# MAT- 110 - Assignment 1

[AFE] Given, $y=\left(\frac{\sin x}{1+\cos x}\right)^2$. Find $\frac{dy}{dx}$ by using chain rule.

Soln $\quad y=\left(\frac{\sin x}{1+\cos x}\right)^2$

$$\frac{dy}{dx}=2.\frac{\sin x}{1+\cos x}\cdot\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$$

$$=\frac{2\sin x}{1+\cos x}\cdot\frac{(1+\cos x)\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

20 Marks

$$=\frac{2\sin x}{1+\cos x}\cdot\frac{(1+\cos x)(\cos x)-\sin x(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{1+\cos x}\cdot\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{1+\cos x}\cdot\frac{1+\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}\quad\text{(Ans).}$$

30 Marks

**If the whole method is correct but the final answer is incorrect give total 40 marks**

[AFE] Given, $y = \left(\frac{1+\cos x}{\sin x}\right)^{-1}$. Find $\frac{dy}{dx}$ by using chain rule.

Solⁿ

$$y = \left(\frac{1+\cos x}{\sin x}\right)^{-1}$$

$$\frac{dy}{dx} = -\left(\frac{1+\cos x}{\sin x}\right)^{-2} \cdot \frac{d}{dx}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= -\frac{\sin^2 x}{(1+\cos x)^2} \cdot \frac{\sin x \frac{d}{dx}(1+\cos x) - (1+\cos x)\frac{d}{dx}\sin x}{\sin^2 x}$$

.................... **20 Marks**

$$= -\frac{\sin x(-\sin x) - (1+\cos x)\cos x}{(1+\cos x)^2}$$

$$= -\frac{-\sin^2 x - \cos x - \cos^2 x}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x} \quad \text{(Ans)}$$

.................... **30 Marks**

**If the whole method is correct but the final answer is incorrect give total 40 marks**

[AFE] Given, $y = \sin\left(\frac{x}{\sqrt{x+1}}\right)$. Find $\frac{dy}{dx}$ by using chain rule.

<u>Sol$^n$</u> $y = \sin\left(\frac{x}{\sqrt{x+1}}\right)$

$$\frac{dy}{dx} = \cos\left(\frac{x}{\sqrt{x+1}}\right) \cdot \frac{d}{dx}\left(\frac{x}{\sqrt{x+1}}\right)$$

$$= \cos\left(\frac{x}{\sqrt{x+1}}\right) \cdot \frac{\sqrt{x+1} \cdot \frac{d}{dx}(x) - x \frac{d}{dx}\left(\sqrt{x+1}\right)}{\left(\sqrt{x+1}\right)^2}$$

**20 Marks**

$$= \cos\left(\frac{x}{\sqrt{x+1}}\right) \cdot \frac{\sqrt{x+1} - x \cdot \frac{1}{2\sqrt{x+1}}}{x+1}$$

$$= \cos\left(\frac{x}{\sqrt{x+1}}\right) \cdot \frac{2(x+1) - x}{2(x+1)^{3/2}}$$

$$= \cos\left(\frac{x}{\sqrt{x+1}}\right) \cdot \frac{x+2}{2(x+1)^{3/2}} \quad \text{(Ans)}$$

**30 Marks**

**If the whole method is correct but the final answer is incorrect give total 40 marks**

[AFE] Given, $y = \sqrt{1 + \cos(x^2)}$. Find $\frac{dy}{dx}$ by using chain rule.

Soln

$$y = \sqrt{1 + \cos(x^2)}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+\cos(x^2)}} \cdot \frac{d}{dx}\left(1+\cos(x^2)\right)$$

**10 Marks**

$$= \frac{1}{2\sqrt{1+\cos(x^2)}} \cdot \left(-\sin(x^2)\right) \frac{d}{dx}(x^2)$$

**10 Marks**

$$= \frac{-\sin x^2 \cdot 2x}{2\sqrt{1+\cos(x^2)}} = \frac{-x \sin(x^2)}{\sqrt{1+\cos(x^2)}}$$

**30 Marks**

(Ans)

**If the whole method is correct but the final answer is incorrect give total 40 marks**

$$\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\frac{\pi}{2} - x} \longrightarrow \boxed{\infty^0 \text{ form}}$$

let, $y = (\tan x)^{\frac{\pi}{2} - x}$

Taking the natural logarithm of both sides,

$$\ln y = \ln (\tan x)^{\frac{\pi}{2} - x}$$

$$= \left(\frac{\pi}{2} - x\right) \ln (\tan x)$$

$$\Rightarrow \ln y = \frac{\ln (\tan x)}{\frac{1}{\frac{\pi}{2} - x}} \rightarrow \frac{\infty}{\infty} \text{ form}$$

20 marks

Thus, $\lim_{x \to \frac{\pi}{2}^-} \ln y = \lim_{x \to \frac{\pi}{2}^-} \frac{\ln (\tan x)}{\frac{1}{\frac{\pi}{2} - x}} \rightarrow \frac{\infty}{\infty}$ form

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\sec^2 x}{\tan x}}{\frac{1}{\left(\frac{\pi}{2} - x\right)^2}}$$

If the whole method is correct but the final answer is incorrect, please give 40 marks

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cos x} \cdot \frac{\frac{\pi}{2} - x}{\sin x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cos x} \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\sin x}$$

$$= 1 . 0$$

30 marks

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \ln y = 0 \quad \therefore \boxed{\lim_{x \to \left(\frac{\pi}{2}\right)^-} y = 1} = e^0 \text{ Ans.}$$

