The History & Applications *of* Calculus

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Contents

Introduction

Calculus is a field of mathematics which focuses on the study of infinitesimals. It has taken many years to master, with its precise definitions of limits, derivatives and integrals. This report delves into the history of this study and its applications in the real world.

Chapter 1 (covered by Student 1) discusses the history of integration and how it has been formalised, along with a brief overview of the origin of the concept of. The names of great contributors are mentioned, such as Newton, Leibniz and even Greek mathematician, philosopher and physician, Hippocrates. Useful techniques of integration are demonstrated, including integration by first principles and integration by substitution, with an example of each. This chapter also covers the Fundamental Theorem of Calculus and its importance in the calculation of integrals.

Chapter 2 (covered by Student 2) goes in depth about the history of differentials, and the progress and applications of differential equations. Riemannian geometry and the importance of differential calculus is glossed over as well. Vector calculus is also discussed, along with its applications in meteorology, introducing new terminology, notations and formulae, including the Fundamental Theorem of The Line Integral.

Chapter 3 (covered by Student 3) talks about the applications of calculus in epidemiology, specifically analysing the spread of communicable diseases across a population. Differential equations are applied to the rate at which a disease is spread and solved to formulate different compartments as explicit functions for easy analysis. The author also models these variables and graphs them using Python. Other variables are considered, such as vaccinations, and the limitations of the model are also discussed.

Chapter 4 (covered by Student 4) details how calculus – in particular differentials and their graphical analysis – is used within the economy. The neoclassical theory of profit maximisation led to the development of its mathematical model in which the optimal level of output is calculated using the derivatives of the marginal revenue and marginal cost functions. This chapter focuses in on such, giving an example and discussing its history and variations to the model. Despite this theory being a basis to modern day economics, the limitations of the model are also considered in the following as it appears to be rarely applicable to today's market economy.

Chapter 5 (covered by Student 5) communicates the use of calculus in the study and discovery of celestial bodies. Different laws of mechanics are defined, including Newton's infamous three laws of motion and his law of gravitation, which is discussed in the application of the two-body problem. This chapter also discusses calculus' aid in the discovery of a body in our solar system, Neptune. A brief summary of the applications in astrophysics is given too, in emphasis of the importance of calculus and its advancements towards our modern knowledge about the universe.

(JH, NG)

1 | History: Integration

Student 1

The area of a shape is the exact number of unit squares that fit within the bounds of its perimeter. We have easily been able to find the area of regular shapes for thousands of years, with the concept of area first being recorded in the 5th century BCE by Hippocrates, one of the many great mathematicians of the classical period of Greece. Hippocrates found that the measure of the region within a circle is directly proportional to the square of its radius [1], i.e. $\mathrm{A} = kr^2$ (of course it was later discovered that $k = \pi \approx 3.14159$). Over the years we have found the areas of many 2-dimensional shapes, such as circles, triangles, different families of quadrilaterals, and other polygons and curves. Until the discovery of calculus, namely the Fundamental Theorem of Calculus, the calculation of irregular shapes was tricky.

What we now call 'integration' is the process of breaking down the area bounded by the curve of a function and the axis into rectangles of equal width and taking the sum of their areas. In fact, 'integration' was first referred to as such by Jacob Bernoulli in 1690 (he actually wrote 'integralia', which is Latin for integral) [2]. The modern integral symbol $\int$ was introduced by Leibniz, one of calculus' most significant pioneers, in the 17th century AD. It was an adaptation of the letter ſ which was commonly used in classical Latin texts for a hard, hissing 's' sound. It stands for the word 'ſumma' meaning 'sum' in Latin [3].

A now obsolete notation for the integral of a function was Newton's bar notation [2]. Newton was another mathematician that inspired the development of calculus and he and Leibniz are both credited for the invention/discovery of calculus. The bar notation was a vertical bar above a variable or function, e.g. $\bar{x}$, similar to the symbol for the mean in statistical mathematics. This notation was to be used in contrast to the familiar dot notation, e.g. $\dot{x}$, which expressed the derivative with respect to time (Newton used calculus for his ideas in Physics). The reason for the distinctive notations for the pair of methods was due to the Fundamental Theorem of Calculus, which says that integration and differentiation are inverse operations, proved by both Newton and Leibniz [4].

First Principles & Approximations

Differentiation is a relatively easy operation, in the sense that if a function is differentiable, its derivative can be found by using the relevant rules such as the chain rule or product rule. Integration, on the other hand, is not as simple. Recall that an integral is the sum of the areas of rectangles of equal infinitesimal width to find the total area under a curve. Finding an n^{th} sum can be very difficult for complicated functions, even just the sum of powers. Therefore, before the discovery of the Fundamental Theorem of Calculus, it was incredibly tricky to calculate the exact value of an integral by, what we now call, 'first principles'.

Take the graph of the function $f(x) = x^2$ (*Figure 1*). How would one find the area A under the curve over the interval $[0, 2]$? First, let's split up the graph into rectangles of equal width (*Figure 2*). Now we can estimate A to be the total area of the 5 rectangles (the 1st rectangle has height 0):

$$A \approx \sum_{x=1}^{5} \frac{2}{5} f\left(\frac{2}{5}(x-1)\right) = \sum_{x=1}^{5} \frac{2}{5}\left(\frac{2}{5}(x-1)\right)^2 = \frac{48}{25} = 1.92$$

Equation 1 – Initial integral approximation

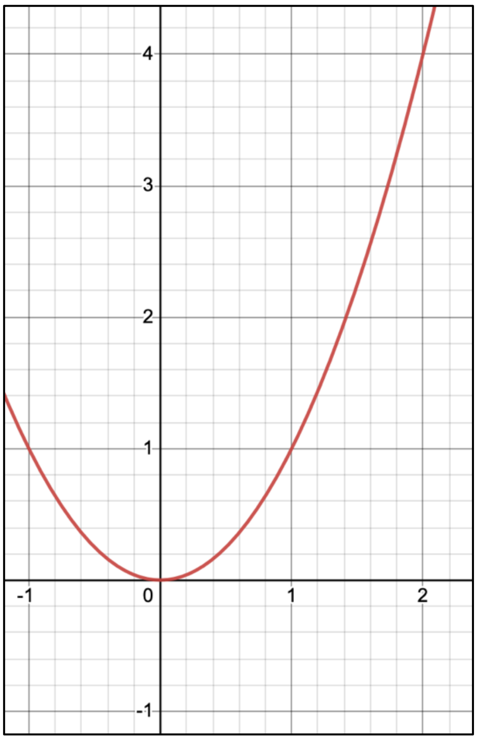

Figure 1 – Graph of f(x)

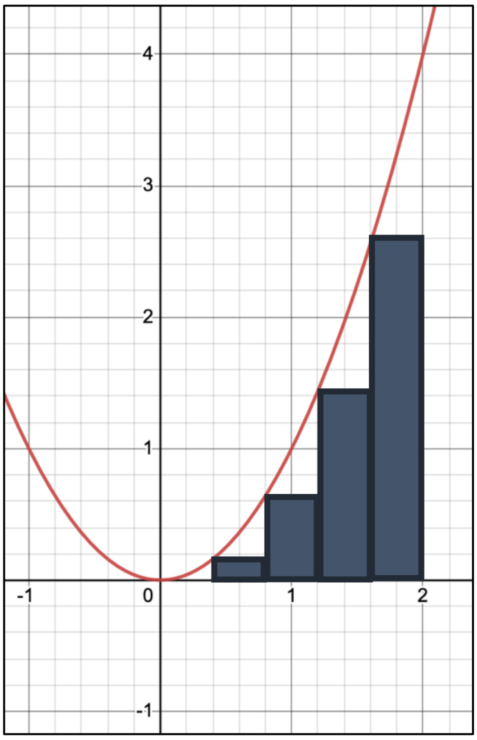

Figure 2 – Region separated into 5 rectangles

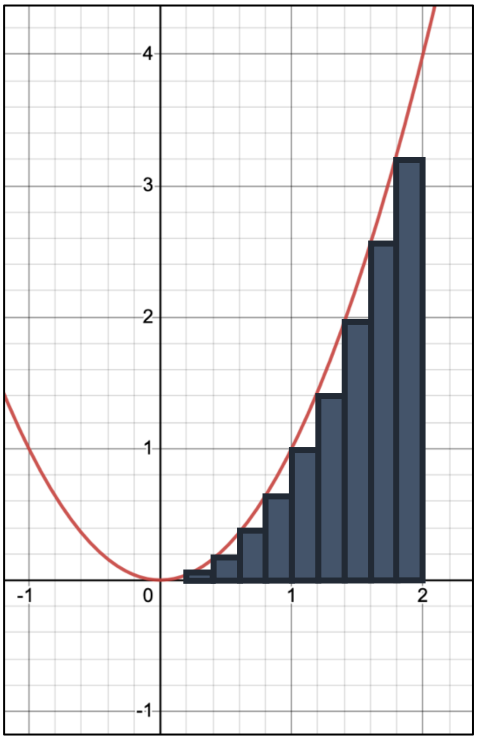

Figure 3 – Region separated into 10 rectangles

These graphs were created using Desmos [5]

If we increase the number of rectangles n, the approximation gets closer to the true value. Let's double n (*Figure 3*) so that there are 10 rectangles in total. Now we estimate:

$$A \approx \sum_{x=1}^{10} \frac{2}{10} f\left(\frac{2}{10}(x-1)\right) = \sum_{x=1}^{10} \frac{1}{5}\left(\frac{1}{5}(x-1)\right)^2 = \frac{57}{25} = 2.28$$

Equation 2 – Second integral approximation

In general, when we divide the region into n rectangles, the approximation is the total area of the rectangles which is given by:

$$A \approx \sum_{x=1}^{n} \frac{2}{n}\left(\frac{2}{n}(x-1)\right)^2$$
$$= \frac{2}{n}\sum_{x=1}^{n} \frac{4}{n^2}(x-1)^2$$
$$= \frac{8}{n^3}\sum_{x=1}^{n}(x^2 - 2x + 1)$$
$$= \frac{8}{n^3}\left(\sum_{x=1}^{n} x^2 - 2\sum_{x=1}^{n} x + \sum_{x=1}^{n} 1\right)$$
$$= \frac{8}{n^3}\left(\left(\frac{n(n+1)(2n+1)}{6}\right) - 2\left(\frac{n(n+1)}{2}\right) + (n)\right)$$
$$= \frac{8}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6} - n^2 - n + n\right)$$
$$= \frac{8}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6} - \frac{6n^2}{6}\right)$$
$$= \frac{8}{n^3}\left(\frac{2n^3 - 3n^2 + n}{6}\right)$$
$$= \frac{8}{n^3}\left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right)$$
$$= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

Equation 3 – General integral approximation for n rectangles

At this point, we can substitute larger and larger values for n into the formula and get a very good approximation for the area under the parabola in *Figure 1*. Before the discovery of calculus, mathematicians relied solely on their intuition to evaluate how the value of an expression changes as a number gets smaller or larger. In *Equation 3*, as n increases the fractions with n in the denominator become smaller and smaller (reduce to zero). This is the sort of intuitive arguments mathematicians would make, which aren't well-justified. In this case, it is correct that the area under the curve is exactly $\frac{8}{3}$ square units (see *The Fundamental Theorem of Calculus*).

Of course, we are now familiar with limits which are much more rigorous. In our current example, we can say that $A = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$. In general, to work out the integral of a function $f(x)$ over a closed interval, the following limit can be used:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \delta x$$

Equation 4 – Riemann Sum

where x_i is the i^{th} sample point (a point along the width of the i^{th} rectangle) and δx is the width of a rectangle. This is called the Riemann Sum [6]. The Riemann Sum gives a precise and rigorous definition to the integral using a limit, a foundation of calculus.

Remark: As the number of rectangles increases, the width of them decreases (δx decreases as n increases). Hence $\delta x \to 0$ and the limit is sometimes written in terms of such, but it means the same thing.

