# Algebra – Practical session 5

**5.1.** Simplify the fraction
$$\frac{4x^3-3x+9}{2x^3+x^2-3x+6},$$
by finding the greatest common divisor of numerator and denominator by means of the Euclidean algorithm, and then dividing both numerator and denominator by it.

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**Solution:** The Euclidean algorithm is:
$$\begin{aligned} 4x^3-3x+9 &= (2x^3+x^2-3x+6)\cdot 2+(-2x^2+3x-3) \\ 2x^3+x^2-3x+6 &= (2x^2-3x+3)(x+2) \end{aligned}$$

So we find that the greatest common divisor is $2x^2-3x+3$. Dividing both numerator and denominator by it we find $4x^3-3x+9=(2x^2-3x+3)(2x+3)$ (by long division), and $2x^3+x^2-3x+6=(2x^2-3x+3)(x+2)$ (already found as part of the Euclidean algorithm). In conclusion we find
$$\frac{4x^3-3x+9}{2x^3+x^2-3x+6}=\frac{(2x^2-3x+3)(2x+3)}{(2x^2-3x+3)(x+2)}=\frac{2x+3}{x+2}.$$

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**5.2.** Find polynomials $u(x)$ and $v(x)$ such that $(2x^2+x+3)\cdot u(x)+(x^2+1)\cdot v(x)=1$.

(Apply Euclid's algorithm to $f(x)=2x^2+x+3$ and $g(x)=x^2+1$, check that their gcd is 1 (otherwise such $u(x)$ and $v(x)$ would not exist), and then do the extended part of the algorithm to find $u(x)$ and $v(x)$.)

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**Solution:** The Euclidean algorithm:
$$\begin{aligned} 2x^2+x+3 &= (x^2+1)\cdot 2+(x+1) \\ x^2+1 &= (x+1)\cdot(x-1)+2. \end{aligned}$$

hence the GCD is 2, or we may say that the GCD is 1. Hence the required polynomials exist because of Bézout's Lemma.

To find them we do the extended part of the Euclidean algorithm:
$$\begin{aligned} 2 &= (x^2+1)-(x+1)\cdot(x-1) \\ &= (x^2+1)+[-(2x^2+x+3)+(x^2+1)\cdot 2]\cdot(x-1) \\ &= (2x^2+x+3)\cdot(-x+1)+(x^2+1)(2x-1) \end{aligned}$$

So we have found $u(x)=-\frac{1}{2}x+\frac{1}{2}$ and $v(x)=x-\frac{1}{2}$.

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**5.3.** Find polynomials $u(x)$ and $v(x)$ such that $(x^3+x+1)\cdot u(x)+(x^2-x-1)\cdot v(x)=1$.

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**Solution:** The Euclidean algorithm:
$$\begin{aligned} x^3+x+1 &= (x^2-x-1)\cdot(x+1)+(3x+2) \\ x^2-x-1 &= (3x+2)\cdot\left(\frac{1}{3}x-\frac{5}{9}\right)+\frac{1}{9}. \end{aligned}$$

hence the GCD is 1/9, or we may say that the GCD is 1. Hence the required polynomials will exist because of Bézout's Lemma.

To find them we do the extended part of the Euclidean algorithm:

$$\begin{aligned}1 &= (x^2 - x - 1) \cdot 9 - (3x + 2) \cdot (3x - 5)\\ &= (x^2 - x - 1) \cdot 9 - [(x^3 + x + 1) - (x^2 - x - 1) \cdot (x + 1)] \cdot (3x - 5)\\ &= (x^3 + x + 1) \cdot (-3x + 5) + (x^2 - x - 1) \cdot [(x + 1)(3x - 5) + 9]\\ &= (x^3 + x + 1) \cdot (-3x + 5) + (x^2 - x - 1) \cdot (3x^2 - 2x + 4)\end{aligned}$$

So we have found $u(x) = -3x + 5$ and $v(x) = 3x^2 - 2x + 4$.

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**5.4.** Factorise the integer $10^6 - 3^6 = 999271$ into a product of primes. (You will need to show that the factors which you find are actually primes.)

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**Solution:** We have

$$\begin{aligned}10^6 - 3^6 &= (10^3 - 3^3) \cdot (10^3 + 3^3)\\ &= (10 - 3) \cdot (10^2 + 10 \cdot 3 + 3^2) \cdot (10 + 3) \cdot (10^2 - 10 \cdot 3 + 3^2)\\ &= 7 \cdot 139 \cdot 13 \cdot 79\end{aligned}$$

Now 2 and 13 are clearly primes. But 79 is also prime because it is not a multiple of any of 2, 3, 5, 7 (which are all primes whose squares do not exceed 79). Similarly, 139 is prime because it is not a multiple of any of 2, 3, 5, 7, 11.

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**5.5.** Consider the polynomial $f(x) = x^4 + 2x^3 - 6x^2 + 8x + 80$.

(a) Check that $2 + 2i$ is a root of $f(x)$. (Use Ruffini's rule.)

(b) Using the information in part (a), find the remaining complex roots of $f(x)$, and write the complete factorisation of $f(x)$ in $\mathbb{C}[x]$.

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**Solution:** (a) The quickest way to compute $f(2 + 2i)$ is applying Ruffini's rule to divide $f(x)$ by $x - 2 - 2i$, as $f(2 + 2i)$ is precisely the remainder of the division:

$$\begin{array}{r|ccccc|c} & 1 & 2 & -6 & 8 & & 80\\ 2+2i & & 2+2i & 4+12i & -28+20i & & -80\\ \hline & 1 & 4+2i & -2+12i & -20+20i & & 0\end{array}$$

This confirms that $f(2 + 2i)$ is a root of $f(x)$, so $2 + 2i$ is a root.

