CHAPTER 2

Geometry

SYSTEM OF CIRCLES

◆ ANGLE BETWEEN TWO CIRCLES ◆ ORTHOGONALITY ◆
◆ RADICAL AXIS ◆ RADICAL CENTRE ◆

2.0 INTRODUCTION

In this chapter, we shall discuss the angle between two intersecting circles and obtain the condition for their orthogonality. We shall also discuss about the radical axis of two circles and radical centre of three circles. We define the coaxal system of circles.

2.1 ANGLE BETWEEN TWO INTERSECTING CIRCLES

Definition :

The angle between two intersecting circles is defined as the angle between the tangents at the point of intersection of the two circles.

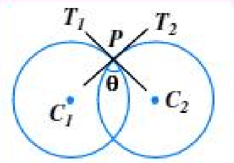

Fig 2.1

$T_1\hat{P}T_2$ is the angle between the circles at P.

Note : *If two circles $S=0$ and $S'=0$ intersect at P and Q, then the angle between the two circles at the points P and Q are equal.*

THEOREM-2.1 **Let C_1, C_2 be the centres of two intersecting circles of radii r_1 and r_2. Let $C_1C_2 = d$. If θ is the angle between the circles then $\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$**

Proof : Let P be the point of intersection of the two circles. Let the tangents drawn to two circles at P intersect the line joining the centres $\overleftrightarrow{C_1C_2}$ at T_1 and T_2. Then $\angle T_1PT_2 = \theta$

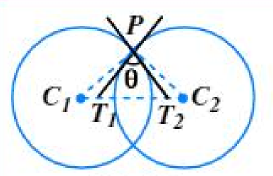

Fig 2.2

Now, $\angle C_1PC_2 = \angle C_1PT_2 + \angle T_2PC_2 = 90^0 + 90^0 - \theta = 180^0 - \theta$

From ΔC_1PC_2, $C_1C_2^2 = C_1P^2 + C_2P^2 - 2(C_1P)(C_2P)\cos\angle C_1PC_2$

i.e., $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(180^0 - \theta)$

$$\therefore \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

Since $\cos\theta$ is independent of the point of intersection (i.e., coordinates of the point of intersection are not involved), the angle at Q is also equal to θ

Note

Let the angle between the circles at Q be ϕ.

Then $\angle C_1QC_2 = 180 - \phi$

$C_1P = C_1Q = r_1$, $C_2P = C_2Q = r_2$

$\therefore$ The triangles C_1PC_2, C_2QC_2 are identical

$\Rightarrow \angle C_1PC_2 = \angle C_1QC_2$

$\Rightarrow 180^0 - \theta = 180^0 - \phi \Rightarrow \theta = \phi$

THEOREM-2.2 **If θ is the angle between the intersecting circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then $\cos\theta = \dfrac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$**

Proof : Let C_1, C_2 be the centres and r_1, r_2 be the radii of the two given circles.

$C_1 = (-g, -f)$, $C_2 = (-g', -f')$

$r_1 = \sqrt{g^2 + f^2 - c}$, $r_2 = \sqrt{g'^2 + f'^2 - c'}$

$$\therefore \cos\theta = \frac{c_1c_2^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{(g'-g)^2 + (f'-f)^2 - (g^2+f^2-c) - (g'^2+f'^2-c')}{2\sqrt{g^2+f^2-c}\sqrt{g'^2+f'^2-c'}}$$

$$\Rightarrow \cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2+f^2-c}\sqrt{g'^2+f'^2-c'}}$$

***1. Find the angle between the circles given by the equations $x^2 + y^2 + 6x - 10y - 135 = 0$, $x^2 + y^2 - 4x + 14y - 116 = 0$.**

Sol. Given circles are $x^2 + y^2 + 6x - 10y - 135 = 0$ -- (1)

$x^2 + y^2 - 4x + 14y - 116 = 0$ -- (2)

Hence $c_1 = (-3, 5)$, $r_1 = \sqrt{9 + 25 + 135} = 13$

$c_2 = (2, -7)$, $r_2 = \sqrt{4 + 49 + 116} = 13$

$d = c_1c_2 = \sqrt{25 + 144} = 13$

Let θ be the angle between (1) & (2)

$$\therefore \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{169 - 169 - 169}{2(169)} = \frac{-1}{2}$$

$$\therefore \theta = \frac{2\pi}{3}$$

2.2 ORTHOGONAL CIRCLES

Definition :

If the angle between two circles is a right angle, then the two circles are said to cut each other orthogonally.

THEOREM-2.3 **Let d be the distance between the centres of two intersecting circles of radii r_1 and r_2. If the two circles cut each other orthogonally, then $d^2 = r_1^2 + r_2^2$.**

Proof : Let θ be the angle between the circles

$\therefore$ The circles are orthogonal $\Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \cos\theta = \cos\frac{\pi}{2} = 0 \Rightarrow d^2 - r_1^2 - r_2^2 = 0$$

$$\Rightarrow d^2 = r_1^2 + r_2^2$$

$$\therefore \theta = \frac{\pi}{2} \Leftrightarrow d^2 = r_1^2 + r_2^2$$

THEOREM-2.4 **The condition that the two circles $S = 0$ and $S' = 0$ may cut each other orthogonally is $2gg' + 2ff' = c + c'$.**

Proof : Let the circles $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

intersect orthogonally at P

$C_1 = (-g, -f)$ and $C_2 = (-g', -f')$ are the centres of the circles whose radii are respectively $r_1 = \sqrt{g^2 + f^2 - c}$ and $r_2 = \sqrt{g'^2 + f'^2 - c'}$

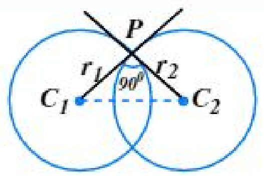

Fig 2.3

From the right angled triangle C_1PC_2, $C_1C_2^2 = C_1P^2 + C_2P^2 = r_1^2 + r_2^2$

$$\Rightarrow (-g + g')^2 + (-f + f')^2 = g^2 + f^2 - c + g'^2 + f'^2 - c'$$

$$\Rightarrow 2gg' + 2ff' = c + c'$$

Note :

If two circles interest orthogonally then $C_1C_2 = r_1^2 + r_2^2$

Note

In the case of orthogonal circles, the tangent of one of the circles at the point of intersection will be a normal to the other circle and hence it passes through the centre of the other circle.

