MTH2004M Differential Equations

Chapter 2 First Order Differential Equations

- Geometrical Interpretation of Solutions
- Separable Differential Equations
- Linear Differential Equations

Separable Differential Equations

First Order ODEs

In this Chapter we study **First Order ODEs.** How do we recognise them?

$$\frac{dy}{dx} = f(x, y)$$

- First order –> highest derivative is one
- Ordinary –> differentiation with respect to one *independent variable*
- Unknown *function* $y \equiv y(x)$ to find

Today, we consider a class of first order ODEs that are called **separable** differential equations.

Motivation Consider solving a first order ODE of the form

$$\frac{dy}{dx} = g(x)$$

Because the RHS depends only on the *independent variable*, x, this is easy to solve!

Solution by Integration

Today, we consider a class of first order ODEs that are called **separable** differential equations.

Motivation Consider solving a first order ODE of the form

$$\frac{dy}{dx} = g(x)$$

Solve simply by integrating ...

$$\int dy = \int g(x)dx,$$
$$\Rightarrow y = G(x) + c$$

where $G(x)$ is called the **anti derivative** of $g(x)$

Solution by Integration

Today, we consider a class of first order ODEs that are called **separable** differential equations.

Motivation Consider solving a first order ODE of the form

$$\frac{dy}{dx} = g(x)$$

Example

$$\frac{dy}{dx} = x,$$
$$\Rightarrow \int dy = \int x dx,$$
$$\Rightarrow y = \frac{1}{2}x^2 + c$$

Here $x^2/2$ is the antiderivative of x

Separable Equations

<u>Definition</u> A **separable** differential equation has the form

$$\frac{dy}{dx} = g(x)h(y)$$

You can separate the

- the *independent* variable, x, and
- the *dependent* variable, y

into the product of two **separate** functions, g and h, respectively.

Some Examples

Definition A **separable** differential equation has the form

$$\frac{dy}{dx} = g(x)h(y)$$

Example

$$\frac{dy}{dx} = y^2 x e^{(3x+4y)}$$

Here we can write

$$\frac{dy}{dx} = \underbrace{xe^{3x}}_{g(x)} \times \underbrace{y^2 e^{4y}}_{h(y)}$$

by laws of exponentials.

Some Examples

Definition A **separable** differential equation has the form

$$\frac{dy}{dx} = g(x)h(y)$$

TASK Are the following equations separable?

$$(1) \qquad \frac{dy}{dx} = y^2(\sin x + 1),$$

$$(2) \qquad \frac{dy}{dx} = y + \sin x$$

Method of Solution

<u>Definition</u> A **separable** differential equation has the form

$$\frac{dy}{dx} = g(x)h(y)$$

We like separable equations, because we can rearrange them and simply **integrate** to solve

$$\begin{aligned} \frac{1}{h(y)}dy &= g(x)dx, \text{ first rearrange} \\ \Rightarrow \int \frac{1}{h(y)}dy &= \int g(x)dx, \text{ then integrate} \\ \Rightarrow H(y) &= G(x) + c \end{aligned}$$

Here $H(y)$ is the anti derivative of $1/h(y)$
and $G(x)$ is the anti derivative of $g(x)$

Constants of Integration

Method of Solution

$$\Rightarrow \int \frac{1}{h(y)}dy = \int g(x)dx,$$
$$\Rightarrow H(y) = G(x) + c$$

REMARK why don't we write the solution as

$H(y) + c_1 = G(x) + c_2$ i.e. two constants of integration

Note that this can be rearranged as

$$H(y) = G(x) + \underbrace{(c_2 - c_1)}_{\text{constant}} = G(x) + c_3$$

So, we often **combine constants of integration**

Example 1

Solve the following differential equation

$$(1 + x)dy - ydx = 0$$

Note this is written in *differential* form

First we need to **classify** the equation so we can choose an appropriate **solution method**

Example 1

Solve the following differential equation

$$(1 + x)dy - ydx = 0$$

Note this is written in *differential* form

Classification Let's re-write

$$\frac{dy}{dx} = \frac{y}{(1 + x)}$$

So we can see straight away this is

- Ordinary - derivatives with respect to one variable
- First order - highest derivative is order one

Can we separate the variables?

Example 1

Solve the following differential equation

$$(1+x)dy - ydx = 0$$

Note this is written in *differential* form

Classification Can we separate the variables?

$$\frac{dy}{dx} = \frac{y}{(1+x)} = \underbrace{\frac{1}{(1+x)}}_{g(x)} \times \underbrace{y}_{h(y)}$$

Yes we can separate the RHS into a **product** of two separate functions

Example 1

Solve the following differential equation

$$(1+x)dy - ydx = 0$$

We have established this is a first order separable ODE, so we can solve by **separation of variables**:

Method of Solution:

$$\frac{dy}{dx} = \frac{y}{(1+x)},$$
$$\Rightarrow \frac{1}{y}dy = \frac{1}{(1+x)}dx, \text{ first rearrange}$$

Example 1

Solve the following differential equation

$$(1+x)dy - ydx = 0$$

We have established this is a first order separable ODE, so we can solve by **separation of variables**:

Method of Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{(1+x)}, \\
\Rightarrow \frac{1}{y}dy &= \frac{1}{(1+x)}dx, \text{ first rearrange} \\
\Rightarrow \int \frac{1}{y}dy &= \int \frac{1}{(1+x)}dx, \text{ then integrate,} \\
\Rightarrow \ln y &= \ln(1+x) + c, \text{ implicit solution}
\end{aligned}$$

Example 1

Solve the following differential equation

$$(1+x)dy - ydx = 0$$

We have established this is a first order separable ODE, so we can solve by **separation of variables**:

Method of Solution:
Finally, we can find an equation for y (explicit solution)

$$\begin{aligned} \ln y &= \ln(1+x) + c, \\ \Rightarrow \exp(\ln y) &= \exp(\ln(1+x) + c), \\ \Rightarrow y &= (1+x)\exp(c), \\ &= A(1+x) \end{aligned}$$

by laws of exponentials and combining constants

Example 1

Solve the following differential equation

$$(1 + x)dy - ydx = 0$$

By separation of variables we have **explicit solution**

$$y = A(1 + x)$$

which is a **one-parameter** family of solutions.

NOTE: An explicit solution may not always be possible!

Task

Solve the differential equation

$$y' = -\frac{x}{y}$$

REMEMBER to **classify** first, then choose your **method of solution**

Solution Curves

The differential equation

$$y' = \frac{x}{y}$$

can be solved by separation of variables to give

$$\begin{aligned} y^2 + x^2 &= c, &\qquad& \text{implicit solution} \\ \Rightarrow y &= \pm\sqrt{c - x^2}, && \text{explicit solution} \end{aligned}$$

This is a **one-parameter** family of solutions.

How can we plot the **solution curves**?

Solution Curves

Trick is to spot the solution looks like an **equation for a circle**:

$$y^2 + x^2 = c$$

where our constant of integration c is related to the **radius**, of the circle. So let's write

$$y^2 + x^2 = r^2$$

By choosing different values for the **parameter** r we can plot a family of **solution curves**

i.e. a number of circles having different radii

Solution Curves

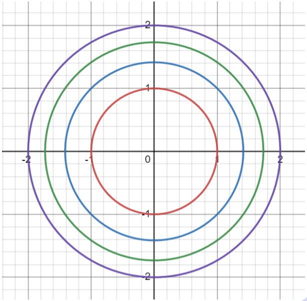
Thus our first order ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

can be solved by separation of variables to give the one-parameter family of solutions

$$y^2 + x^2 = r^2$$

which are plotted as circles centred at $(0, 0)$ with radius r



An initial value problem

What if we have the **initial value problem**

$$\frac{dy}{dx} = -\frac{x}{y}; \qquad y(2) = 2$$

The solution must satisfy the condition $y(2) = 2$.

So, the solution curve must go through the point $(2, 2)$

An initial value problem

What if we have the **initial value problem**

$$\frac{dy}{dx} = -\frac{x}{y}; \qquad y(2) = 2$$

The solution must satisfy the condition $y(2) = 2$

Taking our general solution at $(2, 2)$, we have

$$y^2 + x^2 = r^2 \quad \Rightarrow \quad 2^2 + 2^2 = 8 = r^2$$

So we have a single solution curve

$$y^2 + x^2 = 8$$

Example 2

It is always important to check where the solutions are **valid**

Example Solve the IVP

$$\frac{dy}{dx} = 6y^2 x; \qquad y(0) = 1$$

and specify the **interval of validity**

NOTE The earlier examples we have seen today are valid everywhere

Example 2

Solve the IVP

$$\frac{dy}{dx} = 6y^2 x; \qquad y(0) = 1$$

<u>Classification</u>

- Ordinary ✓
- First Order ✓
- Separable since

$$\frac{dy}{dx} = \underbrace{6x}_{g(x)} \times \underbrace{y^2}_{h(y)}$$

Example 2

Solve the IVP

$$\frac{dy}{dx} = 6y^2x; \qquad y(0) = 1$$

Method of Solution

$$\frac{1}{y^2}dy = 6xdx, \text{ first separate,}$$

$$\Rightarrow \int \frac{1}{y^2}dy = \int 6xdx, \text{ then integrate,}$$

Example 2

Solve the IVP

$$\frac{dy}{dx} = 6y^2x; \qquad y(0) = 1$$

Method of Solution

$$\int \frac{1}{y^2}dy = \int 6xdx,$$

$$\Rightarrow -\frac{1}{y} = \frac{6}{2}x^2 + c, \text{ (implicit)}$$

$$\Rightarrow y = -\frac{1}{3x^2 + c} \text{ (explicit)}$$

Is a one-parameter family of solutions

Example 2

Solve the IVP

$$\frac{dy}{dx} = 6y^2x; \qquad y(0) = 1$$

Method of Solution

$$y = -\frac{1}{3x^2 + c}$$

Applying the initial condition we have

$$-\frac{1}{(3 \times 0) + c} = 1 \Rightarrow c = -1$$

so the particular solution is

$$y = -\frac{1}{3x^2 - 1}$$

Example 2

Solve the IVP

$$\frac{dy}{dx} = 6y^2x; \qquad y(0) = 1$$

By separation of variables we have particular solution

$$y = -\frac{1}{3x^2 - 1}$$

Interval of Validity

- Clearly the *function* is undefined at $x = \pm 1/\sqrt{3}$ so is valid on intervals $(-\infty, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$
- However, the interval of validty must contain $x = 0$, so choose interval $(-1/\sqrt{3}, 1/\sqrt{3})$

Task

Solve the initial value problem

$$e^y \frac{dy}{dx} = -x; \qquad y(1) = 0$$

and determine the interval of validity

REMEMBER **Classify** first, then solve!

