos x + 3\sin x} dx$

$= \frac{18}{25}x + \frac{1}{25}\log|4\cos x + 3\sin x| + k$

Ex 2. **Evaluate** $\int \frac{\cos x}{c\cos x + d\sin x} dx$ **and** $\int \frac{\sin x}{c\cos x + d\sin x} dx$

Sol. Let $I_1 = \int \frac{\cos x}{c\cos x + d\sin x} dx$; $I_2 = \int \frac{\sin x}{c\cos x + d\sin x} dx$

Now, $cI_1 + dI_2 = \int \frac{c\cos x + d\sin x}{c\cos x + d\sin x} dx = \int 1 dx = x + k_1$ -- (1)

Again $dI_1 - cI_2 = \int \frac{d\cos x - c\sin x}{c\cos x + d\sin x} dx = \log|c\cos x + d\sin x| + k_2$ -- (2)

From (1) & (2)

$$I_1 = \frac{1}{c^2 + d^2}\left[c.x + d\log|c\cos x + d\sin x|\right] + k_3$$

$$I_2 = \frac{1}{c^2 + d^2}\left[d.x + c\log|c\cos x + d\sin x|\right] + k_4$$

where $k_3 = \frac{ck_1 + dk_2}{c^2 + d^2}, k_4 = \frac{dk_1 - ck_2}{c^2 + d^2}$

Type 3(a): **To evaluate the integrals of the type** $I = \int \frac{a\cos x + b\sin x + p}{c\cos x + d\sin x + q} dx$

we take $a\cos x + b\sin x + p \equiv A(c\cos x + d\sin x + q) + B\frac{d}{dx}(c\cos x + d\sin x + q) + r$

Equating the coefficients of cos$x$, sin$x$ and the constant terms,

we get $A = \frac{ac + bd}{c^2 + d^2}$, $B = \frac{ad - bc}{c^2 + d^2}$ and $r = p - Aq$

$\therefore I = Ax + B\log|c\cos x + d\sin x + q| + (p - Aq)\int \frac{1}{c\cos x + d\sin x + q} dx$

where the last integral can be evaluated using the substitution $\tan\frac{x}{2} = t$

Ex. **Evaluate** $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx$ **(March-19)**

Sol. Let $I = \int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx$

Let $\cos x + 3\sin x + 7 \equiv A(\cos x + \sin x + 1) + B\frac{d}{dx}(\cos x + \sin x + 1) + C$

$\Rightarrow A + B = 1$, $A - B = 3$ and $A + C = 7 \Rightarrow A = 2, B = -1, C = 5$

$\therefore I = 2\int 1 dx + (-1)\int \frac{(-\sin x + \cos x)}{\cos x + \sin x + 1} dx + 5\int \frac{1}{\cos x + \sin x + 1} dx$

$= 2x - \log|\cos x + \sin x + 1| + 5\log\left|1 + \tan\frac{x}{2}\right| + k$ *(see Ex:3 in Type 2 of this section)*

EXERCISE - 1.5

Evaluate the following integrals.

1. $\int \frac{1}{4\sin^2 x + 9\cos^2 x} dx$

*2. i) $\int \frac{1}{1 + \cos^2 x} dx$ ii) $\int \frac{1}{1 + \sin 2x} dx$

3. $\int \frac{1}{3\sin 2x + 4\cos 2x} dx$

4. $\int \frac{1}{\sin 2x + \sin^2 x} dx$

5. $\int \frac{1}{4\sin^2 x + 3\sin x\cos x + 2\cos^2 x} dx$

6. $\int \frac{1}{3 + 2\cos x} dx$

7. $\int \frac{1}{2 + 3\cos x} dx$

8. $\int \frac{1}{3 + 2\sin x} dx$

9. $\int \frac{1}{2 + 3\sin x} dx$

*10. $\int \frac{1}{4\cos x + 3\sin x} dx$ (May-18)

11. $\int \frac{1}{4 + 5\cos x} dx$

12. $\int \frac{1}{5 + 4\sin x} dx$

13. $\int \frac{1}{5 + 4\sin^2 \theta} d\theta$

**14. $\int \frac{9\cos x - \sin x}{5\cos x + 4\sin x} dx$ (March-17)

**15. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$ (March-18, May-19)

*16. $\int \frac{\cos x}{3\cos x + 4\sin x} dx$

*17. $\int \frac{1}{1 + \tan x} dx$

*18. $\int \frac{1}{1 - \cot x} dx$

19. $\int \frac{1}{d + e\tan x} dx$

**20. $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$ (March-14)

* 21. $\int \frac{1}{3\cos x + 4\sin x + 6} dx$

1.15 INTEGRATION BY PARTS

Some times it might not be possible to directly evaluate the integral of product of two functions using the standard formulae. In such cases the integral can be evaluated by parts with the help of the following rule :

THEOREM-1.5 **If $f(x)$ and $g(x)$ are two differentiable functions on an interval I such that both of them have integrals on I,**

then $\int f(x).g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx + c$

Proof : Since $f(x)$ and $g(x)$ are differentiable,

$[f(x)\int g(x)dx]' = f'(x)\int g(x)dx + f(x)g(x)$

$\Rightarrow f(x)g(x) = [f(x)\int g(x)dx]' - f'(x)\int g(x)\,dx$

On integration w.r.t x

$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(f'(x)\int g(x)dx\right) + c$

Note

The above rule should be remembered as a formula
and the following verbal statement will be helpful to do so.
Integral of [1st function × 2nd function]
= 1st function× integral of the 2nd function
– Integral of [Derivative of the 1st function × integral of the 2nd function]

Example : $\int x\sin x dx = x.\int \sin x dx - \int[(x)'\int \sin x dx]dx = x(-\cos x) - \int 1.(-\cos x)\,dx = -x\cos x + \sin x + c$

Note : The above theorem can also be stated and proved as follows

THEOREM-1.6 **Let u and v be two differentiable functions on an interval I. If $u'v$ has an integral on I then uv' also has an integral on I and**

$\int(uv')(x)\,dx = (uv)(x) - \int(uv')(x)\,dx + c$

Proof: u and v are differentiable on I

$\Rightarrow (uv)' = u'v + uv' \Rightarrow uv' = (uv)' - u'v$

$\therefore \int(uv')(x)\,dx = \int(uv)'(x)dx - \int(u'v)(x)\,dx + c$

$(\because (uv)'$ and $u'v$ have integrals on $I \Rightarrow uv'$ has an integral on I)

$\Rightarrow \int(uv')(x)\,dx = uv - \int(u'v)(x)dx + c$

In a simple notation the above rule can be written as $\int udv = uv - \int vdu + c$

Remark : In the evaluation of some integrals, it becomes necessary to use the rule of integration by parts more than once which can be observed in the following example:

$\int x^3e^x dx = x^3\int e^x dx - \int 3x^2(\int e^x dx)dx$

(Applying integration by parts Ist time)

$= x^3e^x - 3\int x^2e^x dx$

$= x^3e^x - 3[x^2\int e^x dx - \int 2x(\int e^x dx)\,dx]$

(Applying integration by parts 2nd time)

$$= x^3e^x - 3x^2e^x + 6\int xe^x dx$$

$$= x^3e^x - 3x^2e^x + 6[x\int e^x dx - \int 1(\int e^x dx)\, dx]$$

(Applying integration by parts 3rd time)

$$= x^3e^x - 3x^2e^x + 6xe^x dx - 6\int e^x dx$$

$$= -x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c = e^x(x^3 - 3x^2 + 6x - 6) + c$$

Remark : *While performing integration by parts, it is not necessary to add the constant of integration at each of the intermediate stages of integration, as it has no effect on the final result. It is sufficient to add the constant of integration after valuating the last integral in the process*

This is illustrated in the following example.

$$\int x^2 \sin x\, dx = x^2 \int \sin x dx - \int 2x(\int \sin x\, dx)dx$$

$$= x^2[-\cos x + c_1] - \int 2x(-\cos x + c_1)\, dx$$

$$= -x^2 \cos x + c_1x^2 + 2\int (x\cos x - c_1x)dx$$

$$= -x^2 \cos x + c_1x^2 + 2[x\int \cos x dx - \int 1(\int \cos x dx)dx] - c_1x^2 + c_2$$

$$= -x^2 \cos x + 2x(\sin x + c_3) - 2\int (\sin x + c_3)dx + c_2$$

$$= -x^2 \cos x + 2x\sin x + 2c_3x \; -[-2\cos x + 2c_3] + c_4 + c_2$$

$$= -x^2 \cos x + 2x\sin x + 2\cos x + c \quad \text{where } c = c_2 + c_4$$

While performing integration by parts, a doubt that naturally arises is which function in the product is to be considered as the first and which one the second?