(b) Given that $2 + 2i$ is a root of $f(x)$, its conjugate $2 - 2i$ must be a root as well. Hence $2 - 2i$ must be a root of the quotient of the previous division, and applying Ruffini's rule again we find

$$\begin{array}{r|ccc|c} & 1 & 4+2i & -2+12i & -20+20i\\ 2-2i & & 2-2i & 12-12i & 20-20i\\ \hline & 1 & 6 & 10 & 0\end{array}$$

Hence $f(x) = (x - 2 - 2i)(x - 2 + 2i)(x^2 + 6x + 10)$. Finally, the roots of the quadratic factor can easily be found by using the usual formula, and the complete factorisation of $f(x)$ in $\mathbb{C}[x]$ is

$$f(x) = (x - 2 - 2i)(x - 2 + 2i)(x + 3 - i)(x + 3 + i).$$

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**5.6.** Use Ruffini's rule or other means to compute the quotient and the remainder of dividing $x^{100} + 2x^6 + 1$ by $x - 1$.

*Note:* You need not write down individually ALL the terms of the quotient, but you must make it clear that you have found them, and how. (For example, make appropriate use of dots in your calculation and in your final answer.)

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**Solution:** Ruffini's rule reads

$$\begin{array}{c|cccccccccccc|c} & 1 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & & 1 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline & 1 & 1 & 1 & 1 & \cdots & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 4 \end{array}$$

Hence the quotient is

$$q(x) = x^{99} + x^{98} + \cdots + x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 3x + 3$$

and the remainder is $r = 4$. Another way to express this is

$$x^{100} + 2x^6 + 1 = (x-1)(x^{99} + x^{98} + \cdots + x^6 + 3x^5 + 3x^4 + 3x^2 + 3x + 3) + 4,$$

which exhibits the correct quotient and remainder.

Another possible way of answering the question without writing down the whole Ruffini's rule was

$$\begin{aligned} x^{100} + 2x^6 + 1 &= (x^{100} - 1) + 2(x^6 - 1) + 4 \\ &= (x-1)(x^{99} + \cdots + x + 1) + 2(x-1)(x^5 + \cdots + x + 1) + 4 \\ &= (x-1)(x^{99} + \cdots + x^6 + 3x^5 + 3x^4 + 3x^2 + 3x + 3) + 4. \end{aligned}$$

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**5.7.** Find the unique polynomial $f(x)$ of degree at most 3 such that

$$f(-1) = -7, \qquad f(0) = -3, \qquad f(1) = -3, \qquad f(2) = -1.$$

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**Solution:** Writing $f(x) = ax^3 + bx^2 + cx + d$ we have

$$\begin{cases} -a + b - c + d &= -7 \\ d &= -3 \\ a + b + c + d &= -3 \\ 8a + 4b + 2c + d &= 1 \end{cases}$$

Solving this system we find

$$\begin{cases} d &= -3 \\ -a + b - c &= -4 \\ a + b + c &= 0 \\ 4a + 2b + c &= 1 \end{cases} \qquad \begin{cases} d &= -3 \\ b &= -2 \\ a + c &= 2 \\ 4a + c &= 5 \end{cases} \qquad \begin{cases} a &= 1 \\ b &= -2 \\ c &= 1 \\ d &= -3 \end{cases}$$

In conclusion, we find $f(x) = x^3 - 2x^2 + x - 3$.

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**5.8*.** Consider a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$.

(a) Show that the sum of all coefficients of even powers of $x$ in $f(x)$ (which means the coefficients of $x^0 = 1$, $x^2$, $x^4$, etc.) equals $\frac{1}{2}\big(f(1) + f(-1)\big)$.

(b) Find a similar expression for the sum of all coefficients of odd powers of $x$ in $f(x)$.

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**Solution:** (a) We have

$$\begin{aligned} f(1) &= a_0 + a_1 1 + a_2 1^2 + \cdots + a_n 1^n \\ &= a_0 + a_1 + a_2 + \cdots + a_n, \qquad \text{and} \\ f(-1) &= a_0 + a_1(-1) + a_2(-1)^2 + \cdots + a_n(-1)^n \\ &= a_0 - a_1 + a_2 - \cdots + (-1)^n a_n \end{aligned}$$

Adding up the two expressions we get $f(1) + f(-1) = 2(a_0 + a_2 + a_4 + \cdots)$, and the conclusion follows.

(b) Taking the difference instead of the sum, and dividing by 2, we find $a_1 + a_3 + a_7 + \cdots = \frac{1}{2}\big(f(1) - f(-1)\big)$.

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**5.9*.** Compute the sum of the coefficients of the polynomial

$$(2x - 1)(3x^2 - 2)(4x^3 - 3)(5x^4 - 4)(6x^5 - 5)$$

without bringing it to normal form (that is, without explicitly computing the product).

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**Solution:** The sum of the coefficients of a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is $a_n + a_{n-1} + \cdots + a_1 + a_0 = a_n 1^n + a_{n-1} 1^{n-1} + \cdots + a_1 1 + a_0 = f(1)$. Hence if $f(x)$ is the given polynomial we have

$$f(1) = (2 \cdot 1 - 1)(3 \cdot 1^2 - 2)(4 \cdot 1^3 - 3)(5 \cdot 1^4 - 4)(6 \cdot 1^5 - 5) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

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