# $\lim_{x \to 1} (2-x)^{\tan\left(\frac{\pi}{2}x\right)} \longrightarrow \boxed{1^{\infty} \text{ form}}$

let, $y = (2-x)^{\tan\left(\frac{\pi}{2}x\right)}$

Taking the natural logarithm of both sides,

$$\ln y = \ln (2-x)^{\tan\left(\frac{\pi}{2}x\right)}$$

$$= \tan\left(\frac{\pi}{2}x\right) \ln (2-x)$$

$$= \frac{\ln(2-x)}{\frac{1}{\tan\left(\frac{\pi}{2}x\right)}}$$

If the whole method is correct but the final answer is incorrect, please give 40 marks

$$\ln y = \frac{\ln(2-x)}{\cot\left(\frac{\pi}{2}x\right)}$$

Thus, $\lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\ln(2-x)}{\cot\left(\frac{\pi}{2}x\right)} \rightarrow \frac{0}{0}$ form

$$= \lim_{x \to 1} \frac{\frac{-1}{2-x}}{-\frac{\pi}{2} \operatorname{cosec}^2\left(\frac{\pi}{2}x\right)}$$

$$= \lim_{x \to 1} \frac{2 \sin^2\left(\frac{\pi}{2}x\right)}{\pi(2-x)}$$

$\therefore \lim_{x \to 1} \ln y = \frac{2}{\pi}$

30 marks

$\therefore \boxed{\lim_{x \to 1} y = e^{2/\pi}}$ Ans.

#

$$\lim_{x \to 0^+} \left(-\frac{1}{\ln x}\right)^x \longrightarrow \boxed{0^0 \text{ form}}$$

let, $y = \left(-\frac{1}{\ln x}\right)^x$

Taking the natural logarithm of both sides,

$$\ln y = \ln\left(-\frac{1}{\ln x}\right)^x$$

$$= x \ln\left(-\frac{1}{\ln x}\right)$$

If the whole method is correct but the final answer is incorrect, please give 40 marks

$$= \frac{\ln\left(-\frac{1}{\ln x}\right)}{\frac{1}{x}}$$

20 marks

Thus, $\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln\left(-\frac{1}{\ln x}\right)}{\frac{1}{x}} \rightarrow \boxed{\frac{\infty}{\infty} \text{ form}}$

$$= \lim_{x \to 0^+} \left(-\frac{1}{x \ln x}\right)(-x^2)$$

$$= \lim_{x \to 0^+} \frac{x}{\ln x}$$

$$\Rightarrow \lim_{x \to 0^+} \ln y = 0$$

$$\therefore \boxed{\lim_{x \to 0^+} y = e^0 = 1}$$

30 marks

# $\lim_{x \to +\infty} \left[\cos\left(\frac{2}{x}\right)\right]^{x^2} \longrightarrow \boxed{1^\infty \text{ form}}$

let, $y = \left[\cos\left(\frac{2}{x}\right)\right]^{x^2}$

Taking the natural logarithm of both sides,

$$\ln y = \ln\left[\cos\left(\frac{2}{x}\right)\right]^{x^2}$$

$$= x^2 \ln\left[\cos\left(\frac{2}{x}\right)\right]$$

If the whole method is correct but the final answer is incorrect,
please give 40 marks

$$\ln y = \frac{\ln\left[\cos\left(\frac{2}{x}\right)\right]}{\frac{1}{x^2}}$$

20 marks

Thus, $\lim_{x \to +\infty} \ln y = \lim_{x \to +\infty} \frac{\ln\left[\cos\left(\frac{2}{x}\right)\right]}{\frac{1}{x^2}} \rightarrow \boxed{\frac{0}{0} \text{ form}}$

$$= \lim_{x \to +\infty} \frac{\left(-\frac{2}{x^2}\right)\left[-\tan\left(\frac{2}{x}\right)\right]}{-\frac{2}{x^3}}$$

$$= \lim_{x \to +\infty} \frac{-\tan\left(\frac{2}{x}\right)}{\frac{1}{x}} \longrightarrow \boxed{\frac{0}{0} \text{ form}}$$

$$= \lim_{x \to +\infty} \frac{\frac{2}{x^2} \sec^2\left(\frac{2}{x}\right)}{-\frac{1}{x^2}}$$

30 marks

$\Rightarrow \lim_{x \to +\infty} \ln y = -2$ $\quad \boxed{\therefore \lim_{x \to +\infty} y = e^{-2}}$ Ans

If anyone uses correct formula but makes wrong calculation, you can give partial marks

(1) Determine the 2nd derivative of $Q(v) = \frac{2}{(6+2v-v^2)^4}$

Sol$^n$: $Q(v) = 2(6+2v-v^2)^{-4}$

$Q'(v) = -8(2-2v)(6+2v-v^2)^{-5}$ ---- (25 marks)

$Q''(v) = 16(6+2v-v^2)^{-5} + 40(2-2v)^2(6+2v-v^2)^{-6}$

---- (25 marks)

(2) Determine the 4th derivative of $y = e^{-5z} + 8\ln(2z^4)$

Sol$^n$: $\frac{dy}{dz} = -5e^{-5z} + 32z^{-1}$ --- (10 marks)

$\frac{d^2y}{dz^2} = 25e^{-5z} - 32z^{-2}$ ---- (10 marks)

$\frac{d^3y}{dz^3} = -125e^{-5z} + 64z^{-3}$ ---- (10 marks)

$\frac{d^4y}{dz^4} = 625e^{-5z} - 192z^{-4}$ --- (20 marks)

(3) Determine the 2nd derivative of $2x^3 + y^2 = 1 - 4y$. (Implicit)

Sol$^n$: $6x^2 + 2yy' = -4y'$

$\Rightarrow y' = \frac{-3x^2}{y+2}$ --- (20 marks)

$\Rightarrow y'' = -6x(y+2)^{-1} + 3x^2(y+2)^{-2}y'$ --- (20 marks)

$= -6x(y+2)^{-1} - 9x^4(y+2)^{-3}$ --- (10 marks) [Putting the value of y′]

(4) Determine the $2^{nd}$ derivative of $6y - xy^2 = 1$

(Implicit)

Sol$^n$ :-

$$6y' - y^2 - 2xy\,y' = 0$$

$$\Rightarrow y' = \frac{y^2}{6-2xy} \text{ ———— } (20 \text{ marks})$$

$$\Rightarrow y'' = 2yy'(6-2xy)^{-1} - y^2(6-2xy)^{-2}(-2y-2xy')$$

(20 marks)

$$\Rightarrow y'' = 2y^3(6-2xy)^{-2} + 2y^3(6-2xy)^{-2} + 2xy^2y'(6-2xy)^{-2}$$

$$\Rightarrow y'' = 4y^3(6-2xy)^{-2} + 2xy^4(6-2xy)^{-3}$$

(10 marks)

