Algebra – Practical session 9

9.1. Solve each of the following congruences:

(a) $6x \equiv 10 \pmod{19}$;
(b) $6x \equiv 10 \pmod{20}$;
(c) $6x \equiv 10 \pmod{21}$.

Note that *solving* means finding *all solutions,* not just one solution. This includes the possibility that there are no solutions (in which case the set of solutions is empty).

9.2. (a) Find a positive integer x such that $44x$ has 92 as its last two decimal digits.
(b) Now find all integers $0 < x < 100$ such that $44x$ has 92 as its last two decimal digits.

9.3. For each of the following systems of congruences, decide if there are any solutions, and in the affirmative case find all solutions.

(a)
$$\begin{cases} x \equiv 3 & \pmod{12} \\ x \equiv 5 & \pmod{16} \end{cases}$$

(b)
$$\begin{cases} x \equiv 3 & \pmod{13} \\ x \equiv 5 & \pmod{17} \end{cases}$$

(c)
$$\begin{cases} x \equiv 3 & \pmod{14} \\ x \equiv 5 & \pmod{18} \end{cases}$$

9.4. (a) Write the multiplication table of $\mathbb{Z}/11\mathbb{Z}$.
(b) Use the table to check that every class $[a] \neq [0]$ in $\mathbb{Z}/11\mathbb{Z}$ is invertible, and for each such class write its inverse $[a]^{-1}$.
(c) Check that $[a]^{10} = [1]$ for every non-zero class $[a] \in \mathbb{Z}/11\mathbb{Z}$. (You may use the table to compute the power.)
(d) Find all classes $[a] \in \mathbb{Z}/11\mathbb{Z}$ such that $[a]^5 = [1]$.

9.5. Show that $[14]$ is invertible in $\mathbb{Z}/69\mathbb{Z}$, and find its inverse.
Hint: Do not write the multiplication table of $\mathbb{Z}/69\mathbb{Z}$, use the theory instead.

Additional questions, for home practice

9.6. Find all solutions of the system of congruences
$$\begin{cases} x \equiv 2 & \pmod{5} \\ x \equiv 3 & \pmod{7} \\ x \equiv 5 & \pmod{11} \end{cases}$$

9.7. Find how many positive integers, less than ten thousand, have 7 as their right-most digit (in decimal notation), and give remainder 1 when divided by 12.

9.8. Compute $[3]^k$ in $\mathbb{Z}/11\mathbb{Z}$, for $0 \leq k \leq 10$. Now compute $[3]^{333}$ in $\mathbb{Z}/11\mathbb{Z}$.
Hint: After answering the first question note that those powers repeat after a number of steps.