of any one wave crest? (b) Explain why the amplitude of the wave increases as the wave approaches shore. (c) If the wave has amplitude 1.80 m when its speed is 200 m/s, what will be its amplitude where the water is 9.00 m deep? (d) Explain why the amplitude at the shore should be expected to be still greater, but cannot be meaningfully predicted by your model.

55. AMT **Review.** A block of mass $M = 0.450$ kg is attached to one end of a cord of mass 0.003 20 kg; the other end of the cord is attached to a fixed point. The block rotates with constant angular speed in a circle on a frictionless, horizontal table as shown in Figure P16.55. Through what angle does the block rotate in the time interval during which a transverse wave travels along the string from the center of the circle to the block?

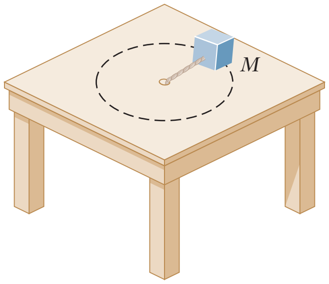

Figure P16.55 Problems 55, 56, and 57.

56. **Review.** A block of mass $M = 0.450$ kg is attached to one end of a cord of mass $m = 0.003$ 20 kg; the other end of the cord is attached to a fixed point. The block rotates with constant angular speed $\omega = 10.0$ rad/s in a circle on a frictionless, horizontal table as shown in Figure P16.55. What time interval is required for a transverse wave to travel along the string from the center of the circle to the block?

57. **Review.** A block of mass M is attached to one end of a cord of mass m; the other end of the cord is attached to a fixed point. The block rotates with constant angular speed ω in a circle on a frictionless, horizontal table as shown in Figure P16.55. What time interval is required for a transverse wave to travel along the string from the center of the circle to the block?

58. A string with linear density 0.500 g/m is held under tension 20.0 N. As a transverse sinusoidal wave propagates on the string, elements of the string move with maximum speed $v_{y,\max}$. (a) Determine the power transmitted by the wave as a function of $v_{y,\max}$. (b) State in words the proportionality between power and $v_{y,\max}$. (c) Find the energy contained in a section of string 3.00 m long as a function of $v_{y,\max}$. (d) Express the answer to part (c) in terms of the mass m of this section. (e) Find the energy that the wave carries past a point in 6.00 s.

59. A wire of density ρ is tapered so that its cross-sectional area varies with x according to

$$A = 1.00 \times 10^{-5}\, x + 1.00 \times 10^{-6}$$

where A is in meters squared and x is in meters. The tension in the wire is T. (a) Derive a relationship for the speed of a wave as a function of position. (b) **What If?** Assume the wire is aluminum and is under a tension $T = 24.0$ N. Determine the wave speed at the origin and at $x = 10.0$ m.

60. A rope of total mass m and length L is suspended vertically. Analysis shows that for short transverse pulses, the waves above a short distance from the free end of the rope can be represented to a good approximation by the linear wave equation discussed in Section 16.6. Show that a transverse pulse travels the length of the rope in a time interval that is given approximately by $\Delta t \approx 2\sqrt{L/g}$. *Suggestion:* First find an expression for the wave speed at any point a distance x from the lower end by considering the rope's tension as resulting from the weight of the segment below that point.

61. A pulse traveling along a string of linear mass density μ is described by the wave function

$$y = [A_0 e^{-bx}] \sin (kx - \omega t)$$

where the factor in brackets is said to be the amplitude. (a) What is the power $P(x)$ carried by this wave at a point x? (b) What is the power $P(0)$ carried by this wave at the origin? (c) Compute the ratio $P(x)/P(0)$.

62. *Why is the following situation impossible?* Tsunamis are ocean surface waves that have enormous wavelengths (100 to 200 km), and the propagation speed for these waves is $v \approx \sqrt{g d_{\text{avg}}}$, where d_{avg} is the average depth of the water. An earthquake on the ocean floor in the Gulf of Alaska produces a tsunami that reaches Hilo, Hawaii, 4 450 km away, in a time interval of 5.88 h. (This method was used in 1856 to estimate the average depth of the Pacific Ocean long before soundings were made to give a direct determination.)

63. M **Review.** An aluminum wire is held between two clamps under zero tension at room temperature. Reducing the temperature, which results in a decrease in the wire's equilibrium length, increases the tension in the wire. Taking the cross-sectional area of the wire to be 5.00×10^{-6} m^2, the density to be 2.70×10^3 kg/m^3, and Young's modulus to be 7.00×10^{10} N/m^2, what strain ($\Delta L/L$) results in a transverse wave speed of 100 m/s?

Challenge Problems

64. Assume an object of mass M is suspended from the bottom of the rope of mass m and length L in Problem 60. (a) Show that the time interval for a transverse pulse to travel the length of the rope is

$$\Delta t = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M + m} - \sqrt{M}\right)$$

(b) **What If?** Show that the expression in part (a) reduces to the result of Problem 60 when $M = 0$. (c) Show that for $m << M$, the expression in part (a) reduces to

$$\Delta t = \sqrt{\frac{mL}{Mg}}$$

65. A rope of total mass m and length L is suspended vertically. As shown in Problem 60, a pulse travels from the bottom to the top of the rope in an approximate time interval $\Delta t = 2\sqrt{L/g}$ with a speed that varies with position x measured from the bottom of the rope as $v = \sqrt{gx}$. Assume the linear wave equation in Section 16.6 describes waves at all locations on the rope. (a) Over what time interval does a pulse travel halfway up the rope? Give your answer as a fraction of the quantity $2\sqrt{L/g}$. (b) A pulse starts traveling up the rope. How far has it traveled after a time interval $\sqrt{L/g}$?

66. A string on a musical instrument is held under tension T and extends from the point $x = 0$ to the point $x = L$. The string is overwound with wire in such a way that its mass per unit length $\mu(x)$ increases uniformly from μ_0 at $x = 0$ to μ_L at $x = L$. (a) Find an expression for $\mu(x)$ as a function of x over the range $0 \le x \le L$. (b) Find an expression for the time interval required for a transverse pulse to travel the length of the string.

67. If a loop of chain is spun at high speed, it can roll along the ground like a circular hoop without collapsing. Consider a chain of uniform linear mass density μ whose center of mass travels to the right at a high speed v_0 as shown in Figure P16.67. (a) Determine the tension in the chain in terms of μ and v_0. Assume the weight of an individual link is negligible compared to the tension. (b) If the loop rolls over a small bump, the resulting deformation of the chain causes two transverse pulses to propagate along the chain, one moving clockwise and one moving counterclockwise. What is the speed of the pulses traveling along the chain? (c) Through what angle does each pulse travel during the time interval over which the loop makes one revolution?

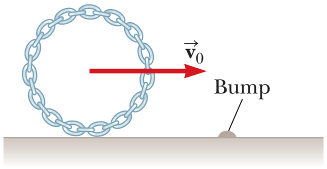

Figure P16.67

Sound Waves

CHAPTER
17

Three musicians play the alpenhorn in Valais, Switzerland. In this chapter, we explore the behavior of sound waves such as those coming from these large musical instruments. *(Stefano Cellai/AGE fotostock)*

Most of the waves we studied in Chapter 16 are constrained to move along a one-dimensional medium. For example, the wave in Figure 16.7 is a purely mathematical construct moving along the *x* axis. The wave in Figure 16.10 is constrained to move along the length of the string. We have also seen waves moving through a two-dimensional medium, such as the ripples on the water surface in the introduction to Part 2 on page 449 and the waves moving over the surface of the ocean in Figure 16.4. In this chapter, we investigate mechanical waves that move through three-dimensional bulk media. For example, seismic waves leaving the focus of an earthquake travel through the three-dimensional interior of the Earth.

We will focus our attention on **sound waves,** which travel through any material, but are most commonly experienced as the mechanical waves traveling through air that result in the human perception of hearing. As sound waves travel through air, elements of air are disturbed from their equilibrium positions. Accompanying these movements are changes in density and pressure of the air along the direction of wave motion. If the source of the sound waves vibrates sinusoidally, the density and pressure variations are also sinusoidal. The mathematical description of sinusoidal sound waves is very similar to that of sinusoidal waves on strings, as discussed in Chapter 16.

Sound waves are divided into three categories that cover different frequency ranges. (1) *Audible waves* lie within the range of sensitivity of the human ear. They can be generated in a variety of ways, such as by musical instruments, human voices, or loudspeakers. (2) *Infrasonic waves* have frequencies below the audible range. Elephants can use infrasonic waves to communicate with one another, even when separated by many kilometers. (3) *Ultrasonic waves* have frequencies above the audible range. You may have used a "silent" whistle to retrieve your dog. Dogs easily hear the ultrasonic sound this whistle emits, although humans cannot detect it at all. Ultrasonic waves are also used in medical imaging.

This chapter begins with a discussion of the pressure variations in a sound wave, the speed of sound waves, and wave intensity, which is a function of wave amplitude. We then provide an alternative description of the intensity of sound waves that compresses the wide range of intensities to which the ear is sensitive into a smaller range for convenience. The effects of the motion of sources and listeners on the frequency of a sound are also investigated.

17.1 Pressure Variations in Sound Waves

In Chapter 16, we began our investigation of waves by imagining the creation of a single pulse that traveled down a string (Figure 16.1) or a spring (Figure 16.3). Let's do something similar for sound. We describe pictorially the motion of a one-dimensional longitudinal sound pulse moving through a long tube containing a compressible gas as shown in Figure 17.1. A piston at the left end can be quickly moved to the right to compress the gas and create the pulse. Before the piston is moved, the gas is undisturbed and of uniform density as represented by the uniformly shaded region in Figure 17.1a. When the piston is pushed to the right (Fig. 17.1b), the gas just in front of it is compressed (as represented by the more heavily shaded region); the pressure and density in this region are now higher than they were before the piston moved. When the piston comes to rest (Fig. 17.1c), the compressed region of the gas continues to move to the right, corresponding to a longitudinal pulse traveling through the tube with speed v.

One can produce a one-dimensional *periodic* sound wave in the tube of gas in Figure 17.1 by causing the piston to move in simple harmonic motion. The results are shown in Figure 17.2. The darker parts of the colored areas in this figure represent regions in which the gas is compressed and the density and pressure are above their equilibrium values. A compressed region is formed whenever the pis-

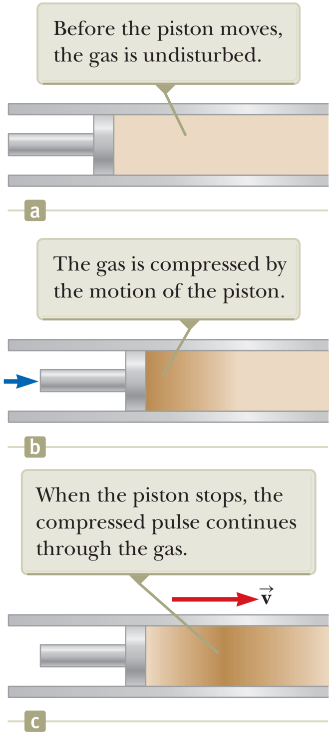

Figure 17.1 Motion of a longitudinal pulse through a compressible gas. The compression (darker region) is produced by the moving piston.

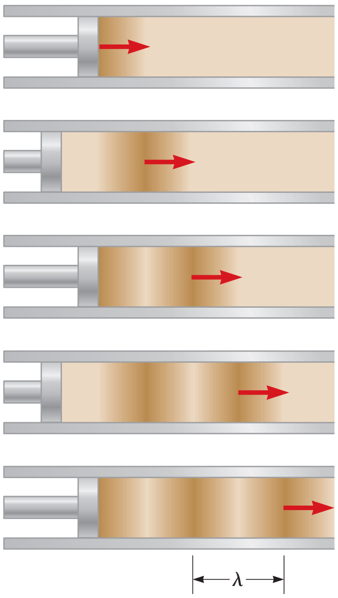

Figure 17.2 A longitudinal wave propagating through a gas-filled tube. The source of the wave is an oscillating piston at the left.

ton is pushed into the tube. This compressed region, called a **compression,** moves through the tube, continuously compressing the region just in front of itself. When the piston is pulled back, the gas in front of it expands and the pressure and density in this region fall below their equilibrium values (represented by the lighter parts of the colored areas in Fig. 17.2). These low-pressure regions, called **rarefactions,** also propagate along the tube, following the compressions. Both regions move at the speed of sound in the medium.

As the piston oscillates sinusoidally, regions of compression and rarefaction are continuously set up. The distance between two successive compressions (or two successive rarefactions) equals the wavelength λ of the sound wave. Because the sound wave is longitudinal, as the compressions and rarefactions travel through the tube, any small element of the gas moves with simple harmonic motion parallel to the direction of the wave. If $s(x, t)$ is the position of a small element relative to its equilibrium position,[1] we can express this harmonic position function as

$$s(x, t) = s_{\max} \cos (kx - \omega t) \tag{17.1}$$

where $s_{\max}$ is the maximum position of the element relative to equilibrium. This parameter is often called the **displacement amplitude** of the wave. The parameter k is the wave number, and ω is the angular frequency of the wave. Notice that the displacement of the element is along x, in the direction of propagation of the sound wave.

The variation in the gas pressure ΔP measured from the equilibrium value is also periodic with the same wave number and angular frequency as for the displacement in Equation 17.1. Therefore, we can write

$$\Delta P = \Delta P_{\max} \sin (kx - \omega t) \tag{17.2}$$

where **the pressure amplitude $\Delta P_{\max}$** is the maximum change in pressure from the equilibrium value.

Notice that we have expressed the displacement by means of a cosine function and the pressure by means of a sine function. We will justify this choice in the procedure that follows and relate the pressure amplitude $P_{\max}$ to the displacement amplitude $s_{\max}$. Consider the piston–tube arrangement of Figure 17.1 once again. In Figure 17.3a, we focus our attention on a small cylindrical element of undisturbed gas of length Δx and area A. The volume of this element is $V_i = A\,\Delta x$.

Figure 17.3b shows this element of gas after a sound wave has moved it to a new position. The cylinder's two flat faces move through different distances s_1 and s_2. The change in volume ΔV of the element in the new position is equal to $A\,\Delta s$, where $\Delta s = s_1 - s_2$.

From the definition of bulk modulus (see Eq. 12.8), we express the pressure variation in the element of gas as a function of its change in volume:

$$\Delta P = -B \frac{\Delta V}{V_i}$$

Let's substitute for the initial volume and the change in volume of the element:

$$\Delta P = -B \frac{A\,\Delta s}{A\,\Delta x}$$

Let the length Δx of the cylinder approach zero so that the ratio $\Delta s/\Delta x$ becomes a partial derivative:

$$\Delta P = -B \frac{\partial s}{\partial x} \tag{17.3}$$

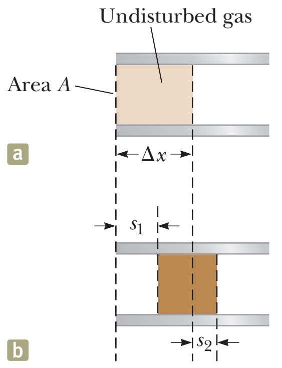

Figure 17.3 (a) An undisturbed element of gas of length Δx in a tube of cross-sectional area A. (b) When a sound wave propagates through the gas, the element is moved to a new position and has a different length. The parameters s_1 and s_2 describe the displacements of the ends of the element from their equilibrium positions.

[1]We use $s(x, t)$ here instead of $y(x, t)$ because the displacement of elements of the medium is not perpendicular to the x direction.

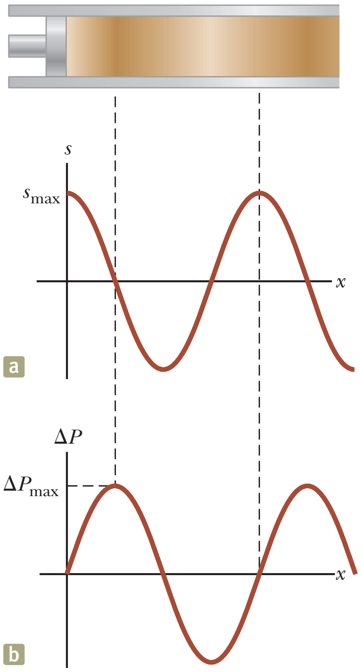

Figure 17.4 (a) Displacement amplitude and (b) pressure amplitude versus position for a sinusoidal longitudinal wave.

Substitute the position function given by Equation 17.1:

$$\Delta P = -B\frac{\partial}{\partial x}[s_{max}\cos(kx - \omega t)] = Bs_{max}k\ \sin(kx - \omega t)$$

From this result, we see that a displacement described by a cosine function leads to a pressure described by a sine function. We also see that the displacement and pressure amplitudes are related by

$$\Delta P_{max} = Bs_{max}k \tag{17.4}$$

This relationship depends on the bulk modulus of the gas, which is not as readily available as is the density of the gas. Once we determine the speed of sound in a gas in Section 17.2, we will be able to provide an expression that relates ΔP_{max} and s_{max} in terms of the density of the gas.

This discussion shows that a sound wave may be described equally well in terms of either pressure or displacement. A comparison of Equations 17.1 and 17.2 shows that the pressure wave is 90° out of phase with the displacement wave. Graphs of these functions are shown in Figure 17.4. The pressure variation is a maximum when the displacement from equilibrium is zero, and the displacement from equilibrium is a maximum when the pressure variation is zero.

Quick Quiz 17.1 If you blow across the top of an empty soft-drink bottle, a pulse of sound travels down through the air in the bottle. At the moment the pulse reaches the bottom of the bottle, what is the correct description of the displacement of elements of air from their equilibrium positions and the pressure of the air at this point? **(a)** The displacement and pressure are both at a maximum. **(b)** The displacement and pressure are both at a minimum. **(c)** The displacement is zero, and the pressure is a maximum. **(d)** The displacement is zero, and the pressure is a minimum.

17.2 Speed of Sound Waves

We now extend the discussion begun in Section 17.1 to evaluate the speed of sound in a gas. In Figure 17.5a, consider the cylindrical element of gas between the piston and the dashed line. This element of gas is in equilibrium under the influence of forces of equal magnitude, from the piston on the left and from the rest of the gas on the right. The magnitude of these forces is PA, where P is the pressure in the gas and A is the cross-sectional area of the tube.

Figure 17.5b shows the situation after a time interval Δt during which the piston moves to the right at a constant speed v_x due to a force from the left on the piston that has increased in magnitude to $(P + \Delta P)A$. By the end of the time interval Δt,

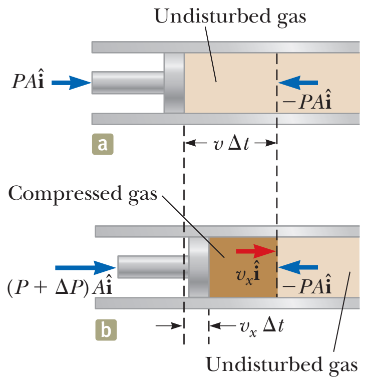

Figure 17.5 (a) An undisturbed element of gas of length $v\ \Delta t$ in a tube of cross-sectional area A. The element is in equilibrium between forces on either end. (b) When the piston moves inward at constant velocity v_x due to an increased force on the left, the element also moves with the same velocity.

every bit of gas in the element is moving with speed v_x. That will not be true in general for a macroscopic element of gas, but it will become true if we shrink the length of the element to an infinitesimal value.

The length of the undisturbed element of gas is chosen to be $v\,\Delta t$, where v is the speed of sound in the gas and Δt is the time interval between the configurations in Figures 17.5a and 17.5b. Therefore, at the end of the time interval Δt, the sound wave will just reach the right end of the cylindrical element of gas. The gas to the right of the element is undisturbed because the sound wave has not reached it yet.

The element of gas is modeled as a nonisolated system in terms of momentum. The force from the piston has provided an impulse to the element, which in turn exhibits a change in momentum. Therefore, we evaluate both sides of the impulse–momentum theorem:

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{I}} \tag{17.5}$$

On the right, the impulse is provided by the constant force due to the increased pressure on the piston:

$$\vec{\mathbf{I}} = \sum \vec{\mathbf{F}}\,\Delta t = (A\,\Delta P\,\Delta t)\,\hat{\mathbf{i}}$$

The pressure change ΔP can be related to the volume change and then to the speeds v and v_x through the bulk modulus:

$$\Delta P = -B\frac{\Delta V}{V_i} = -B\frac{(-v_x A\,\Delta t)}{vA\,\Delta t} = B\frac{v_x}{v}$$

Therefore, the impulse becomes

$$\vec{\mathbf{I}} = \left(AB\frac{v_x}{v}\Delta t\right)\hat{\mathbf{i}} \tag{17.6}$$

On the left-hand side of the impulse–momentum theorem, Equation 17.5, the change in momentum of the element of gas of mass m is as follows:

$$\Delta\vec{\mathbf{p}} = m\,\Delta\vec{\mathbf{v}} = (\rho V_i)(v_x\hat{\mathbf{i}} - 0) = (\rho v v_x A\,\Delta t)\,\hat{\mathbf{i}} \tag{17.7}$$

Substituting Equations 17.6 and 17.7 into Equation 17.5, we find

$$\rho v v_x A\,\Delta t = AB\frac{v_x}{v}\Delta t$$

which reduces to an expression for the speed of sound in a gas:

$$v = \sqrt{\frac{B}{\rho}} \tag{17.8}$$

It is interesting to compare this expression with Equation 16.18 for the speed of transverse waves on a string, $v = \sqrt{T/\mu}$. In both cases, the wave speed depends on an elastic property of the medium (bulk modulus B or string tension T) and on an inertial property of the medium (volume density ρ or linear density μ). In fact, the speed of all mechanical waves follows an expression of the general form

$$v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$$

For longitudinal sound waves in a solid rod of material, for example, the speed of sound depends on Young's modulus Y and the density ρ. Table 17.1 (page 512) provides the speed of sound in several different materials.

The speed of sound also depends on the temperature of the medium. For sound traveling through air, the relationship between wave speed and air temperature is

$$v = 331\sqrt{1 + \frac{T_{\mathrm{C}}}{273}} \tag{17.9}$$

Table 17.1 Speed of Sound in Various Media

Medium	v (m/s)	Medium	v (m/s)	Medium	v (m/s)
Gases		**Liquids at 25°C**		**Solids**[a]	
Hydrogen (0°C)	1 286	Glycerol	1 904	Pyrex glass	5 640
Helium (0°C)	972	Seawater	1 533	Iron	5 950
Air (20°C)	343	Water	1 493	Aluminum	6 420
Air (0°C)	331	Mercury	1 450	Brass	4 700
Oxygen (0°C)	317	Kerosene	1 324	Copper	5 010
		Methyl alcohol	1 143	Gold	3 240
		Carbon tetrachloride	926	Lucite	2 680
				Lead	1 960
				Rubber	1 600

[a]Values given are for propagation of longitudinal waves in bulk media. Speeds for longitudinal waves in thin rods are smaller, and speeds of transverse waves in bulk are smaller yet.

where v is in meters/second, 331 m/s is the speed of sound in air at 0°C, and T_C is the air temperature in degrees Celsius. Using this equation, one finds that at 20°C, the speed of sound in air is approximately 343 m/s.

This information provides a convenient way to estimate the distance to a thunderstorm. First count the number of seconds between seeing the flash of lightning and hearing the thunder. Dividing this time interval by 3 gives the approximate distance to the lightning in kilometers because 343 m/s is approximately $\frac{1}{3}$ km/s. Dividing the time interval in seconds by 5 gives the approximate distance to the lightning in miles because the speed of sound is approximately $\frac{1}{5}$ mi/s.

Having an expression (Eq. 17.8) for the speed of sound, we can now express the relationship between pressure amplitude and displacement amplitude for a sound wave (Eq. 17.4) as

$$\Delta P_{\max} = B s_{\max} k = (\rho v^2) s_{\max}\left(\frac{\omega}{v}\right) = \rho v \omega s_{\max} \tag{17.10}$$

This expression is a bit more useful than Equation 17.4 because the density of a gas is more readily available than is the bulk modulus.

17.3 Intensity of Periodic Sound Waves

In Chapter 16, we showed that a wave traveling on a taut string transports energy, consistent with the notion of energy transfer by mechanical waves in Equation 8.2. Naturally, we would expect sound waves to also represent a transfer of energy. Consider the element of gas acted on by the piston in Figure 17.5. Imagine that the piston is moving back and forth in simple harmonic motion at angular frequency ω. Imagine also that the length of the element becomes very small so that the entire element moves with the same velocity as the piston. Then we can model the element as a particle on which the piston is doing work. The rate at which the piston is doing work on the element at any instant of time is given by Equation 8.19:

$$Power = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}_x$$

where we have used *Power* rather than P so that we don't confuse power P with pressure P! The force $\vec{\mathbf{F}}$ on the element of gas is related to the pressure and the velocity $\vec{\mathbf{v}}_x$ of the element is the derivative of the displacement function, so we find

$$Power = [\Delta P(x, t) A]\hat{\mathbf{i}} \cdot \frac{\partial}{\partial t}[s(x, t)\,\hat{\mathbf{i}}]$$

$$= [\rho v \omega A s_{\max} \sin(kx - \omega t)]\left\{\frac{\partial}{\partial t}[s_{\max} \cos(kx - \omega t)]\right\}$$

$$= \rho v \omega A s_{max} \sin (kx - \omega t)][\omega s_{max} \sin (kx - \omega t)]$$

$$= \rho v \omega^2 A s_{max}^2 \sin^2 (kx - \omega t)$$

We now find the time average power over one period of the oscillation. For any given value of x, which we can choose to be $x = 0$, the average value of $\sin^2 (kx - \omega t)$ over one period T is

$$\frac{1}{T}\int_0^T \sin^2 (0 - \omega t)\, dt = \frac{1}{T}\int_0^T \sin^2 \omega t\, dt = \frac{1}{T}\left(\frac{t}{2} + \frac{\sin 2\omega t}{2\omega}\right)\Bigg|_0^T = \tfrac{1}{2}$$

Therefore,

$$(Power)_{avg} = \tfrac{1}{2}\rho v \omega^2 A s_{max}^2$$

We define the **intensity** I of a wave, or the power per unit area, as the rate at which the energy transported by the wave transfers through a unit area A perpendicular to the direction of travel of the wave:

$$I \equiv \frac{(Power)_{avg}}{A} \qquad \textbf{(17.11)}$$

◀ **Intensity of a sound wave**

In this case, the intensity is therefore

$$I = \tfrac{1}{2}\rho v(\omega s_{max})^2$$

Hence, the intensity of a periodic sound wave is proportional to the square of the displacement amplitude and to the square of the angular frequency. This expression can also be written in terms of the pressure amplitude ΔP_{max}; in this case, we use Equation 17.10 to obtain

$$I = \frac{(\Delta P_{max})^2}{2\rho v} \qquad \textbf{(17.12)}$$

The string waves we studied in Chapter 16 are constrained to move along the one-dimensional string, as discussed in the introduction to this chapter. The sound waves we have studied with regard to Figures 17.1 through 17.3 and 17.5 are constrained to move in one dimension along the length of the tube. As we mentioned in the introduction, however, sound waves can move through three-dimensional bulk media, so let's place a sound source in the open air and study the results.

Consider the special case of a point source emitting sound waves equally in all directions. If the air around the source is perfectly uniform, the sound power radiated in all directions is the same, and the speed of sound in all directions is the same. The result in this situation is called a **spherical wave.** Figure 17.6 shows these spherical waves as a series of circular arcs concentric with the source. Each arc represents a surface over which the phase of the wave is constant. We call such a surface of constant phase a **wave front.** The radial distance between adjacent wave fronts that have the same phase is the wavelength λ of the wave. The radial lines pointing outward from the source, representing the direction of propagation of the waves, are called **rays.**

The average power emitted by the source must be distributed uniformly over each spherical wave front of area $4\pi r^2$. Hence, the wave intensity at a distance r from the source is

$$I = \frac{(Power)_{avg}}{A} = \frac{(Power)_{avg}}{4\pi r^2} \qquad \textbf{(17.13)}$$

The intensity decreases as the square of the distance from the source. This inverse-square law is reminiscent of the behavior of gravity in Chapter 13.

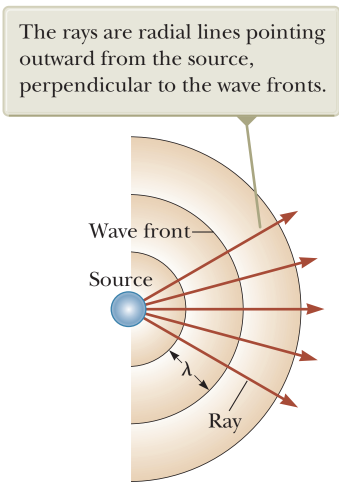

Figure 17.6 Spherical waves emitted by a point source. The circular arcs represent the spherical wave fronts that are concentric with the source.

Quick Quiz 17.2 A vibrating guitar string makes very little sound if it is not mounted on the guitar body. Why does the sound have greater intensity if the string is attached to the guitar body? **(a)** The string vibrates with more energy. **(b)** The energy leaves the guitar at a greater rate. **(c)** The sound power is spread over a larger area at the listener's position. **(d)** The sound power is concentrated over a smaller area at the listener's position. **(e)** The speed of sound is higher in the material of the guitar body. **(f)** None of these answers is correct.

Example 17.1 Hearing Limits

The faintest sounds the human ear can detect at a frequency of 1 000 Hz correspond to an intensity of about 1.00×10^{-12} W/m^2, which is called *threshold of hearing*. The loudest sounds the ear can tolerate at this frequency correspond to an intensity of about 1.00 W/m^2, the *threshold of pain*. Determine the pressure amplitude and displacement amplitude associated with these two limits.

SOLUTION

Conceptualize Think about the quietest environment you have ever experienced. It is likely that the intensity of sound in even this quietest environment is higher than the threshold of hearing.

Categorize Because we are given intensities and asked to calculate pressure and displacement amplitudes, this problem is an analysis problem requiring the concepts discussed in this section.

Analyze To find the amplitude of the pressure variation at the threshold of hearing, use Equation 17.12, taking the speed of sound waves in air to be v = 343 m/s and the density of air to be ρ = 1.20 kg/m^3:

$$\Delta P_{\max} = \sqrt{2\rho v I}$$
$$= \sqrt{2(1.20\ \mathrm{kg/m^3})(343\ \mathrm{m/s})(1.00 \times 10^{-12}\ \mathrm{W/m^2})}$$
$$= \boxed{2.87 \times 10^{-5}\ \mathrm{N/m^2}}$$

Calculate the corresponding displacement amplitude using Equation 17.10, recalling that $\omega = 2\pi f$ (Eq. 16.9):

$$s_{\max} = \frac{\Delta P_{\max}}{\rho v \omega} = \frac{2.87 \times 10^{-5}\ \mathrm{N/m^2}}{(1.20\ \mathrm{kg/m^3})(343\ \mathrm{m/s})(2\pi \times 1\,000\ \mathrm{Hz})}$$
$$= \boxed{1.11 \times 10^{-11}\ \mathrm{m}}$$

In a similar manner, one finds that the loudest sounds the human ear can tolerate (the threshold of pain) correspond to a pressure amplitude of $\boxed{28.7\ \mathrm{N/m^2}}$ and a displacement amplitude equal to $\boxed{1.11 \times 10^{-5}\ \mathrm{m}}$.

Finalize Because atmospheric pressure is about 10^5 N/m^2, the result for the pressure amplitude tells us that the ear is sensitive to pressure fluctuations as small as 3 parts in 10^{10}! The displacement amplitude is also a remarkably small number! If we compare this result for $s_{\max}$ to the size of an atom (about 10^{-10} m), we see that the ear is an extremely sensitive detector of sound waves.

Example 17.2 Intensity Variations of a Point Source

A point source emits sound waves with an average power output of 80.0 W.

(A) Find the intensity 3.00 m from the source.

SOLUTION

Conceptualize Imagine a small loudspeaker sending sound out at an average rate of 80.0 W uniformly in all directions. You are standing 3.00 m away from the speakers. As the sound propagates, the energy of the sound waves is spread out over an ever-expanding sphere, so the intensity of the sound falls off with distance.

Categorize We evaluate the intensity from an equation generated in this section, so we categorize this example as a substitution problem.

17.2 continued

Because a point source emits energy in the form of spherical waves, use Equation 17.13 to find the intensity:

$$I = \frac{(Power)_{avg}}{4\pi r^2} = \frac{80.0\text{ W}}{4\pi(3.00\text{ m})^2} = \boxed{0.707\text{ W/m}^2}$$

This intensity is close to the threshold of pain.

(B) Find the distance at which the intensity of the sound is 1.00×10^{-8} W/m^2.

SOLUTION

Solve for r in Equation 17.13 and use the given value for I:

$$r = \sqrt{\frac{(Power)_{avg}}{4\pi I}} = \sqrt{\frac{80.0\text{ W}}{4\pi(1.00 \times 10^{-8}\text{ W/m}^2)}}$$

$$= \boxed{2.52 \times 10^4\text{ m}}$$

Sound Level in Decibels

Example 17.1 illustrates the wide range of intensities the human ear can detect. Because this range is so wide, it is convenient to use a logarithmic scale, where the **sound level** β (Greek letter beta) is defined by the equation

$$\beta \equiv 10\log\left(\frac{I}{I_0}\right) \quad \textbf{(17.14)}$$

The constant I_0 is the *reference intensity,* taken to be at the threshold of hearing ($I_0 = 1.00 \times 10^{-12}$ W/m^2), and I is the intensity in watts per square meter to which the sound level β corresponds, where β is measured[2] in **decibels** (dB). On this scale, the threshold of pain ($I = 1.00$ W/m^2) corresponds to a sound level of $\beta = 10\log[(1\text{ W/m}^2)/(10^{-12}\text{ W/m}^2)] = 10\log(10^{12}) = 120$ dB, and the threshold of hearing corresponds to $\beta = 10\log[(10^{-12}\text{ W/m}^2)/(10^{-12}\text{ W/m}^2)] = 0$ dB.

Prolonged exposure to high sound levels may seriously damage the human ear. Ear plugs are recommended whenever sound levels exceed 90 dB. Recent evidence suggests that "noise pollution" may be a contributing factor to high blood pressure, anxiety, and nervousness. Table 17.2 gives some typical sound levels.

Table 17.2

Sound Levels

Source of Sound	β (dB)
Nearby jet airplane	150
Jackhammer; machine gun	130
Siren; rock concert	120
Subway; power lawn mower	100
Busy traffic	80
Vacuum cleaner	70
Normal conversation	60
Mosquito buzzing	40
Whisper	30
Rustling leaves	10
Threshold of hearing	0

Quick Quiz 17.3 Increasing the intensity of a sound by a factor of 100 causes the sound level to increase by what amount? **(a)** 100 dB **(b)** 20 dB **(c)** 10 dB **(d)** 2 dB

Example 17.3 Sound Levels

Two identical machines are positioned the same distance from a worker. The intensity of sound delivered by each operating machine at the worker's location is 2.0×10^{-7} W/m^2.

(A) Find the sound level heard by the worker when one machine is operating.

SOLUTION

Conceptualize Imagine a situation in which one source of sound is active and is then joined by a second identical source, such as one person speaking and then a second person speaking at the same time or one musical instrument playing and then being joined by a second instrument.

Categorize This example is a relatively simple analysis problem requiring Equation 17.14.

continued

[2]The unit *bel* is named after the inventor of the telephone, Alexander Graham Bell (1847–1922). The prefix *deci-* is the SI prefix that stands for 10^{-1}.

17.3 continued

Analyze Use Equation 17.14 to calculate the sound level at the worker's location with one machine operating:

$$\beta_1 = 10 \log \left(\frac{2.0 \times 10^{-7}\ \text{W/m}^2}{1.00 \times 10^{-12}\ \text{W/m}^2} \right) = 10 \log (2.0 \times 10^5) = \boxed{53\ \text{dB}}$$

(B) Find the sound level heard by the worker when two machines are operating.

SOLUTION

Use Equation 17.14 to calculate the sound level at the worker's location with double the intensity:

$$\beta_2 = 10 \log \left(\frac{4.0 \times 10^{-7}\ \text{W/m}^2}{1.00 \times 10^{-12}\ \text{W/m}^2} \right) = 10 \log (4.0 \times 10^5) = \boxed{56\ \text{dB}}$$

Finalize These results show that when the intensity is doubled, the sound level increases by only 3 dB. This 3-dB increase is independent of the original sound level. (Prove this to yourself!)

WHAT IF? *Loudness* is a psychological response to a sound. It depends on both the intensity and the frequency of the sound. As a rule of thumb, a doubling in loudness is approximately associated with an increase in sound level of 10 dB. (This rule of thumb is relatively inaccurate at very low or very high frequencies.) If the loudness of the machines in this example is to be doubled, how many machines at the same distance from the worker must be running?

Answer Using the rule of thumb, a doubling of loudness corresponds to a sound level increase of 10 dB. Therefore,

$$\beta_2 - \beta_1 = 10\ \text{dB} = 10 \log \left(\frac{I_2}{I_0} \right) - 10 \log \left(\frac{I_1}{I_0} \right) = 10 \log \left(\frac{I_2}{I_1} \right)$$

$$\log \left(\frac{I_2}{I_1} \right) = 1 \quad \rightarrow \quad I_2 = 10 I_1$$

Therefore, ten machines must be operating to double the loudness.

Loudness and Frequency

The discussion of sound level in decibels relates to a *physical* measurement of the strength of a sound. Let us now extend our discussion from the What If? section of Example 17.3 concerning the *psychological* "measurement" of the strength of a sound.

Of course, we don't have instruments in our bodies that can display numerical values of our reactions to stimuli. We have to "calibrate" our reactions somehow by comparing different sounds to a reference sound, but that is not easy to accomplish. For example, earlier we mentioned that the threshold intensity is 10^{-12} W/m^2, corresponding to an intensity level of 0 dB. In reality, this value is the threshold only for a sound of frequency 1 000 Hz, which is a standard reference frequency in acoustics. If we perform an experiment to measure the threshold intensity at other frequencies, we find a distinct variation of this threshold as a function of frequency. For example, at 100 Hz, a barely audible sound must have an intensity level of about 30 dB! Unfortunately, there is no simple relationship between physical measurements and psychological "measurements." The 100-Hz, 30-dB sound is psychologically "equal" in loudness to the 1 000-Hz, 0-dB sound (both are just barely audible), but they are not physically equal in sound level (30 dB $\neq$ 0 dB).

By using test subjects, the human response to sound has been studied, and the results are shown in the white area of Figure 17.7 along with the approximate frequency and sound-level ranges of other sound sources. The lower curve of the white area corresponds to the threshold of hearing. Its variation with frequency is clear from this diagram. Notice that humans are sensitive to frequencies ranging from about 20 Hz to about 20 000 Hz. The upper bound of the white area is the thresh-

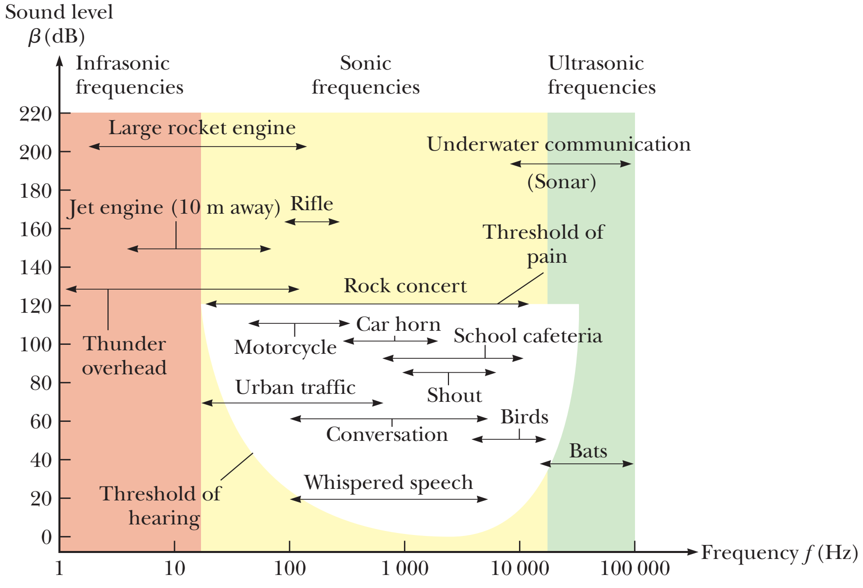

Figure 17.7 Approximate ranges of frequency and sound level of various sources and that of normal human hearing, shown by the white area. (From R. L. Reese, *University Physics*, Pacific Grove, Brooks/Cole, 2000.)

old of pain. Here the boundary of the white area appears straight because the psychological response is relatively independent of frequency at this high sound level.

The most dramatic change with frequency is in the lower left region of the white area, for low frequencies and low intensity levels. Our ears are particularly insensitive in this region. If you are listening to your home entertainment system and the bass (low frequencies) and treble (high frequencies) sound balanced at a high volume, try turning the volume down and listening again. You will probably notice that the bass seems weak, which is due to the insensitivity of the ear to low frequencies at low sound levels as shown in Figure 17.7.

17.4 The Doppler Effect

Perhaps you have noticed how the sound of a vehicle's horn changes as the vehicle moves past you. The frequency of the sound you hear as the vehicle approaches you is higher than the frequency you hear as it moves away from you. This experience is one example of the **Doppler effect.**[3]

To see what causes this apparent frequency change, imagine you are in a boat that is lying at anchor on a gentle sea where the waves have a period of $T = 3.0$ s. Hence, every 3.0 s a crest hits your boat. Figure 17.8a shows this situation, with the water waves moving toward the left. If you set your watch to $t = 0$ just as one crest hits, the watch reads 3.0 s when the next crest hits, 6.0 s when the third crest

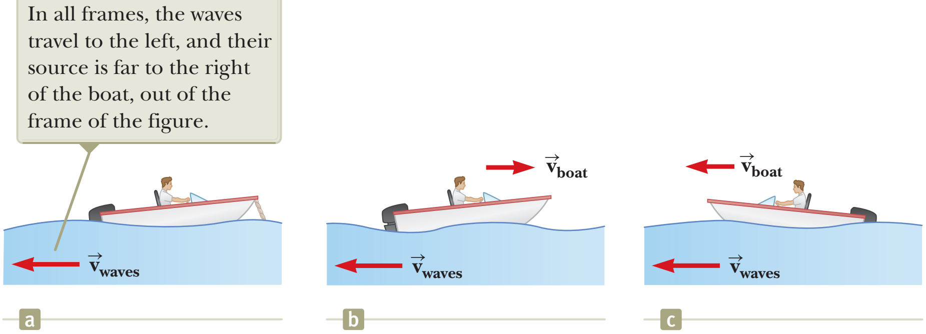

Figure 17.8 (a) Waves moving toward a stationary boat. (b) The boat moving toward the wave source. (c) The boat moving away from the wave source.

[3]Named after Austrian physicist Christian Johann Doppler (1803–1853), who in 1842 predicted the effect for both sound waves and light waves.

hits, and so on. From these observations, you conclude that the wave frequency is $f = 1/T = 1/(3.0 \text{ s}) = 0.33$ Hz. Now suppose you start your motor and head directly into the oncoming waves as in Figure 17.8b. Again you set your watch to $t = 0$ as a crest hits the front (the bow) of your boat. Now, however, because you are moving toward the next wave crest as it moves toward you, it hits you less than 3.0 s after the first hit. In other words, the period you observe is shorter than the 3.0-s period you observed when you were stationary. Because $f = 1/T$, you observe a higher wave frequency than when you were at rest.

If you turn around and move in the same direction as the waves (Fig. 17.8c), you observe the opposite effect. You set your watch to $t = 0$ as a crest hits the back (the stern) of the boat. Because you are now moving away from the next crest, more than 3.0 s has elapsed on your watch by the time that crest catches you. Therefore, you observe a lower frequency than when you were at rest.

These effects occur because the *relative* speed between your boat and the waves depends on the direction of travel and on the speed of your boat. (See Section 4.6.) When you are moving toward the right in Figure 17.8b, this relative speed is higher than that of the wave speed, which leads to the observation of an increased frequency. When you turn around and move to the left, the relative speed is lower, as is the observed frequency of the water waves.

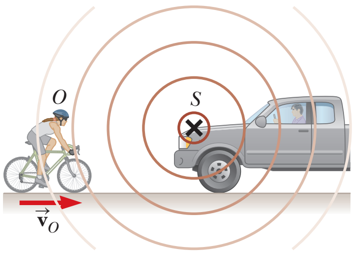

Figure 17.9 An observer O (the cyclist) moves with a speed v_O toward a stationary point source S, the horn of a parked truck. The observer hears a frequency f' that is greater than the source frequency.

Let's now examine an analogous situation with sound waves in which the water waves become sound waves, the water becomes the air, and the person on the boat becomes an observer listening to the sound. In this case, an observer O is moving and a sound source S is stationary. For simplicity, we assume the air is also stationary and the observer moves directly toward the source (Fig. 17.9). The observer moves with a speed v_O toward a stationary point source ($v_S = 0$), where *stationary* means at rest with respect to the medium, air.

If a point source emits sound waves and the medium is uniform, the waves move at the same speed in all directions radially away from the source; the result is a spherical wave as mentioned in Section 17.3. The distance between adjacent wave fronts equals the wavelength λ. In Figure 17.9, the circles are the intersections of these three-dimensional wave fronts with the two-dimensional paper.

We take the frequency of the source in Figure 17.9 to be f, the wavelength to be λ, and the speed of sound to be v. If the observer were also stationary, he would detect wave fronts at a frequency f. (That is, when $v_O = 0$ and $v_S = 0$, the observed frequency equals the source frequency.) When the observer moves toward the source, the speed of the waves relative to the observer is $v' = v + v_O$, as in the case of the boat in Figure 17.8, but the wavelength λ is unchanged. Hence, using Equation 16.12, $v = \lambda f$, we can say that the frequency f' heard by the observer is *increased* and is given by

$$f' = \frac{v'}{\lambda} = \frac{v + v_O}{\lambda}$$

Because $\lambda = v/f$, we can express f' as

$$f' = \left(\frac{v + v_O}{v}\right) f \quad \text{(observer moving toward source)} \tag{17.15}$$

If the observer is moving away from the source, the speed of the wave relative to the observer is $v' = v - v_O$. The frequency heard by the observer in this case is *decreased* and is given by

$$f' = \left(\frac{v - v_O}{v}\right) f \quad \text{(observer moving away from source)} \tag{17.16}$$

These last two equations can be reduced to a single equation by adopting a sign convention. Whenever an observer moves with a speed v_O relative to a stationary source, the frequency heard by the observer is given by Equation 17.15, with v_O interpreted as follows: a positive value is substituted for v_O when the observer moves

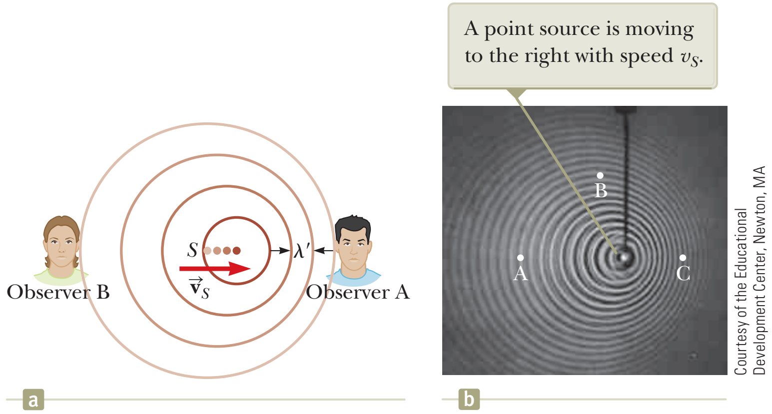

Figure 17.10 (a) A source S moving with a speed v_S toward a stationary observer A and away from a stationary observer B. Observer A hears an increased frequency, and observer B hears a decreased frequency. (b) The Doppler effect in water, observed in a ripple tank. Letters shown in the photo refer to Quick Quiz 17.4.

toward the source, and a negative value is substituted when the observer moves away from the source.

Now suppose the source is in motion and the observer is at rest. If the source moves directly toward observer A in Figure 17.10a, each new wave is emitted from a position to the right of the origin of the previous wave. As a result, the wave fronts heard by the observer are closer together than they would be if the source were not moving. (Fig. 17.10b shows this effect for waves moving on the surface of water.) As a result, the wavelength λ' measured by observer A is shorter than the wavelength λ of the source. During each vibration, which lasts for a time interval T (the period), the source moves a distance $v_S T = v_S/f$ and the wavelength is *shortened* by this amount. Therefore, the observed wavelength λ' is

$$\lambda' = \lambda - \Delta\lambda = \lambda - \frac{v_S}{f}$$

Because $\lambda = v/f$, the frequency f' heard by observer A is

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - (v_S/f)} = \frac{v}{(v/f) - (v_S/f)}$$

$$f' = \left(\frac{v}{v - v_S}\right)f \quad \text{(source moving toward observer)} \qquad \textbf{(17.17)}$$

That is, the observed frequency is *increased* whenever the source is moving toward the observer.

When the source moves away from a stationary observer, as is the case for observer B in Figure 17.10a, the observer measures a wavelength λ' that is *greater* than λ and hears a *decreased* frequency:

$$f' = \left(\frac{v}{v + v_S}\right)f \quad \text{(source moving away from observer)} \qquad \textbf{(17.18)}$$

We can express the general relationship for the observed frequency when a source is moving and an observer is at rest as Equation 17.17, with the same sign convention applied to v_S as was applied to v_O: a positive value is substituted for v_S when the source moves toward the observer, and a negative value is substituted when the source moves away from the observer.

Finally, combining Equations 17.15 and 17.17 gives the following general relationship for the observed frequency that includes all four conditions described by Equations 17.15 through 17.18:

$$f' = \left(\frac{v + v_O}{v - v_S}\right)f \qquad \textbf{(17.19)}$$

◀ **General Doppler-shift expression**

Pitfall Prevention 17.1

Doppler Effect Does Not Depend on Distance Some people think that the Doppler effect depends on the distance between the source and the observer. Although the *intensity* of a sound varies as the distance changes, the apparent *frequency* depends only on the relative speed of source and observer. As you listen to an approaching source, you will detect increasing intensity but constant frequency. As the source passes, you will hear the frequency suddenly drop to a new constant value and the intensity begin to decrease.

In this expression, the signs for the values substituted for v_O and v_S depend on the direction of the velocity. A positive value is used for motion of the observer or the source *toward* the other (associated with an *increase* in observed frequency), and a negative value is used for motion of one *away from* the other (associated with a *decrease* in observed frequency).

Although the Doppler effect is most typically experienced with sound waves, it is a phenomenon common to all waves. For example, the relative motion of source and observer produces a frequency shift in light waves. The Doppler effect is used in police radar systems to measure the speeds of motor vehicles. Likewise, astronomers use the effect to determine the speeds of stars, galaxies, and other celestial objects relative to the Earth.

Quick Quiz 17.4 Consider detectors of water waves at three locations A, B, and C in Figure 17.10b. Which of the following statements is true? **(a)** The wave speed is highest at location A. **(b)** The wave speed is highest at location C. **(c)** The detected wavelength is largest at location B. **(d)** The detected wavelength is largest at location C. **(e)** The detected frequency is highest at location C. **(f)** The detected frequency is highest at location A.

Quick Quiz 17.5 You stand on a platform at a train station and listen to a train approaching the station at a constant velocity. While the train approaches, but before it arrives, what do you hear? **(a)** the intensity and the frequency of the sound both increasing **(b)** the intensity and the frequency of the sound both decreasing **(c)** the intensity increasing and the frequency decreasing **(d)** the intensity decreasing and the frequency increasing **(e)** the intensity increasing and the frequency remaining the same **(f)** the intensity decreasing and the frequency remaining the same

Example 17.4 The Broken Clock Radio AM

Your clock radio awakens you with a steady and irritating sound of frequency 600 Hz. One morning, it malfunctions and cannot be turned off. In frustration, you drop the clock radio out of your fourth-story dorm window, 15.0 m from the ground. Assume the speed of sound is 343 m/s. As you listen to the falling clock radio, what frequency do you hear just before you hear it striking the ground?

SOLUTION

Conceptualize The speed of the clock radio increases as it falls. Therefore, it is a source of sound moving away from you with an increasing speed so the frequency you hear should be less than 600 Hz.

Categorize We categorize this problem as one in which we combine the *particle under constant acceleration* model for the falling radio with our understanding of the frequency shift of sound due to the Doppler effect.

Analyze Because the clock radio is modeled as a particle under constant acceleration due to gravity, use Equation 2.13 to express the speed of the source of sound:

$$(1)\quad v_S = v_{yi} + a_y t = 0 - gt = -gt$$

From Equation 2.16, find the time at which the clock radio strikes the ground:

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2 = 0 + 0 - \tfrac{1}{2}gt^2 \;\rightarrow\; t = \sqrt{-\frac{2y_f}{g}}$$

Substitute into Equation (1):

$$v_S = (-g)\sqrt{-\frac{2y_f}{g}} = -\sqrt{-2gy_f}$$

Use Equation 17.19 to determine the Doppler-shifted frequency heard from the falling clock radio:

$$f' = \left[\frac{v + 0}{v - (-\sqrt{-2gy_f})}\right]f = \left(\frac{v}{v + \sqrt{-2gy_f}}\right)f$$

17.4 continued

Substitute numerical values:

$$f' = \left[\frac{343 \text{ m/s}}{343 \text{ m/s} + \sqrt{-2(9.80 \text{ m/s}^2)(-15.0 \text{ m})}}\right](600 \text{ Hz})$$

$$= \boxed{571 \text{ Hz}}$$

Finalize The frequency is lower than the actual frequency of 600 Hz because the clock radio is moving away from you. If it were to fall from a higher floor so that it passes below $y = -15.0$ m, the clock radio would continue to accelerate and the frequency would continue to drop.

Example 17.5 Doppler Submarines

A submarine (sub A) travels through water at a speed of 8.00 m/s, emitting a sonar wave at a frequency of 1 400 Hz. The speed of sound in the water is 1 533 m/s. A second submarine (sub B) is located such that both submarines are traveling directly toward each other. The second submarine is moving at 9.00 m/s.

(A) What frequency is detected by an observer riding on sub B as the subs approach each other?

SOLUTION

Conceptualize Even though the problem involves subs moving in water, there is a Doppler effect just like there is when you are in a moving car and listening to a sound moving through the air from another car.

Categorize Because both subs are moving, we categorize this problem as one involving the Doppler effect for both a moving source and a moving observer.

Analyze Use Equation 17.19 to find the Doppler-shifted frequency heard by the observer in sub B, being careful with the signs assigned to the source and observer speeds:

$$f' = \left(\frac{v + v_O}{v - v_S}\right)f$$

$$f' = \left[\frac{1\,533 \text{ m/s} + (+9.00 \text{ m/s})}{1\,533 \text{ m/s} - (+8.00 \text{ m/s})}\right](1\,400 \text{ Hz}) = \boxed{1\,416 \text{ Hz}}$$

(B) The subs barely miss each other and pass. What frequency is detected by an observer riding on sub B as the subs recede from each other?

SOLUTION

Use Equation 17.19 to find the Doppler-shifted frequency heard by the observer in sub B, again being careful with the signs assigned to the source and observer speeds:

$$f' = \left(\frac{v + v_O}{v - v_S}\right)f$$

$$f' = \left[\frac{1\,533 \text{ m/s} + (-9.00 \text{ m/s})}{1\,533 \text{ m/s} - (-8.00 \text{ m/s})}\right](1\,400 \text{ Hz}) = \boxed{1\,385 \text{ Hz}}$$

Notice that the frequency drops from 1 416 Hz to 1 385 Hz as the subs pass. This effect is similar to the drop in frequency you hear when a car passes by you while blowing its horn.

(C) While the subs are approaching each other, some of the sound from sub A reflects from sub B and returns to sub A. If this sound were to be detected by an observer on sub A, what is its frequency?

SOLUTION

The sound of apparent frequency 1 416 Hz found in part (A) is reflected from a moving source (sub B) and then detected by a moving observer (sub A). Find the frequency detected by sub A:

$$f'' = \left(\frac{v + v_O}{v - v_S}\right)f'$$

$$= \left[\frac{1\,533 \text{ m/s} + (+8.00 \text{ m/s})}{1\,533 \text{ m/s} - (+9.00 \text{ m/s})}\right](1\,416 \text{ Hz}) = \boxed{1\,432 \text{ Hz}}$$

continued

▶ **17.5** continued

Finalize This technique is used by police officers to measure the speed of a moving car. Microwaves are emitted from the police car and reflected by the moving car. By detecting the Doppler-shifted frequency of the reflected microwaves, the police officer can determine the speed of the moving car.

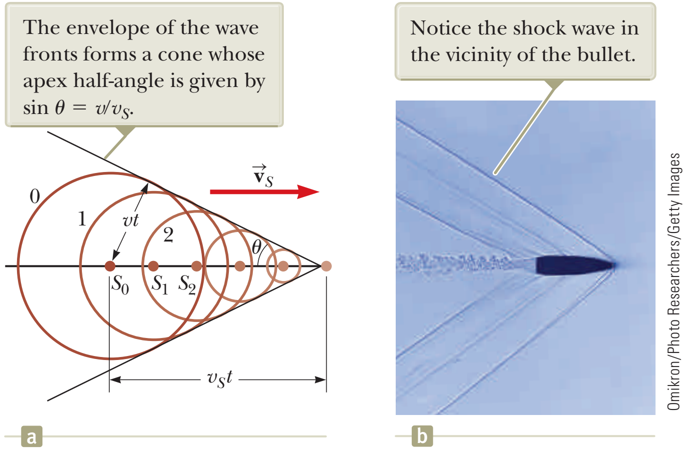

Figure 17.11 (a) A representation of a shock wave produced when a source moves from S_0 to the right with a speed v_S that is greater than the wave speed v in the medium. (b) A stroboscopic photograph of a bullet moving at supersonic speed through the hot air above a candle.

Shock Waves

Now consider what happens when the speed v_S of a source *exceeds* the wave speed v. This situation is depicted graphically in Figure 17.11a. The circles represent spherical wave fronts emitted by the source at various times during its motion. At $t = 0$, the source is at S_0 and moving toward the right. At later times, the source is at S_1, and then S_2, and so on. At the time t, the wave front centered at S_0 reaches a radius of vt. In this same time interval, the source travels a distance $v_S t$. Notice in Figure 17.11a that a straight line can be drawn tangent to all the wave fronts generated at various times. Therefore, the envelope of these wave fronts is a cone whose apex half-angle θ (the "Mach angle") is given by

$$\sin\theta = \frac{vt}{v_S t} = \frac{v}{v_S}$$

The ratio v_S/v is referred to as the *Mach number,* and the conical wave front produced when $v_S > v$ (supersonic speeds) is known as a *shock wave.* An interesting analogy to shock waves is the V-shaped wave fronts produced by a boat (the bow wave) when the boat's speed exceeds the speed of the surface-water waves (Fig. 17.12).

Jet airplanes traveling at supersonic speeds produce shock waves, which are responsible for the loud "sonic boom" one hears. The shock wave carries a great deal of energy concentrated on the surface of the cone, with correspondingly great pressure variations. Such shock waves are unpleasant to hear and can cause damage to buildings when aircraft fly supersonically at low altitudes. In fact, an airplane flying at supersonic speeds produces a double boom because two shock waves are formed, one from the nose of the plane and one from the tail. People near the path of a space shuttle as it glides toward its landing point have reported hearing what sounds like two very closely spaced cracks of thunder.

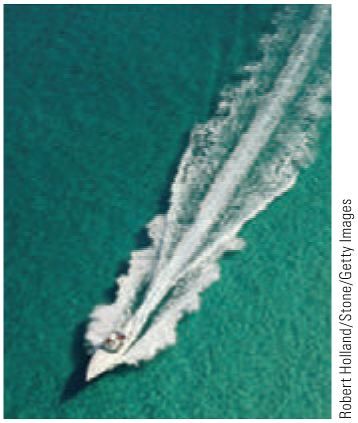

Figure 17.12 The V-shaped bow wave of a boat is formed because the boat speed is greater than the speed of the water waves it generates. A bow wave is analogous to a shock wave formed by an airplane traveling faster than sound.

Quick Quiz 17.6 An airplane flying with a constant velocity moves from a cold air mass into a warm air mass. Does the Mach number **(a)** increase, **(b)** decrease, or **(c)** stay the same?

Summary

Definitions

The **intensity** of a periodic sound wave, which is the power per unit area, is

$$I \equiv \frac{(Power)_{avg}}{A} = \frac{(\Delta P_{max})^2}{2\rho v} \qquad \textbf{(17.11, 17.12)}$$

The **sound level** of a sound wave in decibels is

$$\beta \equiv 10 \log \left(\frac{I}{I_0}\right) \qquad \textbf{(17.14)}$$

The constant I_0 is a reference intensity, usually taken to be at the threshold of hearing (1.00×10^{-12} W/m^2), and I is the intensity of the sound wave in watts per square meter.

Concepts and Principles

Sound waves are longitudinal and travel through a compressible medium with a speed that depends on the elastic and inertial properties of that medium. The speed of sound in a gas having a bulk modulus B and density ρ is

$$v = \sqrt{\frac{B}{\rho}} \qquad \textbf{(17.8)}$$

For sinusoidal sound waves, the variation in the position of an element of the medium is

$$s(x, t) = s_{max} \cos (kx - \omega t) \qquad \textbf{(17.1)}$$

and the variation in pressure from the equilibrium value is

$$\Delta P = \Delta P_{max} \sin (kx - \omega t) \qquad \textbf{(17.2)}$$

where ΔP_{max} is the **pressure amplitude.** The pressure wave is 90° out of phase with the displacement wave. The relationship between s_{max} and ΔP_{max} is

$$\Delta P_{max} = \rho v \omega s_{max} \qquad \textbf{(17.10)}$$

The change in frequency heard by an observer whenever there is relative motion between a source of sound waves and the observer is called the **Doppler effect.** The observed frequency is

$$f' = \left(\frac{v + v_O}{v - v_S}\right) f \qquad \textbf{(17.19)}$$

In this expression, the signs for the values substituted for v_O and v_S depend on the direction of the velocity. A positive value for the speed of the observer or source is substituted if the velocity of one is toward the other, whereas a negative value represents a velocity of one away from the other.

Objective Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Table 17.1 shows the speed of sound is typically an order of magnitude larger in solids than in gases. To what can this higher value be most directly attributed? (a) the difference in density between solids and gases (b) the difference in compressibility between solids and gases (c) the limited size of a solid object compared to a free gas (d) the impossibility of holding a gas under significant tension

2. Two sirens A and B are sounding so that the frequency from A is twice the frequency from B. Compared with the speed of sound from A, is the speed of sound from B (a) twice as fast, (b) half as fast, (c) four times as fast, (d) one-fourth as fast, or (e) the same?

3. As you travel down the highway in your car, an ambulance approaches you from the rear at a high speed (Fig. OQ17.3) sounding its siren at a frequency of 500 Hz. Which statement is correct? (a) You hear a frequency less than 500 Hz. (b) You hear a frequency equal to 500 Hz. (c) You hear a frequency greater

Anthony Redpath/Corbis

Figure OQ17.3

than 500 Hz. (d) You hear a frequency greater than 500 Hz, whereas the ambulance driver hears a frequency lower than 500 Hz. (e) You hear a frequency less than 500 Hz, whereas the ambulance driver hears a frequency of 500 Hz.

4. What happens to a sound wave as it travels from air into water? (a) Its intensity increases. (b) Its wavelength decreases. (c) Its frequency increases. (d) Its frequency remains the same. (e) Its velocity decreases.

5. A church bell in a steeple rings once. At 300 m in front of the church, the maximum sound intensity is 2 $\mu W/m^2$. At 950 m behind the church, the maximum intensity is 0.2 $\mu W/m^2$. What is the main reason for the difference in the intensity? (a) Most of the sound is absorbed by the air before it gets far away from the source. (b) Most of the sound is absorbed by the ground as it travels away from the source. (c) The bell broadcasts the sound mostly toward the front. (d) At a larger distance, the power is spread over a larger area.

6. If a 1.00-kHz sound source moves at a speed of 50.0 m/s toward a listener who moves at a speed of 30.0 m/s in a direction away from the source, what is the apparent frequency heard by the listener? (a) 796 Hz (b) 949 Hz (c) 1 000 Hz (d) 1 068 Hz (e) 1 273 Hz

7. A sound wave can be characterized as (a) a transverse wave, (b) a longitudinal wave, (c) a transverse wave or a longitudinal wave, depending on the nature of its source, (d) one that carries no energy, or (e) a wave that does not require a medium to be transmitted from one place to the other.

8. Assume a change at the source of sound reduces the wavelength of a sound wave in air by a factor of 2. **(i)** What happens to its frequency? (a) It increases by a factor of 4. (b) It increases by a factor of 2. (c) It is unchanged. (d) It decreases by a factor of 2. (e) It changes by an unpredictable factor. **(ii)** What happens to its speed? Choose from the same possibilities as in part (i).

9. A point source broadcasts sound into a uniform medium. If the distance from the source is tripled, how does the intensity change? (a) It becomes one-ninth as large. (b) It becomes one-third as large. (c) It is unchanged. (d) It becomes three times larger. (e) It becomes nine times larger.

10. Suppose an observer and a source of sound are both at rest relative to the ground and a strong wind is blowing away from the source toward the observer. **(i)** What effect does the wind have on the observed frequency? (a) It causes an increase. (b) It causes a decrease. (c) It causes no change. **(ii)** What effect does the wind have on the observed wavelength? Choose from the same possibilities as in part (i). **(iii)** What effect does the wind have on the observed speed of the wave? Choose from the same possibilities as in part (i).

11. A source of sound vibrates with constant frequency. Rank the frequency of sound observed in the following cases from highest to the lowest. If two frequencies are equal, show their equality in your ranking. All the motions mentioned have the same speed, 25 m/s. (a) The source and observer are stationary. (b) The source is moving toward a stationary observer. (c) The source is moving away from a stationary observer. (d) The observer is moving toward a stationary source. (e) The observer is moving away from a stationary source.

12. With a sensitive sound-level meter, you measure the sound of a running spider as −10 dB. What does the negative sign imply? (a) The spider is moving away from you. (b) The frequency of the sound is too low to be audible to humans. (c) The intensity of the sound is too faint to be audible to humans. (d) You have made a mistake; negative signs do not fit with logarithms.

13. Doubling the power output from a sound source emitting a single frequency will result in what increase in decibel level? (a) 0.50 dB (b) 2.0 dB (c) 3.0 dB (d) 4.0 dB (e) above 20 dB

14. Of the following sounds, which one is most likely to have a sound level of 60 dB? (a) a rock concert (b) the turning of a page in this textbook (c) dinner-table conversation (d) a cheering crowd at a football game

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. How can an object move with respect to an observer so that the sound from it is not shifted in frequency?

2. Older auto-focus cameras sent out a pulse of sound and measured the time interval required for the pulse to reach an object, reflect off of it, and return to be detected. Can air temperature affect the camera's focus? New cameras use a more reliable infrared system.

3. A friend sitting in her car far down the road waves to you and beeps her horn at the same moment. How far away must she be for you to calculate the speed of sound to two significant figures by measuring the time interval required for the sound to reach you?

4. How can you determine that the speed of sound is the same for all frequencies by listening to a band or orchestra?

5. Explain how the distance to a lightning bolt (Fig. CQ17.5) can be determined by counting the seconds between the flash and the sound of thunder.

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Figure CQ17.5

6. You are driving toward a cliff and honk your horn. Is there a Doppler shift of the sound when you hear the echo? If so, is it like a moving source or a moving observer? What if the reflection occurs not from a cliff, but from the forward edge of a huge alien spacecraft moving toward you as you drive?

7. The radar systems used by police to detect speeders are sensitive to the Doppler shift of a pulse of microwaves. Discuss how this sensitivity can be used to measure the speed of a car.

8. *The Tunguska event.* On June 30, 1908, a meteor burned up and exploded in the atmosphere above the Tunguska River valley in Siberia. It knocked down trees over thousands of square kilometers and started a forest fire, but produced no crater and apparently caused no human casualties. A witness sitting on his doorstep outside the zone of falling trees recalled events in the following sequence. He saw a moving light in the sky, brighter than the Sun and descending at a low angle to the horizon. He felt his face become warm. He felt the ground shake. An invisible agent picked him up and immediately dropped him about a meter from where he had been seated. He heard a very loud protracted rumbling. Suggest an explanation for these observations and for the order in which they happened.

9. A sonic ranger is a device that determines the distance to an object by sending out an ultrasonic sound pulse and measuring the time interval required for the wave to return by reflection from the object. Typically, these devices cannot reliably detect an object that is less than half a meter from the sensor. Why is that?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Note: Throughout this chapter, pressure variations ΔP are measured relative to atmospheric pressure, 1.013×10^5 Pa.

Section 17.1 Pressure Variations in Sound Waves

1. W A sinusoidal sound wave moves through a medium and is described by the displacement wave function

$$s(x, t) = 2.00 \cos (15.7x - 858t)$$

where s is in micrometers, x is in meters, and t is in seconds. Find (a) the amplitude, (b) the wavelength, and (c) the speed of this wave. (d) Determine the instantaneous displacement from equilibrium of the elements of the medium at the position $x = 0.050\ 0$ m at $t = 3.00$ ms. (e) Determine the maximum speed of the element's oscillatory motion.

2. As a certain sound wave travels through the air, it produces pressure variations (above and below atmospheric pressure) given by $\Delta P = 1.27 \sin (\pi x - 340\pi t)$ in SI units. Find (a) the amplitude of the pressure variations, (b) the frequency, (c) the wavelength in air, and (d) the speed of the sound wave.

3. Write an expression that describes the pressure variation as a function of position and time for a sinusoidal sound wave in air. Assume the speed of sound is 343 m/s, $\lambda = 0.100$ m, and $\Delta P_{max} = 0.200$ Pa.

Section 17.2 Speed of Sound Waves

Problem 85 in Chapter 2 can also be assigned with this section.

Note: In the rest of this chapter, unless otherwise specified, the equilibrium density of air is $\rho = 1.20$ kg/m^3 and the speed of sound in air is $v = 343$ m/s. Use Table 17.1 to find speeds of sound in other media.

4. M An experimenter wishes to generate in air a sound wave that has a displacement amplitude of 5.50×10^{-6} m. The pressure amplitude is to be limited to 0.840 Pa. What is the minimum wavelength the sound wave can have?

5. Calculate the pressure amplitude of a 2.00-kHz sound wave in air, assuming that the displacement amplitude is equal to 2.00×10^{-8} m.

6. Earthquakes at fault lines in the Earth's crust create seismic waves, which are longitudinal (P waves) or transverse (S waves). The P waves have a speed of about 7 km/s. Estimate the average bulk modulus of the Earth's crust given that the density of rock is about 2 500 kg/m^3.

7. A dolphin (Fig. P17.7) in seawater at a temperature of 25°C emits a sound wave directed toward the ocean floor 150 m below. How much time passes before it hears an echo?

Stephen Frink/Photographer's Choice/Getty Images

Figure P17.7

8. A sound wave propagates in air at 27°C with frequency 4.00 kHz. It passes through a region where the temperature gradually changes and then moves through air at 0°C. Give numerical answers to the following questions to the extent possible and state your reasoning about what happens to the wave physically. (a) What happens to the speed of the wave? (b) What happens to its frequency? (c) What happens to its wavelength?

9. Ultrasound is used in medicine both for diagnostic imaging (Fig. P17.9, page 526) and for therapy. For diagnosis, short pulses of ultrasound are passed through the patient's body. An echo reflected from a structure of interest is recorded, and the distance to the structure can be determined from the time delay for the echo's return. To reveal detail, the wavelength of the reflected ultrasound must be small compared to the size of the object reflecting the wave. The speed of ultrasound in human tissue is about 1 500 m/s (nearly the same as the speed of sound in water). (a) What is the wavelength of ultrasound with a frequency of 2.40 MHz? (b) In the whole set of imaging techniques, frequencies in the range 1.00 MHz to 20.0 MHz are used. What is the range of wavelengths corresponding to this range of frequencies?

B. Benoit/Photo Researchers, Inc.

Figure P17.9 A view of a fetus in the uterus made with ultrasound imaging.

10. W A sound wave in air has a pressure amplitude equal to 4.00×10^{-3} Pa. Calculate the displacement amplitude of the wave at a frequency of 10.0 kHz.

$\boxed{\textbf{11.}}$ W Suppose you hear a clap of thunder 16.2 s after seeing the associated lightning strike. The speed of light in air is 3.00×10^8 m/s. (a) How far are you from the lightning strike? (b) Do you need to know the value of the speed of light to answer? Explain.

12. W A rescue plane flies horizontally at a constant speed searching for a disabled boat. When the plane is directly above the boat, the boat's crew blows a loud horn. By the time the plane's sound detector receives the horn's sound, the plane has traveled a distance equal to half its altitude above the ocean. Assuming it takes the sound 2.00 s to reach the plane, determine (a) the speed of the plane and (b) its altitude.

13. AMT W A flowerpot is knocked off a window ledge from a height d = 20.0 m above the sidewalk as shown in Figure P17.13. It falls toward an unsuspecting man of height h = 1.75 m who is standing below. Assume the man requires a time interval of Δt = 0.300 s to respond to the warning. How close to the sidewalk can the flowerpot fall before it is too late for a warning shouted from the balcony to reach the man in time?

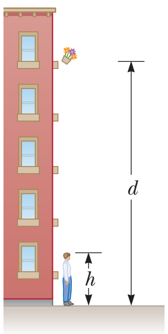


Figure P17.13 Problems 13 and 14.

14. A flowerpot is knocked off a balcony from a height d above the sidewalk as shown in Figure P17.13. It falls toward an unsuspecting man of height h who is standing below. Assume the man requires a time interval of Δt to respond to the warning. How close to the sidewalk can the flowerpot fall before it is too late for a warning shouted from the balcony to reach the man in time? Use the symbol v for the speed of sound.

15. The speed of sound in air (in meters per second) depends on temperature according to the approximate expression

$$v = 331.5 + 0.607T_C$$

where T_C is the Celsius temperature. In dry air, the temperature decreases about 1°C for every 150-m rise in altitude. (a) Assume this change is constant up to an altitude of 9 000 m. What time interval is required for the sound from an airplane flying at 9 000 m to reach the ground on a day when the ground temperature is 30°C? (b) **What If?** Compare your answer with the time interval required if the air were uniformly at 30°C. Which time interval is longer?

16. A sound wave moves down a cylinder as in Figure 17.2. Show that the pressure variation of the wave is described by $\Delta P = \pm \rho v \omega \sqrt{s_{max}^2 - s^2}$, where $s = s(x, t)$ is given by Equation 17.1.

17. A hammer strikes one end of a thick iron rail of length 8.50 m. A microphone located at the opposite end of the rail detects two pulses of sound, one that travels through the air and a longitudinal wave that travels through the rail. (a) Which pulse reaches the microphone first? (b) Find the separation in time between the arrivals of the two pulses.

18. A cowboy stands on horizontal ground between two parallel, vertical cliffs. He is not midway between the cliffs. He fires a shot and hears its echoes. The second echo arrives 1.92 s after the first and 1.47 s before the third. Consider only the sound traveling parallel to the ground and reflecting from the cliffs. (a) What is the distance between the cliffs? (b) **What If?** If he can hear a fourth echo, how long after the third echo does it arrive?

Section 17.3 Intensity of Periodic Sound Waves

$\boxed{\textbf{19.}}$ Calculate the sound level (in decibels) of a sound wave that has an intensity of 4.00 μW/m^2.

20. The area of a typical eardrum is about 5.00×10^{-5} m^2. (a) Calculate the average sound power incident on an eardrum at the threshold of pain, which corresponds to an intensity of 1.00 W/m^2. (b) How much energy is transferred to the eardrum exposed to this sound for 1.00 min?

21. The intensity of a sound wave at a fixed distance from a speaker vibrating at 1.00 kHz is 0.600 W/m^2. (a) Determine the intensity that results if the frequency is increased to 2.50 kHz while a constant displacement amplitude is maintained. (b) Calculate the intensity if the frequency is reduced to 0.500 kHz and the displacement amplitude is doubled.

22. The intensity of a sound wave at a fixed distance from a speaker vibrating at a frequency f is I. (a) Determine the intensity that results if the frequency is increased to f' while a constant displacement amplitude is maintained. (b) Calculate the intensity if the frequency is reduced to $f/2$ and the displacement amplitude is doubled.

23. A person wears a hearing aid that uniformly increases the sound level of all audible frequencies of sound by 30.0 dB. The hearing aid picks up sound having a frequency of 250 Hz at an intensity of 3.0×10^{-11} W/m^2. What is the intensity delivered to the eardrum?

24. The sound intensity at a distance of 16 m from a noisy generator is measured to be 0.25 W/m^2. What is the sound intensity at a distance of 28 m from the generator?

25. W The power output of a certain public-address speaker is 6.00 W. Suppose it broadcasts equally in all directions. (a) Within what distance from the speaker would the sound be painful to the ear? (b) At what distance from the speaker would the sound be barely audible?

26. A sound wave from a police siren has an intensity of 100.0 W/m^2 at a certain point; a second sound wave from a nearby ambulance has an intensity level that is 10 dB greater than the police siren's sound wave at the same point. What is the sound level of the sound wave due to the ambulance?

27. M A train sounds its horn as it approaches an intersection. The horn can just be heard at a level of 50 dB by an observer 10 km away. (a) What is the average power generated by the horn? (b) What intensity level of the horn's sound is observed by someone waiting at an intersection 50 m from the train? Treat the horn as a point source and neglect any absorption of sound by the air.

28. As the people sing in church, the sound level everywhere inside is 101 dB. No sound is transmitted through the massive walls, but all the windows and doors are open on a summer morning. Their total area is 22.0 m^2. (a) How much sound energy is radiated through the windows and doors in 20.0 min? (b) Suppose the ground is a good reflector and sound radiates from the church uniformly in all horizontal and upward directions. Find the sound level 1.00 km away.

29. The most soaring vocal melody is in Johann Sebastian Bach's Mass in B Minor. In one section, the basses, tenors, altos, and sopranos carry the melody from a low D to a high A. In concert pitch, these notes are now assigned frequencies of 146.8 Hz and 880.0 Hz. Find the wavelengths of (a) the initial note and (b) the final note. Assume the chorus sings the melody with a uniform sound level of 75.0 dB. Find the pressure amplitudes of (c) the initial note and (d) the final note. Find the displacement amplitudes of (e) the initial note and (f) the final note.

30. Show that the difference between decibel levels β_1 and β_2 of a sound is related to the ratio of the distances r_1 and r_2 from the sound source by

$$\beta_2 - \beta_1 = 20 \log \left(\frac{r_1}{r_2} \right)$$

31. M A family ice show is held at an enclosed arena. The skaters perform to music with level 80.0 dB. This level is too loud for your baby, who yells at 75.0 dB. (a) What total sound intensity engulfs you? (b) What is the combined sound level?

32. W Two small speakers emit sound waves of different frequencies equally in all directions. Speaker A has an output of 1.00 mW, and speaker B has an output of 1.50 mW. Determine the sound level (in decibels) at point C in Figure P17.32 assuming (a) only speaker A emits sound, (b) only speaker B emits sound, and (c) both speakers emit sound.

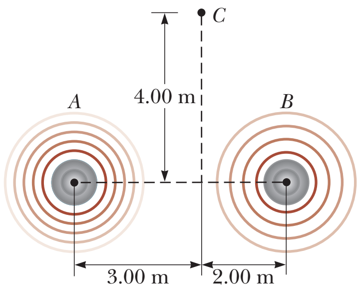


Figure P17.32

33. M A firework charge is detonated many meters above the ground. At a distance of $d_1 = 500$ m from the explosion, the acoustic pressure reaches a maximum of $\Delta P_{max} = 10.0$ Pa (Fig. P17.33). Assume the speed of sound is constant at 343 m/s throughout the atmosphere over the region considered, the ground absorbs all the sound falling on it, and the air absorbs sound energy as described by the rate 7.00 dB/km. What is the sound level (in decibels) at a distance of $d_2 = 4.00 \times 10^3$ m from the explosion?

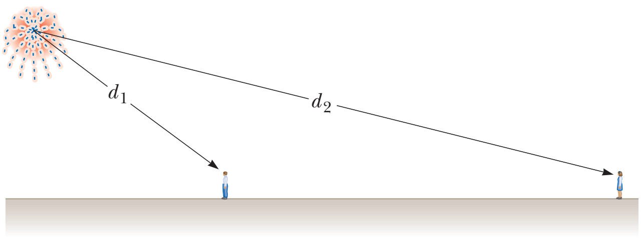


Figure P17.33

34. A fireworks rocket explodes at a height of 100 m above the ground. An observer on the ground directly under the explosion experiences an average sound intensity of 7.00×10^{-2} W/m^2 for 0.200 s. (a) What is the total amount of energy transferred away from the explosion by sound? (b) What is the sound level (in decibels) heard by the observer?

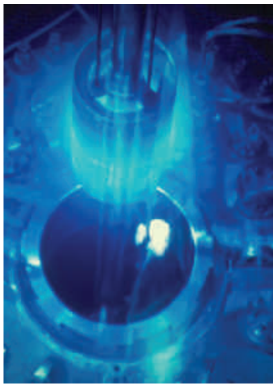
35. The sound level at a distance of 3.00 m from a source is 120 dB. At what distance is the sound level (a) 100 dB and (b) 10.0 dB?

36. *Why is the following situation impossible?* It is early on a Saturday morning, and much to your displeasure your next-door neighbor starts mowing his lawn. As you try to get back to sleep, your next-door neighbor on the other side of your house also begins to mow the lawn with an identical mower the same distance away. This situation annoys you greatly because the total sound now has twice the loudness it had when only one neighbor was mowing.

Section 17.4 The Doppler Effect

37. An ambulance moving at 42 m/s sounds its siren whose frequency is 450 Hz. A car is moving in the same direction as the ambulance at 25 m/s. What frequency does a person in the car hear (a) as the ambulance approaches the car? (b) After the ambulance passes the car?

38. When high-energy charged particles move through a transparent medium with a speed greater than the speed of light in that medium, a shock wave, or bow wave, of light is produced. This phenomenon is called the *Cerenkov effect.* When a nuclear reactor is shielded by a large pool of water, Cerenkov radiation can be seen as a blue glow in the vicinity of the reactor core due to high-speed electrons moving through the water (Fig. 17.38). In a particular case, the Cerenkov radiation produces a wave front with an apex half-angle of 53.0°. Calculate the speed of the electrons in the water. The speed of light in water is 2.25×10^8 m/s.



U.S. Department of Energy/Photo Researchers, Inc.

Figure P17.38

39. A driver travels northbound on a highway at a speed of 25.0 m/s. A police car, traveling southbound at a speed of 40.0 m/s, approaches with its siren producing sound at a frequency of 2 500 Hz. (a) What frequency does the driver observe as the police car approaches? (b) What frequency does the driver detect after the police car passes him? (c) Repeat parts (a) and (b) for the case when the police car is behind the driver and travels northbound.

40. GP Submarine A travels horizontally at 11.0 m/s through ocean water. It emits a sonar signal of frequency $f = 5.27 \times 10^3$ Hz in the forward direction. Submarine B is in front of submarine A and traveling at 3.00 m/s relative to the water in the same direction as submarine A. A crewman in submarine B uses his equipment to detect the sound waves ("pings") from submarine A. We wish to determine what is heard by the crewman in submarine B. (a) An observer on which submarine detects a frequency f' as described by Equation 17.19? (b) In Equation 17.19, should the sign of v_S be positive or negative? (c) In Equation 17.19, should the sign of v_O be positive or negative? (d) In Equation 17.19, what speed of sound should be used? (e) Find the frequency of the sound detected by the crewman on submarine B.

41. AMT **Review.** A block with a speaker bolted to it is connected to a spring having spring constant $k = 20.0$ N/m and oscillates as shown in Figure P17.41. The total mass of the block and speaker is 5.00 kg, and the amplitude of this unit's motion is 0.500 m. The speaker emits sound waves of frequency 440 Hz. Determine (a) the highest and (b) the lowest frequencies heard by the person to the right of the speaker. (c)If the maximum sound level heard by the person is 60.0 dB when the speaker is at its closest distance $d = 1.00$ m from him, what is the minimum sound level heard by the observer?

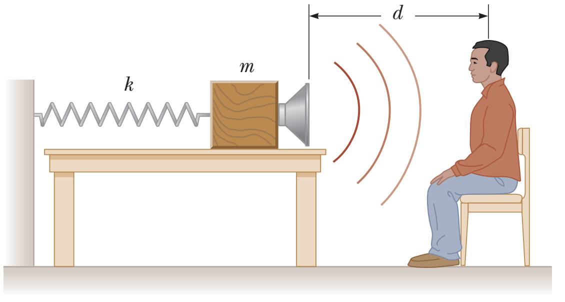


Figure P17.41 Problems 41 and 42.

42. **Review.** A block with a speaker bolted to it is connected to a spring having spring constant k and oscillates as shown in Figure P17.41. The total mass of the block and speaker is m, and the amplitude of this unit's motion is A. The speaker emits sound waves of frequency f. Determine (a) the highest and (b) the lowest frequencies heard by the person to the right of the speaker. (c) If the maximum sound level heard by the person is β when the speaker is at its closest distance d from him, what is the minimum sound level heard by the observer?

43. Expectant parents are thrilled to hear their unborn baby's heartbeat, revealed by an ultrasonic detector that produces beeps of audible sound in synchronization with the fetal heartbeat. Suppose the fetus's ventricular wall moves in simple harmonic motion with an amplitude of 1.80 mm and a frequency of 115 beats per minute. (a) Find the maximum linear speed of the heart wall. Suppose a source mounted on the detector in contact with the mother's abdomen produces sound at 2 000 000.0 Hz, which travels through tissue at 1.50 km/s. (b) Find the maximum change in frequency between the sound that arrives at the wall of the baby's heart and the sound emitted by the source. (c) Find the maximum change in frequency between the reflected sound received by the detector and that emitted by the source.

44. *Why is the following situation impossible?* At the Summer Olympics, an athlete runs at a constant speed down a straight track while a spectator near the edge of the track blows a note on a horn with a fixed frequency. When the athlete passes the horn, she hears the frequency of the horn fall by the musical interval called a minor third. That is, the frequency she hears drops to five-sixths its original value.

45. M Standing at a crosswalk, you hear a frequency of 560 Hz from the siren of an approaching ambulance. After the ambulance passes, the observed frequency of the siren is 480 Hz. Determine the ambulance's speed from these observations.

46. **Review.** A tuning fork vibrating at 512 Hz falls from rest and accelerates at 9.80 m/s^2. How far below the point of release is the tuning fork when waves of frequency 485 Hz reach the release point?

47. A supersonic jet traveling at Mach 3.00 at an altitude of h = 20 000 m is directly over a person at time t = 0 as shown in Figure P17.47. Assume the average speed of sound in air is 335 m/s over the path of the sound. (a) At what time will the person encounter the shock wave due to the sound emitted at t = 0? (b) Where will the plane be when this shock wave is heard?

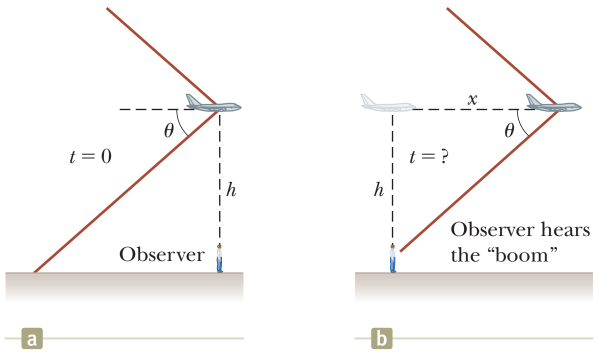


Figure P17.47

Additional Problems

48. A bat (Fig. P17.48) can detect very small objects, such as an insect whose length is approximately equal to one wavelength of the sound the bat makes. If a bat emits chirps at a frequency of 60.0 kHz and the speed of sound in air is 340 m/s, what is the smallest insect the bat can detect?

Hugh Lansdown/Shutterstock.com

Figure P17.48 Problems 48 and 63.

49. Some studies suggest that the upper frequency limit of hearing is determined by the diameter of the eardrum. The diameter of the eardrum is approximately equal to half the wavelength of the sound wave at this upper limit. If the relationship holds exactly, what is the diameter of the eardrum of a person capable of hearing 20 000 Hz? (Assume a body temperature of 37.0°C.)

50. The highest note written for a singer in a published score was F-sharp above high C, 1.480 kHz, for Zerbinetta in the original version of Richard Strauss's opera *Ariadne auf Naxos*. (a) Find the wavelength of this sound in air. (b) Suppose people in the fourth row of seats hear this note with level 81.0 dB. Find the displacement amplitude of the sound. (c) **What If?** In response to complaints, Strauss later transposed the note down to F above high C, 1.397 kHz. By what increment did the wavelength change?

51. Trucks carrying garbage to the town dump form a nearly steady procession on a country road, all traveling at 19.7 m/s in the same direction. Two trucks arrive at the dump every 3 min. A bicyclist is also traveling toward the dump, at 4.47 m/s. (a) With what frequency do the trucks pass the cyclist? (b) **What If?** A hill does not slow down the trucks, but makes the out-of-shape cyclist's speed drop to 1.56 m/s. How often do the trucks whiz past the cyclist now?

52. If a salesman claims a loudspeaker is rated at 150 W, he is referring to the maximum electrical power input to the speaker. Assume a loudspeaker with an input power of 150 W broadcasts sound equally in all directions and produces sound with a level of 103 dB at a distance of 1.60 m from its center. (a) Find its sound power output. (b) Find the efficiency of the speaker, that is, the fraction of input power that is converted into useful output power.

53. An interstate highway has been built through a neighborhood in a city. In the afternoon, the sound level in an apartment in the neighborhood is 80.0 dB as 100 cars pass outside the window every minute. Late at night, the traffic flow is only five cars per minute. What is the average late-night sound level?

54. A train whistle (f = 400 Hz) sounds higher or lower in frequency depending on whether it approaches or recedes. (a) Prove that the difference in frequency between the approaching and receding train whistle is

$$\Delta f = \frac{2u/v}{1 - u^2/v^2} f$$

where u is the speed of the train and v is the speed of sound. (b) Calculate this difference for a train moving at a speed of 130 km/h. Take the speed of sound in air to be 340 m/s.

55. An ultrasonic tape measure uses frequencies above 20 MHz to determine dimensions of structures such as buildings. It does so by emitting a pulse of ultrasound into air and then measuring the time interval for an echo to return from a reflecting surface whose distance away is to be measured. The distance is displayed as a digital readout. For a tape measure that emits a pulse of ultrasound with a frequency of 22.0 MHz, (a) what is the distance to an object from which the echo pulse returns after 24.0 ms when the air temperature is 26°C? (b) What should be the duration of the emitted pulse if it is to include ten cycles of the ultrasonic wave? (c) What is the spatial length of such a pulse?

56. The tensile stress in a thick copper bar is 99.5% of its elastic breaking point of 13.0 × 10^{10} N/m^2. If a 500-Hz sound wave is transmitted through the material, (a) what displacement amplitude will cause the bar to break? (b) What is the maximum speed of the elements of copper at this moment? (c) What is the sound intensity in the bar?

57. **Review.** A 150-g glider moves at $v_1 = 2.30$ m/s on an air track toward an originally stationary 200-g glider as shown in Figure P17.57. The gliders undergo a completely inelastic collision and latch together over a time interval of 7.00 ms. A student suggests roughly half the decrease in mechanical energy of the two-glider system is transferred to the environment by sound. Is this suggestion reasonable? To evaluate the idea, find the implied sound level at a position 0.800 m from the gliders. If the student's idea is unreasonable, suggest a better idea.

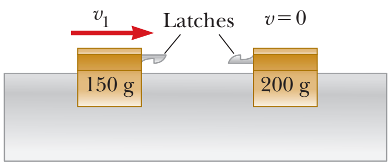


Figure P17.57

58. Consider the following wave function in SI units:

$$\Delta P(r, t) = \left(\frac{25.0}{r}\right) \sin (1.36r - 2\,030t)$$

Explain how this wave function can apply to a wave radiating from a small source, with r being the radial distance from the center of the source to any point outside the source. Give the most detailed description of the wave that you can. Include answers to such questions as the following and give representative values for any quantities that can be evaluated. (a) Does the wave move more toward the right or the left? (b) As it moves away from the source, what happens to its amplitude? (c) Its speed? (d) Its frequency? (e) Its wavelength? (f) Its power? (g) Its intensity?

59. **Review.** For a certain type of steel, stress is always proportional to strain with Young's modulus 20×10^{10} N/m^2. The steel has density 7.86×10^3 kg/m^3. It will fail by bending permanently if subjected to compressive stress greater than its yield strength $\sigma_y = 400$ MPa. A rod 80.0 cm long, made of this steel, is fired at 12.0 m/s straight at a very hard wall. (a) The speed of a one-dimensional compressional wave moving along the rod is given by $v = \sqrt{Y/\rho}$, where Y is Young's modulus for the rod and ρ is the density. Calculate this speed. (b) After the front end of the rod hits the wall and stops, the back end of the rod keeps moving as described by Newton's first law until it is stopped by excess pressure in a sound wave moving back through the rod. What time interval elapses before the back end of the rod receives the message that it should stop? (c) How far has the back end of the rod moved in this time interval? Find (d) the strain and (e) the stress in the rod. (f) If it is not to fail, what is the maximum impact speed a rod can have in terms of σ_y, Y, and ρ?

60. A large set of unoccupied football bleachers has solid seats and risers. You stand on the field in front of the bleachers and sharply clap two wooden boards together once. The sound pulse you produce has no definite frequency and no wavelength. The sound you hear reflected from the bleachers has an identifiable frequency and may remind you of a short toot on a trumpet, buzzer, or kazoo. (a) Explain what accounts for this sound. Compute order-of-magnitude estimates for (b) the frequency, (c) the wavelength, and (d) the duration of the sound on the basis of data you specify.

61. To measure her speed, a skydiver carries a buzzer emitting a steady tone at 1 800 Hz. A friend on the ground at the landing site directly below listens to the amplified sound he receives. Assume the air is calm and the speed of sound is independent of altitude. While the skydiver is falling at terminal speed, her friend on the ground receives waves of frequency 2 150 Hz. (a) What is the skydiver's speed of descent? (b) **What If?** Suppose the skydiver can hear the sound of the buzzer reflected from the ground. What frequency does she receive?

62. Spherical waves of wavelength 45.0 cm propagate outward from a point source. (a) Explain how the intensity at a distance of 240 cm compares with the intensity at a distance of 60.0 cm. (b) Explain how the amplitude at a distance of 240 cm compares with the amplitude at a distance of 60.0 cm. (c) Explain how the phase of the wave at a distance of 240 cm compares with the phase at 60.0 cm at the same moment.

63. A bat (Fig. P17.48), moving at 5.00 m/s, is chasing a flying insect. If the bat emits a 40.0-kHz chirp and receives back an echo at 40.4 kHz, (a) what is the speed of the insect? (b) Will the bat be able to catch the insect? Explain.

64. Two ships are moving along a line due east (Fig. P17.64). The trailing vessel has a speed relative to a land-based observation point of $v_1 = 64.0$ km/h, and the leading ship has a speed of $v_2 = 45.0$ km/h relative to that point. The two ships are in a region of the ocean where the current is moving uniformly due west at $v_{\text{current}} = 10.0$ km/h. The trailing ship transmits a sonar signal at a frequency of 1 200.0 Hz through the water. What frequency is monitored by the leading ship?

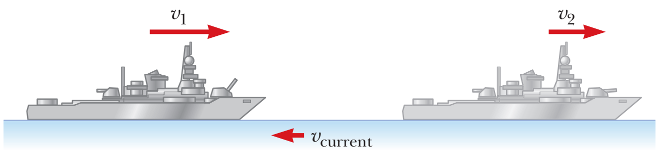


Figure P17.64

65. A police car is traveling east at 40.0 m/s along a straight road, overtaking a car ahead of it moving east at 30.0 m/s. The police car has a malfunctioning siren that is stuck at 1 000 Hz. (a) What would be the wavelength in air of the siren sound if the police car were at rest? (b) What is the wavelength in front of the police car? (c) What is it behind the police car? (d) What is the frequency heard by the driver being chased?

66. The speed of a one-dimensional compressional wave traveling along a thin copper rod is 3.56 km/s. The rod is given a sharp hammer blow at one end. A listener at the far end of the rod hears the sound twice, transmitted through the metal and through air, with a time interval Δt between the two pulses. (a) Which sound arrives first? (b) Find the length of the rod as a function of Δt. (c) Find the length of the rod if $\Delta t = 127$ ms. (d) Imagine that the copper rod is replaced by another material through which the speed of sound is v_r. What is the length of the rod in terms of t and v_r? (e) Would the answer to part (d) go to a well-defined limit as the speed of sound in the rod goes to infinity? Explain your answer.

67. A large meteoroid enters the Earth's atmosphere at a speed of 20.0 km/s and is not significantly slowed before entering the ocean. (a) What is the Mach angle of the shock wave from the meteoroid in the lower atmosphere? (b) If we assume the meteoroid survives the impact with the ocean surface, what is the (initial) Mach angle of the shock wave the meteoroid produces in the water?

68. Three metal rods are located relative to each other as shown in Figure P17.68, where $L_3 = L_1 + L_2$. The speed of sound in a rod is given by $v = \sqrt{Y/\rho}$, where Y is Young's modulus for the rod and ρ is the density. Values of density and Young's modulus for the three materials are $\rho_1 = 2.70 \times 10^3$ kg/m^3, $Y_1 = 7.00 \times 10^{10}$ N/m^2, $\rho_2 = 11.3 \times 10^3$ kg/m^3, $Y_2 = 1.60 \times 10^{10}$ N/m^2, $\rho_3 = 8.80 \times 10^3$ kg/m^3, $Y_3 = 11.0 \times 10^{10}$ N/m^2. If $L_3 = 1.50$ m, what must the ratio L_1/L_2 be if a sound wave is to travel the length of rods 1 and 2 in the same time interval required for the wave to travel the length of rod 3?

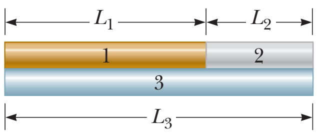


Figure P17.68

69. M With particular experimental methods, it is possible to produce and observe in a long, thin rod both a transverse wave whose speed depends primarily on tension in the rod and a longitudinal wave whose speed is determined by Young's modulus and the density of the material according to the expression $v = \sqrt{Y/\rho}$. The transverse wave can be modeled as a wave in a stretched string. A particular metal rod is 150 cm long and has a radius of 0.200 cm and a mass of 50.9 g. Young's modulus for the material is 6.80×10^{10} N/m^2. What must the tension in the rod be if the ratio of the speed of longitudinal waves to the speed of transverse waves is 8.00?

70. A siren mounted on the roof of a firehouse emits sound at a frequency of 900 Hz. A steady wind is blowing with a speed of 15.0 m/s. Taking the speed of sound in calm air to be 343 m/s, find the wavelength of the sound (a) upwind of the siren and (b) downwind of the siren. Firefighters are approaching the siren from various directions at 15.0 m/s. What frequency does a firefighter hear (c) if she is approaching from an upwind position so that she is moving in the direction in which the wind is blowing and (d) if she is approaching from a downwind position and moving against the wind?

Challenge Problems

71. The Doppler equation presented in the text is valid when the motion between the observer and the source occurs on a straight line so that the source and observer are moving either directly toward or directly away from each other. If this restriction is relaxed, one must use the more general Doppler equation

$$f' = \left(\frac{v + v_O \cos\theta_O}{v - v_S \cos\theta_S}\right) f$$

where θ_O and θ_S are defined in Figure P17.71a. Use the preceding equation to solve the following problem. A train moves at a constant speed of $v = 25.0$ m/s toward the intersection shown in Figure P17.71b. A car is stopped near the crossing, 30.0 m from the tracks. The train's horn emits a frequency of 500 Hz when the train is 40.0 m from the intersection. (a) What is the frequency heard by the passengers in the car? (b) If the train emits this sound continuously and the car is stationary at this position long before the train arrives until long after it leaves, what range of frequencies do passengers in the car hear? (c) Suppose the car is foolishly trying to beat the train to the intersection and is traveling at 40.0 m/s toward the tracks. When the car is 30.0 m from the tracks and the train is 40.0 m from the intersection, what is the frequency heard by the passengers in the car now?

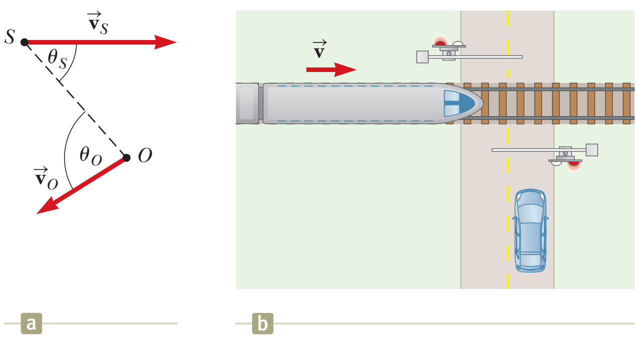


Figure P17.71

72. In Section 17.2, we derived the speed of sound in a gas using the impulse–momentum theorem applied to the cylinder of gas in Figure 17.5. Let us find the speed of sound in a gas using a different approach based on the element of gas in Figure 17.3. Proceed as follows. (a) Draw a force diagram for this element showing the forces exerted on the left and right surfaces due to the pressure of the gas on either side of the element. (b) By applying Newton's second law to the element, show that

$$-\frac{\partial(\Delta P)}{\partial x} A\,\Delta x = \rho A\,\Delta x \frac{\partial^2 s}{\partial t^2}$$

(c) By substituting $\Delta P = -(B\,\partial s/\partial x)$ (Eq. 17.3), derive the following wave equation for sound:

$$\frac{B}{\rho}\frac{\partial^2 s}{\partial x^2} = \frac{\partial^2 s}{\partial t^2}$$

(d) To a mathematical physicist, this equation demonstrates the existence of sound waves and determines their speed. As a physics student, you must take another step or two. Substitute into the wave equation the trial solution $s(x, t) = s_{\max} \cos(kx - \omega t)$. Show that this function satisfies the wave equation, provided $\omega/k = v = \sqrt{B/\rho}$.

73. Equation 17.13 states that at distance r away from a point source with power $(Power)_{\text{avg}}$, the wave intensity is

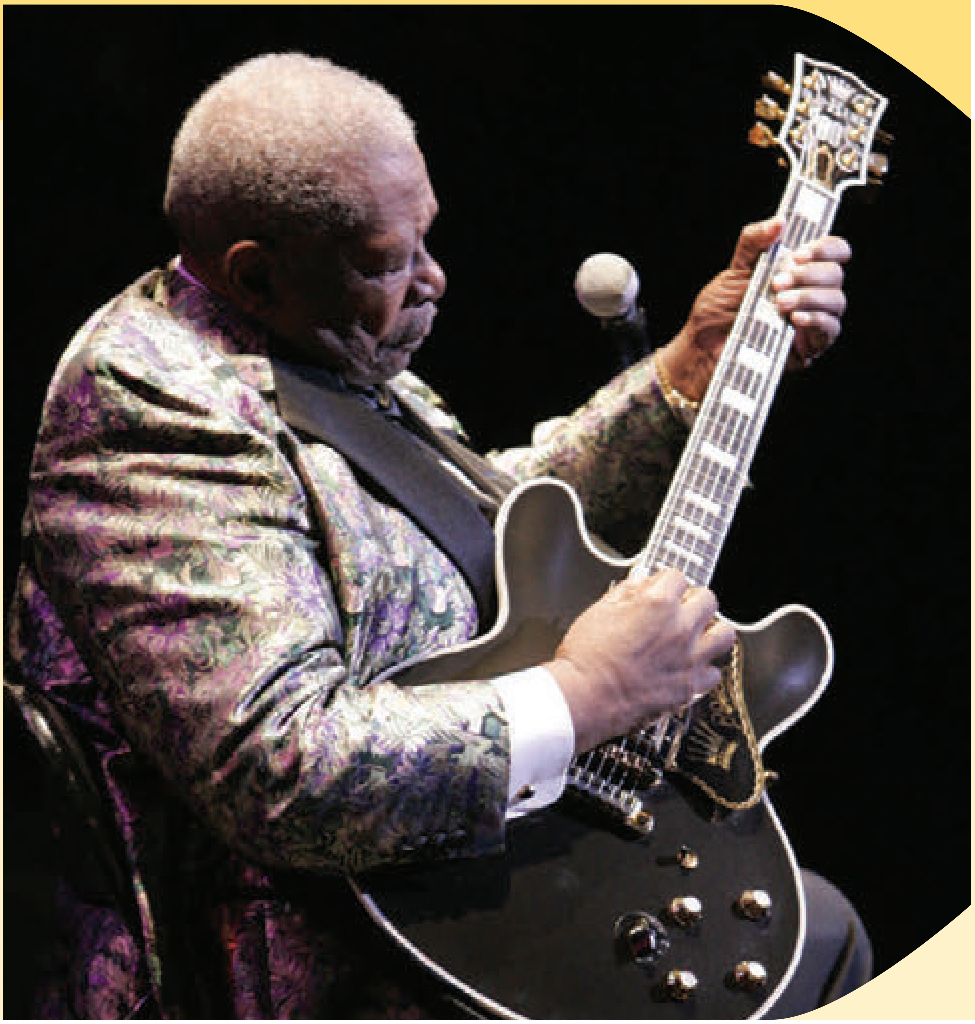
$$I = \frac{(Power)_{\text{avg}}}{4\pi r^2}$$

Study Figure 17.10 and prove that at distance r straight in front of a point source with power $(Power)_{\text{avg}}$ moving with constant speed v_S the wave intensity is

$$I = \frac{(Power)_{\text{avg}}}{4\pi r^2}\left(\frac{v - v_S}{v}\right)$$

CHAPTER 18

Superposition and Standing Waves



Blues master B. B. King takes advantage of standing waves on strings. He changes to higher notes on the guitar by pushing the strings against the frets on the fingerboard, shortening the lengths of the portions of the strings that vibrate. *(AP Photo/Danny Moloshok)*

The wave model was introduced in the previous two chapters. We have seen that waves are very different from particles. A particle is of zero size, whereas a wave has a characteristic size, its wavelength. Another important difference between waves and particles is that we can explore the possibility of two or more waves combining at one point in the same medium. Particles can be combined to form extended objects, but the particles must be at *different* locations. In contrast, two waves can both be present at the same location. The ramifications of this possibility are explored in this chapter.

When waves are combined in systems with boundary conditions, only certain allowed frequencies can exist and we say the frequencies are *quantized.* Quantization is a notion that is at the heart of quantum mechanics, a subject introduced formally in Chapter 40. There we show that analysis of waves under boundary conditions explains many of the quantum phenomena. In this chapter, we use quantization to understand the behavior of the wide array of musical instruments that are based on strings and air columns.

We also consider the combination of waves having different frequencies. When two sound waves having nearly the same frequency interfere, we hear variations in the loudness called *beats.* Finally, we discuss how any nonsinusoidal periodic wave can be described as a sum of sine and cosine functions.

18.1 Analysis Model: Waves in Interference

Many interesting wave phenomena in nature cannot be described by a single traveling wave. Instead, one must analyze these phenomena in terms of a combination of traveling waves. As noted in the introduction, waves have a remarkable difference from particles in that waves can be combined at the *same* location in space. To analyze such wave combinations, we make use of the **superposition principle:**

Superposition principle ▶

> If two or more traveling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves.

Waves that obey this principle are called *linear waves.* (See Section 16.6.) In the case of mechanical waves, linear waves are generally characterized by having amplitudes much smaller than their wavelengths. Waves that violate the superposition principle are called *nonlinear waves* and are often characterized by large amplitudes. In this book, we deal only with linear waves.

One consequence of the superposition principle is that two traveling waves can pass through each other without being destroyed or even altered. For instance, when two pebbles are thrown into a pond and hit the surface at different locations, the expanding circular surface waves from the two locations simply pass through each other with no permanent effect. The resulting complex pattern can be viewed as two independent sets of expanding circles.

> **Pitfall Prevention 18.1**
>
> **Do Waves Actually *Interfere?*** In popular usage, the term *interfere* implies that an agent affects a situation in some way so as to preclude something from happening. For example, in American football, *pass interference* means that a defending player has affected the receiver so that the receiver is unable to catch the ball. This usage is very different from its use in physics, where waves pass through each other and interfere, but do not affect each other in any way. In physics, interference is similar to the notion of *combination* as described in this chapter.

Figure 18.1 is a pictorial representation of the superposition of two pulses. The wave function for the pulse moving to the right is y_1, and the wave function for the pulse moving to the left is y_2. The pulses have the same speed but different shapes, and the displacement of the elements of the medium is in the positive y direction for both pulses. When the waves overlap (Fig. 18.1b), the wave function for the resulting complex wave is given by $y_1 + y_2$. When the crests of the pulses coincide (Fig. 18.1c), the resulting wave given by $y_1 + y_2$ has a larger amplitude than that of the individual pulses. The two pulses finally separate and continue moving in their original directions (Fig. 18.1d). Notice that the pulse shapes remain unchanged after the interaction, as if the two pulses had never met!

The combination of separate waves in the same region of space to produce a resultant wave is called **interference.** For the two pulses shown in Figure 18.1, the displacement of the elements of the medium is in the positive y direction for both pulses, and the resultant pulse (created when the individual pulses overlap) exhibits an amplitude greater than that of either individual pulse. Because the displacements caused by the two pulses are in the same direction, we refer to their superposition as **constructive interference.**

Constructive interference ▶

Now consider two pulses traveling in opposite directions on a taut string where one pulse is inverted relative to the other as illustrated in Figure 18.2. When these pulses begin to overlap, the resultant pulse is given by $y_1 + y_2$, but the values of the function y_2 are negative. Again, the two pulses pass through each other; because the displacements caused by the two pulses are in opposite directions, however, we refer to their superposition as **destructive interference.**

Destructive interference ▶

The superposition principle is the centerpiece of the analysis model called **waves in interference.** In many situations, both in acoustics and optics, waves combine according to this principle and exhibit interesting phenomena with practical applications.

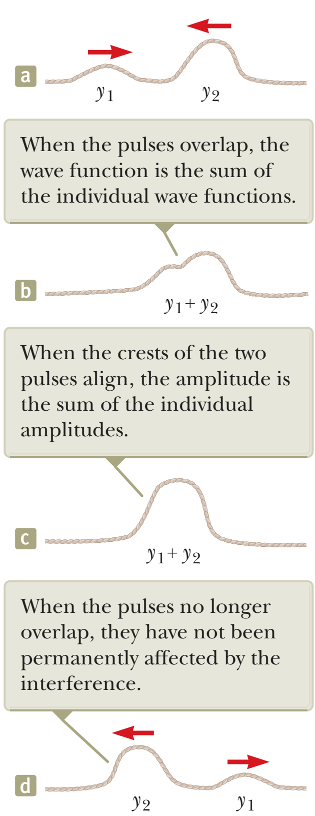


Figure 18.1 Constructive interference. Two positive pulses travel on a stretched string in opposite directions and overlap.

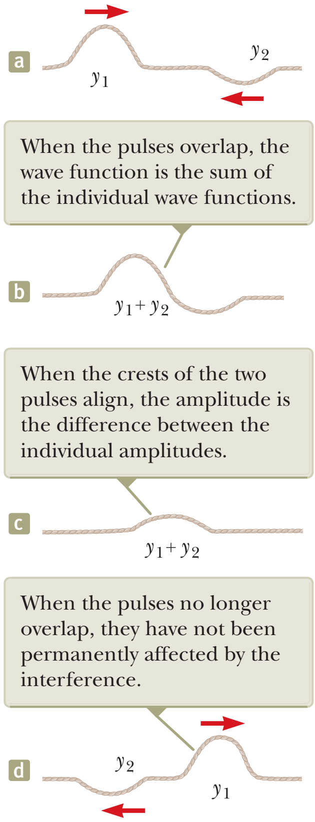


Figure 18.2 Destructive interference. Two pulses, one positive and one negative, travel on a stretched string in opposite directions and overlap.

Quick Quiz 18.1 Two pulses move in opposite directions on a string and are identical in shape except that one has positive displacements of the elements of the string and the other has negative displacements. At the moment the two pulses completely overlap on the string, what happens? **(a)** The energy associated with the pulses has disappeared. **(b)** The string is not moving. **(c)** The string forms a straight line. **(d)** The pulses have vanished and will not reappear.

Superposition of Sinusoidal Waves

Let us now apply the principle of superposition to two sinusoidal waves traveling in the same direction in a linear medium. If the two waves are traveling to the right and have the same frequency, wavelength, and amplitude but differ in phase, we can express their individual wave functions as

$$y_1 = A \sin (kx - \omega t) \qquad y_2 = A \sin (kx - \omega t + \phi)$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant as discussed in Section 16.2. Hence, the resultant wave function y is

$$y = y_1 + y_2 = A\,[\sin (kx - \omega t) + \sin (kx - \omega t + \phi)]$$

To simplify this expression, we use the trigonometric identity

$$\sin a + \sin b = 2 \cos \left(\frac{a - b}{2}\right) \sin \left(\frac{a + b}{2}\right)$$

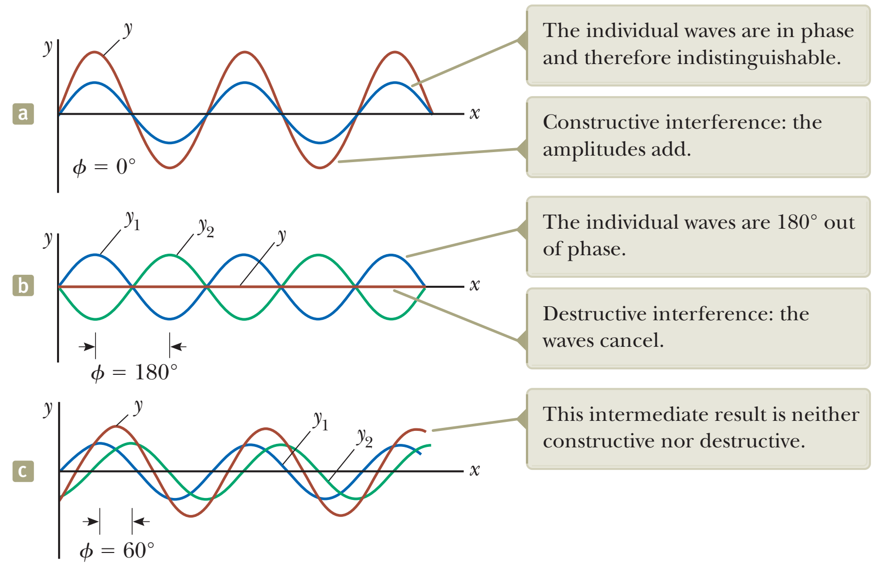


Figure 18.3 The superposition of two identical waves y_1 and y_2 (blue and green, respectively) to yield a resultant wave (red-brown).

Letting $a = kx - \omega t$ and $b = kx - \omega t + \phi$, we find that the resultant wave function y reduces to

Resultant of two traveling sinusoidal waves

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

This result has several important features. The resultant wave function y also is sinusoidal and has the same frequency and wavelength as the individual waves because the sine function incorporates the same values of k and ω that appear in the original wave functions. The amplitude of the resultant wave is $2A\cos(\phi/2)$, and its phase constant is $\phi/2$. If the phase constant ϕ of the original wave equals 0, then $\cos(\phi/2) = \cos 0 = 1$ and the amplitude of the resultant wave is $2A$, twice the amplitude of either individual wave. In this case, the crests of the two waves are at the same locations in space and the waves are said to be everywhere *in phase* and therefore interfere constructively. The individual waves y_1 and y_2 combine to form the red-brown curve y of amplitude $2A$ shown in Figure 18.3a. Because the individual waves are in phase, they are indistinguishable in Figure 18.3a, where they appear as a single blue curve. In general, constructive interference occurs when $\cos(\phi/2) = \pm 1$. That is true, for example, when $\phi = 0, 2\pi, 4\pi, \ldots$ rad, that is, when ϕ is an *even* multiple of π.

When ϕ is equal to π rad or to any *odd* multiple of π, then $\cos(\phi/2) = \cos(\pi/2) = 0$ and the crests of one wave occur at the same positions as the troughs of the second wave (Fig. 18.3b). Therefore, as a consequence of destructive interference, the resultant wave has *zero* amplitude everywhere as shown by the straight red-brown line in Figure 18.3b. Finally, when the phase constant has an arbitrary value other than 0 or an integer multiple of π rad (Fig. 18.3c), the resultant wave has an amplitude whose value is somewhere between 0 and $2A$.

In the more general case in which the waves have the same wavelength but different amplitudes, the results are similar with the following exceptions. In the in-phase case, the amplitude of the resultant wave is not twice that of a single wave, but rather is the sum of the amplitudes of the two waves. When the waves are π rad out of phase, they do not completely cancel as in Figure 18.3b. The result is a wave whose amplitude is the difference in the amplitudes of the individual waves.

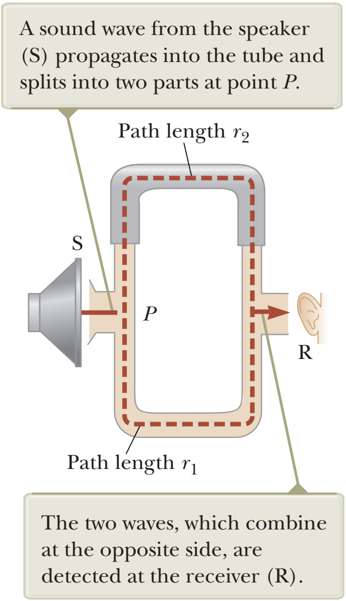


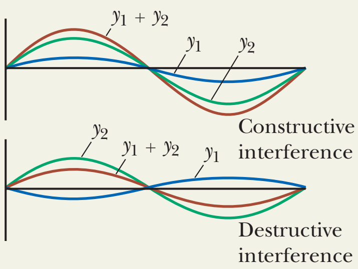
Figure 18.4 An acoustical system for demonstrating interference of sound waves. The upper path length r_2 can be varied by sliding the upper section.

Interference of Sound Waves

One simple device for demonstrating interference of sound waves is illustrated in Figure 18.4. Sound from a loudspeaker S is sent into a tube at point *P*, where there is a T-shaped junction. Half the sound energy travels in one direction, and half travels in the opposite direction. Therefore, the sound waves that reach the receiver R can travel along either of the two paths. The distance along any path from speaker to receiver is called the **path length** r. The lower path length r_1 is fixed, but the upper path length r_2 can be varied by sliding the U-shaped tube, which is similar to that on a slide trombone. When the difference in the path lengths $\Delta r = |r_2 - r_1|$ is either zero or some integer multiple of the wavelength λ (that is, $\Delta r = n\lambda$, where $n = 0, 1, 2, 3, \ldots$), the two waves reaching the receiver at any instant are in phase and interfere constructively as shown in Figure 18.3a. For this case, a maximum in the sound intensity is detected at the receiver. If the path length r_2 is adjusted such that the path difference $\Delta r = \lambda/2, 3\lambda/2, \ldots, n\lambda/2$ (for n odd), the two waves are exactly π rad, or 180°, out of phase at the receiver and hence cancel each other. In this case of destructive interference, no sound is detected at the receiver. This simple experiment demonstrates that a phase difference may arise between two waves generated by the same source when they travel along paths of unequal lengths. This important phenomenon will be indispensable in our investigation of the interference of light waves in Chapter 37.

Analysis Model Waves in Interference

Imagine two waves traveling in the same location through a medium. The displacement of elements of the medium is affected by both waves. According to the **principle of superposition**, the displacement is the sum of the individual displacements that would be caused by each wave. When the waves are in phase, **constructive interference** occurs and the resultant displacement is larger than the individual displacements. **Destructive interference** occurs when the waves are out of phase.



Examples:

- a piano tuner listens to a piano string and a tuning fork vibrating together and notices beats (Section 18.7)
- light waves from two coherent sources combine to form an interference pattern on a screen (Chapter 37)
- a thin film of oil on top of water shows swirls of color (Chapter 37)
- x-rays passing through a crystalline solid combine to form a Laue pattern (Chapter 38)

Example 18.1 Two Speakers Driven by the Same Source AM

Two identical loudspeakers placed 3.00 m apart are driven by the same oscillator (Fig. 18.5). A listener is originally at point O, located 8.00 m from the center of the line connecting the two speakers. The listener then moves to point P, which is a perpendicular distance 0.350 m from O, and she experiences the *first minimum* in sound intensity. What is the frequency of the oscillator?

SOLUTION

Conceptualize In Figure 18.4, a sound wave enters a tube and is then *acoustically* split into two different paths before recombining at the other end. In this example, a signal representing the sound is *electrically* split and sent to two different loudspeakers. After leaving the speakers, the sound waves recombine at the position of the listener. Despite the difference in how the splitting occurs, the path difference discussion related to Figure 18.4 can be applied here.

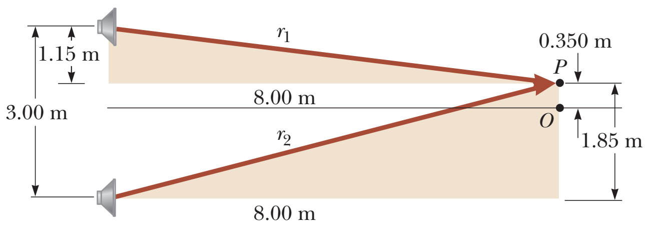


Figure 18.5 (Example 18.1) Two identical loudspeakers emit sound waves to a listener at P.

Categorize Because the sound waves from two separate sources combine, we apply the *waves in interference* analysis model.

continued

18.1 continued

Analyze Figure 18.5 shows the physical arrangement of the speakers, along with two shaded right triangles that can be drawn on the basis of the lengths described in the problem. The first minimum occurs when the two waves reaching the listener at point P are 180° out of phase, in other words, when their path difference Δr equals $\lambda/2$.

From the shaded triangles, find the path lengths from the speakers to the listener:

$$r_1 = \sqrt{(8.00\text{ m})^2 + (1.15\text{ m})^2} = 8.08\text{ m}$$

$$r_2 = \sqrt{(8.00\text{ m})^2 + (1.85\text{ m})^2} = 8.21\text{ m}$$

Hence, the path difference is $r_2 - r_1 = 0.13$ m. Because this path difference must equal $\lambda/2$ for the first minimum, $\lambda = 0.26$ m.

To obtain the oscillator frequency, use Equation 16.12, $v = \lambda f$, where v is the speed of sound in air, 343 m/s:

$$f = \frac{v}{\lambda} = \frac{343\text{ m/s}}{0.26\text{ m}} = \boxed{1.3\text{ kHz}}$$

Finalize This example enables us to understand why the speaker wires in a stereo system should be connected properly. When connected the wrong way—that is, when the positive (or red) wire is connected to the negative (or black) terminal on one of the speakers and the other is correctly wired—the speakers are said to be "out of phase," with one speaker moving outward while the other moves inward. As a consequence, the sound wave coming from one speaker destructively interferes with the wave coming from the other at point O in Figure 18.5. A rarefaction region due to one speaker is superposed on a compression region from the other speaker. Although the two sounds probably do not completely cancel each other (because the left and right stereo signals are usually not identical), a substantial loss of sound quality occurs at point O.

WHAT IF? What if the speakers were connected out of phase? What happens at point P in Figure 18.5?

Answer In this situation, the path difference of $\lambda/2$ combines with a phase difference of $\lambda/2$ due to the incorrect wiring to give a full phase difference of λ. As a result, the waves are in phase and there is a *maximum* intensity at point P.

Figure 18.6 Two identical loudspeakers emit sound waves toward each other. When they overlap, identical waves traveling in opposite directions will combine to form standing waves.

18.2 Standing Waves

The sound waves from the pair of loudspeakers in Example 18.1 leave the speakers in the forward direction, and we considered interference at a point in front of the speakers. Suppose we turn the speakers so that they face each other and then have them emit sound of the same frequency and amplitude. In this situation, two identical waves travel in opposite directions in the same medium as in Figure 18.6. These waves combine in accordance with the waves in interference model.

We can analyze such a situation by considering wave functions for two transverse sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium:

$$y_1 = A\sin(kx - \omega t) \qquad y_2 = A\sin(kx + \omega t)$$

where y_1 represents a wave traveling in the positive x direction and y_2 represents one traveling in the negative x direction. Adding these two functions gives the resultant wave function y:

$$y = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

When we use the trigonometric identity $\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$, this expression reduces to

$$y = (2A\sin kx)\cos\omega t \qquad \textbf{(18.1)}$$

Equation 18.1 represents the wave function of a **standing wave.** A standing wave such as the one on a string shown in Figure 18.7 is an oscillation pattern *with a stationary outline* that results from the superposition of two identical waves traveling in opposite directions.

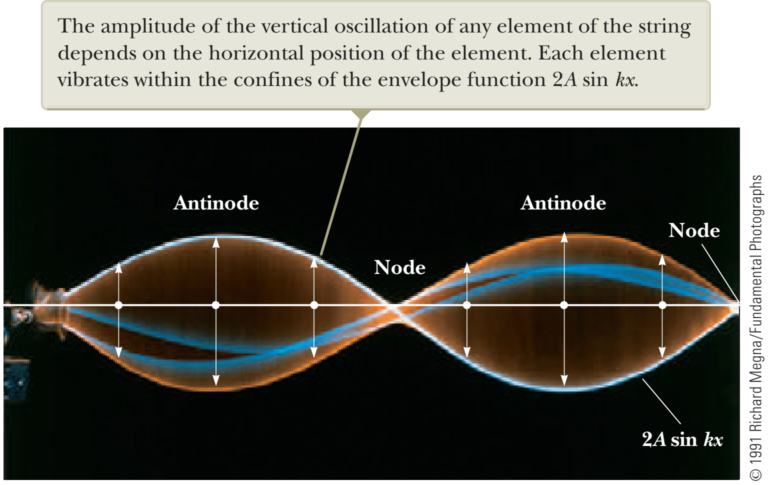


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Figure 18.7 Multiflash photograph of a standing wave on a string. The time behavior of the vertical displacement from equilibrium of an individual element of the string is given by cos ωt. That is, each element vibrates at an angular frequency ω.

Notice that Equation 18.1 does not contain a function of $kx - \omega t$. Therefore, it is not an expression for a single traveling wave. When you observe a standing wave, there is no sense of motion in the direction of propagation of either original wave. Comparing Equation 18.1 with Equation 15.6, we see that it describes a special kind of simple harmonic motion. Every element of the medium oscillates in simple harmonic motion with the same angular frequency ω (according to the cos ωt factor in the equation). The amplitude of the simple harmonic motion of a given element (given by the factor $2A \sin kx$, the coefficient of the cosine function) depends on the location x of the element in the medium, however.

If you can find a noncordless telephone with a coiled cord connecting the handset to the base unit, you can see the difference between a standing wave and a traveling wave. Stretch the coiled cord out and flick it with a finger. You will see a pulse traveling along the cord. Now shake the handset up and down and adjust your shaking frequency until every coil on the cord is moving up at the same time and then down. That is a standing wave, formed from the combination of waves moving away from your hand and reflected from the base unit toward your hand. Notice that there is no sense of traveling along the cord like there was for the pulse. You only see up-and-down motion of the elements of the cord.

Equation 18.1 shows that the amplitude of the simple harmonic motion of an element of the medium has a minimum value of zero when x satisfies the condition $\sin kx = 0$, that is, when

$$kx = 0, \pi, 2\pi, 3\pi, \ldots$$

Because $k = 2\pi/\lambda$, these values for kx give

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots = \frac{n\lambda}{2} \quad n = 0, 1, 2, 3, \ldots \qquad \textbf{(18.2)}$$

◀ Positions of nodes

These points of zero amplitude are called **nodes.**

The element of the medium with the *greatest* possible displacement from equilibrium has an amplitude of $2A$, which we define as the amplitude of the standing wave. The positions in the medium at which this maximum displacement occurs are called **antinodes.** The antinodes are located at positions for which the coordinate x satisfies the condition $\sin kx = \pm 1$, that is, when

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Therefore, the positions of the antinodes are given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots = \frac{n\lambda}{4} \quad n = 1, 3, 5, \ldots \qquad \textbf{(18.3)}$$

◀ Positions of antinodes

Pitfall Prevention 18.2

Three Types of Amplitude We need to distinguish carefully here between the **amplitude of the individual waves,** which is A, and the **amplitude of the simple harmonic motion of the elements of the medium,** which is $2A \sin kx$. A given element in a standing wave vibrates within the constraints of the *envelope* function $2A \sin kx$, where x is that element's position in the medium. Such vibration is in contrast to traveling sinusoidal waves, in which all elements oscillate with the same amplitude and the same frequency and the amplitude A of the wave is the same as the amplitude A of the simple harmonic motion of the elements. Furthermore, we can identify the **amplitude of the standing wave** as $2A$.

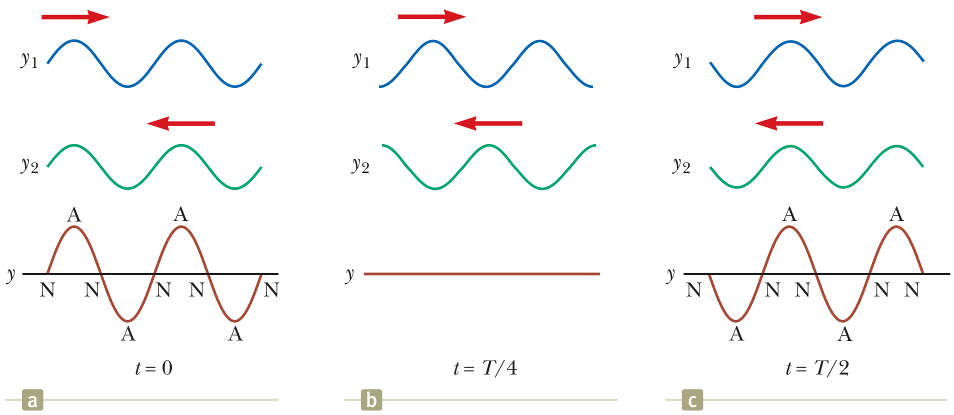

Figure 18.8 Standing-wave patterns produced at various times by two waves of equal amplitude traveling in opposite directions. For the resultant wave y, the nodes (N) are points of zero displacement and the antinodes (A) are points of maximum displacement.

Two nodes and two antinodes are labeled in the standing wave in Figure 18.7. The light blue curve labeled $2A \sin kx$ in Figure 18.7 represents one wavelength of the traveling waves that combine to form the standing wave. Figure 18.7 and Equations 18.2 and 18.3 provide the following important features of the locations of nodes and antinodes:

The distance between adjacent antinodes is equal to $\lambda/2$.
The distance between adjacent nodes is equal to $\lambda/2$.
The distance between a node and an adjacent antinode is $\lambda/4$.

Wave patterns of the elements of the medium produced at various times by two transverse traveling waves moving in opposite directions are shown in Figure 18.8. The blue and green curves are the wave patterns for the individual traveling waves, and the red-brown curves are the wave patterns for the resultant standing wave. At $t = 0$ (Fig. 18.8a), the two traveling waves are in phase, giving a wave pattern in which each element of the medium is at rest and experiencing its maximum displacement from equilibrium. One-quarter of a period later, at $t = T/4$ (Fig. 18.8b), the traveling waves have moved one-fourth of a wavelength (one to the right and the other to the left). At this time, the traveling waves are out of phase, and each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for elements at all values of x; that is, the wave pattern is a straight line. At $t = T/2$ (Fig. 18.8c), the traveling waves are again in phase, producing a wave pattern that is inverted relative to the $t = 0$ pattern. In the standing wave, the elements of the medium alternate in time between the extremes shown in Figures 18.8a and 18.8c.

Quick Quiz 18.2 Consider the waves in Figure 18.8 to be waves on a stretched string. Define the velocity of elements of the string as positive if they are moving upward in the figure. **(i)** At the moment the string has the shape shown by the red-brown curve in Figure 18.8a, what is the instantaneous velocity of elements along the string? (a) zero for all elements (b) positive for all elements (c) negative for all elements (d) varies with the position of the element **(ii)** From the same choices, at the moment the string has the shape shown by the red-brown curve in Figure 18.8b, what is the instantaneous velocity of elements along the string?

Example 18.2 Formation of a Standing Wave

Two waves traveling in opposite directions produce a standing wave. The individual wave functions are

$$y_1 = 4.0 \sin (3.0x - 2.0t)$$
$$y_2 = 4.0 \sin (3.0x + 2.0t)$$

where x and y are measured in centimeters and t is in seconds.

(A) Find the amplitude of the simple harmonic motion of the element of the medium located at $x = 2.3$ cm.

▶ **18.2** continued

SOLUTION

Conceptualize The waves described by the given equations are identical except for their directions of travel, so they indeed combine to form a standing wave as discussed in this section. We can represent the waves graphically by the blue and green curves in Figure 18.8.

Categorize We will substitute values into equations developed in this section, so we categorize this example as a substitution problem.

From the equations for the waves, we see that $A = 4.0$ cm, $k = 3.0$ rad/cm, and $\omega = 2.0$ rad/s. Use Equation 18.1 to write an expression for the standing wave:

$$y = (2A \sin kx) \cos \omega t = 8.0 \sin 3.0x \cos 2.0t$$

Find the amplitude of the simple harmonic motion of the element at the position $x = 2.3$ cm by evaluating the sine function at this position:

$$y_{\max} = (8.0 \text{ cm}) \sin 3.0x \big|_{x = 2.3}$$
$$= (8.0 \text{ cm}) \sin (6.9 \text{ rad}) = \boxed{4.6 \text{ cm}}$$

(B) Find the positions of the nodes and antinodes if one end of the string is at $x = 0$.

SOLUTION

Find the wavelength of the traveling waves:

$$k = \frac{2\pi}{\lambda} = 3.0 \text{ rad/cm} \rightarrow \lambda = \frac{2\pi}{3.0} \text{ cm}$$

Use Equation 18.2 to find the locations of the nodes:

$$x = n\frac{\lambda}{2} = \boxed{n\left(\frac{\pi}{3.0}\right) \text{cm} \quad n = 0, 1, 2, 3, \ldots}$$

Use Equation 18.3 to find the locations of the antinodes:

$$x = n\frac{\lambda}{4} = \boxed{n\left(\frac{\pi}{6.0}\right) \text{cm} \quad n = 1, 3, 5, 7, \ldots}$$

18.3 Analysis Model: Waves Under Boundary Conditions

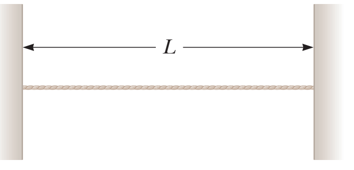

Figure 18.9 A string of length L fixed at both ends.

Consider a string of length L fixed at both ends as shown in Figure 18.9. We will use this system as a model for a guitar string or piano string. Waves can travel in both directions on the string. Therefore, standing waves can be set up in the string by a continuous superposition of waves incident on and reflected from the ends. Notice that there is a *boundary condition* for the waves on the string: because the ends of the string are fixed, they must necessarily have zero displacement and are therefore nodes by definition. The condition that both ends of the string must be nodes fixes the wavelength of the standing wave on the string according to Equation 18.2, which, in turn, determines the frequency of the wave. The boundary condition results in the string having a number of discrete natural patterns of oscillation, called **normal modes,** each of which has a characteristic frequency that is easily calculated. This situation in which only certain frequencies of oscillation are allowed is called **quantization.** Quantization is a common occurrence when waves are subject to boundary conditions and is a central feature in our discussions of quantum physics in the extended version of this text. Notice in Figure 18.8 that there are no boundary conditions, so standing waves of *any* frequency can be established; there is no quantization without boundary conditions. Because boundary conditions occur so often for waves, we identify an analysis model called **waves under boundary conditions** for the discussion that follows.

The normal modes of oscillation for the string in Figure 18.9 can be described by imposing the boundary conditions that the ends be nodes and that the nodes be separated by one-half of a wavelength with antinodes halfway between the nodes. The first normal mode that is consistent with these requirements, shown in Figure 18.10a (page 542), has nodes at its ends and one antinode in the middle. This normal

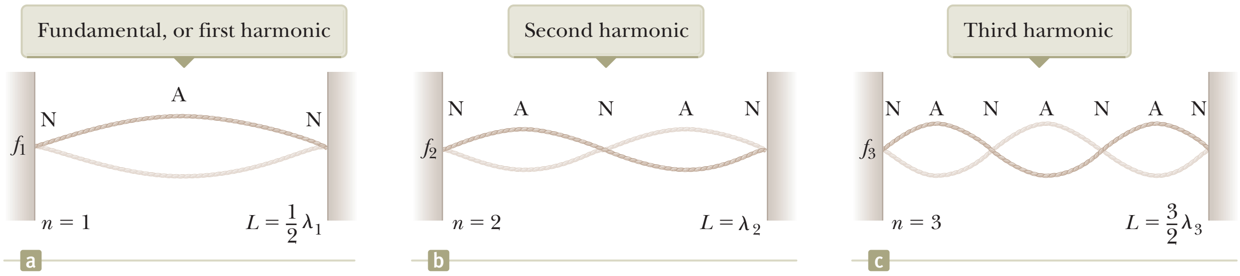

Figure 18.10 The normal modes of vibration of the string in Figure 18.9 form a harmonic series. The string vibrates between the extremes shown.

mode is the longest-wavelength mode that is consistent with our boundary conditions. The first normal mode occurs when the wavelength λ_1 is equal to twice the length of the string, or $\lambda_1 = 2L$. The section of a standing wave from one node to the next node is called a *loop*. In the first normal mode, the string is vibrating in one loop. In the second normal mode (see Fig. 18.10b), the string vibrates in two loops. When the left half of the string is moving upward, the right half is moving downward. In this case, the wavelength λ_2 is equal to the length of the string, as expressed by $\lambda_2 = L$. The third normal mode (see Fig. 18.10c) corresponds to the case in which $\lambda_3 = 2L/3$, and the string vibrates in three loops. In general, the wavelengths of the various normal modes for a string of length L fixed at both ends are

Wavelengths of normal modes

$$\lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, \ldots \tag{18.4}$$

where the index n refers to the nth normal mode of oscillation. These modes are *possible*. The *actual* modes that are excited on a string are discussed shortly.

The natural frequencies associated with the modes of oscillation are obtained from the relationship $f = v/\lambda$, where the wave speed v is the same for all frequencies. Using Equation 18.4, we find that the natural frequencies f_n of the normal modes are

Natural frequencies of normal modes as functions of wave speed and length of string

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \quad n = 1, 2, 3, \ldots \tag{18.5}$$

These natural frequencies are also called the *quantized frequencies* associated with the vibrating string fixed at both ends.

Because $v = \sqrt{T/\mu}$ (see Eq. 16.18) for waves on a string, where T is the tension in the string and μ is its linear mass density, we can also express the natural frequencies of a taut string as

Natural frequencies of normal modes as functions of string tension and linear mass density

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \quad n = 1, 2, 3, \ldots \tag{18.6}$$

The lowest frequency f_1, which corresponds to $n = 1$, is called either the **fundamental** or the **fundamental frequency** and is given by

Fundamental frequency of a taut string

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{18.7}$$

The frequencies of the remaining normal modes are integer multiples of the fundamental frequency (Eq. 18.5). Frequencies of normal modes that exhibit such an integer- multiple relationship form a **harmonic series,** and the normal modes are called **harmonics.** The fundamental frequency f_1 is the frequency of the first harmonic, the frequency $f_2 = 2f_1$ is that of the second harmonic, and the frequency $f_n = nf_1$ is that of the nth harmonic. Other oscillating systems, such as a drumhead, exhibit normal modes, but the frequencies are not related as integer multiples of a fundamental (see Section 18.6). Therefore, we do not use the term *harmonic* in association with those types of systems.

Let us examine further how the various harmonics are created in a string. To excite only a single harmonic, the string would have to be distorted into a shape that corresponds to that of the desired harmonic. After being released, the string would vibrate at the frequency of that harmonic. This maneuver is difficult to perform, however, and is not how a string of a musical instrument is excited. If the string is distorted into a general, nonsinusoidal shape, the resulting vibration includes a combination of various harmonics. Such a distortion occurs in musical instruments when the string is plucked (as in a guitar), bowed (as in a cello), or struck (as in a piano). When the string is distorted into a nonsinusoidal shape, only waves that satisfy the boundary conditions can persist on the string. These waves are the harmonics.

The frequency of a string that defines the musical note that it plays is that of the fundamental, even though other harmonics are present. The string's frequency can be varied by changing the string's tension or its length. For example, the tension in guitar and violin strings is varied by a screw adjustment mechanism or by tun ing pegs located on the neck of the instrument. As the tension is increased, the frequency of the normal modes increases in accordance with Equation 18.6. Once the instrument is "tuned," players vary the frequency by moving their fingers along the neck, thereby changing the length of the oscillating portion of the string. As the length is shortened, the frequency increases because, as Equation 18.6 specifies, the normal-mode frequencies are inversely proportional to string length.

uick Quiz 18.3 When a standing wave is set up on a string fixed at both ends, which of the following statements is true? **(a)** The number of nodes is equal to the number of antinodes. **(b)** The wavelength is equal to the length of the string divided by an integer. **(c)** The frequency is equal to the number of nodes times the fundamental frequency. **(d)** The shape of the string at any instant shows a symmetry about the midpoint of the string.

Analysis Model Waves Under Boundary Conditions

Imagine a wave that is not free to travel throughout all space as in the traveling wave model. If the wave is subject to bound ary conditions, such that cer tain requirements must be met at specific locations in space, the wave is limited to a set of **normal modes** with quantized wavelengths and quantized natural frequencies.

For waves on a string fixed at both ends, the natural frequencies are

$$1, 2, 3, \qquad \textbf{(18.6)}$$

where is the tension in the string and is its linear mass density.

Examples:

- waves traveling back and forth on a guitar string combine to form a standing wave
- sound waves traveling back and forth in a clarinet combine to form standing waves (Section 18.5)
- a microscopic particle confined to small region of space is modeled as a wave and exhibits quantized energies (Chapter 41)
- the Fermi energy of metal is deter mined by modeling electrons as wave-like particles in a box (Chapter 43)

Example 18.3 Give Me a C Note!

The middle C string on a piano has a fundamental frequency of 262 Hz, and the string for the first A above middle C has a fundamental frequency of 440 Hz.

Calculate the frequencies of the next two harmonics of the C string.

continued

18.3 continued

SOLUTION

Conceptualize Remember that the harmonics of a vibrating string have frequencies that are related by integer multiples of the fundamental.

Categorize This first part of the example is a simple substitution problem.

Knowing that the fundamental frequency is $f_1 = 262$ Hz, find the frequencies of the next harmonics by multiplying by integers:

$$f_2 = 2f_1 = \boxed{524 \text{ Hz}}$$

$$f_3 = 3f_1 = \boxed{786 \text{ Hz}}$$

(B) If the A and C strings have the same linear mass density μ and length L, determine the ratio of tensions in the two strings.

SOLUTION

Categorize This part of the example is more of an analysis problem than is part (A) and uses the *waves under boundary conditions* model.

Analyze Use Equation 18.7 to write expressions for the fundamental frequencies of the two strings:

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{T_A}{\mu}} \quad \text{and} \quad f_{1C} = \frac{1}{2L}\sqrt{\frac{T_C}{\mu}}$$

Divide the first equation by the second and solve for the ratio of tensions:

$$\frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}} \rightarrow \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440}{262}\right)^2 = \boxed{2.82}$$

Finalize If the frequencies of piano strings were determined solely by tension, this result suggests that the ratio of tensions from the lowest string to the highest string on the piano would be enormous. Such large tensions would make it difficult to design a frame to support the strings. In reality, the frequencies of piano strings vary due to additional parameters, including the mass per unit length and the length of the string. The What If? below explores a variation in length.

WHAT IF? If you look inside a real piano, you'll see that the assumption made in part (B) is only partially true. The strings are not likely to have the same length. The string densities for the given notes might be equal, but suppose the length of the A string is only 64% of the length of the C string. What is the ratio of their tensions?

Answer Using Equation 18.7 again, we set up the ratio of frequencies:

$$\frac{f_{1A}}{f_{1C}} = \frac{L_C}{L_A}\sqrt{\frac{T_A}{T_C}} \rightarrow \frac{T_A}{T_C} = \left(\frac{L_A}{L_C}\right)^2\left(\frac{f_{1A}}{f_{1C}}\right)^2$$

$$\frac{T_A}{T_C} = (0.64)^2\left(\frac{440}{262}\right)^2 = 1.16$$

Notice that this result represents only a 16% increase in tension, compared with the 182% increase in part (B).

Example 18.4 Changing String Vibration with Water AM

One end of a horizontal string is attached to a vibrating blade, and the other end passes over a pulley as in Figure 18.11a. A sphere of mass 2.00 kg hangs on the end of the string. The string is vibrating in its second harmonic. A container of water is raised under the sphere so that the sphere is completely submerged. In this configuration, the string vibrates in its fifth harmonic as shown in Figure 18.11b. What is the radius of the sphere?

SOLUTION

Conceptualize Imagine what happens when the sphere is immersed in the water. The buoyant force acts upward on the sphere, reducing the tension in the string. The change in tension causes a change in the speed of waves on the

18.4 continued

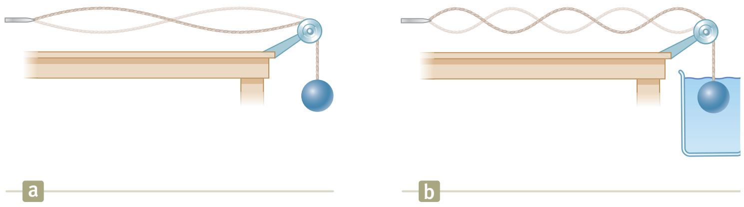

Figure 18.11 (Example 18.4) (a) When the sphere hangs in air, the string vibrates in its second harmonic. (b) When the sphere is immersed in water, the string vibrates in its fifth harmonic.

string, which in turn causes a change in the wavelength. This altered wavelength results in the string vibrating in its fifth normal mode rather than the second.

Categorize The hanging sphere is modeled as a *particle in equilibrium.* One of the forces acting on it is the buoyant force from the water. We also apply the *waves under boundary conditions* model to the string.

Analyze Apply the particle in equilibrium model to the sphere in Figure 18.11a, identifying T_1 as the tension in the string as the sphere hangs in air:

$$\sum F = T_1 - mg = 0$$

$$T_1 = mg$$

Apply the particle in equilibrium model to the sphere in Figure 18.11b, where T_2 is the tension in the string as the sphere is immersed in water:

$$T_2 + B - mg = 0$$

$$(1) \quad B = mg - T_2$$

The desired quantity, the radius of the sphere, will appear in the expression for the buoyant force B. Before proceeding in this direction, however, we must evaluate T_2 from the information about the standing wave.

Write the equation for the frequency of a standing wave on a string (Eq. 18.6) twice, once before the sphere is immersed and once after. Notice that the frequency f is the same in both cases because it is determined by the vibrating blade. In addition, the linear mass density μ and the length L of the vibrating portion of the string are the same in both cases. Divide the equations:

$$\left.\begin{aligned} f &= \frac{n_1}{2L}\sqrt{\frac{T_1}{\mu}} \\ f &= \frac{n_2}{2L}\sqrt{\frac{T_2}{\mu}} \end{aligned}\right\} \rightarrow \quad 1 = \frac{n_1}{n_2}\sqrt{\frac{T_1}{T_2}}$$

Solve for T_2:

$$T_2 = \left(\frac{n_1}{n_2}\right)^2 T_1 = \left(\frac{n_1}{n_2}\right)^2 mg$$

Substitute this result into Equation (1):

$$(2) \quad B = mg - \left(\frac{n_1}{n_2}\right)^2 mg = mg\left[1 - \left(\frac{n_1}{n_2}\right)^2\right]$$

Using Equation 14.5, express the buoyant force in terms of the radius of the sphere:

$$B = \rho_{\text{water}} g V_{\text{sphere}} = \rho_{\text{water}} g \left(\tfrac{4}{3}\pi r^3\right)$$

Solve for the radius of the sphere and substitute from Equation (2):

$$r = \left(\frac{3B}{4\pi\rho_{\text{water}} g}\right)^{1/3} = \left\{\frac{3m}{4\pi\rho_{\text{water}}}\left[1 - \left(\frac{n_1}{n_2}\right)^2\right]\right\}^{1/3}$$

Substitute numerical values:

$$r = \left\{\frac{3(2.00\text{ kg})}{4\pi(1\,000\text{ kg/m}^3)}\left[1 - \left(\frac{2}{5}\right)^2\right]\right\}^{1/3}$$

$$= 0.073\,7\text{ m} = \boxed{7.37\text{ cm}}$$

Finalize Notice that only certain radii of the sphere will result in the string vibrating in a normal mode; the speed of waves on the string must be changed to a value such that the length of the string is an integer multiple of half wavelengths. This limitation is a feature of the *quantization* that was introduced earlier in this chapter: the sphere radii that cause the string to vibrate in a normal mode are *quantized.*

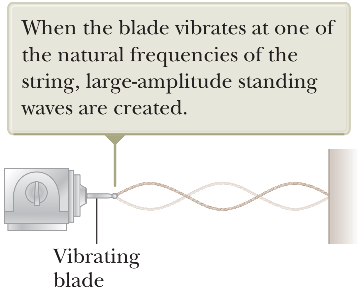

Figure 18.12 Standing waves are set up in a string when one end is connected to a vibrating blade.

18.4 Resonance

We have seen that a system such as a taut string is capable of oscillating in one or more normal modes of oscillation. Suppose we drive such a string with a vibrating blade as in Figure 18.12. We find that if a periodic force is applied to such a system, the amplitude of the resulting motion of the string is greatest when the frequency of the applied force is equal to one of the natural frequencies of the system. This phenomenon, known as *resonance,* was discussed in Section 15.7 with regard to a simple harmonic oscillator. Although a block–spring system or a simple pendulum has only one natural frequency, standing-wave systems have a whole set of natural frequencies, such as that given by Equation 18.6 for a string. Because an oscillating system exhibits a large amplitude when driven at any of its natural frequencies, these frequencies are often referred to as **resonance frequencies.**

Consider the string in Figure 18.12 again. The fixed end is a node, and the end connected to the blade is very nearly a node because the amplitude of the blade's motion is small compared with that of the elements of the string. As the blade oscillates, transverse waves sent down the string are reflected from the fixed end. As we learned in Section 18.3, the string has natural frequencies that are determined by its length, tension, and linear mass density (see Eq. 18.6). When the frequency of the blade equals one of the natural frequencies of the string, standing waves are produced and the string oscillates with a large amplitude. In this resonance case, the wave generated by the oscillating blade is in phase with the reflected wave and the string absorbs energy from the blade. If the string is driven at a frequency that is not one of its natural frequencies, the oscillations are of low amplitude and exhibit no stable pattern.

Resonance is very important in the excitation of musical instruments based on air columns. We shall discuss this application of resonance in Section 18.5.

18.5 Standing Waves in Air Columns

The waves under boundary conditions model can also be applied to sound waves in a column of air such as that inside an organ pipe or a clarinet. Standing waves in this case are the result of interference between longitudinal sound waves traveling in opposite directions.

In a pipe closed at one end, the closed end is a **displacement node** because the rigid barrier at this end does not allow longitudinal motion of the air. Because the pressure wave is 90° out of phase with the displacement wave (see Section 17.1), the closed end of an air column corresponds to a **pressure antinode** (that is, a point of maximum pressure variation).

The open end of an air column is approximately a **displacement antinode**[1] and a pressure node. We can understand why no pressure variation occurs at an open end by noting that the end of the air column is open to the atmosphere; therefore, the pressure at this end must remain constant at atmospheric pressure.

You may wonder how a sound wave can reflect from an open end because there may not appear to be a change in the medium at this point: the medium through which the sound wave moves is air both inside and outside the pipe. Sound can be represented as a pressure wave, however, and a compression region of the sound wave is constrained by the sides of the pipe as long as the region is inside the pipe. As the compression region exits at the open end of the pipe, the constraint of the pipe is removed and the compressed air is free to expand into the atmosphere. Therefore, there is a change in the *character* of the medium between the inside

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[1]Strictly speaking, the open end of an air column is not exactly a displacement antinode. A compression reaching an open end does not reflect until it passes beyond the end. For a tube of circular cross section, an end correction equal to approximately $0.6R$, where R is the tube's radius, must be added to the length of the air column. Hence, the effective length of the air column is longer than the true length L. We ignore this end correction in this discussion.

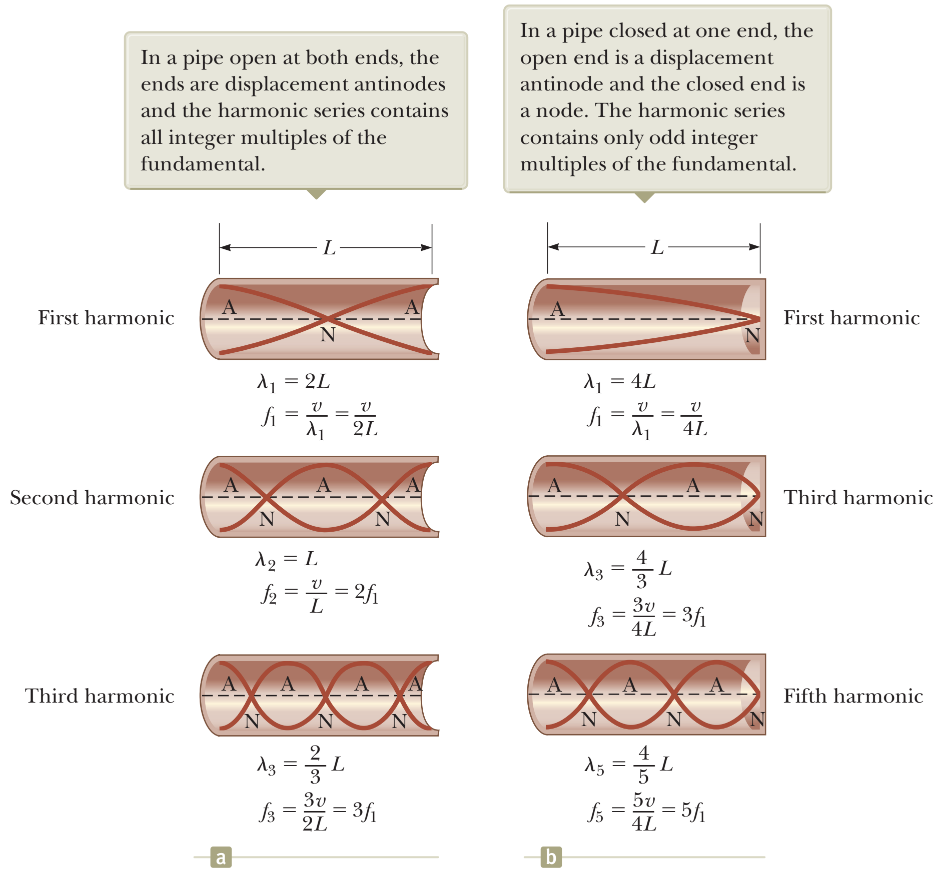


Figure 18.13 Graphical representations of the motion of elements of air in standing longitudinal waves in (a) a column open at both ends and (b) a column closed at one end.

of the pipe and the outside even though there is no change in the *material* of the medium. This change in character is sufficient to allow some reflection.