A judicious selection of first and second functions is essential to apply the rule, otherwise it may lead to more complicated integrals or integrals which can not be evaluated further.

Usually, the function whose integral can be evaluated easily, a number of times, is considered to be the second function and the function which on differentiation, makes further integration process simpler is considered to be the first. This type of selection needs some experience. However, the selection process is made simpler by the (order of the letters in the) word "**ILATE**" where the respective letters stand for **I**nverse trignometric, **L**ogarithmic, **A**lgebraic, **T**rigonometric and **E**xponential functions.

Note : *Some authors suggest the word LIATE. But this does not make any significant difference in the selection of the first and second functions.*

Of the two functions in the given product , the function which appears first (from left to right) in the word **ILATE** is taken as the first function, and that appears latter in the word as the second function.

1.16 SOME STANDARD RESULTS

Some times it might not be possible to directly evaluate the integral of product of two functions using the standard formulae. In such cases the integral can be evaluated by parts with the help of the following rule:

1) $\int [f(x)+xf'(x)]dx = xf(x)+c$

Proof : $(x\,f(x))' = 1.f(x)+xf'(x) \Rightarrow \int [f(x)+x.f'(x)]dx = xf(x)+c$

2) $\int e^x(f(x)+f'(x))dx = e^x f(x)+c$

Proof: $(e^x f(x))' = e^x f(x)+e^x f'(x) = e^x(f(x)+f'(x))$

$\therefore \int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$

3) $\int e^{ax}[af(x)+f'(x)] = e^{ax}f(x)+c$

4) $\int e^{-x}[f(x)-f'(x)] = -e^{-x}f(x)+c$

5) **General rule of integration by parts:**

Let u and v be two functions of x such that

i) the successive derivatives of u i.e., $u', u'', u''',....,$ exist and

ii) the successive integrals of v i.e., $v_1, v_2, v_2,...$ exist then

$\int uv\,dx = u.v_1 - u'.v_2 + u''v_3 - u'''.v_4 + ... + c$

where the evaluation of the integral should end in a finite number of steps.

6) $\int e^x f(x)dx = -e^{-x}[f(x)+f'(x)+f''(x)-f'''(x)+...]+c$

7) $\int e^{-x} f(x)dx = -e^{-x}[f(x)+f'(x)+f''(x)-f'''(x)+...]+c$

8) If the integrand $f(x)$ is such a function that it can not be integrated directly (e.g., $f(x) = \log x$, $\sin^{-1}x$, $\sinh^{-1}x$, $\sqrt{x^2+a^2}$ ... etc) then the rule of integration by parts can be applied to $\int f(x)dx$ by choosing the second function as unity.

That is $\int f(x)dx = \int f(x)1.dx = xf(x) - \int xf'(x)\,dx + c$

Example :

1) $\int \log x\,dx = \int \log x.1\,dx = x\log x - \int x.\frac{1}{x}dx + c = x\log x - x + c = x(\log x - 1)+c = x\log\left(\frac{x}{e}\right)+c$

2) $\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+x^2}.1\,dx = x\sqrt{a^2+x^2} - \int x.\frac{X}{\sqrt{a^2+x^2}}dx$

$= x\sqrt{a^2+x^2} - \int \frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}dx = x\sqrt{a^2+x^2} - \sqrt{a^2+x^2} + a^2\int \frac{1}{\sqrt{a^2+x^2}}dx$

$\Rightarrow 2\int \sqrt{a^2+x^2} = x\sqrt{a^2+x^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right)+c$

$\Rightarrow \int \sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+c$

Note

Using integration by parts we can, similarly, prove that

i) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

ii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c$ ***(proofs are left as an exercise)***

SOLVED EXAMPLES

1. Evaluate $\int (x\sec x + \log|\sec x + \tan x|)\,dx$

Sol. By observation, we find that $(x\log|\sec x+\tan x|)' = 1\log|\sec x+\tan x| + x\sec x$

$\therefore \int (x\sec x + \log|\sec x+\tan x|)\,dx = \int (x\log|\sec x+\tan x|)'\,dx = x\log|\sec x+\tan x| + c$

Aliter: $\int x\sec x\,dx = x\int \sec x dx - \int 1.(\int \sec x\,dx)\,dx$

$= x\log|\sec x+\tan x| - \int \log|\sec x+\tan x|\,dx$

$\Rightarrow \int x\sec x\,dx + \int \log|\sec x+\tan x|\,dx = x\log|\sec x+\tan x| + c$

$\Rightarrow \int (x\sec x + \log|\sec x+\tan x|)\,dx = x\log|\sec x+\tan x| + c$

2. Evaluate $\int e^x \dfrac{1+x}{(2+x)^2}dx$

Sol. $\int e^x \dfrac{1+x}{(2+x)^2}dx = \int e^x\left[\dfrac{(2+x)-1}{(2+x)^2}\right]dx = \int e^x\left[\dfrac{1}{2+x}+\dfrac{-1}{(2+x)^2}\right]dx$

Which is of the form $\int e^x(f(x)+f'(x))dx$

where $f(x)=\dfrac{1}{2+x} = e^x f(x)+c = e^x\left(\dfrac{1}{2+x}\right)+c$

Note :

$\int e^x(f(x)+f'(x))dx$

$= e^x.f(x)+c$

3. Show that $\int \left[\dfrac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right]e^x dx = e^x\sqrt{\dfrac{1+x^n}{1-x^n}}+c$

Sol. $\dfrac{1+n.x^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}} = \dfrac{(1-x^{2n})+n.x^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}} = \dfrac{1+x^n}{\sqrt{1-x^{2n}}}+\dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$

$= \sqrt{\dfrac{1+x^n}{1-x^n}} + \dfrac{d}{dx}\left(\sqrt{\dfrac{1+x^n}{1-x^n}}\right) = f(x)+f'(x)$ where $f(x)=\sqrt{\dfrac{1+x^n}{1-x^n}}$

$\therefore$ Given Integral $= \int e^x[f(x)+f'(x)]dx = e^x f(x)+c = e^x\sqrt{\dfrac{1+x^n}{1-x^n}}+c$

4. Evaluate $\int e^{-2x}[2\tan x - \sec^2 x]dx$

Sol. $(e^{-2x}\tan x)' = -2e^{-2x}\tan x + e^{-2x}\sec^2 x = -e^{-2x}(2\tan x - \sec^2 x)$

$\therefore \int e^{-2x}(2\tan x-\sec^2 x)dx = -e^{-2x}\tan x + c$ (or) $\int e^{-2x}(2\tan x - \sec^2 x)dx =$

$= -\int e^{-2x}[(-2)\tan x + \sec^2 x]dx = -\int e^{ax}[af(x)+f'(x)]\,dx$

where $a=-2$, $f(x)=\tan x$ $= -e^{ax}f(x)+c = -e^{-2x}\tan x + c$

Aliter: $\int e^{-2x}\tan x\,dx = \tan x\left(\dfrac{e^{-2x}}{-2}\right) - \int \sec^2 x\left(\dfrac{e^{-2x}}{-2}\right)dx$

$\Rightarrow \int e^{-2x}2\tan x\,dx = -e^{-2x}\tan x + \int e^{-2x}\sec x^2 x\,dx$

$\Rightarrow \int e^{-2x}(2\tan x - \sec^2 x)\,dx = -e^{-2x}\tan x + c$

5. Evaluate $\int x^3 \sin x\, dx$

Sol. Using the general rule of integration by paths

$$\int x^3 \sin x\, dx = u.v_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

Where $u = x^3$, $v = \sin x$

$$= x^3(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x) + 0(-\cos x) + c$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + c$$

Remember :

$\int e^x . f(x) dx$
$= e^x[f(x) - f'(x) + f''(x)...] + c$

6. Evaluate $\int x^5 e^x dx$

Sol. We known that $\int e^x f(x)\, dx = e^x[f(x) - f'(x) + f''(x) - ......] + c$

$$\therefore \int e^x x^5 dx = e^x[x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120] + c$$

7. Evaluate (i) $\int e^{-x} x^3 dx$ (ii) $\int \frac{x e^x}{(x+1)^2} dx$

Sol. (i) $\int e^{-x} x^3 dx = -e^{-x}[x^3 + 3x^2 + 6x + 6] + c$

(ii) $$\int \frac{x.e^x}{(x+1)^2} dx = \int \left[\frac{x+1-1}{(x+1)^2}\right] e^x dx = \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x dx$$

$$= \int \left[\left(\frac{1}{x+1}\right) + \frac{(-1)}{(x+1)^2}\right] e^x dx = \left(\frac{1}{x+1}\right) e^x + C = \frac{e^x}{x+1} + C$$

