MTH2004M Differential Equations

Chapter 1 Introduction to Differential Equations

- What is a Differential Equation (DE)
- Definitions and Terminology
- Mathematical Modelling using DEs
- Initial Value Problems
- Existence and Uniqueness

Definitions and Terminology

Recap: Differential Equation

Definition:
An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one more independent variables, is a **differential equation** (DE).

Example:

$$y'' + 2y' + 1 = 0$$

- Here $y \equiv y(x)$ is the dependent variable.
- It depends on a single independent variable x.
- We use notation $y' = \frac{dy}{dx}$

Recap: Order

Definition

The **order** of a differential equation is the highest derivative in the equation.

For example

$$\frac{dy}{dx} + 2y + 1 = 0 \text{ is a first order DE}$$

$$y'' = y' - 6y \text{ is a second order DE}$$

$$y''' + 2xy' + 3y^4 = 0 \text{ is a third order DE}$$

Recap: Ordinary vs Partial Differential Equations

- If the function depends on only one variable e.g. $y \equiv y(t)$ then we have an **Ordinary Differential Equation**

$$\frac{dy}{dt} = v(t)$$

- If the function depends on more than one variable e.g. $u \equiv u(x, t)$ then we have a **Partial Differential Equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Recap: Linear ODEs

Any n-th order ordinary differential equation can be written as

$$F(x, y, y', \dots, y^{(n)}) = 0$$

Definition An n-th order ODE is **linear** if the function F is linear in $y, y', \dots y^{(n)}$ i.e. the ODE can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

A **non linear** ODE is one that is not linear.

Week 1 Recap

- Classifying DEs
  - Type
  - Order
  - Linearity
- Verifying solutions to DEs

Example Verify that

$$y = \frac{1}{16}x^4 \quad \text{is a solution to} \quad \frac{dy}{dx} = xy^{1/2}$$

on the interval $(-\infty, \infty)$

$$\text{LHS:} \quad \frac{dy}{dx} = \frac{1}{16}4x^3 = \frac{1}{4}x^3$$

$$\text{RHS:} \quad xy^{1/2} = x\left(\frac{1}{16}x^4\right)^{1/2} = x\left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3$$

Clearly this solution is valid everywhere ... i.e. on the interval $(-\infty, \infty)$

Interval of Validity

Consider the ODE $xy' + y = 0$, which has solution $y = 1/x$

The function $y = 1/x$ is discontinuous and not differentiable at $x = 0$

So the solution is only valid on intervals that don't contain $x = 0$ e.g. the interval $(0, \infty)$ or $(-\infty, 0)$

Solution Curve

The graph of a solution ϕ of an ODE is the **solution curve**

WARNING: Care must be taken when plotting the solution curve

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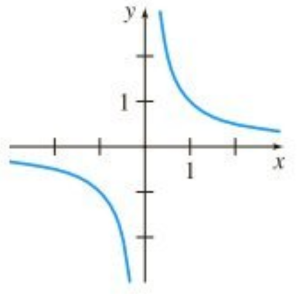

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Explicit vs Implicit Solutions

Definition An **explicit** solution is a solution for which the dependent variable is expressed in terms of only the independent variable e.g.

$$y = xe^x \quad \text{is an explicit solution to} \quad y'' - 2y' + y = 0$$

Explicit solutions can be hard to find!

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Definition If there exists one function ϕ that satisfies the DE and some other relation $G(x, y) = 0$ on interval I, then $G(x, y)$ is an **implicit solution**

Example: Implicit Solution

Show that $x^2 + y^2 = 25$ is an implicit solution to the DE

$$\frac{dy}{dx} = -\frac{x}{y}$$

on the interval $(-5, 5)$.

