1. Dimensions strike back

Using the bracket notation [.] give the dimension of either side of the following equations and conclude on whether or not they correspond to **complete** equations. Note: in the equations below, x and y are components of the position vector, t is a time interval and v_x and v_y are the components of the velocity vector.

> Reminder: Here are the rules we have established so far:
> - $[x] = [y] = L$
> - $[u] = L$ if $u =$ inches, cm, m, yards etc...
> - $[v_x] = [v_y] = L \times T^{-1}$
> - $[\theta] = 1$ if θ is an angle
> - $[u_a] = 1$ if $u_a =$ degrees, radians, seconds of arc etc..
> - $[t] = T$
> - $[u_t] = T$ if $u_t =$ seconds, hours, days, years etc...
> - $[n] = 1$ where n can be any real number like -1, 3.45674 or 5 for example.
> - $[A \times B] = [A] \times [B]$ for any A and B
> - $[A/B] = [A]/[B]$ for any A and B
> - $[A + B] = [A] = [B]$ if $[A] = [B]$

(a) $\sqrt{x^2 + y^2} = 2\,\text{m}$

(b) $\frac{v_x^2}{y} = (63\,\text{km} \cdot \text{h}^{-3})\, t$

(c) $y^{-1/2}\sqrt{v_x^3} = (12\,\text{cm})\, t^{3/2}$

(d) $\frac{\sqrt{v_x^2}}{v_y} = 3$

(e) 1 light year $= (1.17\,\text{rad})\, x$

(f) $\theta^2 v_x^2 = \frac{y^2}{t^2}$

(a) $\sqrt{x^2+y^2} = 2\,m$

$[\sqrt{x^2+y^2}] = [x^2+y^2]^{1/2} = [x]^{2/2} = L$

$[2m] = \underbrace{[2]}_{1}\underbrace{[m]}_{L} = L$

$[LHS] = [RHS]$, $\therefore$ complete equation

(b) $\dfrac{v_x^2}{y} = (63\ \mathrm{km} \cdot \mathrm{h}^{-3})\, t$

$\left[\dfrac{v_x^2}{y} \right] = [v_x]^2 [y]^{-1} = L^2 \cdot T^{-2} \cdot L^{-1} = L \cdot T^{-2}$

$[(63\ \mathrm{km} \cdot \mathrm{h}^{-3})\, t] = \underbrace{[63]}_{1} \underbrace{[\mathrm{km}]}_{L} \underbrace{[\mathrm{h}]^{-3}}_{T^{-3}} \underbrace{[t]}_{T} = L \cdot T^{-2}$

[LHS] = [RHS], ∴ complete equation

(c) $y^{-1/2} \sqrt{v_x^3} = (12\ \mathrm{cm})\, t^{3/2}$

$[y^{-1/2} \sqrt{v_x^3}] = \underbrace{[y]^{-1/2}}_{L^{-1/2}} \underbrace{[v_x]^{3/2}}_{L^{3/2}\, T^{-3/2}} = L \cdot T^{-3/2}$

$[(12\ \mathrm{cm})\, t^{3/2}] = \underbrace{[12]}_{1} \underbrace{[\mathrm{cm}]}_{L} \underbrace{[t]^{3/2}}_{T^{3/2}} = L \cdot T^{3/2}$

[LHS] ≠ [RHS], ∴ not a complete equation

(d) $\dfrac{\sqrt{v_x^2}}{v_y} = 3$.

$\left[\dfrac{\sqrt{v_x^2}}{v_y} \right] = [v_x]^{\frac{2}{2}} [v_y]^{-1} = 1$

$[3] = 1$

[LHS] = [RHS], ∴ complete equation

Continuation Q1

(e) 1 light year $= (1.17 \text{ rad})\, x$

$$[1 \text{ light year}] = \underbrace{[1]}_{1}\underbrace{[\text{light year}]}_{L} = L$$

$$[(1.17 \text{ rad})\, x] = \underbrace{[1.17]}_{1}\underbrace{[\text{rad}]}_{1}\underbrace{[x]}_{L} = L$$

[LHS] = [RHS], ∴ complete equation

(f) $\theta^2 v_x^2 = \dfrac{y^2}{t^2}$

$$[\theta^2 v_x^2] = \underbrace{[\theta]}_{1}{}^2\underbrace{[v_x]}_{(L\cdot T^{-1})^2}{}^2 = L^2 \cdot T^{-2}$$

$$\left[\frac{y^2}{t^2}\right] = \underbrace{[y]^2}_{L^2}\underbrace{[t]^{-2}}_{T^{-2}} = L^2 \cdot T^{-2}$$

[LHS] = [RHS], ∴ complete equation

Find the velocity and the relative position (in 1D) by integrating the following accelerations given that $v_x(t=0)=0$ and $x(t=0)=0$. Optional question: try to guess which type of physical conditions these accelerations correspond to.

(a) $a_x(t)=a_0$, with a_0 a constant quantity.