WARNING Care must be taken when using separation of variables to solve differential equations, as solutions may be lost using this method.

Consider the the first order separable ODE

$$\frac{dy}{dx} = g(x)h(y)$$

with constant solution $y = c$ such that $h(c) = 0$

After the variables are separated we have

$$\frac{1}{h(y)}dy = g(x)dx$$

where the LHS is **undefined** at $y = c$.

Losing a Solution

WARNING Care must be taken when using separation of variables to solve differential equations, as solutions may be lost using this method.

Consider the the first order separable ODE

$$\frac{dy}{dx} = g(x)h(y)$$

with constant solution $y = c$ such that $h(c) = 0$

Since the LHS is **undefined** at $y = c$.

$$\frac{1}{h(y)}dy = g(x)dx$$

you may not find the solution $y = c$ by integrating

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

We can verify the constant solution $y = -2$ satisfies this equation:

$$\begin{aligned} \text{LHS:} \qquad & \frac{dy}{dx} = \frac{d}{dx}(-2) = 0, \\ \text{RHS:} \qquad & y^2 - 4 = (-2)^2 - 4 = 0\checkmark \end{aligned}$$

But, will we find this solution through separation of variables?

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

Classification

- Ordinary ✓
- First Order ✓
- Separable since

$$\frac{dy}{dx} = \underbrace{1}_{g(x)} \times \underbrace{(y^2 - 4)}_{h(y)}$$

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

<u>Method of Solution</u>

$$\begin{aligned} \frac{1}{y^2-4}dy &= dx, \text{ first separate} \\ \Rightarrow \int \frac{1}{y^2-4}dy &= \int dx, \text{ then integrate} \end{aligned}$$

NOTE: you can see that the LHS is **undefined** at $y = 2$

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

<u>Method of Solution</u>

$$\int \frac{1}{y^2 - 4} dy = \int dx$$

Trick is to use **partial fractions** to rewrite LHS

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)} \right) dy = \int dx,$$

$$\Rightarrow \frac{1}{4} \ln(y-2) - \frac{1}{4} \ln(y+2) = x + c$$

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

Method of Solution Now rearrange to find an equation for y

$$\begin{aligned}
\frac{1}{4}\ln(y-2) - \frac{1}{4}\ln(y+2) &= x + \text{const} \\
\Rightarrow \ln\left(\frac{y-2}{y+2}\right) &= 4x + \text{const} \\
\Rightarrow \frac{y-2}{y+2} &= Ae^{4x}
\end{aligned}$$

by laws of logs and collecting constants

Example 3

Solve the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

Method of Solution Now rearrange to find an equation for y

$$\begin{aligned} \frac{y-2}{y+2} &= Ae^{4x} \\ \Rightarrow Ae^{4x}y + 2Ae^{4x} - y + 2 &= 0 \\ \Rightarrow y &= 2\frac{1+Ae^{4x}}{1-Ae^{4x}} \end{aligned}$$

Example 3

We have found by separation of variables that

$$y_{\text{general}} = 2\frac{1 + Ae^{4x}}{1 - Ae^{4x}}$$

is a one-parameter family of solutions.

We have verified that

$$y_{\text{constant}} = -2$$

is a constant solution.

However,

$$\nexists A \text{ such that } y_{\text{general}} = -2$$

We have lost the constant solution!!

Example 3

Summary Solving the differential equation

$$\frac{dy}{dx} = y^2 - 4$$

by separation of variables yields solution

$$y = 2\frac{1 + Ae^{4x}}{1 - Ae^{4x}}$$

However, during separation step:

$$\frac{1}{(y^2 - 4)}dy = dx$$

the LHS is undefined at $y = -2$.

Thus, the constant solution $y = -2$ becomes LOST!

Task

Solve the initial value problem

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x; \qquad y(0) = 0$$

REMEMBER **Classify** first, then solve!

HINT After separating you may need

- a trig identity on the RHS
- integration by parts on the LHS

QUESTION Is an **explicit** solution possible?

Task

For initial value problem

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y} \sin 2x; \qquad y(0) = 0$$

We have first-order, ordinary **separable** differential equation.

We find one-parameter family of solutions

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c$$

For IC $y(0) = 0$ we have **implicit** solution

$$e^{y} + ye^{-y} + e^{-y} = 4 - 2\cos x$$

More Practice . . .

For more practice solving separable differential equations see

Zill text book: Exercises 2.2, page 52