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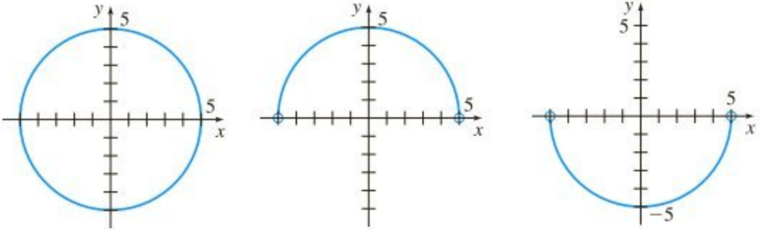

(a) implicit solution $x^2 + y^2 = 25$

(b) explicit solution $y_1 = \sqrt{25 - x^2}, \ -5 < x < 5$

(c) explicit solution $y_2 = -\sqrt{25 - x^2}, \ -5 < x < 5$

Families of Solutions

Indefinite integrals always involve a **constant** of integration e.g.

$$\int x.dx = \frac{x^2}{2} + c$$

In the same way, a first order differential equation $F(x, y, y') = 0$ has a set of solutions $G(x, y, c) = 0$ involving a single constant c. This is a **one parameter family of solutions**

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For higher order differential equations $F(x, y, y,', \ldots, y^{(n)})$, we have **n-parameter family of solutions**

$$G(x, y, c_1, c_2, \ldots, c_n) = 0$$

A solution free of parameters in a **particular solution**

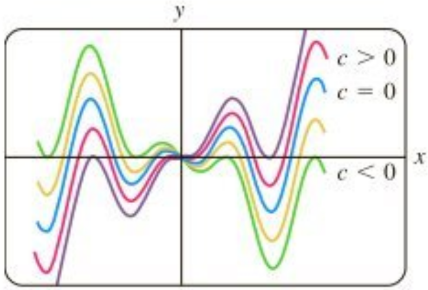
Example: Family of Solutions

Consider the first order ODE

$$xy' - y = x^2 \sin x$$

with explicit one-parameter family of solutions

$$y = cx - x \cos x$$



<u>Definition</u> A **system of ordinary differential equations** is two or more equations containing the derivatives of *two or more unknown functions* of a *single independent variable* e.g.

$$\frac{dx}{dt} = f(t, x, y),$$
$$\frac{dy}{dt} = g(t, x, y)$$

The **solution** will be a pair of differentiable functions

$$x = \phi_1(t); \qquad y = \phi_2(t)$$

defined on interval I.

Task

Verify that the pair of functions

$$x = e^{-2t} + 3e^{6t}$$
$$y = -e^{-2t} + 5e^{6t}$$

is a solution to the following system of differentiatial equations

$$\frac{dx}{dt} = x + 3y,$$
$$\frac{dy}{dt} = 5x + 3y$$

Initial Value Problems

Recall: Families of Solutions

Indefinite integrals always involve a **constant** of integration e.g.

$$\int x.dx = \frac{x^2}{2} + c$$

In the same way, a first order differential equation $F(x, y, y') = 0$ has a set of solutions $G(x, y, c) = 0$ involving a single constant c. This is a **one parameter family of solutions**

A solution free of parameters in a **particular solution**

Initial Value Problems

An initial value problem specifies an **initial condition** that the solution to the ODE must satisfy. e.g.

Solve the ODE

$$\frac{dy}{dx} = f(x, y)$$

subject to the initial condition

$$y(x = x_0) = y_0$$

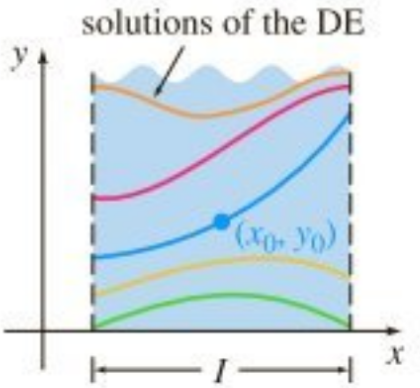
The initial condition allows us to calculate the constant of integration and find a **particular solution**

Geometric Interpretation

Consider the IVP

$$\frac{dy}{dx} = f(x, y) \qquad y(x = x_0) = y_0$$

We seek a **solution curve** on an interval I containing x_0, that passes through (x_0, y_0)



Example: IVP

Consider the ODE

$$\frac{dy}{dx} = y \quad \text{with general solution} \quad y = ce^x$$

Now impose the initial condition

$$y(x = 0) = 3$$

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Then we have

$$y(x = 0) = ce^{x=0} = c = 3$$

So the **particular solution** is

$$y = 3e^x$$

TASK: Plot the solution and check it passes through (0,3) . . .

