MTH 1002 Practical 5 Solutions

1. a) $f'(x) = 15x^4 - 4x - \frac{7}{2}x^{-3/2}$

b) Quotient rule:

$$f'(x) = \frac{(x-1)\frac{d}{dx}(x^3+3x^2+7) - (x^3+3x^2+7)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(3x^2+6x) - (x^3+3x^2+7)}{(x-1)^2}$$

$$= \frac{3x^3+6x^2-3x^2-6x-x^3-3x^2-7}{(x-1)^2}$$

$$= \frac{2x^3-6x-7}{(x-1)^2}$$

c) Product rule

$$f'(x) = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + e^x \sin x$$

$$= e^x(\cos x + \sin x)$$

d) Product rule

$$f'(x) = 4x^4 \frac{d}{dx} e^{cx} + e^{cx} \frac{d}{dx}(4x^4)$$

$$= 4x^4 ce^{cx} + e^{cx} 16x^3$$

$$= 4x^3 e^{cx}(cx+4)$$

e) Quotient rule

$$f'(x) = \frac{x^3 \frac{d}{dx}(\cos ax) - \cos ax \frac{d}{dx}(x^3)}{(x^3)^2}$$

$$= \frac{-ax^3 \sin ax - 3x^2 \cos ax}{x^6} = -\frac{ax \sin ax + 3\cos ax}{x^4}$$

f) Quotient rule

$$f'(x) = a\left[\frac{\sin(cx)\frac{d}{dx}(e^{bx}) - e^{bx}\frac{d}{dx}(\sin(cx))}{\sin^2 cx}\right]$$

$$= ae^{bx}\left(\frac{b\sin(cx) - c\cos(cx)}{\sin^2 cx}\right)$$

2 All these questions use the chain rule,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$$

a) Let $g(x) = 5x + 2$ and $f(g) = \sin(g)$.

$$f'(x) = f'(g(x))\, g'(x) = \cos(g) \cdot 5 = 5\cos(5x + 2)$$

b) Let $g(x) = 1 + \frac{1}{x^2}$ and $f(g) = \ln(g)$

$$f'(x) = f'(g(x))\, g'(x) = \frac{1}{g}\left(-\frac{2}{x^3}\right) = -\frac{2}{x^3}\,\frac{1}{1 + \frac{1}{x^2}}$$

$$= \frac{-2}{x^3 + x}$$

c) $f(x) = \sin^3 x = (\sin x)^3$

Let $g(x) = \sin x$ and $f(g) = g^3$.

$$f'(x) = f'(g(x))\, g'(x) = 3g^2 \cos x$$

$$= 3 \sin^2 x \cdot \cos x$$

3 As in q 2, use the chain rule. Here, the outer function is labelled h and the independent variable is t, so we can write

$$\frac{d}{dt}\left[h(g(t))\right] = h'(g(t))\, g'(t)$$

a) Let $g(t) = t^3 - 1$ and $h(g) = g^{100}$

$$h'(t) = h'(g(t))\, g'(t) = 100 g^{99} \cdot 3t^2$$

$$= 100 (t^3 - 1)^{99} \cdot 3t^2 = 300 t^2 (t^3 - 1)^{99}$$

b) Let $g(t) = a + bt^4$ and $h(g) = \sin(g)$

$$h'(t) = h'(g(t))\, g'(t) = \cos(g) \cdot 4bt^3$$

$$= 4bt^3 \cos(a + bt^4)$$

c) Let $g(t) = b\tan(ct)$ and $h(g) = a\cos(g)$

$$h'(t) = h'(g(t))\,g'(t) = -a\sin(g)\,.\,bc\sec^2(ct)$$

$$= -abc\sec^2(ct)\sin(b\tan(ct))$$

4 a) $y = x^4 e^{3x}\tan x$

$$\Rightarrow \ln y = \ln(x^4) + \ln(e^{3x}) + \ln(\tan x)$$

$$= 4\ln x + 3x + \ln(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right]$$

$$= x^4 e^{3x}\tan x\left[\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right]$$

b) $y = \dfrac{e^{4x}}{x^3\cosh 2x}$

$$\Rightarrow \ln y = \ln(e^{4x}) - \ln(x^3) - \ln(\cosh 2x)$$

$$= 4x - 3\ln x - \ln(\cosh 2x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 4 - \frac{3}{x} - \frac{1}{\cosh 2x}2\sinh 2x$$