1. (i) $g(1) = 1^2+1 = 2$

(ii) $L.H.L = \lim_{t\to 1^-} g(t) = \lim_{t\to 1^-} t^2+1 = 1^2+1 = 2$

$R.H.L = \lim_{t\to 1^+} g(t) = \lim_{t\to 1^+} 2t = 2$

$\lim_{t\to 1} g(t) = 2$

upto this 40 marks

(iii) $g(1) = \lim_{t\to 1} g(t) = 2$

$\therefore g(t)$ is continuous at $t=1$

2. (i) $h(0) = 0^2+1 = 1$

(ii) $L.H.L = \lim_{t\to 0^-} h(t) = \lim_{t\to 0^-} t-1 = -1$

$R.H.L = \lim_{t\to 0^+} h(t) = \lim_{t\to 0^+} t^2+1 = 0+1 = 1$

$\lim_{t\to 0} h(t)$ does not exist

upto this 40 marks

(iii) $\cancel{h(0) \neq \lim_{t\to 0} h(0) =}$. Since $\lim_{t\to 0} h(t)$ does not exist, so $h(t)$ is not continuous at $t=0$

2021/10/31 15:46

3. (i) $f(2) = \sqrt{2+2} = 2$

(ii) $L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$

$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{x+2} = 2$

$\lim_{x\to 2} f(x) = 2$

upto this 40 marks

(iii) $f(2) = \lim_{x\to 2} f(x) = 2$

$f(x)$ is continuous at $x=2$

4. (i) $K(1) = 1+1 = 2$

(ii) $L.H.L = \lim_{x\to 1^-} K(x) = \lim_{x\to 1^-} 1-x^2 = 1-1 = 0$

$R.H.L = \lim_{x\to 1^+} K(x) = \lim_{x\to 1^+} x+1 = 1+1 = 2$

$\therefore \lim_{x\to 1} K(x)$ does not exist

upto this 40 marks

Since $\lim_{x\to 1} K(x)$ does not exist,

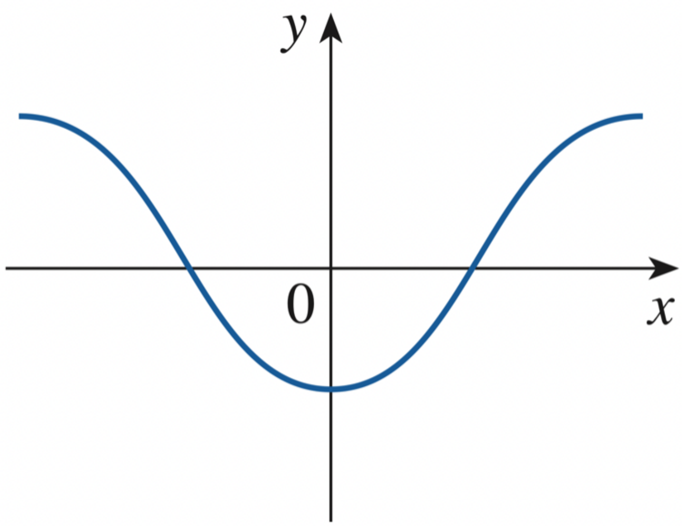
$\therefore K(x)$ is discontinuous at $x=1$

2021/10/31 15:47

## Definition&Interpretation of Derivative (MMRU)

0 points possible (ungraded)

$(a)$Copy the graph of the given function $f$. (Note that the explicit form of $f$ is intentionally not provided.) The derivative can be interpretated as the slope of the graph (you may refer to Stewart's calculus (9th edition) book example 1 in section 2.8 (page 153)). Below the graph of $f$, **sketch** the graph of $f'$ **showing** the **zeros** and the **turning points** clearly with their coordinates. The sketch should have the correct general shape; you do **not** need to use graph paper. You must justify the shape of your graph with some working.




*Sketch of f*

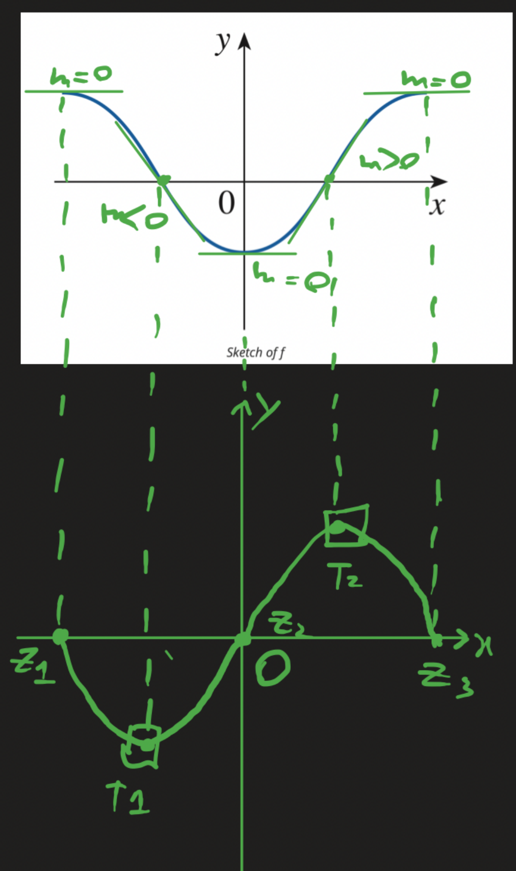
$(b)$Consider the piecewise function

$$g(x) = \begin{cases} (1+x)^4 - 1, & \text{if } x \geq 0 \\ \sin(4x), & \text{if } x < 0. \end{cases}$$

Discover whether $g(x)$ is differentiable at $x = 0$ using the **limit definition** of the derivative. **Without** carrying out any further calculation, what can you conclude about the **continuity** at point $x = 0$?