With the boundary conditions of nodes or antinodes at the ends of the air column, we have a set of normal modes of oscillation as is the case for the string fixed at both ends. Therefore, the air column has quantized frequencies.

The first three normal modes of oscillation of a pipe open at both ends are shown in Figure 18.13a. Notice that both ends are displacement antinodes (approximately). In the first normal mode, the standing wave extends between two adjacent antinodes, which is a distance of half a wavelength. Therefore, the wavelength is twice the length of the pipe, and the fundamental frequency is $f_1 = v/2L$. As Figure 18.13a shows, the frequencies of the higher harmonics are $2f_1, 3f_1, \ldots$.

> In a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.

Pitfall Prevention 18.3

Sound Waves in Air Are Longitudinal, Not Transverse The standing longitudinal waves are drawn as transverse waves in Figure 18.13. Because they are in the same direction as the propagation, it is difficult to draw longitudinal displacements. Therefore, it is best to interpret the red-brown curves in Figure 18.13 as a graphical representation of the waves (our diagrams of string waves are pictorial representations), with the vertical axis representing the horizontal displacement $s(x, t)$ of the elements of the medium.

Because all harmonics are present and because the fundamental frequency is given by the same expression as that for a string (see Eq. 18.5), we can express the natural frequencies of oscillation as

Natural frequencies of a pipe open at both ends

$$f_n = n\frac{v}{2L} \quad n = 1, 2, 3, \ldots \tag{18.8}$$

Despite the similarity between Equations 18.5 and 18.8, you must remember that v in Equation 18.5 is the speed of waves on the string, whereas v in Equation 18.8 is the speed of sound in air.

If a pipe is closed at one end and open at the other, the closed end is a displacement node (see Fig. 18.13b). In this case, the standing wave for the fundamental mode extends from an antinode to the adjacent node, which is one-fourth of a wavelength. Hence, the wavelength for the first normal mode is $4L$, and the fundamental frequency is $f_1 = v/4L$. As Figure 18.13b shows, the higher-frequency waves that satisfy our conditions are those that have a node at the closed end and an antinode at the open end; hence, the higher harmonics have frequencies $3f_1, 5f_1, \ldots$.

> In a pipe closed at one end, the natural frequencies of oscillation form a harmonic series that includes only odd integral multiples of the fundamental frequency.

We express this result mathematically as

Natural frequencies of a pipe closed at one end and open at the other

$$f_n = n\frac{v}{4L} \quad n = 1, 3, 5, \ldots \tag{18.9}$$

It is interesting to investigate what happens to the frequencies of instruments based on air columns and strings during a concert as the temperature rises. The sound emitted by a flute, for example, becomes sharp (increases in frequency) as the flute warms up because the speed of sound increases in the increasingly warmer air inside the flute (consider Eq. 18.8). The sound produced by a violin becomes flat (decreases in frequency) as the strings thermally expand because the expansion causes their tension to decrease (see Eq. 18.6).

Musical instruments based on air columns are generally excited by resonance. The air column is presented with a sound wave that is rich in many frequencies. The air column then responds with a large-amplitude oscillation to the frequencies that match the quantized frequencies in its set of harmonics. In many woodwind instruments, the initial rich sound is provided by a vibrating reed. In brass instruments, this excitation is provided by the sound coming from the vibration of the player's lips. In a flute, the initial excitation comes from blowing over an edge at the mouthpiece of the instrument in a manner similar to blowing across the opening of a bottle with a narrow neck. The sound of the air rushing across the bottle opening has many frequencies, including one that sets the air cavity in the bottle into resonance.

Quick Quiz 18.4 A pipe open at both ends resonates at a fundamental frequency f_{open}. When one end is covered and the pipe is again made to resonate, the fundamental frequency is f_{closed}. Which of the following expressions describes how these two resonant frequencies compare? **(a)** $f_{\text{closed}} = f_{\text{open}}$ **(b)** $f_{\text{closed}} = \frac{1}{2}f_{\text{open}}$ **(c)** $f_{\text{closed}} = 2f_{\text{open}}$ **(d)** $f_{\text{closed}} = \frac{3}{2}f_{\text{open}}$

Quick Quiz 18.5 Balboa Park in San Diego has an outdoor organ. When the air temperature increases, the fundamental frequency of one of the organ pipes **(a)** stays the same, **(b)** goes down, **(c)** goes up, or **(d)** is impossible to determine.

Example 18.5 Wind in a Culvert

A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows across its open ends.

(A) Determine the frequencies of the first three harmonics of the culvert if it is cylindrical in shape and open at both ends. Take $v = 343$ m/s as the speed of sound in air.

SOLUTION

Conceptualize The sound of the wind blowing across the end of the pipe contains many frequencies, and the culvert responds to the sound by vibrating at the natural frequencies of the air column.

Categorize This example is a relatively simple substitution problem.

Find the frequency of the first harmonic of the culvert, modeling it as an air column open at both ends:

$$f_1 = \frac{v}{2L} = \frac{343 \text{ m/s}}{2(1.23 \text{ m})} = \boxed{139 \text{ Hz}}$$

Find the next harmonics by multiplying by integers:

$$f_2 = 2f_1 = \boxed{279 \text{ Hz}}$$

$$f_3 = 3f_1 = \boxed{418 \text{ Hz}}$$

▸ **18.5** continued

(B) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

SOLUTION

Find the frequency of the first harmonic of the culvert, modeling it as an air column closed at one end:

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(1.23 \text{ m})} = \boxed{69.7 \text{ Hz}}$$

Find the next two harmonics by multiplying by odd integers:

$$f_3 = 3f_1 = \boxed{209 \text{ Hz}}$$

$$f_5 = 5f_1 = \boxed{349 \text{ Hz}}$$

Example 18.6 Measuring the Frequency of a Tuning Fork AM

A simple apparatus for demonstrating resonance in an air column is depicted in Figure 18.14. A vertical pipe open at both ends is partially submerged in water, and a tuning fork vibrating at an unknown frequency is placed near the top of the pipe. The length L of the air column can be adjusted by moving the pipe vertically. The sound waves generated by the fork are reinforced when L corresponds to one of the resonance frequencies of the pipe. For a certain pipe, the smallest value of L for which a peak occurs in the sound intensity is 9.00 cm.

(A) What is the frequency of the tuning fork?

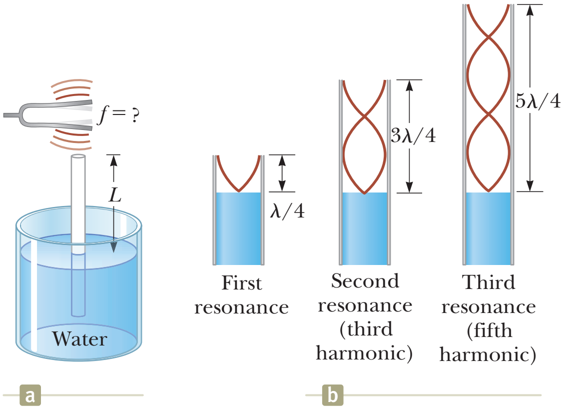


Figure 18.14 (Example 18.6) (a) Apparatus for demonstrating the resonance of sound waves in a pipe closed at one end. The length L of the air column is varied by moving the pipe vertically while it is partially submerged in water. (b) The first three normal modes of the system shown in (a).

SOLUTION

Conceptualize Sound waves from the tuning fork enter the pipe at its upper end. Although the pipe is open at its lower end to allow the water to enter, the water's surface acts like a barrier. The waves reflect from the water surface and combine with those moving downward to form a standing wave.

Categorize Because of the reflection of the sound waves from the water surface, we can model the pipe as open at the upper end and closed at the lower end. Therefore, we can apply the *waves under boundary conditions* model to this situation.

Analyze

Use Equation 18.9 to find the fundamental frequency for $L = 0.090\,0$ m:

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(0.090\,0 \text{ m})} = \boxed{953 \text{ Hz}}$$

Because the tuning fork causes the air column to resonate at this frequency, this frequency must also be that of the tuning fork.

(B) What are the values of L for the next two resonance conditions?

SOLUTION

Use Equation 16.12 to find the wavelength of the sound wave from the tuning fork:

$$\lambda = \frac{v}{f} = \frac{343 \text{ m/s}}{953 \text{ Hz}} = 0.360 \text{ m}$$

Notice from Figure 18.14b that the length of the air column for the second resonance is $3\lambda/4$:

$$L = 3\lambda/4 = \boxed{0.270 \text{ m}}$$

Notice from Figure 18.14b that the length of the air column for the third resonance is $5\lambda/4$:

$$L = 5\lambda/4 = \boxed{0.450 \text{ m}}$$

Finalize Consider how this problem differs from the preceding example. In the culvert, the length was fixed and the air column was presented with a mixture of many frequencies. The pipe in this example is presented with one single frequency from the tuning fork, and the length of the pipe is varied until resonance is achieved.

18.6 Standing Waves in Rods and Membranes

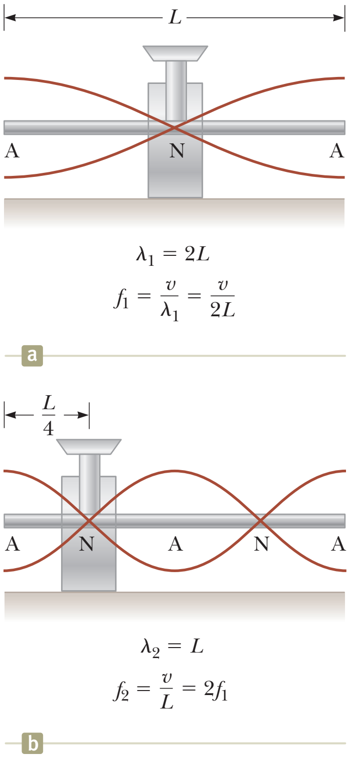


Figure 18.15 Normal-mode longitudinal vibrations of a rod of length L (a) clamped at the middle to produce the first normal mode and (b) clamped at a distance $L/4$ from one end to produce the second normal mode. Notice that the red-brown curves are graphical representations of oscillations parallel to the rod (longitudinal waves).

Standing waves can also be set up in rods and membranes. A rod clamped in the middle and stroked parallel to the rod at one end oscillates as depicted in Figure 18.15a. The oscillations of the elements of the rod are longitudinal, and so the red-brown curves in Figure 18.15 represent *longitudinal* displacements of various parts of the rod. For clarity, the displacements have been drawn in the transverse direction as they were for air columns. The midpoint is a displacement node because it is fixed by the clamp, whereas the ends are displacement antinodes because they are free to oscillate. The oscillations in this setup are analogous to those in a pipe open at both ends. The red-brown lines in Figure 18.15a represent the first normal mode, for which the wavelength is $2L$ and the frequency is $f = v/2L$, where v is the speed of longitudinal waves in the rod. Other normal modes may be excited by clamping the rod at different points. For example, the second normal mode (Fig. 18.15b) is excited by clamping the rod a distance $L/4$ away from one end.

It is also possible to set up transverse standing waves in rods. Musical instruments that depend on transverse standing waves in rods or bars include triangles, marimbas, xylophones, glockenspiels, chimes, and vibraphones. Other devices that make sounds from vibrating bars include music boxes and wind chimes.

Two-dimensional oscillations can be set up in a flexible membrane stretched over a circular hoop such as that in a drumhead. As the membrane is struck at some point, waves that arrive at the fixed boundary are reflected many times. The resulting sound is not harmonic because the standing waves have frequencies that are *not* related by integer multiples. Without this relationship, the sound may be more correctly described as *noise* rather than as music. The production of noise is in contrast to the situation in wind and stringed instruments, which produce sounds that we describe as musical.

Some possible normal modes of oscillation for a two-dimensional circular membrane are shown in Figure 18.16. Whereas nodes are *points* in one-dimensional standing waves on strings and in air columns, a two-dimensional oscillator has *curves* along which there is no displacement of the elements of the medium. The lowest normal mode, which has a frequency f_1, contains only one nodal curve; this curve runs around the outer edge of the membrane. The other possible normal modes show additional nodal curves that are circles and straight lines across the diameter of the membrane.

18.7 Beats: Interference in Time

The interference phenomena we have studied so far involve the superposition of two or more waves having the same frequency. Because the amplitude of the oscil-

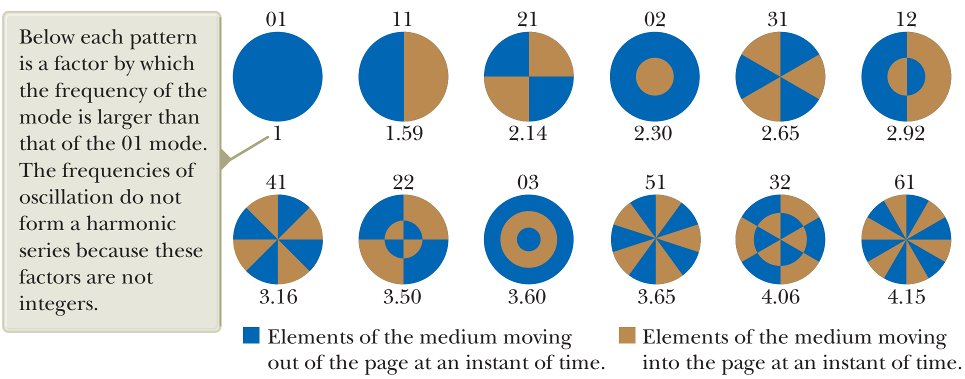


Figure 18.16 Representation of some of the normal modes possible in a circular membrane fixed at its perimeter. The pair of numbers above each pattern corresponds to the number of radial nodes and the number of circular nodes, respectively. In each diagram, elements of the membrane on either side of a nodal line move in opposite directions, as indicated by the colors. *(Adapted from T. D. Rossing,* The Science of Sound, 3rd ed., *Reading, Massachusetts, Addison-Wesley Publishing Co., 2001)*

lation of elements of the medium varies with the position in space of the element in such a wave, we refer to the phenomenon as *spatial interference.* Standing waves in strings and pipes are common examples of spatial interference.

Now let's consider another type of interference, one that results from the superposition of two waves having slightly *different* frequencies. In this case, when the two waves are observed at a point in space, they are periodically in and out of phase. That is, there is a *temporal* (time) alternation between constructive and destructive interference. As a consequence, we refer to this phenomenon as *interference in time* or *temporal interference.* For example, if two tuning forks of slightly different frequencies are struck, one hears a sound of periodically varying amplitude. This phenomenon is called **beating.**

Beating is the periodic variation in amplitude at a given point due to the superposition of two waves having slightly different frequencies.

Definition of beating

The number of amplitude maxima one hears per second, or the *beat frequency,* equals the difference in frequency between the two sources as we shall show below. The maximum beat frequency that the human ear can detect is about 20 beats/s. When the beat frequency exceeds this value, the beats blend indistinguishably with the sounds producing them.

Consider two sound waves of equal amplitude and slightly different frequencies f_1 and f_2 traveling through a medium. We use equations similar to Equation 16.13 to represent the wave functions for these two waves at a point that we identify as $x = 0$. We also choose the phase angle in Equation 16.13 as $\phi = \pi/2$:

$$y_1 = A \sin\left(\frac{\pi}{2} - \omega_1 t\right) = A \cos\left(2\pi f_1 t\right)$$

$$y_2 = A \sin\left(\frac{\pi}{2} - \omega_2 t\right) = A \cos\left(2\pi f_2 t\right)$$

Using the superposition principle, we find that the resultant wave function at this point is

$$y = y_1 + y_2 = A\,(\cos 2\pi f_1 t + \cos 2\pi f_2 t)$$

The trigonometric identity

$$\cos a + \cos b = 2 \cos\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$$

allows us to write the expression for y as

$$y = \left[2A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right] \cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad \textbf{(18.10)}$$

Resultant of two waves of different frequencies but equal amplitude

Graphs of the individual waves and the resultant wave are shown in Figure 18.17. From the factors in Equation 18.10, we see that the resultant wave has an effective

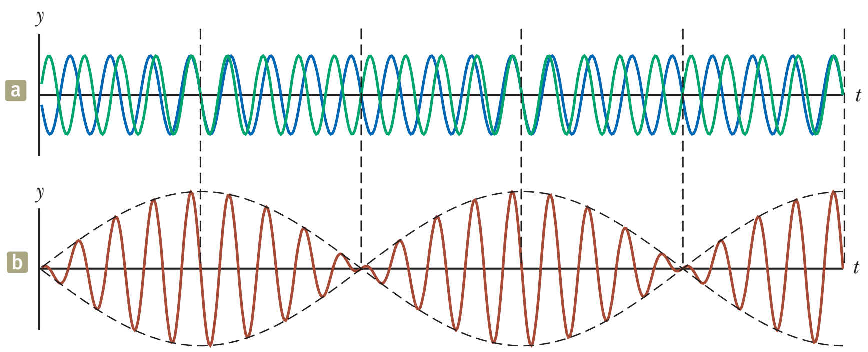


Figure 18.17 Beats are formed by the combination of two waves of slightly different frequencies. (a) The individual waves. (b) The combined wave. The envelope wave (dashed line) represents the beating of the combined sounds.

frequency equal to the average frequency $(f_1 + f_2)/2$. This wave is multiplied by an envelope wave given by the expression in the square brackets:

$$y_{\text{envelope}} = 2A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \quad \textbf{(18.11)}$$

That is, the amplitude and therefore the intensity of the resultant sound vary in time. The dashed black line in Figure 18.17b is a graphical representation of the envelope wave in Equation 18.11 and is a sine wave varying with frequency $(f_1 - f_2)/2$.

A maximum in the amplitude of the resultant sound wave is detected whenever

$$\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pm 1$$

Hence, there are *two* maxima in each period of the envelope wave. Because the amplitude varies with frequency as $(f_1 - f_2)/2$, the number of beats per second, or the **beat frequency** f_{beat}, is twice this value. That is,

Beat frequency ▶

$$f_{\text{beat}} = |f_1 - f_2| \quad \textbf{(18.12)}$$

For instance, if one tuning fork vibrates at 438 Hz and a second one vibrates at 442 Hz, the resultant sound wave of the combination has a frequency of 440 Hz (the musical note A) and a beat frequency of 4 Hz. A listener would hear a 440-Hz sound wave go through an intensity maximum four times every second.

Example 18.7 The Mistuned Piano Strings AM

Two identical piano strings of length 0.750 m are each tuned exactly to 440 Hz. The tension in one of the strings is then increased by 1.0%. If they are now struck, what is the beat frequency between the fundamentals of the two strings?

SOLUTION

Conceptualize As the tension in one of the strings is changed, its fundamental frequency changes. Therefore, when both strings are played, they will have different frequencies and beats will be heard.

Categorize We must combine our understanding of the *waves under boundary conditions* model for strings with our new knowledge of beats.

Analyze Set up a ratio of the fundamental frequencies of the two strings using Equation 18.5:

$$\frac{f_2}{f_1} = \frac{(v_2/2L)}{(v_1/2L)} = \frac{v_2}{v_1}$$

Use Equation 16.18 to substitute for the wave speeds on the strings:

$$\frac{f_2}{f_1} = \frac{\sqrt{T_2/\mu}}{\sqrt{T_1/\mu}} = \sqrt{\frac{T_2}{T_1}}$$

Incorporate that the tension in one string is 1.0% larger than the other; that is, $T_2 = 1.010T_1$:

$$\frac{f_2}{f_1} = \sqrt{\frac{1.010T_1}{T_1}} = 1.005$$

Solve for the frequency of the tightened string:

$$f_2 = 1.005f_1 = 1.005(440 \text{ Hz}) = 442 \text{ Hz}$$

Find the beat frequency using Equation 18.12:

$$f_{\text{beat}} = 442 \text{ Hz} - 440 \text{ Hz} = \boxed{2 \text{ Hz}}$$

Finalize Notice that a 1.0% mistuning in tension leads to an easily audible beat frequency of 2 Hz. A piano tuner can use beats to tune a stringed instrument by "beating" a note against a reference tone of known frequency. The tuner can then adjust the string tension until the frequency of the sound it emits equals the frequency of the reference tone. The tuner does so by tightening or loosening the string until the beats produced by it and the reference source become too infrequent to notice.

18.8 Nonsinusoidal Wave Patterns

It is relatively easy to distinguish the sounds coming from a violin and a saxophone even when they are both playing the same note. On the other hand, a person untrained in music may have difficulty distinguishing a note played on a clarinet from the same note played on an oboe. We can use the pattern of the sound waves from various sources to explain these effects.

When frequencies that are integer multiples of a fundamental frequency are combined to make a sound, the result is a *musical* sound. A listener can assign a pitch to the sound based on the fundamental frequency. Pitch is a psychological reaction to a sound that allows the listener to place the sound on a scale from low to high (bass to treble). Combinations of frequencies that are not integer multiples of a fundamental result in a *noise* rather than a musical sound. It is much harder for a listener to assign a pitch to a noise than to a musical sound.

The wave patterns produced by a musical instrument are the result of the superposition of frequencies that are integer multiples of a fundamental. This superposition results in the corresponding richness of musical tones. The human perceptive response associated with various mixtures of harmonics is the *quality* or *timbre* of the sound. For instance, the sound of the trumpet is perceived to have a "brassy" quality (that is, we have learned to associate the adjective *brassy* with that sound); this quality enables us to distinguish the sound of the trumpet from that of the saxophone, whose quality is perceived as "reedy." The clarinet and oboe, however, both contain air columns excited by reeds; because of this similarity, they have similar mixtures of frequencies and it is more difficult for the human ear to distinguish them on the basis of their sound quality.

The sound wave patterns produced by the majority of musical instruments are nonsinusoidal. Characteristic patterns produced by a tuning fork, a flute, and a clarinet, each playing the same note, are shown in Figure 18.18. Each instrument has its own characteristic pattern. Notice, however, that despite the differences in the patterns, each pattern is periodic. This point is important for our analysis of these waves.

The problem of analyzing nonsinusoidal wave patterns appears at first sight to be a formidable task. If the wave pattern is periodic, however, it can be represented as closely as desired by the combination of a sufficiently large number of sinusoidal waves that form a harmonic series. In fact, we can represent any periodic function as a series of sine and cosine terms by using a mathematical technique based on **Fourier's theorem.**[2] The corresponding sum of terms that represents the periodic wave pattern is called a **Fourier series.** Let $y(t)$ be any function that is periodic in time with period T such that $y(t + T) = y(t)$. Fourier's theorem states that this function can be written as

Fourier's theorem

$$y(t) = \sum (A_n \sin 2\pi f_n t + B_n \cos 2\pi f_n t) \tag{18.13}$$

where the lowest frequency is $f_1 = 1/T$. The higher frequencies are integer multiples of the fundamental, $f_n = nf_1$, and the coefficients A_n and B_n represent the amplitudes of the various waves. Figure 18.19 on page 554 represents a harmonic analysis of the wave patterns shown in Figure 18.18. Each bar in the graph represents one of the terms in the series in Equation 18.13 up to $n = 9$. Notice that a struck tuning fork produces only one harmonic (the first), whereas the flute and clarinet produce the first harmonic and many higher ones.

Notice the variation in relative intensity of the various harmonics for the flute and the clarinet. In general, any musical sound consists of a fundamental frequency f plus other frequencies that are integer multiples of f, all having different intensities.

Pitfall Prevention 18.4

Pitch Versus Frequency Do not confuse the term *pitch* with *frequency*. Frequency is the physical measurement of the number of oscillations per second. Pitch is a psychological reaction to sound that enables a person to place the sound on a scale from high to low or from treble to bass. Therefore, frequency is the stimulus and pitch is the response. Although pitch is related mostly (but not completely) to frequency, they are not the same. A phrase such as "the pitch of the sound" is incorrect because pitch is not a physical property of the sound.

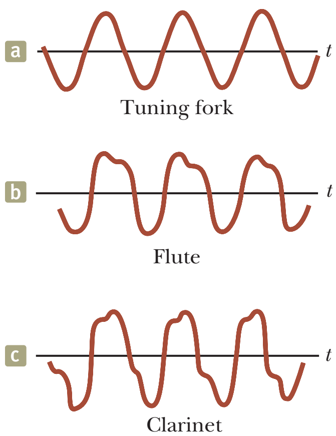


Figure 18.18 Sound wave patterns produced by (a) a tuning fork, (b) a flute, and (c) a clarinet, each at approximately the same frequency.

[2] Developed by Jean Baptiste Joseph Fourier (1786–1830).

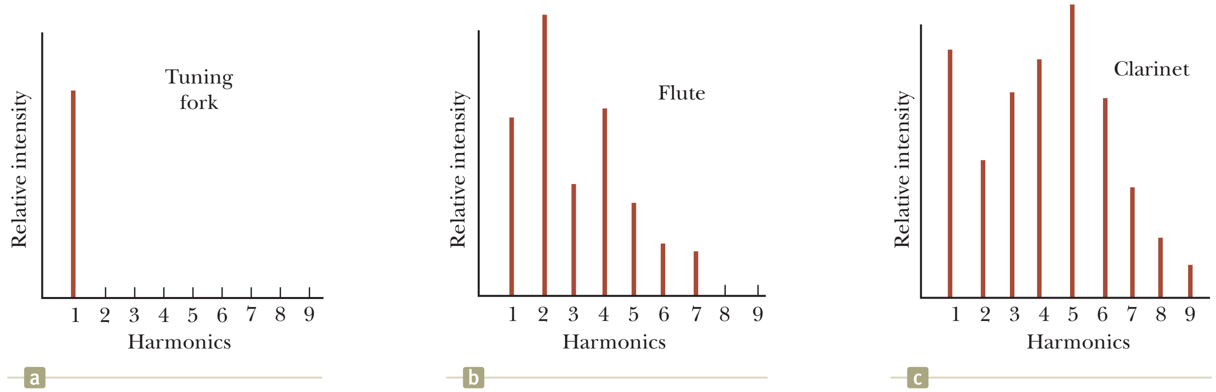


Figure 18.19 Harmonics of the wave patterns shown in Figure 18.18. Notice the variations in intensity of the various harmonics. Parts (a), (b), and (c) correspond to those in Figure 18.18.

We have discussed the *analysis* of a wave pattern using Fourier's theorem. The analysis involves determining the coefficients of the harmonics in Equation 18.13 from a knowledge of the wave pattern. The reverse process, called *Fourier synthesis,* can also be performed. In this process, the various harmonics are added together to form a resultant wave pattern. As an example of Fourier synthesis, consider the building of a square wave as shown in Figure 18.20. The symmetry of the square wave results in only odd multiples of the fundamental frequency combining in its synthesis. In Figure 18.20a, the blue curve shows the combination of f and $3f$. In Figure 18.20b, we have added $5f$ to the combination and obtained the green curve. Notice how the general shape of the square wave is approximated, even though the upper and lower portions are not flat as they should be.

Figure 18.20c shows the result of adding odd frequencies up to $9f$. This approximation (red-brown curve) to the square wave is better than the approximations in Figures 18.20a and 18.20b. To approximate the square wave as closely as possible, we must add all odd multiples of the fundamental frequency, up to infinite frequency.

Using modern technology, musical sounds can be generated electronically by mixing different amplitudes of any number of harmonics. These widely used electronic music synthesizers are capable of producing an infinite variety of musical tones.

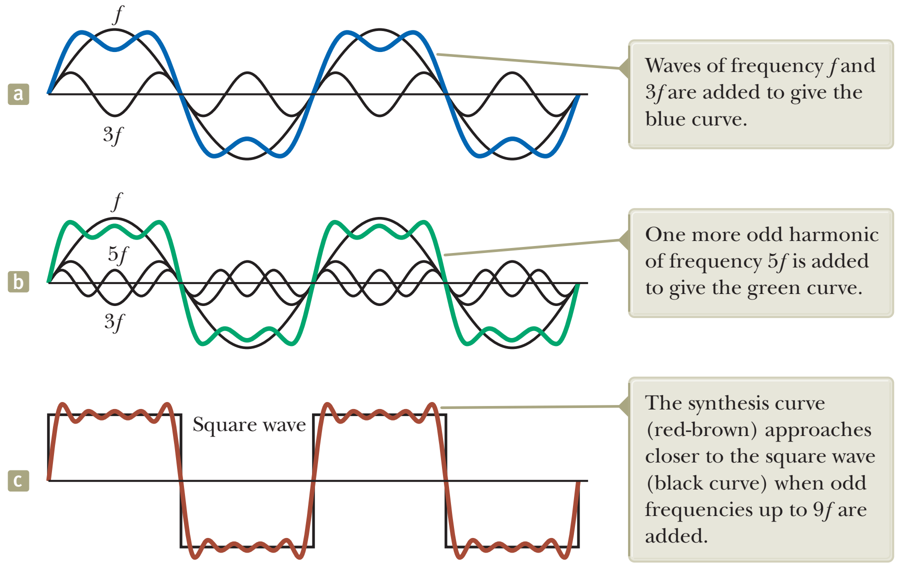


Figure 18.20 Fourier synthesis of a square wave, represented by the sum of odd multiples of the first harmonic, which has frequency f.

Summary

Concepts and Principles

The **superposition principle** specifies that when two or more waves move through a medium, the value of the resultant wave function equals the algebraic sum of the values of the individual wave functions.

The phenomenon of **beating** is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies. The **beat frequency** is

$$f_{\text{beat}} = |f_1 - f_2| \quad \textbf{(18.12)}$$

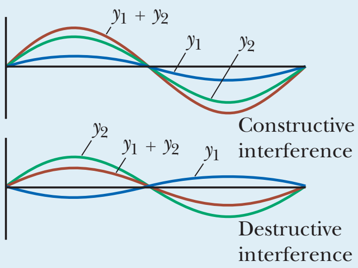
where f_1 and f_2 are the frequencies of the individual waves.

Standing waves are formed from the combination of two sinusoidal waves having the same frequency, amplitude, and wavelength but traveling in opposite directions. The resultant standing wave is described by the wave function

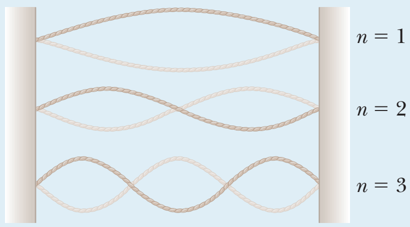
$$y = (2A \sin kx) \cos \omega t \quad \textbf{(18.1)}$$

Hence, the amplitude of the standing wave is $2A$, and the amplitude of the simple harmonic motion of any element of the medium varies according to its position as $2A \sin kx$. The points of zero amplitude (called **nodes**) occur at $x = n\lambda/2$ ($n = 0, 1, 2, 3, \ldots$). The maximum amplitude points (called **antinodes**) occur at $x = n\lambda/4$ ($n = 1, 3, 5, \ldots$). Adjacent antinodes are separated by a distance $\lambda/2$. Adjacent nodes also are separated by a distance $\lambda/2$.

Analysis Models for Problem Solving



Waves in Interference. When two traveling waves having equal frequencies superimpose, the resultant wave is described by the **principle of superposition** and has an amplitude that depends on the phase angle ϕ between the two waves. **Constructive interference** occurs when the two waves are in phase, corresponding to $\phi = 0, 2\pi, 4\pi, \ldots$ rad. **Destructive interference** occurs when the two waves are 180° out of phase, corresponding to $\phi = \pi, 3\pi, 5\pi, \ldots$ rad.



Waves Under Boundary Conditions. When a wave is subject to boundary conditions, only certain natural frequencies are allowed; we say that the frequencies are quantized.

For waves on a string fixed at both ends, the natural frequencies are

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad n = 1, 2, 3, \ldots \quad \textbf{(18.6)}$$

where T is the tension in the string and μ is its linear mass density.

For sound waves with speed v in an air column of length L open at both ends, the natural frequencies are

$$f_n = n\frac{v}{2L} \qquad n = 1, 2, 3, \ldots \quad \textbf{(18.8)}$$

If an air column is open at one end and closed at the other, only odd harmonics are present and the natural frequencies are

$$f_n = n\frac{v}{4L} \qquad n = 1, 3, 5, \ldots \quad \textbf{(18.9)}$$

Objective Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. In Figure OQ18.1 (page 556), a sound wave of wavelength 0.8 m divides into two equal parts that recombine to interfere constructively, with the original difference between their path lengths being $|r_2 - r_1| = 0.8$ m. Rank the following situations according to the intensity of sound at the receiver from the highest to the lowest. Assume the tube walls absorb no sound energy. Give equal ranks to situations in which the intensity is equal.

(a) From its original position, the sliding section is moved out by 0.1 m. (b) Next it slides out an additional 0.1 m. (c) It slides out still another 0.1 m. (d) It slides out 0.1 m more.

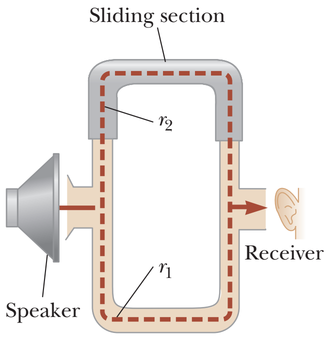


Figure OQ18.1 Objective Question 1 and Problem 6.

2. A string of length L, mass per unit length μ, and tension T is vibrating at its fundamental frequency. **(i)** If the length of the string is doubled, with all other factors held constant, what is the effect on the fundamental frequency? (a) It becomes two times larger. (b) It becomes $\sqrt{2}$ times larger. (c) It is unchanged. (d) It becomes $1/\sqrt{2}$ times as large. (e) It becomes one-half as large. **(ii)** If the mass per unit length is doubled, with all other factors held constant, what is the effect on the fundamental frequency? Choose from the same possibilities as in part (i). **(iii)** If the tension is doubled, with all other factors held constant, what is the effect on the fundamental frequency? Choose from the same possibilities as in part (i).

3. In Example 18.1, we investigated an oscillator at 1.3 kHz driving two identical side-by-side speakers. We found that a listener at point O hears sound with maximum intensity, whereas a listener at point P hears a minimum. What is the intensity at P? (a) less than but close to the intensity at O (b) half the intensity at O (c) very low but not zero (d) zero (e) indeterminate

4. A series of pulses, each of amplitude 0.1 m, is sent down a string that is attached to a post at one end. The pulses are reflected at the post and travel back along the string without loss of amplitude. **(i)** What is the net displacement at a point on the string where two pulses are crossing? Assume the string is rigidly attached to the post. (a) 0.4 m (b) 0.3 m (c) 0.2 m (d) 0.1 m (e) 0 **(ii)** Next assume the end at which reflection occurs is free to slide up and down. Now what is the net displacement at a point on the string where two pulses are crossing? Choose your answer from the same possibilities as in part (i).

5. A flute has a length of 58.0 cm. If the speed of sound in air is 343 m/s, what is the fundamental frequency of the flute, assuming it is a tube closed at one end and open at the other? (a) 148 Hz (b) 296 Hz (c) 444 Hz (d) 591 Hz (e) none of those answers

6. When two tuning forks are sounded at the same time, a beat frequency of 5 Hz occurs. If one of the tuning forks has a frequency of 245 Hz, what is the frequency of the other tuning fork? (a) 240 Hz (b) 242.5 Hz (c) 247.5 Hz (d) 250 Hz (e) More than one answer could be correct.

7. A tuning fork is known to vibrate with frequency 262 Hz. When it is sounded along with a mandolin string, four beats are heard every second. Next, a bit of tape is put onto each tine of the tuning fork, and the tuning fork now produces five beats per second with the same mandolin string. What is the frequency of the string? (a) 257 Hz (b) 258 Hz (c) 262 Hz (d) 266 Hz (e) 267 Hz

8. An archer shoots an arrow horizontally from the center of the string of a bow held vertically. After the arrow leaves it, the string of the bow will vibrate as a superposition of what standing-wave harmonics? (a) It vibrates only in harmonic number 1, the fundamental. (b) It vibrates only in the second harmonic. (c) It vibrates only in the odd-numbered harmonics 1, 3, 5, 7, ... (d) It vibrates only in the even-numbered harmonics 2, 4, 6, 8, ... (e) It vibrates in all harmonics.

9. As oppositely moving pulses of the same shape (one upward, one downward) on a string pass through each other, at one particular instant the string shows no displacement from the equilibrium position at any point. What has happened to the energy carried by the pulses at this instant of time? (a) It was used up in producing the previous motion. (b) It is all potential energy. (c) It is all internal energy. (d) It is all kinetic energy. (e) The positive energy of one pulse adds to zero with the negative energy of the other pulse.

10. A standing wave having three nodes is set up in a string fixed at both ends. If the frequency of the wave is doubled, how many antinodes will there be? (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

11. Suppose all six equal-length strings of an acoustic guitar are played without fingering, that is, without being pressed down at any frets. What quantities are the same for all six strings? Choose all correct answers. (a) the fundamental frequency (b) the fundamental wavelength of the string wave (c) the fundamental wavelength of the sound emitted (d) the speed of the string wave (e) the speed of the sound emitted

12. Assume two identical sinusoidal waves are moving through the same medium in the same direction. Under what condition will the amplitude of the resultant wave be greater than either of the two original waves? (a) in all cases (b) only if the waves have no difference in phase (c) only if the phase difference is less than 90° (d) only if the phase difference is less than 120° (e) only if the phase difference is less than 180°

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. A crude model of the human throat is that of a pipe open at both ends with a vibrating source to introduce the sound into the pipe at one end. Assuming the vibrating source produces a range of frequencies, discuss the effect of changing the pipe's length.

2. When two waves interfere constructively or destructively, is there any gain or loss in energy in the system of the waves? Explain.

3. Explain how a musical instrument such as a piano may be tuned by using the phenomenon of beats.

4. What limits the amplitude of motion of a real vibrating system that is driven at one of its resonant frequencies?

5. A tuning fork by itself produces a faint sound. Explain how each of the following methods can be used to obtain a louder sound from it. Explain also any effect on the time interval for which the fork vibrates audibly. (a) holding the edge of a sheet of paper against one vibrating tine (b) pressing the handle of the tuning fork against a chalkboard or a tabletop (c) holding the tuning fork above a column of air of properly chosen length as in Example 18.6 (d) holding the tuning fork close to an open slot cut in a sheet of foam plastic or cardboard (with the slot similar in size and shape to one tine of the fork and the motion of the tines perpendicular to the sheet)

6. An airplane mechanic notices that the sound from a twin-engine aircraft rapidly varies in loudness when both engines are running. What could be causing this variation from loud to soft?

7. Despite a reasonably steady hand, a person often spills his coffee when carrying it to his seat. Discuss resonance as a possible cause of this difficulty and devise a means for preventing the spills.

8. A soft-drink bottle resonates as air is blown across its top. What happens to the resonance frequency as the level of fluid in the bottle decreases?

9. Does the phenomenon of wave interference apply only to sinusoidal waves?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Note: Unless otherwise specified, assume the speed of sound in air is 343 m/s, its value at an air temperature of 20.0°C. At any other Celsius temperature T_C, the speed of sound in air is described by

$$v = 331\sqrt{1 + \frac{T_C}{273}}$$

where v is in m/s and T is in °C.

Section 18.1 Analysis Model: Waves in Interference

1. W Two waves are traveling in the same direction along a stretched string. The waves are 90.0° out of phase. Each wave has an amplitude of 4.00 cm. Find the amplitude of the resultant wave.

2. Two wave pulses A and B are moving in opposite directions, each with a speed v = 2.00 cm/s. The amplitude of A is twice the amplitude of B. The pulses are shown in Figure P18.2 at t = 0. Sketch the resultant wave at t = 1.00 s, 1.50 s, 2.00 s, 2.50 s, and 3.00 s.

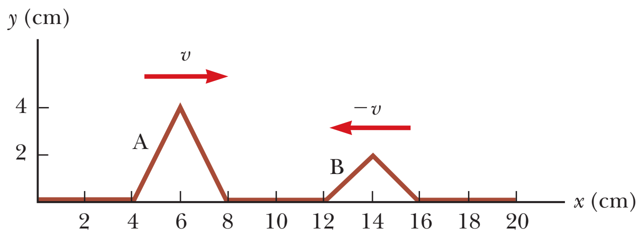


Figure P18.2

3. W Two waves on one string are described by the wave functions

$$y_1 = 3.0 \cos (4.0x - 1.6t) \qquad y_2 = 4.0 \sin (5.0x - 2.0t)$$

where x and y are in centimeters and t is in seconds. Find the superposition of the waves $y_1 + y_2$ at the points (a) x = 1.00, t = 1.00; (b) x = 1.00, t = 0.500; and (c) x = 0.500, t = 0. *Note:* Remember that the arguments of the trigonometric functions are in radians.

4. Two pulses of different amplitudes approach each other, each having a speed of v = 1.00 m/s. Figure P18.4 shows the positions of the pulses at time t = 0. (a) Sketch the resultant wave at t = 2.00 s, 4.00 s, 5.00 s, and 6.00 s. (b) **What If?** If the pulse on the right is inverted so that it is upright, how would your sketches of the resultant wave change?

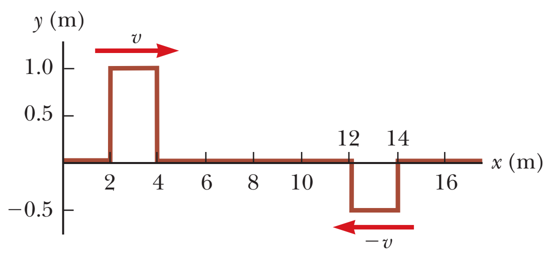


Figure P18.4

5. A tuning fork generates sound waves with a frequency of 246 Hz. The waves travel in opposite directions along a hallway, are reflected by end walls, and return. The hallway is 47.0 m long and the tuning fork is located 14.0 m from one end. What is the phase difference between the reflected waves when they meet at the tuning fork? The speed of sound in air is 343 m/s.

6. The acoustical system shown in Figure OQ18.1 is driven by a speaker emitting sound of frequency 756 Hz. (a) If constructive interference occurs at a particular location of the sliding section, by what minimum amount should the sliding section be moved upward so that destructive interference occurs instead? (b) What minimum distance from the original position of the sliding section will again result in constructive interference?

7. Two pulses traveling on the same string are described by

$$y_1 = \frac{5}{(3x - 4t)^2 + 2} \qquad y_2 = \frac{-5}{(3x + 4t - 6)^2 + 2}$$

(a) In which direction does each pulse travel? (b) At what instant do the two cancel everywhere? (c) At what point do the two pulses always cancel?

8. AMT Two identical loudspeakers are placed on a wall 2.00 m apart. A listener stands 3.00 m from the wall directly in front of one of the speakers. A single oscillator is driving the speakers at a frequency of 300 Hz. (a) What is the phase difference in radians between the waves from the speakers when they reach the observer? (b) **What If?** What is the frequency closest to 300 Hz to which the oscillator may be adjusted such that the observer hears minimal sound?

9. M Two traveling sinusoidal waves are described by the wave functions

$$y_1 = 5.00 \sin [\pi(4.00x - 1\,200t)]$$

$$y_2 = 5.00 \sin [\pi(4.00x - 1\,200t - 0.250)]$$

where x, y_1, and y_2 are in meters and t is in seconds. (a) What is the amplitude of the resultant wave function $y_1 + y_2$? (b) What is the frequency of the resultant wave function?

10. *Why is the following situation impossible?* Two identical loudspeakers are driven by the same oscillator at frequency 200 Hz. They are located on the ground a distance $d = 4.00$ m from each other. Starting far from the speakers, a man walks straight toward the right-hand speaker as shown in Figure P18.10. After passing through three minima in sound intensity, he walks to the next maximum and stops. Ignore any sound reflection from the ground.

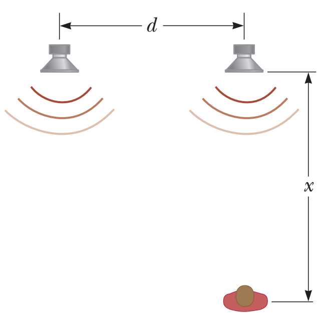


Figure P18.10

11. M Two sinusoidal waves in a string are defined by the wave functions

$$y_1 = 2.00 \sin (20.0x - 32.0t) \qquad y_2 = 2.00 \sin (25.0x - 40.0t)$$

where x, y_1, and y_2 are in centimeters and t is in seconds. (a) What is the phase difference between these two waves at the point $x = 5.00$ cm at $t = 2.00$ s? (b) What is the positive x value closest to the origin for which the two phases differ by $\pm\pi$ at $t = 2.00$ s? (At that location, the two waves add to zero.)

12. Two identical sinusoidal waves with wavelengths of 3.00 m travel in the same direction at a speed of 2.00 m/s. The second wave originates from the same point as the first, but at a later time. The amplitude of the resultant wave is the same as that of each of the two initial waves. Determine the minimum possible time interval between the starting moments of the two waves.

13. Two identical loudspeakers 10.0 m apart are driven by the same oscillator with a frequency of $f = 21.5$ Hz (Fig. P18.13) in an area where the speed of sound is 344 m/s. (a) Show that a receiver at point A records a minimum in sound intensity from the two speakers. (b) If the receiver is moved in the plane of the speakers, show that the path it should take so that the intensity remains at a minimum is along the hyperbola $9x^2 - 16y^2 = 144$ (shown in red-brown in Fig. P18.13). (c) Can the receiver remain at a minimum and move very far away from the two sources? If so, determine the limiting form of the path it must take. If not, explain how far it can go.

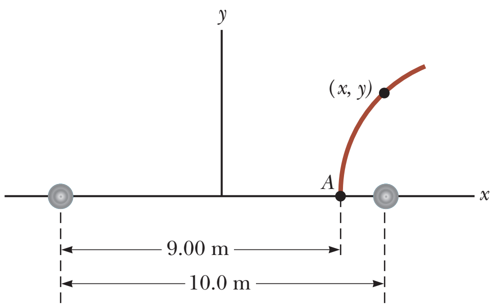


Figure P18.13

Section 18.2 Standing Waves

14. Two waves simultaneously present on a long string have a phase difference ϕ between them so that a standing wave formed from their combination is described by

$$y(x, t) = 2A \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t - \frac{\phi}{2}\right)$$

(a) Despite the presence of the phase angle ϕ, is it still true that the nodes are one-half wavelength apart? Explain. (b) Are the nodes different in any way from the way they would be if ϕ were zero? Explain.

15. W Two sinusoidal waves traveling in opposite directions interfere to produce a standing wave with the wave function

$$y = 1.50 \sin (0.400x) \cos (200t)$$

where x and y are in meters and t is in seconds. Determine (a) the wavelength, (b) the frequency, and (c) the speed of the interfering waves.

16. Verify by direct substitution that the wave function for a standing wave given in Equation 18.1,

$$y = (2A \sin kx) \cos \omega t$$

is a solution of the general linear wave equation, Equation 16.27:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

17. M Two transverse sinusoidal waves combining in a medium are described by the wave functions

$$y_1 = 3.00 \sin \pi(x + 0.600t) \qquad y_2 = 3.00 \sin \pi(x - 0.600t)$$

where x, y_1, and y_2 are in centimeters and t is in seconds. Determine the maximum transverse position of an element of the medium at (a) $x = 0.250$ cm, (b) $x = 0.500$ cm, and (c) $x = 1.50$ cm. (d) Find the three smallest values of x corresponding to antinodes.

18. A standing wave is described by the wave function

$$y = 6 \sin\left(\frac{\pi}{2}x\right) \cos(100\pi t)$$

where x and y are in meters and t is in seconds. (a) Prepare graphs showing y as a function of x for five instants: t = 0, 5 ms, 10 ms, 15 ms, and 20 ms. (b) From the graph, identify the wavelength of the wave and explain how to do so. (c) From the graph, identify the frequency of the wave and explain how to do so. (d) From the equation, directly identify the wavelength of the wave and explain how to do so. (e) From the equation, directly identify the frequency and explain how to do so.

19. M Two identical loudspeakers are driven in phase by a common oscillator at 800 Hz and face each other at a distance of 1.25 m. Locate the points along the line joining the two speakers where relative minima of sound pressure amplitude would be expected.

Section 18.3 Analysis Model: Waves Under Boundary Conditions

20. A standing wave is established in a 120-cm-long string fixed at both ends. The string vibrates in four segments when driven at 120 Hz. (a) Determine the wavelength. (b) What is the fundamental frequency of the string?

21. A string with a mass m = 8.00 g and a length L = 5.00 m has one end attached to a wall; the other end is draped over a small, fixed pulley a distance d = 4.00 m from the wall and attached to a hanging object with a mass M = 4.00 kg as in Figure P18.21. If the horizontal part of the string is plucked, what is the fundamental frequency of its vibration?

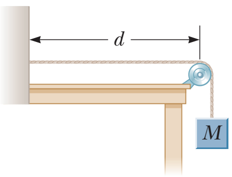


Figure P18.21

22. The 64.0-cm-long string of a guitar has a fundamental frequency of 330 Hz when it vibrates freely along its entire length. A fret is provided for limiting vibration to just the lower two-thirds of the string. (a) If the string is pressed down at this fret and plucked, what is the new fundamental frequency? (b) **What If?** The guitarist can play a "natural harmonic" by gently touching the string at the location of this fret and plucking the string at about one-sixth of the way along its length from the other end. What frequency will be heard then?

23. W The A string on a cello vibrates in its first normal mode with a frequency of 220 Hz. The vibrating segment is 70.0 cm long and has a mass of 1.20 g. (a) Find the tension in the string. (b) Determine the frequency of vibration when the string vibrates in three segments.

24. A taut string has a length of 2.60 m and is fixed at both ends. (a) Find the wavelength of the fundamental mode of vibration of the string. (b) Can you find the frequency of this mode? Explain why or why not.

25. A certain vibrating string on a piano has a length of 74.0 cm and forms a standing wave having two antinodes. (a) Which harmonic does this wave represent? (b) Determine the wavelength of this wave. (c) How many nodes are there in the wave pattern?

26. A string that is 30.0 cm long and has a mass per unit length of 9.00×10^{-3} kg/m is stretched to a tension of 20.0 N. Find (a) the fundamental frequency and (b) the next three frequencies that could cause standing-wave patterns on the string.

27. AMT W In the arrangement shown in Figure P18.27, an object can be hung from a string (with linear mass density μ = 0.002 00 kg/m) that passes over a light pulley. The string is connected to a vibrator (of constant frequency f), and the length of the string between point P and the pulley is L = 2.00 m. When the mass m of the object is either 16.0 kg or 25.0 kg, standing waves are observed; no standing waves are observed with any mass between these values, however. (a) What is the frequency of the vibrator? *Note:* The greater the tension in the string, the smaller the number of nodes in the standing wave. (b) What is the largest object mass for which standing waves could be observed?

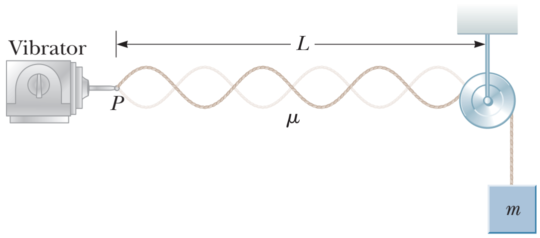


Figure P18.27 Problems 27 and 28.

28. M In the arrangement shown in Figure P18.27, an object of mass m = 5.00 kg hangs from a cord around a light pulley. The length of the cord between point P and the pulley is L = 2.00 m. (a) When the vibrator is set to a frequency of 150 Hz, a standing wave with six loops is formed. What must be the linear mass density of the cord? (b) How many loops (if any) will result if m is changed to 45.0 kg? (c) How many loops (if any) will result if m is changed to 10.0 kg?

29. **Review.** A sphere of mass $M = 1.00$ kg is supported by a string that passes over a pulley at the end of a horizontal rod of length $L = 0.300$ m (Fig. P18.29). The string makes an angle $\theta = 35.0°$ with the rod. The fundamental frequency of standing waves in the portion of the string above the rod is $f = 60.0$ Hz. Find the mass of the portion of the string above the rod.

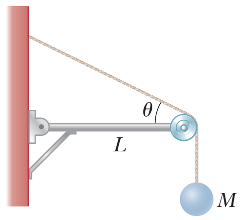


Figure P18.29
Problems 29 and 30.

30. **Review.** A sphere of mass M is supported by a string that passes over a pulley at the end of a horizontal rod of length L (Fig. P18.29). The string makes an angle θ with the rod. The fundamental frequency of standing waves in the portion of the string above the rod is f. Find the mass of the portion of the string above the rod.

31. A violin string has a length of 0.350 m and is tuned to concert G, with $f_G = 392$ Hz. (a) How far from the end of the string must the violinist place her finger to play concert A, with $f_A = 440$ Hz? (b) If this position is to remain correct to one-half the width of a finger (that is, to within 0.600 cm), what is the maximum allowable percentage change in the string tension?

32. **Review.** A solid copper object hangs at the bottom of a steel wire of negligible mass. The top end of the wire is fixed. When the wire is struck, it emits sound with a fundamental frequency of 300 Hz. The copper object is then submerged in water so that half its volume is below the water line. Determine the new fundamental frequency.

33. A standing-wave pattern is observed in a thin wire with a length of 3.00 m. The wave function is

$$y = 0.002\,00 \sin(\pi x) \cos(100\pi t)$$

where x and y are in meters and t is in seconds. (a) How many loops does this pattern exhibit? (b) What is the fundamental frequency of vibration of the wire? (c) **What If?** If the original frequency is held constant and the tension in the wire is increased by a factor of 9, how many loops are present in the new pattern?

Section 18.4 Resonance

34. The Bay of Fundy, Nova Scotia, has the highest tides in the world. Assume in midocean and at the mouth of the bay the Moon's gravity gradient and the Earth's rotation make the water surface oscillate with an amplitude of a few centimeters and a period of 12 h 24 min. At the head of the bay, the amplitude is several meters. Assume the bay has a length of 210 km and a uniform depth of 36.1 m. The speed of long-wavelength water waves is given by $v = \sqrt{gd}$, where d is the water's depth. Argue for or against the proposition that the tide is magnified by standing-wave resonance.

35. An earthquake can produce a *seiche* in a lake in which the water sloshes back and forth from end to end with remarkably large amplitude and long period. Consider a seiche produced in a farm pond. Suppose the pond is 9.15 m long and assume it has a uniform width and depth. You measure that a pulse produced at one end reaches the other end in 2.50 s. (a) What is the wave speed? (b) What should be the frequency of the ground motion during the earthquake to produce a seiche that is a standing wave with antinodes at each end of the pond and one node at the center?

36. High-frequency sound can be used to produce standing-wave vibrations in a wine glass. A standing-wave vibration in a wine glass is observed to have four nodes and four antinodes equally spaced around the 20.0-cm circumference of the rim of the glass. If transverse waves move around the glass at 900 m/s, an opera singer would have to produce a high harmonic with what frequency to shatter the glass with a resonant vibration as shown in Figure P18.36?

Steve Bronstein/Stone/Getty Images

Figure P18.36

Section 18.5 Standing Waves in Air Columns

37. The windpipe of one typical whooping crane is 5.00 feet long. What is the fundamental resonant frequency of the bird's trachea, modeled as a narrow pipe closed at one end? Assume a temperature of 37°C.

38. If a human ear canal can be thought of as resembling an organ pipe, closed at one end, that resonates at a fundamental frequency of 3 000 Hz, what is the length of the canal? Use a normal body temperature of 37°C for your determination of the speed of sound in the canal.

39. Calculate the length of a pipe that has a fundamental frequency of 240 Hz assuming the pipe is (a) closed at one end and (b) open at both ends.

40. W The overall length of a piccolo is 32.0 cm. The resonating air column is open at both ends. (a) Find the frequency of the lowest note a piccolo can sound. (b) Opening holes in the side of a piccolo effectively shortens the length of the resonant column. Assume the highest note a piccolo can sound is 4 000 Hz. Find the distance between adjacent antinodes for this mode of vibration.

41. The fundamental frequency of an open organ pipe corresponds to middle C (261.6 Hz on the chromatic musical scale). The third resonance of a closed organ pipe has the same frequency. What is the length of (a) the open pipe and (b) the closed pipe?

42. The longest pipe on a certain organ is 4.88 m. What is the fundamental frequency (at 0.00°C) if the pipe is (a) closed at one end and (b) open at each end? (c) What will be the frequencies at 20.0°C?

43. An air column in a glass tube is open at one end and closed at the other by a movable piston. The air in the tube is warmed above room temperature, and a 384-Hz tuning fork is held at the open end. Resonance is heard when the piston is at a distance $d_1 = 22.8$ cm from the open end and again when it is at a distance $d_2 = 68.3$ cm from the open end. (a) What speed of sound is implied by these data? (b) How far from the open end will the piston be when the next resonance is heard?

44. A tuning fork with a frequency of $f = 512$ Hz is placed near the top of the tube shown in Figure P18.44. The water level is lowered so that the length L slowly increases from an initial value of 20.0 cm. Determine the next two values of L that correspond to resonant modes.

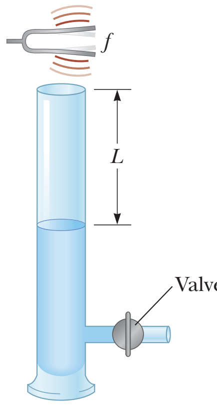


Figure P18.44

45. With a particular fingering, a flute produces a note with frequency 880 Hz at 20.0°C. The flute is open at both ends. (a) Find the air column length. (b) At the beginning of the halftime performance at a late-season football game, the ambient temperature is −5.00°C and the flutist has not had a chance to warm up her instrument. Find the frequency the flute produces under these conditions.

46. A shower stall has dimensions 86.0 cm × 86.0 cm × 210 cm. Assume the stall acts as a pipe closed at both ends, with nodes at opposite sides. Assume singing voices range from 130 Hz to 2 000 Hz and let the speed of sound in the hot air be 355 m/s. For someone singing in this shower, which frequencies would sound the richest (because of resonance)?

47. A glass tube (open at both ends) of length L is positioned near an audio speaker of frequency $f = 680$ Hz. For what values of L will the tube resonate with the speaker?

48. A tunnel under a river is 2.00 km long. (a) At what frequencies can the air in the tunnel resonate? (b) Explain whether it would be good to make a rule against blowing your car horn when you are in the tunnel.

49. As shown in Figure P18.49, water is pumped into a tall, vertical cylinder at a volume flow rate $R = 1.00$ L/min. The radius of the cylinder is $r = 5.00$ cm, and at the open top of the cylinder a tuning fork is vibrating with a frequency $f = 512$ Hz. As the water rises, what time interval elapses between successive resonances?

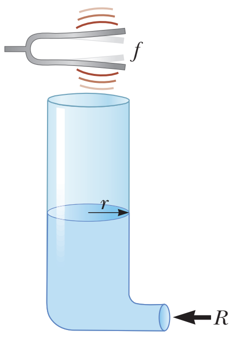


Figure P18.49
Problems 49 and 50.

50. As shown in Figure P18.49, water is pumped into a tall, vertical cylinder at a volume flow rate R. The radius of the cylinder is r, and at the open top of the cylinder a tuning fork is vibrating with a frequency f. As the water rises, what time interval elapses between successive resonances?

51. AMT M Two adjacent natural frequencies of an organ pipe are determined to be 550 Hz and 650 Hz. Calculate (a) the fundamental frequency and (b) the length of this pipe.

52. *Why is the following situation impossible?* A student is listening to the sounds from an air column that is 0.730 m long. He doesn't know if the column is open at both ends or open at only one end. He hears resonance from the air column at frequencies 235 Hz and 587 Hz.

53. A student uses an audio oscillator of adjustable frequency to measure the depth of a water well. The student reports hearing two successive resonances at 51.87 Hz and 59.85 Hz. (a) How deep is the well? (b) How many antinodes are in the standing wave at 51.87 Hz?

Section 18.6 Standing Waves in Rods and Membranes

54. An aluminum rod is clamped one-fourth of the way along its length and set into longitudinal vibration by a variable-frequency driving source. The lowest frequency that produces resonance is 4 400 Hz. The speed of sound in an aluminum rod is 5 100 m/s. Determine the length of the rod.

55. An aluminum rod 1.60 m long is held at its center. It is stroked with a rosin-coated cloth to set up a longitudinal vibration. The speed of sound in a thin rod of aluminum is 5 100 m/s. (a) What is the fundamental frequency of the waves established in the rod? (b) What harmonics are set up in the rod held in this manner? (c) **What If?** What would be the fundamental frequency if the rod were copper, in which the speed of sound is 3 560 m/s?

Section 18.7 Beats: Interference in Time

56. W While attempting to tune the note C at 523 Hz, a piano tuner hears 2.00 beats/s between a reference oscillator and the string. (a) What are the possible frequencies of the string? (b) When she tightens the string slightly, she hears 3.00 beats/s. What is the frequency of the string now? (c) By what percentage should the piano tuner now change the tension in the string to bring it into tune?

57. M In certain ranges of a piano keyboard, more than one string is tuned to the same note to provide extra loudness. For example, the note at 110 Hz has two strings at this frequency. If one string slips from its normal tension of 600 N to 540 N, what beat frequency is heard when the hammer strikes the two strings simultaneously?

58. **Review.** Jane waits on a railroad platform while two trains approach from the same direction at equal speeds of 8.00 m/s. Both trains are blowing their whistles (which have the same frequency), and one train is some distance behind the other. After the first train passes Jane but before the second train passes her, she hears beats of frequency 4.00 Hz. What is the frequency of the train whistles?

59. M **Review.** A student holds a tuning fork oscillating at 256 Hz. He walks toward a wall at a constant speed of 1.33 m/s. (a) What beat frequency does he observe between the tuning fork and its echo? (b) How fast must he walk away from the wall to observe a beat frequency of 5.00 Hz?

Section 18.8 Nonsinusoidal Wave Patterns

60. An A-major chord consists of the notes called A, C#, and E. It can be played on a piano by simultaneously striking strings with fundamental frequencies of 440.00 Hz, 554.37 Hz, and 659.26 Hz. The rich consonance of the chord is associated with near equality of the frequencies of some of the higher harmonics of the three tones. Consider the first five harmonics of each string and determine which harmonics show near equality.

61. Suppose a flutist plays a 523-Hz C note with first harmonic displacement amplitude $A_1 = 100$ nm. From Figure 18.19b read, by proportion, the displacement amplitudes of harmonics 2 through 7. Take these as the values A_2 through A_7 in the Fourier analysis of the sound and assume $B_1 = B_2 = \cdots = B_7 = 0$. Construct a graph of the waveform of the sound. Your waveform will not look exactly like the flute waveform in Figure 18.18b because you simplify by ignoring cosine terms; nevertheless, it produces the same sensation to human hearing.

Additional Problems

62. M A pipe open at both ends has a fundamental frequency of 300 Hz when the temperature is 0°C. (a) What is the length of the pipe? (b) What is the fundamental frequency at a temperature of 30.0°C?

63. A string is 0.400 m long and has a mass per unit length of 9.00×10^{-3} kg/m. What must be the tension in the string if its second harmonic has the same frequency as the second resonance mode of a 1.75-m-long pipe open at one end?

64. Two strings are vibrating at the same frequency of 150 Hz. After the tension in one of the strings is decreased, an observer hears four beats each second when the strings vibrate together. Find the new frequency in the adjusted string.

65. The ship in Figure P18.65 travels along a straight line parallel to the shore and a distance $d = 600$ m from it. The ship's radio receives simultaneous signals of the same frequency from antennas A and B, separated by a distance $L = 800$ m. The signals interfere constructively at point C, which is equidistant from A and B. The signal goes through the first minimum at point D, which is directly outward from the shore from point B. Determine the wavelength of the radio waves.

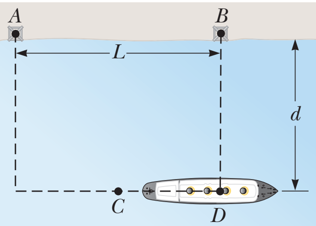


Figure P18.65

66. A 2.00-m-long wire having a mass of 0.100 kg is fixed at both ends. The tension in the wire is maintained at 20.0 N. (a) What are the frequencies of the first three allowed modes of vibration? (b) If a node is observed at a point 0.400 m from one end, in what mode and with what frequency is it vibrating?

67. The fret closest to the bridge on a guitar is 21.4 cm from the bridge as shown in Figure P18.67. When the thinnest string is pressed down at this first fret, the string produces the highest frequency that can be played on that guitar, 2 349 Hz. The next lower note that is produced on the string has frequency 2 217 Hz. How far away from the first fret should the next fret be?

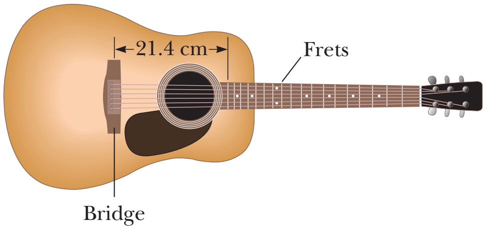


Figure P18.67

68. A string fixed at both ends and having a mass of 4.80 g, a length of 2.00 m, and a tension of 48.0 N vibrates in its second ($n = 2$) normal mode. (a) Is the wavelength in air of the sound emitted by this vibrating string larger or smaller than the wavelength of the wave on the string? (b) What is the ratio of the wavelength in air of the sound emitted by this vibrating string and the wavelength of the wave on the string?

69. A quartz watch contains a crystal oscillator in the form of a block of quartz that vibrates by contracting and expanding. An electric circuit feeds in energy to maintain the oscillation and also counts the voltage pulses to keep time. Two opposite faces of the block, 7.05 mm apart, are antinodes, moving alternately toward each other and away from each other. The plane halfway between these two faces is a node of the vibration. The speed of sound in quartz is equal to 3.70×10^3 m/s. Find the frequency of the vibration.

70. GP **Review.** For the arrangement shown in Figure P18.70, the inclined plane and the small pulley are frictionless; the string supports the object of mass M at the bottom of the plane; and the string has mass m. The system is in equilibrium, and the vertical part of the string has a length h. We wish to study standing waves set up in the vertical section of the string. (a) What analysis model describes the object of mass M? (b) What analysis model describes the waves on the vertical part of the

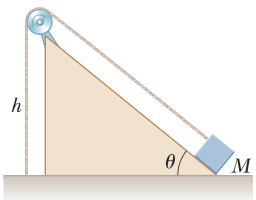


Figure P18.70

string? (c) Find the tension in the string. (d) Model the shape of the string as one leg and the hypotenuse of a right triangle. Find the whole length of the string. (e) Find the mass per unit length of the string. (f) Find the speed of waves on the string. (g) Find the lowest frequency for a standing wave on the vertical section of the string. (h) Evaluate this result for $M = 1.50$ kg, $m = 0.750$ g, $h = 0.500$ m, and $\theta = 30.0°$. (i) Find the numerical value for the lowest frequency for a standing wave on the sloped section of the string.

71. A 0.010 0-kg wire, 2.00 m long, is fixed at both ends and vibrates in its simplest mode under a tension of 200 N. When a vibrating tuning fork is placed near the wire, a beat frequency of 5.00 Hz is heard. (a) What could be the frequency of the tuning fork? (b) What should the tension in the wire be if the beats are to disappear?

72. Two speakers are driven by the same oscillator of frequency f. They are located a distance d from each other on a vertical pole. A man walks straight toward the lower speaker in a direction perpendicular to the pole as shown in Figure P18.72. (a) How many times will he hear a minimum in sound intensity? (b) How far is he from the pole at these moments? Let v represent the speed of sound and assume that the ground does not reflect sound. The man's ears are at the same level as the lower speaker.

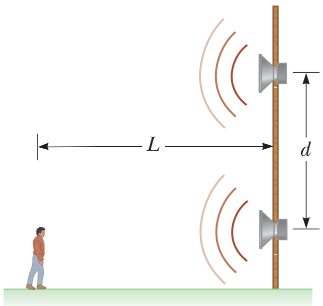


Figure P18.72

73. Review. Consider the apparatus shown in Figure 18.11 and described in Example 18.4. Suppose the number of antinodes in Figure 18.11b is an arbitrary value n. (a) Find an expression for the radius of the sphere in the water as a function of only n. (b) What is the minimum allowed value of n for a sphere of nonzero size? (c) What is the radius of the largest sphere that will produce a standing wave on the string? (d) What happens if a larger sphere is used?

74. Review. The top end of a yo-yo string is held stationary. The yo-yo itself is much more massive than the string. It starts from rest and moves down with constant acceleration 0.800 m/s^2 as it unwinds from the string. The rubbing of the string against the edge of the yo-yo excites transverse standing-wave vibrations in the string. Both ends of the string are nodes even as the length of the string increases. Consider the instant 1.20 s after the motion begins from rest. (a) Show that the rate of change with time of the wavelength of the fundamental mode of oscillation is 1.92 m/s. (b) **What if?** Is the rate of change of the wavelength of the second harmonic also 1.92 m/s at this moment? Explain your answer. (c) **What if?** The experiment is repeated after more mass has been added to the yo-yo body. The mass distribution is kept the same so that the yo-yo still moves with downward acceleration 0.800 m/s^2. At the 1.20-s point in this case, is the rate of change of the fundamental wavelength of the string vibration still equal to 1.92 m/s? Explain. (d) Is the rate of change of the second harmonic wavelength the same as in part (b)? Explain.

75. On a marimba (Fig. P18.75), the wooden bar that sounds a tone when struck vibrates in a transverse standing wave having three antinodes and two nodes. The lowest-frequency note is 87.0 Hz, produced by a bar 40.0 cm long. (a) Find the speed of transverse waves on the bar. (b) A resonant pipe suspended vertically below the center of the bar enhances the loudness of the emitted sound. If the pipe is open at the top end only, what length of the pipe is required to resonate with the bar in part (a)?

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Figure P18.75

76. A nylon string has mass 5.50 g and length $L = 86.0$ cm. The lower end is tied to the floor, and the upper end is tied to a small set of wheels through a slot in a track on which the wheels move (Fig. P18.76). The wheels have a mass that is negligible compared with that of the string, and they roll without friction on the track so that the upper end of the string is essentially free. At equilibrium, the string is vertical and motionless. When it is carrying a small-amplitude wave, you may assume the string is always under uniform tension 1.30 N. (a) Find the speed of transverse waves on the string. (b) The string's vibration possibilities are a set of standing-wave states, each with a node at the fixed bottom end and an antinode at the free top end. Find the node–antinode distances for each of the three simplest states. (c) Find the frequency of each of these states.

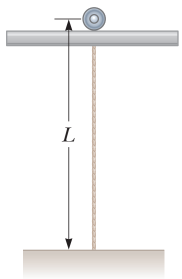

Figure P18.76

77. M Two train whistles have identical frequencies of 180 Hz. When one train is at rest in the station and the other is moving nearby, a commuter standing on the station platform hears beats with a frequency of 2.00 beats/s when the whistles operate together. What are the two possible speeds and directions the moving train can have?

78. **Review.** A loudspeaker at the front of a room and an identical loudspeaker at the rear of the room are being driven by the same oscillator at 456 Hz. A student walks at a uniform rate of 1.50 m/s along the length of the room. She hears a single tone repeatedly becoming louder and softer. (a) Model these variations as beats between the Doppler-shifted sounds the student receives. Calculate the number of beats the student hears each second. (b) Model the two speakers as producing a standing wave in the room and the student as walking between antinodes. Calculate the number of intensity maxima the student hears each second.

79. **Review.** Consider the copper object hanging from the steel wire in Problem 32. The top end of the wire is fixed. When the wire is struck, it emits sound with a fundamental frequency of 300 Hz. The copper object is then submerged in water. If the object can be positioned with any desired fraction of its volume submerged, what is the lowest possible new fundamental frequency?

80. M Two wires are welded together end to end. The wires are made of the same material, but the diameter of one is twice that of the other. They are subjected to a tension of 4.60 N. The thin wire has a length of 40.0 cm and a linear mass density of 2.00 g/m. The combination is fixed at both ends and vibrated in such a way that two antinodes are present, with the node between them being right at the weld. (a) What is the frequency of vibration? (b) What is the length of the thick wire?

81. A string of linear density 1.60 g/m is stretched between clamps 48.0 cm apart. The string does not stretch appreciably as the tension in it is steadily raised from 15.0 N at $t = 0$ to 25.0 N at $t = 3.50$ s. Therefore, the tension as a function of time is given by the expression $T = 15.0 + 10.0t/3.50$, where T is in newtons and t is in seconds. The string is vibrating in its fundamental mode throughout this process. Find the number of oscillations it completes during the 3.50-s interval.

82. A standing wave is set up in a string of variable length and tension by a vibrator of variable frequency. Both ends of the string are fixed. When the vibrator has a frequency f, in a string of length L and under tension T, n antinodes are set up in the string. (a) If the length of the string is doubled, by what factor should the frequency be changed so that the same number of antinodes is produced? (b) If the frequency and length are held constant, what tension will produce $n + 1$ antinodes? (c) If the frequency is tripled and the length of the string is halved, by what factor should the tension be changed so that twice as many antinodes are produced?

83. Two waves are described by the wave functions

$$y_1(x, t) = 5.00 \sin (2.00x - 10.0t)$$

$$y_2(x, t) = 10.0 \cos (2.00x - 10.0t)$$

where x, y_1, and y_2 are in meters and t is in seconds. (a) Show that the wave resulting from their superposition can be expressed as a single sine function. (b) Determine the amplitude and phase angle for this sinusoidal wave.

84. A flute is designed so that it produces a frequency of 261.6 Hz, middle C, when all the holes are covered and the temperature is 20.0°C. (a) Consider the flute as a pipe that is open at both ends. Find the length of the flute, assuming middle C is the fundamental. (b) A second player, nearby in a colder room, also attempts to play middle C on an identical flute. A beat frequency of 3.00 Hz is heard when both flutes are playing. What is the temperature of the second room?

85. AMT **Review.** A 12.0-kg object hangs in equilibrium from a string with a total length of $L = 5.00$ m and a linear mass density of $\mu = 0.001\ 00$ kg/m. The string is wrapped around two light, frictionless pulleys that are separated by a distance of $d = 2.00$ m (Fig. P18.85a). (a) Determine the tension in the string. (b) At what frequency must the string between the pulleys vibrate to form the standing-wave pattern shown in Figure P18.85b?

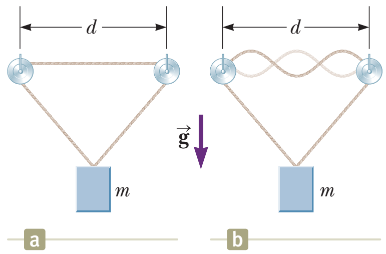

Figure P18.85 Problems 85 and 86.

86. **Review.** An object of mass m hangs in equilibrium from a string with a total length L and a linear mass density μ. The string is wrapped around two light, frictionless pulleys that are separated by a distance d (Fig. P18.85a). (a) Determine the tension in the string. (b) At what frequency must the string between the pulleys vibrate to form the standing-wave pattern shown in Figure P18.85b?

Challenge Problems

87. **Review.** Consider the apparatus shown in Figure P18.87a, where the hanging object has mass M and the string is vibrating in its second harmonic. The vibrating blade at the left maintains a constant frequency. The wind begins to blow to the right, applying a con-

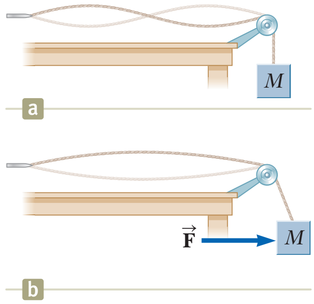

Figure P18.87

stant horizontal force $\vec{\mathbf{F}}$ on the hanging object. What is the magnitude of the force the wind must apply to the hanging object so that the string vibrates in its first harmonic as shown in Figure 18.87b?

88. In Figures 18.20a and 18.20b, notice that the amplitude of the component wave for frequency f is large, that for $3f$ is smaller, and that for $5f$ smaller still. How do we know exactly how much amplitude to assign to each frequency component to build a square wave? This problem helps us find the answer to that question. Let the square wave in Figure 18.20c have an amplitude A and let $t = 0$ be at the extreme left of the figure. So, one period T of the square wave is described by

$$y(t) = \begin{cases} A & 0 < t < \dfrac{T}{2} \\ -A & \dfrac{T}{2} < t < T \end{cases}$$

Express Equation 18.13 with angular frequencies:

$$y(t) = \sum_n (A_n \sin n\omega t + B_n \cos n\omega t)$$

Now proceed as follows. (a) Multiply both sides of Equation 18.13 by $\sin m\omega t$ and integrate both sides over one period T. Show that the left-hand side of the resulting equation is equal to 0 if m is even and is equal to $4A/m\omega$ if m is odd. (b) Using trigonometric identities, show that all terms on the right-hand side involving B_n are equal to zero. (c) Using trigonometric identities, show that all terms on the right-hand side involving A_n are equal to zero *except* for the one case of $m = n$. (d) Show that the entire right-hand side of the equation reduces to $\frac{1}{2}A_m T$. (e) Show that the Fourier series expansion for a square wave is

$$y(t) = \sum_n \frac{4A}{n\pi} \sin n\omega t$$

Thermodynamics

PART 3

A bubble in one of the many mud pots in Yellowstone National Park is caught just at the moment of popping. A mud pot is a pool of bubbling hot mud that demonstrates the existence of thermodynamic processes below the Earth's surface. *(© Adambooth/ Dreamstime.com)*

We now direct our attention to the study of thermodynamics, which involves situ- ations in which the temperature or state (solid, liquid, gas) of a system changes due to energy transfers. As we shall see, thermodynamics is very successful in explaining the bulk properties of matter and the correlation between these properties and the mechanics of atoms and molecules.

Historically, the development of thermodynamics paralleled the development of the atomic theory of matter. By the 1820s, chemical experiments had provided solid evidence for the existence of atoms. At that time, scientists recognized that a connection between thermodynamics and the structure of matter must exist. In 1827, botanist Robert Brown reported that grains of pollen suspended in a liquid move erratically from one place to another as if under constant agitation. In 1905, Albert Einstein used kinetic theory to explain the cause of this erratic motion, known today as *Brownian motion.* Einstein explained this phenomenon by assuming the grains are under constant bombardment by "invisible" molecules in the liquid, which themselves move erratically. This explanation gave scientists insight into the concept of molecular motion and gave credence to the idea that matter is made up of atoms. A connection was thus forged between the everyday world and the tiny, invisible building blocks that make up this world.

Thermodynamics also addresses more practical questions. Have you ever wondered how a refrigerator is able to cool its contents, or what types of transformations occur in a power plant or in the engine of your automobile, or what happens to the kinetic energy of a moving object when the object comes to rest? The laws of thermodynamics can be used to provide explanations for these and other phenomena. ■

CHAPTER

19

Temperature

Why would someone designing a pipeline include these strange loops? Pipelines carrying liquids often contain such loops to allow for expansion and contraction as the temperature changes. We will study thermal expansion in this chapter. *(© Lowell Georgia/CORBIS)*

In our study of mechanics, we carefully defined such concepts as *mass, force,* and *kinetic energy* to facilitate our quantitative approach. Likewise, a quantitative description of thermal phenomena requires careful definitions of such important terms as *temperature, heat,* and *internal energy.* This chapter begins with a discussion of temperature.

Next, when studying thermal phenomena, we consider the importance of the particular substance we are investigating. For example, gases expand appreciably when heated, whereas liquids and solids expand only slightly.

This chapter concludes with a study of ideal gases on the macroscopic scale. Here, we are concerned with the relationships among such quantities as pressure, volume, and temperature of a gas. In Chapter 21, we shall examine gases on a microscopic scale, using a model that represents the components of a gas as small particles.

19.1 Temperature and the Zeroth Law of Thermodynamics

We often associate the concept of temperature with how hot or cold an object feels when we touch it. In this way, our senses provide us with a qualitative indication of temperature. Our senses, however, are unreliable and often mislead us. For exam-

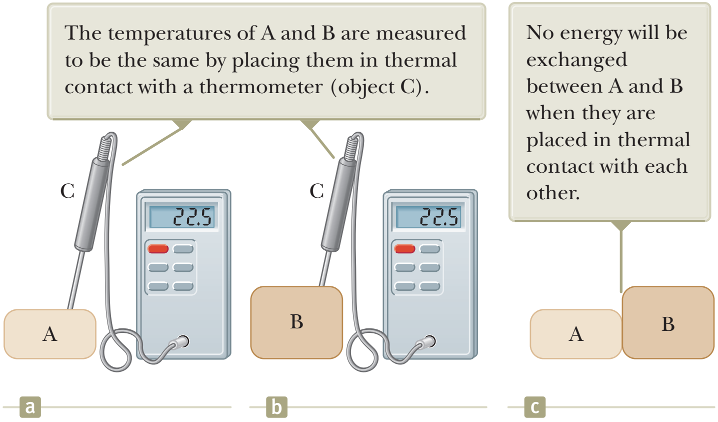

Figure 19.1 The zeroth law of thermodynamics.

ple, if you stand in bare feet with one foot on carpet and the other on an adjacent tile floor, the tile feels colder than the carpet *even though both are at the same temperature.* The two objects feel different because tile transfers energy by heat at a higher rate than carpet does. Your skin "measures" the rate of energy transfer by heat rather than the actual temperature. What we need is a reliable and reproducible method for measuring the relative hotness or coldness of objects rather than the rate of energy transfer. Scientists have developed a variety of thermometers for making such quantitative measurements.

Two objects at different initial temperatures eventually reach some intermediate temperature when placed in contact with each other. For example, when hot water and cold water are mixed in a bathtub, energy is transferred from the hot water to the cold water and the final temperature of the mixture is somewhere between the initial hot and cold temperatures.

Imagine that two objects are placed in an insulated container such that they interact with each other but not with the environment. If the objects are at different temperatures, energy is transferred between them, even if they are initially not in physical contact with each other. The energy-transfer mechanisms from Chapter 8 that we will focus on are heat and electromagnetic radiation. For purposes of this discussion, let's assume two objects are in **thermal contact** with each other if energy can be exchanged between them by these processes due to a temperature difference. **Thermal equilibrium** is a situation in which two objects would not exchange energy by heat or electromagnetic radiation if they were placed in thermal contact.

Let's consider two objects A and B, which are not in thermal contact, and a third object C, which is our thermometer. We wish to determine whether A and B are in thermal equilibrium with each other. The thermometer (object C) is first placed in thermal contact with object A until thermal equilibrium is reached[1] as shown in Figure 19.1a. From that moment on, the thermometer's reading remains constant and we record this reading. The thermometer is then removed from object A and placed in thermal contact with object B as shown in Figure 19.1b. The reading is again recorded after thermal equilibrium is reached. If the two readings are the same, we can conclude that object A and object B are in thermal equilibrium with each other. If they are placed in contact with each other as in Figure 19.1c, there is no exchange of energy between them.

[1]We assume a negligible amount of energy transfers between the thermometer and object A in the time interval during which they are in thermal contact. Without this assumption, which is also made for the thermometer and object B, the measurement of the temperature of an object disturbs the system so that the measured temperature is different from the initial temperature of the object. In practice, whenever you measure a temperature with a thermometer, you measure the disturbed system, not the original system.

We can summarize these results in a statement known as the **zeroth law of thermodynamics** (the law of equilibrium):

Zeroth law of thermodynamics

> If objects A and B are separately in thermal equilibrium with a third object C, then A and B are in thermal equilibrium with each other.

This statement can easily be proved experimentally and is very important because it enables us to define temperature. We can think of **temperature** as the property that determines whether an object is in thermal equilibrium with other objects. Two objects in thermal equilibrium with each other are at the same temperature. Conversely, if two objects have different temperatures, they are not in thermal equilibrium with each other. We now know that temperature is something that determines whether or not energy will transfer between two objects in thermal contact. In Chapter 21, we will relate temperature to the mechanical behavior of molecules.

Quick Quiz 19.1 Two objects, with different sizes, masses, and temperatures, are placed in thermal contact. In which direction does the energy travel? **(a)** Energy travels from the larger object to the smaller object. **(b)** Energy travels from the object with more mass to the one with less mass. **(c)** Energy travels from the object at higher temperature to the object at lower temperature.

19.2 Thermometers and the Celsius Temperature Scale

Thermometers are devices used to measure the temperature of a system. All thermometers are based on the principle that some physical property of a system changes as the system's temperature changes. Some physical properties that change with temperature are (1) the volume of a liquid, (2) the dimensions of a solid, (3) the pressure of a gas at constant volume, (4) the volume of a gas at constant pressure, (5) the electric resistance of a conductor, and (6) the color of an object.

A common thermometer in everyday use consists of a mass of liquid—usually mercury or alcohol—that expands into a glass capillary tube when heated (Fig. 19.2). In this case, the physical property that changes is the volume of a liquid. Any temperature change in the range of the thermometer can be defined as being proportional to the change in length of the liquid column. The thermometer can be calibrated by placing it in thermal contact with a natural system that remains

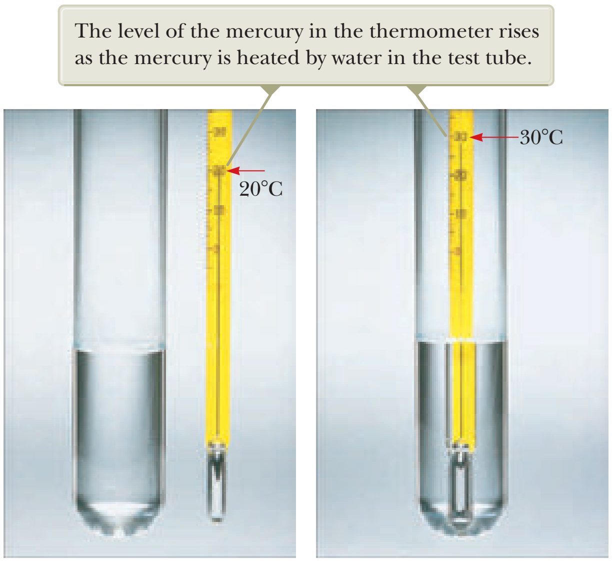

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Figure 19.2 A mercury thermometer before and after increasing its temperature.

at constant temperature. One such system is a mixture of water and ice in thermal equilibrium at atmospheric pressure. On the **Celsius temperature scale,** this mixture is defined to have a temperature of zero degrees Celsius, which is written as 0°C; this temperature is called the *ice point* of water. Another commonly used system is a mixture of water and steam in thermal equilibrium at atmospheric pressure; its temperature is defined as 100°C, which is the *steam point* of water. Once the liquid levels in the thermometer have been established at these two points, the length of the liquid column between the two points is divided into 100 equal segments to create the Celsius scale. Therefore, each segment denotes a change in temperature of one Celsius degree.

Thermometers calibrated in this way present problems when extremely accurate readings are needed. For instance, the readings given by an alcohol thermometer calibrated at the ice and steam points of water might agree with those given by a mercury thermometer only at the calibration points. Because mercury and alcohol have different thermal expansion properties, when one thermometer reads a temperature of, for example, 50°C, the other may indicate a slightly different value. The discrepancies between thermometers are especially large when the temperatures to be measured are far from the calibration points.[2]

An additional practical problem of any thermometer is the limited range of temperatures over which it can be used. A mercury thermometer, for example, cannot be used below the freezing point of mercury, which is −39°C, and an alcohol thermometer is not useful for measuring temperatures above 85°C, the boiling point of alcohol. To surmount this problem, we need a universal thermometer whose readings are independent of the substance used in it. The gas thermometer, discussed in the next section, approaches this requirement.

19.3 The Constant-Volume Gas Thermometer and the Absolute Temperature Scale

One version of a gas thermometer is the constant-volume apparatus shown in Figure 19.3. The physical change exploited in this device is the variation of pressure of a fixed volume of gas with temperature. The flask is immersed in an ice-water bath, and mercury reservoir B is raised or lowered until the top of the mercury in column A is at the zero point on the scale. The height h, the difference between the mercury levels in reservoir B and column A, indicates the pressure in the flask at 0°C by means of Equation 14.4, $P = P_0 + \rho g h$.

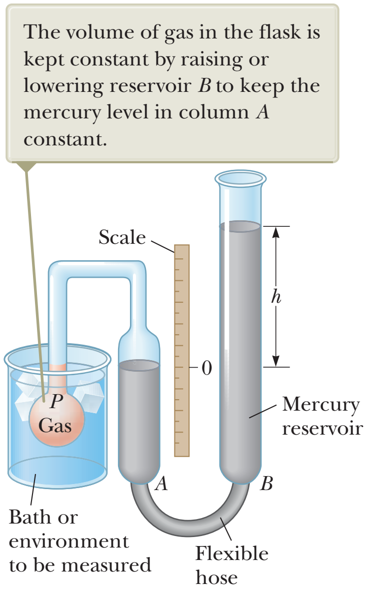

Figure 19.3 A constant-volume gas thermometer measures the pressure of the gas contained in the flask immersed in the bath.

The flask is then immersed in water at the steam point. Reservoir B is readjusted until the top of the mercury in column A is again at zero on the scale, which ensures that the gas's volume is the same as it was when the flask was in the ice bath (hence the designation "constant-volume"). This adjustment of reservoir B gives a value for the gas pressure at 100°C. These two pressure and temperature values are then plotted as shown in Figure 19.4. The line connecting the two points serves as a calibration curve for unknown temperatures. (Other experiments show that a linear relationship between pressure and temperature is a very good assumption.) To measure the temperature of a substance, the gas flask of Figure 19.3 is placed in thermal contact with the substance and the height of reservoir B is adjusted until the top of the mercury column in A is at zero on the scale. The height of the mercury column in B indicates the pressure of the gas; knowing the pressure, the temperature of the substance is found using the graph in Figure 19.4.

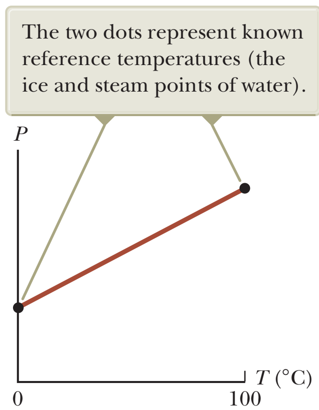

Figure 19.4 A typical graph of pressure versus temperature taken with a constant-volume gas thermometer.

Now suppose temperatures of different gases at different initial pressures are measured with gas thermometers. Experiments show that the thermometer readings are nearly independent of the type of gas used as long as the gas pressure is low and the temperature is well above the point at which the gas liquefies

[2]Two thermometers that use the same liquid may also give different readings, due in part to difficulties in constructing uniform-bore glass capillary tubes.

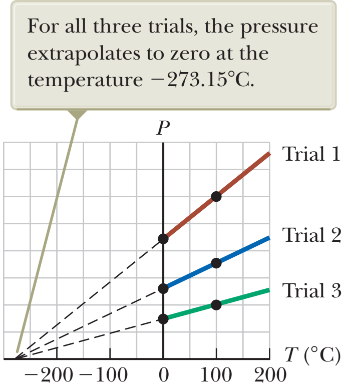

Figure 19.5 Pressure versus temperature for experimental trials in which gases have different pressures in a constant-volume gas thermometer.

(Fig. 19.5). The agreement among thermometers using various gases improves as the pressure is reduced.

If we extend the straight lines in Figure 19.5 toward negative temperatures, we find a remarkable result: **in every case, the pressure is zero when the temperature is −273.15°C!** This finding suggests some special role that this particular temperature must play. It is used as the basis for the **absolute temperature scale,** which sets −273.15°C as its zero point. This temperature is often referred to as **absolute zero.** It is indicated as a zero because at a lower temperature, the pressure of the gas would become negative, which is meaningless. The size of one degree on the absolute temperature scale is chosen to be identical to the size of one degree on the Celsius scale. Therefore, the conversion between these temperatures is

$$T_C = T - 273.15 \tag{19.1}$$

where T_C is the Celsius temperature and T is the absolute temperature.

Because the ice and steam points are experimentally difficult to duplicate and depend on atmospheric pressure, an absolute temperature scale based on two new fixed points was adopted in 1954 by the International Committee on Weights and Measures. The first point is absolute zero. The second reference temperature for this new scale was chosen as the **triple point of water,** which is the single combination of temperature and pressure at which liquid water, gaseous water, and ice (solid water) coexist in equilibrium. This triple point occurs at a temperature of 0.01°C and a pressure of 4.58 mm of mercury. On the new scale, which uses the unit *kelvin,* the temperature of water at the triple point was set at 273.16 kelvins, abbreviated 273.16 K. This choice was made so that the old absolute temperature scale based on the ice and steam points would agree closely with the new scale based on the triple point. This new absolute temperature scale (also called the **Kelvin scale**) employs the SI unit of absolute temperature, the **kelvin,** which is defined to be 1/273.16 of the difference between absolute zero and the temperature of the triple point of water.

Pitfall Prevention 19.1

A Matter of Degree Notations for temperatures in the Kelvin scale do not use the degree sign. The unit for a Kelvin temperature is simply "kelvins" and not "degrees Kelvin."

Figure 19.6 gives the absolute temperature for various physical processes and structures. The temperature of absolute zero (0 K) cannot be achieved, although laboratory experiments have come very close, reaching temperatures of less than one nanokelvin.

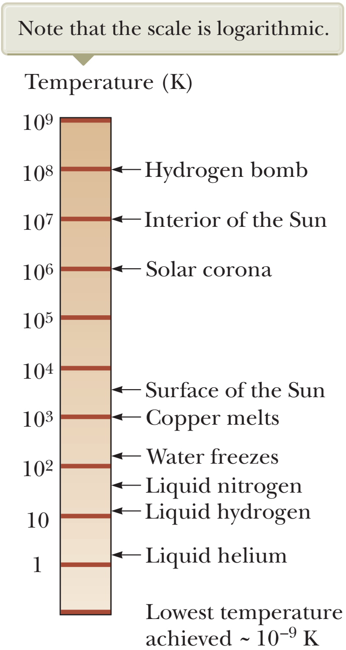

Figure 19.6 Absolute temperatures at which various physical processes occur.

The Celsius, Fahrenheit, and Kelvin Temperature Scales[3]

Equation 19.1 shows that the Celsius temperature T_C is shifted from the absolute (Kelvin) temperature T by 273.15°. Because the size of one degree is the same on the two scales, a temperature difference of 5°C is equal to a temperature difference of 5 K. The two scales differ only in the choice of the zero point. Therefore, the ice-point temperature on the Kelvin scale, 273.15 K, corresponds to 0.00°C, and the Kelvin-scale steam point, 373.15 K, is equivalent to 100.00°C.

A common temperature scale in everyday use in the United States is the **Fahrenheit scale.** This scale sets the temperature of the ice point at 32°F and the temperature of the steam point at 212°F. The relationship between the Celsius and Fahrenheit temperature scales is

$$T_F = \tfrac{9}{5}T_C + 32°\mathrm{F} \tag{19.2}$$

We can use Equations 19.1 and 19.2 to find a relationship between changes in temperature on the Celsius, Kelvin, and Fahrenheit scales:

$$\Delta T_C = \Delta T = \tfrac{5}{9}\Delta T_F \tag{19.3}$$

Of these three temperature scales, only the Kelvin scale is based on a true zero value of temperature. The Celsius and Fahrenheit scales are based on an arbitrary zero associated with one particular substance, water, on one particular planet, the

[3]Named after Anders Celsius (1701–1744), Daniel Gabriel Fahrenheit (1686–1736), and William Thomson, Lord Kelvin (1824–1907), respectively.

Earth. Therefore, if you encounter an equation that calls for a temperature T or that involves a ratio of temperatures, you *must* convert all temperatures to kelvins. If the equation contains a change in temperature ΔT, using Celsius temperatures will give you the correct answer, in light of Equation 19.3, but it is always *safest* to convert temperatures to the Kelvin scale.

Quick Quiz 19.2 Consider the following pairs of materials. Which pair represents two materials, one of which is twice as hot as the other? **(a)** boiling water at 100°C, a glass of water at 50°C **(b)** boiling water at 100°C, frozen methane at −50°C **(c)** an ice cube at −20°C, flames from a circus fire-eater at 233°C **(d)** none of those pairs

Example 19.1 Converting Temperatures

On a day when the temperature reaches 50°F, what is the temperature in degrees Celsius and in kelvins?

SOLUTION

Conceptualize In the United States, a temperature of 50°F is well understood. In many other parts of the world, however, this temperature might be meaningless because people are familiar with the Celsius temperature scale.

Categorize This example is a simple substitution problem.

Solve Equation 19.2 for the Celsius temperature and substitute numerical values:

$$T_C = \tfrac{5}{9}(T_F - 32) = \tfrac{5}{9}(50 - 32) = \boxed{10°C}$$

Use Equation 19.1 to find the Kelvin temperature:

$$T = T_C + 273.15 = 10°C + 273.15 = \boxed{283\ K}$$

A convenient set of weather-related temperature equivalents to keep in mind is that 0°C is (literally) freezing at 32°F, 10°C is cool at 50°F, 20°C is room temperature, 30°C is warm at 86°F, and 40°C is a hot day at 104°F.

19.4 Thermal Expansion of Solids and Liquids

Our discussion of the liquid thermometer makes use of one of the best-known changes in a substance: as its temperature increases, its volume increases. This phenomenon, known as **thermal expansion,** plays an important role in numerous engineering applications. For example, thermal-expansion joints such as those shown in Figure 19.7 must be included in buildings, concrete highways, railroad tracks,

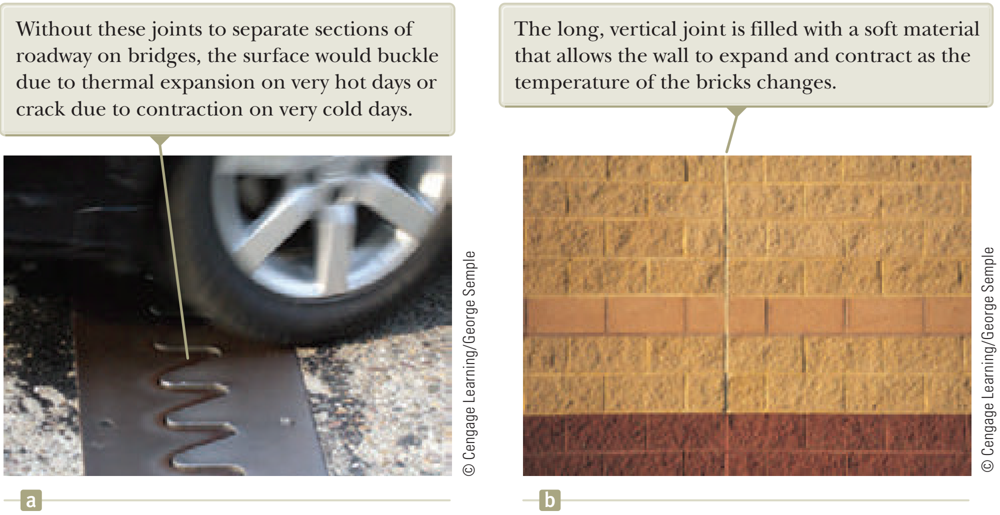

Figure 19.7 Thermal-expansion joints in (a) bridges and (b) walls.

brick walls, and bridges to compensate for dimensional changes that occur as the temperature changes.

Thermal expansion is a consequence of the change in the *average* separation between the atoms in an object. To understand this concept, let's model the atoms as being connected by stiff springs as discussed in Section 15.3 and shown in Figure 15.11b. At ordinary temperatures, the atoms in a solid oscillate about their equilibrium positions with an amplitude of approximately 10^{-11} m and a frequency of approximately 10^{13} Hz. The average spacing between the atoms is about 10^{-10} m. As the temperature of the solid increases, the atoms oscillate with greater amplitudes; as a result, the average separation between them increases.[4] Consequently, the object expands.

If thermal expansion is sufficiently small relative to an object's initial dimensions, the change in any dimension is, to a good approximation, proportional to the first power of the temperature change. Suppose an object has an initial length L_i along some direction at some temperature and the length changes by an amount ΔL for a change in temperature ΔT. Because it is convenient to consider the fractional change in length per degree of temperature change, we define the **average coefficient of linear expansion** as

$$\alpha \equiv \frac{\Delta L / L_i}{\Delta T}$$

Experiments show that α is constant for small changes in temperature. For purposes of calculation, this equation is usually rewritten as

Thermal expansion in one dimension ▶

$$\Delta L = \alpha L_i \Delta T \tag{19.4}$$

or as

$$L_f - L_i = \alpha L_i (T_f - T_i) \tag{19.5}$$

Pitfall Prevention 19.2

Do Holes Become Larger or Smaller? When an object's temperature is raised, every linear dimension increases in size. That includes any holes in the material, which expand in the same way as if the hole were filled with the material as shown in Figure 19.8.

where L_f is the final length, T_i and T_f are the initial and final temperatures, respectively, and the proportionality constant α is the average coefficient of linear expansion for a given material and has units of $(°C)^{-1}$. Equation 19.4 can be used for both thermal expansion, when the temperature of the material increases, and thermal contraction, when its temperature decreases.

It may be helpful to think of thermal expansion as an effective magnification or as a photographic enlargement of an object. For example, as a metal washer is heated (Fig. 19.8), all dimensions, including the radius of the hole, increase according to Equation 19.4. A cavity in a piece of material expands in the same way as if the cavity were filled with the material.

Table 19.1 lists the average coefficients of linear expansion for various materials. For these materials, α is positive, indicating an increase in length with increasing temperature. That is not always the case, however. Some substances—calcite ($CaCO_3$) is one example—expand along one dimension (positive α) and contract along another (negative α) as their temperatures are increased.

Because the linear dimensions of an object change with temperature, it follows that surface area and volume change as well. The change in volume is proportional to the initial volume V_i and to the change in temperature according to the relationship

Thermal expansion in three dimensions ▶

$$\Delta V = \beta V_i \Delta T \tag{19.6}$$

where β is the **average coefficient of volume expansion.** To find the relationship between β and α, assume the average coefficient of linear expansion of the solid is the same in all directions; that is, assume the material is *isotropic*. Consider a solid box of dimensions ℓ, w, and h. Its volume at some temperature T_i is $V_i = \ell w h$. If the

[4]More precisely, thermal expansion arises from the *asymmetrical* nature of the potential energy curve for the atoms in a solid as shown in Figure 15.11a. If the oscillators were truly harmonic, the average atomic separations would not change regardless of the amplitude of vibration.

Table 19.1 Average Expansion Coefficients for Some Materials Near Room Temperature

Material (Solids)	Average Linear Expansion Coefficient $(\alpha)(^\circ\text{C})^{-1}$	Material (Liquids and Gases)	Average Volume Expansion Coefficient $(\beta)(^\circ\text{C})^{-1}$
Aluminum	24×10^{-6}	Acetone	1.5×10^{-4}
Brass and bronze	19×10^{-6}	Alcohol, ethyl	1.12×10^{-4}
Concrete	12×10^{-6}	Benzene	1.24×10^{-4}
Copper	17×10^{-6}	Gasoline	9.6×10^{-4}
Glass (ordinary)	9×10^{-6}	Glycerin	4.85×10^{-4}
Glass (Pyrex)	3.2×10^{-6}	Mercury	1.82×10^{-4}
Invar (Ni–Fe alloy)	0.9×10^{-6}	Turpentine	9.0×10^{-4}
Lead	29×10^{-6}	Air[a] at 0°C	3.67×10^{-3}
Steel	11×10^{-6}	Helium[a]	3.665×10^{-3}

[a]Gases do not have a specific value for the volume expansion coefficient because the amount of expansion depends on the type of process through which the gas is taken. The values given here assume the gas undergoes an expansion at constant pressure.

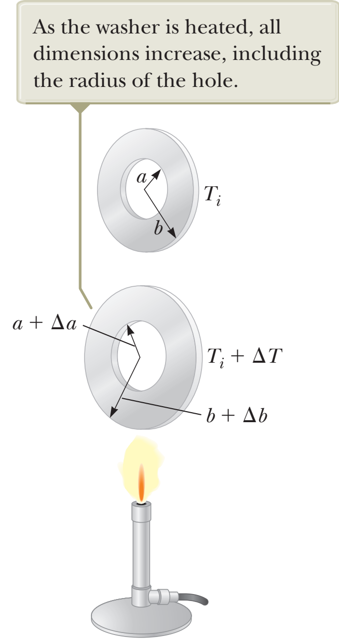

Figure 19.8 Thermal expansion of a homogeneous metal washer. (The expansion is exaggerated in this figure.)

temperature changes to $T_i + \Delta T$, its volume changes to $V_i + \Delta V$, where each dimension changes according to Equation 19.4. Therefore,

$$\begin{aligned} V_i + \Delta V &= (\ell + \Delta\ell)(w + \Delta w)(h + \Delta h) \\ &= (\ell + \alpha\ell\,\Delta T)(w + \alpha w\,\Delta T)(h + \alpha h\,\Delta T) \\ &= \ell wh(1 + \alpha\,\Delta T)^3 \\ &= V_i[1 + 3\alpha\,\Delta T + 3(\alpha\,\Delta T)^2 + (\alpha\,\Delta T)^3] \end{aligned}$$

Dividing both sides by V_i and isolating the term $\Delta V/V_i$, we obtain the fractional change in volume:

$$\frac{\Delta V}{V_i} = 3\alpha\,\Delta T + 3(\alpha\,\Delta T)^2 + (\alpha\,\Delta T)^3$$

Because $\alpha\,\Delta T << 1$ for typical values of ΔT ($< \sim 100^\circ$C), we can neglect the terms $3(\alpha\,\Delta T)^2$ and $(\alpha\,\Delta T)^3$. Upon making this approximation, we see that

$$\frac{\Delta V}{V_i} = 3\alpha\,\Delta T \;\rightarrow\; \Delta V = (3\alpha)V_i\,\Delta T$$

Comparing this expression to Equation 19.6 shows that

$$\beta = 3\alpha$$

In a similar way, you can show that the change in area of a rectangular plate is given by $\Delta A = 2\alpha A_i\,\Delta T$ (see Problem 61).

A simple mechanism called a *bimetallic strip,* found in practical devices such as mechanical thermostats, uses the difference in coefficients of expansion for different materials. It consists of two thin strips of dissimilar metals bonded together. As the temperature of the strip increases, the two metals expand by different amounts and the strip bends as shown in Figure 19.9.

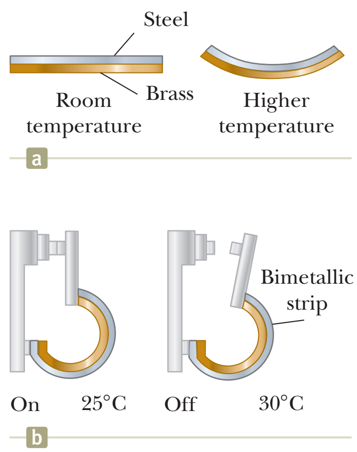

Figure 19.9 (a) A bimetallic strip bends as the temperature changes because the two metals have different expansion coefficients. (b) A bimetallic strip used in a thermostat to break or make electrical contact.

Quick Quiz 19.3 If you are asked to make a very sensitive glass thermometer, which of the following working liquids would you choose? **(a)** mercury **(b)** alcohol **(c)** gasoline **(d)** glycerin

Quick Quiz 19.4 Two spheres are made of the same metal and have the same radius, but one is hollow and the other is solid. The spheres are taken through the same temperature increase. Which sphere expands more? **(a)** The solid sphere expands more. **(b)** The hollow sphere expands more. **(c)** They expand by the same amount. **(d)** There is not enough information to say.

Example 19.2 Expansion of a Railroad Track

A segment of steel railroad track has a length of 30.000 m when the temperature is 0.0°C.

(A) What is its length when the temperature is 40.0°C?

SOLUTION

Conceptualize Because the rail is relatively long, we expect to obtain a measurable increase in length for a 40°C temperature increase.

Categorize We will evaluate a length increase using the discussion of this section, so this part of the example is a substitution problem.

Use Equation 19.4 and the value of the coefficient of linear expansion from Table 19.1:

$$\Delta L = \alpha L_i \Delta T = [11 \times 10^{-6}\ (°\text{C})^{-1}](30.000\ \text{m})(40.0°\text{C}) = 0.013\ \text{m}$$

Find the new length of the track:

$$L_f = 30.000\ \text{m} + 0.013\ \text{m} = \boxed{30.013\ \text{m}}$$

(B) Suppose the ends of the rail are rigidly clamped at 0.0°C so that expansion is prevented. What is the thermal stress set up in the rail if its temperature is raised to 40.0°C?

SOLUTION

Categorize This part of the example is an analysis problem because we need to use concepts from another chapter.

Analyze The thermal stress is the same as the tensile stress in the situation in which the rail expands freely and is then compressed with a mechanical force F back to its original length.

Find the tensile stress from Equation 12.6 using Young's modulus for steel from Table 12.1:

$$\text{Tensile stress} = \frac{F}{A} = Y\frac{\Delta L}{L_i}$$

$$\frac{F}{A} = (20 \times 10^{10}\ \text{N/m}^2)\left(\frac{0.013\ \text{m}}{30.000\ \text{m}}\right) = \boxed{8.7 \times 10^7\ \text{N/m}^2}$$

Finalize The expansion in part (A) is 1.3 cm. This expansion is indeed measurable as predicted in the Conceptualize step. The thermal stress in part (B) can be avoided by leaving small expansion gaps between the rails.

WHAT IF? What if the temperature drops to −40.0°C? What is the length of the unclamped segment?

Answer The expression for the change in length in Equation 19.4 is the same whether the temperature increases or decreases. Therefore, if there is an increase in length of 0.013 m when the temperature increases by 40°C, there is a decrease in length of 0.013 m when the temperature decreases by 40°C. (We assume α is constant over the entire range of temperatures.) The new length at the colder temperature is 30.000 m − 0.013 m = 29.987 m.

Example 19.3 The Thermal Electrical Short

A poorly designed electronic device has two bolts attached to different parts of the device that almost touch each other in its interior as in Figure 19.10. The steel and brass bolts are at different electric potentials, and if they touch, a short circuit will develop, damaging the device. (We will study electric potential in Chapter 25.) The initial gap between the ends of the bolts is $d = 5.0\ \mu$m at 27°C. At what temperature will the bolts touch? Assume the distance between the walls of the device is not affected by the temperature change.

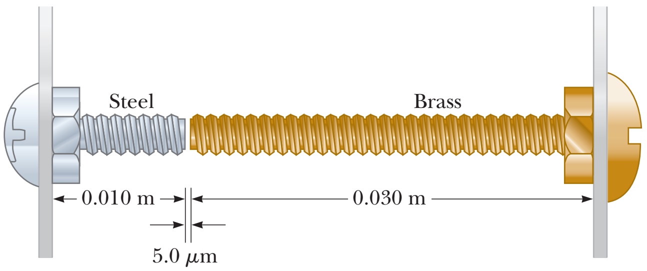

Figure 19.10 (Example 19.3) Two bolts attached to different parts of an electrical device are almost touching when the temperature is 27°C. As the temperature increases, the ends of the bolts move toward each other.

SOLUTION

Conceptualize Imagine the ends of both bolts expanding into the gap between them as the temperature rises.

19.3 continued

Categorize We categorize this example as a thermal expansion problem in which the *sum* of the changes in length of the two bolts must equal the length of the initial gap between the ends.

Analyze Set the sum of the length changes equal to the width of the gap:

$$\Delta L_{\text{br}} + \Delta L_{\text{st}} = \alpha_{\text{br}} L_{i,\text{br}}\,\Delta T + \alpha_{\text{st}} L_{i,\text{st}}\,\Delta T = d$$

Solve for ΔT:

$$\Delta T = \frac{d}{\alpha_{\text{br}} L_{i,\text{br}} + \alpha_{\text{st}} L_{i,\text{st}}}$$

Substitute numerical values:

$$\Delta T = \frac{5.0 \times 10^{-6}\text{ m}}{[19 \times 10^{-6}\,(°\text{C})^{-1}](0.030\text{ m}) + [11 \times 10^{-6}\,(°\text{C})^{-1}](0.010\text{ m})} = 7.4°\text{C}$$

Find the temperature at which the bolts touch:

$$T = 27°\text{C} + 7.4°\text{C} = \boxed{34°\text{C}}$$

Finalize This temperature is possible if the air conditioning in the building housing the device fails for a long period on a very hot summer day.

The Unusual Behavior of Water

Liquids generally increase in volume with increasing temperature and have average coefficients of volume expansion about ten times greater than those of solids. Cold water is an exception to this rule as you can see from its density-versus-temperature curve shown in Figure 19.11. As the temperature increases from 0°C to 4°C, water contracts and its density therefore increases. Above 4°C, water expands with increasing temperature and so its density decreases. Therefore, the density of water reaches a maximum value of 1.000 g/cm^3 at 4°C.

We can use this unusual thermal-expansion behavior of water to explain why a pond begins freezing at the surface rather than at the bottom. When the air temperature drops from, for example, 7°C to 6°C, the surface water also cools and consequently decreases in volume. The surface water is denser than the water below it, which has not cooled and decreased in volume. As a result, the surface water sinks, and warmer water from below moves to the surface. When the air temperature is between 4°C and 0°C, however, the surface water expands as it cools, becoming less dense than the water below it. The mixing process stops, and eventually the surface water freezes. As the water freezes, the ice remains on the surface because ice is less dense than water. The ice continues to build up at the surface, while water near the

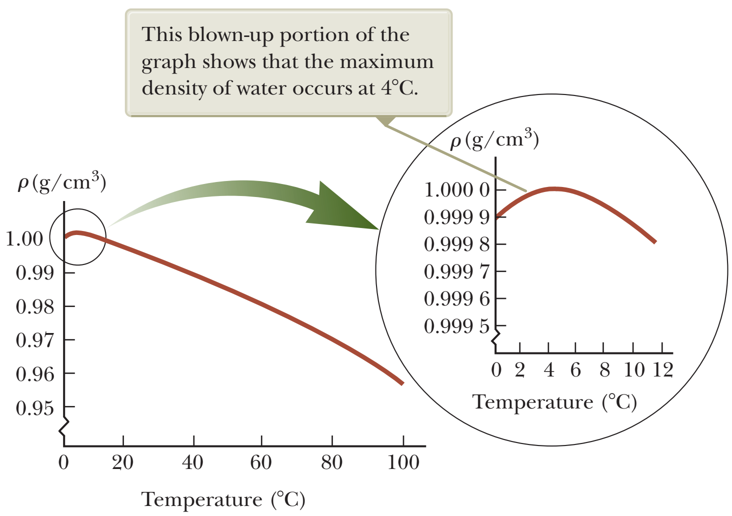

Figure 19.11 The variation in the density of water at atmospheric pressure with temperature.

bottom remains at 4°C. If that were not the case, fish and other forms of marine life would not survive.

19.5 Macroscopic Description of an Ideal Gas

The volume expansion equation $\Delta V = \beta V_i \, \Delta T$ is based on the assumption that the material has an initial volume V_i before the temperature change occurs. Such is the case for solids and liquids because they have a fixed volume at a given temperature.

The case for gases is completely different. The interatomic forces within gases are very weak, and, in many cases, we can imagine these forces to be nonexistent and still make very good approximations. Therefore, *there is no equilibrium separation* for the atoms and no "standard" volume at a given temperature; the volume depends on the size of the container. As a result, we cannot express changes in volume ΔV in a process on a gas with Equation 19.6 because we have no defined volume V_i at the beginning of the process. Equations involving gases contain the volume V, rather than a *change* in the volume from an initial value, as a variable.

For a gas, it is useful to know how the quantities volume V, pressure P, and temperature T are related for a sample of gas of mass m. In general, the equation that interrelates these quantities, called the *equation of state,* is very complicated. If the gas is maintained at a very low pressure (or low density), however, the equation of state is quite simple and can be determined from experimental results. Such a low-density gas is commonly referred to as an **ideal gas.**[5] We can use the **ideal gas model** to make predictions that are adequate to describe the behavior of real gases at low pressures.

It is convenient to express the amount of gas in a given volume in terms of the number of moles n. One **mole** of any substance is that amount of the substance that contains **Avogadro's number** $N_A = 6.022 \times 10^{23}$ of constituent particles (atoms or molecules). The number of moles n of a substance is related to its mass m through the expression

$$n = \frac{m}{M} \tag{19.7}$$

where M is the molar mass of the substance. The molar mass of each chemical element is the atomic mass (from the periodic table; see Appendix C) expressed in grams per mole. For example, the mass of one He atom is 4.00 u (atomic mass units), so the molar mass of He is 4.00 g/mol.

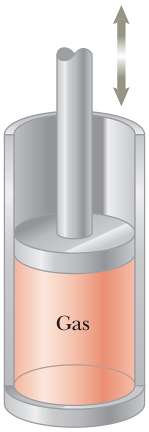

Figure 19.12 An ideal gas confined to a cylinder whose volume can be varied by means of a movable piston.

Now suppose an ideal gas is confined to a cylindrical container whose volume can be varied by means of a movable piston as in Figure 19.12. If we assume the cylinder does not leak, the mass (or the number of moles) of the gas remains constant. For such a system, experiments provide the following information:

- When the gas is kept at a constant temperature, its pressure is inversely proportional to the volume. (This behavior is described historically as Boyle's law.)
- When the pressure of the gas is kept constant, the volume is directly proportional to the temperature. (This behavior is described historically as Charles's law.)
- When the volume of the gas is kept constant, the pressure is directly proportional to the temperature. (This behavior is described historically as Gay–Lussac's law.)

These observations are summarized by the **equation of state for an ideal gas:**

Equation of state for an ideal gas

$$PV = nRT \tag{19.8}$$

[5]To be more specific, the assumptions here are that the temperature of the gas must not be too low (the gas must not condense into a liquid) or too high and that the pressure must be low. The concept of an ideal gas implies that the gas molecules do not interact except upon collision and that the molecular volume is negligible compared with the volume of the container. In reality, an ideal gas does not exist. The concept of an ideal gas is nonetheless very useful because real gases at low pressures are well-modeled as ideal gases.

In this expression, also known as the **ideal gas law,** n is the number of moles of gas in the sample and R is a constant. Experiments on numerous gases show that as the pressure approaches zero, the quantity PV/nT approaches the same value R for all gases. For this reason, R is called the **universal gas constant.** In SI units, in which pressure is expressed in pascals ($1 \text{ Pa} = 1 \text{ N/m}^2$) and volume in cubic meters, the product PV has units of newton · meters, or joules, and R has the value

$$R = 8.314 \text{ J/mol} \cdot \text{K} \qquad \textbf{(19.9)}$$

If the pressure is expressed in atmospheres and the volume in liters ($1 \text{ L} = 10^3 \text{ cm}^3 = 10^{-3} \text{ m}^3$), then R has the value

$$R = 0.082\,06 \text{ L} \cdot \text{atm/mol} \cdot \text{K}$$

Using this value of R and Equation 19.8 shows that the volume occupied by 1 mol of *any* gas at atmospheric pressure and at 0°C (273 K) is 22.4 L.

The ideal gas law states that if the volume and temperature of a fixed amount of gas do not change, the pressure also remains constant. Consider a bottle of champagne that is shaken and then spews liquid when opened as shown in Figure 19.13. A common misconception is that the pressure inside the bottle is increased when the bottle is shaken. On the contrary, because the temperature of the bottle and its contents remains constant as long as the bottle is sealed, so does the pressure, as can be shown by replacing the cork with a pressure gauge. The correct explanation is as follows. Carbon dioxide gas resides in the volume between the liquid surface and the cork. The pressure of the gas in this volume is set higher than atmospheric pressure in the bottling process. Shaking the bottle displaces some of the carbon dioxide gas into the liquid, where it forms bubbles, and these bubbles become attached to the inside of the bottle. (No new gas is generated by shaking.) When the bottle is opened, the pressure is reduced to atmospheric pressure, which causes the volume of the bubbles to increase suddenly. If the bubbles are attached to the bottle (beneath the liquid surface), their rapid expansion expels liquid from the bottle. If the sides and bottom of the bottle are first tapped until no bubbles remain beneath the surface, however, the drop in pressure does not force liquid from the bottle when the champagne is opened.

The ideal gas law is often expressed in terms of the total number of molecules N. Because the number of moles n equals the ratio of the total number of molecules and Avogadro's number N_A, we can write Equation 19.8 as

$$PV = nRT = \frac{N}{N_A}RT$$

$$PV = Nk_BT \qquad \textbf{(19.10)}$$

where k_B is **Boltzmann's constant,** which has the value

$$k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K} \qquad \textbf{(19.11)}$$

◀ **Boltzmann's constant**

It is common to call quantities such as P, V, and T the **thermodynamic variables** of an ideal gas. If the equation of state is known, one of the variables can always be expressed as some function of the other two.

Quick Quiz 19.5 A common material for cushioning objects in packages is made by trapping bubbles of air between sheets of plastic. Is this material more effective at keeping the contents of the package from moving around inside the package on **(a)** a hot day, **(b)** a cold day, or **(c)** either hot or cold days?

Quick Quiz 19.6 On a winter day, you turn on your furnace and the temperature of the air inside your home increases. Assume your home has the normal amount of leakage between inside air and outside air. Is the number of moles of air in your room at the higher temperature **(a)** larger than before, **(b)** smaller than before, or **(c)** the same as before?

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Figure 19.13 A bottle of champagne is shaken and opened. Liquid spews out of the opening. A common misconception is that the pressure inside the bottle is increased by the shaking.

Pitfall Prevention 19.3

So Many *k*s There are a variety of physical quantities for which the letter k is used. Two we have seen previously are the force constant for a spring (Chapter 15) and the wave number for a mechanical wave (Chapter 16). Boltzmann's constant is another k, and we will see k used for thermal conductivity in Chapter 20 and for an electrical constant in Chapter 23. To make some sense of this confusing state of affairs, we use a subscript B for Boltzmann's constant to help us recognize it. In this book, you will see Boltzmann's constant as k_B, but you may see Boltzmann's constant in other resources as simply k.

Example 19.4 Heating a Spray Can

A spray can containing a propellant gas at twice atmospheric pressure (202 kPa) and having a volume of 125.00 cm^3 is at 22°C. It is then tossed into an open fire. (*Warning:* Do not do this experiment; it is very dangerous.) When the temperature of the gas in the can reaches 195°C, what is the pressure inside the can? Assume any change in the volume of the can is negligible.

SOLUTION

Conceptualize Intuitively, you should expect that the pressure of the gas in the container increases because of the increasing temperature.

Categorize We model the gas in the can as ideal and use the ideal gas law to calculate the new pressure.

Analyze Rearrange Equation 19.8:

$$(1) \quad \frac{PV}{T} = nR$$

No air escapes during the compression, so n, and therefore nR, remains constant. Hence, set the initial value of the left side of Equation (1) equal to the final value:

$$(2) \quad \frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f}$$

Because the initial and final volumes of the gas are assumed to be equal, cancel the volumes:

$$(3) \quad \frac{P_i}{T_i} = \frac{P_f}{T_f}$$

Solve for P_f:

$$P_f = \left(\frac{T_f}{T_i}\right)P_i = \left(\frac{468\text{ K}}{295\text{ K}}\right)(202\text{ kPa}) = \boxed{320\text{ kPa}}$$

Finalize The higher the temperature, the higher the pressure exerted by the trapped gas as expected. If the pressure increases sufficiently, the can may explode. Because of this possibility, you should never dispose of spray cans in a fire.

WHAT IF? Suppose we include a volume change due to thermal expansion of the steel can as the temperature increases. Does that alter our answer for the final pressure significantly?

Answer Because the thermal expansion coefficient of steel is very small, we do not expect much of an effect on our final answer.

Find the change in the volume of the can using Equation 19.6 and the value for α for steel from Table 19.1:

$$\Delta V = \beta V_i \Delta T = 3\alpha V_i \Delta T$$
$$= 3[11 \times 10^{-6}\ (°\text{C})^{-1}](125.00\text{ cm}^3)(173°\text{C}) = 0.71\text{ cm}^3$$

Start from Equation (2) again and find an equation for the final pressure:

$$P_f = \left(\frac{T_f}{T_i}\right)\left(\frac{V_i}{V_f}\right)P_i$$

This result differs from Equation (3) only in the factor V_i/V_f. Evaluate this factor:

$$\frac{V_i}{V_f} = \frac{125.00\text{ cm}^3}{(125.00\text{ cm}^3 + 0.71\text{ cm}^3)} = 0.994 = 99.4\%$$

Therefore, the final pressure will differ by only 0.6% from the value calculated without considering the thermal expansion of the can. Taking 99.4% of the previous final pressure, the final pressure including thermal expansion is 318 kPa.

Summary

Definitions

Two objects are in **thermal equilibrium** with each other if they do not exchange energy when in thermal contact.

Temperature is the property that determines whether an object is in thermal equilibrium with other objects. Two objects in thermal equilibrium with each other are at the same temperature. The SI unit of absolute temperature is the **kelvin,** which is defined to be 1/273.16 of the difference between absolute zero and the temperature of the triple point of water.

Concepts and Principles

The **zeroth law of thermodynamics** states that if objects A and B are separately in thermal equilibrium with a third object C, then objects A and B are in thermal equilibrium with each other.

When the temperature of an object is changed by an amount ΔT, its length changes by an amount ΔL that is proportional to ΔT and to its initial length L_i:

$$\Delta L = \alpha L_i \Delta T \quad \textbf{(19.4)}$$

where the constant α is the **average coefficient of linear expansion.** The **average coefficient of volume expansion** β for a solid is approximately equal to 3α.

An **ideal gas** is one for which PV/nT is constant. An ideal gas is described by the **equation of state,**

$$PV = nRT \quad \textbf{(19.8)}$$

where n equals the number of moles of the gas, P is its pressure, V is its volume, R is the universal gas constant (8.314 J/mol · K), and T is the absolute temperature of the gas. A real gas behaves approximately as an ideal gas if it has a low density.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Markings to indicate length are placed on a steel tape in a room that is at a temperature of 22°C. Measurements are then made with the same tape on a day when the temperature is 27°C. Assume the objects you are measuring have a smaller coefficient of linear expansion than steel. Are the measurements (a) too long, (b) too short, or (c) accurate?

2. When a certain gas under a pressure of 5.00×10^6 Pa at 25.0°C is allowed to expand to 3.00 times its original volume, its final pressure is 1.07×10^6 Pa. What is its final temperature? (a) 450 K (b) 233 K (c) 212 K (d) 191 K (e) 115 K

3. If the volume of an ideal gas is doubled while its temperature is quadrupled, does the pressure (a) remain the same, (b) decrease by a factor of 2, (c) decrease by a factor of 4, (d) increase by a factor of 2, or (e) increase by a factor of 4

4. The pendulum of a certain pendulum clock is made of brass. When the temperature increases, what happens to the period of the clock? (a) It increases. (b) It decreases. (c) It remains the same.

5. A temperature of 162°F is equivalent to what temperature in kelvins? (a) 373 K (b) 288 K (c) 345 K (d) 201 K (e) 308 K

6. A cylinder with a piston holds 0.50 m^3 of oxygen at an absolute pressure of 4.0 atm. The piston is pulled outward, increasing the volume of the gas until the pressure drops to 1.0 atm. If the temperature stays constant, what new volume does the gas occupy? (a) 1.0 m^3 (b) 1.5 m^3 (c) 2.0 m^3 (d) 0.12 m^3 (e) 2.5 m^3

7. What would happen if the glass of a thermometer expanded more on warming than did the liquid in the tube? (a) The thermometer would break. (b) It could be used only for temperatures below room temperature. (c) You would have to hold it with the bulb on top. (d) The scale on the thermometer is reversed so that higher temperature values would be found closer to the bulb. (e) The numbers would not be evenly spaced.

8. A cylinder with a piston contains a sample of a thin gas. The kind of gas and the sample size can be changed. The cylinder can be placed in different constant-temperature baths, and the piston can be held in different positions. Rank the following cases according to the pressure of the gas from the highest to the lowest, displaying any cases of equality. (a) A 0.002-mol sample of oxygen is held at 300 K in a 100-cm^3 container. (b) A 0.002-mol sample of oxygen is held at 600 K in a 200-cm^3 container. (c) A 0.002-mol sample of oxygen is held at 600 K in a 300-cm^3 container. (d) A 0.004-mol sample of helium is held at 300 K in a 200-cm^3 container. (e) A 0.004-mol sample of helium is held at 250 K in a 200-cm^3 container.

9. Two cylinders A and B at the same temperature contain the same quantity of the same kind of gas. Cylinder A has three times the volume of cylinder B. What can you conclude about the pressures the gases exert? (a) We can conclude nothing about the pressures.

(b) The pressure in A is three times the pressure in B. (c) The pressures must be equal. (d) The pressure in A must be one-third the pressure in B.

10. A rubber balloon is filled with 1 L of air at 1 atm and 300 K and is then put into a cryogenic refrigerator at 100 K. The rubber remains flexible as it cools. **(i)** What happens to the volume of the balloon? (a) It decreases to $\frac{1}{3}$ L. (b) It decreases to $1/\sqrt{3}$ L. (c) It is constant. (d) It increases to $\sqrt{3}$ L. (e) It increases to 3 L. **(ii)** What happens to the pressure of the air in the balloon? (a) It decreases to $\frac{1}{3}$ atm. (b) It decreases to $1/\sqrt{3}$ atm. (c) It is constant. (d) It increases to $\sqrt{3}$ atm. (e) It increases to 3 atm.

11. The average coefficient of linear expansion of copper is $17 \times 10^{-6}\ (°C)^{-1}$. The Statue of Liberty is 93 m tall on a summer morning when the temperature is 25°C. Assume the copper plates covering the statue are mounted edge to edge without expansion joints and do not buckle or bind on the framework supporting them as the day grows hot. What is the order of magnitude of the statue's increase in height? (a) 0.1 mm (b) 1 mm (c) 1 cm (d) 10 cm (e) 1 m

12. Suppose you empty a tray of ice cubes into a bowl partly full of water and cover the bowl. After one-half hour, the contents of the bowl come to thermal equilibrium, with more liquid water and less ice than you started with. Which of the following is true? (a) The temperature of the liquid water is higher than the temperature of the remaining ice. (b) The temperature of the liquid water is the same as that of the ice. (c) The temperature of the liquid water is less than that of the ice. (d) The comparative temperatures of the liquid water and ice depend on the amounts present.

13. A hole is drilled in a metal plate. When the metal is raised to a higher temperature, what happens to the diameter of the hole? (a) It decreases. (b) It increases. (c) It remains the same. (d) The answer depends on the initial temperature of the metal. (e) None of those answers is correct.

14. On a very cold day in upstate New York, the temperature is −25°C, which is equivalent to what Fahrenheit temperature? (a) −46°F (b) −77°F (c) 18°F (d) −25°F (e) −13°F

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Common thermometers are made of a mercury column in a glass tube. Based on the operation of these thermometers, which has the larger coefficient of linear expansion, glass or mercury? (Don't answer the question by looking in a table.)

2. A piece of copper is dropped into a beaker of water. (a) If the water's temperature rises, what happens to the temperature of the copper? (b) Under what conditions are the water and copper in thermal equilibrium?

3. (a) What does the ideal gas law predict about the volume of a sample of gas at absolute zero? (b) Why is this prediction incorrect?

4. Some picnickers stop at a convenience store to buy some food, including bags of potato chips. They then drive up into the mountains to their picnic site. When they unload the food, they notice that the bags of chips are puffed up like balloons. Why did that happen?

5. In describing his upcoming trip to the Moon, and as portrayed in the movie *Apollo 13* (Universal, 1995), astronaut Jim Lovell said, "I'll be walking in a place where there's a 400-degree difference between sunlight and shadow." Suppose an astronaut standing on the Moon holds a thermometer in his gloved hand. (a) Is the thermometer reading the temperature of the vacuum at the Moon's surface? (b) Does it read any temperature? If so, what object or substance has that temperature?

6. Metal lids on glass jars can often be loosened by running hot water over them. Why does that work?

7. An automobile radiator is filled to the brim with water when the engine is cool. (a) What happens to the water when the engine is running and the water has been raised to a high temperature? (b) What do modern automobiles have in their cooling systems to prevent the loss of coolants?

8. When the metal ring and metal sphere in Figure CQ19.8 are both at room temperature, the sphere can barely be passed through the ring. (a) After the sphere is warmed in a flame, it cannot be passed through the ring. Explain. (b) **What If?** What if the ring is warmed and the sphere is left at room temperature? Does the sphere pass through the ring?

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Figure CQ19.8

9. Is it possible for two objects to be in thermal equilibrium if they are not in contact with each other? Explain.

10. Use a periodic table of the elements (see Appendix C) to determine the number of grams in one mole of (a) hydrogen, which has diatomic molecules; (b) helium; and (c) carbon monoxide.

Problems

The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 19.2 Thermometers and the Celsius Temperature Scale

Section 19.3 The Constant-Volume Gas Thermometer and the Absolute Temperature Scale

1. A nurse measures the temperature of a patient to be 41.5°C. (a) What is this temperature on the Fahrenheit scale? (b) Do you think the patient is seriously ill? Explain.

2. The temperature difference between the inside and the outside of a home on a cold winter day is 57.0°F. Express this difference on (a) the Celsius scale and (b) the Kelvin scale.

3. Convert the following temperatures to their values on the Fahrenheit and Kelvin scales: (a) the sublimation point of dry ice, −78.5°C; (b) human body temperature, 37.0°C.

4. The boiling point of liquid hydrogen is 20.3 K at atmospheric pressure. What is this temperature on (a) the Celsius scale and (b) the Fahrenheit scale?

5. Liquid nitrogen has a boiling point of −195.81°C at atmospheric pressure. Express this temperature (a) in degrees Fahrenheit and (b) in kelvins.

6. Death Valley holds the record for the highest recorded temperature in the United States. On July 10, 1913, at a place called Furnace Creek Ranch, the temperature rose to 134°F. The lowest U.S. temperature ever recorded occurred at Prospect Creek Camp in Alaska on January 23, 1971, when the temperature plummeted to −79.8° F. (a) Convert these temperatures to the Celsius scale. (b) Convert the Celsius temperatures to Kelvin.

7. M In a student experiment, a constant-volume gas thermometer is calibrated in dry ice (−78.5°C) and in boiling ethyl alcohol (78.0°C). The separate pressures are 0.900 atm and 1.635 atm. (a) What value of absolute zero in degrees Celsius does the calibration yield? What pressures would be found at (b) the freezing and (c) the boiling points of water? *Hint:* Use the linear relationship $P = A + BT$, where A and B are constants.

Section 19.4 Thermal Expansion of Solids and Liquids

Note: Table 19.1 is available for use in solving problems in this section.

8. W The concrete sections of a certain superhighway are designed to have a length of 25.0 m. The sections are poured and cured at 10.0°C. What minimum spacing should the engineer leave between the sections to eliminate buckling if the concrete is to reach a temperature of 50.0°C?

9. M The active element of a certain laser is made of a glass rod 30.0 cm long and 1.50 cm in diameter. Assume the average coefficient of linear expansion of the glass is equal to 9.00×10^{-6} (°C)$^{-1}$. If the temperature of the rod increases by 65.0°C, what is the increase in (a) its length, (b) its diameter, and (c) its volume?

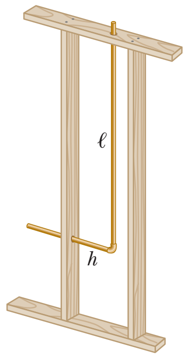

Figure P19.10

10. **Review.** Inside the wall of a house, an L-shaped section of hot-water pipe consists of three parts: a straight, horizontal piece h = 28.0 cm long; an elbow; and a straight, vertical piece ℓ = 134 cm long (Fig. P19.10). A stud and a second-story floorboard hold the ends of this section of copper pipe stationary. Find the magnitude and direction of the displacement of the pipe elbow when the water flow is turned on, raising the temperature of the pipe from 18.0°C to 46.5°C.

11. M A copper telephone wire has essentially no sag between poles 35.0 m apart on a winter day when the temperature is −20.0°C. How much longer is the wire on a summer day when the temperature is 35.0°C?

12. A pair of eyeglass frames is made of epoxy plastic. At room temperature (20.0°C), the frames have circular lens holes 2.20 cm in radius. To what temperature must the frames be heated if lenses 2.21 cm in radius are to be inserted in them? The average coefficient of linear expansion for epoxy is 1.30×10^{-4} (°C)$^{-1}$.

13. The Trans-Alaska pipeline is 1 300 km long, reaching from Prudhoe Bay to the port of Valdez. It experiences temperatures from −73°C to +35°C. How much does the steel pipeline expand because of the difference in temperature? How can this expansion be compensated for?

14. Each year thousands of children are badly burned by hot tap water. Figure P19.14 (page 584) shows a cross-sectional view of an antiscalding faucet attachment designed to prevent such accidents. Within the device, a spring made of material with a high coefficient of thermal expansion controls a movable plunger. When the water temperature rises above a preset safe value, the expansion of the spring causes the plunger to shut off the water flow. Assuming that the initial length L of the unstressed spring is 2.40 cm and its coefficient of linear expansion is 22.0×10^{-6} $(°C)^{-1}$, determine the increase in length of the spring when the water temperature rises by 30.0°C. (You will find the increase in length to be small. Therefore, to provide a greater variation in valve opening for the temperature change anticipated, actual devices have a more complicated mechanical design.)

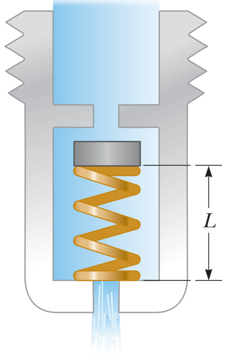

Figure P19.14

15. A square hole 8.00 cm along each side is cut in a sheet of copper. (a) Calculate the change in the area of this hole resulting when the temperature of the sheet is increased by 50.0 K. (b) Does this change represent an increase or a decrease in the area enclosed by the hole?

16. The average coefficient of volume expansion for carbon tetrachloride is 5.81×10^{-4} $(°C)^{-1}$. If a 50.0-gal steel container is filled completely with carbon tetrachloride when the temperature is 10.0°C, how much will spill over when the temperature rises to 30.0°C?

17. W At 20.0°C, an aluminum ring has an inner diameter of 5.000 0 cm and a brass rod has a diameter of 5.050 0 cm. (a) If only the ring is warmed, what temperature must it reach so that it will just slip over the rod? (b) **What If?** If both the ring and the rod are warmed together, what temperature must they both reach so that the ring barely slips over the rod? (c) Would this latter process work? Explain. *Hint:* Consult Table 20.2 in the next chapter.

18. W *Why is the following situation impossible?* A thin brass ring has an inner diameter 10.00 cm at 20.0°C. A solid aluminum cylinder has diameter 10.02 cm at 20.0°C. Assume the average coefficients of linear expansion of the two metals are constant. Both metals are cooled together to a temperature at which the ring can be slipped over the end of the cylinder.

19. A volumetric flask made of Pyrex is calibrated at 20.0°C. It is filled to the 100-mL mark with 35.0°C acetone. After the flask is filled, the acetone cools and the flask warms so that the combination of acetone and flask reaches a uniform temperature of 32.0°C. The combination is then cooled back to 20.0°C. (a) What is the volume of the acetone when it cools to 20.0°C? (b) At the temperature of 32.0°C, does the level of acetone lie above or below the 100-mL mark on the flask? Explain.

20. **Review.** On a day that the temperature is 20.0°C, a concrete walk is poured in such a way that the ends of the walk are unable to move. Take Young's modulus for concrete to be 7.00×10^9 N/m^2 and the compressive strength to be 2.00×10^9 N/m^2. (a) What is the stress in the cement on a hot day of 50.0°C? (b) Does the concrete fracture?

21. A hollow aluminum cylinder 20.0 cm deep has an internal capacity of 2.000 L at 20.0°C. It is completely filled with turpentine at 20.0°C. The turpentine and the aluminum cylinder are then slowly warmed together to 80.0°C. (a) How much turpentine overflows? (b) What is the volume of turpentine remaining in the cylinder at 80.0°C? (c) If the combination with this amount of turpentine is then cooled back to 20.0°C, how far below the cylinder's rim does the turpentine's surface recede?

22. **Review.** The Golden Gate Bridge in San Francisco has a main span of length 1.28 km, one of the longest in the world. Imagine that a steel wire with this length and a cross-sectional area of 4.00×10^{-6} m^2 is laid in a straight line on the bridge deck with its ends attached to the towers of the bridge. On a summer day the temperature of the wire is 35.0°C. (a) When winter arrives, the towers stay the same distance apart and the bridge deck keeps the same shape as its expansion joints open. When the temperature drops to −10.0°C, what is the tension in the wire? Take Young's modulus for steel to be 20.0×10^{10} N/m^2. (b) Permanent deformation occurs if the stress in the steel exceeds its elastic limit of 3.00×10^8 N/m^2. At what temperature would the wire reach its elastic limit? (c) **What If?** Explain how your answers to parts (a) and (b) would change if the Golden Gate Bridge were twice as long.

23. A sample of lead has a mass of 20.0 kg and a density of 11.3×10^3 kg/m^3 at 0°C. (a) What is the density of lead at 90.0°C? (b) What is the mass of the sample of lead at 90.0°C?

24. A sample of a solid substance has a mass m and a density ρ_0 at a temperature T_0. (a) Find the density of the substance if its temperature is increased by an amount ΔT in terms of the coefficient of volume expansion β. (b) What is the mass of the sample if the temperature is raised by an amount ΔT?

25. M An underground gasoline tank can hold 1.00×10^3 gallons of gasoline at 52.0°F. Suppose the tank is being filled on a day when the outdoor temperature (and the temperature of the gasoline in a tanker truck) is 95.0°F. When the underground tank registers that it is full, how many gallons have been transferred from the truck, according to a non-temperature-compensated gauge on the truck? Assume the temperature of the gasoline quickly cools from 95.0°F to 52.0°F upon entering the tank.

Section 19.5 Macroscopic Description of an Ideal Gas

26. A rigid tank contains 1.50 moles of an ideal gas. Determine the number of moles of gas that must be withdrawn from the tank to lower the pressure of the gas from 25.0 atm to 5.00 atm. Assume the volume of the tank and the temperature of the gas remain constant during this operation.

27. Gas is confined in a tank at a pressure of 11.0 atm and a temperature of 25.0°C. If two-thirds of the gas is withdrawn and the temperature is raised to 75.0°C, what is the pressure of the gas remaining in the tank?

28. Your father and your younger brother are confronted with the same puzzle. Your father's garden sprayer and your brother's water cannon both have tanks with a capacity of 5.00 L (Fig. P19.28). Your father puts a negligible amount of concentrated fertilizer into his tank. They both pour in 4.00 L of water and seal up their tanks, so the tanks also contain air at atmospheric pressure. Next, each uses a hand-operated pump to inject more air until the absolute pressure in the tank reaches 2.40 atm. Now each uses his device to spray out water—not air—until the stream becomes feeble, which it does when the pressure in the tank reaches 1.20 atm. To accomplish spraying out all the water, each finds he must pump up the tank three times. Here is the puzzle: most of the water sprays out after the second pumping. The first and the third pumping-up processes seem just as difficult as the second but result in a much smaller amount of water coming out. Account for this phenomenon.

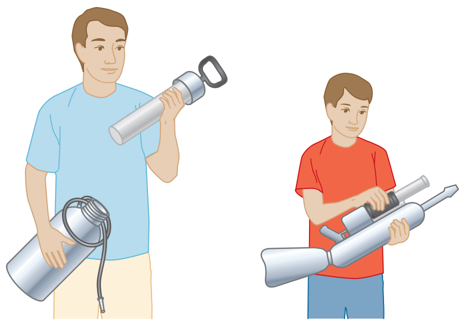

Figure P19.28

29. W Gas is contained in an 8.00-L vessel at a temperature of 20.0°C and a pressure of 9.00 atm. (a) Determine the number of moles of gas in the vessel. (b) How many molecules are in the vessel?

30. A container in the shape of a cube 10.0 cm on each edge contains air (with equivalent molar mass 28.9 g/mol) at atmospheric pressure and temperature 300 K. Find (a) the mass of the gas, (b) the gravitational force exerted on it, and (c) the force it exerts on each face of the cube. (d) Why does such a small sample exert such a great force?

31. M An auditorium has dimensions 10.0 m × 20.0 m × 30.0 m. How many molecules of air fill the auditorium at 20.0°C and a pressure of 101 kPa (1.00 atm)?

32. M The pressure gauge on a tank registers the gauge pressure, which is the difference between the interior pressure and exterior pressure. When the tank is full of oxygen (O_2), it contains 12.0 kg of the gas at a gauge pressure of 40.0 atm. Determine the mass of oxygen that has been withdrawn from the tank when the pressure reading is 25.0 atm. Assume the temperature of the tank remains constant.

33. (a) Find the number of moles in one cubic meter of an ideal gas at 20.0°C and atmospheric pressure. (b) For air, Avogadro's number of molecules has mass 28.9 g. Calculate the mass of one cubic meter of air. (c) State how this result compares with the tabulated density of air at 20.0°C.

34. Use the definition of Avogadro's number to find the mass of a helium atom.

35. A popular brand of cola contains 6.50 g of carbon dioxide dissolved in 1.00 L of soft drink. If the evaporating carbon dioxide is trapped in a cylinder at 1.00 atm and 20.0°C, what volume does the gas occupy?

36. W In state-of-the-art vacuum systems, pressures as low as 1.00×10^{-9} Pa are being attained. Calculate the number of molecules in a 1.00-m^3 vessel at this pressure and a temperature of 27.0°C.

37. M An automobile tire is inflated with air originally at 10.0°C and normal atmospheric pressure. During the process, the air is compressed to 28.0% of its original volume and the temperature is increased to 40.0°C. (a) What is the tire pressure? (b) After the car is driven at high speed, the tire's air temperature rises to 85.0°C and the tire's interior volume increases by 2.00%. What is the new tire pressure (absolute)?

38. **Review.** To measure how far below the ocean surface a bird dives to catch a fish, a scientist uses a method originated by Lord Kelvin. He dusts the interiors of plastic tubes with powdered sugar and then seals one end of each tube. He captures the bird at nighttime in its nest and attaches a tube to its back. He then catches the same bird the next night and removes the tube. In one trial, using a tube 6.50 cm long, water washes away the sugar over a distance of 2.70 cm from the open end of the tube. Find the greatest depth to which the bird dived, assuming the air in the tube stayed at constant temperature.

39. AMT M **Review.** The mass of a hot-air balloon and its cargo (not including the air inside) is 200 kg. The air outside is at 10.0°C and 101 kPa. The volume of the balloon is 400 m^3. To what temperature must the air in the balloon be warmed before the balloon will lift off? (Air density at 10.0°C is 1.244 kg/m^3.)

40. A room of volume V contains air having equivalent molar mass M (in g/mol). If the temperature of the room is raised from T_1 to T_2, what mass of air will leave the room? Assume that the air pressure in the room is maintained at P_0.

41. **Review.** At 25.0 m below the surface of the sea, where the temperature is 5.00°C, a diver exhales an air bubble having a volume of 1.00 cm^3. If the surface temperature of the sea is 20.0°C, what is the volume of the bubble just before it breaks the surface?

42. Estimate the mass of the air in your bedroom. State the quantities you take as data and the value you measure or estimate for each.

43. W A cook puts 9.00 g of water in a 2.00-L pressure cooker that is then warmed to 500°C. What is the pressure inside the container?

44. The pressure gauge on a cylinder of gas registers the gauge pressure, which is the difference between the interior pressure and the exterior pressure P_0. Let's call the gauge pressure P_g. When the cylinder is full, the mass of the gas in it is m_i at a gauge pressure of P_{gi}. Assuming the temperature of the cylinder remains constant, show that the mass of the gas *remaining* in the cylinder when the pressure reading is P_{gf} is given by

$$m_f = m_i\left(\frac{P_{gf} + P_0}{P_{gi} + P_0}\right)$$

Additional Problems

45. Long-term space missions require reclamation of the oxygen in the carbon dioxide exhaled by the crew. In one method of reclamation, 1.00 mol of carbon dioxide produces 1.00 mol of oxygen and 1.00 mol of methane as a byproduct. The methane is stored in a tank under pressure and is available to control the attitude of the spacecraft by controlled venting. A single astronaut exhales 1.09 kg of carbon dioxide each day. If the methane generated in the respiration recycling of three astronauts during one week of flight is stored in an originally empty 150-L tank at −45.0°C, what is the final pressure in the tank?

46. A steel beam being used in the construction of a skyscraper has a length of 35.000 m when delivered on a cold day at a temperature of 15.000°F. What is the length of the beam when it is being installed later on a warm day when the temperature is 90.000°F?

47. A spherical steel ball bearing has a diameter of 2.540 cm at 25.00°C. (a) What is its diameter when its temperature is raised to 100.0°C? (b) What temperature change is required to increase its volume by 1.000%?

48. A bicycle tire is inflated to a gauge pressure of 2.50 atm when the temperature is 15.0°C. While a man rides the bicycle, the temperature of the tire rises to 45.0°C. Assuming the volume of the tire does not change, find the gauge pressure in the tire at the higher temperature.

49. In a chemical processing plant, a reaction chamber of fixed volume V_0 is connected to a reservoir chamber of fixed volume $4V_0$ by a passage containing a thermally insulating porous plug. The plug permits the chambers to be at different temperatures. The plug allows gas to pass from either chamber to the other, ensuring that the pressure is the same in both. At one point in the processing, both chambers contain gas at a pressure of 1.00 atm and a temperature of 27.0°C. Intake and exhaust valves to the pair of chambers are closed. The reservoir is maintained at 27.0°C while the reaction chamber is heated to 400°C. What is the pressure in both chambers after that is done?

50. *Why is the following situation impossible?* An apparatus is designed so that steam initially at T = 150°C, P = 1.00 atm, and V = 0.500 m^3 in a piston–cylinder apparatus undergoes a process in which (1) the volume remains constant and the pressure drops to 0.870 atm, followed by (2) an expansion in which the pressure remains constant and the volume increases to 1.00 m^3, followed by (3) a return to the initial conditions. It is important that the pressure of the gas never fall below 0.850 atm so that the piston will support a delicate and very expensive part of the apparatus. Without such support, the delicate apparatus can be severely damaged and rendered useless. When the design is turned into a working prototype, it operates perfectly.

51. M A mercury thermometer is constructed as shown in Figure P19.51. The Pyrex glass capillary tube has a diameter of 0.004 00 cm, and the bulb has a diameter of 0.250 cm. Find the change in height of the mercury column that occurs with a temperature change of 30.0°C.

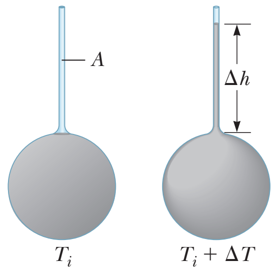

Figure P19.51
Problems 51 and 52.

52. A liquid with a coefficient of volume expansion β just fills a spherical shell of volume V (Fig. P19.51). The shell and the open capillary of area A projecting from the top of the sphere are made of a material with an average coefficient of linear expansion α. The liquid is free to expand into the capillary. Assuming the temperature increases by ΔT, find the distance Δh the liquid rises in the capillary.

53. AMT **Review.** An aluminum pipe is open at both ends and used as a flute. The pipe is cooled to 5.00°C, at which its length is 0.655 m. As soon as you start to play it, the pipe fills with air at 20.0°C. After that, by how much does its fundamental frequency change as the metal rises in temperature to 20.0°C?

54. Two metal bars are made of invar and a third bar is made of aluminum. At 0°C, each of the three bars is drilled with two holes 40.0 cm apart. Pins are put through the holes to assemble the bars into an equilateral triangle as in Figure P19.54. (a) First ignore the expansion of the invar. Find the angle between the invar bars as a function of Celsius temperature. (b) Is your answer accurate for negative as well as positive temperatures? (c) Is it accurate for 0°C? (d) Solve the problem again, including the expansion of the invar. Aluminum melts at 660°C and invar at 1 427°C. Assume the tabulated expansion coefficients are constant. What are (e) the greatest and (f) the smallest attainable angles between the invar bars?

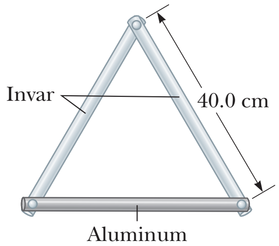

Figure P19.54

55. A student measures the length of a brass rod with a steel tape at 20.0°C. The reading is 95.00 cm. What will the tape indicate for the length of the rod when the rod and the tape are at (a) −15.0°C and (b) 55.0°C?

56. The density of gasoline is 730 kg/m^3 at 0°C. Its average coefficient of volume expansion is 9.60×10^{-4} (°C)$^{-1}$. Assume 1.00 gal of gasoline occupies 0.003 80 m^3.

How many extra kilograms of gasoline would you receive if you bought 10.0 gal of gasoline at 0°C rather than at 20.0°C from a pump that is not temperature compensated?

57. A liquid has a density ρ. (a) Show that the fractional change in density for a change in temperature ΔT is $\Delta\rho/\rho = -\beta\,\Delta T$. (b) What does the negative sign signify? (c) Fresh water has a maximum density of 1.000 0 g/cm^3 at 4.0°C. At 10.0°C, its density is 0.999 7 g/cm^3. What is β for water over this temperature interval? (d) At 0°C, the density of water is 0.999 9 g/cm^3. What is the value for β over the temperature range 0°C to 4.00°C?

58. (a) Take the definition of the coefficient of volume expansion to be

$$\beta = \frac{1}{V}\left.\frac{dV}{dT}\right|_{P=\text{constant}} = \frac{1}{V}\frac{\partial V}{\partial T}$$

Use the equation of state for an ideal gas to show that the coefficient of volume expansion for an ideal gas at constant pressure is given by $\beta = 1/T$, where T is the absolute temperature. (b) What value does this expression predict for β at 0°C? State how this result compares with the experimental values for (c) helium and (d) air in Table 19.1. *Note:* These values are much larger than the coefficients of volume expansion for most liquids and solids.

59. Review. A clock with a brass pendulum has a period of 1.000 s at 20.0°C. If the temperature increases to 30.0°C, (a) by how much does the period change and (b) how much time does the clock gain or lose in one week?

60. A bimetallic strip of length L is made of two ribbons of different metals bonded together. (a) First assume the strip is originally straight. As the strip is warmed, the metal with the greater average coefficient of expansion expands more than the other, forcing the strip into an arc with the outer radius having a greater circumference (Fig. P19.60). Derive an expression for the angle of bending θ as a function of the initial length of the strips, their average coefficients of linear expansion, the change in temperature, and the separation of the centers of the strips ($\Delta r = r_2 - r_1$). (b) Show that the angle of bending decreases to zero when ΔT decreases to zero and also when the two average coefficients of expansion become equal. (c) **What If?** What happens if the strip is cooled?

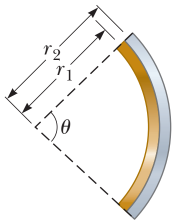

Figure P19.60

61. The rectangular plate shown in Figure P19.61 has an area A_i equal to ℓw. If the temperature increases by ΔT, each dimension increases according to Equation 19.4, where α is the average coefficient of linear expansion. (a) Show that the increase in area is $\Delta A = 2\alpha A_i\,\Delta T$. (b) What approximation does this expression assume?

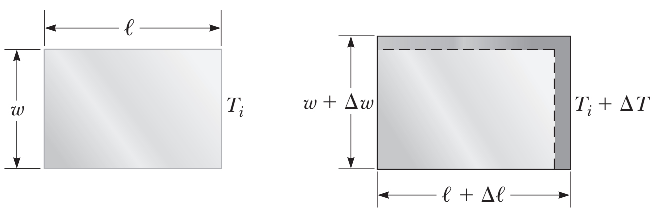

Figure P19.61

62. The measurement of the average coefficient of volume expansion β for a liquid is complicated because the container also changes size with temperature. Figure P19.62 shows a simple means for measuring β despite the expansion of the container. With this apparatus, one arm of a U-tube is maintained at 0°C in a water–ice bath, and the other arm is maintained at a different temperature T_C in a constant-temperature bath. The connecting tube is horizontal. A difference in the length or diameter of the tube between the two arms of the U-tube has no effect on the pressure balance at the bottom of the tube because the pressure depends only on the depth of the liquid. Derive an expression for β for the liquid in terms of h_0, h_t, and T_C.