***Ex. Find k if the circles $x^2+y^2-5x-14y-34=0$ and $x^2+y^2+2x+4y+k=0$ are orthogonal to each other. (March-18)**

Sol. Given circles are $x^2+y^2-5x-14y-34=0$ ---(1)

$x^2+y^2+2x+4y+k=0$ ---(2)

(1) & (2) are orthogonal to each other

$\Rightarrow 2gg'+2ff'=c+c' \Rightarrow 2\left(\frac{-5}{2}\right)(1)+2(-7)(2)=-34+k$

$\Rightarrow -5-28=-34+k \Rightarrow k=1$

THEOREM-2.5

i) If $S=0$, $S'=0$ are two circles intersecting at two distinct points, then $S-S'=0$ (or $S'-S=0$) represents the common chord of these circles.

ii) If $S=0$, $S'=0$ are two circles touching each other, then $S-S'=0$ (or $S'-S=0$) is a common tangent at the point of contact of the two circles.

Proof : Let $S=x^2+y^2+2gx+2fy+c=0$ -- (1) and

$S'=x^2+y^2+2g'x+2f'y+c'=0$ -- (2)

i) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of intersection of (1) & (2).

Consider $S-S'=0$

$\Rightarrow 2(g-g')x+2(f-f')y+(c-c')=0$ -- (3)

Clearly the points P, Q lie on (3) since $S_{11}=0,\ S_{22}=0,\ S_{11}^1=0,\ S_{22}^1=0$.

Equation (3) a linear in x and y and hence it represents a line.

$\therefore$ $S-S'=0$ is the equation of the common chord of (1) & (2).

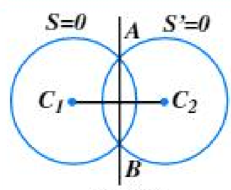

Fig 2.4

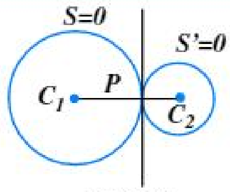

Fig 2.5

ii) Let (1) & (2) touch each other at $P(x_1, y_1)$

Consider $S-S'=0$

$\Rightarrow 2(g-g')x+2(f-f')y+(c-c')=0$ --- (4)

$P(x_1, y_1)$ is a point on (4) and it represents a line and the slope of (4) is $-\frac{(g-g')}{(f-f')}$.

Slope of the line joining the centres of the circles is $\frac{f-f'}{g-g'}$.

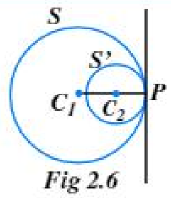

Fig 2.6

$\therefore$ The line (4) is perpendicular to the line of centres and it passes through the point of contact of the two circles. Hence it is a tangent common to both the circles.

THEOREM-2.6

i) If a circle $S \equiv x^2+y^2+2gx+2fy+c=0$ and a line $L \equiv lx + my + n = 0$ intersect each other then the equation of the circle passing through the points of intersection of the circle and the line is given by $S + kL = 0$ where k is a real parameter.

ii) If the circ les $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y+c'= 0$ intersect each other then the equation of the circle passing through the points of intersection of $S = 0$ and $S' = 0$ is given by $\lambda S + \mu S' = 0$ where λ and μ are parameters such that $\lambda + \mu \neq 0$.

Proof :

i) Consider the equation $(x^2 + y^2 + 2gx + 2fy + c) + k(lx + my + n) = 0$

i.e., $S + kL = 0$ -- (1)

Clearly, this equation represents a circle

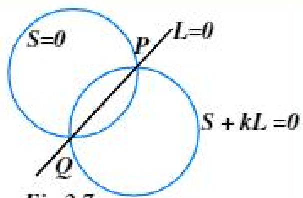

Fig 2.7

In order to prove that (1) represents a circle passing through the points of intersection of the circle $S = 0$ and the line $L = 0$, it is sufficient to show that their points of intersection satisfy equation (1)

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of intersection of $S = 0$ and $L = 0$.

Then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$; $x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$

$lx_1 + my_1 + n = 0$; $lx_2 + my_2 + n = 0$

$\therefore (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) + k(\ell x_1 + my_1 + n) = 0$ -- (2)

and $(x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c) + k(lx_2 + my_2 + n) = 0$ -- (3)

From (2) & (3) it follows that the points P and Q satisfy the equation (1).

$\therefore$ (1) represents a circle passing through the points of intersection of S=0 & L=0.

ii) Consider $\lambda S + \mu S' = 0$ --- (4)

where λ, μ are any real numbers such that $\lambda + \mu \neq 0$. Equation (4) clearly represents a circle and it passes through the points of intersection of $S = 0$ and $S' = 0$.

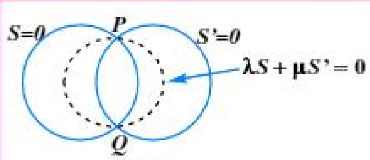

Fig 2.8

Further, equation (4) is equivalent to S+KL=0 where $k = \dfrac{-\mu}{(\lambda + \mu)}$ and $L \equiv S - S' = 0$.

Note

i) The equation $\lambda S + \mu S' = 0$ can also be written as $S + kS' = 0$. where $k = \dfrac{\mu}{\lambda} \neq -1$ (assuming $\lambda \neq 0$)

ii) If $k = -1$, then $S + kS' = S - S' = 0$ represents the common chord of the circles $S = 0$ and $S' = 0$. If the points of intersection coincide, then $S - S' = 0$ is a common tangent to the circles.

