Logarithmic Differentiation

0 points possible (ungraded)

[SADT]Differentiate

$$f(x) = \sqrt{x}^{\sqrt{x}} e^{x^2}$$

with respect to x.

SOLUTION : $\ln f(x) = \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right)$

$= \ln(\sqrt{x})^{\sqrt{x}} + \ln e^{x^2}$

$= \sqrt{x} \ln \sqrt{x} + x^2 \ln e$

$= \frac{1}{2}\sqrt{x} \ln x + x^2$

$\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}\left[\frac{1}{2}\sqrt{x}\ln x + x^2\right]$

$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{2}\sqrt{x}.\frac{1}{x} + \frac{1}{2}\ln x . \frac{1}{2}x^{-1/2} + 2x$

$= \frac{1}{2\sqrt{x}}\left(1 + \frac{1}{2}\ln x\right) + 2x$

$\Rightarrow \boxed{f'(x) = f(x)\left[\frac{1}{2\sqrt{x}}\left(1 + \frac{1}{2}\ln x\right) + 2x\right].}$

Parametric Differentiation

0 points possible (ungraded)

[SADT] The curve

$$\frac{x^2}{4} + \frac{y^2}{49} = 1$$

can be parametrized by $x = a\cos t, y = b\sin t, 0 \leq t < 2\pi$. Find a and b. Then using parametric differentiation, calculate $\frac{dy}{dx}$ in terms of t. Using your answer, show that the curve intersects the coordinate axes at right angles.

SOLUTION : $x = a\cos t$, $y = b\sin t$

$\Rightarrow \frac{a^2\cos^2 t}{4} + \frac{b^2 \sin^2 t}{49} = 1$

$\left.\begin{array}{l} a \text{ can be } \pm 2 \\ b \text{ can be } \pm 7 \end{array}\right\}$ using $\cos^2 t + \sin^2 t = 1$

Take $a=2$ and $b=7$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} = -\frac{7}{2}\cot t.$$

When $x=0$, $a\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$.

At these points, $\frac{dy}{dx} = 0$.

The tangent is horizontal and it intersects the y axis at right angles.

When $y=0$, $b\sin t = 0 \Rightarrow t = 0, \pi$.

$\frac{dy}{dx} \to \pm\infty$ and the tangent is vertical, i.e. perpendicular to the x-axis.

$\therefore$ the curve intersects the coordinate axes at right angles. (shown)

Tangent

0 points possible (ungraded)

[SADT] Let $f(x) = (x+2)^2$. Find the equation of *every* line that is tangent to $y = f(x)$ that passes through the point $(1,1)$. (N.B.: The curve $y = f(x)$ does *NOT* pass through $(1,1)$).

SOLUTION: Let $y = mx + c$ pass through $(1,1)$

$\Rightarrow 1 = m + c$ — ①

$f(x) = (x+2)^2 \Rightarrow f'(x) = 2(x+2)$.

The tangent has gradient m always: $m = 2(x+2) \Rightarrow x = \frac{m}{2} - 2$ at the pt. of intersection

and $y = \frac{m^2}{4}$.

The line passes through

$$\left(\frac{m}{2} - 2, \frac{m^2}{4}\right)$$

$y = mx + c = mx + 1 - m.$ — (2)

$\frac{m^2}{4} = m\left(\frac{m}{2} - 2\right) + 1 - m = \frac{m^2}{2} - 2m + 1 - m$