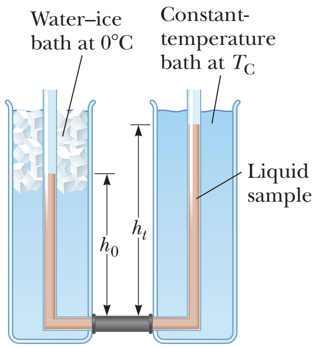

Figure P19.62

63. A copper rod and a steel rod are different in length by 5.00 cm at 0°C. The rods are warmed and cooled together. (a) Is it possible that the length difference remains constant at all temperatures? Explain. (b) If so, describe the lengths at 0°C as precisely as you can. Can you tell which rod is longer? Can you tell the lengths of the rods?

64. AMT GP A vertical cylinder of cross-sectional area A is fitted with a tight-fitting, frictionless piston of mass m (Fig. P19.64). The piston is not restricted in its motion in any way and is supported by the gas at pressure P below it. Atmospheric pressure is P_0. We wish to find the height h in Figure P19.64. (a) What analysis model is appropriate to describe the piston? (b) Write an appropriate force equation for the piston from this analysis model in terms of P, P_0, m, A, and g. (c) Suppose n moles of an ideal gas are in the cylinder at a temperature of T. Substitute for P in your answer to part (b) to find the height h of the piston above the bottom of the cylinder.

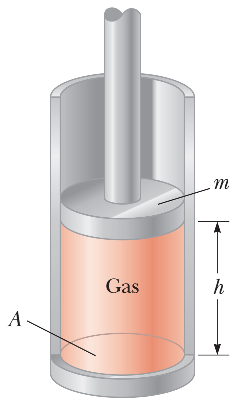

Figure P19.64

65. Review. Consider an object with any one of the shapes displayed in Table 10.2. What is the percentage increase in the moment of inertia of the object when it is warmed from 0°C to 100°C if it is composed of (a) copper or (b) aluminum? Assume the average linear expansion coefficients shown in Table 19.1 do not vary between 0°C and 100°C. (c) Why are the answers for parts (a) and (b) the same for all the shapes?

66. (a) Show that the density of an ideal gas occupying a volume V is given by $\rho = PM/RT$, where M is the molar mass. (b) Determine the density of oxygen gas at atmospheric pressure and 20.0°C.

67. Two concrete spans of a 250-m-long bridge are placed end to end so that no room is allowed for expansion (Fig. P19.67a). If a temperature increase of 20.0°C occurs, what is the height y to which the spans rise when they buckle (Fig. P19.67b)?

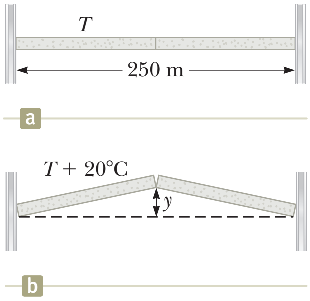

Figure P19.67 Problems 67 and 68.

68. Two concrete spans that form a bridge of length L are placed end to end so that no room is allowed for expansion (Fig. P19.67a). If a temperature increase of ΔT occurs, what is the height y to which the spans rise when they buckle (Fig. P19.67b)?

69. **Review.** (a) Derive an expression for the buoyant force on a spherical balloon, submerged in water, as a function of the depth h below the surface, the volume V_i of the balloon at the surface, the pressure P_0 at the surface, and the density ρ_w of the water. Assume the water temperature does not change with depth. (b) Does the *buoyant* force increase or decrease as the balloon is submerged? (c) At what depth is the buoyant force one-half the surface value?

70. AMT **Review.** Following a collision in outer space, a copper disk at 850°C is rotating about its axis with an angular speed of 25.0 rad/s. As the disk radiates infrared light, its temperature falls to 20.0°C. No external torque acts on the disk. (a) Does the angular speed change as the disk cools? Explain how it changes or why it does not. (b) What is its angular speed at the lower temperature?

71. Starting with Equation 19.10, show that the total pressure P in a container filled with a mixture of several ideal gases is $P = P_1 + P_2 + P_3 + \cdots$, where $P_1, P_2, \cdots$ are the pressures that each gas would exert if it alone filled the container. (These individual pressures are called the *partial pressures* of the respective gases.) This result is known as *Dalton's law of partial pressures.*

Challenge Problems

72. **Review.** A steel wire and a copper wire, each of diameter 2.000 mm, are joined end to end. At 40.0°C, each has an unstretched length of 2.000 m. The wires are connected between two fixed supports 4.000 m apart on a tabletop. The steel wire extends from $x = -2.000$ m to $x = 0$, the copper wire extends from $x = 0$ to $x = 2.000$ m, and the tension is negligible. The temperature is then lowered to 20.0°C. Assume the average coefficient of linear expansion of steel is 11.0×10^{-6} (°C)$^{-1}$ and that of copper is 17.0×10^{-6} (°C)$^{-1}$. Take Young's modulus for steel to be 20.0×10^{10} N/m^2 and that for copper to be 11.0×10^{10} N/m^2. At this lower temperature, find (a) the tension in the wire and (b) the x coordinate of the junction between the wires.

73. **Review.** A steel guitar string with a diameter of 1.00 mm is stretched between supports 80.0 cm apart. The temperature is 0.0°C. (a) Find the mass per unit length of this string. (Use the value 7.86×10^3 kg/m^3 for the density.) (b) The fundamental frequency of transverse oscillations of the string is 200 Hz. What is the tension in the string? Next, the temperature is raised to 30.0°C. Find the resulting values of (c) the tension and (d) the fundamental frequency. Assume both the Young's modulus of 20.0×10^{10} N/m^2 and the average coefficient of expansion $\alpha = 11.0 \times 10^{-6}$ (°C)$^{-1}$ have constant values between 0.0°C and 30.0°C.

74. W A cylinder is closed by a piston connected to a spring of constant 2.00×10^3 N/m (see Fig. P19.74). With the spring relaxed, the cylinder is filled with 5.00 L of gas at a pressure of 1.00 atm and a temperature of 20.0°C. (a) If the piston has a cross-sectional area of 0.010 0 m^2 and negligible mass, how high will it rise when the temperature is raised to 250°C? (b) What is the pressure of the gas at 250°C?

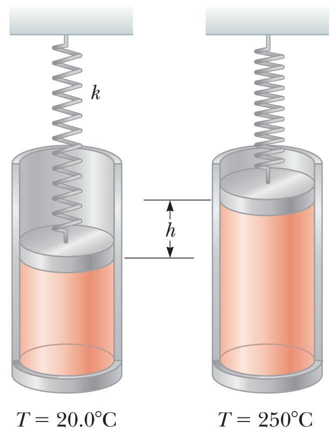

Figure P19.74

75. Helium gas is sold in steel tanks that will rupture if subjected to tensile stress greater than its yield strength of 5×10^8 N/m^2. If the helium is used to inflate a balloon, could the balloon lift the spherical tank the helium came in? Justify your answer. *Suggestion:* You may consider a spherical steel shell of radius r and thickness t having the density of iron and on the verge of breaking apart into two hemispheres because it contains helium at high pressure.

76. A cylinder that has a 40.0-cm radius and is 50.0 cm deep is filled with air at 20.0°C and 1.00 atm (Fig. P19.76a). A 20.0-kg piston is now lowered into the cylinder, compressing the air trapped inside as it takes equilibrium height h_i (Fig. P19.76b). Finally, a 25.0-kg dog stands on the piston, further compressing the air, which remains at 20°C (Fig. P19.76c). (a) How far down

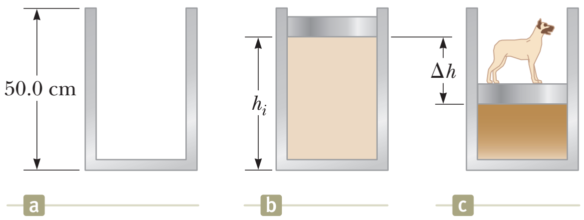

Figure P19.76

(Δh) does the piston move when the dog steps onto it? (b) To what temperature should the gas be warmed to raise the piston and dog back to h_i?

77. The relationship $L = L_i + \alpha L_i \Delta T$ is a valid approximation when $\alpha \Delta T$ is small. If $\alpha \Delta T$ is large, one must integrate the relationship $dL = \alpha L\, dT$ to determine the final length. (a) Assuming the coefficient of linear expansion of a material is constant as L varies, determine a general expression for the final length of a rod made of the material. Given a rod of length 1.00 m and a temperature change of 100.0°C, determine the error caused by the approximation when (b) $\alpha = 2.00 \times 10^{-5}$ (°C)$^{-1}$ (a typical value for a metal) and (c) when $\alpha = 0.020\,0$ (°C)$^{-1}$ (an unrealistically large value for comparison). (d) Using the equation from part (a), solve Problem 21 again to find more accurate results.

78. Review. A house roof is a perfectly flat plane that makes an angle θ with the horizontal. When its temperature changes, between T_c before dawn each day and T_h in the middle of each afternoon, the roof expands and contracts uniformly with a coefficient of thermal expansion α_1. Resting on the roof is a flat, rectangular metal plate with expansion coefficient α_2, greater than α_1. The length of the plate is L, measured along the slope of the roof. The component of the plate's weight perpendicular to the roof is supported by a normal force uniformly distributed over the area of the plate. The coefficient of kinetic friction between the plate and the roof is μ_k. The plate is always at the same temperature as the roof, so we assume its temperature is continuously changing. Because of the difference in expansion coefficients, each bit of the plate is moving relative to the roof below it, except for points along a certain horizontal line running across the plate called the stationary line. If the temperature is rising, parts of the plate below the stationary line are moving down relative to the roof and feel a force of kinetic friction acting up the roof. Elements of area above the stationary line are sliding up the roof, and on them kinetic friction acts downward parallel to the roof. The stationary line occupies no area, so we assume no force of static friction acts on the plate while the temperature is changing. The plate as a whole is very nearly in equilibrium, so the net friction force on it must be equal to the component of its weight acting down the incline. (a) Prove that the stationary line is at a distance of

$$\frac{L}{2}\left(1 - \frac{\tan\theta}{\mu_k}\right)$$

below the top edge of the plate. (b) Analyze the forces that act on the plate when the temperature is falling and prove that the stationary line is at that same distance above the bottom edge of the plate. (c) Show that the plate steps down the roof like an inchworm, moving each day by the distance

$$\frac{L}{\mu_k}(\alpha_2 - \alpha_1)(T_h - T_c)\tan\theta$$

(d) Evaluate the distance an aluminum plate moves each day if its length is 1.20 m, the temperature cycles between 4.00°C and 36.0°C, and if the roof has slope 18.5°, coefficient of linear expansion 1.50×10^{-5} (°C)$^{-1}$, and coefficient of friction 0.420 with the plate. (e) **What If?** What if the expansion coefficient of the plate is less than that of the roof? Will the plate creep up the roof?

79. A 1.00-km steel railroad rail is fastened securely at both ends when the temperature is 20.0°C. As the temperature increases, the rail buckles, taking the shape of an arc of a vertical circle. Find the height h of the center of the rail when the temperature is 25.0°C. (You will need to solve a transcendental equation.)

CHAPTER

20

The First Law of Thermodynamics

In this photograph of the Mt. Baker area near Bellingham, Washington, we see evidence of water in all three phases. In the lake is liquid water, and solid water in the form of snow appears on the ground. The clouds in the sky consist of liquid water droplets that have condensed from the gaseous water vapor in the air. Changes of a substance from one phase to another are a result of energy transfer. *(©iStockphoto.com/KingWu)*

Until about 1850, the fields of thermodynamics and mechanics were considered to be two distinct branches of science. The principle of conservation of energy seemed to describe only certain kinds of mechanical systems. Mid-19th-century experiments performed by Englishman James Joule and others, however, showed a strong connection between the transfer of energy by heat in thermal processes and the transfer of energy by work in mechanical processes. Today we know that mechanical energy can be transformed to internal energy, which is formally defined in this chapter. Once the concept of energy was generalized from mechanics to include internal energy, the principle of conservation of energy as discussed in Chapter 8 emerged as a universal law of nature.

This chapter focuses on the concept of internal energy, the first law of thermodynamics, and some important applications of the first law. The first law of thermodynamics describes systems in which the only energy change is that of internal energy and the transfers of energy are by heat and work. A major difference in our discussion of work in this chapter from that in most of the chapters on mechanics is that we will consider work done on *deformable* systems.

20.1 Heat and Internal Energy

At the outset, it is important to make a major distinction between internal energy and heat, terms that are often incorrectly used interchangeably in popular language.

Internal energy is all the energy of a system that is associated with its microscopic components—atoms and molecules—when viewed from a reference frame at rest with respect to the center of mass of the system.

The last part of this sentence ensures that any bulk kinetic energy of the system due to its motion through space is not included in internal energy. Internal energy includes kinetic energy of random translational, rotational, and vibrational motion of molecules; vibrational potential energy associated with forces between atoms in molecules; and electric potential energy associated with forces between molecules. It is useful to relate internal energy to the temperature of an object, but this relationship is limited. We show in Section 20.3 that internal energy changes can also occur in the absence of temperature changes. In that discussion, we will investigate the internal energy of the system when there is a *physical change,* most often related to a phase change, such as melting or boiling. We assign energy associated with *chemical changes,* related to chemical reactions, to the potential energy term in Equation 8.2, not to internal energy. Therefore, we discuss the *chemical potential energy* in, for example, a human body (due to previous meals), the gas tank of a car (due to an earlier transfer of fuel), and a battery of an electric circuit (placed in the battery during its construction in the manufacturing process).

Heat is defined as a process of transferring energy across the boundary of a system because of a temperature difference between the system and its surroundings. It is also the amount of energy Q transferred by this process.

When you *heat* a substance, you are transferring energy into it by placing it in contact with surroundings that have a higher temperature. Such is the case, for example, when you place a pan of cold water on a stove burner. The burner is at a higher temperature than the water, and so the water gains energy by heat.

Read this definition of heat (Q in Eq. 8.2) very carefully. In particular, notice what heat is *not* in the following common quotes. (1) Heat is *not* energy in a hot substance. For example, "The boiling water has a lot of heat" is incorrect; the boiling water has *internal energy* E_{int}. (2) Heat is *not* radiation. For example, "It was so hot because the sidewalk was radiating heat" is incorrect; energy is leaving the sidewalk by *electromagnetic radiation,* T_{ER} in Equation 8.2. (3) Heat is *not* warmth of an environment. For example, "The heat in the air was so oppressive" is incorrect; on a hot day, the air has a high *temperature* T.

As an analogy to the distinction between heat and internal energy, consider the distinction between work and mechanical energy discussed in Chapter 7. The work done on a system is a measure of the amount of energy transferred to the system from its surroundings, whereas the mechanical energy (kinetic energy plus potential energy) of a system is a consequence of the motion and configuration of the system. Therefore, when a person does work on a system, energy is transferred from the person to the system. It makes no sense to talk about the work *of* a system; one can refer only to the work done *on* or *by* a system when some process has occurred in which energy has been transferred to or from the system. Likewise, it makes no sense to talk about the heat *of* a system; one can refer to heat only when energy has been transferred as a result of a temperature difference. Both heat and work are ways of transferring energy between a system and its surroundings.

Units of Heat

Early studies of heat focused on the resultant increase in temperature of a substance, which was often water. Initial notions of heat were based on a fluid called *caloric* that flowed from one substance to another and caused changes in temperature. From the name of this mythical fluid came an energy unit related to thermal processes, the **calorie (cal),** which is defined as the amount of energy transfer

Pitfall Prevention 20.1

Internal Energy, Thermal Energy, and Bond Energy When reading other physics books, you may see terms such as *thermal energy* and *bond energy*. Thermal energy can be interpreted as that part of the internal energy associated with random motion of molecules and therefore related to temperature. Bond energy is the intermolecular potential energy. Therefore,

Internal energy = thermal energy + bond energy

Although this breakdown is presented here for clarification with regard to other books, we will not use these terms because there is no need for them.

Pitfall Prevention 20.2

Heat, Temperature, and Internal Energy Are Different As you read the newspaper or explore the Internet, be alert for incorrectly used phrases including the word *heat* and think about the proper word to be used in place of *heat*. Incorrect examples include "As the truck braked to a stop, a large amount of heat was generated by friction" and "The heat of a hot summer day . . . ."

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James Prescott Joule

British physicist (1818–1889)

Joule received some formal education in mathematics, philosophy, and chemistry from John Dalton but was in large part self-educated. Joule's research led to the establishment of the principle of conservation of energy. His study of the quantitative relationship among electrical, mechanical, and chemical effects of heat culminated in his announcement in 1843 of the amount of work required to produce a unit of energy, called the mechanical equivalent of heat.

necessary to raise the temperature of 1 g of water from 14.5°C to 15.5°C.[1] (The "Calorie," written with a capital "C" and used in describing the energy content of foods, is actually a kilocalorie.) The unit of energy in the U.S. customary system is the **British thermal unit (Btu),** which is defined as the amount of energy transfer required to raise the temperature of 1 lb of water from 63°F to 64°F.

Once the relationship between energy in thermal and mechanical processes became clear, there was no need for a separate unit related to thermal processes. The *joule* has already been defined as an energy unit based on mechanical processes. Scientists are increasingly turning away from the calorie and the Btu and are using the joule when describing thermal processes. In this textbook, heat, work, and internal energy are usually measured in joules.

The Mechanical Equivalent of Heat

In Chapters 7 and 8, we found that whenever friction is present in a mechanical system, the mechanical energy in the system decreases; in other words, mechanical energy is not conserved in the presence of nonconservative forces. Various experiments show that this mechanical energy does not simply disappear but is transformed into internal energy. You can perform such an experiment at home by hammering a nail into a scrap piece of wood. What happens to all the kinetic energy of the hammer once you have finished? Some of it is now in the nail as internal energy, as demonstrated by the nail being measurably warmer. Notice that there is *no* transfer of energy by heat in this process. For the nail and board as a nonisolated system, Equation 8.2 becomes $\Delta E_{\text{int}} = W + T_{\text{MW}}$, where W is the work done by the hammer on the nail and T_{MW} is the energy leaving the system by sound waves when the nail is struck. Although this connection between mechanical and internal energy was first suggested by Benjamin Thompson, it was James Prescott Joule who established the equivalence of the decrease in mechanical energy and the increase in internal energy.

A schematic diagram of Joule's most famous experiment is shown in Figure 20.1. The system of interest is the Earth, the two blocks, and the water in a thermally insulated container. Work is done within the system on the water by a rotating paddle wheel, which is driven by heavy blocks falling at a constant speed. If the energy transformed in the bearings and the energy passing through the walls by heat are neglected, the decrease in potential energy of the system as the blocks fall equals the work done by the paddle wheel on the water and, in turn, the increase in internal energy of the water. If the two blocks fall through a distance h, the decrease in potential energy of the system is $2mgh$, where m is the mass of one block; this energy causes the temperature of the water to increase. By varying the conditions of the experiment, Joule found that the decrease in mechanical energy is proportional to the product of the mass of the water and the increase in water temperature. The proportionality constant was found to be approximately 4.18 J/g · °C. Hence, 4.18 J of mechanical energy raises the temperature of 1 g of water by 1°C. More precise measurements taken later demonstrated the proportionality to be 4.186 J/g · °C when the temperature of the water was raised from 14.5°C to 15.5°C. We adopt this "15-degree calorie" value:

$$1 \text{ cal} = 4.186 \text{ J} \tag{20.1}$$

This equality is known, for purely historical reasons, as the **mechanical equivalent of heat.** A more proper name would be *equivalence between mechanical energy and internal energy,* but the historical name is well entrenched in our language, despite the incorrect use of the word *heat.*

The falling blocks rotate the paddles, causing the temperature of the water to increase.

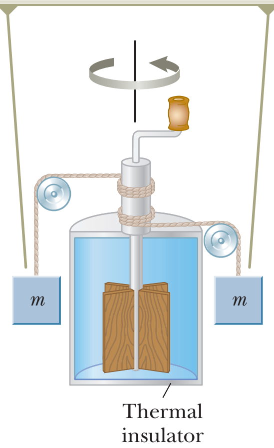

Figure 20.1 Joule's experiment for determining the mechanical equivalent of heat.

[1]Originally, the calorie was defined as the energy transfer necessary to raise the temperature of 1 g of water by 1°C. Careful measurements, however, showed that the amount of energy required to produce a 1°C change depends somewhat on the initial temperature; hence, a more precise definition evolved.

Example 20.1 Losing Weight the Hard Way AM

A student eats a dinner rated at 2 000 Calories. He wishes to do an equivalent amount of work in the gymnasium by lifting a 50.0-kg barbell. How many times must he raise the barbell to expend this much energy? Assume he raises the barbell 2.00 m each time he lifts it and he regains no energy when he lowers the barbell.

SOLUTION

Conceptualize Imagine the student raising the barbell. He is doing work on the system of the barbell and the Earth, so energy is leaving his body. The total amount of work that the student must do is 2 000 Calories.

Categorize We model the system of the barbell and the Earth as a *nonisolated system* for *energy*.

Analyze Reduce the conservation of energy equation, Equation 8.2, to the appropriate expression for the system of the barbell and the Earth:

$$(1)\quad \Delta U_{\text{total}} = W_{\text{total}}$$

Express the change in gravitational potential energy of the system after the barbell is raised once:

$$\Delta U = mgh$$

Express the total amount of energy that must be transferred into the system by work for lifting the barbell n times, assuming energy is not regained when the barbell is lowered:

$$(2)\quad \Delta U_{\text{total}} = nmgh$$

Substitute Equation (2) into Equation (1):

$$nmgh = W_{\text{total}}$$

Solve for n:

$$n = \frac{W_{\text{total}}}{mgh}$$

Substitute numerical values:

$$n = \frac{(2\,000\text{ Cal})}{(50.0\text{ kg})(9.80\text{ m/s}^2)(2.00\text{ m})}\left(\frac{1.00\times10^3\text{ cal}}{\text{Calorie}}\right)\left(\frac{4.186\text{ J}}{1\text{ cal}}\right)$$

$$= \boxed{8.54\times10^3\text{ times}}$$

Finalize If the student is in good shape and lifts the barbell once every 5 s, it will take him about 12 h to perform this feat. Clearly, it is much easier for this student to lose weight by dieting.

In reality, the human body is not 100% efficient. Therefore, not all the energy transformed within the body from the dinner transfers out of the body by work done on the barbell. Some of this energy is used to pump blood and perform other functions within the body. Therefore, the 2 000 Calories can be worked off in less time than 12 h when these other energy processes are included.

20.2 Specific Heat and Calorimetry

When energy is added to a system and there is no change in the kinetic or potential energy of the system, the temperature of the system usually rises. (An exception to this statement is the case in which a system undergoes a change of state—also called a *phase transition*—as discussed in the next section.) If the system consists of a sample of a substance, we find that the quantity of energy required to raise the temperature of a given mass of the substance by some amount varies from one substance to another. For example, the quantity of energy required to raise the temperature of 1 kg of water by 1°C is 4 186 J, but the quantity of energy required to raise the temperature of 1 kg of copper by 1°C is only 387 J. In the discussion that follows, we shall use heat as our example of energy transfer, but keep in mind that the temperature of the system could be changed by means of any method of energy transfer.

The **heat capacity** C of a particular sample is defined as the amount of energy needed to raise the temperature of that sample by 1°C. From this definition, we see that if energy Q produces a change ΔT in the temperature of a sample, then

$$Q = C\,\Delta T \qquad \textbf{(20.2)}$$

Table 20.1 Specific Heats of Some Substances at 25°C and Atmospheric Pressure

Substance	Specific Heat (J/kg · °C)	Substance	Specific Heat (J/kg · °C)
Elemental solids		*Other solids*	
Aluminum	900	Brass	380
Beryllium	1 830	Glass	837
Cadmium	230	Ice (−5°C)	2 090
Copper	387	Marble	860
Germanium	322	Wood	1 700
Gold	129	*Liquids*	
Iron	448	Alcohol (ethyl)	2 400
Lead	128	Mercury	140
Silicon	703	Water (15°C)	4 186
Silver	234	*Gas*	
		Steam (100°C)	2 010

Note: To convert values to units of cal/g · °C, divide by 4 186.

The **specific heat** c of a substance is the heat capacity per unit mass. Therefore, if energy Q transfers to a sample of a substance with mass m and the temperature of the sample changes by ΔT, the specific heat of the substance is

Specific heat ▶

$$c \equiv \frac{Q}{m\,\Delta T} \tag{20.3}$$

Specific heat is essentially a measure of how thermally insensitive a substance is to the addition of energy. The greater a material's specific heat, the more energy must be added to a given mass of the material to cause a particular temperature change. Table 20.1 lists representative specific heats.

From this definition, we can relate the energy Q transferred between a sample of mass m of a material and its surroundings to a temperature change ΔT as

$$Q = mc\,\Delta T \tag{20.4}$$

For example, the energy required to raise the temperature of 0.500 kg of water by 3.00°C is $Q = (0.500\text{ kg})(4\,186\text{ J/kg}\cdot{}^\circ\text{C})(3.00^\circ\text{C}) = 6.28 \times 10^3$ J. Notice that when the temperature increases, Q and ΔT are taken to be positive and energy transfers into the system. When the temperature decreases, Q and ΔT are negative and energy transfers out of the system.

We can identify $mc\,\Delta T$ as the change in internal energy of the system if we ignore any thermal expansion or contraction of the system. (Thermal expansion or contraction would result in a very small amount of work being done on the system by the surrounding air.) Then, Equation 20.4 is a reduced form of Equation 8.2: $\Delta E_{\text{int}} = Q$. The internal energy of the system can be changed by transferring energy into the system by any mechanism. For example, if the system is a baked potato in a microwave oven, Equation 8.2 reduces to the following analog to Equation 20.4: $\Delta E_{\text{int}} = T_{\text{ER}} = mc\,\Delta T$, where T_{ER} is the energy transferred to the potato from the microwave oven by electromagnetic radiation. If the system is the air in a bicycle pump, which becomes hot when the pump is operated, Equation 8.2 reduces to the following analog to Equation 20.4: $\Delta E_{\text{int}} = W = mc\,\Delta T$, where W is the work done on the pump by the operator. By identifying $mc\,\Delta T$ as ΔE_{int}, we have taken a step toward a better understanding of temperature: temperature is related to the energy of the molecules of a system. We will learn more details of this relationship in Chapter 21.

Specific heat varies with temperature. If, however, temperature intervals are not too great, the temperature variation can be ignored and c can be treated as a constant.[2]

Pitfall Prevention 20.3

An Unfortunate Choice of Terminology The name *specific heat* is an unfortunate holdover from the days when thermodynamics and mechanics developed separately. A better name would be *specific energy transfer,* but the existing term is too entrenched to be replaced.

Pitfall Prevention 20.4

Energy Can Be Transferred by Any Method The symbol Q represents the amount of energy transferred, but keep in mind that the energy transfer in Equation 20.4 could be by *any* of the methods introduced in Chapter 8; it does not have to be heat. For example, repeatedly bending a wire coat hanger raises the temperature at the bending point by *work.*

[2] The definition given by Equation 20.4 assumes the specific heat does not vary with temperature over the interval $\Delta T = T_f - T_i$. In general, if c varies with temperature over the interval, the correct expression for Q is $Q = m\int_{T_i}^{T_f} c\,dT$.

For example, the specific heat of water varies by only about 1% from 0°C to 100°C at atmospheric pressure. Unless stated otherwise, we shall neglect such variations.

Quick Quiz 20.1 Imagine you have 1 kg each of iron, glass, and water, and all three samples are at 10°C. **(a)** Rank the samples from highest to lowest temperature after 100 J of energy is added to each sample. **(b)** Rank the samples from greatest to least amount of energy transferred by heat if each sample increases in temperature by 20°C.

Notice from Table 20.1 that water has the highest specific heat of common materials. This high specific heat is in part responsible for the moderate climates found near large bodies of water. As the temperature of a body of water decreases during the winter, energy is transferred from the cooling water to the air by heat, increasing the internal energy of the air. Because of the high specific heat of water, a relatively large amount of energy is transferred to the air for even modest temperature changes of the water. The prevailing winds on the West Coast of the United States are toward the land (eastward). Hence, the energy liberated by the Pacific Ocean as it cools keeps coastal areas much warmer than they would otherwise be. As a result, West Coast states generally have more favorable winter weather than East Coast states, where the prevailing winds do not tend to carry the energy toward land.

Calorimetry

One technique for measuring specific heat involves heating a sample to some known temperature T_x, placing it in a vessel containing water of known mass and temperature $T_w < T_x$, and measuring the temperature of the water after equilibrium has been reached. This technique is called **calorimetry,** and devices in which this energy transfer occurs are called **calorimeters.** Figure 20.2 shows the hot sample in the cold water and the resulting energy transfer by heat from the high-temperature part of the system to the low-temperature part. If the system of the sample and the water is isolated, the principle of conservation of energy requires that the amount of energy Q_{hot} that leaves the sample (of unknown specific heat) equal the amount of energy Q_{cold} that enters the water.[3] Conservation of energy allows us to write the mathematical representation of this energy statement as

$$Q_{\text{cold}} = -Q_{\text{hot}} \tag{20.5}$$

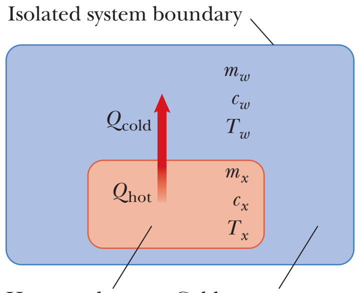

Figure 20.2 In a calorimetry experiment, a hot sample whose specific heat is unknown is placed in cold water in a container that isolates the system from the environment.

Pitfall Prevention 20.5

Remember the Negative Sign It is *critical* to include the negative sign in Equation 20.5. The negative sign in the equation is necessary for consistency with our sign convention for energy transfer. The energy transfer Q_{hot} has a negative value because energy is leaving the hot substance. The negative sign in the equation ensures that the right side is a positive number, consistent with the left side, which is positive because energy is entering the cold water.

Suppose m_x is the mass of a sample of some substance whose specific heat we wish to determine. Let's call its specific heat c_x and its initial temperature T_x as shown in Figure 20.2. Likewise, let m_w, c_w, and T_w represent corresponding values for the water. If T_f is the final temperature after the system comes to equilibrium, Equation 20.4 shows that the energy transfer for the water is $m_w c_w (T_f - T_w)$, which is positive because $T_f > T_w$, and that the energy transfer for the sample of unknown specific heat is $m_x c_x (T_f - T_x)$, which is negative. Substituting these expressions into Equation 20.5 gives

$$m_w c_w (T_f - T_w) = -m_x c_x (T_f - T_x)$$

This equation can be solved for the unknown specific heat c_x.

Example 20.2 Cooling a Hot Ingot

A 0.050 0-kg ingot of metal is heated to 200.0°C and then dropped into a calorimeter containing 0.400 kg of water initially at 20.0°C. The final equilibrium temperature of the mixed system is 22.4°C. Find the specific heat of the metal.

continued

[3]For precise measurements, the water container should be included in our calculations because it also exchanges energy with the sample. Doing so would require that we know the container's mass and composition, however. If the mass of the water is much greater than that of the container, we can neglect the effects of the container.

20.2 continued

SOLUTION

Conceptualize Imagine the process occurring in the isolated system of Figure 20.2. Energy leaves the hot ingot and goes into the cold water, so the ingot cools off and the water warms up. Once both are at the same temperature, the energy transfer stops.

Categorize We use an equation developed in this section, so we categorize this example as a substitution problem.

Use Equation 20.4 to evaluate each side of Equation 20.5:

$$m_w c_w (T_f - T_w) = -m_x c_x (T_f - T_x)$$

Solve for c_x:

$$c_x = \frac{m_w c_w (T_f - T_w)}{m_x (T_x - T_f)}$$

Substitute numerical values:

$$c_x = \frac{(0.400\text{ kg})(4\,186\text{ J/kg}\cdot°\text{C})(22.4°\text{C} - 20.0°\text{C})}{(0.050\,0\text{ kg})(200.0°\text{C} - 22.4°\text{C})}$$

$$= \boxed{453\text{ J/kg}\cdot°\text{C}}$$

The ingot is most likely iron as you can see by comparing this result with the data given in Table 20.1. The temperature of the ingot is initially above the steam point. Therefore, some of the water may vaporize when the ingot is dropped into the water. We assume the system is sealed and this steam cannot escape. Because the final equilibrium temperature is lower than the steam point, any steam that does result recondenses back into water.

WHAT IF? Suppose you are performing an experiment in the laboratory that uses this technique to determine the specific heat of a sample and you wish to decrease the overall uncertainty in your final result for c_x. Of the data given in this example, changing which value would be most effective in decreasing the uncertainty?

Answer The largest experimental uncertainty is associated with the small difference in temperature of 2.4°C for the water. For example, using the rules for propagation of uncertainty in Appendix Section B.8, an uncertainty of 0.1°C in each of T_f and T_w leads to an 8% uncertainty in their difference. For this temperature difference to be larger experimentally, the most effective change is to *decrease the amount of water.*

Example 20.3 Fun Time for a Cowboy AM

A cowboy fires a silver bullet with a muzzle speed of 200 m/s into the pine wall of a saloon. Assume all the internal energy generated by the impact remains with the bullet. What is the temperature change of the bullet?

SOLUTION

Conceptualize Imagine similar experiences you may have had in which mechanical energy is transformed to internal energy when a moving object is stopped. For example, as mentioned in Section 20.1, a nail becomes warm after it is hit a few times with a hammer.

Categorize The bullet is modeled as an *isolated system*. No work is done on the system because the force from the wall moves through no displacement. This example is similar to the skateboarder pushing off a wall in Section 9.7. There, no work is done on the skateboarder by the wall, and potential energy stored in the body from previous meals is transformed to kinetic energy. Here, no work is done by the wall on the bullet, and kinetic energy is transformed to internal energy.

Analyze Reduce the conservation of energy equation, Equation 8.2, to the appropriate expression for the system of the bullet:

$$(1)\quad \Delta K + \Delta E_{\text{int}} = 0$$

The change in the bullet's internal energy is related to its change in temperature:

$$(2)\quad \Delta E_{\text{int}} = mc\,\Delta T$$

Substitute Equation (2) into Equation (1):

$$(0 - \tfrac{1}{2}mv^2) + mc\,\Delta T = 0$$

20.3 continued

Solve for ΔT, using 234 J/kg · °C as the specific heat of silver (see Table 20.1):

$$(3)\quad \Delta T = \frac{\frac{1}{2}mv^2}{mc} = \frac{v^2}{2c} = \frac{(200\text{ m/s})^2}{2(234\text{ J/kg}\cdot{}^\circ\text{C})} = \boxed{85.5^\circ\text{C}}$$

Finalize Notice that the result does not depend on the mass of the bullet.

WHAT IF? Suppose the cowboy runs out of silver bullets and fires a lead bullet at the same speed into the wall. Will the temperature change of the bullet be larger or smaller?

Answer Table 20.1 shows that the specific heat of lead is 128 J/kg · °C, which is smaller than that for silver. Therefore, a given amount of energy input or transformation raises lead to a higher temperature than silver and the final temperature of the lead bullet will be larger. In Equation (3), let's substitute the new value for the specific heat:

$$\Delta T = \frac{v^2}{2c} = \frac{(200\text{ m/s})^2}{2(128\text{ J/kg}\cdot{}^\circ\text{C})} = 156^\circ\text{C}$$

There is no requirement that the silver and lead bullets have the same mass to determine this change in temperature. The only requirement is that they have the same speed.

20.3 Latent Heat

As we have seen in the preceding section, a substance can undergo a change in temperature when energy is transferred between it and its surroundings. In some situations, however, the transfer of energy does not result in a change in temperature. That is the case whenever the physical characteristics of the substance change from one form to another; such a change is commonly referred to as a **phase change.** Two common phase changes are from solid to liquid (melting) and from liquid to gas (boiling); another is a change in the crystalline structure of a solid. All such phase changes involve a change in the system's internal energy but no change in its temperature. The increase in internal energy in boiling, for example, is represented by the breaking of bonds between molecules in the liquid state; this bond breaking allows the molecules to move farther apart in the gaseous state, with a corresponding increase in intermolecular potential energy.

As you might expect, different substances respond differently to the addition or removal of energy as they change phase because their internal molecular arrangements vary. Also, the amount of energy transferred during a phase change depends on the amount of substance involved. (It takes less energy to melt an ice cube than it does to thaw a frozen lake.) When discussing two phases of a material, we will use the term *higher-phase material* to mean the material existing at the higher temperature. So, for example, if we discuss water and ice, water is the higher-phase material, whereas steam is the higher-phase material in a discussion of steam and water. Consider a system containing a substance in two phases in equilibrium such as water and ice. The initial amount of the higher-phase material, water, in the system is m_i. Now imagine that energy Q enters the system. As a result, the final amount of water is m_f due to the melting of some of the ice. Therefore, the amount of ice that melted, equal to the amount of *new* water, is $\Delta m = m_f - m_i$. We define the **latent heat** for this phase change as

$$L \equiv \frac{Q}{\Delta m} \qquad \textbf{(20.6)}$$

This parameter is called latent heat (literally, the "hidden" heat) because this added or removed energy does not result in a temperature change. The value of L for a substance depends on the nature of the phase change as well as on the properties of the substance. If the entire amount of the lower-phase material undergoes a phase change, the change in mass Δm of the higher-phase material is equal to the initial mass of the lower-phase material. For example, if an ice cube of mass m on a

Table 20.2 Latent Heats of Fusion and Vaporization

Substance	Melting Point (°C)	Latent Heat of Fusion (J/kg)	Boiling Point (°C)	Latent Heat of Vaporization (J/kg)
Helium[a]	−272.2	5.23×10^3	−268.93	2.09×10^4
Oxygen	−218.79	1.38×10^4	−182.97	2.13×10^5
Nitrogen	−209.97	2.55×10^4	−195.81	2.01×10^5
Ethyl alcohol	−114	1.04×10^5	78	8.54×10^5
Water	0.00	3.33×10^5	100.00	2.26×10^6
Sulfur	119	3.81×10^4	444.60	3.26×10^5
Lead	327.3	2.45×10^4	1 750	8.70×10^5
Aluminum	660	3.97×10^5	2 450	1.14×10^7
Silver	960.80	8.82×10^4	2 193	2.33×10^6
Gold	1 063.00	6.44×10^4	2 660	1.58×10^6
Copper	1 083	1.34×10^5	1 187	5.06×10^6

[a]Helium does not solidify at atmospheric pressure. The melting point given here corresponds to a pressure of 2.5 MPa.

plate melts completely, the change in mass of the water is $m_f - 0 = m$, which is the mass of new water and is also equal to the initial mass of the ice cube.

From the definition of latent heat, and again choosing heat as our energy transfer mechanism, the energy required to change the phase of a pure substance is

Energy transferred to a substance during a phase change

$$Q = L\,\Delta m \tag{20.7}$$

where Δm is the change in mass of the higher-phase material.

Latent heat of fusion L_f is the term used when the phase change is from solid to liquid (*to fuse* means "to combine by melting"), and **latent heat of vaporization** L_v is the term used when the phase change is from liquid to gas (the liquid "vaporizes").[4] The latent heats of various substances vary considerably as data in Table 20.2 show. When energy enters a system, causing melting or vaporization, the amount of the higher-phase material increases, so Δm is positive and Q is positive, consistent with our sign convention. When energy is extracted from a system, causing freezing or condensation, the amount of the higher-phase material decreases, so Δm is negative and Q is negative, again consistent with our sign convention. Keep in mind that Δm in Equation 20.7 always refers to the higher-phase material.

Pitfall Prevention 20.6

Signs Are Critical Sign errors occur very often when students apply calorimetry equations. For phase changes, remember that Δm in Equation 20.7 is always the change in mass of the higher-phase material. In Equation 20.4, be sure your ΔT is *always* the final temperature minus the initial temperature. In addition, you must *always* include the negative sign on the right side of Equation 20.5.

To understand the role of latent heat in phase changes, consider the energy required to convert a system consisting of a 1.00-g cube of ice at −30.0°C to steam at 120.0°C. Figure 20.3 indicates the experimental results obtained when energy is gradually added to the ice. The results are presented as a graph of temperature of the system versus energy added to the system. Let's examine each portion of the red-brown curve, which is divided into parts A through E.

Part A. On this portion of the curve, the temperature of the system changes from −30.0°C to 0.0°C. Equation 20.4 indicates that the temperature varies linearly with the energy added, so the experimental result is a straight line on the graph. Because the specific heat of ice is 2 090 J/kg · °C, we can calculate the amount of energy added by using Equation 20.4:

$$Q = m_i c_i \,\Delta T = (1.00 \times 10^{-3}\text{ kg})(2\,090\text{ J/kg}\cdot{}^\circ\text{C})(30.0^\circ\text{C}) = 62.7\text{ J}$$

Part B. When the temperature of the system reaches 0.0°C, the ice–water mixture remains at this temperature—even though energy is being added—until all the ice melts. The energy required to melt 1.00 g of ice at 0.0°C is, from Equation 20.7,

$$Q = L_f\,\Delta m_w = L_f m_i = (3.33 \times 10^5\text{ J/kg})(1.00 \times 10^{-3}\text{ kg}) = 333\text{ J}$$

[4]When a gas cools, it eventually *condenses;* that is, it returns to the liquid phase. The energy given up per unit mass is called the *latent heat of condensation* and is numerically equal to the latent heat of vaporization. Likewise, when a liquid cools, it eventually solidifies, and the *latent heat of solidification* is numerically equal to the latent heat of fusion.

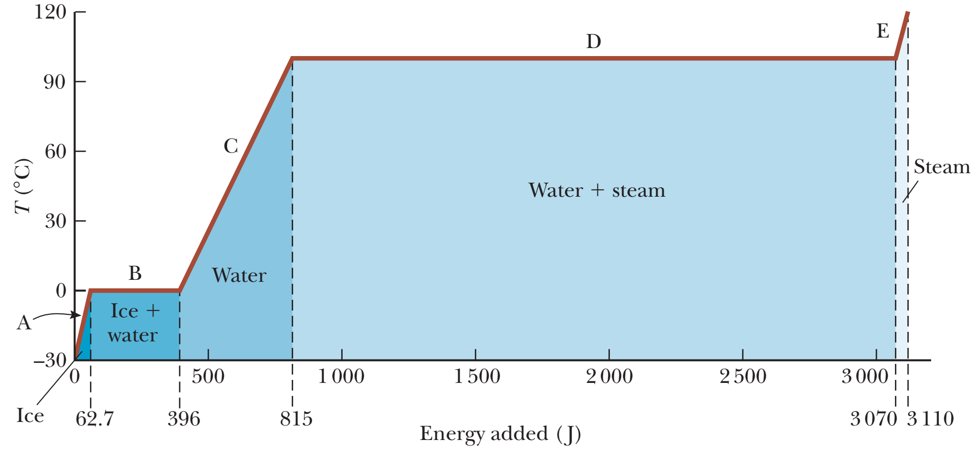

Figure 20.3 A plot of temperature versus energy added when a system initially consisting of 1.00 g of ice at −30.0°C is converted to steam at 120.0°C.

At this point, we have moved to the 396 J (= 62.7 J + 333 J) mark on the energy axis in Figure 20.3.

Part C. Between 0.0°C and 100.0°C, nothing surprising happens. No phase change occurs, and so all energy added to the system, which is now water, is used to increase its temperature. The amount of energy necessary to increase the temperature from 0.0°C to 100.0°C is

$$Q = m_w c_w \, \Delta T = (1.00 \times 10^{-3}\text{ kg})(4.19 \times 10^3\text{ J/kg}\cdot{}^\circ\text{C})(100.0^\circ\text{C}) = 419\text{ J}$$

where m_w is the mass of the water in the system, which is the same as the mass m_i of the original ice.

Part D. At 100.0°C, another phase change occurs as the system changes from water at 100.0°C to steam at 100.0°C. Similar to the ice–water mixture in part B, the water–steam mixture remains at 100.0°C—even though energy is being added—until all the liquid has been converted to steam. The energy required to convert 1.00 g of water to steam at 100.0°C is

$$Q = L_v \, \Delta m_s = L_v m_w = (2.26 \times 10^6\text{ J/kg})(1.00 \times 10^{-3}\text{ kg}) = 2.26 \times 10^3\text{ J}$$

Part E. On this portion of the curve, as in parts A and C, no phase change occurs; therefore, all energy added is used to increase the temperature of the system, which is now steam. The energy that must be added to raise the temperature of the steam from 100.0°C to 120.0°C is

$$Q = m_s c_s \, \Delta T = (1.00 \times 10^{-3}\text{ kg})(2.01 \times 10^3\text{ J/kg}\cdot{}^\circ\text{C})(20.0^\circ\text{C}) = 40.2\text{ J}$$

The total amount of energy that must be added to the system to change 1 g of ice at −30.0°C to steam at 120.0°C is the sum of the results from all five parts of the curve, which is 3.11×10^3 J. Conversely, to cool 1 g of steam at 120.0°C to ice at −30.0°C, we must remove 3.11×10^3 J of energy.

Notice in Figure 20.3 the relatively large amount of energy that is transferred into the water to vaporize it to steam. Imagine reversing this process, with a large amount of energy transferred out of steam to condense it into water. That is why a burn to your skin from steam at 100°C is much more damaging than exposure of your skin to water at 100°C. A very large amount of energy enters your skin from the steam, and the steam remains at 100°C for a long time while it condenses. Conversely, when your skin makes contact with water at 100°C, the water immediately begins to drop in temperature as energy transfers from the water to your skin.

If liquid water is held perfectly still in a very clean container, it is possible for the water to drop below 0°C without freezing into ice. This phenomenon, called **supercooling,** arises because the water requires a disturbance of some sort for the molecules to move apart and start forming the large, open ice structure that makes the density of ice lower than that of water as discussed in Section 19.4. If supercooled water is disturbed, it suddenly freezes. The system drops into the lower-energy configuration of bound molecules of the ice structure, and the energy released raises the temperature back to 0°C.

Commercial hand warmers consist of liquid sodium acetate in a sealed plastic pouch. The solution in the pouch is in a stable supercooled state. When a disk in the pouch is clicked by your fingers, the liquid solidifies and the temperature increases, just like the supercooled water just mentioned. In this case, however, the freezing point of the liquid is higher than body temperature, so the pouch feels warm to the touch. To reuse the hand warmer, the pouch must be boiled until the solid liquefies. Then, as it cools, it passes below its freezing point into the supercooled state.

It is also possible to create **superheating.** For example, clean water in a very clean cup placed in a microwave oven can sometimes rise in temperature beyond 100°C without boiling because the formation of a bubble of steam in the water requires scratches in the cup or some type of impurity in the water to serve as a nucleation site. When the cup is removed from the microwave oven, the superheated water can become explosive as bubbles form immediately and the hot water is forced upward out of the cup.

Quick Quiz 20.2 Suppose the same process of adding energy to the ice cube is performed as discussed above, but instead we graph the internal energy of the system as a function of energy input. What would this graph look like?

Example 20.4 Cooling the Steam AM

What mass of steam initially at 130°C is needed to warm 200 g of water in a 100-g glass container from 20.0°C to 50.0°C?

SOLUTION

Conceptualize Imagine placing water and steam together in a closed insulated container. The system eventually reaches a uniform state of water with a final temperature of 50.0°C.

Categorize Based on our conceptualization of this situation, we categorize this example as one involving calorimetry in which a phase change occurs. The calorimeter is an *isolated system* for *energy:* energy transfers between the components of the system but does not cross the boundary between the system and the environment.

Analyze Write Equation 20.5 to describe the calorimetry process:

$$(1)\quad Q_{\text{cold}} = -Q_{\text{hot}}$$

The steam undergoes three processes: first a decrease in temperature to 100°C, then condensation into liquid water, and finally a decrease in temperature of the water to 50.0°C. Find the energy transfer in the first process using the unknown mass m_s of the steam:

$$Q_1 = m_s c_s \,\Delta T_s$$

Find the energy transfer in the second process:

$$Q_2 = L_v \,\Delta m_s = L_v(0 - m_s) = -m_s L_v$$

Find the energy transfer in the third process:

$$Q_3 = m_s c_w \,\Delta T_{\text{hot water}}$$

Add the energy transfers in these three stages:

$$(2)\quad Q_{\text{hot}} = Q_1 + Q_2 + Q_3 = m_s(c_s\,\Delta T_s - L_v + c_w\,\Delta T_{\text{hot water}})$$

The 20.0°C water and the glass undergo only one process, an increase in temperature to 50.0°C. Find the energy transfer in this process:

$$(3)\quad Q_{\text{cold}} = m_w c_w \,\Delta T_{\text{cold water}} + m_g c_g \,\Delta T_{\text{glass}}$$

Substitute Equations (2) and (3) into Equation (1):

$$m_w c_w\,\Delta T_{\text{cold water}} + m_g c_g\,\Delta T_{\text{glass}} = -m_s(c_s\,\Delta T_s - L_v + c_w\,\Delta T_{\text{hot water}})$$

Solve for m_s:

$$m_s = -\frac{m_w c_w\,\Delta T_{\text{cold water}} + m_g c_g\,\Delta T_{\text{glass}}}{c_s\,\Delta T_s - L_v + c_w\,\Delta T_{\text{hot water}}}$$

20.4 continued

Substitute numerical values:

$$m_s = -\frac{(0.200\text{ kg})(4\,186\text{ J/kg}\cdot°\text{C})(50.0°\text{C} - 20.0°\text{C}) + (0.100\text{ kg})(837\text{ J/kg}\cdot°\text{C})(50.0°\text{C} - 20.0°\text{C})}{(2\,010\text{ J/kg}\cdot°\text{C})(100°\text{C} - 130°\text{C}) - (2.26 \times 10^6\text{ J/kg}) + (4\,186\text{ J/kg}\cdot°\text{C})(50.0°\text{C} - 100°\text{C})}$$

$$= 1.09 \times 10^{-2}\text{ kg} = \boxed{10.9\text{ g}}$$

WHAT IF? What if the final state of the system is water at 100°C? Would we need more steam or less steam? How would the analysis above change?

Answer More steam would be needed to raise the temperature of the water and glass to 100°C instead of 50.0°C. There would be two major changes in the analysis. First, we would not have a term Q_3 for the steam because the water that condenses from the steam does not cool below 100°C. Second, in Q_{cold}, the temperature change would be 80.0°C instead of 30.0°C. For practice, show that the result is a required mass of steam of 31.8 g.

20.4 Work and Heat in Thermodynamic Processes

In thermodynamics, we describe the *state* of a system using such variables as pressure, volume, temperature, and internal energy. As a result, these quantities belong to a category called **state variables.** For any given configuration of the system, we can identify values of the state variables. (For mechanical systems, the state variables include kinetic energy K and potential energy U.) A state of a system can be specified only if the system is in thermal equilibrium internally. In the case of a gas in a container, internal thermal equilibrium requires that every part of the gas be at the same pressure and temperature.

A second category of variables in situations involving energy is **transfer variables.** These variables are those that appear on the right side of the conservation of energy equation, Equation 8.2. Such a variable has a nonzero value if a process occurs in which energy is transferred across the system's boundary. The transfer variable is positive or negative, depending on whether energy is entering or leaving the system. Because a transfer of energy across the boundary represents a change in the system, transfer variables are not associated with a given state of the system, but rather with a *change* in the state of the system.

In the previous sections, we discussed heat as a transfer variable. In this section, we study another important transfer variable for thermodynamic systems, work. Work performed on particles was studied extensively in Chapter 7, and here we investigate the work done on a deformable system, a gas. Consider a gas contained in a cylinder fitted with a movable piston (Fig. 20.4). At equilibrium, the gas occupies a volume V and exerts a uniform pressure P on the cylinder's walls and on the piston. If the piston has a cross-sectional area A, the magnitude of the force exerted by the gas on the piston is $F = PA$. By Newton's third law, the magnitude of the force exerted by the piston on the gas is also PA. Now let's assume we push the piston inward and compress the gas **quasi-statically,** that is, slowly enough to allow the system to remain essentially in internal thermal equilibrium at all times. The point of application of the force on the gas is the bottom face of the piston. As the piston is pushed downward by an external force $\vec{\mathbf{F}} = -F\hat{\mathbf{j}}$ through a displacement of $d\vec{\mathbf{r}} = dy\,\hat{\mathbf{j}}$ (Fig. 20.4b), the work done on the gas is, according to our definition of work in Chapter 7,

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -F\hat{\mathbf{j}} \cdot dy\,\hat{\mathbf{j}} = -F\,dy = -PA\,dy$$

The mass of the piston is assumed to be negligible in this discussion. Because $A\,dy$ is the change in volume of the gas dV, we can express the work done on the gas as

$$dW = -P\,dV \qquad \textbf{(20.8)}$$

If the gas is compressed, dV is negative and the work done on the gas is positive. If the gas expands, dV is positive and the work done on the gas is negative. If the

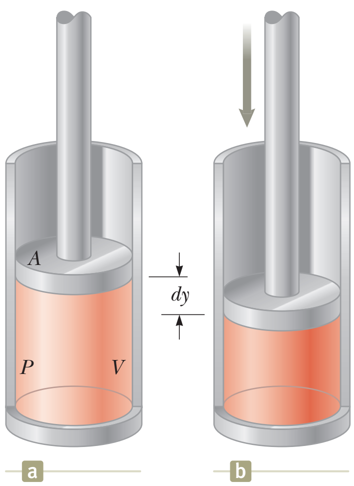

Figure 20.4 Work is done on a gas contained in a cylinder at a pressure P as the piston is pushed downward so that the gas is compressed.

volume remains constant, the work done on the gas is zero. The total work done on the gas as its volume changes from V_i to V_f is given by the integral of Equation 20.8:

Work done on a gas ▶

$$W = -\int_{V_i}^{V_f} P\,dV \tag{20.9}$$

To evaluate this integral, you must know how the pressure varies with volume during the process.

In general, the pressure is not constant during a process followed by a gas, but depends on the volume and temperature. If the pressure and volume are known at each step of the process, the state of the gas at each step can be plotted on an important graphical representation called a ***PV* diagram** as in Figure 20.5. This type of diagram allows us to visualize a process through which a gas is progressing. The curve on a *PV* diagram is called the *path* taken between the initial and final states.

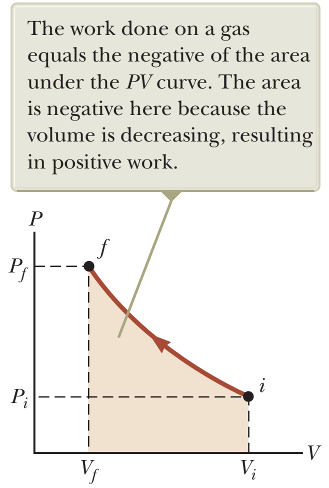

Figure 20.5 A gas is compressed quasi-statically (slowly) from state i to state f. An outside agent must do positive work on the gas to compress it.

Notice that the integral in Equation 20.9 is equal to the area under a curve on a *PV* diagram. Therefore, we can identify an important use for *PV* diagrams:

> The work done on a gas in a quasi-static process that takes the gas from an initial state to a final state is the negative of the area under the curve on a *PV* diagram, evaluated between the initial and final states.

For the process of compressing a gas in a cylinder, the work done depends on the particular path taken between the initial and final states as Figure 20.5 suggests. To illustrate this important point, consider several different paths connecting i and f (Fig. 20.6). In the process depicted in Figure 20.6a, the volume of the gas is first reduced from V_i to V_f at constant pressure P_i and the pressure of the gas then increases from P_i to P_f by heating at constant volume V_f. The work done on the gas along this path is $-P_i(V_f - V_i)$. In Figure 20.6b, the pressure of the gas is increased from P_i to P_f at constant volume V_i and then the volume of the gas is reduced from V_i to V_f at constant pressure P_f. The work done on the gas is $-P_f(V_f - V_i)$. This value is greater than that for the process described in Figure 20.6a because the piston is moved through the same displacement by a larger force. Finally, for the process described in Figure 20.6c, where both P and V change continuously, the work done on the gas has some value between the values obtained in the first two processes. To evaluate the work in this case, the function $P(V)$ must be known so that we can evaluate the integral in Equation 20.9.

The energy transfer Q into or out of a system by heat also depends on the process. Consider the situations depicted in Figure 20.7. In each case, the gas has the same initial volume, temperature, and pressure, and is assumed to be ideal. In Figure 20.7a, the gas is thermally insulated from its surroundings except at the bottom of the gas-filled region, where it is in thermal contact with an energy reservoir. An *energy reservoir* is a source of energy that is considered to be so great that a finite transfer of energy to or from the reservoir does not change its temperature. The piston is held

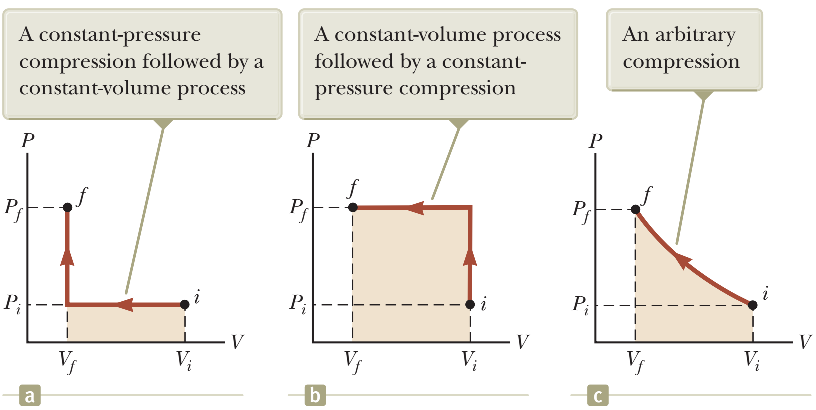

Figure 20.6 The work done on a gas as it is taken from an initial state to a final state depends on the path between these states.

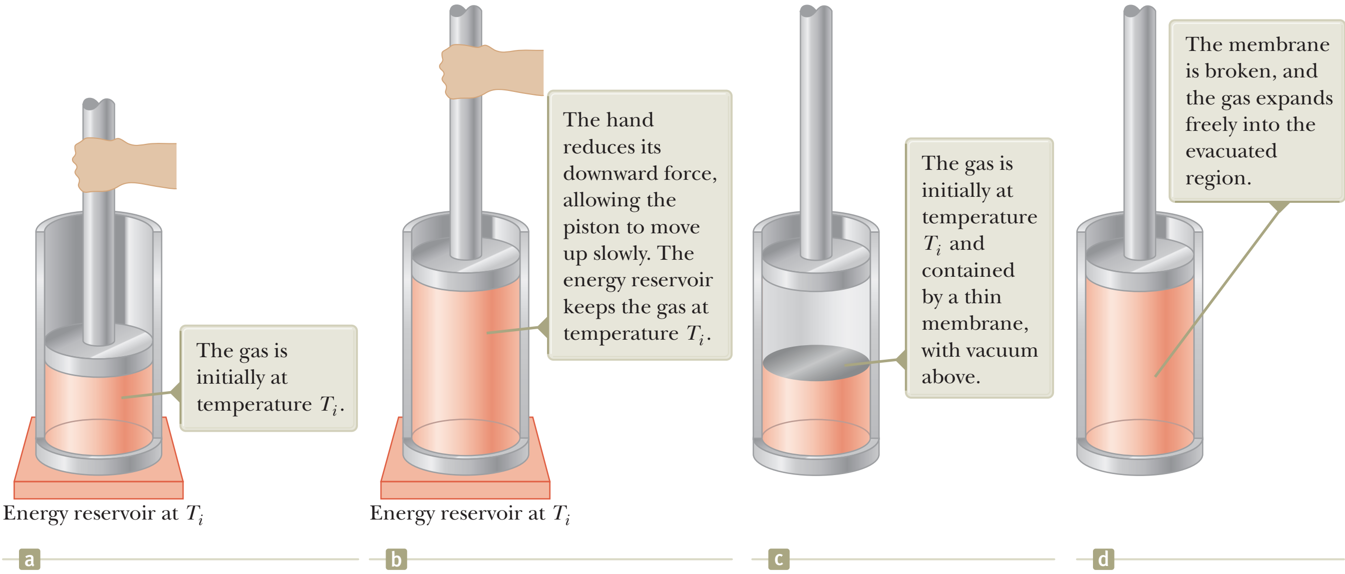

Figure 20.7 Gas in a cylinder. (a) The gas is in contact with an energy reservoir. The walls of the cylinder are perfectly insulating, but the base in contact with the reservoir is conducting. (b) The gas expands slowly to a larger volume. (c) The gas is contained by a membrane in half of a volume, with vacuum in the other half. The entire cylinder is perfectly insulating. (d) The gas expands freely into the larger volume.

at its initial position by an external agent such as a hand. When the force holding the piston is reduced slightly, the piston rises very slowly to its final position shown in Figure 20.7b. Because the piston is moving upward, the gas is doing work on the piston. During this expansion to the final volume V_f, just enough energy is transferred by heat from the reservoir to the gas to maintain a constant temperature T_i.

Now consider the completely thermally insulated system shown in Figure 20.7c. When the membrane is broken, the gas expands rapidly into the vacuum until it occupies a volume V_f and is at a pressure P_f. The final state of the gas is shown in Figure 20.7d. In this case, the gas does no work because it does not apply a force; no force is required to expand into a vacuum. Furthermore, no energy is transferred by heat through the insulating wall.

As we discuss in Section 20.5, experiments show that the temperature of the ideal gas does not change in the process indicated in Figures 20.7c and 20.7d. Therefore, the initial and final states of the ideal gas in Figures 20.7a and 20.7b are identical to the initial and final states in Figures 20.7c and 20.7d, but the paths are different. In the first case, the gas does work on the piston and energy is transferred slowly to the gas by heat. In the second case, no energy is transferred by heat and the value of the work done is zero. Therefore, energy transfer by heat, like work done, depends on the particular process occurring in the system. In other words, because heat and work both depend on the path followed on a *PV* diagram between the initial and final states, neither quantity is determined solely by the endpoints of a thermodynamic process.

20.5 The First Law of Thermodynamics

When we introduced the law of conservation of energy in Chapter 8, we stated that the change in the energy of a system is equal to the sum of all transfers of energy across the system's boundary (Eq. 8.2). The **first law of thermodynamics** is a special case of the law of conservation of energy that describes processes in which only the internal energy[5] changes and the only energy transfers are by heat and work:

$$\Delta E_{\text{int}} = Q + W \tag{20.10}$$

◀ **First law of thermodynamics**

[5]It is an unfortunate accident of history that the traditional symbol for internal energy is U, which is also the traditional symbol for potential energy as introduced in Chapter 7. To avoid confusion between potential energy and internal energy, we use the symbol E_{int} for internal energy in this book. If you take an advanced course in thermodynamics, however, be prepared to see U used as the symbol for internal energy in the first law.

Pitfall Prevention 20.7

Dual Sign Conventions Some physics and engineering books present the first law as $\Delta E_{\text{int}} = Q - W$, with a minus sign between the heat and work. The reason is that work is defined in these treatments as the work done *by* the gas rather than *on* the gas, as in our treatment. The equivalent equation to Equation 20.9 in these treatments defines work as $W = \int_{V_i}^{V_f} P\,dV$. Therefore, if positive work is done by the gas, energy is leaving the system, leading to the negative sign in the first law.

In your studies in other chemistry or engineering courses, or in your reading of other physics books, be sure to note which sign convention is being used for the first law.

Pitfall Prevention 20.8

The First Law With our approach to energy in this book, the first law of thermodynamics is a special case of Equation 8.2. Some physicists argue that the first law is the general equation for energy conservation, equivalent to Equation 8.2. In this approach, the first law is applied to a closed system (so that there is no matter transfer), heat is interpreted so as to include electromagnetic radiation, and work is interpreted so as to include electrical transmission ("electrical work") and mechanical waves ("molecular work"). Keep that in mind if you run across the first law in your reading of other physics books.

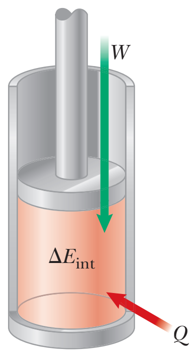

Figure 20.8 The first law of thermodynamics equates the change in internal energy E_{int} in a system to the net energy transfer to the system by heat Q and work W. In the situation shown here, the internal energy of the gas increases.

Look back at Equation 8.2 to see that the first law of thermodynamics is contained within that more general equation.

Let us investigate some special cases in which the first law can be applied. First, consider an *isolated system,* that is, one that does not interact with its surroundings, as we have seen before. In this case, no energy transfer by heat takes place and the work done on the system is zero; hence, the internal energy remains constant. That is, because $Q = W = 0$, it follows that $\Delta E_{\text{int}} = 0$; therefore, $E_{\text{int},i} = E_{\text{int},f}$. We conclude that the internal energy E_{int} of an isolated system remains constant.

Next, consider the case of a system that can exchange energy with its surroundings and is taken through a **cyclic process,** that is, a process that starts and ends at the same state. In this case, the change in the internal energy must again be zero because E_{int} is a state variable; therefore, the energy Q added to the system must equal the negative of the work W done on the system during the cycle. That is, in a cyclic process,

$$\Delta E_{\text{int}} = 0 \quad \text{and} \quad Q = -W \quad \text{(cyclic process)}$$

On a PV diagram for a gas, a cyclic process appears as a closed curve. (The processes described in Figure 20.6 are represented by open curves because the initial and final states differ.) It can be shown that in a cyclic process for a gas, the net work done on the system per cycle equals the area enclosed by the path representing the process on a PV diagram.

20.6 Some Applications of the First Law of Thermodynamics

In this section, we consider additional applications of the first law to processes through which a gas is taken. As a model, let's consider the sample of gas contained in the piston–cylinder apparatus in Figure 20.8. This figure shows work being done on the gas and energy transferring in by heat, so the internal energy of the gas is rising. In the following discussion of various processes, refer back to this figure and mentally alter the directions of the transfer of energy to reflect what is happening in the process.

Before we apply the first law of thermodynamics to specific systems, it is useful to first define some idealized thermodynamic processes. An **adiabatic process** is one during which no energy enters or leaves the system by heat; that is, $Q = 0$. An adiabatic process can be achieved either by thermally insulating the walls of the system or by performing the process rapidly so that there is negligible time for energy to transfer by heat. Applying the first law of thermodynamics to an adiabatic process gives

$$\Delta E_{\text{int}} = W \quad \text{(adiabatic process)} \tag{20.11}$$

This result shows that if a gas is compressed adiabatically such that W is positive, then ΔE_{int} is positive and the temperature of the gas increases. Conversely, the temperature of a gas decreases when the gas expands adiabatically.

Adiabatic processes are very important in engineering practice. Some common examples are the expansion of hot gases in an internal combustion engine, the liquefaction of gases in a cooling system, and the compression stroke in a diesel engine.

The process described in Figures 20.7c and 20.7d, called an **adiabatic free expansion,** is unique. The process is adiabatic because it takes place in an insulated container. Because the gas expands into a vacuum, it does not apply a force on a piston as does the gas in Figures 20.7a and 20.7b, so no work is done on or by the gas. Therefore, in this adiabatic process, both $Q = 0$ and $W = 0$. As a result, $\Delta E_{\text{int}} = 0$ for this process as can be seen from the first law. That is, the initial and final internal energies of a gas are equal in an adiabatic free expansion. As we shall see in Chapter 21, the internal energy of an ideal gas depends only on its temperature. Therefore, we expect no change in temperature during an adiabatic free expansion. This prediction is in accord with the results of experiments performed at low pressures. (Experiments performed at high pressures for real gases show a slight change in temperature after the expansion due to intermolecular interactions, which represent a deviation from the model of an ideal gas.)

A process that occurs at constant pressure is called an **isobaric process.** In Figure 20.8, an isobaric process could be established by allowing the piston to move freely so that it is always in equilibrium between the net force from the gas pushing upward and the weight of the piston plus the force due to atmospheric pressure pushing downward. The first process in Figure 20.6a and the second process in Figure 20.6b are both isobaric.

In such a process, the values of the heat and the work are both usually nonzero. The work done on the gas in an isobaric process is simply

$$W = -P(V_f - V_i) \quad \text{(isobaric process)} \tag{20.12}$$

◀ **Isobaric process**

where P is the constant pressure of the gas during the process.

A process that takes place at constant volume is called an **isovolumetric process.** In Figure 20.8, clamping the piston at a fixed position would ensure an isovolumetric process. The second process in Figure 20.6a and the first process in Figure 20.6b are both isovolumetric.

Because the volume of the gas does not change in such a process, the work given by Equation 20.9 is zero. Hence, from the first law we see that in an isovolumetric process, because $W = 0$,

$$\Delta E_{\text{int}} = Q \quad \text{(isovolumetric process)} \tag{20.13}$$

◀ **Isovolumetric process**

This expression specifies that if energy is added by heat to a system kept at constant volume, all the transferred energy remains in the system as an increase in its internal energy. For example, when a can of spray paint is thrown into a fire, energy enters the system (the gas in the can) by heat through the metal walls of the can. Consequently, the temperature, and therefore the pressure, in the can increases until the can possibly explodes.

◀ **Isothermal process**

A process that occurs at constant temperature is called an **isothermal process.** This process can be established by immersing the cylinder in Figure 20.8 in an ice–water bath or by putting the cylinder in contact with some other constant-temperature reservoir. A plot of P versus V at constant temperature for an ideal gas yields a hyperbolic curve called an *isotherm*. The internal energy of an ideal gas is a function of temperature only. Hence, because the temperature does not change in an isothermal process involving an ideal gas, we must have $\Delta E_{\text{int}} = 0$. For an isothermal process, we conclude from the first law that the energy transfer Q must be equal to the negative of the work done on the gas; that is, $Q = -W$. Any energy that enters the system by heat is transferred out of the system by work; as a result, no change in the internal energy of the system occurs in an isothermal process.

Pitfall Prevention 20.9

$Q \neq 0$ in an Isothermal Process

Do not fall into the common trap of thinking there must be no transfer of energy by heat if the temperature does not change as is the case in an isothermal process. Because the cause of temperature change can be either heat *or* work, the temperature can remain constant even if energy enters the gas by heat, which can only happen if the energy entering the gas by heat leaves by work.

Quick Quiz 20.3 In the last three columns of the following table, fill in the boxes with the correct signs (−, +, or 0) for Q, W, and ΔE_{int}. For each situation, the system to be considered is identified.

Situation	System	Q	W	ΔE_{int}
(a) Rapidly pumping up a bicycle tire	Air in the pump			
(b) Pan of room-temperature water sitting on a hot stove	Water in the pan			
(c) Air quickly leaking out of a balloon	Air originally in the balloon			

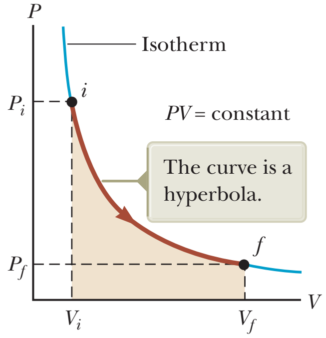

Figure 20.9 The PV diagram for an isothermal expansion of an ideal gas from an initial state to a final state.

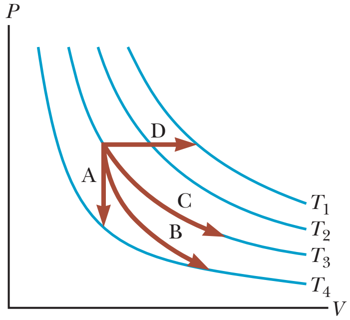

Figure 20.10 (Quick Quiz 20.4) Identify the nature of paths A, B, C, and D.

Isothermal Expansion of an Ideal Gas

Suppose an ideal gas is allowed to expand quasi-statically at constant temperature. This process is described by the PV diagram shown in Figure 20.9. The curve is a hyperbola (see Appendix B, Eq. B.23), and the ideal gas law (Eq. 19.8) with T constant indicates that the equation of this curve is $PV = nRT =$ constant.

Let's calculate the work done on the gas in the expansion from state i to state f. The work done on the gas is given by Equation 20.9. Because the gas is ideal and the process is quasi-static, the ideal gas law is valid for each point on the path. Therefore,

$$W = -\int_{V_i}^{V_f} P\,dV = -\int_{V_i}^{V_f} \frac{nRT}{V}\,dV$$

Because T is constant in this case, it can be removed from the integral along with n and R:

$$W = -nRT\int_{V_i}^{V_f} \frac{dV}{V} = -nRT\ln V\Big|_{V_i}^{V_f}$$

To evaluate the integral, we used $\int(dx/x) = \ln x$. (See Appendix B.) Evaluating the result at the initial and final volumes gives

$$\boxed{W = nRT\ln\left(\frac{V_i}{V_f}\right)} \qquad \textbf{(20.14)}$$

Numerically, this work W equals the negative of the shaded area under the PV curve shown in Figure 20.9. Because the gas expands, $V_f > V_i$ and the value for the work done on the gas is negative as we expect. If the gas is compressed, then $V_f < V_i$ and the work done on the gas is positive.

Quick Quiz 20.4 Characterize the paths in Figure 20.10 as isobaric, isovolumetric, isothermal, or adiabatic. For path B, $Q = 0$. The blue curves are isotherms.

Example 20.5 An Isothermal Expansion

A 1.0-mol sample of an ideal gas is kept at 0.0°C during an expansion from 3.0 L to 10.0 L.

(A) How much work is done on the gas during the expansion?

SOLUTION

Conceptualize Run the process in your mind: the cylinder in Figure 20.8 is immersed in an ice-water bath, and the piston moves outward so that the volume of the gas increases. You can also use the graphical representation in Figure 20.9 to conceptualize the process.

Categorize We will evaluate parameters using equations developed in the preceding sections, so we categorize this example as a substitution problem. Because the temperature of the gas is fixed, the process is isothermal.

Substitute the given values into Equation 20.14:

$$W = nRT\ln\left(\frac{V_i}{V_f}\right)$$

$$= (1.0\text{ mol})(8.31\text{ J/mol}\cdot\text{K})(273\text{ K})\ln\left(\frac{3.0\text{ L}}{10.0\text{ L}}\right)$$

$$= \boxed{-2.7\times10^{3}\text{ J}}$$

(B) How much energy transfer by heat occurs between the gas and its surroundings in this process?

SOLUTION

Find the heat from the first law:

$$\Delta E_{\text{int}} = Q + W$$

$$0 = Q + W$$

$$Q = -W = \boxed{2.7\times10^{3}\text{ J}}$$

20.5 continued

(C) If the gas is returned to the original volume by means of an isobaric process, how much work is done on the gas?

SOLUTION

Use Equation 20.12. The pressure is not given, so incorporate the ideal gas law:

$$W = -P(V_f - V_i) = -\frac{nRT_i}{V_i}(V_f - V_i)$$

$$= -\frac{(1.0 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})(273 \text{ K})}{10.0 \times 10^{-3} \text{ m}^3}(3.0 \times 10^{-3} \text{ m}^3 - 10.0 \times 10^{-3} \text{ m}^3)$$

$$= \boxed{1.6 \times 10^3 \text{ J}}$$

We used the initial temperature and volume to calculate the work done because the final temperature was unknown. The work done on the gas is positive because the gas is being compressed.

Example 20.6 Boiling Water

Suppose 1.00 g of water vaporizes isobarically at atmospheric pressure (1.013×10^5 Pa). Its volume in the liquid state is $V_i = V_{\text{liquid}} = 1.00 \text{ cm}^3$, and its volume in the vapor state is $V_f = V_{\text{vapor}} = 1\,671 \text{ cm}^3$. Find the work done in the expansion and the change in internal energy of the system. Ignore any mixing of the steam and the surrounding air; imagine that the steam simply pushes the surrounding air out of the way.

SOLUTION

Conceptualize Notice that the temperature of the system does not change. There is a phase change occurring as the water evaporates to steam.

Categorize Because the expansion takes place at constant pressure, we categorize the process as isobaric. We will use equations developed in the preceding sections, so we categorize this example as a substitution problem.

Use Equation 20.12 to find the work done on the system as the air is pushed out of the way:

$$W = -P(V_f - V_i)$$

$$= -(1.013 \times 10^5 \text{ Pa})(1\,671 \times 10^{-6} \text{ m}^3 - 1.00 \times 10^{-6} \text{ m}^3)$$

$$= \boxed{-169 \text{ J}}$$

Use Equation 20.7 and the latent heat of vaporization for water to find the energy transferred into the system by heat:

$$Q = L_v \Delta m_s = m_s L_v = (1.00 \times 10^{-3} \text{ kg})(2.26 \times 10^6 \text{ J/kg})$$

$$= 2\,260 \text{ J}$$

Use the first law to find the change in internal energy of the system:

$$\Delta E_{\text{int}} = Q + W = 2\,260 \text{ J} + (-169 \text{ J}) = \boxed{2.09 \text{ kJ}}$$

The positive value for ΔE_{int} indicates that the internal energy of the system increases. The largest fraction of the energy (2 090 J/ 2260 J = 93%) transferred to the liquid goes into increasing the internal energy of the system. The remaining 7% of the energy transferred leaves the system by work done by the steam on the surrounding atmosphere.

Example 20.7 Heating a Solid

A 1.0-kg bar of copper is heated at atmospheric pressure so that its temperature increases from 20°C to 50°C.

(A) What is the work done on the copper bar by the surrounding atmosphere?

SOLUTION

Conceptualize This example involves a solid, whereas the preceding two examples involved liquids and gases. For a solid, the change in volume due to thermal expansion is very small.

continued

20.7 continued

Categorize Because the expansion takes place at constant atmospheric pressure, we categorize the process as isobaric.

Analyze Find the work done on the copper bar using Equation 20.12:

$$W = -P\,\Delta V$$

Express the change in volume using Equation 19.6 and that $\beta = 3\alpha$:

$$W = -P(\beta V_i\,\Delta T) = -P(3\alpha V_i\,\Delta T) = -3\alpha P V_i\,\Delta T$$

Substitute for the volume in terms of the mass and density of the copper:

$$W = -3\alpha P\left(\frac{m}{\rho}\right)\Delta T$$

Substitute numerical values:

$$W = -3[1.7 \times 10^{-5}\,(°\text{C})^{-1}](1.013 \times 10^5\ \text{N/m}^2)\left(\frac{1.0\ \text{kg}}{8.92 \times 10^3\ \text{kg/m}^3}\right)(50°\text{C} - 20°\text{C})$$

$$= \boxed{-1.7 \times 10^{-2}\ \text{J}}$$

Because this work is negative, work is done *by* the copper bar on the atmosphere.

(B) How much energy is transferred to the copper bar by heat?

SOLUTION

Use Equation 20.4 and the specific heat of copper from Table 20.1:

$$Q = mc\,\Delta T = (1.0\ \text{kg})(387\ \text{J/kg}\cdot°\text{C})(50°\text{C} - 20°\text{C})$$
$$= \boxed{1.2 \times 10^4\ \text{J}}$$

(C) What is the increase in internal energy of the copper bar?

SOLUTION

Use the first law of thermodynamics:

$$\Delta E_{\text{int}} = Q + W = 1.2 \times 10^4\ \text{J} + (-1.7 \times 10^{-2}\ \text{J})$$
$$= \boxed{1.2 \times 10^4\ \text{J}}$$

Finalize Most of the energy transferred into the system by heat goes into increasing the internal energy of the copper bar. The fraction of energy used to do work on the surrounding atmosphere is only about 10^{-6}. Hence, when the thermal expansion of a solid or a liquid is analyzed, the small amount of work done on or by the system is usually ignored.

20.7 Energy Transfer Mechanisms in Thermal Processes

In Chapter 8, we introduced a global approach to the energy analysis of physical processes through Equation 8.1, $\Delta E_{\text{system}} = \Sigma\,T$, where T represents energy transfer, which can occur by several mechanisms. Earlier in this chapter, we discussed two of the terms on the right side of this equation, work W and heat Q. In this section, we explore more details about heat as a means of energy transfer and two other energy transfer methods often related to temperature changes: convection (a form of matter transfer T_{MT}) and electromagnetic radiation T_{ER}.

Thermal Conduction

The process of energy transfer by heat (Q in Eq. 8.2) can also be called **conduction** or **thermal conduction.** In this process, the transfer can be represented on an atomic scale as an exchange of kinetic energy between microscopic particles—molecules, atoms, and free electrons—in which less-energetic particles gain energy in collisions with more-energetic particles. For example, if you hold one end of a long metal bar and insert the other end into a flame, you will find that the temperature of the metal in your hand soon increases. The energy reaches your hand by means of conduction. Initially, before the rod is inserted into the flame, the microscopic particles in the metal are vibrating about their equilibrium positions. As the flame raises the temperature of the rod, the particles near the flame begin to vibrate with greater and greater amplitudes. These particles, in turn, collide with their neighbors and transfer some of their energy in the collisions. Slowly, the amplitudes of vibration of metal atoms and electrons farther and farther from the flame increase until eventually those in the metal near your hand are affected. This increased vibration is detected by an increase in the temperature of the metal and of your potentially burned hand.

The rate of thermal conduction depends on the properties of the substance being heated. For example, it is possible to hold a piece of asbestos in a flame indefinitely, which implies that very little energy is conducted through the asbestos. In general, metals are good thermal conductors and materials such as asbestos, cork, paper, and fiberglass are poor conductors. Gases also are poor conductors because the separation distance between the particles is so great. Metals are good thermal conductors because they contain large numbers of electrons that are relatively free to move through the metal and so can transport energy over large distances. Therefore, in a good conductor such as copper, conduction takes place by means of both the vibration of atoms and the motion of free electrons.

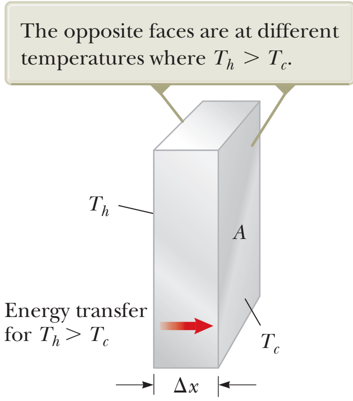

Figure 20.11 Energy transfer through a conducting slab with a cross-sectional area A and a thickness Δx.

Conduction occurs only if there is a difference in temperature between two parts of the conducting medium. Consider a slab of material of thickness Δx and cross-sectional area A. One face of the slab is at a temperature T_c, and the other face is at a temperature $T_h > T_c$ (Fig. 20.11). Experimentally, it is found that energy Q transfers in a time interval Δt from the hotter face to the colder one. The rate $P = Q/\Delta t$ at which this energy transfer occurs is found to be proportional to the cross-sectional area and the temperature difference $\Delta T = T_h - T_c$ and inversely proportional to the thickness:

$$P = \frac{Q}{\Delta t} \propto A\frac{\Delta T}{\Delta x}$$

Notice that P has units of watts when Q is in joules and Δt is in seconds. That is not surprising because P is power, the rate of energy transfer by heat. For a slab of infinitesimal thickness dx and temperature difference dT, we can write the **law of thermal conduction** as

$$P = kA\left|\frac{dT}{dx}\right| \qquad \textbf{(20.15)}$$

where the proportionality constant k is the **thermal conductivity** of the material and $|dT/dx|$ is the **temperature gradient** (the rate at which temperature varies with position).

Substances that are good thermal conductors have large thermal conductivity values, whereas good thermal insulators have low thermal conductivity values. Table 20.3 lists thermal conductivities for various substances. Notice that metals are generally better thermal conductors than nonmetals.

Suppose a long, uniform rod of length L is thermally insulated so that energy cannot escape by heat from its surface except at the ends as shown in Figure 20.12 (page 610). One end is in thermal contact with an energy reservoir at temperature T_c, and the other end is in thermal contact with a reservoir at temperature $T_h > T_c$. When a steady state has been reached, the temperature at each point along the rod is constant in time. In this case, if we assume k is not a function of temperature, the temperature gradient is the same everywhere along the rod and is

$$\left|\frac{dT}{dx}\right| = \frac{T_h - T_c}{L}$$

Table 20.3

Thermal Conductivities

Substance	Thermal Conductivity (W/m · °C)
Metals (at 25°C)	
Aluminum	238
Copper	397
Gold	314
Iron	79.5
Lead	34.7
Silver	427
Nonmetals (approximate values)	
Asbestos	0.08
Concrete	0.8
Diamond	2 300
Glass	0.8
Ice	2
Rubber	0.2
Water	0.6
Wood	0.08
Gases (at 20°C)	
Air	0.023 4
Helium	0.138
Hydrogen	0.172
Nitrogen	0.023 4
Oxygen	0.023 8

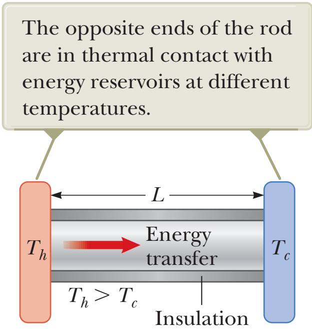

Figure 20.12 Conduction of energy through a uniform, insulated rod of length L.

Therefore, the rate of energy transfer by conduction through the rod is

$$P = kA\left(\frac{T_h - T_c}{L}\right) \quad \textbf{(20.16)}$$

For a compound slab containing several materials of thicknesses $L_1, L_2, \ldots$ and thermal conductivities $k_1, k_2, \ldots$, the rate of energy transfer through the slab at steady state is

$$P = \frac{A(T_h - T_c)}{\sum_i (L_i/k_i)} \quad \textbf{(20.17)}$$

where T_h and T_c are the temperatures of the outer surfaces (which are held constant) and the summation is over all slabs. Example 20.8 shows how Equation 20.17 results from a consideration of two thicknesses of materials.

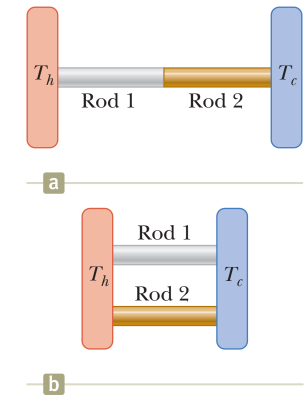

Figure 20.13 (Quick Quiz 20.5) In which case is the rate of energy transfer larger?

Quick Quiz 20.5 You have two rods of the same length and diameter, but they are formed from different materials. The rods are used to connect two regions at different temperatures so that energy transfers through the rods by heat. They can be connected in series as in Figure 20.13a or in parallel as in Figure 20.13b. In which case is the rate of energy transfer by heat larger? **(a)** The rate is larger when the rods are in series. **(b)** The rate is larger when the rods are in parallel. **(c)** The rate is the same in both cases.

Example 20.8 Energy Transfer Through Two Slabs

Two slabs of thickness L_1 and L_2 and thermal conductivities k_1 and k_2 are in thermal contact with each other as shown in Figure 20.14. The temperatures of their outer surfaces are T_c and T_h, respectively, and $T_h > T_c$. Determine the temperature at the interface and the rate of energy transfer by conduction through an area A of the slabs in the steady-state condition.

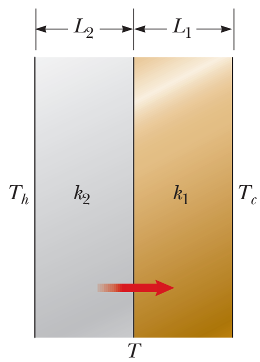

Figure 20.14 (Example 20.8) Energy transfer by conduction through two slabs in thermal contact with each other. At steady state, the rate of energy transfer through slab 1 equals the rate of energy transfer through slab 2.

SOLUTION

Conceptualize Notice the phrase "in the steady-state condition." We interpret this phrase to mean that energy transfers through the compound slab at the same rate at all points. Otherwise, energy would be building up or disappearing at some point. Furthermore, the temperature varies with position in the two slabs, most likely at different rates in each part of the compound slab. When the system is in steady state, the interface is at some fixed temperature T.

Categorize We categorize this example as a thermal conduction problem and impose the condition that the power is the same in both slabs of material.

20.8 continued

Analyze Use Equation 20.16 to express the rate at which energy is transferred through an area A of slab 1:

$$(1) \quad P_1 = k_1 A\left(\frac{T - T_c}{L_1}\right)$$

Express the rate at which energy is transferred through the same area of slab 2:

$$(2) \quad P_2 = k_2 A\left(\frac{T_h - T}{L_2}\right)$$

Set these two rates equal to represent the steady-state situation:

$$k_1 A\left(\frac{T - T_c}{L_1}\right) = k_2 A\left(\frac{T_h - T}{L_2}\right)$$

Solve for T:

$$(3) \quad T = \boxed{\frac{k_1 L_2 T_c + k_2 L_1 T_h}{k_1 L_2 + k_2 L_1}}$$

Substitute Equation (3) into either Equation (1) or Equation (2):

$$(4) \quad P = \boxed{\frac{A(T_h - T_c)}{(L_1/k_1) + (L_2/k_2)}}$$

Finalize Extension of this procedure to several slabs of materials leads to Equation 20.17.

WHAT IF? Suppose you are building an insulated container with two layers of insulation and the rate of energy transfer determined by Equation (4) turns out to be too high. You have enough room to increase the thickness of one of the two layers by 20%. How would you decide which layer to choose?

Answer To decrease the power as much as possible, you must increase the denominator in Equation (4) as much as possible. Whichever thickness you choose to increase, L_1 or L_2, you increase the corresponding term L/k in the denominator by 20%. For this percentage change to represent the largest absolute change, you want to take 20% of the larger term. Therefore, you should increase the thickness of the layer that has the larger value of L/k.

Home Insulation

In engineering practice, the term L/k for a particular substance is referred to as the ***R*-value** of the material. Therefore, Equation 20.17 reduces to

$$P = \frac{A(T_h - T_c)}{\sum_i R_i} \qquad \textbf{(20.18)}$$

where $R_i = L_i/k_i$. The R-values for a few common building materials are given in Table 20.4. In the United States, the insulating properties of materials used in buildings are usually expressed in U.S. customary units, not SI units. Therefore, in

Table 20.4 R-Values for Some Common Building Materials

Material	R-value (ft$^2 \cdot$ °F $\cdot$ h/Btu)
Hardwood siding (1 in. thick)	0.91
Wood shingles (lapped)	0.87
Brick (4 in. thick)	4.00
Concrete block (filled cores)	1.93
Fiberglass insulation (3.5 in. thick)	10.90
Fiberglass insulation (6 in. thick)	18.80
Fiberglass board (1 in. thick)	4.35
Cellulose fiber (1 in. thick)	3.70
Flat glass (0.125 in. thick)	0.89
Insulating glass (0.25-in. space)	1.54
Air space (3.5 in. thick)	1.01
Stagnant air layer	0.17
Drywall (0.5 in. thick)	0.45
Sheathing (0.5 in. thick)	1.32

Table 20.4, R-values are given as a combination of British thermal units, feet, hours, and degrees Fahrenheit.

At any vertical surface open to the air, a very thin stagnant layer of air adheres to the surface. One must consider this layer when determining the R-value for a wall. The thickness of this stagnant layer on an outside wall depends on the speed of the wind. Energy transfer through the walls of a house on a windy day is greater than that on a day when the air is calm. A representative R-value for this stagnant layer of air is given in Table 20.4.

Example 20.9 The *R*-Value of a Typical Wall

Calculate the total R-value for a wall constructed as shown in Figure 20.15a. Starting outside the house (toward the front in the figure) and moving inward, the wall consists of 4 in. of brick, 0.5 in. of sheathing, an air space 3.5 in. thick, and 0.5 in. of drywall.

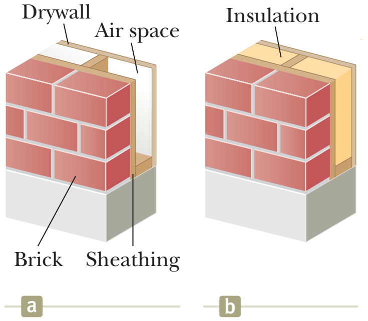

Figure 20.15 (Example 20.9) An exterior house wall containing (a) an air space and (b) insulation.

SOLUTION

Conceptualize Use Figure 20.15 to help conceptualize the structure of the wall. Do not forget the stagnant air layers inside and outside the house.

Categorize We will use specific equations developed in this section on home insulation, so we categorize this example as a substitution problem.

Use Table 20.4 to find the R-value of each layer:

$$R_1 \text{ (outside stagnant air layer)} = 0.17 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$
$$R_2 \text{ (brick)} = 4.00 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$
$$R_3 \text{ (sheathing)} = 1.32 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$
$$R_4 \text{ (air space)} = 1.01 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$
$$R_5 \text{ (drywall)} = 0.45 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$
$$R_6 \text{ (inside stagnant air layer)} = 0.17 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$$

Add the R-values to obtain the total R-value for the wall:

$$R_{\text{total}} = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 = \boxed{7.12 \text{ ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}}$$

WHAT IF? Suppose you are not happy with this total R-value for the wall. You cannot change the overall structure, but you can fill the air space as in Figure 20.15b. To *maximize* the total R-value, what material should you choose to fill the air space?

Answer Looking at Table 20.4, we see that 3.5 in. of fiberglass insulation is more than ten times as effective as 3.5 in. of air. Therefore, we should fill the air space with fiberglass insulation. The result is that we add 10.90 $\text{ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$ of R-value, and we lose 1.01 $\text{ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$ due to the air space we have replaced. The new total R-value is equal to 7.12 $\text{ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$ + 9.89 $\text{ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$ = 17.01 $\text{ft}^2 \cdot {}^\circ\text{F} \cdot \text{h/Btu}$.

Convection

At one time or another, you probably have warmed your hands by holding them over an open flame. In this situation, the air directly above the flame is heated and expands. As a result, the density of this air decreases and the air rises. This hot air warms your hands as it flows by. Energy transferred by the movement of a warm substance is said to have been transferred by **convection,** which is a form of matter transfer, T_{MT} in Equation 8.2. When resulting from differences in density, as with air around a fire, the process is referred to as *natural convection.* Airflow at a beach is an example of natural convection, as is the mixing that occurs as surface water in a lake cools and sinks (see Section 19.4). When the heated substance is forced to move by a fan or pump, as in some hot-air and hot-water heating systems, the process is called *forced convection*.

If it were not for convection currents, it would be very difficult to boil water. As water is heated in a teakettle, the lower layers are warmed first. This water expands and rises to the top because its density is lowered. At the same time, the denser, cool water at the surface sinks to the bottom of the kettle and is heated.

The same process occurs when a room is heated by a radiator. The hot radiator warms the air in the lower regions of the room. The warm air expands and rises to the ceiling because of its lower density. The denser, cooler air from above sinks, and the continuous air current pattern shown in Figure 20.16 is established.

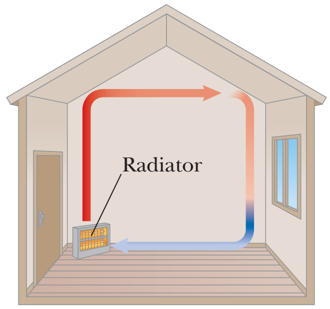

Figure 20.16 Convection currents are set up in a room warmed by a radiator.

Radiation

The third means of energy transfer we shall discuss is **thermal radiation,** T_{ER} in Equation 8.2. All objects radiate energy continuously in the form of electromagnetic waves (see Chapter 34) produced by thermal vibrations of the molecules. You are likely familiar with electromagnetic radiation in the form of the orange glow from an electric stove burner, an electric space heater, or the coils of a toaster.

The rate at which the surface of an object radiates energy is proportional to the fourth power of the absolute temperature of the surface. Known as **Stefan's law,** this behavior is expressed in equation form as

$$P = \sigma A e T^4 \tag{20.19}$$

◀ **Stefan's law**

where P is the power in watts of electromagnetic waves radiated from the surface of the object, σ is a constant equal to $5.669\,6 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$, A is the surface area of the object in square meters, e is the **emissivity,** and T is the surface temperature in kelvins. The value of e can vary between zero and unity depending on the properties of the surface of the object. The emissivity is equal to the **absorptivity,** which is the fraction of the incoming radiation that the surface absorbs. A mirror has very low absorptivity because it reflects almost all incident light. Therefore, a mirror surface also has a very low emissivity. At the other extreme, a black surface has high absorptivity and high emissivity. An **ideal absorber** is defined as an object that absorbs all the energy incident on it, and for such an object, $e = 1$. An object for which $e = 1$ is often referred to as a **black body.** We shall investigate experimental and theoretical approaches to radiation from a black body in Chapter 40.

Every second, approximately 1 370 J of electromagnetic radiation from the Sun passes perpendicularly through each 1 $\mathrm{m^2}$ at the top of the Earth's atmosphere. This radiation is primarily visible and infrared light accompanied by a significant amount of ultraviolet radiation. We shall study these types of radiation in detail in Chapter 34. Enough energy arrives at the surface of the Earth each day to supply all our energy needs on this planet hundreds of times over, if only it could be captured and used efficiently. The growth in the number of solar energy–powered houses and proposals for solar energy "farms" in the United States reflects the increasing efforts being made to use this abundant energy.

What happens to the atmospheric temperature at night is another example of the effects of energy transfer by radiation. If there is a cloud cover above the Earth, the water vapor in the clouds absorbs part of the infrared radiation emitted by the Earth and re-emits it back to the surface. Consequently, temperature levels at the surface remain moderate. In the absence of this cloud cover, there is less in the way to prevent this radiation from escaping into space; therefore, the temperature decreases more on a clear night than on a cloudy one.

As an object radiates energy at a rate given by Equation 20.19, it also absorbs electromagnetic radiation from the surroundings, which consist of other objects that radiate energy. If the latter process did not occur, an object would eventually radiate all its energy and its temperature would reach absolute zero. If an object is at a temperature T and its surroundings are at an average temperature T_0, the net rate of energy gained or lost by the object as a result of radiation is

$$P_{\text{net}} = \sigma A e(T^4 - T_0{}^4) \tag{20.20}$$

When an object is in equilibrium with its surroundings, it radiates and absorbs energy at the same rate and its temperature remains constant. When an object is hotter than its surroundings, it radiates more energy than it absorbs and its temperature decreases.

The Dewar Flask

The *Dewar flask*[6] is a container designed to minimize energy transfers by conduction, convection, and radiation. Such a container is used to store cold or hot liquids for long periods of time. (An insulated bottle, such as a Thermos, is a common household equivalent of a Dewar flask.) The standard construction (Fig. 20.17) consists of a double-walled Pyrex glass vessel with silvered walls. The space between the walls is evacuated to minimize energy transfer by conduction and convection. The silvered surfaces minimize energy transfer by radiation because silver is a very good reflector and has very low emissivity. A further reduction in energy loss is obtained by reducing the size of the neck. Dewar flasks are commonly used to store liquid nitrogen (boiling point 77 K) and liquid oxygen (boiling point 90 K).

To confine liquid helium (boiling point 4.2 K), which has a very low heat of vaporization, it is often necessary to use a double Dewar system in which the Dewar flask containing the liquid is surrounded by a second Dewar flask. The space between the two flasks is filled with liquid nitrogen.

Newer designs of storage containers use "superinsulation" that consists of many layers of reflecting material separated by fiberglass. All this material is in a vacuum, and no liquid nitrogen is needed with this design.

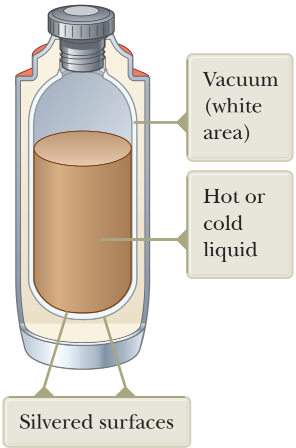

Figure 20.17 A cross-sectional view of a Dewar flask, which is used to store hot or cold substances.

[6]Invented by Sir James Dewar (1842–1923).

Summary

Definitions

Internal energy is a system's energy associated with its temperature and its physical state (solid, liquid, gas). Internal energy includes kinetic energy of random translation, rotation, and vibration of molecules; vibrational potential energy within molecules; and potential energy between molecules.

Heat is the process of energy transfer across the boundary of a system resulting from a temperature difference between the system and its surroundings. The symbol Q represents the amount of energy transferred by this process.

A **calorie** is the amount of energy necessary to raise the temperature of 1 g of water from 14.5°C to 15.5°C.

The **heat capacity** C of any sample is the amount of energy needed to raise the temperature of the sample by 1°C.

The **specific heat** c of a substance is the heat capacity per unit mass:

$$c \equiv \frac{Q}{m\,\Delta T} \tag{20.3}$$

The **latent heat** of a substance is defined as the ratio of the energy input to a substance to the change in mass of the higher-phase material:

$$L \equiv \frac{Q}{\Delta m} \tag{20.6}$$

Concepts and Principles

The energy Q required to change the temperature of a mass m of a substance by an amount ΔT is

$$Q = mc\,\Delta T \qquad (20.4)$$

where c is the specific heat of the substance.

The energy required to change the phase of a pure substance is

$$Q = L\,\Delta m \qquad (20.7)$$

where L is the latent heat of the substance, which depends on the nature of the phase change and the substance, and Δm is the change in mass of the higher-phase material.

The **work** done on a gas as its volume changes from some initial value V_i to some final value V_f is

$$W = -\int_{V_i}^{V_f} P\,dV \qquad (20.9)$$

where P is the pressure of the gas, which may vary during the process. To evaluate W, the process must be fully specified; that is, P and V must be known during each step. The work done depends on the path taken between the initial and final states.

The **first law of thermodynamics** is a specific reduction of the conservation of energy equation (Eq. 8.2) and states that when a system undergoes a change from one state to another, the change in its internal energy is

$$\Delta E_{\text{int}} = Q + W \qquad (20.10)$$

where Q is the energy transferred into the system by heat and W is the work done on the system. Although Q and W both depend on the path taken from the initial state to the final state, the quantity ΔE_{int} does not depend on the path.

In a **cyclic process** (one that originates and terminates at the same state), $\Delta E_{\text{int}} = 0$ and therefore $Q = -W$. That is, the energy transferred into the system by heat equals the negative of the work done on the system during the process.

In an **adiabatic process,** no energy is transferred by heat between the system and its surroundings ($Q = 0$). In this case, the first law gives $\Delta E_{\text{int}} = W$. In the **adiabatic free expansion** of a gas, $Q = 0$ and $W = 0$, so $\Delta E_{\text{int}} = 0$. That is, the internal energy of the gas does not change in such a process.

An **isobaric process** is one that occurs at constant pressure. The work done on a gas in such a process is $W = -P(V_f - V_i)$.

An **isovolumetric process** is one that occurs at constant volume. No work is done in such a process, so $\Delta E_{\text{int}} = Q$.

An **isothermal process** is one that occurs at constant temperature. The work done on an ideal gas during an isothermal process is

$$W = nRT \ln\left(\frac{V_i}{V_f}\right) \qquad (20.14)$$

Conduction can be viewed as an exchange of kinetic energy between colliding molecules or electrons. The rate of energy transfer by conduction through a slab of area A is

$$P = kA\left|\frac{dT}{dx}\right| \qquad (20.15)$$

where k is the **thermal conductivity** of the material from which the slab is made and $|dT/dx|$ is the **temperature gradient.**

In **convection,** a warm substance transfers energy from one location to another.

All objects emit **thermal radiation** in the form of electromagnetic waves at the rate

$$P = \sigma A e T^4 \qquad (20.19)$$

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. An ideal gas is compressed to half its initial volume by means of several possible processes. Which of the following processes results in the most work done on the gas? (a) isothermal (b) adiabatic (c) isobaric (d) The work done is independent of the process.

2. A poker is a stiff, nonflammable rod used to push burning logs around in a fireplace. For safety and comfort of use, should the poker be made from a material with (a) high specific heat and high thermal conductivity, (b) low specific heat and low thermal conductivity,

(c) low specific heat and high thermal conductivity, or (d) high specific heat and low thermal conductivity?

3. Assume you are measuring the specific heat of a sample of originally hot metal by using a calorimeter containing water. Because your calorimeter is not perfectly insulating, energy can transfer by heat between the contents of the calorimeter and the room. To obtain the most accurate result for the specific heat of the metal, you should use water with which initial temperature? (a) slightly lower than room temperature (b) the same as room temperature (c) slightly higher than room temperature (d) whatever you like because the initial temperature makes no difference

4. An amount of energy is added to ice, raising its temperature from −10°C to −5°C. A larger amount of energy is added to the same mass of water, raising its temperature from 15°C to 20°C. From these results, what would you conclude? (a) Overcoming the latent heat of fusion of ice requires an input of energy. (b) The latent heat of fusion of ice delivers some energy to the system. (c) The specific heat of ice is less than that of water. (d) The specific heat of ice is greater than that of water. (e) More information is needed to draw any conclusion.

5. How much energy is required to raise the temperature of 5.00 kg of lead from 20.0°C to its melting point of 327°C? The specific heat of lead is 128 J/kg · °C. (a) 4.04×10^5 J (b) 1.07×10^5 J (c) 8.15×10^4 J (d) 2.13×10^4 J (e) 1.96×10^5 J

6. Ethyl alcohol has about one-half the specific heat of water. Assume equal amounts of energy are transferred by heat into equal-mass liquid samples of alcohol and water in separate insulated containers. The water rises in temperature by 25°C. How much will the alcohol rise in temperature? (a) It will rise by 12°C. (b) It will rise by 25°C. (c) It will rise by 50°C. (d) It depends on the rate of energy transfer. (e) It will not rise in temperature.

7. The specific heat of substance A is greater than that of substance B. Both A and B are at the same initial temperature when equal amounts of energy are added to them. Assuming no melting or vaporization occurs, which of the following can be concluded about the final temperature T_A of substance A and the final temperature T_B of substance B? (a) $T_A > T_B$ (b) $T_A < T_B$ (c) $T_A = T_B$ (d) More information is needed.

8. Beryllium has roughly one-half the specific heat of water (H_2O). Rank the quantities of energy input required to produce the following changes from the largest to the smallest. In your ranking, note any cases of equality. (a) raising the temperature of 1 kg of H_2O from 20°C to 26°C (b) raising the temperature of 2 kg of H_2O from 20°C to 23°C (c) raising the temperature of 2 kg of H_2O from 1°C to 4°C (d) raising the temperature of 2 kg of beryllium from −1°C to 2°C (e) raising the temperature of 2 kg of H_2O from −1°C to 2°C

9. A person shakes a sealed insulated bottle containing hot coffee for a few minutes. **(i)** What is the change in the temperature of the coffee? (a) a large decrease (b) a slight decrease (c) no change (d) a slight increase (e) a large increase **(ii)** What is the change in the internal energy of the coffee? Choose from the same possibilities.

10. A 100-g piece of copper, initially at 95.0°C, is dropped into 200 g of water contained in a 280-g aluminum can; the water and can are initially at 15.0°C. What is the final temperature of the system? (Specific heats of copper and aluminum are 0.092 and 0.215 cal/g · °C, respectively.) (a) 16°C (b) 18°C (c) 24°C (d) 26°C (e) none of those answers

11. Star *A* has twice the radius and twice the absolute surface temperature of star *B*. The emissivity of both stars can be assumed to be 1. What is the ratio of the power output of star *A* to that of star *B*? (a) 4 (b) 8 (c) 16 (d) 32 (e) 64

12. If a gas is compressed isothermally, which of the following statements is true? (a) Energy is transferred into the gas by heat. (b) No work is done on the gas. (c) The temperature of the gas increases. (d) The internal energy of the gas remains constant. (e) None of those statements is true.

13. When a gas undergoes an adiabatic expansion, which of the following statements is true? (a) The temperature of the gas does not change. (b) No work is done by the gas. (c) No energy is transferred to the gas by heat. (d) The internal energy of the gas does not change. (e) The pressure increases.

14. If a gas undergoes an isobaric process, which of the following statements is true? (a) The temperature of the gas doesn't change. (b) Work is done on or by the gas. (c) No energy is transferred by heat to or from the gas. (d) The volume of the gas remains the same. (e) The pressure of the gas decreases uniformly.

15. How long would it take a 1 000 W heater to melt 1.00 kg of ice at −20.0°C, assuming all the energy from the heater is absorbed by the ice? (a) 4.18 s (b) 41.8 s (c) 5.55 min (d) 6.25 min (e) 38.4 min

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Rub the palm of your hand on a metal surface for about 30 seconds. Place the palm of your other hand on an unrubbed portion of the surface and then on the rubbed portion. The rubbed portion will feel warmer. Now repeat this process on a wood surface. Why does the temperature difference between the rubbed and unrubbed portions of the wood surface seem larger than for the metal surface?

2. You need to pick up a very hot cooking pot in your kitchen. You have a pair of cotton oven mitts. To pick up the pot most comfortably, should you soak them in cold water or keep them dry?

3. What is wrong with the following statement: "Given any two bodies, the one with the higher temperature contains more heat."

4. Why is a person able to remove a piece of dry aluminum foil from a hot oven with bare fingers, whereas a burn results if there is moisture on the foil?

5. Using the first law of thermodynamics, explain why the *total* energy of an isolated system is always constant.

6. In 1801, Humphry Davy rubbed together pieces of ice inside an icehouse. He made sure that nothing in the environment was at a higher temperature than the rubbed pieces. He observed the production of drops of liquid water. Make a table listing this and other experiments or processes to illustrate each of the following situations. (a) A system can absorb energy by heat, increase in internal energy, and increase in temperature. (b) A system can absorb energy by heat and increase in internal energy without an increase in temperature. (c) A system can absorb energy by heat without increasing in temperature or in internal energy. (d) A system can increase in internal energy and in temperature without absorbing energy by heat. (e) A system can increase in internal energy without absorbing energy by heat or increasing in temperature.

7. It is the morning of a day that will become hot. You just purchased drinks for a picnic and are loading them, with ice, into a chest in the back of your car. (a) You wrap a wool blanket around the chest. Does doing so help to keep the beverages cool, or should you expect the wool blanket to warm them up? Explain your answer. (b) Your younger sister suggests you wrap her up in another wool blanket to keep her cool on the hot day like the ice chest. Explain your response to her.

8. In usually warm climates that experience a hard freeze, fruit growers will spray the fruit trees with water, hoping that a layer of ice will form on the fruit. Why would such a layer be advantageous?

9. Suppose you pour hot coffee for your guests, and one of them wants it with cream. He wants the coffee to be as warm as possible several minutes later when he drinks it. To have the warmest coffee, should the person add the cream just after the coffee is poured or just before drinking? Explain.

10. When camping in a canyon on a still night, a camper notices that as soon as the sun strikes the surrounding peaks, a breeze begins to stir. What causes the breeze?

11. Pioneers stored fruits and vegetables in underground cellars. In winter, why did the pioneers place an open barrel of water alongside their produce?

12. Is it possible to convert internal energy to mechanical energy? Explain with examples.

Problems

Enhanced WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 20.1 Heat and Internal Energy

1. A 55.0-kg woman eats a 540 Calorie (540 kcal) jelly doughnut for breakfast. (a) How many joules of energy are the equivalent of one jelly doughnut? (b) How many steps must the woman climb on a very tall stairway to change the gravitational potential energy of the woman–Earth system by a value equivalent to the food energy in one jelly doughnut? Assume the height of a single stair is 15.0 cm. (c) If the human body is only 25.0% efficient in converting chemical potential energy to mechanical energy, how many steps must the woman climb to work off her breakfast?

Section 20.2 Specific Heat and Calorimetry

2. AMT W Consider Joule's apparatus described in Figure 20.1. The mass of each of the two blocks is 1.50 kg, and the insulated tank is filled with 200 g of water. What is the increase in the water's temperature after the blocks fall through a distance of 3.00 m?

3. A combination of 0.250 kg of water at 20.0°C, 0.400 kg of aluminum at 26.0°C, and 0.100 kg of copper at 100°C is mixed in an insulated container and allowed to come to thermal equilibrium. Ignore any energy transfer to or from the container. What is the final temperature of the mixture?

4. The highest waterfall in the world is the Salto Angel in Venezuela. Its longest single falls has a height of 807 m. If water at the top of the falls is at 15.0°C, what is the maximum temperature of the water at the bottom of the falls? Assume all the kinetic energy of the water as it reaches the bottom goes into raising its temperature.

5. What mass of water at 25.0°C must be allowed to come to thermal equilibrium with a 1.85-kg cube of aluminum initially at 150°C to lower the temperature of the aluminum to 65.0°C? Assume any water turned to steam subsequently condenses.

6. M The temperature of a silver bar rises by 10.0°C when it absorbs 1.23 kJ of energy by heat. The mass of the bar is

525 g. Determine the specific heat of silver from these data.

7. In cold climates, including the northern United States, a house can be built with very large windows facing south to take advantage of solar heating. Sunlight shining in during the daytime is absorbed by the floor, interior walls, and objects in the room, raising their temperature to 38.0°C. If the house is well insulated, you may model it as losing energy by heat steadily at the rate 6 000 W on a day in April when the average exterior temperature is 4°C and when the conventional heating system is not used at all. During the period between 5:00 p.m. and 7:00 a.m., the temperature of the house drops and a sufficiently large "thermal mass" is required to keep it from dropping too far. The thermal mass can be a large quantity of stone (with specific heat 850 J/kg · °C) in the floor and the interior walls exposed to sunlight. What mass of stone is required if the temperature is not to drop below 18.0°C overnight?

8. A 50.0-g sample of copper is at 25.0°C. If 1 200 J of energy is added to it by heat, what is the final temperature of the copper?

9. An aluminum cup of mass 200 g contains 800 g of water in thermal equilibrium at 80.0°C. The combination of cup and water is cooled uniformly so that the temperature decreases by 1.50°C per minute. At what rate is energy being removed by heat? Express your answer in watts.

10. If water with a mass m_h at temperature T_h is poured into an aluminum cup of mass m_{Al} containing mass m_c of water at T_c, where $T_h > T_c$, what is the equilibrium temperature of the system?

11. M A 1.50-kg iron horseshoe initially at 600°C is dropped into a bucket containing 20.0 kg of water at 25.0°C. What is the final temperature of the water–horseshoe system? Ignore the heat capacity of the container and assume a negligible amount of water boils away.

12. An electric drill with a steel drill bit of mass m = 27.0 g and diameter 0.635 cm is used to drill into a cubical steel block of mass M = 240 g. Assume steel has the same properties as iron. The cutting process can be modeled as happening at one point on the circumference of the bit. This point moves in a helix at constant tangential speed 40.0 m/s and exerts a force of constant magnitude 3.20 N on the block. As shown in Figure P20.12, a groove in the bit carries the chips up to the top of the block, where they form a pile around the hole. The drill is turned on and drills into the block for a time interval of 15.0 s. Let's assume this time interval is long enough for conduction within the steel to bring it all to a uniform temperature. Furthermore, assume the steel objects lose a negligible amount of energy by conduction, convection, and radiation into their environment. (a) Suppose the drill bit cuts three-quarters of the way through the block during 15.0 s. Find the temperature change of the whole quantity of steel. (b) **What If?** Now suppose the drill bit is dull and cuts only one-eighth of the way through the block in 15.0 s. Identify the temperature change of the whole quantity of steel in this case. (c) What pieces of data, if any, are unnecessary for the solution? Explain.

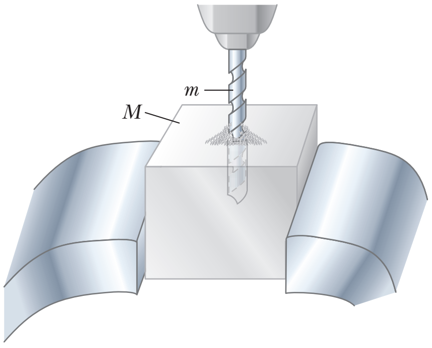

Figure P20.12

13. W An aluminum calorimeter with a mass of 100 g contains 250 g of water. The calorimeter and water are in thermal equilibrium at 10.0°C. Two metallic blocks are placed into the water. One is a 50.0-g piece of copper at 80.0°C. The other has a mass of 70.0 g and is originally at a temperature of 100°C. The entire system stabilizes at a final temperature of 20.0°C. (a) Determine the specific heat of the unknown sample. (b) Using the data in Table 20.1, can you make a positive identification of the unknown material? Can you identify a possible material? (c) Explain your answers for part (b).

14. A 3.00-g copper coin at 25.0°C drops 50.0 m to the ground. (a) Assuming 60.0% of the change in gravitational potential energy of the coin–Earth system goes into increasing the internal energy of the coin, determine the coin's final temperature. (b) **What If?** Does the result depend on the mass of the coin? Explain.

15. Two thermally insulated vessels are connected by a narrow tube fitted with a valve that is initially closed as shown in Figure P20.15. One vessel of volume 16.8 L contains oxygen at a temperature of 300 K and a pressure of 1.75 atm. The other vessel of volume 22.4 L contains oxygen at a temperature of 450 K and a pressure of 2.25 atm. When the valve is opened, the gases in the two vessels mix and the temperature and pressure become uniform throughout. (a) What is the final temperature? (b) What is the final pressure?

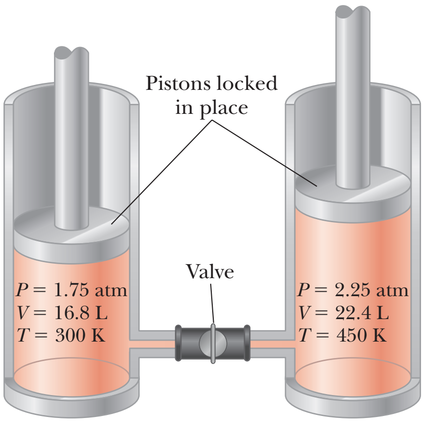

Figure P20.15

Section 20.3 Latent Heat

16. A 50.0-g copper calorimeter contains 250 g of water at 20.0°C. How much steam at 100°C must be condensed into the water if the final temperature of the system is to reach 50.0°C?

17. M A 75.0-kg cross-country skier glides over snow as in Figure P20.17. The coefficient of friction between skis and snow is 0.200. Assume all the snow beneath his skis is at 0°C and that all the internal energy generated by friction is added to snow, which sticks to his skis until it melts. How far would he have to ski to melt 1.00 kg of snow?

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Figure P20.17

18. W How much energy is required to change a 40.0-g ice cube from ice at −10.0°C to steam at 110°C?

19. A 75.0-g ice cube at 0°C is placed in 825 g of water at 25.0°C. What is the final temperature of the mixture?

20. AMT M A 3.00-g lead bullet at 30.0°C is fired at a speed of 240 m/s into a large block of ice at 0°C, in which it becomes embedded. What quantity of ice melts?

21. Steam at 100°C is added to ice at 0°C. (a) Find the amount of ice melted and the final temperature when the mass of steam is 10.0 g and the mass of ice is 50.0 g. (b) **What If?** Repeat when the mass of steam is 1.00 g and the mass of ice is 50.0 g.

22. W A 1.00-kg block of copper at 20.0°C is dropped into a large vessel of liquid nitrogen at 77.3 K. How many kilograms of nitrogen boil away by the time the copper reaches 77.3 K? (The specific heat of copper is 0.092 0 cal/g · °C, and the latent heat of vaporization of nitrogen is 48.0 cal/g.)

23. In an insulated vessel, 250 g of ice at 0°C is added to 600 g of water at 18.0°C. (a) What is the final temperature of the system? (b) How much ice remains when the system reaches equilibrium?

24. An automobile has a mass of 1 500 kg, and its aluminum brakes have an overall mass of 6.00 kg. (a) Assume all the mechanical energy that transforms into internal energy when the car stops is deposited in the brakes and no energy is transferred out of the brakes by heat. The brakes are originally at 20.0°C. How many times can the car be stopped from 25.0 m/s before the brakes start to melt? (b) Identify some effects ignored in part (a) that are important in a more realistic assessment of the warming of the brakes.

Section 20.4 Work and Heat in Thermodynamic Processes

25. An ideal gas is enclosed in a cylinder with a movable piston on top of it. The piston has a mass of 8 000 g and an area of 5.00 cm^2 and is free to slide up and down, keeping the pressure of the gas constant. How much work is done on the gas as the temperature of 0.200 mol of the gas is raised from 20.0°C to 300°C?

26. An ideal gas is enclosed in a cylinder that has a movable piston on top. The piston has a mass m and an area A and is free to slide up and down, keeping the pressure of the gas constant. How much work is done on the gas as the temperature of n mol of the gas is raised from T_1 to T_2?

27. One mole of an ideal gas is warmed slowly so that it goes from the PV state (P_i, V_i) to ($3P_i$, $3V_i$) in such a way that the pressure of the gas is directly proportional to the volume. (a) How much work is done on the gas in the process? (b) How is the temperature of the gas related to its volume during this process?

28. W (a) Determine the work done on a gas that expands from i to f as indicated in Figure P20.28. (b) **What If?** How much work is done on the gas if it is compressed from f to i along the same path?

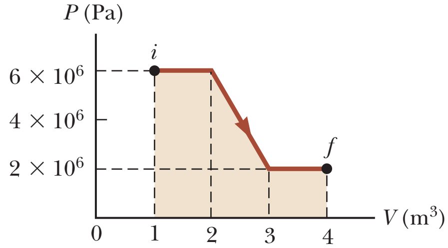


Figure P20.28

29. M An ideal gas is taken through a quasi-static process described by $P = \alpha V^2$, with $\alpha = 5.00$ atm/m^6, as shown in Figure P20.29. The gas is expanded to twice its original volume of 1.00 m^3. How much work is done on the expanding gas in this process?

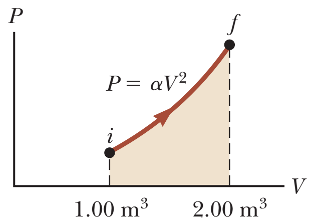


Figure P20.29

Section 20.5 The First Law of Thermodynamics

30. W A gas is taken through the cyclic process described in Figure P20.30. (a) Find the net energy transferred to the system by heat during one complete cycle. (b) **What If?** If the cycle is reversed—that is, the process follows the path $ACBA$—what is the net energy input per cycle by heat?

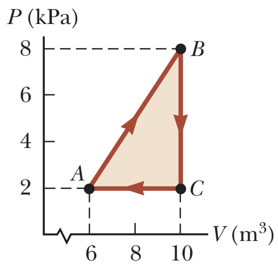


Figure P20.30
Problems 30 and 31.

31. Consider the cyclic process depicted in Figure P20.30. If Q is negative for the process BC and ΔE_{int} is negative for the process CA, what are the signs of Q, W, and ΔE_{int} that are associated with each of the three processes?

32. *Why is the following situation impossible?* An ideal gas undergoes a process with the following parameters: $Q = 10.0$ J, $W = 12.0$ J, and $\Delta T = -2.00°$C.

33. A thermodynamic system undergoes a process in which its internal energy decreases by 500 J. Over the same time interval, 220 J of work is done on the system. Find the energy transferred from it by heat.

34. A sample of an ideal gas goes through the process shown in Figure P20.34. From A to B, the process is adiabatic; from B to C, it is isobaric with 345 kJ of energy entering the system by heat; from C to D, the process is isothermal; and from D to A, it is isobaric with 371 kJ of energy leaving the system by heat. Determine the difference in internal energy $E_{\text{int},B} - E_{\text{int},A}$.

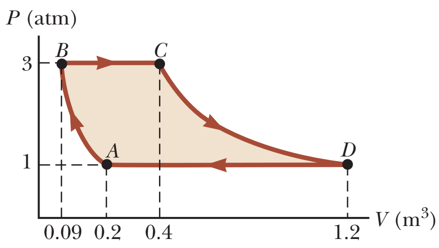


Figure P20.34

Section 20.6 Some Applications of the First Law of Thermodynamics

35. A 2.00-mol sample of helium gas initially at 300 K, and 0.400 atm is compressed isothermally to 1.20 atm. Noting that the helium behaves as an ideal gas, find (a) the final volume of the gas, (b) the work done on the gas, and (c) the energy transferred by heat.

36. (a) How much work is done on the steam when 1.00 mol of water at 100°C boils and becomes 1.00 mol of steam at 100°C at 1.00 atm pressure? Assume the steam to behave as an ideal gas. (b) Determine the change in internal energy of the system of the water and steam as the water vaporizes.

37. An ideal gas initially at 300 K undergoes an isobaric expansion at 2.50 kPa. If the volume increases from 1.00 m^3 to 3.00 m^3 and 12.5 kJ is transferred to the gas by heat, what are (a) the change in its internal energy and (b) its final temperature?

38. One mole of an ideal gas does 3 000 J of work on its surroundings as it expands isothermally to a final pressure of 1.00 atm and volume of 25.0 L. Determine (a) the initial volume and (b) the temperature of the gas.

39. A 1.00-kg block of aluminum is warmed at atmospheric pressure so that its temperature increases from 22.0°C to 40.0°C. Find (a) the work done on the aluminum, (b) the energy added to it by heat, and (c) the change in its internal energy.

40. In Figure P20.40, the change in internal energy of a gas that is taken from A to C along the blue path is +800 J. The work done on the gas along the red path ABC is −500 J. (a) How much energy must be added to the system by heat as it goes from A through B to C? (b) If the pressure at point A is five times that of point C, what is the work done on the system in going from C to D? (c) What is the energy exchanged with the surroundings by heat as the gas goes from C to A along the green path? (d) If the change in internal energy in going from point D to point A is +500 J, how much energy must be added to the system by heat as it goes from point C to point D?

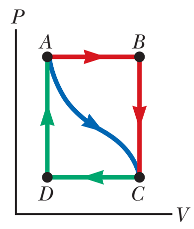


Figure P20.40

41. An ideal gas initially at P_i, V_i, and T_i is taken through a cycle as shown in Figure P20.41. (a) Find the net work done on the gas per cycle for 1.00 mol of gas initially at 0°C. (b) What is the net energy added by heat to the gas per cycle?

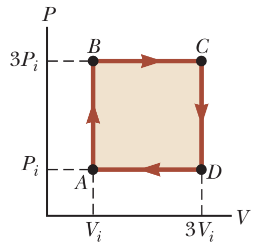


Figure P20.41
Problems 41 and 42.

42. An ideal gas initially at P_i, V_i, and T_i is taken through a cycle as shown in Figure P20.41. (a) Find the net work done on the gas per cycle. (b) What is the net energy added by heat to the system per cycle?

Section 20.7 Energy Transfer Mechanisms in Thermal Processes

43. A glass windowpane in a home is 0.620 cm thick and has dimensions of 1.00 m × 2.00 m. On a certain day, the temperature of the interior surface of the glass is 25.0°C and the exterior surface temperature is 0°C. (a) What is the rate at which energy is transferred by heat through the glass? (b) How much energy is transferred through the window in one day, assuming the temperatures on the surfaces remain constant?

44. A concrete slab is 12.0 cm thick and has an area of 5.00 m^2. Electric heating coils are installed under the slab to melt the ice on the surface in the winter months. What minimum power must be supplied to the coils to maintain a temperature difference of 20.0°C between the bottom of the slab and its surface? Assume all the energy transferred is through the slab.

45. A student is trying to decide what to wear. His bedroom is at 20.0°C. His skin temperature is 35.0°C. The area of his exposed skin is 1.50 m^2. People all over the world have skin that is dark in the infrared, with emissivity about 0.900. Find the net energy transfer from his body by radiation in 10.0 min.

46. The surface of the Sun has a temperature of about 5 800 K. The radius of the Sun is 6.96×10^8 m. Calculate the total energy radiated by the Sun each second. Assume the emissivity of the Sun is 0.986.

47. The tungsten filament of a certain 100-W lightbulb radiates 2.00 W of light. (The other 98 W is carried away by convection and conduction.) The filament has a surface area of 0.250 mm^2 and an emissivity of 0.950. Find the filament's temperature. (The melting point of tungsten is 3 683 K.)

48. At high noon, the Sun delivers 1 000 W to each square meter of a blacktop road. If the hot asphalt transfers energy only by radiation, what is its steady-state temperature?

49. Two lightbulbs have cylindrical filaments much greater in length than in diameter. The evacuated bulbs are identical except that one operates at a filament temperature of 2 100°C and the other operates at 2 000°C. (a) Find the ratio of the power emitted by the hotter lightbulb to that emitted by the cooler lightbulb. (b) With the bulbs operating at the same respective temperatures, the cooler lightbulb is to be altered by making its filament thicker so that it emits the same power as the hotter one. By what factor should the radius of this filament be increased?

50. The human body must maintain its core temperature inside a rather narrow range around 37°C. Metabolic processes, notably muscular exertion, convert chemical energy into internal energy deep in the interior. From the interior, energy must flow out to the skin or lungs to be expelled to the environment. During moderate exercise, an 80-kg man can metabolize food energy at the rate 300 kcal/h, do 60 kcal/h of mechanical work, and put out the remaining 240 kcal/h of energy by heat. Most of the energy is carried from the body interior out to the skin by forced convection (as a plumber would say), whereby blood is warmed in the interior and then cooled at the skin, which is a few degrees cooler than the body core. Without blood flow, living tissue is a good thermal insulator, with thermal conductivity about 0.210 W/m · °C. Show that blood flow is essential to cool the man's body by calculating the rate of energy conduction in kcal/h through the tissue layer under his skin. Assume that its area is 1.40 m^2, its thickness is 2.50 cm, and it is maintained at 37.0°C on one side and at 34.0°C on the other side.

51. M A copper rod and an aluminum rod of equal diameter are joined end to end in good thermal contact. The temperature of the free end of the copper rod is held constant at 100°C and that of the far end of the aluminum rod is held at 0°C. If the copper rod is 0.150 m long, what must be the length of the aluminum rod so that the temperature at the junction is 50.0°C?

52. A box with a total surface area of 1.20 m^2 and a wall thickness of 4.00 cm is made of an insulating material. A 10.0-W electric heater inside the box maintains the inside temperature at 15.0°C above the outside temperature. Find the thermal conductivity k of the insulating material.

53. (a) Calculate the R-value of a thermal window made of two single panes of glass each 0.125 in. thick and separated by a 0.250-in. air space. (b) By what factor is the transfer of energy by heat through the window reduced by using the thermal window instead of the single-pane window? Include the contributions of inside and outside stagnant air layers.

54. At our distance from the Sun, the intensity of solar radiation is 1 370 W/m^2. The temperature of the Earth is affected by the *greenhouse effect* of the atmosphere. This phenomenon describes the effect of absorption of infrared light emitted by the surface so as to make the surface temperature of the Earth higher than if it were airless. For comparison, consider a spherical object of radius r with no atmosphere at the same distance from the Sun as the Earth. Assume its emissivity is the same for all kinds of electromagnetic waves and its temperature is uniform over its surface. (a) Explain why the projected area over which it absorbs sunlight is πr^2 and the surface area over which it radiates is $4\pi r^2$. (b) Compute its steady-state temperature. Is it chilly?

55. A bar of gold (Au) is in thermal contact with a bar of silver (Ag) of the same length and area (Fig. P20.55). One end of the compound bar is maintained at 80.0°C, and the opposite end is at 30.0°C. When the energy transfer reaches steady state, what is the temperature at the junction?

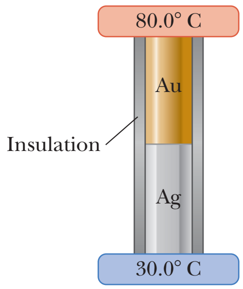


Figure P20.55

56. For bacteriological testing of water supplies and in medical clinics, samples must routinely be incubated for 24 h at 37°C. Peace Corps volunteer and MIT engineer Amy Smith invented a low-cost, low-maintenance incubator. The incubator consists of a foam-insulated box containing a waxy material that melts at 37.0°C interspersed among tubes, dishes, or bottles containing the test samples and growth medium (bacteria food). Outside the box, the waxy material is first melted by a stove or solar energy collector. Then the waxy material is put into the box to keep the test samples warm as the material solidifies. The heat of fusion of the phase-change material is 205 kJ/kg. Model the insulation as a panel with surface area 0.490 m^2, thickness 4.50 cm, and conductivity 0.012 0 W/m · °C. Assume the exterior temperature is 23.0°C for 12.0 h and 16.0°C for 12.0 h. (a) What mass of the waxy material is required to conduct the bacteriological test? (b) Explain why your calculation can be done without knowing the mass of the test samples or of the insulation.

57. A large, hot pizza floats in outer space after being jettisoned as refuse from a spacecraft. What is the order of magnitude (a) of its rate of energy loss and (b) of its rate of temperature change? List the quantities you estimate and the value you estimate for each.

Additional Problems

58. M A gas expands from I to F in Figure P20.58 (page 622). The energy added to the gas by heat is 418 J when the gas goes from I to F along the diagonal path. (a) What is the change in internal energy of the gas? (b) How much energy must be added to the gas by heat along the indirect path *IAF*?

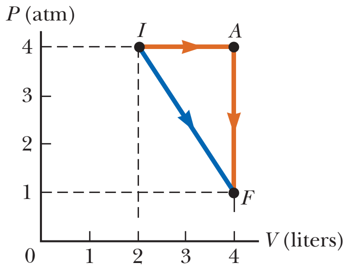


Figure P20.58

59. M Gas in a container is at a pressure of 1.50 atm and a volume of 4.00 m^3. What is the work done on the gas (a) if it expands at constant pressure to twice its initial volume, and (b) if it is compressed at constant pressure to one-quarter its initial volume?

60. Liquid nitrogen has a boiling point of 77.3 K and a latent heat of vaporization of 2.01×10^5 J/kg. A 25.0-W electric heating element is immersed in an insulated vessel containing 25.0 L of liquid nitrogen at its boiling point. How many kilograms of nitrogen are boiled away in a period of 4.00 h?

61. M An aluminum rod 0.500 m in length and with a cross-sectional area of 2.50 cm^2 is inserted into a thermally insulated vessel containing liquid helium at 4.20 K. The rod is initially at 300 K. (a) If one-half of the rod is inserted into the helium, how many liters of helium boil off by the time the inserted half cools to 4.20 K? Assume the upper half does not yet cool. (b) If the circular surface of the upper end of the rod is maintained at 300 K, what is the approximate boil-off rate of liquid helium in liters per second after the lower half has reached 4.20 K? (Aluminum has thermal conductivity of 3 100 W/m · K at 4.20 K; ignore its temperature variation. The density of liquid helium is 125 kg/m^3.)

62. AMT GP **Review.** Two speeding lead bullets, one of mass 12.0 g moving to the right at 300 m/s and one of mass 8.00 g moving to the left at 400 m/s, collide head-on, and all the material sticks together. Both bullets are originally at temperature 30.0°C. Assume the change in kinetic energy of the system appears entirely as increased internal energy. We would like to determine the temperature and phase of the bullets after the collision. (a) What two analysis models are appropriate for the system of two bullets for the time interval from before to after the collision? (b) From one of these models, what is the speed of the combined bullets after the collision? (c) How much of the initial kinetic energy has transformed to internal energy in the system after the collision? (d) Does all the lead melt due to the collision? (e) What is the temperature of the combined bullets after the collision? (f) What is the phase of the combined bullets after the collision?

63. A *flow calorimeter* is an apparatus used to measure the specific heat of a liquid. The technique of flow calorimetry involves measuring the temperature difference between the input and output points of a flowing stream of the liquid while energy is added by heat at a known rate. A liquid of density 900 kg/m^3 flows through the calorimeter with volume flow rate of 2.00 L/min. At steady state, a temperature difference 3.50°C is established between the input and output points when energy is supplied at the rate of 200 W. What is the specific heat of the liquid?

64. A *flow calorimeter* is an apparatus used to measure the specific heat of a liquid. The technique of flow calorimetry involves measuring the temperature difference between the input and output points of a flowing stream of the liquid while energy is added by heat at a known rate. A liquid of density ρ flows through the calorimeter with volume flow rate R. At steady state, a temperature difference ΔT is established between the input and output points when energy is supplied at the rate P. What is the specific heat of the liquid?

65. AMT **Review.** Following a collision between a large spacecraft and an asteroid, a copper disk of radius 28.0 m and thickness 1.20 m at a temperature of 850°C is floating in space, rotating about its symmetry axis with an angular speed of 25.0 rad/s. As the disk radiates infrared light, its temperature falls to 20.0°C. No external torque acts on the disk. (a) Find the change in kinetic energy of the disk. (b) Find the change in internal energy of the disk. (c) Find the amount of energy it radiates.

66. An ice-cube tray is filled with 75.0 g of water. After the filled tray reaches an equilibrium temperature of 20.0°C, it is placed in a freezer set at −8.00°C to make ice cubes. (a) Describe the processes that occur as energy is being removed from the water to make ice. (b) Calculate the energy that must be removed from the water to make ice cubes at −8.00°C.

67. On a cold winter day, you buy roasted chestnuts from a street vendor. Into the pocket of your down parka you put the change he gives you: coins constituting 9.00 g of copper at –12.0°C. Your pocket already contains 14.0 g of silver coins at 30.0°C. A short time later the temperature of the copper coins is 4.00°C and is increasing at a rate of 0.500°C/s. At this time, (a) what is the temperature of the silver coins and (b) at what rate is it changing?

68. The rate at which a resting person converts food energy is called one's *basal metabolic rate* (BMR). Assume that the resulting internal energy leaves a person's body by radiation and convection of dry air. When you jog, most of the food energy you burn above your BMR becomes internal energy that would raise your body temperature if it were not eliminated. Assume that evaporation of perspiration is the mechanism for eliminating this energy. Suppose a person is jogging for "maximum fat burning," converting food energy at the rate 400 kcal/h above his BMR, and putting out energy by work at the rate 60.0 W. Assume that the heat of evaporation of water at body temperature is equal to its heat of vaporization at 100°C. (a) Determine the hourly rate at which water must evaporate from his skin. (b) When you metabolize fat, the hydrogen atoms in the fat molecule are transferred to oxygen to form water. Assume that metabolism of 1.00 g of fat generates 9.00 kcal of energy and produces 1.00 g of water. What fraction of the water the jogger needs is provided by fat metabolism?

69. An iron plate is held against an iron wheel so that a kinetic friction force of 50.0 N acts between the two pieces of metal. The relative speed at which the two surfaces slide over each other is 40.0 m/s. (a) Calculate the rate at which mechanical energy is converted to internal energy. (b) The plate and the wheel each have a mass of 5.00 kg, and each receives 50.0% of the internal energy. If the system is run as described for 10.0 s and each object is then allowed to reach a uniform internal temperature, what is the resultant temperature increase?

70. A resting adult of average size converts chemical energy in food into internal energy at the rate 120 W, called her *basal metabolic rate*. To stay at constant temperature, the body must put out energy at the same rate. Several processes exhaust energy from your body. Usually, the most important is thermal conduction into the air in contact with your exposed skin. If you are not wearing a hat, a convection current of warm air rises vertically from your head like a plume from a smokestack. Your body also loses energy by electromagnetic radiation, by your exhaling warm air, and by evaporation of perspiration. In this problem, consider still another pathway for energy loss: moisture in exhaled breath. Suppose you breathe out 22.0 breaths per minute, each with a volume of 0.600 L. Assume you inhale dry air and exhale air at 37.0°C containing water vapor with a vapor pressure of 3.20 kPa. The vapor came from evaporation of liquid water in your body. Model the water vapor as an ideal gas. Assume its latent heat of evaporation at 37.0°C is the same as its heat of vaporization at 100°C. Calculate the rate at which you lose energy by exhaling humid air.

71. M A 40.0-g ice cube floats in 200 g of water in a 100-g copper cup; all are at a temperature of 0°C. A piece of lead at 98.0°C is dropped into the cup, and the final equilibrium temperature is 12.0°C. What is the mass of the lead?

72. One mole of an ideal gas is contained in a cylinder with a movable piston. The initial pressure, volume, and temperature are P_i, V_i, and T_i, respectively. Find the work done on the gas in the following processes. In operational terms, describe how to carry out each process and show each process on a *PV* diagram. (a) an isobaric compression in which the final volume is one-half the initial volume (b) an isothermal compression in which the final pressure is four times the initial pressure (c) an isovolumetric process in which the final pressure is three times the initial pressure

73. **Review.** A 670-kg meteoroid happens to be composed of aluminum. When it is far from the Earth, its temperature is −15.0°C and it moves at 14.0 km/s relative to the planet. As it crashes into the Earth, assume the internal energy transformed from the mechanical energy of the meteoroid–Earth system is shared equally between the meteoroid and the Earth and all the material of the meteoroid rises momentarily to the same final temperature. Find this temperature. Assume the specific heat of liquid and of gaseous aluminum is 1 170 J/kg · °C.

74. *Why is the following situation impossible?* A group of campers arises at 8:30 a.m. and uses a solar cooker, which consists of a curved, reflecting surface that concentrates sunlight onto the object to be warmed (Fig. P20.74). During the day, the maximum solar intensity reaching the Earth's surface at the cooker's location is $I = 600$ W/m^2. The cooker faces the Sun and has a face diameter of $d = 0.600$ m. Assume a fraction f of 40.0% of the incident energy is transferred to 1.50 L of water in an open container, initially at 20.0°C. The water comes to a boil, and the campers enjoy hot coffee for breakfast before hiking ten miles and returning by noon for lunch.

Figure P20.74

75. During periods of high activity, the Sun has more sunspots than usual. Sunspots are cooler than the rest of the luminous layer of the Sun's atmosphere (the photosphere). Paradoxically, the total power output of the active Sun is not lower than average but is the same or slightly higher than average. Work out the details of the following crude model of this phenomenon. Consider a patch of the photosphere with an area of 5.10×10^{14} m^2. Its emissivity is 0.965. (a) Find the power it radiates if its temperature is uniformly 5 800 K, corresponding to the quiet Sun. (b) To represent a sunspot, assume 10.0% of the patch area is at 4 800 K and the other 90.0% is at 5 890 K. Find the power output of the patch. (c) State how the answer to part (b) compares with the answer to part (a). (d) Find the average temperature of the patch. Note that this cooler temperature results in a higher power output.

76. (a) In air at 0°C, a 1.60-kg copper block at 0°C is set sliding at 2.50 m/s over a sheet of ice at 0°C. Friction brings the block to rest. Find the mass of the ice that melts. (b) As the block slows down, identify its energy input Q, its change in internal energy ΔE_{int}, and the change in mechanical energy for the block–ice system. (c) For the ice as a system, identify its energy input Q and its change in internal energy ΔE_{int}. (d) A 1.60-kg block of ice at 0°C is set sliding at 2.50 m/s over a sheet of copper at 0°C. Friction brings the block to rest. Find the mass of the ice that melts. (e) Evaluate Q and ΔE_{int} for the block of ice as a system and ΔE_{mech} for the block–ice system. (f) Evaluate Q and ΔE_{int} for the metal sheet as a system. (g) A thin, 1.60-kg slab of copper at 20°C is set sliding at 2.50 m/s over an identical stationary slab at the same temperature. Friction quickly stops the motion. Assuming no energy is transferred to the environment by heat, find the change in temperature of both objects. (h) Evaluate Q and ΔE_{int} for the sliding slab and ΔE_{mech} for the two-slab system. (i) Evaluate Q and ΔE_{int} for the stationary slab.

77. M Water in an electric teakettle is boiling. The power absorbed by the water is 1.00 kW. Assuming the pressure of vapor in the kettle equals atmospheric pressure, determine the speed of effusion of vapor from the kettle's spout if the spout has a cross-sectional area of 2.00 cm^2. Model the steam as an ideal gas.

78. The average thermal conductivity of the walls (including the windows) and roof of the house depicted in Figure P20.78 is 0.480 W/m · °C, and their average thickness is 21.0 cm. The house is kept warm with natural gas having a heat of combustion (that is, the energy provided per cubic meter of gas burned) of 9 300 kcal/m^3. How many cubic meters of gas must be burned each day to maintain an inside temperature of 25.0°C if the outside temperature is 0.0°C? Disregard radiation and the energy transferred by heat through the ground.

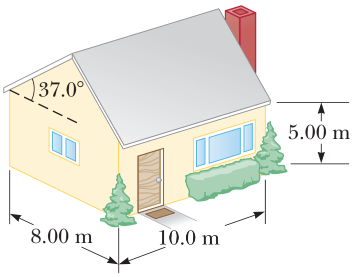


Figure P20.78

79. A cooking vessel on a slow burner contains 10.0 kg of water and an unknown mass of ice in equilibrium at 0°C at time t = 0. The temperature of the mixture is measured at various times, and the result is plotted in Figure P20.79. During the first 50.0 min, the mixture remains at 0°C. From 50.0 min to 60.0 min, the temperature increases to 2.00°C. Ignoring the heat capacity of the vessel, determine the initial mass of the ice.

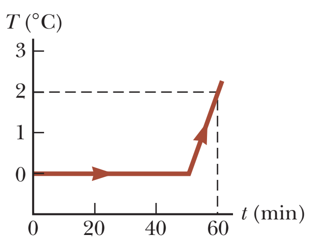


Figure P20.79

80. A student measures the following data in a calorimetry experiment designed to determine the specific heat of aluminum:

Initial temperature of water and calorimeter:	70.0°C
Mass of water:	0.400 kg
Mass of calorimeter:	0.040 kg
Specific heat of calorimeter:	0.63 kJ/kg · °C
Initial temperature of aluminum:	27.0°C
Mass of aluminum:	0.200 kg
Final temperature of mixture:	66.3°C

(a) Use these data to determine the specific heat of aluminum. (b) Explain whether your result is within 15% of the value listed in Table 20.1.

Challenge Problems

81. Consider the piston–cylinder apparatus shown in Figure P20.81. The bottom of the cylinder contains 2.00 kg of water at just under 100.0°C. The cylinder has a radius of r = 7.50 cm. The piston of mass m = 3.00 kg sits on the surface of the water. An electric heater in the cylinder base transfers energy into the water at a rate of 100 W. Assume the cylinder is much taller than shown in the figure, so we don't need to be concerned about the piston reaching the top of the cylinder. (a) Once the water begins boiling, how fast is the piston rising? Model the steam as an ideal gas. (b) After the water has completely turned to steam and the heater continues to transfer energy to the steam at the same rate, how fast is the piston rising?

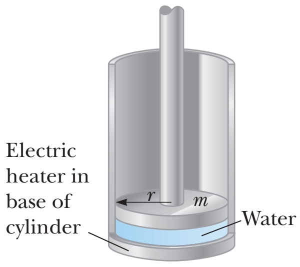


Figure P20.81

82. A spherical shell has inner radius 3.00 cm and outer radius 7.00 cm. It is made of material with thermal conductivity k = 0.800 W/m · °C. The interior is maintained at temperature 5°C and the exterior at 40°C. After an interval of time, the shell reaches a steady state with the temperature at each point within it remaining constant in time. (a) Explain why the rate of energy transfer P must be the same through each spherical surface, of radius r, within the shell and must satisfy

$$\frac{dT}{dr} = \frac{P}{4\pi k r^2}$$

(b) Next, prove that

$$\int_5^{40} dT = \frac{P}{4\pi k}\int_{0.03}^{0.07} r^{-2}\,dr$$

where T is in degrees Celsius and r is in meters. (c) Find the rate of energy transfer through the shell. (d) Prove that

$$\int_5^{T} dT = 1.84\int_{0.03}^{r} r^{-2}\,dr$$

where T is in degrees Celsius and r is in meters. (e) Find the temperature within the shell as a function of radius. (f) Find the temperature at r = 5.00 cm, halfway through the shell.

83. A pond of water at 0°C is covered with a layer of ice 4.00 cm thick. If the air temperature stays constant at −10.0°C, what time interval is required for the ice thickness to increase to 8.00 cm? *Suggestion:* Use Equation 20.16 in the form

$$\frac{dQ}{dt} = kA\,\frac{\Delta T}{x}$$

and note that the incremental energy dQ extracted from the water through the thickness x of ice is the amount required to freeze a thickness dx of ice. That is, $dQ = L_f \rho A\,dx$, where ρ is the density of the ice, A is the area, and L_f is the latent heat of fusion.

84. (a) The inside of a hollow cylinder is maintained at a temperature T_a, and the outside is at a lower temperature, T_b (Fig. P20.84). The wall of the cylinder has a thermal conductivity k. Ignoring end effects, show that the rate of energy conduction from the inner surface to the outer surface in the radial direction is

$$\frac{dQ}{dt} = 2\pi Lk\left[\frac{T_a - T_b}{\ln(b/a)}\right]$$

Suggestions: The temperature gradient is dT/dr. A radial energy current passes through a concentric cylinder of area $2\pi rL$. (b) The passenger section of a jet airliner is in the shape of a cylindrical tube with a length of 35.0 m and an inner radius of 2.50 m. Its walls are lined with an insulating material 6.00 cm in thickness and having a thermal conductivity of 4.00×10^{-5} cal/s · cm · °C. A heater must maintain the interior temperature at 25.0°C while the outside temperature is −35.0°C. What power must be supplied to the heater?

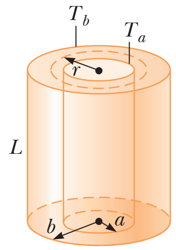


Figure P20.84

CHAPTER

21

The Kinetic Theory of Gases

A boy inflates his bicycle tire with a hand-operated pump. Kinetic theory helps to describe the details of the air in the pump. *(© Cengage Learning/George Semple)*

In Chapter 19, we discussed the properties of an ideal gas by using such macroscopic variables as pressure, volume, and temperature. Such large-scale properties can be related to a description on a microscopic scale, where matter is treated as a collection of molecules. Applying Newton's laws of motion in a statistical manner to a collection of particles provides a reasonable description of thermodynamic processes. To keep the mathematics relatively simple, we shall consider primarily the behavior of gases because in gases the interactions between molecules are much weaker than they are in liquids or solids.

We shall begin by relating pressure and temperature directly to the details of molecular motion in a sample of gas. Based on these results, we will make predictions of molar specific heats of gases. Some of these predictions will be correct and some will not. We will extend our model to explain those values that are not predicted correctly by the simpler model. Finally, we discuss the distribution of molecular speeds in a gas.

21.1 Molecular Model of an Ideal Gas

In this chapter, we will investigate a *structural model* for an ideal gas. A **structural model** is a theoretical construct designed to represent a system that cannot be observed directly because it is too large or too small. For example, we can only observe the solar system from the inside; we cannot travel outside the solar system and look back to see how it works. This restricted vantage point has led to different historical structural models of the solar system: the *geocentric model,* with the Earth at the center, and the *heliocentric model,* with the Sun at the center. Of course, the latter has been shown to be correct. An example of a system too small to observe directly is the hydrogen atom. Various structural models of this system have been developed, including the *Bohr model* (Section 42.3) and the *quantum model* (Section 42.4). Once a structural model is developed, various predictions are made for experimental observations. For example, the geocentric model of the solar system makes predictions of how the movement of Mars should appear from the Earth. It turns out that those predictions do not match the actual observations. When that occurs with a structural model, the model must be modified or replaced with another model.

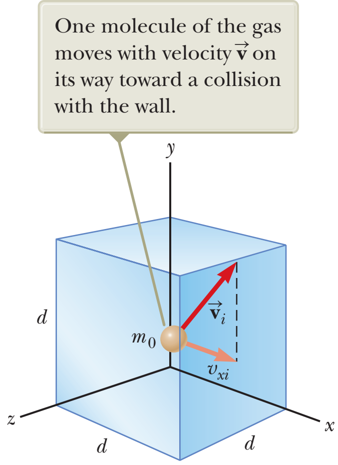


Figure 21.1 A cubical box with sides of length d containing an ideal gas.

The structural model that we will develop for an ideal gas is called **kinetic theory.** This model treats an ideal gas as a collection of molecules with the following properties:

1. *Physical components:*
   The gas consists of a number of identical molecules within a cubic container of side length d. The number of molecules in the gas is large, and the average separation between them is large compared with their dimensions. Therefore, the molecules occupy a negligible volume in the container. This assumption is consistent with the ideal gas model, in which we imagine the molecules to be point-like.
2. *Behavior of the components:*
   (a) The molecules obey Newton's laws of motion, but as a whole their motion is isotropic: any molecule can move in any direction with any speed.
   (b) The molecules interact only by short-range forces during elastic collisions. This assumption is consistent with the ideal gas model, in which the molecules exert no long-range forces on one another.
   (c) The molecules make elastic collisions with the walls.

Although we often picture an ideal gas as consisting of single atoms, the behavior of molecular gases approximates that of ideal gases rather well at low pressures. Usually, molecular rotations or vibrations have no effect on the motions considered here.

For our first application of kinetic theory, let us relate the macroscope variable of pressure P to microscopic quantities. Consider a collection of N molecules of an ideal gas in a container of volume V. As indicated above, the container is a cube with edges of length d (Fig. 21.1). We shall first focus our attention on one of these molecules of mass m_0 and assume it is moving so that its component of velocity in the x direction is v_{xi} as in Figure 21.2. (The subscript i here refers to the ith molecule in the collection, not to an initial value. We will combine the effects of all the molecules shortly.) As the molecule collides elastically with any wall (property 2(c) above), its velocity component perpendicular to the wall is reversed because the mass of the wall is far greater than the mass of the molecule. The molecule is modeled as a nonisolated system for which the impulse from the wall causes a change in the molecule's momentum. Because the momentum component p_{xi} of the molecule is m_0v_{xi} before the collision and $-m_0v_{xi}$ after the collision, the change in the x component of the momentum of the molecule is

$$\Delta p_{xi} = -m_0v_{xi} - (m_0v_{xi}) = -2m_0v_{xi} \quad \textbf{(21.1)}$$

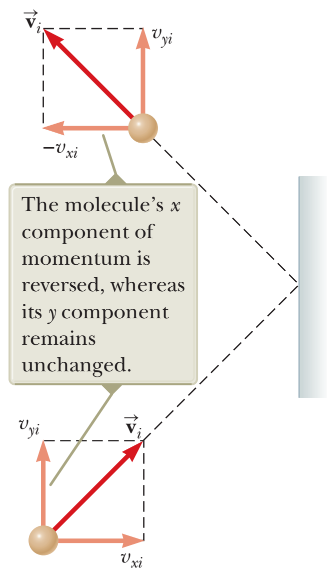


Figure 21.2 A molecule makes an elastic collision with the wall of the container. In this construction, we assume the molecule moves in the xy plane.

From the nonisolated system model for momentum, we can apply the impulse-momentum theorem (Eqs. 9.11 and 9.13) to the molecule to give

$$\overline{F}_{i,\text{on molecule}}\,\Delta t_{\text{collision}} = \Delta p_{xi} = -2m_0 v_{xi} \tag{21.2}$$

where $\overline{F}_{i,\text{on molecule}}$ is the x component of the average force[1] the wall exerts on the molecule during the collision and $\Delta t_{\text{collision}}$ is the duration of the collision. For the molecule to make another collision with the same wall after this first collision, it must travel a distance of $2d$ in the x direction (across the cube and back). Therefore, the time interval between two collisions with the same wall is

$$\Delta t = \frac{2d}{v_{xi}} \tag{21.3}$$

The force that causes the change in momentum of the molecule in the collision with the wall occurs only during the collision. We can, however, find the long-term average force for many back-and-forth trips across the cube by averaging the force in Equation 21.2 over the time interval for the molecule to move across the cube and back once, Equation 21.3. The average change in momentum per trip for the time interval for many trips is the same as that for the short duration of the collision. Therefore, we can rewrite Equation 21.2 as

$$\overline{F}_i\,\Delta t = -2m_0 v_{xi} \tag{21.4}$$

where $\overline{F}_i$ is the average force component over the time interval for the molecule to move across the cube and back. Because exactly one collision occurs for each such time interval, this result is also the long-term average force on the molecule over long time intervals containing any number of multiples of Δt.

