Assignment 4 Questions

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Vector Calculus

0 points possible (ungraded)

[SADT8] If $\mathbf{A}$ and $\mathbf{B}$ are vector fields, prove the following:

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\,\mathbf{A} + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})\,.$$

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Vector Calculus

0 points possible (ungraded)

[SADT4] If $\mathbf{V}$ is a vector field, prove that:

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}.$$

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Vector Calculus

0 points possible (ungraded)

[SADT3] For scalar functions u and v, show that

$$\mathbf{B} = (\nabla u)\times(\nabla v)$$

is solenoidal and that

$$\mathbf{A} = \frac{1}{2}(u\nabla v - v\nabla u)$$

is a vector potential for $\mathbf{B}$, i.e. $\mathbf{B} = \nabla\times\mathbf{A}$

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Vector Calculus

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[SADT5] Let $\mathbf{E} := E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$ and $\mathbf{H} := H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$ be two vectors assumed to have continuous partial derivatives (of second order at least) with

respect to position and time. Suppose further that $\mathbf{E}$ and $\mathbf{H}$ satisfy the equations:

$$\nabla\cdot\mathbf{E} = 0, \nabla\cdot\mathbf{H} = 0, \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

prove that $\mathbf{E}$ and $\mathbf{H}$ satisfy the equation

$$\nabla^2 E_i = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \text{ and } \nabla^2 H_i = \frac{1}{c^2}\frac{\partial^2 H_i}{\partial t^2}$$

Here, $i = x, y$ or z.

Hint: Use the fact that

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}.$$

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[5ADTB] Solution:

Considering, $\Delta(\vec{A}\cdot\vec{B}) = \sum \hat{i}\frac{d}{du}(\vec{A}\cdot\vec{B})$

$$= \sum \hat{i}\left(\frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}\right)$$

$$= \sum \hat{i}\left(\frac{d\vec{A}}{du}\cdot\vec{B}\right) + \sum \hat{i}\left(\vec{A}\cdot\frac{d\vec{B}}{d\vec{u}}\right) \quad \text{--- (1)}$$

Again, $\left(\vec{B}\cdot\frac{d\vec{A}}{du}\right)\hat{i} = \left(\vec{B}\cdot\frac{d\vec{A}}{du}\right)\hat{i} - (B\cdot\hat{i})\frac{d\vec{A}}{du}$

$$\Rightarrow \left(\vec{B}\cdot\frac{d\vec{A}}{du}\right)\hat{i} = \vec{B}\left(\hat{i}\cdot\frac{d\vec{A}}{du}\right) + (B\cdot\hat{i})\frac{d\vec{A}}{du}$$

$$\Rightarrow \sum\left(\frac{d\vec{A}}{du}\cdot\vec{B}\right)\hat{i} = \vec{B}\cdot\left(\hat{i}\cdot\frac{d\vec{A}}{du}\right) + (\vec{B}\cdot\hat{i})\frac{d\vec{A}}{du}$$

$$\Rightarrow \sum\hat{i}\left(\frac{d\vec{A}}{du}\cdot\vec{B}\right) = \vec{B}\cdot\sum\left(\hat{i}\cdot\frac{d\vec{A}}{du}\right) + \left(\vec{B}\cdot\sum\hat{i}\frac{d}{du}\right)\vec{A}$$

$$\Rightarrow \sum\hat{i}\left(\frac{d\vec{A}}{du}\cdot\vec{B}\right) = \vec{B}\cdot(\Delta\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

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$$\Rightarrow \sum\hat{i}\left(\frac{d\vec{A}}{du}\cdot\vec{B}\right) = \vec{B}\cdot(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} \quad \text{--- (2)}$$

Interchanging the role of $\vec{A}$ & $\vec{B}$

similarly can prove,

$$\sum \hat{i}\left(\vec{A}\cdot\frac{d\vec{B}}{du}\right) = \vec{A}\cdot(\nabla\cdot\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} \quad \text{— (3)}$$

~~(1) − (2) − (3)~~

Substituting equation 2, 3 & 1,

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\cdot(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla\cdot\vec{A})$$

[Proved]

[SADT 4] Solution:

$$L.H.S = \nabla \times (\nabla \times v)$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times$$

$$\left[\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right)\hat{i} - \left(\frac{\delta v_3}{\delta x} - \frac{\delta v_1}{\delta z}\right)\hat{j} + \left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)\hat{k}\right]$$

$$= \left(\frac{\delta^2 v_2}{\delta x \delta y} - \frac{\delta^2 v_1}{\delta y^2} - \frac{\delta^2 v_1}{\delta z^2} + \frac{\delta^2 v_3}{\delta z \delta x}\right)\hat{i}$$

$$- \left(\frac{\delta^2 v_2}{\delta x} - \frac{\delta^2 v_1}{\delta y \delta x} + \frac{\delta^2 v_2}{\delta z^2} - \frac{\delta^2 A_3}{\delta y \delta z}\right)\hat{j}$$

$$+ \left(\frac{\delta^2 v_2}{\delta y^2} - \frac{\delta^2 v_3}{\delta x^2} + \frac{\delta^2 v_1}{\delta z \delta x} - \frac{\delta^2 v_3}{\delta y^2}\right)\hat{k}$$

$$R.H.S = \nabla(\nabla \cdot V) - \nabla^2 \cdot V$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \left(\frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z^2} - \frac{\partial^2 V_1}{\partial y^2}\right)\hat{i}$$