8. Evaluate

Sol. $\int x^2 \cos x\, dx = x^2 \sin x - \int \sin x (x^2)' dx = x^2 \sin x - 2\int x \sin x\, dx + c_1$ ------(i)

Consider $\int x \sin x\, dx = -x\cos x - \int (-\cos x) dx = -x\cos x + \sin x + c_2$

$$\therefore \int x^2 \cos x\, dx = x^2 \sin x - 2(\sin x - x\cos x) + c = (x^2 - 2)\sin x + 2x\cos x + c$$

Application of the rule : $\int f(x)\, dx = xf(x) - \int xf'(x)\, dx$

1) $\int \sin^{-1} x\, dx = x\sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} dx = x\sin^{-1} x + \sqrt{1-x^2} + c$

2) $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1-x^2} + c$

3) $\int \tan^{-1} x\, dx = x\tan^{-1} x - \log\sqrt{1+x^2} + c$

4) $\int \cot^{-1} x\, dx = x\cot^{-1} x + \log\sqrt{1+x^2} + c$

5) $\int \sec^{-1} x\, dx = x\sec^{-1} x - \log(x + \sqrt{x^2-1}) + c, x \in (1, \infty)$

6) $\int \text{cosec}^{-1} x\, dx = x\text{cosec}^{-1} x + \log(x + \sqrt{x^2-1}) + c, x \in (1, \infty)$

7) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \sqrt{x^2+1} + c$

8) $\int \cosh^{-1} x\, dx = x \cosh^{-1} x - \sqrt{x^2-1} + c$

9) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x + \frac{1}{2}\log\left|1-x^2\right| + c$

(Proofs of results 2 to 9 are left as an exercise)

1.17 A PAIR OF STANDARD INTEGRALS

THEOREM-1.7

* i) $\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin x(bx+c) - b\cos(bx+c)] + k$ **(March-19)**

ii) $\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos(bx+c) + b\sin(bx+c)] + k$

where a, b, are non - zero real numbers and k is the constant of integration.

Proof : Let $I = \int e^{ax} \sin(bx+c)\, dx$

$$= \sin(bx+c)\frac{e^{ax}}{a} - \int [b\cos(bx+c)]\frac{e^{ax}}{a}dx \;= \frac{e^{ax}}{a}\sin(bx+c) - \frac{b}{a}\int e^{ax}\cos(bx+c)dx$$

$$= \frac{e^{ax}}{a}\sin(bx+c) - \frac{b}{a}\left[\cos(bx+c)\frac{e^{ax}}{a} - \int -b\sin(bx+c)\frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}}{a}\sin(bx+c) - \frac{b}{a^2}e^{ax}\cos(bx+c) - \frac{b^2}{a^2}\int e^{ax}\sin(bx+c)dx$$

$$= \frac{e^{ax}}{a^2}[a\sin(bx+c) - b\cos(bx+c)] - \frac{b^2}{a^2}.I$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + k$$

The second integral can be evaluated in a similar way. In particular, we have

> *i)* $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + k = \frac{e^{ax}}{r}\sin(bx-\theta) + k$ *and*
>
> *ii)* $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + k = \frac{e^{ax}}{r}\cos(bx-\theta) + k$
>
> *where* $r = \sqrt{a^2+b^2}$ *and* θ *is the solution of the simultaneous equations :*
>
> $a = r\cos\theta$ *and* $b = r\cos\theta$ *(i.e.,* $\theta = \tan^{-1}\left(\frac{b}{a}\right)$*)*

Caution : One should be cautious in finding $\theta (0 \le \theta < 2\pi)$ using $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ as the signs of b and a decide the quadrant in which θ lies.

*iii) $\int e^x \sin dx = \frac{e^x}{2}(\sin x - \cos x) + k \;\; = \frac{e^x}{\sqrt{2}}\sin\left(x - \frac{\pi}{4}\right) + k$

*iv) $\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x) + k \;\; = \frac{e^x}{\sqrt{2}}\cos\left(x - \frac{\pi}{4}\right) + k$

Further, if $P = \int e^{ax}\cos bx\, dx$ and $Q = \int e^{ax}\sin bx\, dx$ then we can show that $(P^2 + Q^2)(a^2 + b^2) = e^{2ax}$ and $\tan^{-1}\left(\frac{Q}{P}\right) + \tan^{-1}\left(\frac{b}{a}\right) = bx$ assuming the constants integration to be zero.

Corollaries

1) $\int a^x \sin(bx + c)dx = \int e^{(\log a)x}\sin(bx + c)\, dx = \frac{a^x}{(\log a)^2 + b^2}$
$[(\log a)\sin(bx + c) - b\cos(bx + c)] + k$

2) $\int a^x \cos(bx + c)\, dx = \frac{a^x}{(\log a)^2 + b^2}[(\log a)\cos(bx + c) + b\cos(bx + c)] + k$

EXERCISE - 1.6

Evaluate the following integrals

1. *a) $\int xe^x dx$ *b) $\int e^x(1 + x^2)dx$ *c) $\int x^2 e^{-3x} dx$

2. *a) $\int x\cos x\, dx$ *b) $\int x\sin x\, dx$ *c) $\int x\sec^2 x\, dx$

   *d) $\int x\cot^2 x\, dx$ *e) $\int x\sec^2 2x\, dx$ *f) $\int x\sin^2 x\, dx$

   g) $\int x\left(\frac{\sec 2x - 1}{\sec 2x + 1}\right)dx$ h) $\int \frac{x}{1 + \cos x}dx$ i) $\int \frac{x + \sin x}{1 + \cos x}dx$

3. *a) $\int e^x(\sin x + \cos x)\, dx$ *b) $\int e^x(\tan x + \sec^2 x)\, dx$ *c) $\int e^x \sec x(1 + \tan x)dx$

   *d) $\int e^x(\tan x + \log\sec x)\, dx$ **e) $\int e^x\left(\frac{x\log x + 1}{x}\right)dx$ (March-18, 19)

4. *a) $\int \log x\, dx$ *b) $\int x\log x\, dx$ *c) $\int x^n \log x\, dx$

   *d) $\int \frac{\log x}{x^2}dx$ *e) $\int \sqrt{x}\log x\, dx$ f) $\int \log(1 + x^2)\, dx$

   *g) $\int x\log(1 + x)dx$ *h) $\int (\log x)^2 dx$

5. *a) $\int x\sin^{-1} x\, dx$ *b) $\int x\cos^{-1} x\, dx$ *c) $\int x\tan^{-1} x\, dx$

   d) $\int x\sec^{-1} x\, dx$ *e) $\int x^2\sin^{-1} x\, dx$ f) $\int x^2\cos^{-1} x\, dx$

   *g) $\int x^2\tan^{-1} x\, dx$ *h) $\int \frac{1}{x^2}\tan^{-1} x\, dx$

6. a) $\int x\sinh^{-1} x\,dx$ b) $\int x\cosh^{-1} x\,dx$ c) $\int x\tanh^{-1} x\,dx, |x|<1$

7. *a) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$ *b) $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$ *c) $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

d) $\int x\tan^{-1}(x^2)\,dx$ e) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ f) $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

g) $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ h) $\int \frac{x\tan^{-1}x}{(1+x^2)^{\frac{3}{2}}}dx$

8. *a) $\int e^{\sqrt{x}}dx$ b) $\int \sin\sqrt{x}\,dx$ *c) $\int \cos\sqrt{x}\,dx$

d) $\int \tan^{-1}\sqrt{x}\,dx$ e) $\int \sec^{-1}\sqrt{x}\,dx$

9. *a) $\int e^x \sin x\,dx$ *b) $\int e^x \cos x\,dx$ *c) $\int e^{ax}\sin bx\,dx$

*d) $\int e^{ax}\cos bx\,dx$ e) $\int 2^x \cos x\,dx$ f) $\int a^x \cos 2x\,dx$

g) $\int e^{2x}\cos x\cos 3x\,dx$ h) $\int e^x \sin^2 x\,dx$ i) $\int e^x \sin 3x\cos 3x\,dx$

*j) $\int \frac{x^2}{\sqrt{1-x^2}}e^{m\sin^{-1}x}dx$

10. a) $\int[\sin(\log x)+\cos(\log x)]dx$ b) $\int \sin(\log x)\,dx$ c) $\int \cos(\log x)\,dx$

d) $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$ e) $\int \frac{\log x}{(1+\log x)^2}dx$ f) $\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$

11. *a) $\int \frac{e^x(x+1)}{(x+2)^2}dx$ b) $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$ *c) $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

d) $\int e^x\left(\frac{\cos x-\sin x}{1-\cos 2x}\right)dx$ e) $\int \frac{e^x(x-1)}{(x+1)^3}dx$ *f) $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$

12. a) $\int e^x \frac{2-x^2}{(1-x)\sqrt{1-x^2}}dx$ b) $\int e^{\sin x}(x\cos x-\sec x\tan x)\,dx$

1.18 INTEGRATION BY SUCCESSIVE REDUCTION (Reduction formulae)

An integral involving a real function $f(x)$ and a variable $n (n \in N)$ as a parameter is usually denoted by I_n. That is $I_n = \int R(f(x);n)\,dx$ where R is a real function of $f(x)$ and n. This may be referred to as an integral of index n (order n).