(b) $a_x(t)=\gamma\, t$, with γ a constant quantity.

(c) $a_x(t)=\frac{v_\infty}{\tau}e^{-t/\tau}$, with v_∞ and τ being constants.

(d) $a_x(t)=a_0\cos(\omega t)$, with a_0 and ω being constants.

(a) $a_x(t) = a_0 = \text{constant}$ (e.g. free fall)

- $a_x(t) \equiv \dot{v}_x(t)$

$$\Rightarrow \underbrace{\int_0^t a_0\, dt'}_{a_0 \times (t-0)} = \underbrace{\int_0^t \dot{v}_x(t')\, dt'}_{v_x(t) - \underbrace{v_x(0)}_{=0 \text{ from given information}} \text{ from FTC}}$$

$\therefore$ $v_x(t) = a_0 t$

- $v_x(t) \equiv \dot{x}(t)$

$$\Rightarrow \underbrace{\int_0^t a_0 t'\, dt'}_{\frac{1}{2} a_0 \times (t^2 - 0^2)} = \underbrace{\int_0^t \dot{x}(t')\, dt'}_{x(t) - \underbrace{x(0)}_{=0 \text{ from information given}} \text{ from FTC}}$$

$\therefore$ $x(t) = \frac{1}{2} a_0 t^2$

(b) $a_x(t) = \gamma t$ with γ constant (e.g. jet)

• $a_x(t) \equiv \dot{v}_x(t)$

$$\Rightarrow \underbrace{\int_0^t \gamma t' dt'}_{\frac{1}{2}\gamma(t^2 - 0^2)} = \underbrace{\int_0^t \dot{v}_x(t') dt'}_{v_x(t) - \underbrace{v_x(0)}_{=0 \text{ from given information}}}$$

$$\therefore v_x(t) = \frac{1}{2}\gamma t^2$$

• $v_x(t) \equiv \dot{x}(t)$

$$\Rightarrow \underbrace{\int_0^t \frac{1}{2}\gamma t'^2 dt'}_{\frac{1}{6}\gamma(t^3 - 0^3)} = \underbrace{\int_0^t \dot{x}(t') dt'}_{x(t) - \underbrace{x(0)}_{=0 \text{ from given information}}}$$

$$\therefore x(t) = \frac{1}{6}\gamma t^3$$

(c) $a_x(t) = \frac{v_\infty}{\tau} e^{-t/\tau}$, v_∞ and τ positive constants

• $a_x(t) \equiv \dot{v}_x(t)$ (e.g. fall with fluid friction)

$$\Rightarrow \underbrace{\int_0^t \frac{v_\infty}{\tau} e^{-\frac{t'}{\tau}} dt'}_{\left[-v_\infty e^{-\frac{t'}{\tau}}\right]_0^t = v_\infty\left(1 - e^{-\frac{t}{\tau}}\right)} = \underbrace{\int_0^t \dot{v}_x(t') dt'}_{v_x(t) - \underbrace{v_x(0)}_{=0 \text{ from given information}}}$$

Continuation Q2(c)

$$\therefore \quad v_x(t) = v_\infty\left(1 - e^{-t/\tau}\right)$$

- $v_x(t) \equiv \dot{x}(t)$

$$\Rightarrow \underbrace{\int_0^t v_\infty\left(1 - e^{-\frac{t'}{\tau}}\right) dt'}_{\left[v_\infty t' + v_\infty \tau e^{-\frac{t'}{\tau}}\right]_0^t = v_\infty\left(t - \tau + \tau e^{-t/\tau}\right)} = \underbrace{\int_0^t \dot{x}(t')\,dt'}_{x(t) - \underbrace{x(0)}_{=0}}$$

$=0$ from given information

$$\therefore \quad x(t) = v_\infty\left(t - \tau\left(1 - e^{-\frac{t}{\tau}}\right)\right)$$

(d) $a_x(t) = a_0 \cos(\omega t)$, a_0 and ω constant

- $a_x(t) \equiv \dot{v}_x(t)$ (e.g. harmonic oscillator)

$$\Rightarrow \underbrace{\int_0^t a_0 \cos(\omega t')\,dt}_{\left[\frac{a_0}{\omega}\sin(\omega t')\right]_0^t = \frac{a_0}{\omega}\sin(\omega t)} = \underbrace{\int_0^t \dot{v}_x(t')\,dt'}_{v_x(t) - \underbrace{v_x(0)}_{=0}}$$

$=0$ from given information

$$\therefore \quad v_x(t) = \frac{a_0}{\omega}\sin(\omega t)$$

Continuation Q2 (d)

- $v_x(t) \equiv \dot{x}(t)$

$$\Rightarrow \underbrace{\int_0^t \frac{a_0}{\omega} \sin(\omega t')\,dt'}_{\left[-\frac{a_0}{\omega^2}\cos(\omega t')\right]_0^t = \frac{a_0}{\omega^2}\left(1-\cos(\omega t)\right)} = \underbrace{\int_0^t \dot{x}(t')\,dt'}_{x(t) - \underbrace{x(0)}_{=0 \text{ from given information}}}$$

$$\therefore \quad x(t) = \frac{a_0}{\omega^2}\left(1-\cos(\omega t)\right)$$

Give the instantaneous velocity and acceleration vectors in the orthonormal frame $(O, \hat{\imath}, \hat{\jmath})$ given the following position vectors in that frame:

(a) $\vec{r}(t) = (10\,\mathrm{m}\cdot\mathrm{s}^{-1})\, t\ \hat{\imath} + (-2.5\,\mathrm{m}\cdot\mathrm{s}^{-2})\, t^2\ \hat{\jmath}$

(b) $\vec{r}(t) = R\cos(\omega t)\,\hat{\imath} + R\sin(\omega t)\,\hat{\jmath}$

(c) $\vec{r}(t) = R(\omega t - \sin(\omega t))\,\hat{\imath} + R(\omega t - \cos(\omega t)\,\hat{\jmath}$

(a) $\vec{r}(t) = (10\,\mathrm{m}\cdot\mathrm{s}^{-1})\, t\,\hat{\imath} + (-2.5\,\mathrm{m}\cdot\mathrm{s}^{-2})\, t^2\,\hat{\jmath}$

$$\vec{v}(t) \equiv \dot{\vec{r}}(t) = (10\,\mathrm{m}\cdot\mathrm{s}^{-1})\,\hat{\imath} + (-5\,\mathrm{m}\cdot\mathrm{s}^{-2})\,t\,\hat{\jmath}$$

$$\vec{a}(t) \equiv \dot{\vec{v}}(t) = (-5\,\mathrm{m}\cdot\mathrm{s}^{-2})\,\hat{\jmath}$$

In the above we can simply differentiate each component independently of the others.

(b) $\vec{r}(t) = R\cos(\omega t)\,\hat{\imath} + R\sin(\omega t)\,\hat{\jmath}$

$$\vec{v}(t) \equiv \dot{\vec{r}}(t) = \frac{d}{dt}\Big(R\cos(\omega t)\Big)\,\hat{\imath} + \frac{d}{dt}\big(R\sin(\omega t)\big)\,\hat{\jmath}$$

$$\vec{v}(t) = -R\omega\sin(\omega t)\,\hat{\imath} + R\omega\cos(\omega t)\,\hat{\jmath}$$

$$\vec{a}(t) = \dot{\vec{v}}(t) = -R\omega^2\cos(\omega t)\,\hat{\imath} - R\omega^2\sin(\omega t)\,\hat{\jmath}$$

$$\vec{a}(t) = -\omega^2\,\vec{r}(t)$$

Continuation Q3

(c) $\vec{r}(t) = R(\omega t - \sin(\omega t))\hat{i} + R(\omega t + \cos(\omega t))\hat{j}$

$$\vec{v}(t) \equiv \dot{\vec{r}}(t) = R\omega(1-\cos(\omega t))\hat{i} + R\omega(1-\sin(\omega t))\hat{j}$$

$$\vec{a}(t) \equiv \dot{\vec{v}}(t) = R\omega^2 \sin(\omega t)\,\hat{i} - R\omega^2 \cos(\omega t)\,\hat{j}$$

(a) State the definition of the average velocity in 1D as seen from the lectures.

(b) From the answer of question (a), show that the average velocity $v_{x,avg}$ can also be written as:

$$v_{x,avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v_x(t)\, dt$$

(c) Using the equation given in question (b) show that if the 1D acceleration is constant then we have that the average velocity is equal to the arithmetic average:

$$v_{x,avg} = \frac{1}{2}(v_x(t_1) + v_x(t_2))$$

(a) From the lectures $v_{x,avg} \equiv \dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$.

(b) $v_{x,avg} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$. From FTC we have that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t)\, dt$$

. But since $v_x(t) \equiv \dot{x}(t)$

we get $x(t_2) - x(t_1) = \displaystyle\int_{t_1}^{t_2} v_x(t)\, dt$

$$\therefore \quad v_{x,avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v_x(t)\, dt$$

QED

Continuation Q4

(c) If $a_x(t) = a_x$ then by integration

$$\underbrace{\int_0^t a_x \, dt'}_{a_x \times (t-0)} = \underbrace{\int_0^t \dot{v}_x(t') \, dt}_{v_x(t) - v_x(0)}$$

$$\Rightarrow \quad v_x(t) = v_x(0) + a_x t$$

$$\int_{t_1}^{t_2} v_x(t) \, dt = v_x(0)(t_2 - t_1) + \frac{1}{2} a_x (t_2^2 - t_1^2)$$

now, $t_2^2 - t_1^2 = (t_1 - t_2)(t_1 + t_2)$ so that

$$v_{x,avg} = v_x(0) + \frac{1}{2} a_x (t_2 + t_1)$$

$$v_{x,avg} = \frac{1}{2}\left(v_x(t_2) + v_x(t_1) \right)$$

QED