(a)

* m is the slope




$z_1$, $z_2$ & $z_3$ are zeros

$T_1$ & $T_2$ are turning points

Sketch of $f'$

* 10 marks for the general correct shape
* 5 marks for correct nature of gradient of tangent found (i.e. the graph of $f'$ is positive when $m > 0$)
* 5 marks for finding the positions of zeros & turning points

(b)

$$g(x) = \begin{cases} (1+x)^4 - 1 & \text{if } x \geq 0 \\ \sin 4x & \text{if } x < 0 \end{cases}$$

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{\sin 4h - \overbrace{[(1+0)^4 - 1]}^{\text{Deduct points if } \sin 4x \text{ is used here}}}{h}$$

$$= 4 \lim_{h \to 0^-} \frac{\sin 4h}{4h}$$

$$= 4(1)$$

$$= 4$$

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^+} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h)^4 - 1 - \overbrace{[(1+0)^4 - 1]}^{\text{Deduct points if } \sin 4x \text{ is used here}}}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h)^4 - 1}{h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0^+} \frac{4(1+h)^3}{1}$$

$$= 4$$

$$\therefore \lim_{h \to 0^-} \frac{g(x+h)-g(x)}{h} = \lim_{h \to 0^+} \frac{g(x+h)-g(x)}{h}$$

Hence $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ exists at $x=0$

and so $g$ is differentiable at $x=0$.

$\boxed{* \text{ 20 marks}}$

Differentiability implies continuity

$\Rightarrow g$ is continuous at $x=0$.

(This should be infered from differentiality test above)

* 10 marks

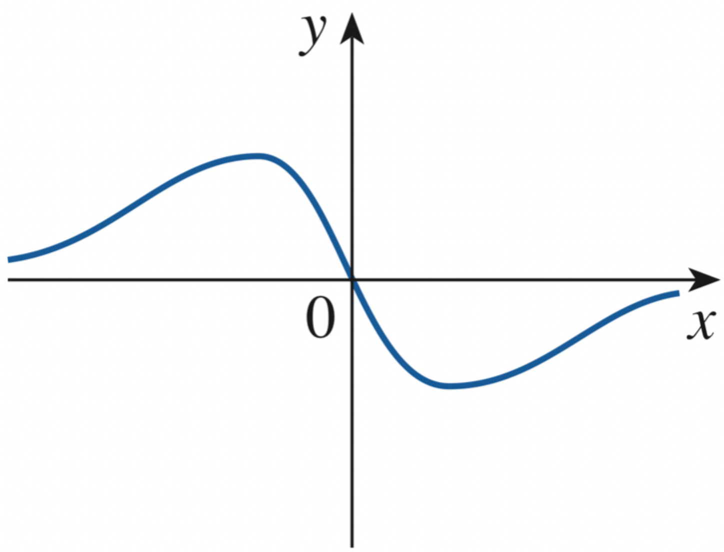
"Any" calculation in this step = 0 out of 10

Total = $10+5+5+20+10=50$

## Definition&Interpretation of Derivative (MMRU)

0 points possible (ungraded)

$(a)$Copy the graph of the given function $f$. (Note that the explicit form of $f$ is intentionally not provided.) The derivative can be interpretated as the slope of the graph (you may refer to Stewart's calculus (9th edition) book example 1 in section 2.8 (page 153)). Below the graph of $f$, $\textbf{sketch}$ the graph of $f'$ $\textbf{showing}$ the $\textbf{zeros}$ and the $\textbf{turning points}$ clearly with their coordinates. The sketch should have the correct general shape; you do $\textbf{not}$ need to use graph paper. You must justify the shape of your graph with some working.



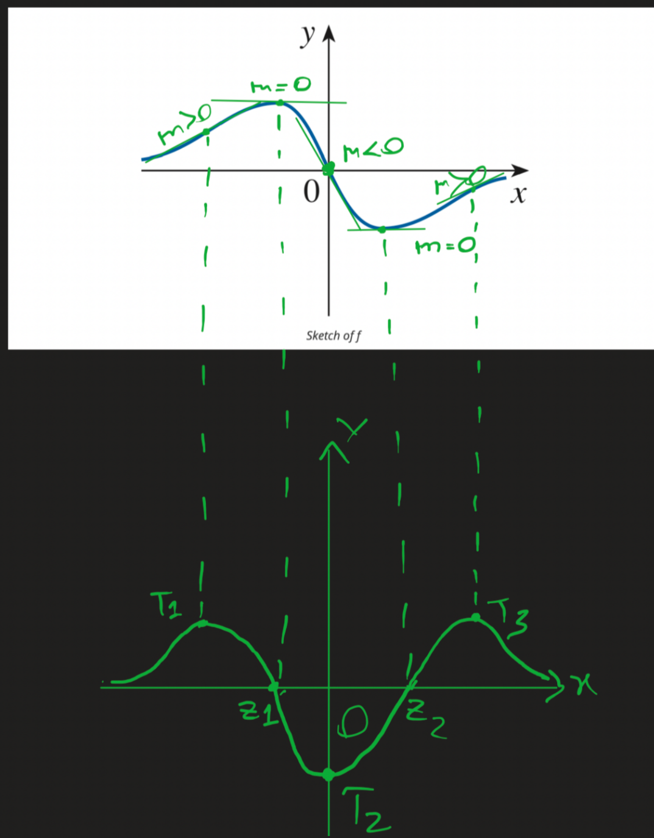
*Sketch of f*

$(b)$Consider the piecewise function

$$g(x) = \begin{cases} (1+x)^2 - 1, & \text{if } x \leq 0 \\ \sin(2x), & \text{if } x > 0. \end{cases}$$

Discover whether $g(x)$ is differentiable at $x = 0$ using the $\textbf{limit definition}$ of the derivative. $\textbf{Without}$ carrying out any further calculation, what can you conclude about the $\textbf{continuity}$ at point $x = 0$?