An integral involving two or more real functions and two variables m and n $(m,n \in N)$ as parameters, is denoted by $I_{m,n}$ or $I_{n,m}$.

That is $I_{m,n} = \int R(f(x), g(x), ...; m,n)\, dx$

where R is a real function of $f(x)$, $g(x)$,.... and m,n. This may be referred to as integral of indices m and n.

If an integral I_n (or $I_{m,n}$) cannot be evaluated directly using the methods discussed so far, then we look for the possibility of connecting it to an integral of the same type but with a lower index (say I_{n-1} and / or I_{n-2}), using integration by parts. This gives rise to an algebraic relation which expresses a higher index integral in terms of a lower index integral. Such an algebraic relation is called a ***reduction formula***. Using a reduction formula for integral I_n, it can be completely evaluated by successively reducing the values of the index n. This process is called ***integration by successive reduction***.

In this section we derive various reduction formulae and illustrate the process of integration by successive reduction through some examples.

THEOREM-1.8 * If $I_n = \int x^n e^{ax} dx$ then $I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$ $(n \in N)$.

Proof : For $n \in N$, $I_n = \int x^n e^{ax} dx = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \left(\frac{e^{ax}}{a}\right) dx$

$$= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$$

Note that $I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{e^{ax}}{a} + c$

Ex. **Evaluate $\int x^3 e^{5x} dx$**

Sol. $I_3 = \int x^3 e^{5x} dx \quad (\because n = 3, a = 5)$

$$= \frac{1}{5} x^3 e^{5x} - \frac{3}{5} I_2 = \frac{1}{5} x^3 e^{5x} - \frac{3}{5}\left[\frac{1}{5} x^2 e^{5x} - \frac{2}{5} I_1\right]$$

$$= \frac{1}{5} x^3 e^{5x} - \frac{3}{5}.\frac{1}{5} x^2 e^{5x} + \frac{3}{5}.\frac{2}{5}\left[\frac{1}{5} x e^{5x} - \frac{1}{5} I_0\right]$$

$$= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125}\left(\frac{e^{5x}}{5}\right) + c$$

$$\therefore \int x^3 e^{5x} dx = \frac{e^{5x}}{625}[125x^3 - 75x^2 + 30x - 6] + c$$

Note : i) The above integral can also be evaluated using the general rule of integration by parts.

ii) $\int x^3 e^{ax} dx = \frac{e^{ax}}{a^4}[(ax)^3 - 3(ax)^2 + 6ax - 6] + c$

THEOREM-1.9 **If $I_n = \int \sin^n x\, dx$, $n \in N$ then $I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} (n \geq 2)$

(March-2014)

Proof : $I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\, dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\, dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x\, dx$$

$$= -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow I_n = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n} I_{n-2}$$

Ex. **Evaluate $\int Sin^5 x\, dx$**

Sol. $I_5 = \int \text{Sin}^5 x\, dx = -\frac{\sin^4 x\cos x}{5} + \frac{4}{5} I_3$

$$= -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x\cos x}{3} + \frac{2}{3} I_1\right]$$

$$= -\frac{\sin^4 x\cos x}{5} - \frac{4}{15}\sin^2 x\cos x + \frac{8}{15}\int \sin x\, dx$$

$$= -\frac{\sin^4 x\cos x}{5} - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + c$$

THEOREM-1.10 ***If $I_n = \int \cos^n x\, dx$ then $I_n = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n} I_{n-2}$** **(March-17, May-19)**

Proof: $I_n = \int \cos^{n-1} x\cos x\, dx$

$$= \cos^{n-1} x\sin x - \int \sin x(n-1)\cos^{n-2} x(-\sin x)dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\, dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x\, dx$$

$$= \cos^{n-1} x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Ex. **Evaluate $\int \cos^5 x\, dx$**

Sol. $I_5 = \int \cos^5 x dx = \frac{\cos^4 x\sin x}{5} + \frac{4}{5} I_3$

$$I_5 = \frac{\cos^4 x\sin x}{5} + \frac{4}{5}\left[\frac{\cos^2 x\sin x}{3} + \frac{2}{3} I_1\right]$$

$$= \frac{\cos^4 x\sin x}{5} + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\int \cos x\, dx$$

$$= \frac{\cos^4 x\sin x}{5} + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + c$$

THEOREM-1.11 **If $I_n = \int \tan^n x\, dx$ then $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ (March-18)

Sol: $I_n = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x(\sec^2 x - 1)\, dx$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

***Ex.** **Evaluate $\int \tan^6 x\, dx$**

Sol. $\int \tan^6 x\, dx$

$$= I_6 = \frac{\tan^5 x}{5} - I_4 = \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - I_2\right] = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x + c$$

THEOREM-1.12 *If $I_n = \int \cot^n x\, dx$ then $I_n = \dfrac{-\cot^{n-1} x}{n-1} - I_{n-2}$ (March-19)

Proof : $I_n = \int \cot^{n-2} x \cot^2 x\, dx = \int \cot^{n-2} x(\text{cosec}^2 x - 1) dx$

$$= \int \cot^{n-2} x \text{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

Ex. **Evaluate $\int \cot^4 x\, dx$**

Sol. $I_4 = \int \cot^4 x\, dx = \dfrac{-\cot^3 x}{3} - I_2 = \dfrac{-\cot^3 x}{3} - \left[\dfrac{-\cot x}{1} - I_0\right] = \dfrac{-\cot^3 x}{3} + \cot x + x + c$

THEOREM-1.13 *If $I_n = \int \sec^n x\, dx$ then $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$. (March-17)

Proof : $I_n = \int \sec^n x\, dx = \int \sec^{n-2} \sec^2 x\, dx$

$$= \sec^{n-2} x \tan x - \int \tan x(n-2) \sec^{n-3} \sec x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x(\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n (1 + n - 2) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Ex. **Evaluate $\int \sec^5 x\, dx$**

Sol. $I_5 = \int \sec^5 x\, dx = \dfrac{\sec^3 x \tan x}{4} + \dfrac{3}{4}$

$$I_3 = \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\left[\frac{\sec x \tan x}{2} + \frac{1}{2} I_1\right] = \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \tan x + \frac{3}{8} \int \sec x\, dx$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log|\sec x + \tan x| + c$$

THEOREM-1.14 *If $I_n = \int \operatorname{cosec}^n x\,dx$ then $I_n = \dfrac{-\operatorname{cosec}^{n-2}x\cot x}{n-1} + \dfrac{n-2}{n-1}I_{n-2}$ (March-19)

Proof : $I_n = \int \operatorname{cosec}^{n-2}x\operatorname{cosec}^2 x\,dx$

$$= -\operatorname{cosec}^{n-2}x\cot x - \int -\cot x(n-2)\operatorname{cosec}^{n-3}x(-\operatorname{cosec}x\cot x)dx$$

$$= -\operatorname{cosec}^{n-2}x\cot x - (n-2)\int \operatorname{cosec}^{n-2}x\cot^2 x\,dx$$

$$= -\operatorname{cosec}^{n-2}x\cot x - (n-2)\int \operatorname{cosec}^{n-2}x(\operatorname{cosec}^2 x - 1)dx$$

$$= -\operatorname{cosec}^{n-2}x\cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = -\operatorname{cosec}^{n-2}x\cot x + (n-2)I_{n-2}$$

$$I_n = -\frac{\operatorname{cosec}^{n-2}x\cot x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

Ex. **Evaluate $\int \operatorname{cosec}^5 x\,dx$**

Sol. $\int \operatorname{cosec}^5 x\,dx$

$$I_5 = \frac{-\operatorname{cosec}^3 x\cot x}{4} + \frac{3}{4}I_3 = -\frac{\operatorname{cosec}^3 x\cot x}{4} + \frac{3}{4}\left[\frac{-\operatorname{cosec}x\cot x}{2} + \frac{1}{2}I_1\right]$$

$$= -\frac{\operatorname{cosec}^3 x\cot x}{4} - \frac{3\operatorname{cosec}x\cot x}{8} + \frac{3}{8}\int \operatorname{cosec}x\,dx$$

$$= -\frac{\operatorname{cosec}^3 x\cot x}{4} - \frac{3\operatorname{cosec}x\cot x}{8} + \frac{3}{8}\log|\operatorname{cosec}x - \cot x| + c$$