Equation 21.3 and 21.4 enable us to express the x component of the long-term average force exerted by the wall on the molecule as

$$\overline{F}_i = -\frac{2m_0 v_{xi}}{\Delta t} = -\frac{2m_0 v_{xi}^2}{2d} = -\frac{m_0 v_{xi}^2}{d} \tag{21.5}$$

Now, by Newton's third law, the x component of the long-term average force exerted by the *molecule* on the *wall* is equal in magnitude and opposite in direction:

$$\overline{F}_{i,\text{on wall}} = -\overline{F}_i = -\left(-\frac{m_0 v_{xi}^2}{d}\right) = \frac{m_0 v_{xi}^2}{d} \tag{21.6}$$

The total average force $\overline{F}$ exerted by the gas on the wall is found by adding the average forces exerted by the individual molecules. Adding terms such as those in Equation 21.6 for all molecules gives

$$\overline{F} = \sum_{i=1}^{N} \frac{m_0 v_{xi}^2}{d} = \frac{m_0}{d}\sum_{i=1}^{N} v_{xi}^2 \tag{21.7}$$

where we have factored out the length of the box and the mass m_0 because property 1 tells us that all the molecules are the same. We now impose an additional feature from property 1, that the number of molecules is large. For a small number of molecules, the actual force on the wall would vary with time. It would be nonzero during the short interval of a collision of a molecule with the wall and zero when no molecule happens to be hitting the wall. For a very large number of molecules such as Avogadro's number, however, these variations in force are smoothed out so that the average force given above is the same over *any* time interval. Therefore, the *constant* force F on the wall due to the molecular collisions is

$$F = \frac{m_0}{d}\sum_{i=1}^{N} v_{xi}^2 \tag{21.8}$$

[1]For this discussion, we use a bar over a variable to represent the average value of the variable, such as $\overline{F}$ for the average force, rather than the subscript "avg" that we have used before. This notation is to save confusion because we already have a number of subscripts on variables.

To proceed further, let's consider how to express the average value of the square of the x component of the velocity for N molecules. The traditional average of a set of values is the sum of the values over the number of values:

$$\overline{v_x^{\,2}} = \frac{\sum_{i=1}^{N} v_{xi}^{\,2}}{N} \rightarrow \sum_{i=1}^{N} v_{xi}^{\,2} = N\overline{v_x^{\,2}} \tag{21.9}$$

Using Equation 21.9 to substitute for the sum in Equation 21.8 gives

$$F = \frac{m_0}{d} N\overline{v_x^{\,2}} \tag{21.10}$$

Now let's focus again on one molecule with velocity components v_{xi}, v_{yi}, and v_{zi}. The Pythagorean theorem relates the square of the speed of the molecule to the squares of the velocity components:

$$v_i^2 = v_{xi}^{\,2} + v_{yi}^{\,2} + v_{zi}^{\,2} \tag{21.11}$$

Hence, the average value of v^2 for all the molecules in the container is related to the average values of $v_x^{\,2}$, $v_y^{\,2}$, and $v_z^{\,2}$ according to the expression

$$\overline{v^2} = \overline{v_x^{\,2}} + \overline{v_y^{\,2}} + \overline{v_z^{\,2}} \tag{21.12}$$

Because the motion is isotropic (property 2(a) above), the average values $\overline{v_x^{\,2}}$, $\overline{v_y^{\,2}}$, and $\overline{v_z^{\,2}}$ are equal to one another. Using this fact and Equation 21.12, we find that

$$\overline{v^2} = 3\overline{v_x^{\,2}} \tag{21.13}$$

Therefore, from Equation 21.10, the total force exerted on the wall is

$$F = \tfrac{1}{3}N\frac{m_0\overline{v^2}}{d} \tag{21.14}$$

Using this expression, we can find the total pressure exerted on the wall:

$$P = \frac{F}{A} = \frac{F}{d^2} = \tfrac{1}{3}N\frac{m_0\overline{v^2}}{d^3} = \tfrac{1}{3}\left(\frac{N}{V}\right)m_0\overline{v^2}$$

$$P = \tfrac{2}{3}\left(\frac{N}{V}\right)\left(\tfrac{1}{2}m_0\overline{v^2}\right) \tag{21.15}$$

Relationship between pressure and molecular kinetic energy

where we have recognized the volume V of the cube as d^3.

Equation 21.15 indicates that the pressure of a gas is proportional to (1) the number of molecules per unit volume and (2) the average translational kinetic energy of the molecules, $\tfrac{1}{2}m_0\overline{v^2}$. In analyzing this structural model of an ideal gas, we obtain an important result that relates the macroscopic quantity of pressure to a microscopic quantity, the average value of the square of the molecular speed. Therefore, a key link between the molecular world and the large-scale world has been established.

Notice that Equation 21.15 verifies some features of pressure with which you are probably familiar. One way to increase the pressure inside a container is to increase the number of molecules per unit volume N/V in the container. That is what you do when you add air to a tire. The pressure in the tire can also be raised by increasing the average translational kinetic energy of the air molecules in the tire. That can be accomplished by increasing the temperature of that air, which is why the pressure inside a tire increases as the tire warms up during long road trips. The continuous flexing of the tire as it moves along the road surface results in work done on the rubber as parts of the tire distort, causing an increase in internal energy of the rubber. The increased temperature of the rubber results in the transfer of energy by heat into the air inside the tire. This transfer increases the air's temperature, and this increase in temperature in turn produces an increase in pressure.

Molecular Interpretation of Temperature

Let's now consider another macroscopic variable, the temperature T of the gas. We can gain some insight into the meaning of temperature by first writing Equation 21.15 in the form

$$PV = \frac{2}{3}N\left(\frac{1}{2}m_0\overline{v^2}\right) \tag{21.16}$$

Let's now compare this expression with the equation of state for an ideal gas (Eq. 19.10):

$$PV = Nk_BT \tag{21.17}$$

Equating the right sides of Equations 21.16 and 21.17 and solving for T gives

Relationship between temperature and molecular kinetic energy

$$T = \frac{2}{3k_B}\left(\frac{1}{2}m_0\overline{v^2}\right) \tag{21.18}$$

This result tells us that temperature is a direct measure of average molecular kinetic energy. By rearranging Equation 21.18, we can relate the translational molecular kinetic energy to the temperature:

Average kinetic energy per molecule

$$\frac{1}{2}m_0\overline{v^2} = \frac{3}{2}k_BT \tag{21.19}$$

That is, the average translational kinetic energy per molecule is $\frac{3}{2}k_BT$. Because $\overline{v_x^2} = \frac{1}{3}\overline{v^2}$ (Eq. 21.13), it follows that

$$\frac{1}{2}m_0\overline{v_x^2} = \frac{1}{2}k_BT \tag{21.20}$$

In a similar manner, for the y and z directions,

$$\frac{1}{2}m_0\overline{v_y^2} = \frac{1}{2}k_BT \quad \text{and} \quad \frac{1}{2}m_0\overline{v_z^2} = \frac{1}{2}k_BT$$

Therefore, each translational degree of freedom contributes an equal amount of energy, $\frac{1}{2}k_BT$, to the gas. (In general, a "degree of freedom" refers to an independent means by which a molecule can possess energy.) A generalization of this result, known as the **theorem of equipartition of energy,** is as follows:

Theorem of equipartition of energy

Each degree of freedom contributes $\frac{1}{2}k_BT$ to the energy of a system, where possible degrees of freedom are those associated with translation, rotation, and vibration of molecules.

The total translational kinetic energy of N molecules of gas is simply N times the average energy per molecule, which is given by Equation 21.19:

Total translational kinetic energy of *N* molecules

$$K_{\text{tot trans}} = N\left(\frac{1}{2}m_0\overline{v^2}\right) = \frac{3}{2}Nk_BT = \frac{3}{2}nRT \tag{21.21}$$

where we have used $k_B = R/N_A$ for Boltzmann's constant and $n = N/N_A$ for the number of moles of gas. If the gas molecules possess only translational kinetic energy, Equation 21.21 represents the internal energy of the gas. This result implies that the internal energy of an ideal gas depends *only* on the temperature. We will follow up on this point in Section 21.2.

The square root of $\overline{v^2}$ is called the **root-mean-square (rms) speed** of the molecules. From Equation 21.19, we find that the rms speed is

Root-mean-square speed

$$v_{\text{rms}} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_BT}{m_0}} = \sqrt{\frac{3RT}{M}} \tag{21.22}$$

where M is the molar mass in kilograms per mole and is equal to m_0N_A. This expression shows that, at a given temperature, lighter molecules move faster, on the average, than do heavier molecules. For example, at a given temperature, hydrogen molecules, whose molar mass is 2.02×10^{-3} kg/mol, have an average speed approximately four times that of oxygen molecules, whose molar mass is 32.0×10^{-3} kg/mol. Table 21.1 lists the rms speeds for various molecules at 20°C.

Table 21.1 Some Root-Mean-Square (rms) Speeds

Gas	Molar Mass (g/mol)	v_{rms} at 20°C (m/s)	Gas	Molar Mass (g/mol)	v_{rms} at 20°C (m/s)
H_2	2.02	1902	NO	30.0	494
He	4.00	1352	O_2	32.0	478
H_2O	18.0	637	CO_2	44.0	408
Ne	20.2	602	SO_2	64.1	338
N_2 or CO	28.0	511			

Pitfall Prevention 21.1

The Square Root of the Square? Taking the square root of $\overline{v^2}$ does not "undo" the square because we have taken an average *between* squaring and taking the square root. Although the square root of $(\overline{v})^2$ is $\overline{v} = v_{avg}$ because the squaring is done after the averaging, the square root of $\overline{v^2}$ is *not* v_{avg}, but rather v_{rms}.

Quick Quiz 21.1 Two containers hold an ideal gas at the same temperature and pressure. Both containers hold the same type of gas, but container B has twice the volume of container A. **(i)** What is the average translational kinetic energy per molecule in container B? (a) twice that of container A (b) the same as that of container A (c) half that of container A (d) impossible to determine **(ii)** From the same choices, describe the internal energy of the gas in container B.

Example 21.1 A Tank of Helium

A tank used for filling helium balloons has a volume of 0.300 m^3 and contains 2.00 mol of helium gas at 20.0°C. Assume the helium behaves like an ideal gas.

(A) What is the total translational kinetic energy of the gas molecules?

SOLUTION

Conceptualize Imagine a microscopic model of a gas in which you can watch the molecules move about the container more rapidly as the temperature increases. Because the gas is monatomic, the total translational kinetic energy of the molecules is the internal energy of the gas.

Categorize We evaluate parameters with equations developed in the preceding discussion, so this example is a substitution problem.

Use Equation 21.21 with $n = 2.00$ mol and $T = 293$ K:

$$E_{\text{int}} = \tfrac{3}{2}nRT = \tfrac{3}{2}(2.00\text{ mol})(8.31\text{ J/mol}\cdot\text{K})(293\text{ K})$$
$$= \boxed{7.30 \times 10^3\text{ J}}$$

(B) What is the average kinetic energy per molecule?

SOLUTION

Use Equation 21.19:

$$\tfrac{1}{2}m_0\overline{v^2} = \tfrac{3}{2}k_BT = \tfrac{3}{2}(1.38 \times 10^{-23}\text{ J/K})(293\text{ K})$$
$$= \boxed{6.07 \times 10^{-21}\text{ J}}$$

WHAT IF? What if the temperature is raised from 20.0°C to 40.0°C? Because 40.0 is twice as large as 20.0, is the total translational energy of the molecules of the gas twice as large at the higher temperature?

Answer The expression for the total translational energy depends on the temperature, and the value for the temperature must be expressed in kelvins, not in degrees Celsius. Therefore, the ratio of 40.0 to 20.0 is *not* the appropriate ratio. Converting the Celsius temperatures to kelvins, 20.0°C is 293 K and 40.0°C is 313 K. Therefore, the total translational energy increases by a factor of only 313 K/293 K = 1.07.

21.2 Molar Specific Heat of an Ideal Gas

Consider an ideal gas undergoing several processes such that the change in temperature is $\Delta T = T_f - T_i$ for all processes. The temperature change can be achieved

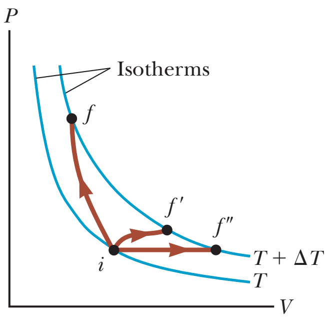


Figure 21.3 An ideal gas is taken from one isotherm at temperature T to another at temperature $T + \Delta T$ along three different paths.

by taking a variety of paths from one isotherm to another as shown in Figure 21.3. Because ΔT is the same for all paths, the change in internal energy ΔE_{int} is the same for all paths. The work W done on the gas (the negative of the area under the curves), however, is different for each path. Therefore, from the first law of thermodynamics, we can argue that the heat $Q = \Delta E_{\text{int}} - W$ associated with a given change in temperature does *not* have a unique value as discussed in Section 20.4.

We can address this difficulty by defining specific heats for two special processes that we have studied: isovolumetric and isobaric. Because the number of moles n is a convenient measure of the amount of gas, we define the **molar specific heats** associated with these processes as follows:

$$Q = nC_V \Delta T \quad \text{(constant volume)} \tag{21.23}$$

$$Q = nC_P \Delta T \quad \text{(constant pressure)} \tag{21.24}$$

where C_V is the **molar specific heat at constant volume** and C_P is the **molar specific heat at constant pressure.** When energy is added to a gas by heat at constant pressure, not only does the internal energy of the gas increase, but (negative) work is done on the gas because of the change in volume required to keep the pressure constant. Therefore, the heat Q in Equation 21.24 must account for both the increase in internal energy and the transfer of energy out of the system by work. For this reason, Q is greater in Equation 21.24 than in Equation 21.23 for given values of n and ΔT. Therefore, C_P is greater than C_V.

In the previous section, we found that the temperature of a gas is a measure of the average translational kinetic energy of the gas molecules. This kinetic energy is associated with the motion of the center of mass of each molecule. It does not include the energy associated with the internal motion of the molecule, namely, vibrations and rotations about the center of mass. That should not be surprising because the simple kinetic theory model assumes a structureless molecule.

So, let's first consider the simplest case of an ideal monatomic gas, that is, a gas containing one atom per molecule such as helium, neon, or argon. When energy is added to a monatomic gas in a container of fixed volume, all the added energy goes into increasing the translational kinetic energy of the atoms. There is no other way to store the energy in a monatomic gas. Therefore, from Equation 21.21, we see that the internal energy E_{int} of N molecules (or n mol) of an ideal monatomic gas is

Internal energy of an ideal monatomic gas

$$E_{\text{int}} = K_{\text{tot trans}} = \tfrac{3}{2}Nk_{\text{B}}T = \tfrac{3}{2}nRT \tag{21.25}$$

For a monatomic ideal gas, E_{int} is a function of T only and the functional relationship is given by Equation 21.25. In general, the internal energy of any ideal gas is a function of T only and the exact relationship depends on the type of gas.

If energy is transferred by heat to a system at constant volume, no work is done on the system. That is, $W = -\int P\,dV = 0$ for a constant-volume process. Hence, from the first law of thermodynamics,

$$Q = \Delta E_{\text{int}} \tag{21.26}$$

In other words, all the energy transferred by heat goes into increasing the internal energy of the system. A constant-volume process from i to f for an ideal gas is described in Figure 21.4, where ΔT is the temperature difference between the two isotherms. Substituting the expression for Q given by Equation 21.23 into Equation 21.26, we obtain

$$\boxed{\Delta E_{\text{int}} = nC_V\,\Delta T} \tag{21.27}$$

This equation applies to all ideal gases, those gases having more than one atom per molecule as well as monatomic ideal gases.

In the limit of infinitesimal changes, we can use Equation 21.27 to express the molar specific heat at constant volume as

$$C_V = \frac{1}{n}\frac{dE_{\text{int}}}{dT} \tag{21.28}$$

Let's now apply the results of this discussion to a monatomic gas. Substituting the internal energy from Equation 21.25 into Equation 21.28 gives

$$C_V = \tfrac{3}{2}R = 12.5\ \text{J/mol}\cdot\text{K} \qquad \textbf{(21.29)}$$

This expression predicts a value of $C_V = \frac{3}{2}R$ for *all* monatomic gases. This prediction is in excellent agreement with measured values of molar specific heats for such gases as helium, neon, argon, and xenon over a wide range of temperatures (Table 21.2). Small variations in Table 21.2 from the predicted values are because real gases are not ideal gases. In real gases, weak intermolecular interactions occur, which are not addressed in our ideal gas model.

Now suppose the gas is taken along the constant-pressure path $i \rightarrow f'$ shown in Figure 21.4. Along this path, the temperature again increases by ΔT. The energy that must be transferred by heat to the gas in this process is $Q = nC_P\,\Delta T$. Because the volume changes in this process, the work done on the gas is $W = -P\,\Delta V$, where P is the constant pressure at which the process occurs. Applying the first law of thermodynamics to this process, we have

$$\Delta E_{\text{int}} = Q + W = nC_P\,\Delta T + (-P\,\Delta V) \qquad \textbf{(21.30)}$$

In this case, the energy added to the gas by heat is channeled as follows. Part of it leaves the system by work (that is, the gas moves a piston through a displacement), and the remainder appears as an increase in the internal energy of the gas. The change in internal energy for the process $i \rightarrow f'$, however, is equal to that for the process $i \rightarrow f$ because E_{int} depends only on temperature for an ideal gas and ΔT is the same for both processes. In addition, because $PV = nRT$, note that for a constant-pressure process, $P\,\Delta V = nR\,\Delta T$. Substituting this value for $P\,\Delta V$ into Equation 21.30 with $\Delta E_{\text{int}} = nC_V\,\Delta T$ (Eq. 21.27) gives

$$nC_V\,\Delta T = nC_P\,\Delta T - nR\,\Delta T$$

$$C_P - C_V = R \qquad \textbf{(21.31)}$$

This expression applies to *any* ideal gas. It predicts that the molar specific heat of an ideal gas at constant pressure is greater than the molar specific heat at constant volume by an amount R, the universal gas constant (which has the value 8.31 J/mol · K). This expression is applicable to real gases as the data in Table 21.2 show.

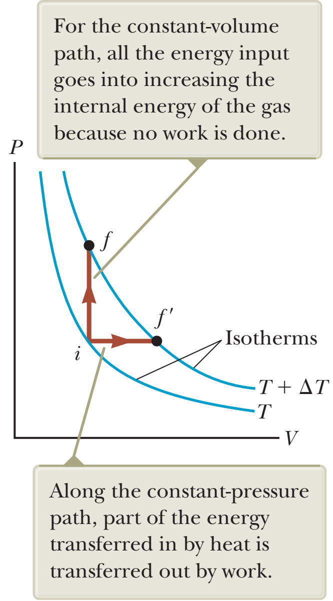


Figure 21.4 Energy is transferred by heat to an ideal gas in two ways.

Table 21.2 Molar Specific Heats of Various Gases

Gas	Molar Specific Heat (J/mol · K)[a] C_P	C_V	$C_P - C_V$	$\gamma = C_P/C_V$
Monatomic gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Ne	20.8	12.7	8.12	1.64
Kr	20.8	12.3	8.49	1.69
Diatomic gases				
H_2	28.8	20.4	8.33	1.41
N_2	29.1	20.8	8.33	1.40
O_2	29.4	21.1	8.33	1.40
CO	29.3	21.0	8.33	1.40
Cl_2	34.7	25.7	8.96	1.35
Polyatomic gases				
CO_2	37.0	28.5	8.50	1.30
SO_2	40.4	31.4	9.00	1.29
H_2O	35.4	27.0	8.37	1.30
CH_4	35.5	27.1	8.41	1.31

[a] All values except that for water were obtained at 300 K.

Because $C_V = \frac{3}{2}R$ for a monatomic ideal gas, Equation 21.31 predicts a value $C_P = \frac{5}{2}R = 20.8$ J/mol · K for the molar specific heat of a monatomic gas at constant pressure. The ratio of these molar specific heats is a dimensionless quantity γ (Greek letter gamma):

Ratio of molar specific heats for a monatomic ideal gas

$$\gamma = \frac{C_P}{C_V} = \frac{5R/2}{3R/2} = \frac{5}{3} = 1.67 \qquad \textbf{(21.32)}$$

Theoretical values of C_V, C_P, and γ are in excellent agreement with experimental values obtained for monatomic gases, but they are in serious disagreement with the values for the more complex gases (see Table 21.2). That is not surprising; the value $C_V = \frac{3}{2}R$ was derived for a monatomic ideal gas, and we expect some additional contribution to the molar specific heat from the internal structure of the more complex molecules. In Section 21.3, we describe the effect of molecular structure on the molar specific heat of a gas. The internal energy—and hence the molar specific heat—of a complex gas must include contributions from the rotational and the vibrational motions of the molecule.

In the case of solids and liquids heated at constant pressure, very little work is done during such a process because the thermal expansion is small. Consequently, C_P and C_V are approximately equal for solids and liquids.

Quick Quiz 21.2 **(i)** How does the internal energy of an ideal gas change as it follows path $i \rightarrow f$ in Figure 21.4? (a) E_{int} increases. (b) E_{int} decreases. (c) E_{int} stays the same. (d) There is not enough information to determine how E_{int} changes. **(ii)** From the same choices, how does the internal energy of an ideal gas change as it follows path $f \rightarrow f'$ along the isotherm labeled $T + \Delta T$ in Figure 21.4?

Example 21.2 Heating a Cylinder of Helium

A cylinder contains 3.00 mol of helium gas at a temperature of 300 K.

(A) If the gas is heated at constant volume, how much energy must be transferred by heat to the gas for its temperature to increase to 500 K?

SOLUTION

Conceptualize Run the process in your mind with the help of the piston–cylinder arrangement in Figure 19.12. Imagine that the piston is clamped in position to maintain the constant volume of the gas.

Categorize We evaluate parameters with equations developed in the preceding discussion, so this example is a substitution problem.

Use Equation 21.23 to find the energy transfer:

$$Q_1 = nC_V\,\Delta T$$

Substitute the given values:

$$Q_1 = (3.00\text{ mol})(12.5\text{ J/mol}\cdot\text{K})(500\text{ K} - 300\text{ K})$$
$$= \boxed{7.50 \times 10^3\text{ J}}$$

(B) How much energy must be transferred by heat to the gas at constant pressure to raise the temperature to 500 K?

SOLUTION

Use Equation 21.24 to find the energy transfer:

$$Q_2 = nC_P\,\Delta T$$

Substitute the given values:

$$Q_2 = (3.00\text{ mol})(20.8\text{ J/mol}\cdot\text{K})(500\text{ K} - 300\text{ K})$$
$$= \boxed{12.5 \times 10^3\text{ J}}$$

This value is larger than Q_1 because of the transfer of energy out of the gas by work to raise the piston in the constant pressure process.

21.3 The Equipartition of Energy

Predictions based on our model for molar specific heat agree quite well with the behavior of monatomic gases, but not with the behavior of complex gases (see Table 21.2). The value predicted by the model for the quantity $C_P - C_V = R$, however, is the same for all gases. This similarity is not surprising because this difference is the result of the work done on the gas, which is independent of its molecular structure.

To clarify the variations in C_V and C_P in gases more complex than monatomic gases, let's explore further the origin of molar specific heat. So far, we have assumed the sole contribution to the internal energy of a gas is the translational kinetic energy of the molecules. The internal energy of a gas, however, includes contributions from the translational, vibrational, and rotational motion of the molecules. The rotational and vibrational motions of molecules can be activated by collisions and therefore are "coupled" to the translational motion of the molecules. The branch of physics known as *statistical mechanics* has shown that, for a large number of particles obeying the laws of Newtonian mechanics, the available energy is, on average, shared equally by each independent degree of freedom. Recall from Section 21.1 that the equipartition theorem states that, at equilibrium, each degree of freedom contributes $\frac{1}{2}k_BT$ of energy per molecule.

Let's consider a diatomic gas whose molecules have the shape of a dumbbell (Fig. 21.5). In this model, the center of mass of the molecule can translate in the x, y, and z directions (Fig. 21.5a). In addition, the molecule can rotate about three mutually perpendicular axes (Fig. 21.5b). The rotation about the y axis can be neglected because the molecule's moment of inertia I_y and its rotational energy $\frac{1}{2}I_y\omega^2$ about this axis are negligible compared with those associated with the x and z axes. (If the two atoms are modeled as particles, then I_y is identically zero.) Therefore, there are five degrees of freedom for translation and rotation: three associated with the translational motion and two associated with the rotational motion. Because each degree of freedom contributes, on average, $\frac{1}{2}k_BT$ of energy per molecule, the internal energy for a system of N molecules, ignoring vibration for now, is

$$E_{int} = 3N(\tfrac{1}{2}k_BT) + 2N(\tfrac{1}{2}k_BT) = \tfrac{5}{2}Nk_BT = \tfrac{5}{2}nRT$$

We can use this result and Equation 21.28 to find the molar specific heat at constant volume:

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT} = \frac{1}{n}\frac{d}{dT}(\tfrac{5}{2}nRT) = \tfrac{5}{2}R = 20.8\ \text{J/mol}\cdot\text{K} \qquad \textbf{(21.33)}$$

From Equations 21.31 and 21.32, we find that

$$C_P = C_V + R = \tfrac{7}{2}R = 29.1\ \text{J/mol}\cdot\text{K}$$

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = \frac{7}{5} = 1.40$$

These results agree quite well with most of the data for diatomic molecules given in Table 21.2. That is rather surprising because we have not yet accounted for the possible vibrations of the molecule.

In the model for vibration, the two atoms are joined by an imaginary spring (see Fig. 21.5c). The vibrational motion adds two more degrees of freedom, which correspond to the kinetic energy and the potential energy associated with vibrations along the length of the molecule. Hence, a model that includes all three types of motion predicts a total internal energy of

$$E_{int} = 3N(\tfrac{1}{2}k_BT) + 2N(\tfrac{1}{2}k_BT) + 2N(\tfrac{1}{2}k_BT) = \tfrac{7}{2}Nk_BT = \tfrac{7}{2}nRT$$

and a molar specific heat at constant volume of

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT} = \frac{1}{n}\frac{d}{dT}(\tfrac{7}{2}nRT) = \tfrac{7}{2}R = 29.1\ \text{J/mol}\cdot\text{K} \qquad \textbf{(21.34)}$$

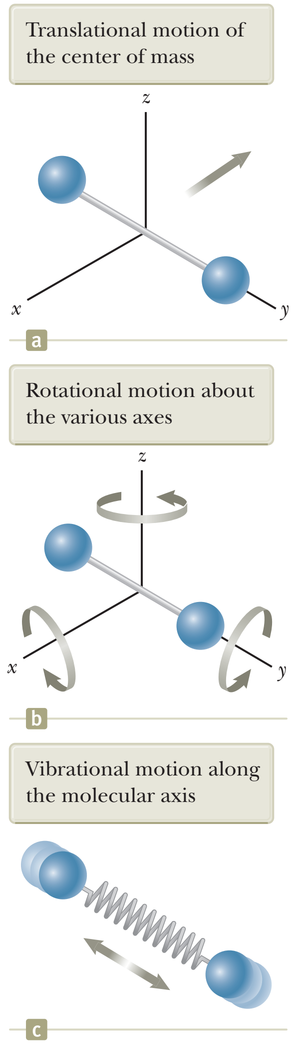


Figure 21.5 Possible motions of a diatomic molecule.

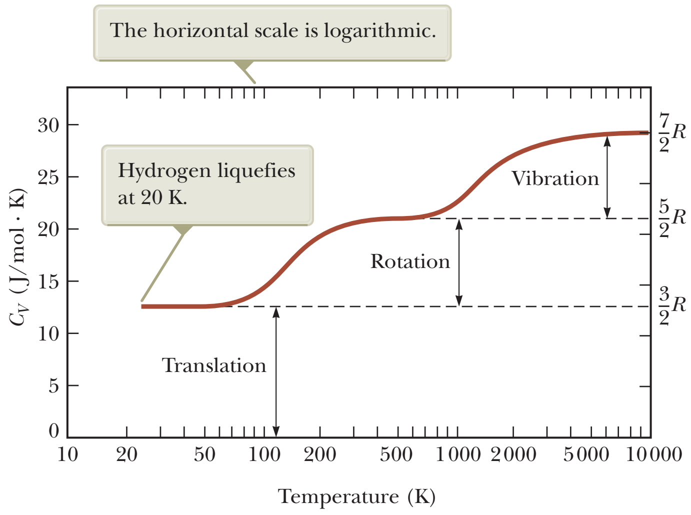


Figure 21.6 The molar specific heat of hydrogen as a function of temperature.

This value is inconsistent with experimental data for molecules such as H_2 and N_2 (see Table 21.2) and suggests a breakdown of our model based on classical physics.

It might seem that our model is a failure for predicting molar specific heats for diatomic gases. We can claim some success for our model, however, if measurements of molar specific heat are made over a wide temperature range rather than at the single temperature that gives us the values in Table 21.2. Figure 21.6 shows the molar specific heat of hydrogen as a function of temperature. The remarkable feature about the three plateaus in the graph's curve is that they are at the values of the molar specific heat predicted by Equations 21.29, 21.33, and 21.34! For low temperatures, the diatomic hydrogen gas behaves like a monatomic gas. As the temperature rises to room temperature, its molar specific heat rises to a value for a diatomic gas, consistent with the inclusion of rotation but not vibration. For high temperatures, the molar specific heat is consistent with a model including all types of motion.

Before addressing the reason for this mysterious behavior, let's make some brief remarks about polyatomic gases. For molecules with more than two atoms, three axes of rotation are available. The vibrations are more complex than for diatomic molecules. Therefore, the number of degrees of freedom is even larger. The result is an even higher predicted molar specific heat, which is in qualitative agreement with experiment. The molar specific heats for the polyatomic gases in Table 21.2 are higher than those for diatomic gases. The more degrees of freedom available to a molecule, the more "ways" there are to store energy, resulting in a higher molar specific heat.

A Hint of Energy Quantization

Our model for molar specific heats has been based so far on purely classical notions. It predicts a value of the specific heat for a diatomic gas that, according to Figure 21.6, only agrees with experimental measurements made at high temperatures. To explain why this value is only true at high temperatures and why the plateaus in Figure 21.6 exist, we must go beyond classical physics and introduce some quantum physics into the model. In Chapter 18, we discussed quantization of frequency for vibrating strings and air columns; only certain frequencies of standing waves can exist. That is a natural result whenever waves are subject to boundary conditions.

Quantum physics (Chapters 40 through 43) shows that atoms and molecules can be described by the waves under boundary conditions analysis model. Consequently, these waves have quantized frequencies. Furthermore, in quantum physics, the energy of a system is proportional to the frequency of the wave representing the system. Hence, **the energies of atoms and molecules are quantized.**

For a molecule, quantum physics tells us that the rotational and vibrational energies are quantized. Figure 21.7 shows an **energy-level diagram** for the rotational and vibrational quantum states of a diatomic molecule. The lowest allowed state is called the **ground state.** The black lines show the energies allowed for the molecule. Notice that allowed vibrational states are separated by larger energy gaps than are rotational states.

At low temperatures, the energy a molecule gains in collisions with its neighbors is generally not large enough to raise it to the first excited state of either rotation or vibration. Therefore, even though rotation and vibration are allowed according to classical physics, they do not occur in reality at low temperatures. All molecules are in the ground state for rotation and vibration. The only contribution to the molecules' average energy is from translation, and the specific heat is that predicted by Equation 21.29.

As the temperature is raised, the average energy of the molecules increases. In some collisions, a molecule may have enough energy transferred to it from another molecule to excite the first rotational state. As the temperature is raised further, more molecules can be excited to this state. The result is that rotation begins to contribute to the internal energy, and the molar specific heat rises. At about room temperature in Figure 21.6, the second plateau has been reached and rotation contributes fully to the molar specific heat. The molar specific heat is now equal to the value predicted by Equation 21.33.

There is no contribution at room temperature from vibration because the molecules are still in the ground vibrational state. The temperature must be raised even further to excite the first vibrational state, which happens in Figure 21.6 between 1 000 K and 10 000 K. At 10 000 K on the right side of the figure, vibration is contributing fully to the internal energy and the molar specific heat has the value predicted by Equation 21.34.

The predictions of this model are supportive of the theorem of equipartition of energy. In addition, the inclusion in the model of energy quantization from quantum physics allows a full understanding of Figure 21.6.

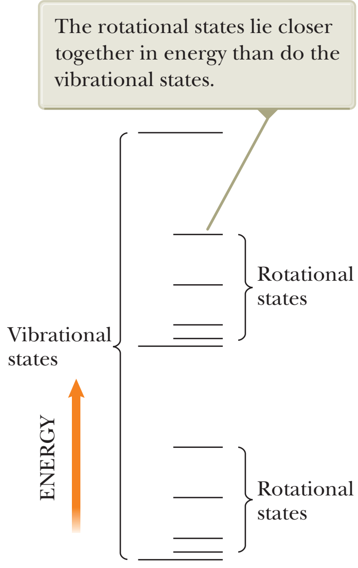


Figure 21.7 An energy-level diagram for vibrational and rotational states of a diatomic molecule.

Quick Quiz 21.3 The molar specific heat of a diatomic gas is measured at constant volume and found to be 29.1 J/mol · K. What are the types of energy that are contributing to the molar specific heat? **(a)** translation only **(b)** translation and rotation only **(c)** translation and vibration only **(d)** translation, rotation, and vibration

Quick Quiz 21.4 The molar specific heat of a gas is measured at constant volume and found to be $11R/2$. Is the gas most likely to be **(a)** monatomic, **(b)** diatomic, or **(c)** polyatomic?

21.4 Adiabatic Processes for an Ideal Gas

As noted in Section 20.6, an **adiabatic process** is one in which no energy is transferred by heat between a system and its surroundings. For example, if a gas is compressed (or expanded) rapidly, very little energy is transferred out of (or into) the system by heat, so the process is nearly adiabatic. Such processes occur in the cycle of a gasoline engine, which is discussed in detail in Chapter 22. Another example of an adiabatic process is the slow expansion of a gas that is thermally insulated from its surroundings. All three variables in the ideal gas law—P, V, and T—change during an adiabatic process.

Let's imagine an adiabatic gas process involving an infinitesimal change in volume dV and an accompanying infinitesimal change in temperature dT. The work done on the gas is $-P\,dV$. Because the internal energy of an ideal gas depends only on temperature, the change in the internal energy in an adiabatic process is the same as that for an isovolumetric process between the same temperatures, $dE_{\text{int}} = nC_V\,dT$ (Eq. 21.27). Hence, the first law of thermodynamics, $\Delta E_{\text{int}} = Q + W$, with $Q = 0$, becomes the infinitesimal form

$$dE_{\text{int}} = nC_V\,dT = -P\,dV \tag{21.35}$$

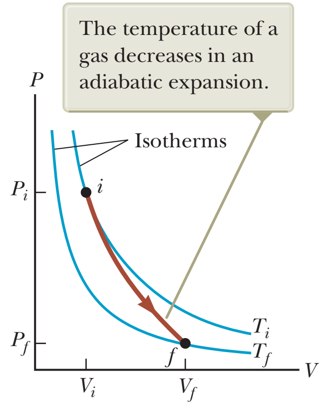


Figure 21.8 The PV diagram for an adiabatic expansion of an ideal gas.

Taking the total differential of the equation of state of an ideal gas, $PV = nRT$, gives

$$P\,dV + V\,dP = nR\,dT \tag{21.36}$$

Eliminating dT from Equations 21.35 and 21.36, we find that

$$P\,dV + V\,dP = -\frac{R}{C_V}P\,dV$$

Substituting $R = C_P - C_V$ and dividing by PV gives

$$\frac{dV}{V} + \frac{dP}{P} = -\left(\frac{C_P - C_V}{C_V}\right)\frac{dV}{V} = (1-\gamma)\frac{dV}{V}$$

$$\frac{dP}{P} + \gamma\frac{dV}{V} = 0$$

Integrating this expression, we have

$$\ln P + \gamma \ln V = \text{constant}$$

which is equivalent to

Relationship between P and V for an adiabatic process involving an ideal gas

$$PV^\gamma = \text{constant} \tag{21.37}$$

The PV diagram for an adiabatic expansion is shown in Figure 21.8. Because $\gamma > 1$, the PV curve is steeper than it would be for an isothermal expansion, for which $PV =$ constant. By the definition of an adiabatic process, no energy is transferred by heat into or out of the system. Hence, from the first law, we see that ΔE_{int} is negative (work is done *by* the gas, so its internal energy decreases) and so ΔT also is negative. Therefore, the temperature of the gas decreases ($T_f < T_i$) during an adiabatic expansion.[2] Conversely, the temperature increases if the gas is compressed adiabatically. Applying Equation 21.37 to the initial and final states, we see that

$$P_iV_i^\gamma = P_fV_f^\gamma \tag{21.38}$$

Using the ideal gas law, we can express Equation 21.37 as

Relationship between T and V for an adiabatic process involving an ideal gas

$$TV^{\gamma-1} = \text{constant} \tag{21.39}$$

Example 21.3 A Diesel Engine Cylinder

Air at 20.0°C in the cylinder of a diesel engine is compressed from an initial pressure of 1.00 atm and volume of 800.0 cm^3 to a volume of 60.0 cm^3. Assume air behaves as an ideal gas with $\gamma = 1.40$ and the compression is adiabatic. Find the final pressure and temperature of the air.

SOLUTION

Conceptualize Imagine what happens if a gas is compressed into a smaller volume. Our discussion above and Figure 21.8 tell us that the pressure and temperature both increase.

Categorize We categorize this example as a problem involving an adiabatic process.

Analyze Use Equation 21.38 to find the final pressure:

$$P_f = P_i\left(\frac{V_i}{V_f}\right)^\gamma = (1.00\text{ atm})\left(\frac{800.0\text{ cm}^3}{60.0\text{ cm}^3}\right)^{1.40}$$

$$= \boxed{37.6\text{ atm}}$$

[2]In the adiabatic free expansion discussed in Section 20.6, the temperature remains constant. In this unique process, no work is done because the gas expands into a vacuum. In general, the temperature decreases in an adiabatic expansion in which work is done.

21.3 continued

Use the ideal gas law to find the final temperature:

$$\frac{P_iV_i}{T_i} = \frac{P_fV_f}{T_f}$$

$$T_f = \frac{P_f V_f}{P_iV_i}T_i = \frac{(37.6 \text{ atm})(60.0 \text{ cm}^3)}{(1.00 \text{ atm})(800.0 \text{ cm}^3)}(293 \text{ K})$$

$$= 826 \text{ K} = \boxed{553°\text{C}}$$

Finalize The temperature of the gas increases by a factor of 826 K/293 K = 2.82. The high compression in a diesel engine raises the temperature of the gas enough to cause the combustion of fuel without the use of spark plugs.

21.5 Distribution of Molecular Speeds

Thus far, we have considered only average values of the energies of all the molecules in a gas and have not addressed the distribution of energies among individual molecules. The motion of the molecules is extremely chaotic. Any individual molecule collides with others at an enormous rate, typically a billion times per second. Each collision results in a change in the speed and direction of motion of each of the participant molecules. Equation 21.22 shows that rms molecular speeds increase with increasing temperature. At a given time, what is the relative number of molecules that possess some characteristic such as energy within a certain range?

We shall address this question by considering the **number density** $n_V(E)$. This quantity, called a *distribution function,* is defined so that $n_V(E)\ dE$ is the number of molecules per unit volume with energy between E and $E + dE$. (The ratio of the number of molecules that have the desired characteristic to the total number of molecules is the probability that a particular molecule has that characteristic.) In general, the number density is found from statistical mechanics to be

Pitfall Prevention 21.2

The Distribution Function

The distribution function $n_V(E)$ is defined in terms of the number of molecules with energy in the range E to $E + dE$ rather than in terms of the number of molecules with energy E. Because the number of molecules is finite and the number of possible values of the energy is infinite, the number of molecules with an *exact* energy E may be zero.

$$n_V(E) = n_0e^{-E/k_BT} \qquad \textbf{(21.40)}$$

◀ **Boltzmann distribution law**

where n_0 is defined such that $n_0\ dE$ is the number of molecules per unit volume having energy between $E = 0$ and $E = dE$. This equation, known as the **Boltzmann distribution law,** is important in describing the statistical mechanics of a large number of molecules. It states that the probability of finding the molecules in a particular energy state varies exponentially as the negative of the energy divided by k_BT. All the molecules would fall into the lowest energy level if the thermal agitation at a temperature T did not excite the molecules to higher energy levels.

Example 21.4 Thermal Excitation of Atomic Energy Levels

As discussed in Section 21.4, atoms can occupy only certain discrete energy levels. Consider a gas at a temperature of 2 500 K whose atoms can occupy only two energy levels separated by 1.50 eV, where 1 eV (electron volt) is an energy unit equal to 1.60×10^{-19} J (Fig. 21.9). Determine the ratio of the number of atoms in the higher energy level to the number in the lower energy level.

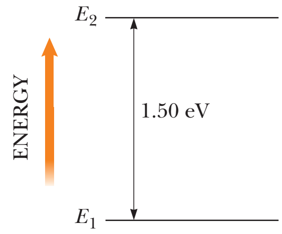


Figure 21.9 (Example 21.4) Energy-level diagram for a gas whose atoms can occupy two energy states.

SOLUTION

Conceptualize In your mental representation of this example, remember that only two possible states are allowed for the system of the atom. Figure 21.9 helps you visualize the two states on an energy-level diagram. In this case, the atom has two possible energies, E_1 and E_2, where $E_1 < E_2$.

continued

21.4 continued

Categorize We categorize this example as one in which we focus on particles in a two-state quantized system. We will apply the Boltzmann distribution law to this system.

Analyze Set up the ratio of the number of atoms in the higher energy level to the number in the lower energy level and use Equation 21.40 to express each number:

$$(1) \quad \frac{n_V(E_2)}{n_V(E_1)} = \frac{n_0 e^{-E_2/k_B T}}{n_0 e^{-E_1/k_B T}} = e^{-(E_2 - E_1)/k_B T}$$

Evaluate $k_B T$ in the exponent:

$$k_B T = (1.38 \times 10^{-23}\ \mathrm{J/K})(2\,500\ \mathrm{K})\left(\frac{1\ \mathrm{eV}}{1.60 \times 10^{-19}\ \mathrm{J}}\right) = 0.216\ \mathrm{eV}$$

Substitute this value into Equation (1):

$$\frac{n_V(E_2)}{n_V(E_1)} = e^{-1.50\ \mathrm{eV}/0.216\ \mathrm{eV}} = e^{-6.96} = \boxed{9.52 \times 10^{-4}}$$

Finalize This result indicates that at $T = 2\,500$ K, only a small fraction of the atoms are in the higher energy level. In fact, for every atom in the higher energy level, there are about 1 000 atoms in the lower level. The number of atoms in the higher level increases at even higher temperatures, but the distribution law specifies that at equilibrium there are always more atoms in the lower level than in the higher level.

WHAT IF? What if the energy levels in Figure 21.9 were closer together in energy? Would that increase or decrease the fraction of the atoms in the upper energy level?

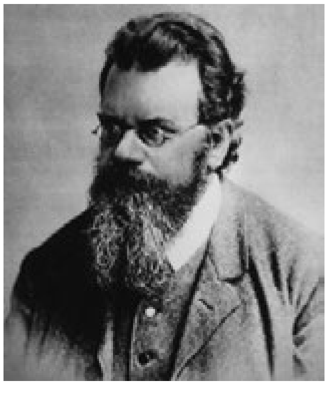
Answer If the excited level is lower in energy than that in Figure 21.9, it would be easier for thermal agitation to excite atoms to this level and the fraction of atoms in this energy level would be larger, which we can see mathematically by expressing Equation (1) as

$$r_2 = e^{-(E_2 - E_1)/k_B T}$$

where r_2 is the ratio of atoms having energy E_2 to those with energy E_1. Differentiating with respect to E_2, we find

$$\frac{dr_2}{dE_2} = \frac{d}{dE_2}\left[e^{-(E_2 - E_1)/k_B T}\right] = -\frac{1}{k_B T} e^{-(E_2 - E_1)/k_B T} < 0$$

Because the derivative has a negative value, as E_2 decreases, r_2 increases.



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Ludwig Boltzmann
Austrian physicist (1844–1906)
Boltzmann made many important contributions to the development of the kinetic theory of gases, electromagnetism, and thermodynamics. His pioneering work in the field of kinetic theory led to the branch of physics known as statistical mechanics.

Now that we have discussed the distribution of energies among molecules in a gas, let's think about the distribution of molecular speeds. In 1860, James Clerk Maxwell (1831–1879) derived an expression that describes the distribution of molecular speeds in a very definite manner. His work and subsequent developments by other scientists were highly controversial because direct detection of molecules could not be achieved experimentally at that time. About 60 years later, however, experiments were devised that confirmed Maxwell's predictions.

Let's consider a container of gas whose molecules have some distribution of speeds. Suppose we want to determine how many gas molecules have a speed in the range from, for example, 400 to 401 m/s. Intuitively, we expect the speed distribution to depend on temperature. Furthermore, we expect the distribution to peak in the vicinity of v_{rms}. That is, few molecules are expected to have speeds much less than or much greater than v_{rms} because these extreme speeds result only from an unlikely chain of collisions.

The observed speed distribution of gas molecules in thermal equilibrium is shown in Figure 21.10. The quantity N_v, called the **Maxwell–Boltzmann speed distribution function,** is defined as follows. If N is the total number of molecules, the number of molecules with speeds between v and $v + dv$ is $dN = N_v\,dv$. This number is also equal to the area of the shaded rectangle in Figure 21.10. Furthermore, the fraction of molecules with speeds between v and $v + dv$ is $(N_v\,dv)/N$. This fraction is also equal to the probability that a molecule has a speed in the range v to $v + dv$.

The fundamental expression that describes the distribution of speeds of N gas molecules is

$$N_v = 4\pi N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} v^2 e^{-m_0 v^2/2k_B T} \quad \textbf{(21.41)}$$

where m_0 is the mass of a gas molecule, k_B is Boltzmann's constant, and T is the absolute temperature.[3] Observe the appearance of the Boltzmann factor $e^{-E/k_B T}$ with $E = \frac{1}{2}m_0 v^2$.

As indicated in Figure 21.10, the average speed is somewhat lower than the rms speed. The *most probable speed* v_{mp} is the speed at which the distribution curve reaches a peak. Using Equation 21.41, we find that

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_B T}{m_0}} = 1.73\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.42)}$$

$$v_{avg} = \sqrt{\frac{8k_B T}{\pi m_0}} = 1.60\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.43)}$$

$$v_{mp} = \sqrt{\frac{2k_B T}{m_0}} = 1.41\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.44)}$$

Equation 21.42 has previously appeared as Equation 21.22. The details of the derivations of these equations from Equation 21.41 are left for the end-of-chapter problems (see Problems 42 and 69). From these equations, we see that

$$v_{rms} > v_{avg} > v_{mp}$$

Figure 21.11 represents speed distribution curves for nitrogen, N_2. The curves were obtained by using Equation 21.41 to evaluate the distribution function at various speeds and at two temperatures. Notice that the peak in each curve shifts to the right as T increases, indicating that the average speed increases with increasing temperature, as expected. Because the lowest speed possible is zero and the upper classical limit of the speed is infinity, the curves are asymmetrical. (In Chapter 39, we show that the actual upper limit is the speed of light.)

Equation 21.41 shows that the distribution of molecular speeds in a gas depends both on mass and on temperature. At a given temperature, the fraction of molecules with speeds exceeding a fixed value increases as the mass decreases. Hence,

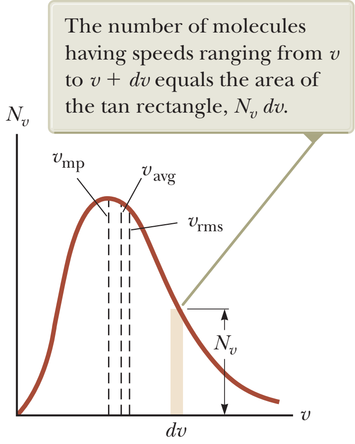


Figure 21.10 The speed distribution of gas molecules at some temperature. The function N_v approaches zero as v approaches infinity.

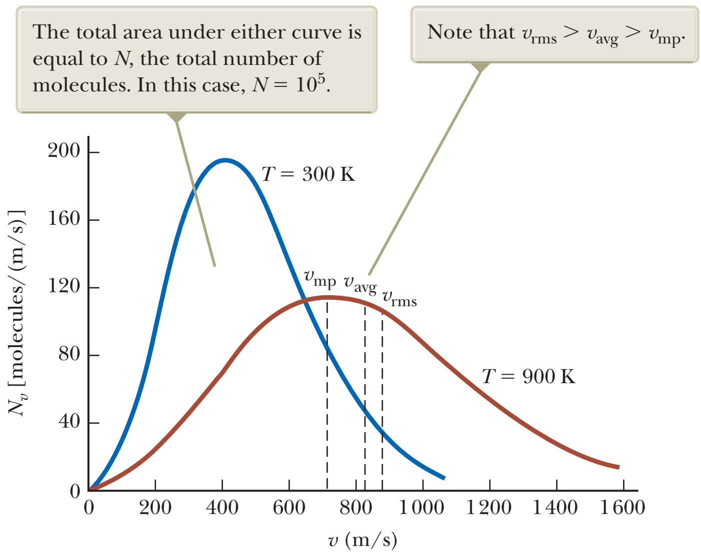


Figure 21.11 The speed distribution function for 10^5 nitrogen molecules at 300 K and 900 K.

[3] For the derivation of this expression, see an advanced textbook on thermodynamics.

lighter molecules such as H_2 and He escape into space more readily from the Earth's atmosphere than do heavier molecules such as N_2 and O_2. (See the discussion of escape speed in Chapter 13. Gas molecules escape even more readily from the Moon's surface than from the Earth's because the escape speed on the Moon is lower than that on the Earth.)

The speed distribution curves for molecules in a liquid are similar to those shown in Figure 21.11. We can understand the phenomenon of evaporation of a liquid from this distribution in speeds, given that some molecules in the liquid are more energetic than others. Some of the faster-moving molecules in the liquid penetrate the surface and even leave the liquid at temperatures well below the boiling point. The molecules that escape the liquid by evaporation are those that have sufficient energy to overcome the attractive forces of the molecules in the liquid phase. Consequently, the molecules left behind in the liquid phase have a lower average kinetic energy; as a result, the temperature of the liquid decreases. Hence, evaporation is a cooling process. For example, an alcohol-soaked cloth can be placed on a feverish head to cool and comfort a patient.

Example 21.5 A System of Nine Particles

Nine particles have speeds of 5.00, 8.00, 12.0, 12.0, 12.0, 14.0, 14.0, 17.0, and 20.0 m/s.

(A) Find the particles' average speed.

SOLUTION

Conceptualize Imagine a small number of particles moving in random directions with the few speeds listed. This situation is not representative of the large number of molecules in a gas, so we should not expect the results to be consistent with those from statistical mechanics.

Categorize Because we are dealing with a small number of particles, we can calculate the average speed directly.

Analyze Find the average speed of the particles by dividing the sum of the speeds by the total number of particles:

$$v_{avg} = \frac{(5.00 + 8.00 + 12.0 + 12.0 + 12.0 + 14.0 + 14.0 + 17.0 + 20.0)\text{ m/s}}{9}$$

$$= \boxed{12.7\text{ m/s}}$$

(B) What is the rms speed of the particles?

SOLUTION

Find the average speed squared of the particles by dividing the sum of the speeds squared by the total number of particles:

$$\overline{v^2} = \frac{(5.00^2 + 8.00^2 + 12.0^2 + 12.0^2 + 12.0^2 + 14.0^2 + 14.0^2 + 17.0^2 + 20.0^2)\text{ m}^2/\text{s}^2}{9}$$

$$= 178\text{ m}^2/\text{s}^2$$

Find the rms speed of the particles by taking the square root:

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{178\text{ m}^2/\text{s}^2} = \boxed{13.3\text{ m/s}}$$

(C) What is the most probable speed of the particles?

SOLUTION

Three of the particles have a speed of 12.0 m/s, two have a speed of 14.0 m/s, and the remaining four have different speeds. Hence, the most probable speed v_{mp} is 12.0 m/s.

Finalize Compare this example, in which the number of particles is small and we know the individual particle speeds, with the next example.

Example 21.6 Molecular Speeds in a Hydrogen Gas

A 0.500-mol sample of hydrogen gas is at 300 K.

(A) Find the average speed, the rms speed, and the most probable speed of the hydrogen molecules.

SOLUTION

Conceptualize Imagine a huge number of particles in a real gas, all moving in random directions with different speeds.

Categorize We cannot calculate the averages as was done in Example 21.5 because the individual speeds of the particles are not known. We are dealing with a very large number of particles, however, so we can use the Maxwell-Boltzmann speed distribution function.

Analyze Use Equation 21.43 to find the average speed:

$$v_{avg} = 1.60\sqrt{\frac{k_B T}{m_0}} = 1.60\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.78 \times 10^3\text{ m/s}}$$

Use Equation 21.42 to find the rms speed:

$$v_{rms} = 1.73\sqrt{\frac{k_B T}{m_0}} = 1.73\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.93 \times 10^3\text{ m/s}}$$

Use Equation 21.44 to find the most probable speed:

$$v_{mp} = 1.41\sqrt{\frac{k_B T}{m_0}} = 1.41\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.57 \times 10^3\text{ m/s}}$$

(B) Find the number of molecules with speeds between 400 m/s and 401 m/s.

SOLUTION

Use Equation 21.41 to evaluate the number of molecules in a narrow speed range between v and $v + dv$:

$$(1)\quad N_v\,dv = 4\pi N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} v^2 e^{-m_0 v^2/2k_B T}\,dv$$

Evaluate the constant in front of v^2:

$$4\pi N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} = 4\pi n N_A\left(\frac{m_0}{2\pi k_B T}\right)^{3/2}$$

$$= 4\pi(0.500\text{ mol})(6.02 \times 10^{23}\text{ mol}^{-1})\left[\frac{2(1.67 \times 10^{-27}\text{ kg})}{2\pi(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}\right]^{3/2}$$

$$= 1.74 \times 10^{14}\text{ s}^3/\text{m}^3$$

Evaluate the exponent of e that appears in Equation (1):

$$-\frac{m_0 v^2}{2k_B T} = -\frac{2(1.67 \times 10^{-27}\text{ kg})(400\text{ m/s})^2}{2(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})} = -0.064\,5$$

Evaluate $N_v\,dv$ using these values in Equation (1):

$$N_v\,dv = (1.74 \times 10^{14}\text{ s}^3/\text{m}^3)(400\text{ m/s})^2 e^{-0.064\,5}(1\text{ m/s})$$

$$= \boxed{2.61 \times 10^{19}\text{ molecules}}$$

Finalize In this evaluation, we could calculate the result without integration because $dv = 1$ m/s is much smaller than $v = 400$ m/s. Had we sought the number of particles between, say, 400 m/s and 500 m/s, we would need to integrate Equation (1) between these speed limits.

Summary

Concepts and Principles

The pressure of N molecules of an ideal gas contained in a volume V is

$$P = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m_0\overline{v^2}\right) \quad \textbf{(21.15)}$$

The average translational kinetic energy per molecule of a gas, $\frac{1}{2}m_0\overline{v^2}$, is related to the temperature T of the gas through the expression

$$\frac{1}{2}m_0\overline{v^2} = \frac{3}{2}k_B T \quad \textbf{(21.19)}$$

where k_B is Boltzmann's constant. Each translational degree of freedom (x, y, or z) has $\frac{1}{2}k_B T$ of energy associated with it.

The internal energy of N molecules (or n mol) of an ideal monatomic gas is

$$E_{int} = \frac{3}{2}Nk_B T = \frac{3}{2}nRT \quad \textbf{(21.25)}$$

The change in internal energy for n mol of any ideal gas that undergoes a change in temperature ΔT is

$$\Delta E_{int} = nC_V\,\Delta T \quad \textbf{(21.27)}$$

where C_V is the **molar specific heat at constant volume.**

The molar specific heat of an ideal monatomic gas at constant volume is $C_V = \frac{3}{2}R$; the molar specific heat at constant pressure is $C_P = \frac{5}{2}R$. The ratio of specific heats is given by $\gamma = C_P/C_V = \frac{5}{3}$.

If an ideal gas undergoes an adiabatic expansion or compression, the first law of thermodynamics, together with the equation of state, shows that

$$PV^{\gamma} = \text{constant} \quad \textbf{(21.37)}$$

The **Boltzmann distribution law** describes the distribution of particles among available energy states. The relative number of particles having energy between E and $E + dE$ is $n_V(E)\,dE$, where

$$n_V(E) = n_0 e^{-E/k_B T} \quad \textbf{(21.40)}$$

The **Maxwell–Boltzmann speed distribution function** describes the distribution of speeds of molecules in a gas:

$$N_v = 4\pi N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} v^2 e^{-m_0 v^2/2k_B T} \quad \textbf{(21.41)}$$

Equation 21.41 enables us to calculate the **root-mean-square speed,** the **average speed,** and the **most probable speed** of molecules in a gas:

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_B T}{m_0}} = 1.73\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.42)}$$

$$v_{avg} = \sqrt{\frac{8k_B T}{\pi m_0}} = 1.60\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.43)}$$

$$v_{mp} = \sqrt{\frac{2k_B T}{m_0}} = 1.41\sqrt{\frac{k_B T}{m_0}} \quad \textbf{(21.44)}$$

Objective Questions

$\boxed{\textbf{1.}}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Cylinder A contains oxygen (O_2) gas, and cylinder B contains nitrogen (N_2) gas. If the molecules in the two cylinders have the same rms speeds, which of the following statements is *false*? (a) The two gases have different temperatures. (b) The temperature of cylinder B is less than the temperature of cylinder A. (c) The temperature of cylinder B is greater than the temperature of cylinder A. (d) The average kinetic energy of the nitrogen molecules is less than the average kinetic energy of the oxygen molecules.

2. An ideal gas is maintained at constant pressure. If the temperature of the gas is increased from 200 K to 600 K, what happens to the rms speed of the molecules? (a) It increases by a factor of 3. (b) It remains the same. (c) It is one-third the original speed. (d) It is $\sqrt{3}$ times the original speed. (e) It increases by a factor of 6.

3. Two samples of the same ideal gas have the same pressure and density. Sample B has twice the volume of sample A. What is the rms speed of the molecules in sample B? (a) twice that in sample A (b) equal to that in sample A (c) half that in sample A (d) impossible to determine

$\boxed{\textbf{4.}}$ A helium-filled latex balloon initially at room temperature is placed in a freezer. The latex remains flexible. **(i)** Does the balloon's volume (a) increase, (b) decrease, or (c) remain the same? **(ii)** Does the pressure of the helium gas (a) increase significantly, (b) decrease significantly, or (c) remain approximately the same?

5. A gas is at 200 K. If we wish to double the rms speed of the molecules of the gas, to what value must we raise its temperature? (a) 283 K (b) 400 K (c) 566 K (d) 800 K (e) 1130 K

6. Rank the following from largest to smallest, noting any cases of equality. (a) the average speed of molecules in a particular sample of ideal gas (b) the most probable speed (c) the root-mean-square speed (d) the average vector velocity of the molecules

7. A sample of gas with a thermometer immersed in the gas is held over a hot plate. A student is asked to give a step-by-step account of what makes our observation of the temperature of the gas increase. His response includes the following steps. (a) The molecules speed up. (b) Then the molecules collide with one another more often. (c) Internal friction makes the collisions inelastic. (d) Heat is produced in the collisions. (e) The molecules of the gas transfer more energy to the thermometer when they strike it, so we observe that the temperature has gone up. (f) The same process can take place without the use of a hot plate if you quickly push in the piston in an insulated cylinder containing the gas. **(i)** Which of the parts (a) through (f) of this account are correct statements necessary for a clear and complete explanation? **(ii)** Which are correct statements that are not necessary to account for the higher thermometer reading? **(iii)** Which are incorrect statements?

8. An ideal gas is contained in a vessel at 300 K. The temperature of the gas is then increased to 900 K. **(i)** By what factor does the average kinetic energy of the molecules change, (a) a factor of 9, (b) a factor of 3, (c) a factor of $\sqrt{3}$, (d) a factor of 1, or (e) a factor of $\frac{1}{3}$? Using the same choices as in part (i), by what factor does each of the following change: **(ii)** the rms molecular speed of the molecules, **(iii)** the average momentum change that one molecule undergoes in a collision with one particular wall, **(iv)** the rate of collisions of molecules with walls, and **(v)** the pressure of the gas.

9. Which of the assumptions below is *not* made in the kinetic theory of gases? (a) The number of molecules is very large. (b) The molecules obey Newton's laws of motion. (c) The forces between molecules are long range. (d) The gas is a pure substance. (e) The average separation between molecules is large compared to their dimensions.

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Hot air rises, so why does it generally become cooler as you climb a mountain? *Note:* Air has low thermal conductivity.

2. Why does a diatomic gas have a greater energy content per mole than a monatomic gas at the same temperature?

$\boxed{3.}$ When alcohol is rubbed on your body, it lowers your skin temperature. Explain this effect.

$\boxed{4.}$ What happens to a helium-filled latex balloon released into the air? Does it expand or contract? Does it stop rising at some height?

5. Which is denser, dry air or air saturated with water vapor? Explain.

6. One container is filled with helium gas and another with argon gas. Both containers are at the same temperature. Which molecules have the higher rms speed? Explain.

7. Dalton's law of partial pressures states that the total pressure of a mixture of gases is equal to the sum of the pressures that each gas in the mixture would exert if it were alone in the container. Give a convincing argument for this law based on the kinetic theory of gases.

Problems

The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

$\boxed{1.}$ full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 21.1 Molecular Model of an Ideal Gas

Problem 30 in Chapter 19 can be assigned with this section.

$\boxed{1.}$ M (a) How many atoms of helium gas fill a spherical balloon of diameter 30.0 cm at 20.0°C and 1.00 atm? (b) What is the average kinetic energy of the helium atoms? (c) What is the rms speed of the helium atoms?

$\boxed{2.}$ M A cylinder contains a mixture of helium and argon gas in equilibrium at 150°C. (a) What is the average kinetic energy for each type of gas molecule? (b) What is the rms speed of each type of molecule?

3. In a 30.0-s interval, 500 hailstones strike a glass window of area 0.600 m^2 at an angle of 45.0° to the window surface. Each hailstone has a mass of 5.00 g and a speed of 8.00 m/s. Assuming the collisions are elastic, find (a) the average force and (b) the average pressure on the window during this interval.

4. In an ultrahigh vacuum system (with typical pressures lower than 10^{-7} pascal), the pressure is measured to be 1.00×10^{-10} torr (where 1 torr = 133 Pa). Assuming the temperature is 300 K, find the number of molecules in a volume of 1.00 m^3.

5. A spherical balloon of volume 4.00×10^3 cm^3 contains helium at a pressure of 1.20×10^5 Pa. How many moles of helium are in the balloon if the average kinetic energy of the helium atoms is 3.60×10^{-22} J?

6. A spherical balloon of volume V contains helium at a pressure P. How many moles of helium are in the balloon if the average kinetic energy of the helium atoms is $\overline{K}$?

7. A 2.00-mol sample of oxygen gas is confined to a 5.00-L vessel at a pressure of 8.00 atm. Find the average translational kinetic energy of the oxygen molecules under these conditions.

8. Oxygen, modeled as an ideal gas, is in a container and has a temperature of 77.0°C. What is the rms-average magnitude of the momentum of the gas molecules in the container?

9. Calculate the mass of an atom of (a) helium, (b) iron, and (c) lead. Give your answers in kilograms. The atomic masses of these atoms are 4.00 u, 55.9 u, and 207 u, respectively.

10. The rms speed of an oxygen molecule (O_2) in a container of oxygen gas is 625 m/s. What is the temperature of the gas?

11. A 5.00-L vessel contains nitrogen gas at 27.0°C and 3.00 atm. Find (a) the total translational kinetic energy of the gas molecules and (b) the average kinetic energy per molecule.

12. A 7.00-L vessel contains 3.50 moles of gas at a pressure of 1.60×10^6 Pa. Find (a) the temperature of the gas and (b) the average kinetic energy of the gas molecules in the vessel. (c) What additional information would you need if you were asked to find the average speed of the gas molecules?

13. In a period of 1.00 s, 5.00×10^{23} nitrogen molecules strike a wall with an area of 8.00 cm^2. Assume the molecules move with a speed of 300 m/s and strike the wall head-on in elastic collisions. What is the pressure exerted on the wall? *Note:* The mass of one N_2 molecule is 4.65×10^{-26} kg.

Section 21.2 Molar Specific Heat of an Ideal Gas

> *Note:* You may use data in Table 21.2 about particular gases. Here we define a "monatomic ideal gas" to have molar specific heats $C_V = \frac{3}{2}R$ and $C_P = \frac{5}{2}R$, and a "diatomic ideal gas" to have $C_V = \frac{5}{2}R$ and $C_P = \frac{7}{2}R$.

14. In a constant-volume process, 209 J of energy is transferred by heat to 1.00 mol of an ideal monatomic gas initially at 300 K. Find (a) the work done on the gas, (b) the increase in internal energy of the gas, and (c) its final temperature.

15. A sample of a diatomic ideal gas has pressure P and volume V. When the gas is warmed, its pressure triples and its volume doubles. This warming process includes two steps, the first at constant pressure and the second at constant volume. Determine the amount of energy transferred to the gas by heat.

16. **Review.** A house has well-insulated walls. It contains a volume of 100 m^3 of air at 300 K. (a) Calculate the energy required to increase the temperature of this diatomic ideal gas by 1.00°C. (b) **What If?** If all this energy could be used to lift an object of mass m through a height of 2.00 m, what is the value of m?

17. A 1.00-mol sample of hydrogen gas is heated at constant pressure from 300 K to 420 K. Calculate (a) the energy transferred to the gas by heat, (b) the increase in its internal energy, and (c) the work done on the gas.

18. A vertical cylinder with a heavy piston contains air at 300 K. The initial pressure is 2.00×10^5 Pa, and the initial volume is 0.350 m^3. Take the molar mass of air as 28.9 g/mol and assume $C_V = \frac{5}{2}R$. (a) Find the specific heat of air at constant volume in units of J/kg · °C. (b) Calculate the mass of the air in the cylinder. (c) Suppose the piston is held fixed. Find the energy input required to raise the temperature of the air to 700 K. (d) **What If?** Assume again the conditions of the initial state and assume the heavy piston is free to move. Find the energy input required to raise the temperature to 700 K.

19. Calculate the change in internal energy of 3.00 mol of helium gas when its temperature is increased by 2.00 K.

20. A 1.00-L insulated bottle is full of tea at 90.0°C. You pour out one cup of tea and immediately screw the stopper back on the bottle. Make an order-of-magnitude estimate of the change in temperature of the tea remaining in the bottle that results from the admission of air at room temperature. State the quantities you take as data and the values you measure or estimate for them.

21. **Review.** This problem is a continuation of Problem 39 in Chapter 19. A hot-air balloon consists of an envelope of constant volume 400 m^3. Not including the air inside, the balloon and cargo have mass 200 kg. The air outside and originally inside is a diatomic ideal gas at 10.0°C and 101 kPa, with density 1.25 kg/m^3. A propane burner at the center of the spherical envelope injects energy into the air inside. The air inside stays at constant pressure. Hot air, at just the temperature required to make the balloon lift off, starts to fill the envelope at its closed top, rapidly enough so that negligible energy flows by heat to the cool air below it or out through the wall of the balloon. Air at 10°C leaves through an opening at the bottom of the envelope until the whole balloon is filled with hot air at uniform temperature. Then the burner is shut off and the balloon rises from the ground. (a) Evaluate the quantity of energy the burner must transfer to the air in the balloon. (b) The "heat value" of propane—the internal energy released by burning each kilogram—is 50.3 MJ/kg. What mass of propane must be burned?

Section 21.3 The Equipartition of Energy

22. A certain molecule has f degrees of freedom. Show that an ideal gas consisting of such molecules has the following properties: (a) its total internal energy is $fnRT/2$, (b) its molar specific heat at constant volume is $fR/2$, (c) its molar specific heat at constant pressure is $(f + 2)R/2$, and (d) its specific heat ratio is $\gamma = C_P/C_V = (f + 2)/f$.

23. In a crude model (Fig. P21.23) of a rotating diatomic chlorine molecule (Cl_2), the two Cl atoms are 2.00×10^{-10} m apart and rotate about their center of mass with angular speed $\omega = 2.00 \times 10^{12}$ rad/s. What is the rotational kinetic energy of one molecule of Cl_2, which has a molar mass of 70.0 g/mol?

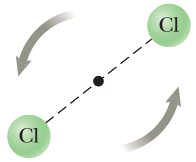


Figure P21.23

24. *Why is the following situation impossible?* A team of researchers discovers a new gas, which has a value of $\gamma = C_P/C_V$ of 1.75.

25. M The relationship between the heat capacity of a sample and the specific heat of the sample material is discussed in Section 20.2. Consider a sample containing 2.00 mol of an ideal diatomic gas. Assuming the molecules rotate but do not vibrate, find (a) the total heat capacity of the sample at constant volume and (b) the total heat capacity at constant pressure. (c) **What If?** Repeat parts (a) and (b), assuming the molecules both rotate and vibrate.

Section 21.4 Adiabatic Processes for an Ideal Gas

26. M A 2.00-mol sample of a diatomic ideal gas expands slowly and adiabatically from a pressure of 5.00 atm and a volume of 12.0 L to a final volume of 30.0 L. (a) What is the final pressure of the gas? (b) What are the initial and final temperatures? Find (c) Q, (d) ΔE_{int}, and (e) W for the gas during this process.

27. W During the compression stroke of a certain gasoline engine, the pressure increases from 1.00 atm to 20.0 atm. If the process is adiabatic and the air–fuel mixture behaves as a diatomic ideal gas, (a) by what factor does the volume change and (b) by what factor does the temperature change? Assuming the compression starts with 0.016 0 mol of gas at 27.0°C, find the values of (c) Q, (d) ΔE_{int}, and (e) W that characterize the process.

28. W How much work is required to compress 5.00 mol of air at 20.0°C and 1.00 atm to one-tenth of the original volume (a) by an isothermal process? (b) **What If?** How much work is required to produce the same compression in an adiabatic process? (c) What is the final pressure in part (a)? (d) What is the final pressure in part (b)?

29. M Air in a thundercloud expands as it rises. If its initial temperature is 300 K and no energy is lost by thermal conduction on expansion, what is its temperature when the initial volume has doubled?

30. *Why is the following situation impossible?* A new diesel engine that increases fuel economy over previous models is designed. Automobiles fitted with this design become incredible best sellers. Two design features are responsible for the increased fuel economy: (1) the engine is made entirely of aluminum to reduce the weight of the automobile, and (2) the exhaust of the engine is used to prewarm the air to 50°C before it enters the cylinder to increase the final temperature of the compressed gas. The engine has a *compression ratio*—that is, the ratio of the initial volume of the air to its final volume after compression—of 14.5. The compression process is adiabatic, and the air behaves as a diatomic ideal gas with $\gamma = 1.40$.

31. During the power stroke in a four-stroke automobile engine, the piston is forced down as the mixture of combustion products and air undergoes an adiabatic expansion. Assume (1) the engine is running at 2 500 cycles/min; (2) the gauge pressure immediately before the expansion is 20.0 atm; (3) the volumes of the mixture immediately before and after the expansion are 50.0 cm^3 and 400 cm^3, respectively (Fig. P21.31); (4) the time interval for the expansion is one-fourth that of the total cycle; and (5) the mixture behaves like an ideal gas with specific heat ratio 1.40. Find the average power generated during the power stroke.

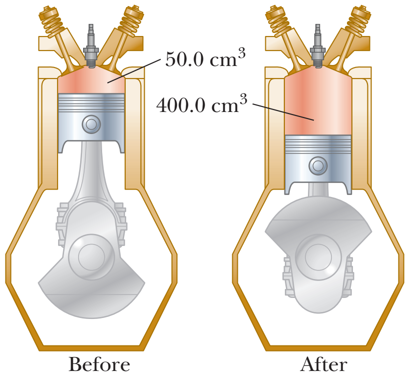


Figure P21.31

32. GP Air (a diatomic ideal gas) at 27.0°C and atmospheric pressure is drawn into a bicycle pump (see the chapter-opening photo on page 626) that has a cylinder with an inner diameter of 2.50 cm and length 50.0 cm. The downstroke adiabatically compresses the air, which reaches a gauge pressure of 8.00×10^5 Pa before entering the tire. We wish to investigate the temperature increase of the pump. (a) What is the initial volume of the air in the pump? (b) What is the number of moles of air in the pump? (c) What is the absolute pressure of the compressed air? (d) What is the volume of the compressed air? (e) What is the temperature of the compressed air? (f) What is the increase in internal energy of the gas during the compression? **What If?** The pump is made of steel that is 2.00 mm thick. Assume 4.00 cm of the cylinder's length is allowed to come to thermal equilibrium with the air. (g) What is the volume of steel in this 4.00-cm length? (h) What is the mass of steel in this 4.00-cm length? (i) Assume the pump is compressed once. After the adiabatic expansion, conduction results in the energy increase in part (f) being shared between the gas and the 4.00-cm length of steel. What will be the increase in temperature of the steel after one compression?

33. A 4.00-L sample of a diatomic ideal gas with specific heat ratio 1.40, confined to a cylinder, is carried through a closed cycle. The gas is initially at 1.00 atm and 300 K. First, its pressure is tripled under constant volume. Then, it expands adiabatically to its original pressure. Finally, the gas is compressed isobarically to its original volume. (a) Draw a *PV* diagram of this cycle. (b) Determine the volume of the gas at the end of the adiabatic expansion. (c) Find the temperature of the gas at the start of the adiabatic expansion. (d) Find the temperature at the end of the cycle. (e) What was the net work done on the gas for this cycle?

34. An ideal gas with specific heat ratio γ confined to a cylinder is put through a closed cycle. Initially, the gas is at P_i, V_i, and T_i. First, its pressure is tripled under constant volume. It then expands adiabatically to its original pressure and finally is compressed isobarically to its original volume. (a) Draw a *PV* diagram of this cycle. (b) Determine the volume at the end of the adiabatic expansion. Find (c) the temperature of the gas at the start of the adiabatic expansion and (d) the temperature at the end of the cycle. (e) What was the net work done on the gas for this cycle?

Section 21.5 Distribution of Molecular Speeds

35. Helium gas is in thermal equilibrium with liquid helium at 4.20 K. Even though it is on the point of condensation, model the gas as ideal and determine the most probable speed of a helium atom (mass = 6.64×10^{-27} kg) in it.

36. M Fifteen identical particles have various speeds: one has a speed of 2.00 m/s, two have speeds of 3.00 m/s, three have speeds of 5.00 m/s, four have speeds of 7.00 m/s, three have speeds of 9.00 m/s, and two have speeds of 12.0 m/s. Find (a) the average speed, (b) the rms speed, and (c) the most probable speed of these particles.

37. W One cubic meter of atomic hydrogen at 0°C at atmospheric pressure contains approximately 2.70×10^{25} atoms. The first excited state of the hydrogen atom has an energy of 10.2 eV above that of the lowest state, called the ground state. Use the Boltzmann factor to find the number of atoms in the first excited state (a) at 0°C and at (b) (1.00×10^4)°C.

38. Two gases in a mixture diffuse through a filter at rates proportional to their rms speeds. (a) Find the ratio of speeds for the two isotopes of chlorine, ^{35}Cl and ^{37}Cl, as they diffuse through the air. (b) Which isotope moves faster?

39. **Review.** At what temperature would the average speed of helium atoms equal (a) the escape speed from the Earth, 1.12×10^4 m/s, and (b) the escape speed from the Moon, 2.37×10^3 m/s? *Note:* The mass of a helium atom is 6.64×10^{-27} kg.

40. Consider a container of nitrogen gas molecules at 900 K. Calculate (a) the most probable speed, (b) the average speed, and (c) the rms speed for the molecules. (d) State how your results compare with the values displayed in Figure 21.11.

41. Assume the Earth's atmosphere has a uniform temperature of 20.0°C and uniform composition, with an effective molar mass of 28.9 g/mol. (a) Show that the number density of molecules depends on height y above sea level according to

$$n_V(y) = n_0 e^{-m_0 g y / k_B T}$$

where n_0 is the number density at sea level (where $y = 0$). This result is called the *law of atmospheres.* (b) Commercial jetliners typically cruise at an altitude of 11.0 km. Find the ratio of the atmospheric density there to the density at sea level.

42. From the Maxwell–Boltzmann speed distribution, show that the most probable speed of a gas molecule is given by Equation 21.44. *Note:* The most probable speed corresponds to the point at which the slope of the speed distribution curve dN_v/dv is zero.

43. The law of atmospheres states that the number density of molecules in the atmosphere depends on height y above sea level according to

$$n_V(y) = n_0 e^{-m_0 g y / k_B T}$$

where n_0 is the number density at sea level (where $y = 0$). The average height of a molecule in the Earth's atmosphere is given by

$$y_{\text{avg}} = \frac{\int_0^\infty y n_V(y)\, dy}{\int_0^\infty n_V(y)\, dy} = \frac{\int_0^\infty y e^{-m_0 g y / k_B T}\, dy}{\int_0^\infty e^{-m_0 g y / k_B T}\, dy}$$

(a) Prove that this average height is equal to $k_B T / m_0 g$. (b) Evaluate the average height, assuming the temperature is 10.0°C and the molecular mass is 28.9 u, both uniform throughout the atmosphere.

Additional Problems

44. Eight molecules have speeds of 3.00 km/s, 4.00 km/s, 5.80 km/s, 2.50 km/s, 3.60 km/s, 1.90 km/s, 3.80 km/s, and 6.60 km/s. Find (a) the average speed of the molecules and (b) the rms speed of the molecules.

45. A small oxygen tank at a gauge pressure of 125 atm has a volume of 6.88 L at 21.0°C. (a) If an athlete breathes oxygen from this tank at the rate of 8.50 L/min when measured at atmospheric pressure and the temperature remains at 21.0°C, how long will the tank last before it is empty? (b) At a particular moment during this process, what is the ratio of the rms speed of the molecules remaining in the tank to the rms speed of those being released at atmospheric pressure?

46. The dimensions of a classroom are 4.20 m × 3.00 m × 2.50 m. (a) Find the number of molecules of air in the classroom at atmospheric pressure and 20.0°C. (b) Find the mass of this air, assuming the air consists of diatomic molecules with molar mass 28.9 g/mol. (c) Find the average kinetic energy of the molecules. (d) Find the rms molecular speed. (e) **What If?** Assume the molar specific heat of the air is independent of temperature. Find the change in internal energy of the air in the room as the temperature is raised to 25.0°C. (f) Explain how you could convince a fellow student that your answer to part (e) is correct, even though it sounds surprising.

47. The Earth's atmosphere consists primarily of oxygen (21%) and nitrogen (78%). The rms speed of oxygen molecules (O_2) in the atmosphere at a certain location is 535 m/s. (a) What is the temperature of the atmosphere at this location? (b) Would the rms speed of nitrogen molecules (N_2) at this location be higher, equal to, or lower than 535 m/s? Explain. (c) Determine the rms speed of N_2 at his location.

48. AMT The *mean free path* ℓ of a molecule is the average distance that a molecule travels before colliding with another molecule. It is given by

$$\ell = \frac{1}{\sqrt{2}\pi d^2 N_V}$$

where d is the diameter of the molecule and N_V is the number of molecules per unit volume. The number of collisions that a molecule makes with other molecules per unit time, or *collision frequency f*, is given by

$$f = \frac{v_{\text{avg}}}{\ell}$$

(a) If the diameter of an oxygen molecule is 2.00×10^{-10} m, find the mean free path of the molecules in a scuba tank that has a volume of 12.0 L and is filled with oxygen at a gauge pressure of 100 atm at a temperature of 25.0°C. (b) What is the average time interval between molecular collisions for a molecule of this gas?

49. AMT An air rifle shoots a lead pellet by allowing high-pressure air to expand, propelling the pellet down the rifle barrel. Because this process happens very quickly, no appreciable thermal conduction occurs and the expansion is essentially adiabatic. Suppose the rifle starts with 12.0 cm^3 of compressed air, which behaves as an ideal gas with $\gamma = 1.40$. The expanding air pushes a 1.10-g pellet as a piston with cross-sectional area 0.030 0 cm^2 along the 50.0-cm-long gun barrel. What initial pressure is required to eject the pellet with a muzzle speed of 120 m/s? Ignore the effects of the air in front of the bullet and friction with the inside walls of the barrel.

50. In a sample of a solid metal, each atom is free to vibrate about some equilibrium position. The atom's energy consists of kinetic energy for motion in the x, y, and z directions plus elastic potential energy associated with the Hooke's law forces exerted by neighboring atoms on it in the x, y, and z directions. According to the theorem of equipartition of energy, assume the average energy of each atom is $\frac{1}{2}k_BT$ for each degree of freedom. (a) Prove that the molar specific heat of the solid is $3R$. The *Dulong–Petit law* states that this result generally describes pure solids at sufficiently high temperatures. (You may ignore the difference between the specific heat at constant pressure and the specific heat at constant volume.) (b) Evaluate the specific heat c of iron. Explain how it compares with the value listed in Table 20.1. (c) Repeat the evaluation and comparison for gold.

51. A certain ideal gas has a molar specific heat of $C_V = \frac{7}{2}R$. A 2.00-mol sample of the gas always starts at pressure 1.00×10^5 Pa and temperature 300 K. For each of the following processes, determine (a) the final pressure, (b) the final volume, (c) the final temperature, (d) the change in internal energy of the gas, (e) the energy added to the gas by heat, and (f) the work done on the gas. **(i)** The gas is heated at constant pressure to 400 K. **(ii)** The gas is heated at constant volume to 400 K. **(iii)** The gas is compressed at constant temperature to 1.20×10^5 Pa. **(iv)** The gas is compressed adiabatically to 1.20×10^5 Pa.

52. The compressibility κ of a substance is defined as the fractional change in volume of that substance for a given change in pressure:

$$\kappa = -\frac{1}{V}\frac{dV}{dP}$$

(a) Explain why the negative sign in this expression ensures κ is always positive. (b) Show that if an ideal gas is compressed isothermally, its compressibility is given by $\kappa_1 = 1/P$. (c) **What If?** Show that if an ideal gas is compressed adiabatically, its compressibility is given by $\kappa_2 = 1/(\gamma P)$. Determine values for (d) κ_1 and (e) κ_2 for a monatomic ideal gas at a pressure of 2.00 atm.

53. **Review.** Oxygen at pressures much greater than 1 atm is toxic to lung cells. Assume a deep-sea diver breathes a mixture of oxygen (O_2) and helium (He). By weight, what ratio of helium to oxygen must be used if the diver is at an ocean depth of 50.0 m?

54. Examine the data for polyatomic gases in Table 21.2 and give a reason why sulfur dioxide has a higher specific heat at constant volume than the other polyatomic gases at 300 K.

55. Model air as a diatomic ideal gas with $M = 28.9$ g/mol. A cylinder with a piston contains 1.20 kg of air at 25.0°C and 2.00×10^5 Pa. Energy is transferred by heat into the system as it is permitted to expand, with the pressure rising to 4.00×10^5 Pa. Throughout the expansion, the relationship between pressure and volume is given by

$$P = CV^{1/2}$$

where C is a constant. Find (a) the initial volume, (b) the final volume, (c) the final temperature, (d) the work done on the air, and (e) the energy transferred by heat.

56. Review. As a sound wave passes through a gas, the compressions are either so rapid or so far apart that thermal conduction is prevented by a negligible time interval or by effective thickness of insulation. The compressions and rarefactions are adiabatic. (a) Show that the speed of sound in an ideal gas is

$$v = \sqrt{\frac{\gamma RT}{M}}$$

where M is the molar mass. The speed of sound in a gas is given by Equation 17.8; use that equation and the definition of the bulk modulus from Section 12.4. (b) Compute the theoretical speed of sound in air at 20.0°C and state how it compares with the value in Table 17.1. Take $M = 28.9$ g/mol. (c) Show that the speed of sound in an ideal gas is

$$v = \sqrt{\frac{\gamma k_B T}{m_0}}$$

where m_0 is the mass of one molecule. (d) State how the result in part (c) compares with the most probable, average, and rms molecular speeds.

57. Twenty particles, each of mass m_0 and confined to a volume V, have various speeds: two have speed v, three have speed $2v$, five have speed $3v$, four have speed $4v$, three have speed $5v$, two have speed $6v$, and one has speed $7v$. Find (a) the average speed, (b) the rms speed, (c) the most probable speed, (d) the average pressure the particles exert on the walls of the vessel, and (e) the average kinetic energy per particle.

58. In a cylinder, a sample of an ideal gas with number of moles n undergoes an adiabatic process. (a) Starting with the expression $W = -\int P\,dV$ and using the condition PV^γ = constant, show that the work done on the gas is

$$W = \left(\frac{1}{\gamma - 1}\right)(P_f V_f - P_i V_i)$$

(b) Starting with the first law of thermodynamics, show that the work done on the gas is equal to $nC_V(T_f - T_i)$. (c) Are these two results consistent with each other? Explain.

59. As a 1.00-mol sample of a monatomic ideal gas expands adiabatically, the work done on it is -2.50×10^3 J. The initial temperature and pressure of the gas are 500 K and 3.60 atm. Calculate (a) the final temperature and (b) the final pressure.

60. A sample consists of an amount n in moles of a monatomic ideal gas. The gas expands adiabatically, with work W done on it. (Work W is a negative number.) The initial temperature and pressure of the gas are T_i and P_i. Calculate (a) the final temperature and (b) the final pressure.

61. When a small particle is suspended in a fluid, bombardment by molecules makes the particle jitter about at random. Robert Brown discovered this motion in 1827 while studying plant fertilization, and the motion has become known as *Brownian motion*. The particle's average kinetic energy can be taken as $\frac{3}{2}k_B T$, the same as that of a molecule in an ideal gas. Consider a spherical particle of density 1.00×10^3 kg/m^3 in water at 20.0°C. (a) For a particle of diameter d, evaluate the rms speed. (b) The particle's actual motion is a random walk, but imagine that it moves with constant velocity equal in magnitude to its rms speed. In what time interval would it move by a distance equal to its own diameter? (c) Evaluate the rms speed and the time interval for a particle of diameter 3.00 μm. (d) Evaluate the rms speed and the time interval for a sphere of mass 70.0 kg, modeling your own body.

62. A vessel contains 1.00×10^4 oxygen molecules at 500 K. (a) Make an accurate graph of the Maxwell speed distribution function versus speed with points at speed intervals of 100 m/s. (b) Determine the most probable speed from this graph. (c) Calculate the average and rms speeds for the molecules and label these points on your graph. (d) From the graph, estimate the fraction of molecules with speeds in the range 300 m/s to 600 m/s.

63. AMT A pitcher throws a 0.142-kg baseball at 47.2 m/s. As it travels 16.8 m to home plate, the ball slows down to 42.5 m/s because of air resistance. Find the change in temperature of the air through which it passes. To find the greatest possible temperature change, you may make the following assumptions. Air has a molar specific heat of $C_P = \frac{7}{2}R$ and an equivalent molar mass of 28.9 g/mol. The process is so rapid that the cover of the baseball acts as thermal insulation and the temperature of the ball itself does not change. A change in temperature happens initially only for the air in a cylinder 16.8 m in length and 3.70 cm in radius. This air is initially at 20.0°C.

64. The latent heat of vaporization for water at room temperature is 2 430 J/g. Consider one particular molecule at the surface of a glass of liquid water, moving upward with sufficiently high speed that it will be the next molecule to join the vapor. (a) Find its translational kinetic energy. (b) Find its speed. Now consider a thin gas made only of molecules like that one. (c) What is its temperature? (d) Why are you not burned by water evaporating from a vessel at room temperature?

65. A sample of a monatomic ideal gas occupies 5.00 L at atmospheric pressure and 300 K (point A in Fig. P21.65). It is warmed at constant volume to 3.00 atm (point B). Then it is allowed to expand isothermally to 1.00 atm (point C) and at last compressed isobarically to its original state. (a) Find the number of moles in the sample.

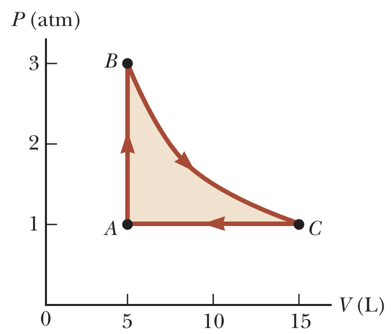


Figure P21.65

Find (b) the temperature at point B, (c) the temperature at point C, and (d) the volume at point C. (e) Now consider the processes $A \to B$, $B \to C$, and $C \to A$. Describe how to carry out each process experimentally. (f) Find Q, W, and ΔE_{int} for each of the processes. (g) For the whole cycle $A \to B \to C \to A$, find Q, W, and ΔE_{int}.

66. Consider the particles in a gas centrifuge, a device used to separate particles of different mass by whirling them in a circular path of radius r at angular speed ω. The force acting on a gas molecule toward the center of the centrifuge is $m_0\omega^2 r$. (a) Discuss how a gas centrifuge can be used to separate particles of different mass. (b) Suppose the centrifuge contains a gas of particles of identical mass. Show that the density of the particles as a function of r is

$$n(r) = n_0 e^{m_0 r^2 \omega^2 / 2k_B T}$$

67. For a Maxwellian gas, use a computer or programmable calculator to find the numerical value of the ratio $N_v(v)/N_v(v_{\text{mp}})$ for the following values of v: (a) $v = (v_{\text{mp}}/50.0)$, (b) $(v_{\text{mp}}/10.0)$, (c) $(v_{\text{mp}}/2.00)$, (d) v_{mp}, (e) $2.00v_{\text{mp}}$, (f) $10.0v_{\text{mp}}$, and (g) $50.0v_{\text{mp}}$. Give your results to three significant figures.

68. A triatomic molecule can have a linear configuration, as does CO_2 (Fig. P21.68a), or it can be nonlinear, like H_2O (Fig. P21.68b). Suppose the temperature of a gas of triatomic molecules is sufficiently low that vibrational motion is negligible. What is the molar specific heat at constant volume, expressed as a multiple of the universal gas constant, (a) if the molecules are linear and (b) if the molecules are nonlinear? At high temperatures, a triatomic molecule has two modes of vibration, and each contributes $\frac{1}{2}R$ to the molar specific heat for its kinetic energy and another $\frac{1}{2}R$ for its potential energy. Identify the high-temperature molar specific heat at constant volume for a triatomic ideal gas of (c) linear molecules and (d) nonlinear molecules. (e) Explain how specific heat data can be used to determine whether a triatomic molecule is linear or nonlinear. Are the data in Table 21.2 sufficient to make this determination?

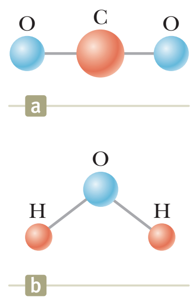


Figure P21.68

69. Using the Maxwell–Boltzmann speed distribution function, verify Equations 21.42 and 21.43 for (a) the rms speed and (b) the average speed of the molecules of a gas at a temperature T. The average value of v^n is

$$\overline{v^n} = \frac{1}{N}\int_0^\infty v^n N_v \, dv$$

Use the table of integrals B.6 in Appendix B.

70. On the PV diagram for an ideal gas, one isothermal curve and one adiabatic curve pass through each point as shown in Figure P21.70. Prove that the slope of the adiabatic curve is steeper than the slope of the isotherm at that point by the factor γ.

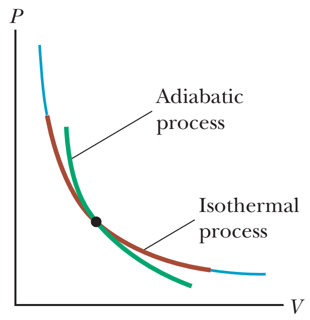


Figure P21.70

71. In Beijing, a restaurant keeps a pot of chicken broth simmering continuously. Every morning, it is topped up to contain 10.0 L of water along with a fresh chicken, vegetables, and spices. The molar mass of water is 18.0 g/mol. (a) Find the number of molecules of water in the pot. (b) During a certain month, 90.0% of the broth was served each day to people who then emigrated immediately. Of the water molecules in the pot on the first day of the month, when was the last one likely to have been ladled out of the pot? (c) The broth has been simmering for centuries, through wars, earthquakes, and stove repairs. Suppose the water that was in the pot long ago has thoroughly mixed into the Earth's hydrosphere, of mass 1.32×10^{21} kg. How many of the water molecules originally in the pot are likely to be present in it again today?

72. **Review.** (a) If it has enough kinetic energy, a molecule at the surface of the Earth can "escape the Earth's gravitation" in the sense that it can continue to move away from the Earth forever as discussed in Section 13.6. Using the principle of conservation of energy, show that the minimum kinetic energy needed for "escape" is $m_0 g R_E$, where m_0 is the mass of the molecule, g is the free-fall acceleration at the surface, and R_E is the radius of the Earth. (b) Calculate the temperature for which the minimum escape kinetic energy is ten times the average kinetic energy of an oxygen molecule.

73. Using multiple laser beams, physicists have been able to cool and trap sodium atoms in a small region. In one experiment, the temperature of the atoms was reduced to 0.240 mK. (a) Determine the rms speed of the sodium atoms at this temperature. The atoms can be trapped for about 1.00 s. The trap has a linear dimension of roughly 1.00 cm. (b) Over what approximate time interval would an atom wander out of the trap region if there were no trapping action?

Challenge Problems

74. Equations 21.42 and 21.43 show that $v_{\text{rms}} > v_{\text{avg}}$ for a collection of gas particles, which turns out to be true whenever the particles have a distribution of speeds. Let us explore this inequality for a two-particle gas.

Let the speed of one particle be $v_1 = av_{avg}$ and the other particle have speed $v_2 = (2 - a)v_{avg}$. (a) Show that the average of these two speeds is v_{avg}. (b) Show that

$$v_{rms}^2 = v_{avg}^2\,(2 - 2a + a^2)$$

(c) Argue that the equation in part (b) proves that, in general, $v_{rms} > v_{avg}$. (d) Under what special condition will $v_{rms} = v_{avg}$ for the two-particle gas?

75. AMT A cylinder is closed at both ends and has insulating walls. It is divided into two compartments by an insulating piston that is perpendicular to the axis of the cylinder as shown in Figure P21.75a. Each compartment contains 1.00 mol of oxygen that behaves as an ideal gas with $\gamma = 1.40$. Initially, the two compartments have equal volumes and their temperatures are 550 K and 250 K. The piston is then allowed to move slowly parallel to the axis of the cylinder until it comes to rest at an equilibrium position (Fig. P21.75b). Find the final temperatures in the two compartments.

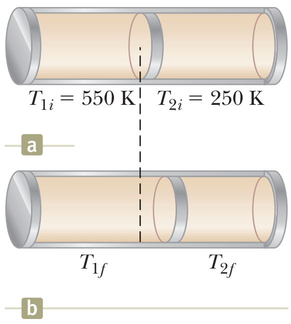


Figure P21.75

Heat Engines, Entropy, and the Second Law of Thermodynamics

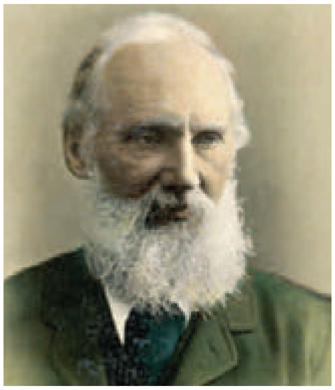
CHAPTER

22

A Stirling engine from the early nineteenth century. Air is heated in the lower cylinder using an external source. As this happens, the air expands and pushes against a piston, causing it to move. The air is then cooled, allowing the cycle to begin again. This is one example of a heat engine, which we study in this chapter. *(© SSPL/The Image Works)*

The first law of thermodynamics, which we studied in Chapter 20, is a statement of conservation of energy and is a special-case reduction of Equation 8.2. This law states that a change in internal energy in a system can occur as a result of energy transfer by heat, by work, or by both. Although the first law of thermodynamics is very important, it makes no distinction between processes that occur spontaneously and those that do not. Only certain types of energy transformation and energy transfer processes actually take place in nature, however. The *second law of thermodynamics,* the major topic in this chapter, establishes which processes do and do not occur. The following are examples of processes that do not violate the first law of thermodynamics if they proceed in either direction, but are observed in reality to proceed in only one direction:

- When two objects at different temperatures are placed in thermal contact with each other, the net transfer of energy by heat is always from the warmer object to the cooler object, never from the cooler to the warmer.
- A rubber ball dropped to the ground bounces several times and eventually comes to rest, but a ball lying on the ground never gathers internal energy from the ground and begins bouncing on its own.
- An oscillating pendulum eventually comes to rest because of collisions with air molecules and friction at the point of suspension. The mechanical energy of the system is converted to internal energy in the air, the pendulum, and the suspension; the reverse conversion of energy never occurs.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Lord Kelvin
British physicist and mathematician (1824–1907)
Born William Thomson in Belfast, Kelvin was the first to propose the use of an absolute scale of temperature. The Kelvin temperature scale is named in his honor. Kelvin's work in thermodynamics led to the idea that energy cannot pass spontaneously from a colder object to a hotter object.

All these processes are *irreversible;* that is, they are processes that occur naturally in one direction only. No irreversible process has ever been observed to run backward. If it were to do so, it would violate the second law of thermodynamics.[1]

22.1 Heat Engines and the Second Law of Thermodynamics

A **heat engine** is a device that takes in energy by heat[2] and, operating in a cyclic process, expels a fraction of that energy by means of work. For instance, in a typical process by which a power plant produces electricity, a fuel such as coal is burned and the high-temperature gases produced are used to convert liquid water to steam. This steam is directed at the blades of a turbine, setting it into rotation. The mechanical energy associated with this rotation is used to drive an electric generator. Another device that can be modeled as a heat engine is the internal combustion engine in an automobile. This device uses energy from a burning fuel to perform work on pistons that results in the motion of the automobile.

Let us consider the operation of a heat engine in more detail. A heat engine carries some working substance through a cyclic process during which (1) the working substance absorbs energy by heat from a high-temperature energy reservoir, (2) work is done by the engine, and (3) energy is expelled by heat to a lower-temperature reservoir. As an example, consider the operation of a steam engine (Fig. 22.1), which uses water as the working substance. The water in a boiler absorbs energy from burning fuel and evaporates to steam, which then does work by expanding against a piston. After the steam cools and condenses, the liquid water produced returns to the boiler and the cycle repeats.


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Figure 22.1 A steam-driven locomotive obtains its energy by burning wood or coal. The generated energy vaporizes water into steam, which powers the locomotive. Modern locomotives use diesel fuel instead of wood or coal. Whether old-fashioned or modern, such locomotives can be modeled as heat engines, which extract energy from a burning fuel and convert a fraction of it to mechanical energy.

It is useful to represent a heat engine schematically as in Figure 22.2. The engine absorbs a quantity of energy $|Q_h|$ from the hot reservoir. For the mathematical discussion of heat engines, we use absolute values to make all energy transfers by heat positive, and the direction of transfer is indicated with an explicit positive or negative sign. The engine does work W_{eng} (so that *negative* work $W = -W_{\text{eng}}$ is done *on* the engine) and then gives up a quantity of energy $|Q_c|$ to the cold reservoir.

[1]Although a process occurring in the time-reversed sense has never been *observed,* it is *possible* for it to occur. As we shall see later in this chapter, however, the probability of such a process occurring is infinitesimally small. From this viewpoint, processes occur with a vastly greater probability in one direction than in the opposite direction.

[2]We use heat as our model for energy transfer into a heat engine. Other methods of energy transfer are possible in the model of a heat engine, however. For example, the Earth's atmosphere can be modeled as a heat engine in which the input energy transfer is by means of electromagnetic radiation from the Sun. The output of the atmospheric heat engine causes the wind structure in the atmosphere.

Because the working substance goes through a cycle, its initial and final internal energies are equal: $\Delta E_{\text{int}} = 0$. Hence, from the first law of thermodynamics, $\Delta E_{\text{int}} = Q + W = Q - W_{\text{eng}} = 0$, and the net work W_{eng} done by a heat engine is equal to the net energy Q_{net} transferred to it. As you can see from Figure 22.2, $Q_{\text{net}} = |Q_h| - |Q_c|$; therefore,

$$W_{\text{eng}} = |Q_h| - |Q_c| \tag{22.1}$$

The **thermal efficiency** e of a heat engine is defined as the ratio of the net work done by the engine during one cycle to the energy input at the higher temperature during the cycle:

$$e \equiv \frac{W_{\text{eng}}}{|Q_h|} = \frac{|Q_h| - |Q_c|}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|} \tag{22.2}$$

Thermal efficiency of a heat engine

You can think of the efficiency as the ratio of what you gain (work) to what you give (energy transfer at the higher temperature). In practice, all heat engines expel only a fraction of the input energy Q_h by mechanical work; consequently, their efficiency is always less than 100%. For example, a good automobile engine has an efficiency of about 20%, and diesel engines have efficiencies ranging from 35% to 40%.

Equation 22.2 shows that a heat engine has 100% efficiency ($e = 1$) only if $|Q_c| = 0$, that is, if no energy is expelled to the cold reservoir. In other words, a heat engine with perfect efficiency would have to expel all the input energy by work. Because efficiencies of real engines are well below 100%, the **Kelvin–Planck form of the second law of thermodynamics** states the following:

> It is impossible to construct a heat engine that, operating in a cycle, produces no effect other than the input of energy by heat from a reservoir and the performance of an equal amount of work.

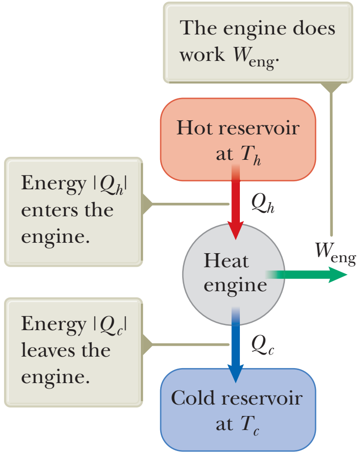

Figure 22.2 Schematic representation of a heat engine.

This statement of the second law means that during the operation of a heat engine, W_{eng} can never be equal to $|Q_h|$ or, alternatively, that some energy $|Q_c|$ *must* be rejected to the environment. Figure 22.3 is a schematic diagram of the impossible "perfect" heat engine.

Quick Quiz 22.1 The energy input to an engine is 4.00 times greater than the work it performs. **(i)** What is its thermal efficiency? (a) 4.00 (b) 1.00 (c) 0.250 (d) impossible to determine **(ii)** What fraction of the energy input is expelled to the cold reservoir? (a) 0.250 (b) 0.750 (c) 1.00 (d) impossible to determine

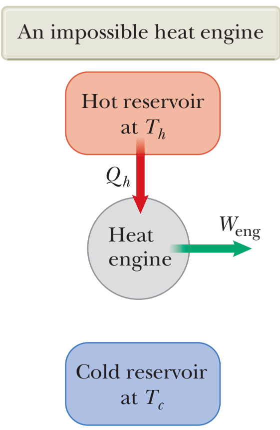

Figure 22.3 Schematic diagram of a heat engine that takes in energy from a hot reservoir and does an equivalent amount of work. It is impossible to construct such a perfect engine.

Pitfall Prevention 22.1

The First and Second Laws Notice the distinction between the first and second laws of thermodynamics. If a gas undergoes a *one-time isothermal process*, then $\Delta E_{\text{int}} = Q + W = 0$ and $W = -Q$. Therefore, the first law allows *all* energy input by heat to be expelled by work. In a heat engine, however, in which a substance undergoes a *cyclic* process, only a *portion* of the energy input by heat can be expelled by work according to the second law.

Example 22.1 The Efficiency of an Engine

An engine transfers 2.00×10^3 J of energy from a hot reservoir during a cycle and transfers 1.50×10^3 J as exhaust to a cold reservoir.

(A) Find the efficiency of the engine.

SOLUTION

Conceptualize Review Figure 22.2; think about energy going into the engine from the hot reservoir and splitting, with part coming out by work and part by heat into the cold reservoir.

Categorize This example involves evaluation of quantities from the equations introduced in this section, so we categorize it as a substitution problem.

Find the efficiency of the engine from Equation 22.2:

$$e = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{1.50 \times 10^3\ \text{J}}{2.00 \times 10^3\ \text{J}} = \boxed{0.250,\text{ or } 25.0\%}$$

(B) How much work does this engine do in one cycle?

SOLUTION

Find the work done by the engine by taking the difference between the input and output energies:

$$W_{\text{eng}} = |Q_h| - |Q_c| = 2.00 \times 10^3\ \text{J} - 1.50 \times 10^3\ \text{J}$$
$$= \boxed{5.0 \times 10^2\ \text{J}}$$

WHAT IF? Suppose you were asked for the power output of this engine. Do you have sufficient information to answer this question?

Answer No, you do not have enough information. The power of an engine is the *rate* at which work is done by the engine. You know how much work is done per cycle, but you have no information about the time interval associated with one cycle. If you were told that the engine operates at 2 000 rpm (revolutions per minute), however, you could relate this rate to the period of rotation T of the mechanism of the engine. Assuming there is one thermodynamic cycle per revolution, the power is

$$P = \frac{W_{\text{eng}}}{T} = \frac{5.0 \times 10^2\ \text{J}}{\left(\frac{1}{2\,000}\ \text{min}\right)}\left(\frac{1\ \text{min}}{60\ \text{s}}\right) = 1.7 \times 10^4\ \text{W}$$

22.2 Heat Pumps and Refrigerators

In a heat engine, the direction of energy transfer is from the hot reservoir to the cold reservoir, which is the natural direction. The role of the heat engine is to process the energy from the hot reservoir so as to do useful work. What if we wanted to transfer energy from the cold reservoir to the hot reservoir? Because that is not the natural direction of energy transfer, we must put some energy into a device to be successful. Devices that perform this task are called **heat pumps** and **refrigerators.** For example, homes in summer are cooled using heat pumps called *air conditioners.* The air conditioner transfers energy from the cool room in the home to the warm air outside.

In a refrigerator or a heat pump, the engine takes in energy $|Q_c|$ from a cold reservoir and expels energy $|Q_h|$ to a hot reservoir (Fig. 22.4), which can be accomplished only if work is done *on* the engine. From the first law, we know that the energy given up to the hot reservoir must equal the sum of the work done and the energy taken in from the cold reservoir. Therefore, the refrigerator or heat pump transfers energy from a colder body (for example, the contents of a kitchen refrigerator or the winter air outside a building) to a hotter body (the air in the kitchen or a room in the building). In practice, it is desirable to carry out this process with

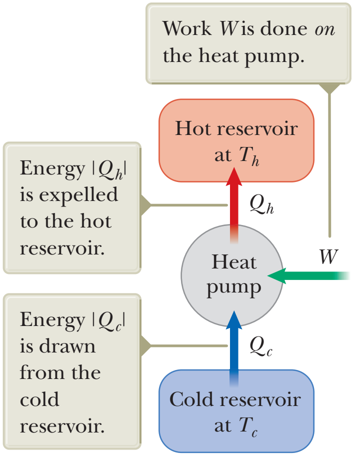

Figure 22.4 Schematic representation of a heat pump.

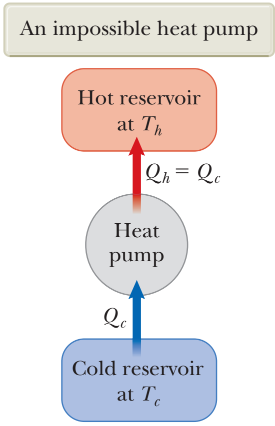

Figure 22.5 Schematic diagram of an impossible heat pump or refrigerator, that is, one that takes in energy from a cold reservoir and expels an equivalent amount of energy to a hot reservoir without the input of energy by work.

a minimum of work. If the process could be accomplished without doing any work, the refrigerator or heat pump would be "perfect" (Fig. 22.5). Again, the existence of such a device would be in violation of the second law of thermodynamics, which in the form of the **Clausius statement**[3] states:

> It is impossible to construct a cyclical machine whose sole effect is to transfer energy continuously by heat from one object to another object at a higher temperature without the input of energy by work.

In simpler terms, energy does not transfer spontaneously by heat from a cold object to a hot object. Work input is required to run a refrigerator.

The Clausius and Kelvin–Planck statements of the second law of thermodynamics appear at first sight to be unrelated, but in fact they are equivalent in all respects. Although we do not prove so here, if either statement is false, so is the other.[4]

In practice, a heat pump includes a circulating fluid that passes through two sets of metal coils that can exchange energy with the surroundings. The fluid is cold and at low pressure when it is in the coils located in a cool environment, where it absorbs energy by heat. The resulting warm fluid is then compressed and enters the other coils as a hot, high-pressure fluid. There it releases its stored energy to the warm surroundings. In an air conditioner, energy is absorbed into the fluid in coils located in a building's interior; after the fluid is compressed, energy leaves the fluid through coils located outdoors. In a refrigerator, the external coils are behind the unit (Fig. 22.6) or underneath the unit. The internal coils are in the walls of the refrigerator and absorb energy from the food.

The effectiveness of a heat pump is described in terms of a number called the **coefficient of performance** (COP). The COP is similar to the thermal efficiency for a heat engine in that it is a ratio of what you gain (energy transferred to or from a reservoir) to what you give (work input). For a heat pump operating in the cooling mode, "what you gain" is energy removed from the cold reservoir. The most effective refrigerator or air conditioner is one that removes the greatest amount of energy

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Figure 22.6 The back of a household refrigerator. The air surrounding the coils is the hot reservoir.

[3]First expressed by Rudolf Clausius (1822–1888).

[4]See an advanced textbook on thermodynamics for this proof.

from the cold reservoir in exchange for the least amount of work. Therefore, for these devices operating in the cooling mode, we define the COP in terms of $|Q_c|$:

$$\text{COP (cooling mode)} = \frac{\text{energy transferred at low temperature}}{\text{work done on heat pump}} = \frac{|Q_c|}{W} \quad \textbf{(22.3)}$$

A good refrigerator should have a high COP, typically 5 or 6.

In addition to cooling applications, heat pumps are becoming increasingly popular for heating purposes. The energy-absorbing coils for a heat pump are located outside a building, in contact with the air or buried in the ground. The other set of coils are in the building's interior. The circulating fluid flowing through the coils absorbs energy from the outside and releases it to the interior of the building from the interior coils.

In the heating mode, the COP of a heat pump is defined as the ratio of the energy transferred to the hot reservoir to the work required to transfer that energy:

$$\text{COP (heating mode)} = \frac{\text{energy transferred at high temperature}}{\text{work done on heat pump}} = \frac{|Q_h|}{W} \quad \textbf{(22.4)}$$

If the outside temperature is 25°F (−4°C) or higher, a typical value of the COP for a heat pump is about 4. That is, the amount of energy transferred to the building is about four times greater than the work done by the motor in the heat pump. As the outside temperature decreases, however, it becomes more difficult for the heat pump to extract sufficient energy from the air and so the COP decreases. Therefore, the use of heat pumps that extract energy from the air, although satisfactory in moderate climates, is not appropriate in areas where winter temperatures are very low. It is possible to use heat pumps in colder areas by burying the external coils deep in the ground. In that case, the energy is extracted from the ground, which tends to be warmer than the air in the winter.

Quick Quiz 22.2 The energy entering an electric heater by electrical transmission can be converted to internal energy with an efficiency of 100%. By what factor does the cost of heating your home change when you replace your electric heating system with an electric heat pump that has a COP of 4.00? Assume the motor running the heat pump is 100% efficient. **(a)** 4.00 **(b)** 2.00 **(c)** 0.500 **(d)** 0.250

Example 22.2 Freezing Water

A certain refrigerator has a COP of 5.00. When the refrigerator is running, its power input is 500 W. A sample of water of mass 500 g and temperature 20.0°C is placed in the freezer compartment. How long does it take to freeze the water to ice at 0°C? Assume all other parts of the refrigerator stay at the same temperature and there is no leakage of energy from the exterior, so the operation of the refrigerator results only in energy being extracted from the water.

SOLUTION

Conceptualize Energy leaves the water, reducing its temperature and then freezing it into ice. The time interval required for this entire process is related to the rate at which energy is withdrawn from the water, which, in turn, is related to the power input of the refrigerator.

Categorize We categorize this example as one that combines our understanding of temperature changes and phase changes from Chapter 20 and our understanding of heat pumps from this chapter.

Analyze Use the power rating of the refrigerator to find the time interval Δt required for the freezing process to occur:

$$P = \frac{W}{\Delta t} \rightarrow \Delta t = \frac{W}{P}$$

22.2 continued

Use Equation 22.3 to relate the work W done on the heat pump to the energy $|Q_c|$ extracted from the water:

$$\Delta t = \frac{|Q_c|}{P(\text{COP})}$$

Use Equations 20.4 and 20.7 to substitute the amount of energy $|Q_c|$ that must be extracted from the water of mass m:

$$\Delta t = \frac{|mc\,\Delta T + L_f\,\Delta m|}{P(\text{COP})}$$

Recognize that the amount of water that freezes is $\Delta m = -m$ because all the water freezes:

$$\Delta t = \frac{|m(c\,\Delta T - L_f)|}{P(\text{COP})}$$

Substitute numerical values:

$$\Delta t = \frac{|(0.500\text{ kg})[(4\,186\text{ J/kg}\cdot{}^\circ\text{C})(-20.0^\circ\text{C}) - 3.33\times10^5\text{ J/kg}]|}{(500\text{ W})(5.00)}$$

$$= \boxed{83.3\text{ s}}$$

Finalize In reality, the time interval for the water to freeze in a refrigerator is much longer than 83.3 s, which suggests that the assumptions of our model are not valid. Only a small part of the energy extracted from the refrigerator interior in a given time interval comes from the water. Energy must also be extracted from the container in which the water is placed, and energy that continuously leaks into the interior from the exterior must be extracted.

22.3 Reversible and Irreversible Processes

In the next section, we will discuss a theoretical heat engine that is the most efficient possible. To understand its nature, we must first examine the meaning of reversible and irreversible processes. In a **reversible** process, the system undergoing the process can be returned to its initial conditions along the same path on a PV diagram, and every point along this path is an equilibrium state. A process that does not satisfy these requirements is **irreversible.**

Pitfall Prevention 22.2

All Real Processes Are Irreversible The reversible process is an idealization; all real processes on the Earth are irreversible.

All natural processes are known to be irreversible. Let's examine the adiabatic free expansion of a gas, which was already discussed in Section 20.6, and show that it cannot be reversible. Consider a gas in a thermally insulated container as shown in Figure 22.7. A membrane separates the gas from a vacuum. When the membrane is punctured, the gas expands freely into the vacuum. As a result of the puncture, the system has changed because it occupies a greater volume after the expansion. Because the gas does not exert a force through a displacement, it does no work on the surroundings as it expands. In addition, no energy is transferred to or from the gas by heat because the container is insulated from its surroundings. Therefore, in this adiabatic process, the system has changed but the surroundings have not.

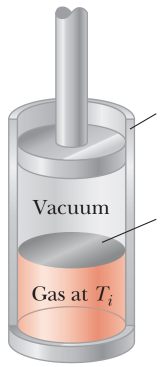

Figure 22.7 Adiabatic free expansion of a gas.

For this process to be reversible, we must return the gas to its original volume and temperature without changing the surroundings. Imagine trying to reverse the process by compressing the gas to its original volume. To do so, we fit the container with a piston and use an engine to force the piston inward. During this process, the surroundings change because work is being done by an outside agent on the system. In addition, the system changes because the compression increases the temperature of the gas. The temperature of the gas can be lowered by allowing it to come into contact with an external energy reservoir. Although this step returns the gas to its original conditions, the surroundings are again affected because energy is being added to the surroundings from the gas. If this

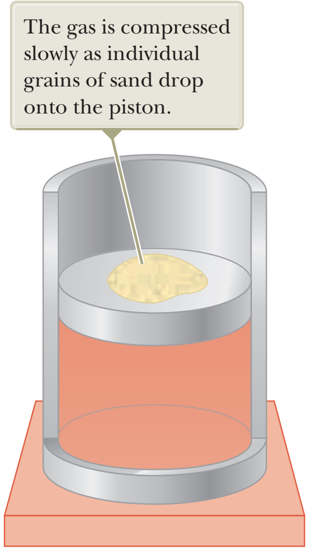

Figure 22.8 A method for compressing a gas in an almost reversible isothermal process.

energy could be used to drive the engine that compressed the gas, the net energy transfer to the surroundings would be zero. In this way, the system and its surroundings could be returned to their initial conditions and we could identify the process as reversible. The Kelvin–Planck statement of the second law, however, specifies that the energy removed from the gas to return the temperature to its original value cannot be completely converted to mechanical energy by the process of work done by the engine in compressing the gas. Therefore, we must conclude that the process is irreversible.

We could also argue that the adiabatic free expansion is irreversible by relying on the portion of the definition of a reversible process that refers to equilibrium states. For example, during the sudden expansion, significant variations in pressure occur throughout the gas. Therefore, there is no well-defined value of the pressure for the entire system at any time between the initial and final states. In fact, the process cannot even be represented as a path on a *PV* diagram. The *PV* diagram for an adiabatic free expansion would show the initial and final conditions as points, but these points would not be connected by a path. Therefore, because the intermediate conditions between the initial and final states are not equilibrium states, the process is irreversible.

Although all real processes are irreversible, some are almost reversible. If a real process occurs very slowly such that the system is always very nearly in an equilibrium state, the process can be approximated as being reversible. Suppose a gas is compressed isothermally in a piston–cylinder arrangement in which the gas is in thermal contact with an energy reservoir and we continuously transfer just enough energy from the gas to the reservoir to keep the temperature constant. For example, imagine that the gas is compressed very slowly by dropping grains of sand onto a frictionless piston as shown in Figure 22.8. As each grain lands on the piston and compresses the gas a small amount, the system deviates from an equilibrium state, but it is so close to one that it achieves a new equilibrium state in a relatively short time interval. Each grain added represents a change to a new equilibrium state, but the differences between states are so small that the entire process can be approximated as occurring through continuous equilibrium states. The process can be reversed by slowly removing grains from the piston.

A general characteristic of a reversible process is that no nonconservative effects (such as turbulence or friction) that transform mechanical energy to internal energy can be present. Such effects can be impossible to eliminate completely. Hence, it is not surprising that real processes in nature are irreversible.

22.4 The Carnot Engine

Pitfall Prevention 22.3

Don't Shop for a Carnot Engine

The Carnot engine is an idealization; do not expect a Carnot engine to be developed for commercial use. We explore the Carnot engine only for theoretical considerations.

In 1824, a French engineer named Sadi Carnot described a theoretical engine, now called a **Carnot engine,** that is of great importance from both practical and theoretical viewpoints. He showed that a heat engine operating in an ideal, reversible cycle—called a **Carnot cycle—**between two energy reservoirs is the most efficient engine possible. Such an ideal engine establishes an upper limit on the efficiencies of all other engines. That is, the net work done by a working substance taken through the Carnot cycle is the greatest amount of work possible for a given amount of energy supplied to the substance at the higher temperature. **Carnot's theorem** can be stated as follows:

> No real heat engine operating between two energy reservoirs can be more efficient than a Carnot engine operating between the same two reservoirs.

In this section, we will show that the efficiency of a Carnot engine depends only on the temperatures of the reservoirs. In turn, that efficiency represents the

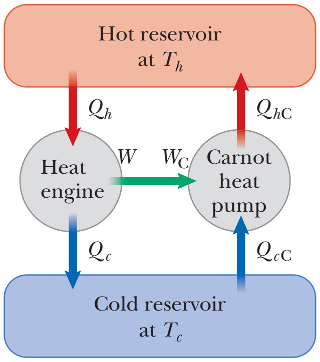

Figure 22.9 A Carnot engine operated as a heat pump and another engine with a proposed higher efficiency operate between two energy reservoirs. The work output and input are matched.

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Sadi Carnot
French engineer (1796–1832)
Carnot was the first to show the quantitative relationship between work and heat. In 1824, he published his only work, *Reflections on the Motive Power of Heat,* which reviewed the industrial, political, and economic importance of the steam engine. In it, he defined work as "weight lifted through a height."

maximum possible efficiency for real engines. Let us confirm that the Carnot engine is the most efficient. We imagine a hypothetical engine with an efficiency greater than that of the Carnot engine. Consider Figure 22.9, which shows the hypothetical engine with $e > e_C$ on the left connected between hot and cold reservoirs. In addition, let us attach a Carnot engine between the same reservoirs. Because the Carnot cycle is reversible, the Carnot engine can be run in reverse as a Carnot heat pump as shown on the right in Figure 22.9. We match the output work of the engine to the input work of the heat pump, $W = W_C$, so there is no exchange of energy by work between the surroundings and the engine–heat pump combination.

Because of the proposed relation between the efficiencies, we must have

$$e > e_C \rightarrow \frac{|W|}{|Q_h|} > \frac{|W_C|}{|Q_{hC}|}$$

The numerators of these two fractions cancel because the works have been matched. This expression requires that

$$|Q_{hC}| > |Q_h| \tag{22.5}$$

From Equation 22.1, the equality of the works gives us

$$|W| = |W_C| \rightarrow |Q_h| - |Q_c| = |Q_{hC}| - |Q_{cC}|$$

which can be rewritten to put the energies exchanged with the cold reservoir on the left and those with the hot reservoir on the right:

$$|Q_{hC}| - |Q_h| = |Q_{cC}| - |Q_c| \tag{22.6}$$

Note that the left side of Equation 22.6 is positive, so the right side must be positive also. We see that the net energy exchange with the hot reservoir is equal to the net energy exchange with the cold reservoir. As a result, for the combination of the heat engine and the heat pump, energy is transferring from the cold reservoir to the hot reservoir by heat with no input of energy by work.

This result is in violation of the Clausius statement of the second law. Therefore, our original assumption that $e > e_C$ must be incorrect, and we must conclude that the Carnot engine represents the highest possible efficiency for an engine. The key feature of the Carnot engine that makes it the most efficient is its *reversibility;* it can be run in reverse as a heat pump. All real engines are less efficient than the Carnot engine because they do not operate through a reversible cycle. The efficiency of a real engine is further reduced by such practical difficulties as friction and energy losses by conduction.

To describe the Carnot cycle taking place between temperatures T_c and T_h, let's assume the working substance is an ideal gas contained in a cylinder fitted with a movable piston at one end. The cylinder's walls and the piston are thermally nonconducting. Four stages of the Carnot cycle are shown in Figure 22.10(page 662),

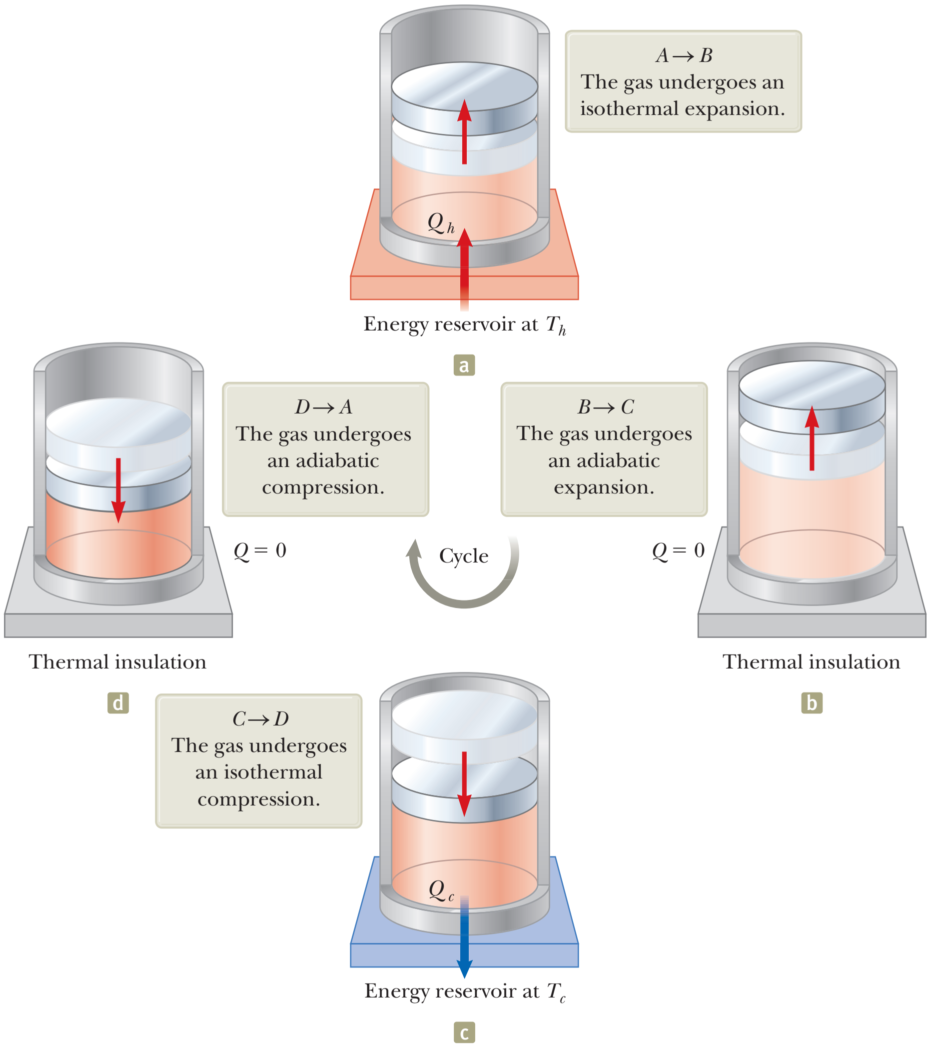


Figure 22.10 The Carnot cycle. The letters *A*, *B*, *C*, and *D* refer to the states of the gas shown in Figure 22.11. The arrows on the piston indicate the direction of its motion during each process.

and the *PV* diagram for the cycle is shown in Figure 22.11. The Carnot cycle consists of two adiabatic processes and two isothermal processes, all reversible:

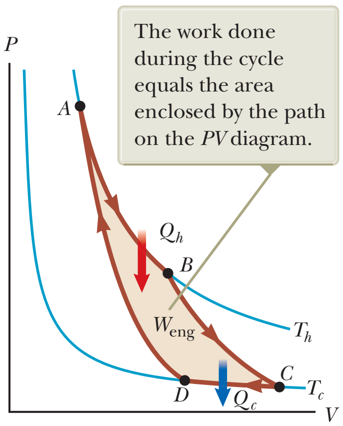


Figure 22.11 *PV* diagram for the Carnot cycle. The net work done W_{eng} equals the net energy transferred into the Carnot engine in one cycle, $|Q_h| - |Q_c|$.

1. Process $A \to B$ (Fig. 22.10a) is an isothermal expansion at temperature T_h. The gas is placed in thermal contact with an energy reservoir at temperature T_h. During the expansion, the gas absorbs energy $|Q_h|$ from the reservoir through the base of the cylinder and does work W_{AB} in raising the piston.
2. In process $B \to C$ (Fig. 22.10b), the base of the cylinder is replaced by a thermally nonconducting wall and the gas expands adiabatically; that is, no energy enters or leaves the system by heat. During the expansion, the temperature of the gas decreases from T_h to T_c and the gas does work W_{BC} in raising the piston.
3. In process $C \to D$ (Fig. 22.10c), the gas is placed in thermal contact with an energy reservoir at temperature T_c and is compressed isothermally at temperature T_c. During this time, the gas expels energy $|Q_c|$ to the reservoir and the work done by the piston on the gas is W_{CD}.
4. In the final process $D \to A$ (Fig. 22.10d), the base of the cylinder is replaced by a nonconducting wall and the gas is compressed adiabatically. The temperature of the gas increases to T_h, and the work done by the piston on the gas is W_{DA}.

The thermal efficiency of the engine is given by Equation 22.2:

$$e = 1 - \frac{|Q_c|}{|Q_h|}$$

In Example 22.3, we show that for a Carnot cycle,

$$\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h} \tag{22.7}$$

Hence, the thermal efficiency of a Carnot engine is

$$\boxed{e_C = 1 - \frac{T_c}{T_h}} \tag{22.8}$$

Efficiency of a Carnot engine

This result indicates that all Carnot engines operating between the same two temperatures have the same efficiency.[5]

Equation 22.8 can be applied to any working substance operating in a Carnot cycle between two energy reservoirs. According to this equation, the efficiency is zero if $T_c = T_h$, as one would expect. The efficiency increases as T_c is lowered and T_h is raised. The efficiency can be unity (100%), however, only if $T_c = 0$ K. Such reservoirs are not available; therefore, the maximum efficiency is always less than 100%. In most practical cases, T_c is near room temperature, which is about 300 K. Therefore, one usually strives to increase the efficiency by raising T_h.

Theoretically, a Carnot-cycle heat engine run in reverse constitutes the most effective heat pump possible, and it determines the maximum COP for a given combination of hot and cold reservoir temperatures. Using Equations 22.1 and 22.4, we see that the maximum COP for a heat pump in its heating mode is

$$\begin{aligned}\text{COP}_C\,(\text{heating mode}) &= \frac{|Q_h|}{W} \\ &= \frac{|Q_h|}{|Q_h| - |Q_c|} = \frac{1}{1 - \dfrac{|Q_c|}{|Q_h|}} = \frac{1}{1 - \dfrac{T_c}{T_h}} = \frac{T_h}{T_h - T_c}\end{aligned}$$

The Carnot COP for a heat pump in the cooling mode is

$$\text{COP}_C\,(\text{cooling mode}) = \frac{T_c}{T_h - T_c}$$

As the difference between the temperatures of the two reservoirs approaches zero in this expression, the theoretical COP approaches infinity. In practice, the low temperature of the cooling coils and the high temperature at the compressor limit the COP to values below 10.

Quick Quiz 22.3 Three engines operate between reservoirs separated in temperature by 300 K. The reservoir temperatures are as follows: Engine A: $T_h = 1\,000$ K, $T_c = 700$ K; Engine B: $T_h = 800$ K, $T_c = 500$ K; Engine C: $T_h = 600$ K, $T_c = 300$ K. Rank the engines in order of theoretically possible efficiency from highest to lowest.

[5]For the processes in the Carnot cycle to be reversible, they must be carried out infinitesimally slowly. Therefore, although the Carnot engine is the most efficient engine possible, it has zero power output because it takes an infinite time interval to complete one cycle! For a real engine, the short time interval for each cycle results in the working substance reaching a high temperature lower than that of the hot reservoir and a low temperature higher than that of the cold reservoir. An engine undergoing a Carnot cycle between this narrower temperature range was analyzed by F. L. Curzon and B. Ahlborn ("Efficiency of a Carnot engine at maximum power output," *Am. J. Phys.* **43**(1), 22, 1975), who found that the efficiency at maximum power output depends only on the reservoir temperatures T_c and T_h and is given by $e_{\text{C-A}} = 1 - (T_c/T_h)^{1/2}$. The Curzon–Ahlborn efficiency $e_{\text{C-A}}$ provides a closer approximation to the efficiencies of real engines than does the Carnot efficiency.

Example 22.3 Efficiency of the Carnot Engine

Show that the ratio of energy transfers by heat in a Carnot engine is equal to the ratio of reservoir temperatures, as given by Equation 22.7.

SOLUTION

Conceptualize Make use of Figures 22.10 and 22.11 to help you visualize the processes in the Carnot cycle.

Categorize Because of our understanding of the Carnot cycle, we can categorize the processes in the cycle as isothermal and adiabatic.

Analyze For the isothermal expansion (process $A \rightarrow B$ in Fig. 22.10), find the energy transfer by heat from the hot reservoir using Equation 20.14 and the first law of thermodynamics:

$$|Q_h| = |\Delta E_{\text{int}} - W_{AB}| = |0 - W_{AB}| = nRT_h \ln \frac{V_B}{V_A}$$

In a similar manner, find the energy transfer to the cold reservoir during the isothermal compression $C \rightarrow D$:

$$|Q_c| = |\Delta E_{\text{int}} - W_{CD}| = |0 - W_{CD}| = nRT_c \ln \frac{V_C}{V_D}$$

Divide the second expression by the first:

$$(1)\quad \frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}\frac{\ln(V_C/V_D)}{\ln(V_B/V_A)}$$

Apply Equation 21.39 to the adiabatic processes $B \rightarrow C$ and $D \rightarrow A$:

$$T_h V_B^{\gamma-1} = T_c V_C^{\gamma-1}$$
$$T_h V_A^{\gamma-1} = T_c V_D^{\gamma-1}$$

Divide the first equation by the second:

$$\left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1}$$

$$(2)\quad \frac{V_B}{V_A} = \frac{V_C}{V_D}$$

Substitute Equation (2) into Equation (1):

$$\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}\frac{\ln(V_C/V_D)}{\ln(V_B/V_A)} = \frac{T_c}{T_h}\frac{\ln(V_C/V_D)}{\ln(V_C/V_D)} = \frac{T_c}{T_h}$$

Finalize This last equation is Equation 22.7, the one we set out to prove.

Example 22.4 The Steam Engine

A steam engine has a boiler that operates at 500 K. The energy from the burning fuel changes water to steam, and this steam then drives a piston. The cold reservoir's temperature is that of the outside air, approximately 300 K. What is the maximum thermal efficiency of this steam engine?

SOLUTION

Conceptualize In a steam engine, the gas pushing on the piston in Figure 22.10 is steam. A real steam engine does not operate in a Carnot cycle, but, to find the maximum possible efficiency, imagine a Carnot steam engine.

Categorize We calculate an efficiency using Equation 22.8, so we categorize this example as a substitution problem.

Substitute the reservoir temperatures into Equation 22.8:

$$e_C = 1 - \frac{T_c}{T_h} = 1 - \frac{300\text{ K}}{500\text{ K}} = \boxed{0.400} \quad \text{or} \quad \boxed{40.0\%}$$

This result is the highest *theoretical* efficiency of the engine. In practice, the efficiency is considerably lower.

22.4 continued

WHAT IF? Suppose we wished to increase the theoretical efficiency of this engine. This increase can be achieved by raising T_h by ΔT or by decreasing T_c by the same ΔT. Which would be more effective?

Answer A given ΔT would have a larger fractional effect on a smaller temperature, so you would expect a larger change in efficiency if you alter T_c by ΔT. Let's test that numerically. Raising T_h by 50 K, corresponding to $T_h = 550$ K, would give a maximum efficiency of

$$e_C = 1 - \frac{T_c}{T_h} = 1 - \frac{300\text{ K}}{550\text{ K}} = 0.455$$

Decreasing T_c by 50 K, corresponding to $T_c = 250$ K, would give a maximum efficiency of

$$e_C = 1 - \frac{T_c}{T_h} = 1 - \frac{250\text{ K}}{500\text{ K}} = 0.500$$

Although changing T_c is *mathematically* more effective, often changing T_h is *practically* more feasible.

22.5 Gasoline and Diesel Engines

In a gasoline engine, six processes occur in each cycle; they are illustrated in Figure 22.12. In this discussion, let's consider the interior of the cylinder above the piston to be the system that is taken through repeated cycles in the engine's operation. For a given cycle, the piston moves up and down twice, which represents a four-stroke cycle consisting of two upstrokes and two downstrokes. The processes in the cycle can be approximated by the **Otto cycle** shown in the *PV* diagram in Figure 22.13 (page 666). In the following discussion, refer to Figure 22.12 for the pictorial representation of the strokes and Figure 22.13 for the significance on the *PV* diagram of the letter designations below:

1. During the *intake stroke* (Fig. 22.12a and $O \rightarrow A$ in Figure 22.13), the piston moves downward and a gaseous mixture of air and fuel is drawn into the

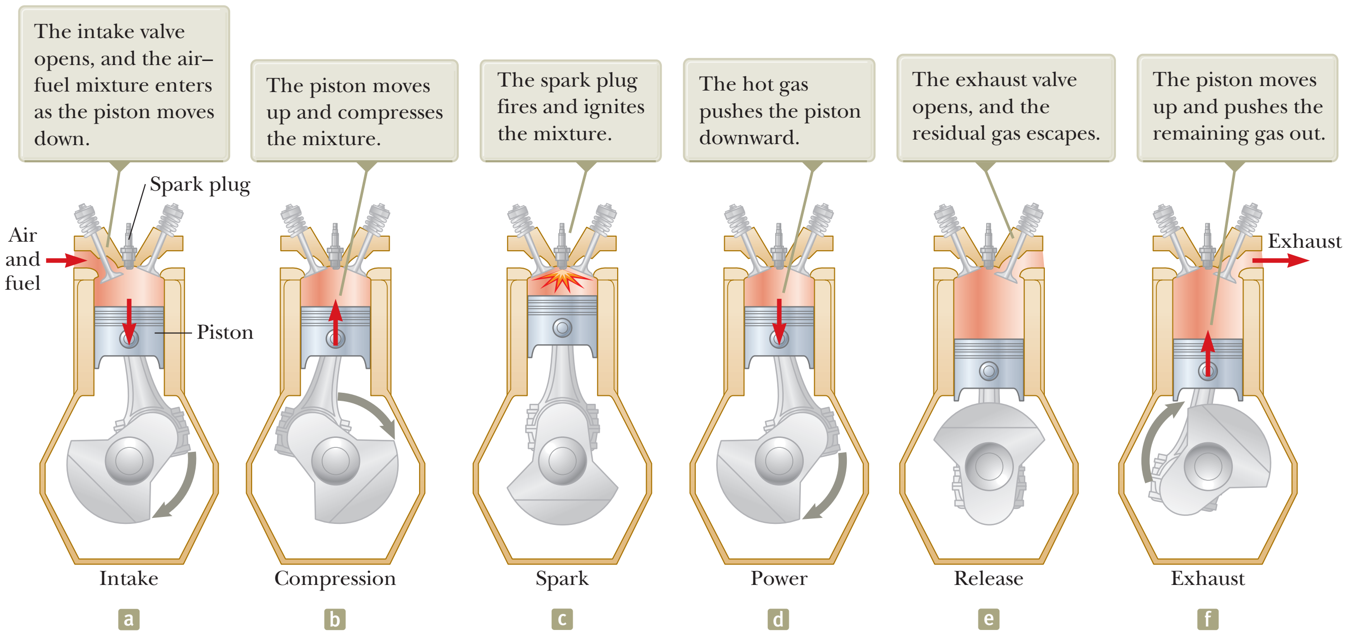


Figure 22.12 The four-stroke cycle of a conventional gasoline engine. The arrows on the piston indicate the direction of its motion during each process.

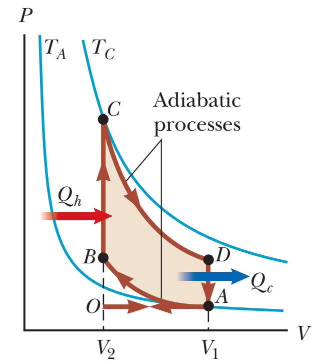


Figure 22.13 *PV* diagram for the Otto cycle, which approximately represents the processes occurring in an internal combustion engine.

cylinder at atmospheric pressure. That is the energy input part of the cycle: energy enters the system (the interior of the cylinder) by matter transfer as potential energy stored in the fuel. In this process, the volume increases from V_2 to V_1. This apparent backward numbering is based on the compression stroke (process 2 below), in which the air–fuel mixture is compressed from V_1 to V_2.

2. During the *compression stroke* (Fig. 22.12b and $A \rightarrow B$ in Fig. 22.13), the piston moves upward, the air–fuel mixture is compressed adiabatically from volume V_1 to volume V_2, and the temperature increases from T_A to T_B. The work done on the gas is positive, and its value is equal to the negative of the area under the curve AB in Figure 22.13.
3. Combustion occurs when the spark plug fires (Fig. 22.12c and $B \rightarrow C$ in Fig. 22.13). That is not one of the strokes of the cycle because it occurs in a very short time interval while the piston is at its highest position. The combustion represents a rapid energy transformation from potential energy stored in chemical bonds in the fuel to internal energy associated with molecular motion, which is related to temperature. During this time interval, the mixture's pressure and temperature increase rapidly, with the temperature rising from T_B to T_C. The volume, however, remains approximately constant because of the short time interval. As a result, approximately no work is done on or by the gas. We can model this process in the *PV* diagram (Fig. 22.13) as that process in which the energy $|Q_h|$ enters the system. (In reality, however, this process is a *transformation* of energy already in the cylinder from process $O \rightarrow A$.)
4. In the *power stroke* (Fig. 22.12d and $C \rightarrow D$ in Fig. 22.13), the gas expands adiabatically from V_2 to V_1. This expansion causes the temperature to drop from T_C to T_D. Work is done by the gas in pushing the piston downward, and the value of this work is equal to the area under the curve CD.
5. Release of the residual gases occurs when an exhaust valve is opened (Fig. 22.12e and $D \rightarrow A$ in Fig. 22.13). The pressure suddenly drops for a short time interval. During this time interval, the piston is almost stationary and the volume is approximately constant. Energy is expelled from the interior of the cylinder and continues to be expelled during the next process.
6. In the final process, the *exhaust stroke* (Fig. 22.12e and $A \rightarrow O$ in Fig. 22.13), the piston moves upward while the exhaust valve remains open. Residual gases are exhausted at atmospheric pressure, and the volume decreases from V_1 to V_2. The cycle then repeats.

If the air–fuel mixture is assumed to be an ideal gas, the efficiency of the Otto cycle is

$$e = 1 - \frac{1}{(V_1/V_2)^{\gamma - 1}} \quad \text{(Otto cycle)} \tag{22.9}$$

where V_1/V_2 is the **compression ratio** and γ is the ratio of the molar specific heats C_P/C_V for the air–fuel mixture. Equation 22.9, which is derived in Example 22.5, shows that the efficiency increases as the compression ratio increases. For a typical compression ratio of 8 and with $\gamma = 1.4$, Equation 22.9 predicts a theoretical efficiency of 56% for an engine operating in the idealized Otto cycle. This value is much greater than that achieved in real engines (15% to 20%) because of such effects as friction, energy transfer by conduction through the cylinder walls, and incomplete combustion of the air–fuel mixture.

Diesel engines operate on a cycle similar to the Otto cycle, but they do not employ a spark plug. The compression ratio for a diesel engine is much greater than that for a gasoline engine. Air in the cylinder is compressed to a very small volume, and, as a consequence, the cylinder temperature at the end of the compression stroke is very high. At this point, fuel is injected into the cylinder. The temperature is high enough for the air–fuel mixture to ignite without the assistance of a spark plug. Diesel engines are more efficient than gasoline engines because of their greater compression ratios and resulting higher combustion temperatures.

Example 22.5 Efficiency of the Otto Cycle

Show that the thermal efficiency of an engine operating in an idealized Otto cycle (see Figs. 22.12 and 22.13) is given by Equation 22.9. Treat the working substance as an ideal gas.

SOLUTION

Conceptualize Study Figures 22.12 and 22.13 to make sure you understand the working of the Otto cycle.

Categorize As seen in Figure 22.13, we categorize the processes in the Otto cycle as isovolumetric and adiabatic.

Analyze Model the energy input and output as occurring by heat in processes $B \rightarrow C$ and $D \rightarrow A$. (In reality, most of the energy enters and leaves by matter transfer as the air–fuel mixture enters and leaves the cylinder.) Use Equation 21.23 to find the energy transfers by heat for these processes, which take place at constant volume:

$$B \rightarrow C \quad |Q_h| = nC_V(T_C - T_B)$$

$$D \rightarrow A \quad |Q_c| = nC_V(T_D - T_A)$$

Substitute these expressions into Equation 22.2:

$$(1) \quad e = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{T_D - T_A}{T_C - T_B}$$

Apply Equation 21.39 to the adiabatic processes $A \rightarrow B$ and $C \rightarrow D$:

$$A \rightarrow B \quad T_A V_A^{\gamma-1} = T_B V_B^{\gamma-1}$$

$$C \rightarrow D \quad T_C V_C^{\gamma-1} = T_D V_D^{\gamma-1}$$

Solve these equations for the temperatures T_A and T_D, noting that $V_A = V_D = V_1$ and $V_B = V_C = V_2$:

$$(2) \quad T_A = T_B\left(\frac{V_B}{V_A}\right)^{\gamma-1} = T_B\left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$(3) \quad T_D = T_C\left(\frac{V_C}{V_D}\right)^{\gamma-1} = T_C\left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

Subtract Equation (2) from Equation (3) and rearrange:

$$(4) \quad \frac{T_D - T_A}{T_C - T_B} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

Substitute Equation (4) into Equation (1):

$$e = 1 - \frac{1}{(V_1/V_2)^{\gamma-1}}$$

Finalize This final expression is Equation 22.9.

22.6 Entropy

The zeroth law of thermodynamics involves the concept of temperature, and the first law involves the concept of internal energy. Temperature and internal energy are both state variables; that is, the value of each depends only on the thermodynamic state of a system, not on the process that brought it to that state. Another state variable—this one related to the second law of thermodynamics—is *entropy*.

Entropy was originally formulated as a useful concept in thermodynamics. Its importance grew, however, as the field of statistical mechanics developed because the analytical techniques of statistical mechanics provide an alternative means of interpreting entropy and a more global significance to the concept. In statistical

Pitfall Prevention 22.4

Entropy Is Abstract Entropy is one of the most abstract notions in physics, so follow the discussion in this and the subsequent sections very carefully. Do not confuse energy with entropy. Even though the names sound similar, they are very different concepts. On the other hand, energy and entropy are intimately related, as we shall see in this discussion.

mechanics, the behavior of a substance is described in terms of the statistical behavior of its atoms and molecules.

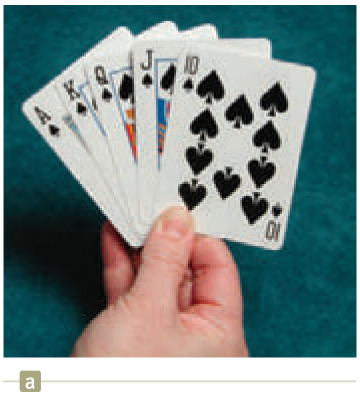
We will develop our understanding of entropy by first considering some nonthermodynamic systems, such as a pair of dice and poker hands. We will then expand on these ideas and use them to understand the concept of entropy as applied to thermodynamic systems.

We begin this process by distinguishing between *microstates* and *macrostates* of a system. A **microstate** is a particular configuration of the individual constituents of the system. A **macrostate** is a description of the system's conditions from a macroscopic point of view.

For any given macrostate of the system, a number of microstates are possible. For example, the macrostate of a 4 on a pair of dice can be formed from the possible microstates 1–3, 2–2, and 3–1. The macrostate of 2 has only one microstate, 1–1. It is assumed all microstates are equally probable. We can compare these two macrostates in three ways: (1) *Uncertainty:* If we know that a macrostate of 4 exists, there is some uncertainty as to the microstate that exists, because there are multiple microstates that will result in a 4. In comparison, there is lower uncertainty (in fact, *zero* uncertainty) for a macrostate of 2 because there is only one microstate. (2) *Choice:* There are more choices of microstates for a 4 than for a 2. (3) *Probability:* The macrostate of 4 has a higher probability than a macrostate of 2 because there are more ways (microstates) of achieving a 4. The notions of uncertainty, choice, and probability are central to the concept of entropy, as we discuss below.

Let's look at another example related to a poker hand. There is only one microstate associated with the macrostate of a royal flush of five spades, laid out in order from ten to ace (Fig. 22.14a). Figure 22.14b shows another poker hand. The macrostate here is "worthless hand." The *particular* hand (the microstate) in Figure 22.14b and the hand in Figure 22.14a are equally probable. There are, however, *many* other hands similar to that in Figure 22.14b; that is, there are many microstates that also qualify as worthless hands. If you, as a poker player, are told your opponent holds a macrostate of a royal flush in spades, there is *zero uncertainty* as to what five cards are in the hand, only *one choice* of what those cards are, and *low probability* that the hand actually occurred. In contrast, if you are told that your opponent has the macrostate of "worthless hand," there is *high uncertainty* as to what the five cards are, *many choices* of what they could be, and a *high probability* that a worthless hand occurred. Another variable in poker, of course, is the value of the hand, related to the probability: the higher the probability, the lower the value. The important point to take away from this discussion is that uncertainty, choice, and probability are related in these situations: if one is high, the others are high, and vice versa.

Another way of describing macrostates is by means of "missing information." For high-probability macrostates with many microstates, there is a large amount



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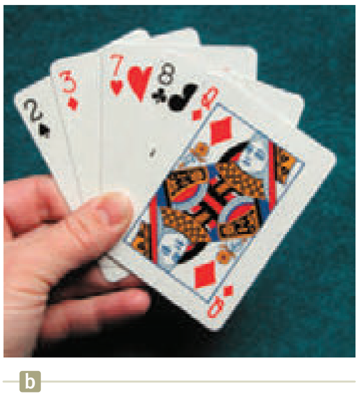
a



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b

Figure 22.14 (a) A royal flush has low probability of occurring. (b) A worthless poker hand, one of many.

of missing information, meaning we have very little information about what microstate actually exists. For a macrostate of a 2 on a pair of dice, we have no missing information; we *know* the microstate is 1–1. For a macrostate of a worthless poker hand, however, we have lots of missing information, related to the large number of choices we could make as to the actual hand that is held.

Quick Quiz 22.4 **(a)** Suppose you select four cards at random from a standard deck of playing cards and end up with a macrostate of four deuces. How many microstates are associated with this macrostate? **(b)** Suppose you pick up two cards and end up with a macrostate of two aces. How many microstates are associated with this macrostate?

For thermodynamic systems, the variable **entropy** S is used to represent the level of uncertainty, choice, probability, or missing information in the system. Consider a configuration (a macrostate) in which all the oxygen molecules in your room are located in the west half of the room and the nitrogen molecules in the east half. Compare that macrostate to the more common configuration of the air molecules distributed uniformly throughout the room. The latter configuration has the higher uncertainty and more missing information as to where the molecules are located because they could be anywhere, not just in one half of the room according to the type of molecule. The configuration with a uniform distribution also represents more choices as to where to locate molecules. It also has a much higher probability of occurring; have you ever noticed your half of the room suddenly being empty of oxygen? Therefore, the latter configuration represents a higher entropy.

For systems of dice and poker hands, the comparisons between probabilities for various macrostates involve relatively small numbers. For example, a macrostate of a 4 on a pair of dice is only three times as probable as a macrostate of 2. The ratio of probabilities of a worthless hand and a royal flush is significantly larger. When we are talking about a macroscopic thermodynamic system containing on the order of Avogadro's number of molecules, however, the ratios of probabilities can be astronomical.

Let's explore this concept by considering 100 molecules in a container. Half of the molecules are oxygen and the other half are nitrogen. At any given moment, the probability of one molecule being in the left part of the container shown in Figure 22.15a as a result of random motion is $\frac{1}{2}$. If there are two molecules as shown in Figure 22.15b, the probability of both being in the left part is $\left(\frac{1}{2}\right)^2$, or 1 in 4. If there are three molecules (Fig. 22.15c), the probability of them all being in the left portion at the same moment is $\left(\frac{1}{2}\right)^3$, or 1 in 8. For 100 independently moving molecules, the probability that the 50 oxygen molecules will be found in the left part at any moment is $\left(\frac{1}{2}\right)^{50}$. Likewise, the probability that the remaining 50 nitrogen molecules will be found in the right part at any moment is $\left(\frac{1}{2}\right)^{50}$. Therefore, the probability of

Pitfall Prevention 22.5

Entropy Is for Thermodynamic Systems We are not applying the word *entropy* to describe systems of dice or cards. We are only discussing dice and cards to set up the notions of microstates, macrostates, uncertainty, choice, probability, and missing information. Entropy can *only* be used to describe thermodynamic systems that contain many particles, allowing the system to store energy as internal energy.

Pitfall Prevention 22.6

Entropy and Disorder Some textbook treatments of entropy relate entropy to the *disorder* of a system. While this approach has some merit, it is not entirely successful. For example, consider two samples of the same solid material at the same temperature. One sample has volume V and the other volume $2V$. The larger sample has higher entropy than the smaller one simply because there are more molecules in it. But there is no sense in which it is more disordered than the smaller sample. We will not use the disorder approach in this text, but watch for it in other sources.

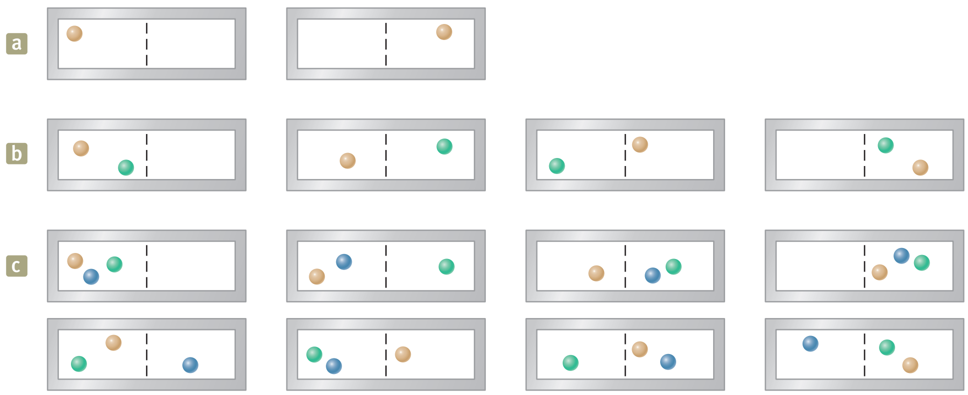

Figure 22.15 Possible distributions of identical molecules in a container. The colors used here exist only to allow us to distinguish among the molecules. (a) One molecule in a container has a 1-in-2 chance of being on the left side. (b) Two molecules have a 1-in-4 chance of being on the left side at the same time. (c) Three molecules have a 1-in-8 chance of being on the left side at the same time.

finding this oxygen–nitrogen separation as a result of random motion is the product $(\frac{1}{2})^{50}(\frac{1}{2})^{50} = (\frac{1}{2})^{100}$, which corresponds to about 1 in 10^{30}. When this calculation is extrapolated from 100 molecules to the number in 1 mol of gas (6.02×10^{23}), the separated arrangement is found to be *extremely* improbable!

Conceptual Example 22.6 Let's Play Marbles!

Suppose you have a bag of 100 marbles of which 50 are red and 50 are green. You are allowed to draw four marbles from the bag according to the following rules. Draw one marble, record its color, and return it to the bag. Shake the bag and then draw another marble. Continue this process until you have drawn and returned four marbles. What are the possible macrostates for this set of events? What is the most likely macrostate? What is the least likely macrostate?