Remember :
Angle between two intersecting circles with centres C_1, C_2 and radii r_1, r_2 is $Cos^{-1}\left[\frac{C_1C_2^2 - r_1^2 - r_2^2}{2r_1r_2}\right]$

SOLVED EXAMPLES

***1.** **If the angle between the circles $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + kx + 6y - 59 = 0$ is 45^0 find k.** **(May-18)**

Sol. Here $g = -6, f = -3, c = 41,\ g' = \frac{k}{2}, f' = 3, c' = -59$

Given $\theta = 45^0$

$$\therefore \cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$$

$$\Rightarrow \cos 45^0 = \left|\frac{41 - 59 - 2(-6)\left(\frac{k}{2}\right) - 2(-3)(3)}{2\sqrt{36 + 9 - 41}\sqrt{\frac{k^2}{4} + 9 + 59}}\right|$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{|3k|}{2\sqrt{\frac{k^2}{4} + 68}} \Rightarrow k = \pm 4$$

***2.** **Find the equation of the circle which passes through the point (0,–3) and intersect the circles given by the equations $x^2 + y^2 - 6x + 3y + 5 = 0$ and $x^2+y^2-x-7y = 0$ orthogonally.**

Sol. Let $A = (0, -3)$

Given circles are $x^2 + y^2 - 6x + 3y + 5 = 0$ -- (1)

$x^2 + y^2 - x - 7y = 0$ -- (2)

Let the equation of the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ -- (3)

(3) cuts (1) orthogonally

$$\Rightarrow 2g(-3) + 2f\left(\frac{3}{2}\right) = c + 5$$

$\Rightarrow -6g + 3f - c = 5$ -- (4)

(3) cuts (2) orthogonally

$$\Rightarrow 2g\left(\frac{-1}{2}\right) + 2f\left(\frac{-7}{2}\right) = c$$

$\Rightarrow -g - 7f - c = 0$ -- (5)

(3) Passes through $A \Rightarrow 0 + 9 + 0 - 6f + c = 0$

$\Rightarrow 6f - c = 9$ -- (6)

Solving (4), (5) and (6) we get $g = \frac{1}{3}, f = \frac{2}{3}$ and $c = -5$

$\therefore$ Equation of the required circle is $x^2 + y^2 + \frac{2}{3}x + \frac{4}{3}y - 5 = 0$

$\Rightarrow 3(x^2 + y^2) + 2x + 4y - 15 = 0$

*** 3. If P and Q are conjugate points w.r.t a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, then prove that the circle PQ as diameter cuts the circle $S = 0$ orthogonally.**

Sol. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, $S = x^2 + y^2 + 2gx + 2fy + c = 0$

The polar of P w.r.t $S = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

P and Q are conjugate w.r.t $S = 0 \Rightarrow S_{12} = 0$

$\Rightarrow x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$ -- (1)

Equation of the circle with $\overline{PQ}$ as diameter is

$S' = (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$S' = x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$

$\therefore 2g' = -(x_1 + x_2)$ and $2f' = -(y_1 + y_2)$, $c' = x_1x_2 + y_1y_2$

$2gg' + 2ff' = -g(x_1 + x_2) - f(y_1 + y_2)$

$= c + (x_1x_2 + y_1y_2) = c + c'$

$\therefore$ The circles $S = 0$ and $S' = 0$ cut each other orthogonally.

***4. If the equations of two circles whose radii are a and a' are $S = 0$ and $S' = 0$, then show that the circles $\frac{S}{a} + \frac{S'}{a'} = 0$ and $\frac{S}{a} - \frac{S'}{a'} = 0$ intersect orthogonally.**

Sol. With out loss of generality, let the line of centres be choosen as x-axis and the distance between them be $2p$, origin being the mid - point.

The centres of the circles are $(p, 0)$ and $(-p, 0)$

$\therefore S = (x - p)^2 + y^2 = a^2$

$\Rightarrow S = x^2 + y^2 - 2px + p^2 - a^2 = 0$

$S' = (x + p)^2 + y^2 = a'^2$

$\Rightarrow S' = x^2 + y^2 + 2px + p^2 - a'^2 = 0$

Circles are $\frac{S}{a} \pm \frac{S'}{a'} = 0 \Rightarrow a'S \pm aS' = 0$

$a'S + aS' = 0$

$\Rightarrow (a' + a)(x^2 + y^2 + p^2) - 2px(a' - a) - aa'(a' + a) = 0$

$\Rightarrow x^2 + y^2 - 2p\left(\frac{a' - a}{a' + a}\right)x + p^2 - aa' = 0$

$a'S - aS' = 0$

$\Rightarrow x^2 + y^2 - 2p\left(\frac{a' + a}{a' - a}\right)x + p^2 + aa' = 0$

$2gg' + 2ff' = 2\left(-p\left(\frac{a' - a}{a' + a}\right)\right)\left(-p\left(\frac{a' + a}{a' - a}\right)\right) + 0$

$= 2p^2 = c + c'$

$\therefore$ The circles $\frac{S}{a} \pm \frac{S'}{a'} = 0$ cut each other orthogonally

***5. Find the equation of the circle which cuts orthogonally the three circles $x^2+y^2+2x+4y+1=0$, $2(x^2+y^2)+6x+8y-3=0$ and $x^2+y^2-2x+6y-3=0$**

Sol. Let the equation of the required circle be $x^2+y^2+2gx+2fy+c=0$ --(1)

Given circles are $x^2+y^2+2x+4y+1=0$ --(2)

$$x^2+y^2+3x+4y-\frac{3}{2}=0 \quad \text{--(3)}$$

$$x^2+y^2-2x+6y-3=0 \quad \text{--(4)}$$

(1) cuts (2) orthogonally $\Rightarrow 2g+4f-c=1$ --(5)

(1) cuts (3) orthogonally $\Rightarrow 3g+4f-c=\frac{-3}{2}$ --(6)

(1) cuts (4) orthgonally $\Rightarrow -2g+6f-c=-3$ --(7)

Solving (5), (6) and (7), we get

$g=\frac{-5}{2}, f=-7$ and $c=-34$

$\therefore$ Equation of the required circle is $g=\frac{-5}{2}, f=-7$ $x^2+y^2-5x-14y-34=0$

***6. Find the equation of the circle passing through the intersection of the circles $x^2+y^2=2ax$, $x^2+y^2=2by$ and having its centre on the line $\frac{x}{a}-\frac{y}{b}=2$.**

Sol. Given circles are $S=x^2+y^2-2ax=0$ --(1)

and $S'=x^2+y^2-2by=0$ --(2)

The equation of the common chord of (1) & (2) is

$L=S'-S=0 \Rightarrow L=ax-by=0$ --(3)

The equation of the circle passing through the intersection of (1) & (2) is

$S+\lambda L=0 \Rightarrow x^2+y^2-2ax+\lambda(ax-by)=0$

$\Rightarrow x^2+y^2+(\lambda-2)ax-\lambda by=0$ --(4)

λ is a parameter.