The short-hand notation for an integral is using the $\int$ symbol along with a differential, which denotes the variable that the function is being integrated with respect to. A definite integral contains limits – the interval in which the function is being integrated:

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$$

Equation 5 – Definite integral notation

Indefinite integrals denote the operation of finding the function F. The limits are absent in these types of integrals:

$$\int f(x)\ \mathrm{d}x = F(x)$$

Equation 6 – Indefinite integral notation

The function F can be found by finding the area of $f(x)$ over an arbitrary interval. We take a parameter t and use x as the upper limit. The lower limit can just be set to a constant.

$$\int f(x)\ \mathrm{d}x = \int_a^x f(t)\ \mathrm{d}t = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\ \delta t = F(x)$$

Equation 7 – Indefinite integral definition

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus was a huge breakthrough in mathematics. Integrals no longer required all of the hard work as shown previously. The theorem states that the operations integration and differentiation are the inverses of each other. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\ \mathrm{d}x\right) = f(x)$$

Equation 8 – The Fundamental Theorem of Calculus

This is commonly thought to be proved by Newton and/or Leibniz, however it is said to be first proved by Isaac Barrow [7].

Take *Equation 6* and substitute $F(x)$ into *Equation 8*:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(F(x)\big) = f(x)$$

Equation 9 – F(x) substitution

Whence we see that $F(x)$ is an anti-derivative of $f(x)$. This specific fact allowed for the computation of integrals to be a piece of cake – if you know a function whose derivative is the function you wish to find the integral of, then you have already found the integral! Of course, this is elementary now, and is taught at A-Level, even before or in the absence of the method of first principles.

Take the previously computed function $f(x) = x^2$ (*Figure 1*). Let's find the integral of the function over the same interval, $[0, 2]$, using our new-found knowledge.
Differentiating x^3, we get $3x^2$ by the power rule. This result is a multiple of $f(x)$, namely $3 \times f(x)$. So let us differentiate $\frac{1}{3}x^3$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}x^3\right) = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = \frac{1}{3} \times 3x^2 = x^2$$

Aha! We've found a function whose derivative is our integrand $f(x)$. Hence,

$$\int_0^2 f(x)\,\mathrm{d}x = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}$$

Equation 10 – Applying the Fundamental Theorem of Calculus

This is the same result (see *Equation 3*).

Sometimes, you may see the anti-derivate placed in square brackets along with the limits before the substitution, i.e.

$$\int_a^b x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_a^b$$

Equation 11 – Common 'pre-substitution' notation

For any function, there is an infinity of anti-derivatives which we call the family of anti-derivatives. This is because constants (numbers with no relation to the differential variable) differentiate to zero. So, when we calculate an indefinite integral, we find a function which lacks a constant, and add an arbitrary constant to it (usually denoted c). This constant is commonly referred to as 'the constant of integration'. For example:

$$\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c$$

Equation 12 – Demonstration of the constant of integration

Manipulations of Integrals

Over the years, mathematicians have tamed the 'beast' that is calculus. Methods have been discovered which aid in the calculation of integrals. A lot of whom have been thanks to the differential – an infinitesimal change in a variable. Differentials are denoted with a 'd' preceding a variable name, for example $\mathrm{d}x$. These handy values can be used not only in differentiation but also in integration. An example of this is with what is sometimes called 'integration by substitution' or the 'u substitution' [8]. Take the function $f(x) = x(2x^2 + 1)^3$. How would one go about integrating this? We wish to find

$$I = \int x(2x^2 + 1)^3\,\mathrm{d}x$$

Equation 13 – Substitution integral example

Let $u = 2x^2 + 1$ then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 4x$$

Then by rearranging, we see that

$$\mathrm{d}x = \frac{1}{4x}\mathrm{d}u$$

If we substitute this new differential into our integral, we can change the variable we are integrating with respect to:

$$I = \int x(2x^2 + 1)^3 \times \frac{1}{4x}\mathrm{d}u$$

$$I = \int \frac{1}{4}(2x^2 + 1)^3 \,\mathrm{d}u$$

Recall, $u = 2x^2 + 1$, therefore

$$I = \int \frac{1}{4}u^3 \,\mathrm{d}u$$

This looks a lot better than the original integral (*Equation 13*). We can integrate this with ease, using the knowledge of the power rule:

$$I = \int \frac{1}{4}u^3 \,\mathrm{d}u = \frac{1}{16}u^4 + c$$

But $u = 2x^2 + 1$, hence

$$I = \frac{1}{16}(2x^2 + 1)^4 + c$$

Equation 14 – Substitution integral solution

Nothing's Impossible! ...right?

Some integrals have been known (and proved) not to be able to be computed using the standard methods of integration, such as the infamous integral:

$$\int e^{x^2}\mathrm{d}x$$

Also, not at all obviously,

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$$

Bizarre but quite stunning. This integral is used in probability theory for the Normal distribution, a phenomenon which seems to occur throughout nature time and time again.

2 | History: Differentials

Student 2

Differential calculus is defined as a section of calculus focused on the problem of finding the rate of change of a function [9]. This has been the main function of differential calculus throughout history and into the modern era having uses all over the STEM fields like meteorology and finance.

The major groundwork for this being laid by Gregorius Saint-Vincent with his work. He improved the theory of conic sections using the basis of Commandinos' editions of Archimedes in 1558, he also developed a method of infinitesimals [10]. This greatly improved the understanding of the groundwork for differential calculus and led to many great mathematicians building off his work.

The next main contributor to the development of differential calculus was Pierre de Fermat. A French lawyer and government official, born in 1601, He is most remembered for coming up with his last theorem; however, his main contribution to the development of differential calculus was his work on finding the minima and maxima, in 1630, which was:

If a real-valued function $f(x)$ is differentiable on an interval (a, b) and $f(x)$ has a maximum or minimum at $c \in (a, b)$, then $f'(c) = 0$ [11]. The main result of this was an elaboration on the main rational behind finding the minima and maxima values of a continuous function. This had taken the idea of looking at the change in a function and applied it to lines on a graph to come up with a standardized and consistent method to finding the minima and maxima.

Differential Equations

The next big step in the progression of differential calculus is Daniel Bernoulli work on differential equations in 1680, his work was on differential equations in the form:

$$y' + p(x)y = q(x)y^n$$

Equation 15 – Differential equation

where $p(x)$ and $q(x)$ are continuous functions on the interval we're working on, and n is a real number. Later differential equations in this form would be referred to as Bernoulli Equations [12]. This discovery helped mathematicians throughout time and into the modern era. It is vital in finding if there has been a change in a fluid or if it has remained static, because of this it can be used in the modern day to look at water pressure from a boiler or to explain how quicky a liquid flows.

The next major player in the progress of differential calculus was Michel Rolle, a French mathematician who worked on, in the 1690s, a special case of the mean value theorem which would later be named after him called Rolles's theorem: Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$. Informally the theorem states that if the output of the differential equation is the same at the endpoints, then there must be another point c where $f'(c) = 0$ [13]. This can be used to find the minima and maxima of a function. His work built off the groundwork left by Fermat, he used his ideas to create a more intuitive and useful theorem.

The Riemann Integral

The next major contributor was Riemann, he developed what would become known as the Riemann integral: The function f is said to be Riemann integrable if its lower and upper integral are the same. When this happens, we define:

$$\int_a^b f(x)\,\mathrm{d}x = L(a,b) = U(f,a,b)$$

Equation 16 – Riemann-integrable function

A function $f:[a,b] \to R$ is Riemann integrable if for every $\varepsilon > 0$ there exist step functions $s,t:[a,b] \to R$ for which $s(x) \le f(x) \le t(x)\ \forall\, x \in [a,b]$ and

$$\int_a^b t(x)\,\mathrm{d}x - \int_a^b s(x)\,\mathrm{d}x < \varepsilon$$

holds.

The theorem states if $f:[a,b] \to R$ is continuous, then f is Riemann integrable [14]. However, his main contribution to differential calculus was his work on differential geometry. Riemannian geometry is a section of geometry that looks at Riemannian manifolds. Riemannian manifold is a Riemannian metric on a smooth manifold M is a choice $x \in M$ of a positive definite inner product h, i on $T_x M$, the inner products varying smoothly with x. Then M is known as a Riemannian manifold [15]. This means that local angles, length of curves, surface area and volume can be found by integrating local variables. Riemannian stated that Riemannian Geometry was a broad and abstract generalization of the differential geometry of surfaces in $\mathbb{R}^3$. This would then be later used in many different fields of science, figures such as Einstein as it had a profound impact on group theory, differential topology and other areas of STEM, a direct result of this it's its major use in data analysis and the transformation of shapes.

Vector Calculus & Meteorology

A modern-day application of calculus is in meteorology where it is used to help predict weather patterns and storms. In 1838 Francois Arago, a French mathematician and physicist developed a method to calculate long term averages for rainfall and temperature used data collected across a long period of time. His method was instrumental for the development of later methods in the field.

The main method used in modern day is called vector calculus; it can be used to display pressure and humidity as a scalar field and winds as a vector field these variables can come together to create a mass of spinning air called a supercell. This is instrumental in helping to predict tornados [16].

Vector Calculus is the study of the integration and differentiation of the vector field in the three-dimensional Euclidean space. A vector field is the representation of the distribution of a given vector to each point in the subset of space. Vector Calculus or Vector analysis deals with two types of integral Line integrals and Surface integrals. A line integral is an integral when the function that will be integrated can be calculated along a line. A Surface integral is like a line integral expect that the integration is done over a surface rather than a line.

In vector calculus there are three important formulas that take:

$$F(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

Fundamental Theorem of the Line Integral
Consider $F = \nabla f$ and the curve C that has endpoints A and B

$$\int_C F \cdot \mathrm{d}r = f(B) - f(A)$$

Equation 17 – The Fundamental Theorem of the Line Integral

The Circulation Curl Form
According to Green's Theorem,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \oint_C F \cdot \mathrm{d}r$$

According to Stoke's Theorem,

$$\iint_D \nabla \times F \cdot n \mathrm{d}\sigma = \oint_C F \cdot \mathrm{d}r$$

Equation 18 – Circulation Curl Form

Here C refers to the edge of curve S.

Flux Divergence Form
According to Greens theorem,

$$\iint_D \nabla \cdot F \, \mathrm{d}A = \oint_C F \cdot n \mathrm{d}s$$

According to Divergence theorem,

$$\iiint_D \nabla \cdot F \, \mathrm{d}V = \oiint_S F \cdot n \mathrm{d}\sigma$$

Equation 19 – Flux Divergence Form

[17]

This section of vector calculus can be used to predict the weather. These next equations are used to increase the accuracy of predictions; because vectors have both direction and magnitude, they can be used to model the wind in a tornado. With this relationship they can be graphed with line segments used to show vectors. The wind vector w can be found using the equation

$$w = G - A$$

with A representing the motion of the aircraft as it travels through the air compared with G representing the movement of the aircraft on the ground
[18]

This will enable forecasters (people who predict the weather) to know the speed as well as direction of the wind. The speed of the wind is important as it can be used to predict weather changes at low pressure.

Using these ideas meteorologists were able to come up with multiple different equations and formulas to help predict the weather. One of the major ones used is called the shallow water equations. They are used to help describe many features of large-scale circulation patterns of different winds. These equations are used by many meteorologists and oceanographers. This is seen as a valid approximation due to the information that the troposphere is a very thin layer of water (10km deep) that spans the entire radius of the earth.

The Shallow water equations on a domain of $\mathbb{R}^2$ with the local cartesian coordinates (x, y) with the constant frequency $\frac{f}{2}$ are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + g\frac{\partial h}{\partial x} = 0, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla \cdot u = 0$$

Where t denotes time, and the constant g is the acceleration due to gravity of the fluid. The horizontal velocity has two components $u(x, y, t) = (u, v)$ and the Lagrangian derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u \cdot \nabla$$

Lagrangian derivative shows the rate of change of the dependent variable. It is a directional derivative with the trajectory of a fluid particle. $u \cdot \nabla$ if often cited as the advection of transport term. This means it is the transport of a property of the atmosphere such as fluid. If any variable $A(x, t)$ satisfies $\frac{\mathrm{D}A}{\mathrm{D}t} = 0$, then we say that $A(x, t)$ is conserved in the Lagrangian sense. These laws are of fundamental importance to meteorology and oceanography.