SOLUTION

Table 22.1 Possible Results of Drawing Four Marbles from a Bag

Macrostate	Possible Microstates	Total Number of Microstates
All R	RRRR	1
1G, 3R	RRRG, RRGR, RGRR, GRRR	4
2G, 2R	RRGG, RGRG, GRRG, RGGR, GRGR, GGRR	6
3G, 1R	GGGR, GGRG, GRGG, RGGG	4
All G	GGGG	1

Because each marble is returned to the bag before the next one is drawn and the bag is then shaken, the probability of drawing a red marble is always the same as the probability of drawing a green one. All the possible microstates and macrostates are shown in Table 22.1. As this table indicates, there is only one way to draw a macrostate of four red marbles, so there is only one microstate for that macrostate. There are, however, four possible microstates that correspond to the macrostate of one green marble and three red marbles, six microstates that correspond to two green marbles and two red marbles, four microstates that correspond to three green marbles and one red marble, and one microstate that corresponds to four green marbles. The most likely macrostate—two red marbles and two green marbles—corresponds to the largest number of choices of microstates, and, therefore, the most uncertainty as to what the exact microstate is. The least likely macrostates—four red marbles or four green marbles—correspond to only one choice of microstate and, therefore, zero uncertainty. There is no missing information for the least likely states: we know the colors of all four marbles.

We have investigated the notions of uncertainty, number of choices, probability, and missing information for some non-thermodynamic systems and have argued that the concept of entropy can be related to these notions for thermodynamic systems. We have not yet indicated how to evaluate entropy numerically for a thermodynamic system. This evaluation was done through statistical means by Boltzmann in the 1870s and appears in its currently accepted form as

$$S = k_B \ln W \tag{22.10}$$

where k_B is Boltzmann's constant. Boltzmann intended W, standing for *Wahrscheinlichkeit,* the German word for probability, to be proportional to the probability that a given macrostate exists. It is equivalent to let W be the number of microstates associated with the macrostate, so we can interpret W as representing the number of "ways" of achieving the macrostate. Therefore, macrostates with larger numbers of microstates have higher probability and, equivalently, higher entropy.

In the kinetic theory of gases, gas molecules are represented as particles moving randomly. Suppose the gas is confined to a volume V. For a uniform distribution of gas in the volume, there are a large number of equivalent microstates, and the entropy of the gas can be related to the number of microstates corresponding to a given macrostate. Let us count the number of microstates by considering the variety of molecular locations available to the molecules. Let us assume each molecule occupies some microscopic volume V_m. The total number of possible locations of a single molecule in a macroscopic volume V is the ratio $w = V/V_m$, which is a huge number. We use lowercase w here to represent the number of ways a single molecule can be placed in the volume or the number of microstates for a single molecule, which is equivalent to the number of available locations. We assume the probabilities of a molecule occupying any of these locations are equal. As more molecules are added to the system, the number of possible ways the molecules can be positioned in the volume multiplies, as we saw in Figure 22.15. For example, if you consider two molecules, for every possible placement of the first, all possible placements of the second are available. Therefore, there are w ways of locating the first molecule, and for each way, there are w ways of locating the second molecule. The total number of ways of locating the two molecules is $W = w \times w = w^2 = (V/V_m)^2$. (Uppercase W represents the number of ways of putting multiple molecules into the volume and is not to be confused with work.)

Now consider placing N molecules of gas in the volume V. Neglecting the very small probability of having two molecules occupy the same location, each molecule may go into any of the V/V_m locations, and so the number of ways of locating N molecules in the volume becomes $W = w^N = (V/V_m)^N$. Therefore, the spatial part of the entropy of the gas, from Equation 22.10, is

$$S = k_B \ln W = k_B \ln \left(\frac{V}{V_m}\right)^N = Nk_B \ln \left(\frac{V}{V_m}\right) = nR \ln \left(\frac{V}{V_m}\right) \qquad \textbf{(22.11)}$$

We will use this expression in the next section as we investigate changes in entropy for processes occurring in thermodynamic systems.

Notice that we have indicated Equation 22.11 as representing only the *spatial* portion of the entropy of the gas. There is also a temperature-dependent portion of the entropy that the discussion above does not address. For example, imagine an isovolumetric process in which the temperature of the gas increases. Equation 22.11 above shows no change in the spatial portion of the entropy for this situation. There *is* a change in entropy, however, associated with the increase in temperature. We can understand this by appealing again to a bit of quantum physics. Recall from Section 21.3 that the energies of the gas molecules are quantized. When the temperature of a gas changes, the distribution of energies of the gas molecules changes according to the Boltzmann distribution law, discussed in Section 21.5. Therefore, as the temperature of the gas increases, there is more uncertainty about the particular microstate that exists as gas molecules distribute themselves into higher available quantum states. We will see the entropy change associated with an isovolumetric process in Example 22.8.

22.7 Changes in Entropy for Thermodynamic Systems

Thermodynamic systems are constantly in flux, changing continuously from one microstate to another. If the system is in equilibrium, a given macrostate exists, and the system fluctuates from one microstate associated with that macrostate to another. This change is unobservable because we are only able to detect the macrostate. Equilibrium states have tremendously higher probability than nonequilibrium states, so it is highly unlikely that an equilibrium state will spontaneously change to a nonequilibrium state. For example, we do not observe a spontaneous split into the oxygen–nitrogen separation discussed in Section 22.6.

What if the system begins in a low-probability macrostate, however? What if the room *begins* with an oxygen–nitrogen separation? In this case, the system will progress from this low-probability macrostate to the much-higher probability state: the gases will disperse and mix throughout the room. Because entropy is related to probability, a spontaneous increase in entropy, such as in the latter situation, is natural. If the oxygen and nitrogen molecules were initially spread evenly throughout the room, a decrease in entropy would occur if the spontaneous splitting of molecules occurred.

One way of conceptualizing a change in entropy is to relate it to *energy spreading*. A natural tendency is for energy to undergo spatial spreading in time, representing an increase in entropy. If a basketball is dropped onto a floor, it bounces several times and eventually comes to rest. The initial gravitational potential energy in the basketball–Earth system has been transformed to internal energy in the ball and the floor. That energy is spreading outward by heat into the air and into regions of the floor farther from the drop point. In addition, some of the energy has spread throughout the room by sound. It would be unnatural for energy in the room and floor to reverse this motion and concentrate into the stationary ball so that it spontaneously begins to bounce again.

In the adiabatic free expansion of Section 22.3, the spreading of energy accompanies the spreading of the molecules as the gas rushes into the evacuated half of the container. If a warm object is placed in thermal contact with a cool object, energy transfers from the warm object to the cool one by heat, representing a spread of energy until it is distributed more evenly between the two objects.

Now consider a mathematical representation of this spreading of energy or, equivalently, the change in entropy. The original formulation of entropy in thermodynamics involves the transfer of energy by heat during a reversible process. Consider any infinitesimal process in which a system changes from one equilibrium state to another. If dQ_r is the amount of energy transferred by heat when the system follows a reversible path between the states, the change in entropy dS is equal to this amount of energy divided by the absolute temperature of the system:

Change in entropy for an infinitesimal process

$$dS = \frac{dQ_r}{T} \qquad (22.12)$$

We have assumed the temperature is constant because the process is infinitesimal. Because entropy is a state variable, the change in entropy during a process depends only on the endpoints and therefore is independent of the actual path followed. Consequently, the entropy change for an irreversible process can be determined by calculating the entropy change for a *reversible* process that connects the same initial and final states.

The subscript r on the quantity dQ_r is a reminder that the transferred energy is to be measured along a reversible path even though the system may actually have followed some irreversible path. When energy is absorbed by the system, dQ_r is positive and the entropy of the system increases. When energy is expelled by the system, dQ_r is negative and the entropy of the system decreases. Notice that Equation 22.12 does not define entropy but rather the *change* in entropy. Hence, the meaningful quantity in describing a process is the *change* in entropy.

To calculate the change in entropy for a *finite* process, first recognize that T is generally not constant during the process. Therefore, we must integrate Equation 22.12:

Change in entropy for a finite process

$$\Delta S = \int_i^f dS = \int_i^f \frac{dQ_r}{T} \qquad (22.13)$$

As with an infinitesimal process, the change in entropy ΔS of a system going from one state to another has the same value for *all* paths connecting the two states. That is, the finite change in entropy ΔS of a system depends only on the properties of the initial and final equilibrium states. Therefore, we are free to choose any convenient reversible path over which to evaluate the entropy in place of the actual path as long as the initial and final states are the same for both paths. This point is explored further on in this section.

From Equation 22.10, we see that a change in entropy is represented in the Boltzmann formulation as

$$\Delta S = k_{\mathrm{B}} \ln \left(\frac{W_f}{W_i} \right) \quad \textbf{(22.14)}$$

where W_i and W_f represent the inital and final numbers of microstates, respectively, for the initial and final configurations of the system. If $W_f > W_i$, the final state is more probable than the the initial state (there are more choices of microstates), and the entropy increases.

Quick Quiz 22.5 An ideal gas is taken from an initial temperature T_i to a higher final temperature T_f along two different reversible paths. Path A is at constant pressure, and path B is at constant volume. What is the relation between the entropy changes of the gas for these paths? **(a)** $\Delta S_{\mathrm{A}} > \Delta S_{\mathrm{B}}$ **(b)** $\Delta S_{\mathrm{A}} = \Delta S_{\mathrm{B}}$ **(c)** $\Delta S_{\mathrm{A}} < \Delta S_{\mathrm{B}}$

Quick Quiz 22.6 True or False: The entropy change in an adiabatic process must be zero because $Q = 0$.

Example 22.7 Change in Entropy: Melting

A solid that has a latent heat of fusion L_f melts at a temperature T_m. Calculate the change in entropy of this substance when a mass m of the substance melts.

SOLUTION

Conceptualize We can choose any convenient reversible path to follow that connects the initial and final states. It is not necessary to identify the process or the path because, whatever it is, the effect is the same: energy enters the substance by heat and the substance melts. The mass m of the substance that melts is equal to Δm, the change in mass of the higher-phase (liquid) substance.

Categorize Because the melting takes place at a fixed temperature, we categorize the process as isothermal.

Analyze Use Equation 20.7 in Equation 22.13, noting that the temperature remains fixed:

$$\Delta S = \int \frac{dQ_r}{T} = \frac{1}{T_m} \int dQ_r = \frac{Q_r}{T_m} = \frac{L_f \Delta m}{T_m} = \boxed{\frac{L_f m}{T_m}}$$

Finalize Notice that Δm is positive so that ΔS is positive, representing that energy is added to the substance.

Entropy Change in a Carnot Cycle

Let's consider the changes in entropy that occur in a Carnot heat engine that operates between the temperatures T_c and T_h. In one cycle, the engine takes in energy $|Q_h|$ from the hot reservoir and expels energy $|Q_c|$ to the cold reservoir. These energy transfers occur only during the isothermal portions of the Carnot cycle; therefore, the constant temperature can be brought out in front of the integral sign in Equation 22.13. The integral then simply has the value of the total amount of energy transferred by heat. Therefore, the total change in entropy for one cycle is

$$\Delta S = \frac{|Q_h|}{T_h} - \frac{|Q_c|}{T_c} \quad \textbf{(22.15)}$$

where the minus sign represents that energy is leaving the engine at temperature T_c. In Example 22.3, we showed that for a Carnot engine,

$$\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}$$

Using this result in Equation 22.15, we find that the total change in entropy for a Carnot engine operating in a cycle is *zero:*

$$\Delta S = 0$$

Now consider a system taken through an arbitrary (non-Carnot) reversible cycle. Because entropy is a state variable—and hence depends only on the properties of a given equilibrium state—we conclude that $\Delta S = 0$ for *any* reversible cycle. In general, we can write this condition as

$$\oint \frac{dQ_r}{T} = 0 \quad \text{(reversible cycle)} \tag{22.16}$$

where the symbol $\oint$ indicates that the integration is over a closed path.

Entropy Change in a Free Expansion

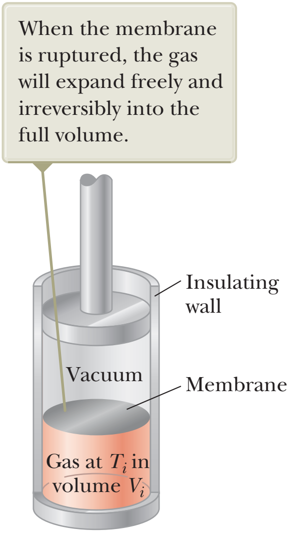

Figure 22.16 Adiabatic free expansion of a gas. The container is thermally insulated from its surroundings; therefore, $Q = 0$.

Let's again consider the adiabatic free expansion of a gas occupying an initial volume V_i (Fig. 22.16). In this situation, a membrane separating the gas from an evacuated region is broken and the gas expands to a volume V_f. This process is irreversible; the gas would not spontaneously crowd into half the volume after filling the entire volume. What is the change in entropy of the gas during this process? The process is neither reversible nor quasi-static. As shown in Section 20.6, the initial and final temperatures of the gas are the same.

To apply Equation 22.13, we cannot take $Q = 0$, the value for the irreversible process, but must instead find Q_r; that is, we must find an equivalent reversible path that shares the same initial and final states. A simple choice is an isothermal, reversible expansion in which the gas pushes slowly against a piston while energy enters the gas by heat from a reservoir to hold the temperature constant. Because T is constant in this process, Equation 22.13 gives

$$\Delta S = \int_i^f \frac{dQ_r}{T} = \frac{1}{T}\int_i^f dQ_r$$

For an isothermal process, the first law of thermodynamics specifies that $\int_i^f dQ_r$ is equal to the negative of the work done on the gas during the expansion from V_i to V_f, which is given by Equation 20.14. Using this result, we find that the entropy change for the gas is

$$\Delta S = nR\ln\left(\frac{V_f}{V_i}\right) \tag{22.17}$$

Because $V_f > V_i$, we conclude that ΔS is positive. This positive result indicates that the entropy of the gas *increases* as a result of the irreversible, adiabatic expansion.

It is easy to see that the energy has spread after the expansion. Instead of being concentrated in a relatively small space, the molecules and the energy associated with them are scattered over a larger region.

Entropy Change in Thermal Conduction

Let us now consider a system consisting of a hot reservoir and a cold reservoir that are in thermal contact with each other and isolated from the rest of the Universe. A process occurs during which energy Q is transferred by heat from the hot reservoir at temperature T_h to the cold reservoir at temperature T_c. The process as described is irreversible (energy would not spontaneously flow from cold to hot), so we must find an equivalent reversible process. The overall process is a combination of two processes: energy leaving the hot reservoir and energy entering the cold reservoir. We will calculate the entropy change for the reservoir in each process and add to obtain the overall entropy change.

Consider first the process of energy entering the cold reservoir. Although the reservoir has absorbed some energy, the temperature of the reservoir has not changed. The energy that has entered the reservoir is the same as that which would enter by means of a reversible, isothermal process. The same is true for energy leaving the hot reservoir.

Because the cold reservoir absorbs energy Q, its entropy increases by Q/T_c. At the same time, the hot reservoir loses energy Q, so its entropy change is $-Q/T_h$. Therefore, the change in entropy of the system is

$$\Delta S = \frac{Q}{T_c} + \frac{-Q}{T_h} = Q\left(\frac{1}{T_c} - \frac{1}{T_h}\right) > 0 \quad \textbf{(22.18)}$$

This increase is consistent with our interpretation of entropy changes as representing the spreading of energy. In the initial configuration, the hot reservoir has excess internal energy relative to the cold reservoir. The process that occurs spreads the energy into a more equitable distribution between the two reservoirs.

Example 22.8 Adiabatic Free Expansion: Revisited

Let's verify that the macroscopic and microscopic approaches to the calculation of entropy lead to the same conclusion for the adiabatic free expansion of an ideal gas. Suppose the ideal gas in Figure 22.16 expands to four times its initial volume. As we have seen for this process, the initial and final temperatures are the same.

(A) Using a macroscopic approach, calculate the entropy change for the gas.

SOLUTION

Conceptualize Look back at Figure 22.16, which is a diagram of the system before the adiabatic free expansion. Imagine breaking the membrane so that the gas moves into the evacuated area. The expansion is irreversible.

Categorize We can replace the irreversible process with a reversible isothermal process between the same initial and final states. This approach is macroscopic, so we use a thermodynamic variable, in particular, the volume V.

Analyze Use Equation 22.17 to evaluate the entropy change:

$$\Delta S = nR\ln\left(\frac{V_f}{V_i}\right) = nR\ln\left(\frac{4V_i}{V_i}\right) = \boxed{nR\ln 4}$$

(B) Using statistical considerations, calculate the change in entropy for the gas and show that it agrees with the answer you obtained in part (A).

SOLUTION

Categorize This approach is microscopic, so we use variables related to the individual molecules.

Analyze As in the discussion leading to Equation 22.11, the number of microstates available to a single molecule in the initial volume V_i is $w_i = V_i/V_m$, where V_i is the initial volume of the gas and V_m is the microscopic volume occupied by the molecule. Use this number to find the number of available microstates for N molecules:

$$W_i = w_i^N = \left(\frac{V_i}{V_m}\right)^N$$

Find the number of available microstates for N molecules in the final volume $V_f = 4V_i$:

$$W_f = \left(\frac{V_f}{V_m}\right)^N = \left(\frac{4V_i}{V_m}\right)^N$$

continued

22.8 continued

Use Equation 22.14 to find the entropy change:

$$\Delta S = k_B \ln\left(\frac{W_f}{W_i}\right)$$

$$= k_B \ln\left(\frac{4V_i}{V_i}\right)^N = k_B \ln(4^N) = Nk_B \ln 4 = \boxed{nR \ln 4}$$

Finalize The answer is the same as that for part (A), which dealt with macroscopic parameters.

WHAT IF? In part (A), we used Equation 22.17, which was based on a reversible isothermal process connecting the initial and final states. Would you arrive at the same result if you chose a different reversible process?

Answer You *must* arrive at the same result because entropy is a state variable. For example, consider the two-step process in Figure 22.17: a reversible adiabatic expansion from V_i to $4V_i$ ($A \rightarrow B$) during which the temperature drops from T_1 to T_2 and a reversible isovolumetric process ($B \rightarrow C$) that takes the gas back to the initial temperature T_1. During the reversible adiabatic process, $\Delta S = 0$ because $Q_r = 0$.

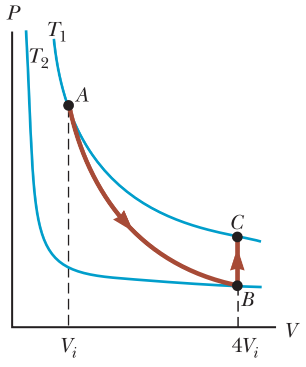

Figure 22.17 (Example 22.8) A gas expands to four times its initial volume and back to the initial temperature by means of a two-step process.

For the reversible isovolumetric process ($B \rightarrow C$), use Equation 22.13:

$$\Delta S = \int_i^f \frac{dQ_r}{T} = \int_{T_2}^{T_1} \frac{nC_V dT}{T} = nC_V \ln\left(\frac{T_1}{T_2}\right)$$

Find the ratio of temperature T_1 to T_2 from Equation 21.39 for the adiabatic process:

$$\frac{T_1}{T_2} = \left(\frac{4V_i}{V_i}\right)^{\gamma-1} = (4)^{\gamma-1}$$

Substitute to find ΔS:

$$\Delta S = nC_V \ln(4)^{\gamma-1} = nC_V(\gamma - 1)\ln 4$$

$$= nC_V\left(\frac{C_P}{C_V} - 1\right)\ln 4 = n(C_P - C_V)\ln 4 = nR \ln 4$$

We do indeed obtain the exact same result for the entropy change.

22.8 Entropy and the Second Law

If we consider a system and its surroundings to include the entire Universe, the Universe is always moving toward a higher-probability macrostate, corresponding to the continuous spreading of energy. An alternative way of stating this behavior is as follows:

Entropy statement of the second law of thermodynamics

The entropy of the Universe increases in all real processes.

This statement is yet another wording of the second law of thermodynamics that can be shown to be equivalent to the Kelvin-Planck and Clausius statements.

Let us show this equivalence first for the Clausius statement. Looking at Figure 22.5, we see that, if the heat pump operates in this manner, energy is spontaneously flowing from the cold reservoir to the hot reservoir without an input of energy by work. As a result, the energy in the system is not spreading evenly between the two reservoirs, but is *concentrating* in the hot reservoir. Consequently, if the Clausius statement of the second law is not true, then the entropy statement is also not true, demonstrating their equivalence.

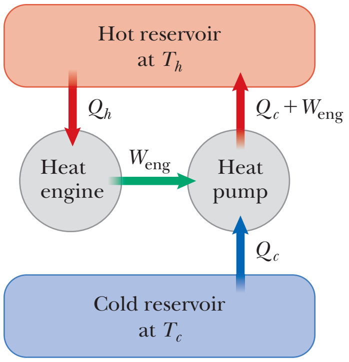

Figure 22.18 The impossible engine of Figure 22.3 transfers energy by work to a heat pump operating between two energy reservoirs. This situation is forbidden by the Clausius statement of the second law of thermodynamics.

For the equivalence of the Kelvin–Planck statement, consider Figure 22.18, which shows the impossible engine of Figure 22.3 connected to a heat pump operating between the same reservoirs. The output work of the engine is used to drive the heat pump. The net effect of this combination is that energy leaves the cold reservoir and is delivered to the hot reservoir without the input of work. (The work done by the engine on the heat pump is *internal* to the system of both devices.) This is forbidden by the Clausius statement of the second law, which we have shown to be equivalent to the entropy statement. Therefore, the Kelvin–Planck statement of the second law is also equivalent to the entropy statement.

When dealing with a system that is not isolated from its surroundings, remember that the increase in entropy described in the second law is that of the system *and* its surroundings. When a system and its surroundings interact in an irreversible process, the increase in entropy of one is greater than the decrease in entropy of the other. Hence, the change in entropy of the Universe must be greater than zero for an irreversible process and equal to zero for a reversible process.

We can check this statement of the second law for the calculations of entropy change that we made in Section 22.7. Consider first the entropy change in a free expansion, described by Equation 22.17. Because the free expansion takes place in an insulated container, no energy is transferred by heat from the surroundings. Therefore, Equation 22.17 represents the entropy change of the entire Universe. Because $V_f > V_i$, the entropy change of the Universe is positive, consistent with the second law.

Now consider the entropy change in thermal conduction, described by Equation 22.18. Let each reservoir be half the Universe. (The larger the reservoir, the better is the assumption that its temperature remains constant!) Then the entropy change of the Universe is represented by Equation 22.18. Because $T_h > T_c$, this entropy change is positive, again consistent with the second law. The positive entropy change is also consistent with the notion of energy spreading. The warm portion of the Universe has excess internal energy relative to the cool portion. Thermal conduction represents a spreading of the energy more equitably throughout the Universe.

Finally, let us look at the entropy change in a Carnot cycle, given by Equation 22.15. The entropy change of the engine itself is zero. The entropy change of the reservoirs is

$$\Delta S = \frac{|Q_c|}{T_c} - \frac{|Q_h|}{T_h}$$

In light of Equation 22.7, this entropy change is also zero. Therefore, the entropy change of the Universe is only that associated with the work done by the engine. A portion of that work will be used to change the mechanical energy of a system external to the engine: speed up the shaft of a machine, raise a weight, and so on. There is no change in internal energy of the external system due to this portion of the work, or, equivalently, no energy spreading, so the entropy change is again zero. The other portion of the work will be used to overcome various friction forces or other nonconservative forces in the external system. This process will cause an increase in internal energy of that system. That same increase in internal energy could have happened via a reversible thermodynamic process in which energy Q_r is transferred by heat, so the entropy change associated with that part of the work is positive. As a result, the overall entropy change of the Universe for the operation of the Carnot engine is positive, again consistent with the second law.

Ultimately, because real processes are irreversible, the entropy of the Universe should increase steadily and eventually reach a maximum value. At this value, assuming that the second law of thermodynamics, as formulated here on Earth, applies to the entire expanding Universe, the Universe will be in a state of uniform temperature and density. The total energy of the Universe will have spread more evenly throughout the Universe. All physical, chemical, and biological processes will have ceased at this time. This gloomy state of affairs is sometimes referred to as the *heat death* of the Universe.

Summary

Definitions

The **thermal efficiency** e of a heat engine is

$$e \equiv \frac{W_{\text{eng}}}{|Q_h|} = \frac{|Q_h| - |Q_c|}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|} \quad \textbf{(22.2)}$$

The **microstate** of a system is the description of its individual components. The **macrostate** is a description of the system from a macroscopic point of view. A given macrostate can have many microstates.

From a microscopic viewpoint, the **entropy** of a given macrostate is defined as

$$S \equiv k_B \ln W \quad \textbf{(22.10)}$$

where k_B is Boltzmann's constant and W is the number of microstates of the system corresponding to the macrostate.

In a **reversible** process, the system can be returned to its initial conditions along the same path on a PV diagram, and every point along this path is an equilibrium state. A process that does not satisfy these requirements is **irreversible.**

Concepts and Principles

A **heat engine** is a device that takes in energy by heat and, operating in a cyclic process, expels a fraction of that energy by means of work. The net work done by a heat engine in carrying a working substance through a cyclic process ($\Delta E_{\text{int}} = 0$) is

$$W_{\text{eng}} = |Q_h| - |Q_c| \quad \textbf{(22.1)}$$

where $|Q_h|$ is the energy taken in from a hot reservoir and $|Q_c|$ is the energy expelled to a cold reservoir.

Two ways the **second law of thermodynamics** can be stated are as follows:

- It is impossible to construct a heat engine that, operating in a cycle, produces no effect other than the input of energy by heat from a reservoir and the performance of an equal amount of work (the Kelvin–Planck statement).
- It is impossible to construct a cyclical machine whose sole effect is to transfer energy continuously by heat from one object to another object at a higher temperature without the input of energy by work (the Clausius statement).

Carnot's theorem states that no real heat engine operating (irreversibly) between the temperatures T_c and T_h can be more efficient than an engine operating reversibly in a Carnot cycle between the same two temperatures.

The thermal efficiency of a heat engine operating in the Carnot cycle is

$$e_C = 1 - \frac{T_c}{T_h} \quad \textbf{(22.8)}$$

The macroscopic state of a system that has a large number of microstates has four qualities that are all related: (1) *uncertainty:* because of the large number of microstates, there is a large uncertainty as to which one actually exists; (2) *choice:* again because of the large number of microstates, there is a large number of choices from which to select as to which one exists; (3) *probability:* a macrostate with a large number of microstates is more likely to exist than one with a small number of microstates; (4) *missing information:* because of the large number of microstates, there is a high amount of missing information as to which one exists. For a thermodynamic system, all four of these can be related to the state variable of **entropy.**

The second law of thermodynamics states that when real (irreversible) processes occur, there is a spatial spreading of energy. This spreading of energy is related to a thermodynamic state variable called **entropy** S. Therefore, yet another way the second law can be stated is as follows:

- The entropy of the Universe increases in all real processes.

The **change in entropy** dS of a system during a process between two infinitesimally separated equilibrium states is

$$dS = \frac{dQ_r}{T} \quad \textbf{(22.12)}$$

where dQ_r is the energy transfer by heat for the system for a reversible process that connects the initial and final states.

The change in entropy of a system during an arbitrary finite process between an initial state and a final state is

$$\Delta S = \int_i^f \frac{dQ_r}{T} \quad \textbf{(22.13)}$$

The value of ΔS for the system is the same for all paths connecting the initial and final states. The change in entropy for a system undergoing any reversible, cyclic process is zero.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. The second law of thermodynamics implies that the coefficient of performance of a refrigerator must be what? (a) less than 1 (b) less than or equal to 1 (c) greater than or equal to 1 (d) finite (e) greater than 0

2. Assume a sample of an ideal gas is at room temperature. What action will *necessarily* make the entropy of the sample increase? (a) Transfer energy into it by heat. (b) Transfer energy into it irreversibly by heat. (c) Do work on it. (d) Increase either its temperature or its volume, without letting the other variable decrease. (e) None of those choices is correct.

3. A refrigerator has 18.0 kJ of work done on it while 115 kJ of energy is transferred from inside its interior. What is its coefficient of performance? (a) 3.40 (b) 2.80 (c) 8.90 (d) 6.40 (e) 5.20

4. Of the following, which is *not* a statement of the second law of thermodynamics? (a) No heat engine operating in a cycle can absorb energy from a reservoir and use it entirely to do work. (b) No real engine operating between two energy reservoirs can be more efficient than a Carnot engine operating between the same two reservoirs. (c) When a system undergoes a change in state, the change in the internal energy of the system is the sum of the energy transferred to the system by heat and the work done on the system. (d) The entropy of the Universe increases in all natural processes. (e) Energy will not spontaneously transfer by heat from a cold object to a hot object.

5. Consider cyclic processes completely characterized by each of the following net energy inputs and outputs. In each case, the energy transfers listed are the *only* ones occurring. Classify each process as (a) possible, (b) impossible according to the first law of thermodynamics, (c) impossible according to the second law of thermodynamics, or (d) impossible according to both the first and second laws. **(i)** Input is 5 J of work, and output is 4 J of work. **(ii)** Input is 5 J of work, and output is 5 J of energy transferred by heat. **(iii)** Input is 5 J of energy transferred by electrical transmission, and output is 6 J of work. **(iv)** Input is 5 J of energy transferred by heat, and output is 5 J of energy transferred by heat. **(v)** Input is 5 J of energy transferred by heat, and output is 5 J of work. **(vi)** Input is 5 J of energy transferred by heat, and output is 3 J of work plus 2 J of energy transferred by heat.

6. A compact air-conditioning unit is placed on a table inside a well-insulated apartment and is plugged in and turned on. What happens to the average temperature of the apartment? (a) It increases. (b) It decreases. (c) It remains constant. (d) It increases until the unit warms up and then decreases. (e) The answer depends on the initial temperature of the apartment.

7. A steam turbine operates at a boiler temperature of 450 K and an exhaust temperature of 300 K. What is the maximum theoretical efficiency of this system? (a) 0.240 (b) 0.500 (c) 0.333 (d) 0.667 (e) 0.150

8. A thermodynamic process occurs in which the entropy of a system changes by −8 J/K. According to the second law of thermodynamics, what can you conclude about the entropy change of the environment? (a) It must be +8 J/K or less. (b) It must be between +8 J/K and 0. (c) It must be equal to +8 J/K. (d) It must be +8 J/K or more. (e) It must be zero.

9. A sample of a monatomic ideal gas is contained in a cylinder with a piston. Its state is represented by the dot in the *PV* diagram shown in Figure OQ22.9. Arrows *A* through *E* represent isobaric, isothermal, adiabatic, and isovolumetric processes that the sample can undergo. In each process except *D*, the volume changes by a factor of 2. All five processes are reversible. Rank the processes according to the change in entropy of the gas from the largest positive value to the largest-magnitude negative value. In your rankings, display any cases of equality.

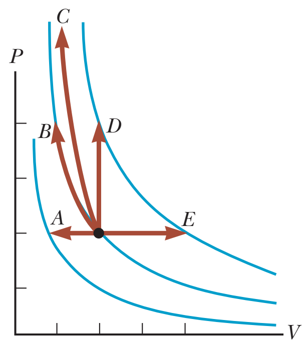

Figure OQ22.9

10. An engine does 15.0 kJ of work while exhausting 37.0 kJ to a cold reservoir. What is the efficiency of the engine? (a) 0.150 (b) 0.288 (c) 0.333 (d) 0.450 (e) 1.20

11. The arrow *OA* in the *PV* diagram shown in Figure OQ22.11 represents a reversible adiabatic expansion of an ideal gas. The same sample of gas, starting from the same state *O*, now undergoes an adiabatic free expansion to the same final volume. What point on the diagram could represent the final state of the gas? (a) the same point *A* as for the reversible expansion (b) point *B* (c) point *C* (d) any of those choices (e) none of those choices

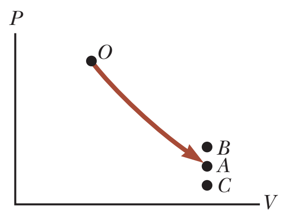

Figure OQ22.11

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. The energy exhaust from a certain coal-fired electric generating station is carried by "cooling water" into Lake Ontario. The water is warm from the viewpoint of living things in the lake. Some of them congregate around the outlet port and can impede the water flow. (a) Use the theory of heat engines to explain why this action can reduce the electric power output of the station. (b) An engineer says that the electric output is reduced because of "higher back pressure on the turbine blades." Comment on the accuracy of this statement.

2. Discuss three different common examples of natural processes that involve an increase in entropy. Be sure to account for all parts of each system under consideration.

3. Does the second law of thermodynamics contradict or correct the first law? Argue for your answer.

4. "The first law of thermodynamics says you can't really win, and the second law says you can't even break even." Explain how this statement applies to a particular device or process; alternatively, argue against the statement.

5. "Energy is the mistress of the Universe, and entropy is her shadow." Writing for an audience of general readers, argue for this statement with at least two examples. Alternatively, argue for the view that entropy is like an executive who instantly determines what will happen, whereas energy is like a bookkeeper telling us how little we can afford. (Arnold Sommerfeld suggested the idea for this question.)

6. (a) Give an example of an irreversible process that occurs in nature. (b) Give an example of a process in nature that is nearly reversible.

$\boxed{7.}$ The device shown in Figure CQ22.7, called a thermoelectric converter, uses a series of semiconductor cells to transform internal energy to electric potential energy, which we will study in Chapter 25. In the photograph on the left, both legs of the device are at the same temperature and no electric potential energy is produced. When one leg is at a higher temperature than the other as shown in the photograph on the right, however, electric potential energy is produced as the device extracts energy from the hot reservoir and drives a small electric motor. (a) Why is the difference in temperature necessary to produce electric potential energy in this demonstration? (b) In what sense does this intriguing experiment demonstrate the second law of thermodynamics?

Courtesy of PASCO Scientific Company

Figure CQ22.7

8. A steam-driven turbine is one major component of an electric power plant. Why is it advantageous to have the temperature of the steam as high as possible?

9. Discuss the change in entropy of a gas that expands (a) at constant temperature and (b) adiabatically.

10. Suppose your roommate cleans and tidies up your messy room after a big party. Because she is creating more order, does this process represent a violation of the second law of thermodynamics?

11. Is it possible to construct a heat engine that creates no thermal pollution? Explain.

12. (a) If you shake a jar full of jelly beans of different sizes, the larger beans tend to appear near the top and the smaller ones tend to fall to the bottom. Why? (b) Does this process violate the second law of thermodynamics?

13. What are some factors that affect the efficiency of automobile engines?

Problems

The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 22.1 Heat Engines and the Second Law of Thermodynamics

1. (M) A particular heat engine has a mechanical power output of 5.00 kW and an efficiency of 25.0%. The engine expels 8.00×10^3 J of exhaust energy in each cycle. Find (a) the energy taken in during each cycle and (b) the time interval for each cycle.

2. The work done by an engine equals one-fourth the energy it absorbs from a reservoir. (a) What is its thermal efficiency? (b) What fraction of the energy absorbed is expelled to the cold reservoir?

3. (W) A heat engine takes in 360 J of energy from a hot reservoir and performs 25.0 J of work in each cycle. Find (a) the efficiency of the engine and (b) the energy expelled to the cold reservoir in each cycle.

4. A gun is a heat engine. In particular, it is an internal combustion piston engine that does not operate in a cycle, but comes apart during its adiabatic expansion process. A certain gun consists of 1.80 kg of iron. It fires one 2.40-g bullet at 320 m/s with an energy efficiency of 1.10%. Assume the body of the gun absorbs all the energy exhaust—the other 98.9%—and increases uniformly in temperature for a short time interval before it loses any energy by heat into the environment. Find its temperature increase.

5. An engine absorbs 1.70 kJ from a hot reservoir at 277°C and expels 1.20 kJ to a cold reservoir at 27°C in each cycle. (a) What is the engine's efficiency? (b) How much work is done by the engine in each cycle? (c) What is the power output of the engine if each cycle lasts 0.300 s?

6. A multicylinder gasoline engine in an airplane, operating at 2.50×10^3 rev/min, takes in energy 7.89×10^3 J and exhausts 4.58×10^3 J for each revolution of the crankshaft. (a) How many liters of fuel does it consume in 1.00 h of operation if the heat of combustion of the fuel is equal to 4.03×10^7 J/L? (b) What is the mechanical power output of the engine? Ignore friction and express the answer in horsepower. (c) What is the torque exerted by the crankshaft on the load? (d) What power must the exhaust and cooling system transfer out of the engine?

7. (W) Suppose a heat engine is connected to two energy reservoirs, one a pool of molten aluminum (660°C) and the other a block of solid mercury (−38.9°C). The engine runs by freezing 1.00 g of aluminum and melting 15.0 g of mercury during each cycle. The heat of

fusion of aluminum is 3.97×10^5 J/kg; the heat of fusion of mercury is 1.18×10^4 J/kg. What is the efficiency of this engine?

Section 22.2 Heat Pumps and Refrigerators

8. W A refrigerator has a coefficient of performance equal to 5.00. The refrigerator takes in 120 J of energy from a cold reservoir in each cycle. Find (a) the work required in each cycle and (b) the energy expelled to the hot reservoir.

9. During each cycle, a refrigerator ejects 625 kJ of energy to a high-temperature reservoir and takes in 550 kJ of energy from a low-temperature reservoir. Determine (a) the work done on the refrigerant in each cycle and (b) the coefficient of performance of the refrigerator.

10. A heat pump has a coefficient of performance of 3.80 and operates with a power consumption of 7.03×10^3 W. (a) How much energy does it deliver into a home during 8.00 h of continuous operation? (b) How much energy does it extract from the outside air?

11. A refrigerator has a coefficient of performance of 3.00. The ice tray compartment is at -20.0°C, and the room temperature is 22.0°C. The refrigerator can convert 30.0 g of water at 22.0°C to 30.0 g of ice at -20.0°C each minute. What input power is required? Give your answer in watts.

12. A heat pump has a coefficient of performance equal to 4.20 and requires a power of 1.75 kW to operate. (a) How much energy does the heat pump add to a home in one hour? (b) If the heat pump is reversed so that it acts as an air conditioner in the summer, what would be its coefficient of performance?

13. A freezer has a coefficient of performance of 6.30. It is advertised as using electricity at a rate of 457 kWh/yr. (a) On average, how much energy does it use in a single day? (b) On average, how much energy does it remove from the refrigerator in a single day? (c) What maximum mass of water at 20.0°C could the freezer freeze in a single day? *Note:* One kilowatt-hour (kWh) is an amount of energy equal to running a 1-kW appliance for one hour.

Section 22.3 Reversible and Irreversible Processes

Section 22.4 The Carnot Engine

14. A heat engine operates between a reservoir at 25.0°C and one at 375°C. What is the maximum efficiency possible for this engine?

15. M One of the most efficient heat engines ever built is a coal-fired steam turbine in the Ohio River valley, operating between 1 870°C and 430°C. (a) What is its maximum theoretical efficiency? (b) The actual efficiency of the engine is 42.0%. How much mechanical power does the engine deliver if it absorbs 1.40×10^5 J of energy each second from its hot reservoir?

16. *Why is the following situation impossible?* An inventor comes to a patent office with the claim that her heat engine, which employs water as a working substance, has a thermodynamic efficiency of 0.110. Although this efficiency is low compared with typical automobile engines, she explains that her engine operates between an energy reservoir at room temperature and a water–ice mixture at atmospheric pressure and therefore requires no fuel other than that to make the ice. The patent is approved, and working prototypes of the engine prove the inventor's efficiency claim.

17. W A Carnot engine has a power output of 150 kW. The engine operates between two reservoirs at 20.0°C and 500°C. (a) How much energy enters the engine by heat per hour? (b) How much energy is exhausted by heat per hour?

18. A Carnot engine has a power output P. The engine operates between two reservoirs at temperature T_c and T_h. (a) How much energy enters the engine by heat in a time interval Δt? (b) How much energy is exhausted by heat in the time interval Δt?

19. What is the coefficient of performance of a refrigerator that operates with Carnot efficiency between temperatures -3.00°C and $+27.0$°C?

20. An ideal refrigerator or ideal heat pump is equivalent to a Carnot engine running in reverse. That is, energy $|Q_c|$ is taken in from a cold reservoir and energy $|Q_h|$ is rejected to a hot reservoir. (a) Show that the work that must be supplied to run the refrigerator or heat pump is

$$W = \frac{T_h - T_c}{T_c}|Q_c|$$

(b) Show that the coefficient of performance (COP) of the ideal refrigerator is

$$\text{COP} = \frac{T_c}{T_h - T_c}$$

21. What is the maximum possible coefficient of performance of a heat pump that brings energy from outdoors at -3.00°C into a 22.0°C house? *Note:* The work done to run the heat pump is also available to warm the house.

22. M How much work does an ideal Carnot refrigerator require to remove 1.00 J of energy from liquid helium at 4.00 K and expel this energy to a room-temperature (293-K) environment?

23. If a 35.0%-efficient Carnot heat engine (Fig. 22.2) is run in reverse so as to form a refrigerator (Fig. 22.4), what would be this refrigerator's coefficient of performance?

24. A power plant operates at a 32.0% efficiency during the summer when the seawater used for cooling is at 20.0°C. The plant uses 350°C steam to drive turbines. If the plant's efficiency changes in the same proportion as the ideal efficiency, what would be the plant's efficiency in the winter, when the seawater is at 10.0°C?

25. A heat engine is being designed to have a Carnot efficiency of 65.0% when operating between two energy reservoirs. (a) If the temperature of the cold reservoir is 20.0°C, what must be the temperature of the hot reservoir? (b) Can the actual efficiency of the engine be equal to 65.0%? Explain.

26. A Carnot heat engine operates between temperatures T_h and T_c. (a) If $T_h = 500$ K and $T_c = 350$ K, what is the efficiency of the engine? (b) What is the change in its efficiency for each degree of increase in T_h above 500 K? (c) What is the change in its efficiency for each degree of change in T_c? (d) Does the answer to part (c) depend on T_c? Explain.

27. M An ideal gas is taken through a Carnot cycle. The isothermal expansion occurs at 250°C, and the isothermal compression takes place at 50.0°C. The gas takes in 1.20×10^3 J of energy from the hot reservoir during the isothermal expansion. Find (a) the energy expelled to the cold reservoir in each cycle and (b) the net work done by the gas in each cycle.

28. W An electric power plant that would make use of the temperature gradient in the ocean has been proposed. The system is to operate between 20.0°C (surface-water temperature) and 5.00°C (water temperature at a depth of about 1 km). (a) What is the maximum efficiency of such a system? (b) If the electric power output of the plant is 75.0 MW, how much energy is taken in from the warm reservoir per hour? (c) In view of your answer to part (a), explain whether you think such a system is worthwhile. Note that the "fuel" is free.

29. M A heat engine operates in a Carnot cycle between 80.0°C and 350°C. It absorbs 21 000 J of energy per cycle from the hot reservoir. The duration of each cycle is 1.00 s. (a) What is the mechanical power output of this engine? (b) How much energy does it expel in each cycle by heat?

30. Suppose you build a two-engine device with the exhaust energy output from one heat engine supplying the input energy for a second heat engine. We say that the two engines are running *in series*. Let e_1 and e_2 represent the efficiencies of the two engines. (a) The overall efficiency of the two-engine device is defined as the total work output divided by the energy put into the first engine by heat. Show that the overall efficiency e is given by

$$e = e_1 + e_2 - e_1 e_2$$

What If? For parts (b) through (e) that follow, assume the two engines are Carnot engines. Engine 1 operates between temperatures T_h and T_i. The gas in engine 2 varies in temperature between T_i and T_c. In terms of the temperatures, (b) what is the efficiency of the combination engine? (c) Does an improvement in net efficiency result from the use of two engines instead of one? (d) What value of the intermediate temperature T_i results in equal work being done by each of the two engines in series? (e) What value of T_i results in each of the two engines in series having the same efficiency?

31. W Argon enters a turbine at a rate of 80.0 kg/min, a temperature of 800°C, and a pressure of 1.50 MPa. It expands adiabatically as it pushes on the turbine blades and exits at pressure 300 kPa. (a) Calculate its temperature at exit. (b) Calculate the (maximum) power output of the turning turbine. (c) The turbine is one component of a model closed-cycle gas turbine engine. Calculate the maximum efficiency of the engine.

32. At point A in a Carnot cycle, 2.34 mol of a monatomic ideal gas has a pressure of 1 400 kPa, a volume of 10.0 L, and a temperature of 720 K. The gas expands isothermally to point B and then expands adiabatically to point C, where its volume is 24.0 L. An isothermal compression brings it to point D, where its volume is 15.0 L. An adiabatic process returns the gas to point A. (a) Determine all the unknown pressures, volumes, and temperatures as you fill in the following table:

	P	V	T
A	1 400 kPa	10.0 L	720 K
B			
C		24.0 L	
D		15.0 L	

(b) Find the energy added by heat, the work done by the engine, and the change in internal energy for each of the steps $A \to B$, $B \to C$, $C \to D$, and $D \to A$. (c) Calculate the efficiency $W_{\text{net}}/|Q_h|$. (d) Show that the efficiency is equal to $1 - T_C/T_A$, the Carnot efficiency.

33. An electric generating station is designed to have an electric output power of 1.40 MW using a turbine with two-thirds the efficiency of a Carnot engine. The exhaust energy is transferred by heat into a cooling tower at 110°C. (a) Find the rate at which the station exhausts energy by heat as a function of the fuel combustion temperature T_h. (b) If the firebox is modified to run hotter by using more advanced combustion technology, how does the amount of energy exhaust change? (c) Find the exhaust power for $T_h = 800$°C. (d) Find the value of T_h for which the exhaust power would be only half as large as in part (c). (e) Find the value of T_h for which the exhaust power would be one-fourth as large as in part (c).

34. An ideal (Carnot) freezer in a kitchen has a constant temperature of 260 K, whereas the air in the kitchen has a constant temperature of 300 K. Suppose the insulation for the freezer is not perfect but rather conducts energy into the freezer at a rate of 0.150 W. Determine the average power required for the freezer's motor to maintain the constant temperature in the freezer.

35. A heat pump used for heating shown in Figure P22.35 is essentially an air conditioner installed backward. It

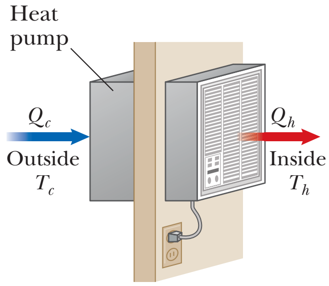

Figure P22.35

extracts energy from colder air outside and deposits it in a warmer room. Suppose the ratio of the actual energy entering the room to the work done by the device's motor is 10.0% of the theoretical maximum ratio. Determine the energy entering the room per joule of work done by the motor given that the inside temperature is 20.0°C and the outside temperature is −5.00°C.

Section 22.5 Gasoline and Diesel Engines

Note: For problems in this section, assume the gas in the engine is diatomic with $\gamma = 1.40$.

36. A gasoline engine has a compression ratio of 6.00. (a) What is the efficiency of the engine if it operates in an idealized Otto cycle? (b) **What If?** If the actual efficiency is 15.0%, what fraction of the fuel is wasted as a result of friction and energy transfers by heat that could be avoided in a reversible engine? Assume complete combustion of the air–fuel mixture.

37. M In a cylinder of an automobile engine, immediately after combustion the gas is confined to a volume of 50.0 cm^3 and has an initial pressure of 3.00×10^6 Pa. The piston moves outward to a final volume of 300 cm^3, and the gas expands without energy transfer by heat. (a) What is the final pressure of the gas? (b) How much work is done by the gas in expanding?

38. An idealized diesel engine operates in a cycle known as the *air-standard diesel cycle* shown in Figure P22.38. Fuel is sprayed into the cylinder at the point of maximum compression, *B*. Combustion occurs during the expansion $B \rightarrow C$, which is modeled as an isobaric process. Show that the efficiency of an engine operating in this idealized diesel cycle is

$$e = 1 - \frac{1}{\gamma}\left(\frac{T_D - T_A}{T_C - T_B}\right)$$

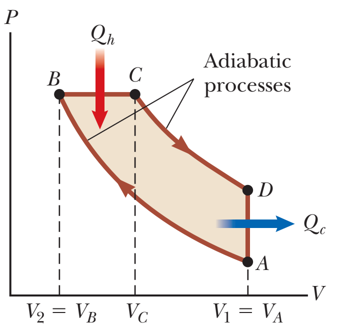

Figure P22.38

Section 22.6 Entropy

39. Prepare a table like Table 22.1 by using the same procedure (a) for the case in which you draw three marbles from your bag rather than four and (b) for the case in which you draw five marbles rather than four.

40. (a) Prepare a table like Table 22.1 for the following occurrence. You toss four coins into the air simultaneously and then record the results of your tosses in terms of the numbers of heads (H) and tails (T) that result. For example, HHTH and HTHH are two possible ways in which three heads and one tail can be achieved. (b) On the basis of your table, what is the most probable result recorded for a toss?

41. If you roll two dice, what is the total number of ways in which you can obtain (a) a 12 and (b) a 7?

Section 22.7 Changes in Entropy for Thermodynamic Systems

Section 22.8 Entropy and the Second Law

42. W An ice tray contains 500 g of liquid water at 0°C. Calculate the change in entropy of the water as it freezes slowly and completely at 0°C.

43. A Styrofoam cup holding 125 g of hot water at 100°C cools to room temperature, 20.0°C. What is the change in entropy of the room? Neglect the specific heat of the cup and any change in temperature of the room.

44. A 1.00-kg iron horseshoe is taken from a forge at 900°C and dropped into 4.00 kg of water at 10.0°C. Assuming that no energy is lost by heat to the surroundings, determine the total entropy change of the horseshoe-plus-water system. (*Suggestion:* Note that $dQ = mc\,dT$.)

45. AMT A 1 500-kg car is moving at 20.0 m/s. The driver brakes to a stop. The brakes cool off to the temperature of the surrounding air, which is nearly constant at 20.0°C. What is the total entropy change?

46. AMT Two 2.00×10^3-kg cars both traveling at 20.0 m/s undergo a head-on collision and stick together. Find the change in entropy of the surrounding air resulting from the collision if the air temperature is 23.0°C. Ignore the energy carried away from the collision by sound.

47. AMT A 70.0-kg log falls from a height of 25.0 m into a lake. If the log, the lake, and the air are all at 300 K, find the change in entropy of the air during this process.

48. A 1.00-mol sample of H_2 gas is contained in the left side of the container shown in Figure P22.48, which has equal volumes on the left and right. The right side is evacuated. When the valve is opened, the gas streams into the right side. (a) What is the entropy change of the gas? (b) Does the temperature of the gas change? Assume the container is so large that the hydrogen behaves as an ideal gas.

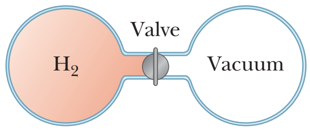

Figure P22.48

49. A 2.00-L container has a center partition that divides it into two equal parts as shown in Figure P22.49. The left side contains 0.044 0 mol of H_2 gas, and the right side contains 0.044 0 mol of O_2 gas. Both gases are at room temperature and at atmospheric pressure. The partition is removed, and the gases are allowed to mix. What is the entropy increase of the system?

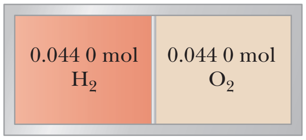

Figure P22.49

50. What change in entropy occurs when a 27.9-g ice cube at −12°C is transformed into steam at 115°C?

51. Calculate the change in entropy of 250 g of water warmed slowly from 20.0°C to 80.0°C.

52. How fast are you personally making the entropy of the Universe increase right now? Compute an order-of-magnitude estimate, stating what quantities you take as data and the values you measure or estimate for them.

53. When an aluminum bar is connected between a hot reservoir at 725 K and a cold reservoir at 310 K, 2.50 kJ of energy is transferred by heat from the hot reservoir to the cold reservoir. In this irreversible process, calculate the change in entropy of (a) the hot reservoir, (b) the cold reservoir, and (c) the Universe, neglecting any change in entropy of the aluminum rod.

54. When a metal bar is connected between a hot reservoir at T_h and a cold reservoir at T_c, the energy transferred by heat from the hot reservoir to the cold reservoir is Q. In this irreversible process, find expressions for the change in entropy of (a) the hot reservoir, (b) the cold reservoir, and (c) the Universe, neglecting any change in entropy of the metal rod.

55. The temperature at the surface of the Sun is approximately 5 800 K, and the temperature at the surface of the Earth is approximately 290 K. What entropy change of the Universe occurs when 1.00×10^3 J of energy is transferred by radiation from the Sun to the Earth?

Additional Problems

56. Calculate the increase in entropy of the Universe when you add 20.0 g of 5.00°C cream to 200 g of 60.0°C coffee. Assume that the specific heats of cream and coffee are both 4.20 J/g · °C.

57. How much work is required, using an ideal Carnot refrigerator, to change 0.500 kg of tap water at 10.0°C into ice at −20.0°C? Assume that the freezer compartment is held at −20.0°C and that the refrigerator exhausts energy into a room at 20.0°C.

58. A steam engine is operated in a cold climate where the exhaust temperature is 0°C. (a) Calculate the theoretical maximum efficiency of the engine using an intake steam temperature of 100°C. (b) If, instead, superheated steam at 200°C is used, find the maximum possible efficiency.

59. The energy absorbed by an engine is three times greater than the work it performs. (a) What is its thermal efficiency? (b) What fraction of the energy absorbed is expelled to the cold reservoir?

60. Every second at Niagara Falls, some 5.00×10^3 m^3 of water falls a distance of 50.0 m. What is the increase in entropy of the Universe per second due to the falling water? Assume the mass of the surroundings is so great that its temperature and that of the water stay nearly constant at 20.0°C. Also assume a negligible amount of water evaporates.

61. Find the maximum (Carnot) efficiency of an engine that absorbs energy from a hot reservoir at 545°C and exhausts energy to a cold reservoir at 185°C.

62. In 1993, the U.S. government instituted a requirement that all room air conditioners sold in the United States must have an energy efficiency ratio (EER) of 10 or higher. The EER is defined as the ratio of the cooling capacity of the air conditioner, measured in British thermal units per hour, or Btu/h, to its electrical power requirement in watts. (a) Convert the EER of 10.0 to dimensionless form, using the conversion 1 Btu = 1 055 J. (b) What is the appropriate name for this dimensionless quantity? (c) In the 1970s, it was common to find room air conditioners with EERs of 5 or lower. State how the operating costs compare for 10 000-Btu/h air conditioners with EERs of 5.00 and 10.0. Assume each air conditioner operates for 1 500 h during the summer in a city where electricity costs 17.0¢ per kWh.

63. Energy transfers by heat through the exterior walls and roof of a house at a rate of 5.00×10^3 J/s = 5.00 kW when the interior temperature is 22.0°C and the outside temperature is −5.00°C. (a) Calculate the electric power required to maintain the interior temperature at 22.0°C if the power is used in electric resistance heaters that convert all the energy transferred in by electrical transmission into internal energy. (b) **What If?** Calculate the electric power required to maintain the interior temperature at 22.0°C if the power is used to drive an electric motor that operates the compressor of a heat pump that has a coefficient of performance equal to 60.0% of the Carnot-cycle value.

64. One mole of neon gas is heated from 300 K to 420 K at constant pressure. Calculate (a) the energy Q transferred to the gas, (b) the change in the internal energy of the gas, and (c) the work done *on* the gas. Note that neon has a molar specific heat of $C_P = 20.79$ J/mol · K for a constant-pressure process.

65. An airtight freezer holds n moles of air at 25.0°C and 1.00 atm. The air is then cooled to −18.0°C. (a) What is the change in entropy of the air if the volume is held constant? (b) What would the entropy change be if the pressure were maintained at 1.00 atm during the cooling?

66. Suppose an ideal (Carnot) heat pump could be constructed for use as an air conditioner. (a) Obtain an expression for the coefficient of performance (COP) for such an air conditioner in terms of T_h and T_c. (b) Would such an air conditioner operate on a smaller energy input if the difference in the operating temperatures were greater or smaller? (c) Compute the COP for such an air conditioner if the indoor temperature is 20.0°C and the outdoor temperature is 40.0°C.

67. GP In 1816, Robert Stirling, a Scottish clergyman, patented the *Stirling engine,* which has found a wide variety of applications ever since, including current use in solar energy collectors to transform sunlight into electricity. Fuel is burned externally to warm one of the engine's two cylinders. A fixed quantity of inert gas moves cyclically between the cylinders, expanding in the hot one and contracting in the cold one. Figure P22.67 represents a model for its thermodynamic cycle. Consider n moles of an ideal monatomic gas being taken once through the cycle, consisting of two isothermal processes at temperatures $3T_i$ and T_i and two constant-volume processes. Let us find the efficiency of this engine. (a) Find the energy transferred by heat into the gas during the isovolumetric process *AB*. (b) Find the energy transferred by heat into the gas during the isothermal process *BC*. (c) Find the energy transferred by heat into the gas during the isovolumetric process *CD*. (d) Find the energy transferred by heat into the gas during the isothermal process *DA*. (e) Identify which of the results from parts (a) through (d) are positive and evaluate the energy input to the engine by heat. (f) From the first law of thermodynamics, find the work done by the engine. (g) From the results of parts (e) and (f), evaluate the efficiency of the engine. A Stirling engine is easier to manufacture than an internal combustion engine or a turbine. It can run on burning garbage. It can run on the energy transferred by sunlight and produce no material exhaust. Stirling engines are not currently used in automobiles due to long startup times and poor acceleration response.

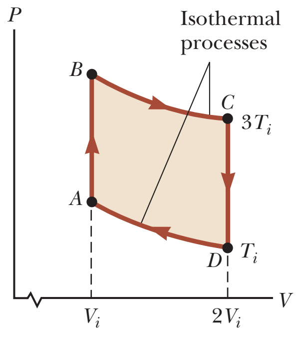

Figure P22.67

68. A firebox is at 750 K, and the ambient temperature is 300 K. The efficiency of a Carnot engine doing 150 J of work as it transports energy between these constant-temperature baths is 60.0%. The Carnot engine must take in energy 150 J/0.600 = 250 J from the hot reservoir and must put out 100 J of energy by heat into the environment. To follow Carnot's reasoning, suppose some other heat engine S could have an efficiency of 70.0%. (a) Find the energy input and exhaust energy output of engine S as it does 150 J of work. (b) Let engine S operate as in part (a) and run the Carnot engine in reverse between the same reservoirs. The output work of engine S is the input work for the Carnot refrigerator. Find the total energy transferred to or from the firebox and the total energy transferred to or from the environment as both engines operate together. (c) Explain how the results of parts (a) and (b) show that the Clausius statement of the second law of thermodynamics is violated. (d) Find the energy input and work output of engine S as it puts out exhaust energy of 100 J. Let engine S operate as in part (c) and contribute 150 J of its work output to running the Carnot engine in reverse. Find (e) the total energy the firebox puts out as both engines operate together, (f) the total work output, and (g) the total energy transferred to the environment. (h) Explain how the results show that the Kelvin–Planck statement of the second law is violated. Therefore, our assumption about the efficiency of engine S must be false. (i) Let the engines operate together through one cycle as in part (d). Find the change in entropy of the Universe. (j) Explain how the result of part (i) shows that the entropy statement of the second law is violated.

69. **Review.** This problem complements Problem 88 in Chapter 10. In the operation of a single-cylinder internal combustion piston engine, one charge of fuel explodes to drive the piston outward in the *power stroke.* Part of its energy output is stored in a turning flywheel. This energy is then used to push the piston inward to compress the next charge of fuel and air. In this compression process, assume an original volume of 0.120 L of a diatomic ideal gas at atmospheric pressure is compressed adiabatically to one-eighth of its original volume. (a) Find the work input required to compress the gas. (b) Assume the flywheel is a solid disk of mass 5.10 kg and radius 8.50 cm, turning freely without friction between the power stroke and the compression stroke. How fast must the flywheel turn immediately after the power stroke? This situation represents the minimum angular speed at which the engine can operate without stalling. (c) When the engine's operation is well above the point of stalling, assume the flywheel puts 5.00% of its maximum energy into compressing the next charge of fuel and air. Find its maximum angular speed in this case.

70. A biology laboratory is maintained at a constant temperature of 7.00°C by an air conditioner, which is vented to the air outside. On a typical hot summer day, the outside temperature is 27.0°C and the air-conditioning unit emits energy to the outside at a rate of 10.0 kW. Model the unit as having a coefficient of performance (COP) equal to 40.0% of the COP of an ideal Carnot device. (a) At what rate does the air conditioner remove energy from the laboratory? (b) Calculate the power required for the work input. (c) Find the change in entropy of the Universe produced by the air conditioner in 1.00 h. (d) **What If?** The outside temperature increases to 32.0°C. Find the fractional change in the COP of the air conditioner.

71. A power plant, having a Carnot efficiency, produces 1.00 GW of electrical power from turbines that take in steam at 500 K and reject water at 300 K into a flowing river. The water downstream is 6.00 K warmer due to the output of the power plant. Determine the flow rate of the river.

72. A power plant, having a Carnot efficiency, produces electric power P from turbines that take in energy from steam at temperature T_h and discharge energy at temperature T_c through a heat exchanger into a flowing river. The water downstream is warmer by ΔT due to the output of the power plant. Determine the flow rate of the river.

73. A 1.00-mol sample of an ideal monatomic gas is taken through the cycle shown in Figure P22.73. The process $A \rightarrow B$ is a reversible isothermal expansion. Calculate (a) the net work done by the gas, (b) the energy added to the gas by heat, (c) the energy exhausted from the gas by heat, and (d) the efficiency of the cycle. (e) Explain how the efficiency compares with that of a Carnot engine operating between the same temperature extremes.

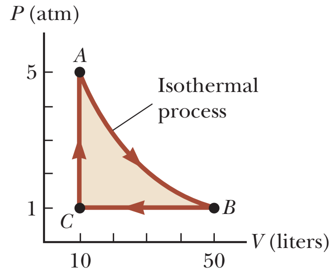

Figure P22.73

74. A system consisting of n moles of an ideal gas with molar specific heat at constant pressure C_P undergoes two reversible processes. It starts with pressure P_i and volume V_i, expands isothermally, and then contracts adiabatically to reach a final state with pressure P_i and volume $3V_i$. (a) Find its change in entropy in the isothermal process. (The entropy does not change in the adiabatic process.) (b) **What If?** Explain why the answer to part (a) must be the same as the answer to Problem 77. (You do not need to solve Problem 77 to answer this question.)

75. A heat engine operates between two reservoirs at $T_2 =$ 600 K and $T_1 = 350$ K. It takes in 1.00×10^3 J of energy from the higher-temperature reservoir and performs 250 J of work. Find (a) the entropy change of the Universe ΔS_U for this process and (b) the work W that could have been done by an ideal Carnot engine operating between these two reservoirs. (c) Show that the difference between the amounts of work done in parts (a) and (b) is $T_1 \, \Delta S_U$.

76. A 1.00-mol sample of a monatomic ideal gas is taken through the cycle shown in Figure P22.76. At point A, the pressure, volume, and temperature are P_i, V_i, and T_i, respectively. In terms of R and T_i, find (a) the total energy entering the system by heat per cycle, (b) the total energy leaving the system by heat per cycle, and (c) the efficiency of an engine operating in this cycle. (d) Explain how the efficiency compares with that of an engine operating in a Carnot cycle between the same temperature extremes.

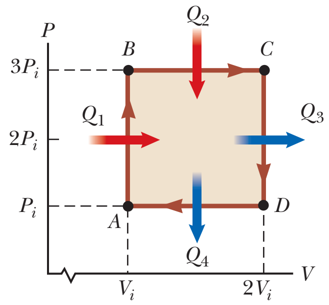

Figure P22.76

77. A sample consisting of n moles of an ideal gas undergoes a reversible isobaric expansion from volume V_i to volume $3V_i$. Find the change in entropy of the gas by calculating $\int_i^f dQ/T$, where $dQ = nC_P \, dT$.

78. An athlete whose mass is 70.0 kg drinks 16.0 ounces (454 g) of refrigerated water. The water is at a temperature of 35.0°F. (a) Ignoring the temperature change of the body that results from the water intake (so that the body is regarded as a reservoir always at 98.6°F), find the entropy increase of the entire system. (b) **What If?** Assume the entire body is cooled by the drink and the average specific heat of a person is equal to the specific heat of liquid water. Ignoring any other energy transfers by heat and any metabolic energy release, find the athlete's temperature after she drinks the cold water given an initial body temperature of 98.6°F. (c) Under these assumptions, what is the entropy increase of the entire system? (d) State how this result compares with the one you obtained in part (a).

79. A sample of an ideal gas expands isothermally, doubling in volume. (a) Show that the work done on the gas in expanding is $W = -nRT \ln 2$. (b) Because the internal energy E_{int} of an ideal gas depends solely on its temperature, the change in internal energy is zero during the expansion. It follows from the first law that the energy input to the gas by heat during the expansion is equal to the energy output by work. Does this process have 100% efficiency in converting energy input by heat into work output? (c) Does this conversion violate the second law? Explain.

80. *Why is the following situation impossible?* Two samples of water are mixed at constant pressure inside an insulated container: 1.00 kg of water at 10.0°C and 1.00 kg of water at 30.0°C. Because the container is insulated, there is no exchange of energy by heat between the water and the environment. Furthermore, the amount of energy that leaves the warm water by heat is equal to the amount that enters the cool water by heat. Therefore, the entropy change of the Universe is zero for this process.

Challenge Problems

81. A 1.00-mol sample of an ideal gas ($\gamma = 1.40$) is carried through the Carnot cycle described in Figure 22.11. At point A, the pressure is 25.0 atm and the temperature is 600 K. At point C, the pressure is 1.00 atm and the temperature is 400 K. (a) Determine the pressures and volumes at points A, B, C, and D. (b) Calculate the net work done per cycle.

82. The compression ratio of an Otto cycle as shown in Figure 22.13 is $V_A/V_B = 8.00$. At the beginning A of the compression process, 500 cm^3 of gas is at 100 kPa and 20.0°C. At the beginning of the adiabatic expansion, the temperature is $T_C = 750$°C. Model the working fluid as an ideal gas with $\gamma = 1.40$. (a) Fill in this table to follow the states of the gas:

	T (K)	P (kPa)	V (cm^3)
A	293	100	500
B			
C	1 023		
D			

(b) Fill in this table to follow the processes:

	Q	W	ΔE_{int}
$A \rightarrow B$			
$B \rightarrow C$			
$C \rightarrow D$			
$D \rightarrow A$			
$ABCDA$			

(c) Identify the energy input $|Q_h|$, (d) the energy exhaust $|Q_c|$, and (e) the net output work W_{eng}. (f) Calculate the thermal efficiency. (g) Find the number of crankshaft revolutions per minute required for a one-cylinder engine to have an output power of 1.00 kW = 1.34 hp. *Note:* The thermodynamic cycle involves four piston strokes.

Electricity and Magnetism

PART

4

A Transrapid maglev train pulls into a station in Shanghai, China. The word *maglev* is an abbreviated form of *magnetic levitation.* This train makes no physical contact with its rails; its weight is totally supported by electromagnetic forces. In this part of the book, we will study these forces. *(OTHK/Asia Images/Jupiterimages)*

We now study the branch of physics concerned with electric and magnetic phe-nomena. The laws of electricity and magnetism play a central role in the operation of such devices as smartphones, televisions, electric motors, computers, high-energy accelerators, and other electronic devices. More fundamentally, the interatomic and intermolecular forces responsible for the formation of solids and liquids are electric in origin.

Evidence in Chinese documents suggests magnetism was observed as early as 2000 BC. The ancient Greeks observed electric and magnetic phenomena possibly as early as 700 BC. The Greeks knew about magnetic forces from observations that the naturally occurring stone *magnetite* (Fe_3O_4) is attracted to iron. (The word *electric* comes from *elecktron,* the Greek word for "amber." The word *magnetic* comes from *Magnesia,* the name of the district of Greece where magnetite was first found.)

Not until the early part of the nineteenth century did scientists establish that electricity and magnetism are related phenomena. In 1819, Hans Oersted discovered that a compass needle is deflected when placed near a circuit carrying an electric current. In 1831, Michael Faraday and, almost simultaneously, Joseph Henry showed that when a wire is moved near a magnet (or, equivalently, when a magnet is moved near a wire), an electric current is established in the wire. In 1873, James Clerk Maxwell used these observations and other experimental facts as a basis for formulating the laws of electromagnetism as we know them today. (*Electromagnetism* is a name given to the combined study of electricity and magnetism.)

Maxwell's contributions to the field of electromagnetism were especially significant because the laws he formulated are basic to *all* forms of electromagnetic phenomena. His work is as important as Newton's work on the laws of motion and the theory of gravitation. ■

CHAPTER

23

Electric Fields

This young woman is enjoying the effects of electrically charging her body. Each individual hair on her head becomes charged and exerts a repulsive force on the other hairs, resulting in the "stand-up" hairdo seen here. *(Ted Kinsman / Photo Researchers, Inc.)*

In this chapter, we begin the study of electromagnetism. The first link that we will make to our previous study is through the concept of *force.* The electromagnetic force between charged particles is one of the fundamental forces of nature. We begin by describing some basic properties of one manifestation of the electromagnetic force, the electric force. We then discuss Coulomb's law, which is the fundamental law governing the electric force between any two charged particles. Next, we introduce the concept of an electric field associated with a charge distribution and describe its effect on other charged particles. We then show how to use Coulomb's law to calculate the electric field for a given charge distribution. The chapter concludes with a discussion of the motion of a charged particle in a uniform electric field.

The second link between electromagnetism and our previous study is through the concept of *energy.* We will discuss that connection in Chapter 25.

23.1 Properties of Electric Charges

A number of simple experiments demonstrate the existence of electric forces. For example, after rubbing a balloon on your hair on a dry day, you will find that the balloon attracts bits of paper. The attractive force is often strong enough to suspend the paper from the balloon.

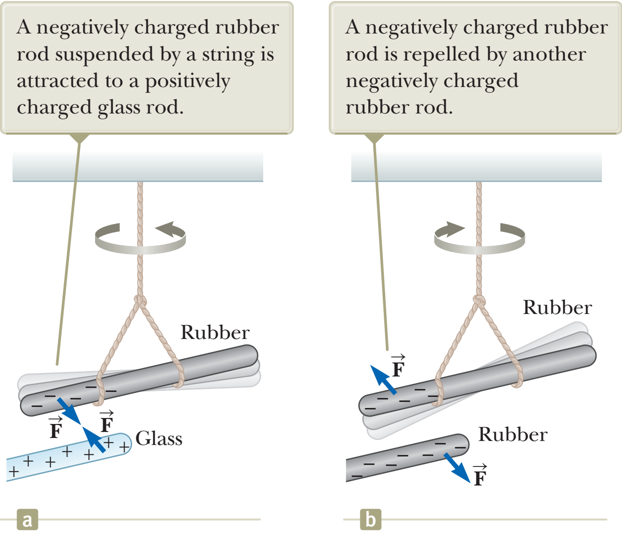

Figure 23.1 The electric force between (a) oppositely charged objects and (b) like-charged objects.

When materials behave in this way, they are said to be *electrified* or to have become **electrically charged.** You can easily electrify your body by vigorously rubbing your shoes on a wool rug. Evidence of the electric charge on your body can be detected by lightly touching (and startling) a friend. Under the right conditions, you will see a spark when you touch and both of you will feel a slight tingle. (Experiments such as these work best on a dry day because an excessive amount of moisture in the air can cause any charge you build up to "leak" from your body to the Earth.)

In a series of simple experiments, it was found that there are two kinds of electric charges, which were given the names **positive** and **negative** by Benjamin Franklin (1706–1790). Electrons are identified as having negative charge, and protons are positively charged. To verify that there are two types of charge, suppose a hard rubber rod that has been rubbed on fur is suspended by a string as shown in Figure 23.1. When a glass rod that has been rubbed on silk is brought near the rubber rod, the two attract each other (Fig. 23.1a). On the other hand, if two charged rubber rods (or two charged glass rods) are brought near each other as shown in Figure 23.1b, the two repel each other. This observation shows that the rubber and glass have two different types of charge on them. On the basis of these observations, we conclude that **charges of the same sign repel one another and charges with opposite signs attract one another.**

Using the convention suggested by Franklin, the electric charge on the glass rod is called positive and that on the rubber rod is called negative. Therefore, any charged object attracted to a charged rubber rod (or repelled by a charged glass rod) must have a positive charge, and any charged object repelled by a charged rubber rod (or attracted to a charged glass rod) must have a negative charge.

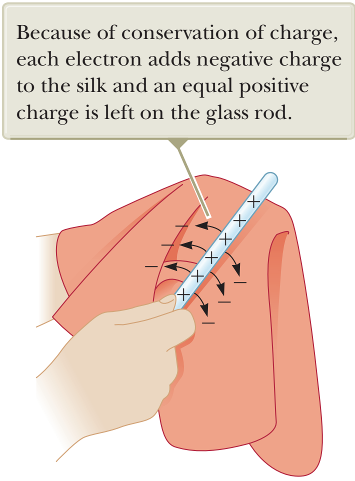

Figure 23.2 When a glass rod is rubbed with silk, electrons are transferred from the glass to the silk.

◀ **Electric charge is conserved**

Another important aspect of electricity that arises from experimental observations is that **electric charge is always conserved** in an isolated system. That is, when one object is rubbed against another, charge is not created in the process. The electrified state is due to a *transfer* of charge from one object to the other. One object gains some amount of negative charge while the other gains an equal amount of positive charge. For example, when a glass rod is rubbed on silk as in Figure 23.2, the silk obtains a negative charge equal in magnitude to the positive charge on the glass rod. We now know from our understanding of atomic structure that electrons are transferred in the rubbing process from the glass to the silk. Similarly, when rubber is rubbed on fur, electrons are transferred from the fur to the rubber, giving the rubber a net negative charge and the fur a net positive charge. This process works because neutral, uncharged matter contains as many positive charges (protons within atomic nuclei)

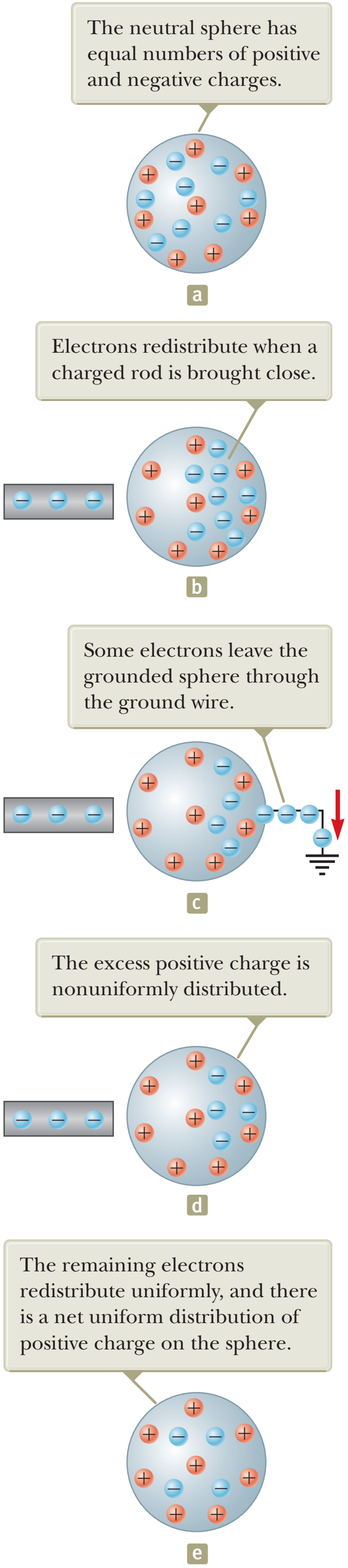

Figure 23.3 Charging a metallic object by *induction*. (a) A neutral metallic sphere. (b) A charged rubber rod is placed near the sphere. (c) The sphere is grounded. (d) The ground connection is removed. (e) The rod is removed.

as negative charges (electrons). Conservation of electric charge for an isolated system is like conservation of energy, momentum, and angular momentum, but we don't identify an analysis model for this conservation principle because it is not used often enough in the mathematical solution to problems.

In 1909, Robert Millikan (1868–1953) discovered that electric charge always occurs as integral multiples of a fundamental amount of charge e (see Section 25.7). In modern terms, the electric charge q is said to be **quantized,** where q is the standard symbol used for charge as a variable. That is, electric charge exists as discrete "packets," and we can write $q = \pm Ne$, where N is some integer. Other experiments in the same period showed that the electron has a charge $-e$ and the proton has a charge of equal magnitude but opposite sign $+e$. Some particles, such as the neutron, have no charge.

Quick Quiz 23.1 Three objects are brought close to each other, two at a time. When objects A and B are brought together, they repel. When objects B and C are brought together, they also repel. Which of the following are true? **(a)** Objects A and C possess charges of the same sign. **(b)** Objects A and C possess charges of opposite sign. **(c)** All three objects possess charges of the same sign. **(d)** One object is neutral. **(e)** Additional experiments must be performed to determine the signs of the charges.

23.2 Charging Objects by Induction

It is convenient to classify materials in terms of the ability of electrons to move through the material:

> Electrical **conductors** are materials in which some of the electrons are free electrons[1] that are not bound to atoms and can move relatively freely through the material; electrical **insulators** are materials in which all electrons are bound to atoms and cannot move freely through the material.

Materials such as glass, rubber, and dry wood fall into the category of electrical insulators. When such materials are charged by rubbing, only the area rubbed becomes charged and the charged particles are unable to move to other regions of the material.

In contrast, materials such as copper, aluminum, and silver are good electrical conductors. When such materials are charged in some small region, the charge readily distributes itself over the entire surface of the material.

Semiconductors are a third class of materials, and their electrical properties are somewhere between those of insulators and those of conductors. Silicon and germanium are well-known examples of semiconductors commonly used in the fabrication of a variety of electronic chips used in computers, cellular telephones, and home theater systems. The electrical properties of semiconductors can be changed over many orders of magnitude by the addition of controlled amounts of certain atoms to the materials.

To understand how to charge a conductor by a process known as **induction,** consider a neutral (uncharged) conducting sphere insulated from the ground as shown in Figure 23.3a. There are an equal number of electrons and protons in the sphere if the charge on the sphere is exactly zero. When a negatively charged rubber rod is brought near the sphere, electrons in the region nearest the rod experience a repulsive force and migrate to the opposite side of the sphere. This migration leaves

[1] A metal atom contains one or more outer electrons, which are weakly bound to the nucleus. When many atoms combine to form a metal, the *free electrons* are these outer electrons, which are not bound to any one atom. These electrons move about the metal in a manner similar to that of gas molecules moving in a container.

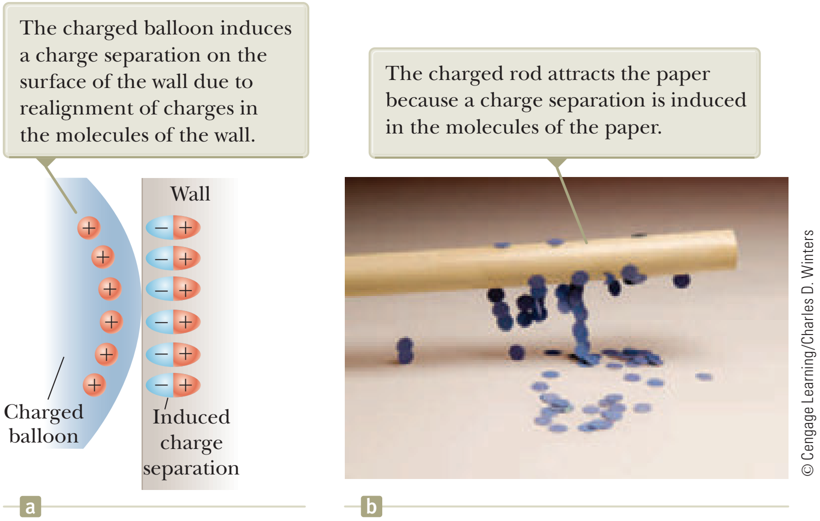

Figure 23.4 (a) A charged balloon is brought near an insulating wall. (b) A charged rod is brought close to bits of paper.

the side of the sphere near the rod with an effective positive charge because of the diminished number of electrons as in Figure 23.3b. (The left side of the sphere in Figure 23.3b is positively charged *as if* positive charges moved into this region, but remember that only electrons are free to move.) This process occurs even if the rod never actually touches the sphere. If the same experiment is performed with a conducting wire connected from the sphere to the Earth (Fig. 23.3c), some of the electrons in the conductor are so strongly repelled by the presence of the negative charge in the rod that they move out of the sphere through the wire and into the Earth. The symbol ⏚ at the end of the wire in Figure 23.3c indicates that the wire is connected to **ground,** which means a reservoir, such as the Earth, that can accept or provide electrons freely with negligible effect on its electrical characteristics. If the wire to ground is then removed (Fig. 23.3d), the conducting sphere contains an excess of *induced* positive charge because it has fewer electrons than it needs to cancel out the positive charge of the protons. When the rubber rod is removed from the vicinity of the sphere (Fig. 23.3e), this induced positive charge remains on the ungrounded sphere. Notice that the rubber rod loses none of its negative charge during this process.

Charging an object by induction requires no contact with the object inducing the charge. That is in contrast to charging an object by rubbing (that is, by *conduction*), which does require contact between the two objects.

A process similar to induction in conductors takes place in insulators. In most neutral molecules, the center of positive charge coincides with the center of negative charge. In the presence of a charged object, however, these centers inside each molecule in an insulator may shift slightly, resulting in more positive charge on one side of the molecule than on the other. This realignment of charge within individual molecules produces a layer of charge on the surface of the insulator as shown in Figure 23.4a. The proximity of the positive charges on the surface of the object and the negative charges on the surface of the insulator results in an attractive force between the object and the insulator. Your knowledge of induction in insulators should help you explain why a charged rod attracts bits of electrically neutral paper as shown in Figure 23.4b.

Quick Quiz 23.2 Three objects are brought close to one another, two at a time. When objects A and B are brought together, they attract. When objects B and C are brought together, they repel. Which of the following are necessarily true? **(a)** Objects A and C possess charges of the same sign. **(b)** Objects A and C possess charges of opposite sign. **(c)** All three objects possess charges of the same sign. **(d)** One object is neutral. **(e)** Additional experiments must be performed to determine information about the charges on the objects.

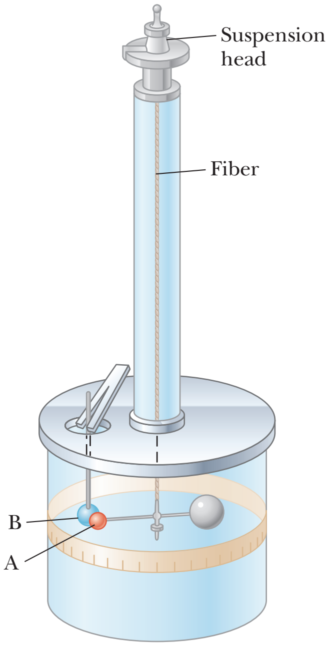

Figure 23.5 Coulomb's balance, used to establish the inverse-square law for the electric force.

23.3 Coulomb's Law

Charles Coulomb measured the magnitudes of the electric forces between charged objects using the torsion balance, which he invented (Fig. 23.5). The operating principle of the torsion balance is the same as that of the apparatus used by Cavendish to measure the density of the Earth (see Section 13.1), with the electrically neutral spheres replaced by charged ones. The electric force between charged spheres A and B in Figure 23.5 causes the spheres to either attract or repel each other, and the resulting motion causes the suspended fiber to twist. Because the restoring torque of the twisted fiber is proportional to the angle through which the fiber rotates, a measurement of this angle provides a quantitative measure of the electric force of attraction or repulsion. Once the spheres are charged by rubbing, the electric force between them is very large compared with the gravitational attraction, and so the gravitational force can be neglected.

From Coulomb's experiments, we can generalize the properties of the **electric force** (sometimes called the *electrostatic force*) between two stationary charged particles. We use the term **point charge** to refer to a charged particle of zero size. The electrical behavior of electrons and protons is very well described by modeling them as point charges. From experimental observations, we find that the magnitude of the electric force (sometimes called the *Coulomb force*) between two point charges is given by **Coulomb's law.**

Coulomb's law ▶

$$F_e = k_e \frac{|q_1||q_2|}{r^2} \tag{23.1}$$

where k_e is a constant called the **Coulomb constant.** In his experiments, Coulomb was able to show that the value of the exponent of r was 2 to within an uncertainty of a few percent. Modern experiments have shown that the exponent is 2 to within an uncertainty of a few parts in 10^{16}. Experiments also show that the electric force, like the gravitational force, is conservative.

The value of the Coulomb constant depends on the choice of units. The SI unit of charge is the **coulomb** (C). The Coulomb constant k_e in SI units has the value

Coulomb constant ▶

$$k_e = 8.987\,6 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \tag{23.2}$$

This constant is also written in the form

$$k_e = \frac{1}{4\pi\epsilon_0} \tag{23.3}$$

where the constant ϵ_0 (Greek letter epsilon) is known as the **permittivity of free space** and has the value

$$\epsilon_0 = 8.854\,2 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2 \tag{23.4}$$

The smallest unit of free charge e known in nature,[2] the charge on an electron $(-e)$ or a proton $(+e)$, has a magnitude

$$e = 1.602\,18 \times 10^{-19} \text{ C} \tag{23.5}$$

Therefore, 1 C of charge is approximately equal to the charge of 6.24×10^{18} electrons or protons. This number is very small when compared with the number of free electrons in 1 cm^3 of copper, which is on the order of 10^{23}. Nevertheless, 1 C is a substantial amount of charge. In typical experiments in which a rubber or glass rod is charged by friction, a net charge on the order of 10^{-6} C is obtained. In other


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Charles Coulomb
French physicist (1736–1806)
Coulomb's major contributions to science were in the areas of electrostatics and magnetism. During his lifetime, he also investigated the strengths of materials, thereby contributing to the field of structural mechanics. In ergonomics, his research provided an understanding of the ways in which people and animals can best do work.

[2]No unit of charge smaller than e has been detected on a free particle; current theories, however, propose the existence of particles called *quarks* having charges $-e/3$ and $2e/3$. Although there is considerable experimental evidence for such particles inside nuclear matter, *free* quarks have never been detected. We discuss other properties of quarks in Chapter 46.

Table 23.1 Charge and Mass of the Electron, Proton, and Neutron

Particle	Charge (C)	Mass (kg)
Electron (e)	$-1.602\,176\,5 \times 10^{-19}$	$9.109\,4 \times 10^{-31}$
Proton (p)	$+1.602\,176\,5 \times 10^{-19}$	$1.672\,62 \times 10^{-27}$
Neutron (n)	0	$1.674\,93 \times 10^{-27}$

words, only a very small fraction of the total available charge is transferred between the rod and the rubbing material.

The charges and masses of the electron, proton, and neutron are given in Table 23.1. Notice that the electron and proton are identical in the magnitude of their charge but vastly different in mass. On the other hand, the proton and neutron are similar in mass but vastly different in charge. Chapter 46 will help us understand these interesting properties.

Example 23.1 The Hydrogen Atom

The electron and proton of a hydrogen atom are separated (on the average) by a distance of approximately 5.3×10^{-11} m. Find the magnitudes of the electric force and the gravitational force between the two particles.

SOLUTION

Conceptualize Think about the two particles separated by the very small distance given in the problem statement. In Chapter 13, we mentioned that the gravitational force between an electron and a proton is very small compared to the electric force between them, so we expect this to be the case with the results of this example.

Categorize The electric and gravitational forces will be evaluated from universal force laws, so we categorize this example as a substitution problem.

Use Coulomb's law to find the magnitude of the electric force:

$$F_e = k_e \frac{|e||-e|}{r^2} = (8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \frac{(1.60 \times 10^{-19} \text{ C})^2}{(5.3 \times 10^{-11} \text{ m})^2}$$

$$= \boxed{8.2 \times 10^{-8} \text{ N}}$$

Use Newton's law of universal gravitation and Table 23.1 (for the particle masses) to find the magnitude of the gravitational force:

$$F_g = G \frac{m_e m_p}{r^2}$$

$$= (6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2) \frac{(9.11 \times 10^{-31} \text{ kg})(1.67 \times 10^{-27} \text{ kg})}{(5.3 \times 10^{-11} \text{ m})^2}$$

$$= \boxed{3.6 \times 10^{-47} \text{ N}}$$

The ratio $F_e/F_g \approx 2 \times 10^{39}$. Therefore, the gravitational force between charged atomic particles is negligible when compared with the electric force. Notice the similar forms of Newton's law of universal gravitation and Coulomb's law of electric forces. Other than the magnitude of the forces between elementary particles, what is a fundamental difference between the two forces?

When dealing with Coulomb's law, remember that force is a vector quantity and must be treated accordingly. Coulomb's law expressed in vector form for the electric force exerted by a charge q_1 on a second charge q_2, written $\vec{\mathbf{F}}_{12}$, is

$$\vec{\mathbf{F}}_{12} = k_e \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}_{12} \quad \text{(23.6)}$$

◀ Vector form of Coulomb's law

where $\hat{\mathbf{r}}_{12}$ is a unit vector directed from q_1 toward q_2 as shown in Figure 23.6a (page 696). Because the electric force obeys Newton's third law, the electric force exerted by q_2 on q_1 is equal in magnitude to the force exerted by q_1 on q_2 and in the opposite direction; that is, $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$. Finally, Equation 23.6 shows that if q_1 and q_2 have the

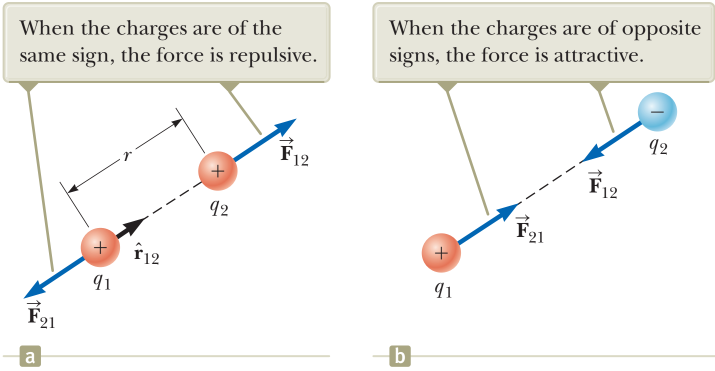


Figure 23.6 Two point charges separated by a distance r exert a force on each other that is given by Coulomb's law. The force $\vec{\mathbf{F}}_{21}$ exerted by q_2 on q_1 is equal in magnitude and opposite in direction to the force $\vec{\mathbf{F}}_{12}$ exerted by q_1 on q_2.

same sign as in Figure 23.6a, the product q_1q_2 is positive and the electric force on one particle is directed away from the other particle. If q_1 and q_2 are of opposite sign as shown in Figure 23.6b, the product q_1q_2 is negative and the electric force on one particle is directed toward the other particle. These signs describe the *relative* direction of the force but not the *absolute* direction. A negative product indicates an attractive force, and a positive product indicates a repulsive force. The *absolute* direction of the force on a charge depends on the location of the other charge. For example, if an x axis lies along the two charges in Figure 23.6a, the product q_1q_2 is positive, but $\vec{\mathbf{F}}_{12}$ points in the positive x direction and $\vec{\mathbf{F}}_{21}$ points in the negative x direction.

When more than two charges are present, the force between any pair of them is given by Equation 23.6. Therefore, the resultant force on any one of them equals the vector sum of the forces exerted by the other individual charges. For example, if four charges are present, the resultant force exerted by particles 2, 3, and 4 on particle 1 is

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{41}$$

Quick Quiz 23.3 Object A has a charge of $+2\ \mu$C, and object B has a charge of $+6\ \mu$C. Which statement is true about the electric forces on the objects? **(a)** $\vec{\mathbf{F}}_{AB} = -3\vec{\mathbf{F}}_{BA}$ **(b)** $\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}$ **(c)** $3\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}$ **(d)** $\vec{\mathbf{F}}_{AB} = 3\vec{\mathbf{F}}_{BA}$ **(e)** $\vec{\mathbf{F}}_{AB} = \vec{\mathbf{F}}_{BA}$ **(f)** $3\vec{\mathbf{F}}_{AB} = \vec{\mathbf{F}}_{BA}$

Example 23.2 Find the Resultant Force

Consider three point charges located at the corners of a right triangle as shown in Figure 23.7, where $q_1 = q_3 = 5.00\ \mu$C, $q_2 = -2.00\ \mu$C, and $a = 0.100$ m. Find the resultant force exerted on q_3.

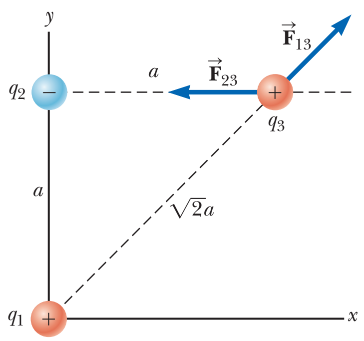


Figure 23.7 (Example 23.2) The force exerted by q_1 on q_3 is $\vec{\mathbf{F}}_{13}$. The force exerted by q_2 on q_3 is $\vec{\mathbf{F}}_{23}$. The resultant force $\vec{\mathbf{F}}_3$ exerted on q_3 is the vector sum $\vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23}$.

SOLUTION

Conceptualize Think about the net force on q_3. Because charge q_3 is near two other charges, it will experience two electric forces. These forces are exerted in different directions as shown in Figure 23.7. Based on the forces shown in the figure, estimate the direction of the net force vector.

Categorize Because two forces are exerted on charge q_3, we categorize this example as a vector addition problem.

Analyze The directions of the individual forces exerted by q_1 and q_2 on q_3 are shown in Figure 23.7. The force $\vec{\mathbf{F}}_{23}$ exerted by q_2 on q_3 is attractive because q_2 and q_3 have opposite signs. In the coordinate system shown in Figure 23.7, the attractive force $\vec{\mathbf{F}}_{23}$ is to the left (in the negative x direction).

The force $\vec{\mathbf{F}}_{13}$ exerted by q_1 on q_3 is repulsive because both charges are positive. The repulsive force $\vec{\mathbf{F}}_{13}$ makes an angle of 45.0° with the x axis.

23.2 continued

Use Equation 23.1 to find the magnitude of $\vec{\mathbf{F}}_{23}$:

$$F_{23} = k_e \frac{|q_2||q_3|}{a^2}$$
$$= (8.988 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)\frac{(2.00 \times 10^{-6}\text{ C})(5.00 \times 10^{-6}\text{ C})}{(0.100\text{ m})^2} = 8.99\text{ N}$$

Find the magnitude of the force $\vec{\mathbf{F}}_{13}$:

$$F_{13} = k_e \frac{|q_1||q_3|}{(\sqrt{2}\,a)^2}$$
$$= (8.988 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)\frac{(5.00 \times 10^{-6}\text{ C})(5.00 \times 10^{-6}\text{ C})}{2(0.100\text{ m})^2} = 11.2\text{ N}$$

Find the x and y components of the force $\vec{\mathbf{F}}_{13}$:

$$F_{13x} = (11.2\text{ N})\cos 45.0° = 7.94\text{ N}$$
$$F_{13y} = (11.2\text{ N})\sin 45.0° = 7.94\text{ N}$$

Find the components of the resultant force acting on q_3:

$$F_{3x} = F_{13x} + F_{23x} = 7.94\text{ N} + (-8.99\text{ N}) = -1.04\text{ N}$$
$$F_{3y} = F_{13y} + F_{23y} = 7.94\text{ N} + 0 = 7.94\text{ N}$$

Express the resultant force acting on q_3 in unit-vector form:

$$\vec{\mathbf{F}}_3 = \boxed{(-1.04\hat{\mathbf{i}} + 7.94\hat{\mathbf{j}})\text{ N}}$$

Finalize The net force on q_3 is upward and toward the left in Figure 23.7. If q_3 moves in response to the net force, the distances between q_3 and the other charges change, so the net force changes. Therefore, if q_3 is free to move, it can be modeled as a particle under a net force as long as it is recognized that the force exerted on q_3 is *not* constant. As a reminder, we display most numerical values to three significant figures, which leads to operations such as 7.94 N + (−8.99 N) = −1.04 N above. If you carry all intermediate results to more significant figures, you will see that this operation is correct.

WHAT IF? What if the signs of all three charges were changed to the opposite signs? How would that affect the result for $\vec{\mathbf{F}}_3$?

Answer The charge q_3 would still be attracted toward q_2 and repelled from q_1 with forces of the same magnitude. Therefore, the final result for $\vec{\mathbf{F}}_3$ would be the same.

Example 23.3 Where Is the Net Force Zero? AM

Three point charges lie along the x axis as shown in Figure 23.8. The positive charge $q_1 = 15.0\ \mu$C is at $x = 2.00$ m, the positive charge $q_2 = 6.00\ \mu$C is at the origin, and the net force acting on q_3 is zero. What is the x coordinate of q_3?

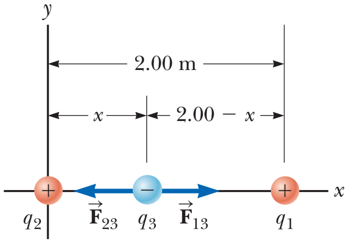


Figure 23.8 (Example 23.3) Three point charges are placed along the x axis. If the resultant force acting on q_3 is zero, the force $\vec{\mathbf{F}}_{13}$ exerted by q_1 on q_3 must be equal in magnitude and opposite in direction to the force $\vec{\mathbf{F}}_{23}$ exerted by q_2 on q_3.

SOLUTION

Conceptualize Because q_3 is near two other charges, it experiences two electric forces. Unlike the preceding example, however, the forces lie along the same line in this problem as indicated in Figure 23.8. Because q_3 is negative and q_1 and q_2 are positive, the forces $\vec{\mathbf{F}}_{13}$ and $\vec{\mathbf{F}}_{23}$ are both attractive. Because q_2 is the smaller charge, the position of q_3 at which the force is zero should be closer to q_2 than to q_1.

Categorize Because the net force on q_3 is zero, we model the point charge as a *particle in equilibrium.*

Analyze Write an expression for the net force on charge q_3 when it is in equilibrium:

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_{13} = -k_e\frac{|q_2||q_3|}{x^2}\hat{\mathbf{i}} + k_e\frac{|q_1||q_3|}{(2.00 - x)^2}\hat{\mathbf{i}} = 0$$

Move the second term to the right side of the equation and set the coefficients of the unit vector $\hat{\mathbf{i}}$ equal:

$$k_e\frac{|q_2||q_3|}{x^2} = k_e\frac{|q_1||q_3|}{(2.00 - x)^2}$$

continued

23.3 continued

Eliminate k_e and $|q_3|$ and rearrange the equation: $(2.00 - x)^2|q_2| = x^2|q_1|$

Take the square root of both sides of the equation: $(2.00 - x)\sqrt{|q_2|} = \pm x\sqrt{|q_1|}$

Solve for x: $x = \dfrac{2.00\sqrt{|q_2|}}{\sqrt{|q_2|} \pm \sqrt{|q_1|}}$

Substitute numerical values, choosing the plus sign: $x = \dfrac{2.00\sqrt{6.00 \times 10^{-6}\text{ C}}}{\sqrt{6.00 \times 10^{-6}\text{ C}} + \sqrt{15.0 \times 10^{-6}\text{ C}}} = 0.775\text{ m}$

Finalize Notice that the movable charge is indeed closer to q_2 as we predicted in the Conceptualize step. The second solution to the equation (if we choose the negative sign) is $x = -3.44$ m. That is another location where the *magnitudes* of the forces on q_3 are equal, but both forces are in the same direction, so they do not cancel.

WHAT IF? Suppose q_3 is constrained to move only along the x axis. From its initial position at $x = 0.775$ m, it is pulled a small distance along the x axis. When released, does it return to equilibrium, or is it pulled farther from equilibrium? That is, is the equilibrium stable or unstable?

Answer If q_3 is moved to the right, $\vec{\mathbf{F}}_{13}$ becomes larger and $\vec{\mathbf{F}}_{23}$ becomes smaller. The result is a net force to the right, in the same direction as the displacement. Therefore, the charge q_3 would continue to move to the right and the equilibrium is *unstable.* (See Section 7.9 for a review of stable and unstable equilibria.)

If q_3 is constrained to stay at a *fixed* x coordinate but allowed to move up and down in Figure 23.8, the equilibrium is stable. In this case, if the charge is pulled upward (or downward) and released, it moves back toward the equilibrium position and oscillates about this point.

Example 23.4 Find the Charge on the Spheres AM

Two identical small charged spheres, each having a mass of 3.00×10^{-2} kg, hang in equilibrium as shown in Figure 23.9a. The length L of each string is 0.150 m, and the angle θ is 5.00°. Find the magnitude of the charge on each sphere.

SOLUTION

Conceptualize Figure 23.9a helps us conceptualize this example. The two spheres exert repulsive forces on each other. If they are held close to each other and released, they move outward from the center and settle into the configuration in Figure 23.9a after the oscillations have vanished due to air resistance.

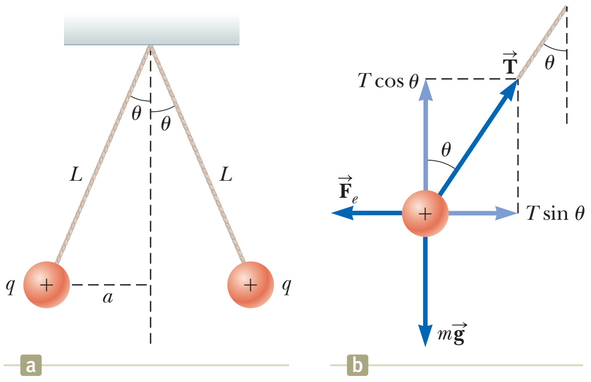


Figure 23.9 (Example 23.4) (a) Two identical spheres, each carrying the same charge q, suspended in equilibrium. (b) Diagram of the forces acting on the sphere on the left part of (a).

Categorize The key phrase "in equilibrium" helps us model each sphere as a *particle in equilibrium.* This example is similar to the particle in equilibrium problems in Chapter 5 with the added feature that one of the forces on a sphere is an electric force.

Analyze The force diagram for the left-hand sphere is shown in Figure 23.9b. The sphere is in equilibrium under the application of the force $\vec{\mathbf{T}}$ from the string, the electric force $\vec{\mathbf{F}}_e$ from the other sphere, and the gravitational force $m\vec{\mathbf{g}}$.

From the particle in equilibrium model, set the net force on the left-hand sphere equal to zero for each component:

$$(1)\quad \sum F_x = T\sin\theta - F_e = 0 \rightarrow T\sin\theta = F_e$$

$$(2)\quad \sum F_y = T\cos\theta - mg = 0 \rightarrow T\cos\theta = mg$$

Divide Equation (1) by Equation (2) to find F_e:

$$(3)\quad \tan\theta = \frac{F_e}{mg} \rightarrow F_e = mg\tan\theta$$

Use the geometry of the right triangle in Figure 23.9a to find a relationship between , and

(4) $\sin\theta = -$ sin

Solve Coulomb's law (Eq. 23.1) for the charge on each sphere and substitute from Equations (3) and (4):

mg tan sin

Substitute numerical values:

$$\frac{3.00 \quad 10 \quad \text{kg})(9.80 \text{ m} \quad \tan \; 5.00\;)[\; 0.150 \text{ m} \; \sin \; 5.00\;)]}{8.988 \quad 10 \; \text{N}}$$

$4.42 \quad 10 \quad \text{C}$

Finalize If the sign of the charges were not given in Figure 23.9, we could not determine them. In fact, the sign of the charge is not important. The situation is the same whether both spheres are positively charged or negatively charged.

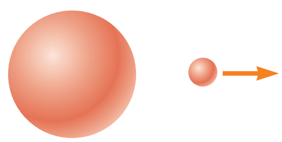
WHAT IF? Suppose your roommate proposes solving this problem without the assumption that the charges are of equal magnitude. She claims the symmetry of the problem is destroyed if the charges are not equal, so the strings would make two different angles with the vertical and the problem would be much more complicated. How would you respond?

Answer The symmetry is not destroyed and the angles are not different. Newton's third law requires the magnitudes of the electric forces on the two spheres to be the same, regardless of the equality or nonequality of the charges. The solu tion to the example remains the same with one change: the value of in the solution is replaced by in the new situation, where and are the values of the charges on the two spheres. The symmetry of the problem would be destroyed if the *masses* of the spheres were not the same. In this case, the strings would make different angles with the vertical and the problem would be more complicated.

23.4 Analysis Model: Particle in a Field (Electric)

In Section 5.1, we discussed the differences between contact forces and field forces. Two field forces—the gravitational force in Chapter 13 and the electric force here—have been introduced into our discussions so far. As pointed out earlier, field forces can act through space, producing an effect even when no physical contact occurs between interacting objects. Such an interaction can be modeled as a two-step pro cess: a source particle establishes a field, and then a charged particle interacts with the field and experiences a force. The gravitational field at a point in space due to a source particle was defined in Section 13.4 to be equal to the gravitational force acting on a test particle of mass divided by that mass: Then the force exerted by the field is (Eq. 5.5).

The concept of a field was developed by Michael Faraday (1791–1867) in the con text of electric forces and is of such practical value that we shall devote much atten tion to it in the next several chapters. In this approach, an **electric field** is said to exist in the region of space around a charged object, the **source charge.** The presence of the electric field can be detected by placing a **test charge** in the field and noting the electric force on it. As an example, consider Figure 23.10, which shows a small positive test charge placed near a second object carrying a much greater positive charge We define the electric field due to the source charge at the location of the test charge to be the electric force on the test charge *per unit charge*, or, to be more specific, the **electric field vector** at a point in space is defined as the electric force act ing on a positive test charge placed at that point divided by the test charge:



igure 23.10 A small positive test charge placed at point near an object carrying a much larger positive charge expe riences an electric field at point established by the source charge Q. We will *always* assume that the test charge is so small that the field of the source charge is unaffected by its presence.

$$-\qquad (23.7)$$

◀ **Definition of electric field**

When using Equation 23.7, we must assume the test charge is small enough that it does not disturb the charge distri bution responsible for the electric field. If the test charge is great enough, the charge on the metallic sphere is redistrib uted and the electric field it sets up is different from the field it sets up in the presence of the much smaller test charge.

This dramatic photograph captures a lightning bolt striking a tree near some rural homes. Lightning is associated with very strong electric fields in the atmosphere.

Pitfall Prevention 23.1

Particles Only Equation 23.8 is valid only for a *particle* of charge q, that is, an object of zero size. For a charged *object* of finite size in an electric field, the field may vary in magnitude and direction over the size of the object, so the corresponding force equation may be more complicated.

The vector $\vec{\mathbf{E}}$ has the SI units of newtons per coulomb (N/C). The direction of $\vec{\mathbf{E}}$ as shown in Figure 23.10 is the direction of the force a positive test charge experiences when placed in the field. Note that $\vec{\mathbf{E}}$ is the field produced by some charge or charge distribution *separate from* the test charge; it is not the field produced by the test charge itself. Also note that the existence of an electric field is a property of its source; the presence of the test charge is not necessary for the field to exist. The test charge serves as a *detector* of the electric field: an electric field exists at a point if a test charge at that point experiences an electric force.

If an arbitrary charge q is placed in an electric field $\vec{\mathbf{E}}$, it experiences an electric force given by

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} \tag{23.8}$$

This equation is the mathematical representation of the electric version of the **particle in a field** analysis model. If q is positive, the force is in the same direction as the field. If q is negative, the force and the field are in opposite directions. Notice the similarity between Equation 23.8 and the corresponding equation from the gravitational version of the particle in a field model, $\vec{\mathbf{F}}_g = m\vec{\mathbf{g}}$ (Section 5.5). Once the magnitude and direction of the electric field are known at some point, the electric force exerted on *any* charged particle placed at that point can be calculated from Equation 23.8.

To determine the direction of an electric field, consider a point charge q as a source charge. This charge creates an electric field at all points in space surrounding it. A test charge q_0 is placed at point P, a distance r from the source charge, as in Figure 23.11a. We imagine using the test charge to determine the direction of the electric force and therefore that of the electric field. According to Coulomb's law, the force exerted by q on the test charge is

$$\vec{\mathbf{F}}_e = k_e \frac{q q_0}{r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from q toward q_0. This force in Figure 23.11a is directed away from the source charge q. Because the electric field at P, the position of the test charge, is defined by $\vec{\mathbf{E}} = \vec{\mathbf{F}}_e/q_0$, the electric field at P created by q is

$$\vec{\mathbf{E}} = k_e \frac{q}{r^2}\hat{\mathbf{r}} \tag{23.9}$$

If the source charge q is positive, Figure 23.11b shows the situation with the test charge removed: the source charge sets up an electric field at P, directed away from q. If q is

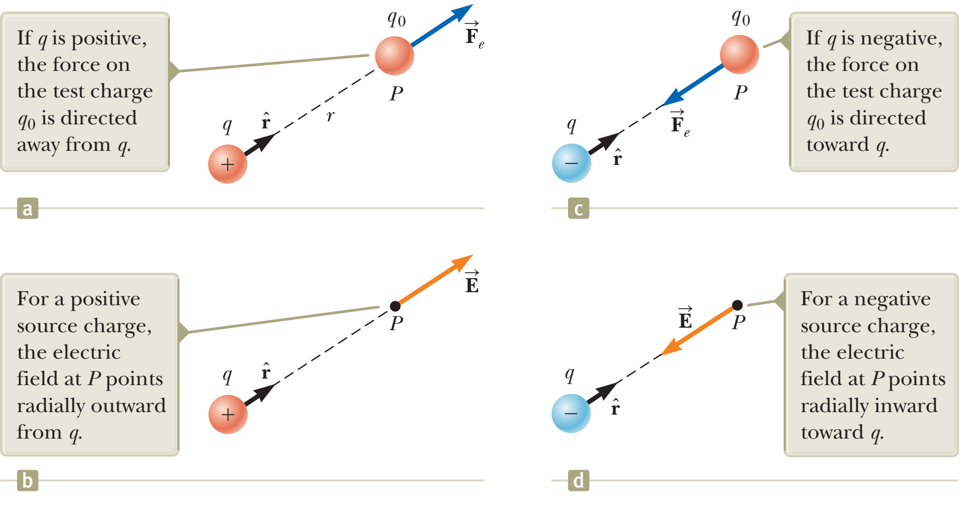


Figure 23.11 (a), (c) When a test charge q_0 is placed near a source charge q, the test charge experiences a force. (b), (d) At a point P near a source charge q, there exists an electric field.

negative as in Figure 23.11c, the force on the test charge is toward the source charge, so the electric field at P is directed toward the source charge as in Figure 23.11d.

To calculate the electric field at a point P due to a small number of point charges, we first calculate the electric field vectors at P individually using Equation 23.9 and then add them vectorially. In other words, at any point P, the total electric field due to a group of source charges equals the vector sum of the electric fields of all the charges. This superposition principle applied to fields follows directly from the vector addition of electric forces. Therefore, the electric field at point P due to a group of source charges can be expressed as the vector sum

$$\vec{\mathbf{E}} = k_e \sum_i \frac{q_i}{r_i^{\,2}} \hat{\mathbf{r}}_i \qquad \textbf{(23.10)}$$

◀ **Electric field due to a finite number of point charges**

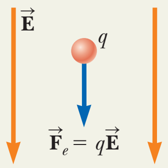
where r_i is the distance from the ith source charge q_i to the point P and $\hat{\mathbf{r}}_i$ is a unit vector directed from q_i toward P.

In Example 23.6, we explore the electric field due to two charges using the superposition principle. Part (B) of the example focuses on an **electric dipole,** which is defined as a positive charge q and a negative charge $-q$ separated by a distance $2a$. The electric dipole is a good model of many molecules, such as hydrochloric acid (HCl). Neutral atoms and molecules behave as dipoles when placed in an external electric field. Furthermore, many molecules, such as HCl, are permanent dipoles. The effect of such dipoles on the behavior of materials subjected to electric fields is discussed in Chapter 26.

Quick Quiz 23.4 A test charge of $+3\ \mu$C is at a point P where an external electric field is directed to the right and has a magnitude of 4×10^6 N/C. If the test charge is replaced with another test charge of $-3\ \mu$C, what happens to the external electric field at P? **(a)** It is unaffected. **(b)** It reverses direction. **(c)** It changes in a way that cannot be determined.

Analysis Model Particle in a Field (Electric)

Imagine an object with charge that we call a *source charge*. The source charge establishes an **electric field $\vec{\mathbf{E}}$** throughout space. Now imagine a particle with charge q is placed in that field. The particle interacts with the electric field so that the particle experiences an electric force given by



$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} \qquad \textbf{(23.8)}$$

Examples:

- an electron moves between the deflection plates of a cathode ray oscilloscope and is deflected from its original path
- charged ions experience an electric force from the electric field in a velocity selector before entering a mass spectrometer (Chapter 29)
- an electron moves around the nucleus in the electric field established by the proton in a hydrogen atom as modeled by the Bohr theory (Chapter 42)
- a hole in a semiconducting material moves in response to the electric field established by applying a voltage to the material (Chapter 43)

Example 23.5 A Suspended Water Droplet AM

A water droplet of mass 3.00×10^{-12} kg is located in the air near the ground during a stormy day. An atmospheric electric field of magnitude 6.00×10^3 N/C points vertically downward in the vicinity of the water droplet. The droplet remains suspended at rest in the air. What is the electric charge on the droplet?

SOLUTION

Conceptualize Imagine the water droplet hovering at rest in the air. This situation is not what is normally observed, so something must be holding the water droplet up.

continued

23.5 continued

Categorize The droplet can be modeled as a particle and is described by two analysis models associated with fields: the *particle in a field (gravitational)* and the *particle in a field (electric)*. Furthermore, because the droplet is subject to forces but remains at rest, it is also described by the *particle in equilibrium* model.

Analyze

Write Newton's second law from the particle in equilibrium model in the vertical direction:

$$(1)\quad \sum F_y = 0 \ \rightarrow\ F_e - F_g = 0$$

Using the two particle in a field models mentioned in the Categorize step, substitute for the forces in Equation (1), recognizing that the vertical component of the electric field is negative:

$$q(-E) - mg = 0$$

Solve for the charge on the water droplet:

$$q = -\frac{mg}{E}$$

Substitute numerical values:

$$q = -\frac{(3.00 \times 10^{-12}\,\mathrm{kg})(9.80\,\mathrm{m/s^2})}{6.00 \times 10^{3}\,\mathrm{N/C}} = \boxed{-4.90 \times 10^{-15}\,\mathrm{C}}$$

Finalize Noting the smallest unit of free charge in Equation 23.5, the charge on the water droplet is a large number of these units. Notice that the electric *force* is upward to balance the downward gravitational force. The problem statement claims that the electric *field* is in the downward direction. Therefore, the charge found above is negative so that the electric force is in the direction opposite to the electric field.

Example 23.6 Electric Field Due to Two Charges

Charges q_1 and q_2 are located on the x axis, at distances a and b, respectively, from the origin as shown in Figure 23.12.

(A) Find the components of the net electric field at the point P, which is at position $(0, y)$.

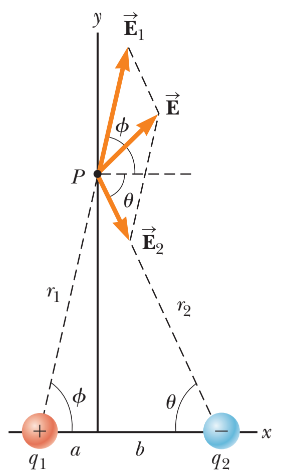


Figure 23.12 (Example 23.6) The total electric field $\vec{\mathbf{E}}$ at P equals the vector sum $\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2$, where $\vec{\mathbf{E}}_1$ is the field due to the positive charge q_1 and $\vec{\mathbf{E}}_2$ is the field due to the negative charge q_2.

SOLUTION

Conceptualize Compare this example with Example 23.2. There, we add vector forces to find the net force on a charged particle. Here, we add electric field vectors to find the net electric field at a point in space. If a charged particle were placed at P, we could use the particle in a field model to find the electric force on the particle.

Categorize We have two source charges and wish to find the resultant electric field, so we categorize this example as one in which we can use the superposition principle represented by Equation 23.10.

Analyze Find the magnitude of the electric field at P due to charge q_1:

$$E_1 = k_e\frac{|q_1|}{r_1^2} = k_e\frac{|q_1|}{a^2 + y^2}$$

Find the magnitude of the electric field at P due to charge q_2:

$$E_2 = k_e\frac{|q_2|}{r_2^2} = k_e\frac{|q_2|}{b^2 + y^2}$$

Write the electric field vectors for each charge in unit-vector form:

$$\vec{\mathbf{E}}_1 = k_e\frac{|q_1|}{a^2 + y^2}\cos\phi\,\hat{\mathbf{i}} + k_e\frac{|q_1|}{a^2 + y^2}\sin\phi\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{E}}_2 = k_e\frac{|q_2|}{b^2 + y^2}\cos\theta\,\hat{\mathbf{i}} - k_e\frac{|q_2|}{b^2 + y^2}\sin\theta\,\hat{\mathbf{j}}$$

23.6 continued

Write the components of the net electric field vector:

$$(1)\quad E_x = E_{1x} + E_{2x} = \boxed{k_e \frac{|q_1|}{a^2 + y^2}\cos\phi + k_e \frac{|q_2|}{b^2 + y^2}\cos\theta}$$

$$(2)\quad E_y = E_{1y} + E_{2y} = \boxed{k_e \frac{|q_1|}{a^2 + y^2}\sin\phi - k_e \frac{|q_2|}{b^2 + y^2}\sin\theta}$$

(B) Evaluate the electric field at point P in the special case that $|q_1| = |q_2|$ and $a = b$.

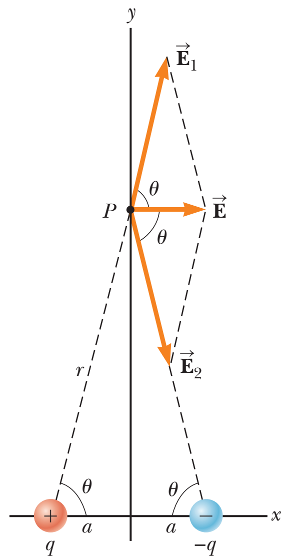


Figure 23.13 (Example 23.6) When the charges in Figure 23.12 are of equal magnitude and equidistant from the origin, the situation becomes symmetric as shown here.

SOLUTION

Conceptualize Figure 23.13 shows the situation in this special case. Notice the symmetry in the situation and that the charge distribution is now an electric dipole.

Categorize Because Figure 23.13 is a special case of the general case shown in Figure 23.12, we can categorize this example as one in which we can take the result of part (A) and substitute the appropriate values of the variables.

Analyze Based on the symmetry in Figure 23.13, evaluate Equations (1) and (2) from part (A) with $a = b$, $|q_1| = |q_2| = q$, and $\phi = \theta$:

$$(3)\quad E_x = k_e \frac{q}{a^2 + y^2}\cos\theta + k_e \frac{q}{a^2 + y^2}\cos\theta = 2k_e \frac{q}{a^2 + y^2}\cos\theta$$

$$E_y = k_e \frac{q}{a^2 + y^2}\sin\theta - k_e \frac{q}{a^2 + y^2}\sin\theta = 0$$

From the geometry in Figure 23.13, evaluate $\cos\theta$:

$$(4)\quad \cos\theta = \frac{a}{r} = \frac{a}{(a^2 + y^2)^{1/2}}$$

Substitute Equation (4) into Equation (3):

$$E_x = 2k_e \frac{q}{a^2 + y^2}\left[\frac{a}{(a^2 + y^2)^{1/2}}\right] = \boxed{k_e \frac{2aq}{(a^2 + y^2)^{3/2}}}$$

(C) Find the electric field due to the electric dipole when point P is a distance $y >> a$ from the origin.

SOLUTION

In the solution to part (B), because $y >> a$, neglect a^2 compared with y^2 and write the expression for E in this case:

$$(5)\quad E \approx \boxed{k_e \frac{2aq}{y^3}}$$

Finalize From Equation (5), we see that at points far from a dipole but along the perpendicular bisector of the line joining the two charges, the magnitude of the electric field created by the dipole varies as $1/r^3$, whereas the more slowly varying field of a point charge varies as $1/r^2$ (see Eq. 23.9). That is because at distant points, the fields of the two charges of equal magnitude and opposite sign almost cancel each other. The $1/r^3$ variation in E for the dipole also is obtained for a distant point along the x axis and for any general distant point.

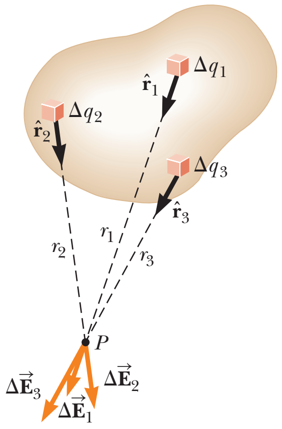


Figure 23.14 The electric field at P due to a continuous charge distribution is the vector sum of the fields $\Delta\vec{\mathbf{E}}_i$ due to all the elements Δq_i of the charge distribution. Three sample elements are shown.

23.5 Electric Field of a Continuous Charge Distribution

Equation 23.10 is useful for calculating the electric field due to a small number of charges. In many cases, we have a continuous distribution of charge rather than a collection of discrete charges. The charge in these situations can be described as continuously distributed along some line, over some surface, or throughout some volume.

To set up the process for evaluating the electric field created by a continuous charge distribution, let's use the following procedure. First, divide the charge distribution into small elements, each of which contains a small charge Δq as shown in Figure 23.14. Next, use Equation 23.9 to calculate the electric field due to one of these elements at a point P. Finally, evaluate the total electric field at P due to the charge distribution by summing the contributions of all the charge elements (that is, by applying the superposition principle).

The electric field at P due to one charge element carrying charge Δq is

$$\Delta\vec{\mathbf{E}} = k_e \frac{\Delta q}{r^2}\hat{\mathbf{r}}$$

where r is the distance from the charge element to point P and $\hat{\mathbf{r}}$ is a unit vector directed from the element toward P. The total electric field at P due to all elements in the charge distribution is approximately

$$\vec{\mathbf{E}} \approx k_e \sum_i \frac{\Delta q_i}{r_i^2}\hat{\mathbf{r}}_i$$

where the index i refers to the ith element in the distribution. Because the number of elements is very large and the charge distribution is modeled as continuous, the total field at P in the limit $\Delta q_i \to 0$ is

Electric field due to a continuous charge distribution ▶

$$\vec{\mathbf{E}} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^2}\hat{\mathbf{r}}_i = k_e \int \frac{dq}{r^2}\hat{\mathbf{r}} \quad \textbf{(23.11)}$$

where the integration is over the entire charge distribution. The integration in Equation 23.11 is a vector operation and must be treated appropriately.

Let's illustrate this type of calculation with several examples in which the charge is distributed on a line, on a surface, or throughout a volume. When performing such calculations, it is convenient to use the concept of a *charge density* along with the following notations:

- If a charge Q is uniformly distributed throughout a volume V, the **volume charge density** ρ is defined by

Volume charge density ▶

$$\rho \equiv \frac{Q}{V}$$

where ρ has units of coulombs per cubic meter (C/m^3).

- If a charge Q is uniformly distributed on a surface of area A, the **surface charge density** σ (Greek letter sigma) is defined by

Surface charge density ▶

$$\sigma \equiv \frac{Q}{A}$$

where σ has units of coulombs per square meter (C/m^2).

- If a charge Q is uniformly distributed along a line of length ℓ, the **linear charge density** λ is defined by

Linear charge density ▶

$$\lambda \equiv \frac{Q}{\ell}$$

where λ has units of coulombs per meter (C/m).

- If the charge is nonuniformly distributed over a volume, surface, or line, the amounts of charge dq in a small volume, surface, or length element are

$$dq = \rho\, dV \qquad dq = \sigma\, dA \qquad dq = \lambda\, d\ell$$

Problem-Solving Strategy Calculating the Electric Field

The following procedure is recommended for solving problems that involve the determination of an electric field due to individual charges or a charge distribution.

1. Conceptualize. Establish a mental representation of the problem: think carefully about the individual charges or the charge distribution and imagine what type of electric field it would create. Appeal to any symmetry in the arrangement of charges to help you visualize the electric field.

2. Categorize. Are you analyzing a group of individual charges or a continuous charge distribution? The answer to this question tells you how to proceed in the Analyze step.

3. Analyze.

(a) If you are analyzing a group of individual charges, use the superposition principle: when several point charges are present, the resultant field at a point in space is the *vector sum* of the individual fields due to the individual charges (Eq. 23.10). Be very careful in the manipulation of vector quantities. It may be useful to review the material on vector addition in Chapter 3. Example 23.6 demonstrated this procedure.

(b) If you are analyzing a continuous charge distribution, the superposition principle is applied by replacing the vector sums for evaluating the total electric field from individual charges by vector integrals. The charge distribution is divided into infinitesimal pieces, and the vector sum is carried out by integrating over the entire charge distribution (Eq. 23.11). Examples 23.7 through 23.9 demonstrate such procedures.

Consider symmetry when dealing with either a distribution of point charges or a continuous charge distribution. Take advantage of any symmetry in the system you observed in the Conceptualize step to simplify your calculations. The cancellation of field components perpendicular to the axis in Example 23.8 is an example of the application of symmetry.

4. Finalize. Check to see if your electric field expression is consistent with the mental representation and if it reflects any symmetry that you noted previously. Imagine varying parameters such as the distance of the observation point from the charges or the radius of any circular objects to see if the mathematical result changes in a reasonable way.

Example 23.7 The Electric Field Due to a Charged Rod

A rod of length ℓ has a uniform positive charge per unit length λ and a total charge Q. Calculate the electric field at a point P that is located along the long axis of the rod and a distance a from one end (Fig. 23.15).

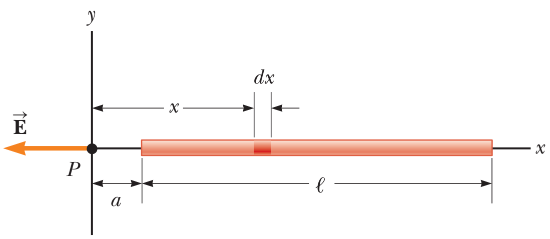


Figure 23.15 (Example 23.7) The electric field at P due to a uniformly charged rod lying along the x axis.

SOLUTION

Conceptualize The field $d\vec{\mathbf{E}}$ at P due to each segment of charge on the rod is in the negative x direction because every segment carries a positive charge. Figure 23.15 shows the appropriate geometry. In our result, we expect the electric field to become smaller as the distance a becomes larger because point P is farther from the charge distribution.

continued

23.7 continued

Categorize Because the rod is continuous, we are evaluating the field due to a continuous charge distribution rather than a group of individual charges. Because every segment of the rod produces an electric field in the negative x direction, the sum of their contributions can be handled without the need to add vectors.

Analyze Let's assume the rod is lying along the x axis, dx is the length of one small segment, and dq is the charge on that segment. Because the rod has a charge per unit length λ, the charge dq on the small segment is $dq = \lambda\, dx$.

Find the magnitude of the electric field at P due to one segment of the rod having a charge dq:

$$dE = k_e \frac{dq}{x^2} = k_e \frac{\lambda\, dx}{x^2}$$

Find the total field at P using[4] Equation 23.11:

$$E = \int_a^{\ell+a} k_e \lambda \frac{dx}{x^2}$$

Noting that k_e and $\lambda = Q/\ell$ are constants and can be removed from the integral, evaluate the integral:

$$E = k_e \lambda \int_a^{\ell+a} \frac{dx}{x^2} = k_e \lambda \left[-\frac{1}{x}\right]_a^{\ell+a}$$

$$(1) \quad E = k_e \frac{Q}{\ell}\left(\frac{1}{a} - \frac{1}{\ell + a}\right) = \boxed{\frac{k_e Q}{a(\ell + a)}}$$

Finalize We see that our prediction is correct; if a becomes larger, the denominator of the fraction grows larger, and E becomes smaller. On the other hand, if $a \to 0$, which corresponds to sliding the bar to the left until its left end is at the origin, then $E \to \infty$. That represents the condition in which the observation point P is at zero distance from the charge at the end of the rod, so the field becomes infinite. We explore large values of a below.

WHAT IF? Suppose point P is very far away from the rod. What is the nature of the electric field at such a point?

Answer If P is far from the rod ($a >> \ell$), then ℓ in the denominator of Equation (1) can be neglected and $E \approx k_e Q/a^2$. That is exactly the form you would expect for a point charge. Therefore, at large values of a/ℓ, the charge distribution appears to be a point charge of magnitude Q; the point P is so far away from the rod we cannot distinguish that it has a size. The use of the limiting technique ($a/\ell \to \infty$) is often a good method for checking a mathematical expression.

Example 23.8 The Electric Field of a Uniform Ring of Charge

A ring of radius a carries a uniformly distributed positive total charge Q. Calculate the electric field due to the ring at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring (Fig. 23.16a).

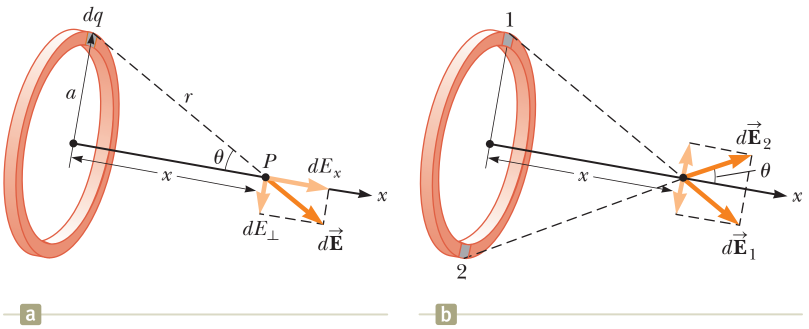


Figure 23.16 (Example 23.8) A uniformly charged ring of radius a. (a) The field at P on the x axis due to an element of charge dq. (b) The total electric field at P is along the x axis. The perpendicular component of the field at P due to segment 1 is canceled by the perpendicular component due to segment 2.

SOLUTION

Conceptualize Figure 23.16a shows the electric field contribution $d\vec{\mathbf{E}}$ at P due to a single segment of charge at the top of the ring. This field vector can be resolved into components dE_x parallel to

[4] To carry out integrations such as this one, first express the charge element dq in terms of the other variables in the integral. (In this example, there is one variable, x, so we made the change $dq = \lambda\, dx$.) The integral must be over scalar quantities; therefore, express the electric field in terms of components, if necessary. (In this example, the field has only an x component, so this detail is of no concern.) Then, reduce your expression to an integral over a single variable (or to multiple integrals, each over a single variable). In examples that have spherical or cylindrical symmetry, the single variable is a radial coordinate.

23.8 continued

the axis of the ring and $dE_\perp$ perpendicular to the axis. Figure 23.16b shows the electric field contributions from two segments on opposite sides of the ring. Because of the symmetry of the situation, the perpendicular components of the field cancel. That is true for all pairs of segments around the ring, so we can ignore the perpendicular component of the field and focus solely on the parallel components, which simply add.

Categorize Because the ring is continuous, we are evaluating the field due to a continuous charge distribution rather than a group of individual charges.

Analyze Evaluate the parallel component of an electric field contribution from a segment of charge dq on the ring:

$$(1)\quad dE_x = k_e \frac{dq}{r^2}\cos\theta = k_e \frac{dq}{a^2 + x^2}\cos\theta$$

From the geometry in Figure 23.16a, evaluate $\cos\theta$:

$$(2)\quad \cos\theta = \frac{x}{r} = \frac{x}{(a^2 + x^2)^{1/2}}$$

Substitute Equation (2) into Equation (1):

$$dE_x = k_e \frac{dq}{a^2 + x^2}\left[\frac{x}{(a^2 + x^2)^{1/2}}\right] = \frac{k_e x}{(a^2 + x^2)^{3/2}}\,dq$$

All segments of the ring make the same contribution to the field at P because they are all equidistant from this point. Integrate over the circumference of the ring to obtain the total field at P:

$$E_x = \int \frac{k_e x}{(a^2 + x^2)^{3/2}}\,dq = \frac{k_e x}{(a^2 + x^2)^{3/2}}\int dq$$

$$(3)\quad E = \boxed{\frac{k_e x}{(a^2 + x^2)^{3/2}}\,Q}$$

Finalize This result shows that the field is zero at $x = 0$. Is that consistent with the symmetry in the problem? Furthermore, notice that Equation (3) reduces to k_eQ/x^2 if $x >> a$, so the ring acts like a point charge for locations far away from the ring. From a faraway point, we cannot distinguish the ring shape of the charge.

WHAT IF? Suppose a negative charge is placed at the center of the ring in Figure 23.16 and displaced slightly by a distance $x << a$ along the x axis. When the charge is released, what type of motion does it exhibit?

Answer In the expression for the field due to a ring of charge, let $x << a$, which results in

$$E_x = \frac{k_e Q}{a^3}x$$

Therefore, from Equation 23.8, the force on a charge $-q$ placed near the center of the ring is

$$F_x = -\frac{k_e q Q}{a^3}x$$

Because this force has the form of Hooke's law (Eq. 15.1), the motion of the negative charge is described with the *particle in simple harmonic motion model!*

Example 23.9 The Electric Field of a Uniformly Charged Disk

A disk of radius R has a uniform surface charge density σ. Calculate the electric field at a point P that lies along the central perpendicular axis of the disk and a distance x from the center of the disk (Fig. 23.17).

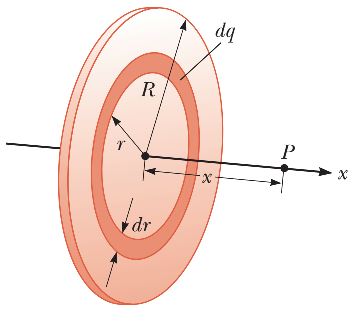


Figure 23.17 (Example 23.9) A uniformly charged disk of radius R. The electric field at an axial point P is directed along the central axis, perpendicular to the plane of the disk.

SOLUTION

Conceptualize If the disk is considered to be a set of concentric rings, we can use our result from Example 23.8—which gives the field created by a single ring of radius a—and sum the contributions of all rings making up the disk. By symmetry, the field at an axial point must be along the central axis.

continued

▶ 23.9 continued

Categorize Because the disk is continuous, we are evaluating the field due to a continuous charge distribution rather than a group of individual charges.

Analyze Find the amount of charge dq on the surface area of a ring of radius r and width dr as shown in Figure 23.17:

$$dq = \sigma\, dA = \sigma(2\pi r\, dr) = 2\pi\sigma r\, dr$$

Use this result in the equation given for E_x in Example 23.8 (with a replaced by r and Q replaced by dq) to find the field due to the ring:

$$dE_x = \frac{k_e x}{(r^2 + x^2)^{3/2}}(2\pi\sigma r\, dr)$$

To obtain the total field at P, integrate this expression over the limits $r = 0$ to $r = R$, noting that x is a constant in this situation:

$$E_x = k_e x\pi\sigma \int_0^R \frac{2r\, dr}{(r^2 + x^2)^{3/2}}$$

$$= k_e x\pi\sigma \int_0^R (r^2 + x^2)^{-3/2} d(r^2)$$

$$= k_e x\pi\sigma \left[\frac{(r^2 + x^2)^{-1/2}}{-1/2}\right]_0^R = \boxed{2\pi k_e\sigma\left[1 - \frac{x}{(R^2 + x^2)^{1/2}}\right]}$$

Finalize This result is valid for all values of $x > 0$. For large values of x, the result above can be evaluated by a series expansion and shown to be equivalent to the electric field of a point charge Q. We can calculate the field close to the disk along the axis by assuming $x << R$; in this case, the expression in brackets reduces to unity to give us the near-field approximation

$$E = 2\pi k_e\sigma = \frac{\sigma}{2\epsilon_0}$$

where ϵ_0 is the permittivity of free space. In Chapter 24, we obtain the same result for the field created by an infinite plane of charge with uniform surface charge density.

WHAT IF? What if we let the radius of the disk grow so that the disk becomes an infinite plane of charge?

Answer The result of letting $R \to \infty$ in the final result of the example is that the magnitude of the electric field becomes

$$E = 2\pi k_e\sigma = \frac{\sigma}{2\epsilon_0}$$

This is the same expression that we obtained for $x << R$. If $R \to \infty$, *everywhere* is near-field—the result is independent of the position at which you measure the electric field. Therefore, the electric field due to an infinite plane of charge is uniform throughout space.

An infinite plane of charge is impossible in practice. If two planes of charge are placed close to each other, however, with one plane positively charged, and the other negatively, the electric field between the plates is very close to uniform at points far from the edges. Such a configuration will be investigated in Chapter 26.

23.6 Electric Field Lines

We have defined the electric field in the mathematical representation with Equation 23.7. Let's now explore a means of visualizing the electric field in a pictorial representation. A convenient way of visualizing electric field patterns is to draw lines, called **electric field lines** and first introduced by Faraday, that are related to the electric field in a region of space in the following manner:

- The electric field vector $\vec{\mathbf{E}}$ is tangent to the electric field line at each point. The line has a direction, indicated by an arrowhead, that is the same as that of the electric field vector. The direction of the line is that of the force on a positive charge placed in the field according to the particle in a field model.

- The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of the electric field in that region. Therefore, the field lines are close together where the electric field is strong and far apart where the field is weak.

These properties are illustrated in Figure 23.18. The density of field lines through surface A is greater than the density of lines through surface B. Therefore, the magnitude of the electric field is larger on surface A than on surface B. Furthermore, because the lines at different locations point in different directions, the field is nonuniform.

Is this relationship between strength of the electric field and the density of field lines consistent with Equation 23.9, the expression we obtained for E using Coulomb's law? To answer this question, consider an imaginary spherical surface of radius r concentric with a point charge. From symmetry, we see that the magnitude of the electric field is the same everywhere on the surface of the sphere. The number of lines N that emerge from the charge is equal to the number that penetrate the spherical surface. Hence, the number of lines per unit area on the sphere is $N/4\pi r^2$ (where the surface area of the sphere is $4\pi r^2$). Because E is proportional to the number of lines per unit area, we see that E varies as $1/r^2$; this finding is consistent with Equation 23.9.

Representative electric field lines for the field due to a single positive point charge are shown in Figure 23.19a. This two-dimensional drawing shows only the field lines that lie in the plane containing the point charge. The lines are actually directed radially outward from the charge in all directions; therefore, instead of the flat "wheel" of lines shown, you should picture an entire spherical distribution of lines. Because a positive charge placed in this field would be repelled by the positive source charge, the lines are directed radially away from the source charge. The electric field lines representing the field due to a single negative point charge are directed toward the charge (Fig. 23.19b). In either case, the lines are along the radial direction and extend all the way to infinity. Notice that the lines become closer together as they approach the charge, indicating that the strength of the field increases as we move toward the source charge.

The rules for drawing electric field lines are as follows:

- The lines must begin on a positive charge and terminate on a negative charge. In the case of an excess of one type of charge, some lines will begin or end infinitely far away.

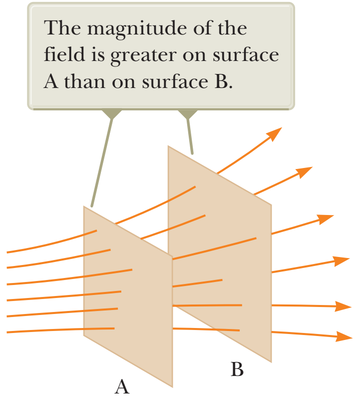


Figure 23.18 Electric field lines penetrating two surfaces.

Pitfall Prevention 23.2
Electric Field Lines Are Not Paths of Particles! Electric field lines represent the field at various locations. Except in very special cases, they *do not* represent the path of a charged particle moving in an electric field.

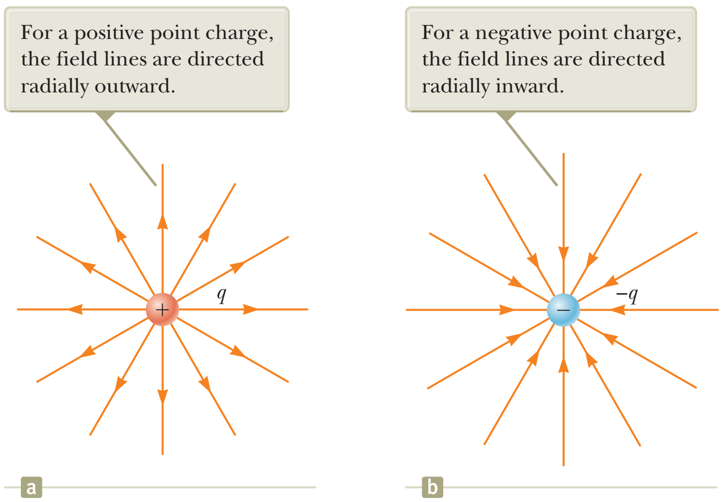


Figure 23.19 The electric field lines for a point charge. Notice that the figures show only those field lines that lie in the plane of the page.

Pitfall Prevention 23.3

Electric Field Lines Are Not Real Electric field lines are not material objects. They are used only as a pictorial representation to provide a qualitative description of the electric field. Only a finite number of lines from each charge can be drawn, which makes it appear as if the field were quantized and exists only in certain parts of space. The field, in fact, is continuous, existing at every point. You should avoid obtaining the wrong impression from a two-dimensional drawing of field lines used to describe a three-dimensional situation.

- The number of lines drawn leaving a positive charge or approaching a negative charge is proportional to the magnitude of the charge.
- No two field lines can cross.

We choose the number of field lines starting from any object with a positive charge q_+ to be Cq_+ and the number of lines ending on any object with a negative charge q_- to be $C|q_-|$, where C is an arbitrary proportionality constant. Once C is chosen, the number of lines is fixed. For example, in a two-charge system, if object 1 has charge Q_1 and object 2 has charge Q_2, the ratio of number of lines in contact with the charges is $N_2/N_1 = |Q_2/Q_1|$. The electric field lines for two point charges of equal magnitude but opposite signs (an electric dipole) are shown in Figure 23.20. Because the charges are of equal magnitude, the number of lines that begin at the positive charge must equal the number that terminate at the negative charge. At points very near the charges, the lines are nearly radial, as for a single isolated charge. The high density of lines between the charges indicates a region of strong electric field.

Figure 23.21 shows the electric field lines in the vicinity of two equal positive point charges. Again, the lines are nearly radial at points close to either charge, and the same number of lines emerges from each charge because the charges are equal in magnitude. Because there are no negative charges available, the electric field lines end infinitely far away. At great distances from the charges, the field is approximately equal to that of a single point charge of magnitude $2q$.

Finally, in Figure 23.22, we sketch the electric field lines associated with a positive charge $+2q$ and a negative charge $-q$. In this case, the number of lines leaving $+2q$ is twice the number terminating at $-q$. Hence, only half the lines that leave the positive charge reach the negative charge. The remaining half terminate on a negative charge we assume to be at infinity. At distances much greater than the charge separation, the electric field lines are equivalent to those of a single charge $+q$.

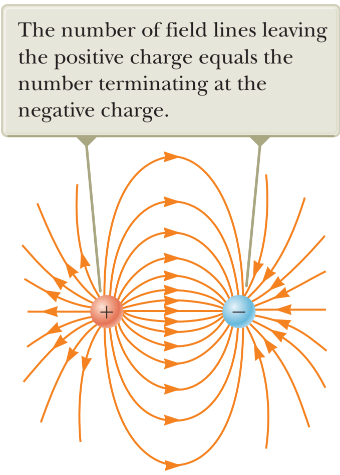


Figure 23.20 The electric field lines for two point charges of equal magnitude and opposite sign (an electric dipole).

Quick Quiz 23.5 Rank the magnitudes of the electric field at points A, B, and C shown in Figure 23.21 (greatest magnitude first).

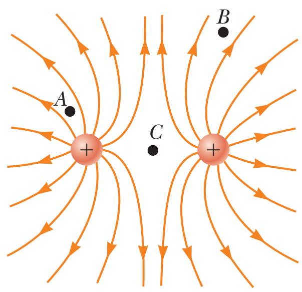


Figure 23.21 The electric field lines for two positive point charges. (The locations A, B, and C are discussed in Quick Quiz 23.5.)

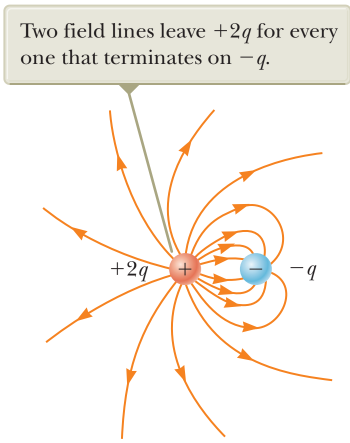


Figure 23.22 The electric field lines for a point charge $+2q$ and a second point charge $-q$.

23.7 Motion of a Charged Particle in a Uniform Electric Field

When a particle of charge q and mass m is placed in an electric field $\vec{\mathbf{E}}$, the electric force exerted on the charge is $q\vec{\mathbf{E}}$ according to Equation 23.8 in the particle in a field model. If that is the only force exerted on the particle, it must be the net force, and it causes the particle to accelerate according to the particle under a net force model. Therefore,

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = m\vec{\mathbf{a}}$$

and the acceleration of the particle is

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m} \quad \textbf{(23.12)}$$

If $\vec{\mathbf{E}}$ is uniform (that is, constant in magnitude and direction), and the particle is free to move, the electric force on the particle is constant and we can apply the particle under constant acceleration model to the motion of the particle. Therefore, the particle in this situation is described by *three* analysis models: particle in a field, particle under a net force, and particle under constant acceleration! If the particle has a positive charge, its acceleration is in the direction of the electric field. If the particle has a negative charge, its acceleration is in the direction opposite the electric field.

Pitfall Prevention 23.4

Just Another Force Electric forces and fields may seem abstract to you. Once $\vec{\mathbf{F}}_e$ is evaluated, however, it causes a particle to move according to our well-established models of forces and motion from Chapters 2 through 6. Keeping this link with the past in mind should help you solve problems in this chapter.

Example 23.10 An Accelerating Positive Charge: Two Models AM

A uniform electric field $\vec{\mathbf{E}}$ is directed along the x axis between parallel plates of charge separated by a distance d as shown in Figure 23.23. A positive point charge q of mass m is released from rest at a point Ⓐ next to the positive plate and accelerates to a point Ⓑ next to the negative plate.

(A) Find the speed of the particle at Ⓑ by modeling it as a particle under constant acceleration.

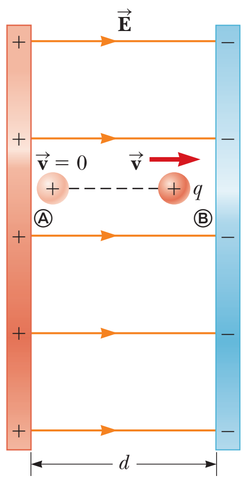


Figure 23.23 (Example 23.10) A positive point charge q in a uniform electric field $\vec{\mathbf{E}}$ undergoes constant acceleration in the direction of the field.

SOLUTION

Conceptualize When the positive charge is placed at Ⓐ, it experiences an electric force toward the right in Figure 23.23 due to the electric field directed toward the right. As a result, it will accelerate to the right and arrive at Ⓑ with some speed.

Categorize Because the electric field is uniform, a constant electric force acts on the charge. Therefore, as suggested in the discussion preceding the example and in the problem statement, the point charge can be modeled as a charged *particle under constant acceleration.*

Analyze Use Equation 2.17 to express the velocity of the particle as a function of position:

$$v_f^2 = v_i^2 + 2a(x_f - x_i) = 0 + 2a(d - 0) = 2ad$$

Solve for v_f and substitute for the magnitude of the acceleration from Equation 23.12:

$$v_f = \sqrt{2ad} = \sqrt{2\left(\frac{qE}{m}\right)d} = \boxed{\sqrt{\frac{2qEd}{m}}}$$

(B) Find the speed of the particle at Ⓑ by modeling it as a nonisolated system in terms of energy.

SOLUTION

Categorize The problem statement tells us that the charge is a *nonisolated system* for *energy*. The electric force, like any force, can do work on a system. Energy is transferred to the system of the charge by work done by the electric force exerted on the charge. The initial configuration of the system is when the particle is at rest at Ⓐ, and the final configuration is when it is moving with some speed at Ⓑ.

continued

23.10 continued

Analyze Write the appropriate reduction of the conservation of energy equation, Equation 8.2, for the system of the charged particle:

$$W = \Delta K$$

Replace the work and kinetic energies with values appropriate for this situation:

$$F_e\,\Delta x = K_{Ⓑ} - K_{Ⓐ} = \tfrac{1}{2}mv_f^{\,2} - 0 \;\rightarrow\; v_f = \sqrt{\frac{2F_e\,\Delta x}{m}}$$

Substitute for the magnitude of the electric force F_e from the particle in a field model and the displacement Δx:

$$v_f = \sqrt{\frac{2(qE)(d)}{m}} = \boxed{\sqrt{\frac{2qEd}{m}}}$$

Finalize The answer to part (B) is the same as that for part (A), as we expect. This problem can be solved with different approaches. We saw the same possibilities with mechanical problems.

Example 23.11 An Accelerated Electron AM

An electron enters the region of a uniform electric field as shown in Figure 23.24, with $v_i = 3.00 \times 10^6$ m/s and $E = 200$ N/C. The horizontal length of the plates is $\ell = 0.100$ m.

(A) Find the acceleration of the electron while it is in the electric field.

SOLUTION

Conceptualize This example differs from the preceding one because the velocity of the charged particle is initially perpendicular to the electric field lines. (In Example 23.10, the velocity of the charged particle is always parallel to the electric field lines.) As a result, the electron in this example follows a curved path as shown in Figure 23.24. The motion of the electron is the same as that of a massive particle projected horizontally in a gravitational field near the surface of the Earth.

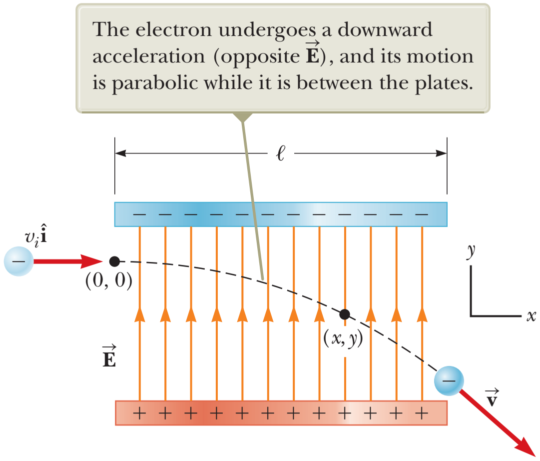


Figure 23.24 (Example 23.11) An electron is projected horizontally into a uniform electric field produced by two charged plates.

Categorize The electron is a *particle in a field (electric)*. Because the electric field is uniform, a constant electric force is exerted on the electron. To find the acceleration of the electron, we can model it as a *particle under a net force*.

Analyze From the particle in a field model, we know that the direction of the electric force on the electron is downward in Figure 23.24, opposite the direction of the electric field lines. From the particle under a net force model, therefore, the acceleration of the electron is downward.

The particle under a net force model was used to develop Equation 23.12 in the case in which the electric force on a particle is the only force. Use this equation to evaluate the y component of the acceleration of the electron:

$$a_y = -\frac{eE}{m_e}$$

Substitute numerical values:

$$a_y = -\frac{(1.60 \times 10^{-19}\text{ C})(200\text{ N/C})}{9.11 \times 10^{-31}\text{ kg}} = \boxed{-3.51 \times 10^{13}\text{ m/s}^2}$$

(B) Assuming the electron enters the field at time $t = 0$, find the time at which it leaves the field.

SOLUTION

Categorize Because the electric force acts only in the vertical direction in Figure 23.24, the motion of the particle in the horizontal direction can be analyzed by modeling it as a *particle under constant velocity*.

23.11 continued

Analyze Solve Equation 2.7 for the time at which the electron arrives at the right edges of the plates:

$$x_f = x_i + v_x t \rightarrow t = \frac{x_f - x_i}{v_x}$$

Substitute numerical values:

$$t = \frac{\ell - 0}{v_x} = \frac{0.100 \text{ m}}{3.00 \times 10^6 \text{ m/s}} = \boxed{3.33 \times 10^{-8} \text{ s}}$$

(C) Assuming the vertical position of the electron as it enters the field is $y_i = 0$, what is its vertical position when it leaves the field?

SOLUTION

Categorize Because the electric force is constant in Figure 23.24, the motion of the particle in the vertical direction can be analyzed by modeling it as a *particle under constant acceleration*.

Analyze Use Equation 2.16 to describe the position of the particle at any time t:

$$y_f = y_i + v_{yi} t + \tfrac{1}{2} a_y t^2$$

Substitute numerical values:

$$y_f = 0 + 0 + \tfrac{1}{2}(-3.51 \times 10^{13} \text{ m/s}^2)(3.33 \times 10^{-8} \text{ s})^2$$

$$= -0.019\,5 \text{ m} = \boxed{-1.95 \text{ cm}}$$

Finalize If the electron enters just below the negative plate in Figure 23.24 and the separation between the plates is less than the value just calculated, the electron will strike the positive plate.

Notice that we have used *four* analysis models to describe the electron in the various parts of this problem. We have neglected the gravitational force acting on the electron, which represents a good approximation when dealing with atomic particles. For an electric field of 200 N/C, the ratio of the magnitude of the electric force eE to the magnitude of the gravitational force mg is on the order of 10^{12} for an electron and on the order of 10^9 for a proton.

Summary

Definitions

The **electric field** $\vec{\mathbf{E}}$ at some point in space is defined as the electric force $\vec{\mathbf{F}}_e$ that acts on a small positive test charge placed at that point divided by the magnitude q_0 of the test charge:

$$\vec{\mathbf{E}} \equiv \frac{\vec{\mathbf{F}}_e}{q_0} \quad \textbf{(23.7)}$$

Concepts and Principles

Electric charges have the following important properties:

- Charges of opposite sign attract one another, and charges of the same sign repel one another.
- The total charge in an isolated system is conserved.
- Charge is quantized.

Conductors are materials in which electrons move freely. **Insulators** are materials in which electrons do not move freely.

continued

Coulomb's law states that the electric force exerted by a point charge q_1 on a second point charge q_2 is

$$\vec{\mathbf{F}}_{12} = k_e \frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}_{12} \qquad \textbf{(23.6)}$$

where r is the distance between the two charges and $\hat{\mathbf{r}}_{12}$ is a unit vector directed from q_1 toward q_2. The constant k_e, which is called the **Coulomb constant,** has the value $k_e = 8.988 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2$.

At a distance r from a point charge q, the electric field due to the charge is

$$\vec{\mathbf{E}} = k_e \frac{q}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(23.9)}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the charge toward the point in question. The electric field is directed radially outward from a positive charge and radially inward toward a negative charge.

The electric field due to a group of point charges can be obtained by using the superposition principle. That is, the total electric field at some point equals the vector sum of the electric fields of all the charges:

$$\vec{\mathbf{E}} = k_e \sum_i \frac{q_i}{r_i^2}\,\hat{\mathbf{r}}_i \qquad \textbf{(23.10)}$$

The electric field at some point due to a continuous charge distribution is

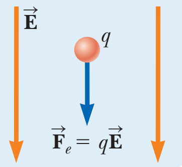
$$\vec{\mathbf{E}} = k_e \int \frac{dq}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(23.11)}$$

where dq is the charge on one element of the charge distribution and r is the distance from the element to the point in question.

Analysis Models for Problem Solving

Particle in a Field (Electric) A source particle with some electric charge establishes an **electric field $\vec{\mathbf{E}}$** throughout space. When a particle with charge q is placed in that field, it experiences an electric force given by

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} \qquad \textbf{(23.8)}$$



Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. A free electron and a free proton are released in identical electric fields. **(i)** How do the magnitudes of the electric force exerted on the two particles compare? (a) It is millions of times greater for the electron. (b) It is thousands of times greater for the electron. (c) They are equal. (d) It is thousands of times smaller for the electron. (e) It is millions of times smaller for the electron. **(ii)** Compare the magnitudes of their accelerations. Choose from the same possibilities as in part (i).