Centre of (4), $C=\left(\frac{a(2-\lambda)}{2},\frac{\lambda b}{2}\right)$

Since C lies on $\frac{x}{a}-\frac{y}{b}=2$, $\frac{(2-\lambda)a}{2a}-\frac{\lambda b}{2b}=2$

$\Rightarrow 2-\lambda-\lambda=4 \Rightarrow 2\lambda=-2 \Rightarrow \lambda=-1$

$\therefore$ Equation of the required circle is $x^2+y^2-3ax+by=0$

***7. If the straight line represented by $x\cos\alpha+y\sin\alpha=p$ intersect the circle $x^2+y^2=a^2$ at the points A and B, then show that the equation of the circle with $\overline{AB}$ as diameter is $(x^2+y^2-a^2)-2p(x\cos\alpha+y\sin\alpha-p)=0$**

Sol. The equation of the circle passing through the points A and B is

$(x^2+y^2-a^2)+\lambda(x\cos\alpha+y\sin\alpha-p)=0$ --(1) λ is a parameter

Centre of (1), $C=\left(\frac{-\lambda\cos\alpha}{2},\frac{-\lambda\sin\alpha}{2}\right)$

Since the line $x\cos\alpha+y\sin\alpha-p=0$ is a diameter of (1) C lies on the line

$\therefore \left(\frac{-\lambda\cos\alpha}{2}\right)\cos\alpha+\left(\frac{-\lambda\sin\alpha}{2}\right)\sin\alpha=p \Rightarrow \lambda=-2p$

$\therefore$ Equation of the required circle is $(x^2+y^2-a^2)-2p(x\cos\alpha+y\sin\alpha-p)=0$

Remember :
Any circle passing through points of Intersections of given circle S=0 and given line L=0 is of the form $S+\lambda L=0$

8. Find the equations of the circles which cut orthogonally the circles $x^2+y^2-6y+1=0$, $x^2+y^2-4y+1=0$ and touch the line $3x+4y+5=0$.

Sol. Let the required circle be $x^2+y^2+2gx+2fy+c=0$ -- (1)

Given circles are $x^2+y^2-6y+1=0$ -- (2)

$x^2+y^2-4y+1=0$ -- (3)

(1) cuts (2) and (3) orthogonally

$2g(0)+2f(-3)=c+1 \Rightarrow -6f=c+1$

and $2g(0)+2f(-2)=c+1 \Rightarrow -4f=c+1$

$\therefore f=0$ and $c=-1$

(1) Touches the line $3x+4y+5=0$

$$\therefore \frac{|3(-g)+4(-f)+5|}{5}=\sqrt{g^2+f^2-c}$$

$$\Rightarrow \frac{|-3g+5|}{5}=\sqrt{g^2+1} \Rightarrow (-3g+5)^2=25(g^2+1)$$

$$\Rightarrow 16g^2+30g=0 \Rightarrow g=0,\frac{-15}{8}$$

$\therefore$ The equation of the required circles are

$$x^2+y^2-1=0 \text{ and } 4(x^2+y^2)-15x-4=0$$

EXERCISE - 2.1

1. Find the acute angle or angle of intersection of the following circles.

***a) $x^2+y^2-12x-6y+41=0$, $x^2+y^2+4x+6y-59=0$ **(March-2017)** [Ans : $\theta=\frac{\pi}{4}$]

b) $x^2+y^2+4x-14y+28=0$, $x^2+y^2+4x-5=0$ **(March-2017) [Ans : $\theta=60^0=\frac{\pi}{3}$]

*2. Show that the angle between the circles $x^2+y^2=a^2$, $x^2+y^2=ax+ay$ is $\frac{3\pi}{4}$

3. Show that the circles given by the following equations intersect each other orthogonally

*a) $x^2+y^2+2my-g=0$, $x^2+y^2+2lx+g=0$

*b) $3x^2+3y^2-8x+29y=0$, $x^2+y^2-2x-2y-7=0$

*c) $x^2+y^2+4x-2y-11=0$, $x^2+y^2-4x-8y+11=0$

*d) $x^2+y^2-2x+4y+4=0$, $x^2+y^2+3x+4y+1=0$

4. Find k if the following pairs of circles are orthogonal

*a) $x^2+y^2+2by-k=0$, $x^2+y^2+2ax+8=0$ [Ans : $k=8$]

*b) $x^2+y^2-6x-8y+12=0$, $x^2+y^2-4x+6y+k=0$ [Ans : $k=-24$]

*c) $x^2+y^2-16y+k=0$, $x^2+y^2+4x+8=0$ [Ans : $k=-8$]

*5. Find the equation of the circle which passes through the origin and intersects each of the following circles orthogonally.

a) $x^2 + y^2 - 4x + 6y + 10 = 0$, $x^2 + y^2 + 12y + 6 = 0$ **[Ans :** $2(x^2 + y^2) - 7x + 2y = 0$]

b) $x^2 + y^2 - 4x - 6y - 3 = 0$, $x^2 + y^2 - 8y + 12 = 0$ **[Ans ;** $x^2 + y^2 + 6x - 3y = 0$]

*6. Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.