(a) $m$ is the slope




Sketch of $f'$

$z_1$ & $z_2$ are zeros

$T_1, T_2$ & $T_3$ are turning points

* 10 marks for the general correct shape
* 5 marks for correct nature of gradient of tangent found (i.e. the graph of $f'$ is positive when $m>0$)
* 5 marks for finding the positions of zeros & turning points

(b) $g(x) = \begin{cases} (1+x)^2 - 1, & \text{if } x \le 0 \\ \sin(2x), & \text{if } x > 0 \end{cases}$

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{g(h) - g(0)}{h}$$

Deduct points if $\sin 2x$ is used here

$$= \lim_{h \to 0^-} \frac{\sin 2h - \overbrace{[(1+0)^2 - 1]}}{h}$$

$$= 2 \lim_{h \to 0^-} \frac{\sin 2h}{2h}$$

$$= 2(1)$$

$$= 2$$

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^+} \frac{g(h) - g(0)}{h}$$

Deduct points if $\sin 2x$ is used here

$$= \lim_{h \to 0^+} \frac{(1+h)^2 - 1 - \overbrace{[(1+0)^2 - 1]}}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0^+} \frac{2(1+h)}{1}$$

$$= 2$$

$$\therefore \lim_{h \to 0^-} \frac{g(x+h)-g(x)}{h} = \lim_{h \to 0^+} \frac{g(x+h)-g(x)}{h}$$

Hence $\displaystyle\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ exists at $x=0$

and so $g$ is differentiable at $x=0$.

$\boxed{* \text{ 20 marks}}$

Differentiability implies continuity

$\Rightarrow g$ is continuous at $x=0$.

(This should be infered from differentiality test above)

* 10 marks

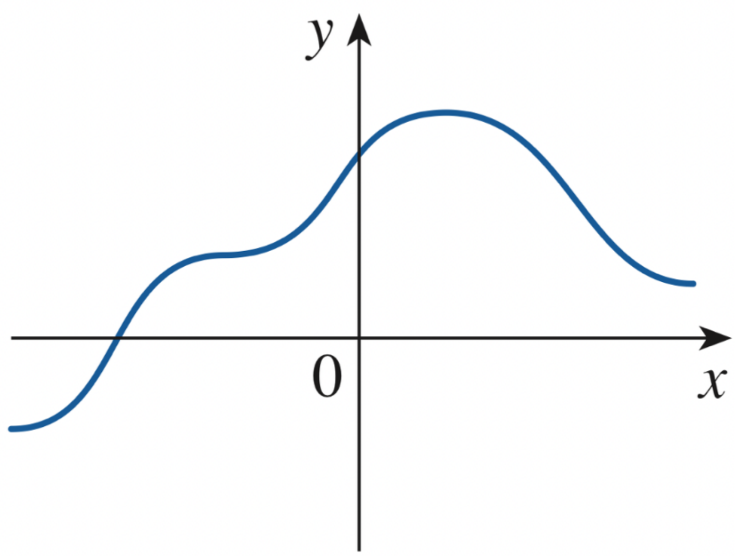
"Any" calculation in this step = 0 out of 10

Total = $10+5+5+20+10=50$

## Definition&Interpretation of Derivative (MMRU)

0 points possible (ungraded)

$(a)$Copy the graph of the given function $f$. (Note that the explicit form of $f$ is intentionally not provided.) The derivative can be interpretated as the slope of the graph (you may refer to Stewart's calculus (9th edition) book example 1 in section 2.8 (page 153)). Below the graph of $f$, **sketch** the graph of $f'$ **showing** the **zeros** and the **turning points** clearly with their coordinates. The sketch should have the correct general shape; you do **not** need to use graph paper. You must justify the shape of your graph with some working.




*Sketch of f*

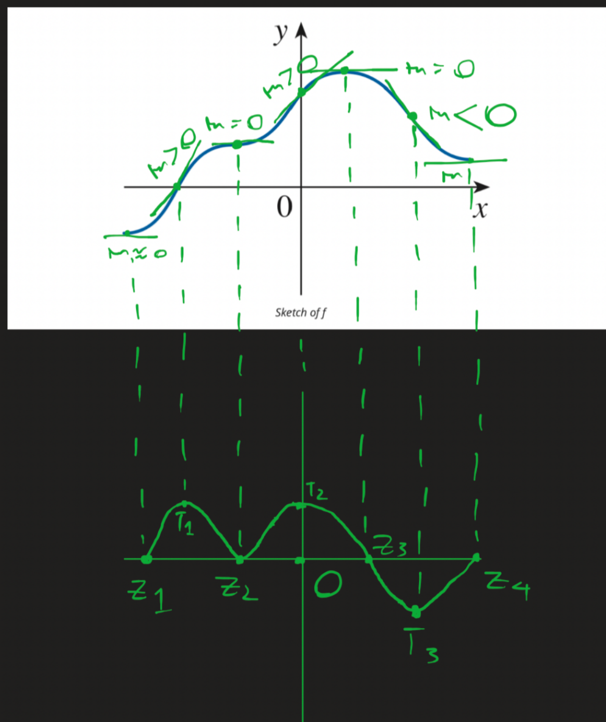
$(b)$Consider the function

$$g(x) = x|x+2| - \frac{5}{2}$$

Discover whether $g(x)$ is differentiable at $x = -2$ using the **limit definition** of the derivative **and** state the domain of $g'$.

$(c)$Can you tell anything about the continuity of a function at a point where it is **not** differentiable? Explain in a few words.