Task: IVP

Consider the ODE

$$\frac{dy}{dx} = y \quad \text{with general solution} \quad y = ce^x$$

What is the solution curve that passes through the point $(1, -2)$?

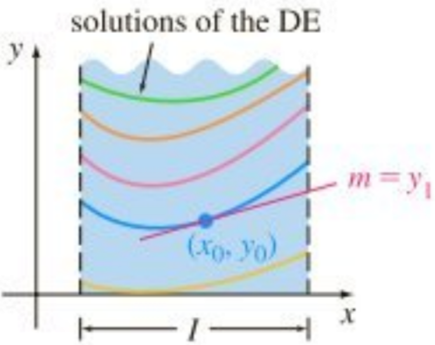
HINT: In the previous example the solution curve passed through the point $(0, 3)$.

Second-Order IVP

Consider the second order IVP

$$\frac{d^2y}{dx^2} = f(x, y, y'); \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

To solve this would require integrating TWICE resulting in TWO constants of integration. So you need TWO conditions.



Example: Second-order IVP

Consider the second-order IVP

$$y'' + 16y = 0, \qquad y(\pi 2) = -2, \quad y'(\pi/2) = 1$$

with general solution

$$y = c_1 \cos 4x + c_2 \sin 4x$$

Imposing the **first condition** gives

$$\begin{aligned} y = c_1 \cos(4 \times \pi/2) + c_2 \sin(4 \times \pi/2) &= -2, \\ \Rightarrow c_1 \times 1 + c_2 \times 0 &= -2, \\ \Rightarrow c_1 &= -2 \end{aligned}$$

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Imposing the **second condition** gives

$$\begin{aligned} y' = -4c_1 \sin(4 \times \pi/2) + 4c_2 \cos(4 \times \pi/2) &= 1, \\ \Rightarrow 4c_1 \times 0 + 4c_2 \times 1 &= 1, \\ \Rightarrow c_2 &= 1/4 \end{aligned}$$

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$$y'' + 16y = 0, \qquad y(\pi/2) = -2, \quad y'(\pi/2) = 1$$

with general solution

$$y = c_1 \cos 4x + c_2 \sin 4x$$

So we have **particular** solution

$$y = -2\cos 4x + \frac{1}{4}\sin 4x$$

Interval of Validity

It is important to consider where solutions to IVP are **valid**

For example, consider the IVP

$$y' + 2xy^2 = 0; \qquad y(0) = 1$$

with particular solution

$$y = \frac{1}{x^2 - 1}$$

Interval of Validity

It is important to consider where solutions to IVP are **valid**

For example, consider the IVP

$$y' + 2xy^2 = 0; \qquad y(0) = 1$$

with particular solution

$$y = \frac{1}{x^2 - 1}$$

- This *function* is defined on intervals $(\infty, -1), (-1, 1), (1, \infty)$ i.e. everywhere except $x = \pm 1$
- However, the *interval of validity* for the IVP must contain $x = 0$ i.e. the interval $(-1, 1)$

Task

Consider the IVP

$$\frac{dy}{dx} = -2xe^{-y}; \qquad y(1) = 0$$

with particular solution

$$y = \ln(2 - x^2)$$

What is the interval of validity for this particular solution?

Existence and Uniqueness

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Two fundamental questions arise when considering an initial value problem

- Does a solution exist?
- If so, is the solution unique?

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Existence Does the differential equation

$$\frac{dy}{dx} = f(x, y)$$

possess solutions? Do any of the solutions curves pass through the point (x_0, y_0)?

Existence and Uniqueness

Two fundamental questions arise when considering an initial value problem

- Does a solution exist?
- If so, is the solution unique?

Uniqueness Can we be certain that only ONE solution curve passes through the point (x_0, y_0)

Counter Example

Consider the initial value problem

$$\frac{dy}{dx} = xy^{1/2} \quad y(x=0) = 0$$

We have already seen this has solution $y = \frac{1}{16}x^4$, which satisfies $y(x=0) = 0$.