THEOREM-1.15 If $I_{m,n} = \int \sin^m x\cos^n x\,dx$ then

i) $I_{m,n} = \dfrac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \dfrac{n-1}{m+n}I_{m,n-2}$

ii) $I_{m,n} = -\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \dfrac{m-1}{m+n}I_{m-2,n}$ $(m,n \in N,\ m \ge 2, n \ge 2)$

Proof : (i) $I_{m,n} = \int \sin^m x\cos^n x\,dx = \int(\sin^m x\cos^{n-1}x)\cos x\,dx$

$$= \sin^m x\cos^{n-1}x\sin x - \int\begin{bmatrix}\sin^m x(n-1)\cos^{n-2}x(-\sin x)\\ +\cos^{n-1}x\sin^{m-1}x\cos x\end{bmatrix}\sin x\,dx$$

$$= \sin^{m+1}x\cos^{n-1}x + (n-1)\int \sin^m x\cos^{n-2}x\sin^2 x\,dx - m\int \sin^m x\cos^n x\,dx$$

$$= \sin^{m+1}x\cos^{n-1}x + (n-1)\int \sin^m x\cos^{n-2}x(1-\cos^2 x)dx - mI_{m,n}$$

$$= \sin^{m+1}x\cos^{n-1}x + (n-1)\int \sin^m x\cos^{n-2}x\,dx - (n-1)\int \sin^m x\cos^{n-2}x\,dx - mI_{m,n}$$

$$= \sin^{m+1}x\cos^{n-1}x + (n-1)I_{m,n-2} - (n-1)I_{m,n} - mI_{m,n}$$

$$\Rightarrow (m+m)I_{m,n} = \sin^{m+1}x\cos^{n-1}x + (n-1)I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$$

ii) $I_{m,n} = \int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x\, dx$

$$= \sin^{m-1} x \cos^n x(-\cos x) - \int \begin{bmatrix} \sin^{m-1} x\, n\cos^{n-1} x(-\sin x) \\ -\cos^n x(m-1)\sin^{m-2} x \cos x \end{bmatrix} (-\cos x) dx$$

$$= -\sin^{m-1} x \cos^{n+1} x \;\; - n\int \sin^m x \cos^n x\, dx \;\; + (m-1)\int \sin^{m-2} x \cos^n x(1-\sin^2 x) dx$$

$$= \sin^{m-1} x \cos^{n+1} x - nI_{m,n} \;\; + (m-1)\int \sin^{m-2} x \cos^n x\, dx \;\; - (m-1)\int \sin^m x \cos^n x\, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} \;\; + (m-1)I_{m-2,n} - (m-1)I_{m,n} \Rightarrow (m+n)I_{m,n}$$

$$= -\sin^{m-1} x \cos^{n+1} x + (m-1)I_{m-2,n} \Rightarrow I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

Ex. **Evaluate $\int \sin^5 x \cos^4 x\, dx$**

Sol. $\int \sin^5 x \cos^4 x\, dx = I_{5,4} = \frac{-\sin^4 x \cos^5 x}{9} + \frac{4}{9} I_{3,4} = -\frac{\sin^4 x \cos^5 x}{9} + \frac{4}{9}\left[-\frac{\sin^2 x \cos^5 x}{7} + \frac{2}{7} I_{1,4} \right]$

$$= -\frac{\sin^4 x \cos^5 x}{9} - \frac{4\sin^2 x \cos^5 x}{63} + \frac{4}{63}\int \sin x \cos^4 x\, dx = -\frac{\sin^4 x \cos^5 x}{9} - \frac{4\sin^2 x \cos^5 x}{63} - \frac{\cos^5 x}{315} + c$$

THEOREM-1.16 **If $I_n = \int \frac{\sin nx}{\sin x} dx (n \in N)$ then $I_n = \frac{2}{n-1}\sin(n-1)x + I_{n-2}$.**

Proof : We know that $\sin nx - \sin(n-2)x = 2\cos(n-1)x \sin x$

$$\Rightarrow \frac{\sin nx}{\sin x} = 2\cos(n-1)x + \frac{\sin(n-2)x}{\sin x} \Rightarrow \int \frac{\sin nx}{\sin x} dx$$

$$= 2\int \cos(n-1)x\, dx + \int \frac{\sin(n-2)x}{\sin x} dx \Rightarrow I_n = 2\frac{\sin(n-1)x}{n-1} + I_{n-2}$$

$$\therefore I_n = \frac{2}{n-1}\sin(n-1)x + I_{n-2},\; n \geq 2$$

Note

i) *$\sin nx + \sin(n-2)x = 2\sin(n-1)x \cos x$ leads to the reduction formula*

$$I_n = \int \frac{\sin nx}{\cos x} dx \Rightarrow I_n = -\frac{2}{n-1}\cos(n-1)x - I_{n-2}$$

ii) *$\cos nx - \cos(n-2)x = -2\sin(n-1)x \sin x$ leads to the reduction formula*

$$I_n = \int \frac{\cos nx}{\sin x} dx \Rightarrow I_n = \frac{2}{n-1}\cos(n-1)x + I_{n-2}$$

THEOREM-1.17 **If $I_{m,n} = \int x^m (\log x)^n dx \;\; (m,n \in N)$ then $I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$**

Proof : $I_{m,n} = \int x^m (\log x)^n dx = (\log x)^n \frac{x^{m+1}}{m+1} - \int \left[n(\log x)^{n-1} \frac{1}{x} \right] \frac{x^{m+1}}{m+1} dx$

$$= \frac{x^{m+1}}{m+1}(\log x)^m - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

Note : If $I_n = \int (\log x)^n dx \; (n \in N)$ then $I_n = x(\log x)^n - n.I_{n-1}$.

THEOREM-1.18 If $I_n = \int \frac{t^n}{t^2+1}\,dt \ (n \in N)$ then $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$

Proof : $I_{n+2} = \int \frac{t^{n+2}}{t^2+1}\,dt = \int \left(\frac{t^2}{1+t^2}\right) t^n\,dt = \int \left(\frac{t^2+1-1}{t^2+1}\right) t^n\,dt$

$$= \int t^n\,dt - \int \frac{t^n}{t^2+1}\,dt = \frac{t^{n+1}}{n+1} - I_n \quad \therefore I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$$

Note

Replacing n by n – 2 in two above reduction formula, we get $I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$

THEREOM-1.19 If $I_n = \int \frac{dx}{(x^2+a^2)^n}, (n \in N)$, then $I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\frac{1}{a^2} I_n$.

Proof : $I_n = \int \frac{1}{(x^2+a^2)^n}\,dx = \int \frac{1}{(x^2+a^2)^n} 1\,dx$

$$= \frac{x}{(x^2+a^2)^n} - \int \left[(-n)\frac{1}{(x^2+a^2)^{n+1}} 2x\right] x\,dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\,dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{1}{(x^2+a^2)^n}\,dx - 2na^2\int \frac{1}{(x^2+a^2)^{n+1}}$$

$$= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1} \Rightarrow 2na^2 I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1)I_n$$

$$\Rightarrow I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{(2n-1)}{2n}\frac{1}{a^2} I_n$$

Note

Replacing n with (n–1) the above formula reduces to $I_n = \frac{1}{(2n-2)a^2}\frac{x}{(x^2+a^2)^{n-1}} + \frac{2n-3}{2n-2}\frac{1}{a^2} I_{n-1} (n \geq 2)$

For instance, for $n = 2$,

$$I_2 = \int \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^2}\frac{x}{(x^2+a^2)} + \frac{1}{2}\frac{1}{a^2}\int \frac{1}{x^2+a^2}\,dx$$

$$\Rightarrow \int \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^2}\frac{x}{(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + c$$

For $n = 3$,

$$I_3 = \int \frac{1}{(x^2+a^2)^3}\,dx = \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_2$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\left[\frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\frac{x}{x^2+a^2} + \frac{3}{8a^5}\tan^{-1}\left(\frac{x}{a}\right) + c$$

EXERCISE - 1.7

1. Evaluate

a) $\int x^3 e^{2x} dx$ b) $\int \sin^4 x\, dx$ c) $\int \cos^5 x\, dx$

d) $\int \tan^5 x\, dx$ e) $\int \cot^5 x\, dx$ f) $\int \sec^4 x\, dx$

g) $\int \operatorname{cosec}^4 x\, dx$ h) $\int \sin^4 x \cos^5 x\, dx$

*2. If $I_n = \int x^n e^{-x} dx$ prove that $I_n = -x^n e^{-x} + nI_{n-1}$.

