1) (i) The linear combinations of the vectors

$$\{ e_1,\ e_1+e_2\ , e_1+e_2+e_3 \}$$

are: $c_1 e_1 + c_2(e_1+e_2) + c_3(e_1+e_2+e_3) =$

$$(c_1+c_2+c_3)e_1 + (c_2+c_3)e_2 + c_3 e_3 ,\ \text{where}\ c_1, c_2, c_3 \in \mathbb{R}$$

The span of these vectors is $\mathbb{R}^3$ when every vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

is a linear combination of $e_1, e_1+e_2, e_1+e_2+e_3$. Let x be a linear combination of those vectors, then

$$\begin{bmatrix} c_1+c_2+c_3 \\ c_2+c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \text{or}\ \begin{cases} c_1+c_2+c_3 = x_1 \\ c_2+c_3 = x_2 \\ c_3 = x_3 \end{cases}$$

Which has a unique solution in terms of c_1, c_2, c_3 :

$$\begin{cases} c_1 = x_1 - x_2 \\ c_2 = x_2 - x_3 \\ c_3 = x_3 \end{cases}$$

So, $\operatorname{span}(e_1, e_1+e_2, e_1+e_2+e_3) = \mathbb{R}^3$.

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1) (ii) The linear combinations of the vectors $\mathbb{R}^3$

$$\{e_1, -2e_2, e_1+e_2-e_3\} \text{ are :}$$

$$c_1e_1 - 2c_2e_2 + c_3(e_1+e_2-e_3) =$$

$$(c_1+c_3)e_1 + (c_3-2c_2)e_2 - c_3e_3 = \begin{bmatrix} c_1+c_3 \\ c_3-2c_2 \\ -c_3 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

The span of these vectors is $\mathbb{R}^3$ when every vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

is a linear combination of $e_1, -2e_2, e_1+e_2-e_3$. Let x be a linear combination of those vectors, then

$$\begin{bmatrix} c_1+c_3 \\ c_3-2c_2 \\ -c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ or } \begin{cases} c_1+c_3 = x_1 \\ c_3-2c_2 = x_2 \\ -c_3 = x_3 \end{cases}$$

Which has a unique solution in terms of c_1, c_2, c_3:

$$\begin{cases} c_1 = x_1 + x_3 \\ c_2 = -(x_2+x_3)/2 \\ c_3 = -x_3 \end{cases}$$

So, $\text{span}(e_1, -2e_2, e_1+e_2-e_3) = \mathbb{R}^3$

Remark: Alternatively you can show that

$$\text{span}(e_1, -2e_2, e_1+e_2-e_3) = \text{span}(e_1, e_2, e_3) = \mathbb{R}^3.$$

So, $\operatorname{span}(e_1, -2e_2, e_1+e_2-e_3) = \{d_1e_1 + d_2e_2 + d_3e_3 : d_1, d_2, d_3 \in \mathbb{R}\}$
$= \operatorname{span}(e_1, e_2, e_3) = \mathbb{R}^3$.

(iii) Same as above, the linear combinations of the three vectors e_1+e_2, e_2, e_2-3e_1 are:

$$c_1(e_1+e_2) + c_2 e_2 + c_3(e_2 - 3e_1) =$$
$$(c_1 - 3c_3)e_1 + (c_1 + c_2 + c_3)e_2 =$$
$$d_1 e_1 + d_2 e_2, \text{ where } \begin{cases} d_1 = c_1 - 3c_3 \\ d_2 = c_1 + c_2 + c_3 \end{cases}$$

So, $\operatorname{span}(e_1+e_2, e_2, e_2-3e_1) = \{d_1e_1 + d_2e_2 : d_1, d_2 \in \mathbb{R}\}$
$= \operatorname{span}(e_1, e_2) \neq \mathbb{R}^3$.

These vectors do not span $\mathbb{R}^3$ but only the xy-plane.

2) Let the vector u be a linear combination of v, w.
Then, there are nonzero scalars d_1, d_2 such that:

$$u = d_1 v + d_2 w$$

or $$u - d_1 v - d_2 w = 0 \quad (1)$$

So, there are nonzero scalars

$$c_1 = 1,\ c_2 = -d_1,\ c_3 = -d_2$$

that satisfy equation (1). But this is impossible because the vectors u, v, w are linearly independent. Hence, u cannot be a linear combination of v, w.