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$$+ \left(\frac{\partial^2 V_1}{\partial x \partial y} + \frac{\partial^2 V_3}{\partial z \partial y} - \frac{\partial^2 V_2}{\partial z^2} - \frac{\partial^2 V_2}{\partial x^2}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 V_1}{\partial x \partial z} + \frac{\partial^2 V_2}{\partial x \partial z} - \frac{\partial^2 V_3}{\partial y^2} - \frac{\partial^2 V_3}{\partial x^2}\right)\hat{k}$$

$$\therefore L.H.S = R.H.S$$

[Proved]

[SADT 3] Solution:

~~Given~~ we know, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Given, $B = (\nabla u) \times (\nabla v)$

$$= \left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) \times \left(\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right)$$

$$\nabla . B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}\right) - \hat{j}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x}\right) + \hat{k}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y}\right)$$

$$= 0$$

Thus, for solenoidal vector field, $\nabla . B = 0$

Now given,

$$A = \tfrac{1}{2}(u\nabla v - v\nabla u)$$

$$= \tfrac{1}{2}\left(u\frac{\delta v}{\delta x}\hat{i} + u\frac{\delta v}{\delta y}\hat{j} + u\frac{\delta v}{\delta z}\hat{k} - v\frac{\delta u}{\delta x}\hat{i} - v\frac{\delta u}{\delta y}\hat{j} - v\frac{\delta u}{\delta z}\hat{k}\right)$$

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \left(u\frac{\delta v}{\delta x} - v\frac{\delta u}{\delta x}\right) & \left(u\frac{\delta v}{\delta y} - v\frac{\delta u}{\delta y}\right) & \left(u\frac{\delta v}{\delta y} - v\frac{\delta u}{\delta z}\right) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\delta}{\delta y}\left(u\frac{\delta v}{\delta z} - v\frac{\delta u}{\delta z}\right) - \frac{\delta}{\delta z}\left(u\frac{\delta v}{\delta y} - v\frac{\delta u}{\delta y}\right)\right\}$$

$$- \hat{j}\left\{\frac{\delta}{\delta x}\left(u\frac{\delta v}{\delta z} - v\frac{\delta u}{\delta z}\right) - \frac{\delta}{\delta z}\left(u\frac{\delta v}{\delta x} - v\frac{\delta u}{\delta x}\right)\right\}$$

$$+ \hat{k}\left\{\frac{\delta}{\delta x}\left(u\frac{\delta v}{\delta y} - v\frac{\delta u}{\delta y}\right) - \frac{\delta}{\delta y}\left(u\frac{\delta v}{\delta x} - v\frac{\delta u}{\delta x}\right)\right\}$$

$$= 0$$

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$\therefore B = \nabla \times A$ [Showed]

[SADT5] Solution:

Given, $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$

& $\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{H} = 0$, $\nabla \times \vec{E} = -1/c \frac{\delta \vec{H}}{\delta t}$

& $\nabla \times H = 1/c \frac{\delta \vec{E}}{\delta t}$.

Now, $\nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\frac{1}{c} \cdot \frac{\delta E}{\delta t}\right)$

$\Rightarrow \nabla \cdot (\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \frac{1}{c} \frac{\delta}{\delta t} (\nabla \cdot \vec{E})$

$\Rightarrow \nabla (0) - \nabla^2 \vec{H} = 1/c \frac{\delta}{\delta t} \left(-1/c \cdot \frac{\delta \vec{H}}{\delta t}\right)$

$\Rightarrow -\nabla^2 H = -1/c^2 \cdot \frac{\delta^2 \vec{H}}{\delta t^2}$

$\Rightarrow \nabla^2 \vec{H} = \frac{1}{c^2} \cdot \frac{\delta^2 \vec{H}}{\delta t^2}$

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$\Rightarrow \nabla^2 = \frac{1}{c^2} \cdot \frac{\delta^2}{\delta t^2}$ $[\vec{H} = 0]$

So, we can write,

$$\nabla^2 = \frac{1}{c^2} \cdot \frac{\delta^2}{\delta t^2} = 0 \quad \text{——— (i)}$$

By equating with components of zero vector and generating components in equation (i),

$$\nabla^2 H_i - \frac{1}{c^2} \cdot \frac{\delta^2 H_i}{\delta t^2} = 0$$

$$\Rightarrow \nabla^2 H_i = 1/c^2 \cdot \frac{\delta^2 H_i}{\delta t^2}$$

Also,

$$\nabla^2 = - \frac{1}{c^2} \frac{\delta^2}{\delta t^2} = 0$$

$$\Rightarrow \nabla^2 E_i = \frac{1}{c^2} \cdot \frac{\delta^2 E_i}{\delta t^2}$$

Thus, E & H satisfy the given equations.

[Proved]