Solutions - Practical 6

1) One way is to use the definition;

Another - easier way - is to use the Proposition

Proposition

Let

$$S = span(v_1, v_2, \ldots, v_k)$$

be the span of k vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$. Then S is a vector subspace of the vector space $\mathbb{R}^n$.

So,

$$S = \left\{ \begin{bmatrix} x \\ x-y \\ x+2y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ -y \\ 2y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right)$$

Hence, S is a subspace of $\mathbb{R}^3$. The vectors $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ is a basis for S if they span S, which they do, and are linearly independent. Let

$$c_1 u + c_2 v = 0 \quad , \quad c_1, c_2 \text{ scalars},$$

Then, $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_1 - c_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\rightsquigarrow$ which yields that $c_1 = 0$ and $c_2 = 0$.

2)(a) $S_1 = \{ x \text{ in } \mathbb{R}^3 : x_1 = x_2 = x_3 \} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

By Proposition, the span of a set of vectors is a vector subspace of the larger space, where the vectors belong.

Here, we have the span of one vector, so it forms a basis for S_1 and $\dim S_1 = 1$.

(b) $S_2 = \{ x \text{ in } \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

Again, by Proposition, the span of these two vectors is a vector subspace of $\mathbb{R}^3$.
Also, $\dim S_2 = 2$.

(c) $S_3 = \{ x \text{ in } \mathbb{R}^3 : x_1 \geq 0 \}$.

We have that the zero vector belongs in S_3, however $c \cdot u = \begin{bmatrix} c x_1 \\ c x_2 \\ c x_3 \end{bmatrix}$ does not always belong to S_3. For example, when $c=-1$, we have:

$$-u = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}$$ that does not belong to S_3.

3) By Gauss-Jordan elimination method:

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 4 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 4 & 1 & 0 \end{array}\right]}_{\text{Red. Row Echelon Form}}$$

Hence, the system has infinitely many solutions:

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 4x_3 + x_4 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -4x_3 - x_4 \end{cases}$$

The solution set is:

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ in } \mathbb{R}^4 : x_1 = x_3 + 2x_4 ,\ x_2 = -4x_3 - x_4 \right\}$$

$$S = \left\{ \begin{bmatrix} x_3 + 2x_4 \\ -4x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_3 \\ -4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} : x_3, x_4 \text{ in } \mathbb{R} \right\}$$

$$= \left\{ x_3 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \text{ in } \mathbb{R} \right\} = \text{span}\Bigg(\underbrace{\begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}}_{= u}, \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{= v} \Bigg)$$

So, u and v span S. If u and v are also linearly independent, they form basis for S. Let $c_1 u + c_2 v = 0$, c_1, c_2 scalars

then
$$c_1 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so
$$\begin{bmatrix} c_1 + 2c_2 \\ -4c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
which yields $c_1 = c_2 = 0$.

4) The vectors u, v, w are a basis for $\mathbb{R}^3$ *when the are linearly independent, ...by*

Theorem

Any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

So, let $c_1 u + c_2 v + c_3 w = 0$ for some unknown scalars.

Then,
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $\rightsquigarrow$ homogeneous system

$$A \cdot X = 0$$

If $\det A \neq 0$, then it has the unique solution $X = 0$

It is $\det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$

$$= (-2+1) - 2(-2-2) + (-1-2)$$

$$= -1 + 8 - 3 = 4 \neq 0$$

Hence, $c_1 = c_2 = c_3 = 0$ *and therefore the vectors u, v, w are linearly independent.*

5) Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Column space: $\mathrm{col}(A) = \mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

These vectors span the column space. If, in addition, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, they consist a basis for $\mathrm{col}(A)$. It is therefore enough to show that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ implies for the scalars that $c_1 = c_2 = c_3 = 0$.

This homogeneous system can be re-written as

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Its determinant is equal to

$$\det\begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix} = 1\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix} - 1\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix} + (-1)\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix} = (1-(-1)) - (-1-1) - (1-1) = 4$$

Moreover, it is $\dim(\mathrm{col}(A)) = 3$.

Row space: $\mathrm{row}(A) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where

$$\mathbf{v}_1 = \begin{bmatrix}1\\ 1\\ -1\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}-1\\ 1\\ 1\end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

We don't need to test again the linear independence of the vectors. We know that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent because the column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ were independent. (the linear system here has matrix the transpose of A and $\det A^T = \det A = 4$). So, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathrm{row}(A)$.
Moreover, it is $\dim(\mathrm{row}(A)) = 3$.

Null space: $\mathrm{null}(A)$ is the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. The system

$$Ax = \begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

has unique solution the trivial solution $x = y = z = 0$ because A is invertible ($\det A = 4$). So, $\mathrm{null}(A) = \mathbf{0}$ and $\mathrm{nullity}(A) = 0$.

It is $\mathrm{rank}(A) + \mathrm{nullity}(A) = \dim(\mathrm{col}(A)) + \mathrm{nullity}(A) = 3 + 0 = 3$