*3. If $I_n = \int (\log x)^n dx$ then prove that $I_n = x(\log x)^n - nI_{n-1}$ and hence evaluate $\int (\log x)^4 dx$

4. If $I_n = \int x^{-n} e^{ax} dx$ then prove that $I_n = \dfrac{-e^{ax}}{(n-1)x^{n-1}} + \dfrac{a}{n-1} I_{n-1}$

5. If $I_n = \int \dfrac{\sin nx}{\cos x} dx$, prove that $I_n = -\dfrac{2}{n-1}\cos(n-1)x - I_{n-2}$

6. If $I_n = \int \dfrac{\cos nx}{\sin x} dx$, prove that $I_n = \dfrac{2}{n-1}\cos(n-1)x + I_{n-2}$

7. If $I_n = \int \dfrac{x^n}{1+x^2} dx$ prove that $I_n = \dfrac{x^{n-1}}{n-1} - I_{n-2} (n \ge 2)$ and hence show that $I_6 = \dfrac{x^5}{5} - \dfrac{x^3}{3} + x - \tan^{-1} x + c$

8. Show that $\int \dfrac{1}{(x^2+a^2)^3} dx = \dfrac{1}{4a^2} \dfrac{x}{(x^2+a^2)^2} + \dfrac{3}{8a^4} \dfrac{x}{x^2+a^2} + \dfrac{3}{8a^5} \tan^{-1}\left(\dfrac{x}{a}\right) + c$

ANSWERS

EXERCISE - 1.1

1. a) $\dfrac{x^8}{4} + c$ b) $\dfrac{4}{5} x^{5/2} + c$ c) $\sqrt[3]{2}\dfrac{3}{5} x^{5/3} + c$ d) $\dfrac{x^3}{3} + c$ e) $-\cos x + c$

f) $x + c$ g) $-\cot x - x + c$ h) $\log|x| + c$ i) $\tan x + c$

2. a) $\dfrac{x^4}{4} - 2\dfrac{x^3}{3} + 3x + c$ b) $6\sqrt{x} - 2\log|x| - \dfrac{1}{3x} + c$ c) $\dfrac{x^2}{4} + \dfrac{3}{2}x - \dfrac{1}{2}\log|x| + c$

d) $\dfrac{x^2}{2} - \dfrac{3}{2}\log|x| - \dfrac{5}{2x} + c$ e) $\log|x| - 2\sqrt{x} + c$ f) $x + 2\log|x| + \dfrac{3}{x} + c$

g) $\dfrac{x^2}{2} + 4\tan^{-1} x + c$ h) $e^x - \log|x| + 2\log\left|x + \sqrt{x^2-1}\right| + c$ i) $\tanh^{-1} x + \tan^{-1} x + c$

j) $\sin^{-1} x + \sinh^{-1} x + c$

3. a) $x - x^3 + \frac{3}{5}x^5 - \frac{x^7}{7} + c$ b) $\log x - \frac{4}{\sqrt{x}} - \frac{1}{x} + c$ c) $\frac{9}{4}x^2 + 3x + \frac{1}{2}\log|x| + c$

d) $\frac{4}{18}x^2 - \frac{4}{9}x + \frac{1}{9}\log|x| + c$ e) $2\sqrt{x} + 2\cosh^{-1}x + \frac{3}{2x} + c$

4. a) $\tan x - \sin x + \frac{x^3}{3} + c$ b) $\sec x + 3\log|x| - 4x + c$ c) $\frac{x^4}{4} - \sin x + 4\sinh^{-1}x + c$

d) $\cosh x + \log\left|x + \sqrt{x^2 - 1}\right| + c$ e) $\sinh x + \sinh^{-1}x + c$ f) $\sinh x - \cosh x + c$

5. a) $-\cot x - \frac{x}{2} + c$ b) $\tan x - \cot x + c$ c) $-\sqrt{2}\cos x + c$

d) $-\cot x + \operatorname{cosec} x + c$ e) $\tan x - \sec x + c$ f) $-\cot x - \operatorname{cosec} x + c$

g) $x - \tan x + \sec x + c$ h) $-2\operatorname{cosec} x + 3\sec x + c$ i) $\tan x - 2\cot x - \frac{\cot^3 x}{3} + c$

6. a) $-\cot x - \operatorname{cosec} x + c$ b) $-(\sin x + \cos x + c)$ c) $\sin x + \cos x + c$

d) $\cos x - \sin x + c$

7. a) $a^2\tan x - b^2\cot x - (a+b)^2 x + c$ b) $2(\sin x + x\cos\alpha) + c$ c) $\pm x + c$

d) $\pm\frac{x^2}{2} + c$ e) $x - \sinh^{-1}x + c$

EXERCISE - 1.2

1. a) $\frac{e^{2x}}{2} + c$ b) $-\frac{\cos 7x}{7} + c$ c) $\sin(x^2 + 1) + c$ d) $\frac{1}{2}\log(x^2 + 1) + c$

e) $\frac{1}{3}(\log x)^3 + c$ f) $e^{\tan^{-1}x} + c$ g) $\log|e^x + 1| + c$ h) $\log|3x^2 - 2| + c$

i) $\frac{1}{2}\tan^{-1}(2x + 1) + c$ j) $-\frac{1}{4}\cos^4 x + c$ k) $-\cos(\tan^{-1}x) + c$ l) $\tan^{-1}(x^3) + c$

m) $2\sqrt{\sin^{-1}x} + c$ n) $\log(x^2 + x + 1) + c$ o) $\log|\sin x + \cos x| + c$ p) $\frac{1}{7}\log|7x + 3| + c$

q) $\frac{2}{5}\sqrt{1 + 5x} + c$ r) $\frac{2}{9}(3x - 2)^{3/2} + c$ s) $\frac{1}{12}(3x^2 - 4)^2 + c$ t) $-\frac{1}{12}(1 - 2x^3)^2 + c$

2. a) $\frac{1}{2}[\log(1 + x)]^2 + c$ b) $-\frac{1}{2}\frac{1}{(1 + \tan x)^2} + c$ c) $-\frac{1}{4}\cos(x^4) + c$ d) $-\frac{1}{1 + \sin x} + c$

e) $\frac{3}{4}\sin^{4/3}x + c$ f) $e^{x^2} + c$ g) $x + c$ h) $\frac{1}{3}\sin^{-1}(x^3) + c$

i) $\frac{1}{2}\tan^{-1}(x^4) + C$ j) $\frac{1}{9}\tan^{-1}(x^9) + c$ k) $\frac{1}{4b(a + b\cot x)^4} + c$ l) $-\cos(e^x) + c$

m) $-\cos(\log x)+c$ n) $\log|\log x|+c$ o) $\frac{1}{n+1}(1+\log x)^{n+1}+c$ p) $\sin(\log x)+c$

q) $\frac{n}{ab}\log(bx^{n}+c)+k$ r) $\log|\sinh x|+c$ s) $2\tan^{-1}(\sqrt{x+2})+c$ t) $2\tan^{-1}(\sqrt{x+4})+c$

3. a) $\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{\alpha}{2}\right)\right|+c$ where $\cos\alpha=\frac{a}{r},\ \sin\alpha=\frac{b}{r},\ r=\sqrt{a^2+b^2}$

b) $\frac{1}{2}\log\left|\frac{1+\sqrt{3}\tan\left(\frac{x}{2}\right)}{\sqrt{3}-\tan\left(\frac{x}{2}\right)}\right|+c$ c) $\frac{1}{b-a}\log\left|a\cos^2 x+b\sin^2 x\right|+c$

d) $\log|\sin(\log x)|+c$ e) $\log\left|\sin(e^x)\right|+c$ f) $\log\left|\tan\left(\frac{\pi}{4}+\frac{1}{2}\tan x\right)\right|+c$

g) $\frac{2}{3}(\sin x)^{3/2}+c$ h) $\frac{\tan^5 x}{5}+c$ i) $2\sqrt{x^2+3x-4}+c$

j) $-\frac{2}{3}(\cot x)^{3/2}+c$ k) $\frac{1}{12}(\cos 3x-9\cos x)+c$ l) $\frac{1}{12}(\sin 3x+9\sin x)+c$

m) $\frac{1}{6}(\sin 3x+3\sin x)+c$ n) $\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+c$ o) $\frac{1}{40}(4x+3)^{5/2}-\frac{1}{8}(4x+3)^{3/2}+c$

p) $\frac{1}{c}\sin^{-1}\left(\frac{b+cx}{a}\right)+k$ q) $\frac{1}{ac}\tan^{-1}\left(\frac{b+cx}{a}\right)+k$ r) $x-\log(1+e^x)+c$

s) $\frac{4}{3}(1-x)^{3/2}-\frac{2}{5}(1-x)^{5/2}-2\sqrt{1-x}+c$ t) $2\left[\frac{1}{7}t^7-\frac{3}{5}t^5-t^3-t\right]+c$ where $t=\sqrt{x+1}$