[Ans : $3(x^2 + y^2) - 14x + 23y - 15 = 0$]

*7. Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having the centre at (2, 3). **[Ans ;** $x^2 + y^2 - 4x - 6y + 9 = 0$]

*8. Find the equation of the circle which pass through the points (2, 0) (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$. **[Ans :** $7x^2 + 7y^2 - 8x - 8y - 12 = 0$]

*9. Find the equation of circle which intersect the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ orthogonally and passes through the point (3, 0) and touches Y-axis. **[Ans :** $x^2 + y^2 - 6x - 6y + 9 = 0$]

*10. Find the equation of the circle which cuts the circles $x^2 + y^2 - 4x - 6y + 11 = 0$ and $x^2 + y^2 - 10x - 4y + 21 = 0$ orthogonally and has the diameter along the line $2x + 3y = 7$.

[Ans ; $x^2 + y^2 - 4x - 2y + 3 = 0$]

*11. Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally. **[Ans :** $x^2 + y^2 - 4x - 4y = 0$]

12. Find the equation of the circle orthogonal to each of the circles

*a) $x^2 + y^2 + 4x + 2y + 1 = 0$, $2(x^2 + y^2) + 8x + 6y - 3 = 0$, $x^2 + y^2 + 6x - 2y - 3 = 0$

[Ans : $x^2 + y^2 + 28x - 5y + 50 = 0$]

*b) $x^2 + y^2 + 2x + 4y + 1 = 0$, $2(x^2 + y^2) + 6x + 8y - 3 = 0$, $x^2 + y^2 - 2x + 6y - 3 = 0$

[Ans : $x^2 + y^2 - 5x - 14y - 34 = 0$]

*13. Find the equation of the circle passing through the points of intersections of the circles $x^2 + y^2 - 8x - 6y + 21 = 0$, $x^2 + y^2 - 2x - 15 = 0$ and (1, 2). **[Ans :** $3(x^2 + y^2) - 18x - 12y + 27 = 0$]

*14. a) If the straight line $2x + 3y = 1$ intersects the circle $x^2 + y^2 = 4$ at the points A and B, then find the equation of the circle having AB as diameter. **[Ans :** $13(x^2 + y^2) - 4x - 6y - 50 = 0$]

b) If $x + y = 3$ is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, find the equation of the circle having $\overline{AB}$ as diameter. **(March-2017) **[Ans :** $x^2 + y^2 - 6x + 4 = 0$]

2.3 RADICAL AXIS OF TWO CIRCLES

Definition :

The locus of a point, the powers of which with respect to two given circles are equal, is a straight line, called the radical axis of the two circles.

THEOREM-2.7 **The radical axis of two circles $S = 0$ and $S' = 0$ is the straight line $S - S' = 0$.**

Proof : Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$

Let $P(x_1, y_1)$ be any point on the radical axis. Then the powers of P with respect to the circles are equal $\Rightarrow S_{11} = S'_{11} \Rightarrow S_{11} - S'_{11} = 0$

$\Rightarrow 2(g - g')x_1 + 2(f - f')y_1 + (c - c') = 0$

Locus of $P(x_1, y_1)$ is $2(g - g')x + 2(f - f')y + (c - c') = 0$)

i.e., $S - S' = 0$.

$\therefore$ The equation of the radical axis of the circles $S = 0$ & $S' = 0$ is $S - S' = 0$.

Equation (1) is a first degree equation in x and y, so it represents a straight line

$\therefore$ Radical axis of two circles is a straight line.

Note :

Radical axis of two circles is nearer to the centre of the smaller circle than to the centre of the larger circle.

Note

i) *For concentric circles, the radical axis doesnot exist, since there is no point whose powers with respect to two distinct concentric circles are equal. (However, if their radii are equal, then the locus is the whole plane). So whenever we consider the radical axis of two circles, it means that two circles are non - concentric.*

ii) *While using the formula $S-S' = 0$ to find the equation of the radical axis, the equations of the circles must be in the general form.*

iii) *The lengths of the tangents, drawn to two circles from any point (outside the two circles) on their radical axis, are equal.*

THOEREM-2.8 **The radical axis of any two circles is perpendicular to their line of centres.**

Note :

Radical axis doesn't exists for concentric circles of different radii.

Proof : Let the circles be $S = x^2 + y^2 + 2gx + 2fy + c = 0$,

$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$

Hence $C_1 = (-g, -f)$ & $C_2 = (-g', -f')$

Equation of the radical axis is $S-S' = 0$

$\Rightarrow 2(g - g')x + 2(f - f')y + (c - c') = 0$ --(1)

Slope of (1) = $\dfrac{-(g - g')}{(f - f')}$

Slope of $C_1C_2 = \dfrac{(f - f')}{(g - g')}$

Since the product of the slopes is -1, the radical axis is perpendicular to the line of centres.

THEOREM-2.9 **The radical axis of any two intersecting circles is their common chord.**

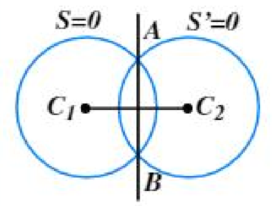

Fig 2.9

Proof : Let the two circles $S = 0$ and $S' = 0$ intersect at $A(x_1, y_1)$ and $B(x_2, y_2)$

Since $A(x_1, y_1)$ lies on both the circles, then $S_{11} = 0$ and $S'_{11} = 0 \Rightarrow S_{11} - S'_{11} = 0$ -- (1)

Similarly $B(x_2, y_2)$ lies on both the circles, then $S_{22} = 0$ and $S'_{22} = 0 \Rightarrow S_{22} - S'_{22} = 0$ -- (2)

Hence $A(x_1, y_1)$ and $B(x_2, y_2)$ satisfy the equation $S - S' = 0$, which is the radical axis of two circles.

$\therefore$ The common chord AB (produced) is the radical axis of the circles.

THEOREM-2.10 **The radical axis of two circles touching each other is the common tangent at their point of contact.**

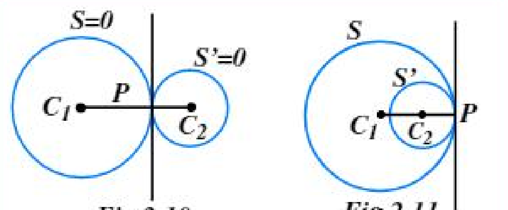

Fig 2.10 *Fig 2.11*

Proof : Let $P(x_1, y_1)$ be the point of contact of the two circles $S = 0$ and $S' = 0$ touching each other.