2. What prevents gravity from pulling you through the ground to the center of the Earth? Choose the best answer. (a) The density of matter is too great. (b) The positive nuclei of your body's atoms repel the positive nuclei of the atoms of the ground. (c) The density of the ground is greater than the density of your body. (d) Atoms are bound together by chemical bonds. (e) Electrons on the ground's surface and the surface of your feet repel one another.

3. A very small ball has a mass of 5.00×10^{-3} kg and a charge of 4.00 μC. What magnitude electric field directed upward will balance the weight of the ball so that the ball is suspended motionless above the ground? (a) 8.21×10^2 N/C (b) 1.22×10^4 N/C (c) 2.00×10^{-2} N/C (d) 5.11×10^6 N/C (e) 3.72×10^3 N/C

4. An electron with a speed of 3.00×10^6 m/s moves into a uniform electric field of magnitude 1.00×10^3 N/C. The field lines are parallel to the electron's velocity and pointing in the same direction as the velocity. How far does the electron travel before it is brought to rest? (a) 2.56 cm (b) 5.12 cm (c) 11.2 cm (d) 3.34 m (e) 4.24 m

5. A point charge of -4.00 nC is located at (0, 1.00) m. What is the x component of the electric field due to the point charge at (4.00, -2.00) m? (a) 1.15 N/C (b) -0.864 N/C (c) 1.44 N/C (d) -1.15 N/C (e) 0.864 N/C

6. A circular ring of charge with radius b has total charge q uniformly distributed around it. What is the magnitude of the electric field at the center of the ring? (a) 0 (b) $k_e q/b^2$ (c) $k_e q^2/b^2$ (d) $k_e q^2/b$ (e) none of those answers

7. What happens when a charged insulator is placed near an uncharged metallic object? (a) They repel each other. (b) They attract each other. (c) They may attract or repel each other, depending on whether the charge on the insulator is positive or negative. (d) They exert no electrostatic force on each other. (e) The charged insulator always spontaneously discharges.

8. Estimate the magnitude of the electric field due to the proton in a hydrogen atom at a distance of 5.29×10^{-11} m, the expected position of the electron in the atom. (a) 10^{-11} N/C (b) 10^8 N/C (c) 10^{14} N/C (d) 10^6 N/C (e) 10^{12} N/C

9. **(i)** A metallic coin is given a positive electric charge. Does its mass (a) increase measurably, (b) increase by an amount too small to measure directly, (c) remain unchanged, (d) decrease by an amount too small to measure directly, or (e) decrease measurably? **(ii)** Now the coin is given a negative electric charge. What happens to its mass? Choose from the same possibilities as in part (i).

10. Assume the charged objects in Figure OQ23.10 are fixed. Notice that there is no sight line from the location of q_2 to the location of q_1. If you were at q_1, you would be unable to see q_2 because it is behind q_3. How would you calculate the electric force exerted on the object with charge q_1? (a) Find only the force exerted by q_2 on charge q_1. (b) Find only the force exerted by q_3 on charge q_1. (c) Add the force that q_2 would exert by itself on charge q_1 to the force that q_3 would exert by itself on charge q_1. (d) Add the force that q_3 would exert by itself to a certain fraction of the force that q_2 would exert by itself. (e) There is no definite way to find the force on charge q_1.

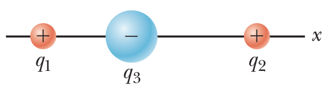


Figure OQ23.10

11. Three charged particles are arranged on corners of a square as shown in Figure OQ23.11, with charge $-Q$ on both the particle at the upper left corner and the particle at the lower right corner and with charge $+2Q$ on the particle at the lower left corner. **(i)** What is the direction of the electric field at the upper right corner, which is a point in empty space? (a) It is upward and to the right. (b) It is straight to the right. (c) It is straight downward. (d) It is downward and to the left. (e) It is perpendicular to the plane of the picture and outward. **(ii)** Suppose the $+2Q$ charge at the lower left corner is removed. Then does the magnitude of the field at the upper right corner (a) become larger, (b) become smaller, (c) stay the same, or (d) change unpredictably?

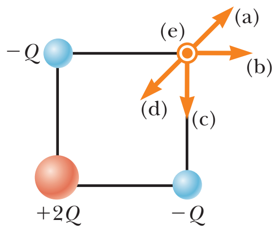


Figure OQ23.11

12. Two point charges attract each other with an electric force of magnitude F. If the charge on one of the particles is reduced to one-third its original value and the distance between the particles is doubled, what is the resulting magnitude of the electric force between them? (a) $\frac{1}{12}F$ (b) $\frac{1}{3}F$ (c) $\frac{1}{6}F$ (d) $\frac{3}{4}F$ (e) $\frac{3}{2}F$

13. Assume a uniformly charged ring of radius R and charge Q produces an electric field E_{ring} at a point P on its axis, at distance x away from the center of the ring as in Figure OQ23.13a. Now the same charge Q is spread uniformly over the circular area the ring encloses, forming a flat disk of charge with the same radius as in Figure OQ23.13b. How does the field E_{disk} produced by the disk at P compare with the field produced by the ring at the same point? (a) $E_{\text{disk}} < E_{\text{ring}}$ (b) $E_{\text{disk}} = E_{\text{ring}}$ (c) $E_{\text{disk}} > E_{\text{ring}}$ (d) impossible to determine

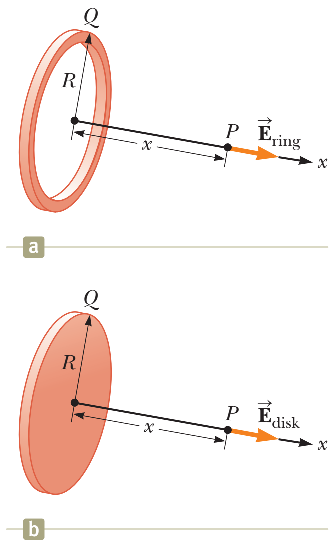


Figure OQ23.13

14. An object with negative charge is placed in a region of space where the electric field is directed vertically upward. What is the direction of the electric force exerted on this charge? (a) It is up. (b) It is down. (c) There is no force. (d) The force can be in any direction.

15. The magnitude of the electric force between two protons is 2.30×10^{-26} N. How far apart are they? (a) 0.100 m (b) 0.022 0 m (c) 3.10 m (d) 0.005 70 m (e) 0.480 m

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

$\boxed{1.}$ (a) Would life be different if the electron were positively charged and the proton were negatively charged? (b) Does the choice of signs have any bearing on physical and chemical interactions? Explain your answers.

2. A charged comb often attracts small bits of dry paper that then fly away when they touch the comb. Explain why that occurs.

3. A person is placed in a large, hollow, metallic sphere that is insulated from ground. If a large charge is placed on the sphere, will the person be harmed upon touching the inside of the sphere?

4. A student who grew up in a tropical country and is studying in the United States may have no experience with static electricity sparks and shocks until his or her first American winter. Explain.

5. If a suspended object A is attracted to a charged object B, can we conclude that A is charged? Explain.

6. Consider point *A* in Figure CQ23.6 located an arbitrary distance from two positive point charges in otherwise empty space. (a) Is it possible for an electric field to exist at point *A* in empty space? Explain. (b) Does charge exist at this point? Explain. (c) Does a force exist at this point? Explain.

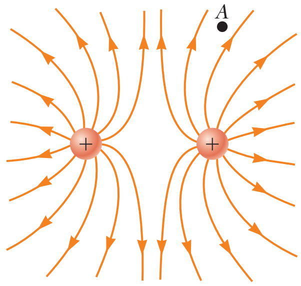


Figure CQ23.6

7. In fair weather, there is an electric field at the surface of the Earth, pointing down into the ground. What is the sign of the electric charge on the ground in this situation?

8. Why must hospital personnel wear special conducting shoes while working around oxygen in an operating room? What might happen if the personnel wore shoes with rubber soles?

9. A balloon clings to a wall after it is negatively charged by rubbing. (a) Does that occur because the wall is positively charged? (b) Why does the balloon eventually fall?

10. Consider two electric dipoles in empty space. Each dipole has zero net charge. (a) Does an electric force exist between the dipoles; that is, can two objects with zero net charge exert electric forces on each other? (b) If so, is the force one of attraction or of repulsion?

11. A glass object receives a positive charge by rubbing it with a silk cloth. In the rubbing process, have protons been added to the object or have electrons been removed from it?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 23.1 Properties of Electric Charges

1. Find to three significant digits the charge and the mass of the following particles. *Suggestion:* Begin by looking up the mass of a neutral atom on the periodic table of the elements in Appendix C. (a) an ionized hydrogen atom, represented as H^+ (b) a singly ionized sodium atom, Na^+ (c) a chloride ion Cl^- (d) a doubly ionized calcium atom, $Ca^{++} = Ca^{2+}$ (e) the center of an ammonia molecule, modeled as an N^{3-} ion (f) quadruply ionized nitrogen atoms, N^{4+}, found in plasma in a hot star (g) the nucleus of a nitrogen atom (h) the molecular ion H_2O^-

2. **W** (a) Calculate the number of electrons in a small, electrically neutral silver pin that has a mass of 10.0 g. Silver has 47 electrons per atom, and its molar mass is 107.87 g/mol. (b) Imagine adding electrons to the pin until the negative charge has the very large value 1.00 mC. How many electrons are added for every 10^9 electrons already present?

Section 23.2 Charging Objects by Induction
Section 23.3 Coulomb's Law

3. Two protons in an atomic nucleus are typically separated by a distance of 2×10^{-15} m. The electric repulsive force between the protons is huge, but the attractive nuclear force is even stronger and keeps the nucleus from bursting apart. What is the magnitude of the electric force between two protons separated by 2.00×10^{-15} m?

4. A charged particle *A* exerts a force of 2.62 μN to the right on charged particle *B* when the particles are 13.7 mm apart. Particle *B* moves straight away from *A* to make the distance between them 17.7 mm. What vector force does it then exert on *A*?

5. In a thundercloud, there may be electric charges of +40.0 C near the top of the cloud and −40.0 C near the bottom of the cloud. These charges are separated by 2.00 km. What is the electric force on the top charge?

6. (a) Find the magnitude of the electric force between a Na^+ ion and a Cl^- ion separated by 0.50 nm. (b) Would the answer change if the sodium ion were replaced by Li^+ and the chloride ion by Br^-? Explain.

7. **Review.** A molecule of DNA (deoxyribonucleic acid) is 2.17 μm long. The ends of the molecule become singly ionized: negative on one end, positive on the other. The helical molecule acts like a spring and compresses 1.00% upon becoming charged. Determine the effective spring constant of the molecule.

8. Nobel laureate Richard Feynman (1918–1988) once said that if two persons stood at arm's length from each other and each person had 1% more electrons than protons, the force of repulsion between them would be enough to lift a "weight" equal to that of the entire Earth. Carry out an order-of-magnitude calculation to substantiate this assertion.

9. A 7.50-nC point charge is located 1.80 m from a 4.20-nC point charge. (a) Find the magnitude of the

electric force that one particle exerts on the other. (b) Is the force attractive or repulsive?

10. W (a) Two protons in a molecule are 3.80×10^{-10} m apart. Find the magnitude of the electric force exerted by one proton on the other. (b) State how the magnitude of this force compares with the magnitude of the gravitational force exerted by one proton on the other. (c) **What If?** What must be a particle's charge-to-mass ratio if the magnitude of the gravitational force between two of these particles is equal to the magnitude of electric force between them?

11. M Three point charges are arranged as shown in Figure P23.11. Find (a) the magnitude and (b) the direction of the electric force on the particle at the origin.

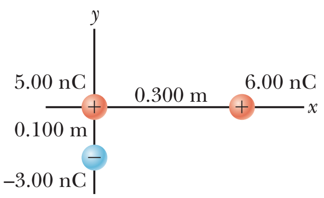


Figure P23.11 Problems 11 and 35.

12. Three point charges lie along a straight line as shown in Figure P23.12, where $q_1 = 6.00\ \mu$C, $q_2 = 1.50\ \mu$C, and $q_3 = -2.00\ \mu$C. The separation distances are $d_1 = 3.00$ cm and $d_2 = 2.00$ cm. Calculate the magnitude and direction of the net electric force on (a) q_1, (b) q_2, and (c) q_3.

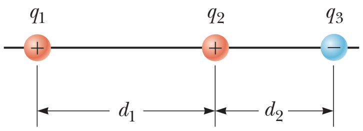


Figure P23.12

13. W Two small beads having positive charges $q_1 = 3q$ and $q_2 = q$ are fixed at the opposite ends of a horizontal insulating rod of length $d = 1.50$ m. The bead with charge q_1 is at the origin. As shown in Figure P23.13, a third small, charged bead is free to slide on the rod. (a) At what position x is the third bead in equilibrium? (b) Can the equilibrium be stable?

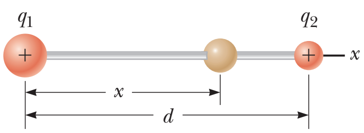


Figure P23.13 Problems 13 and 14.

14. Two small beads having charges q_1 and q_2 of the same sign are fixed at the opposite ends of a horizontal insulating rod of length d. The bead with charge q_1 is at the origin. As shown in Figure P23.13, a third small, charged bead is free to slide on the rod. (a) At what position x is the third bead in equilibrium? (b) Can the equilibrium be stable?

15. M Three charged particles are located at the corners of an equilateral triangle as shown in Figure P23.15. Calculate the total electric force on the 7.00-μC charge.

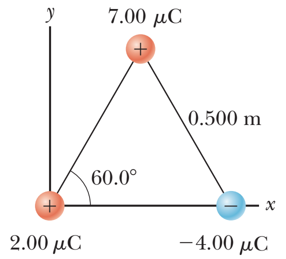


Figure P23.15 Problems 15 and 30.

16. Two small metallic spheres, each of mass $m = 0.200$ g, are suspended as pendulums by light strings of length L as shown in Figure P23.16. The spheres are given the same electric charge of 7.2 nC, and they come to equilibrium when each string is at an angle of $\theta = 5.00°$ with the vertical. How long are the strings?

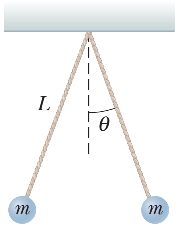


Figure P23.16

17. **Review.** In the Bohr theory of the hydrogen atom, an electron moves in a circular orbit about a proton, where the radius of the orbit is 5.29×10^{-11} m. (a) Find the magnitude of the electric force exerted on each particle. (b) If this force causes the centripetal acceleration of the electron, what is the speed of the electron?

18. GP Particle A of charge 3.00×10^{-4} C is at the origin, particle B of charge -6.00×10^{-4} C is at (4.00 m, 0), and particle C of charge 1.00×10^{-4} C is at (0, 3.00 m). We wish to find the net electric force on C. (a) What is the x component of the electric force exerted by A on C? (b) What is the y component of the force exerted by A on C? (c) Find the magnitude of the force exerted by B on C. (d) Calculate the x component of the force exerted by B on C. (e) Calculate the y component of the force exerted by B on C. (f) Sum the two x components from parts (a) and (d) to obtain the resultant x component of the electric force acting on C. (g) Similarly, find the y component of the resultant force vector acting on C. (h) Find the magnitude and direction of the resultant electric force acting on C.

19. A point charge $+2Q$ is at the origin and a point charge $-Q$ is located along the x axis at $x = d$ as in Figure P23.19. Find a symbolic expression for the net force on a third point charge $+Q$ located along the y axis at $y = d$.

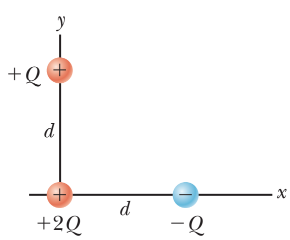


Figure P23.19

20. **Review.** Two identical particles, each having charge $+q$, are fixed in space and separated by a distance d. A third particle with charge $-Q$ is free to move and lies initially at rest on the perpendicular bisector of the two fixed charges a distance x from the midpoint between those charges (Fig. P23.20). (a) Show that if x is small compared with d, the motion of $-Q$ is simple harmonic along the perpendicular bisector. (b) Determine the period of that motion. (c) How fast will the charge $-Q$ be moving when it is at the midpoint between the two fixed charges if initially it is released at a distance $a << d$ from the midpoint?

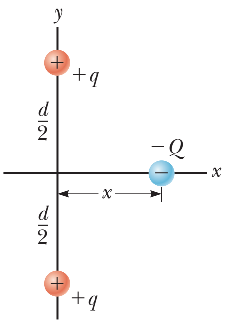


Figure P23.20

21. W Two identical conducting small spheres are placed with their centers 0.300 m apart. One is given a charge of 12.0 nC and the other a charge of −18.0 nC. (a) Find the electric force exerted by one sphere on the other. (b) **What If?** The spheres are connected by a conducting wire. Find the electric force each exerts on the other after they have come to equilibrium.

22. *Why is the following situation impossible?* Two identical dust particles of mass 1.00 μg are floating in empty space, far from any external sources of large gravitational or electric fields, and at rest with respect to each other. Both particles carry electric charges that are identical in magnitude and sign. The gravitational and electric forces between the particles happen to have the same magnitude, so each particle experiences zero net force and the distance between the particles remains constant.

Section 23.4 Analysis Model: Particle in a Field (Electric)

23. What are the magnitude and direction of the electric field that will balance the weight of (a) an electron and (b) a proton? (You may use the data in Table 23.1.)

24. A small object of mass 3.80 g and charge −18.0 μC is suspended motionless above the ground when immersed in a uniform electric field perpendicular to the ground. What are the magnitude and direction of the electric field?

25. Four charged particles are at the corners of a square of side a as shown in Figure P23.25. Determine (a) the electric field at the location of charge q and (b) the total electric force exerted on q.

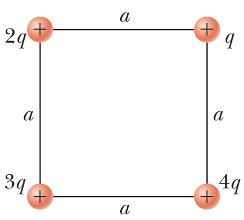


Figure P23.25

26. Three point charges lie along a circle of radius r at angles of 30°, 150°, and 270° as shown in Figure P23.26. Find a symbolic expression for the resultant electric field at the center of the circle.

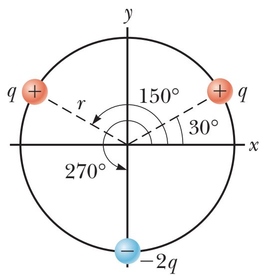


Figure P23.26

27. Two equal positively charged particles are at opposite corners of a trapezoid as shown in Figure P23.27. Find symbolic expressions for the total electric field at (a) the point P and (b) the point P'.

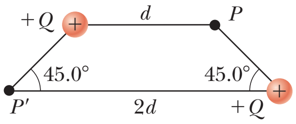


Figure P23.27

28. Consider n equal positively charged particles each of magnitude Q/n placed symmetrically around a circle of radius a. (a) Calculate the magnitude of the electric field at a point a distance x from the center of the circle and on the line passing through the center and perpendicular to the plane of the circle. (b) Explain why this result is identical to the result of the calculation done in Example 23.8.

29. M In Figure P23.29, determine the point (other than infinity) at which the electric field is zero.

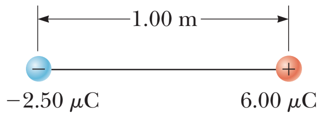


Figure P23.29

30. W Three charged particles are at the corners of an equilateral triangle as shown in Figure P23.15. (a) Calculate the electric field at the position of the 2.00-μC charge due to the 7.00-μC and −4.00-μC charges. (b) Use your answer to part (a) to determine the force on the 2.00-μC charge.

31. Three point charges are located on a circular arc as shown in Figure P23.31. (a) What is the total electric field at P, the center of the arc? (b) Find the electric force that would be exerted on a −5.00-nC point charge placed at P.

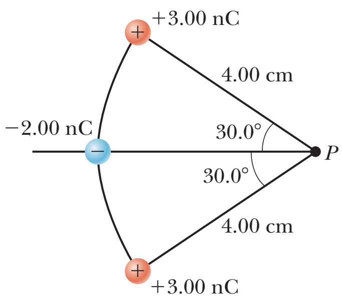


Figure P23.31

32. Two charged particles are located on the x axis. The first is a charge $+Q$ at $x = -a$. The second is an unknown charge located at $x = +3a$. The net electric field these charges produce at the origin has a magnitude of $2k_eQ/a^2$. Explain how many values are possible for the unknown charge and find the possible values.

33. AMT A small, 2.00-g plastic ball is suspended by a 20.0-cm-long string in a uniform electric field as shown in Figure P23.33. If the ball is in equilibrium when the string makes a 15.0° angle with the vertical, what is the net charge on the ball?

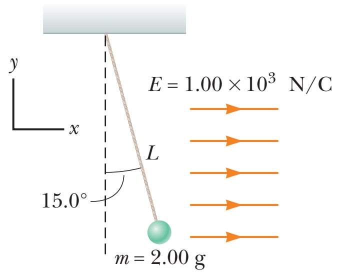


Figure P23.33

34. Two 2.00-μC point charges are located on the x axis. One is at $x = 1.00$ m, and the other is at $x = -1.00$ m. (a) Determine the electric field on the y axis at $y = 0.500$ m. (b) Calculate the electric force on a -3.00-μC charge placed on the y axis at $y = 0.500$ m.

35. Three point charges are arranged as shown in Figure P23.11. (a) Find the vector electric field that the 6.00-nC and -3.00-nC charges together create at the origin. (b) Find the vector force on the 5.00-nC charge.

36. Consider the electric dipole shown in Figure P23.36. Show that the electric field at a *distant* point on the $+x$ axis is $E_x \approx 4k_eqa/x^3$.

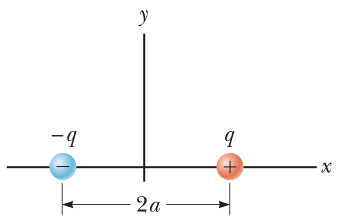


Figure P23.36

Section 23.5 Electric Field of a Continuous Charge Distribution

37. W A rod 14.0 cm long is uniformly charged and has a total charge of -22.0 μC. Determine (a) the magnitude and (b) the direction of the electric field along the axis of the rod at a point 36.0 cm from its center.

38. A uniformly charged disk of radius 35.0 cm carries charge with a density of 7.90×10^{-3} C/m^2. Calculate the electric field on the axis of the disk at (a) 5.00 cm, (b) 10.0 cm, (c) 50.0 cm, and (d) 200 cm from the center of the disk.

39. M A uniformly charged ring of radius 10.0 cm has a total charge of 75.0 μC. Find the electric field on the axis of the ring at (a) 1.00 cm, (b) 5.00 cm, (c) 30.0 cm, and (d) 100 cm from the center of the ring.

40. The electric field along the axis of a uniformly charged disk of radius R and total charge Q was calculated in Example 23.9. Show that the electric field at distances x that are large compared with R approaches that of a particle with charge $Q = \sigma\pi R^2$. *Suggestion:* First show that $x/(x^2 + R^2)^{1/2} = (1 + R^2/x^2)^{-1/2}$ and use the binomial expansion $(1 + \delta)^n \approx 1 + n\delta$, when $\delta << 1$.

41. Example 23.9 derives the exact expression for the electric field at a point on the axis of a uniformly charged disk. Consider a disk of radius $R = 3.00$ cm having a uniformly distributed charge of $+5.20$ μC. (a) Using the result of Example 23.9, compute the electric field at a point on the axis and 3.00 mm from the center. (b) **What If?** Explain how the answer to part (a) compares with the field computed from the near-field approximation $E = \sigma/2\epsilon_0$. (We derived this expression in Example 23.9.) (c) Using the result of Example 23.9, compute the electric field at a point on the axis and 30.0 cm from the center of the disk. (d) **What If?** Explain how the answer to part (c) compares with the electric field obtained by treating the disk as a $+5.20$-μC charged particle at a distance of 30.0 cm.

42. A uniformly charged rod of length L and total charge Q lies along the x axis as shown in Figure P23.42. (a) Find the components of the electric field at the point P on the y axis a distance d from the origin. (b) What are the approximate values of the field components when $d >> L$? Explain why you would expect these results.

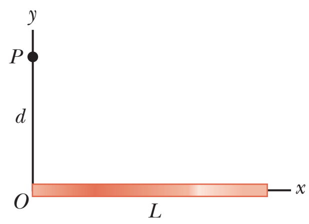


Figure P23.42

43. W A continuous line of charge lies along the x axis, extending from $x = +x_0$ to positive infinity. The line carries positive charge with a uniform linear charge density λ_0. What are (a) the magnitude and (b) the direction of the electric field at the origin?

44. A thin rod of length ℓ and uniform charge per unit length λ lies along the x axis as shown in Figure P23.44. (a) Show that the electric field at P, a distance d from the rod along its perpendicular bisector, has no x

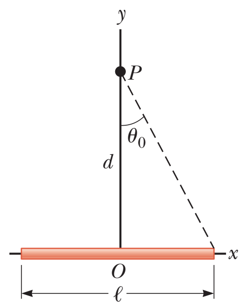


Figure P23.44

component and is given by $E = 2k_e\lambda \sin \theta_0/d$. (b) **What If?** Using your result to part (a), show that the field of a rod of infinite length is $E = 2k_e\lambda/d$.

45. M A uniformly charged insulating rod of length 14.0 cm is bent into the shape of a semicircle as shown in Figure P23.45. The rod has a total charge of $-7.50\ \mu$C. Find (a) the magnitude and (b) the direction of the electric field at *O*, the center of the semicircle.

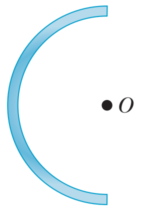


Figure P23.45

46. (a) Consider a uniformly charged, thin-walled, right circular cylindrical shell having total charge Q, radius R, and length ℓ. Determine the electric field at a point a distance d from the right side of the cylinder as shown in Figure P23.46. *Suggestion:* Use the result of Example 23.8 and treat the cylinder as a collection of ring charges. (b) **What If?** Consider now a solid cylinder with the same dimensions and carrying the same charge, uniformly distributed through its volume. Use the result of Example 23.9 to find the field it creates at the same point.

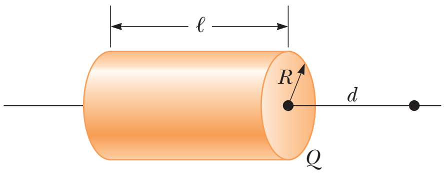


Figure P23.46

Section 23.6 Electric Field Lines

47. A negatively charged rod of finite length carries charge with a uniform charge per unit length. Sketch the electric field lines in a plane containing the rod.

48. A positively charged disk has a uniform charge per unit area σ as described in Example 23.9. Sketch the electric field lines in a plane perpendicular to the plane of the disk passing through its center.

49. W Figure P23.49 shows the electric field lines for two charged particles separated by a small distance. (a) Determine the ratio q_1/q_2. (b) What are the signs of q_1 and q_2?

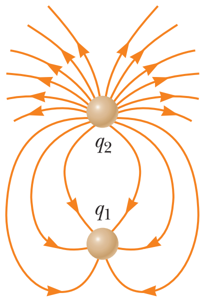


Figure P23.49

50. Three equal positive charges q are at the corners of an equilateral triangle of side a as shown in Figure P23.50. Assume the three charges together create an electric field. (a) Sketch the field lines in the plane of the charges. (b) Find the location of one point (other than ∞) where the electric field is zero. What are (c) the magnitude and (d) the direction of the electric field at P due to the two charges at the base?

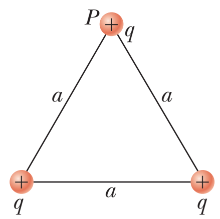


Figure P23.50

Section 23.7 Motion of a Charged Particle in a Uniform Electric Field

51. AMT M A proton accelerates from rest in a uniform electric field of 640 N/C. At one later moment, its speed is 1.20 Mm/s (nonrelativistic because v is much less than the speed of light). (a) Find the acceleration of the proton. (b) Over what time interval does the proton reach this speed? (c) How far does it move in this time interval? (d) What is its kinetic energy at the end of this interval?

52. W A proton is projected in the positive x direction into a region of a uniform electric field $\vec{\mathbf{E}} = (-6.00 \times 10^5)\,\hat{\mathbf{i}}$ N/C at $t = 0$. The proton travels 7.00 cm as it comes to rest. Determine (a) the acceleration of the proton, (b) its initial speed, and (c) the time interval over which the proton comes to rest.

53. An electron and a proton are each placed at rest in a uniform electric field of magnitude 520 N/C. Calculate the speed of each particle 48.0 ns after being released.

54. GP Protons are projected with an initial speed $v_i = 9.55$ km/s from a field-free region through a plane and into a region where a uniform electric field $\vec{\mathbf{E}} = -720\hat{\mathbf{j}}$ N/C is present above the plane as shown in Figure P23.54. The initial velocity vector of the protons makes an angle θ with the plane. The protons are to hit a target that lies at a horizontal distance of $R = 1.27$ mm from the point where the protons cross the plane and enter the electric field. We wish to find the angle θ at which the protons must pass through the plane to strike the target. (a) What analysis model describes the horizontal motion of the protons above the plane? (b) What analysis model describes the vertical motion of the protons above the plane? (c) Argue that Equation 4.13 would be applicable to the protons in this situation. (d) Use Equation 4.13 to write an expression for R in terms of v_i, E, the charge and mass of the proton, and the angle θ. (e) Find the two possible values of the angle θ. (f) Find the time interval during which the proton is above the plane in Figure P23.54 for each of the two possible values of θ.

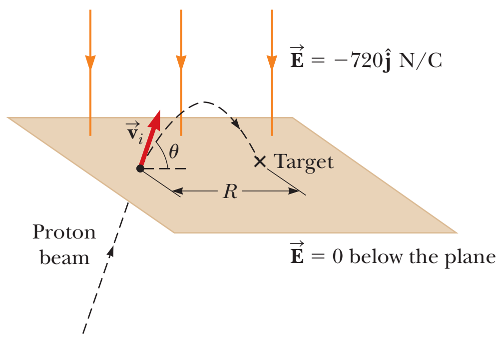


Figure P23.54

55. The electrons in a particle beam each have a kinetic energy K. What are (a) the magnitude and (b) the direction of the electric field that will stop these electrons in a distance d?

56. Two horizontal metal plates, each 10.0 cm square, are aligned 1.00 cm apart with one above the other. They are given equal-magnitude charges of opposite sign so that a uniform downward electric field of 2.00×10^3 N/C exists in the region between them. A particle of mass 2.00×10^{-16} kg and with a positive charge of 1.00×10^{-6} C leaves the center of the bottom negative plate with an initial speed of 1.00×10^5 m/s at an angle of 37.0° above the horizontal. (a) Describe the trajectory of the particle. (b) Which plate does it strike? (c) Where does it strike, relative to its starting point?

57. M A proton moves at 4.50×10^5 m/s in the horizontal direction. It enters a uniform vertical electric field with a magnitude of 9.60×10^3 N/C. Ignoring any gravitational effects, find (a) the time interval required for the proton to travel 5.00 cm horizontally, (b) its vertical displacement during the time interval in which it travels 5.00 cm horizontally, and (c) the horizontal and vertical components of its velocity after it has traveled 5.00 cm horizontally.

Additional Problems

58. Three solid plastic cylinders all have radius 2.50 cm and length 6.00 cm. Find the charge of each cylinder given the following additional information about each one. Cylinder (a) carries charge with uniform density 15.0 nC/m² everywhere on its surface. Cylinder (b) carries charge with uniform density 15.0 nC/m² on its curved lateral surface only. Cylinder (c) carries charge with uniform density 500 nC/m³ throughout the plastic.

59. Consider an infinite number of identical particles, each with charge q, placed along the x axis at distances a, $2a$, $3a$, $4a$, . . . from the origin. What is the electric field at the origin due to this distribution? *Suggestion:* Use

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

60. A particle with charge −3.00 nC is at the origin, and a particle with negative charge of magnitude Q is at $x = 50.0$ cm. A third particle with a positive charge is in equilibrium at $x = 20.9$ cm. What is Q?

61. AMT A small block of mass m and charge Q is placed on an insulated, frictionless, inclined plane of angle θ as in Figure P23.61. An electric field is applied parallel to the incline. (a) Find an expression for the magnitude of the electric field that enables the block to remain at rest. (b) If $m = 5.40$ g, $Q = -7.00\ \mu$C, and $\theta = 25.0°$, determine the magnitude and the direction of the electric field that enables the block to remain at rest on the incline.

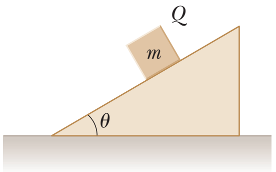


Figure P23.61

62. A small sphere of charge $q_1 = 0.800\ \mu$C hangs from the end of a spring as in Figure P23.62a. When another small sphere of charge $q_2 = -0.600\ \mu$C is held beneath the first sphere as in Figure P23.62b, the spring stretches by $d = 3.50$ cm from its original length and reaches a new equilibrium position with a separation between the charges of $r = 5.00$ cm. What is the force constant of the spring?

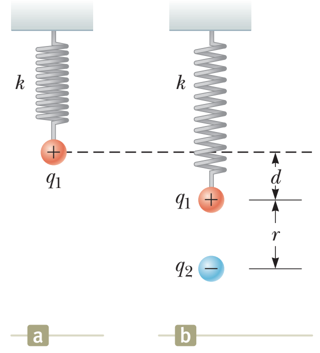


Figure P23.62

63. A line of charge starts at $x = +x_0$ and extends to positive infinity. The linear charge density is $\lambda = \lambda_0 x_0 / x$, where λ_0 is a constant. Determine the electric field at the origin.

64. A small sphere of mass $m = 7.50$ g and charge $q_1 = 32.0$ nC is attached to the end of a string and hangs vertically as in Figure P23.64. A second charge of equal mass and charge $q_2 = -58.0$ nC is located below the first charge a distance $d = 2.00$ cm below the first charge as in Figure P23.64. (a) Find the tension in the string. (b) If the string can withstand a maximum tension of 0.180 N, what is the smallest value d can have before the string breaks?

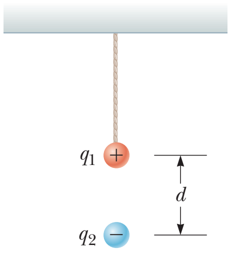


Figure P23.64

65. AMT A uniform electric field of magnitude 640 N/C exists between two parallel plates that are 4.00 cm apart. A proton is released from rest at the positive plate at the same instant an electron is released from rest at the negative plate. (a) Determine the distance from the positive plate at which the two pass each other. Ignore the electrical attraction between the proton and electron. (b) **What If?** Repeat part (a) for a sodium ion (Na^+) and a chloride ion (Cl^-).

66. Two small silver spheres, each with a mass of 10.0 g, are separated by 1.00 m. Calculate the fraction of the electrons in one sphere that must be transferred to the other to produce an attractive force of 1.00×10^4 N (about 1 ton) between the spheres. The number of electrons per atom of silver is 47.

67. M A charged cork ball of mass 1.00 g is suspended on a light string in the presence of a uniform electric field as shown in Figure P23.67. When $\vec{\mathbf{E}} = (3.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}}) \times 10^5$ N/C, the ball is in equilibrium at $\theta = 37.0°$. Find (a) the charge on the ball and (b) the tension in the string.

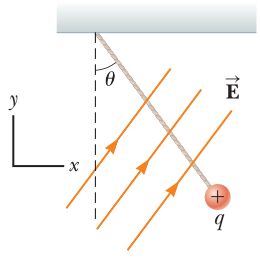


Figure P23.67 Problems 67 and 68.

68. A charged cork ball of mass m is suspended on a light string in the presence of a uniform electric field as shown in Figure P23.67. When $\vec{\mathbf{E}} = A\hat{\mathbf{i}} + B\hat{\mathbf{j}}$, where A and B are positive quantities, the ball is in equilibrium at the angle θ. Find (a) the charge on the ball and (b) the tension in the string.

69. Three charged particles are aligned along the x axis as shown in Figure P23.69. Find the electric field at (a) the position (2.00 m, 0) and (b) the position (0, 2.00 m).

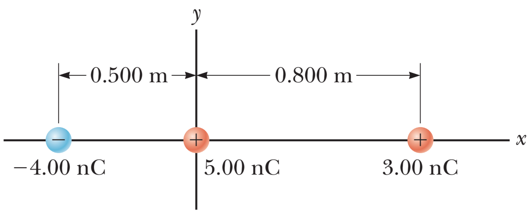


Figure P23.69

70. Two point charges $q_A = -12.0\ \mu$C and $q_B = 45.0\ \mu$C and a third particle with unknown charge q_C are located on the x axis. The particle q_A is at the origin, and q_B is at $x = 15.0$ cm. The third particle is to be placed so that each particle is in equilibrium under the action of the electric forces exerted by the other two particles. (a) Is this situation possible? If so, is it possible in more than one way? Explain. Find (b) the required location and (c) the magnitude and the sign of the charge of the third particle.

71. A line of positive charge is formed into a semicircle of radius $R = 60.0$ cm as shown in Figure P23.71. The charge per unit length along the semicircle is described by the expression $\lambda = \lambda_0 \cos\theta$. The total charge on the semicircle is 12.0 μC. Calculate the total force on a charge of 3.00 μC placed at the center of curvature P.

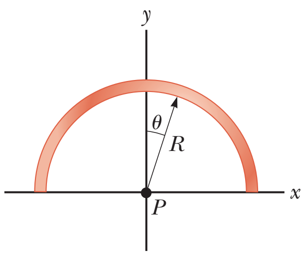


Figure P23.71

72. Four identical charged particles ($q = +10.0\ \mu$C) are located on the corners of a rectangle as shown in Figure P23.72. The dimensions of the rectangle are $L = 60.0$ cm and $W = 15.0$ cm. Calculate (a) the magnitude and (b) the direction of the total electric force exerted on the charge at the lower left corner by the other three charges.

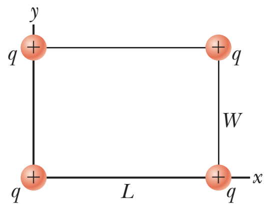


Figure P23.72

73. Two small spheres hang in equilibrium at the bottom ends of threads, 40.0 cm long, that have their top ends tied to the same fixed point. One sphere has mass 2.40 g and charge +300 nC. The other sphere has the same mass and charge +200 nC. Find the distance between the centers of the spheres.

74. *Why is the following situation impossible?* An electron enters a region of uniform electric field between two parallel plates. The plates are used in a cathode-ray tube to adjust the position of an electron beam on a distant fluorescent screen. The magnitude of the electric field between the plates is 200 N/C. The plates are 0.200 m in length and are separated by 1.50 cm. The electron enters the region at a speed of 3.00×10^6 m/s, traveling parallel to the plane of the plates in the direction of their length. It leaves the plates heading toward its correct location on the fluorescent screen.

75. **Review.** Two identical blocks resting on a frictionless, horizontal surface are connected by a light spring having a spring constant $k = 100$ N/m and an unstretched length $L_i = 0.400$ m as shown in Figure P23.75a. A charge Q is slowly placed on each block, causing the spring to stretch to an equilibrium length $L = 0.500$ m as shown in Figure P23.75b. Determine the value of Q, modeling the blocks as charged particles.

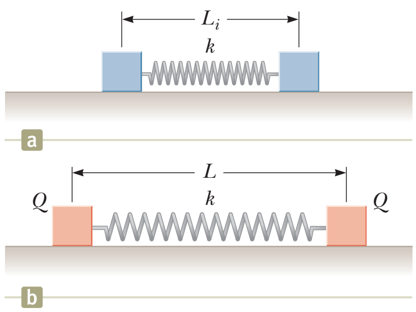


Figure P23.75 Problems 75 and 76.

76. **Review.** Two identical blocks resting on a frictionless, horizontal surface are connected by a light spring having a spring constant k and an unstretched length L_i as shown in Figure P23.75a. A charge Q is slowly placed on each block, causing the spring to stretch to an equilibrium length L as shown in Figure P23.75b. Determine the value of Q, modeling the blocks as charged particles.

77. Three identical point charges, each of mass $m = 0.100$ kg, hang from three strings as shown in Figure

P23.77. If the lengths of the left and right strings are each $L = 30.0$ cm and the angle θ is 45.0°, determine the value of q.

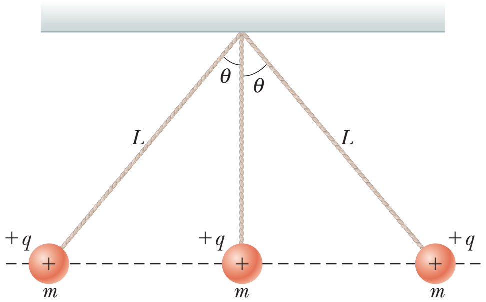


Figure P23.77

78. Show that the maximum magnitude $E_{\max}$ of the electric field along the axis of a uniformly charged ring occurs at $x = a/\sqrt{2}$ (see Fig. 23.16) and has the value $Q/(6\sqrt{3}\pi\epsilon_0 a^2)$.

79. Two hard rubber spheres, each of mass $m = 15.0$ g, are rubbed with fur on a dry day and are then suspended with two insulating strings of length $L = 5.00$ cm whose support points are a distance $d = 3.00$ cm from each other as shown in Figure P23.79. During the rubbing process, one sphere receives exactly twice the charge of the other. They are observed to hang at equilibrium, each at an angle of $\theta = 10.0°$ with the vertical. Find the amount of charge on each sphere.

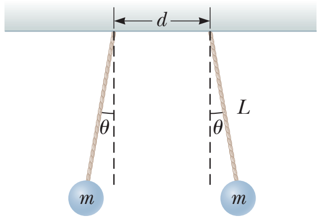


Figure P23.79

80. Two identical beads each have a mass m and charge q. When placed in a hemispherical bowl of radius R with frictionless, nonconducting walls, the beads move, and at equilibrium, they are a distance d apart (Fig. P23.80). (a) Determine the charge q on each bead. (b) Determine the charge required for d to become equal to $2R$.

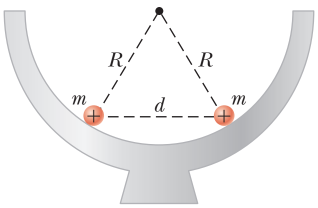


Figure P23.80

81. Two small spheres of mass m are suspended from strings of length ℓ that are connected at a common point. One sphere has charge Q and the other charge $2Q$. The strings make angles θ_1 and θ_2 with the vertical. (a) Explain how θ_1 and θ_2 are related. (b) Assume θ_1 and θ_2 are small. Show that the distance r between the spheres is approximately

$$r \approx \left(\frac{4k_e Q^2 \ell}{mg}\right)^{1/3}$$

82. **Review.** A negatively charged particle $-q$ is placed at the center of a uniformly charged ring, where the ring has a total positive charge Q as shown in Figure P23.82. The particle, confined to move along the x axis, is moved a small distance x along the axis (where $x << a$) and released. Show that the particle oscillates in simple harmonic motion with a frequency given by

$$f = \frac{1}{2\pi}\left(\frac{k_e q Q}{ma^3}\right)^{1/2}$$

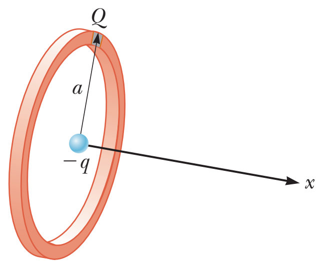


Figure P23.82

83. **Review.** A 1.00-g cork ball with charge 2.00 μC is suspended vertically on a 0.500-m-long light string in the presence of a uniform, downward-directed electric field of magnitude $E = 1.00 \times 10^5$ N/C. If the ball is displaced slightly from the vertical, it oscillates like a simple pendulum. (a) Determine the period of this oscillation. (b) Should the effect of gravitation be included in the calculation for part (a)? Explain.

Challenge Problems

84. Identical thin rods of length $2a$ carry equal charges $+Q$ uniformly distributed along their lengths. The rods lie along the x axis with their centers separated by a distance $b > 2a$ (Fig. P23.84). Show that the magnitude of the force exerted by the left rod on the right one is

$$F = \left(\frac{k_e Q^2}{4a^2}\right) \ln\left(\frac{b^2}{b^2 - 4a^2}\right)$$

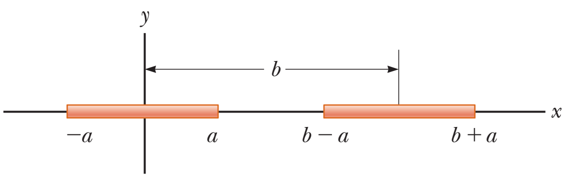


Figure P23.84

85. Eight charged particles, each of magnitude q, are located on the corners of a cube of edge s as shown in Figure P23.85 (page 724). (a) Determine the x, y, and z components of the total force exerted by the other charges on the charge located at point A. What are

(b) the magnitude and (c) the direction of this total force?

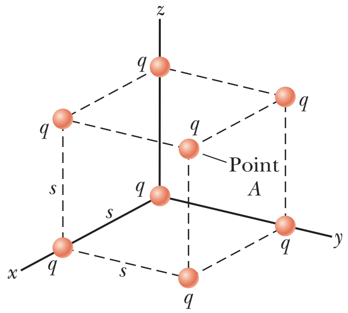


Figure P23.85 Problems 85 and 86.

86. Consider the charge distribution shown in Figure P23.85. (a) Show that the magnitude of the electric field at the center of any face of the cube has a value of $2.18k_eq/s^2$. (b) What is the direction of the electric field at the center of the top face of the cube?

87. Review. An electric dipole in a uniform horizontal electric field is displaced slightly from its equilibrium position as shown in Figure P23.87, where θ is small. The separation of the charges is $2a$, and each of the two particles has mass m. (a) Assuming the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{qE}{ma}}$$

What If? (b) Suppose the masses of the two charged particles in the dipole are not the same even though each particle continues to have charge q. Let the masses of the particles be m_1 and m_2. Show that the frequency of the oscillation in this case is

$$f = \frac{1}{2\pi}\sqrt{\frac{qE(m_1 + m_2)}{2am_1m_2}}$$

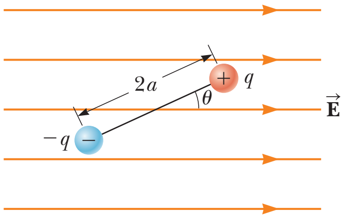


Figure P23.87

88. Inez is putting up decorations for her sister's quinceañera (fifteenth birthday party). She ties three light silk ribbons together to the top of a gateway and hangs a rubber balloon from each ribbon (Fig. P23.88). To include the effects of the gravitational and buoyant forces on it, each balloon can be modeled as a particle of mass 2.00 g, with its center 50.0 cm from the point of support. Inez rubs the whole surface of each balloon with her woolen scarf, making the balloons hang separately with gaps between them. Looking directly upward from below the balloons, Inez notices that the centers of the hanging balloons form a horizontal equilateral triangle with sides 30.0 cm long. What is the common charge each balloon carries?

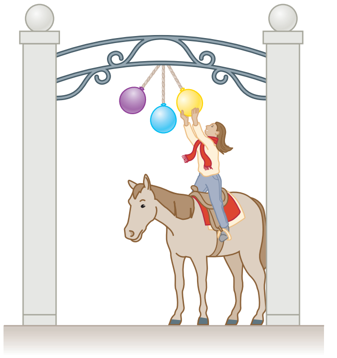


Figure P23.88

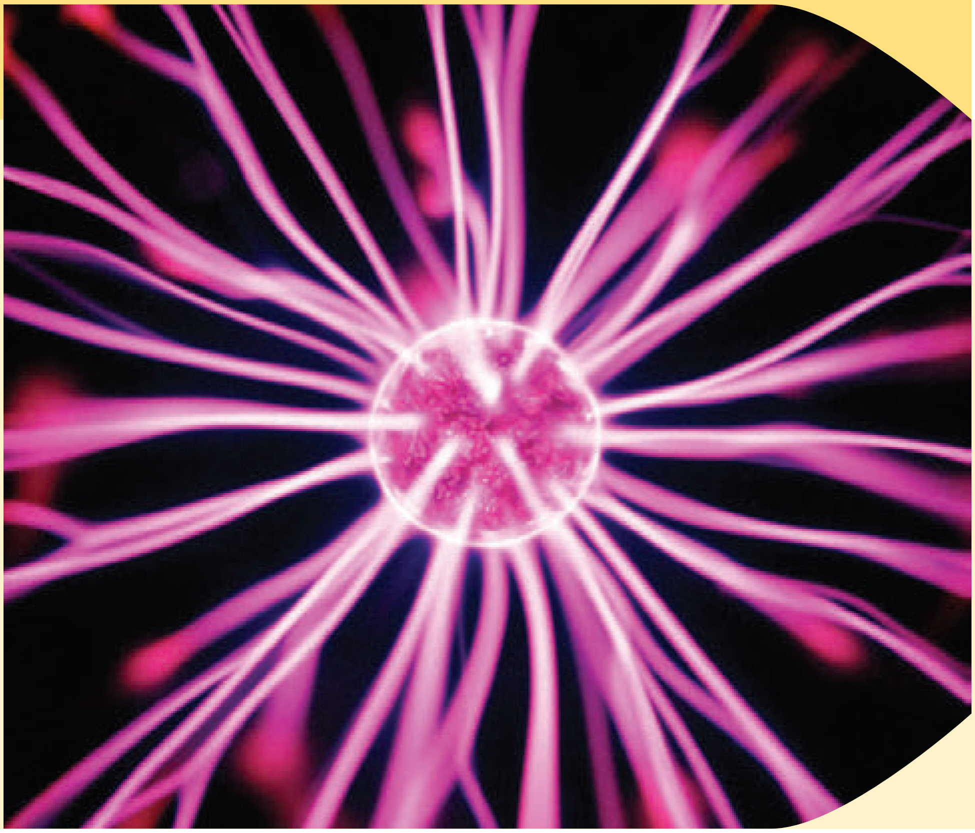
89. A line of charge with uniform density 35.0 nC/m lies along the line $y = -15.0$ cm between the points with coordinates $x = 0$ and $x = 40.0$ cm. Find the electric field it creates at the origin.

90. A particle of mass m and charge q moves at high speed along the x axis. It is initially near $x = -\infty$, and it ends up near $x = +\infty$. A second charge Q is fixed at the point $x = 0$, $y = -d$. As the moving charge passes the stationary charge, its x component of velocity does not change appreciably, but it acquires a small velocity in the y direction. Determine the angle through which the moving charge is deflected from the direction of its initial velocity.

91. Two particles, each with charge 52.0 nC, are located on the y axis at $y = 25.0$ cm and $y = -25.0$ cm. (a) Find the vector electric field at a point on the x axis as a function of x. (b) Find the field at $x = 36.0$ cm. (c) At what location is the field $1.00\hat{\mathbf{i}}$ kN/C? You may need a computer to solve this equation. (d) At what location is the field $16.0\hat{\mathbf{i}}$ kN/C?

CHAPTER

Gauss's Law



In a tabletop plasma ball, the colorful lines emanating from the sphere give evidence of strong electric fields. Using Gauss's law, we show in this chapter that the electric field surrounding a uniformly charged sphere is identical to that of a point charge. *(Steve Cole/Getty Images)*

In Chapter 23, we showed how to calculate the electric field due to a given charge distribution by integrating over the distribution. In this chapter, we describe *Gauss's law* and an alternative procedure for calculating electric fields. Gauss's law is based on the inverse-square behavior of the electric force between point charges. Although Gauss's law is a direct consequence of Coulomb's law, it is more convenient for calculating the electric fields of highly symmetric charge distributions and makes it possible to deal with complicated problems using qualitative reasoning. As we show in this chapter, Gauss's law is important in understanding and verifying the properties of conductors in electrostatic equilibrium.

4.1 Electric Flux

The concept of electric field lines was described qualitatively in Chapter 23. We now treat electric field lines in a more quantitative way.

Consider an electric field that is uniform in both magnitude and direction as shown in Figure 24.1. The field lines penetrate a rectangular surface of area whose plane is oriented perpendicular to the field. Recall from Section 23.6 that the number of lines per unit area (in other words, the *line density*) is proportional to the magnitude of the electric field. Therefore, the total number of lines penetrating the surface is proportional to the product EA. This product of the magnitude of the electric field and surface area perpendicular to the field is called the **electric flux** (uppercase Greek letter phi):

(24.1)

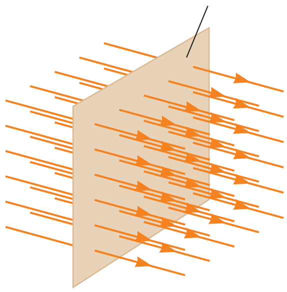


Figure 24.1 Field lines representing a uniform electric field penetrating a plane of area perpendicular to the field.

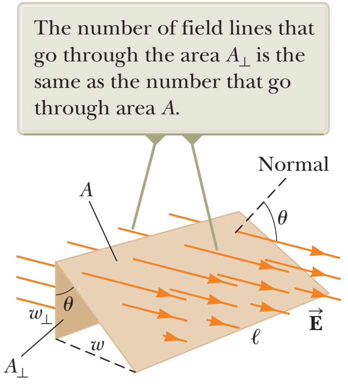


Figure 24.2 Field lines representing a uniform electric field penetrating an area A whose normal is at an angle θ to the field.

From the SI units of E and A, we see that Φ_E has units of newton meters squared per coulomb ($\text{N}\cdot\text{m}^2/\text{C}$). Electric flux is proportional to the number of electric field lines penetrating some surface.

If the surface under consideration is not perpendicular to the field, the flux through it must be less than that given by Equation 24.1. Consider Figure 24.2, where the normal to the surface of area A is at an angle θ to the uniform electric field. Notice that the number of lines that cross this area A is equal to the number of lines that cross the area $A_\perp$, which is a projection of area A onto a plane oriented perpendicular to the field. The area A is the product of the length and the width of the surface: $A = \ell w$. At the left edge of the figure, we see that the widths of the surfaces are related by $w_\perp = w\cos\theta$. The area $A_\perp$ is given by $A_\perp = \ell w_\perp = \ell w\cos\theta$ and we see that the two areas are related by $A_\perp = A\cos\theta$. Because the flux through A equals the flux through $A_\perp$, the flux through A is

$$\Phi_E = EA_\perp = EA\cos\theta \tag{24.2}$$

From this result, we see that the flux through a surface of fixed area A has a maximum value EA when the surface is perpendicular to the field (when the normal to the surface is parallel to the field, that is, when $\theta = 0°$ in Fig. 24.2); the flux is zero when the surface is parallel to the field (when the normal to the surface is perpendicular to the field, that is, when $\theta = 90°$).

In this discussion, the angle θ is used to describe the orientation of the surface of area A. We can also interpret the angle as that between the electric field vector and the normal to the surface. In this case, the product $E\cos\theta$ in Equation 24.2 is the component of the electric field perpendicular to the surface. The flux through the surface can then be written $\Phi_E = (E\cos\theta)A = E_nA$, where we use E_n as the component of the electric field normal to the surface.

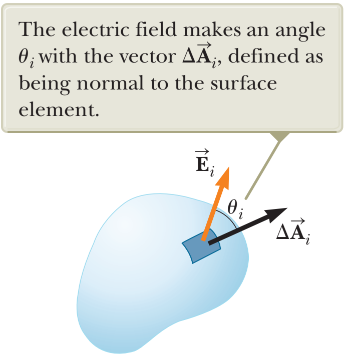


Figure 24.3 A small element of surface area ΔA_i in an electric field.

We assumed a uniform electric field in the preceding discussion. In more general situations, the electric field may vary over a large surface. Therefore, the definition of flux given by Equation 24.2 has meaning only for a small element of area over which the field is approximately constant. Consider a general surface divided into a large number of small elements, each of area ΔA_i. It is convenient to define a vector $\Delta\vec{\mathbf{A}}_i$ whose magnitude represents the area of the ith element of the large surface and whose direction is defined to be *perpendicular* to the surface element as shown in Figure 24.3. The electric field $\vec{\mathbf{E}}_i$ at the location of this element makes an angle θ_i with the vector $\Delta\vec{\mathbf{A}}_i$. The electric flux $\Phi_{E,i}$ through this element is

$$\Phi_{E,i} = E_i\,\Delta A_i\,\cos\theta_i = \vec{\mathbf{E}}_i\cdot\Delta\vec{\mathbf{A}}_i$$

where we have used the definition of the scalar product of two vectors ($\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} \equiv AB\cos\theta$; see Chapter 7). Summing the contributions of all elements gives an approximation to the total flux through the surface:

$$\Phi_E \approx \sum \vec{\mathbf{E}}_i\cdot\Delta\vec{\mathbf{A}}_i$$

If the area of each element approaches zero, the number of elements approaches infinity and the sum is replaced by an integral. Therefore, the general definition of electric flux is

Definition of electric flux ▶

$$\Phi_E \equiv \int_{\text{surface}} \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} \tag{24.3}$$

Equation 24.3 is a *surface integral,* which means it must be evaluated over the surface in question. In general, the value of Φ_E depends both on the field pattern and on the surface.

We are often interested in evaluating the flux through a *closed surface,* defined as a surface that divides space into an inside and an outside region so that one cannot move from one region to the other without crossing the surface. The surface of a sphere, for example, is a closed surface. By convention, if the area element in Equa-

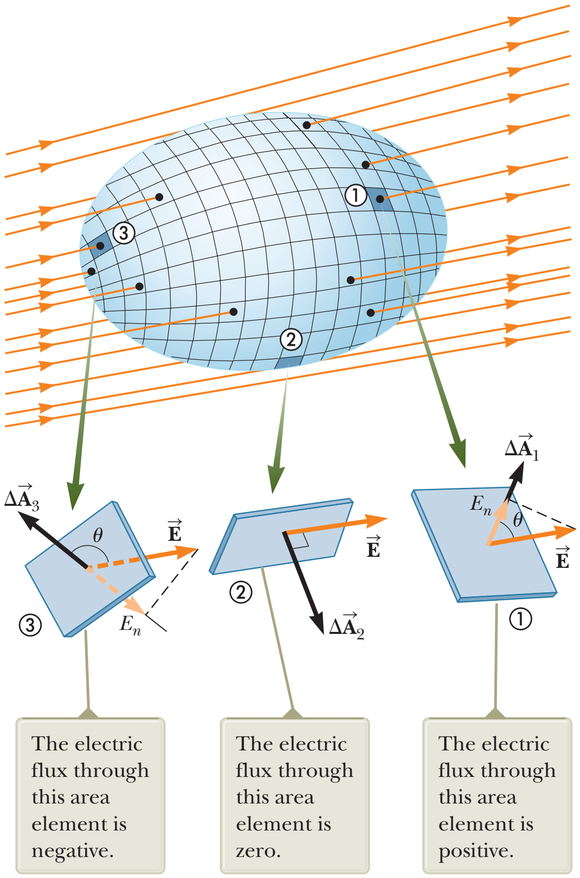


Figure 24.4 A closed surface in an electric field. The area vectors are, by convention, normal to the surface and point outward.

tion 24.3 is part of a closed surface, the direction of the area vector is chosen so that the vector points outward from the surface. If the area element is not part of a closed surface, the direction of the area vector is chosen so that the angle between the area vector and the electric field vector is less than or equal to 90°.

Consider the closed surface in Figure 24.4. The vectors $\Delta\vec{\mathbf{A}}_i$ point in different directions for the various surface elements, but for each element they are normal to the surface and point outward. At the element labeled ①, the field lines are crossing the surface from the inside to the outside and $\theta < 90°$; hence, the flux $\Phi_{E,1} = \vec{\mathbf{E}} \cdot \Delta\vec{\mathbf{A}}_1$ through this element is positive. For element ②, the field lines graze the surface (perpendicular to $\Delta\vec{\mathbf{A}}_2$); therefore, $\theta = 90°$ and the flux is zero. For elements such as ③, where the field lines are crossing the surface from outside to inside, $180° > \theta > 90°$ and the flux is negative because $\cos\theta$ is negative. The *net* flux through the surface is proportional to the net number of lines leaving the surface, where the net number means *the number of lines leaving the surface minus the number of lines entering the surface.* If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative. Using the symbol $\oint$ to represent an integral over a closed surface, we can write the net flux Φ_E through a closed surface as

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n \, dA \tag{24.4}$$

where E_n represents the component of the electric field normal to the surface.

Quick Quiz 24.1 Suppose a point charge is located at the center of a spherical surface. The electric field at the surface of the sphere and the total flux through the sphere are determined. Now the radius of the sphere is halved.

What happens to the flux through the sphere and the magnitude of the electric field at the surface of the sphere? **(a)** The flux and field both increase. **(b)** The flux and field both decrease. **(c)** The flux increases, and the field decreases. **(d)** The flux decreases, and the field increases. **(e)** The flux remains the same, and the field increases. **(f)** The flux decreases, and the field remains the same.

Example 24.1 Flux Through a Cube

Consider a uniform electric field $\vec{\mathbf{E}}$ oriented in the x direction in empty space. A cube of edge length ℓ is placed in the field, oriented as shown in Figure 24.5. Find the net electric flux through the surface of the cube.

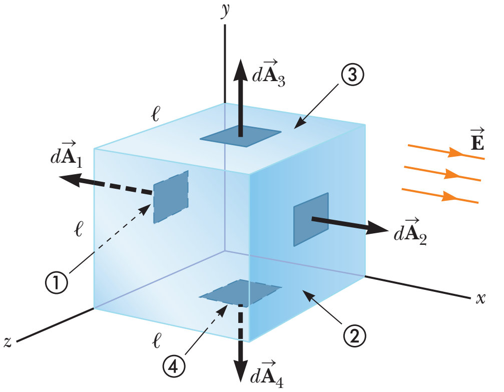


Figure 24.5 (Example 24.1) A closed surface in the shape of a cube in a uniform electric field oriented parallel to the x axis. Side ④ is the bottom of the cube, and side ① is opposite side ②.

SOLUTION

Conceptualize Examine Figure 24.5 carefully. Notice that the electric field lines pass through two faces perpendicularly and are parallel to four other faces of the cube.

Categorize We evaluate the flux from its definition, so we categorize this example as a substitution problem.

The flux through four of the faces (③, ④, and the unnumbered faces) is zero because $\vec{\mathbf{E}}$ is parallel to the four faces and therefore perpendicular to $d\vec{\mathbf{A}}$ on these faces.

Write the integrals for the net flux through faces ① and ②:

$$\Phi_E = \int_1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \int_2 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

For face ①, $\vec{\mathbf{E}}$ is constant and directed inward but $d\vec{\mathbf{A}}_1$ is directed outward ($\theta = 180°$). Find the flux through this face:

$$\int_1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_1 E(\cos 180°)\, dA = -E\int_1 dA = -EA = -E\ell^2$$

For face ②, $\vec{\mathbf{E}}$ is constant and outward and in the same direction as $d\vec{\mathbf{A}}_2$ ($\theta = 0°$). Find the flux through this face:

$$\int_2 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_2 E(\cos 0°)\, dA = E\int_2 dA = +EA = E\ell^2$$

Find the net flux by adding the flux over all six faces:

$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = \boxed{0}$$

24.2 Gauss's Law

In this section, we describe a general relationship between the net electric flux through a closed surface (often called a *gaussian surface*) and the charge enclosed by the surface. This relationship, known as *Gauss's law,* is of fundamental importance in the study of electric fields.

Consider a positive point charge q located at the center of a sphere of radius r as shown in Figure 24.6. From Equation 23.9, we know that the magnitude of the electric field everywhere on the surface of the sphere is $E = k_e q/r^2$. The field lines are directed radially outward and hence are perpendicular to the surface at every point on the surface. That is, at each surface point, $\vec{\mathbf{E}}$ is parallel to the vector $\Delta\vec{\mathbf{A}}_i$ representing a local element of area ΔA_i surrounding the surface point. Therefore,

$$\vec{\mathbf{E}} \cdot \Delta\vec{\mathbf{A}}_i = E\,\Delta A_i$$

and, from Equation 24.4, we find that the net flux through the gaussian surface is

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = E\oint dA$$

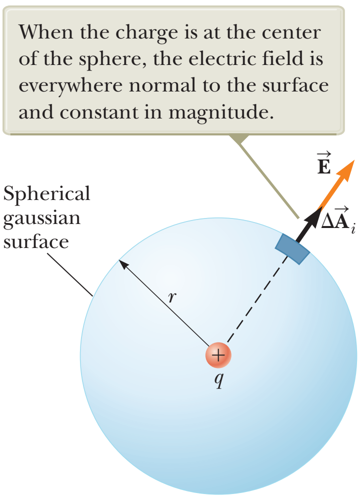


Figure 24.6 A spherical gaussian surface of radius r surrounding a positive point charge q.

where we have moved E outside of the integral because, by symmetry, E is constant over the surface. The value of E is given by $E = k_e q/r^2$. Furthermore, because the surface is spherical, $\oint dA = A = 4\pi r^2$. Hence, the net flux through the gaussian surface is

$$\Phi_E = k_e \frac{q}{r^2}(4\pi r^2) = 4\pi k_e q$$

Recalling from Equation 23.3 that $k_e = 1/4\pi\epsilon_0$, we can write this equation in the form

$$\Phi_E = \frac{q}{\epsilon_0} \quad \text{(24.5)}$$

Equation 24.5 shows that the net flux through the spherical surface is proportional to the charge inside the surface. The flux is independent of the radius r because the area of the spherical surface is proportional to r^2, whereas the electric field is proportional to $1/r^2$. Therefore, in the product of area and electric field, the dependence on r cancels.

Now consider several closed surfaces surrounding a charge q as shown in Figure 24.7. Surface S_1 is spherical, but surfaces S_2 and S_3 are not. From Equation 24.5, the flux that passes through S_1 has the value q/ϵ_0. As discussed in the preceding section, flux is proportional to the number of electric field lines passing through a surface. The construction shown in Figure 24.7 shows that the number of lines through S_1 is equal to the number of lines through the nonspherical surfaces S_2 and S_3. Therefore,

the net flux through *any* closed surface surrounding a point charge q is given by q/ϵ_0 and is independent of the shape of that surface.

Now consider a point charge located *outside* a closed surface of arbitrary shape as shown in Figure 24.8. As can be seen from this construction, any electric field line entering the surface leaves the surface at another point. The number of electric field lines entering the surface equals the number leaving the surface. Therefore, the net electric flux through a closed surface that surrounds no charge is zero. Applying this result to Example 24.1, we see that the net flux through the cube is zero because there is no charge inside the cube.

Let's extend these arguments to two generalized cases: (1) that of many point charges and (2) that of a continuous distribution of charge. We once again use the superposition principle, which states that the electric field due to many charges is

Karl Friedrich Gauss
German mathematician and astronomer (1777–1855)
Gauss received a doctoral degree in mathematics from the University of Helmstedt in 1799. In addition to his work in electromagnetism, he made contributions to mathematics and science in number theory, statistics, non-Euclidean geometry, and cometary orbital mechanics. He was a founder of the German Magnetic Union, which studies the Earth's magnetic field on a continual basis.

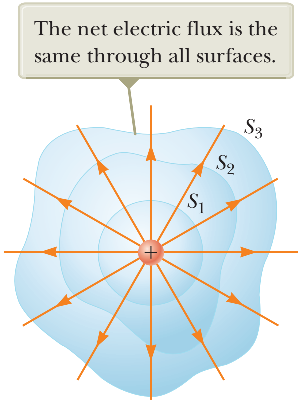


Figure 24.7 Closed surfaces of various shapes surrounding a positive charge.

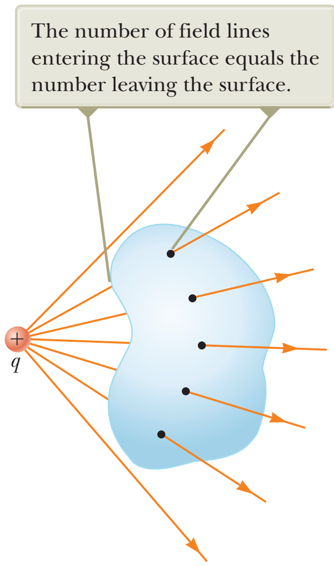


Figure 24.8 A point charge located *outside* a closed surface.

the vector sum of the electric fields produced by the individual charges. Therefore, the flux through any closed surface can be expressed as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint (\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \cdots) \cdot d\vec{\mathbf{A}}$$

where $\vec{\mathbf{E}}$ is the total electric field at any point on the surface produced by the vector addition of the electric fields at that point due to the individual charges. Consider the system of charges shown in Figure 24.9. The surface S surrounds only one charge, q_1; hence, the net flux through S is q_1/ϵ_0. The flux through S due to charges q_2, q_3, and q_4 outside it is zero because each electric field line from these charges that enters S at one point leaves it at another. The surface S' surrounds charges q_2 and q_3; hence, the net flux through it is $(q_2 + q_3)/\epsilon_0$. Finally, the net flux through surface S'' is zero because there is no charge inside this surface. That is, *all* the electric field lines that enter S'' at one point leave at another. Charge q_4 does not contribute to the net flux through any of the surfaces.

Charge q_4 does not contribute to the flux through any surface because it is outside all surfaces.

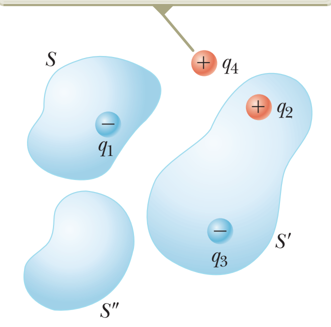


Figure 24.9 The net electric flux through any closed surface depends only on the charge *inside* that surface. The net flux through surface S is q_1/ϵ_0, the net flux through surface S' is $(q_2 + q_3)/\epsilon_0$, and the net flux through surface S'' is zero.

The mathematical form of **Gauss's law** is a generalization of what we have just described and states that the net flux through *any* closed surface is

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon_0} \tag{24.6}$$

where $\vec{\mathbf{E}}$ represents the electric field at any point on the surface and q_{in} represents the net charge inside the surface.

When using Equation 24.6, you should note that although the charge q_{in} is the net charge inside the gaussian surface, $\vec{\mathbf{E}}$ represents the *total electric field,* which includes contributions from charges both inside and outside the surface.

In principle, Gauss's law can be solved for $\vec{\mathbf{E}}$ to determine the electric field due to a system of charges or a continuous distribution of charge. In practice, however, this type of solution is applicable only in a limited number of highly symmetric situations. In the next section, we use Gauss's law to evaluate the electric field for charge distributions that have spherical, cylindrical, or planar symmetry. If one chooses the gaussian surface surrounding the charge distribution carefully, the integral in Equation 24.6 can be simplified and the electric field determined.

Pitfall Prevention 24.1

Zero Flux Is Not Zero Field

In two situations, there is zero flux through a closed surface: either (1) there are no charged particles enclosed by the surface or (2) there are charged particles enclosed, but the net charge inside the surface is zero. For either situation, it is *incorrect* to conclude that the electric field on the surface is zero. Gauss's law states that the electric *flux* is proportional to the enclosed charge, not the electric *field.*

Quick Quiz 24.2 If the net flux through a gaussian surface is *zero,* the following four statements *could be true.* Which of the statements *must be true?* **(a)** There are no charges inside the surface. **(b)** The net charge inside the surface is zero. **(c)** The electric field is zero everywhere on the surface. **(d)** The number of electric field lines entering the surface equals the number leaving the surface.

Conceptual Example 24.2 Flux Due to a Point Charge

A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if **(A)** the charge is tripled, **(B)** the radius of the sphere is doubled, **(C)** the surface is changed to a cube, and **(D)** the charge is moved to another location inside the surface.

SOLUTION

(A) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(B) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(C) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(D) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

24.3 Application of Gauss's Law to Various Charge Distributions

As mentioned earlier, Gauss's law is useful for determining electric fields when the charge distribution is highly symmetric. The following examples demonstrate ways of choosing the gaussian surface over which the surface integral given by Equation 24.6 can be simplified and the electric field determined. In choosing the surface, always take advantage of the symmetry of the charge distribution so that E can be removed from the integral. The goal in this type of calculation is to determine a surface for which each portion of the surface satisfies one or more of the following conditions:

Pitfall Prevention 24.2

Gaussian Surfaces Are Not Real

A gaussian surface is an imaginary surface you construct to satisfy the conditions listed here. It does not have to coincide with a physical surface in the situation.

1. The value of the electric field can be argued by symmetry to be constant over the portion of the surface.
2. The dot product in Equation 24.6 can be expressed as a simple algebraic product $E\,dA$ because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are parallel.
3. The dot product in Equation 24.6 is zero because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are perpendicular.
4. The electric field is zero over the portion of the surface.

Different portions of the gaussian surface can satisfy different conditions as long as every portion satisfies at least one condition. All four conditions are used in examples throughout the remainder of this chapter and will be identified by number. If the charge distribution does not have sufficient symmetry such that a gaussian surface that satisfies these conditions can be found, Gauss's law is still true, but is not useful for determining the electric field for that charge distribution.

Example 24.3 A Spherically Symmetric Charge Distribution

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 24.10).

(A) Calculate the magnitude of the electric field at a point outside the sphere.

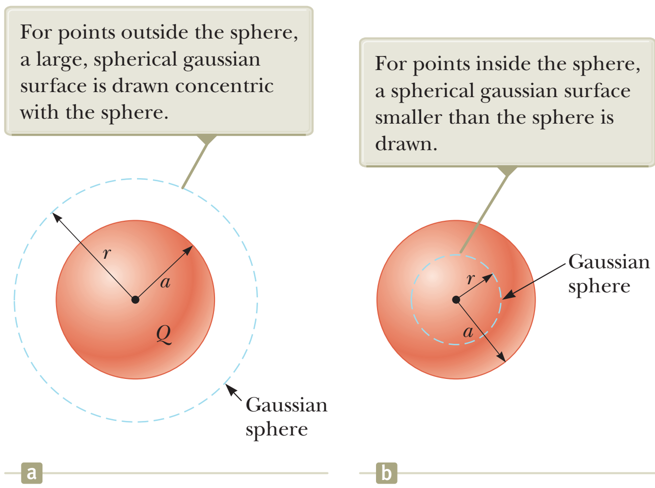


Figure 24.10 (Example 24.3) A uniformly charged insulating sphere of radius a and total charge Q. In diagrams such as this one, the dotted line represents the intersection of the gaussian surface with the plane of the page.

SOLUTION

Conceptualize Notice how this problem differs from our previous discussion of Gauss's law. The electric field due to point charges was discussed in Section 24.2. Now we are considering the electric field due to a distribution of charge. We found the field for various distributions of charge in Chapter 23 by integrating over the distribution. This example demonstrates a difference from our discussions in Chapter 23. In this chapter, we find the electric field using Gauss's law.

Categorize Because the charge is distributed uniformly throughout the sphere, the charge distribution has spherical symmetry and we can apply Gauss's law to find the electric field.

Analyze To reflect the spherical symmetry, let's choose a spherical gaussian surface of radius r, concentric with the sphere, as shown in Figure 24.10a. For this choice, condition (2) is satisfied everywhere on the surface and $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\,dA$.

continued

24.3 continued

Replace $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ in Gauss's law with $E\,dA$:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\,dA = \frac{Q}{\epsilon_0}$$

By symmetry, E has the same value everywhere on the surface, which satisfies condition (1), so we can remove E from the integral:

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

Solve for E:

$$(1) \quad E = \frac{Q}{4\pi\epsilon_0 r^2} = \boxed{k_e\frac{Q}{r^2} \quad (\text{for } r > a)}$$

Finalize This field is identical to that for a point charge. Therefore, **the electric field due to a uniformly charged sphere in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.**

(B) Find the magnitude of the electric field at a point inside the sphere.

SOLUTION

Analyze In this case, let's choose a spherical gaussian surface having radius $r < a$, concentric with the insulating sphere (Fig. 24.10b). Let V' be the volume of this smaller sphere. To apply Gauss's law in this situation, recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q.

Calculate q_{in} by using $q_{\text{in}} = \rho V'$:

$$q_{\text{in}} = \rho V' = \rho\left(\tfrac{4}{3}\pi r^3\right)$$

Notice that conditions (1) and (2) are satisfied everywhere on the gaussian surface in Figure 24.10b. Apply Gauss's law in the region $r < a$:

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

Solve for E and substitute for q_{in}:

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}r$$

Substitute $\rho = Q/\frac{4}{3}\pi a^3$ and $\epsilon_0 = 1/4\pi k_e$:

$$(2) \quad E = \frac{Q/\frac{4}{3}\pi a^3}{3(1/4\pi k_e)}r = \boxed{k_e\frac{Q}{a^3}r \quad (\text{for } r < a)}$$

Finalize This result for E differs from the one obtained in part (A). It shows that $E \rightarrow 0$ as $r \rightarrow 0$. Therefore, the result eliminates the problem that would exist at $r = 0$ if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if $E \propto 1/r^2$ for $r < a$, the field would be infinite at $r = 0$, which is physically impossible.

WHAT IF? Suppose the radial position $r = a$ is approached from inside the sphere and from outside. Do we obtain the same value of the electric field from both directions?

Answer Equation (1) shows that the electric field approaches a value from the outside given by

$$E = \lim_{r\rightarrow a}\left(k_e\frac{Q}{r^2}\right) = k_e\frac{Q}{a^2}$$

From the inside, Equation (2) gives

$$E = \lim_{r\rightarrow a}\left(k_e\frac{Q}{a^3}r\right) = k_e\frac{Q}{a^3}a = k_e\frac{Q}{a^2}$$

Therefore, the value of the field is the same as the surface is approached from both directions. A plot of E versus r is shown in Figure 24.11. Notice that the magnitude of the field is continuous.

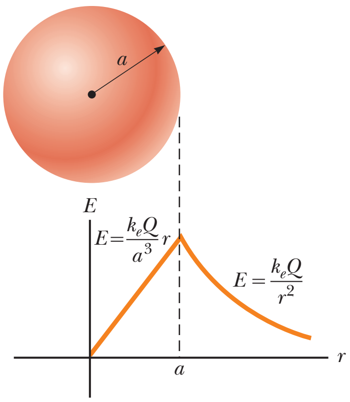


Figure 24.11 (Example 24.3) A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

Example 24.4 A Cylindrically Symmetric Charge Distribution

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (Fig. 24.12a).

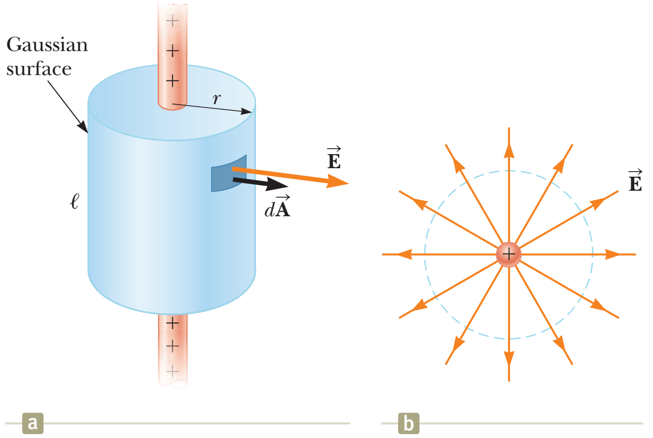


Figure 24.12 (Example 24.4) (a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

SOLUTION

Conceptualize The line of charge is *infinitely* long. Therefore, the field is the same at all points equidistant from the line, regardless of the vertical position of the point in Figure 24.12a. We expect the field to become weaker as we move farther away from the line of charge.

Categorize Because the charge is distributed uniformly along the line, the charge distribution has cylindrical symmetry and we can apply Gauss's law to find the electric field.

Analyze The symmetry of the charge distribution requires that $\vec{\mathbf{E}}$ be perpendicular to the line charge and directed outward as shown in Figure 24.12b. To reflect the symmetry of the charge distribution, let's choose a cylindrical gaussian surface of radius r and length ℓ that is coaxial with the line charge. For the curved part of this surface, $\vec{\mathbf{E}}$ is constant in magnitude and perpendicular to the surface at each point, satisfying conditions (1) and (2). Furthermore, the flux through the ends of the gaussian cylinder is zero because $\vec{\mathbf{E}}$ is parallel to these surfaces. That is the first application we have seen of condition (3).

We must take the surface integral in Gauss's law over the entire gaussian surface. Because $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ is zero for the flat ends of the cylinder, however, we restrict our attention to only the curved surface of the cylinder.

Apply Gauss's law and conditions (1) and (2) for the curved surface, noting that the total charge inside our gaussian surface is $\lambda\ell$:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda\ell}{\epsilon_0}$$

Substitute the area $A = 2\pi r\ell$ of the curved surface:

$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

Solve for the magnitude of the electric field:

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \boxed{2k_e \frac{\lambda}{r}} \qquad \textbf{(24.7)}$$

Finalize This result shows that the electric field due to a cylindrically symmetric charge distribution varies as $1/r$, whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$. Equation 24.7 can also be derived by direct integration over the charge distribution. (See Problem 44 in Chapter 23.)

WHAT IF? What if the line segment in this example were not infinitely long?

Answer If the line charge in this example were of finite length, the electric field would not be given by Equation 24.7. A finite line charge does not possess sufficient symmetry to make use of Gauss's law because the magnitude of the electric field is no longer constant over the surface of the gaussian cylinder: the field near the ends of the line would be different from that far from the ends. Therefore, condition (1) would not be satisfied in this situation. Furthermore, $\vec{\mathbf{E}}$ is not perpendicular to the cylindrical surface at all points: the field vectors near the ends would have a component parallel to the line. Therefore, condition (2) would not be satisfied. For points close to a finite line charge and far from the ends, Equation 24.7 gives a good approximation of the value of the field.

It is left for you to show (see Problem 33) that the electric field inside a uniformly charged rod of finite radius and infinite length is proportional to r.

Example 24.5 A Plane of Charge

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

SOLUTION

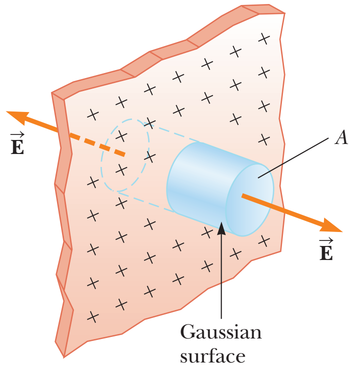


Figure 24.13 (Example 24.5) A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

Conceptualize Notice that the plane of charge is *infinitely* large. Therefore, the electric field should be the same at all points equidistant from the plane. How would you expect the electric field to depend on the distance from the plane?

Categorize Because the charge is distributed uniformly on the plane, the charge distribution is symmetric; hence, we can use Gauss's law to find the electric field.

Analyze By symmetry, $\vec{\mathbf{E}}$ must be perpendicular to the plane at all points. The direction of $\vec{\mathbf{E}}$ is away from positive charges, indicating that the direction of $\vec{\mathbf{E}}$ on one side of the plane must be opposite its direction on the other side as shown in Figure 24.13. A gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane. Because $\vec{\mathbf{E}}$ is parallel to the curved surface of the cylinder—and therefore perpendicular to $d\vec{\mathbf{A}}$ at all points on this surface—condition (3) is satisfied and there is no contribution to the surface integral from this surface. For the flat ends of the cylinder, conditions (1) and (2) are satisfied. The flux through each end of the cylinder is EA; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.

Write Gauss's law for this surface, noting that the enclosed charge is $q_{\text{in}} = \sigma A$:

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Solve for E:

$$E = \boxed{\frac{\sigma}{2\epsilon_0}} \qquad \textbf{(24.8)}$$

Finalize Because the distance from each flat end of the cylinder to the plane does not appear in Equation 24.8, we conclude that $E = \sigma/2\epsilon_0$ at *any* distance from the plane. That is, the field is uniform everywhere. Figure 24.14 shows this uniform field due to an infinite plane of charge, seen edge-on.

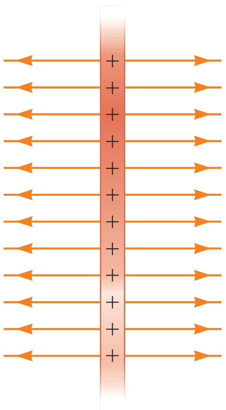


Figure 24.14 (Example 24.5) The electric field lines due to an infinite plane of positive charge.

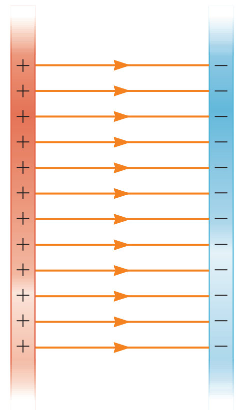


Figure 24.15 (Example 24.5) The electric field lines between two infinite planes of charge, one positive and one negative. In practice, the field lines near the edges of finite-sized sheets of charge will curve outward.

WHAT IF? Suppose two infinite planes of charge are parallel to each other, one positively charged and the other negatively charged. The surface charge densities of both planes are of the same magnitude. What does the electric field look like in this situation?

Answer We first addressed this configuration in the **What If?** section of Example 23.9. The electric fields due to the two planes add in the region between the planes, resulting in a uniform field of magnitude σ/ϵ_0, and cancel elsewhere to give a field of zero. Figure 24.15 shows the field lines for such a configuration. This method is a practical way to achieve uniform electric fields with finite-sized planes placed close to each other.

Conceptual Example 24.6 Don't Use Gauss's Law Here!

Explain why Gauss's law cannot be used to calculate the electric field near an electric dipole, a charged disk, or a triangle with a point charge at each corner.

24.6 continued

SOLUTION

The charge distributions of all these configurations do not have sufficient symmetry to make the use of Gauss's law practical. We cannot find a closed surface surrounding any of these distributions for which all portions of the surface satisfy one or more of conditions (1) through (4) listed at the beginning of this section.

24.4 Conductors in Electrostatic Equilibrium

As we learned in Section 23.2, a good electrical conductor contains charges (electrons) that are not bound to any atom and therefore are free to move about within the material. When there is no net motion of charge within a conductor, the conductor is in **electrostatic equilibrium.** A conductor in electrostatic equilibrium has the following properties:

Properties of a conductor in electrostatic equilibrium

1. The electric field is zero everywhere inside the conductor, whether the conductor is solid or hollow.
2. If the conductor is isolated and carries a charge, the charge resides on its surface.
3. The electric field at a point just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

We verify the first three properties in the discussion that follows. The fourth property is presented here (but not verified until we have studied the appropriate material in Chapter 25) to provide a complete list of properties for conductors in electrostatic equilibrium.

We can understand the first property by considering a conducting slab placed in an external field $\vec{\mathbf{E}}$ (Fig. 24.16). The electric field inside the conductor *must* be zero, assuming electrostatic equilibrium exists. If the field were not zero, free electrons in the conductor would experience an electric force ($\vec{\mathbf{F}} = q\vec{\mathbf{E}}$) and would accelerate due to this force. This motion of electrons, however, would mean that the conductor is not in electrostatic equilibrium. Therefore, the existence of electrostatic equilibrium is consistent only with a zero field in the conductor.

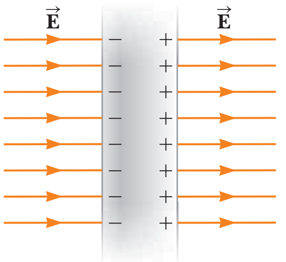


Figure 24.16 A conducting slab in an external electric field $\vec{\mathbf{E}}$. The charges induced on the two surfaces of the slab produce an electric field that opposes the external field, giving a resultant field of zero inside the slab.

Let's investigate how this zero field is accomplished. Before the external field is applied, free electrons are uniformly distributed throughout the conductor. When the external field is applied, the free electrons accelerate to the left in Figure 24.16, causing a plane of negative charge to accumulate on the left surface. The movement of electrons to the left results in a plane of positive charge on the right surface. These planes of charge create an additional electric field inside the conductor that opposes the external field. As the electrons move, the surface charge densities on the left and right surfaces increase until the magnitude of the internal field equals that of the external field, resulting in a net field of zero inside the conductor. The time it takes a good conductor to reach equilibrium is on the order of 10^{-16} s, which for most purposes can be considered instantaneous.

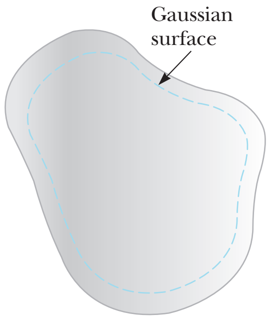


Figure 24.17 A conductor of arbitrary shape. The broken line represents a gaussian surface that can be just inside the conductor's surface.

If the conductor is hollow, the electric field inside the conductor is also zero, whether we consider points in the conductor or in the cavity within the conductor. The zero value of the electric field in the cavity is easiest to argue with the concept of electric potential, so we will address this issue in Section 25.6.

Gauss's law can be used to verify the second property of a conductor in electrostatic equilibrium. Figure 24.17 shows an arbitrarily shaped conductor. A gaussian surface is drawn inside the conductor and can be very close to the conductor's surface. As we have just shown, the electric field everywhere inside the conductor is zero when it is in electrostatic equilibrium. Therefore, the electric field must be zero at every point on the gaussian surface, in accordance with condition (4) in Section 24.3, and the net flux through this gaussian surface is zero. From this result and Gauss's law, we conclude that the net charge inside the gaussian surface is zero. Because there can be no net charge inside the gaussian surface (which is arbitrarily close to the conductor's surface), any net charge on the conductor must reside on its surface. Gauss's law does not indicate how this excess charge is distributed on the conductor's surface, only that it resides exclusively on the surface.

To verify the third property, let's begin with the perpendicularity of the field to the surface. If the field vector $\vec{\mathbf{E}}$ had a component parallel to the conductor's surface, free electrons would experience an electric force and move along the surface; in such a case, the conductor would not be in equilibrium. Therefore, the field vector must be perpendicular to the surface.

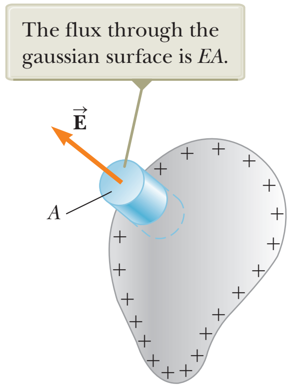


Figure 24.18 A gaussian surface in the shape of a small cylinder is used to calculate the electric field immediately outside a charged conductor.

To determine the magnitude of the electric field, we use Gauss's law and draw a gaussian surface in the shape of a small cylinder whose end faces are parallel to the conductor's surface (Fig. 24.18). Part of the cylinder is just outside the conductor, and part is inside. The field is perpendicular to the conductor's surface from the condition of electrostatic equilibrium. Therefore, condition (3) in Section 24.3 is satisfied for the curved part of the cylindrical gaussian surface: there is no flux through this part of the gaussian surface because $\vec{\mathbf{E}}$ is parallel to the surface. There is no flux through the flat face of the cylinder inside the conductor because here $\vec{\mathbf{E}} = 0$, which satisfies condition (4). Hence, the net flux through the gaussian surface is equal to that through only the flat face outside the conductor, where the field is perpendicular to the gaussian surface. Using conditions (1) and (2) for this face, the flux is EA, where E is the electric field just outside the conductor and A is the area of the cylinder's face. Applying Gauss's law to this surface gives

$$\Phi_E = \oint E\,dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

where we have used $q_{\text{in}} = \sigma A$. Solving for E gives for the electric field immediately outside a charged conductor:

$$E = \frac{\sigma}{\epsilon_0} \qquad \textbf{(24.9)}$$

Quick Quiz 24.3 Your younger brother likes to rub his feet on the carpet and then touch you to give you a shock. While you are trying to escape the shock treatment, you discover a hollow metal cylinder in your basement, large enough to climb inside. In which of the following cases will you *not* be shocked? **(a)** You climb inside the cylinder, making contact with the inner surface, and your charged brother touches the outer metal surface. **(b)** Your charged brother is inside touching the inner metal surface and you are outside, touching the outer metal surface. **(c)** Both of you are outside the cylinder, touching its outer metal surface but not touching each other directly.

Example 24.7 A Sphere Inside a Spherical Shell

A solid insulating sphere of radius a carries a net positive charge Q uniformly distributed throughout its volume. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-2Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in Figure 24.19 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

24.7 continued

SOLUTION

Conceptualize Notice how this problem differs from Example 24.3. The charged sphere in Figure 24.10 appears in Figure 24.19, but it is now surrounded by a shell carrying a charge $-2Q$. Think about how the presence of the shell will affect the electric field of the sphere.

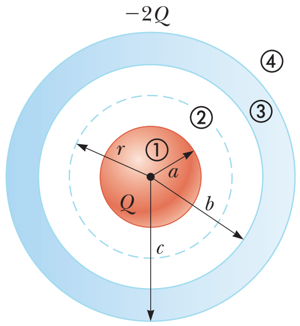


Figure 24.19 (Example 24.7) An insulating sphere of radius a and carrying a charge Q surrounded by a conducting spherical shell carrying a charge $-2Q$.

Categorize The charge is distributed uniformly throughout the sphere, and we know that the charge on the conducting shell distributes itself uniformly on the surfaces. Therefore, the system has spherical symmetry and we can apply Gauss's law to find the electric field in the various regions.

Analyze In region ②—between the surface of the solid sphere and the inner surface of the shell—we construct a spherical gaussian surface of radius r, where $a < r < b$, noting that the charge inside this surface is $+Q$ (the charge on the solid sphere). Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the gaussian surface.

The charge on the conducting shell creates zero electric field in the region $r < b$, so the shell has no effect on the field in region ② due to the sphere. Therefore, write an expression for the field in region ② as that due to the sphere from part (A) of Example 24.3:

$$E_2 = \boxed{k_e \frac{Q}{r^2} \quad (\text{for } a < r < b)}$$

Because the conducting shell creates zero field inside itself, it also has no effect on the field inside the sphere. Therefore, write an expression for the field in region ① as that due to the sphere from part (B) of Example 24.3:

$$E_1 = \boxed{k_e \frac{Q}{a^3} r \quad (\text{for } r < a)}$$

In region ④, where $r > c$, construct a spherical gaussian surface; this surface surrounds a total charge $q_{in} = Q + (-2Q) = -Q$. Therefore, model the charge distribution as a sphere with charge $-Q$ and write an expression for the field in region ④ from part (A) of Example 24.3:

$$E_4 = \boxed{-k_e \frac{Q}{r^2} \quad (\text{for } r > c)}$$

In region ③, the electric field must be zero because the spherical shell is a conductor in equilibrium:

$$E_3 = \boxed{0 \quad (\text{for } b < r < c)}$$

Construct a gaussian surface of radius r in region ③, where $b < r < c$, and note that q_{in} must be zero because $E_3 = 0$. Find the amount of charge q_{inner} on the inner surface of the shell:

$$q_{in} = q_{sphere} + q_{inner}$$
$$q_{inner} = q_{in} - q_{sphere} = 0 - Q = -Q$$

Finalize The charge on the inner surface of the spherical shell must be $-Q$ to cancel the charge $+Q$ on the solid sphere and give zero electric field in the material of the shell. Because the net charge on the shell is $-2Q$, its outer surface must carry a charge $-Q$.

WHAT IF? How would the results of this problem differ if the sphere were conducting instead of insulating?

Answer The only change would be in region ①, where $r < a$. Because there can be no charge inside a conductor in electrostatic equilibrium, $q_{in} = 0$ for a gaussian surface of radius $r < a$; therefore, on the basis of Gauss's law and symmetry, $E_1 = 0$. In regions ②, ③, and ④, there would be no way to determine from observations of the electric field whether the sphere is conducting or insulating.

Summary

Definition

Electric flux is proportional to the number of electric field lines that penetrate a surface. If the electric field is uniform and makes an angle θ with the normal to a surface of area A, the electric flux through the surface is

$$\Phi_E = EA\cos\theta \tag{24.2}$$

In general, the electric flux through a surface is

$$\Phi_E \equiv \int_{\text{surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \tag{24.3}$$

Concepts and Principles

Gauss's law says that the net electric flux Φ_E through any closed gaussian surface is equal to the *net* charge q_{in} inside the surface divided by ϵ_0:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon_0} \tag{24.6}$$

Using Gauss's law, you can calculate the electric field due to various symmetric charge distributions.

A conductor in **electrostatic equilibrium** has the following properties:

1. The electric field is zero everywhere inside the conductor, whether the conductor is solid or hollow.
2. If the conductor is isolated and carries a charge, the charge resides on its surface.
3. The electric field at a point just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. A cubical gaussian surface surrounds a long, straight, charged filament that passes perpendicularly through two opposite faces. No other charges are nearby. **(i)** Over how many of the cube's faces is the electric field zero? (a) 0 (b) 2 (c) 4 (d) 6 **(ii)** Through how many of the cube's faces is the electric flux zero? Choose from the same possibilities as in part (i).

2. A coaxial cable consists of a long, straight filament surrounded by a long, coaxial, cylindrical conducting shell. Assume charge Q is on the filament, zero net charge is on the shell, and the electric field is $E_1\hat{\mathbf{i}}$ at a particular point P midway between the filament and the inner surface of the shell. Next, you place the cable into a uniform external field $-E\hat{\mathbf{i}}$. What is the x component of the electric field at P then? (a) 0 (b) between 0 and E_1 (c) E_1 (d) between 0 and $-E_1$ (e) $-E_1$

3. In which of the following contexts can Gauss's law *not* be readily applied to find the electric field? (a) near a long, uniformly charged wire (b) above a large, uniformly charged plane (c) inside a uniformly charged ball (d) outside a uniformly charged sphere (e) Gauss's law can be readily applied to find the electric field in all these contexts.

4. A particle with charge q is located inside a cubical gaussian surface. No other charges are nearby. **(i)** If the particle is at the center of the cube, what is the flux through each one of the faces of the cube? (a) 0 (b) $q/2\epsilon_0$ (c) $q/6\epsilon_0$ (d) $q/8\epsilon_0$ (e) depends on the size of the cube **(ii)** If the particle can be moved to any point within the cube, what maximum value can the flux through one face approach? Choose from the same possibilities as in part (i).

5. Charges of 3.00 nC, −2.00 nC, −7.00 nC, and 1.00 nC are contained inside a rectangular box with length 1.00 m, width 2.00 m, and height 2.50 m. Outside the box are charges of 1.00 nC and 4.00 nC. What is the electric flux through the surface of the box? (a) 0 (b) $-5.64 \times 10^2\ \text{N}\cdot\text{m}^2/\text{C}$ (c) $-1.47 \times 10^3\ \text{N}\cdot\text{m}^2/\text{C}$ (d) $1.47 \times 10^3\ \text{N}\cdot\text{m}^2/\text{C}$ (e) $5.64 \times 10^2\ \text{N}\cdot\text{m}^2/\text{C}$

6. A large, metallic, spherical shell has no net charge. It is supported on an insulating stand and has a small hole at the top. A small tack with charge Q is lowered on a silk thread through the hole into the interior of the shell. **(i)** What is the charge on the inner surface of the shell, (a) Q (b) $Q/2$ (c) 0 (d) $-Q/2$ or (e) $-Q$? Choose your answers to the following questions from

the same possibilities. **(ii)** What is the charge on the outer surface of the shell? **(iii)** The tack is now allowed to touch the interior surface of the shell. After this contact, what is the charge on the tack? **(iv)** What is the charge on the inner surface of the shell now? **(v)** What is the charge on the outer surface of the shell now?

7. Two solid spheres, both of radius 5 cm, carry identical total charges of 2 μC. Sphere A is a good conductor. Sphere B is an insulator, and its charge is distributed uniformly throughout its volume. **(i)** How do the magnitudes of the electric fields they separately create at a radial distance of 6 cm compare? (a) $E_A > E_B = 0$ (b) $E_A > E_B > 0$ (c) $E_A = E_B > 0$ (d) $0 < E_A < E_B$ (e) $0 = E_A < E_B$ **(ii)** How do the magnitudes of the electric fields they separately create at radius 4 cm compare? Choose from the same possibilities as in part (i).

8. A uniform electric field of 1.00 N/C is set up by a uniform distribution of charge in the *xy* plane. What is the electric field inside a metal ball placed 0.500 m above the *xy* plane? (a) 1.00 N/C (b) −1.00 N/C (c) 0 (d) 0.250 N/C (e) varies depending on the position inside the ball

9. A solid insulating sphere of radius 5 cm carries electric charge uniformly distributed throughout its volume. Concentric with the sphere is a conducting spherical shell with no net charge as shown in Figure OQ24.9. The inner radius of the shell is 10 cm, and the outer radius is 15 cm. No other charges are nearby. (a) Rank the magnitude of the electric field at points *A* (at radius 4 cm), *B* (radius 8 cm), *C* (radius 12 cm), and *D* (radius 16 cm) from largest to smallest. Display any cases of equality in your ranking. (b) Similarly rank the electric flux through concentric spherical surfaces through points *A*, *B*, *C*, and *D*.

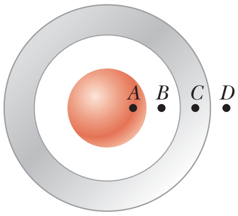


Figure OQ24.9

10. A cubical gaussian surface is bisected by a large sheet of charge, parallel to its top and bottom faces. No other charges are nearby. **(i)** Over how many of the cube's faces is the electric field zero? (a) 0 (b) 2 (c) 4 (d) 6 **(ii)** Through how many of the cube's faces is the electric flux zero? Choose from the same possibilities as in part (i).

11. Rank the electric fluxes through each gaussian surface shown in Figure OQ24.11 from largest to smallest. Display any cases of equality in your ranking.

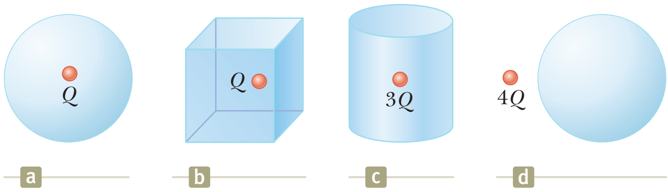


Figure OQ24.11

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Consider an electric field that is uniform in direction throughout a certain volume. Can it be uniform in magnitude? Must it be uniform in magnitude? Answer these questions (a) assuming the volume is filled with an insulating material carrying charge described by a volume charge density and (b) assuming the volume is empty space. State reasoning to prove your answers.

2. A cubical surface surrounds a point charge *q*. Describe what happens to the total flux through the surface if (a) the charge is doubled, (b) the volume of the cube is doubled, (c) the surface is changed to a sphere, (d) the charge is moved to another location inside the surface, and (e) the charge is moved outside the surface.

3. A uniform electric field exists in a region of space containing no charges. What can you conclude about the net electric flux through a gaussian surface placed in this region of space?

$\boxed{4.}$ If the total charge inside a closed surface is known but the distribution of the charge is unspecified, can you use Gauss's law to find the electric field? Explain.

5. Explain why the electric flux through a closed surface with a given enclosed charge is independent of the size or shape of the surface.

6. If more electric field lines leave a gaussian surface than enter it, what can you conclude about the net charge enclosed by that surface?

$\boxed{7.}$ A person is placed in a large, hollow, metallic sphere that is insulated from ground. (a) If a large charge is placed on the sphere, will the person be harmed upon touching the inside of the sphere? (b) Explain what will happen if the person also has an initial charge whose sign is opposite that of the charge on the sphere.

8. Consider two identical conducting spheres whose surfaces are separated by a small distance. One sphere is given a large net positive charge, and the other is given a small net positive charge. It is found that the force between the spheres is attractive even though they both have net charges of the same sign. Explain how this attraction is possible.

$\boxed{9.}$ A common demonstration involves charging a rubber balloon, which is an insulator, by rubbing it on your hair and then touching the balloon to a ceiling or wall, which is also an insulator. Because of the electrical attraction between the charged balloon and the neutral wall, the balloon sticks to the wall. Imagine now that we have two infinitely large, flat sheets of insulating material. One is charged, and the other is neutral. If these sheets are brought into contact, does an attractive force exist between them as there was for the balloon and the wall?

10. On the basis of the repulsive nature of the force between like charges and the freedom of motion of charge within a conductor, explain why excess charge on an isolated conductor must reside on its surface.

11. The Sun is lower in the sky during the winter than it is during the summer. (a) How does this change affect the flux of sunlight hitting a given area on the surface of the Earth? (b) How does this change affect the weather?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 24.1 Electric Flux

1. A flat surface of area 3.20 m^2 is rotated in a uniform electric field of magnitude $E = 6.20 \times 10^5$ N/C. Determine the electric flux through this area (a) when the electric field is perpendicular to the surface and (b) when the electric field is parallel to the surface.

2. W A vertical electric field of magnitude 2.00×10^4 N/C exists above the Earth's surface on a day when a thunderstorm is brewing. A car with a rectangular size of 6.00 m by 3.00 m is traveling along a dry gravel roadway sloping downward at 10.0°. Determine the electric flux through the bottom of the car.

3. M A 40.0-cm-diameter circular loop is rotated in a uniform electric field until the position of maximum electric flux is found. The flux in this position is measured to be 5.20×10^5 N · m²/C. What is the magnitude of the electric field?

4. W Consider a closed triangular box resting within a horizontal electric field of magnitude $E = 7.80 \times 10^4$ N/C as shown in Figure P24.4. Calculate the electric flux through (a) the vertical rectangular surface, (b) the slanted surface, and (c) the entire surface of the box.

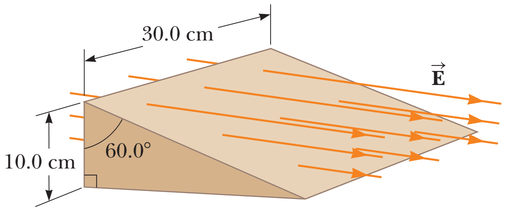


Figure P24.4

5. M An electric field of magnitude 3.50 kN/C is applied along the x axis. Calculate the electric flux through a rectangular plane 0.350 m wide and 0.700 m long (a) if the plane is parallel to the yz plane, (b) if the plane is parallel to the xy plane, and (c) if the plane contains the y axis and its normal makes an angle of 40.0° with the x axis.

6. A nonuniform electric field is given by the expression

$$\vec{\mathbf{E}} = ay\hat{\mathbf{i}} + bz\hat{\mathbf{j}} + cx\hat{\mathbf{k}}$$

where a, b, and c are constants. Determine the electric flux through a rectangular surface in the xy plane, extending from $x = 0$ to $x = w$ and from $y = 0$ to $y = h$.

Section 24.2 Gauss's Law

7. An uncharged, nonconducting, hollow sphere of radius 10.0 cm surrounds a 10.0-μC charge located at the origin of a Cartesian coordinate system. A drill with a radius of 1.00 mm is aligned along the z axis, and a hole is drilled in the sphere. Calculate the electric flux through the hole.

8. Find the net electric flux through the spherical closed surface shown in Figure P24.8. The two charges on the right are inside the spherical surface.

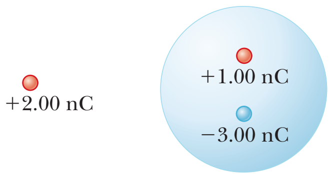


Figure P24.8

9. M The following charges are located inside a submarine: 5.00 μC, −9.00 μC, 27.0 μC, and −84.0 μC. (a) Calculate the net electric flux through the hull of the submarine. (b) Is the number of electric field lines leaving the submarine greater than, equal to, or less than the number entering it?

10. W The electric field everywhere on the surface of a thin, spherical shell of radius 0.750 m is of magnitude 890 N/C and points radially toward the center of the sphere. (a) What is the net charge within the sphere's surface? (b) What is the distribution of the charge inside the spherical shell?

11. Four closed surfaces, S_1 through S_4, together with the charges $-2Q$, Q, and $-Q$ are sketched in Figure P24.11. (The colored lines are the intersections of the surfaces with the page.) Find the electric flux through each surface.

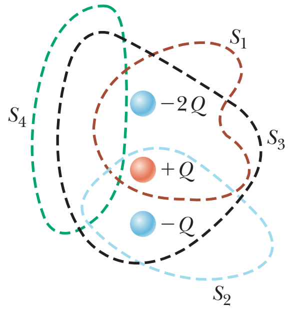


Figure P24.11

12. A charge of 170 μC is at the center of a cube of edge 80.0 cm. No other charges are nearby. (a) Find the flux through each face of the cube. (b) Find the flux through the whole surface of the cube. (c) **What If?** Would your answers to either part (a) or part (b) change if the charge were not at the center? Explain.

13. In the air over a particular region at an altitude of 500 m above the ground, the electric field is 120 N/C directed downward. At 600 m above the ground, the electric field is 100 N/C downward. What is the average volume charge density in the layer of air between these two elevations? Is it positive or negative?

14. A particle with charge of 12.0 μC is placed at the center of a spherical shell of radius 22.0 cm. What is the total electric flux through (a) the surface of the shell and (b) any hemispherical surface of the shell? (c) Do the results depend on the radius? Explain.

15. (a) Find the net electric flux through the cube shown in Figure P24.15. (b) Can you use Gauss's law to find the electric field on the surface of this cube? Explain.

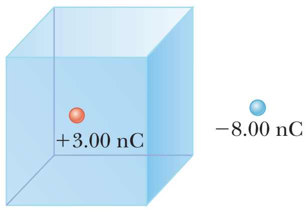


Figure P24.15

16. (a) A particle with charge q is located a distance d from an infinite plane. Determine the electric flux through the plane due to the charged particle. (b) **What If?** A particle with charge q is located a *very small* distance from the center of a *very large* square on the line perpendicular to the square and going through its center. Determine the approximate electric flux through the square due to the charged particle. (c) How do the answers to parts (a) and (b) compare? Explain.

17. An infinitely long line charge having a uniform charge per unit length λ lies a distance d from point O as shown in Figure P24.17. Determine the total electric flux through the surface of a sphere of radius R centered at O resulting from this line charge. Consider both cases, where (a) $R < d$ and (b) $R > d$.

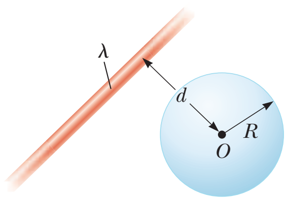


Figure P24.17

18. Find the net electric flux through (a) the closed spherical surface in a uniform electric field shown in Figure P24.18a and (b) the closed cylindrical surface shown in Figure P24.18b. (c) What can you conclude about the charges, if any, inside the cylindrical surface?

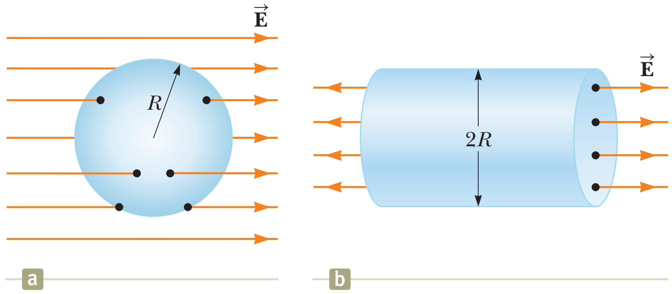


Figure P24.18

19. A particle with charge $Q = 5.00\ \mu$C is located at the center of a cube of edge $L = 0.100$ m. In addition, six other identical charged particles having $q = -1.00\ \mu$C are positioned symmetrically around Q as shown in Figure P24.19. Determine the electric flux through one face of the cube.

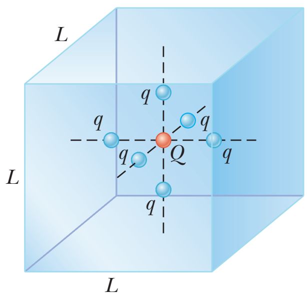


Figure P24.19
Problems 19 and 20.

20. A particle with charge Q is located at the center of a cube of edge L. In addition, six other identical charged particles q are positioned symmetrically around Q as shown in Figure P24.19. For each of these particles, q is a negative number. Determine the electric flux through one face of the cube.

21. A particle with charge Q is located a small distance δ immediately above the center of the flat face of a hemisphere of radius R as shown in Figure P24.21. What is the electric flux (a) through the curved surface and (b) through the flat face as $\delta \to 0$?

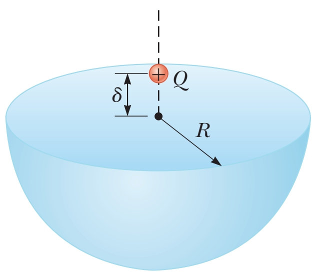


Figure P24.21

22. Figure P24.22 (page 742) represents the top view of a cubic gaussian surface in a uniform electric field $\vec{\mathbf{E}}$ oriented parallel to the top and bottom faces of the cube. The field makes an angle θ with side ①, and the area of each face is A. In symbolic form, find the electric flux through (a) face ①, (b) face ②, (c) face ③, (d) face ④, and (e) the top and bottom faces of the cube. (f) What is the net electric flux through the cube? (g) How much charge is enclosed within the gaussian surface?

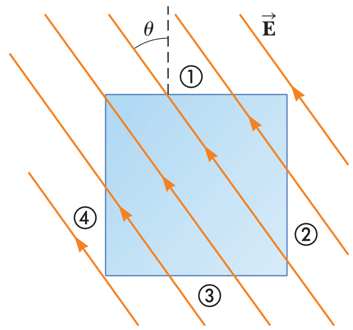


Figure P24.22

Section 24.3 Application of Gauss's Law to Various Charge Distributions

23. In nuclear fission, a nucleus of uranium-238, which contains 92 protons, can divide into two smaller spheres, each having 46 protons and a radius of 5.90×10^{-15} m. What is the magnitude of the repulsive electric force pushing the two spheres apart?

24. W The charge per unit length on a long, straight filament is $-90.0\ \mu$C/m. Find the electric field (a) 10.0 cm, (b) 20.0 cm, and (c) 100 cm from the filament, where distances are measured perpendicular to the length of the filament.

25. AMT A 10.0-g piece of Styrofoam carries a net charge of $-0.700\ \mu$C and is suspended in equilibrium above the center of a large, horizontal sheet of plastic that has a uniform charge density on its surface. What is the charge per unit area on the plastic sheet?

26. Determine the magnitude of the electric field at the surface of a lead-208 nucleus, which contains 82 protons and 126 neutrons. Assume the lead nucleus has a volume 208 times that of one proton and consider a proton to be a sphere of radius 1.20×10^{-15} m.

27. M A large, flat, horizontal sheet of charge has a charge per unit area of $9.00\ \mu$C/m^2. Find the electric field just above the middle of the sheet.

28. Suppose you fill two rubber balloons with air, suspend both of them from the same point, and let them hang down on strings of equal length. You then rub each with wool or on your hair so that the balloons hang apart with a noticeable separation between them. Make order-of-magnitude estimates of (a) the force on each, (b) the charge on each, (c) the field each creates at the center of the other, and (d) the total flux of electric field created by each balloon. In your solution, state the quantities you take as data and the values you measure or estimate for them.

29. M Consider a thin, spherical shell of radius 14.0 cm with a total charge of $32.0\ \mu$C distributed uniformly on its surface. Find the electric field (a) 10.0 cm and (b) 20.0 cm from the center of the charge distribution.

30. W A nonconducting wall carries charge with a uniform density of $8.60\ \mu$C/cm^2. (a) What is the electric field 7.00 cm in front of the wall if 7.00 cm is small compared with the dimensions of the wall? (b) Does your result change as the distance from the wall varies? Explain.

31. M A uniformly charged, straight filament 7.00 m in length has a total positive charge of $2.00\ \mu$C. An uncharged cardboard cylinder 2.00 cm in length and 10.0 cm in radius surrounds the filament at its center, with the filament as the axis of the cylinder. Using reasonable approximations, find (a) the electric field at the surface of the cylinder and (b) the total electric flux through the cylinder.

32. Assume the magnitude of the electric field on each face of the cube of edge $L = 1.00$ m in Figure P24.32 is uniform and the directions of the fields on each face are as indicated. Find (a) the net electric flux through the cube and (b) the net charge inside the cube. (c) Could the net charge be a single point charge?

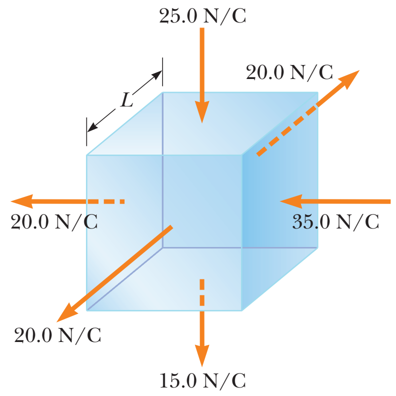


Figure P24.32

33. Consider a long, cylindrical charge distribution of radius R with a uniform charge density ρ. Find the electric field at distance r from the axis, where $r < R$.

34. W A cylindrical shell of radius 7.00 cm and length 2.40 m has its charge uniformly distributed on its curved surface. The magnitude of the electric field at a point 19.0 cm radially outward from its axis (measured from the midpoint of the shell) is 36.0 kN/C. Find (a) the net charge on the shell and (b) the electric field at a point 4.00 cm from the axis, measured radially outward from the midpoint of the shell.

35. W A solid sphere of radius 40.0 cm has a total positive charge of $26.0\ \mu$C uniformly distributed throughout its volume. Calculate the magnitude of the electric field (a) 0 cm, (b) 10.0 cm, (c) 40.0 cm, and (d) 60.0 cm from the center of the sphere.

36. AMT **Review.** A particle with a charge of -60.0 nC is placed at the center of a nonconducting spherical shell of inner radius 20.0 cm and outer radius 25.0 cm. The spherical shell carries charge with a uniform density of $-1.33\ \mu$C/m^3. A proton moves in a circular orbit just outside the spherical shell. Calculate the speed of the proton.

Section 24.4 Conductors in Electrostatic Equilibrium

37. M A long, straight metal rod has a radius of 5.00 cm and a charge per unit length of 30.0 nC/m. Find the electric field (a) 3.00 cm, (b) 10.0 cm, and (c) 100 cm from the axis of the rod, where distances are measured perpendicular to the rod's axis.

38. *Why is the following situation impossible?* A solid copper sphere of radius 15.0 cm is in electrostatic equilibrium and carries a charge of 40.0 nC. Figure P24.38 shows the magnitude of the electric field as a function of radial position r measured from the center of the sphere.

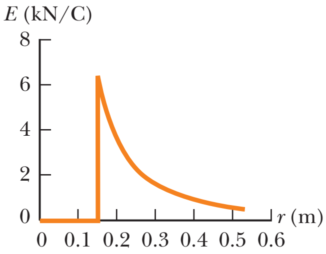


Figure P24.38

39. W A solid metallic sphere of radius a carries total charge Q. No other charges are nearby. The electric field just outside its surface is k_eQ/a^2 radially outward. At this close point, the uniformly charged surface of the sphere looks exactly like a uniform flat sheet of charge. Is the electric field here given by σ/ϵ_0 or by $\sigma/2\epsilon_0$?

40. A positively charged particle is at a distance $R/2$ from the center of an uncharged thin, conducting, spherical shell of radius R. Sketch the electric field lines set up by this arrangement both inside and outside the shell.

41. A very large, thin, flat plate of aluminum of area A has a total charge Q uniformly distributed over its surfaces. Assuming the same charge is spread uniformly over the *upper* surface of an otherwise identical glass plate, compare the electric fields just above the center of the upper surface of each plate.

42. In a certain region of space, the electric field is $\vec{\mathbf{E}} = 6.00 \times 10^3\, x^2\hat{\mathbf{i}}$, where $\vec{\mathbf{E}}$ is in newtons per coulomb and x is in meters. Electric charges in this region are at rest and remain at rest. (a) Find the volume density of electric charge at $x = 0.300$ m. *Suggestion:* Apply Gauss's law to a box between $x = 0.300$ m and $x = 0.300\text{ m} + dx$. (b) Could this region of space be inside a conductor?

43. AMT Two identical conducting spheres each having a radius of 0.500 cm are connected by a light, 2.00-m-long conducting wire. A charge of 60.0 μC is placed on one of the conductors. Assume the surface distribution of charge on each sphere is uniform. Determine the tension in the wire.

44. A square plate of copper with 50.0-cm sides has no net charge and is placed in a region of uniform electric field of 80.0 kN/C directed perpendicularly to the plate. Find (a) the charge density of each face of the plate and (b) the total charge on each face.

45. A long, straight wire is surrounded by a hollow metal cylinder whose axis coincides with that of the wire. The wire has a charge per unit length of λ, and the cylinder has a net charge per unit length of 2λ. From this information, use Gauss's law to find (a) the charge per unit length on the inner surface of the cylinder, (b) the charge per unit length on the outer surface of the cylinder, and (c) the electric field outside the cylinder a distance r from the axis.

46. M A thin, square, conducting plate 50.0 cm on a side lies in the xy plane. A total charge of 4.00×10^{-8} C is placed on the plate. Find (a) the charge density on each face of the plate, (b) the electric field just above the plate, and (c) the electric field just below the plate. You may assume the charge density is uniform.

47. M A solid conducting sphere of radius 2.00 cm has a charge of 8.00 μC. A conducting spherical shell of inner radius 4.00 cm and outer radius 5.00 cm is concentric with the solid sphere and has a charge of -4.00 μC. Find the electric field at (a) $r = 1.00$ cm, (b) $r = 3.00$ cm, (c) $r = 4.50$ cm, and (d) $r = 7.00$ cm from the center of this charge configuration.

Additional Problems

48. Consider a plane surface in a uniform electric field as in Figure P24.48, where $d = 15.0$ cm and $\theta = 70.0°$. If the net flux through the surface is 6.00 N $\cdot$ m^2/C, find the magnitude of the electric field.

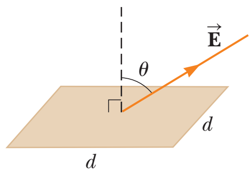


Figure P24.48 Problems 48 and 49.

49. Find the electric flux through the plane surface shown in Figure P24.48 if $\theta = 60.0°$, $E = 350$ N/C, and $d = 5.00$ cm. The electric field is uniform over the entire area of the surface.

50. A hollow, metallic, spherical shell has exterior radius 0.750 m, carries no net charge, and is supported on an insulating stand. The electric field everywhere just outside its surface is 890 N/C radially toward the center of the sphere. Explain what you can conclude about (a) the amount of charge on the exterior surface of the sphere and the distribution of this charge, (b) the amount of charge on the interior surface of the sphere and its distribution, and (c) the amount of charge inside the shell and its distribution.

51. A sphere of radius $R = 1.00$ m surrounds a particle with charge $Q = 50.0$ μC located at its center as shown in Figure P24.51. Find the electric flux through a circular cap of half-angle $\theta = 45.0°$.

52. A sphere of radius R surrounds a particle with charge Q located at its center as shown in Figure P24.51. Find the electric flux through a circular cap of half-angle θ.

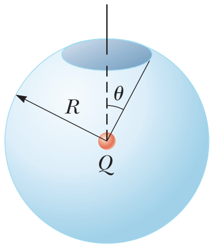


Figure P24.51 Problems 51 and 52.

53. A very large conducting plate lying in the xy plane carries a charge per unit area of σ. A second such plate located above the first plate at $z = z_0$ and oriented parallel to the xy plane carries a charge per unit area of -2σ. Find the electric field for (a) $z < 0$, (b) $0 < z < z_0$, and (c) $z > z_0$.

54. GP A solid, insulating sphere of radius a has a uniform charge density throughout its volume and a total charge Q. Concentric with this sphere is an uncharged, conducting, hollow sphere whose inner and outer radii are b and c as shown in Figure P24.54 (page 744). We wish to understand completely the charges and electric fields at all locations. (a) Find the charge contained within a sphere of radius $r < a$. (b) From this value, find the magnitude of the electric field for $r < a$. (c) What charge is contained within a sphere of radius r when $a < r < b$? (d) From this value, find the magnitude of the electric field for r when $a < r < b$. (e) Now consider r when $b < r < c$. What is the magnitude of the electric field for this range of values of r? (f) From this value, what must be the charge on the inner surface of the hollow sphere? (g) From part (f), what must be the charge on the outer surface of the hollow sphere? (h) Consider the three spherical surfaces of radii a, b, and c. Which of these surfaces has the largest magnitude of surface charge density?

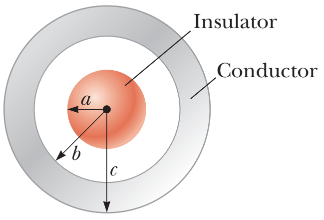


Figure P24.54
Problems 54, 55, and 57.

55. A solid insulating sphere of radius $a = 5.00$ cm carries a net positive charge of $Q = 3.00\ \mu$C uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius $b = 10.0$ cm and outer radius $c = 15.0$ cm as shown in Figure P24.54, having net charge $q = -1.00\ \mu$C. Prepare a graph of the magnitude of the electric field due to this configuration versus r for $0 < r < 25.0$ cm.

56. Two infinite, nonconducting sheets of charge are parallel to each other as shown in Figure P24.56. The sheet on the left has a uniform surface charge density σ, and the one on the right has a uniform charge density $-\sigma$. Calculate the electric field at points (a) to the left of, (b) in between, and (c) to the right of the two sheets. (d) **What If?** Find the electric fields in all three regions if both sheets have *positive* uniform surface charge densities of value σ.

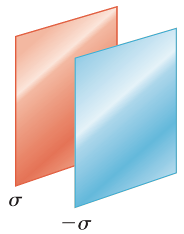


Figure P24.56

57. W For the configuration shown in Figure P24.54, suppose $a = 5.00$ cm, $b = 20.0$ cm, and $c = 25.0$ cm. Furthermore, suppose the electric field at a point 10.0 cm from the center is measured to be 3.60×10^3 N/C radially inward and the electric field at a point 50.0 cm from the center is of magnitude 200 N/C and points radially outward. From this information, find (a) the charge on the insulating sphere, (b) the net charge on the hollow conducting sphere, (c) the charge on the inner surface of the hollow conducting sphere, and (d) the charge on the outer surface of the hollow conducting sphere.

58. An insulating solid sphere of radius a has a uniform volume charge density and carries a total positive charge Q. A spherical gaussian surface of radius r, which shares a common center with the insulating sphere, is inflated starting from $r = 0$. (a) Find an expression for the electric flux passing through the surface of the gaussian sphere as a function of r for $r < a$. (b) Find an expression for the electric flux for $r > a$. (c) Plot the flux versus r.

59. A uniformly charged spherical shell with positive surface charge density σ contains a circular hole in its surface. The radius r of the hole is small compared with the radius R of the sphere. What is the electric field at the center of the hole? *Suggestion:* This problem can be solved by using the principle of superposition.

60. An infinitely long, cylindrical, insulating shell of inner radius a and outer radius b has a uniform volume charge density ρ. A line of uniform linear charge density λ is placed along the axis of the shell. Determine the electric field for (a) $r < a$, (b) $a < r < b$, and (c) $r > b$.

Challenge Problems

61. A slab of insulating material has a nonuniform positive charge density $\rho = Cx^2$, where x is measured from the center of the slab as shown in Figure P24.61 and C is a constant. The slab is infinite in the y and z directions. Derive expressions for the electric field in (a) the exterior regions ($|x| > d/2$) and (b) the interior region of the slab ($-d/2 < x < d/2$).

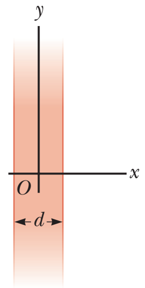


Figure P24.61
Problems 61 and 69.

62. AMT **Review.** An early (incorrect) model of the hydrogen atom, suggested by J. J. Thomson, proposed that a positive cloud of charge $+e$ was uniformly distributed throughout the volume of a sphere of radius R, with the electron (an equal-magnitude negatively charged particle $-e$) at the center. (a) Using Gauss's law, show that the electron would be in equilibrium at the center and, if displaced from the center a distance $r < R$, would experience a restoring force of the form $F = -Kr$, where K is a constant. (b) Show that $K = k_e e^2/R^3$. (c) Find an expression for the frequency f of simple harmonic oscillations that an electron of mass m_e would undergo if displaced a small distance ($< R$) from the center and released. (d) Calculate a numerical value for R that would result in a frequency of 2.47×10^{15} Hz, the frequency of the light radiated in the most intense line in the hydrogen spectrum.

63. A closed surface with dimensions $a = b = 0.400$ m and $c = 0.600$ m is located as shown in Figure P24.63. The left edge of the closed surface is located at position $x = a$. The electric field throughout the region is nonuniform and is given by $\vec{\mathbf{E}} = (3.00 + 2.00x^2)\hat{\mathbf{i}}$ N/C, where x is in meters. (a) Calculate the net electric flux

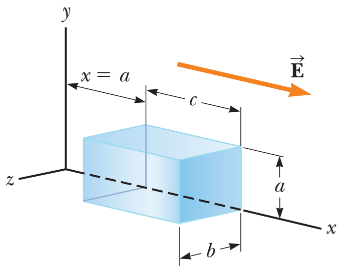


Figure P24.63

leaving the closed surface. (b) What net charge is enclosed by the surface?

64. A sphere of radius $2a$ is made of a nonconducting material that has a uniform volume charge density ρ. Assume the material does not affect the electric field. A spherical cavity of radius a is now removed from the sphere as shown in Figure P24.64. Show that the electric field within the cavity is uniform and is given by $E_x = 0$ and $E_y = \rho a/3\epsilon_0$.

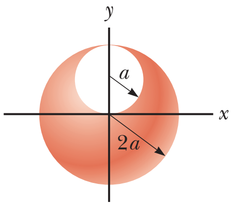


Figure P24.64

65. A spherically symmetric charge distribution has a charge density given by $\rho = a/r$, where a is constant. Find the electric field within the charge distribution as a function of r. *Note:* The volume element dV for a spherical shell of radius r and thickness dr is equal to $4\pi r^2 dr$.

66. A solid insulating sphere of radius R has a nonuniform charge density that varies with r according to the expression $\rho = Ar^2$, where A is a constant and $r < R$ is measured from the center of the sphere. (a) Show that the magnitude of the electric field outside $(r > R)$ the sphere is $E = AR^5/5\epsilon_0 r^2$. (b) Show that the magnitude of the electric field inside $(r < R)$ the sphere is $E = Ar^3/5\epsilon_0$. *Note:* The volume element dV for a spherical shell of radius r and thickness dr is equal to $4\pi r^2 dr$.

67. An infinitely long insulating cylinder of radius R has a volume charge density that varies with the radius as

$$\rho = \rho_0\left(a - \frac{r}{b}\right)$$

where ρ_0, a, and b are positive constants and r is the distance from the axis of the cylinder. Use Gauss's law to determine the magnitude of the electric field at radial distances (a) $r < R$ and (b) $r > R$.

68. A particle with charge Q is located on the axis of a circle of radius R at a distance b from the plane of the circle (Fig. P24.68). Show that if one-fourth of the electric flux from the charge passes through the circle, then $R = \sqrt{3}b$.

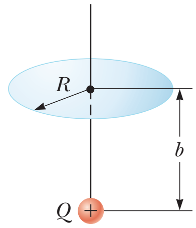


Figure P24.68

69. **Review.** A slab of insulating material (infinite in the y and z directions) has a thickness d and a uniform positive charge density ρ. An edge view of the slab is shown in Figure P24.61. (a) Show that the magnitude of the electric field a distance x from its center and inside the slab is $E = \rho x/\epsilon_0$. (b) **What If?** Suppose an electron of charge $-e$ and mass m_e can move freely within the slab. It is released from rest at a distance x from the center. Show that the electron exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \epsilon_0}}$$

CHAPTER

25

Electric Potential

Processes occurring during thunderstorms cause large differences in electric potential between a thundercloud and the ground. The result of this potential difference is an electrical discharge that we call lightning, such as this display. Notice at the left that a downward channel of lightning (a *stepped leader*) is about to make contact with a channel coming up from the ground (a *return stroke*). *(Costazzurra/Shutterstock.com)*

In Chapter 23, we linked our new study of electromagnetism to our earlier studies of *force*. Now we make a new link to our earlier investigations into *energy.* The concept of potential energy was introduced in Chapter 7 in connection with such conservative forces as the gravitational force and the elastic force exerted by a spring. By using the law of conservation of energy, we could solve various problems in mechanics that were not solvable with an approach using forces. The concept of potential energy is also of great value in the study of electricity. Because the electrostatic force is conservative, electrostatic phenomena can be conveniently described in terms of an electric potential energy. This idea enables us to define a quantity known as *electric potential.* Because the electric potential at any point in an electric field is a scalar quantity, we can use it to describe electrostatic phenomena more simply than if we were to rely only on the electric field and electric forces. The concept of electric potential is of great practical value in the operation of electric circuits and devices that we will study in later chapters.

25.1 Electric Potential and Potential Difference

When a charge q is placed in an electric field $\vec{\mathbf{E}}$ created by some source charge distribution, the particle in a field model tells us that there is an electric force $q\vec{\mathbf{E}}$

acting on the charge. This force is conservative because the force between charges described by Coulomb's law is conservative. Let us identify the charge and the field as a system. If the charge is free to move, it will do so in response to the electric force. Therefore, the electric field will be doing work on the charge. This work is *internal* to the system. This situation is similar to that in a gravitational system: When an object is released near the surface of the Earth, the gravitational force does work on the object. This work is internal to the object–Earth system as discussed in Sections 7.7 and 7.8.

When analyzing electric and magnetic fields, it is common practice to use the notation $d\vec{\mathbf{s}}$ to represent an infinitesimal displacement vector that is oriented tangent to a path through space. This path may be straight or curved, and an integral performed along this path is called either a *path integral* or a *line integral* (the two terms are synonymous).

For an infinitesimal displacement $d\vec{\mathbf{s}}$ of a point charge q immersed in an electric field, the work done within the charge–field system by the electric field on the charge is $W_{\text{int}} = \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{s}} = q\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$. Recall from Equation 7.26 that internal work done in a system is equal to the negative of the change in the potential energy of the system: $W_{\text{int}} = -\Delta U$. Therefore, as the charge q is displaced, the electric potential energy of the charge–field system is changed by an amount $dU = -W_{\text{int}} = -q\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$. For a finite displacement of the charge from some point Ⓐ in space to some other point Ⓑ, the change in electric potential energy of the system is

$$\Delta U = -q\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \qquad (25.1)$$

◀ **Change in electric potential energy of a system**

The integration is performed along the path that q follows as it moves from Ⓐ to Ⓑ. Because the force $q\vec{\mathbf{E}}$ is conservative, this line integral does not depend on the path taken from Ⓐ to Ⓑ.

For a given position of the charge in the field, the charge–field system has a potential energy U relative to the configuration of the system that is defined as $U = 0$. Dividing the potential energy by the charge gives a physical quantity that depends only on the source charge distribution and has a value at every point in an electric field. This quantity is called the **electric potential** (or simply the **potential**) V:

$$V = \frac{U}{q} \qquad (25.2)$$

Because potential energy is a scalar quantity, electric potential also is a scalar quantity.

The **potential difference** $\Delta V = V_{Ⓑ} - V_{Ⓐ}$ between two points Ⓐ and Ⓑ in an electric field is defined as the change in electric potential energy of the system when a charge q is moved between the points (Eq. 25.1) divided by the charge:

$$\Delta V \equiv \frac{\Delta U}{q} = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \qquad (25.3)$$

◀ **Potential difference between two points**

In this definition, the infinitesimal displacement $d\vec{\mathbf{s}}$ is interpreted as the displacement between two points in space rather than the displacement of a point charge as in Equation 25.1.

Just as with potential energy, only *differences* in electric potential are meaningful. We often take the value of the electric potential to be zero at some convenient point in an electric field.

Potential difference should not be confused with difference in potential energy. The potential *difference* between Ⓐ and Ⓑ exists solely because of a source charge and depends on the source charge distribution (consider points Ⓐ and Ⓑ in the discussion above *without* the presence of the charge q). For a potential *energy* to exist, we must have a system of two or more charges. The potential

Pitfall Prevention 25.1

Potential and Potential Energy

The *potential is characteristic of the field only,* independent of a charged particle that may be placed in the field. *Potential energy is characteristic of the charge-field system* due to an interaction between the field and a charged particle placed in the field.

energy belongs to the system and changes only if a charge is moved relative to the rest of the system. This situation is similar to that for the electric field. An electric *field* exists solely because of a source charge. An electric *force* requires two charges: the source charge to set up the field and another charge placed within that field.

Pitfall Prevention 25.2

Voltage A variety of phrases are used to describe the potential difference between two points, the most common being **voltage,** arising from the unit for potential. A voltage *applied* to a device, such as a television, or *across* a device is the same as the potential difference across the device. Despite popular language, voltage is *not* something that moves *through* a device.

Pitfall Prevention 25.3

The Electron Volt The electron volt is a unit of *energy,* NOT of potential. The energy of any system may be expressed in eV, but this unit is most convenient for describing the emission and absorption of visible light from atoms. Energies of nuclear processes are often expressed in MeV.

Let's now consider the situation in which an external agent moves the charge in the field. If the agent moves the charge from Ⓐ to Ⓑ without changing the kinetic energy of the charge, the agent performs work that changes the potential energy of the system: $W = \Delta U$. From Equation 25.3, the work done by an external agent in moving a charge q through an electric field at constant velocity is

$$W = q\,\Delta V \tag{25.4}$$

Because electric potential is a measure of potential energy per unit charge, the SI unit of both electric potential and potential difference is joules per coulomb, which is defined as a **volt** (V):

$$1\text{ V} \equiv 1\text{ J/C}$$

That is, as we can see from Equation 25.4, 1 J of work must be done to move a 1-C charge through a potential difference of 1 V.

Equation 25.3 shows that potential difference also has units of electric field times distance. It follows that the SI unit of electric field (N/C) can also be expressed in volts per meter:

$$1\text{ N/C} = 1\text{ V/m}$$

Therefore, we can state a new interpretation of the electric field:

The electric field is a measure of the rate of change of the electric potential with respect to position.

A unit of energy commonly used in atomic and nuclear physics is the **electron volt** (eV), which is defined as the energy a charge–field system gains or loses when a charge of magnitude e (that is, an electron or a proton) is moved through a potential difference of 1 V. Because 1 V = 1 J/C and the fundamental charge is equal to 1.60×10^{-19} C, the electron volt is related to the joule as follows:

$$1\text{ eV} = 1.60 \times 10^{-19}\text{ C}\cdot\text{V} = 1.60 \times 10^{-19}\text{ J} \tag{25.5}$$

For instance, an electron in the beam of a typical dental x-ray machine may have a speed of 1.4×10^{8} m/s. This speed corresponds to a kinetic energy 1.1×10^{-14} J (using relativistic calculations as discussed in Chapter 39), which is equivalent to 6.7×10^{4} eV. Such an electron has to be accelerated from rest through a potential difference of 67 kV to reach this speed.

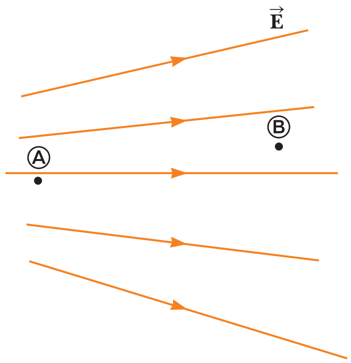


Figure 25.1 (Quick Quiz 25.1) Two points in an electric field.

Quick Quiz 25.1 In Figure 25.1, two points Ⓐ and Ⓑ are located within a region in which there is an electric field. **(i)** How would you describe the potential difference $\Delta V = V_{\text{Ⓑ}} - V_{\text{Ⓐ}}$? (a) It is positive. (b) It is negative. (c) It is zero. **(ii)** A negative charge is placed at Ⓐ and then moved to Ⓑ. How would you describe the change in potential energy of the charge–field system for this process? Choose from the same possibilities.

25.2 Potential Difference in a Uniform Electric Field

Equations 25.1 and 25.3 hold in all electric fields, whether uniform or varying, but they can be simplified for the special case of a uniform field. First, consider a uniform electric field directed along the negative y axis as shown in Figure 25.2a. Let's calculate the potential difference between two points Ⓐ and Ⓑ separated by a dis-

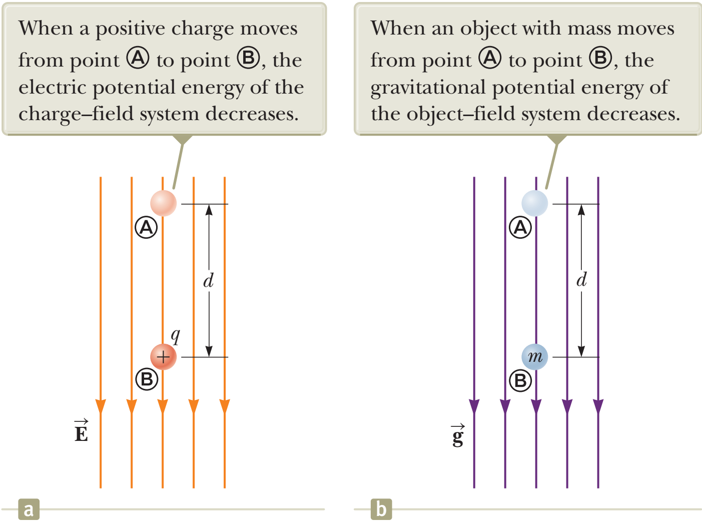


Figure 25.2 (a) When the electric field $\vec{\mathbf{E}}$ is directed downward, point Ⓑ is at a lower electric potential than point Ⓐ. (b) A gravitational analog to the situation in (a).

tance d, where the displacement $\vec{\mathbf{s}}$ points from Ⓐ toward Ⓑ and is parallel to the field lines. Equation 25.3 gives

$$V_{\text{Ⓑ}} - V_{\text{Ⓐ}} = \Delta V = -\int_{\text{Ⓐ}}^{\text{Ⓑ}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_{\text{Ⓐ}}^{\text{Ⓑ}} E\,ds\,(\cos 0°) = -\int_{\text{Ⓐ}}^{\text{Ⓑ}} E\,ds$$

Because E is constant, it can be removed from the integral sign, which gives

$$\Delta V = -E\int_{\text{Ⓐ}}^{\text{Ⓑ}} ds$$

$$\boxed{\Delta V = -Ed} \qquad \textbf{(25.6)}$$

◀ **Potential difference between two points in a uniform electric field**

The negative sign indicates that the electric potential at point Ⓑ is lower than at point Ⓐ; that is, $V_{\text{Ⓑ}} < V_{\text{Ⓐ}}$. Electric field lines *always* point in the direction of decreasing electric potential as shown in Figure 25.2a.

Now suppose a charge q moves from Ⓐ to Ⓑ. We can calculate the change in the potential energy of the charge–field system from Equations 25.3 and 25.6:

$$\Delta U = q\,\Delta V = -qEd \qquad \textbf{(25.7)}$$

This result shows that if q is positive, then ΔU is negative. Therefore, in a system consisting of a positive charge and an electric field, the electric potential energy of the system decreases when the charge moves in the direction of the field. If a positive charge is released from rest in this electric field, it experiences an electric force $q\vec{\mathbf{E}}$ in the direction of $\vec{\mathbf{E}}$ (downward in Fig. 25.2a). Therefore, it accelerates downward, gaining kinetic energy. As the charged particle gains kinetic energy, the electric potential energy of the charge–field system decreases by an equal amount. This equivalence should not be surprising; it is simply conservation of mechanical energy in an isolated system as introduced in Chapter 8.

Figure 25.2b shows an analogous situation with a gravitational field. When a particle with mass m is released in a gravitational field, it accelerates downward, gaining kinetic energy. At the same time, the gravitational potential energy of the object–field system decreases.

The comparison between a system of a positive charge residing in an electrical field and an object with mass residing in a gravitational field in Figure 25.2 is useful for conceptualizing electrical behavior. The electrical situation, however, has one feature that the gravitational situation does not: the charge can be negative. If q is negative, then ΔU in Equation 25.7 is positive and the situation is reversed.

Pitfall Prevention 25.4

The Sign of ΔV The negative sign in Equation 25.6 is due to the fact that we started at point Ⓐ and moved to a new point in the *same* direction as the electric field lines. If we started from Ⓑ and moved to Ⓐ, the potential difference would be $+Ed$. In a uniform electric field, the magnitude of the potential difference is Ed and the sign can be determined by the direction of travel.

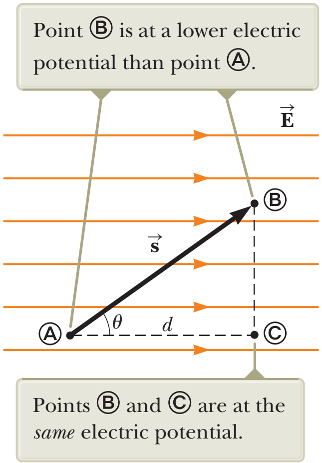


Figure 25.3 A uniform electric field directed along the positive x axis. Three points in the electric field are labeled.

A system consisting of a negative charge and an electric field *gains* electric potential energy when the charge moves in the direction of the field. If a negative charge is released from rest in an electric field, it accelerates in a direction *opposite* the direction of the field. For the negative charge to move in the direction of the field, an external agent must apply a force and do positive work on the charge.

Now consider the more general case of a charged particle that moves between Ⓐ and Ⓑ in a uniform electric field such that the vector $\vec{\mathbf{s}}$ is *not* parallel to the field lines as shown in Figure 25.3. In this case, Equation 25.3 gives

Change in potential between two points in a uniform electric field

$$\Delta V = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\vec{\mathbf{E}} \cdot \int_{Ⓐ}^{Ⓑ} d\vec{\mathbf{s}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{s}} \tag{25.8}$$

where again $\vec{\mathbf{E}}$ was removed from the integral because it is constant. The change in potential energy of the charge–field system is

$$\Delta U = q\Delta V = -q\vec{\mathbf{E}} \cdot \vec{\mathbf{s}} \tag{25.9}$$

Finally, we conclude from Equation 25.8 that all points in a plane perpendicular to a uniform electric field are at the same electric potential. We can see that in Figure 25.3, where the potential difference $V_Ⓑ - V_Ⓐ$ is equal to the potential difference $V_Ⓒ - V_Ⓐ$. (Prove this fact to yourself by working out two dot products for $\vec{\mathbf{E}} \cdot \vec{\mathbf{s}}$: one for $\vec{\mathbf{s}}_{Ⓐ\rightarrow Ⓑ}$, where the angle θ between $\vec{\mathbf{E}}$ and $\vec{\mathbf{s}}$ is arbitrary as shown in Figure 25.3, and one for $\vec{\mathbf{s}}_{Ⓐ\rightarrow Ⓒ}$, where $\theta = 0$.) Therefore, $V_Ⓑ = V_Ⓒ$. The name **equipotential surface** is given to any surface consisting of a continuous distribution of points having the same electric potential.

The equipotential surfaces associated with a uniform electric field consist of a family of parallel planes that are all perpendicular to the field. Equipotential surfaces associated with fields having other symmetries are described in later sections.

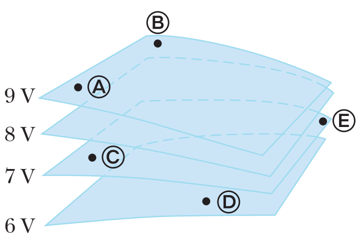


Figure 25.4 (Quick Quiz 25.2) Four equipotential surfaces.

Quick Quiz 25.2 The labeled points in Figure 25.4 are on a series of equipotential surfaces associated with an electric field. Rank (from greatest to least) the work done by the electric field on a positively charged particle that moves from Ⓐ to Ⓑ, from Ⓑ to Ⓒ, from Ⓒ to Ⓓ, and from Ⓓ to Ⓔ.

Example 25.1 The Electric Field Between Two Parallel Plates of Opposite Charge

A battery has a specified potential difference ΔV between its terminals and establishes that potential difference between conductors attached to the terminals. A 12-V battery is connected between two parallel plates as shown in Figure 25.5. The separation between the plates is $d = 0.30$ cm, and we assume the electric field between the plates to be uniform. (This assumption is reasonable if the plate separation is small relative to the plate dimensions and we do not consider locations near the plate edges.) Find the magnitude of the electric field between the plates.

25.1 continued

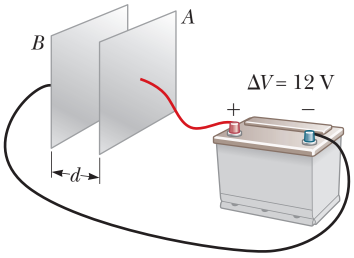


Figure 25.5 (Example 25.1) A 12-V battery connected to two parallel plates. The electric field between the plates has a magnitude given by the potential difference ΔV divided by the plate separation d.

SOLUTION

Conceptualize In Example 24.5, we illustrated the uniform electric field between parallel plates. The new feature to this problem is that the electric field is related to the new concept of electric potential.

Categorize The electric field is evaluated from a relationship between field and potential given in this section, so we categorize this example as a substitution problem.

Use Equation 25.6 to evaluate the magnitude of the electric field between the plates:

$$E = \frac{|V_B - V_A|}{d} = \frac{12\text{ V}}{0.30 \times 10^{-2}\text{ m}} = \boxed{4.0 \times 10^3\text{ V/m}}$$

The configuration of plates in Figure 25.5 is called a *parallel-plate capacitor* and is examined in greater detail in Chapter 26.

Example 25.2 Motion of a Proton in a Uniform Electric Field AM

A proton is released from rest at point Ⓐ in a uniform electric field that has a magnitude of 8.0×10^4 V/m (Fig. 25.6). The proton undergoes a displacement of magnitude $d = 0.50$ m to point Ⓑ in the direction of $\vec{\mathbf{E}}$. Find the speed of the proton after completing the displacement.

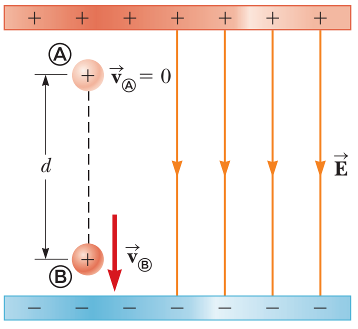


Figure 25.6 (Example 25.2) A proton accelerates from Ⓐ to Ⓑ in the direction of the electric field.

SOLUTION

Conceptualize Visualize the proton in Figure 25.6 moving downward through the potential difference. The situation is analogous to an object falling through a gravitational field. Also compare this example to Example 23.10 where a positive charge was moving in a uniform electric field. In that example, we applied the particle under constant acceleration and nonisolated system models. Now that we have investigated electric potential energy, what model can we use here?

Categorize The system of the proton and the two plates in Figure 25.6 does not interact with the environment, so we model it as an *isolated system* for *energy*.

Analyze

Write the appropriate reduction of Equation 8.2, the conservation of energy equation, for the isolated system of the charge and the electric field:

$$\Delta K + \Delta U = 0$$

Substitute the changes in energy for both terms:

$$\left(\tfrac{1}{2}mv^2 - 0\right) + e\,\Delta V = 0$$

Solve for the final speed of the proton and substitute for ΔV from Equation 25.6:

$$v = \sqrt{\frac{-2e\,\Delta V}{m}} = \sqrt{\frac{-2e(-Ed)}{m}} = \sqrt{\frac{2eEd}{m}}$$

Substitute numerical values:

$$v = \sqrt{\frac{2(1.6 \times 10^{-19}\text{ C})(8.0 \times 10^4\text{ V})(0.50\text{ m})}{1.67 \times 10^{-27}\text{ kg}}}$$

$$= \boxed{2.8 \times 10^6\text{ m/s}}$$

continued

▸ **25.2** continued

Finalize Because ΔV is negative for the field, ΔU is also negative for the proton–field system. The negative value of ΔU means the potential energy of the system decreases as the proton moves in the direction of the electric field. As the proton accelerates in the direction of the field, it gains kinetic energy while the electric potential energy of the system decreases at the same time.

Figure 25.6 is oriented so that the proton moves downward. The proton's motion is analogous to that of an object falling in a gravitational field. Although the gravitational field is always downward at the surface of the Earth, an electric field can be in any direction, depending on the orientation of the plates creating the field. Therefore, Figure 25.6 could be rotated 90° or 180° and the proton could move horizontally or upward in the electric field!

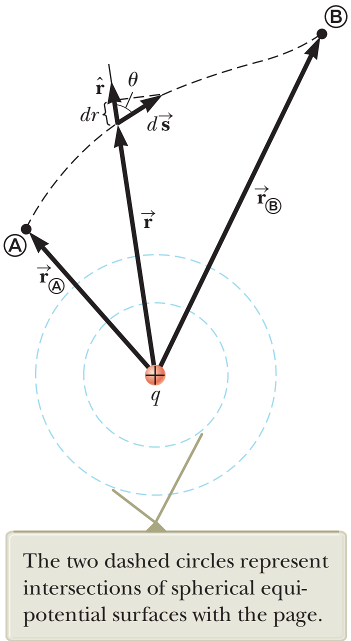


Figure 25.7 The potential difference between points Ⓐ and Ⓑ due to a point charge q depends *only* on the initial and final radial coordinates $r_{\text{Ⓐ}}$ and $r_{\text{Ⓑ}}$.

Pitfall Prevention 25.5

Similar Equation Warning Do not confuse Equation 25.11 for the electric potential of a point charge with Equation 23.9 for the electric field of a point charge. Potential is proportional to $1/r$, whereas the magnitude of the field is proportional to $1/r^2$. The effect of a charge on the space surrounding it can be described in two ways. The charge sets up a vector electric field $\vec{\mathbf{E}}$, which is related to the force experienced by a charge placed in the field. It also sets up a scalar potential V, which is related to the potential energy of the two-charge system when a charge is placed in the field.

25.3 Electric Potential and Potential Energy Due to Point Charges

As discussed in Section 23.4, an isolated positive point charge q produces an electric field directed radially outward from the charge. To find the electric potential at a point located a distance r from the charge, let's begin with the general expression for potential difference, Equation 25.3,

$$V_{\text{Ⓑ}} - V_{\text{Ⓐ}} = -\int_{\text{Ⓐ}}^{\text{Ⓑ}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

where Ⓐ and Ⓑ are the two arbitrary points shown in Figure 25.7. At any point in space, the electric field due to the point charge is $\vec{\mathbf{E}} = (k_e q/r^2)\hat{\mathbf{r}}$ (Eq. 23.9), where $\hat{\mathbf{r}}$ is a unit vector directed radially outward from the charge. Therefore, the quantity $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ can be expressed as

$$\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = k_e \frac{q}{r^2}\,\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}}$$

Because the magnitude of $\hat{\mathbf{r}}$ is 1, the dot product $\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}} = ds\cos\theta$, where θ is the angle between $\hat{\mathbf{r}}$ and $d\vec{\mathbf{s}}$. Furthermore, $ds\cos\theta$ is the projection of $d\vec{\mathbf{s}}$ onto $\hat{\mathbf{r}}$; therefore, $ds\cos\theta = dr$. That is, any displacement $d\vec{\mathbf{s}}$ along the path from point Ⓐ to point Ⓑ produces a change dr in the magnitude of $\vec{\mathbf{r}}$, the position vector of the point relative to the charge creating the field. Making these substitutions, we find that $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = (k_e q/r^2)dr$; hence, the expression for the potential difference becomes

$$V_{\text{Ⓑ}} - V_{\text{Ⓐ}} = -k_e q \int_{r_{\text{Ⓐ}}}^{r_{\text{Ⓑ}}} \frac{dr}{r^2} = k_e \frac{q}{r}\Bigg|_{r_{\text{Ⓐ}}}^{r_{\text{Ⓑ}}}$$

$$V_{\text{Ⓑ}} - V_{\text{Ⓐ}} = k_e q\left[\frac{1}{r_{\text{Ⓑ}}} - \frac{1}{r_{\text{Ⓐ}}}\right] \qquad \textbf{(25.10)}$$

Equation 25.10 shows us that the integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ is *independent* of the path between points Ⓐ and Ⓑ. Multiplying by a charge q_0 that moves between points Ⓐ and Ⓑ, we see that the integral of $q_0\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ is also independent of path. This latter integral, which is the work done by the electric force on the charge q_0, shows that the electric force is conservative (see Section 7.7). We define a field that is related to a conservative force as a **conservative field.** Therefore, Equation 25.10 tells us that the electric field of a fixed point charge q is conservative. Furthermore, Equation 25.10 expresses the important result that the potential difference between any two points Ⓐ and Ⓑ in a field created by a point charge depends only on the radial coordinates $r_{\text{Ⓐ}}$ and $r_{\text{Ⓑ}}$. It is customary to choose the reference of electric potential for a point charge to be $V = 0$ at $r_{\text{Ⓐ}} = \infty$. With this reference choice, the electric potential due to a point charge at any distance r from the charge is

$$V = k_e \frac{q}{r} \qquad \textbf{(25.11)}$$

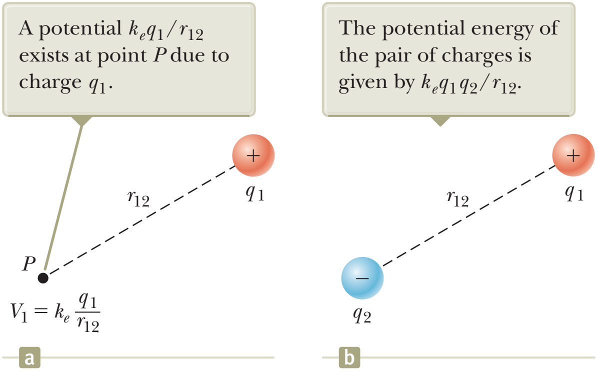


Figure 25.8 (a) Charge q_1 establishes an electric potential V_1 at point P. (b) Charge q_2 is brought from infinity to point P.