4. a) $\frac{1}{4}\left[x+\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 6x}{6}\right]+c$ b) $\frac{1}{4}\left[\frac{\cos 6x}{2}-\frac{\cos 4x}{4}-\frac{\cos 2x}{2}\right]+c$

c) $\frac{1}{\sin(b-a)}\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+c$ d) $\frac{1}{\sin(b-a)}\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right|+c$

e) $\frac{-1}{n(\sec x+\tan x)^n}+c$

5. a) $\frac{1}{2}\tan^{-1}(2x)+c$ b) $\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)+c$ c) $\frac{x}{2}\sqrt{4x^2+9}+\frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right)+c$

d) $\frac{x}{2}\sqrt{9x^2-25}-\frac{25}{6}\cosh^{-1}\left(\frac{3x}{5}\right)+c$ e) $\frac{x}{2}\sqrt{16-25x^2}+\frac{8}{5}\sin^{-1}\left(\frac{5x}{4}\right)+c$

f) $\cosh^{-1}3x+C$ g) $\frac{1}{2}\sin^{-1}(2x)+C$

EXERCISE - 1.3

1. $2\log|x-3|-\log|x-2|+c$

2. $\log\left|\dfrac{x+1}{x+2}\right|+c$

3. $2\log|x-1|-\log(x^2+1)-\tan^{-1}x+c$

4. $\dfrac{1}{b^2-a^2}\left[\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)-\dfrac{1}{b}\tan^{-1}\left(\dfrac{x}{b}\right)\right]+c$

5. $\log\left(\dfrac{e^x+1}{e^x}\right)-e^{-x}+c$

6. $\log\left(\dfrac{e^x-1}{e^x}\right)+c$

7. $\log|x+1|+\dfrac{4}{x+2}+c$

8. $-\dfrac{1}{5}\log|1-x|+\dfrac{1}{10}\log(x^2+4)+\dfrac{1}{10}\tan^{-1}\left(\dfrac{x}{2}\right)+c$

9. $-\dfrac{3}{2}\log|x|-\dfrac{1}{6}\log|x+2|+\dfrac{5}{3}\log|x-1|+c$

10. $\log\left|\dfrac{2x-1}{3x-1}\right|+c$

11. $\dfrac{1}{2}\log|x|-\log|x+1|+\dfrac{1}{2}\log|x+2|+c$

12. $-\dfrac{1}{6}\log|x-1|-\dfrac{8}{15}\log|x+2|+\dfrac{7}{10}\log|x-3|+c$

13. $2\log|x-1|-\dfrac{1}{x-1}-2\log|x+2|+c$

14. $\dfrac{1}{(a-b)(a-c)}\log|x-a|+\dfrac{1}{(b-a)(b-c)}\log|x-b|+\dfrac{1}{(c-a)(c-b)}\log(x-c)+c$

EXERCISE - 1.4

Type 1 :

1. $\log\left|\dfrac{x-2}{x+1}\right|+c$

2. $\log\left|\dfrac{2x-1}{3x-1}\right|+c$

3. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)+c$

4. $\tan^{-1}(x+3)+c$

5. $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right)+c$

6. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$

7. $\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{6x+1}{\sqrt{11}}\right)+c$

8. $\dfrac{1}{8\sqrt{2}}\log\left|\dfrac{2x-1-2\sqrt{2}}{2x-1+2\sqrt{2}}\right|+c$

9. $\dfrac{1}{2\sqrt{14}}\log\left|\dfrac{x-1+\sqrt{\frac{7}{2}}}{x-1-\sqrt{\frac{7}{2}}}\right|+c$

10. $\dfrac{1}{\sqrt{31}}\log\left|\dfrac{4x+3-\sqrt{31}}{4x+3+\sqrt{31}}\right|+c$

Type 2:

11. $\sinh^{-1}\left(\dfrac{x+1}{3}\right)+c$

12. $\sin^{-1}\left(\dfrac{2x-1}{\sqrt{5}}\right)+c$

13. $\sin^{-1}(2x-5)+c$

14. $\cosh^{-1}\left(\dfrac{2x+1}{3}\right)+c$

15. $\sinh^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)+c$

16. $\sinh^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)+c$

Type 3:

17. $\dfrac{3x-4}{6}\sqrt{3+8x-3x^2}+\dfrac{25}{6\sqrt{3}}\sin^{-1}\left(\dfrac{3x-4}{5}\right)+c$

18. $\dfrac{2x-3}{4}\sqrt{1+3x-x^2}+\dfrac{13}{8}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{13}}\right)+c$

Type 4:

19. $\frac{11}{48}\log|3x-1|+\frac{7}{16}\log|x+5|+c$

20. $\log|x^2+x+1|+\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

21. $\frac{1}{2}\log|x^2+3x+12|-\frac{1}{\sqrt{39}}\tan^{-1}\left(\frac{2x+3}{\sqrt{39}}\right)+c$

22. $-\frac{1}{2}\log|6x-7-x^2|-\frac{7}{2\sqrt{2}}\log\left|\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}\right|+c$

23. $\frac{3}{4}\log|2x^2-4x+3|+\sqrt{2}\tan^{-1}(\sqrt{2}(x-1))+c$

24. $\frac{3}{4}\log|2x^2-4x+3|+\sqrt{2}\tan^{-1}(\sqrt{2}(x-1))+c$

25. $-\frac{5}{4}\log|x|+\frac{5}{8}\log|x^2+2x+4|+\frac{17}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$

Type 5:

26. $2\sqrt{2x^2+x-3}+c$

27. $\sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

28. $\sqrt{2x^2+5x-6}-\frac{11}{2\sqrt{2}}\cosh^{-1}\left(\frac{4x+5}{\sqrt{73}}\right)+c$

Type 6:

29. $\frac{1}{2}(2x^2-x+1)^{3/2}-\frac{5}{4\sqrt{2}}\left(x-\frac{1}{4}\right)\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}-\frac{35}{64\sqrt{2}}\sinh^{-1}\left(\frac{4x-1}{\sqrt{7}}\right)+c$

30. $(6+x-2x^2)^{3/2}+\frac{13}{16}(4x-1)\sqrt{6+x-2x^2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+c$

31. $\frac{5}{3}(x^2-6x+13)^{3/2}+6(x-3)\sqrt{x^2-6x+13}+24\sinh^{-1}\left(\frac{x-3}{2}\right)$

Type 7:

32. $2\sqrt{\frac{2x+1}{x+1}}+c$

33. $-\sinh^{-1}\left(\frac{2+x}{x\sqrt{3}}\right)+c$

34. $-\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{4x+7}{6x+3}\right)+c$

Type 8:

35. $\sqrt{2}\tan^{-1}\left(\sqrt{\frac{x-1}{2}}\right)+c$

36. $2\tan^{-1}(\sqrt{x+1})+c$

37. $\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2x+4}-1}{\sqrt{2x+4}+1}\right|+c$

38. $\sqrt{2}\tan^{-1}\left(\sqrt{\frac{x}{2}}\right)+c$

39. $\frac{2}{15}(3x+8)(x+1)^{3/2}+c$

40. $\frac{1}{3}(x+3)\sqrt{4x+3}+c$

Type 9:

41. $-\frac{1}{x}\sqrt{1+x^2}+c$

42. $-\frac{1}{4x}\sqrt{4+x^2}+c$

43. $\frac{1}{18\sqrt{2}}\log\left|\frac{x\sqrt{2}+\sqrt{x^2-9}}{x\sqrt{2}-\sqrt{x^2-9}}\right|+c$

44. $-\tan^{-1}\left(\frac{\sqrt{x^2+2}}{x}\right)+c$

EXERCISE - 1.5

1. $\frac{1}{6}\tan^{-1}\left(\frac{3}{2}\tan x\right)+c$

2. i) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)+c$ ii) $-\frac{1}{1+\tan x}+c$

3. $\frac{1}{10}\log\left|\frac{2\tan x+1}{2-\tan x}\right|+c$

4. $\frac{1}{2}\log\left|\frac{\tan x}{\tan x+2}\right|+c$

5. $\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{8\tan x+3}{\sqrt{23}}\right)+c$

6. $\frac{2}{\sqrt{5}}\tan^{-1}\left[\frac{1}{\sqrt{5}}\tan\left(\frac{x}{2}\right)+c\right]$

7. $\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}+\tan\left(x/2\right)}{\sqrt{5}-\tan\left(x/2\right)}\right|+c$

8. $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan\left(x/2\right)+2}{\sqrt{5}}\right)+c$

9. $\frac{1}{\sqrt{5}}\log\left|\frac{2\tan\left(\frac{x}{2}\right)+3-\sqrt{5}}{2\tan\left(\frac{x}{2}\right)+3\sqrt{5}}\right|+c$