$\Rightarrow \frac{m^2}{4} - 3m + 1 = 0 \Rightarrow m^2 - 12m + 4 = 0$

$\Rightarrow m = \frac{12 \pm \sqrt{144 - 16}}{2}$

$= 6 \pm 4\sqrt{2}.$

$c = 1 - (6 \pm 4\sqrt{2}) = -5 \mp 4\sqrt{2}.$

The tangents are $\boxed{y = (6 + 4\sqrt{2})x - 5 - 4\sqrt{2}}$

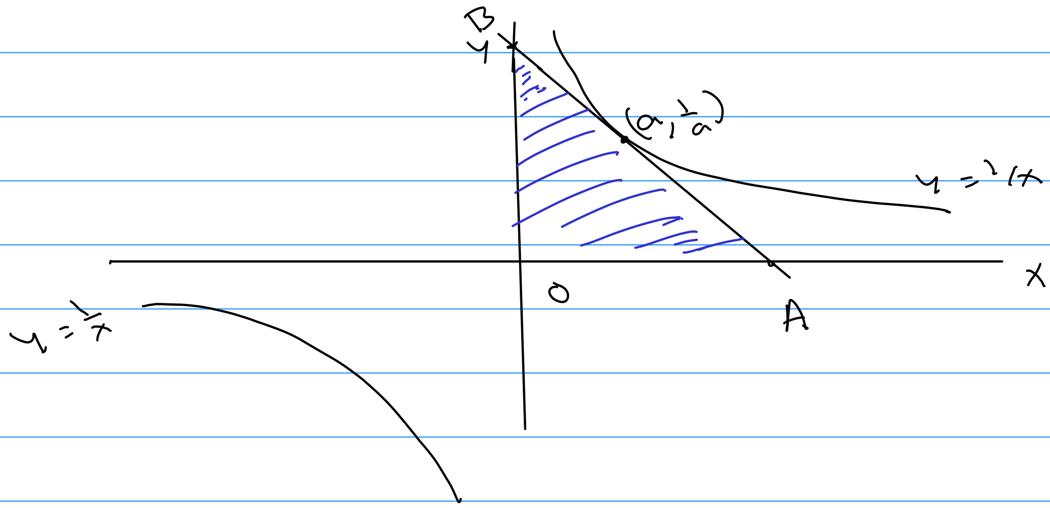
and $\boxed{y = (6 - 4\sqrt{2})x - 5 + 4\sqrt{2}}$

Tangent

0 points possible (ungraded)

[SADT] Find the area of the right angled triangle formed by the tangent to the graph $y = \frac{1}{x}$ at any point on the curve in the first quadrant and the coordinate axes. (The answer is a constant number.)

SOLUTION



We wish to find the area of the blue triangle.

at $x = a$, the equation of the tangent is:

$y - \frac{1}{a} = \frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=a} (x - a)$

$$\Rightarrow y - \frac{1}{a} = -\frac{1}{a^2}(x-a)$$

$y = 0 \Rightarrow x - a = a \Rightarrow x = 2a$

point $A(2a, 0)$.

$x = 0, \Rightarrow y = \frac{2}{a}$ point $B(0, \frac{2}{a})$.

Blue area $= \frac{1}{2}(OA)(OB) = \frac{1}{\cancel{2}} \cdot \cancel{2}\cancel{a} \cdot \frac{2}{\cancel{a}}$

$= \underline{\underline{2}}$.

$$\boxed{\text{area} = 2.}$$

Logarithmic Differentiation

0 points possible (ungraded)

[SADT] Let $h(t) = \dfrac{\sqrt{5t+8}\ \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}$. Find $h'(t)$ using logarithmic differentiation.

SOLUTION: $h(t) = \dfrac{(5t+8)^{1/2}(1-9\cos 4t)^{1/3}}{(t^2+10t)^{1/4}}$

Take ln on both sides:

$\ln h(t) = \ln(5t+8)^{1/2} + \ln(1-9\cos 4t)^{1/3} - \ln(t^2+10t)^{1/4}$

$= \frac{1}{2}\ln(5t+8) + \frac{1}{3}\ln(1-9\cos 4t) - \frac{1}{4}(\ln t + \ln(t+10))$

Take the derivative on both sides with respect to t:

$\dfrac{h'(t)}{h(t)} = \dfrac{1}{2}\dfrac{5}{(5t+8)} + \dfrac{36\sin 4t}{3(1-9\cos 4t)} - \dfrac{1}{4}\left(\dfrac{1}{t} + \dfrac{1}{t+10}\right)$

Multiply both sides by h(t) and you get:

$$\boxed{h'(t) = h(t)\left[\frac{5}{2(5t+8)} + \frac{12\sin 4t}{1-9\cos 4t} - \frac{1}{4}\left(\frac{1}{t} + \frac{1}{t+10}\right)\right]}$$

0/50 if the student does not take logs.

Logarithmic Differentiation

0 points possible (ungraded)

[SADT] Using logarithmic differentiation, Show that $y(x) = xa^{2x}e^{x^2}$ has no stationary points other than $x = 0$, if $e^{-\sqrt{2}} < a < e^{\sqrt{2}}$.

SOLUTION: $y = xa^{2x}e^{x^2} \Rightarrow \ln y = \ln x + \ln a^{2x} + \ln e^{x^2}$

$= \ln x + 2x\ln a + x^2 \ln e$

$= \ln x + 2x\ln a + x^2.$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2\ln a + 2x$$

$$\Rightarrow \frac{dy}{dx} = x a^{2x} e^{x^2}\left(\frac{1}{x} + 2\ln a + 2x\right)$$

$$= a^{2x} e^{x^2}\left(1 + 2x\ln a + 2x^2\right)$$

25/50 up to this point.