These semi-geostrophic equations result in the following approximations to the shallow water equations

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - fv + g\frac{\partial h}{\partial x} = 0, \qquad \frac{\mathrm{D}v_g}{\mathrm{D}t} + fu + g\frac{\partial h}{\partial y} = 0$$

where u_g and v_g are the two components of the geostrophic wind

$$g\nabla h = \left(fv_g - fu_g\right)$$

The geostrophic flow is parallel to h (the contours of the constant) [19].

The main difference between the semi-geostrophic equations and the shallow water equations is the use change to geostrophic wind (u_g, v_g) from the values of fluid velocity (u, v) in the derivatives of u and v while leaving the operator unchanged and the continuity equation constant. This is known as the geostrophic momentum operator developed by Hoskins in 1975. The main implication of which is the fact that the flow in the rate of change of momentum is a lot smaller than the Coriolis force. due to the semi geostrophic theory the advecting velocity is not approximated. Ultimately the use and incorporation of two velocity fields is to account for the idea that multiple atmospheric flows, like fronts or jet streams and have two different distinct length scales. To sum it up, the geostrophic momentum approximation is modernly now understood in the eyes of mechanics. Potential vorticity:

$$q = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial\left(v_g, u_g\right)}{\partial(x, y)}\right)$$

In atmospheric dynamics (including thermal dynamics), the vector dot product of vorticity is directly proportional to the potential vorticity, and that the flow can only be altered by a large frictional process. Potential vorticity is a very important fundamental process for the understating and comprehension of cyclogenesis. Cyclogenesis is the birth, creation and development of a cyclone, this is incredibly useful for meteorologists to be able to predict when a cyclone will appear and help minimize damage. This is especially effective along the polar front and when looking at analysing the flow of the ocean. This method was instrumental as it was able to help with understanding the progression and evolution of hurricanes and provide context on how they behave. Whereas semi geostrophic equations play a pivotal role in understanding fronts and studying them. If we define new coordinates as:

$$X \equiv (X, Y) \equiv \left(x + \frac{v_g}{f}, y - \frac{u_g}{f}\right)$$

We can make the statement that

$$\frac{\mathrm{D}X}{\mathrm{D}t} = u_g \equiv (u_g, v_g)$$

Because of this the motion of these new coordinates is geostrophic and it is directly proportional to the potential vorticity

$$\frac{\partial(X, Y)}{\partial(x, y)} = \frac{hq}{f}$$

The vector X may be shown as the gradient of the scalar function $P(x)$,

$$X = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right)$$

Which, when given a constant that is unique it becomes defined by

$$P(X, t) = \frac{1}{2}(x^2 + y^2) + \frac{gh(x, t)}{f^2}$$

We observe the fact that by implementing $g\nabla h = (fv_g - fu_g)$ has created a special equation called a Monge-Ampere equation. The Monge-Ampere equation is a fully nonlinear partial differential equation, where the highest order term is the determinate of the Hessian of the function to be solved for [20]. The Monge-Ampere equation is for P, given $q(x, y, t)$ and suitable conditions on the boundary

$$q = \frac{f}{h}\left(P_{xx}P_{yy} - P_{xy}^2\right)$$

When the previous equation is shown as non-singular, we can represent $x(X)$ by using a scalar function of $R(X)$

$$x = \left(\frac{\partial R}{\partial X}, \frac{\partial R}{\partial Y}\right)$$

where R has another constraint imposed on it

$$R(x) = x \cdot X - P(x)$$

The reason for the condition and transformation between $R(x)$ and $P(x)$ as it can be used by meteorologist to help convey, through the local singularities of the map, as atmospheric fronts. The semi geostrophic equations can be integrated by using the conservation of vorticity. This integration starts by finding the solution of dual Monge-Ampere equations for R and next finding out the advecting velocities.

The semi-geostrophic equations contain two very important types of geometry, sympletic geometry and contract geometry. These tow geometric properties play a fundamental role in Monge Ampere equations which in turn play a vital role in fluid mechanics which is used in looking at the atmospheric changes.

In conclusion there is a long and complex history around the development of differential calculus and how that fact that is can be used to find the rate of change of a function can be applied to unuseral areas such as meteorology and that it plays a vital role in looking at change in the atmosphere and to help further our understanding of and track the progression of more violent weather such as hurricanes and cyclones.

3 | Application: Epidemiology

Student 3

The study of epidemiology is thought to have been born from Greek physician Hippocrates, who believed that sicknesses were the result of imbalances of the four humors (black bile, yellow bile, blood, and phlegm) [21]. The field has developed significantly thanks to breakthroughs, thoughts, and ideas surrounding diseases as well as scientific capabilities. Modern epidemiology utilises tools such as advanced statistics applied to machine learning to create predictive epidemiological models to study the spread and measures necessary to prevent it [22] [23]. This chapter will focus on a primitive model that utilises calculus in order to predict the spread of a disease through a generalised population.

SIR Model

One of the most basic models possible contains three compartments in which the population are sorted into: Susceptible, Infectious, and Removed [24]. The model is named after these compartments, SIR. A scientific model is necessarily a simplification of the real world, so we will assume the following:

- All individuals in the population can be sorted into only one of the following:
    - Susceptible – able to contract the disease;
    - Infectious – able to transmit the disease;
    - Removed – unable to contract the disease;
- The population is large, homogeneous, and of constant size;
- Travel is confined to within the model;
- All individuals encounter the same proportion of the population in each compartment every ‘day’.

We show the movement of the population through compartments by the diagram:

Figure 4 - Diagram showing movement through compartments

We will now assign variables corresponding to the population in each compartment and governing compartment movement. $S, I,$ and R correspond to the number of susceptible, infectious, and removed individuals, respectively. Fix two variables, α and β, such that, if $S, I,$ and R people are susceptible, infectious, and removed, respectively, on a given day, βI is the number of newly removed individuals the next day, while the number of new infections is $\alpha I \frac{S}{N}$. The proportion of susceptible individuals in the population appears in the expression since an infectious individual will encounter individuals in all compartments, but only those interactions with susceptible individuals result in new cases. We can express the size of each compartment the next day as a set of equations:

$$\begin{cases} S(t+1) = S(t) - \alpha I(t)\dfrac{S(t)}{N} \\ I(t+1) = I(t) + \alpha I(t)\dfrac{S(t)}{N} - \beta I(t) \\ R(t+1) = R(t) + \beta I(t) \end{cases}$$

Equation 20 – System representing the size of each compartment after 1 day

Using Ideas from Calculus

We can make steps towards transforming our discrete system into a continuous one by considering the changes in compartments over Δt days and rearranging so the equations become:

$$\begin{cases} S(t+\Delta t) - S(t) = -\alpha I(t)\dfrac{S(t)}{N}\Delta t \\ I(t+\Delta t) - I(t) = \left(\alpha I(t)\dfrac{S(t)}{N} - \beta I(t)\right)\Delta t \\ R(t+\Delta t) - R(t) = \beta I(t)\Delta t \end{cases}$$

Equation 21 – Changes in compartments over time

We can define new variables representative of the proportion of the population each compartment contributes as follows:

$$s(t) = \frac{S(t)}{N}, i(t) = \frac{I(t)}{N}, r(t) = \frac{R(t)}{N}.$$

Equation 22 – Proportion functions for each compartment

By dividing our previous expressions by N and Δt, they simplify to

$$\begin{cases} \dfrac{S(t+\Delta t) - S(t)}{N\Delta t} = -\alpha i(t)s(t) \\ \dfrac{I(t+\Delta t) - I(t)}{N\Delta t} = \alpha i(t)s(t) - \beta i(t) \\ \dfrac{R(t+\Delta t) - R(t)}{N\Delta t} = \beta(t) \end{cases}$$

Equation 23 – Simplification of system

Consider the expression $\frac{C(t+\Delta t)-C(t)}{\Delta t}$ and its similarity to the difference quotient, $\frac{f(x+h)-f(x)}{h}$. By rewriting the difference quotient, we can show that the two expressions are equivalent, so $x \to t$, $h \to \Delta t$, and the function of x, $f(x)$, is replaced by the function of t, $C(t)$, which calculates the number of individuals in a given compartment at time t.

It is commonly denoted that $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{\mathrm{d}f}{\mathrm{d}x}$ using Leibniz's notation. Applying this to the compartment, we find that $\lim_{h\to 0}\frac{C(t+\Delta t)-C(t)}{\Delta t} = \frac{\mathrm{d}C}{\mathrm{d}t}$. In our expressions, the LHS of the equations are divided by N, so the LHS represents the change in a given compartment as a proportion of the total proportion, which we defined using the lowercase version of a compartment's letter.

The expressions can be finally simplified to the following:

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = -\alpha si \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = \alpha si - \beta i \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \beta i \end{cases}$$

Equation 24 – Derivative notation for instantaneous change

where s, i, and r are functions of t, and α and β are real constants.

It is important to note that the SIR model, being a simplification, is inherently flawed. The model is deterministic, meaning any set of input parameters will always yield the same simulation every time it is run, so there is no variation in runs, unlike real life. Stochastic models, those that incorporate probability and randomness, are more realistic, however they typically require a greater number of parameters to control the transmission behaviours.

There are a couple of interesting characteristics of the model that we can now study, one being the sizes of compartments as $t \to \infty$. Common sense dictates that $i \to 0$ as $t \to 0$ as there is a positive recovery rate. Another interesting result is found if we start to analyse relations between compartments, starting with the relationship between the susceptible and infectious compartments:

$$\frac{\mathrm{d}i/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t} = \frac{\mathrm{d}i}{\mathrm{d}s} = \frac{\alpha si - \beta i}{-\alpha si} = \frac{\beta i}{\alpha si} - \frac{\alpha si}{\alpha si} = \frac{\beta}{\alpha s} - 1.$$

Equation 25 – Rate of change of the infectious against the susceptible

We can simplify this further:

$$\frac{\mathrm{d}i}{\mathrm{d}s} = \frac{\beta}{\alpha s} - 1$$
$$\mathrm{d}i = \left(\frac{\beta}{\alpha s} - 1\right)\mathrm{d}s$$
$$\int \mathrm{d}i = \int \left(\frac{\beta}{\alpha s} - 1\right)\mathrm{d}s$$
$$i = \frac{\beta}{\alpha}\int \frac{1}{s}\mathrm{d}s - \int 1\ \mathrm{d}s$$
$$i = \frac{\beta}{\alpha}\ln s - s + C$$

Equation 26 – Solved differential equation for the infected as a function of the susceptible

Now looking at the relationship between the susceptible and removed,

$$\frac{\mathrm{d}r/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t} = \frac{\beta i}{-\alpha si} = -\frac{\beta}{\alpha s}$$

Equation 27 – Rate of change of the removed against the susceptible

Which can be further manipulated:

$$\frac{\mathrm{d}r}{\mathrm{d}s} = -\frac{\beta}{\alpha s}$$
$$\mathrm{d}r = -\frac{\beta}{\alpha s}\mathrm{d}s$$

$$\int \mathrm{d}r = \int -\frac{\beta}{\alpha s}\mathrm{d}s$$
$$r = -\frac{\beta}{\alpha}\int \frac{1}{s}\mathrm{d}s$$
$$r = -\frac{\beta}{\alpha}\ln s + C$$

Equation 28 – Solution for the removed as a function of the susceptible

Finally, looking at the relationship between the infectious and removed compartments, we form the expression:

$$\frac{\mathrm{d}i/\mathrm{d}t}{\mathrm{d}r/\mathrm{d}t} = \frac{\alpha s i - \beta i}{\beta i} = \frac{\alpha s}{\beta} - 1$$

Equation 29 – Rate of change of the infected against the removed

By integrating we can find:

$$\frac{\mathrm{d}i}{\mathrm{d}r} = \frac{\alpha s}{\beta} - 1$$
$$\mathrm{d}i = \left(\frac{\alpha s}{\beta} - 1\right)\mathrm{d}r$$
$$\int \mathrm{d}i = \frac{\alpha}{\beta}\int s\,\mathrm{d}r - \int \mathrm{d}r$$