$$= 4 - \frac{3}{x} - 2\tanh 2x$$

$$\frac{dy}{dx} = y\left[4 - \frac{3}{x} - 2\tanh 2x\right]$$

$$= \frac{e^{4x}}{x^3 \cosh 2x}\left[4 - \frac{3}{x} - 2\tanh 2x\right]$$

5 a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^3 + b(x+h) - (ax^3 + bx)}{h}$$

$$= \lim_{h \to 0} \frac{a(\cancel{x^3} + x^2 h + xh^2 + h^3) + b(\cancel{x} + h) - \cancel{ax^3} - \cancel{bx}}{h}$$

$$= \lim_{h \to 0} \frac{a(x^2 h + xh^2 + h^3) + bh}{h}$$

$$= \lim_{h \to 0} ax^2 + xh + h^2 + b = ax^2 + b$$

b) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{c(x+h)+d} - \sqrt{cx+d}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{c(x+h)+d} - \sqrt{cx+d}}{h} \cdot \frac{\sqrt{c(x+h)+d} + \sqrt{cx+d}}{\sqrt{c(x+h)+d} + \sqrt{cx+d}}$$

$$= \lim_{h \to 0} \frac{c(x+h) + d - (cx+d)}{h(\sqrt{c(x+h)+d} + \sqrt{cx+d})}$$

$$= \lim_{h \to 0} \frac{ch}{h(\sqrt{c(x+h)+d} + \sqrt{cx+d})} = \lim_{h \to 0} \frac{c}{\sqrt{c(x+h)+d} + \sqrt{cx+d}}$$

$$= \frac{c}{\sqrt{cx+d} + \sqrt{cx+d}} = \frac{c}{2\sqrt{cx+d}}$$

c) $$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sqrt{e(x+h)+f}} - \frac{1}{\sqrt{ex+f}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\sqrt{ex+f} - \sqrt{e(x+h)+f}}{\sqrt{e(x+h)+f}\,\sqrt{ex+f}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\left(\sqrt{ex+f} - \sqrt{e(x+h)+f}\right)\left(\sqrt{ex+f} + \sqrt{e(x+h)+f}\right)}{\sqrt{e(x+h)+f}\,\sqrt{ex+f}\,\left(\sqrt{ex+f} + \sqrt{e(x+h)+f}\right)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{(ex+f) - (e(x+h)+f)}{\sqrt{e(x+h)+f}\,\sqrt{ex+f}\,\left(\sqrt{ex+f} + \sqrt{e(x+h)+f}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \frac{-e\cancel{h}}{\sqrt{e(x+h)+f}\,\sqrt{ex+f}\,\left(\sqrt{ex+f} + \sqrt{e(x+h)+f}\right)}$$

$$= \lim_{h \to 0} \frac{-e}{\sqrt{e(x+h)+f}\,\sqrt{ex+f}\,\left(\sqrt{ex+f} + \sqrt{e(x+h)+f}\right)}$$

$$= \frac{-e}{\sqrt{ex+f} \cdot \sqrt{ex+f}\,\left(\sqrt{ex+f} + \sqrt{ex+f}\right)}$$

$$= \frac{-e}{2(ex+f)^{3/2}}$$

6. $f'(x) = \frac{d}{dx}[(x-a)](x-b)(x-c)$

$+ (x-a)\frac{d}{dx}[(x-b)](x-c) + (x-a)(x-b)\frac{d}{dx}[(x-c)]$

$= (x-b)(x-c) + (x-a)(x-c) + (x-b)(x-c)$

Then,

$\frac{f'(x)}{f(x)} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)}$

$= \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$

7 a) $\frac{d}{dx}(\cos 2x) = \frac{d}{dx}[\cos^2 x - \sin^2 x]$

$-2\sin 2x = 2\cos x.(-\sin x) - 2\sin x.\cos x$

$\sin 2x = \sin x.\cos x + \sin x.\cos x$

$= 2\sin x.\cos x$

b) $\frac{d}{da}[\sin(a+b)] = \frac{d}{da}[\sin a.\cos b + \cos a.\sin b]$

$= \cos a.\cos b - \sin a.\sin b$