$\therefore S_{11} = 0$ and $\therefore S'_{11} = 0 \Rightarrow S_{11} - S'_{11} = 0$

Hence the point $P(x_1, y_1)$ lies on the radical axis $S - S' = 0$.

The radical axis passing through the point of contact also being perpendicular to the line of centres will be the common tangent.

THEOREM-2.11 **The radical axes of three circles whose centres are non-collinear, taken in pairs, are concurrent.**

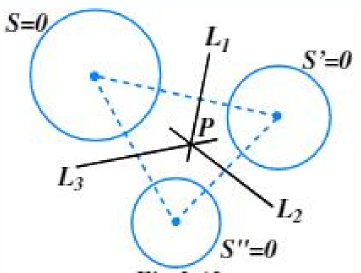

Fig 2.12

Proof : Let the equations of three circles (whose centres are not collinear) be

$$S = x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{-- (1)}$$

$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \text{-- (2)}$$

$$S'' = x^2 + y^2 + 2g''x + 2f''y + c'' = 0 \quad \text{-- (3)}$$

The radical axis L_1(say) of (1) and (2) is

$$L_1 = 2(g - g')(x) + 2(f - f')y + (c - c') = 0 \quad \text{-- (4)}$$

Similarly, the radical axis L_2 (say) of (2)&(3) is

$$L_2 = 2(g' - g'')x + 2(f' - f'')y + (c' - c'') = 0 \quad \text{-- (5)}$$

and the radical axis L_3 (say) of (3) & (1) is

$$L_3 = 2(g'' - g)x + 2(f'' - f)y + (c'' - c) = 0 \quad \text{-- (6)}$$

Now $L_1+L_2+L_3= 0$ shows that L_1, L_2 and L_3 are concurrent.

2.4 RADICAL CENTRE

Definition :

The point of concurrence of the radical axes of three circles whose centres are non-collinear, taken in pairs, is called the radical centre of the three circles.

Note

The powers of the radical centre w.r.t each of the three circles are equal.

THEOREM-2.12 **The radical axis of the two circles bisects the line common tangent of the circles.**

Proof : Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ -- (1)

$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \text{-- (2)}$$

be two circles and T_1, T_2 be the points of contact of common tangent to the circles $S = 0$ & $S' = 0$.

We know that radical axis of two circles is perpendicular to the line joining the centres of circles. The common tangent is not perpendicular to the line joining the centres. Therefore common tangent and radical axis intersect at a point.

Let T_1T_2 (common tangent) intersect the radical axis of (1) and (2) at $P(x_1, y_1)$.

The powers of P with respect to the circles $S = 0$ and $S' = 0$ are equal.

$$\therefore PT_1.PT_1 = PT_2.PT_2 \Rightarrow PT_1^2 = PT_2^2 \Rightarrow \overline{PT_1} = \overline{PT_2}$$

i.e., P is the mid point of T_1 and T_2.

$\therefore$ The radical axis of the two circles bisects each of their common tangents.

THEOREM-2.13 **If a circle cuts two given circles orthogonally then its centre lies on the radical axis of the circles**

Proof : *Let* $x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle which cuts the given circles say $x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0$, $i = 1, 2$ orthogonally

Then $2gg_1 + 2ff_1 = c + c_1$ ---- (1)

and $2gg_2 + 2ff_2 = c + c_2$ ---- (2)

from (1), (2) $2g(g_1 - g_2) + 2f(f_1 - f_2) = c_1 - c_2$

i.e $2(g_1 - g_2)(-g) + 2(f_1 - f_2)(-f) + c_1 - c_2 = 0$ ----(3)

Since the equation of radical axis of the given circles is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$, clearly the point $(-g, -f)$ lies on it from (3)

THEOREM-2.14 **Let $S' = 0$, $S'' = 0$, $S''' = 0$ be three pairwise nonintersecting circles, whose centres are non collinear, then a circle with radical centre of the circles as centre and having radius equal to the length of tangent from radical centre to any of these circles cuts the given circles orthogonally.**

Note : *A circle with radical centre of the circles as centre and having radius equal to the length of tangent from radical centre to any of these circles cuts the given circles orthogonally.*

Proof : Let C be the radical centre of three given circles. As no two of the three circles are intersecting, the point C is exterior to these circles. Choose C as the origin.

Let the equations of the three circles be

$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ -- (1)

$S'' = x^2 + y^2 + 2g''x + 2f''y + c'' = 0$ -- (2)

$S''' = x^2 + y^2 + 2g'''x + 2f'''y + c''' = 0$ -- (3)

The length of the tangent from C to (1) is $\sqrt{c'}$. Since the origin is an external point to these circles, we have c', c'', c''' are positive.

Now, the equation of the circle having the centre as C and radius $\sqrt{c'}$ is

$x^2 + y^2 - c' = 0$ -- (4)

Now, we prove that (4) is orthogonal to (1), (2) and (3). The lengths of the tangents from C to (2) & (3) are $\sqrt{c''}$, $\sqrt{c'''}$ respectively.

The lengths of the tangents from the radical centre to these three circles are equal.

$\therefore \sqrt{c'} = \sqrt{c''} = \sqrt{c'''} \Rightarrow c' = c'' = c'''$

The circles (1) & (4) are orthogonal since the condition of orthogonally is satisfied.

i.e., $2((g')(0) + (f')(0)) = -c' + c' \Rightarrow 0 = 0$

Similarly, the condition of orthogonality is satisfied by (2) and (4) and (3) and (4).

$\therefore$ The circle having radical centre of three circles as the centre and having the length of tangent from the radical centre to any one of these circles as radius cuts the given three circles orthogonally.