(a) m is the slope




$z_1, z_2, z_3$ & $z_4$ are zeros
$T_1, T_2$ & $T_3$ are turning points

Sketch of $f'$

* 10 marks for the general correct shape
* 5 marks for correct nature of gradient of tangent found (i.e. the graph of $f'$ is positive when $m>0$)
* 5 marks for finding the positions of zeros & turning points

(b) $$g(x) = x|x+2| - \frac{5}{2}$$

$$= x\left(\begin{cases} x+2 & x+2 \geqslant 0 \\ -(x+2) & x+2 < 0 \end{cases}\right) - \frac{5}{2}$$

$$= \begin{cases} x^2 + 2x - \frac{5}{2} & x \geqslant -2 \\ -x^2 - 2x - \frac{5}{2} & x < -2 \end{cases}$$

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^-} \frac{g(-2+h) - g(-2)}{h}$$

$$= \lim_{h \to 0^-} \frac{-(-2+h)^2 - 2(-2+h) - \frac{5}{2} - \left(-\frac{5}{2}\right)}{h}$$

$$= \lim_{h \to 0^-} \frac{2h - h^2}{h}$$

$$= \lim_{h \to 0^-} (2 - h)$$

$$= 2$$

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^+} \frac{g(-2+h) - g(-2)}{h}$$

$$= \lim_{h \to 0^+} \frac{(-2+h)^2 + 2(-2+h) - \frac{5}{2} - \left(-\frac{5}{2}\right)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 - 2h}{h}$$

$$= \lim_{h \to 0^+} (h - 2)$$

$$= -2$$

$$\lim_{h \to 0^-} \frac{g(x+h)-g(x)}{h} \neq \lim_{h \to 0^+} \frac{g(x+h)-g(x)}{h}$$

$$\therefore \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \text{ does not exist}$$

and so $g$ is not differentiable at $x=-2$.

$\boxed{* \text{ 15 marks}}$

Domain of $g' = (-\infty, -2) \cup (-2, \infty)$

$\boxed{* \text{ 5 marks}}$

(c) If a function is not differentiable, we cannot infer anything about its continuity, i.e, it "may" be continuous at that point or may "not".

* 10 marks

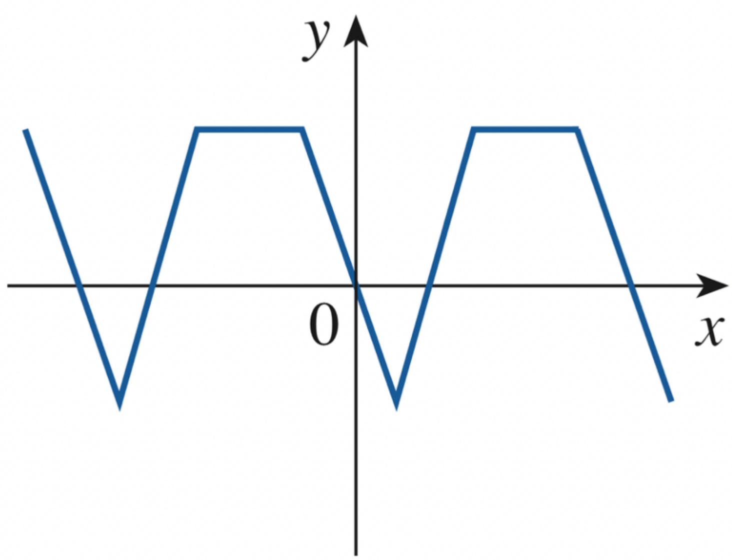
"Any" calculation in this step = 0 out of 10

Total = 10+5+5+15+5+10 = 50

## Definition&Interpretation of Derivative (MMRU)

0 points possible (ungraded)

$(a)$Copy the graph of the given function $f$. (Note that the explicit form of $f$ is intentionally not provided.) The derivative can be interpretated as the slope of the graph (you may refer to Stewart's calculus (9th edition) book example 1 in section 2.8 (page 153)). Below the graph of $f$, **sketch** the graph of $f'$ **showing** the **zeros** and the **turning points** clearly with their coordinates. The sketch should have the correct general shape; you do **not** need to use graph paper. You must justify the shape of your graph with some working.




*Sketch of f*

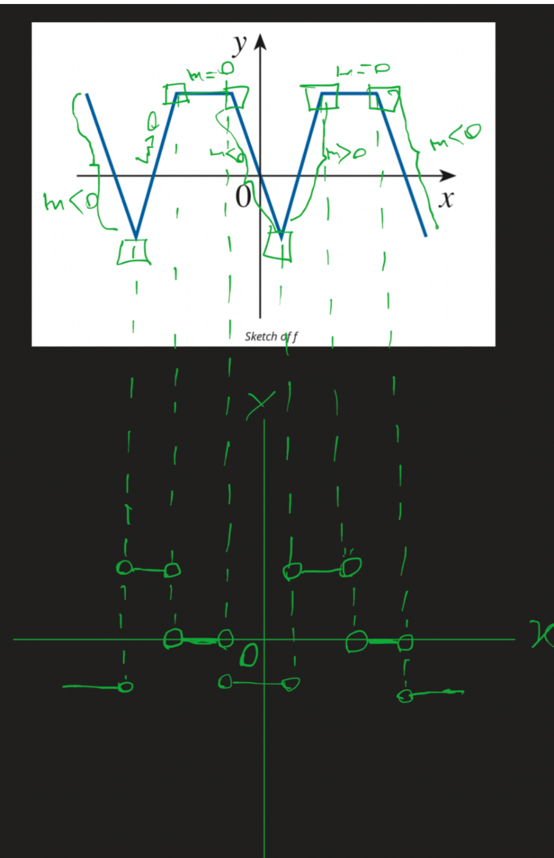
$(b)$Consider the function

$$g(x) = \frac{1}{2}x^2|x + 4|$$

Discover whether $g(x)$ is differentiable at $x = -4$ using the **limit definition** of the derivative **and** state the domain of $g'$.

$(c)$Can you tell anything about the continuity of a function at a point where it is **not** differentiable? Explain in a few words.

