Algebra – Practical session 8

8.1. Find all complex solutions of the following symmetric system:

$$\begin{cases} x^3 + y^3 = 37 \\ x + y = 3 \end{cases}$$

8.2. Find a polynomial $g(y)$ such that $\quad g\Big(x + \dfrac{1}{x}\Big) = x^3 + \dfrac{1}{x^3}$.

8.3. Let α and β be the complex roots of the polynomial $x^2 + x + 2$. Without computing α and β separately, compute $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$, and $\alpha^4 + \beta^4$. (That is, use only your knowledge of $\alpha + \beta$ and $\alpha\beta$.)

8.4. Let α, β, γ be the complex roots of the polynomial $2x^3 - 4x - 6$. Find a polynomial of degree three, with integer coefficients, having roots $1/\alpha, 1/\beta, 1/\gamma$.

8.5. Let α, β and γ be the complex roots of the polynomial $x^3 + 2x^2 - x + 3$. Compute $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$.

Questions to take home for further practice, also on earlier material

8.6. Find a polynomial whose roots are the squares of the roots of $x^2 - 2x - 5$.

(One way is finding the roots using the formula for quadratic equations, squaring them etc. Do not do that, use symmetric polynomials instead to work out the coefficients of the desired polynomial.)

8.7. Use the extended Euclidean algorithm to write the rational expression

$$\frac{1}{(x^3 - x^2 - 1)(x^2 + 1)}$$

in the form

$$\frac{f(x)}{x^3 - x^2 - 1} + \frac{g(x)}{x^2 + 1},$$

where $f(x)$ and $g(x)$ are polynomials (that is, find $f(x)$ and $g(x)$).

8.8*. A 1-meter long ruler has 10 black marks on it, which divide it into 11 equal parts. It also has 18 red marks on it, which divide it into 19 equal parts. Find the smallest distance between a black mark and a red mark. Say also where this distance is found on the ruler (such as between the mth black mark and the nth mark, starting from the same end).

Note: Give an exact answer as a fraction, not an approximation. You may draw a picture for guidance, but then preferably solve without reference to that (pretending that the numbers involved are too large for an accurate picture.)