Remember : *The centres of the circles which cuts two circles orthogonally lies on their radical axis.*

SOLVED EXAMPLES

> **Remember :**
> *If $S=0, S^{1}=0$ are two circles in standard form, then their radical axis is $S-S^{1}=0$.*
> *If $S=x^2+y^2+2g_1x+2f_1y+c_1=0$, $S^1=x^2+y^2+2g_2x+2f_2y+c_2=0$ then it is $2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0$*

***1. Find the equation of the radical axis of the circles represented by $2x^2+2y^2+3x+6y-5=0$ and $3x^2+3y^2-7x+8y-11=0$**

Sol. The given circles are $S = x^2+y^2+\frac{3}{2}x+3y-\frac{5}{2}=0$ -- (1)

and $S' = x^2+y^2-\frac{7}{3}x+\frac{8}{3}y-\frac{11}{3}=0$ -- (2)

Radical axis of (1) & (2) is $S-S'=0$

$\Rightarrow \left(\frac{3}{2}+\frac{7}{3}\right)x+\left(3-\frac{8}{3}\right)y+\left(\frac{-5}{2}+\frac{11}{3}\right)=0$

$\Rightarrow 23x+2y+7=0$

***2. If the circles $x^2+y^2+2gx+2fy=0$ and $x^2+y^2+2g'x+2f'y=0$ touch each other, then show that $gf'=g'f$.**

Sol. Since the two circles touch each other and both of them pass through origin, origin is the point of contact of the two circles

Equation of the tangent at (0, 0) to $x^2+y^2+2gx+2fy=0$ is

$x(0)+y(0)+g(x+0)+f(y+0)=0 \Rightarrow gx+fy=0$ -- (1)

The equation of the tangent at (0,0) to $x^2+y^2+2g'x+2f'y=0$ is $g'x+f'y=0$ -- (2)

Since (1) & (2) represent the same tangent to the two circles, $\frac{g}{g'}=\frac{f}{f'} \Rightarrow gf'=g'f$

***3. Find the radical centre of the circles $x^2+y^2+4x-7=0$, $2x^2+2y^2+3x+5y-9=0$ and $x^2+y^2+y=0$.**

> **Remember :**
> *If centres of three circles are non-collinear, then the point of concurrence of the three radical axes is called as radical centre.*

Sol. Given circles are $S=x^2+y^2+4x-7=0$ -- (1)

$S' = x^2+y^2+\frac{3}{2}x+\frac{5}{2}y-\frac{9}{2}=0$ -- (2)

$S'' = x^2+y^2+y=0$ -- (3)

Radical axis of (1) & (2) is $S-S'=0$

$\Rightarrow \left(4-\frac{3}{2}\right)x-\frac{5}{2}y+\left(\frac{9}{2}-7\right)=0$

$\Rightarrow \frac{5}{2}x-\frac{5}{2}y-\frac{5}{2}=0 \Rightarrow x-y-1=0$ -- (4)

Radical axis of (1) & (3) is $S-S''=0$

$\Rightarrow 4x-y-7=0$ -- (5)

Solving (4) & (5), we get Radical centre of the circles = (2, 1)

***4. Show that the common chord of the circles $x^2+y^2-6x-4y+9=0$ and $x^2+y^2-8x-6y+23=0$ is the diameter of the second circle and also find its length.**

Sol. Given circles are $S=x^2+y^2-6x-4y+9=0$ -- (1)

$S'=x^2+y^2-8x-6y+23=0$ -- (2)

The equation of the common chord of (1) & (2) is

$S-S'=0 \Rightarrow 2x+2y-14=0 \Rightarrow x+y-7=0$ -- (3)

Centre of (2), $C=(4, 3)$

Clearly C satisifies (3) $\therefore$ (3) is a diameter of (2)

Length of common chord = Diameter of (2) $=2\sqrt{16+9-23}=2\sqrt{2}$

> **Remember :**
> *A circle $S=0$ is said to bisect the circumference of a circle $S'=0$ if the common chord of the two circles passes through the centre of $S'=0$ i.e., if the common chord is a diameter of $S'=0$.*

***5. Prove that the radical axis of the circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g'x+2f'y+c=0$ is the diameter of the later circle (or the former bisects the circumference of the later) if $2g'(g-g')+2f'(f-f')=c-c'$**

Sol. Given circles are

$$S=x^2+y^2+2gx+2fy+c=0 \quad \text{-- (1)}$$

$$S'=x^2+y^2+2g'x+2f'y+c'=0 \quad \text{-- (2)}$$

The radical axis of (1) & (2) is $S-S'=0$

$$\Rightarrow 2(g-g')x+2(f-f')y+(c-c')=0 \quad \text{-- (3)}$$

Centre of (2), $C=(-g',-f')$

Radical axis is the diameter of (2) if C lies on (3)

i.e., if $2(g-g')(-g')+2(f-f')(-f')+c-c'=0$

i.e., if $2g'(g-g')+2f'(f-f')=c-c'$, which is true.

> **Remember :**
> *The circle with radical centre as centre and length of tangent from radical centre to any of the circles as radius cuts the three circles orthogonally.*

***6. Find the equation of the circle which cuts the circles $x^2+y^2+4x+2y+1=0$, $2(x^2+y^2)+8x+6y-3=0$ and $x^2+y^2+6x-2y-3=0$ orthogonally.**

Sol. Given circles are

$$S=x^2+y^2+4x+2y+1=0 \quad \text{-- (1)}$$

$$S'=x^2+y^2+4x+3y-\frac{3}{2}=0 \quad \text{-- (2)}$$

$$S''=x^2+y^2+6x-2y-3=0 \quad \text{-- (3)}$$

Equation of the radical axis of (1) & (2) is

$$S'-S=0 \Rightarrow y-\frac{5}{2}=0 \Rightarrow y=\frac{5}{2} \quad \text{-- (4)}$$

$\therefore$ Equation of the radical axis of (1) & (3) is

$$S''-S=0 \Rightarrow 2x-4y-4=0$$

$$\Rightarrow x-2y-2=0 \quad \text{-- (5)}$$

Solving (4) & (5), we get $x=7$

$\therefore$ Radical centre of (1), (2) & (3) is $C\left(7,\frac{5}{2}\right)$

Length of the tangent from C to (1) $=\sqrt{S_{11}}$

$$=\sqrt{49+\frac{25}{4}+28+5+1}=\sqrt{\frac{357}{4}}=\frac{\sqrt{357}}{2}$$

$\therefore$ Equation of the circle which is orthogonal to (1), (2) & (3) is

$$(x-7)^2+\left(y-\frac{5}{2}\right)^2=\frac{357}{4}$$

$$\Rightarrow x^2+y^2-14x-5y-34=0$$

2.5 COAXAL SYSTEM OF CIRCLES

Definition :

A system of circles in which every pair of circles has the same radial axis is called a system of coaxal circles or a coaxal system.