Using the expression for r we found previously in terms of s, we can rearrange to find $s = e^{\frac{\alpha}{\beta}(C-r)}$. Substituting this expression into the integrand gives:

$$\int \mathrm{d}i = \frac{\alpha}{\beta}\int e^{\frac{\alpha}{\beta}(C-r)}\,\mathrm{d}r - \int \mathrm{d}r$$

Let $\frac{\alpha}{\beta} = \kappa$,

$$i = \kappa \int e^{\kappa C - \kappa r}\mathrm{d}r - r$$
$$i = \kappa e^{\kappa C}\int e^{-\kappa r}\mathrm{d}r - r$$
$$i = \kappa e^{\kappa C}\left(\frac{e^{-\kappa r}}{-\kappa}\right) - r$$
$$i = -e^{\kappa C}e^{-\kappa r} - r$$
$$i = -e^{\kappa(C-r)} - r$$

Substituting κ back in for $\frac{\alpha}{\beta}$:

$$i = -e^{\frac{\alpha}{\beta}(C-r)} - r$$

Equation 30 – Solution for the infected as a function of the removed

Python Model

This model can be coded rather simply in Python and produces graphs that look like the following:

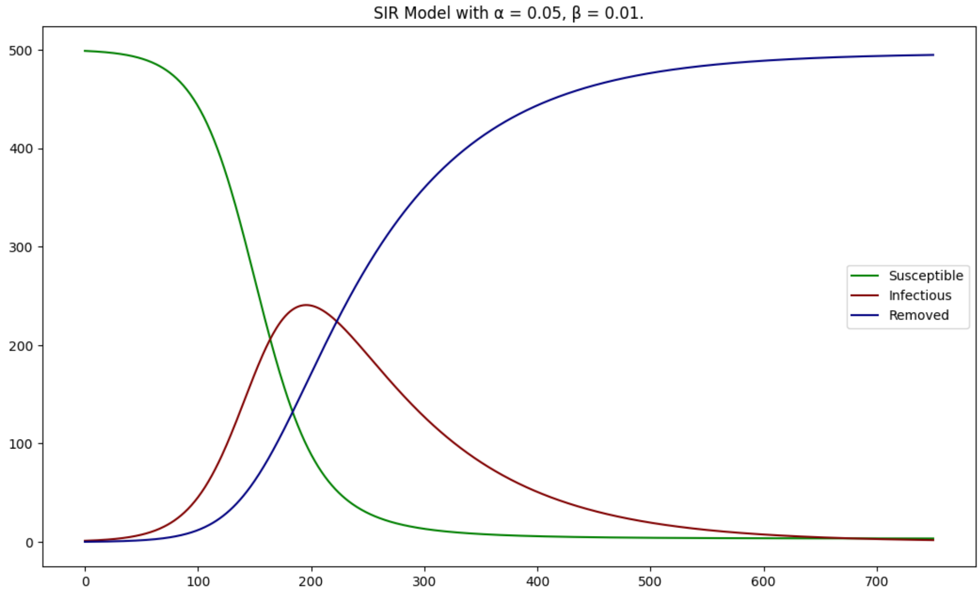

Figure 5 – Model output for α = 0.05, β = 0.01

The characteristic shape of the graph, particularly the 'Infectious' line, may be familiar with the general public after the SARS-CoV-2 pandemic brought epidemiology into the forefront of everyday life [25]. There was a large focus on 'flattening the curve' by limiting contact between individuals [26]. If we reduce α to 0.025, representative of every infectious individual encountering $\frac{1}{40}$ of the susceptible population, we see the peak of the red line is decreased, however the tail seems to be much longer.

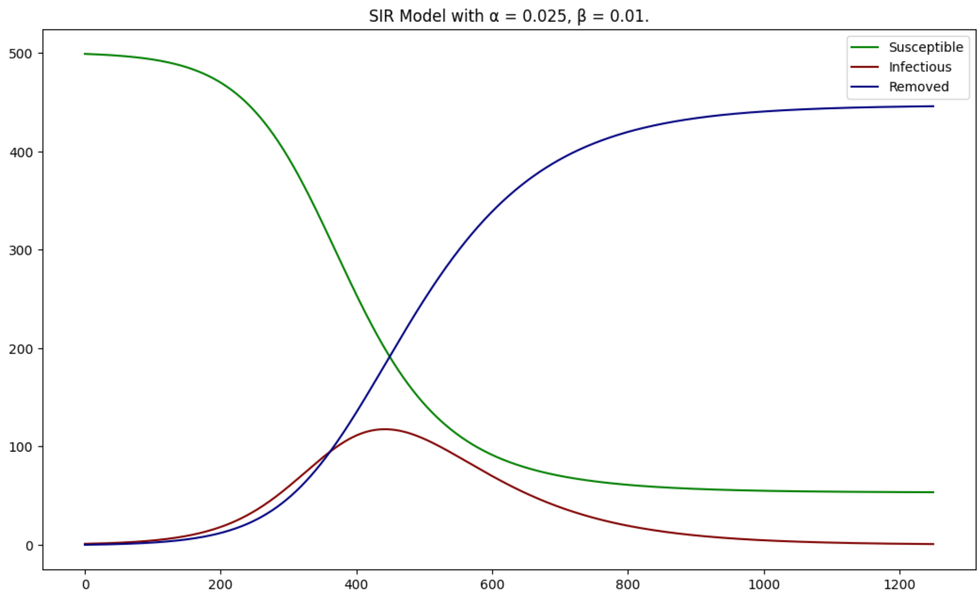

Figure 6 – Model output for α = 0.025, β = 0.01

In this simulation, not all individuals in the population became infected, evident by the susceptible line tending towards a positive value around 53. There exists a metric whose purpose is to quantify how fast a disease is growing within a population called the R-value, or reproductive number.

R-value

As mentioned previously, there exists a metric for the speed at which a disease permeates through a population, the R-value, or reproductive number [27]. The most basic form of this is the basic reproductive number, denoted R_0, which is the estimated number of cases caused by a single infection within a solely susceptible population [27]. As time progresses, we denote the effective reproductive number, which is variable, as R_e or R_t; R_e will be used for this document. If $R_e < 1$, then the spread of the disease will slow. Conversely, if $R_e > 1$, then the number of infectious cases will increase [27].

Using the Python model created previously, we can calculate R_e and plot it on the graph for varying α and β:

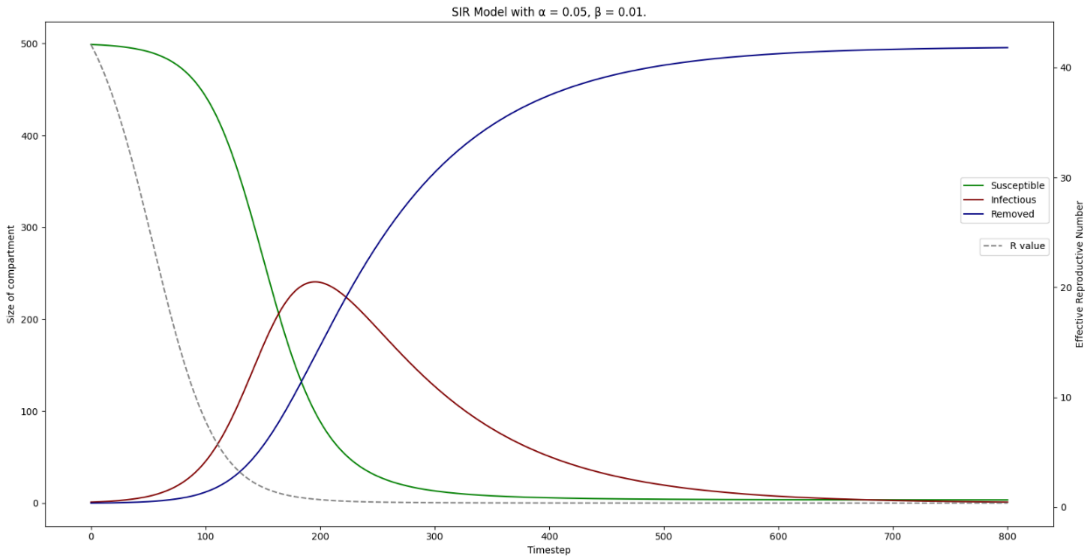

Figure 7 – Model output for α = 0.05, β = 0.01

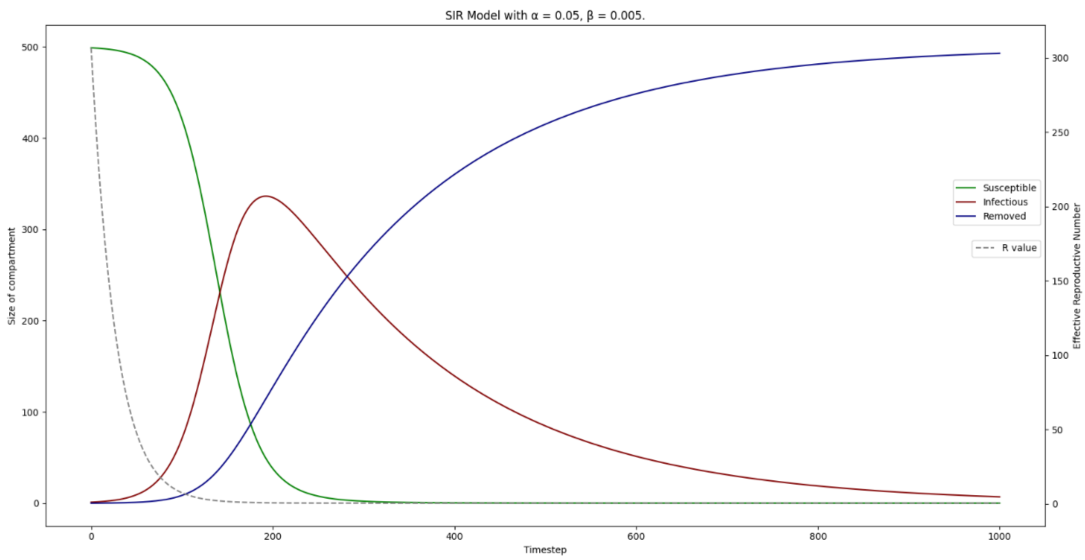

Figure 8 – Model output for α = 0.05, β = 0.005

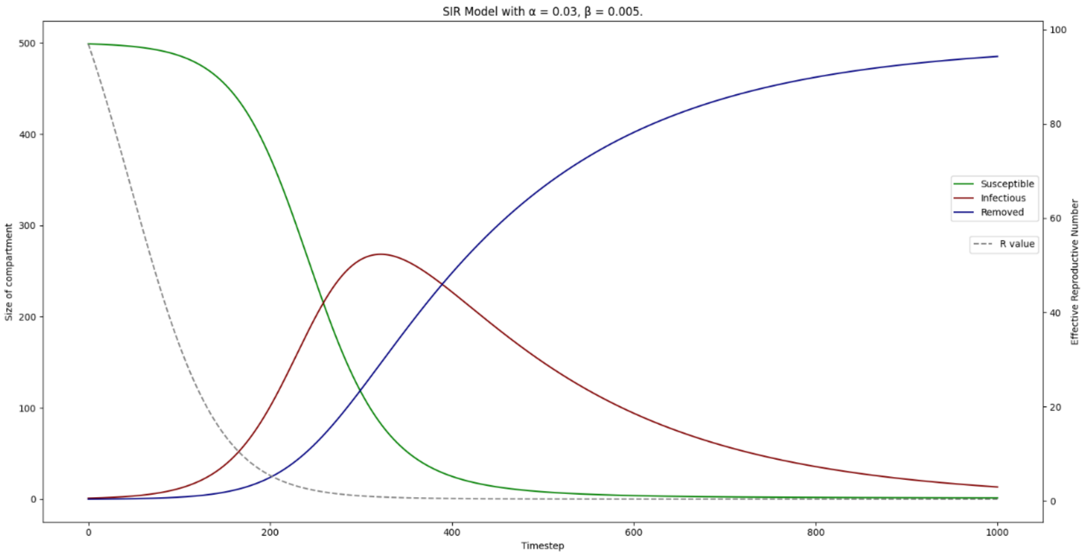

Figure 9 – Model output for α = 0.03, β = 0.005

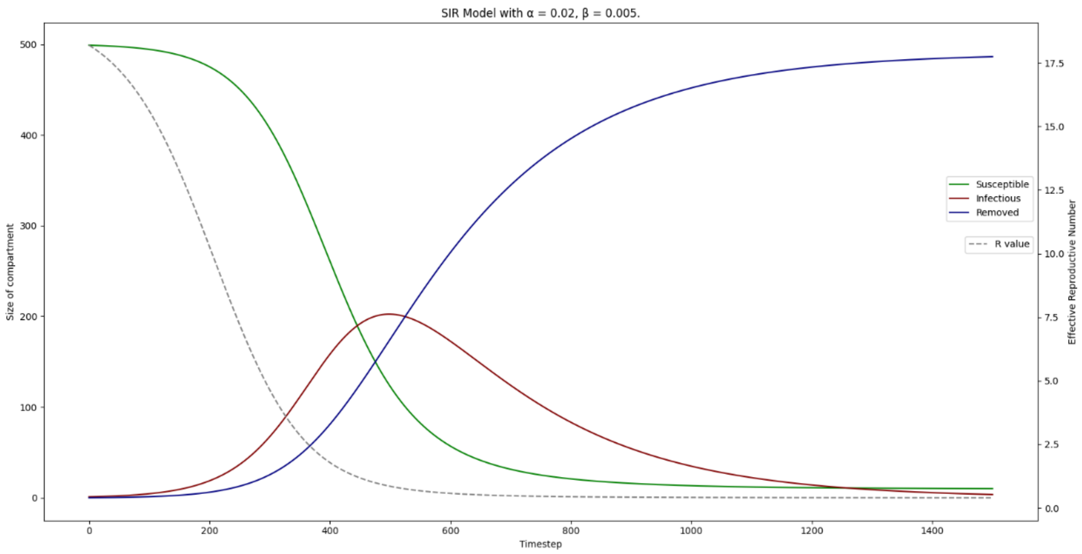

Figure 10 – Model output for α = 0.02, β = 0.005

Looking at the y axis on the right, we can see that the effective reproductive number, R_e, reached values around 300 in the simulation before dropping to more appropriate values. R_e only appears to reach unreasonable values when both α is high and β is low, meaning that infectious people encounter a larger proportion of the susceptible population, while infectious individuals remain infectious for longer before recovering.

SEIR Model

The SIR model can be altered by inserting a fourth compartment between the susceptible and infectious compartments, called exposed. This forms the structure of an SEIR model. In this new model, the movement of the population through compartments can be shown using a similar diagram as before:

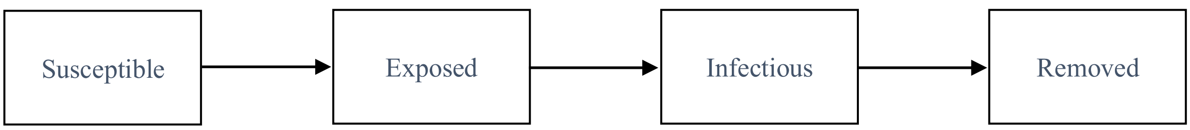

Figure 11 - Diagram showing population movement through the model

The equations for the SIR model must be adjusted to accommodate the additional compartment. We will simplify the model to the following:

$$\text{Susceptible} \xrightarrow{\alpha} \text{Exposed} \xrightarrow{\beta} \text{Infectious} \xrightarrow{\gamma} \text{Removed}$$

Figure 12 – Markov diagram for an SEIR model

We can refer to α as the 'infection coefficient', β as the 'latent infection coefficient', and γ as the 'recovery coefficient'. The model described above relies on the system of equations:

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = -\alpha s i \\ \dfrac{\mathrm{d}e}{\mathrm{d}t} = \alpha i - \beta e \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = \beta e - \gamma i \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \gamma i \end{cases}$$

Equation 31 – Instantaneous rates of change against t (derivatives)

In theory, we should expect the peak of the infectious compartment to be slightly flattened and follow a peak in the exposed compartment. This is due to the population having to become exposed prior to becoming infectious, and the fact that only a proportion of the exposed population become infectious at a time, inducing a wave of infectious individuals much more gradually than the previous models.

Setting $\alpha = \beta = \gamma = 0.005$ provides the following:

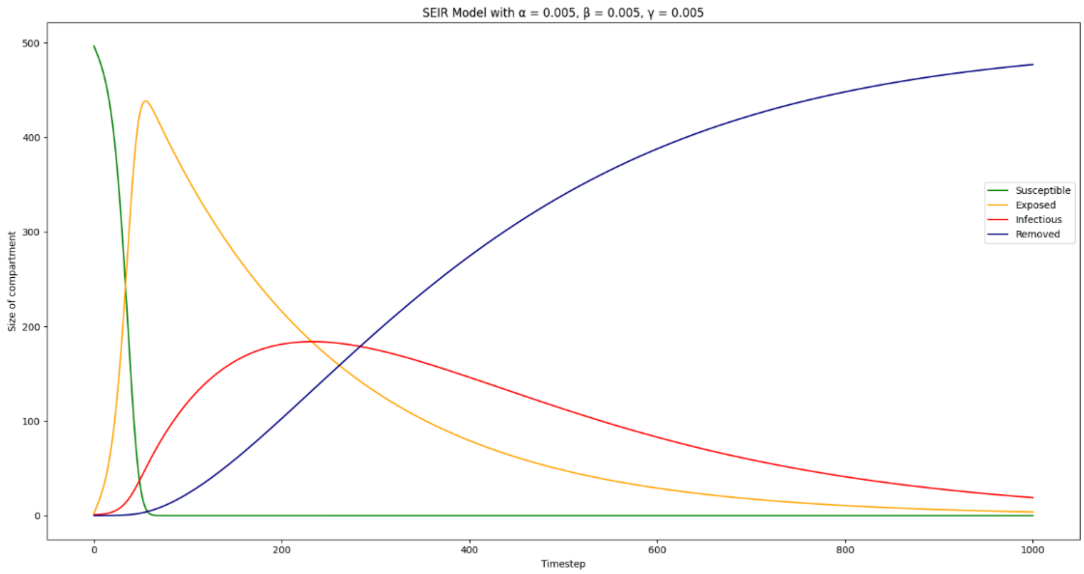

Figure 13 – Model output for α = 0.005, β = 0.005, γ = 0.005

The behaviour of the graph is as expected, although it takes much longer for the infection to die out in the population.

Vaccination

It is possible that the population in the model are able to prevent infection by way of vaccinations. This provides a direct route from the susceptible compartment to the removed compartment. Suppose that, in a population N, $\frac{N}{1000}$ people get vaccinated per timestep, so in a purely susceptible population, all individuals would be in the removed compartment by timestep 1000. This model is described by the following:

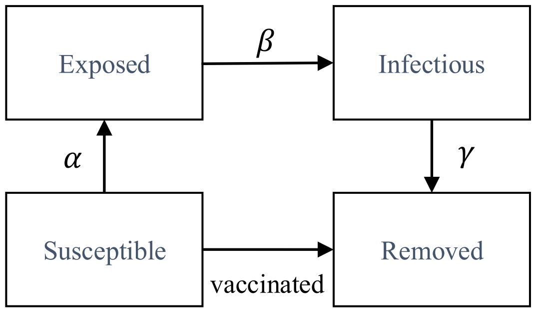

Figure 14 – Markov diagram for an SEIR model with vaccinations

Here is a comparison between the model with and without vaccinations, with $\alpha = 0.005, \beta = 0.005, \gamma = 0.01$:

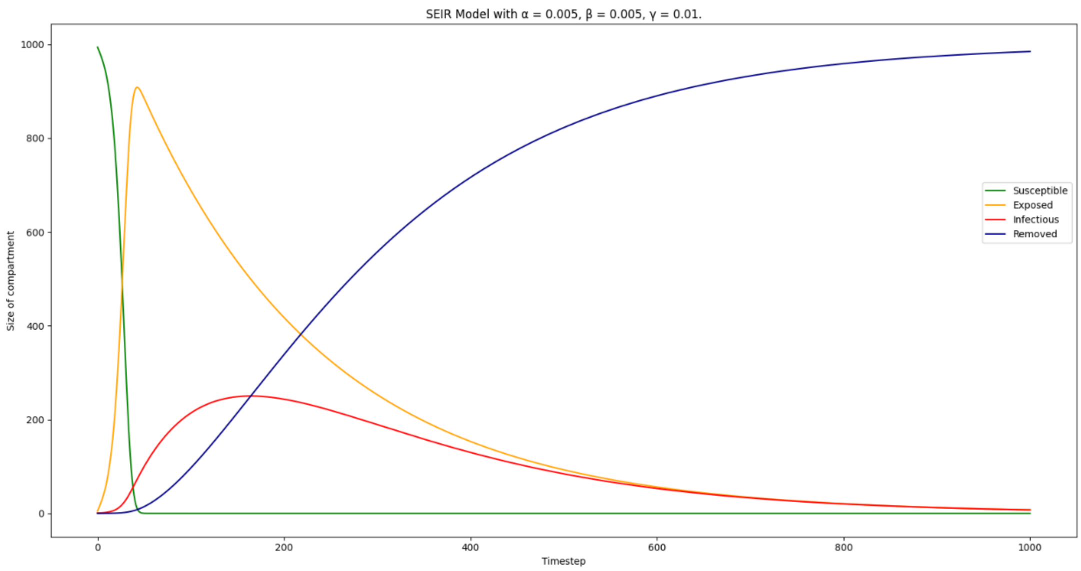

Figure 15 – Model output without vaccinations

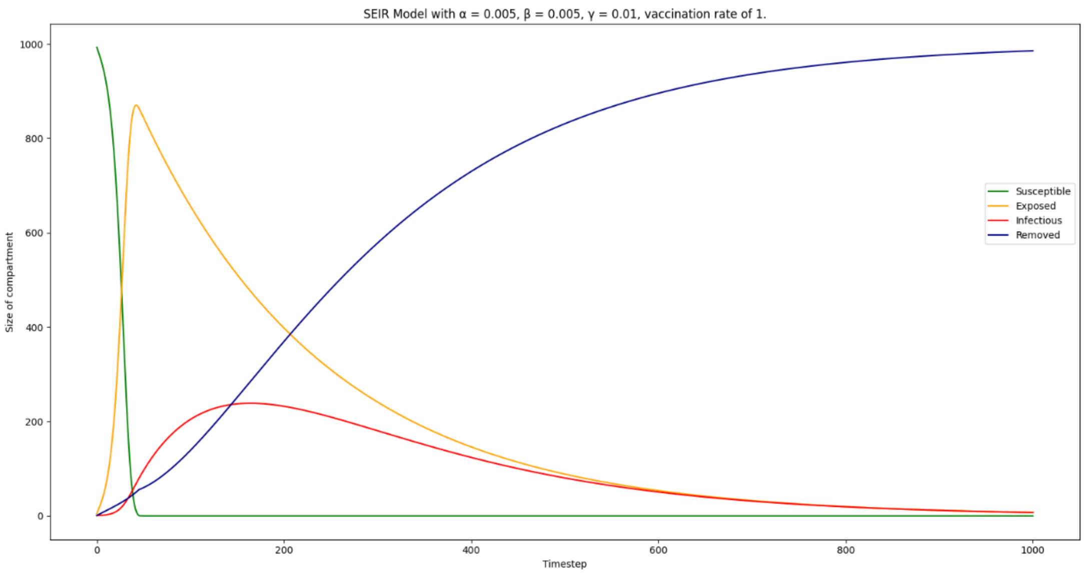

Figure 16 – Model output with vaccinations

From these graphs, it is evident that relatively small-scale vaccination does not have a profound effect on the transmission of the disease. However, starting the disease spread while a proportion of the population is already vaccinated does have a greater impact.