□ no derivative at a sharp points




Sketch of $f'$

* 10 marks for the general correct shape
* 5 marks for correct nature of gradient of tangent found
  (i.e. the graph of $f'$ is positive when $m > 0$)
* 5 marks for finding the positions of the 'holes'

(b) $$g(x) = \frac{1}{2}x^2 |x+4|$$

$$= \frac{1}{2}x^2 \begin{cases} x+4 & x+4 \geq 0 \\ -(x+4) & x+4 < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}x^3 + 2x^2 & x \geq -4 \\ -\frac{1}{2}x^3 - 2x^2 & x < -4 \end{cases}$$

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^-} \frac{g(-4+h) - g(-4)}{h}$$

$$= \lim_{h \to 0^-} \frac{-\frac{1}{2}(-4+h)^3 - 2(-4+h)^2}{h}$$

$$= \lim_{h \to 0^-} \frac{-\frac{h^3}{2} + 4h^2 - 8h}{h}$$

$$= -8$$

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0^+} \frac{g(-4+h) - g(-4)}{h}$$

$$= \lim_{h \to 0^+} \frac{\frac{1}{2}(-4+h)^3 + 2(-4+h)^2}{h}$$

$$= \lim_{h \to 0^+} \frac{\frac{h^3}{2} - 4h^2 + 8h}{h}$$

$$= 8$$

$$\lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} \neq \lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h}$$

$\therefore \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ does not exist

and so $g$ is not differentiable at $x=-4$

$\boxed{* \text{ 15 marks}}$

Domain of $g' = (-\infty,-4)\cup(-4,\infty)$

$\boxed{* \text{ 5 marks}}$

(c) If a function is not differentiable, we cannot infer anything about its continuity, i.e, it "may" be continuous at that point or may "not".

* 10 marks

"Any" calculation in this step = 0 out of 10

Total = $10+5+5+15+5+10 = 50$

## Greatest Integer Function

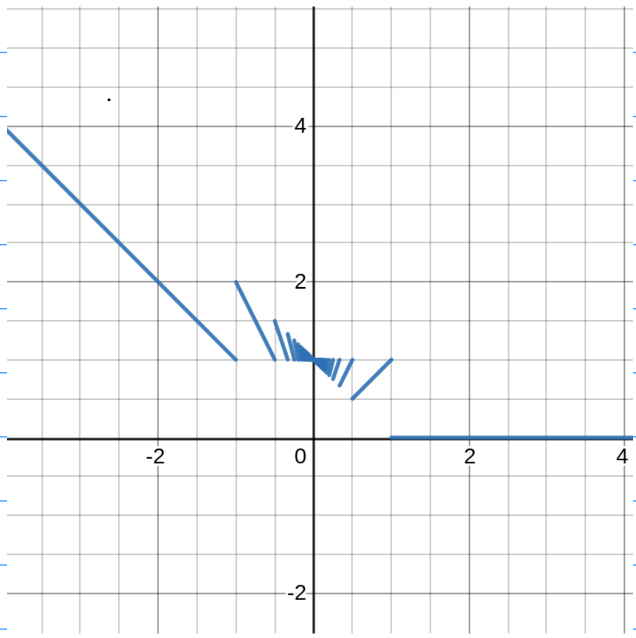
0 points possible (ungraded)

(SADT) Sketch $f(x) = x\lfloor \frac{1}{x} \rfloor, -3 \leq x \leq 3$ and evaluate the following limit:

$$\lim_{x \to 0} f(x),$$

if it exists. (*Hint*: Use the squeeze theorem to evaluate left hand and right hand limits.) Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

<u>SOLUTION:</u> The sketch is given below:




20 for the correct sketch

For the floor function, it must be true that $x \leq \lfloor x \rfloor < x+1$ and therefore,

$$\frac{1}{x} \leq \lfloor \tfrac{1}{x} \rfloor < \frac{1}{x} + 1$$

If $x > 0$, $x \cdot \frac{1}{x} \leq x \cdot \lfloor \frac{1}{x} \rfloor < x \cdot \left(\frac{1}{x} + 1\right)$

$$\Rightarrow 1 \leq x \cdot \lfloor \tfrac{1}{x} \rfloor < 1 + x$$

$\lim_{x \to 0^+} 1 = 1$ and $\lim_{x \to 0^+} (1+x) = 1$

By the squeeze theorem, $\lim_{x \to 0^+} x \lfloor \frac{1}{x} \rfloor = 1$ — ①

If $x < 0$,

$$x \cdot \frac{1}{x} \geq x \cdot \left\lfloor \frac{1}{x} \right\rfloor > x\left(\frac{1}{x} + 1\right)$$

$$\Rightarrow 1 + x < x \cdot \left\lfloor \frac{1}{x} \right\rfloor \leq 1$$

Also, $\lim\limits_{x \to 0^-} (1+x) = 1$ and $\lim\limits_{x \to 0^-} 1 = 1$

By the squeeze theorem,

$$\lim_{x \to 0^-} x \cdot \left\lfloor \frac{1}{x} \right\rfloor = 1 \quad — ②$$

If we combine the left hand limit of ② and the right hand limit of ①, we get

$$\boxed{\lim_{x \to 0} x \left\lfloor \frac{1}{x} \right\rfloor = 1}$$

30 for finding the limit ~~without using the graph.~~

· Student must find both left and right hand limits.

If a student uses advanced methods (e.g. $\epsilon - \delta$, call them for a viva).

Squeeze theorem and continuity

0 points possible (ungraded)

(SADT)Assume that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), \text{ if } x \neq 0 \\ 0, \text{ if } x = 0. \end{cases}$$

Show that $f$ is continuous at $x = 0$.

SOLUTION: The student must demonstrate the following:

(i) $f(0)$ exists: This is clear from the fact that $f(0) = 0$

(ii) $\lim_{x\to 0} f(x)$ exists: We find this using the squeeze theorem:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \underline{x^2 \sin\left(\tfrac{1}{x}\right)}$$

must use this piece of the function

But $-1 \leq \sin\left(\frac{1}{x}\right) \leq +1$

$\because x^2 \geqslant 0$, $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

$$\lim_{x\to 0} (\pm x^2) = 0$$

$\Rightarrow \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, by the squeeze theorem.

$$\Rightarrow \lim_{x\to 0} f(x) = 0$$

and hence (iii) $\lim_{x\to 0} f(x) = f(0) \Rightarrow f$ is continuous at $x = 0$.