We obtain the electric potential resulting from two or more point charges by applying the superposition principle. That is, the total electric potential at some point P due to several point charges is the sum of the potentials due to the individual charges. For a group of point charges, we can write the total electric potential at P as

$$V = k_e \sum_i \frac{q_i}{r_i} \quad \textbf{(25.12)}$$

◀ **Electric potential due to several point charges**

Figure 25.8a shows a charge q_1, which sets up an electric field throughout space. The charge also establishes an electric potential at all points, including point P, where the electric potential is V_1. Now imagine that an external agent brings a charge q_2 from infinity to point P. The work that must be done to do this is given by Equation 25.4, $W = q_2 \Delta V$. This work represents a transfer of energy across the boundary of the two-charge system, and the energy appears in the system as potential energy U when the particles are separated by a distance r_{12} as in Figure 25.8b. From Equation 8.2, we have $W = \Delta U$. Therefore, the **electric potential energy** of a pair of point charges[1] can be found as follows:

$$\Delta U = W = q_2 \Delta V \quad \rightarrow \quad U - 0 = q_2\left(k_e \frac{q_1}{r_{12}} - 0\right)$$

$$U = k_e \frac{q_1 q_2}{r_{12}} \quad \textbf{(25.13)}$$

If the charges are of the same sign, then U is positive. Positive work must be done by an external agent on the system to bring the two charges near each other (because charges of the same sign repel). If the charges are of opposite sign, as in Figure 25.8b, then U is negative. Negative work is done by an external agent against the attractive force between the charges of opposite sign as they are brought near each other; a force must be applied opposite the displacement to prevent q_2 from accelerating toward q_1.

If the system consists of more than two charged particles, we can obtain the total potential energy of the system by calculating U for every *pair* of charges and summing the terms algebraically. For example, the total potential energy of the system of three charges shown in Figure 25.9 is

$$U = k_e\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \quad \textbf{(25.14)}$$

Physically, this result can be interpreted as follows. Imagine q_1 is fixed at the position shown in Figure 25.9 but q_2 and q_3 are at infinity. The work an external agent must do to bring q_2 from infinity to its position near q_1 is $k_e q_1 q_2/r_{12}$, which is the first term in Equation 25.14. The last two terms represent the work required to bring q_3 from infinity to its position near q_1 and q_2. (The result is independent of the order in which the charges are transported.)

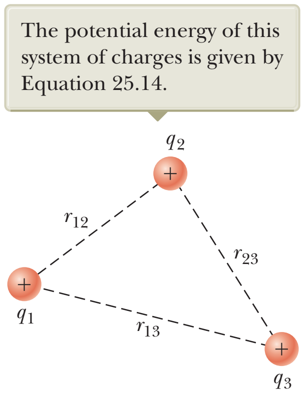


Figure 25.9 Three point charges are fixed at the positions shown.

[1]The expression for the electric potential energy of a system made up of two point charges, Equation 25.13, is of the *same* form as the equation for the gravitational potential energy of a system made up of two point masses, $-Gm_1m_2/r$ (see Chapter 13). The similarity is not surprising considering that both expressions are derived from an inverse-square force law.

Quick Quiz 25.3 In Figure 25.8b, take q_2 to be a negative source charge and q_1 to be a second charge whose sign can be changed. **(i)** If q_1 is initially positive and is changed to a charge of the same magnitude but negative, what happens to the potential at the position of q_1 due to q_2? (a) It increases. (b) It decreases. (c) It remains the same. **(ii)** When q_1 is changed from positive to negative, what happens to the potential energy of the two-charge system? Choose from the same possibilities.

Example 25.3 The Electric Potential Due to Two Point Charges

As shown in Figure 25.10a, a charge $q_1 = 2.00\ \mu C$ is located at the origin and a charge $q_2 = -6.00\ \mu C$ is located at (0, 3.00) m.

(A) Find the total electric potential due to these charges at the point P, whose coordinates are (4.00, 0) m.

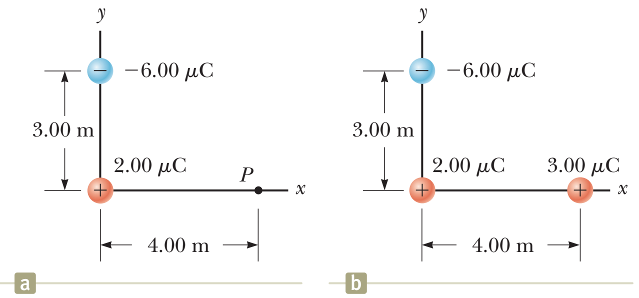


Figure 25.10 (Example 25.3) (a) The electric potential at P due to the two charges q_1 and q_2 is the algebraic sum of the potentials due to the individual charges. (b) A third charge $q_3 = 3.00\ \mu C$ is brought from infinity to point P.

SOLUTION

Conceptualize Recognize first that the 2.00-μC and −6.00-μC charges are source charges and set up an electric field as well as a potential at all points in space, including point P.

Categorize The potential is evaluated using an equation developed in this chapter, so we categorize this example as a substitution problem.

Use Equation 25.12 for the system of two source charges:

$$V_P = k_e\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

Substitute numerical values:

$$V_P = (8.988 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left(\frac{2.00 \times 10^{-6}\ \text{C}}{4.00\ \text{m}} + \frac{-6.00 \times 10^{-6}\ \text{C}}{5.00\ \text{m}}\right)$$

$$= \boxed{-6.29 \times 10^3\ \text{V}}$$

(B) Find the change in potential energy of the system of two charges plus a third charge $q_3 = 3.00\ \mu C$ as the latter charge moves from infinity to point P (Fig. 25.10b).

SOLUTION

Assign $U_i = 0$ for the system to the initial configuration in which the charge q_3 is at infinity. Use Equation 25.2 to evaluate the potential energy for the configuration in which the charge is at P:

$$U_f = q_3 V_P$$

Substitute numerical values to evaluate ΔU:

$$\Delta U = U_f - U_i = q_3 V_P - 0 = (3.00 \times 10^{-6}\ \text{C})(-6.29 \times 10^3\ \text{V})$$

$$= \boxed{-1.89 \times 10^{-2}\ \text{J}}$$

Therefore, because the potential energy of the system has decreased, an external agent has to do positive work to remove the charge q_3 from point P back to infinity.

WHAT IF? You are working through this example with a classmate and she says, "Wait a minute! In part (B), we ignored the potential energy associated with the pair of charges q_1 and q_2!" How would you respond?

Answer Given the statement of the problem, it is not necessary to include this potential energy because part (B) asks for the *change* in potential energy of the system as q_3 is brought in from infinity. Because the configuration of charges q_1 and q_2 does not change in the process, there is no ΔU associated with these charges. Had part (B) asked to find the change in potential energy when *all three* charges start out infinitely far apart and are then brought to the positions in Figure 25.10b, however, you would have to calculate the change using Equation 25.14.

25.4 Obtaining the Value of the Electric Field from the Electric Potential

The electric field $\vec{\mathbf{E}}$ and the electric potential V are related as shown in Equation 25.3, which tells us how to find ΔV if the electric field $\vec{\mathbf{E}}$ is known. What if the situation is reversed? How do we calculate the value of the electric field if the electric potential is known in a certain region?

From Equation 25.3, the potential difference dV between two points a distance ds apart can be expressed as

$$dV = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \tag{25.15}$$

If the electric field has only one component E_x, then $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_x\,dx$. Therefore, Equation 25.15 becomes $dV = -E_x\,dx$, or

$$E_x = -\frac{dV}{dx} \tag{25.16}$$

That is, the x component of the electric field is equal to the negative of the derivative of the electric potential with respect to x. Similar statements can be made about the y and z components. Equation 25.16 is the mathematical statement of the electric field being a measure of the rate of change with position of the electric potential as mentioned in Section 25.1.

Experimentally, electric potential and position can be measured easily with a voltmeter (a device for measuring potential difference) and a meterstick. Consequently, an electric field can be determined by measuring the electric potential at several positions in the field and making a graph of the results. According to Equation 25.16, the slope of a graph of V versus x at a given point provides the magnitude of the electric field at that point.

Imagine starting at a point and then moving through a displacement $d\vec{\mathbf{s}}$ along an equipotential surface. For this motion, $dV = 0$ because the potential is constant along an equipotential surface. From Equation 25.15, we see that $dV = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$; therefore, because the dot product is zero, $\vec{\mathbf{E}}$ must be perpendicular to the displacement along the equipotential surface. This result shows that the equipotential surfaces must always be perpendicular to the electric field lines passing through them.

As mentioned at the end of Section 25.2, the equipotential surfaces associated with a uniform electric field consist of a family of planes perpendicular to the field lines. Figure 25.11a shows some representative equipotential surfaces for this situation.

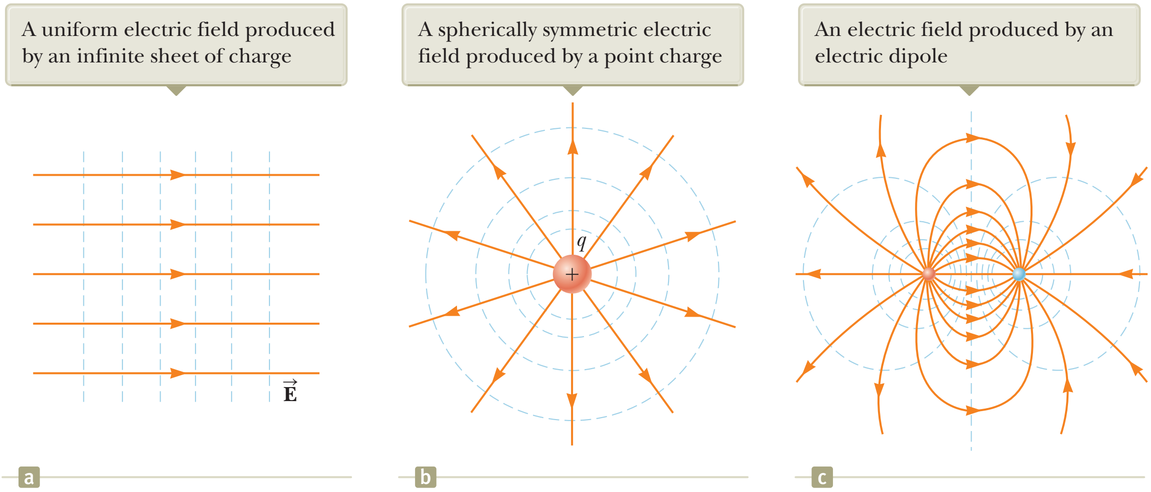


Figure 25.11 Equipotential surfaces (the dashed blue lines are intersections of these surfaces with the page) and electric field lines. In all cases, the equipotential surfaces are *perpendicular* to the electric field lines at every point.

If the charge distribution creating an electric field has spherical symmetry such that the volume charge density depends only on the radial distance r, the electric field is radial. In this case, $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_r\,dr$, and we can express dV as $dV = -E_r\,dr$. Therefore,

$$E_r = -\frac{dV}{dr} \tag{25.17}$$

For example, the electric potential of a point charge is $V = k_e q/r$. Because V is a function of r only, the potential function has spherical symmetry. Applying Equation 25.17, we find that the magnitude of the electric field due to the point charge is $E_r = k_e q/r^2$, a familiar result. Notice that the potential changes only in the radial direction, not in any direction perpendicular to r. Therefore, V (like E_r) is a function only of r, which is again consistent with the idea that equipotential surfaces are perpendicular to field lines. In this case, the equipotential surfaces are a family of spheres concentric with the spherically symmetric charge distribution (Fig. 25.11b). The equipotential surfaces for an electric dipole are sketched in Figure 25.11c.

In general, the electric potential is a function of all three spatial coordinates. If $V(r)$ is given in terms of the Cartesian coordinates, the electric field components E_x, E_y, and E_z can readily be found from $V(x, y, z)$ as the partial derivatives[2]

Finding the electric field from the potential

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \tag{25.18}$$

Quick Quiz 25.4 In a certain region of space, the electric potential is zero everywhere along the x axis. **(i)** From this information, you can conclude that the x component of the electric field in this region is (a) zero, (b) in the positive x direction, or (c) in the negative x direction. **(ii)** Suppose the electric potential is +2 V everywhere along the x axis. From the same choices, what can you conclude about the x component of the electric field now?

25.5 Electric Potential Due to Continuous Charge Distributions

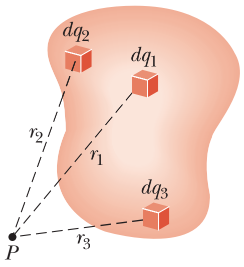


Figure 25.12 The electric potential at point P due to a continuous charge distribution can be calculated by dividing the charge distribution into elements of charge dq and summing the electric potential contributions over all elements. Three sample elements of charge are shown.

In Section 25.3, we found how to determine the electric potential due to a small number of charges. What if we wish to find the potential due to a continuous distribution of charge? The electric potential in this situation can be calculated using two different methods. The first method is as follows. If the charge distribution is known, we consider the potential due to a small charge element dq, treating this element as a point charge (Fig. 25.12). From Equation 25.11, the electric potential dV at some point P due to the charge element dq is

$$dV = k_e \frac{dq}{r} \tag{25.19}$$

where r is the distance from the charge element to point P. To obtain the total potential at point P, we integrate Equation 25.19 to include contributions from all elements of the charge distribution. Because each element is, in general, a different distance from point P and k_e is constant, we can express V as

Electric potential due to a continuous charge distribution

$$\boxed{V = k_e \int \frac{dq}{r}} \tag{25.20}$$

[2]In vector notation, $\vec{\mathbf{E}}$ is often written in Cartesian coordinate systems as

$$\vec{\mathbf{E}} = -\nabla V = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)V$$

where ∇ is called the *gradient operator.*

In effect, we have replaced the sum in Equation 25.12 with an integral. In this expression for V, the electric potential is taken to be zero when point P is infinitely far from the charge distribution.

The second method for calculating the electric potential is used if the electric field is already known from other considerations such as Gauss's law. If the charge distribution has sufficient symmetry, we first evaluate $\vec{\mathbf{E}}$ using Gauss's law and then substitute the value obtained into Equation 25.3 to determine the potential difference ΔV between any two points. We then choose the electric potential V to be zero at some convenient point.

Problem-Solving Strategy Calculating Electric Potential

The following procedure is recommended for solving problems that involve the determination of an electric potential due to a charge distribution.

1. Conceptualize. Think carefully about the individual charges or the charge distribution you have in the problem and imagine what type of potential would be created. Appeal to any symmetry in the arrangement of charges to help you visualize the potential.

2. Categorize. Are you analyzing a group of individual charges or a continuous charge distribution? The answer to this question will tell you how to proceed in the *Analyze* step.

3. Analyze. When working problems involving electric potential, remember that it is a *scalar quantity*, so there are no components to consider. Therefore, when using the superposition principle to evaluate the electric potential at a point, simply take the algebraic sum of the potentials due to each charge. You must keep track of signs, however.

As with potential energy in mechanics, only *changes* in electric potential are significant; hence, the point where the potential is set at zero is arbitrary. When dealing with point charges or a finite-sized charge distribution, we usually define $V = 0$ to be at a point infinitely far from the charges. If the charge distribution itself extends to infinity, however, some other nearby point must be selected as the reference point.

(a) *If you are analyzing a group of individual charges:* Use the superposition principle, which states that when several point charges are present, the resultant potential at a point P in space is the *algebraic sum* of the individual potentials at P due to the individual charges (Eq. 25.12). Example 25.4 below demonstrates this procedure.

(b) *If you are analyzing a continuous charge distribution:* Replace the sums for evaluating the total potential at some point P from individual charges by integrals (Eq. 25.20). The total potential at P is obtained by integrating over the entire charge distribution. For many problems, it is possible in performing the integration to express dq and r in terms of a single variable. To simplify the integration, give careful consideration to the geometry involved in the problem. Examples 25.5 through 25.7 demonstrate such a procedure.

To obtain the potential from the electric field: Another method used to obtain the potential is to start with the definition of the potential difference given by Equation 25.3. If $\vec{\mathbf{E}}$ is known or can be obtained easily (such as from Gauss's law), the line integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ can be evaluated.

4. Finalize. Check to see if your expression for the potential is consistent with your mental representation and reflects any symmetry you noted previously. Imagine varying parameters such as the distance of the observation point from the charges or the radius of any circular objects to see if the mathematical result changes in a reasonable way.

Example 25.4 The Electric Potential Due to a Dipole

An electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance $2a$ as shown in Figure 25.13. The dipole is along the x axis and is centered at the origin.

(A) Calculate the electric potential at point P on the y axis.

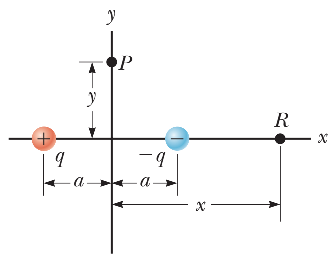


Figure 25.13 (Example 25.4) An electric dipole located on the x axis.

SOLUTION

Conceptualize Compare this situation to that in part (B) of Example 23.6. It is the same situation, but here we are seeking the electric potential rather than the electric field.

Categorize We categorize the problem as one in which we have a small number of particles rather than a continuous distribution of charge. The electric potential can be evaluated by summing the potentials due to the individual charges.

Analyze Use Equation 25.12 to find the electric potential at P due to the two charges:

$$V_P = k_e \sum_i \frac{q_i}{r_i} = k_e\left(\frac{q}{\sqrt{a^2+y^2}} + \frac{-q}{\sqrt{a^2+y^2}}\right) = \boxed{0}$$

(B) Calculate the electric potential at point R on the positive x axis.

SOLUTION

Use Equation 25.12 to find the electric potential at R due to the two charges:

$$V_R = k_e \sum_i \frac{q_i}{r_i} = k_e\left(\frac{-q}{x-a} + \frac{q}{x+a}\right) = \boxed{-\frac{2k_e qa}{x^2-a^2}}$$

(C) Calculate V and E_x at a point on the x axis far from the dipole.

SOLUTION

For point R far from the dipole such that $x >> a$, neglect a^2 in the denominator of the answer to part (B) and write V in this limit:

$$V_R = \lim_{x>>a}\left(-\frac{2k_e qa}{x^2-a^2}\right) \approx \boxed{-\frac{2k_e qa}{x^2} \quad (x >> a)}$$

Use Equation 25.16 and this result to calculate the x component of the electric field at a point on the x axis far from the dipole:

$$E_x = -\frac{dV}{dx} = -\frac{d}{dx}\left(-\frac{2k_e qa}{x^2}\right)$$

$$= 2k_e qa \frac{d}{dx}\left(\frac{1}{x^2}\right) = \boxed{-\frac{4k_e qa}{x^3} \quad (x >> a)}$$

Finalize The potentials in parts (B) and (C) are negative because points on the positive x axis are closer to the negative charge than to the positive charge. For the same reason, the x component of the electric field is negative. Notice that we have a $1/r^3$ falloff of the electric field with distance far from the dipole, similar to the behavior of the electric field on the y axis in Example 23.6.

WHAT IF? Suppose you want to find the electric field at a point P on the y axis. In part (A), the electric potential was found to be zero for all values of y. Is the electric field zero at all points on the y axis?

Answer No. That there is no change in the potential along the y axis tells us only that the y component of the electric field is zero. Look back at Figure 23.13 in Example 23.6. We showed there that the electric field of a dipole on the y axis has only an x component. We could not find the x component in the current example because we do not have an expression for the potential near the y axis as a function of x.

Example 25.5 Electric Potential Due to a Uniformly Charged Ring

(A) Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q.

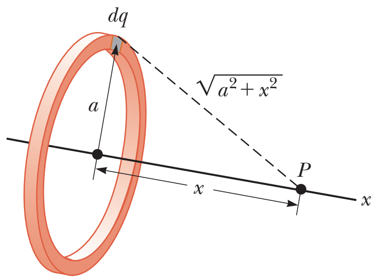


Figure 25.14 (Example 25.5) A uniformly charged ring of radius a lies in a plane perpendicular to the x axis. All elements dq of the ring are the same distance from a point P lying on the x axis.

SOLUTION

Conceptualize Study Figure 25.14, in which the ring is oriented so that its plane is perpendicular to the x axis and its center is at the origin. Notice that the symmetry of the situation means that all the charges on the ring are the same distance from point P. Compare this example to Example 23.8. Notice that no vector considerations are necessary here because electric potential is a scalar.

Categorize Because the ring consists of a continuous distribution of charge rather than a set of discrete charges, we must use the integration technique represented by Equation 25.20 in this example.

Analyze We take point P to be at a distance x from the center of the ring as shown in Figure 25.14.

Use Equation 25.20 to express V in terms of the geometry:

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{a^2 + x^2}}$$

Noting that a and x do not vary for an integration over the ring, bring $\sqrt{a^2 + x^2}$ in front of the integral sign and integrate over the ring:

$$V = \frac{k_e}{\sqrt{a^2 + x^2}} \int dq = \boxed{\frac{k_e Q}{\sqrt{a^2 + x^2}}} \tag{25.21}$$

(B) Find an expression for the magnitude of the electric field at point P.

SOLUTION

From symmetry, notice that along the x axis $\vec{\mathbf{E}}$ can have only an x component. Therefore, apply Equation 25.16 to Equation 25.21:

$$E_x = -\frac{dV}{dx} = -k_e Q \frac{d}{dx}(a^2 + x^2)^{-1/2}$$

$$= -k_e Q(-\tfrac{1}{2})(a^2 + x^2)^{-3/2}(2x)$$

$$E_x = \boxed{\frac{k_e x}{(a^2 + x^2)^{3/2}} Q} \tag{25.22}$$

Finalize The only variable in the expressions for V and E_x is x. That is not surprising because our calculation is valid only for points along the x axis, where y and z are both zero. This result for the electric field agrees with that obtained by direct integration (see Example 23.8). For practice, use the result of part (B) in Equation 25.3 to verify that the potential is given by the expression in part (A).

Example 25.6 Electric Potential Due to a Uniformly Charged Disk

A uniformly charged disk has radius R and surface charge density σ.

(A) Find the electric potential at a point P along the perpendicular central axis of the disk.

SOLUTION

Conceptualize If we consider the disk to be a set of concentric rings, we can use our result from Example 25.5—which gives the potential due to a ring of radius a—and sum the contributions of all rings making up the disk. Figure

continued

25.6 continued

25.15 shows one such ring. Because point P is on the central axis of the disk, symmetry again tells us that all points in a given ring are the same distance from P.

Categorize Because the disk is continuous, we evaluate the potential due to a continuous charge distribution rather than a group of individual charges.

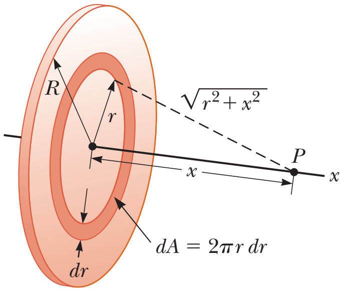


Figure 25.15 (Example 25.6) A uniformly charged disk of radius R lies in a plane perpendicular to the x axis. The calculation of the electric potential at any point P on the x axis is simplified by dividing the disk into many rings of radius r and width dr, with area $2\pi r\,dr$.

Analyze Find the amount of charge dq on a ring of radius r and width dr as shown in Figure 25.15:

$$dq = \sigma\,dA = \sigma(2\pi r\,dr) = 2\pi\sigma r\,dr$$

Use this result in Equation 25.21 in Example 25.5 (with a replaced by the variable r and Q replaced by the differential dq) to find the potential due to the ring:

$$dV = \frac{k_e\,dq}{\sqrt{r^2+x^2}} = \frac{k_e 2\pi\sigma r\,dr}{\sqrt{r^2+x^2}}$$

To obtain the total potential at P, integrate this expression over the limits $r = 0$ to $r = R$, noting that x is a constant:

$$V = \pi k_e\sigma \int_0^R \frac{2r\,dr}{\sqrt{r^2+x^2}} = \pi k_e\sigma \int_0^R (r^2+x^2)^{-1/2}\,2r\,dr$$

This integral is of the common form $\int u^n\,du$, where $n = -\frac{1}{2}$ and $u = r^2 + x^2$, and has the value $u^{n+1}/(n+1)$. Use this result to evaluate the integral:

$$V = \boxed{2\pi k_e\sigma\left[(R^2+x^2)^{1/2} - x\right]} \tag{25.23}$$

(B) Find the x component of the electric field at a point P along the perpendicular central axis of the disk.

SOLUTION

As in Example 25.5, use Equation 25.16 to find the electric field at any axial point:

$$E_x = -\frac{dV}{dx} = \boxed{2\pi k_e\sigma\left[1 - \frac{x}{(R^2+x^2)^{1/2}}\right]} \tag{25.24}$$

Finalize Compare Equation 25.24 with the result of Example 23.9. They are the same. The calculation of V and $\vec{\mathbf{E}}$ for an arbitrary point off the x axis is more difficult to perform because of the absence of symmetry and we do not treat that situation in this book.

Example 25.7 Electric Potential Due to a Finite Line of Charge

A rod of length ℓ located along the x axis has a total charge Q and a uniform linear charge density λ. Find the electric potential at a point P located on the y axis a distance a from the origin (Fig. 25.16).

SOLUTION

Conceptualize The potential at P due to every segment of charge on the rod is positive because every segment carries a positive charge. Notice that we have no symmetry to appeal to here, but the simple geometry should make the problem solvable.

Categorize Because the rod is continuous, we evaluate the potential due to a continuous charge distribution rather than a group of individual charges.

Analyze In Figure 25.16, the rod lies along the x axis, dx is the length of one small segment, and dq is the charge on that segment. Because the rod has a charge per unit length λ, the charge dq on the small segment is $dq = \lambda\,dx$.

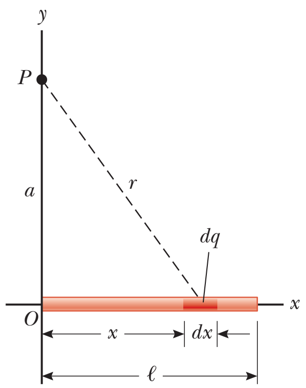


Figure 25.16 (Example 25.7) A uniform line charge of length ℓ located along the x axis. To calculate the electric potential at P, the line charge is divided into segments each of length dx and each carrying a charge $dq = \lambda\,dx$.

25.7 continued

Find the potential at P due to one segment of the rod at an arbitrary position x:

$$dV = k_e \frac{dq}{r} = k_e \frac{\lambda\, dx}{\sqrt{a^2 + x^2}}$$

Find the total potential at P by integrating this expression over the limits $x = 0$ to $x = \ell$:

$$V = \int_0^{\ell} k_e \frac{\lambda\, dx}{\sqrt{a^2 + x^2}}$$

Noting that k_e and $\lambda = Q/\ell$ are constants and can be removed from the integral, evaluate the integral with the help of Appendix B:

$$V = k_e \lambda \int_0^{\ell} \frac{dx}{\sqrt{a^2 + x^2}} = k_e \frac{Q}{\ell} \ln\left(x + \sqrt{a^2 + x^2}\right)\Big|_0^{\ell}$$

Evaluate the result between the limits:

$$V = k_e \frac{Q}{\ell}\left[\ln\left(\ell + \sqrt{a^2 + \ell^2}\right) - \ln a\right] = \boxed{k_e \frac{Q}{\ell} \ln\left(\frac{\ell + \sqrt{a^2 + \ell^2}}{a}\right)} \quad \textbf{(25.25)}$$

Finalize If $\ell << a$, the potential at P should approach that of a point charge because the rod is very short compared to the distance from the rod to P. By using a series expansion for the natural logarithm from Appendix B.5, it is easy to show that Equation 25.25 becomes $V = k_eQ/a$.

WHAT IF? What if you were asked to find the electric field at point P? Would that be a simple calculation?

Answer Calculating the electric field by means of Equation 23.11 would be a little messy. There is no symmetry to appeal to, and the integration over the line of charge would represent a vector addition of electric fields at point P. Using Equation 25.18, you could find E_y by replacing a with y in Equation 25.25 and performing the differentiation with respect to y. Because the charged rod in Figure 25.16 lies entirely to the right of $x = 0$, the electric field at point P would have an x component to the left if the rod is charged positively. You cannot use Equation 25.18 to find the x component of the field, however, because the potential due to the rod was evaluated at a specific value of x ($x = 0$) rather than a general value of x. You would have to find the potential as a function of both x and y to be able to find the x and y components of the electric field using Equation 25.18.

25.6 Electric Potential Due to a Charged Conductor

In Section 24.4, we found that when a solid conductor in equilibrium carries a net charge, the charge resides on the conductor's outer surface. Furthermore, the electric field just outside the conductor is perpendicular to the surface and the field inside is zero.

We now generate another property of a charged conductor, related to electric potential. Consider two points Ⓐ and Ⓑ on the surface of a charged conductor as shown in Figure 25.17. Along a surface path connecting these points, $\vec{\mathbf{E}}$ is always

Pitfall Prevention 25.6

Potential May Not Be Zero

The electric potential inside the conductor is not necessarily zero in Figure 25.17, even though the electric field is zero. Equation 25.15 shows that a zero value of the field results in no *change* in the potential from one point to another inside the conductor. Therefore, the potential everywhere inside the conductor, including the surface, has the same value, which may or may not be zero, depending on where the zero of potential is defined.

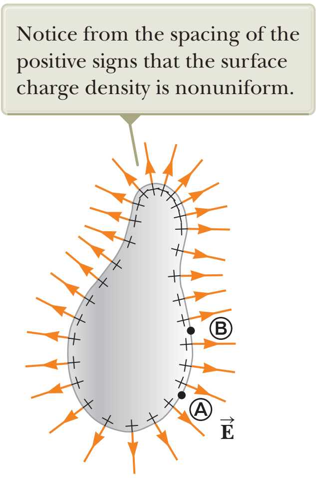


Figure 25.17 An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all the charge resides at the surface, $\vec{\mathbf{E}} = 0$ inside the conductor, and the direction of $\vec{\mathbf{E}}$ immediately outside the conductor is perpendicular to the surface. The electric potential is constant inside the conductor and is equal to the potential at the surface.

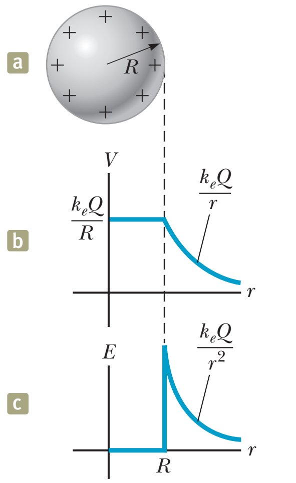


Figure 25.18 (a) The excess charge on a conducting sphere of radius R is uniformly distributed on its surface. (b) Electric potential versus distance r from the center of the charged conducting sphere. (c) Electric field magnitude versus distance r from the center of the charged conducting sphere.

perpendicular to the displacement $d\vec{\mathbf{s}}$; therefore, $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$. Using this result and Equation 25.3, we conclude that the potential difference between Ⓐ and Ⓑ is necessarily zero:

$$V_{Ⓑ} - V_{Ⓐ} = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$$

This result applies to any two points on the surface. Therefore, V is constant everywhere on the surface of a charged conductor in equilibrium. That is,

> the surface of any charged conductor in electrostatic equilibrium is an equipotential surface: every point on the surface of a charged conductor in equilibrium is at the same electric potential. Furthermore, because the electric field is zero inside the conductor, the electric potential is constant everywhere inside the conductor and equal to its value at the surface.

Because of the constant value of the potential, no work is required to move a charge from the interior of a charged conductor to its surface.

Consider a solid metal conducting sphere of radius R and total positive charge Q as shown in Figure 25.18a. As determined in part (A) of Example 24.3, the electric field outside the sphere is k_eQ/r^2 and points radially outward. Because the field outside a spherically symmetric charge distribution is identical to that of a point charge, we expect the potential to also be that of a point charge, k_eQ/r. At the surface of the conducting sphere in Figure 25.18a, the potential must be k_eQ/R. Because the entire sphere must be at the same potential, the potential at any point within the sphere must also be k_eQ/R. Figure 25.18b is a plot of the electric potential as a function of r, and Figure 25.18c shows how the electric field varies with r.

When a net charge is placed on a spherical conductor, the surface charge density is uniform as indicated in Figure 25.18a. If the conductor is nonspherical as in Figure 25.17, however, the surface charge density is high where the radius of curvature is small (as noted in Section 24.4) and low where the radius of curvature is large. Because the electric field immediately outside the conductor is proportional to the surface charge density, the electric field is large near convex points having small radii of curvature and reaches very high values at sharp points. In Example 25.8, the relationship between electric field and radius of curvature is explored mathematically.

Example 25.8 Two Connected Charged Spheres

Two spherical conductors of radii r_1 and r_2 are separated by a distance much greater than the radius of either sphere. The spheres are connected by a conducting wire as shown in Figure 25.19. The charges on the spheres in equilibrium are q_1 and q_2, respectively, and they are uniformly charged. Find the ratio of the magnitudes of the electric fields at the surfaces of the spheres.

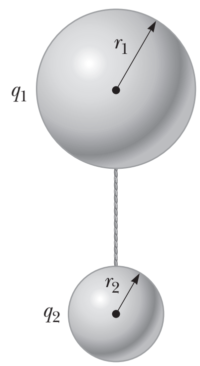


Figure 25.19 (Example 25.8) Two charged spherical conductors connected by a conducting wire. The spheres are at the *same* electric potential V.

SOLUTION

Conceptualize Imagine the spheres are much farther apart than shown in Figure 25.19. Because they are so far apart, the field of one does not affect the charge distribution on the other. The conducting wire between them ensures that both spheres have the same electric potential.

Categorize Because the spheres are so far apart, we model the charge distribution on them as spherically symmetric, and we can model the field and potential outside the spheres to be that due to point charges.

Analyze Set the electric potentials at the surfaces of the spheres equal to each other:

$$V = k_e \frac{q_1}{r_1} = k_e \frac{q_2}{r_2}$$

25.8 continued

Solve for the ratio of charges on the spheres: (1) $\dfrac{q_1}{q_2} = \dfrac{r_1}{r_2}$

Write expressions for the magnitudes of the electric fields at the surfaces of the spheres: $E_1 = k_e \dfrac{q_1}{r_1^{\,2}}$ and $E_2 = k_e \dfrac{q_2}{r_2^{\,2}}$

Evaluate the ratio of these two fields: $\dfrac{E_1}{E_2} = \dfrac{q_1}{q_2}\dfrac{r_2^{\,2}}{r_1^{\,2}}$

Substitute for the ratio of charges from Equation (1): (2) $\dfrac{E_1}{E_2} = \dfrac{r_1}{r_2}\dfrac{r_2^{\,2}}{r_1^{\,2}} = \boxed{\dfrac{r_2}{r_1}}$

Finalize The field is stronger in the vicinity of the smaller sphere even though the electric potentials at the surfaces of both spheres are the same. If $r_2 \to 0$, then $E_2 \to \infty$, verifying the statement above that the electric field is very large at sharp points.

A Cavity Within a Conductor

Suppose a conductor of arbitrary shape contains a cavity as shown in Figure 25.20. Let's assume no charges are inside the cavity. In this case, the electric field inside the cavity must be *zero* regardless of the charge distribution on the outside surface of the conductor as we mentioned in Section 24.4. Furthermore, the field in the cavity is zero even if an electric field exists outside the conductor.

To prove this point, remember that every point on the conductor is at the same electric potential; therefore, any two points Ⓐ and Ⓑ on the cavity's surface must be at the same potential. Now imagine a field $\vec{\mathbf{E}}$ exists in the cavity and evaluate the potential difference $V_Ⓑ - V_Ⓐ$ defined by Equation 25.3:

$$V_Ⓑ - V_Ⓐ = -\int_Ⓐ^Ⓑ \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

Because $V_Ⓑ - V_Ⓐ = 0$, the integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ must be zero for all paths between any two points Ⓐ and Ⓑ on the conductor. The only way that can be true for *all* paths is if $\vec{\mathbf{E}}$ is zero *everywhere* in the cavity. Therefore, a cavity surrounded by conducting walls is a field-free region as long as no charges are inside the cavity.

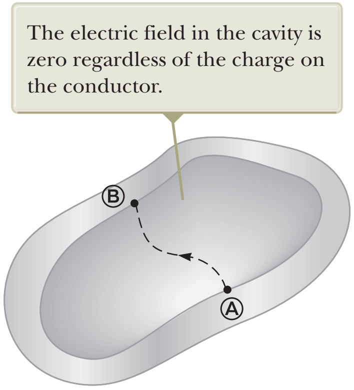


Figure 25.20 A conductor in electrostatic equilibrium containing a cavity.

Corona Discharge

A phenomenon known as **corona discharge** is often observed near a conductor such as a high-voltage power line. When the electric field in the vicinity of the conductor is sufficiently strong, electrons resulting from random ionizations of air molecules near the conductor accelerate away from their parent molecules. These rapidly moving electrons can ionize additional molecules near the conductor, creating more free electrons. The observed glow (or corona discharge) results from the recombination of these free electrons with the ionized air molecules. If a conductor has an irregular shape, the electric field can be very high near sharp points or edges of the conductor; consequently, the ionization process and corona discharge are most likely to occur around such points.

Corona discharge is used in the electrical transmission industry to locate broken or faulty components. For example, a broken insulator on a transmission tower has sharp edges where corona discharge is likely to occur. Similarly, corona discharge will occur at the sharp end of a broken conductor strand. Observation of these discharges is difficult because the visible radiation emitted is weak and most of the radiation is in the ultraviolet. (We will discuss ultraviolet radiation and other portions of the electromagnetic spectrum in Section 34.7.) Even use of traditional ultraviolet cameras is of little help because the radiation from the corona

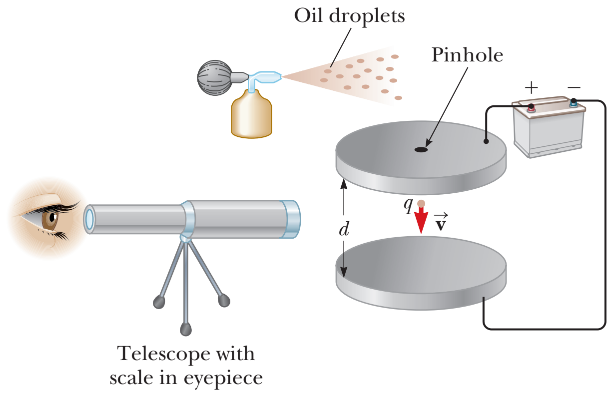


Figure 25.21 Schematic drawing of the Millikan oil-drop apparatus.

discharge is overwhelmed by ultraviolet radiation from the Sun. Newly developed dual-spectrum devices combine a narrow-band ultraviolet camera with a visible-light camera to show a daylight view of the corona discharge in the actual location on the transmission tower or cable. The ultraviolet part of the camera is designed to operate in a wavelength range in which radiation from the Sun is very weak.

25.7 The Millikan Oil-Drop Experiment

Robert Millikan performed a brilliant set of experiments from 1909 to 1913 in which he measured e, the magnitude of the elementary charge on an electron, and demonstrated the quantized nature of this charge. His apparatus, diagrammed in Figure 25.21, contains two parallel metallic plates. Oil droplets from an atomizer are allowed to pass through a small hole in the upper plate. Millikan used x-rays to ionize the air in the chamber so that freed electrons would adhere to the oil drops, giving them a negative charge. A horizontally directed light beam is used to illuminate the oil droplets, which are viewed through a telescope whose long axis is perpendicular to the light beam. When viewed in this manner, the droplets appear as shining stars against a dark background and the rate at which individual drops fall can be determined.

Let's assume a single drop having a mass m and carrying a charge q is being viewed and its charge is negative. If no electric field is present between the plates, the two forces acting on the charge are the gravitational force $m\vec{\mathbf{g}}$ acting downward[3] and a viscous drag force $\vec{\mathbf{F}}_D$ acting upward as indicated in Figure 25.22a. The drag force is proportional to the drop's speed as discussed in Section 6.4. When the drop reaches its terminal speed v_T the two forces balance each other ($mg = F_D$).

Now suppose a battery connected to the plates sets up an electric field between the plates such that the upper plate is at the higher electric potential. In this case, a third force $q\vec{\mathbf{E}}$ acts on the charged drop. The particle in a field model applies twice to the particle: it is in a gravitational field and an electric field. Because q is negative and $\vec{\mathbf{E}}$ is directed downward, this electric force is directed upward as shown in Figure 25.22b. If this upward force is strong enough, the drop moves upward and the drag force $\vec{\mathbf{F}}'_D$ acts downward. When the upward electric force $q\vec{\mathbf{E}}$ balances the sum of the gravitational force and the downward drag force $\vec{\mathbf{F}}'_D$, the drop reaches a new terminal speed v'_T in the upward direction.

With the field turned on, a drop moves slowly upward, typically at rates of hundredths of a centimeter per second. The rate of fall in the absence of a field is comparable. Hence, one can follow a single droplet for hours, alternately rising and falling, by simply turning the electric field on and off.

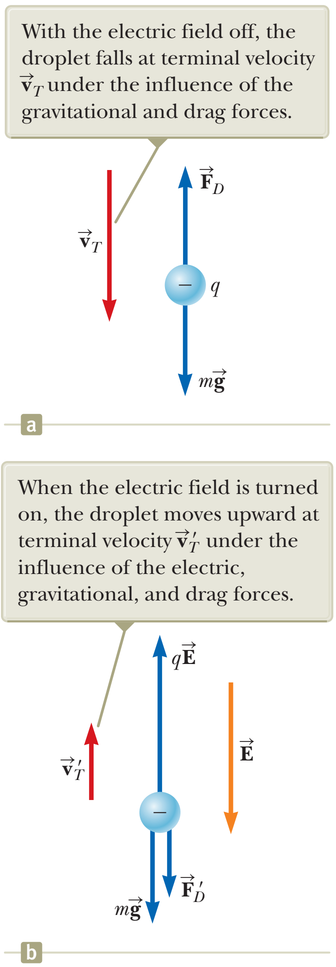


Figure 25.22 The forces acting on a negatively charged oil droplet in the Millikan experiment.

[3]There is also a buoyant force on the oil drop due to the surrounding air. This force can be incorporated as a correction in the gravitational force $m\vec{\mathbf{g}}$ on the drop, so we will not consider it in our analysis.

After recording measurements on thousands of droplets, Millikan and his coworkers found that all droplets, to within about 1% precision, had a charge equal to some integer multiple of the elementary charge e:

$$q = ne \quad n = 0, -1, -2, -3, \ldots$$

where $e = 1.60 \times 10^{-19}$ C. Millikan's experiment yields conclusive evidence that charge is quantized. For this work, he was awarded the Nobel Prize in Physics in 1923.

25.8 Applications of Electrostatics

The practical application of electrostatics is represented by such devices as lightning rods and electrostatic precipitators and by such processes as xerography and the painting of automobiles. Scientific devices based on the principles of electrostatics include electrostatic generators, the field-ion microscope, and ion-drive rocket engines. Details of two devices are given below.

The Van de Graaff Generator

Experimental results show that when a charged conductor is placed in contact with the inside of a hollow conductor, all the charge on the charged conductor is transferred to the hollow conductor. In principle, the charge on the hollow conductor and its electric potential can be increased without limit by repetition of the process.

In 1929, Robert J. Van de Graaff (1901–1967) used this principle to design and build an electrostatic generator, and a schematic representation of it is given in Figure 25.23. This type of generator was once used extensively in nuclear physics research. Charge is delivered continuously to a high-potential electrode by means of a moving belt of insulating material. The high-voltage electrode is a hollow metal dome mounted on an insulating column. The belt is charged at point Ⓐ by means of a corona discharge between comb-like metallic needles and a grounded grid. The needles are maintained at a positive electric potential of typically 10^4 V. The positive charge on the moving belt is transferred to the dome by a second comb of needles at point Ⓑ. Because the electric field inside the dome is negligible, the positive charge on the belt is easily transferred to the conductor regardless of its potential. In practice, it is possible to increase the electric potential of the dome until electrical discharge occurs through the air. Because the "breakdown" electric field in air is about 3×10^6 V/m, a sphere 1.00 m in radius can be raised to a maximum potential of 3×10^6 V. The potential can be increased further by increasing the dome's radius and placing the entire system in a container filled with high-pressure gas.

Van de Graaff generators can produce potential differences as large as 20 million volts. Protons accelerated through such large potential differences receive enough energy to initiate nuclear reactions between themselves and various target nuclei. Smaller generators are often seen in science classrooms and museums. If a person insulated from the ground touches the sphere of a Van de Graaff generator, his or her body can be brought to a high electric potential. The person's hair acquires a net positive charge, and each strand is repelled by all the others as in the opening photograph of Chapter 23.

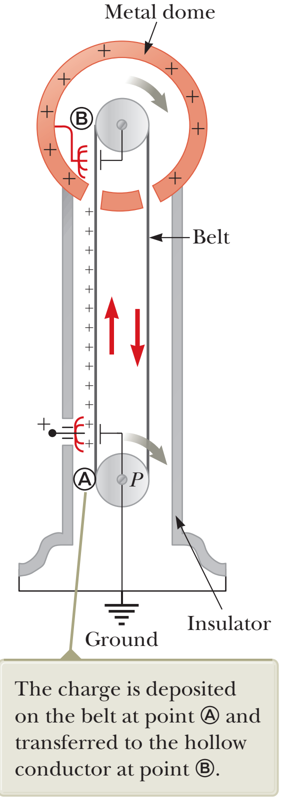


Figure 25.23 Schematic diagram of a Van de Graaff generator. Charge is transferred to the metal dome at the top by means of a moving belt.

The Electrostatic Precipitator

One important application of electrical discharge in gases is the *electrostatic precipitator.* This device removes particulate matter from combustion gases, thereby reducing air pollution. Precipitators are especially useful in coal-burning power plants and industrial operations that generate large quantities of smoke. Current systems are able to eliminate more than 99% of the ash from smoke.

Figure 25.24a (page 766) shows a schematic diagram of an electrostatic precipitator. A high potential difference (typically 40 to 100 kV) is maintained between

Figure 25.24 (a) Schematic diagram of an electrostatic precipitator. Compare the air pollution when the electrostatic precipitator is (b) operating and (c) turned off.

a wire running down the center of a duct and the walls of the duct, which are grounded. The wire is maintained at a negative electric potential with respect to the walls, so the electric field is directed toward the wire. The values of the field near the wire become high enough to cause a corona discharge around the wire; the air near the wire contains positive ions, electrons, and such negative ions as O_2^-. The air to be cleaned enters the duct and moves near the wire. As the electrons and negative ions created by the discharge are accelerated toward the outer wall by the electric field, the dirt particles in the air become charged by collisions and ion capture. Because most of the charged dirt particles are negative, they too are drawn to the duct walls by the electric field. When the duct is periodically shaken, the particles break loose and are collected at the bottom.

In addition to reducing the level of particulate matter in the atmosphere (compare Figs. 25.24b and c), the electrostatic precipitator recovers valuable materials in the form of metal oxides.

Summary

Definitions

The **potential difference** ΔV between points Ⓐ and Ⓑ in an electric field $\vec{\mathbf{E}}$ is defined as

$$\Delta V \equiv \frac{\Delta U}{q} = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \qquad \textbf{(25.3)}$$

where ΔU is given by Equation 25.1 on page 767. The **electric potential** $V = U/q$ is a scalar quantity and has the units of joules per coulomb, where $1 \text{ J/C} \equiv 1 \text{ V}$.

An **equipotential surface** is one on which all points are at the same electric potential. Equipotential surfaces are perpendicular to electric field lines.

Concepts and Principles

When a positive charge q is moved between points Ⓐ and Ⓑ in an electric field $\vec{\mathbf{E}}$, the change in the potential energy of the charge–field system is

$$\Delta U = -q \int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \quad \textbf{(25.1)}$$

The potential difference between two points separated by a distance d in a uniform electric field $\vec{\mathbf{E}}$ is

$$\Delta V = -Ed \quad \textbf{(25.6)}$$

if the direction of travel between the points is in the same direction as the electric field.

If we define $V = 0$ at $r = \infty$, the electric potential due to a point charge at any distance r from the charge is

$$V = k_e \frac{q}{r} \quad \textbf{(25.11)}$$

The electric potential associated with a group of point charges is obtained by summing the potentials due to the individual charges.

The **electric potential energy** associated with a pair of point charges separated by a distance r_{12} is

$$U = k_e \frac{q_1 q_2}{r_{12}} \quad \textbf{(25.13)}$$

We obtain the potential energy of a distribution of point charges by summing terms like Equation 25.13 over all pairs of particles.

If the electric potential is known as a function of coordinates x, y, and z, we can obtain the components of the electric field by taking the negative derivative of the electric potential with respect to the coordinates. For example, the x component of the electric field is

$$E_x = -\frac{dV}{dx} \quad \textbf{(25.16)}$$

The electric potential due to a continuous charge distribution is

$$V = k_e \int \frac{dq}{r} \quad \textbf{(25.20)}$$

Every point on the surface of a charged conductor in electrostatic equilibrium is at the same electric potential. The potential is constant everywhere inside the conductor and equal to its value at the surface.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. In a certain region of space, the electric field is zero. From this fact, what can you conclude about the electric potential in this region? (a) It is zero. (b) It does not vary with position. (c) It is positive. (d) It is negative. (e) None of those answers is necessarily true.

2. Consider the equipotential surfaces shown in Figure 25.4. In this region of space, what is the approximate direction of the electric field? (a) It is out of the page. (b) It is into the page. (c) It is toward the top of the page. (d) It is toward the bottom of the page. (e) The field is zero.

3. **(i)** A metallic sphere A of radius 1.00 cm is several centimeters away from a metallic spherical shell B of radius 2.00 cm. Charge 450 nC is placed on A, with no charge on B or anywhere nearby. Next, the two objects are joined by a long, thin, metallic wire (as shown in Fig. 25.19), and finally the wire is removed. How is the charge shared between A and B? (a) 0 on A, 450 nC on B (b) 90.0 nC on A and 360 nC on B, with equal surface charge densities (c) 150 nC on A and 300 nC on B (d) 225 nC on A and 225 nC on B (e) 450 nC on A and 0 on B **(ii)** A metallic sphere A of radius 1 cm with charge 450 nC hangs on an insulating thread inside an uncharged thin metallic spherical shell B of radius 2 cm. Next, A is made temporarily to touch the inner surface of B. How is the charge then shared between them? Choose from the same possibilities. Arnold Arons, the only physics teacher yet to have his picture on the cover of *Time* magazine, suggested the idea for this question.

4. The electric potential at $x = 3.00$ m is 120 V, and the electric potential at $x = 5.00$ m is 190 V. What is the x component of the electric field in this region, assuming the field is uniform? (a) 140 N/C (b) −140 N/C (c) 35.0 N/C (d) −35.0 N/C (e) 75.0 N/C

5. Rank the potential energies of the four systems of particles shown in Figure OQ25.5 from largest to smallest. Include equalities if appropriate.

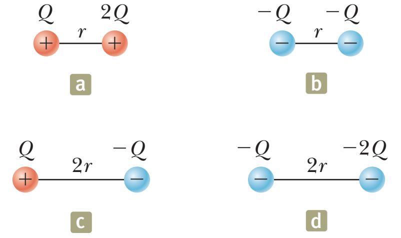


Figure OQ25.5

6. In a certain region of space, a uniform electric field is in the x direction. A particle with negative charge is carried from $x = 20.0$ cm to $x = 60.0$ cm. **(i)** Does the electric potential energy of the charge–field system (a) increase, (b) remain constant, (c) decrease, or (d) change unpredictably? **(ii)** Has the particle moved to a position where the electric potential is (a) higher than before, (b) unchanged, (c) lower than before, or (d) unpredictable?

7. Rank the electric potentials at the four points shown in Figure OQ25.7 from largest to smallest.

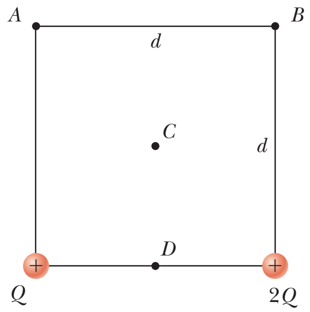


Figure OQ25.7

8. An electron in an x-ray machine is accelerated through a potential difference of 1.00×10^4 V before it hits the target. What is the kinetic energy of the electron in electron volts? (a) 1.00×10^4 eV (b) 1.60×10^{-15} eV (c) 1.60×10^{-22} eV (d) 6.25×10^{22} eV (e) 1.60×10^{-19} eV

9. Rank the electric potential energies of the systems of charges shown in Figure OQ25.9 from largest to smallest. Indicate equalities if appropriate.

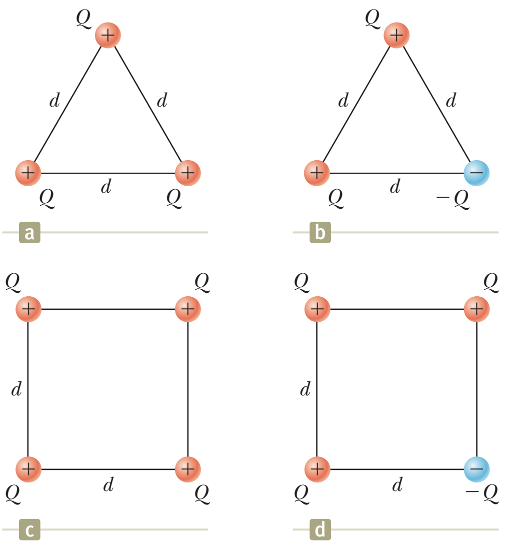


Figure OQ25.9

10. Four particles are positioned on the rim of a circle. The charges on the particles are $+0.500\ \mu$C, $+1.50\ \mu$C, $-1.00\ \mu$C, and $-0.500\ \mu$C. If the electric potential at the center of the circle due to the $+0.500\ \mu$C charge alone is 4.50×10^4 V, what is the total electric potential at the center due to the four charges? (a) 18.0×10^4 V (b) 4.50×10^4 V (c) 0 (d) -4.50×10^4 V (e) 9.00×10^4 V

11. A proton is released from rest at the origin in a uniform electric field in the positive x direction with magnitude 850 N/C. What is the change in the electric potential energy of the proton–field system when the proton travels to $x = 2.50$ m? (a) 3.40×10^{-16} J (b) -3.40×10^{-16} J (c) 2.50×10^{-16} J (d) -2.50×10^{-16} J (e) -1.60×10^{-19} J

$\boxed{12.}$ A particle with charge -40.0 nC is on the x axis at the point with coordinate $x = 0$. A second particle, with charge -20.0 nC, is on the x axis at $x = 0.500$ m. **(i)** Is the point at a finite distance where the electric field is zero (a) to the left of $x = 0$, (b) between $x = 0$ and $x = 0.500$ m, or (c) to the right of $x = 0.500$ m? **(ii)** Is the electric potential zero at this point? (a) No; it is positive. (b) Yes. (c) No; it is negative. **(iii)** Is there a point at a finite distance where the electric potential is zero? (a) Yes; it is to the left of $x = 0$. (b) Yes; it is between $x = 0$ and $x = 0.500$ m. (c) Yes; it is to the right of $x = 0.500$ m. (d) No.

13. A filament running along the x axis from the origin to $x = 80.0$ cm carries electric charge with uniform density. At the point P with coordinates ($x = 80.0$ cm, $y = 80.0$ cm), this filament creates electric potential 100 V. Now we add another filament along the y axis, running from the origin to $y = 80.0$ cm, carrying the same amount of charge with the same uniform density. At the same point P, is the electric potential created by the pair of filaments (a) greater than 200 V, (b) 200 V, (c) 100 V, (d) between 0 and 200 V, or (e) 0?

14. In different experimental trials, an electron, a proton, or a doubly charged oxygen atom (O^{--}), is fired within a vacuum tube. The particle's trajectory carries it through a point where the electric potential is 40.0 V and then through a point at a different potential. Rank each of the following cases according to the change in kinetic energy of the particle over this part of its flight from the largest increase to the largest decrease in kinetic energy. In your ranking, display any cases of equality. (a) An electron moves from 40.0 V to 60.0 V. (b) An electron moves from 40.0 V to 20.0 V. (c) A proton moves from 40.0 V to 20.0 V. (d) A proton moves from 40.0 V to 10.0 V. (e) An O^{--} ion moves from 40.0 V to 60.0 V.

15. A helium nucleus (charge = $2e$, mass = 6.63×10^{-27} kg) traveling at 6.20×10^5 m/s enters an electric field, traveling from point Ⓐ, at a potential of 1.50×10^3 V, to point Ⓑ, at 4.00×10^3 V. What is its speed at point Ⓑ? (a) 7.91×10^5 m/s (b) 3.78×10^5 m/s (c) 2.13×10^5 m/s (d) 2.52×10^6 m/s (e) 3.01×10^8 m/s

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. What determines the maximum electric potential to which the dome of a Van de Graaff generator can be raised?

2. Describe the motion of a proton (a) after it is released from rest in a uniform electric field. Describe the changes (if any) in (b) its kinetic energy and (c) the electric potential energy of the proton–field system.

$\boxed{3.}$ When charged particles are separated by an infinite distance, the electric potential energy of the pair is zero. When the particles are brought close, the electric potential energy of a pair with the same sign is positive, whereas the electric potential energy of a pair with opposite signs is negative. Give a physical explanation of this statement.

4. Study Figure 23.3 and the accompanying text discussion of charging by induction. When the grounding wire is touched to the rightmost point on the sphere in Figure 23.3c, electrons are drained away from the sphere to leave the sphere positively charged. Suppose the grounding wire is touched to the leftmost point on the sphere instead. (a) Will electrons still drain away, moving closer to the negatively charged rod as they do so? (b) What kind of charge, if any, remains on the sphere?

5. Distinguish between electric potential and electric potential energy.

6. Describe the equipotential surfaces for (a) an infinite line of charge and (b) a uniformly charged sphere.

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 25.1 Electric Potential and Potential Difference

Section 25.2 Potential Difference in a Uniform Electric Field

1. M Oppositely charged parallel plates are separated by 5.33 mm. A potential difference of 600 V exists between the plates. (a) What is the magnitude of the electric field between the plates? (b) What is the magnitude of the force on an electron between the plates? (c) How much work must be done on the electron to move it to the negative plate if it is initially positioned 2.90 mm from the positive plate?

2. A uniform electric field of magnitude 250 V/m is directed in the positive x direction. A $+12.0$-μC charge moves from the origin to the point $(x, y) = (20.0$ cm, 50.0 cm). (a) What is the change in the potential energy of the charge–field system? (b) Through what potential difference does the charge move?

3. M (a) Calculate the speed of a proton that is accelerated from rest through an electric potential difference of 120 V. (b) Calculate the speed of an electron that is accelerated through the same electric potential difference.

4. W How much work is done (by a battery, generator, or some other source of potential difference) in moving Avogadro's number of electrons from an initial point where the electric potential is 9.00 V to a point where the electric potential is -5.00 V? (The potential in each case is measured relative to a common reference point.)

5. W A uniform electric field of magnitude 325 V/m is directed in the negative y direction in Figure P25.5. The coordinates of point Ⓐ are $(-0.200, -0.300)$ m, and those of point Ⓑ are $(0.400, 0.500)$ m. Calculate the electric potential difference $V_{Ⓑ} - V_{Ⓐ}$ using the dashed-line path.

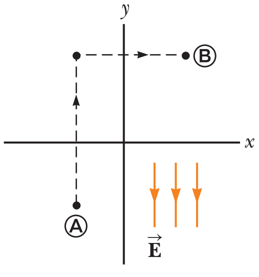


Figure P25.5

6. Starting with the definition of work, prove that at every point on an equipotential surface, the surface must be perpendicular to the electric field there.

7. AMT M An electron moving parallel to the x axis has an initial speed of 3.70×10^6 m/s at the origin. Its speed is reduced to 1.40×10^5 m/s at the point $x = 2.00$ cm. (a) Calculate the electric potential difference between the origin and that point. (b) Which point is at the higher potential?

8. (a) Find the electric potential difference ΔV_e required to stop an electron (called a "stopping potential") moving with an initial speed of 2.85×10^7 m/s. (b) Would a proton traveling at the same speed require a greater or lesser magnitude of electric potential difference? Explain. (c) Find a symbolic expression for the ratio of the proton stopping potential and the electron stopping potential, $\Delta V_p / \Delta V_e$.

9. AMT A particle having charge $q = +2.00$ μC and mass $m = 0.010\,0$ kg is connected to a string that is $L = 1.50$ m long and tied to the pivot point P in Figure P25.9. The particle, string, and pivot point all lie on a frictionless,

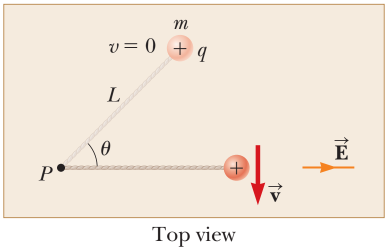


Figure P25.9

horizontal table. The particle is released from rest when the string makes an angle $\theta = 60.0°$ with a uniform electric field of magnitude $E = 300$ V/m. Determine the speed of the particle when the string is parallel to the electric field.

10. **GP** **Review.** A block having mass m and charge $+Q$ is connected to an insulating spring having a force constant k. The block lies on a frictionless, insulating, horizontal track, and the system is immersed in a uniform electric field of magnitude E directed as shown in Figure P25.10. The block is released from rest when the spring is unstretched (at $x = 0$). We wish to show that the ensuing motion of the block is simple harmonic. (a) Consider the system of the block, the spring, and the electric field. Is this system isolated or nonisolated? (b) What kinds of potential energy exist within this system? (c) Call the initial configuration of the system that existing just as the block is released from rest. The final configuration is when the block momentarily comes to rest again. What is the value of x when the block comes to rest momentarily? (d) At some value of x we will call $x = x_0$, the block has zero net force on it. What analysis model describes the particle in this situation? (e) What is the value of x_0? (f) Define a new coordinate system x' such that $x' = x - x_0$. Show that x' satisfies a differential equation for simple harmonic motion. (g) Find the period of the simple harmonic motion. (h) How does the period depend on the electric field magnitude?

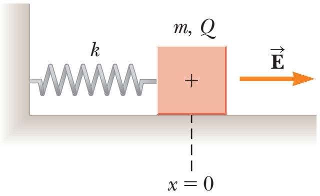


Figure P25.10

11. An insulating rod having linear charge density $\lambda = 40.0\ \mu$C/m and linear mass density $\mu = 0.100$ kg/m is released from rest in a uniform electric field $E = 100$ V/m directed perpendicular to the rod (Fig. P25.11). (a) Determine the speed of the rod after it has traveled 2.00 m. (b) **What If?** How does your answer to part (a) change if the electric field is not perpendicular to the rod? Explain.

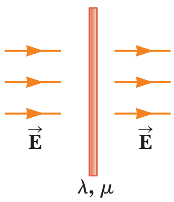


Figure P25.11

Section 25.3 Electric Potential and Potential Energy Due to Point Charges

> *Note:* Unless stated otherwise, assume the reference level of potential is $V = 0$ at $r = \infty$.

12. (a) Calculate the electric potential 0.250 cm from an electron. (b) What is the electric potential difference between two points that are 0.250 cm and 0.750 cm from an electron? (c) How would the answers change if the electron were replaced with a proton?

13. Two point charges are on the y axis. A 4.50-μC charge is located at $y = 1.25$ cm, and a -2.24-μC charge is located at $y = -1.80$ cm. Find the total electric potential at (a) the origin and (b) the point whose coordinates are (1.50 cm, 0).

14. The two charges in Figure P25.14 are separated by $d = 2.00$ cm. Find the electric potential at (a) point A and (b) point B, which is halfway between the charges.

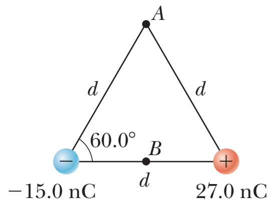


Figure P25.14

15. Three positive charges are located at the corners of an equilateral triangle as in Figure P25.15. Find an expression for the electric potential at the center of the triangle.

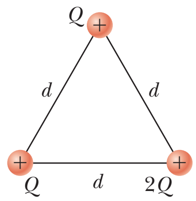


Figure P25.15

16. **M** Two point charges $Q_1 = +5.00$ nC and $Q_2 = -3.00$ nC are separated by 35.0 cm. (a) What is the electric potential at a point midway between the charges? (b) What is the potential energy of the pair of charges? What is the significance of the algebraic sign of your answer?

17. Two particles, with charges of 20.0 nC and −20.0 nC, are placed at the points with coordinates (0, 4.00 cm) and (0, −4.00 cm) as shown in Figure P25.17. A particle with charge 10.0 nC is located at the origin. (a) Find the electric potential energy of the configuration of the three fixed charges. (b) A fourth particle, with a mass of 2.00×10^{-13} kg and a charge of 40.0 nC, is released from rest at the point (3.00 cm, 0). Find its speed after it has moved freely to a very large distance away.

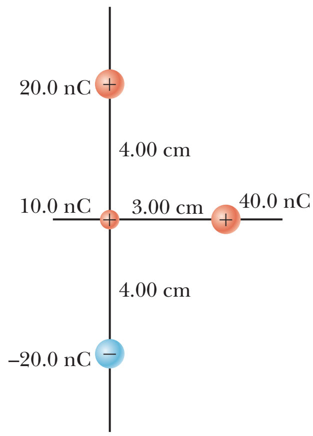


Figure P25.17

18. The two charges in Figure P25.18 are separated by a distance $d = 2.00$ cm, and $Q = +5.00$ nC. Find (a) the electric potential at A, (b) the electric potential at B, and (c) the electric potential difference between B and A.

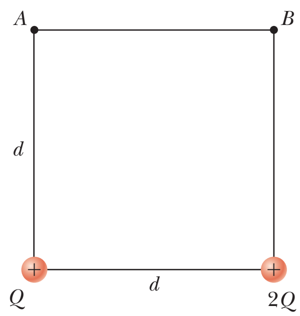


Figure P25.18

19. **W** Given two particles with 2.00-μC charges as shown in Figure P25.19 and a particle with charge $q = 1.28 \times 10^{-18}$ C at the origin, (a) what is the net force exerted by the two 2.00-μC charges on the charge q? (b) What is the electric field at the origin due to the two 2.00-μC particles? (c) What is the electric potential at the origin due to the two 2.00-μC particles?

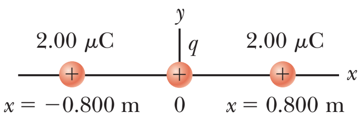


Figure P25.19

20. M At a certain distance from a charged particle, the magnitude of the electric field is 500 V/m and the electric potential is −3.00 kV. (a) What is the distance to the particle? (b) What is the magnitude of the charge?

21. Four point charges each having charge Q are located at the corners of a square having sides of length a. Find expressions for (a) the total electric potential at the center of the square due to the four charges and (b) the work required to bring a fifth charge q from infinity to the center of the square.

22. M The three charged particles in Figure P25.22 are at the vertices of an isosceles triangle (where d = 2.00 cm). Taking q = 7.00 μC, calculate the electric potential at point A, the midpoint of the base.

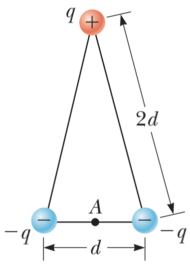


Figure P25.22

23. A particle with charge $+q$ is at the origin. A particle with charge $-2q$ is at x = 2.00 m on the x axis. (a) For what finite value(s) of x is the electric field zero? (b) For what finite value(s) of x is the electric potential zero?

24. Show that the amount of work required to assemble four identical charged particles of magnitude Q at the corners of a square of side s is $5.41k_eQ^2/s$.

25. Two particles each with charge +2.00 μC are located on the x axis. One is at x = 1.00 m, and the other is at x = −1.00 m. (a) Determine the electric potential on the y axis at y = 0.500 m. (b) Calculate the change in electric potential energy of the system as a third charged particle of −3.00 μC is brought from infinitely far away to a position on the y axis at y = 0.500 m.

26. Two charged particles of equal magnitude are located along the y axis equal distances above and below the x axis as shown in Figure P25.26. (a) Plot a graph of the electric potential at points along the x axis over the interval $-3a < x < 3a$. You should plot the potential in units of k_eQ/a. (b) Let the charge of the particle located at $y = -a$ be negative. Plot the potential along the y axis over the interval $-4a < y < 4a$.

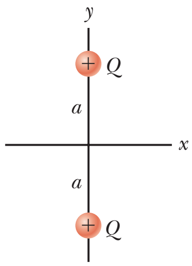


Figure P25.26

27. W Four identical charged particles (q = +10.0 μC) are located on the corners of a rectangle as shown in Figure P25.27. The dimensions of the rectangle are L = 60.0 cm and W = 15.0 cm. Calculate the change in electric potential energy of the system as the particle at the lower left corner in Figure P25.27 is brought to this position from infinitely far away. Assume the other three particles in Figure P25.27 remain fixed in position.

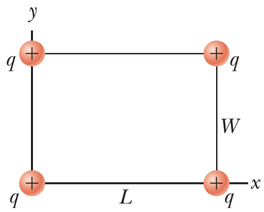


Figure P25.27

28. Three particles with equal positive charges q are at the corners of an equilateral triangle of side a as shown in Figure P25.28. (a) At what point, if any, in the plane of the particles is the electric potential zero? (b) What is the electric potential at the position of one of the particles due to the other two particles in the triangle?

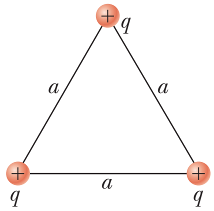


Figure P25.28

29. Five particles with equal negative charges $-q$ are placed symmetrically around a circle of radius R. Calculate the electric potential at the center of the circle.

30. **Review.** A light, unstressed spring has length d. Two identical particles, each with charge q, are connected to the opposite ends of the spring. The particles are held stationary a distance d apart and then released at the same moment. The system then oscillates on a frictionless, horizontal table. The spring has a bit of internal kinetic friction, so the oscillation is damped. The particles eventually stop vibrating when the distance between them is $3d$. Assume the system of the spring and two charged particles is isolated. Find the increase in internal energy that appears in the spring during the oscillations.

31. AMT **Review.** Two insulating spheres have radii 0.300 cm and 0.500 cm, masses 0.100 kg and 0.700 kg, and uniformly distributed charges −2.00 μC and 3.00 μC. They are released from rest when their centers are separated by 1.00 m. (a) How fast will each be moving when they collide? (b) **What If?** If the spheres were conductors, would the speeds be greater or less than those calculated in part (a)? Explain.

32. **Review.** Two insulating spheres have radii r_1 and r_2, masses m_1 and m_2, and uniformly distributed charges $-q_1$ and q_2. They are released from rest when their centers are separated by a distance d. (a) How fast is each moving when they collide? (b) **What If?** If the spheres were conductors, would their speeds be greater or less than those calculated in part (a)? Explain.

33. How much work is required to assemble eight identical charged particles, each of magnitude q, at the corners of a cube of side s?

34. Four identical particles, each having charge q and mass m, are released from rest at the vertices of a square of side L. How fast is each particle moving when their distance from the center of the square doubles?

35. AMT In 1911, Ernest Rutherford and his assistants Geiger and Marsden conducted an experiment in which they scattered alpha particles (nuclei of helium atoms) from thin sheets of gold. An alpha particle, having charge $+2e$ and mass 6.64×10^{-27} kg, is a product of certain radioactive decays. The results of the experiment led Rutherford to the idea that most of an atom's mass is in a very small nucleus, with electrons in orbit around it. (This is the planetary model of the atom, which we'll study in Chapter 42.) Assume an alpha particle, initially very far from a stationary gold nucleus, is fired with a velocity of 2.00×10^7 m/s directly toward the nucleus (charge $+79e$). What is the smallest distance between the alpha particle and the nucleus before the alpha particle reverses direction? Assume the gold nucleus remains stationary.

Section 25.4 Obtaining the Value of the Electric Field from the Electric Potential

36. Figure P25.36 represents a graph of the electric potential in a region of space versus position x, where the electric field is parallel to the x axis. Draw a graph of the x component of the electric field versus x in this region.

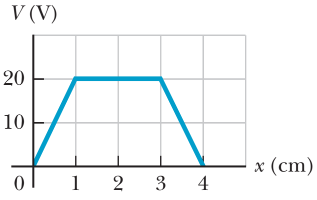


Figure P25.36

37. The potential in a region between $x = 0$ and $x = 6.00$ m is $V = a + bx$, where $a = 10.0$ V and $b = -7.00$ V/m. Determine (a) the potential at $x = 0$, 3.00 m, and 6.00 m and (b) the magnitude and direction of the electric field at $x = 0$, 3.00 m, and 6.00 m.

38. An electric field in a region of space is parallel to the x axis. The electric potential varies with position as shown in Figure P25.38. Graph the x component of the electric field versus position in this region of space.

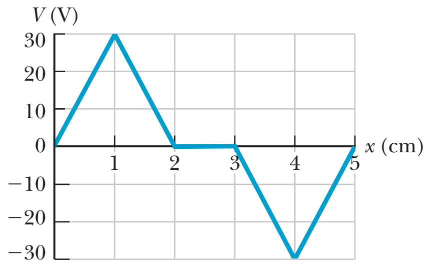
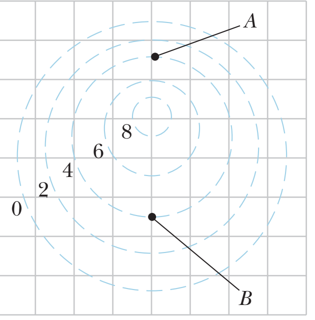


Figure P25.38

39. Over a certain region of space, the electric potential is $V = 5x - 3x^2y + 2yz^2$. (a) Find the expressions for the x, y, and z components of the electric field over this region. (b) What is the magnitude of the field at the point P that has coordinates (1.00, 0, −2.00) m?

40. Figure P25.40 shows several equipotential lines, each labeled by its potential in volts. The distance between the lines of the square grid represents 1.00 cm. (a) Is the magnitude of the field larger at A or at B? Explain how you can tell. (b) Explain what you can determine about $\vec{\mathbf{E}}$ at B. (c) Represent what the electric field looks like by drawing at least eight field lines.



Numerical values are in volts.

Figure P25.40

41. The electric potential inside a charged spherical conductor of radius R is given by $V = k_eQ/R$, and the potential outside is given by $V = k_eQ/r$. Using $E_r = -dV/dr$, derive the electric field (a) inside and (b) outside this charge distribution.

42. It is shown in Example 25.7 that the potential at a point P a distance a above one end of a uniformly charged rod of length ℓ lying along the x axis is

$$V = k_e \frac{Q}{\ell} \ln\left(\frac{\ell + \sqrt{a^2 + \ell^2}}{a}\right)$$

Use this result to derive an expression for the y component of the electric field at P.

Section 25.5 Electric Potential Due to Continuous Charge Distributions

43. Consider a ring of radius R with the total charge Q spread uniformly over its perimeter. What is the potential difference between the point at the center of the ring and a point on its axis a distance $2R$ from the center?

44. A uniformly charged insulating rod of length 14.0 cm is bent into the shape of a semicircle as shown in Figure P25.44. The rod has a total charge of $-7.50\ \mu$C. Find the electric potential at O, the center of the semicircle.

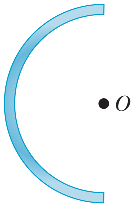


Figure P25.44

45. A rod of length L (Fig. P25.45) lies along the x axis with its left end at the origin. It has a nonuniform charge

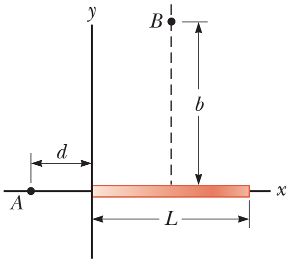


Figure P25.45 Problems 45 and 46.

density $\lambda = \alpha x$, where α is a positive constant. (a) What are the units of α? (b) Calculate the electric potential at A.

46. For the arrangement described in Problem 45, calculate the electric potential at point B, which lies on the perpendicular bisector of the rod a distance b above the x axis.

47. W A wire having a uniform linear charge density λ is bent into the shape shown in Figure P25.47. Find the electric potential at point O.

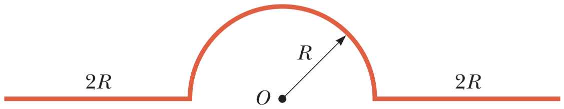


Figure P25.47

Section 25.6 Electric Potential Due to a Charged Conductor

48. The electric field magnitude on the surface of an irregularly shaped conductor varies from 56.0 kN/C to 28.0 kN/C. Can you evaluate the electric potential on the conductor? If so, find its value. If not, explain why not.

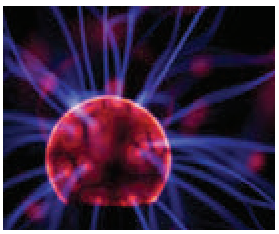
49. How many electrons should be removed from an initially uncharged spherical conductor of radius 0.300 m to produce a potential of 7.50 kV at the surface?

50. M A spherical conductor has a radius of 14.0 cm and a charge of 26.0 μC. Calculate the electric field and the electric potential at (a) r = 10.0 cm, (b) r = 20.0 cm, and (c) r = 14.0 cm from the center.

51. Electric charge can accumulate on an airplane in flight. You may have observed needle-shaped metal extensions on the wing tips and tail of an airplane. Their purpose is to allow charge to leak off before much of it accumulates. The electric field around the needle is much larger than the field around the body of the airplane and can become large enough to produce dielectric breakdown of the air, discharging the airplane. To model this process, assume two charged spherical conductors are connected by a long conducting wire and a 1.20-μC charge is placed on the combination. One sphere, representing the body of the airplane, has a radius of 6.00 cm; the other, representing the tip of the needle, has a radius of 2.00 cm. (a) What is the electric potential of each sphere? (b) What is the electric field at the surface of each sphere?

Section 25.8 Applications of Electrostatics

52. M Lightning can be studied with a Van de Graaff generator, which consists of a spherical dome on which charge is continuously deposited by a moving belt. Charge can be added until the electric field at the surface of the dome becomes equal to the dielectric strength of air. Any more charge leaks off in sparks as shown in Figure P25.52. Assume the dome has a diameter of 30.0 cm and is surrounded by dry air with a "breakdown" electric field of 3.00×10^6 V/m. (a) What is the maximum potential of the dome? (b) What is the maximum charge on the dome?



David Evison/Shutterstock.com

Figure P25.52

Additional Problems

53. *Why is the following situation impossible?* In the Bohr model of the hydrogen atom, an electron moves in a circular orbit about a proton. The model states that the electron can exist only in certain allowed orbits around the proton: those whose radius r satisfies $r = n^2(0.052\,9 \text{ nm})$, where $n = 1, 2, 3, \ldots$. For one of the possible allowed states of the atom, the electric potential energy of the system is -13.6 eV.

54. **Review.** In fair weather, the electric field in the air at a particular location immediately above the Earth's surface is 120 N/C directed downward. (a) What is the surface charge density on the ground? Is it positive or negative? (b) Imagine the surface charge density is uniform over the planet. What then is the charge of the whole surface of the Earth? (c) What is the Earth's electric potential due to this charge? (d) What is the difference in potential between the head and the feet of a person 1.75 m tall? (Ignore any charges in the atmosphere.) (e) Imagine the Moon, with 27.3% of the radius of the Earth, had a charge 27.3% as large, with the same sign. Find the electric force the Earth would then exert on the Moon. (f) State how the answer to part (e) compares with the gravitational force the Earth exerts on the Moon.

55. **Review.** From a large distance away, a particle of mass 2.00 g and charge 15.0 μC is fired at $21.0\hat{\mathbf{i}}$ m/s straight toward a second particle, originally stationary but free to move, with mass 5.00 g and charge 8.50 μC. Both particles are constrained to move only along the x axis. (a) At the instant of closest approach, both particles will be moving at the same velocity. Find this velocity. (b) Find the distance of closest approach. After the interaction, the particles will move far apart again. At this time, find the velocity of (c) the 2.00-g particle and (d) the 5.00-g particle.

56. **Review.** From a large distance away, a particle of mass m_1 and positive charge q_1 is fired at speed v in the positive x direction straight toward a second particle, originally stationary but free to move, with mass m_2 and positive charge q_2. Both particles are constrained to move only along the x axis. (a) At the instant of closest approach, both particles will be moving at the same velocity. Find this velocity. (b) Find the distance of closest approach. After the interaction, the particles will move far apart again. At this time, find the velocity of (c) the particle of mass m_1 and (d) the particle of mass m_2.

57. M The liquid-drop model of the atomic nucleus suggests high-energy oscillations of certain nuclei can split the nucleus into two unequal fragments plus a few neutrons. The fission products acquire kinetic energy from their mutual Coulomb repulsion. Assume the charge is distributed uniformly throughout the volume of each spherical fragment and, immediately before separating, each fragment is at rest and their surfaces are in contact. The electrons surrounding the nucleus can be ignored. Calculate the electric potential energy (in electron volts) of two spherical fragments from a uranium nucleus having the following charges and radii: $38e$ and 5.50×10^{-15} m, and $54e$ and 6.20×10^{-15} m.

58. On a dry winter day, you scuff your leather-soled shoes across a carpet and get a shock when you extend the tip of one finger toward a metal doorknob. In a dark room, you see a spark perhaps 5 mm long. Make order-of-magnitude estimates of (a) your electric potential and (b) the charge on your body before you touch the doorknob. Explain your reasoning.

59. The electric potential immediately outside a charged conducting sphere is 200 V, and 10.0 cm farther from the center of the sphere the potential is 150 V. Determine (a) the radius of the sphere and (b) the charge on it. The electric potential immediately outside another charged conducting sphere is 210 V, and 10.0 cm farther from the center the magnitude of the electric field is 400 V/m. Determine (c) the radius of the sphere and (d) its charge on it. (e) Are the answers to parts (c) and (d) unique?

60. (a) Use the exact result from Example 25.4 to find the electric potential created by the dipole described in the example at the point $(3a, 0)$. (b) Explain how this answer compares with the result of the approximate expression that is valid when x is much greater than a.

61. Calculate the work that must be done on charges brought from infinity to charge a spherical shell of radius $R = 0.100$ m to a total charge $Q = 125\ \mu$C.

62. Calculate the work that must be done on charges brought from infinity to charge a spherical shell of radius R to a total charge Q.

63. The electric potential everywhere on the xy plane is

$$V = \frac{36}{\sqrt{(x+1)^2 + y^2}} - \frac{45}{\sqrt{x^2 + (y-2)^2}}$$

where V is in volts and x and y are in meters. Determine the position and charge on each of the particles that create this potential.

64. *Why is the following situation impossible?* You set up an apparatus in your laboratory as follows. The x axis is the symmetry axis of a stationary, uniformly charged ring of radius $R = 0.500$ m and charge $Q = 50.0\ \mu$C (Fig. P25.64). You place a particle with charge $Q = 50.0\ \mu$C and mass $m = 0.100$ kg at the center of the ring and arrange for it to be constrained to move only along the x axis. When it is displaced slightly, the particle is repelled by the ring and accelerates along the x axis. The particle moves faster than you expected and strikes the opposite wall of your laboratory at 40.0 m/s.

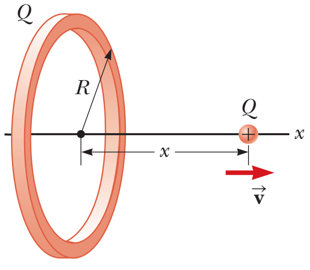


Figure P25.64

65. From Gauss's law, the electric field set up by a uniform line of charge is

$$\vec{\mathbf{E}} = \left(\frac{\lambda}{2\pi\epsilon_0 r}\right)\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially away from the line and λ is the linear charge density along the line. Derive an expression for the potential difference between $r = r_1$ and $r = r_2$.

66. A uniformly charged filament lies along the x axis between $x = a = 1.00$ m and $x = a + \ell = 3.00$ m as shown in Figure P25.66. The total charge on the filament is 1.60 nC. Calculate successive approximations for the electric potential at the origin by modeling the filament as (a) a single charged particle at $x = 2.00$ m, (b) two 0.800-nC charged particles at $x = 1.5$ m and $x = 2.5$ m, and (c) four 0.400-nC charged particles at $x = 1.25$ m, $x = 1.75$ m, $x = 2.25$ m, and $x = 2.75$ m. (d) Explain how the results compare with the potential given by the exact expression

$$V = \frac{k_e Q}{\ell} \ln\left(\frac{\ell + a}{a}\right)$$

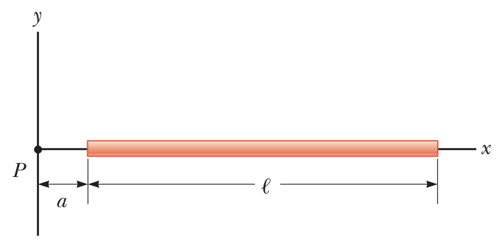


Figure P25.66

67. The thin, uniformly charged rod shown in Figure P25.67 has a linear charge density λ. Find an expression for the electric potential at P.

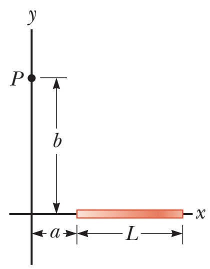


Figure P25.67

68. A Geiger–Mueller tube is a radiation detector that consists of a closed, hollow, metal cylinder (the cathode) of inner radius r_a and a coaxial cylindrical wire (the anode) of radius r_b (Fig. P25.68a). The charge per unit length on the anode is λ, and the charge per unit length on the cathode is $-\lambda$. A gas fills the space between the electrodes. When the tube is in use (Fig. P25.68b) and a high-energy elementary particle passes through this space, it can ionize an atom of the gas. The strong electric field makes the resulting ion and electron accelerate in opposite directions. They strike other molecules of the gas to ionize them, producing an avalanche of electrical discharge. The pulse of electric current between the wire and the cylinder is counted by an external circuit. (a) Show that the magnitude of the electric potential difference between the wire and the cylinder is

$$\Delta V = 2k_e\lambda \ln\left(\frac{r_a}{r_b}\right)$$

(b) Show that the magnitude of the electric field in the space between cathode and anode is

$$E = \frac{\Delta V}{\ln(r_a/r_b)}\left(\frac{1}{r}\right)$$

where r is the distance from the axis of the anode to the point where the field is to be calculated.

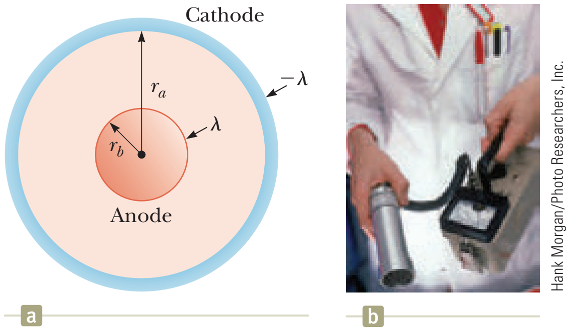


Figure P25.68

69. **Review.** Two parallel plates having charges of equal magnitude but opposite sign are separated by 12.0 cm. Each plate has a surface charge density of 36.0 nC/m^2. A proton is released from rest at the positive plate. Determine (a) the magnitude of the electric field between the plates from the charge density, (b) the potential difference between the plates, (c) the kinetic energy of the proton when it reaches the negative plate, (d) the speed of the proton just before it strikes the negative plate, (e) the acceleration of the proton, and (f) the force on the proton. (g) From the force, find the magnitude of the electric field. (h) How does your value of the electric field compare with that found in part (a)?

70. When an uncharged conducting sphere of radius a is placed at the origin of an xyz coordinate system that lies in an initially uniform electric field $\vec{\mathbf{E}} = E_0\hat{\mathbf{k}}$, the resulting electric potential is $V(x, y, z) = V_0$ for points inside the sphere and

$$V(x, y, z) = V_0 - E_0 z + \frac{E_0 a^3 z}{(x^2 + y^2 + z^2)^{3/2}}$$

for points outside the sphere, where V_0 is the (constant) electric potential on the conductor. Use this equation to determine the x, y, and z components of the resulting electric field (a) inside the sphere and (b) outside the sphere.

Challenge Problems

71. An electric dipole is located along the y axis as shown in Figure P25.71. The magnitude of its electric dipole moment is defined as $p = 2aq$. (a) At a point P, which is far from the dipole ($r >> a$), show that the electric potential is

$$V = \frac{k_e p \cos\theta}{r^2}$$

(b) Calculate the radial component E_r and the perpendicular component E_θ of the associated electric field. Note that $E_\theta = -(1/r)(\partial V/\partial\theta)$. Do these results seem reasonable for (c) $\theta = 90°$ and 0°? (d) For $r = 0$? (e) For the dipole arrangement shown in Figure P25.71, express V in terms of Cartesian coordinates using $r = (x^2 + y^2)^{1/2}$ and

$$\cos\theta = \frac{y}{(x^2 + y^2)^{1/2}}$$

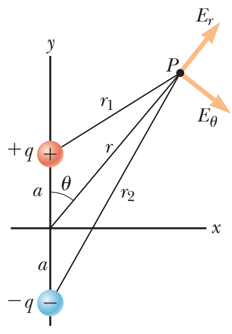


Figure P25.71

(f) Using these results and again taking $r >> a$, calculate the field components E_x and E_y.

72. A solid sphere of radius R has a uniform charge density ρ and total charge Q. Derive an expression for its total electric potential energy. *Suggestion:* Imagine the sphere is constructed by adding successive layers of concentric shells of charge $dq = (4\pi r^2\, dr)\rho$ and use $dU = V\, dq$.

73. A disk of radius R (Fig. P25.73) has a nonuniform surface charge density $\sigma = Cr$, where C is a constant and r is measured from the center of the disk to a point on the surface of the disk. Find (by direct integration) the electric potential at P.

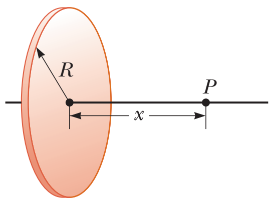


Figure P25.73

74. Four balls, each with mass m, are connected by four nonconducting strings to form a square with side a as shown in Figure P25.74. The assembly is placed on a nonconducting, frictionless, horizontal surface. Balls 1 and 2 each have charge q, and balls 3 and 4 are uncharged. After the string connecting balls 1 and 2 is cut, what is the maximum speed of balls 3 and 4?

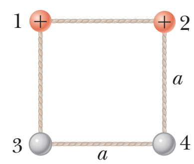


Figure P25.74

75. (a) A uniformly charged cylindrical shell with no end caps has total charge Q, radius R, and length h. Determine the electric potential at a point a distance d from the right end of the cylinder as shown in Figure P25.75.

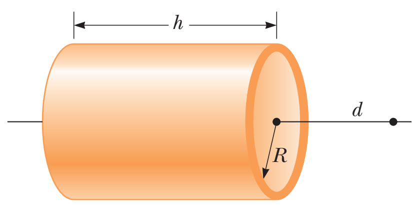


Figure P25.75

Suggestion: Use the result of Example 25.5 by treating the cylinder as a collection of ring charges. (b) **What If?** Use the result of Example 25.6 to solve the same problem for a solid cylinder.

76. As shown in Figure P25.76, two large, parallel, vertical conducting plates separated by distance d are charged so that their potentials are $+V_0$ and $-V_0$. A small conducting ball of mass m and radius R (where $R << d$) hangs midway between the plates. The thread of length L supporting the ball is a conducting wire connected to ground, so the potential of the ball is fixed at $V = 0$. The ball hangs straight down in stable equilibrium when V_0 is sufficiently small. Show that the equilibrium of the ball is unstable if V_0 exceeds the critical value $[k_e d^2 mg/(4RL)]^{1/2}$. *Suggestion:* Consider the forces on the ball when it is displaced a distance $x << L$.

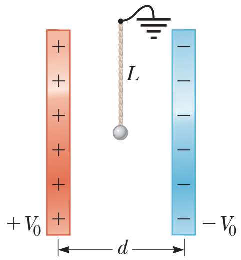


Figure P25.76

77. A particle with charge q is located at $x = -R$, and a particle with charge $-2q$ is located at the origin. Prove that the equipotential surface that has zero potential is a sphere centered at $(-4R/3, 0, 0)$ and having a radius $r = \frac{2}{3}R$.

Capacitance and Dielectrics

CHAPTER

26

When a patient receives a shock from a defibrillator, the energy delivered to the patient is initially stored in a *capacitor.* We will study capacitors and capacitance in this chapter. *(Andrew Olney/Getty Images)*

In this chapter, we introduce the first of three simple circuit elements that can be connected with wires to form an electric circuit. Electric circuits are the basis for the vast majority of the devices used in our society. Here we shall discuss *capacitors,* devices that store electric charge. This discussion is followed by the study of *resistors* in Chapter 27 and *inductors* in Chapter 32. In later chapters, we will study more sophisticated circuit elements such as *diodes* and *transistors.*

Capacitors are commonly used in a variety of electric circuits. For instance, they are used to tune the frequency of radio receivers, as filters in power supplies, to eliminate sparking in automobile ignition systems, and as energy-storing devices in electronic flash units.

26.1 Definition of Capacitance

Consider two conductors as shown in Figure 26.1 (page 778). Such a combination of two conductors is called a **capacitor.** The conductors are called *plates.* If the conductors carry charges of equal magnitude and opposite sign, a potential difference ΔV exists between them.

Pitfall Prevention 26.1

Capacitance Is a Capacity To understand capacitance, think of similar notions that use a similar word. The *capacity* of a milk carton is the volume of milk it can store. The *heat capacity* of an object is the amount of energy an object can store per unit of temperature difference. The *capacitance* of a capacitor is the amount of charge the capacitor can store per unit of potential difference.

Pitfall Prevention 26.2

Potential Difference Is ΔV, Not V We use the symbol ΔV for the potential difference across a circuit element or a device because this notation is consistent with our definition of potential difference and with the meaning of the delta sign. It is a common but confusing practice to use the symbol V without the delta sign for both a potential and a potential difference! Keep that in mind if you consult other texts.

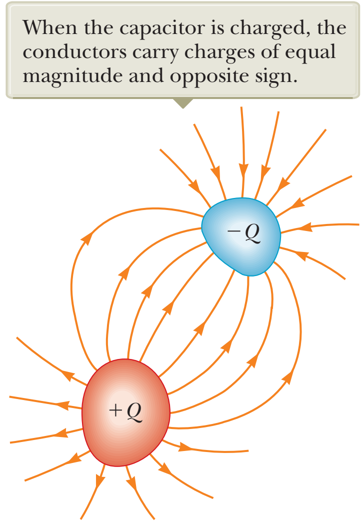


Figure 26.1 A capacitor consists of two conductors.

What determines how much charge is on the plates of a capacitor for a given voltage? Experiments show that the quantity of charge Q on a capacitor[1] is linearly proportional to the potential difference between the conductors; that is, $Q \propto \Delta V$. The proportionality constant depends on the shape and separation of the conductors.[2] This relationship can be written as $Q = C\,\Delta V$ if we define capacitance as follows:

Definition of capacitance ▶

The **capacitance** C of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors:

$$C \equiv \frac{Q}{\Delta V} \tag{26.1}$$

By definition *capacitance is always a positive quantity*. Furthermore, the charge Q and the potential difference ΔV are always expressed in Equation 26.1 as positive quantities.

From Equation 26.1, we see that capacitance has SI units of coulombs per volt. Named in honor of Michael Faraday, the SI unit of capacitance is the **farad** (F):

$$1\text{ F} = 1\text{ C/V}$$

The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarads (10^{-6} F) to picofarads (10^{-12} F). We shall use the symbol μF to represent microfarads. In practice, to avoid the use of Greek letters, physical capacitors are often labeled "mF" for microfarads and "mmF" for micromicrofarads or, equivalently, "pF" for picofarads.

Let's consider a capacitor formed from a pair of parallel plates as shown in Figure 26.2. Each plate is connected to one terminal of a battery, which acts as a source of potential difference. If the capacitor is initially uncharged, the battery establishes an electric field in the connecting wires when the connections are made. Let's focus on the plate connected to the negative terminal of the battery. The electric field in the wire applies a force on electrons in the wire immediately outside this plate; this force causes the electrons to move onto the plate. The movement continues until the plate, the wire, and the terminal are all at the same electric potential. Once this equilibrium situation is attained, a potential difference no longer exists between the terminal and the plate; as a result, no electric field is present in the wire and

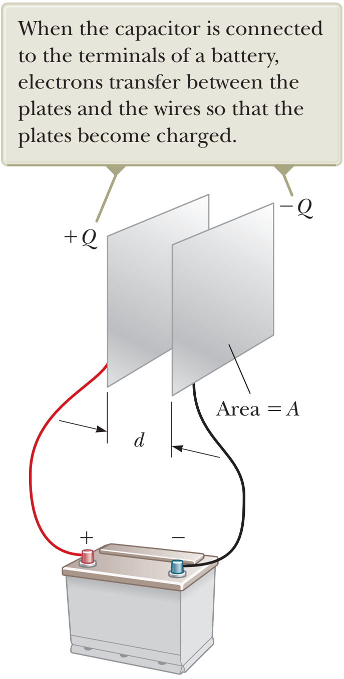


Figure 26.2 A parallel-plate capacitor consists of two parallel conducting plates, each of area A, separated by a distance d.

[1]Although the total charge on the capacitor is zero (because there is as much excess positive charge on one conductor as there is excess negative charge on the other), it is common practice to refer to the magnitude of the charge on either conductor as "the charge on the capacitor."

[2]The proportionality between Q and ΔV can be proven from Coulomb's law or by experiment.

the electrons stop moving. The plate now carries a negative charge. A similar process occurs at the other capacitor plate, where electrons move from the plate to the wire, leaving the plate positively charged. In this final configuration, the potential difference across the capacitor plates is the same as that between the terminals of the battery.

Quick Quiz 26.1 A capacitor stores charge Q at a potential difference ΔV. What happens if the voltage applied to the capacitor by a battery is doubled to $2\,\Delta V$? **(a)** The capacitance falls to half its initial value, and the charge remains the same. **(b)** The capacitance and the charge both fall to half their initial values. **(c)** The capacitance and the charge both double. **(d)** The capacitance remains the same, and the charge doubles.

26.2 Calculating Capacitance

We can derive an expression for the capacitance of a pair of oppositely charged conductors having a charge of magnitude Q in the following manner. First we calculate the potential difference using the techniques described in Chapter 25. We then use the expression $C = Q/\Delta V$ to evaluate the capacitance. The calculation is relatively easy if the geometry of the capacitor is simple.

Although the most common situation is that of two conductors, a single conductor also has a capacitance. For example, imagine a single spherical, charged conductor. The electric field lines around this conductor are exactly the same as if there were a conducting, spherical shell of infinite radius, concentric with the sphere and carrying a charge of the same magnitude but opposite sign. Therefore, we can identify the imaginary shell as the second conductor of a two-conductor capacitor. The electric potential of the sphere of radius a is simply k_eQ/a (see Section 25.6), and setting $V = 0$ for the infinitely large shell gives

$$C = \frac{Q}{\Delta V} = \frac{Q}{k_eQ/a} = \frac{a}{k_e} = 4\pi\epsilon_0 a \qquad \textbf{(26.2)}$$

◀ **Capacitance of an isolated charged sphere**

This expression shows that the capacitance of an isolated, charged sphere is proportional to its radius and is independent of both the charge on the sphere and its potential, as is the case with all capacitors. Equation 26.1 is the general definition of capacitance in terms of electrical parameters, but the capacitance of a given capacitor will depend only on the geometry of the plates.

The capacitance of a pair of conductors is illustrated below with three familiar geometries, namely, parallel plates, concentric cylinders, and concentric spheres. In these calculations, we assume the charged conductors are separated by a vacuum.

> **Pitfall Prevention 26.3**
> **Too Many Cs** Do not confuse an italic C for capacitance with a non-italic C for the unit coulomb.

Parallel-Plate Capacitors

Two parallel, metallic plates of equal area A are separated by a distance d as shown in Figure 26.2. One plate carries a charge $+Q$, and the other carries a charge $-Q$. The surface charge density on each plate is $\sigma = Q/A$. If the plates are very close together (in comparison with their length and width), we can assume the electric field is uniform between the plates and zero elsewhere. According to the What If? feature of Example 24.5, the value of the electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Because the field between the plates is uniform, the magnitude of the potential difference between the plates equals Ed (see Eq. 25.6); therefore,

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

Substituting this result into Equation 26.1, we find that the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{Q}{Qd/\epsilon_0 A}$$

Capacitance of parallel plates

$$C = \frac{\epsilon_0 A}{d} \tag{26.3}$$

That is, the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation.

Let's consider how the geometry of these conductors influences the capacity of the pair of plates to store charge. As a capacitor is being charged by a battery, electrons flow into the negative plate and out of the positive plate. If the capacitor plates are large, the accumulated charges are able to distribute themselves over a substantial area and the amount of charge that can be stored on a plate for a given potential difference increases as the plate area is increased. Therefore, it is reasonable that the capacitance is proportional to the plate area A as in Equation 26.3.

Now consider the region that separates the plates. Imagine moving the plates closer together. Consider the situation before any charges have had a chance to move in response to this change. Because no charges have moved, the electric field between the plates has the same value but extends over a shorter distance. Therefore, the magnitude of the potential difference between the plates $\Delta V = Ed$ (Eq. 25.6) is smaller. The difference between this new capacitor voltage and the terminal voltage of the battery appears as a potential difference across the wires connecting the battery to the capacitor, resulting in an electric field in the wires that drives more charge onto the plates and increases the potential difference between the plates. When the potential difference between the plates again matches that of the battery, the flow of charge stops. Therefore, moving the plates closer together causes the charge on the capacitor to increase. If d is increased, the charge decreases. As a result, the inverse relationship between C and d in Equation 26.3 is reasonable.

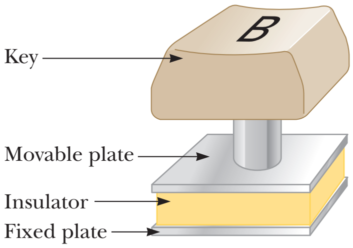


Figure 26.3 (Quick Quiz 26.2) One type of computer keyboard button.

Quick Quiz 26.2 Many computer keyboard buttons are constructed of capacitors as shown in Figure 26.3. When a key is pushed down, the soft insulator between the movable plate and the fixed plate is compressed. When the key is pressed, what happens to the capacitance? **(a)** It increases. **(b)** It decreases. **(c)** It changes in a way you cannot determine because the electric circuit connected to the keyboard button may cause a change in ΔV.

Example 26.1 The Cylindrical Capacitor

A solid cylindrical conductor of radius a and charge Q is coaxial with a cylindrical shell of negligible thickness, radius $b > a$, and charge $-Q$ (Fig. 26.4a). Find the capacitance of this cylindrical capacitor if its length is ℓ.

SOLUTION

Conceptualize Recall that any pair of conductors qualifies as a capacitor, so the system described in this example therefore qualifies. Figure 26.4b helps visualize the electric field between the conductors. We expect the capacitance to depend only on geometric factors, which, in this case, are a, b, and ℓ.

Categorize Because of the cylindrical symmetry of the system, we can use results from previous studies of cylindrical systems to find the capacitance.

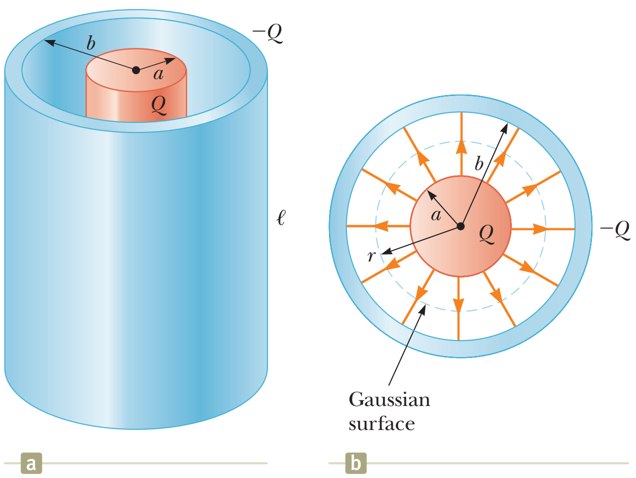


Figure 26.4 (Example 26.1) (a) A cylindrical capacitor consists of a solid cylindrical conductor of radius a and length ℓ surrounded by a coaxial cylindrical shell of radius b. (b) End view. The electric field lines are radial. The dashed line represents the end of a cylindrical gaussian surface of radius r and length ℓ.

26.1 continued

Analyze Assuming ℓ is much greater than a and b, we can neglect end effects. In this case, the electric field is perpendicular to the long axis of the cylinders and is confined to the region between them (Fig. 26.4b).

Write an expression for the potential difference between the two cylinders from Equation 25.3:

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

Apply Equation 24.7 for the electric field outside a cylindrically symmetric charge distribution and notice from Figure 26.4b that $\vec{\mathbf{E}}$ is parallel to $d\vec{\mathbf{s}}$ along a radial line:

$$V_b - V_a = -\int_a^b E_r\,dr = -2k_e\lambda \int_a^b \frac{dr}{r} = -2k_e\lambda \ln\left(\frac{b}{a}\right)$$

Substitute the absolute value of ΔV into Equation 26.1 and use $\lambda = Q/\ell$:

$$C = \frac{Q}{\Delta V} = \frac{Q}{(2k_e Q/\ell)\ln(b/a)} = \boxed{\frac{\ell}{2k_e \ln(b/a)}} \qquad \textbf{(26.4)}$$

Finalize The capacitance depends on the radii a and b and is proportional to the length of the cylinders. Equation 26.4 shows that the capacitance per unit length of a combination of concentric cylindrical conductors is

$$\frac{C}{\ell} = \frac{1}{2k_e \ln(b/a)} \qquad \textbf{(26.5)}$$

An example of this type of geometric arrangement is a *coaxial cable,* which consists of two concentric cylindrical conductors separated by an insulator. You probably have a coaxial cable attached to your television set if you are a subscriber to cable television. The coaxial cable is especially useful for shielding electrical signals from any possible external influences.

WHAT IF? Suppose $b = 2.00a$ for the cylindrical capacitor. You would like to increase the capacitance, and you can do so by choosing to increase either ℓ by 10% or a by 10%. Which choice is more effective at increasing the capacitance?

Answer According to Equation 26.4, C is proportional to ℓ, so increasing ℓ by 10% results in a 10% increase in C. For the result of the change in a, let's use Equation 26.4 to set up a ratio of the capacitance C' for the enlarged cylinder radius a' to the original capacitance:

$$\frac{C'}{C} = \frac{\ell/2k_e \ln(b/a')}{\ell/2k_e \ln(b/a)} = \frac{\ln(b/a)}{\ln(b/a')}$$

We now substitute $b = 2.00a$ and $a' = 1.10a$, representing a 10% increase in a:

$$\frac{C'}{C} = \frac{\ln(2.00a/a)}{\ln(2.00a/1.10a)} = \frac{\ln 2.00}{\ln 1.82} = 1.16$$

which corresponds to a 16% increase in capacitance. Therefore, it is more effective to increase a than to increase ℓ.

Note two more extensions of this problem. First, it is advantageous to increase a only for a range of relationships between a and b. If $b > 2.85a$, increasing ℓ by 10% is more effective than increasing a (see Problem 70). Second, if b decreases, the capacitance increases. Increasing a or decreasing b has the effect of bringing the plates closer together, which increases the capacitance.

Example 26.2 The Spherical Capacitor

A spherical capacitor consists of a spherical conducting shell of radius b and charge $-Q$ concentric with a smaller conducting sphere of radius a and charge Q (Fig. 26.5, page 782). Find the capacitance of this device.

SOLUTION

Conceptualize As with Example 26.1, this system involves a pair of conductors and qualifies as a capacitor. We expect the capacitance to depend on the spherical radii a and b.

continued

26.2 continued

Categorize Because of the spherical symmetry of the system, we can use results from previous studies of spherical systems to find the capacitance.

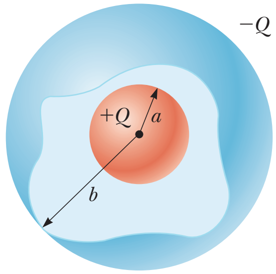


Figure 26.5 (Example 26.2) A spherical capacitor consists of an inner sphere of radius a surrounded by a concentric spherical shell of radius b. The electric field between the spheres is directed radially outward when the inner sphere is positively charged.

Analyze As shown in Chapter 24, the direction of the electric field outside a spherically symmetric charge distribution is radial and its magnitude is given by the expression $E = k_e Q/r^2$. In this case, this result applies to the field *between* the spheres ($a < r < b$).

Write an expression for the potential difference between the two conductors from Equation 25.3:

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

Apply the result of Example 24.3 for the electric field outside a spherically symmetric charge distribution and note that $\vec{\mathbf{E}}$ is parallel to $d\vec{\mathbf{s}}$ along a radial line:

$$V_b - V_a = -\int_a^b E_r\,dr = -k_e Q \int_a^b \frac{dr}{r^2} = k_e Q\left[\frac{1}{r}\right]_a^b$$

$$(1)\quad V_b - V_a = k_e Q\left(\frac{1}{b} - \frac{1}{a}\right) = k_e Q\,\frac{a-b}{ab}$$

Substitute the absolute value of ΔV into Equation 26.1:

$$C = \frac{Q}{\Delta V} = \frac{Q}{|V_b - V_a|} = \boxed{\frac{ab}{k_e(b-a)}} \qquad \textbf{(26.6)}$$

Finalize The capacitance depends on a and b as expected. The potential difference between the spheres in Equation (1) is negative because Q is positive and $b > a$. Therefore, in Equation 26.6, when we take the absolute value, we change $a - b$ to $b - a$. The result is a positive number.

WHAT IF? If the radius b of the outer sphere approaches infinity, what does the capacitance become?

Answer In Equation 26.6, we let $b \to \infty$:

$$C = \lim_{b\to\infty} \frac{ab}{k_e(b-a)} = \frac{ab}{k_e(b)} = \frac{a}{k_e} = 4\pi\epsilon_0 a$$

Notice that this expression is the same as Equation 26.2, the capacitance of an isolated spherical conductor.

26.3 Combinations of Capacitors

Two or more capacitors often are combined in electric circuits. We can calculate the equivalent capacitance of certain combinations using methods described in this section. Throughout this section, we assume the capacitors to be combined are initially uncharged.

In studying electric circuits, we use a simplified pictorial representation called a **circuit diagram.** Such a diagram uses **circuit symbols** to represent various circuit elements. The circuit symbols are connected by straight lines that represent the wires between the circuit elements. The circuit symbols for capacitors, batteries, and switches as well as the color codes used for them in this text are given in Figure 26.6. The symbol for the capacitor reflects the geometry of the most common model for a capacitor, a pair of parallel plates. The positive terminal of the battery is at the higher potential and is represented in the circuit symbol by the longer line.

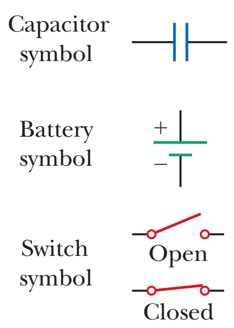


Figure 26.6 Circuit symbols for capacitors, batteries, and switches. Notice that capacitors are in blue, batteries are in green, and switches are in red. The closed switch can carry current, whereas the open one cannot.

Parallel Combination

Two capacitors connected as shown in Figure 26.7a are known as a **parallel combination** of capacitors. Figure 26.7b shows a circuit diagram for this combination of capacitors. The left plates of the capacitors are connected to the positive terminal of the battery by a conducting wire and are therefore both at the same electric potential

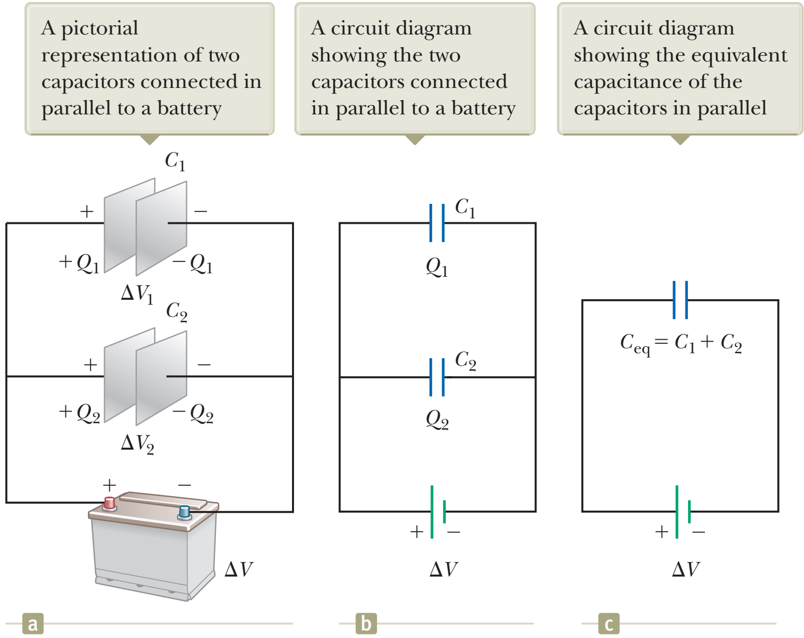


Figure 26.7 Two capacitors connected in parallel. All three diagrams are equivalent.

as the positive terminal. Likewise, the right plates are connected to the negative terminal and so are both at the same potential as the negative terminal. Therefore, the individual potential differences across capacitors connected in parallel are the same and are equal to the potential difference applied across the combination. That is,

$$\Delta V_1 = \Delta V_2 = \Delta V$$

where ΔV is the battery terminal voltage.

After the battery is attached to the circuit, the capacitors quickly reach their maximum charge. Let's call the maximum charges on the two capacitors Q_1 and Q_2, where $Q_1 = C_1\Delta V_1$ and $Q_2 = C_2\Delta V_2$. The *total charge* Q_{tot} stored by the two capacitors is the sum of the charges on the individual capacitors:

$$Q_{tot} = Q_1 + Q_2 = C_1\Delta V_1 + C_2\Delta V_2 \qquad \textbf{(26.7)}$$

Suppose you wish to replace these two capacitors by one *equivalent capacitor* having a capacitance C_{eq} as in Figure 26.7c. The effect this equivalent capacitor has on the circuit must be exactly the same as the effect of the combination of the two individual capacitors. That is, the equivalent capacitor must store charge Q_{tot} when connected to the battery. Figure 26.7c shows that the voltage across the equivalent capacitor is ΔV because the equivalent capacitor is connected directly across the battery terminals. Therefore, for the equivalent capacitor,

$$Q_{tot} = C_{eq}\,\Delta V$$

Substituting this result into Equation 26.7 gives

$$C_{eq}\,\Delta V = C_1\,\Delta V_1 + C_2\,\Delta V_2$$

$$C_{eq} = C_1 + C_2 \quad \text{(parallel combination)}$$

where we have canceled the voltages because they are all the same. If this treatment is extended to three or more capacitors connected in parallel, the **equivalent capacitance** is found to be

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots \quad \text{(parallel combination)}} \qquad \textbf{(26.8)}$$

◀ **Equivalent capacitance for capacitors in parallel**

Therefore, the equivalent capacitance of a parallel combination of capacitors is (1) the algebraic sum of the individual capacitances and (2) greater than any of

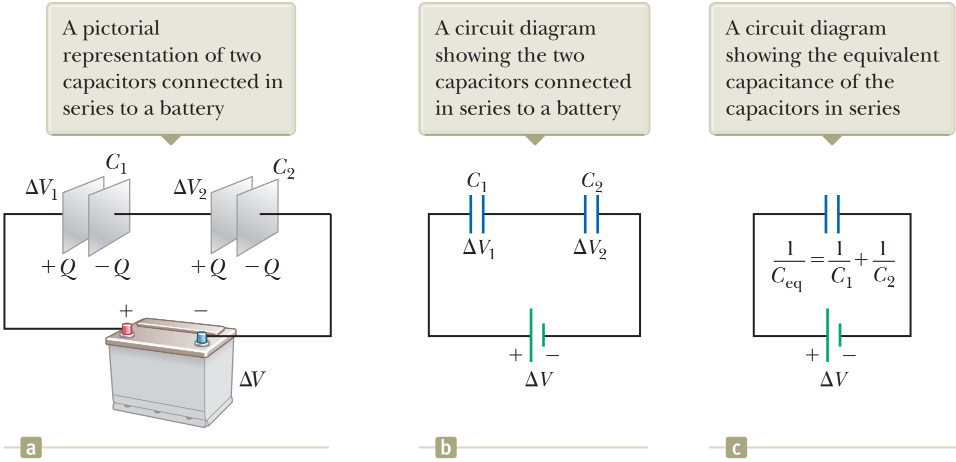


Figure 26.8 Two capacitors connected in series. All three diagrams are equivalent.

the individual capacitances. Statement (2) makes sense because we are essentially combining the areas of all the capacitor plates when they are connected with conducting wire, and capacitance of parallel plates is proportional to area (Eq. 26.3).

Series Combination

Two capacitors connected as shown in Figure 26.8a and the equivalent circuit diagram in Figure 26.8b are known as a **series combination** of capacitors. The left plate of capacitor 1 and the right plate of capacitor 2 are connected to the terminals of a battery. The other two plates are connected to each other and to nothing else; hence, they form an isolated system that is initially uncharged and must continue to have zero net charge. To analyze this combination, let's first consider the uncharged capacitors and then follow what happens immediately after a battery is connected to the circuit. When the battery is connected, electrons are transferred out of the left plate of C_1 and into the right plate of C_2. As this negative charge accumulates on the right plate of C_2, an equivalent amount of negative charge is forced off the left plate of C_2, and this left plate therefore has an excess positive charge. The negative charge leaving the left plate of C_2 causes negative charges to accumulate on the right plate of C_1. As a result, both right plates end up with a charge $-Q$ and both left plates end up with a charge $+Q$. Therefore, the charges on capacitors connected in series are the same:

$$Q_1 = Q_2 = Q$$

where Q is the charge that moved between a wire and the connected outside plate of one of the capacitors.

Figure 26.8a shows the individual voltages ΔV_1 and ΔV_2 across the capacitors. These voltages add to give the total voltage ΔV_{tot} across the combination:

$$\Delta V_{tot} = \Delta V_1 + \Delta V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} \tag{26.9}$$

In general, the total potential difference across any number of capacitors connected in series is the sum of the potential differences across the individual capacitors.

Suppose the equivalent single capacitor in Figure 26.8c has the same effect on the circuit as the series combination when it is connected to the battery. After it is fully charged, the equivalent capacitor must have a charge of $-Q$ on its right plate and a charge of $+Q$ on its left plate. Applying the definition of capacitance to the circuit in Figure 26.8c gives

$$\Delta V_{tot} = \frac{Q}{C_{eq}}$$

Substituting this result into Equation 26.9, we have

$$\frac{Q}{C_{\text{eq}}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

Canceling the charges because they are all the same gives

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{(series combination)}$$

When this analysis is applied to three or more capacitors connected in series, the relationship for the **equivalent capacitance** is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \quad \text{(series combination)} \qquad \textbf{(26.10)}$$

Equivalent capacitance for capacitors in series

This expression shows that (1) the inverse of the equivalent capacitance is the algebraic sum of the inverses of the individual capacitances and (2) the equivalent capacitance of a series combination is always less than any individual capacitance in the combination.

Quick Quiz 26.3 Two capacitors are identical. They can be connected in series or in parallel. If you want the *smallest* equivalent capacitance for the combination, how should you connect them? **(a)** in series **(b)** in parallel **(c)** either way because both combinations have the same capacitance

Example 26.3 Equivalent Capacitance

Find the equivalent capacitance between *a* and *b* for the combination of capacitors shown in Figure 26.9a. All capacitances are in microfarads.

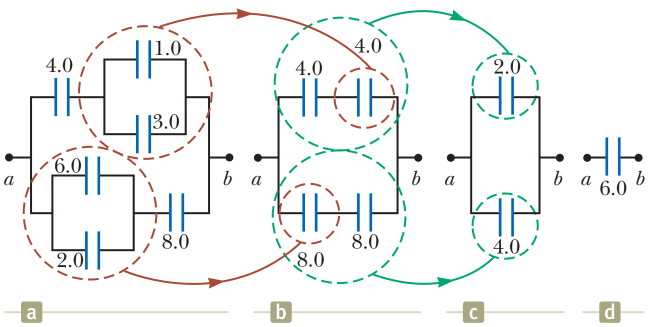


Figure 26.9 (Example 26.3) To find the equivalent capacitance of the capacitors in (a), we reduce the various combinations in steps as indicated in (b), (c), and (d), using the series and parallel rules described in the text. All capacitances are in microfarads.

SOLUTION

Conceptualize Study Figure 26.9a carefully and make sure you understand how the capacitors are connected. Verify that there are only series and parallel connections between capacitors.

Categorize Figure 26.9a shows that the circuit contains both series and parallel connections, so we use the rules for series and parallel combinations discussed in this section.

Analyze Using Equations 26.8 and 26.10, we reduce the combination step by step as indicated in the figure. As you follow along below, notice that in each step we replace the combination of two capacitors in the circuit diagram with a single capacitor having the equivalent capacitance.

The 1.0-μF and 3.0-μF capacitors (upper red-brown circle in Fig. 26.9a) are in parallel. Find the equivalent capacitance from Equation 26.8:

$$C_{\text{eq}} = C_1 + C_2 = 4.0\ \mu\text{F}$$

The 2.0-μF and 6.0-μF capacitors (lower red-brown circle in Fig. 26.9a) are also in parallel:

$$C_{\text{eq}} = C_1 + C_2 = 8.0\ \mu\text{F}$$

The circuit now looks like Figure 26.9b. The two 4.0-μF capacitors (upper green circle in Fig. 26.9b) are in series. Find the equivalent capacitance from Equation 26.10:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{4.0\ \mu\text{F}} + \frac{1}{4.0\ \mu\text{F}} = \frac{1}{2.0\ \mu\text{F}}$$

$$C_{\text{eq}} = 2.0\ \mu\text{F}$$

continued

26.3 continued

The two 8.0-μF capacitors (lower green circle in Fig. 26.9b) are also in series. Find the equivalent capacitance from Equation 26.10:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{8.0\ \mu\text{F}} + \frac{1}{8.0\ \mu\text{F}} = \frac{1}{4.0\ \mu\text{F}}$$

$$C_{eq} = 4.0\ \mu\text{F}$$

The circuit now looks like Figure 26.9c. The 2.0-μF and 4.0-μF capacitors are in parallel:

$$C_{eq} = C_1 + C_2 = \boxed{6.0\ \mu\text{F}}$$

Finalize This final value is that of the single equivalent capacitor shown in Figure 26.9d. For further practice in treating circuits with combinations of capacitors, imagine a battery is connected between points *a* and *b* in Figure 26.9a so that a potential difference ΔV is established across the combination. Can you find the voltage across and the charge on each capacitor?

26.4 Energy Stored in a Charged Capacitor

Because positive and negative charges are separated in the system of two conductors in a capacitor, electric potential energy is stored in the system. Many of those who work with electronic equipment have at some time verified that a capacitor can store energy. If the plates of a charged capacitor are connected by a conductor such as a wire, charge moves between each plate and its connecting wire until the capacitor is uncharged. The discharge can often be observed as a visible spark. If you accidentally touch the opposite plates of a charged capacitor, your fingers act as a pathway for discharge and the result is an electric shock. The degree of shock you receive depends on the capacitance and the voltage applied to the capacitor. Such a shock could be dangerous if high voltages are present as in the power supply of a home theater system. Because the charges can be stored in a capacitor even when the system is turned off, unplugging the system does not make it safe to open the case and touch the components inside.

Figure 26.10a shows a battery connected to a single parallel-plate capacitor with a switch in the circuit. Let us identify the circuit as a system. When the switch is closed (Fig. 26.10b), the battery establishes an electric field in the wires and charges

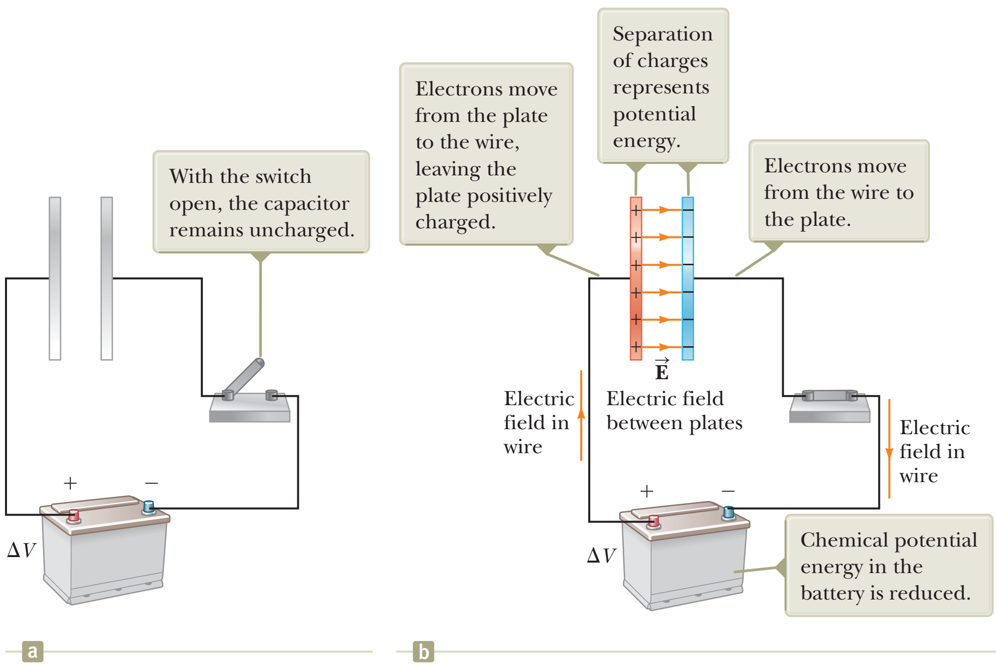


Figure 26.10 (a) A circuit consisting of a capacitor, a battery, and a switch. (b) When the switch is closed, the battery establishes an electric field in the wire and the capacitor becomes charged.

flow between the wires and the capacitor. As that occurs, there is a transformation of energy within the system. Before the switch is closed, energy is stored as chemical potential energy in the battery. This energy is transformed during the chemical reaction that occurs within the battery when it is operating in an electric circuit. When the switch is closed, some of the chemical potential energy in the battery is transformed to electric potential energy associated with the separation of positive and negative charges on the plates.

To calculate the energy stored in the capacitor, we shall assume a charging process that is different from the actual process described in Section 26.1 but that gives the same final result. This assumption is justified because the energy in the final configuration does not depend on the actual charge-transfer process.[3] Imagine the plates are disconnected from the battery and you transfer the charge mechanically through the space between the plates as follows. You grab a small amount of positive charge on one plate and apply a force that causes this positive charge to move over to the other plate. Therefore, you do work on the charge as it is transferred from one plate to the other. At first, no work is required to transfer a small amount of charge dq from one plate to the other,[4] but once this charge has been transferred, a small potential difference exists between the plates. Therefore, work must be done to move additional charge through this potential difference. As more and more charge is transferred from one plate to the other, the potential difference increases in proportion and more work is required. The overall process is described by the nonisolated system model for energy. Equation 8.2 reduces to $W = \Delta U_E$; the work done on the system by the external agent appears as an increase in electric potential energy in the system.

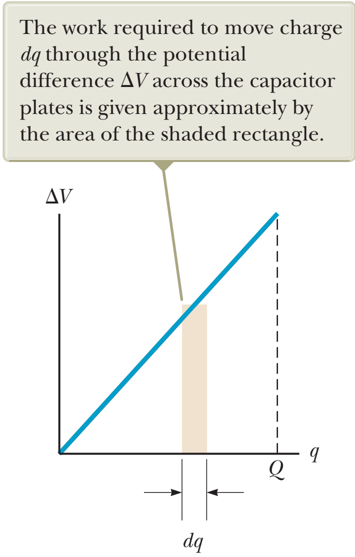


Figure 26.11 A plot of potential difference versus charge for a capacitor is a straight line having slope $1/C$.

Suppose q is the charge on the capacitor at some instant during the charging process. At the same instant, the potential difference across the capacitor is $\Delta V = q/C$. This relationship is graphed in Figure 26.11. From Section 25.1, we know that the work necessary to transfer an increment of charge dq from the plate carrying charge $-q$ to the plate carrying charge q (which is at the higher electric potential) is

$$dW = \Delta V\, dq = \frac{q}{C}\, dq$$

The work required to transfer the charge dq is the area of the tan rectangle in Figure 26.11. Because 1 V = 1 J/C, the unit for the area is the joule. The total work required to charge the capacitor from $q = 0$ to some final charge $q = Q$ is

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

The work done in charging the capacitor appears as electric potential energy U_E stored in the capacitor. Using Equation 26.1, we can express the potential energy stored in a charged capacitor as

$$U_E = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2 \qquad \textbf{(26.11)}$$

◀ **Energy stored in a charged capacitor**

Because the curve in Figure 26.11 is a straight line, the total area under the curve is that of a triangle of base Q and height ΔV.

Equation 26.11 applies to any capacitor, regardless of its geometry. For a given capacitance, the stored energy increases as the charge and the potential difference increase. In practice, there is a limit to the maximum energy (or charge) that can be stored because, at a sufficiently large value of ΔV, discharge ultimately occurs

[3]This discussion is similar to that of state variables in thermodynamics. The change in a state variable such as temperature is independent of the path followed between the initial and final states. The potential energy of a capacitor (or any system) is also a state variable, so its change does not depend on the process followed to charge the capacitor.

[4]We shall use lowercase q for the time-varying charge on the capacitor while it is charging to distinguish it from uppercase Q, which is the total charge on the capacitor after it is completely charged.

Pitfall Prevention 26.4

Not a New Kind of Energy The energy given by Equation 26.12 is not a new kind of energy. The equation describes familiar electric potential energy associated with a system of separated source charges. Equation 26.12 provides a new *interpretation*, or a new way of *modeling* the energy. Furthermore, Equation 26.13 correctly describes the energy density associated with *any* electric field, regardless of the source.

between the plates. For this reason, capacitors are usually labeled with a maximum operating voltage.

We can consider the energy in a capacitor to be stored in the electric field created between the plates as the capacitor is charged. This description is reasonable because the electric field is proportional to the charge on the capacitor. For a parallel-plate capacitor, the potential difference is related to the electric field through the relationship $\Delta V = Ed$. Furthermore, its capacitance is $C = \epsilon_0 A/d$ (Eq. 26.3). Substituting these expressions into Equation 26.11 gives

$$U_E = \tfrac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \tfrac{1}{2}(\epsilon_0 Ad)E^2 \tag{26.12}$$

Because the volume occupied by the electric field is Ad, the *energy per unit volume* $u_E = U_E/Ad$, known as the *energy density*, is

Energy density in an electric field

$$u_E = \tfrac{1}{2}\epsilon_0 E^2 \tag{26.13}$$

Although Equation 26.13 was derived for a parallel-plate capacitor, the expression is generally valid regardless of the source of the electric field. That is, the energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.

Quick Quiz 26.4 You have three capacitors and a battery. In which of the following combinations of the three capacitors is the maximum possible energy stored when the combination is attached to the battery? **(a)** series **(b)** parallel **(c)** no difference because both combinations store the same amount of energy

Example 26.4 Rewiring Two Charged Capacitors

Two capacitors C_1 and C_2 (where $C_1 > C_2$) are charged to the same initial potential difference ΔV_i. The charged capacitors are removed from the battery, and their plates are connected with opposite polarity as in Figure 26.12a. The switches S_1 and S_2 are then closed as in Figure 26.12b.

(A) Find the final potential difference ΔV_f between a and b after the switches are closed.

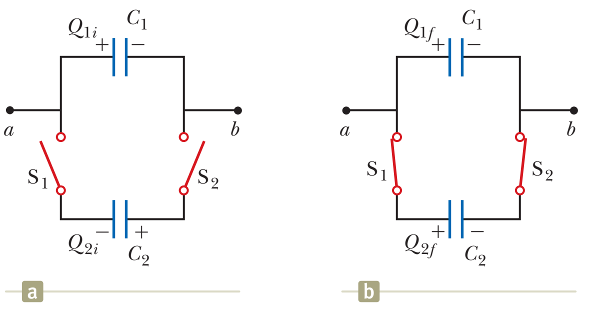


Figure 26.12 (Example 26.4) (a) Two capacitors are charged to the same initial potential difference and connected together with plates of opposite sign to be in contact when the switches are closed. (b) When the switches are closed, the charges redistribute.

SOLUTION

Conceptualize Figure 26.12 helps us understand the initial and final configurations of the system. When the switches are closed, the charge on the system will redistribute between the capacitors until both capacitors have the same potential difference. Because $C_1 > C_2$, more charge exists on C_1 than on C_2, so the final configuration will have positive charge on the left plates as shown in Figure 26.12b.

Categorize In Figure 26.12b, it might appear as if the capacitors are connected in parallel, but there is no battery in this circuit to apply a voltage across the combination. Therefore, we *cannot* categorize this problem as one in which capacitors are connected in parallel. We *can* categorize it as a problem involving an isolated system for electric charge. The left-hand plates of the capacitors form an isolated system because they are not connected to the right-hand plates by conductors.

Analyze Write an expression for the total charge on the left-hand plates of the system before the switches are closed, noting that a negative sign for Q_{2i} is necessary because the charge on the left plate of capacitor C_2 is negative:

$$(1)\quad Q_i = Q_{1i} + Q_{2i} = C_1\,\Delta V_i - C_2\,\Delta V_i = (C_1 - C_2)\Delta V_i$$

26.4 continued

After the switches are closed, the charges on the individual capacitors change to new values Q_{1f} and Q_{2f} such that the potential difference is again the same across both capacitors, with a value of ΔV_f. Write an expression for the total charge on the left-hand plates of the system after the switches are closed:

$$(2)\quad Q_f = Q_{1f} + Q_{2f} = C_1\,\Delta V_f + C_2\,\Delta V_f = (C_1 + C_2)\Delta V_f$$

Because the system is isolated, the initial and final charges on the system must be the same. Use this condition and Equations (1) and (2) to solve for ΔV_f:

$$Q_f = Q_i \;\rightarrow\; (C_1 + C_2)\,\Delta V_f = (C_1 - C_2)\,\Delta V_i$$

$$(3)\quad \Delta V_f = \boxed{\left(\frac{C_1 - C_2}{C_1 + C_2}\right)\Delta V_i}$$

(B) Find the total energy stored in the capacitors before and after the switches are closed and determine the ratio of the final energy to the initial energy.

SOLUTION

Use Equation 26.11 to find an expression for the total energy stored in the capacitors before the switches are closed:

$$(4)\quad U_i = \tfrac{1}{2}C_1(\Delta V_i)^2 + \tfrac{1}{2}C_2(\Delta V_i)^2 = \boxed{\tfrac{1}{2}(C_1 + C_2)(\Delta V_i)^2}$$

Write an expression for the total energy stored in the capacitors after the switches are closed:

$$U_f = \tfrac{1}{2}C_1(\Delta V_f)^2 + \tfrac{1}{2}C_2(\Delta V_f)^2 = \tfrac{1}{2}(C_1 + C_2)(\Delta V_f)^2$$

Use the results of part (A) to rewrite this expression in terms of ΔV_i:

$$(5)\quad U_f = \tfrac{1}{2}(C_1 + C_2)\left[\left(\frac{C_1 - C_2}{C_1 + C_2}\right)\Delta V_i\right]^2 = \boxed{\tfrac{1}{2}\frac{(C_1 - C_2)^2(\Delta V_i)^2}{C_1 + C_2}}$$

Divide Equation (5) by Equation (4) to obtain the ratio of the energies stored in the system:

$$\frac{U_f}{U_i} = \frac{\tfrac{1}{2}(C_1 - C_2)^2(\Delta V_i)^2/(C_1 + C_2)}{\tfrac{1}{2}(C_1 + C_2)(\Delta V_i)^2}$$

$$(6)\quad \frac{U_f}{U_i} = \boxed{\left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2}$$

Finalize The ratio of energies is *less* than unity, indicating that the final energy is *less* than the initial energy. At first, you might think the law of energy conservation has been violated, but that is not the case. The "missing" energy is transferred out of the system by the mechanism of electromagnetic waves (T_{ER} in Eq. 8.2), as we shall see in Chapter 34. Therefore, this system is isolated for electric charge, but nonisolated for energy.

WHAT IF? What if the two capacitors have the same capacitance? What would you expect to happen when the switches are closed?

Answer Because both capacitors have the same initial potential difference applied to them, the charges on the identical capacitors have the same magnitude. When the capacitors with opposite polarities are connected together, the equal-magnitude charges should cancel each other, leaving the capacitors uncharged.

Let's test our results to see if that is the case mathematically. In Equation (1), because the capacitances are equal, the initial charge Q_i on the system of left-hand plates is zero. Equation (3) shows that $\Delta V_f = 0$, which is consistent with uncharged capacitors. Finally, Equation (5) shows that $U_f = 0$, which is also consistent with uncharged capacitors.

One device in which capacitors have an important role is the portable *defibrillator* (see the chapter-opening photo on page 777). When cardiac fibrillation (random contractions) occurs, the heart produces a rapid, irregular pattern of beats. A fast discharge of energy through the heart can return the organ to its normal beat pattern. Emergency medical teams use portable defibrillators that contain batteries capable of charging a capacitor to a high voltage. (The circuitry actually permits the capacitor to be charged to a much higher voltage than that of the battery.) Up to 360 J is stored in the electric field of a large capacitor in a defibrillator when it is fully charged. The stored energy is released through the heart by conducting electrodes, called paddles, which are placed on both sides of the victim's chest. The defibrillator can deliver the energy to a patient in about 2 ms (roughly equivalent to 3 000 times the power delivered to a 60-W lightbulb!). The paramedics must wait between applications of the energy because of the time interval necessary for the capacitors to become fully charged. In this application and others (e.g., camera flash units and lasers used for fusion experiments), capacitors serve as energy reservoirs that can be slowly charged and then quickly discharged to provide large amounts of energy in a short pulse.

26.5 Capacitors with Dielectrics

Pitfall Prevention 26.5

Is the Capacitor Connected to a Battery? For problems in which a capacitor is modified (by insertion of a dielectric, for example), you must note whether modifications to the capacitor are being made while the capacitor is connected to a battery or after it is disconnected. If the capacitor remains connected to the battery, the voltage across the capacitor necessarily remains the same. If you disconnect the capacitor from the battery before making any modifications to the capacitor, the capacitor is an isolated system for electric charge and its charge remains the same.

A **dielectric** is a nonconducting material such as rubber, glass, or waxed paper. We can perform the following experiment to illustrate the effect of a dielectric in a capacitor. Consider a parallel-plate capacitor that without a dielectric has a charge Q_0 and a capacitance C_0. The potential difference across the capacitor is $\Delta V_0 = Q_0/C_0$. Figure 26.13a illustrates this situation. The potential difference is measured by a device called a *voltmeter.* Notice that no battery is shown in the figure; also, we must assume no charge can flow through an ideal voltmeter. Hence, there is no path by which charge can flow and alter the charge on the capacitor. If a dielectric is now inserted between the plates as in Figure 26.13b, the voltmeter indicates that the voltage between the plates decreases to a value ΔV. The voltages with and without the dielectric are related by a factor κ as follows:

$$\Delta V = \frac{\Delta V_0}{\kappa}$$

Because $\Delta V < \Delta V_0$, we see that $\kappa > 1$. The dimensionless factor κ is called the **dielectric constant** of the material. The dielectric constant varies from one material to another. In this section, we analyze this change in capacitance in terms of electrical parameters such as electric charge, electric field, and potential difference; Section 26.7 describes the microscopic origin of these changes.

Because the charge Q_0 on the capacitor does not change, the capacitance must change to the value

$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{\Delta V_0/\kappa} = \kappa\frac{Q_0}{\Delta V_0}$$

Capacitance of a capacitor filled with a material of dielectric constant κ

$$C = \kappa C_0 \tag{26.14}$$

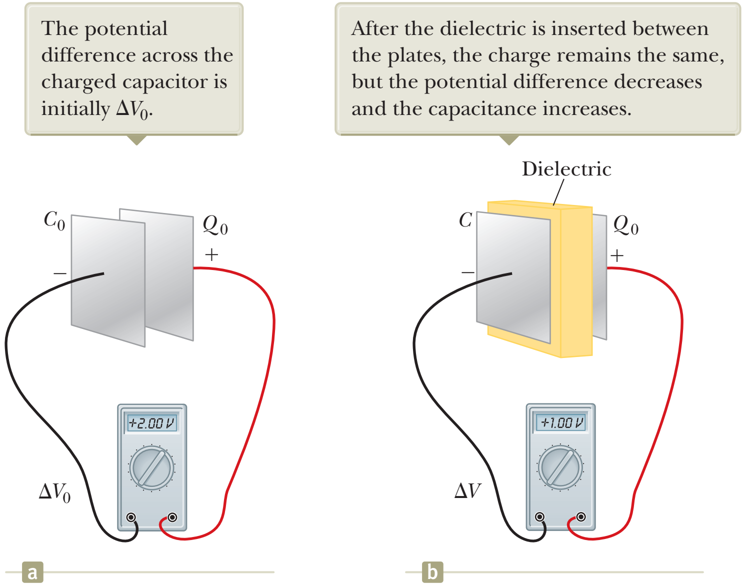


Figure 26.13 A charged capacitor (a) before and (b) after insertion of a dielectric between the plates.

That is, the capacitance *increases* by the factor κ when the dielectric completely fills the region between the plates.[5] Because $C_0 = \epsilon_0 A/d$ (Eq. 26.3) for a parallel-plate capacitor, we can express the capacitance of a parallel-plate capacitor filled with a dielectric as

$$C = \kappa \frac{\epsilon_0 A}{d} \tag{26.15}$$

From Equation 26.15, it would appear that the capacitance could be made very large by inserting a dielectric between the plates and decreasing d. In practice, the lowest value of d is limited by the electric discharge that could occur through the dielectric medium separating the plates. For any given separation d, the maximum voltage that can be applied to a capacitor without causing a discharge depends on the **dielectric strength** (maximum electric field) of the dielectric. If the magnitude of the electric field in the dielectric exceeds the dielectric strength, the insulating properties break down and the dielectric begins to conduct.

Physical capacitors have a specification called by a variety of names, including *working voltage, breakdown voltage,* and *rated voltage.* This parameter represents the largest voltage that can be applied to the capacitor without exceeding the dielectric strength of the dielectric material in the capacitor. Consequently, when selecting a capacitor for a given application, you must consider its capacitance as well as the expected voltage across the capacitor in the circuit, making sure the expected voltage is smaller than the rated voltage of the capacitor.

Insulating materials have values of κ greater than unity and dielectric strengths greater than that of air as Table 26.1 indicates. Therefore, a dielectric provides the following advantages:

- An increase in capacitance
- An increase in maximum operating voltage
- Possible mechanical support between the plates, which allows the plates to be close together without touching, thereby decreasing d and increasing C

Table 26.1 Approximate Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Material	Dielectric Constant κ	Dielectric Strength[a] (10^6 V/m)
Air (dry)	1.000 59	3
Bakelite	4.9	24
Fused quartz	3.78	8
Mylar	3.2	7
Neoprene rubber	6.7	12
Nylon	3.4	14
Paper	3.7	16
Paraffin-impregnated paper	3.5	11
Polystyrene	2.56	24
Polyvinyl chloride	3.4	40
Porcelain	6	12
Pyrex glass	5.6	14
Silicone oil	2.5	15
Strontium titanate	233	8
Teflon	2.1	60
Vacuum	1.000 00	—
Water	80	—

[a] The dielectric strength equals the maximum electric field that can exist in a dielectric without electrical breakdown. These values depend strongly on the presence of impurities and flaws in the materials.

[5] If the dielectric is introduced while the potential difference is held constant by a battery, the charge increases to a value $Q = \kappa Q_0$. The additional charge comes from the wires attached to the capacitor, and the capacitance again increases by the factor κ.

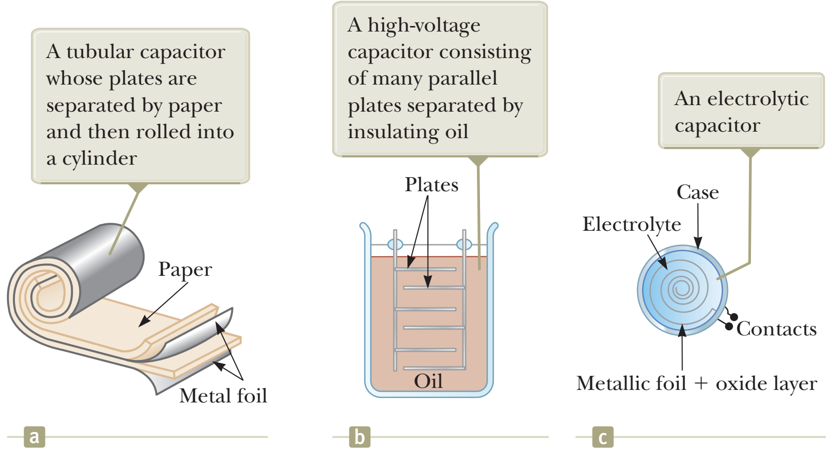


Figure 26.14 Three commercial capacitor designs.

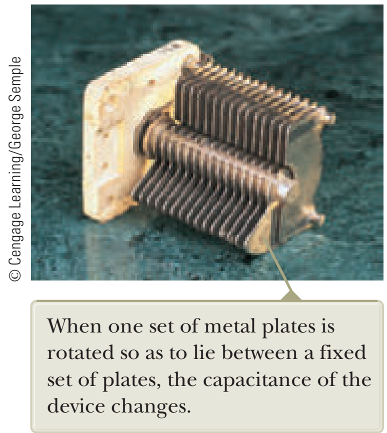


Figure 26.15 A variable capacitor.

Types of Capacitors

Many capacitors are built into integrated circuit chips, but some electrical devices still use stand-alone capacitors. Commercial capacitors are often made from metallic foil interlaced with thin sheets of either paraffin-impregnated paper or Mylar as the dielectric material. These alternate layers of metallic foil and dielectric are rolled into a cylinder to form a small package (Fig. 26.14a). High-voltage capacitors commonly consist of a number of interwoven metallic plates immersed in silicone oil (Fig. 26.14b). Small capacitors are often constructed from ceramic materials.

Often, an *electrolytic capacitor* is used to store large amounts of charge at relatively low voltages. This device, shown in Figure 26.14c, consists of a metallic foil in contact with an *electrolyte,* a solution that conducts electricity by virtue of the motion of ions contained in the solution. When a voltage is applied between the foil and the electrolyte, a thin layer of metal oxide (an insulator) is formed on the foil, and this layer serves as the dielectric. Very large values of capacitance can be obtained in an electrolytic capacitor because the dielectric layer is very thin and therefore the plate separation is very small.

Electrolytic capacitors are not reversible as are many other capacitors. They have a polarity, which is indicated by positive and negative signs marked on the device. When electrolytic capacitors are used in circuits, the polarity must be correct. If the polarity of the applied voltage is the opposite of what is intended, the oxide layer is removed and the capacitor conducts electricity instead of storing charge.

Variable capacitors (typically 10 to 500 pF) usually consist of two interwoven sets of metallic plates, one fixed and the other movable, and contain air as the dielectric (Fig. 26.15). These types of capacitors are often used in radio tuning circuits.

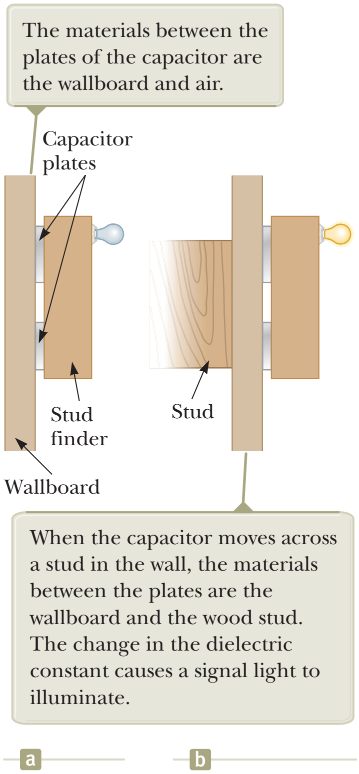


Figure 26.16 (Quick Quiz 26.5) A stud finder.

Quick Quiz 26.5 If you have ever tried to hang a picture or a mirror, you know it can be difficult to locate a wooden stud in which to anchor your nail or screw. A carpenter's stud finder is a capacitor with its plates arranged side by side instead of facing each other as shown in Figure 26.16. When the device is moved over a stud, does the capacitance **(a)** increase or **(b)** decrease?

Example 26.5 Energy Stored Before and After AM

A parallel-plate capacitor is charged with a battery to a charge Q_0. The battery is then removed, and a slab of material that has a dielectric constant κ is inserted between the plates. Identify the system as the capacitor and the dielectric. Find the energy stored in the system before and after the dielectric is inserted.

26.5 continued

SOLUTION

Conceptualize Think about what happens when the dielectric is inserted between the plates. Because the battery has been removed, the charge on the capacitor must remain the same. We know from our earlier discussion, however, that the capacitance must change. Therefore, we expect a change in the energy of the system.

Categorize Because we expect the energy of the system to change, we model it as a *nonisolated system* for *energy* involving a capacitor and a dielectric.

Analyze From Equation 26.11, find the energy stored in the absence of the dielectric:

$$U_0 = \frac{Q_0^{\ 2}}{2C_0}$$

Find the energy stored in the capacitor after the dielectric is inserted between the plates:

$$U = \frac{Q_0^{\ 2}}{2C}$$

Use Equation 26.14 to replace the capacitance C:

$$U = \frac{Q_0^{\ 2}}{2\kappa C_0} = \frac{U_0}{\kappa}$$

Finalize Because $\kappa > 1$, the final energy is less than the initial energy. We can account for the decrease in energy of the system by performing an experiment and noting that the dielectric, when inserted, is pulled into the device. To keep the dielectric from accelerating, an external agent must do negative work on the dielectric. Equation 8.2 becomes $\Delta U = W$, where both sides of the equation are negative.

26.6 Electric Dipole in an Electric Field

We have discussed the effect on the capacitance of placing a dielectric between the plates of a capacitor. In Section 26.7, we shall describe the microscopic origin of this effect. Before we can do so, however, let's expand the discussion of the electric dipole introduced in Section 23.4 (see Example 23.6). The electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance $2a$ as shown in Figure 26.17. The **electric dipole moment** of this configuration is defined as the vector $\vec{\mathbf{p}}$ directed from $-q$ toward $+q$ along the line joining the charges and having magnitude

$$p \equiv 2aq \tag{26.16}$$

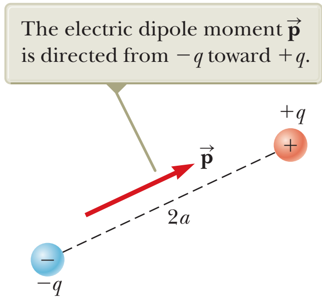


Figure 26.17 An electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance of $2a$.

Now suppose an electric dipole is placed in a uniform electric field $\vec{\mathbf{E}}$ and makes an angle θ with the field as shown in Figure 26.18. We identify $\vec{\mathbf{E}}$ as the field *external* to the dipole, established by some other charge distribution, to distinguish it from the field *due to* the dipole, which we discussed in Section 23.4.

Each of the charges is modeled as a particle in an electric field. The electric forces acting on the two charges are equal in magnitude ($F = qE$) and opposite in direction as shown in Figure 26.18. Therefore, the net force on the dipole is zero. The two forces produce a net torque on the dipole, however; the dipole is therefore described by the rigid object under a net torque model. As a result, the dipole rotates in the direction that brings the dipole moment vector into greater alignment with the field. The torque due to the force on the positive charge about an axis through O in Figure 26.18 has magnitude $Fa \sin \theta$, where $a \sin \theta$ is the moment arm of F about O. This force tends to produce a clockwise rotation. The torque about O on the negative charge is also of magnitude $Fa \sin \theta$; here again, the force tends to produce a clockwise rotation. Therefore, the magnitude of the net torque about O is

$$\tau = 2Fa \sin \theta$$

Because $F = qE$ and $p = 2aq$, we can express τ as

$$\tau = 2aqE \sin \theta = pE \sin \theta \tag{26.17}$$

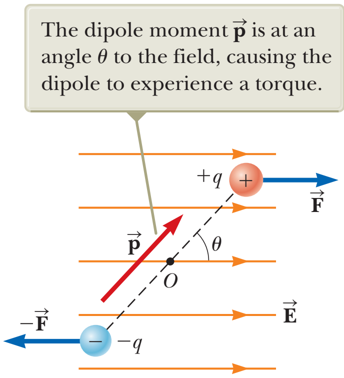


Figure 26.18 An electric dipole in a uniform external electric field.