It is clear that the trivial solution $y = 0$ also satisfies this IVP.

So the solution curve is not unique.

Theorem: Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by

$$a \leq x \leq b \qquad c \leq y \leq d$$

that contains point (x_0, y_0).

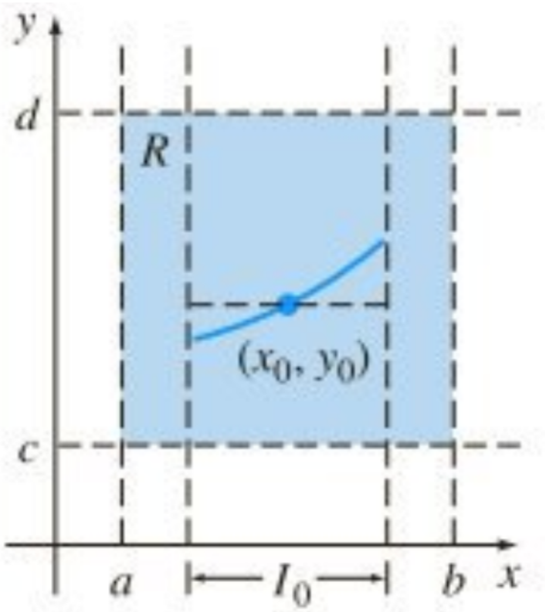
If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists

- some interval $I_0 : (x_0 - h, x_0 + h), h > 0$, contained in $[a, b]$, and
- a unique function $y(x)$ defined on I_0

that is a solution of the IVP

$$\frac{dy}{dx} = f(x, y) \qquad y(x = x_0) = y_0$$

Theorem: Existence of a Unique Solution



How to check Existence of a Unique Solution

So the IVP

$$\frac{dy}{dx} = f(x, y) \qquad y(x = x_0) = y_0$$

has a unique solution if

- $f(x, y)$ is continuous
- $\partial f/\partial y$ is continuous

in some region containing (x_0, y_0).

This usually easy to check!

Example

Consider the initial value problem

$$\frac{dy}{dx} = y; \qquad y(0) = 3$$

Here we have

$$f(x, y) = y \qquad \frac{\partial f}{\partial y} = 1$$

and we require a rectangle R containing $(0, 3)$ e.g.

$$-\infty \leq x \leq \infty \qquad -\infty \leq y \leq \infty$$

Both f and $\partial f/\partial y$ are continuous on R
So a unique solution exists.

Counter Example

Consider the initial value problem

$$\frac{dy}{dx} = xy^{1/2}; \qquad y(0) = 0$$

Here we have

$$f(x, y) = xy^{1/2} \qquad \frac{\partial f}{\partial y} = \frac{1}{2}xy^{-1/2} = \frac{x}{2\sqrt{y}}$$

and we require a rectangle R containing $(0, 0)$ e.g.

$$-1 \leq x \leq 1 \qquad \underbrace{-1 \leq y \leq 1}_{\text{need y=0 in R}}$$

$\Rightarrow \partial f/\partial y$ is NOT continuous on R.
So a unique solution does not exist.

Task

(a) Show that

$$y = \tan(x + c)$$

is a one parameter family of solutions to the ODE

$$y' = 1 + y^2$$

(b) Verify if a unique solution exists for the initial condition

$$y(0) = 0$$

(c) Explain why the solution is not valid in the region $(-2, 2)$

Necessary but not Sufficient

The **Theorem of Existence and Uniqueness** states that $f(x, y)$ and $\partial f/\partial y$ must be continuous of some rectangle R containing the initial condition.

These conditions are **necessary** for the existence of a unique solution to an IVP.

However, the are **NOT sufficient**. If these conditions do not hold then ANYTHING could happen!

- there may be no solution
- there may be many solutions
- there may be a unique solution ...

Something to consider ....

Suppose that a first order differential equation

$$\frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0$$

has a one-parameter family of solutions. Suppose f and $\partial f / \partial y$ are continuous in some region such that $(x_0, y_0) \in R$.

Explain why two different solution curves cannot intersect or be tangent to each other at (x_0, y_0).