Lagrangian and Hamiltonian Mechanics

1. Given the function *f(x,y,z) = 2y –sin(xz)* where $x = 3t,\ y = e^{t-1},\ z = \ln t.$
calculate the derivative *df/dt.*

2. A mechanical system made of two masses m_1 and m_2 has two *degrees of freedom* (the *generalized coordinates* θ_1 and θ_2). l_1 and l_2 are constants. The system has *kinetic energy T:*

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

and *potential energy V:*

$$V = -(m_1 + m_2)g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

(a) Find the *Lagrangian* of the system.
(b) Calculate the *generalized force* and the *generalized momentum* relative to the coordinate θ_2.

Solution:

1.

$f(x, y, z) = 2y - \sin(xz),\ x = 3t,\ y = e^{t-1},\ z = \ln t.$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

$$\frac{\partial f}{\partial x} = -z\cos(xz); \quad \frac{\partial f}{\partial y} = 2; \quad \frac{\partial f}{\partial z} = -x\cos(xz).$$

$$\frac{dx}{dt} = 3; \quad \frac{dy}{dt} = e^{t-1}; \quad \frac{dz}{dt} = \frac{1}{t}.$$

$$\frac{df}{dt} = -3z\cos(xz) + 2e^{t-1} - \frac{x\cos(xz)}{t}$$

2.

(a) The Lagrangian is $\mathcal{L} = T - V$

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^{\ 2} + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^{\ 2} + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2 cos(\theta_1 - \theta_2) +$$

$$+ (m_1 + m_2)g\ell_1 cos\theta_1 + m_2 g\ell_2 cos\theta_2$$

(b) The generalized force (relative to θ_2) is defined as $\frac{\partial \mathcal{L}}{\partial \theta_2}$.

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2 sin(\theta_1 - \theta_2) - m_2 g\ell_2 sin\theta_2$$

The generalized momentum (relative to θ_2) is defined as $\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}$.

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2\ell_2^2\dot{\theta}_2 + m_2\ell_1\ell_2\dot{\theta}_1 cos(\theta_1 - \theta_2)$$