## Squeeze Theorem

0 points possible (ungraded)

UPDATED QUESTION: By applying the squeeze theorem, find:

(SADT) ~~Find~~

$$\lim_{x\to+\infty} \frac{x^2\left(2+\sin^2 x\right)}{x+100}.$$

SOLUTION: $-1 \le \sin x \le 1 \Rightarrow 0 \le \sin^2 x \le 1$

$$\Rightarrow \frac{2x^2}{x+100} \le \frac{x^2(2+\sin^2 x)}{x+100} \le \frac{3x^2}{x+100}$$

$$a>0 \Rightarrow \lim_{x\to\infty} \frac{ax^2}{x+100} = a\lim_{x\to\infty} \frac{x}{1+\frac{100}{x}} = \infty$$

Hence, $\lim\limits_{x\to\infty} \dfrac{x^2(2+\sin^2 x)}{x+100} = \infty$, by the squeeze theorem.

Do NOT accept solutions using L'Hôpital's rule. The student MUST use the squeeze theorem.

## Squeeze Theorem

0 points possible (ungraded)

(SADT) Using a geometric construction and the squeeze theorem, find

$$\lim_{\theta \to 0^+} \frac{\tan\theta}{\theta}$$

For this problem you may NOT use L'Hospital's rule. You may NOT assume that

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$$

SOLUTION:

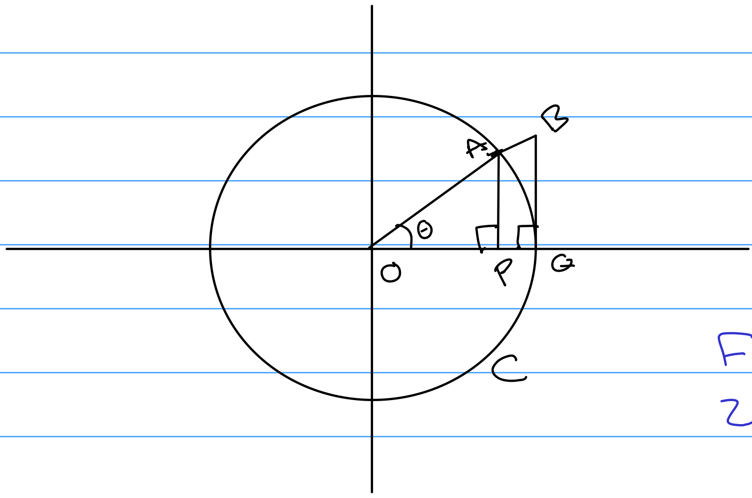



FIGURE: 20 Marks

Consider the circle $C$ of radius 1. $OA = OQ = 1$

$O$, $A$ and $B$ are collinear.

CALCULATION: 30 MARKS.

$A\hat{P}O = B\hat{Q}O = 90^\circ$

$A\hat{O}P = \theta$

area of $\Delta$ OBQ $>$ area of sector OAQ $>$ area of $\Delta$ OAP

$$\Rightarrow \tfrac{1}{2}(1)\cos\theta\sin\theta < \tfrac{1}{2}(1)(\theta) < \tfrac{1}{2}(1)(\tan\theta)$$

For $\theta > 0$, $\tan\theta > 0$

$$\Rightarrow \frac{\cos\theta\sin\theta}{\frac{\sin\theta}{\cos\theta}} < \frac{\theta}{\tan\theta} < 1$$

Taking reciprocals, we have

$$1 < \frac{\tan\theta}{\theta} < \sec^2\theta$$

$$\lim_{\theta \to 0^+} 1 = 1 \quad \text{and} \quad \lim_{\theta \to 0^+} \sec^2\theta = \lim_{\theta \to 0^+} \frac{1}{\cos^2\theta}$$

$$= \frac{1}{1} = 1$$

Hence $\displaystyle\lim_{\theta \to 0^+} \frac{\tan\theta}{\theta} = 1$, as required. ∎

N.B. Any student who uses L'Hospital's rule or the fact that $\displaystyle\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$ will not be given any marks out of 30.

# Check whether $\lim_{x\to -1} f(x)$ exists or not, where $f(x) = \begin{cases} -x-8, & x \le -1 \\ -x^2-4x-4, & x > -1 \end{cases}$.

Solution: L.H.L.= $\lim_{x\to -1^-} -x-8$ [10 marks]

$=-(-1)-8$

$=-7.$ [10 marks]

R.H.L.= $\lim_{x\to -1^+} -x^2-4x-4$ [10 marks]

$= -(-1)^2-4(-1)-4$

$= -1.$ [10 marks]

Limit doesn't exist. [10 marks]

# Check whether $\lim_{x\to -2} f(x)$ exists or not, where $f(x) = \begin{cases} 3x+2, & x < -2 \\ x^2+3x-1, & x \ge -2 \end{cases}$.

Solution: L.H.L.= $\lim_{x\to -2^-} 3x+2$ [10 marks]

$=3(-2)+2$

$= -4.$ [10 marks]

R.H.L.= $\lim_{x\to -2^+} x^2+3x-1$ [10 marks]

$=(-2)^2+3(-2)-1$

$=-3.$ [10 marks]

Limit doesn't exist. [10 marks]

# Check whether $\lim_{x\to 2} f(x)$ exists or not, where $f(x) = \begin{cases} x^2-4x+6, & x < 2 \\ -x^2+4x-2, & x \ge 2 \end{cases}$.

Solution: L.H.L.= $\lim_{x\to 2^-} x^2-4x+6.$ [10 marks]

$=2^2-(4\times 2)+6$

$= 2.$ [10 marks]

R.H.L.= $\lim_{x\to 2^+} -x^2+4x-2$ [10 marks]

$=-2^2+(4\times 2)-2$

$=2.$ [10 marks]

Limit exists. [10 marks]

# Check whether $\lim_{x\to 2} f(x)$ exists or not, where $f(x) = \begin{cases} x^2-3x, & x \le 2 \\ \frac{x^2-8}{x}, & x > 2 \end{cases}$.

Solution: L.H.L.= $\lim_{x\to 2^-} x^2-3x$ [10 marks]

$=2^2-(3\times 2)$

$= -2.$ [10 marks]

R.H.L.= $\lim_{x\to 2^+} \frac{x^2-8}{x}$ [10 marks]

$=\frac{2^2-8}{2}$

$=-2$ [10 marks]

Limit exists. [10 marks]