1) (i) The augmented matrix is already in row echelon form.

We bring the augmented matrix in reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_2/2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$$

That is equivalent to the system:

$$\begin{cases} x = 0 \\ y + \frac{1}{2} z = 0 \end{cases}, \quad \text{so} \quad \begin{cases} x = 0 \\ y = -\frac{1}{2} z \end{cases}$$

The system has infinitely many solutions and its solution set then is:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = 0,\ y = -\frac{1}{2} z,\ z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ -1/2\, z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$= \left\{ z \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} \right)$$

1) (ii) Again the matrix is in row echelon form, and we bring it in reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 3 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_2/3]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right]$$

That corresponds to a system of 2 equations with 4 unknowns:

$$\begin{cases} x + z + 2w = 0 \\ y - \frac{z}{3} - \frac{w}{3} = 0 \end{cases}$$

, so we solve it in terms of 2 variables. Here we choose x and y.

$$\downarrow$$

$$\begin{cases} x = -z - 2w \\ y = \frac{z}{3} + \frac{w}{3} \end{cases}$$

The system has infinitely many solutions and its solution set is:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \ x = -z - 2w, \ y = \frac{z}{3} + \frac{w}{3}, \ z, w \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -z - 2w \\ \frac{z}{3} + \frac{w}{3} \\ z \\ w \end{bmatrix} : z, w \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} -1 \\ 1/3 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} : z, w \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} -1 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} \right).$$

2) *We first write the augmented matrix and we follow the 2 steps: We bring the matrix in its row echelon form and then in its reduced row echelon form.*

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 1 \\ 2 & -4 & 4 & 6 & 0 \\ 1 & 2 & -2 & 3 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{} \left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 1 \\ 2 & -4 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2/2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 1 \\ 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 1 \\ 0 & -4 & 4 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

and this is the row echelon form of the augmented matrix.

$$\xrightarrow{R_2/(-4)} \left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_1 - 2R_3 \\ R_2 + \frac{1}{4}R_3}} \left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Which is the reduced row echelon form of the matrix.

equivalently we write:

$$\begin{cases} x_1 = 3 \\ x_2 - x_3 = 0 \\ x_4 = -1 \end{cases}$$

The solution set then is:

$$S = \left\{ \begin{bmatrix} 3 \\ x_2 \\ x_2 \\ -1 \end{bmatrix} : x_2 \in \mathbb{R} \right\}.$$

Remark: this solution set, due to the constant terms, cannot be expressed as the span of vectors. This can only happen in homogenous systems.

3) (i) To show that the vectors are linearly independent, we need to solve the linear system

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$$

In terms of the scalars c_1, c_2, c_3.

The above vector equation becomes the homogeneous system:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad c_1, c_2, c_3 \in \mathbb{R}$$

That can be represented by the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Which is already in row echelon form

We will bring it in its reduced row echelon form

$$\xrightarrow{\substack{R_1 - R_3 \\ R_2 - R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the solution of the homogeneous system is:

$$c_1 = c_2 = c_3 = 0 .$$

Therefore the vectors are linearly independent.

3) (ii) The vector u belongs to span(v_1, v_2, v_3) when

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3$$

for some scalars c_1, c_2, c_3. This is a non-homogeneous system
In augmented matrix notation it becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So the solution is unique and it is $c_1 = -1$, $c_2 = 0$, $c_3 = 1$.

Let's go back in the initial equation: $-1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 = u$

or $\quad u = v_3 - v_1$

Which can be also verified $\quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

4) We have the matrices:

$$B_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 0 & 0 & -2 & 5 \\ 0 & 1 & 0 & 9 & -2 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix}.$$

B_1 is not in a row echelon form, the last row should be zero;

B_2 is in reduced row echelon form because above and under the pivots 1 (first row) and 1 (second row) the elements are zero, the second pivot is on the right of the first pivot and finally the zero row is the last one;

B_3 is in row echelon form, since under the pivots 1 (first row) and 1 (second row) the elements are zero, the second pivot is on the right of the first pivot and finally the zero row is the last one;

B_4 is in reduced row echelon form like B_2, all three pivots are ordered correctly, and the matrix elements above and below the pivots are equal to zero.