Since the radical axis is perpendicular to the line of centres, it follows that the centres of the circles forming a coaxal system are collinear.

Note-1 :

If $S = 0$, $S' = 0$ are two circles, then $\lambda_1 S + \lambda_2 S' = 0$ (λ_1, λ_2 are parameters not both zero and $\lambda_1 + \lambda_2 \neq 0$) represents a coaxal system of circles of which $S = 0$, $S' = 0$ are its members.

Note-2 :

If $S = 0$ is a circle and $L = 0$ is a line, then $S + \lambda L = 0$ (λ is a parameter) represents a coaxal system of circles of which the circle $S = 0$ is a member and $L = 0$ is the radical axis of the system.

EXERCISE - 2.2

I. 1. Find the equation of the radical axis of the following circles

i) $x^2+y^2-3x-4y+5=0,\ 3(x^2+y^2)-7x+8y-11=0$

ii) $x^2+y^2+2x+4y+1=0,\ x^2+y^2+4x+y=0$

iii) $x^2+y^2+4x+6y-7=0, 4(x^2+y^2)+8x+12y-9=0$

iv) $x^2+y^2-2x-4y-1=0, x^2+y^2-4x-6y+5=0$

[**Ans :** (i) $x+10y-13=0$; (ii) $2x-3y-1=0$; (iii) $8x+12y-19=0$; (iv) $x+y-3=0$]

2. Find the equation of common chord of the following pair of circles

i) $x^2+y^2-4x-4y+3=0,\ x^2+y^2-5x-6y+4=0$

ii) $x^2+y^2+2x+3y+1=0,\ x^2+y^2+4x+3y+2=0$

iii) $(x-a)^2+(y-b)^2=c^2,\ \ (x-b)^2+(y-a)^2=c^2 (a \neq b)$

[**Ans :** (i) $x+2y-1=0$; (ii) $2x+1=0$; (iii) $x-y=0$]

3. Find the equation of the common tangent of the following circles at their point of contact

i) $x^2+y^2+10x-2y+22=0,\ x^2+y^2+2x-8y+8=0$

ii) $x^2+y^2-8y-4=0,\ x^2+y^2-2x-4y=0$ [**Ans :** (i) $4x+3y+7=0$; (ii) $x-2y-2=0$]

4. Show that the circles $x^2+y^2-8x-2y+8=0$ and $x^2+y^2-2x+6y+6=0$ touch each other and find the point of contact. [**Ans :** $\left(\frac{11}{5}, -\frac{7}{5}\right)$]

5. If the two circles $x^2+y^2+2gx+2fy=0$ and $x^2+y^2+2g'x+2f'y=0$ touch each other then show that $f'g = fg'$

6. Find the radical centre of the following circles.

*i) $x^2+y^2-4x-6y+5=0, x^2+y^2-2x-4y-1=0, x^2+y^2-6x-2y=0$ (**March-18**)

ii) $x^2+y^2+4x-7=0, 2x^2+2y^2+3x+5y-9=0, x^2+y^2+y=0$

[**Ans :** (i) $\left(\frac{7}{6}, \frac{11}{6}\right)$; (ii) $(2, 1)$]

7. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ is the diameter of the second circle and also find its length. [Ans : $2\sqrt{2}$]

8. Find the equation and length of the common chord of the following circles.

*i) $x^2 + y^2 + 2x + 2y + 1 = 0,\ x^2 + y^2 + 4x + 3y + 2 = 0$ **(March-17)** [Ans : $2x + y + 1 = 0,\ \dfrac{2}{\sqrt{5}}$]

ii) $x^2 + y^2 - 5x - 6y + 4 = 0,\ x^2 + y^2 - 2x - 2 = 0$ [Ans : $x + 2y - 2 = 0,\ 2\sqrt{\dfrac{14}{5}}$]

9. Prove that the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is the diameter of the latter circle (or the former bisects the circumference of the latter) if $2g'(g - g') + 2f'(f - f') = c - c'$

10. Show that thecircles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}$

11. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch each other. Find the coordinates of the point of contact. Is the point of contact external or internal ?

[Ans : $\left(\dfrac{1}{5}, \dfrac{3}{5}\right)$, touches externally]

12. Find the equation of the circle which cuts the following circles orthogonally

i) $x^2 + y^2 + 4x - 7 = 0,\ 2x^2 + 2y^2 + 3x + 5y - 9 = 0,\ x^2 + y^2 + y = 0$

ii) $x^2 + y^2 + 2x + 4y + 1 = 0,\ 2x^2 + 2y^2 + 6x + 8y - 3 = 0,\ x^2 + y^2 - 2x + 6y - 3 = 0$

iii) $x^2 + y^2 + 2x + 17y + 4 = 0,\ x^2 + y^2 + 7x + 6y + 11 = 0,\ x^2 + y^2 - x + 22y + 3 = 0$

iv) $x^2 + y^2 + 4x + 2y + 1 = 0,\ 2(x^2 + y^2) + 8x + 6y - 3 = 0,\ x^2 + y^2 + 6x - 2y - 3 = 0$

[Ans : (i) $x^2 + y^2 - 4x - 2y - 1 = 0$; (ii) $x^2 + y^2 - 5x - 14y - 34 = 0$
(iii) $x^2 + y^2 - 6x - 4y - 44 = 0$; (iv) $x^2 + y^2 - 14x - 5y - 34 = 0$]