Solutions - Practical 5

Solution to Problem 1. The standard formula for the inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

By applying this formula, we find:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Solution to Problem 2.

To find the inverse of the matrix, we use Gauss–Jordan elimination to bring the matrix in its reduced row-echelon form:

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1/4} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ 0 & 1 & 2 & 1/4 & 3/4 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ 0 & 1 & 2 & 1/4 & 3/4 & 0 \\ 0 & 1 & 3 & -1/2 & 1/2 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ 0 & 1 & 2 & 1/4 & 3/4 & 0 \\ 0 & 0 & 1 & -3/4 & -1/4 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 1/4 + 6/4 & 3/4 + 2/4 & -2 \\ 0 & 0 & 1 & -3/4 & -1/4 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 7/4 & 5/4 & -2 \\ 0 & 0 & 1 & -3/4 & -1/4 & 1 \end{array}\right]$$

Therefore, the inverse of the matrix B is

$$B^{-1} = \begin{bmatrix} 1/4 & -1/4 & 0 \\ 7/4 & 5/4 & -2 \\ -3/4 & -1/4 & 1 \end{bmatrix}$$

and the rank of the matrix B is $\text{rank}(B) = 3$.

Solution to Problem 3.
(a) If A is invertible then exists the matrix A^{-1} such that $A^{-1}A = I$. So, we can multiply the matrix equation from the left with A^{-1}:

$$\begin{aligned} AB &= O \\ A^{-1}AB &= A^{-1}O \\ IB &= O \\ B &= O \end{aligned}$$

(b) A is invertible, so the matrix A^{-1} exists and $A^{-1}A = I$. Again, as in Pr.4(a), we can multiply the matrix equation from the left with A^{-1}:

$$\begin{aligned} AB &= AC \\ A^{-1}AB &= A^{-1}AC \\ IB &= IC \\ B &= C \end{aligned}$$

(c) A satisfies the matrix equation $A^2 - 2A + I = O$. So It is

$$\begin{aligned} A^2 - 2A + I &= O \\ A^2 - 2A &= -I \\ -A^2 + 2A &= I \\ A(-A + 2I) &= I \end{aligned}$$

which implies that

$$AA^{-1} = I \text{ where } A^{-1} = -A + 2I$$

So A is invertible, with inverse the matrix $A^{-1} = -A + 2I$.

(d) A satisfies the matrix equation $A^3 + \alpha A^2 + \beta A - I = O$, so

$$\begin{aligned} A^3 + \alpha A^2 + \beta A - I &= O \\ A^3 + \alpha A^2 + \beta A &= I \\ A(A^2 + \alpha A + \beta I) &= I \end{aligned}$$

which implies that

$$AA^{-1} = I \text{ where } A^{-1} = A^2 + \alpha A + \beta I$$

So A is invertible, with inverse the matrix $A^{-1} = A^2 + \alpha A + \beta I$.

4) (a)

It is $\det A = \begin{vmatrix} 2 & -3 & 2 \\ 4 & 1 & 0 \\ 0 & 0 & 6 \end{vmatrix} = +6 \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = 6[2 \cdot 1 - (-3) \cdot 4]$

$= 6(2 + 12) = 84$

$\det B = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -2 \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + 0$

$= -2\,[(-2)\cdot(-2) - 1] - [1 \cdot (-2) - 1 \cdot 0]$

$= -2(4-1) + 2 = -4$

(b) It is $\det A = 84$, so $\det(A^n) = (\det A)^n = 84^n$

5)

(a) It is $\det(P^{-1}AP) = \det P^{-1} \det A \det P = (\det P)^{-1} \det A \det P$

$= \det A$

(b) It is $\det(P^T AP) = \det P^T \det A \det P = \det P \det A \det P$

$= \det A (\det P)^2$

(c) If $\det A=1$ and $ABA^T=B^2$ then

$$\det(ABA^T)=\det(B^2)$$
$$\det A \det B \det A^T=(\det B)^2$$
$$\det A \det B \det A =(\det B)^2$$
$$\det B=(\det B)^2$$

So, $\det B(\det B-1)=0$ and we get that

either $\det B=0$ or $\det B=1$.

6) (a) We construct the following matrices

$$A=\begin{bmatrix}2 & 1 & -1\\ -1 & 2 & 1\\ 1 & -1 & 2\end{bmatrix}, \quad B_1=\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ 1 & -1 & 2\end{bmatrix},$$

$$B_2=\begin{bmatrix}2 & 1 & -1\\ -1 & 1 & 1\\ 1 & 1 & 2\end{bmatrix}, \quad B_3=\begin{bmatrix}2 & 1 & 1\\ -1 & 2 & 1\\ 1 & -1 & 1\end{bmatrix}$$

It is $\det A=\begin{vmatrix}2 & 1 & -1\\ -1 & 2 & 1\\ 1 & -1 & 2\end{vmatrix}=2\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix}-\begin{vmatrix}-1 & 1\\ 1 & 2\end{vmatrix}+(-1)\begin{vmatrix}-1 & 2\\ 1 & -1\end{vmatrix}=$

$$= 2(4+1) - (-2-1) - (1-2) = 2\cdot 5 + 3 + 1 = 14.$$

$$\det B_1 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= (4+1) - (2-1) - (-1-2)$$

$$= 5 - 1 + 3 = 7$$

$$\det B_2 = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 2(2-1) - (-2-1) - (-1-1)$$

$$= 2 + 3 + 2 = 7$$

$$\det B_3 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(2-(-1)) - (-1-1) + (1-2)$$

$$= 2\cdot 3 + 2 - 1 = 7.$$

So the solution of the system using <u>Cramer's rule</u> is

$$x = \frac{\det B_1}{\det A} = \frac{7}{14} = \frac{1}{2}, \quad y = \frac{\det B_2}{\det A} = \frac{7}{14} = \frac{1}{2},$$

and $$z = \frac{\det B_3}{\det A} = \frac{7}{14} = \frac{1}{2}.$$

(b) For this second linear system it is:

$$A = \begin{bmatrix} 1 & a & a \\ 1 & -1 & a \\ 1 & 1 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & a & a \\ 0 & -1 & a \\ 0 & 1 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 0 & a \\ 1 & 0 & a \\ 1 & 0 & 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & a & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Since B_1, B_2 and B_3 have all a zero column, their determinants are all equal to zero: $\det B_1 = \det B_2 = \det B_3 = 0$

The determinant of the matrix A is:

$$\det A = \begin{vmatrix} 1 & a & a \\ 1 & -1 & a \\ 1 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & a \\ 1 & 1 \end{vmatrix} - a \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} + a \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= (-1-a) - a(1-a) + a(1+1)$$

$$= -1 - a - a + a^2 + 2a = a^2 - 1.$$

The matrix A is invertible when $\det A \neq 0$, hence when $a^2 \neq 1$ or $a \neq \pm 1$.
Then by Cramer's rule the systems has a unique solution, the zero solution:

$$x = \frac{\det B_1}{\det A} = 0 \quad , \quad y = \frac{\det B_2}{\det A} = 0 \quad , \quad z = \frac{\det B_3}{\det A} = 0$$

Remark: Remember that homogeneous systems like this one can have infinitely many solutions or a unique solution the zero solution
When $\det A \neq 0$ any system $A \cdot x = 0$ yields $x = 0$

vector