In these graphs, the model used parameters $\alpha = 0.005, \beta = 0.005, \gamma = 0.01$:

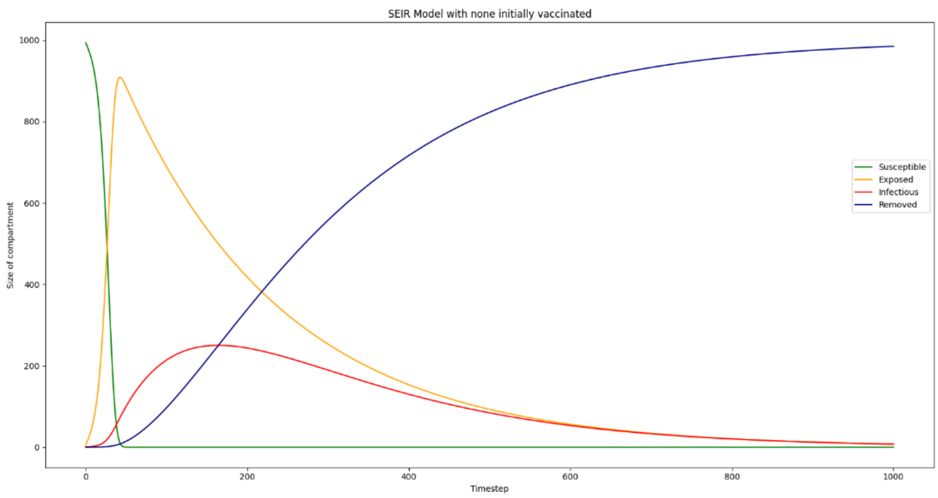

Figure 17 – No initial vaccinations

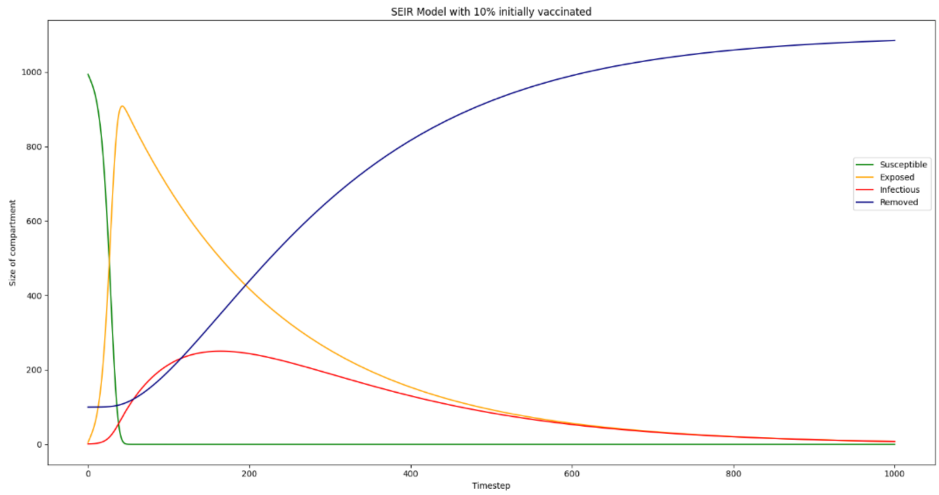

Figure 18 – 10% initially vaccinated

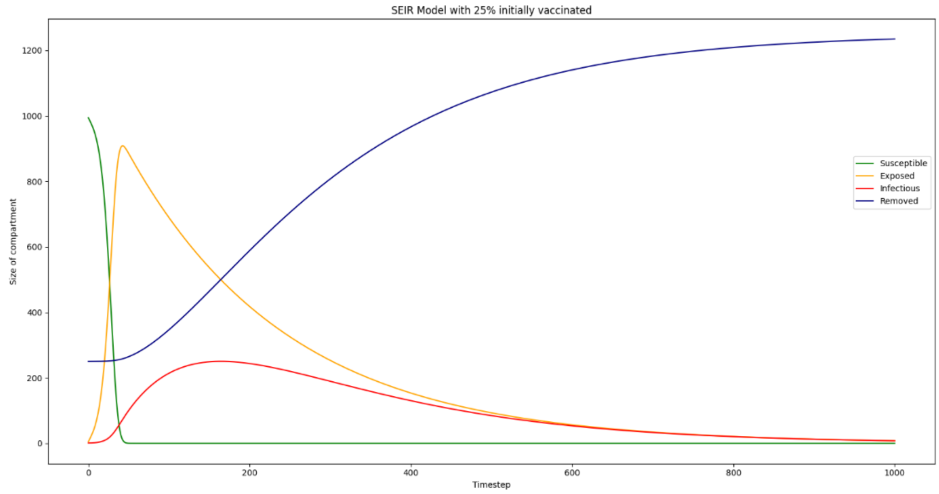

Figure 19 – 25% initially vaccinated

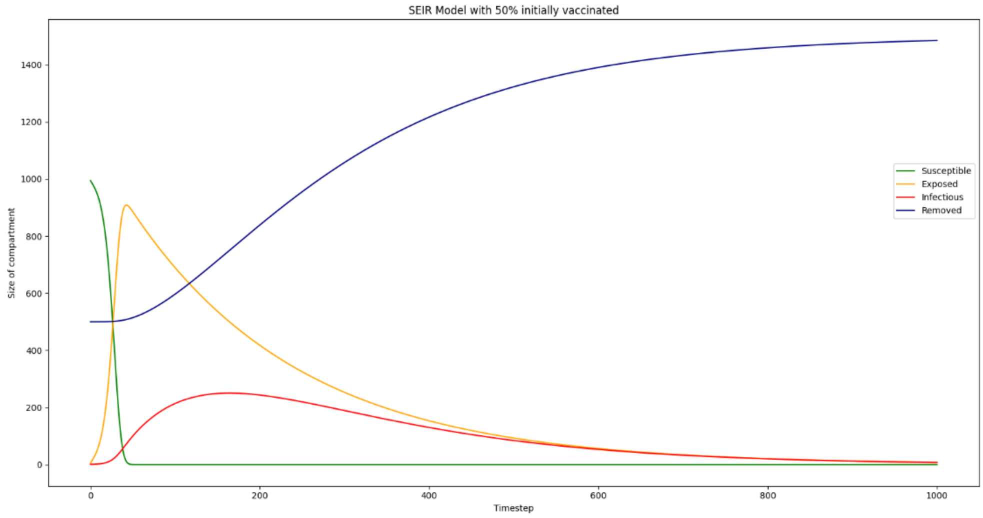

Figure 20 – 50% initially vaccinated

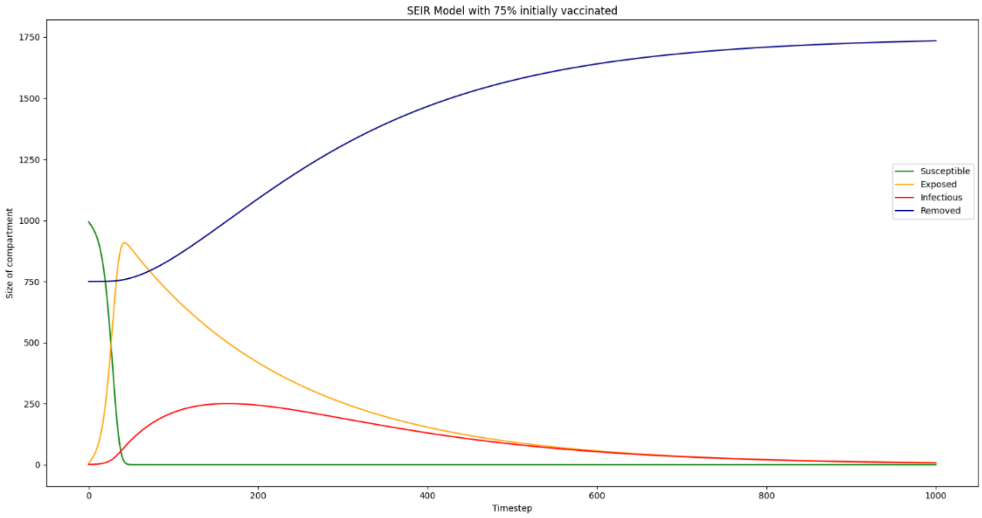

Figure 21 – 75% initially vaccinated

It is quite clear from these graphs that vaccination of a population before an infection outbreak can significantly reduce the impact it has.

Model Limitations

As the model becomes more complex, we start to realise that it may not be displaying the same behaviours that we would expect from a real-life disease outbreak. As previously stated, this compartmental model is necessarily a simplification of the real world. Of course, in real life people are not confined to an enclosed space and people do not come into contact with the whole population. An agent-based stochastic model has the potential to be much more accurate given that it can, provided it has sufficient parameters, replicate the behaviours of a population with greater accuracy; for example, produce realistic population movement within a simulated space [28], include probability-based disease transmission affected by individuals' characteristics [29], and simulate airflow to determine the location of airborne disease particles [30].

The assumption that the population is well mixed rules out the possibility of any space-dependent behaviours of the transmission of the disease. This includes, most notably, herd immunity and 'super spreaders', as commonly referred to during the SARS-CoV-2 pandemic [31]. These emergent behaviours, in a large enough population, have a significant effect on how the disease spreads, further criticising the accuracy of the deterministic compartmental model we observed previously. We expected vaccination to have a much more pronounced effect on the results of the simulation, but the lessened impact can be explained by the lack of a spatial aspect to the model and, thus, no herd immunity. So-called 'super spreaders' also require some metric of socialness or popularity of heterogeneous individuals – something that is simply not implementable within our continuous deterministic model. However, as a population becomes large enough, it is appropriate to imagine all individuals would average and form a homogeneous population, to which our model can be applied to.

We have only touched upon the most primitive of disease modelling techniques and acknowledged their weaknesses. Thankfully, years of technological advancements have allowed epidemiologists to craft substantially more sophisticated models capable of providing vital information about the behaviours of outbreaks. This information can then be used to mitigate damage, put restrictions in place, and even attempt to prevent future outbreaks entirely.

One of the most recent and significant examples of the applications of scientific modelling was the SARS-CoV-2 pandemic that pulled epidemiology to the front of public perception. During this time, masses of information were being processed in order to simulate, track, and learn about the virus [28] - [31].

With all things considered, epidemiology and the modelling of disease outbreaks proves the usefulness and importance of utilising differential calculus, an area of mathematics that has been around for millennia and applying it to modern problems. It serves to show us that even the most basic of mathematical tools can be the most helpful.

4 | Application: Profit Margins & Maximisation

Student 4

At the centre of modern-day society lies the economy, of which is propelled by the drive for profit – and its maximisation – amongst firms (organisations that produce goods). Profit entails the financial gain obtained when the revenue generated through sales exceeds the expenses associated with sustaining such a business venture (e.g. production and labour costs) [32]. A firm will then either reinvest their accumulated profit back into itself, distribute it amongst shareholders as dividends or it will be pocketed by the owners. Irrespective of the path of redistribution, profit is the linchpin to ensure the longevity and growth of a firm, hence the desire for profit maximisation. The following chapter details when the motivation for profit maximisation was realised and the mathematical model used to calculate the necessary conditions to achieve profit maximisation. More specifically, it describes how calculus is used to derive the optimal conditions using a given marginal revenue (the total sales plus that generated by the last unit of output) and marginal cost (the total cost plus that associated with producing the next unit of output). Also covered are the variations and limitations to this model.

The desire for profit maximisation was originally focussed in on upon the development of neoclassical economics, as first coined by American economist Thorstein Veblen in his 1900 article 'Preconceptions of Economic Science' [33]. The shift from Classical to neoclassical economics came about in 1870, introducing an approach now centring on the determination of outputs and income through supply and demand rather than deriving product value from production cost [34]. Often referred to as the Marginal Revolution, this shift was spearheaded by the following economists and their writings: Englishman William Stanley Jevons' 1871 'Theory of Political Economy', Austrian Carl Menger's 1871 'Principle of Economics' and Frenchman Léon Walras' 1874 'Elements of Pure Economics' [35]. Across such texts, it is said that neoclassical economics relies on three assumptions, one of which being that individuals maximise utility (personal satisfaction) and firms maximise profits [36]. This is reiterated through the theory of the firm; a microeconomic approach that suggests a firm exists and make decisions solely to maximise profits [37]. The mathematical model used by firms to ensure profit maximisation is detailed in the following.