10. $\frac{1}{5}\log\left|\frac{2\tan\frac{x}{2}+1}{2\left(2-\tan\frac{x}{2}\right)}\right|+c$

11. $\frac{1}{3}\log\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right|+c$

12. $\frac{2}{3}\tan^{-1}\left(\frac{5\tan\left(x/2\right)+4}{3}\right)+c$

13. $\frac{1}{3}\tan^{-1}\left(\frac{5\tan x+4}{3}\right)+c$

14. $x+\log|5\cos x+4\sin x|+c$

15. $\frac{23}{41}x-\frac{2}{41}\log|4\cos x+5\sin x|+c$

16. $\frac{3}{25}x+\frac{4}{25}\log|3\cos x+4\sin x|+c$

17. $\frac{1}{2}x+\frac{1}{2}\log|\sin x+\cos x|+c$

18. $\frac{1}{2}x+\frac{1}{2}\log|\sin x-\cos x|+c$

19. $\frac{1}{d^2+e^2}\{dx+e\log|d\cos x+e\sin x|\}+k$

20. $\frac{18}{25}x+\frac{1}{25}\log|3\sin x+4\cos x+5|-\frac{4}{5\left(3+\tan\frac{x}{2}\right)}+c$

21. $\frac{2}{\sqrt{11}}\tan^{-1}\left[\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right]+C$

EXERCISE - 1.6

Evaluate the following integrals

1. a) $e^x(x-1)+c$ b) $e^x(x^2-2x+3)+c$ c) $-\frac{e^{-3x}}{27}(9x^2+6x+2)+c$

2. a) $x\sin x+\cos x+c$ b) $-x\cos x+\sin x+c$ c) $x\tan x-\log|\sec x|+c$

   d) $\log|\sin x|-x\cot x-\frac{1}{2}x^2+c$ e) $\frac{1}{2}x\tan 2x-\frac{1}{4}\log|\sec 2x|+c$

   f) $\frac{1}{4}\left[x^2-x\sin 2x-\frac{1}{2}\cos 2x\right]+c$ g) $x\tan x-\log|\sec x|-\frac{x^2}{2}+c$

   h) $x\tan\frac{x}{2}-2\log\sec\frac{x}{2}+C$ i) $x\tan\frac{x}{2}+c$

3. a) $e^x\sin x+c$ b) $e^x\tan x+c$ c) $e^x\sec x+c$ 1

   d) $e^x\log(\sec x)+c$ e) $e^x\log x+c$

4. a) $x(\log x-1)+c$ b) $\frac{x^2}{2}\log x-\frac{x^2}{4}+c$ c) $\frac{x^{n+1}}{n+1}\left[\log x-\frac{1}{n+1}\right]+c$

   d) $-\frac{1}{x}[\log x+1]+c$ e) $\frac{2}{3}x^{\frac{3}{2}}\left[\log x-\frac{2}{3}\right]+c$ f) $x\log(1+x)^2-2x+2\tan^{-1}x+c$

   g) $\frac{x^2}{2}\log(1+x)-\frac{x^2}{4}+\frac{x}{2}-\frac{1}{2}\log(1+x)+c$ h) $x((\log x)^2-2\log x+2)+C$

5. a) $\frac{1}{4}[(2x^2-1)\sin^{-1}x+x\sqrt{1-x^2}]+c$ b) $\frac{1}{4}[(2x^2-1)\cos^{-1}x-x\sqrt{1-x^2}]+c$

   c) $\frac{1}{2}[(x^2+1)\tan^{-1}x-x]+c$ d) $\frac{1}{2}[x^2\sec^{-1}x-\sqrt{x^2-1}]+c$

   e) $\frac{x^3}{3}\sin^{-1}x+\frac{1}{3}\sqrt{1-x^2}-\frac{1}{9}(1-x^2)^{\frac{3}{2}}+c$ f) $\frac{x^3}{3}\cos^{-1}x-\frac{1}{3}\sqrt{1-x^2}+\frac{1}{9}(1-x^2)^{\frac{3}{2}}+c$

   g) $\frac{x^3}{3}\tan^{-1}x-\frac{x^3}{6}+\frac{1}{6}\log(1+x^2)$ h) $-\frac{1}{x}\tan^{-1}x+\log x-\frac{1}{2}\log(1+x^2)+c$

6. a) $\frac{1}{4}[(2x^2+1)\sinh^{-1}x-x\sqrt{1+x^2}]+c$ b) $\frac{1}{4}[(2x^2+1)\cosh^{-1}x-x\sqrt{x^2-1}]+c$

   c) $\frac{x^2}{2}\tanh^{-1}x-\frac{1}{4}\log\left|\frac{1+x}{1-x}\right|+\frac{x}{2}+c$

7. a) $2x\tan^{-1}x-\log(1+x^2)+c$ b) $\frac{1}{2}[x\cos^{-1}x-\sqrt{1-x^2}]+c$

c) $3\left[x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\right]+c$ d) $\frac{x^2}{2}\tan^{-1}(x^2)-\frac{1}{4}\log(1+x^4)+c$

e) $x-\sqrt{1-x^2}\sin^{-1}x+c$ f) $2x\tan^{-1}x-\log(1+x^4)+c$

g) $2x\tan^{-1}x-\log(1+x^4)+c$ h) $\frac{1}{\sqrt{1+x^2}}[x-\tan^{-1}x]+c$

8. a) $2e^{\sqrt{x}}(\sqrt{x}-1)+c$ b) $2[\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}]+c$

c) $2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+c$ d) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+c$

e) $x\sec^{-1}\sqrt{x}-\sqrt{x-1}+c$

9. a) $\frac{e^x}{\sqrt{2}}(\sin x-\cos x)+C$ b) $\frac{e^x}{\sqrt{2}}(\cos x+\sin x)+C$

c) $\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+C$ d) $\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

e) $\frac{2^x}{(\log 2)^2+1}[\log 2.\cos x+\sin x]+C$ f) $\frac{2a^x\sin 2x+(\log a)a^x\cos 2x}{(\log a)^2+4}$

g) $e^{2x}\left[\frac{\cos 4x+2\sin 4x}{10}+\frac{\cos 2x+\sin 2x}{8}\right]+C$ h) $\frac{e^x}{2}\left[1-\frac{\cos 2x+2\sin 2x}{5}\right]+C$

i) $\frac{e^x}{74}[\sin 6x-6\cos 6x]+C$

j) $\begin{cases}\frac{1}{2}\sin^{-1}x-\frac{1}{4}\sin(2\sin^{-1}x)+C; m=0\\ \frac{1}{2}e^{m\sin^{-1}x}\left[\frac{1}{m}-\frac{1}{m^2+4}(m\cos(2\sin^{-1}x)+2\sin(2\sin^{-1}x))+C; m\neq 0\right]\end{cases}$

10. a) $x\sin(\log x)+C$ b) $\frac{x}{2}[\sin(\log x)-\cos(\log x)]+C$

c) $\frac{x}{2}[\cos(\log x)+\sin(\log x)]+C$ d) $\frac{x}{\log x}+C$

e) $\frac{x}{1+\log x}+C$ f) $x\left[\log(\log x)-\frac{1}{\log x}\right]+C$

11. a) $\frac{e^x}{x+2}+C$ b) $e^x\tan\frac{x}{2}+C$ c) $-e^x\cot\frac{x}{2}+C$

d) $-\frac{e^x}{2}\operatorname{cosec}x+C$ e) $\frac{e^x}{(x+1)^2}+C$ f) $e^x\tan^{-1}x+C$

12. a) $e^x\sqrt{\frac{1+x}{1-x}}+C$ b) $e^{\sin x}(x-\sec x)+C$

EXERCISE - 1.7

1. a) $\frac{e^{2x}}{16}[(2x)^3-3(2x)^2+6(2x)-6]+c$ b) $-\frac{\sin^3 x\cos x}{4}-\frac{3}{8}\sin x\cos x+\frac{3}{8}x+c$

c) $\frac{\cos^4 x\sin x}{5}+\frac{4}{15}\cos^2 x\sin x+\frac{8}{15}\sin x+c$ d) $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\log|\sec x|+c$

e) $-\frac{\cot^4 x}{4}+\frac{\cot^2 x}{2}+\log|\sin x|+c$ f) $\frac{\sec^2 x\tan x}{3}+\frac{2}{3}\tan x+c$

g) $-\frac{\operatorname{cosec}^2 x\cot x}{3}-\frac{2}{3}\cot x+c$ h) $-\frac{\sin^2 x\cos^5 x}{7}-\frac{2}{35}\cos^5 x+c$

3. $x[(\log x)^4-4(\log x)^3+12(\log x)^2-24\log x+24+c$