3) (i) The span of **u**, **v** is the set of all linear combinations of **u** and **v**:

$$c_1 u + c_2 v = c_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ c_2 \\ 3c_1 + c_2 \end{bmatrix}$$

so

$$\text{span}(u,v) = \left\{ \begin{bmatrix} 2c_1 \\ c_2 \\ 3c_1 + c_2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

* Remark: To understand what this span implies, we can set:

$$x = 2c_1 \quad , y = c_2$$

and these new scalars give:

$$\text{span}(u,v) = \left\{ \begin{bmatrix} x \\ y \\ \frac{3x}{2} + y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

the plane $\quad z = \frac{3}{2}x + y \quad$ in $\mathbb{R}^3$.

(ii) We have two vectors in $\mathbb{R}^4$. Let c_1, c_2 scalars, then all linear combinations of **u** and **v** are:

$$c_1 u + c_2 v = c_1 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_2 \\ -2c_1 + c_2 \\ c_2 \\ 5c_1 \end{bmatrix}$$

So, $$\operatorname{span}(u,v) = \left\{ \begin{bmatrix} 3c_2 \\ -2c_1 + c_2 \\ c_2 \\ 5c_1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

↳ the span contains $\mathbb{R}^4$ vectors of this type determined by 2 free scalars → so it consists a plane in $\mathbb{R}^4$.

*(iii) To find the span of **u, v, w** we need to find the form of the linear combinations of the vectors. Let* c_1, c_2, c_3 *scalars, then*

$$c_1 u + c_2 v + c_3 w = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1/2 \\ 3/2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 - c_3 \\ -c_1 - c_2/2 \\ 3c_1/2 + c_3 \\ 2c_1 + c_2 \end{bmatrix}$$

$$\operatorname{span}(u,v,w) = \left\{ \begin{bmatrix} c_1 - c_3 \\ -c_1 - c_2/2 \\ 3c_1/2 + c_3 \\ 2c_1 + c_2 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}$$

4) (i) u, v are linearly independent when $c_1 u + c_2 v = 0$ implies $c_1 = c_2 = 0$.

It is: $$c_1 u + c_2 v = c_1 \begin{bmatrix} 1 \\ 1-c \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1(1-c) \end{bmatrix}$$

By setting $c_1 u + c_2 v = 0$ we get $\begin{bmatrix} c_1 + c_2 \\ c_1(1-c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or equivalently $$\begin{cases} c_2 = -c_1 \\ c_1(1-c) = 0 \end{cases}.$$

- If $c=1$ then $c_1(1-c) = c_1 \cdot 0 = 0$ implies that c_1 is any real number, hence the vectors u, v are linearly dependent.
- If $c \neq 1$ then we get that $c_1 = 0$ and $c_2 = -c_1 = 0$, as well.

So, only when $c \neq 1$ u, v are linearly independent.

(ii) Same process as above. The linear combinations of u, v give

$$c_1 u + c_2 v = c_1 \begin{bmatrix} 1 \\ c \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2/3 \\ c \cdot c_1 - c_2/3 \\ 3c_1 + c_2 \end{bmatrix}$$

Now, by setting that equal to the zero vector, we get the system

$$\begin{cases} c_1 + \dfrac{c_2}{3} = 0 \\ c_1 \cdot c - \dfrac{c_2}{3} = 0 \\ 3c_1 + c_2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} c_1 = -\dfrac{c_2}{3} \\ c \cdot c_1 + c_1 = 0 \\ c_1 = -\dfrac{c_2}{3} \end{cases}$$

Which reduces to a system of 2 equations

$$\begin{cases} c_1 = -\dfrac{c_2}{3} \\ c_1(c+1) = 0 \end{cases}$$

- If $c=-1$ then $c_1(-1+1) = c_1 \cdot 0 = 0$ implies that c_1 is any real number, hence the vectors u, v are linearly dependent.
- If $c \neq -1$ then we get that $c_1 = 0$ and $c_2 = -c_1 = 0$, as well.

So, only when $c \neq -1$ u, v are linearly independent.

5) Let $c_1 u + c_2 v + c_3 w = 0$.

We check whether that condition implies $c_1 = c_2 = c_3 = 0$.