Calculating Profit Maximisation

For maximum profit, the first and second order conditions must be met. The first order condition depicts that the marginal revenue MR must be equal to the marginal cost MC.
Net profit is calculated using the total revenue minus the total cost, as shown in the equation below [38]:

$$\pi(Q) = TR(Q) - TC(Q)$$

Equation 32 – Net profit formula

where $\pi(Q)$ is the net profit function, $TR(Q)$ is the total revenue function, $TC(Q)$ is the total cost function and Q is the quantity/output level.
Put simply, the total revenue is the number of units of output multiplied by the average price per unit, as shown below:

$$TR(Q) = P \cdot Q$$

Equation 33 – Total revenue formula

where P is the average price per unit.

The total cost function, however, is slightly more complex as it involves the calculation of the fixed and variable costs. The fixed costs consist of those independent of the business' success, such as rent. Whereas, the variable costs rely solely on production volume, increasing and decreasing alongside it. This includes labour costs, utility expenses and that of the raw materials used in production [39]. The equation for the total cost function is shown below [40]:

$$TC(Q) = FC + (VC \cdot Q)$$

Equation 34 – Total cost formula

where F is the fixed costs and V is the variable costs.
Taking the derivative of the net profit function (*Equation 32*) with respect to Q and setting it equal to zero gives:

$$\frac{\partial \pi(Q)}{\partial Q} = \frac{\partial TR(Q)}{\partial Q} - \frac{\partial TC(Q)}{\partial Q} = 0$$

Equation 35 – Net profit derivative set equal to zero

and rearranges to

$$\frac{\partial TR(Q)}{\partial Q} = \frac{\partial TC(Q)}{\partial Q}$$

Equation 36 – Rearrangement of net profit derivative

The marginal revenue is equal to the derivative of the total revenue function and the marginal cost is equivalent to the derivative of the total cost function as shown below:

$$\frac{\partial TR(Q)}{\partial Q} = MR$$

Equation 37 – Rate of change of total revenue equals marginal revenue

and

$$\frac{\partial TC(Q)}{\partial Q} = MC$$

Equation 38 – Rate of change of total cost equals marginal cost

ergo, this implies that the first order condition is met as:

$$MR = MC$$

Equation 39 – Marginal revenue equal marginal cost

However, this could suggest that (*Equation 35*) is either the maximum or minimum. Therefore to ensure profit maximisation, the second order condition must also be satisfied. This is where the first order condition is fulfilled for decreasing marginal revenue and increasing marginal cost [41].
Thus, the second derivative of the net profit function (*Equation 32*) must be negative [38]:

$$\frac{\partial^2 \pi(Q)}{\partial^2 Q} < 0$$

which implies

$$\frac{\partial^2 TR(Q)}{\partial^2 Q} - \frac{\partial^2 TC(Q)}{\partial^2 Q} < 0$$

Equation 40 – Second derivative of net profit set negative to find maximum

When rearranging, this gives:

$$\frac{\partial MR}{\partial Q} < \frac{\partial MC}{\partial Q}$$

Equation 41 – Rate of change of MR is less than that of MC

Graphically, the gradient for the marginal cost curve must be steeper than that of the marginal revenue curve for the net profit function (*Equation 32*) to be at a maximum. This is shown in *Figure 22* with P_1 and P_2 being the points of intersection along the marginal revenue and marginal cost curves. While at both points the first order condition is satisfied (*Equation 39*), P_1 indicates falling marginal cost whereas P_2 occurs at rising marginal cost. Therefore, P_2 fulfils the second order condition and thus denotes the optimal conditions for profit maximisation. The negative gradient on the marginal cost curve over the domain $[0, Q_2)$ is often explained by the existence of economies of scale; when a firm initially and drastically increases levels of output Q to reduce the cost of production in the long run. The positive gradient over the latter domain (Q_2, ∞) is consequent of the opposite; diseconomies of scale [42]. This occurs when a business grows almost too much that the cost per unit of output increases. Hence why the precise point of Q_2 is the optimum for profit maximisation, being the highest output before the business becomes a diseconomy of scale.

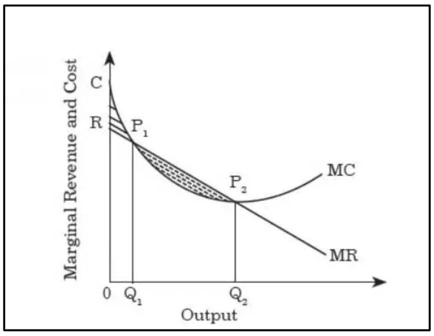

Figure 22 – Marginal conditions of profit maximisation

Example

Suppose $P = 500$ and $TC(Q) = 500 + 125Q - 10Q^2 + Q^3$, what is the optimal output level of this business? [42]
First, we must calculate the total revenue function using *Equation 33*, which gives:

$$TR(Q) = 500Q.$$

Then the total net profit function using *Equation 32*, which gives:

$$\pi(Q) = 500Q - (500 + 125Q - 10Q^2 + Q^3) = -500 + 375Q + 10Q^2 - Q^3.$$

To satisfy the first order condition, we differentiate the total net profit function and set it equal to zero as seen in *Equation 35*:

$$\frac{\partial \pi(Q)}{\partial Q} = 375 + 20Q - 3Q^2 = 0.$$

To solve the above equation for output Q, we must use the quadratic formula shown below:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which gives:

$$Q = \frac{-20 + \sqrt{20^2 - (4(-3)375)}}{2(-3)} = -\frac{25}{3}$$

and

$$Q = \frac{-20 - \sqrt{20^2 - (4(-3)375)}}{2(-3)} = 15.$$

However, the only valid solution is $Q = 15$ as the output value must be positive.

Now to check this satisfies the second order condition (*Equation 40*):

$$\frac{\partial^2 \pi(Q)}{\partial^2 Q} = 20 - 6Q < 0$$
$$20 - 6(15) < 0$$
$$-70 < 0$$

The second order condition holds for $Q = 15$, confirming it is a maximum. Therefore, the optimal output level of this business is at $Q = 15$. This denoted in *Figure 23* as the *x*-coordinate of the intersection of the marginal revenue and marginal cost curve.

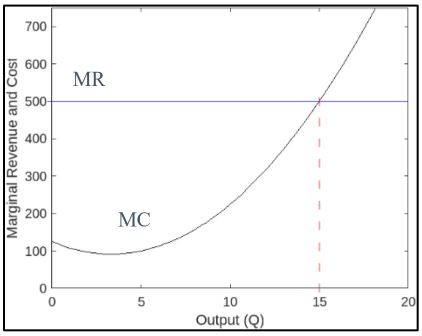

Figure 23 – Graphs of MR and MC (constructed in MATLAB)

Short vs Long Run

This model can be applied to various circumstances, whether the firm is working in the short or long run. When working in the short run, the firm is in a planning period where they keep at least one variable fixed while adjusting others – such as raw materials used in production – in an attempt to meet market demand. The long run, however, accommodates for all inputs to be variable and as such they are able to fully adjust to the market hence reaching an equilibrium [43]. In terms of profit maximisation, the optimal output for the short run can only be determined by past investment decisions and minimal current market information. Whereas, in the long run, all information regarding input and output is available to the firm and there is no inclination for expansion or contraction within. Therefore, it, arguably, provides a more accurate optimal level of output within the profit maximising model.

Within the short run, firms are able to make supernormal profit; excess profit/that above the normal level [44]. This is often pivotal to the initial growth of a firm as it can be invested in market research to better estimate the optimal level of output. Although, this can provide incentive for other firms to enter the market, in turn increasing supply and diminishing the once supernormal profit. In complete contrast, firms in the long run operate with normal profit; the minimum level of profit required for the firm to survive within the market. This is due to the fact that firms in the long run are able to make the necessary adjustments to their capacity to align with the current market conditions.

Limitations of the Model

While mathematics provides a definitive route for calculating profit maximisation through both differential calculus and graphical analysis, the method itself is not without limitations. Arguably the most limiting factor is that, in reality, the marginal revenue and marginal cost is not easy to determine [45]. This is due to the elasticity of demand; fluctuations in the consumption of goods. Perhaps this is epitomised through the current cost of living crisis. As disposable income has decreased amongst consumers, they therefore are compelled to prioritise essentials. Ergo, the luxury market begins to destabilise as the consumption of such products fall. In conjunction with this, the mathematical model does not take into account the time value of money [46] nor the inevitability of inflation. Thus, the same calculation cannot be used continually but instead the model must be regenerated along with the ever-changing economy. Moreover, it cannot be used comparatively in the long run to determine the success of a business venture. Another limitation of this model is that it does not consider the uncertainty of the market or the high-risk factors involved when striving for profit maximisation. Hence, the optimal conditions calculated could in fact be detrimental to the survival of the firm if it is unable to cope with such risk.

The basis of the profit maximising model relies on the neoclassical idea that within a perfectly competitive economy – in which consumers are perfectly rational (motivated to maximise utility) – if the firm maximises their profits, then the outcome of such is said to have desirable qualities [47]. Perfect competition is characterised by the existence of a substantial number of buyers and sellers of a homogenous product which results in supply and demand remaining constant [48]. In this idealised framework for market economy, all firms are price takers (they themselves cannot influence the market price of their product) and market share does not influence product price (no monopoly; one firm does not have absolute control of the market) [49]. Within such a market, firms can easily enter and exit without being bound by regulations or incurring cost and they are not affected by other sectors (for example they rely on cheap and efficient transportation of their products instead of sustaining significant costs). The model also relies on perfect capital mobility; where there are no barriers to the movement of money or assets between countries, for example transaction costs and exchange rates become negligible. Furthermore, consumers must have perfect knowledge; complete information on the product and its previous and current prices throughout the market. This being said, the theory differs drastically from reality. The assumption of a homogenous product and all firms as price takers is particularly unrealistic due to the current market's emphasis on product differentiation and subsequent brand loyalty amongst consumers. Instead the complete antithesis of an imperfectly competitive economy is a more accurate reflection of today's society as companies compete for market share, coinciding with the idea of monopoly. Hence, this theoretical model economy isn't applicable to the current market and thus puts into question the reliability of the profit maximisation model.

In conclusion, calculus is evidently pivotal to society as one of the main motivations within firms, and thus the market, is profit maximisation calculated through differential calculus and its graphical representations. Despite its limitations, the theory and model act as a sturdy basis for economists to work from and hence play a large role in the economy.

5 | Application: Celestial Bodies

Student 6

Nicolaus Copernicus – a well-known Polish astronomer- developed a model of our solar system to demonstrate uniform circular motion [50] as well as heliocentrism. Previously it was believed that celestial bodies revolved around the Earth: the idea that these bodies moved in a uniform circular motion had been developed from ideas put forth by Aristotle.

Kepler's Laws

In 1610 Johannes Kepler determined that the moon would now have the classification of satellite [51]. He also discovered a relationship between the time it takes for a planet to orbit once around the sun [52]. In doing this he discovered that, contrary to Copernicus' belief, planetary trajectories were not circular but in fact elliptical. This was one of the three laws Kepler created in regard to the motion of celestial bodies.

Kepler's laws were:

1. The planets move around the sun in orbits which are ellipses, with the sun at one focus.
2. The radius vector from the sun to a planet sweeps out equal areas in equal times
3. The ratio of the square of the of the orbital period to the cubed of the ellipsis semi major axis is the same for all planets [53]

These laws created what is known as the Kepler problem or the two-body problem.

Newton's Laws

Isaac Newton furthered research on gravity which was applicable to celestial bodies. He developed his laws of motion which are as follows:

Law 1 – Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by external, unbalanced forces to change that state

Law 2 – If a body of mass m is subject to various forces and if a is its acceleration as observed in an inertial co-ordinate frame then

$$\sum F = ma$$

Equation 42 – Newton's 2nd Law

Law 3 – If a body 1 exerts a force F_2 on body 2 and the latter exerts a force F_1 on the former, the regardless of what other forces may be acting on the two bodies these forces are equal and opposite:

$$F_1 = F_2$$

Equation 43 – Newton's 3rd Law

The Two-body Problem

These laws are applicable in both terrestrial and extra-terrestrial scenarios such as motion of celestial bodies. Newtons law of gravitation shown below can be applied to the Kepler problem where F is force, G is gravity, m_1 is the mass of object one, m_2 is the mass of object two and r is the distance between the objects.

$$F = \frac{G(m_1, m_2)}{r^2}$$

Equation 44 – Newton's Law of Gravitation

Solving the two-body problem for the motion of celestial bodies can be done using calculus and Newton's laws. MIT lectures taught this problem in 2016 using this [54]. They show that if body 1's position is denoted as $\overrightarrow{r_1}$ and body 2's is $\overrightarrow{r_2}$ then the difference of $\overrightarrow{r_1} - \overrightarrow{r_2}$ is equal to $\vec{r}$. With magnitude r and $\hat{r}$ is the unit vector of body 1 from body 2.