There are no stationary points for $x \neq 0$ if:

$2x^2 + 2x\ln a + 1 = 0$ has no solutions,

i.e. $(2\ln a)^2 - 4.2.1 < 0$ (DISCRIMINANT)

$$\Rightarrow (\ln a)^2 < 2$$

$$\Rightarrow -\sqrt{2} < \ln a < \sqrt{2}$$

$$\Rightarrow e^{-\sqrt{2}} < a < e^{\sqrt{2}} \quad \text{(shown).}$$

Parametric Differentiation

0 points possible (ungraded)

[SADT] The parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the forms $x = a(\theta - \sin\theta)$,$y = a(1 - \cos\theta)$. Show that the tangent to the curve has slope $\cot\left(\frac{\theta}{2}\right)$. Use this result at a few calculated values of x and y to sketch the form of the particle's trajectory.

SOLUTION:

$$\frac{dx}{d\theta} = a(1 - \cos\theta) = a\left(1 - 1 + 2\sin^2\frac{\theta}{2}\right)$$

$$= 2a\sin^2\left(\frac{\theta}{2}\right)$$

$$\frac{dy}{d\theta} = a\sin\theta = 2a\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\sin\frac{\theta}{2}} = \cot\left(\frac{\theta}{2}\right)$$

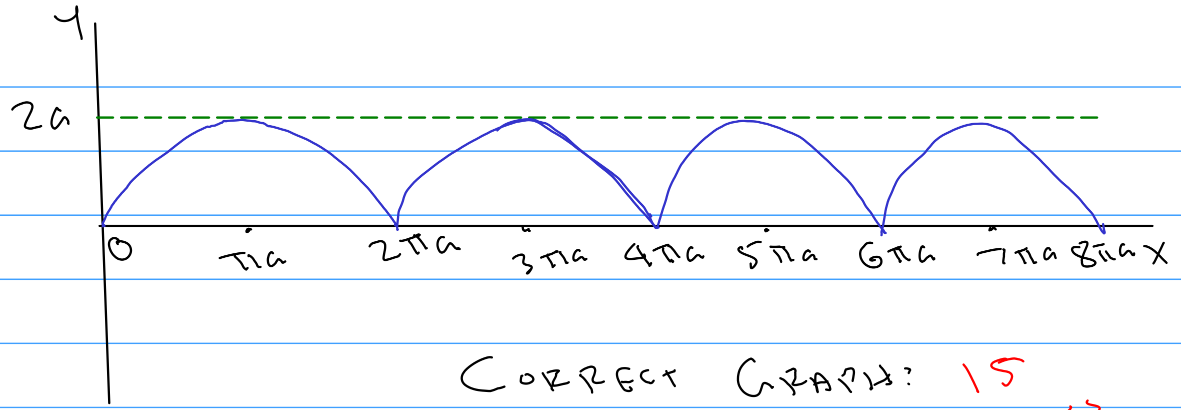
= gradient of tangent

(25 marks)

Set a value for a and plug in as

many values of θ as necessary.

(10 marks)



CORRECT GRAPH: 15 MARKS.

Parametric Differentiation

0 points possible (ungraded)

[SADT] Show that the curve $4y^3 = a^2(x+3y)$ can be parameterised as $x = a\cos 3\theta, y = a\cos\theta$. Find $\frac{dy}{dx}$ first by implicit differentiation and then by parametric differentiation. Show that they are equivalent.

SOLUTION: LHS $= 4y^3 = 4a^3\cos^3\theta$

$$\text{RHS} = a^2(x+3y) = a^2(a\cos 3\theta + 3a\cos\theta)$$
$$= a^3(\cos 3\theta + 3\cos\theta)$$
$$= a^3[\cos(2\theta+\theta) + 3\cos\theta]$$
$$= a^3(\cos 2\theta\cos\theta - \sin 2\theta\sin\theta + 3\cos\theta)$$
$$= a^3[(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta + 3\cos\theta]$$
$$= a^3\cos\theta[2\cos^2\theta - 1 - 2\sin^2\theta + 3]$$
$$= a^3\cos\theta[2\cos^2\theta - 1 - 2(1-\cos^2\theta) + 3]$$
$$= a^3\cos\theta \cdot 4\cos^2\theta = 4a^3\cos^3\theta$$

Hence the parametrization is correct.