It is:

$$c_1 u + c_2 v + c_3 w = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 + c_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $\begin{cases} c_1 = c_2 \\ c_1 + c_2 + c_3 = 0 \end{cases}$ *This system does not have a unique solution. It has the infinitely many solutions:*

$$c_2 = c_1 \ , \ c_3 = -2c_1 \ , \text{ where } c_1 \in \mathbb{R}$$

Hence, u, v, w are linearly dependent. If we replace the solution for the scalars in the initial equation, we get:

$$c_1 u + c_1 v - 2c_1 w = 0 \ , \quad c_1 \in \mathbb{R}$$

So,

$$u + v - 2w = 0 \quad \text{or} \quad w = \frac{1}{2} u + \frac{1}{2} v .$$

The span of these vectors is:

$$\text{span}(u, v, w) = \text{span}\left(u, v, \frac{1}{2} u + \frac{1}{2} v\right) = \dots$$

$$\ldots = \text{span}(u, v) = \left\{ d_1 u + d_2 v \; : \; d_1, d_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} d_1 - d_2 \\ d_1 + d_2 \\ 0 \end{bmatrix} \; : \; d_1, d_2 \in \mathbb{R} \right\}.$$

If we just rename the scalars into x and y, where

$x = d_1 - d_2$, $y = d_1 + d_2$, *the span becomes:*

$$\text{span}(u, v, w) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}, \; x, y \in \mathbb{R} \right\}$$

Which is the x-y plane.

6) (i) $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

The dot product of u *and* v *is*

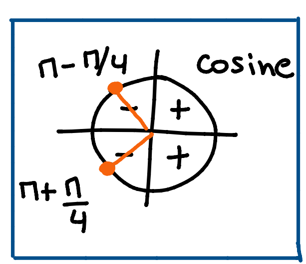
$$u \cdot v = 1 \cdot 0 + (-1) \cdot 3 = -3 .$$

The lengths of u *and* v *are:*

$$\|u\| = \sqrt{1^2 + (-1)^2} = \sqrt{2} , \quad \|v\| = \sqrt{0^2 + 3^2} = 3$$

The angle between the vectors can be found by the angle formula:

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{-3}{\sqrt{2} \cdot 3} = -\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4}$$



So, $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$

We verify the Cauchy - Schwarz inequality: $|u\cdot v| \leq \|u\|\cdot\|v\|$

$|u\cdot v| = |-3| = 3$, $\|u\|\cdot\|v\| = \sqrt{2}\cdot 3$. $\hookrightarrow$ $3 \leq \sqrt{2}\cdot 3$ or $1 \leq \sqrt{2}$, true.

And we also verify the triangle inequality: $\|u+v\| \leq \|u\| + \|v\|$

$$u+v = \begin{bmatrix} 1+0 \\ -1+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\|u+v\| = \sqrt{1^2+2^2} = \sqrt{5}$$

$\hookrightarrow$ $\sqrt{5} \leq \sqrt{2} + 3$

$5 \leq \sqrt{2}^2 + 3^2 + 2\sqrt{2}\cdot 3$

$5 \leq 11 + 6\sqrt{2}$, true.

(ii) Same as (i) for the vectors $u = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$, $v = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

- Dot product $u\cdot v = 0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} + \left(-\frac{1}{2}\right)\cdot\frac{2}{3} = 0 + \frac{1}{3} - \frac{1}{3} = 0$

Hence the vectors are orthogonal and we do not need to use the Angle formula to conclude that $\theta = \frac{\pi}{2}$ $\left(\text{or } \theta = \frac{3\pi}{2}\right)$.

- Lengths $\|u\| = \sqrt{0^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

  $\|v\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{9}{9}} = 1$.

- Cauchy - Schwarz inequality: $|u\cdot v| \leq \|u\|\cdot\|v\|$

  $0 \leq \frac{1}{\sqrt{2}}\cdot 1$, true

For the triangle inequality: $\|u+v\| \leq \|u\| + \|v\|$

We need $u+v = \begin{bmatrix} 1/3 \\ 7/6 \\ 1/6 \end{bmatrix}$ and $\|u+v\| = \sqrt{\left(\frac{2}{6}\right)^2 + \left(\frac{7}{6}\right)^2 + \left(\frac{1}{6}\right)^2} = \sqrt{\frac{54}{36}} = \sqrt{\frac{3}{2}}$

The inequality is verified by: $\sqrt{\frac{3}{2}} \leq \frac{1}{\sqrt{2}} + 1$

$$\downarrow$$

$$\sqrt{3} \leq 1 + \sqrt{2}$$

$$\downarrow$$

$$3 \leq (1+\sqrt{2})^2 = 1 + 2\sqrt{2} + \sqrt{2}^2 = 3 + 2\sqrt{2},$$

true.