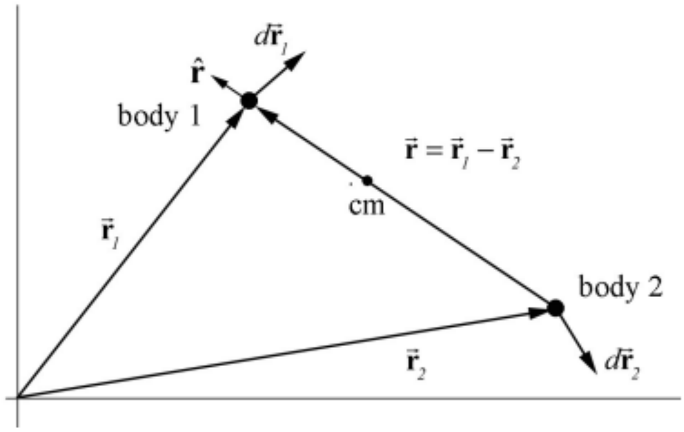

Figure 24 – Two-body problem [54]

They also use the reduce mass μ in a single body scenario to show:

$$\frac{1}{\mu} = \frac{1}{m1} + \frac{1}{m2}$$

$$\mu = \frac{m1m2}{m1 + m2}$$

They continue to apply Newton's laws to show:

"$\overrightarrow{F_{2,1}} = -F_{2,1}\hat{r} = -G\frac{(m1m2)}{r^2}\hat{r} \ldots$
$\frac{F_{2,1}}{m1} = \frac{d^2\overrightarrow{r_1}}{dt^2} \ldots$
$\frac{F_{1,2}}{m2} = \frac{d^2\overrightarrow{r_2}}{dt^2} \ldots$
$\frac{F_{2,1}}{m1} - \frac{F_{1,2}}{m2} = \frac{d^2\overrightarrow{r1}}{dt^2} - \frac{d^2r2}{dt^2} = \frac{d^2\vec{r}}{dt^2} \ldots$
$\overrightarrow{F_{2,1}}\left(\frac{1}{m1} + \frac{1}{m2}\right) = \frac{d^2\vec{r}}{dt^2} \ldots$
$\frac{\overrightarrow{F_{2,1}}}{\mu} = \frac{d^2\vec{r}}{dt^2} \ldots$
$\frac{F_{2,1}}{1} = \mu\frac{d^2\vec{r}}{dt^2}$"

Modern Applications

The application of differentiation is a compulsory component within this method of solution, majorly due to its compliance with Newton's laws. The interaction of both Kepler and Newton's laws are a large aspect of celestial mechanics as they are applicable to majority of situations demonstrated within the field of research. The calculus involved in conjunction with a range of other mathematical applications give an insight into conditions that are not usual/naturally occurring on Earth. Celestial mechanics currently have applications such as the launching and maintain of artificial satellites as the laws are used in the determination of planetary orbits and gravitational fields. In addition to this, the development of celestial mechanics allows agencies interested in defence can track and prevent any celestial bodies such as space debris and asteroids from becoming a threat to Earth.

Discovery of Neptune

In 1846, Le Verrier predicted the existence of the planet Neptune through the use of calculus due to observation made about Uranus' orbit. Uranus' orbit did not match it's predicted orbit- it appeared that the planet was being pulled out of its expected path by another gravitational field. A Harvard paper titled "A short method for the discovery of Neptune" was written by R. A. Lyttleton. This shows the mathematical process that Le Verrier went through to come to this discovery [55]. The formulas used are be shown below:

$$v = nt + \varepsilon + 2e\sin(nt + \varepsilon - \omega)$$

Equation 45 – Elliptic motion equation

This is the equation shows undisturbed elliptic motion for heliocentric (as thought by Kepler) longitude.
Changes in longitude can be calculated with the formula below:

$$\Delta v = m'(a_0 + bt + c\cos(nt) + d\sin(nt))$$

Equation 46 – Change in longitude formula

When $v(t) =$ heliocentric longitude from observation and $v_e(t)$ be the calculated value.
Since m' is unknown we use the formula

$$\delta(t) = v(t) - v_e(t)$$

Let $\omega =$ disturbances caused by m' since t_0

$$\omega(t) = v(t) + \ v_0(t)$$
$$\delta(t) = (v_0 - v_e) + \omega$$

By ignoring m' and letting $W(\tau) = \omega(t) = \omega(t_0 + \tau)$, we can get the equations:

$$\delta(t) = m'[a_0 + bt_0 + b_\tau + c\cos(nt_0 + n_\tau) + d\sin(nt_0 + n_\tau)] + W(\tau)$$
$$\delta(t_0 + \tau) = A + B(I - \cos(n\tau)) + (C\tau + D\sin(n\tau) + W(\tau))$$
$$\delta(t_0 - \tau) = A + B(I - \cos(n\tau)) - (C\tau + D\sin(n\tau) + W(\tau))$$
$$p(\tau) = \frac{\delta(t_0 + \tau) - 2\delta(t_0) + \delta(t_0 - \tau)}{I - \cos(n\tau)}$$

The modern way of relating true longitude and mean longitude is:

$$r^2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right) = h = nab$$
$$\frac{r^2}{ab}\mathrm{d}v = n\mathrm{d}t = \mathrm{d}l$$
$$v = \int hr^{-2}\mathrm{d}t$$
$$l = nt + \varepsilon$$
$$\Delta v = \int (r^{-3}\Delta h - 2hr^{-3}\Delta r)\mathrm{d}t$$
$$\Delta l = t\Delta n + \Delta\varepsilon$$

Equation 47 – True longitude-mean longitude relation

Le Verrier's method gave a longitude of 327.4 degrees whereas the current method gives a longitude of 329.4 degrees. With a difference of 2 degrees, it shows that Le Verrier's predictions were close enough to allow Neptune to be discoverable through a telescope. Le Verrier's calculations were accurate considering the difference in knowledge comparative to modern times- also considering that these calculations may be run by computer software currently.

Application of Calculus in Astrophysics

Calculus has a multitude of uses within the field of astrophysics and further, astrodynamics. Using calculus in conjunction with mechanics and other mathematical methods, has led to the development of our knowledge of a field, long pondered by philosophers and theorists. Further application can develop the laws and previous methods to build upon this knowledge- possibly discover more planets (Pluto had been discovered by a similar method to Neptune) and allow satellites to orbit them using Newtons laws of gravitation. Without the use of mathematics as shown previously, our knowledge of space would have been significantly hindered, it would be possible that Neptune remained undiscovered. Calculus has become such a large part of mathematics that it is taught widely and has many applications even within small fields.

Conclusion

This report discussed the history and applications of calculus. From the Classical Greek era of pure geometry (millennia before calculus' formalisation) through to our current use of the field in sciences, a broad range of interesting ideas have been reviewed.

Integration is one of the most fundamental parts of calculus. Its method was considered many years before the 'invention' of the study itself – the idea of it being an operation had gone straight under the noses of mathematicians for centuries, until the bright minds of Newton and Leibniz formalised calculus with concrete definitions. The Fundamental Theorem of Calculus was a very important advancement in the field as it provided a new range of techniques to calculate integrals, and with it came the differential which is a principal part of the study, in both differentiation and integration. Integration has proved to be useful in many situations, including differential equations and a lot of calculus-applied situations, as described throughout the report.

Differential calculus is yet another important object in calculus. Differentiation is the 'other half' of integration, being the inverse of it, so many methods involve the pair of them hand-in-hand. Differentials allow us to describe the relationships between variables, giving us the ability to define instantaneous rates of change. Derivatives are able to be formulated into equations, even with their further derivatives, to form differential equations, which can be solved to find variables as functions of others.

Calculus is crucial in the modelling of epidemiology, namely the SIR model for the spread of disease, especially within recent years with the global pandemic of 2020. Of course, some assumptions are made and other variables, such as vaccinations, need to be considered, so there are limitations to the model. Communicability of a disease is a factor too, as it can depend on the individual so it could be risky to generalise a whole population. Differential equations are clearly an important part of modelling such scenarios.

The profit maximising model is an important application of calculus as it lies at the centre of the economy, of which is fundamental to modern day society. However, the limitations to such puts its reliability as a realistic model into question. The model's assumption of a perfectly competitive market, void of product differentiation and additional costs incurred during the manufacture and shipment of goods is somewhat unrealistic. In conjunction, the model itself is perhaps flawed given the complexity of calculating an accurate marginal revenue and marginal cost. This is in addition to the idea that the time value of money becomes negligible. Irrespective of this, the theory behind it acts as an incentive within the current market economy and as such the model is a necessary construction, attempting to quantify the conditions required for firms to achieve profit maximisation.

Calculus has proved to be important in the study of space. Without the use of it, Neptune may never have been discovered, as well as Pluto. We could even potentially discover even more bodies in space using the tools in the field. They have also been useful in allowing us to model the movement of such bodies and predict things such as the trajectory of meteorites and planets.

(JH, NG)

Appendices

Minutes

02/02/2023

Attendance: (NG) , (EP) , (ED) , (JH) , (SH)
Apologies: none
A general introduction to the group study project was given by Manuela Maura and we were allocated our topic - 'calculus in history and its applications'.
Roles were confirmed: EP allocated as Chairperson, NG as secretary, ED as research coordinator, JH as report coordinator and IT coordinator and SH as presentation coordinator.
Any other progress: A Snapchat group was created for ease of communication.
Agenda for the next meeting: Narrow in on topic(s), find research sources and assign presenting order.

07/02/2023

Attendance: (NG) , (EP) , (ED) , (JH) , (SH)
Apologies: none
Research: 'A Concise History of Mathematics' – Dirk Jan Struik
Presentation: Sub-topics to be discussed chronologically, collective introduction and conclusion, sections 1 and 2 on development through history (covered by SH and JH), sections 3-5 on its applications (covered by ED, EP and NG).
Agenda for the next meeting: More in-depth research on sub-topics and allocate them accordingly to sections.

14/02/2023

Attendance: (NG) , (EP) , (ED) , (JH) , (SH)
Apologies: none
Research:

- Where has the field of epidemiology come from?
- Simulating An Epidemic – Grant Sanderson
- The R value and growth rate – Government resource

Presentation: The applications to be split into epidemiology (covered by ED), profit margins and maximisation (covered by NG) and meteorology (covered by EP). The coinciding topics of integrals and differentials and the history of such is to be covered by JH and SH respectively.
Agenda for the next meeting: Individual research to begin and be complete by 28/02/2023.

21/02/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: (EP)
Agenda for the next meeting: Continuation of individual research.

28/02/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Reallocation of roles: ED to take over from EP as chairperson.
Agenda for the next meeting: Determine internal deadlines.

07/03/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Internal deadlines: Individual report chapters to be sent via email to JH by the 14th of March, presentation slides to be sent via email to ED by the 19th of March.
Agenda for the next meeting: Complete the introduction and conclusion for the report and book rooms for future meetings.

14/03/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Future meetings: Sunday 19th 8 – 10pm, Thursday 23rd 2 – 3pm and Sunday 26th 2 – 3:30pm, all booked in UL202.
Agenda for the next meeting: Combine presentation slides.

19/03/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Progress: All slides combined and a rough run through completed.
Agenda for the next meeting: Practice presentation.

23/03/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Progress: Practise one of presentation and discussion on introduction.
Agenda for the next meeting: Solidify presentation.

26/03/2023

Attendance: (NG) , (ED) , (JH) , (SH)
Apologies: None
Progress: Presentation fully complete and practised in preparation for the formal presentation on 27/03/2023.

Figures & Equations

Figures

Equations

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