10 points for showing this.

Using implicit differentiation:

$$\frac{d}{dx}(4y^3) = \frac{d}{dx}[a^2(x+3y)]$$

$$\Rightarrow 12y^2\frac{dy}{dx} = a^2\left(1 + 3\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}$$ (15 points)

$$= \frac{a^2}{12a^2\cos^2\theta - 3a^2}$$

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} \quad \text{—} ①$$

Using parametric differentiation, we have:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{-3a\sin 3\theta} = \frac{\sin\theta}{3\sin 3\theta}$$

(15 marks)

$$\frac{dy}{dx} = \frac{\sin\theta}{3\sin(2\theta+\theta)} = \frac{\sin\theta}{3(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)}$$

$$= \frac{\sin\theta}{3[2\sin\theta\cos\theta\cos\theta + (2\cos^2\theta - 1)\sin\theta]}$$

$$= \frac{1}{3(2\cos^2\theta + 2\cos^2\theta - 1)}$$

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3}$$, which agrees with equation ①.

10 marks for this

Mean Value Theorem

0 points possible (ungraded)

[SADT] Use the Mean Value Theorem to establish the fact that, for $x > 0, e^x > 1 + x$. (DO NOT use any other method).

[Hint: Take a such that $0 < a < x$ and apply the mean value theorem.]

SOLUTION: By the MVT, there exists $a \in (0, x)$ such that

$$f'(a) = \frac{f(x) - f(0)}{x - 0}, \text{ where } f(x) = e^x$$

20 marks for logic

$$\Rightarrow e^a = \frac{e^x - e^0}{x}$$

10 marks for this statement

$$\Rightarrow e^x = 1 + x e^a \qquad a > 0 \Rightarrow e^a > e^0 = 1$$

$$\therefore e^x > 1 + x \text{ as required.}$$

20 marks.

Mean Value Theorem

0 points possible (ungraded)

[SADT] Use the Mean Value Theorem to establish the fact that, for $x > 1$, $\frac{x-1}{x} < \ln x < x - 1$. (DO NOT use any other method).

[Hint: Take a such that $1 < a < x$ and apply the mean value theorem.]

SOLUTION: Let $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

There exists $a \in (1, x)$ such that

$$f'(a) = \frac{f(x) - f(1)}{x - 1} \Rightarrow \frac{1}{a} = \frac{\ln x}{x - 1}$$

$$\Rightarrow \ln x = \frac{x-1}{a}$$

20 marks up to this point.

$$1 < a < x \Rightarrow \frac{1}{x} < \frac{1}{a} < 1$$

$$(x - 1 > 0) \Rightarrow \frac{x-1}{x} < \frac{x-1}{a} < x - 1$$

20 marks

$$\Rightarrow \frac{x-1}{x} < \ln x < x - 1 \quad \text{(proved)}$$

10 marks

Mean Value Theorem

0 points possible (ungraded)

[SADT] Suppose two cars start a race at the same time and end it at the same time. Prove that at some point in time, their speeds were the same.

SOLUTIONS: Let $s_1(t)$ and $s_2(t)$ be the distances of cars 1 and 2 from the starting point. Consider $d(t) := s_1(t) - s_2(t)$, the distance between the two cars. Let the race start at $t=a$ and end at $t=b$. We have the following:

20 marks

① $d(a) = d(b) = 0$

② $d(t)$ is continuous on $[a,b]$.

③ $d(t)$ is differentiable on (a,b).

10 marks

Thus, $d(t)$ satisfies the conditions of the mean value theorem and Rolle's theorem. Therefore, at some c, $a<c<b$, $d'(c)=0$. But $d'(c) = s_1'(c) - s_2'(c)$

$= v_1(c) - v_2(c) = 0$

where v_1 and v_2 are the speeds of the two cars. Thus, $v_1(c) = v_2(c) = 0$ and their speeds are the same at $t=c$.

(PROVED)

20 MARKS

Mean Value Theorem

0 points possible (ungraded)

[SADT] Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

SOLUTION: By the mean value theorem, there exists $2 < c < 8$ such that

25 MARKS

$$f'(c) = \frac{f(8) - f(2)}{8-2}$$

$$\Rightarrow \quad f(8) - f(2) = 6f'(c).$$

25 MARKS.

But $3 \le f'(c) \le 5$

$$\Rightarrow 18 \le 6f'(c) \le 30$$

$$\Rightarrow 18 \le f(8) - f(2) \le 30,$$

as required.