Based on this expression, it is convenient to express the torque in vector form as the cross product of the vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{E}}$:

Torque on an electric dipole in an external electric field

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \tag{26.18}$$

We can also model the system of the dipole and the external electric field as an isolated system for energy. Let's determine the potential energy of the system as a function of the dipole's orientation with respect to the field. To do so, recognize that work must be done by an external agent to rotate the dipole through an angle so as to cause the dipole moment vector to become less aligned with the field. The work done is then stored as electric potential energy in the system. Notice that this potential energy is associated with a *rotational* configuration of the system. Previously, we have seen potential energies associated with *translational* configurations: an object with mass was moved in a gravitational field, a charge was moved in an electric field, or a spring was extended. The work dW required to rotate the dipole through an angle $d\theta$ is $dW = \tau\, d\theta$ (see Eq. 10.25). Because $\tau = pE \sin\theta$ and the work results in an increase in the electric potential energy U, we find that for a rotation from θ_i to θ_f, the change in potential energy of the system is

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau\, d\theta = \int_{\theta_i}^{\theta_f} pE \sin\theta\, d\theta = pE \int_{\theta_i}^{\theta_f} \sin\theta\, d\theta$$

$$= pE\left[-\cos\theta\right]_{\theta_i}^{\theta_f} = pE(\cos\theta_i - \cos\theta_f)$$

The term that contains $\cos\theta_i$ is a constant that depends on the initial orientation of the dipole. It is convenient to choose a reference angle of $\theta_i = 90°$ so that $\cos\theta_i = \cos 90° = 0$. Furthermore, let's choose $U_i = 0$ at $\theta_i = 90°$ as our reference value of potential energy. Hence, we can express a general value of $U_E = U_f$ as

$$U_E = -pE\cos\theta \tag{26.19}$$

We can write this expression for the potential energy of a dipole in an electric field as the dot product of the vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{E}}$:

Potential energy of the system of an electric dipole in an external electric field

$$\boxed{U_E = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}} \tag{26.20}$$

To develop a conceptual understanding of Equation 26.19, compare it with the expression for the potential energy of the system of an object in the Earth's gravitational field, $U_g = mgy$ (Eq. 7.19). First, both expressions contain a parameter of the entity placed in the field: mass for the object, dipole moment for the dipole. Second, both expressions contain the field, g for the object, E for the dipole. Finally, both expressions contain a configuration description: translational position y for the object, rotational position θ for the dipole. In both cases, once the configuration is changed, the system tends to return to the original configuration when the object is released: the object of mass m falls toward the ground, and the dipole begins to rotate back toward the configuration in which it is aligned with the field.

Molecules are said to be *polarized* when a separation exists between the average position of the negative charges and the average position of the positive charges in the molecule. In some molecules such as water, this condition is always present; such molecules are called **polar molecules.** Molecules that do not possess a permanent polarization are called **nonpolar molecules.**

We can understand the permanent polarization of water by inspecting the geometry of the water molecule. The oxygen atom in the water molecule is bonded to the hydrogen atoms such that an angle of 105° is formed between the two bonds (Fig. 26.19). The center of the negative charge distribution is near the oxygen atom, and the center of the positive charge distribution lies at a point midway along the line joining the hydrogen atoms (the point labeled × in Fig. 26.19). We can model the water molecule and other polar molecules as dipoles because the average positions of the positive and negative charges act as point charges. As a result, we can apply our discussion of dipoles to the behavior of polar molecules.

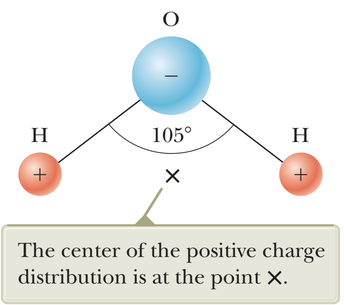


Figure 26.19 The water molecule, H_2O, has a permanent polarization resulting from its nonlinear geometry.

Washing with soap and water is a household scenario in which the dipole structure of water is exploited. Grease and oil are made up of nonpolar molecules, which are generally not attracted to water. Plain water is not very useful for removing this type of grime. Soap contains long molecules called *surfactants.* In a long molecule, the polarity characteristics of one end of the molecule can be different from those at the other end. In a surfactant molecule, one end acts like a nonpolar molecule and the other acts like a polar molecule. The nonpolar end can attach to a grease or oil molecule, and the polar end can attach to a water molecule. Therefore, the soap serves as a chain, linking the dirt and water molecules together. When the water is rinsed away, the grease and oil go with it.

A symmetric molecule (Fig. 26.20a) has no permanent polarization, but polarization can be induced by placing the molecule in an electric field. A field directed to the left as in Figure 26.20b causes the center of the negative charge distribution to shift to the right relative to the positive charges. This *induced polarization* is the effect that predominates in most materials used as dielectrics in capacitors.

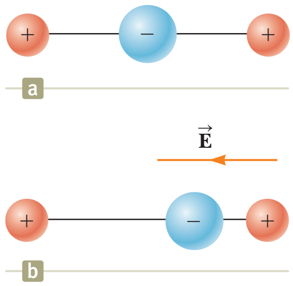


Figure 26.20 (a) A linear symmetric molecule has no permanent polarization. (b) An external electric field induces a polarization in the molecule.

Example 26.6 The H_2O Molecule AM

The water (H_2O) molecule has an electric dipole moment of 6.3×10^{-30} C · m. A sample contains 10^{21} water molecules, with the dipole moments all oriented in the direction of an electric field of magnitude 2.5×10^5 N/C. How much work is required to rotate the dipoles from this orientation ($\theta = 0°$) to one in which all the moments are perpendicular to the field ($\theta = 90°$)?

SOLUTION

Conceptualize When all the dipoles are aligned with the electric field, the dipoles–electric field system has the minimum potential energy. This energy has a negative value given by the product of the right side of Equation 26.19, evaluated at 0°, and the number N of dipoles.

Categorize The combination of the dipoles and the electric field is identified as a system. We use the *nonisolated system* model because an external agent performs work on the system to change its potential energy.

Analyze Write the appropriate reduction of the conservation of energy equation, Equation 8.2, for this situation:

$$(1)\quad \Delta U_E = W$$

Use Equation 26.19 to evaluate the initial and final potential energies of the system and Equation (1) to calculate the work required to rotate the dipoles:

$$W = U_{90°} - U_{0°} = (-NpE\cos 90°) - (-NpE\cos 0°)$$
$$= NpE = (10^{21})(6.3 \times 10^{-30}\ \text{C}\cdot\text{m})(2.5 \times 10^5\ \text{N/C})$$
$$= \boxed{1.6 \times 10^{-3}\ \text{J}}$$

Finalize Notice that the work done on the system is positive because the potential energy of the system has been raised from a negative value to a value of zero.

26.7 An Atomic Description of Dielectrics

In Section 26.5, we found that the potential difference ΔV_0 between the plates of a capacitor is reduced to $\Delta V_0/\kappa$ when a dielectric is introduced. The potential difference is reduced because the magnitude of the electric field decreases between the plates. In particular, if $\vec{\mathbf{E}}_0$ is the electric field without the dielectric, the field in the presence of a dielectric is

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{E}}_0}{\kappa} \qquad \textbf{(26.21)}$$

First consider a dielectric made up of polar molecules placed in the electric field between the plates of a capacitor. The dipoles (that is, the polar molecules making

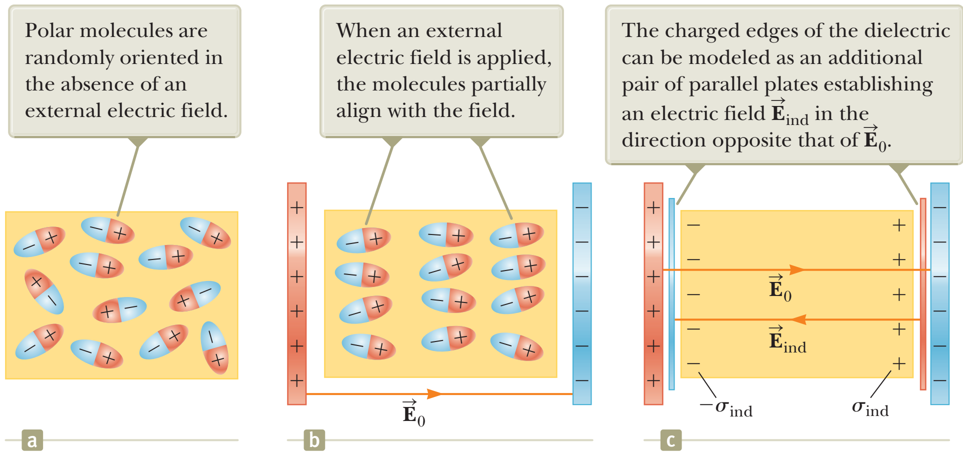


Figure 26.21 (a) Polar molecules in a dielectric. (b) An electric field is applied to the dielectric. (c) Details of the electric field inside the dielectric.

up the dielectric) are randomly oriented in the absence of an electric field as shown in Figure 26.21a. When an external field $\vec{\mathbf{E}}_0$ due to charges on the capacitor plates is applied, a torque is exerted on the dipoles, causing them to partially align with the field as shown in Figure 26.21b. The dielectric is now polarized. The degree of alignment of the molecules with the electric field depends on temperature and the magnitude of the field. In general, the alignment increases with decreasing temperature and with increasing electric field.

If the molecules of the dielectric are nonpolar, the electric field due to the plates produces an induced polarization in the molecule. These induced dipole moments tend to align with the external field, and the dielectric is polarized. Therefore, a dielectric can be polarized by an external field regardless of whether the molecules in the dielectric are polar or nonpolar.

With these ideas in mind, consider a slab of dielectric material placed between the plates of a capacitor so that it is in a uniform electric field $\vec{\mathbf{E}}_0$ as shown in Figure 26.21b. The electric field due to the plates is directed to the right and polarizes the dielectric. The net effect on the dielectric is the formation of an *induced* positive surface charge density σ_{ind} on the right face and an equal-magnitude negative surface charge density $-\sigma_{\text{ind}}$ on the left face as shown in Figure 26.21c. Because we can model these surface charge distributions as being due to charged parallel plates, the induced surface charges on the dielectric give rise to an induced electric field $\vec{\mathbf{E}}_{\text{ind}}$ in the direction opposite the external field $\vec{\mathbf{E}}_0$. Therefore, the net electric field $\vec{\mathbf{E}}$ in the dielectric has a magnitude

$$E = E_0 - E_{\text{ind}} \tag{26.22}$$

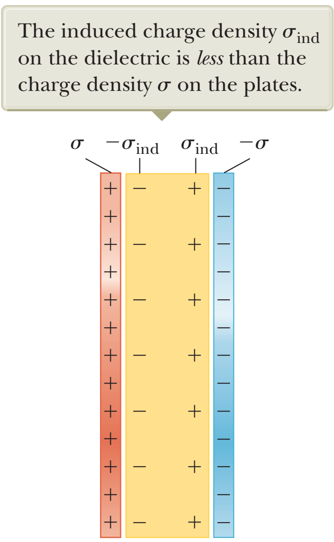


Figure 26.22 Induced charge on a dielectric placed between the plates of a charged capacitor.

In the parallel-plate capacitor shown in Figure 26.22, the external field E_0 is related to the charge density σ on the plates through the relationship $E_0 = \sigma/\epsilon_0$. The induced electric field in the dielectric is related to the induced charge density σ_{ind} through the relationship $E_{\text{ind}} = \sigma_{\text{ind}}/\epsilon_0$. Because $E = E_0/\kappa = \sigma/\kappa\epsilon_0$, substitution into Equation 26.22 gives

$$\frac{\sigma}{\kappa\epsilon_0} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_{\text{ind}}}{\epsilon_0}$$

$$\sigma_{\text{ind}} = \left(\frac{\kappa - 1}{\kappa}\right)\sigma \tag{26.23}$$

Because $\kappa > 1$, this expression shows that the charge density σ_{ind} induced on the dielectric is less than the charge density σ on the plates. For instance, if $\kappa = 3$, the induced charge density is two-thirds the charge density on the plates. If no dielectric is present, then $\kappa = 1$ and $\sigma_{\text{ind}} = 0$ as expected. If the dielectric is replaced by an electrical conductor for which $E = 0$, however, Equation 26.22 indicates that $E_0 = E_{\text{ind}}$, which corresponds to $\sigma_{\text{ind}} = \sigma$. That is, the surface charge induced on the conductor is equal in magnitude but opposite in sign to that on the plates, resulting in a net electric field of zero in the conductor (see Fig. 24.16).

Example 26.7 Effect of a Metallic Slab

A parallel-plate capacitor has a plate separation d and plate area A. An uncharged metallic slab of thickness a is inserted midway between the plates.

(A) Find the capacitance of the device.

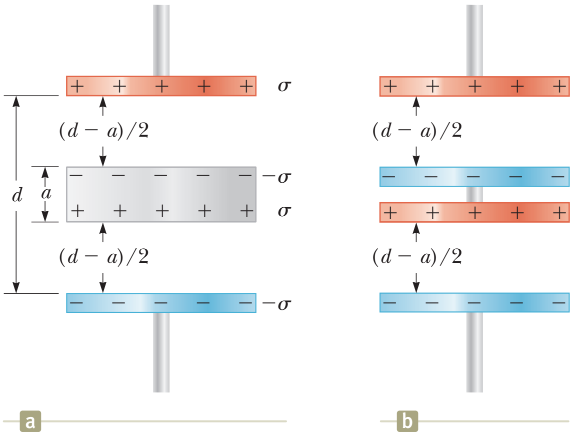


Figure 26.23 (Example 26.7) (a) A parallel-plate capacitor of plate separation d partially filled with a metallic slab of thickness a. (b) The equivalent circuit of the device in (a) consists of two capacitors in series, each having a plate separation $(d - a)/2$.

SOLUTION

Conceptualize Figure 26.23a shows the metallic slab between the plates of the capacitor. Any charge that appears on one plate of the capacitor must induce a charge of equal magnitude and opposite sign on the near side of the slab as shown in Figure 26.23a. Consequently, the net charge on the slab remains zero and the electric field inside the slab is zero.

Categorize The planes of charge on the metallic slab's upper and lower edges are identical to the distribution of charges on the plates of a capacitor. The metal between the slab's edges serves only to make an electrical connection between the edges. Therefore, we can model the edges of the slab as conducting planes and the bulk of the slab as a wire. As a result, the capacitor in Figure 26.23a is equivalent to two capacitors in series, each having a plate separation $(d - a)/2$ as shown in Figure 26.23b.

Analyze Use Equation 26.3 and the rule for adding two capacitors in series (Eq. 26.10) to find the equivalent capacitance in Figure 26.23b:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\dfrac{\epsilon_0 A}{(d-a)/2}} + \frac{1}{\dfrac{\epsilon_0 A}{(d-a)/2}}$$

$$C = \boxed{\frac{\epsilon_0 A}{d - a}}$$

(B) Show that the capacitance of the original capacitor is unaffected by the insertion of the metallic slab if the slab is infinitesimally thin.

SOLUTION

In the result for part (A), let $a \to 0$:

$$C = \lim_{a\to 0}\left(\frac{\epsilon_0 A}{d - a}\right) = \frac{\epsilon_0 A}{d}$$

Finalize The result of part (B) is the original capacitance before the slab is inserted, which tells us that we can insert an infinitesimally thin metallic sheet between the plates of a capacitor without affecting the capacitance. We use this fact in the next example.

WHAT IF? What if the metallic slab in part (A) is not midway between the plates? How would that affect the capacitance?

Answer Let's imagine moving the slab in Figure 26.23a upward so that the distance between the upper edge of the slab and the upper plate is b. Then, the distance between the lower edge of the slab and the lower plate is $d - b - a$. As in part (A), we find the total capacitance of the series combination:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\epsilon_0 A/b} + \frac{1}{\epsilon_0 A/(d - b - a)}$$

$$= \frac{b}{\epsilon_0 A} + \frac{d - b - a}{\epsilon_0 A} = \frac{d - a}{\epsilon_0 A} \rightarrow C = \frac{\epsilon_0 A}{d - a}$$

which is the same result as found in part (A). The capacitance is independent of the value of b, so it does not matter where the slab is located. In Figure 26.23b, when the central structure is moved up or down, the decrease in plate separation of one capacitor is compensated by the increase in plate separation for the other.

Example 26.8 A Partially Filled Capacitor

A parallel-plate capacitor with a plate separation d has a capacitance C_0 in the absence of a dielectric. What is the capacitance when a slab of dielectric material of dielectric constant κ and thickness fd is inserted between the plates (Fig. 26.24a), where f is a fraction between 0 and 1?

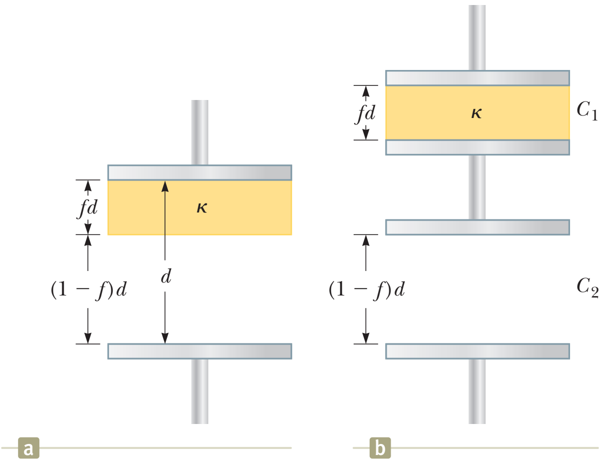


Figure 26.24 (Example 26.8) (a) A parallel-plate capacitor of plate separation d partially filled with a dielectric of thickness fd. (b) The equivalent circuit of the capacitor consists of two capacitors connected in series.

SOLUTION

Conceptualize In our previous discussions of dielectrics between the plates of a capacitor, the dielectric filled the volume between the plates. In this example, only part of the volume between the plates contains the dielectric material.

Categorize In Example 26.7, we found that an infinitesimally thin metallic sheet inserted between the plates of a capacitor does not affect the capacitance. Imagine sliding an infinitesimally thin metallic slab along the bottom face of the dielectric shown in Figure 26.24a. We can model this system as a series combination of two capacitors as shown in Figure 26.24b. One capacitor has a plate separation fd and is filled with a dielectric; the other has a plate separation $(1 - f)d$ and has air between its plates.

Analyze Evaluate the two capacitances in Figure 26.24b from Equation 26.15:

$$C_1 = \frac{\kappa\epsilon_0 A}{fd} \quad \text{and} \quad C_2 = \frac{\epsilon_0 A}{(1-f)d}$$

Find the equivalent capacitance C from Equation 26.10 for two capacitors combined in series:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{fd}{\kappa\epsilon_0 A} + \frac{(1-f)d}{\epsilon_0 A}$$

$$\frac{1}{C} = \frac{fd}{\kappa\epsilon_0 A} + \frac{\kappa(1-f)d}{\kappa\epsilon_0 A} = \frac{f + \kappa(1-f)}{\kappa}\frac{d}{\epsilon_0 A}$$

Invert and substitute for the capacitance without the dielectric, $C_0 = \epsilon_0 A/d$:

$$C = \frac{\kappa}{f + \kappa(1-f)}\frac{\epsilon_0 A}{d} = \boxed{\frac{\kappa}{f + \kappa(1-f)}C_0}$$

Finalize Let's test this result for some known limits. If $f \to 0$, the dielectric should disappear. In this limit, $C \to C_0$, which is consistent with a capacitor with air between the plates. If $f \to 1$, the dielectric fills the volume between the plates. In this limit, $C \to \kappa C_0$, which is consistent with Equation 26.14.

Summary

Definitions

A **capacitor** consists of two conductors carrying charges of equal magnitude and opposite sign. The **capacitance** C of any capacitor is the ratio of the charge Q on either conductor to the potential difference ΔV between them:

$$C \equiv \frac{Q}{\Delta V} \tag{26.1}$$

The capacitance depends only on the geometry of the conductors and not on an external source of charge or potential difference. The SI unit of capacitance is coulombs per volt, or the **farad** (F): 1 F = 1 C/V.

The **electric dipole moment** $\vec{\mathbf{p}}$ of an electric dipole has a magnitude

$$p \equiv 2aq \tag{26.16}$$

where $2a$ is the distance between the charges q and $-q$. The direction of the electric dipole moment vector is from the negative charge toward the positive charge.

Concepts and Principles

If two or more capacitors are connected in parallel, the potential difference is the same across all capacitors. The equivalent capacitance of a **parallel combination** of capacitors is

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \quad \textbf{(26.8)}$$

If two or more capacitors are connected in series, the charge is the same on all capacitors, and the equivalent capacitance of the **series combination** is given by

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \quad \textbf{(26.10)}$$

These two equations enable you to simplify many electric circuits by replacing multiple capacitors with a single equivalent capacitance.

Energy is stored in a charged capacitor because the charging process is equivalent to the transfer of charges from one conductor at a lower electric potential to another conductor at a higher potential. The energy stored in a capacitor of capacitance C with charge Q and potential difference ΔV is

$$U_E = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2 \quad \textbf{(26.11)}$$

When a dielectric material is inserted between the plates of a capacitor, the capacitance increases by a dimensionless factor κ, called the **dielectric constant:**

$$C = \kappa C_0 \quad \textbf{(26.14)}$$

where C_0 is the capacitance in the absence of the dielectric.

The torque acting on an electric dipole in a uniform electric field $\vec{\mathbf{E}}$ is

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \quad \textbf{(26.18)}$$

The potential energy of the system of an electric dipole in a uniform external electric field $\vec{\mathbf{E}}$ is

$$U_E = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}} \quad \textbf{(26.20)}$$

Objective Questions

$\boxed{\textbf{1.}}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. A fully charged parallel-plate capacitor remains connected to a battery while you slide a dielectric between the plates. Do the following quantities (a) increase, (b) decrease, or (c) stay the same? **(i)** C **(ii)** Q **(iii)** ΔV **(iv)** the energy stored in the capacitor

2. By what factor is the capacitance of a metal sphere multiplied if its volume is tripled? (a) 3 (b) $3^{1/3}$ (c) 1 (d) $3^{-1/3}$ (e) $\frac{1}{3}$

3. An electronics technician wishes to construct a parallel-plate capacitor using rutile ($\kappa = 100$) as the dielectric. The area of the plates is 1.00 cm^2. What is the capacitance if the rutile thickness is 1.00 mm? (a) 88.5 pF (b) 177 pF (c) 8.85 μF (d) 100 μF (e) 35.4 μF

4. A parallel-plate capacitor is connected to a battery. What happens to the stored energy if the plate separation is doubled while the capacitor remains connected to the battery? (a) It remains the same. (b) It is doubled. (c) It decreases by a factor of 2. (d) It decreases by a factor of 4. (e) It increases by a factor of 4.

5. If three unequal capacitors, initially uncharged, are connected in series across a battery, which of the following statements is true? (a) The equivalent capacitance is greater than any of the individual capacitances. (b) The largest voltage appears across the smallest capacitance. (c) The largest voltage appears across the largest capacitance. (d) The capacitor with the largest capacitance has the greatest charge. (e) The capacitor with the smallest capacitance has the smallest charge.

6. Assume a device is designed to obtain a large potential difference by first charging a bank of capacitors connected in parallel and then activating a switch arrangement that in effect disconnects the capacitors from the charging source and from each other and reconnects them all in a series arrangement. The group of charged capacitors is then discharged in series. What is the maximum potential difference that can be obtained in this manner by using ten 500-μF capacitors and an 800-V charging source? (a) 500 V (b) 8.00 kV (c) 400 kV (d) 800 V (e) 0

$\boxed{\textbf{7.}}$ **(i)** What happens to the magnitude of the charge on each plate of a capacitor if the potential difference between the conductors is doubled? (a) It becomes four times larger. (b) It becomes two times larger. (c) It is unchanged. (d) It becomes one-half as large. (e) It becomes one-fourth as large. **(ii)** If the potential difference across a capacitor is doubled, what happens to the energy stored? Choose from the same possibilities as in part (i).

8. A capacitor with very large capacitance is in series with another capacitor with very small capacitance. What is the equivalent capacitance of the combination? (a) slightly greater than the capacitance of the large capacitor (b) slightly less than the capacitance of the large capacitor (c) slightly greater than the capacitance of the small capacitor (d) slightly less than the capacitance of the small capacitor

9. A parallel-plate capacitor filled with air carries a charge Q. The battery is disconnected, and a slab of material with dielectric constant $\kappa = 2$ is inserted between the plates. Which of the following statements is true? (a) The voltage across the capacitor decreases by a factor of 2. (b) The voltage across the capacitor is doubled. (c) The charge on the plates is doubled. (d) The charge on the plates decreases by a factor of 2. (e) The electric field is doubled.

10. **(i)** A battery is attached to several different capacitors connected in parallel. Which of the following statements is true? (a) All capacitors have the same charge, and the equivalent capacitance is greater than the capacitance of any of the capacitors in the group. (b) The capacitor with the largest capacitance carries the smallest charge. (c) The potential difference across each capacitor is the same, and the equivalent capacitance is greater than any of the capacitors in the group. (d) The capacitor with the smallest capacitance carries the largest charge. (e) The potential differences across the capacitors are the same only if the capacitances are the same. **(ii)** The capacitors are reconnected in series, and the combination is again connected to the battery. From the same choices, choose the one that is true.

11. A parallel-plate capacitor is charged and then is disconnected from the battery. By what factor does the stored energy change when the plate separation is then doubled? (a) It becomes four times larger. (b) It becomes two times larger. (c) It stays the same. (d) It becomes one-half as large. (e) It becomes one-fourth as large.

12. **(i)** Rank the following five capacitors from greatest to smallest capacitance, noting any cases of equality. (a) a 20-μF capacitor with a 4-V potential difference between its plates (b) a 30-μF capacitor with charges of magnitude 90 μC on each plate (c) a capacitor with charges of magnitude 80 μC on its plates, differing by 2 V in potential, (d) a 10-μF capacitor storing energy 125 μJ (e) a capacitor storing energy 250 μJ with a 10-V potential difference **(ii)** Rank the same capacitors in part (i) from largest to smallest according to the potential difference between the plates. **(iii)** Rank the capacitors in part (i) in the order of the magnitudes of the charges on their plates. **(iv)** Rank the capacitors in part (i) in the order of the energy they store.

13. True or False? (a) From the definition of capacitance $C = Q/\Delta V$, it follows that an uncharged capacitor has a capacitance of zero. (b) As described by the definition of capacitance, the potential difference across an uncharged capacitor is zero.

14. You charge a parallel-plate capacitor, remove it from the battery, and prevent the wires connected to the plates from touching each other. When you increase the plate separation, do the following quantities (a) increase, (b) decrease, or (c) stay the same? **(i)** C **(ii)** Q **(iii)** E between the plates **(iv)** ΔV

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. (a) Why is it dangerous to touch the terminals of a high-voltage capacitor even after the voltage source that charged the capacitor is disconnected from the capacitor? (b) What can be done to make the capacitor safe to handle after the voltage source has been removed?

2. Assume you want to increase the maximum operating voltage of a parallel-plate capacitor. Describe how you can do that with a fixed plate separation.

3. If you were asked to design a capacitor in which small size and large capacitance were required, what would be the two most important factors in your design?

4. Explain why a dielectric increases the maximum operating voltage of a capacitor even though the physical size of the capacitor doesn't change.

5. Explain why the work needed to move a particle with charge Q through a potential difference ΔV is $W = Q\,\Delta V$, whereas the energy stored in a charged capacitor is $U_E = \frac{1}{2}Q\,\Delta V$. Where does the factor $\frac{1}{2}$ come from?

6. An air-filled capacitor is charged, then disconnected from the power supply, and finally connected to a voltmeter. Explain how and why the potential difference changes when a dielectric is inserted between the plates of the capacitor.

7. The sum of the charges on both plates of a capacitor is zero. What does a capacitor store?

8. Because the charges on the plates of a parallel-plate capacitor are opposite in sign, they attract each other. Hence, it would take positive work to increase the plate separation. What type of energy in the system changes due to the external work done in this process?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 26.1 Definition of Capacitance

1. (a) When a battery is connected to the plates of a 3.00-μF capacitor, it stores a charge of 27.0 μC. What is the voltage of the battery? (b) If the same capacitor is connected to another battery and 36.0 μC of charge is stored on the capacitor, what is the voltage of the battery?

2. W Two conductors having net charges of +10.0 μC and −10.0 μC have a potential difference of 10.0 V between them. (a) Determine the capacitance of the system. (b) What is the potential difference between the two conductors if the charges on each are increased to +100 μC and −100 μC?

3. W (a) How much charge is on each plate of a 4.00-μF capacitor when it is connected to a 12.0-V battery? (b) If this same capacitor is connected to a 1.50-V battery, what charge is stored?

Section 26.2 Calculating Capacitance

4. M An air-filled spherical capacitor is constructed with inner- and outer-shell radii of 7.00 cm and 14.0 cm, respectively. (a) Calculate the capacitance of the device. (b) What potential difference between the spheres results in a 4.00-μC charge on the capacitor?

5. M A 50.0-m length of coaxial cable has an inner conductor that has a diameter of 2.58 mm and carries a charge of 8.10 μC. The surrounding conductor has an inner diameter of 7.27 mm and a charge of −8.10 μC. Assume the region between the conductors is air. (a) What is the capacitance of this cable? (b) What is the potential difference between the two conductors?

6. W (a) Regarding the Earth and a cloud layer 800 m above the Earth as the "plates" of a capacitor, calculate the capacitance of the Earth–cloud layer system. Assume the cloud layer has an area of 1.00 km^2 and the air between the cloud and the ground is pure and dry. Assume charge builds up on the cloud and on the ground until a uniform electric field of 3.00×10^6 N/C throughout the space between them makes the air break down and conduct electricity as a lightning bolt. (b) What is the maximum charge the cloud can hold?

7. When a potential difference of 150 V is applied to the plates of a parallel-plate capacitor, the plates carry a surface charge density of 30.0 nC/cm^2. What is the spacing between the plates?

8. An air-filled parallel-plate capacitor has plates of area 2.30 cm^2 separated by 1.50 mm. (a) Find the value of its capacitance. The capacitor is connected to a 12.0-V battery. (b) What is the charge on the capacitor? (c) What is the magnitude of the uniform electric field between the plates?

9. M An air-filled capacitor consists of two parallel plates, each with an area of 7.60 cm^2, separated by a distance of 1.80 mm. A 20.0-V potential difference is applied to these plates. Calculate (a) the electric field between the plates, (b) the surface charge density, (c) the capacitance, and (d) the charge on each plate.

10. A variable air capacitor used in a radio tuning circuit is made of N semicircular plates, each of radius R and positioned a distance d from its neighbors, to which it is electrically connected. As shown in Figure P26.10, a second identical set of plates is enmeshed with the first set. Each plate in the second set is halfway between two plates of the first set. The second set can rotate as a unit. Determine the capacitance as a function of the angle of rotation θ, where $\theta = 0$ corresponds to the maximum capacitance.

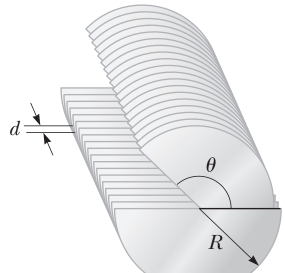


Figure P26.10

11. An isolated, charged conducting sphere of radius 12.0 cm creates an electric field of 4.90×10^4 N/C at a distance 21.0 cm from its center. (a) What is its surface charge density? (b) What is its capacitance?

12. **Review.** A small object of mass m carries a charge q and is suspended by a thread between the vertical plates of a parallel-plate capacitor. The plate separation is d. If the thread makes an angle θ with the vertical, what is the potential difference between the plates?

Section 26.3 Combinations of Capacitors

13. W Two capacitors, $C_1 = 5.00$ μF and $C_2 = 12.0$ μF, are connected in parallel, and the resulting combination is connected to a 9.00-V battery. Find (a) the equivalent capacitance of the combination, (b) the potential difference across each capacitor, and (c) the charge stored on each capacitor.

14. W **What If?** The two capacitors of Problem 13 ($C_1 = 5.00$ μF and $C_2 = 12.0$ μF) are now connected in series and to a 9.00-V battery. Find (a) the equivalent capacitance of the combination, (b) the potential difference across each capacitor, and (c) the charge on each capacitor.

15. Find the equivalent capacitance of a 4.20-μF capacitor and an 8.50-μF capacitor when they are connected (a) in series and (b) in parallel.

16. Given a 2.50-μF capacitor, a 6.25-μF capacitor, and a 6.00-V battery, find the charge on each capacitor if you connect them (a) in series across the battery and (b) in parallel across the battery.

17. According to its design specification, the timer circuit delaying the closing of an elevator door is to have a capacitance of 32.0 μF between two points A and B. When one circuit is being constructed, the inexpensive but durable capacitor installed between these two points is found to have capacitance 34.8 μF. To meet the specification, one additional capacitor can be placed between the two points. (a) Should it be in series or in parallel with the 34.8-μF capacitor? (b) What should be its capacitance? (c) **What If?** The next circuit comes down the assembly line with capacitance 29.8 μF between A and B. To meet the specification, what additional capacitor should be installed in series or in parallel in that circuit?

18. *Why is the following situation impossible?* A technician is testing a circuit that contains a capacitance C. He realizes that a better design for the circuit would include a capacitance $\frac{7}{3}C$ rather than C. He has three additional capacitors, each with capacitance C. By combining these additional capacitors in a certain combination that is then placed in parallel with the original capacitor, he achieves the desired capacitance.

19. For the system of four capacitors shown in Figure P26.19, find (a) the equivalent capacitance of the system, (b) the charge on each capacitor, and (c) the potential difference across each capacitor.

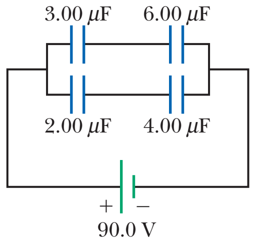


Figure P26.19
Problems 19 and 56.

20. Three capacitors are connected to a battery as shown in Figure P26.20. Their capacitances are $C_1 = 3C$, $C_2 = C$, and $C_3 = 5C$. (a) What is the equivalent capacitance of this set of capacitors? (b) State the ranking of the capacitors according to the charge they store from largest to smallest. (c) Rank the capacitors according to the potential differences across them from largest to smallest. (d) **What If?** Assume C_3 is increased. Explain what happens to the charge stored by each capacitor.

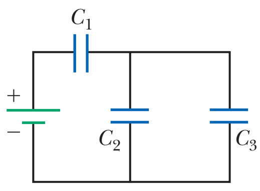


Figure P26.20

21. M A group of identical capacitors is connected first in series and then in parallel. The combined capacitance in parallel is 100 times larger than for the series connection. How many capacitors are in the group?

22. W (a) Find the equivalent capacitance between points a and b for the group of capacitors connected as shown in Figure P26.22. Take $C_1 = 5.00\ \mu\text{F}$, $C_2 = 10.0\ \mu\text{F}$, and $C_3 = 2.00\ \mu\text{F}$. (b) What charge is stored on C_3 if the potential difference between points a and b is 60.0 V?

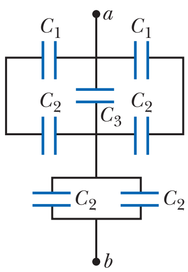


Figure P26.22

23. M Four capacitors are connected as shown in Figure P26.23. (a) Find the equivalent capacitance between points a and b. (b) Calculate the charge on each capacitor, taking $\Delta V_{ab} = 15.0$ V.

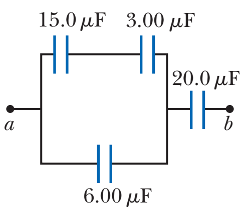


Figure P26.23

24. M Consider the circuit shown in Figure P26.24, where $C_1 = 6.00\ \mu\text{F}$, $C_2 = 3.00\ \mu\text{F}$, and $\Delta V = 20.0$ V. Capacitor C_1 is first charged by closing switch S_1. Switch S_1 is then opened, and the charged capacitor is connected to the uncharged capacitor by closing S_2. Calculate (a) the initial charge acquired by C_1 and (b) the final charge on each capacitor.

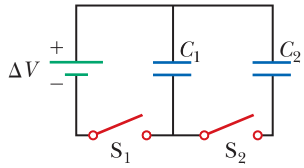


Figure P26.24

25. Find the equivalent capacitance between points a and b in the combination of capacitors shown in Figure P26.25.

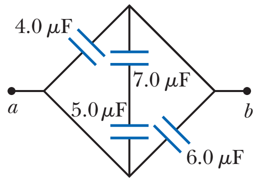


Figure P26.25

26. Find (a) the equivalent capacitance of the capacitors in Figure P26.26, (b) the charge on each capacitor, and (c) the potential difference across each capacitor.

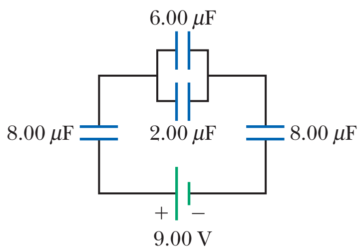


Figure P26.26

27. Two capacitors give an equivalent capacitance of 9.00 pF when connected in parallel and an equivalent capacitance of 2.00 pF when connected in series. What is the capacitance of each capacitor?

28. Two capacitors give an equivalent capacitance of C_p when connected in parallel and an equivalent capacitance of C_s when connected in series. What is the capacitance of each capacitor?

29. Consider three capacitors C_1, C_2, and C_3 and a battery. If only C_1 is connected to the battery, the charge on C_1 is 30.8 μC. Now C_1 is disconnected, discharged, and connected in series with C_2. When the series combination of C_2 and C_1 is connected across the battery, the charge on C_1 is 23.1 μC. The circuit is disconnected, and both capacitors are discharged. Next, C_3, C_1, and the battery are connected in series, resulting in a charge on C_1 of 25.2 μC. If, after being disconnected and discharged, C_1, C_2, and C_3 are connected in series with one another and with the battery, what is the charge on C_1?

Section 26.4 Energy Stored in a Charged Capacitor

30. The immediate cause of many deaths is ventricular fibrillation, which is an uncoordinated quivering of the heart. An electric shock to the chest can cause momentary paralysis of the heart muscle, after which the heart sometimes resumes its proper beating. One type of *defibrillator* (chapter-opening photo, page 777) applies a strong electric shock to the chest over a time interval of a few milliseconds. This device contains a capacitor of several microfarads, charged to several thousand volts. Electrodes called paddles are held against the chest on both sides of the heart, and the capacitor is discharged through the patient's chest. Assume an energy of 300 J is to be delivered from a 30.0-μF capacitor. To what potential difference must it be charged?

31. A 12.0-V battery is connected to a capacitor, resulting in 54.0 μC of charge stored on the capacitor. How much energy is stored in the capacitor?

32. W (a) A 3.00-μF capacitor is connected to a 12.0-V battery. How much energy is stored in the capacitor? (b) Had the capacitor been connected to a 6.00-V battery, how much energy would have been stored?

33. As a person moves about in a dry environment, electric charge accumulates on the person's body. Once it is at high voltage, either positive or negative, the body can discharge via sparks and shocks. Consider a human body isolated from ground, with the typical capacitance 150 pF. (a) What charge on the body will produce a potential of 10.0 kV? (b) Sensitive electronic devices can be destroyed by electrostatic discharge from a person. A particular device can be destroyed by a discharge releasing an energy of 250 μJ. To what voltage on the body does this situation correspond?

34. Two capacitors, $C_1 = 18.0\ \mu$F and $C_2 = 36.0\ \mu$F, are connected in series, and a 12.0-V battery is connected across the two capacitors. Find (a) the equivalent capacitance and (b) the energy stored in this equivalent capacitance. (c) Find the energy stored in each individual capacitor. (d) Show that the sum of these two energies is the same as the energy found in part (b). (e) Will this equality always be true, or does it depend on the number of capacitors and their capacitances? (f) If the same capacitors were connected in parallel, what potential difference would be required across them so that the combination stores the same energy as in part (a)? (g) Which capacitor stores more energy in this situation, C_1 or C_2?

35. Two identical parallel-plate capacitors, each with capacitance 10.0 μF, are charged to potential difference 50.0 V and then disconnected from the battery. They are then connected to each other in parallel with plates of like sign connected. Finally, the plate separation in one of the capacitors is doubled. (a) Find the total energy of the system of two capacitors *before* the plate separation is doubled. (b) Find the potential difference across each capacitor *after* the plate separation is doubled. (c) Find the total energy of the system *after* the plate separation is doubled. (d) Reconcile the difference in the answers to parts (a) and (c) with the law of conservation of energy.

36. Two identical parallel-plate capacitors, each with capacitance C, are charged to potential difference ΔV and then disconnected from the battery. They are then connected to each other in parallel with plates of like sign connected. Finally, the plate separation in one of the capacitors is doubled. (a) Find the total energy of the system of two capacitors *before* the plate separation is doubled. (b) Find the potential difference across each capacitor *after* the plate separation is doubled. (c) Find the total energy of the system *after* the plate separation is doubled. (d) Reconcile the difference in the answers to parts (a) and (c) with the law of conservation of energy.

37. Two capacitors, $C_1 = 25.0\ \mu$F and $C_2 = 5.00\ \mu$F, are connected in parallel and charged with a 100-V power supply. (a) Draw a circuit diagram and (b) calculate the total energy stored in the two capacitors. (c) **What If?** What potential difference would be required across the same two capacitors connected in series for the combination to store the same amount of energy as in part (b)? (d) Draw a circuit diagram of the circuit described in part (c).

38. A parallel-plate capacitor has a charge Q and plates of area A. What force acts on one plate to attract it toward the other plate? Because the electric field between the plates is $E = Q/A\epsilon_0$, you might think the force is $F = QE = Q^2/A\epsilon_0$. This conclusion is wrong because the field E includes contributions from both plates, and the field created by the positive plate cannot exert any force on the positive plate. Show that the force exerted on each plate is actually $F = Q^2/2A\epsilon_0$. *Suggestion:* Let $C = \epsilon_0 A/x$ for an arbitrary plate separation x and note that the work done in separating the two charged plates is $W = \int F\,dx$.

39. AMT **Review.** A storm cloud and the ground represent the plates of a capacitor. During a storm, the capacitor has a potential difference of 1.00×10^8 V between its plates and a charge of 50.0 C. A lightning strike delivers 1.00% of the energy of the capacitor to a tree on the ground. How much sap in the tree can be boiled away? Model the sap as water initially at 30.0°C. Water has a specific heat of 4 186 J/kg · °C, a boiling point of 100°C, and a latent heat of vaporization of 2.26×10^6 J/kg.

40. GP Consider two conducting spheres with radii R_1 and R_2 separated by a distance much greater than either radius. A total charge Q is shared between the spheres. We wish to show that when the electric potential energy of the system has a minimum value, the potential difference between the spheres is zero. The total charge Q is equal to $q_1 + q_2$, where q_1 represents the charge on the first sphere and q_2 the charge on the second. Because the spheres are very far apart, you can assume the charge of each is uniformly distributed over its surface. (a) Show that the energy associated with a single conducting sphere of radius R and charge q surrounded by a vacuum is $U = k_e q^2/2R$. (b) Find the total energy of the system of two spheres in terms of q_1, the total charge Q, and the radii R_1 and R_2. (c) To minimize the energy, differentiate the result to part (b) with respect to q_1 and set the derivative equal to zero. Solve for q_1 in terms of Q and the radii. (d) From the result to part (c), find the charge q_2. (e) Find the potential of each sphere. (f) What is the potential difference between the spheres?

41. AMT **Review.** The circuit in Figure P26.41 (page 804) consists of two identical, parallel metal plates connected to identical metal springs, a switch, and a 100-V battery.

With the switch open, the plates are uncharged, are separated by a distance d = 8.00 mm, and have a capacitance C = 2.00 μF. When the switch is closed, the distance between the plates decreases by a factor of 0.500. (a) How much charge collects on each plate? (b) What is the spring constant for each spring?

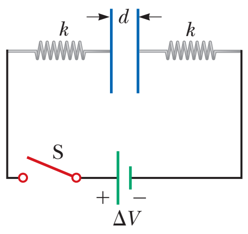


Figure P26.41

Section 26.5 Capacitors with Dielectrics

42. A supermarket sells rolls of aluminum foil, plastic wrap, and waxed paper. (a) Describe a capacitor made from such materials. Compute order-of-magnitude estimates for (b) its capacitance and (c) its breakdown voltage.

43. W (a) How much charge can be placed on a capacitor with air between the plates before it breaks down if the area of each plate is 5.00 cm^2? (b) **What If?** Find the maximum charge if polystyrene is used between the plates instead of air.

44. The voltage across an air-filled parallel-plate capacitor is measured to be 85.0 V. When a dielectric is inserted and completely fills the space between the plates as in Figure P26.44, the voltage drops to 25.0 V. (a) What is the dielectric constant of the inserted material? (b) Can you identify the dielectric? If so, what is it? (c) If the dielectric does not completely fill the space between the plates, what could you conclude about the voltage across the plates?

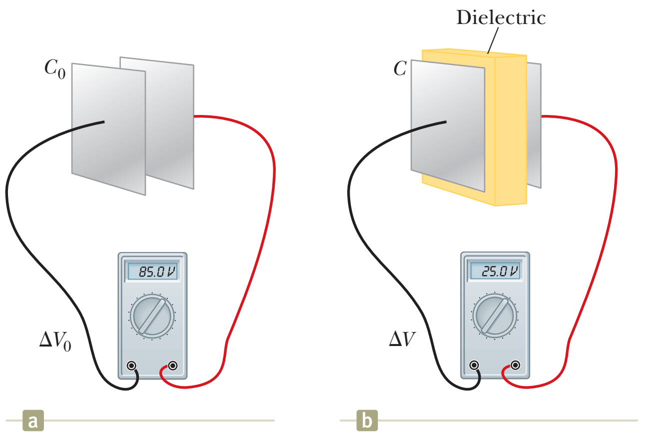


Figure P26.44

45. W Determine (a) the capacitance and (b) the maximum potential difference that can be applied to a Teflon-filled parallel-plate capacitor having a plate area of 1.75 cm^2 and a plate separation of 0.040 0 mm.

46. A commercial capacitor is to be constructed as shown in Figure P26.46. This particular capacitor is made from two strips of aluminum foil separated by a strip of paraffin-coated paper. Each strip of foil and paper is 7.00 cm wide. The foil is 0.004 00 mm thick, and the paper is 0.025 0 mm thick and has a dielectric constant of 3.70. What length should the strips have if a capacitance of 9.50×10^{-8} F is desired before the capacitor is rolled up? (Adding a second strip of paper and rolling the capacitor would effectively double its capacitance by allowing charge storage on both sides of each strip of foil.)

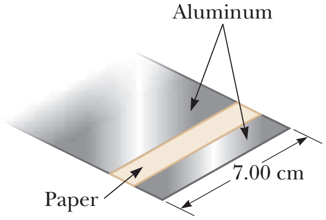


Figure P26.46

47. A parallel-plate capacitor in air has a plate separation of 1.50 cm and a plate area of 25.0 cm^2. The plates are charged to a potential difference of 250 V and disconnected from the source. The capacitor is then immersed in distilled water. Assume the liquid is an insulator. Determine (a) the charge on the plates before and after immersion, (b) the capacitance and potential difference after immersion, and (c) the change in energy of the capacitor.

48. Each capacitor in the combination shown in Figure P26.48 has a breakdown voltage of 15.0 V. What is the breakdown voltage of the combination?

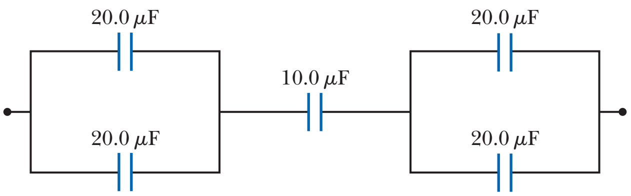


Figure P26.48

49. AMT A 2.00-nF parallel-plate capacitor is charged to an initial potential difference ΔV_i = 100 V and is then isolated. The dielectric material between the plates is mica, with a dielectric constant of 5.00. (a) How much work is required to withdraw the mica sheet? (b) What is the potential difference across the capacitor after the mica is withdrawn?

Section 26.6 Electric Dipole in an Electric Field

50. M A small, rigid object carries positive and negative 3.50-nC charges. It is oriented so that the positive charge has coordinates (−1.20 mm, 1.10 mm) and the negative charge is at the point (1.40 mm, −1.30 mm). (a) Find the electric dipole moment of the object. The object is placed in an electric field $\vec{\mathbf{E}} = (7.80 \times 10^3\,\hat{\mathbf{i}} - 4.90 \times 10^3\,\hat{\mathbf{j}})$ N/C. (b) Find the torque acting on the object. (c) Find the potential energy of the object–field system when the object is in this orientation. (d) Assuming the orientation of the object can change, find the difference between the maximum and minimum potential energies of the system.

51. AMT An infinite line of positive charge lies along the y axis, with charge density λ = 2.00 μC/m. A dipole is placed with its center along the x axis at $x = 25.0$ cm. The dipole consists of two charges $\pm 10.0\ \mu$C separated by 2.00 cm. The axis of the dipole makes an angle of 35.0° with the x axis, and the positive charge is farther from the line of charge than the negative charge. Find the net force exerted on the dipole.

52. A small object with electric dipole moment $\vec{\mathbf{p}}$ is placed in a nonuniform electric field $\vec{\mathbf{E}} = E(x)\hat{\mathbf{i}}$. That is, the field is in the x direction, and its magnitude depends only on the coordinate x. Let θ represent the angle between the dipole moment and the x direction. Prove that the net force on the dipole is

$$F = p\left(\frac{dE}{dx}\right)\cos\theta$$

acting in the direction of increasing field.

Section 26.7 An Atomic Description of Dielectrics

53. The general form of Gauss's law describes how a charge creates an electric field in a material, as well as in vacuum:

$$\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon}$$

where $\epsilon = \kappa\epsilon_0$ is the permittivity of the material. (a) A sheet with charge Q uniformly distributed over its area A is surrounded by a dielectric. Show that the sheet creates a uniform electric field at nearby points with magnitude $E = Q/2A\epsilon$. (b) Two large sheets of area A, carrying opposite charges of equal magnitude Q, are a small distance d apart. Show that they create uniform electric field in the space between them with magnitude $E = Q/A\epsilon$. (c) Assume the negative plate is at zero potential. Show that the positive plate is at potential $Qd/A\epsilon$. (d) Show that the capacitance of the pair of plates is given by $C = A\epsilon/d = \kappa A\epsilon_0/d$.

Additional Problems

54. Find the equivalent capacitance of the group of capacitors shown in Figure P26.54.

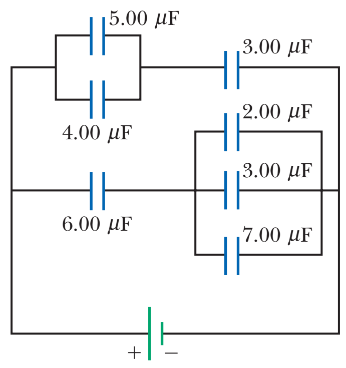


Figure P26.54

55. Four parallel metal plates P_1, P_2, P_3, and P_4, each of area 7.50 cm^2, are separated successively by a distance $d = 1.19$ mm as shown in Figure P26.55. Plate P_1 is connected to the negative terminal of a battery, and P_2 is connected to the positive terminal. The battery maintains a potential difference of 12.0 V. (a) If P_3 is connected to the negative terminal, what is the capacitance of the three-plate system $P_1P_2P_3$? (b) What is the charge on P_2? (c) If P_4 is now connected to the positive terminal, what is the capacitance of the four-plate system $P_1P_2P_3P_4$? (d) What is the charge on P_4?

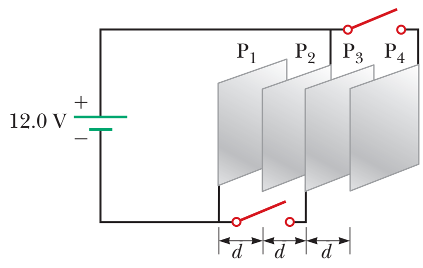


Figure P26.55

56. For the system of four capacitors shown in Figure P26.19, find (a) the total energy stored in the system and (b) the energy stored by each capacitor. (c) Compare the sum of the answers in part (b) with your result to part (a) and explain your observation.

57. A uniform electric field $E = 3\ 000$ V/m exists within a certain region. What volume of space contains an energy equal to 1.00×10^{-7} J? Express your answer in cubic meters and in liters.

58. Two large, parallel metal plates, each of area A, are oriented horizontally and separated by a distance $3d$. A grounded conducting wire joins them, and initially each plate carries no charge. Now a third identical plate carrying charge Q is inserted between the two plates, parallel to them and located a distance d from the upper plate as shown in Figure P26.58. (a) What induced charge appears on each of the two original plates? (b) What potential difference appears between the middle plate and each of the other plates?

Figure P26.58

59. M A parallel-plate capacitor is constructed using a dielectric material whose dielectric constant is 3.00 and whose dielectric strength is 2.00×10^8 V/m. The desired capacitance is 0.250 μF, and the capacitor must withstand a maximum potential difference of 4.00 kV. Find the minimum area of the capacitor plates.

60. *Why is the following situation impossible?* A 10.0-μF capacitor has plates with vacuum between them. The capacitor is charged so that it stores 0.050 0 J of energy. A particle with charge $-3.00\ \mu$C is fired from the positive plate toward the negative plate with an initial kinetic energy equal to 1.00×10^{-4} J. The particle arrives at the negative plate with a reduced kinetic energy.

61. A model of a red blood cell portrays the cell as a capacitor with two spherical plates. It is a positively charged conducting liquid sphere of area A, separated by an insulating membrane of thickness t from the surrounding negatively charged conducting fluid. Tiny electrodes introduced into the cell show a potential difference of 100 mV across the membrane. Take the membrane's thickness as 100 nm and its dielectric constant as 5.00. (a) Assume that a typical red blood cell has a mass of 1.00×10^{-12} kg and density 1 100 kg/m^3. Calculate its volume and its surface area. (b) Find the capacitance of the cell. (c) Calculate the charge on the surfaces of the membrane. How many electronic charges does this charge represent?

62. A parallel-plate capacitor with vacuum between its horizontal plates has a capacitance of 25.0 μF. A nonconducting liquid with dielectric constant 6.50 is poured into the space between the plates, filling up a fraction f of its volume. (a) Find the new capacitance as a function of f. (b) What should you expect the capacitance to be when $f = 0$? Does your expression from part (a) agree with your answer? (c) What capacitance should you expect when $f = 1$? Does the expression from part (a) agree with your answer?

63. A 10.0-μF capacitor is charged to 15.0 V. It is next connected in series with an uncharged 5.00-μF capacitor. The series combination is finally connected across a 50.0-V battery as diagrammed in Figure P26.63. Find the new potential differences across the 5.00-μF and 10.0-μF capacitors after the switch is thrown closed.

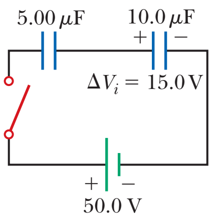


Figure P26.63

64. Assume that the internal diameter of the Geiger–Mueller tube described in Problem 68 in Chapter 25 is 2.50 cm and that the wire along the axis has a diameter of 0.200 mm. The dielectric strength of the gas between the central wire and the cylinder is 1.20×10^6 V/m. Use the result of that problem to calculate the maximum potential difference that can be applied between the wire and the cylinder before breakdown occurs in the gas.

65. Two square plates of sides ℓ are placed parallel to each other with separation d as suggested in Figure P26.65. You may assume d is much less than ℓ. The plates carry uniformly distributed static charges $+Q_0$ and $-Q_0$. A block of metal has width ℓ, length ℓ, and thickness slightly less than d. It is inserted a distance x into the space between the plates. The charges on the plates remain uniformly distributed as the block slides in. In a static situation, a metal prevents an electric field from penetrating inside it. The metal can be thought of as a perfect dielectric, with $\kappa \to \infty$. (a) Calculate the stored energy in the system as a function of x. (b) Find the direction and magnitude of the force that acts on the metallic block. (c) The area of the advancing front face of the block is essentially equal to ℓd. Considering the force on the block as acting on this face, find the stress (force per area) on it. (d) Express the energy density in the electric field between the charged plates in terms of Q_0, ℓ, d, and ϵ_0. (e) Explain how the answers to parts (c) and (d) compare with each other.

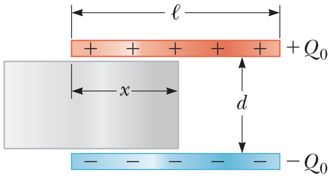


Figure P26.65

66. (a) Two spheres have radii a and b, and their centers are a distance d apart. Show that the capacitance of this system is

$$C = \frac{4\pi\epsilon_0}{\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{2}{d}}$$

provided d is large compared with a and b. *Suggestion:* Because the spheres are far apart, assume the potential of each equals the sum of the potentials due to each sphere. (b) Show that as d approaches infinity, the above result reduces to that of two spherical capacitors in series.

67. A capacitor of unknown capacitance has been charged to a potential difference of 100 V and then disconnected from the battery. When the charged capacitor is then connected in parallel to an uncharged 10.0-μF capacitor, the potential difference across the combination is 30.0 V. Calculate the unknown capacitance.

68. A parallel-plate capacitor of plate separation d is charged to a potential difference ΔV_0. A dielectric slab of thickness d and dielectric constant κ is introduced between the plates while the battery remains connected to the plates. (a) Show that the ratio of energy stored after the dielectric is introduced to the energy stored in the empty capacitor is $U/U_0 = \kappa$. (b) Give a physical explanation for this increase in stored energy. (c) What happens to the charge on the capacitor? *Note:* This situation is not the same as in Example 26.5, in which the battery was removed from the circuit before the dielectric was introduced.

69. Capacitors $C_1 = 6.00$ μF and $C_2 = 2.00$ μF are charged as a parallel combination across a 250-V battery. The capacitors are disconnected from the battery and from each other. They are then connected positive plate to negative plate and negative plate to positive plate. Calculate the resulting charge on each capacitor.

70. Example 26.1 explored a cylindrical capacitor of length ℓ with radii a and b for the two conductors. In the What If? section of that example, it was claimed that increasing ℓ by 10% is more effective in terms of increasing the capacitance than increasing a by 10% if $b > 2.85a$. Verify this claim mathematically.

71. To repair a power supply for a stereo amplifier, an electronics technician needs a 100-μF capacitor capable of withstanding a potential difference of 90 V between the plates. The immediately available supply is a box of five 100-μF capacitors, each having a maximum voltage capability of 50 V. (a) What combination of these capacitors has the proper electrical characteristics? Will the technician use all the capacitors in the box? Explain your answers. (b) In the combination of capacitors obtained in part (a), what will be the maximum voltage across each of the capacitors used?

Challenge Problems

72. The inner conductor of a coaxial cable has a radius of 0.800 mm, and the outer conductor's inside radius is 3.00 mm. The space between the conductors is filled with polyethylene, which has a dielectric constant of 2.30 and a dielectric strength of 18.0×10^6 V/m. What is the maximum potential difference this cable can withstand?

73. Some physical systems possessing capacitance continuously distributed over space can be modeled as an infinite array of discrete circuit elements. Examples are a microwave waveguide and the axon of a nerve cell. To practice analysis of an infinite array, determine the equivalent capacitance C between terminals X and Y of the infinite set of capacitors represented in Figure P26.73. Each capacitor has capacitance C_0. *Suggestions:* Imagine that the ladder is cut at the line AB and note that the equivalent capacitance of the infinite section to the right of AB is also C.

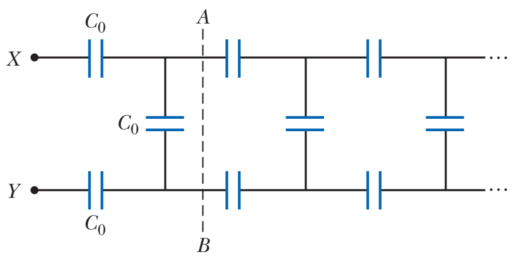


Figure P26.73

74. Consider two long, parallel, and oppositely charged wires of radius r with their centers separated by a distance D that is much larger than r. Assuming the charge is distributed uniformly on the surface of each wire, show that the capacitance per unit length of this pair of wires is

$$\frac{C}{\ell} = \frac{\pi\epsilon_0}{\ln(D/r)}$$

75. Determine the equivalent capacitance of the combination shown in Figure P26.75. *Suggestion:* Consider the symmetry involved.

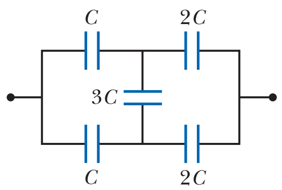


Figure P26.75

76. A parallel-plate capacitor with plates of area LW and plate separation t has the region between its plates filled with wedges of two dielectric materials as shown in Figure P26.76. Assume t is much less than both L and W. (a) Determine its capacitance. (b) Should the capacitance be the same if the labels κ_1 and κ_2 are interchanged? Demonstrate that your expression does or does not have this property. (c) Show that if κ_1 and κ_2 approach equality to a common value κ, your result becomes the same as the capacitance of a capacitor containing a single dielectric: $C = \kappa\epsilon_0 LW/t$.

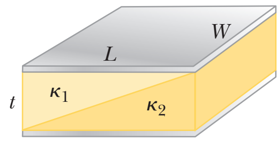


Figure P26.76

77. Calculate the equivalent capacitance between points a and b in Figure P26.77. Notice that this system is not a simple series or parallel combination. *Suggestion:* Assume a potential difference ΔV between points a and b. Write expressions for ΔV_{ab} in terms of the charges and capacitances for the various possible pathways from a to b and require conservation of charge for those capacitor plates that are connected to each other.

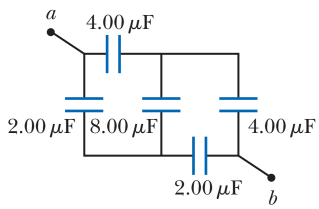


Figure P26.77

78. A capacitor is constructed from two square, metallic plates of sides ℓ and separation d. Charges $+Q$ and $-Q$ are placed on the plates, and the power supply is then removed. A material of dielectric constant κ is inserted a distance x into the capacitor as shown in Figure P26.78. Assume d is much smaller than x. (a) Find the equivalent capacitance of the device. (b) Calculate the energy stored in the capacitor. (c) Find the direction and magnitude of the force exerted by the plates on the dielectric. (d) Obtain a numerical value for the force when $x = \ell/2$, assuming $\ell = 5.00$ cm, $d = 2.00$ mm, the dielectric is glass ($\kappa = 4.50$), and the capacitor was charged to 2.00×10^3 V before the dielectric was inserted. *Suggestion:* The system can be considered as two capacitors connected in parallel.

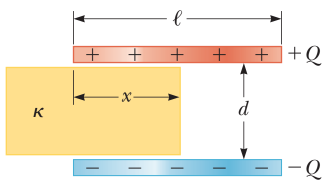


Figure P26.78

CHAPTER

27 Current and Resistance

These two lightbulbs provide similar power output by visible light (electromagnetic radiation). The compact fluorescent bulb on the left, however, produces this light output with far less input by electrical transmission than the incandescent bulb on the right. The fluorescent bulb, therefore, is less costly to operate and saves valuable resources needed to generate electricity. *(Christina Richards/Shutterstock.com)*

We now consider situations involving electric charges that are in motion through some region of space. We use the term *electric current,* or simply *current,* to describe the rate of flow of charge. Most practical applications of electricity deal with electric currents, including a variety of home appliances. For example, the voltage from a wall plug produces a current in the coils of a toaster when it is turned on. In these common situations, current exists in a conductor such as a copper wire. Currents can also exist outside a conductor. For instance, a beam of electrons in a particle accelerator constitutes a current.

This chapter begins with the definition of current. A microscopic description of current is given, and some factors that contribute to the opposition to the flow of charge in conductors are discussed. A classical model is used to describe electrical conduction in metals, and some limitations of this model are cited. We also define electrical resistance and introduce a new circuit element, the resistor. We conclude by discussing the rate at which energy is transferred to a device in an electric circuit. The energy transfer mechanism in Equation 8.2 that corresponds to this process is electrical transmission T_{ET}.

27.1 Electric Current

In this section, we study the flow of electric charges through a piece of material. The amount of flow depends on both the material through which the charges are passing and the potential difference across the material. Whenever there is a net flow of charge through some region, an electric *current* is said to exist.

It is instructive to draw an analogy between water flow and current. The flow of water in a plumbing pipe can be quantified by specifying the amount of water that emerges from a faucet during a given time interval, often measured in liters per minute. A river current can be characterized by describing the rate at which the water flows past a particular location. For example, the flow over the brink at Niagara Falls is maintained at rates between 1 400 m^3/s and 2 800 m^3/s.

There is also an analogy between thermal conduction and current. In Section 20.7, we discussed the flow of energy by heat through a sample of material. The rate of energy flow is determined by the material as well as the temperature difference across the material as described by Equation 20.15.

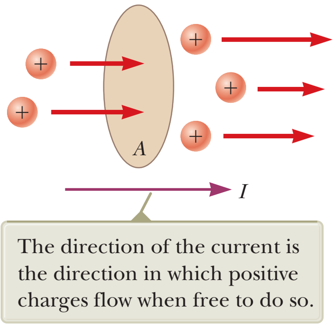


Figure 27.1 Charges in motion through an area A. The time rate at which charge flows through the area is defined as the current I.

To define current quantitatively, suppose charges are moving perpendicular to a surface of area A as shown in Figure 27.1. (This area could be the cross-sectional area of a wire, for example.) The **current** is defined as the rate at which charge flows through this surface. If ΔQ is the amount of charge that passes through this surface in a time interval Δt, the **average current** I_{avg} is equal to the charge that passes through A per unit time:

$$I_{avg} = \frac{\Delta Q}{\Delta t} \tag{27.1}$$

If the rate at which charge flows varies in time, the current varies in time; we define the **instantaneous current** I as the limit of the average current as $\Delta t \to 0$:

$$I \equiv \frac{dQ}{dt} \tag{27.2}$$

◀ **Electric current**

The SI unit of current is the **ampere** (A):

$$1 \text{ A} = 1 \text{ C/s} \tag{27.3}$$

That is, 1 A of current is equivalent to 1 C of charge passing through a surface in 1 s.

The charged particles passing through the surface in Figure 27.1 can be positive, negative, or both. It is conventional to assign to the current the same direction as the flow of positive charge. In electrical conductors such as copper or aluminum, the current results from the motion of negatively charged electrons. Therefore, in an ordinary conductor, the direction of the current is opposite the direction of flow of electrons. For a beam of positively charged protons in an accelerator, however, the current is in the direction of motion of the protons. In some cases—such as those involving gases and electrolytes, for instance—the current is the result of the flow of both positive and negative charges. It is common to refer to a moving charge (positive or negative) as a mobile **charge carrier.**

If the ends of a conducting wire are connected to form a loop, all points on the loop are at the same electric potential; hence, the electric field is zero within and at the surface of the conductor. Because the electric field is zero, there is no net transport of charge through the wire; therefore, there is no current. If the ends of the conducting wire are connected to a battery, however, all points on the loop are not at the same potential. The battery sets up a potential difference between the ends of the loop, creating an electric field within the wire. The electric field exerts forces on the electrons in the wire, causing them to move in the wire and therefore creating a current.

Pitfall Prevention 27.1

"Current Flow" Is Redundant

The phrase *current flow* is commonly used, although it is technically incorrect because current *is* a flow (of charge). This wording is similar to the phrase *heat transfer,* which is also redundant because heat *is* a transfer (of energy). We will avoid this phrase and speak of *flow of charge* or *charge flow.*

Pitfall Prevention 27.2

Batteries Do Not Supply Electrons

A battery does not supply electrons to the circuit. It establishes the electric field that exerts a force on electrons already in the wires and elements of the circuit.

Microscopic Model of Current

We can relate current to the motion of the charge carriers by describing a microscopic model of conduction in a metal. Consider the current in a cylindrical

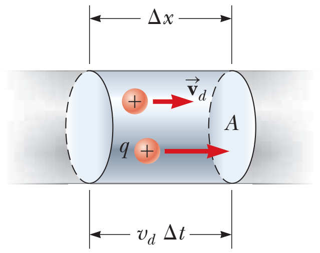


Figure 27.2 A segment of a uniform conductor of cross-sectional area A.

conductor of cross-sectional area A (Fig. 27.2). The volume of a segment of the conductor of length Δx (between the two circular cross sections shown in Fig. 27.2) is $A\,\Delta x$. If n represents the number of mobile charge carriers per unit volume (in other words, the charge carrier density), the number of carriers in the segment is $nA\,\Delta x$. Therefore, the total charge ΔQ in this segment is

$$\Delta Q = (nA\,\Delta x)q$$

where q is the charge on each carrier. If the carriers move with a velocity $\vec{\mathbf{v}}_d$ parallel to the axis of the cylinder, the magnitude of the displacement they experience in the x direction in a time interval Δt is $\Delta x = v_d\,\Delta t$. Let Δt be the time interval required for the charge carriers in the segment to move through a displacement whose magnitude is equal to the length of the segment. This time interval is also the same as that required for all the charge carriers in the segment to pass through the circular area at one end. With this choice, we can write ΔQ as

$$\Delta Q = (nAv_d\,\Delta t)q$$

Dividing both sides of this equation by Δt, we find that the average current in the conductor is

$$I_{\text{avg}} = \frac{\Delta Q}{\Delta t} = nqv_dA \tag{27.4}$$

In reality, the speed of the charge carriers v_d is an average speed called the **drift speed.** To understand the meaning of drift speed, consider a conductor in which the charge carriers are free electrons. If the conductor is isolated—that is, the potential difference across it is zero—these electrons undergo random motion that is analogous to the motion of gas molecules. The electrons collide repeatedly with the metal atoms, and their resultant motion is complicated and zigzagged as in Figure 27.3a. As discussed earlier, when a potential difference is applied across the conductor (for example, by means of a battery), an electric field is set up in the conductor; this field exerts an electric force on the electrons, producing a current. In addition to the zigzag motion due to the collisions with the metal atoms, the electrons move slowly along the conductor (in a direction opposite that of $\vec{\mathbf{E}}$) at the **drift velocity** $\vec{\mathbf{v}}_d$ as shown in Figure 27.3b.

You can think of the atom–electron collisions in a conductor as an effective internal friction (or drag force) similar to that experienced by a liquid's molecules flowing through a pipe stuffed with steel wool. The energy transferred from the electrons to the metal atoms during collisions causes an increase in the atom's vibrational energy and a corresponding increase in the conductor's temperature.

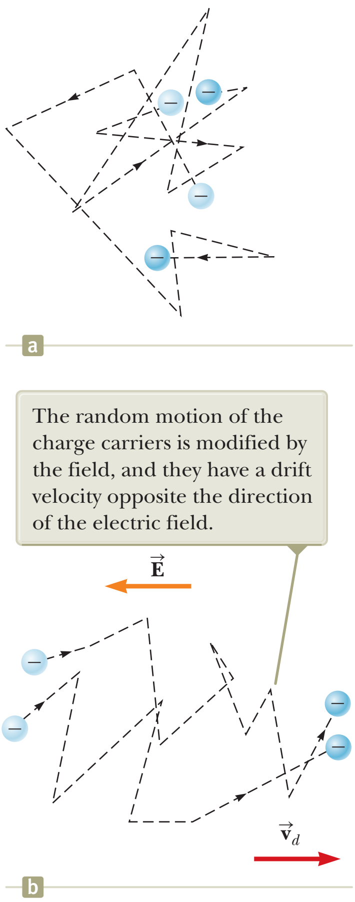


Figure 27.3 (a) A schematic diagram of the random motion of two charge carriers in a conductor in the absence of an electric field. The drift velocity is zero. (b) The motion of the charge carriers in a conductor in the presence of an electric field. Because of the acceleration of the charge carriers due to the electric force, the paths are actually parabolic. The drift speed, however, is much smaller than the average speed, so the parabolic shape is not visible on this scale.

Quick Quiz 27.1 Consider positive and negative charges moving horizontally through the four regions shown in Figure 27.4. Rank the current in these four regions from highest to lowest.

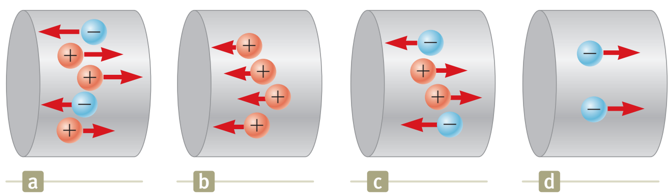


Figure 27.4 (Quick Quiz 27.1) Charges move through four regions.

Example 27.1 Drift Speed in a Copper Wire

The 12-gauge copper wire in a typical residential building has a cross-sectional area of 3.31×10^{-6} m^2. It carries a constant current of 10.0 A. What is the drift speed of the electrons in the wire? Assume each copper atom contributes one free electron to the current. The density of copper is 8.92 g/cm^3.

SOLUTION

Conceptualize Imagine electrons following a zigzag motion such as that in Figure 27.3a, with a drift velocity parallel to the wire superimposed on the motion as in Figure 27.3b. As mentioned earlier, the drift speed is small, and this example helps us quantify the speed.

Categorize We evaluate the drift speed using Equation 27.4. Because the current is constant, the average current during any time interval is the same as the constant current: $I_{\text{avg}} = I$.

Analyze The periodic table of the elements in Appendix C shows that the molar mass of copper is $M = 63.5$ g/mol. Recall that 1 mol of any substance contains Avogadro's number of atoms ($N_A = 6.02 \times 10^{23}$ mol^{-1}).

Use the molar mass and the density of copper to find the volume of 1 mole of copper:

$$V = \frac{M}{\rho}$$

From the assumption that each copper atom contributes one free electron to the current, find the electron density in copper:

$$n = \frac{N_A}{V} = \frac{N_A \rho}{M}$$

Solve Equation 27.4 for the drift speed and substitute for the electron density:

$$v_d = \frac{I_{\text{avg}}}{nqA} = \frac{I}{nqA} = \frac{IM}{qAN_A\rho}$$

Substitute numerical values:

$$v_d = \frac{(10.0 \text{ A})(0.063\,5 \text{ kg/mol})}{(1.60 \times 10^{-19} \text{ C})(3.31 \times 10^{-6} \text{ m}^2)(6.02 \times 10^{23} \text{ mol}^{-1})(8\,920 \text{ kg/m}^3)}$$

$$= \boxed{2.23 \times 10^{-4} \text{ m/s}}$$

Finalize This result shows that typical drift speeds are very small. For instance, electrons traveling with a speed of 2.23×10^{-4} m/s would take about 75 min to travel 1 m! You might therefore wonder why a light turns on almost instantaneously when its switch is thrown. In a conductor, changes in the electric field that drives the free electrons according to the particle in a field model travel through the conductor with a speed close to that of light. So, when you flip on a light switch, electrons already in the filament of the lightbulb experience electric forces and begin moving after a time interval on the order of nanoseconds.

27.2 Resistance

In Section 24.4, we argued that the electric field inside a conductor is zero. This statement is true, however, *only* if the conductor is in static equilibrium as stated in that discussion. The purpose of this section is to describe what happens when there is a nonzero electric field in the conductor. As we saw in Section 27.1, a current exists in the wire in this case.

Consider a conductor of cross-sectional area A carrying a current I. The **current density** J in the conductor is defined as the current per unit area. Because the current $I = nqv_dA$, the current density is

$$J \equiv \frac{I}{A} = nqv_d \qquad (27.5)$$

◀ Current density

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Georg Simon Ohm
German physicist (1789–1854)
Ohm, a high school teacher and later a professor at the University of Munich, formulated the concept of resistance and discovered the proportionalities expressed in Equations 27.6 and 27.7.

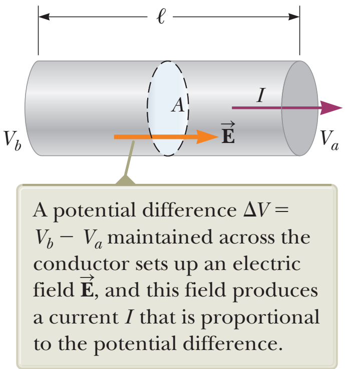

Figure 27.5 A uniform conductor of length ℓ and cross-sectional area A.

where J has SI units of amperes per meter squared. This expression is valid only if the current density is uniform and only if the surface of cross-sectional area A is perpendicular to the direction of the current.

A current density and an electric field are established in a conductor whenever a potential difference is maintained across the conductor. In some materials, the current density is proportional to the electric field:

$$J = \sigma E \tag{27.6}$$

where the constant of proportionality σ is called the **conductivity** of the conductor.[1] Materials that obey Equation 27.6 are said to follow **Ohm's law,** named after Georg Simon Ohm. More specifically, Ohm's law states the following:

> For many materials (including most metals), the ratio of the current density to the electric field is a constant σ that is independent of the electric field producing the current.

Materials and devices that obey Ohm's law and hence demonstrate this simple relationship between E and J are said to be *ohmic*. Experimentally, however, it is found that not all materials and devices have this property. Those that do not obey Ohm's law are said to be *nonohmic*. Ohm's law is not a fundamental law of nature; rather, it is an empirical relationship valid only for certain situations.

We can obtain an equation useful in practical applications by considering a segment of straight wire of uniform cross-sectional area A and length ℓ as shown in Figure 27.5. A potential difference $\Delta V = V_b - V_a$ is maintained across the wire, creating in the wire an electric field and a current. If the field is assumed to be uniform, the magnitude of the potential difference across the wire is related to the field within the wire through Equation 25.6,

$$\Delta V = E\ell$$

Therefore, we can express the current density (Eq. 27.6) in the wire as

$$J = \sigma \frac{\Delta V}{\ell}$$

Because $J = I/A$, the potential difference across the wire is

$$\Delta V = \frac{\ell}{\sigma} J = \left(\frac{\ell}{\sigma A}\right) I = R\,I$$

Pitfall Prevention 27.3

Equation 27.7 Is Not Ohm's Law

Many individuals call Equation 27.7 Ohm's law, but that is incorrect. This equation is simply the definition of resistance, and it provides an important relationship between voltage, current, and resistance. Ohm's law is related to a proportionality of J to E (Eq. 27.6) or, equivalently, of I to ΔV, which, from Equation 27.7, indicates that the resistance is constant, independent of the applied voltage. We will see some devices for which Equation 27.7 correctly describes their resistance, but that do *not* obey Ohm's law.

The quantity $R = \ell/\sigma A$ is called the **resistance** of the conductor. We define the resistance as the ratio of the potential difference across a conductor to the current in the conductor:

$$R \equiv \frac{\Delta V}{I} \tag{27.7}$$

We will use this equation again and again when studying electric circuits. This result shows that resistance has SI units of volts per ampere. One volt per ampere is defined to be one **ohm** (Ω):

$$1\ \Omega \equiv 1\ \text{V/A} \tag{27.8}$$

Equation 27.7 shows that if a potential difference of 1 V across a conductor causes a current of 1 A, the resistance of the conductor is 1 Ω. For example, if an electrical appliance connected to a 120-V source of potential difference carries a current of 6 A, its resistance is 20 Ω.

Most electric circuits use circuit elements called **resistors** to control the current in the various parts of the circuit. As with capacitors in Chapter 26, many resistors are built into integrated circuit chips, but stand-alone resistors are still available and

[1] Do not confuse conductivity σ with surface charge density, for which the same symbol is used.

Table 27.1 Color Coding for Resistors

Color	Number	Multiplier	Tolerance
Black	0	1	
Brown	1	10^1	
Red	2	10^2	
Orange	3	10^3	
Yellow	4	10^4	
Green	5	10^5	
Blue	6	10^6	
Violet	7	10^7	
Gray	8	10^8	
White	9	10^9	
Gold		10^{-1}	5%
Silver		10^{-2}	10%
Colorless			20%

The colored bands on this resistor are yellow, violet, black, and gold.

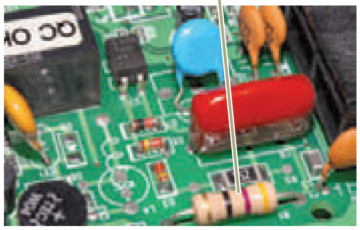

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Figure 27.6 A close-up view of a circuit board shows the color coding on a resistor. The gold band on the left tells us that the resistor is oriented "backward" in this view and we need to read the colors from right to left.

widely used. Two common types are the *composition resistor,* which contains carbon, and the *wire-wound resistor,* which consists of a coil of wire. Values of resistors in ohms are normally indicated by color coding as shown in Figure 27.6 and Table 27.1. The first two colors on a resistor give the first two digits in the resistance value, with the decimal place to the right of the second digit. The third color represents the power of 10 for the multiplier of the resistance value. The last color is the tolerance of the resistance value. As an example, the four colors on the resistor at the bottom of Figure 27.6 are yellow (= 4), violet (= 7), black (= 10^0), and gold (= 5%), and so the resistance value is $47 \times 10^0 = 47\ \Omega$ with a tolerance value of 5% $= 2\ \Omega$.

The inverse of conductivity is **resistivity**[2] ρ:

$$\rho = \frac{1}{\sigma} \tag{27.9}$$

◀ Resistivity is the inverse of conductivity

where ρ has the units ohm · meters ($\Omega \cdot$ m). Because $R = \ell/\sigma A$, we can express the resistance of a uniform block of material along the length ℓ as

$$\boxed{R = \rho\frac{\ell}{A}} \tag{27.10}$$

◀ Resistance of a uniform material along the length ℓ

Every ohmic material has a characteristic resistivity that depends on the properties of the material and on temperature. In addition, as you can see from Equation 27.10, the resistance of a sample of the material depends on the geometry of the sample as well as on the resistivity of the material. Table 27.2 (page 814) gives the resistivities of a variety of materials at 20°C. Notice the enormous range, from very low values for good conductors such as copper and silver to very high values for good insulators such as glass and rubber. An ideal conductor would have zero resistivity, and an ideal insulator would have infinite resistivity.

Equation 27.10 shows that the resistance of a given cylindrical conductor such as a wire is proportional to its length and inversely proportional to its cross-sectional area. If the length of a wire is doubled, its resistance doubles. If its cross-sectional area is doubled, its resistance decreases by one half. The situation is analogous to the flow of a liquid through a pipe. As the pipe's length is increased, the resistance to flow increases. As the pipe's cross-sectional area is increased, more liquid crosses a given cross section of the pipe per unit time interval. Therefore, more liquid flows for the same pressure differential applied to the pipe, and the resistance to flow decreases.

Ohmic materials and devices have a linear current–potential difference relationship over a broad range of applied potential differences (Fig. 27.7a, page 814). The slope of the I-versus-ΔV curve in the linear region yields a value for $1/R$. Nonohmic

Pitfall Prevention 27.4

Resistance and Resistivity Resistivity is a property of a *substance,* whereas resistance is a property of an *object.* We have seen similar pairs of variables before. For example, density is a property of a substance, whereas mass is a property of an object. Equation 27.10 relates resistance to resistivity, and Equation 1.1 relates mass to density.

[2]Do not confuse resistivity ρ with mass density or charge density, for which the same symbol is used.

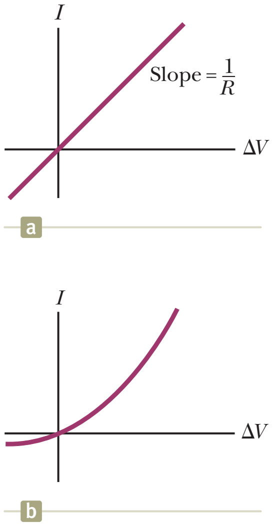

Figure 27.7 (a) The current–potential difference curve for an ohmic material. The curve is linear, and the slope is equal to the inverse of the resistance of the conductor. (b) A nonlinear current–potential difference curve for a junction diode. This device does not obey Ohm's law.

Table 27.2 Resistivities and Temperature Coefficients of Resistivity for Various Materials

Material	Resistivity[a] ($\Omega \cdot m$)	Temperature Coefficient[b] α [$(°C)^{-1}$]
Silver	1.59×10^{-8}	3.8×10^{-3}
Copper	1.7×10^{-8}	3.9×10^{-3}
Gold	2.44×10^{-8}	3.4×10^{-3}
Aluminum	2.82×10^{-8}	3.9×10^{-3}
Tungsten	5.6×10^{-8}	4.5×10^{-3}
Iron	10×10^{-8}	5.0×10^{-3}
Platinum	11×10^{-8}	3.92×10^{-3}
Lead	22×10^{-8}	3.9×10^{-3}
Nichrome[c]	1.00×10^{-6}	0.4×10^{-3}
Carbon	3.5×10^{-5}	-0.5×10^{-3}
Germanium	0.46	-48×10^{-3}
Silicon[d]	2.3×10^{3}	-75×10^{-3}
Glass	10^{10} to 10^{14}	
Hard rubber	$\sim 10^{13}$	
Sulfur	10^{15}	
Quartz (fused)	75×10^{16}	

[a] All values at 20°C. All elements in this table are assumed to be free of impurities.
[b] See Section 27.4.
[c] A nickel–chromium alloy commonly used in heating elements. The resistivity of Nichrome varies with composition and ranges between 1.00×10^{-6} and 1.50×10^{-6} $\Omega \cdot m$.
[d] The resistivity of silicon is very sensitive to purity. The value can be changed by several orders of magnitude when it is doped with other atoms.

materials have a nonlinear current–potential difference relationship. One common semiconducting device with nonlinear I-versus-ΔV characteristics is the *junction diode* (Fig. 27.7b). The resistance of this device is low for currents in one direction (positive ΔV) and high for currents in the reverse direction (negative ΔV). In fact, most modern electronic devices, such as transistors, have nonlinear current–potential difference relationships; their proper operation depends on the particular way they violate Ohm's law.

Quick Quiz 27.2 A cylindrical wire has a radius r and length ℓ. If both r and ℓ are doubled, does the resistance of the wire **(a)** increase, **(b)** decrease, or **(c)** remain the same?

Quick Quiz 27.3 In Figure 27.7b, as the applied voltage increases, does the resistance of the diode **(a)** increase, **(b)** decrease, or **(c)** remain the same?

Example 27.2 The Resistance of Nichrome Wire

The radius of 22-gauge Nichrome wire is 0.32 mm.

(A) Calculate the resistance per unit length of this wire.

SOLUTION

Conceptualize Table 27.2 shows that Nichrome has a resistivity two orders of magnitude larger than the best conductors in the table. Therefore, we expect it to have some special practical applications that the best conductors may not have.

Categorize We model the wire as a cylinder so that a simple geometric analysis can be applied to find the resistance.

Analyze Use Equation 27.10 and the resistivity of Nichrome from Table 27.2 to find the resistance per unit length:

$$\frac{R}{\ell} = \frac{\rho}{A} = \frac{\rho}{\pi r^2} = \frac{1.0 \times 10^{-6}\ \Omega \cdot \text{m}}{\pi(0.32 \times 10^{-3}\ \text{m})^2} = \boxed{3.1\ \Omega/\text{m}}$$

27.2 continued

(B) If a potential difference of 10 V is maintained across a 1.0-m length of the Nichrome wire, what is the current in the wire?

SOLUTION

Analyze Use Equation 27.7 to find the current:

$$I = \frac{\Delta V}{R} = \frac{\Delta V}{(R/\ell)\ell} = \frac{10\text{ V}}{(3.1\ \Omega/\text{m})(1.0\text{ m})} = \boxed{3.2\text{ A}}$$

Finalize Because of its high resistivity and resistance to oxidation, Nichrome is often used for heating elements in toasters, irons, and electric heaters.

WHAT IF? What if the wire were composed of copper instead of Nichrome? How would the values of the resistance per unit length and the current change?

Answer Table 27.2 shows us that copper has a resistivity two orders of magnitude smaller than that for Nichrome. Therefore, we expect the answer to part (A) to be smaller and the answer to part (B) to be larger. Calculations show that a copper wire of the same radius would have a resistance per unit length of only 0.053 Ω/m. A 1.0-m length of copper wire of the same radius would carry a current of 190 A with an applied potential difference of 10 V.

Example 27.3 The Radial Resistance of a Coaxial Cable

Coaxial cables are used extensively for cable television and other electronic applications. A coaxial cable consists of two concentric cylindrical conductors. The region between the conductors is completely filled with polyethylene plastic as shown in Figure 27.8a. Current leakage through the plastic, in the *radial* direction, is unwanted. (The cable is designed to conduct current along its length, but that is *not* the current being considered here.) The radius of the inner conductor is $a = 0.500$ cm, the radius of the outer conductor is $b = 1.75$ cm, and the length is $L = 15.0$ cm. The resistivity of the plastic is $1.0 \times 10^{13}\ \Omega \cdot \text{m}$. Calculate the resistance of the plastic between the two conductors.

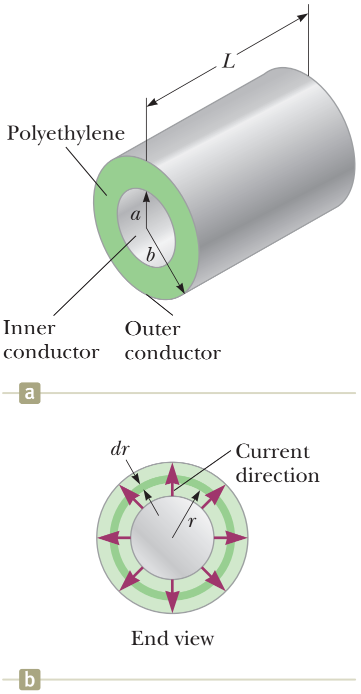

Figure 27.8 (Example 27.3) A coaxial cable. (a) Polyethylene plastic fills the gap between the two conductors. (b) End view, showing current leakage.

SOLUTION

Conceptualize Imagine two currents as suggested in the text of the problem. The desired current is along the cable, carried within the conductors. The undesired current corresponds to leakage through the plastic, and its direction is radial.

Categorize Because the resistivity and the geometry of the plastic are known, we categorize this problem as one in which we find the resistance of the plastic from these parameters. Equation 27.10, however, represents the resistance of a block of material. We have a more complicated geometry in this situation. Because the area through which the charges pass depends on the radial position, we must use integral calculus to determine the answer.

Analyze We divide the plastic into concentric cylindrical shells of infinitesimal thickness dr (Fig. 27.8b). Any charge passing from the inner to the outer conductor must move radially through this shell. Use a differential form of Equation 27.10, replacing ℓ with dr for the length variable: $dR = \rho\ dr/A$, where dR is the resistance of a shell of plastic of thickness dr and surface area A.

Write an expression for the resistance of our hollow cylindrical shell of plastic representing the area as the surface area of the shell:

$$dR = \frac{\rho\ dr}{A} = \frac{\rho}{2\pi r L}\ dr$$

continued

27.3 continued

Integrate this expression from $r = a$ to $r = b$: (1) $R = \int dR = \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} = \frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)$

Substitute the values given: $R = \frac{1.0 \times 10^{13}\ \Omega\cdot\text{m}}{2\pi(0.150\text{ m})}\ln\left(\frac{1.75\text{ cm}}{0.500\text{ cm}}\right) = \boxed{1.33 \times 10^{13}\ \Omega}$

Finalize Let's compare this resistance to that of the inner copper conductor of the cable along the 15.0-cm length.

Use Equation 27.10 to find the resistance of the copper cylinder:

$$R_{\text{Cu}} = \rho\frac{\ell}{A} = (1.7 \times 10^{-8}\ \Omega\cdot\text{m})\left[\frac{0.150\text{ m}}{\pi(5.00 \times 10^{-3}\text{ m})^2}\right]$$
$$= 3.2 \times 10^{-5}\ \Omega$$

This resistance is 18 orders of magnitude smaller than the radial resistance. Therefore, almost all the current corresponds to charge moving along the length of the cable, with a very small fraction leaking in the radial direction.

WHAT IF? Suppose the coaxial cable is enlarged to twice the overall diameter with two possible choices: (1) the ratio b/a is held fixed, or (2) the difference $b - a$ is held fixed. For which choice does the leakage current between the inner and outer conductors increase when the voltage is applied between them?

Answer For the current to increase, the resistance must decrease. For choice (1), in which b/a is held fixed, Equation (1) shows that the resistance is unaffected. For choice (2), we do not have an equation involving the difference $b - a$ to inspect. Looking at Figure 27.8b, however, we see that increasing b and a while holding the difference constant results in charge flowing through the same thickness of plastic but through a larger area perpendicular to the flow. This larger area results in lower resistance and a higher current.

27.3 A Model for Electrical Conduction

In this section, we describe a structural model of electrical conduction in metals that was first proposed by Paul Drude (1863–1906) in 1900. (See Section 21.1 for a review of structural models.) This model leads to Ohm's law and shows that resistivity can be related to the motion of electrons in metals. Although the Drude model described here has limitations, it introduces concepts that are applied in more elaborate treatments.

Following the outline of structural models from Section 21.1, the Drude model for electrical conduction has the following properties:

1. *Physical components:*
   Consider a conductor as a regular array of atoms plus a collection of free electrons, which are sometimes called *conduction* electrons. We identify the system as the combination of the atoms and the conduction electrons. The conduction electrons, although bound to their respective atoms when the atoms are not part of a solid, become free when the atoms condense into a solid.
2. *Behavior of the components:*
   (a) In the absence of an electric field, the conduction electrons move in random directions through the conductor (Fig. 27.3a). The situation is similar to the motion of gas molecules confined in a vessel. In fact, some scientists refer to conduction electrons in a metal as an *electron gas.*
   (b) When an electric field is applied to the system, the free electrons drift slowly in a direction opposite that of the electric field (Fig. 27.3b), with an average drift speed v_d that is much smaller (typically 10^{-4} m/s) than their average speed v_{avg} between collisions (typically 10^6 m/s).
   (c) The electron's motion after a collision is independent of its motion before the collision. The excess energy acquired by the electrons due to

the work done on them by the electric field is transferred to the atoms of the conductor when the electrons and atoms collide.

With regard to property 2(c) above, the energy transferred to the atoms causes the internal energy of the system and, therefore, the temperature of the conductor to increase.

We are now in a position to derive an expression for the drift velocity, using several of our analysis models. When a free electron of mass m_e and charge q $(= -e)$ is subjected to an electric field $\vec{\mathbf{E}}$, it is described by the particle in a field model and experiences a force $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. The electron is a particle under a net force, and its acceleration can be found from Newton's second law, $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$:

$$\vec{\mathbf{a}} = \frac{\sum \vec{\mathbf{F}}}{m} = \frac{q\vec{\mathbf{E}}}{m_e} \qquad \textbf{(27.11)}$$

Because the electric field is uniform, the electron's acceleration is constant, so the electron can be modeled as a particle under constant acceleration. If $\vec{\mathbf{v}}_i$ is the electron's initial velocity the instant after a collision (which occurs at a time defined as $t = 0$), the velocity of the electron at a very short time t later (immediately before the next collision occurs) is, from Equation 4.8,

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = \vec{\mathbf{v}}_i + \frac{q\vec{\mathbf{E}}}{m_e}t \qquad \textbf{(27.12)}$$

Let's now take the average value of $\vec{\mathbf{v}}_f$ for all the electrons in the wire over all possible collision times t and all possible values of $\vec{\mathbf{v}}_i$. Assuming the initial velocities are randomly distributed over all possible directions (property 2(a) above), the average value of $\vec{\mathbf{v}}_i$ is zero. The average value of the second term of Equation 27.12 is $(q\vec{\mathbf{E}}/m_e)\tau$, where τ is the *average time interval between successive collisions*. Because the average value of $\vec{\mathbf{v}}_f$ is equal to the drift velocity,

$$\vec{\mathbf{v}}_{f,\text{avg}} = \vec{\mathbf{v}}_d = \frac{q\vec{\mathbf{E}}}{m_e}\tau \qquad \textbf{(27.13)}$$

◀ **Drift velocity in terms of microscopic quantities**

The value of τ depends on the size of the metal atoms and the number of electrons per unit volume. We can relate this expression for drift velocity in Equation 27.13 to the current in the conductor. Substituting the magnitude of the velocity from Equation 27.13 into Equation 27.4, the average current in the conductor is given by

$$I_{\text{avg}} = nq\left(\frac{qE}{m_e}\tau\right)A = \frac{nq^2E}{m_e}\tau A \qquad \textbf{(27.14)}$$

Because the current density J is the current divided by the area A,

$$J = \frac{nq^2E}{m_e}\tau$$

◀ **Current density in terms of microscopic quantities**

where n is the number of electrons per unit volume. Comparing this expression with Ohm's law, $J = \sigma E$, we obtain the following relationships for conductivity and resistivity of a conductor:

$$\sigma = \frac{nq^2\tau}{m_e} \qquad \textbf{(27.15)}$$

◀ **Conductivity in terms of microscopic quantities**

$$\rho = \frac{1}{\sigma} = \frac{m_e}{nq^2\tau} \qquad \textbf{(27.16)}$$

◀ **Resistivity in terms of microscopic quantities**

According to this classical model, conductivity and resistivity do not depend on the strength of the electric field. This feature is characteristic of a conductor obeying Ohm's law.

The model shows that the resistivity can be calculated from a knowledge of the density of the electrons, their charge and mass, and the average time interval τ between collisions. This time interval is related to the average distance between collisions ℓ_{avg} (the *mean free path*) and the average speed v_{avg} through the expression[3]

$$\tau = \frac{\ell_{avg}}{v_{avg}} \qquad \textbf{(27.17)}$$

Although this structural model of conduction is consistent with Ohm's law, it does not correctly predict the values of resistivity or the behavior of the resistivity with temperature. For example, the results of classical calculations for v_{avg} using the ideal gas model for the electrons are about a factor of ten smaller than the actual values, which results in incorrect predictions of values of resistivity from Equation 27.16. Furthermore, according to Equations 27.16 and 27.17, the resistivity is predicted to vary with temperature as does v_{avg}, which, according to an ideal-gas model (Chapter 21, Eq. 21.43), is proportional to $\sqrt{T}$. This behavior is in disagreement with the experimentally observed linear dependence of resistivity with temperature for pure metals. (See Section 27.4.) Because of these incorrect predictions, we must modify our structural model. We shall call the model that we have developed so far the *classical* model for electrical conduction. To account for the incorrect predictions of the classical model, we develop it further into a *quantum mechanical* model, which we shall describe briefly.

We discussed two important simplification models in earlier chapters, the particle model and the wave model. Although we discussed these two simplification models separately, quantum physics tells us that this separation is not so clear-cut. As we shall discuss in detail in Chapter 40, particles have wave-like properties. The predictions of some models can only be matched to experimental results if the model includes the wave-like behavior of particles. The structural model for electrical conduction in metals is one of these cases.

Let us imagine that the electrons moving through the metal have wave-like properties. If the array of atoms in a conductor is regularly spaced (that is, periodic), the wave-like character of the electrons makes it possible for them to move freely through the conductor and a collision with an atom is unlikely. For an idealized conductor, no collisions would occur, the mean free path would be infinite, and the resistivity would be zero. Electrons are scattered only if the atomic arrangement is irregular (not periodic), as a result of structural defects or impurities, for example. At low temperatures, the resistivity of metals is dominated by scattering caused by collisions between the electrons and impurities. At high temperatures, the resistivity is dominated by scattering caused by collisions between the electrons and the atoms of the conductor, which are continuously displaced as a result of thermal agitation, destroying the perfect periodicity. The thermal motion of the atoms makes the structure irregular (compared with an atomic array at rest), thereby reducing the electron's mean free path.

Although it is beyond the scope of this text to show this modification in detail, the classical model modified with the wave-like character of the electrons results in predictions of resistivity values that are in agreement with measured values and predicts a linear temperature dependence. Quantum notions had to be introduced in Chapter 21 to understand the temperature behavior of molar specific heats of gases. Here we have another case in which quantum physics is necessary for the model to agree with experiment. Although classical physics can explain a tremendous range of phenomena, we continue to see hints that quantum physics must be incorporated into our models. We shall study quantum physics in detail in Chapters 40 through 46.

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[3]Recall that the average speed of a group of particles depends on the temperature of the group (Chapter 21) and is not the same as the drift speed v_d.

27.4 Resistance and Temperature

Over a limited temperature range, the resistivity of a conductor varies approximately linearly with temperature according to the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \quad (27.18)$$

◀ **Variation of ρ with temperature**

where ρ is the resistivity at some temperature T (in degrees Celsius), ρ_0 is the resistivity at some reference temperature T_0 (usually taken to be 20°C), and α is the **temperature coefficient of resistivity.** From Equation 27.18, the temperature coefficient of resistivity can be expressed as

$$\alpha = \frac{1}{\rho_0}\frac{\Delta\rho}{\Delta T} \quad (27.19)$$

◀ **Temperature coefficient of resistivity**

where $\Delta\rho = \rho - \rho_0$ is the change in resistivity in the temperature interval $\Delta T = T - T_0$.

The temperature coefficients of resistivity for various materials are given in Table 27.2. Notice that the unit for α is degrees Celsius^{-1} [$(°C)^{-1}$]. Because resistance is proportional to resistivity (Eq. 27.10), the variation of resistance of a sample is

$$R = R_0[1 + \alpha(T - T_0)] \quad (27.20)$$

where R_0 is the resistance at temperature T_0. Use of this property enables precise temperature measurements through careful monitoring of the resistance of a probe made from a particular material.

For some metals such as copper, resistivity is nearly proportional to temperature as shown in Figure 27.9. A nonlinear region always exists at very low temperatures, however, and the resistivity usually reaches some finite value as the temperature approaches absolute zero. This residual resistivity near absolute zero is caused primarily by the collision of electrons with impurities and imperfections in the metal. In contrast, high-temperature resistivity (the linear region) is predominantly characterized by collisions between electrons and metal atoms.

Notice that three of the α values in Table 27.2 are negative, indicating that the resistivity of these materials decreases with increasing temperature. This behavior is indicative of a class of materials called *semiconductors,* first introduced in Section 23.2, and is due to an increase in the density of charge carriers at higher temperatures.

Because the charge carriers in a semiconductor are often associated with impurity atoms (as we discuss in more detail in Chapter 43), the resistivity of these materials is very sensitive to the type and concentration of such impurities.

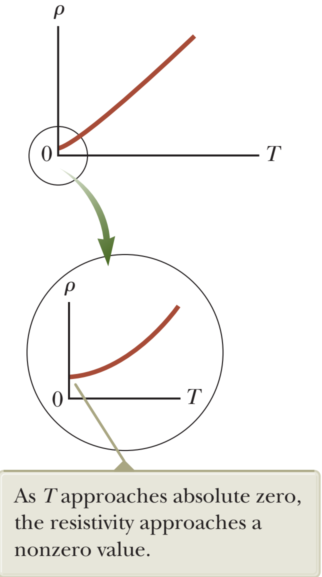


Figure 27.9 Resistivity versus temperature for a metal such as copper. The curve is linear over a wide range of temperatures, and ρ increases with increasing temperature.

Quick Quiz 27.4 When does an incandescent lightbulb carry more current, **(a)** immediately after it is turned on and the glow of the metal filament is increasing or **(b)** after it has been on for a few milliseconds and the glow is steady?

27.5 Superconductors

There is a class of metals and compounds whose resistance decreases to zero when they are below a certain temperature T_c, known as the **critical temperature.** These materials are known as **superconductors.** The resistance–temperature graph for a superconductor follows that of a normal metal at temperatures above T_c (Fig. 27.10). When the temperature is at or below T_c, the resistivity drops suddenly to zero. This phenomenon was discovered in 1911 by Dutch physicist Heike Kamerlingh-Onnes (1853–1926) as he worked with mercury, which is a superconductor below 4.2 K. Measurements have shown that the resistivities of superconductors below their T_c values are less than $4 \times 10^{-25}\ \Omega \cdot \text{m}$, or approximately 10^{17} times smaller than the resistivity of copper. In practice, these resistivities are considered to be zero.

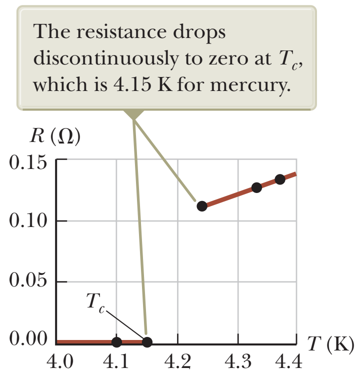


Figure 27.10 Resistance versus temperature for a sample of mercury (Hg). The graph follows that of a normal metal above the critical temperature T_c.

Courtesy of IBM Research Laboratory

A small permanent magnet levitated above a disk of the superconductor $YBa_2Cu_3O_7$, which is in liquid nitrogen at 77 K.

Table 27.3 Critical Temperatures for Various Superconductors

Material	T_c (K)
$HgBa_2Ca_2Cu_3O_8$	134
Tl—Ba—Ca—Cu—O	125
Bi—Sr—Ca—Cu—O	105
$YBa_2Cu_3O_7$	92
Nb_3Ge	23.2
Nb_3Sn	18.05
Nb	9.46
Pb	7.18
Hg	4.15
Sn	3.72
Al	1.19
Zn	0.88

Today, thousands of superconductors are known, and as Table 27.3 illustrates, the critical temperatures of recently discovered superconductors are substantially higher than initially thought possible. Two kinds of superconductors are recognized. The more recently identified ones are essentially ceramics with high critical temperatures, whereas superconducting materials such as those observed by Kamerlingh-Onnes are metals. If a room-temperature superconductor is ever identified, its effect on technology could be tremendous.

The value of T_c is sensitive to chemical composition, pressure, and molecular structure. Copper, silver, and gold, which are excellent conductors, do not exhibit superconductivity.

One truly remarkable feature of superconductors is that once a current is set up in them, it persists *without any applied potential difference* (because $R = 0$). Steady currents have been observed to persist in superconducting loops for several years with no apparent decay!

An important and useful application of superconductivity is in the development of superconducting magnets, in which the magnitudes of the magnetic field are approximately ten times greater than those produced by the best normal electromagnets. Such superconducting magnets are being considered as a means of storing energy. Superconducting magnets are currently used in medical magnetic resonance imaging, or MRI, units, which produce high-quality images of internal organs without the need for excessive exposure of patients to x-rays or other harmful radiation.

27.6 Electrical Power

In typical electric circuits, energy T_{ET} is transferred by electrical transmission from a source such as a battery to some device such as a lightbulb or a radio receiver. Let's determine an expression that will allow us to calculate the rate of this energy transfer. First, consider the simple circuit in Figure 27.11, where energy is delivered to a resistor. (Resistors are designated by the circuit symbol —\/\/\/—.) Because the connecting wires also have resistance, some energy is delivered to the wires and some to the resistor. Unless noted otherwise, we shall assume the resistance of the wires is small compared with the resistance of the circuit element so that the energy delivered to the wires is negligible.

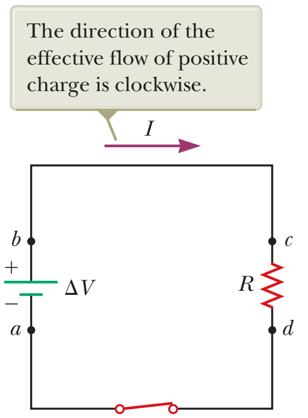


Figure 27.11 A circuit consisting of a resistor of resistance R and a battery having a potential difference ΔV across its terminals.

Imagine following a positive quantity of charge Q moving clockwise around the circuit in Figure 27.11 from point a through the battery and resistor back to point a. We identify the entire circuit as our system. As the charge moves from a to b through the battery, the electric potential energy of the system *increases* by an amount $Q\,\Delta V$

while the chemical potential energy in the battery *decreases* by the same amount. (Recall from Eq. 25.3 that $\Delta U = q\,\Delta V$.) As the charge moves from c to d through the resistor, however, the electric potential energy of the system decreases due to collisions of electrons with atoms in the resistor. In this process, the electric potential energy is transformed to internal energy corresponding to increased vibrational motion of the atoms in the resistor. Because the resistance of the interconnecting wires is neglected, no energy transformation occurs for paths bc and da. When the charge returns to point a, the net result is that some of the chemical potential energy in the battery has been delivered to the resistor and resides in the resistor as internal energy E_{int} associated with molecular vibration.

The resistor is normally in contact with air, so its increased temperature results in a transfer of energy by heat Q into the air. In addition, the resistor emits thermal radiation T_{ER}, representing another means of escape for the energy. After some time interval has passed, the resistor reaches a constant temperature. At this time, the input of energy from the battery is balanced by the output of energy from the resistor by heat and radiation, and the resistor is a nonisolated system in steady state. Some electrical devices include *heat sinks*[4] connected to parts of the circuit to prevent these parts from reaching dangerously high temperatures. Heat sinks are pieces of metal with many fins. Because the metal's high thermal conductivity provides a rapid transfer of energy by heat away from the hot component and the large number of fins provides a large surface area in contact with the air, energy can transfer by radiation and into the air by heat at a high rate.

Let's now investigate the rate at which the electric potential energy of the system decreases as the charge Q passes through the resistor:

$$\frac{dU}{dt} = \frac{d}{dt}(Q\,\Delta V) = \frac{dQ}{dt}\Delta V = I\,\Delta V$$

where I is the current in the circuit. The system regains this potential energy when the charge passes through the battery, at the expense of chemical energy in the battery. The rate at which the potential energy of the system decreases as the charge passes through the resistor is equal to the rate at which the system gains internal energy in the resistor. Therefore, the power P, representing the rate at which energy is delivered to the resistor, is

$$P = I\,\Delta V \tag{27.21}$$

We derived this result by considering a battery delivering energy to a resistor. Equation 27.21, however, can be used to calculate the power delivered by a voltage source to *any* device carrying a current I and having a potential difference ΔV between its terminals.

Using Equation 27.21 and $\Delta V = IR$ for a resistor, we can express the power delivered to the resistor in the alternative forms

$$P = I^2 R = \frac{(\Delta V)^2}{R} \tag{27.22}$$

When I is expressed in amperes, ΔV in volts, and R in ohms, the SI unit of power is the watt, as it was in Chapter 8 in our discussion of mechanical power. The process by which energy is transformed to internal energy in a conductor of resistance R is often called *joule heating;*[5] this transformation is also often referred to as an I^2R loss.

Pitfall Prevention 27.5

Charges Do Not Move All the Way Around a Circuit in a Short Time In terms of understanding the energy transfer in a circuit, it is useful to *imagine* a charge moving all the way around the circuit even though it would take hours to do so.

Pitfall Prevention 27.6

Misconceptions About Current Several common misconceptions are associated with current in a circuit like that in Figure 27.11. One is that current comes out of one terminal of the battery and is then "used up" as it passes through the resistor, leaving current in only one part of the circuit. The current is actually the same *everywhere* in the circuit. A related misconception has the current coming out of the resistor being smaller than that going in because some of the current is "used up." Yet another misconception has current coming out of both terminals of the battery, in opposite directions, and then "clashing" in the resistor, delivering the energy in this manner. That is not the case; charges flow in the same rotational sense at *all* points in the circuit.

Pitfall Prevention 27.7

Energy Is Not "Dissipated" In some books, you may see Equation 27.22 described as the power "dissipated in" a resistor, suggesting that energy disappears. Instead, we say energy is "delivered to" a resistor.

[4]This usage is another misuse of the word *heat* that is ingrained in our common language.

[5]It is commonly called *joule heating* even though the process of heat does not occur when energy delivered to a resistor appears as internal energy. It is another example of incorrect usage of the word *heat* that has become entrenched in our language.

Figure 27.12 These power lines transfer energy from the electric company to homes and businesses. The energy is transferred at a very high voltage, possibly hundreds of thousands of volts in some cases. Even though it makes power lines very dangerous, the high voltage results in less loss of energy due to resistance in the wires.

When transporting energy by electricity through power lines (Fig. 27.12), you should not assume the lines have zero resistance. Real power lines do indeed have resistance, and power is delivered to the resistance of these wires. Utility companies seek to minimize the energy transformed to internal energy in the lines and maximize the energy delivered to the consumer. Because $P = I\,\Delta V$, the same amount of energy can be transported either at high currents and low potential differences or at low currents and high potential differences. Utility companies choose to transport energy at low currents and high potential differences primarily for economic reasons. Copper wire is very expensive, so it is cheaper to use high-resistance wire (that is, wire having a small cross-sectional area; see Eq. 27.10). Therefore, in the expression for the power delivered to a resistor, $P = I^2R$, the resistance of the wire is fixed at a relatively high value for economic considerations. The I^2R loss can be reduced by keeping the current I as low as possible, which means transferring the energy at a high voltage. In some instances, power is transported at potential differences as great as 765 kV. At the destination of the energy, the potential difference is usually reduced to 4 kV by a device called a *transformer*. Another transformer drops the potential difference to 240 V for use in your home. Of course, each time the potential difference decreases, the current increases by the same factor and the power remains the same. We shall discuss transformers in greater detail in Chapter 33.

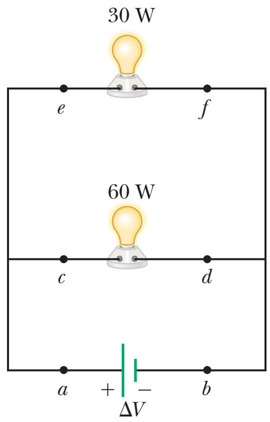


Figure 27.13 (Quick Quiz 27.5) Two lightbulbs connected across the same potential difference.

Quick Quiz 27.5 For the two lightbulbs shown in Figure 27.13, rank the current values at points a through f from greatest to least.

Example 27.4 Power in an Electric Heater

An electric heater is constructed by applying a potential difference of 120 V across a Nichrome wire that has a total resistance of 8.00 Ω. Find the current carried by the wire and the power rating of the heater.

SOLUTION

Conceptualize As discussed in Example 27.2, Nichrome wire has high resistivity and is often used for heating elements in toasters, irons, and electric heaters. Therefore, we expect the power delivered to the wire to be relatively high.

Categorize We evaluate the power from Equation 27.22, so we categorize this example as a substitution problem.

Use Equation 27.7 to find the current in the wire:

$$I = \frac{\Delta V}{R} = \frac{120\text{ V}}{8.00\ \Omega} = \boxed{15.0\text{ A}}$$

Find the power rating using the expression $P = I^2R$ from Equation 27.22:

$$P = I^2R = (15.0\text{ A})^2(8.00\ \Omega) = 1.80 \times 10^3\text{ W} = \boxed{1.80\text{ kW}}$$

WHAT IF? What if the heater were accidentally connected to a 240-V supply? (That is difficult to do because the shape and orientation of the metal contacts in 240-V plugs are different from those in 120-V plugs.) How would that affect the current carried by the heater and the power rating of the heater, assuming the resistance remains constant?

Answer If the applied potential difference were doubled, Equation 27.7 shows that the current would double. According to Equation 27.22, $P = (\Delta V)^2/R$, the power would be four times larger.

Example 27.5 Linking Electricity and Thermodynamics AM

An immersion heater must increase the temperature of 1.50 kg of water from 10.0°C to 50.0°C in 10.0 min while operating at 110 V.

(A) What is the required resistance of the heater?

SOLUTION

Conceptualize An immersion heater is a resistor that is inserted into a container of water. As energy is delivered to the immersion heater, raising its temperature, energy leaves the surface of the resistor by heat, going into the water. When the immersion heater reaches a constant temperature, the rate of energy delivered to the resistance by electrical transmission (T_{ET}) is equal to the rate of energy delivered by heat (Q) to the water.

Categorize This example allows us to link our new understanding of power in electricity with our experience with specific heat in thermodynamics (Chapter 20). The water is a *nonisolated system.* Its internal energy is rising because of energy transferred into the water by heat from the resistor, so Equation 8.2 reduces to $\Delta E_{int} = Q$. In our model, we assume the energy that enters the water from the heater remains in the water.

Analyze To simplify the analysis, let's ignore the initial period during which the temperature of the resistor increases and also ignore any variation of resistance with temperature. Therefore, we imagine a constant rate of energy transfer for the entire 10.0 min.

Set the rate of energy delivered to the resistor equal to the rate of energy Q entering the water by heat:

$$P = \frac{(\Delta V)^2}{R} = \frac{Q}{\Delta t}$$

Use Equation 20.4, $Q = mc\,\Delta T$, to relate the energy input by heat to the resulting temperature change of the water and solve for the resistance:

$$\frac{(\Delta V)^2}{R} = \frac{mc\,\Delta T}{\Delta t} \rightarrow R = \frac{(\Delta V)^2\,\Delta t}{mc\,\Delta T}$$

Substitute the values given in the statement of the problem:

$$R = \frac{(110\text{ V})^2(600\text{ s})}{(1.50\text{ kg})(4\,186\text{ J/kg}\cdot{}^\circ\text{C})(50.0^\circ\text{C} - 10.0^\circ\text{C})} = \boxed{28.9\ \Omega}$$

(B) Estimate the cost of heating the water.

SOLUTION

Multiply the power by the time interval to find the amount of energy transferred to the resistor:

$$T_{ET} = P\,\Delta t = \frac{(\Delta V)^2}{R}\Delta t = \frac{(110\text{ V})^2}{28.9\ \Omega}(10.0\text{ min})\left(\frac{1\text{ h}}{60.0\text{ min}}\right)$$

$$= 69.8\text{ Wh} = 0.069\,8\text{ kWh}$$

Find the cost knowing that energy is purchased at an estimated price of 11¢ per kilowatt-hour:

$$\text{Cost} = (0.069\,8\text{ kWh})(\$0.11/\text{kWh}) = \$0.008 = \boxed{0.8¢}$$

Finalize The cost to heat the water is very low, less than one cent. In reality, the cost is higher because some energy is transferred from the water into the surroundings by heat and electromagnetic radiation while its temperature is increasing. If you have electrical devices in your home with power ratings on them, use this power rating and an approximate time interval of use to estimate the cost for one use of the device.

Summary

Definitions

The electric **current** I in a conductor is defined as

$$I \equiv \frac{dQ}{dt} \quad \textbf{(27.2)}$$

where dQ is the charge that passes through a cross section of the conductor in a time interval dt. The SI unit of current is the **ampere** (A), where 1 A = 1 C/s.

continued

The **current density** J in a conductor is the current per unit area:

$$J \equiv \frac{I}{A} \quad (27.5)$$

The **resistance** R of a conductor is defined as

$$R \equiv \frac{\Delta V}{I} \quad (27.7)$$

where ΔV is the potential difference across the conductor and I is the current it carries. The SI unit of resistance is volts per ampere, which is defined to be 1 **ohm** (Ω); that is, $1\ \Omega = 1\ \text{V/A}$.

Concepts and Principles

The average current in a conductor is related to the motion of the charge carriers through the relationship

$$I_{avg} = nqv_dA \quad (27.4)$$

where n is the density of charge carriers, q is the charge on each carrier, v_d is the drift speed, and A is the cross-sectional area of the conductor.

The current density in an ohmic conductor is proportional to the electric field according to the expression

$$J = \sigma E \quad (27.6)$$

The proportionality constant σ is called the **conductivity** of the material of which the conductor is made. The inverse of σ is known as **resistivity** ρ (that is, $\rho = 1/\sigma$). Equation 27.6 is known as **Ohm's law,** and a material is said to obey this law if the ratio of its current density to its applied electric field is a constant that is independent of the applied field.

For a uniform block of material of cross-sectional area A and length ℓ, the resistance over the length ℓ is

$$R = \rho\frac{\ell}{A} \quad (27.10)$$

where ρ is the resistivity of the material.

In a classical model of electrical conduction in metals, the electrons are treated as molecules of a gas. In the absence of an electric field, the average velocity of the electrons is zero. When an electric field is applied, the electrons move (on average) with a **drift velocity** $\vec{\mathbf{v}}_d$ that is opposite the electric field. The drift velocity is given by

$$\vec{\mathbf{v}}_d = \frac{q\vec{\mathbf{E}}}{m_e}\tau \quad (27.13)$$

where q is the electron's charge, m_e is the mass of the electron, and τ is the average time interval between electron–atom collisions. According to this model, the resistivity of the metal is

$$\rho = \frac{m_e}{nq^2\tau} \quad (27.16)$$

where n is the number of free electrons per unit volume.

The resistivity of a conductor varies approximately linearly with temperature according to the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \quad (27.18)$$

where ρ_0 is the resistivity at some reference temperature T_0 and α is the **temperature coefficient of resistivity.**

If a potential difference ΔV is maintained across a circuit element, the **power,** or rate at which energy is supplied to the element, is

$$P = I\,\Delta V \quad (27.21)$$

Because the potential difference across a resistor is given by $\Delta V = IR$, we can express the power delivered to a resistor as

$$P = I^2R = \frac{(\Delta V)^2}{R} \quad (27.22)$$

The energy delivered to a resistor by electrical transmission T_{ET} appears in the form of internal energy E_{int} in the resistor.

Objective Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Car batteries are often rated in ampere-hours. Does this information designate the amount of (a) current, (b) power, (c) energy, (d) charge, or (e) potential the battery can supply?

$\boxed{\textbf{2.}}$ Two wires A and B with circular cross sections are made of the same metal and have equal lengths, but the resistance of wire A is three times greater than that of wire B. **(i)** What is the ratio of the cross-sectional area of A to that of B? (a) 3 (b) $\sqrt{3}$ (c) 1 (d) $1/\sqrt{3}$ (e) $\frac{1}{3}$ **(ii)** What is the ratio of the radius of A to that of B? Choose from the same possibilities as in part (i).

3. A cylindrical metal wire at room temperature is carrying electric current between its ends. One end is at potential $V_A = 50$ V, and the other end is at potential $V_B = 0$ V. Rank the following actions in terms of the change that each one separately would produce in the current from the greatest increase to the greatest decrease. In your ranking, note any cases of equality. (a) Make $V_A = 150$ V with $V_B = 0$ V. (b) Adjust V_A to triple the power with which the wire converts electrically transmitted energy into internal energy. (c) Double the radius of the wire. (d) Double the length of the wire. (e) Double the Celsius temperature of the wire.

4. A current-carrying ohmic metal wire has a cross-sectional area that gradually becomes smaller from one end of the wire to the other. The current has the same value for each section of the wire, so charge does not accumulate at any one point. **(i)** How does the drift speed vary along the wire as the area becomes smaller? (a) It increases. (b) It decreases. (c) It remains constant. **(ii)** How does the resistance per unit length vary along the wire as the area becomes smaller? Choose from the same possibilities as in part (i).

5. A potential difference of 1.00 V is maintained across a 10.0-Ω resistor for a period of 20.0 s. What total charge passes by a point in one of the wires connected to the resistor in this time interval? (a) 200 C (b) 20.0 C (c) 2.00 C (d) 0.005 00 C (e) 0.050 0 C

6. Three wires are made of copper having circular cross sections. Wire 1 has a length L and radius r. Wire 2 has a length L and radius $2r$. Wire 3 has a length $2L$ and radius $3r$. Which wire has the smallest resistance? (a) wire 1 (b) wire 2 (c) wire 3 (d) All have the same resistance. (e) Not enough information is given to answer the question.

7. A metal wire of resistance R is cut into three equal pieces that are then placed together side by side to form a new cable with a length equal to one-third the original length. What is the resistance of this new cable? (a) $\frac{1}{9}R$ (b) $\frac{1}{3}R$ (c) R (d) $3R$ (e) $9R$

8. A metal wire has a resistance of 10.0 Ω at a temperature of 20.0°C. If the same wire has a resistance of 10.6 Ω at 90.0°C, what is the resistance of this wire when its temperature is −20.0°C? (a) 0.700 Ω (b) 9.66 Ω (c) 10.3 Ω (d) 13.8 Ω (e) 6.59 Ω

9. The current-versus-voltage behavior of a certain electrical device is shown in Figure OQ27.9. When the potential difference across the device is 2 V, what is its resistance? (a) 1 Ω (b) $\frac{3}{4}$ Ω (c) $\frac{4}{3}$ Ω (d) undefined (e) none of those answers

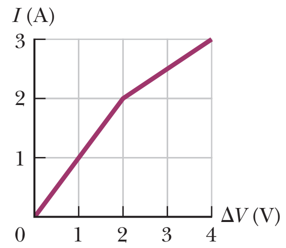


Figure OQ27.9

10. Two conductors made of the same material are connected across the same potential difference. Conductor A has twice the diameter and twice the length of conductor B. What is the ratio of the power delivered to A to the power delivered to B? (a) 8 (b) 4 (c) 2 (d) 1 (e) $\frac{1}{2}$

11. Two conducting wires A and B of the same length and radius are connected across the same potential difference. Conductor A has twice the resistivity of conductor B. What is the ratio of the power delivered to A to the power delivered to B? (a) 2 (b) $\sqrt{2}$ (c) 1 (d) $1/\sqrt{2}$ (e) $\frac{1}{2}$

12. Two lightbulbs both operate on 120 V. One has a power of 25 W and the other 100 W. **(i)** Which lightbulb has higher resistance? (a) The dim 25-W lightbulb does. (b) The bright 100-W lightbulb does. (c) Both are the same. **(ii)** Which lightbulb carries more current? Choose from the same possibilities as in part (i).

13. Wire B has twice the length and twice the radius of wire A. Both wires are made from the same material. If wire A has a resistance R, what is the resistance of wire B? (a) $4R$ (b) $2R$ (c) R (d) $\frac{1}{2}R$ (e) $\frac{1}{4}R$

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. If you were to design an electric heater using Nichrome wire as the heating element, what parameters of the wire could you vary to meet a specific power output such as 1 000 W?

2. What factors affect the resistance of a conductor?

3. When the potential difference across a certain conductor is doubled, the current is observed to increase by a factor of 3. What can you conclude about the conductor?

4. Over the time interval after a difference in potential is applied between the ends of a wire, what would happen to the drift velocity of the electrons in a wire and to the current in the wire if the electrons could move freely without resistance through the wire?

5. How does the resistance for copper and for silicon change with temperature? Why are the behaviors of these two materials different?

$\boxed{6.}$ Use the atomic theory of matter to explain why the resistance of a material should increase as its temperature increases.

$\boxed{7.}$ If charges flow very slowly through a metal, why does it not require several hours for a light to come on when you throw a switch?

8. Newspaper articles often contain statements such as "10 000 volts of electricity surged through the victim's body.'' What is wrong with this statement?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 27.1 Electric Current

1. AMT M A 200-km-long high-voltage transmission line 2.00 cm in diameter carries a steady current of 1 000 A. If the conductor is copper with a free charge density of 8.50×10^{28} electrons per cubic meter, how many years does it take one electron to travel the full length of the cable?

2. A small sphere that carries a charge q is whirled in a circle at the end of an insulating string. The angular frequency of revolution is ω. What average current does this revolving charge represent?

3. W An aluminum wire having a cross-sectional area equal to 4.00×10^{-6} m^2 carries a current of 5.00 A. The density of aluminum is 2.70 g/cm^3. Assume each aluminum atom supplies one conduction electron per atom. Find the drift speed of the electrons in the wire.

4. AMT In the Bohr model of the hydrogen atom (which will be covered in detail in Chapter 42), an electron in the lowest energy state moves at a speed of 2.19×10^6 m/s in a circular path of radius 5.29×10^{-11} m. What is the effective current associated with this orbiting electron?

5. A proton beam in an accelerator carries a current of 125 μA. If the beam is incident on a target, how many protons strike the target in a period of 23.0 s?

6. A copper wire has a circular cross section with a radius of 1.25 mm. (a) If the wire carries a current of 3.70 A, find the drift speed of the electrons in this wire. (b) All other things being equal, what happens to the drift speed in wires made of metal having a larger number of conduction electrons per atom than copper? Explain.

7. Suppose the current in a conductor decreases exponentially with time according to the equation $I(t) = I_0 e^{-t/\tau}$, where I_0 is the initial current (at $t = 0$) and τ is a constant having dimensions of time. Consider a fixed observation point within the conductor. (a) How much charge passes this point between $t = 0$ and $t = \tau$? (b) How much charge passes this point between $t = 0$ and $t = 10\tau$? (c) **What If?** How much charge passes this point between $t = 0$ and $t = \infty$?

8. W Figure P27.8 represents a section of a conductor of nonuniform diameter carrying a current of $I = 5.00$ A. The radius of cross-section A_1 is $r_1 = 0.400$ cm. (a) What is the magnitude of the current density across A_1? The radius r_2 at A_2 is larger than the radius r_1 at A_1. (b) Is the current at A_2 larger, smaller, or the same? (c) Is the current density at A_2 larger, smaller, or the same? Assume $A_2 = 4A_1$. Specify the (d) radius, (e) current, and (f) current density at A_2.

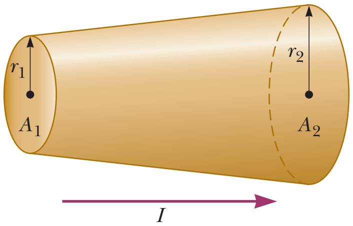


Figure P27.8

9. W The quantity of charge q (in coulombs) that has passed through a surface of area 2.00 cm^2 varies with time according to the equation $q = 4t^3 + 5t + 6$, where t is in seconds. (a) What is the instantaneous current through the surface at $t = 1.00$ s? (b) What is the value of the current density?

10. A Van de Graaff generator produces a beam of 2.00-MeV deuterons, which are heavy hydrogen nuclei containing a proton and a neutron. (a) If the beam current is 10.0 μA, what is the average separation of the deuterons? (b) Is the electrical force of repulsion among them a significant factor in beam stability? Explain.

11. M The electron beam emerging from a certain high-energy electron accelerator has a circular cross section of radius 1.00 mm. (a) The beam current is 8.00 μA. Find the current density in the beam assuming it is uniform throughout. (b) The speed of the electrons is so close to the speed of light that their speed can be taken as 300 Mm/s with negligible error. Find the electron density in the beam. (c) Over what time interval does Avogadro's number of electrons emerge from the accelerator?

12. W An electric current in a conductor varies with time according to the expression $I(t) = 100 \sin(120\pi t)$, where I is in amperes and t is in seconds. What is the total charge passing a given point in the conductor from $t = 0$ to $t = \frac{1}{240}$ s?

13. W A teapot with a surface area of 700 cm^2 is to be plated with silver. It is attached to the negative electrode of an electrolytic cell containing silver nitrate ($Ag^+NO_3^-$). The cell is powered by a 12.0-V battery and has a resistance of 1.80 Ω. If the density of silver is 10.5×10^3 kg/m^3, over what time interval does a 0.133-mm layer of silver build up on the teapot?

Section 27.2 Resistance

14. W A lightbulb has a resistance of 240 Ω when operating with a potential difference of 120 V across it. What is the current in the lightbulb?

15. M A wire 50.0 m long and 2.00 mm in diameter is connected to a source with a potential difference of 9.11 V, and the current is found to be 36.0 A. Assume a temperature of 20.0°C and, using Table 27.2, identify the metal out of which the wire is made.

16. A 0.900-V potential difference is maintained across a 1.50-m length of tungsten wire that has a cross-sectional area of 0.600 mm^2. What is the current in the wire?

17. An electric heater carries a current of 13.5 A when operating at a voltage of 120 V. What is the resistance of the heater?

18. Aluminum and copper wires of equal length are found to have the same resistance. What is the ratio of their radii?

19. M Suppose you wish to fabricate a uniform wire from 1.00 g of copper. If the wire is to have a resistance of $R = 0.500$ Ω and all the copper is to be used, what must be (a) the length and (b) the diameter of this wire?

20. Suppose you wish to fabricate a uniform wire from a mass m of a metal with density ρ_m and resistivity ρ. If the wire is to have a resistance of R and all the metal is to be used, what must be (a) the length and (b) the diameter of this wire?

21. A portion of Nichrome wire of radius 2.50 mm is to be used in winding a heating coil. If the coil must draw a current of 9.25 A when a voltage of 120 V is applied across its ends, find (a) the required resistance of the coil and (b) the length of wire you must use to wind the coil.

Section 27.3 A Model for Electrical Conduction

22. If the current carried by a conductor is doubled, what happens to (a) the charge carrier density, (b) the current density, (c) the electron drift velocity, and (d) the average time interval between collisions?

23. A current density of 6.00×10^{-13} A/m^2 exists in the atmosphere at a location where the electric field is 100 V/m. Calculate the electrical conductivity of the Earth's atmosphere in this region.

24. GP An iron wire has a cross-sectional area equal to 5.00×10^{-6} m^2. Carry out the following steps to determine the drift speed of the conduction electrons in the wire if it carries a current of 30.0 A. (a) How many kilograms are there in 1.00 mole of iron? (b) Starting with the density of iron and the result of part (a), compute the molar density of iron (the number of moles of iron per cubic meter). (c) Calculate the number density of iron atoms using Avogadro's number. (d) Obtain the number density of conduction electrons given that there are two conduction electrons per iron atom. (e) Calculate the drift speed of conduction electrons in this wire.

25. M If the magnitude of the drift velocity of free electrons in a copper wire is 7.84×10^{-4} m/s, what is the electric field in the conductor?

Section 27.4 Resistance and Temperature

26. A certain lightbulb has a tungsten filament with a resistance of 19.0 Ω when at 20.0°C and 140 Ω when hot. Assume the resistivity of tungsten varies linearly with temperature even over the large temperature range involved here. Find the temperature of the hot filament.

27. What is the fractional change in the resistance of an iron filament when its temperature changes from 25.0°C to 50.0°C?

28. While taking photographs in Death Valley on a day when the temperature is 58.0°C, Bill Hiker finds that a certain voltage applied to a copper wire produces a current of 1.00 A. Bill then travels to Antarctica and applies the same voltage to the same wire. What current does he register there if the temperature is −88.0°C? Assume that no change occurs in the wire's shape and size.

29. If a certain silver wire has a resistance of 6.00 Ω at 20.0°C, what resistance will it have at 34.0°C?

30. Plethysmographs are devices used for measuring changes in the volume of internal organs or limbs. In one form of this device, a rubber capillary tube with an inside diameter of 1.00 mm is filled with mercury at 20.0°C. The resistance of the mercury is measured with the aid of electrodes sealed into the ends of the tube. If 100 cm of the tube is wound in a helix around a patient's upper arm, the blood flow during a heartbeat causes the arm to expand, stretching the length of the tube by 0.040 0 cm. From this observation and assuming cylindrical symmetry, you can find the change in volume of the arm, which gives an indication of blood flow. Taking the resistivity of mercury to be 9.58×10^{-7} Ω · m, calculate (a) the resistance of the mercury and (b) the fractional change in resistance during the heartbeat. *Hint:* The fraction by which the cross-sectional area of the mercury column decreases is the fraction by which the length increases because the volume of mercury is constant.

31. M (a) A 34.5-m length of copper wire at 20.0°C has a radius of 0.25 mm. If a potential difference of 9.00 V is applied across the length of the wire, determine the current in the wire. (b) If the wire is heated to 30.0°C while the 9.00-V potential difference is maintained, what is the resulting current in the wire?

32. An engineer needs a resistor with a zero overall temperature coefficient of resistance at 20.0°C. She designs a pair of circular cylinders, one of carbon and one of Nichrome as shown in Figure P27.32 (page 828). The device must have an overall resistance of $R_1 + R_2 = 10.0\ \Omega$ independent of temperature and a uniform radius of $r = 1.50$ mm. Ignore thermal expansion of the cylinders and assume both are always at the same temperature. (a) Can she meet the design goal with this method? (b) If so, state what you can determine about the lengths ℓ_1 and ℓ_2 of each segment. If not, explain.

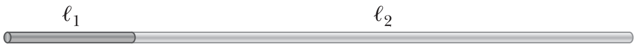


Figure P27.32

33. M An aluminum wire with a diameter of 0.100 mm has a uniform electric field of 0.200 V/m imposed along its entire length. The temperature of the wire is 50.0°C. Assume one free electron per atom. (a) Use the information in Table 27.2 to determine the resistivity of aluminum at this temperature. (b) What is the current density in the wire? (c) What is the total current in the wire? (d) What is the drift speed of the conduction electrons? (e) What potential difference must exist between the ends of a 2.00-m length of the wire to produce the stated electric field?

34. **Review.** An aluminum rod has a resistance of 1.23 Ω at 20.0°C. Calculate the resistance of the rod at 120°C by accounting for the changes in both the resistivity and the dimensions of the rod. The coefficient of linear expansion for aluminum is $2.40 \times 10^{-6}\ (°C)^{-1}$.

35. At what temperature will aluminum have a resistivity that is three times the resistivity copper has at room temperature?

Section 27.6 Electrical Power

36. Assume that global lightning on the Earth constitutes a constant current of 1.00 kA between the ground and an atmospheric layer at potential 300 kV. (a) Find the power of terrestrial lightning. (b) For comparison, find the power of sunlight falling on the Earth. Sunlight has an intensity of 1 370 W/m^2 above the atmosphere. Sunlight falls perpendicularly on the circular projected area that the Earth presents to the Sun.

37. In a hydroelectric installation, a turbine delivers 1 500 hp to a generator, which in turn transfers 80.0% of the mechanical energy out by electrical transmission. Under these conditions, what current does the generator deliver at a terminal potential difference of 2 000 V?

38. A Van de Graaff generator (see Fig. 25.23) is operating so that the potential difference between the high-potential electrode Ⓑ and the charging needles at Ⓐ is 15.0 kV. Calculate the power required to drive the belt against electrical forces at an instant when the effective current delivered to the high-potential electrode is 500 μA.

39. A certain waffle iron is rated at 1.00 kW when connected to a 120-V source. (a) What current does the waffle iron carry? (b) What is its resistance?

40. The potential difference across a resting neuron in the human body is about 75.0 mV and carries a current of about 0.200 mA. How much power does the neuron release?

41. Suppose your portable DVD player draws a current of 350 mA at 6.00 V. How much power does the player require?

42. AMT M **Review.** A well-insulated electric water heater warms 109 kg of water from 20.0°C to 49.0°C in 25.0 min. Find the resistance of its heating element, which is connected across a 240-V potential difference.

43. A 100-W lightbulb connected to a 120-V source experiences a voltage surge that produces 140 V for a moment. By what percentage does its power output increase? Assume its resistance does not change.

44. The cost of energy delivered to residences by electrical transmission varies from \$0.070/kWh to \$0.258/kWh throughout the United States; \$0.110/kWh is the average value. At this average price, calculate the cost of (a) leaving a 40.0-W porch light on for two weeks while you are on vacation, (b) making a piece of dark toast in 3.00 min with a 970-W toaster, and (c) drying a load of clothes in 40.0 min in a 5.20×10^3-W dryer.

45. W Batteries are rated in terms of ampere-hours (A · h). For example, a battery that can produce a current of 2.00 A for 3.00 h is rated at 6.00 A · h. (a) What is the total energy, in kilowatt-hours, stored in a 12.0-V battery rated at 55.0 A · h? (b) At \$0.110 per kilowatt-hour, what is the value of the electricity produced by this battery?

46. W Residential building codes typically require the use of 12-gauge copper wire (diameter 0.205 cm) for wiring receptacles. Such circuits carry currents as large as 20.0 A. If a wire of smaller diameter (with a higher gauge number) carried that much current, the wire could rise to a high temperature and cause a fire. (a) Calculate the rate at which internal energy is produced in 1.00 m of 12-gauge copper wire carrying 20.0 A. (b) **What If?** Repeat the calculation for a 12-gauge aluminum wire. (c) Explain whether a 12-gauge aluminum wire would be as safe as a copper wire.

47. M Assuming the cost of energy from the electric company is \$0.110/kWh, compute the cost per day of operating a lamp that draws a current of 1.70 A from a 110-V line.

48. An 11.0-W energy-efficient fluorescent lightbulb is designed to produce the same illumination as a conventional 40.0-W incandescent lightbulb. Assuming a cost of \$0.110/kWh for energy from the electric company, how much money does the user of the energy-efficient bulb save during 100 h of use?

49. A coil of Nichrome wire is 25.0 m long. The wire has a diameter of 0.400 mm and is at 20.0°C. If it carries a current of 0.500 A, what are (a) the magnitude of the electric field in the wire and (b) the power delivered to it? (c) **What If?** If the temperature is increased to 340°C and the potential difference across the wire remains constant, what is the power delivered?

50. **Review.** A rechargeable battery of mass 15.0 g delivers an average current of 18.0 mA to a portable DVD player at 1.60 V for 2.40 h before the battery must be recharged. The recharger maintains a potential difference of 2.30 V across the battery and delivers a charging current of 13.5 mA for 4.20 h. (a) What is the efficiency of the battery as an energy storage device? (b) How much internal energy is produced in the battery during one charge–discharge cycle? (c) If the battery is surrounded by ideal thermal insulation and has an effective specific heat of 975 J/kg · °C, by how much will its temperature increase during the cycle?

51. A 500-W heating coil designed to operate from 110 V is made of Nichrome wire 0.500 mm in diameter. (a) Assuming the resistivity of the Nichrome remains constant at its 20.0°C value, find the length of wire used. (b) **What If?** Now consider the variation of resistivity with temperature. What power is delivered to the coil of part (a) when it is warmed to 1 200°C?

52. *Why is the following situation impossible?* A politician is decrying wasteful uses of energy and decides to focus on energy used to operate plug-in electric clocks in the United States. He estimates there are 270 million of these clocks, approximately one clock for each person in the population. The clocks transform energy taken in by electrical transmission at the average rate 2.50 W. The politician gives a speech in which he complains that, at today's electrical rates, the nation is losing $100 million every year to operate these clocks.

53. M A certain toaster has a heating element made of Nichrome wire. When the toaster is first connected to a 120-V source (and the wire is at a temperature of 20.0°C), the initial current is 1.80 A. The current decreases as the heating element warms up. When the toaster reaches its final operating temperature, the current is 1.53 A. (a) Find the power delivered to the toaster when it is at its operating temperature. (b) What is the final temperature of the heating element?

54. Make an order-of-magnitude estimate of the cost of one person's routine use of a handheld hair dryer for 1 year. If you do not use a hair dryer yourself, observe or interview someone who does. State the quantities you estimate and their values.

55. M **Review.** The heating element of an electric coffee maker operates at 120 V and carries a current of 2.00 A. Assuming the water absorbs all the energy delivered to the resistor, calculate the time interval during which the temperature of 0.500 kg of water rises from room temperature (23.0°C) to the boiling point.

56. A 120-V motor has mechanical power output of 2.50 hp. It is 90.0% efficient in converting power that it takes in by electrical transmission into mechanical power. (a) Find the current in the motor. (b) Find the energy delivered to the motor by electrical transmission in 3.00 h of operation. (c) If the electric company charges $0.110/kWh, what does it cost to run the motor for 3.00 h?

Additional Problems

57. M A particular wire has a resistivity of $3.0 \times 10^{-8}\ \Omega \cdot \text{m}$ and a cross-sectional area of $4.0 \times 10^{-6}\ \text{m}^2$. A length of this wire is to be used as a resistor that will receive 48 W of power when connected across a 20-V battery. What length of wire is required?

58. Determine the temperature at which the resistance of an aluminum wire will be twice its value at 20.0°C. Assume its coefficient of resistivity remains constant.

59. A car owner forgets to turn off the headlights of his car while it is parked in his garage. If the 12.0-V battery in his car is rated at 90.0 A · h and each headlight requires 36.0 W of power, how long will it take the battery to completely discharge?

60. Lightbulb A is marked "25 W 120 V," and lightbulb B is marked "100 W 120 V." These labels mean that each lightbulb has its respective power delivered to it when it is connected to a constant 120-V source. (a) Find the resistance of each lightbulb. (b) During what time interval does 1.00 C pass into lightbulb A? (c) Is this charge different upon its exit versus its entry into the lightbulb? Explain. (d) In what time interval does 1.00 J pass into lightbulb A? (e) By what mechanisms does this energy enter and exit the lightbulb? Explain. (f) Find the cost of running lightbulb A continuously for 30.0 days, assuming the electric company sells its product at $0.110 per kWh.

61. W One wire in a high-voltage transmission line carries 1 000 A starting at 700 kV for a distance of 100 mi. If the resistance in the wire is 0.500 Ω/mi, what is the power loss due to the resistance of the wire?

62. An experiment is conducted to measure the electrical resistivity of Nichrome in the form of wires with different lengths and cross-sectional areas. For one set of measurements, a student uses 30-gauge wire, which has a cross-sectional area of $7.30 \times 10^{-8}\ \text{m}^2$. The student measures the potential difference across the wire and the current in the wire with a voltmeter and an ammeter, respectively. (a) For each set of measurements given in the table taken on wires of three different lengths, calculate the resistance of the wires and the corresponding values of the resistivity. (b) What is the average value of the resistivity? (c) Explain how this value compares with the value given in Table 27.2.

L (m)	ΔV (V)	I (A)	R (Ω)	ρ (Ω · m)
0.540	5.22	0.72		
1.028	5.82	0.414		
1.543	5.94	0.281		

63. A charge Q is placed on a capacitor of capacitance C. The capacitor is connected into the circuit shown in Figure P27.63, with an open switch, a resistor, and an initially uncharged capacitor of capacitance $3C$. The

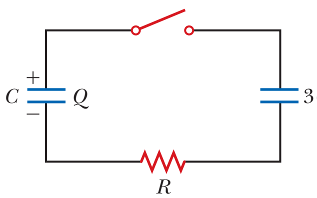


Figure P27.63

switch is then closed, and the circuit comes to equilibrium. In terms of Q and C, find (a) the final potential difference between the plates of each capacitor, (b) the charge on each capacitor, and (c) the final energy stored in each capacitor. (d) Find the internal energy appearing in the resistor.

64. **Review.** An office worker uses an immersion heater to warm 250 g of water in a light, covered, insulated cup from 20.0°C to 100°C in 4.00 min. The heater is a Nichrome resistance wire connected to a 120-V power supply. Assume the wire is at 100°C throughout the 4.00-min time interval. (a) Specify a relationship between a diameter and a length that the wire can have. (b) Can it be made from less than 0.500 cm^3 of Nichrome?

65. An x-ray tube used for cancer therapy operates at 4.00 MV with electrons constituting a beam current of 25.0 mA striking a metal target. Nearly all the power in the beam is transferred to a stream of water flowing through holes drilled in the target. What rate of flow, in kilograms per second, is needed if the rise in temperature of the water is not to exceed 50.0°C?

66. AMT M An all-electric car (not a hybrid) is designed to run from a bank of 12.0-V batteries with total energy storage of 2.00×10^7 J. If the electric motor draws 8.00 kW as the car moves at a steady speed of 20.0 m/s, (a) what is the current delivered to the motor? (b) How far can the car travel before it is "out of juice"?

67. A straight, cylindrical wire lying along the x axis has a length of 0.500 m and a diameter of 0.200 mm. It is made of a material described by Ohm's law with a resistivity of $\rho = 4.00 \times 10^{-8}\ \Omega \cdot$ m. Assume a potential of 4.00 V is maintained at the left end of the wire at $x = 0$. Also assume $V = 0$ at $x = 0.500$ m. Find (a) the magnitude and direction of the electric field in the wire, (b) the resistance of the wire, (c) the magnitude and direction of the electric current in the wire, and (d) the current density in the wire. (e) Show that $E = \rho J$.

68. A straight, cylindrical wire lying along the x axis has a length L and a diameter d. It is made of a material described by Ohm's law with a resistivity ρ. Assume potential V is maintained at the left end of the wire at $x = 0$. Also assume the potential is zero at $x = L$. In terms of L, d, V, ρ, and physical constants, derive expressions for (a) the magnitude and direction of the electric field in the wire, (b) the resistance of the wire, (c) the magnitude and direction of the electric current in the wire, and (d) the current density in the wire. (e) Show that $E = \rho J$.

69. W An electric utility company supplies a customer's house from the main power lines (120 V) with two copper wires, each of which is 50.0 m long and has a resistance of 0.108 Ω per 300 m. (a) Find the potential difference at the customer's house for a load current of 110 A. For this load current, find (b) the power delivered to the customer and (c) the rate at which internal energy is produced in the copper wires.

70. The strain in a wire can be monitored and computed by measuring the resistance of the wire. Let L_i represent the original length of the wire, A_i its original cross-sectional area, $R_i = \rho L_i/A_i$ the original resistance between its ends, and $\delta = \Delta L/L_i = (L - L_i)/L_i$ the strain resulting from the application of tension. Assume the resistivity and the volume of the wire do not change as the wire stretches. (a) Show that the resistance between the ends of the wire under strain is given by $R = R_i(1 + 2\delta + \delta^2)$. (b) If the assumptions are precisely true, is this result exact or approximate? Explain your answer.

71. An oceanographer is studying how the ion concentration in seawater depends on depth. She makes a measurement by lowering into the water a pair of concentric metallic cylinders (Fig. P27.71) at the end of a cable and taking data to determine the resistance between these electrodes as a function of depth. The water between the two cylinders forms a cylindrical shell of inner radius r_a, outer radius r_b, and length L much larger than r_b. The scientist applies a potential difference ΔV between the inner and outer surfaces, producing an outward radial current I. Let ρ represent the resistivity of the water. (a) Find the resistance of the water between the cylinders in terms of L, ρ, r_a, and r_b. (b) Express the resistivity of the water in terms of the measured quantities L, r_a, r_b, ΔV, and I.

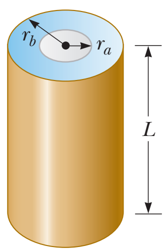


Figure P27.71

72. *Why is the following situation impossible?* An inquisitive physics student takes a 100-W incandescent lightbulb out of its socket and measures its resistance with an ohmmeter. He measures a value of 10.5 Ω. He is able to connect an ammeter to the lightbulb socket to correctly measure the current drawn by the bulb while operating. Inserting the bulb back into the socket and operating the bulb from a 120-V source, he measures the current to be 11.4 A.

73. The temperature coefficients of resistivity α in Table 27.2 are based on a reference temperature T_0 of 20.0°C. Suppose the coefficients were given the symbol α' and were based on a T_0 of 0°C. What would the coefficient α' for silver be? *Note:* The coefficient α satisfies $\rho = \rho_0[1 + \alpha(T - T_0)]$, where ρ_0 is the resistivity of the material at $T_0 = 20.0$°C. The coefficient α' must satisfy the expression $\rho = \rho_0'[1 + \alpha' T]$, where ρ_0' is the resistivity of the material at 0°C.

74. A close analogy exists between the flow of energy by heat because of a temperature difference (see Section 20.7) and the flow of electric charge because of a potential difference. In a metal, energy dQ and electrical charge dq are both transported by free electrons. Consequently, a good electrical conductor is usually a good thermal conductor as well. Consider a thin conducting slab of thickness dx, area A, and electrical conductivity σ, with a potential difference dV between opposite faces. (a) Show that the current $I = dq/dt$ is given by the equation on the left:

Charge conduction	Thermal conduction
$\dfrac{dq}{dt} = \sigma A \left\lvert \dfrac{dV}{dx} \right\rvert$	$\dfrac{dQ}{dt} = kA \left\lvert \dfrac{dT}{dx} \right\rvert$

In the analogous thermal conduction equation on the right (Eq. 20.15), the rate dQ/dt of energy flow by heat (in SI units of joules per second) is due to a temperature gradient dT/dx in a material of thermal conductivity k. (b) State analogous rules relating the direction of the electric current to the change in potential and relating the direction of energy flow to the change in temperature.

75. **Review.** When a straight wire is warmed, its resistance is given by $R = R_0[1 + \alpha(T - T_0)]$ according to Equation 27.20, where α is the temperature coefficient of resistivity. This expression needs to be modified if we include the change in dimensions of the wire due to thermal expansion. For a copper wire of radius 0.100 0 mm and length 2.000 m, find its resistance at 100.0°C, including the effects of both thermal expansion and temperature variation of resistivity. Assume the coefficients are known to four significant figures.

76. **Review.** When a straight wire is warmed, its resistance is given by $R = R_0[1 + \alpha(T - T_0)]$ according to Equation 27.20, where α is the temperature coefficient of resistivity. This expression needs to be modified if we include the change in dimensions of the wire due to thermal expansion. Find a more precise expression for the resistance, one that includes the effects of changes in the dimensions of the wire when it is warmed. Your final expression should be in terms of R_0, T, T_0, the temperature coefficient of resistivity α, and the coefficient of linear expansion α'.

77. **Review.** A parallel-plate capacitor consists of square plates of edge length ℓ that are separated by a distance d, where $d << \ell$. A potential difference ΔV is maintained between the plates. A material of dielectric constant κ fills half the space between the plates. The dielectric slab is withdrawn from the capacitor as shown in Figure P27.77. (a) Find the capacitance when the left edge of the dielectric is at a distance x from the center of the capacitor. (b) If the dielectric is removed at a constant speed v, what is the current in the circuit as the dielectric is being withdrawn?

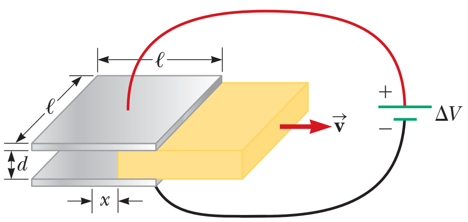


Figure P27.77

78. The dielectric material between the plates of a parallel-plate capacitor always has some nonzero conductivity σ. Let A represent the area of each plate and d the distance between them. Let κ represent the dielectric constant of the material. (a) Show that the resistance R and the capacitance C of the capacitor are related by

$$RC = \frac{\kappa \epsilon_0}{\sigma}$$

(b) Find the resistance between the plates of a 14.0-nF capacitor with a fused quartz dielectric.

79. Gold is the most ductile of all metals. For example, one gram of gold can be drawn into a wire 2.40 km long. The density of gold is 19.3×10^3 kg/m^3, and its resistivity is $2.44 \times 10^{-8}\ \Omega \cdot$ m. What is the resistance of such a wire at 20.0°C?

80. The current–voltage characteristic curve for a semiconductor diode as a function of temperature T is given by

$$I = I_0(e^{e\,\Delta V/k_B T} - 1)$$

Here the first symbol e represents Euler's number, the base of natural logarithms. The second e is the magnitude of the electron charge, the k_B stands for Boltzmann's constant, and T is the absolute temperature. (a) Set up a spreadsheet to calculate I and $R = \Delta V/I$ for $\Delta V = 0.400$ V to 0.600 V in increments of 0.005 V. Assume $I_0 = 1.00$ nA. (b) Plot R versus ΔV for $T = 280$ K, 300 K, and 320 K.

81. The potential difference across the filament of a lightbulb is maintained at a constant value while equilibrium temperature is being reached. The steady-state current in the bulb is only one-tenth of the current drawn by the bulb when it is first turned on. If the temperature coefficient of resistivity for the bulb at 20.0°C is 0.004 50 $(°\text{C})^{-1}$ and the resistance increases linearly with increasing temperature, what is the final operating temperature of the filament?

Challenge Problems

82. A more general definition of the temperature coefficient of resistivity is

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}$$

where ρ is the resistivity at temperature T. (a) Assuming α is constant, show that

$$\rho = \rho_0 e^{\alpha(T - T_0)}$$

where ρ_0 is the resistivity at temperature T_0. (b) Using the series expansion $e^x \approx 1 + x$ for $x << 1$, show that the resistivity is given approximately by the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \quad \text{for } \alpha(T - T_0) << 1$$

83. A spherical shell with inner radius r_a and outer radius r_b is formed from a material of resistivity ρ. It carries current radially, with uniform density in all directions. Show that its resistance is

$$R = \frac{\rho}{4\pi}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

84. Material with uniform resistivity ρ is formed into a wedge as shown in Figure P27.84. Show that the resistance between face A and face B of this wedge is

$$R = \rho \frac{L}{w(y_2 - y_1)} \ln \frac{y_2}{y_1}$$

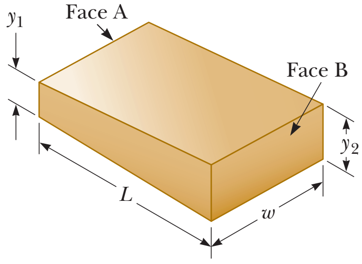


Figure P27.84

85. A material of resistivity ρ is formed into the shape of a truncated cone of height h as shown in Figure P27.85. The bottom end has radius b, and the top end has radius a. Assume the current is distributed uniformly over any circular cross section of the cone so that the current density does not depend on radial position. (The current density does vary with position along the axis of the cone.) Show that the resistance between the two ends is

$$R = \frac{\rho}{\pi}\left(\frac{h}{ab}\right)$$

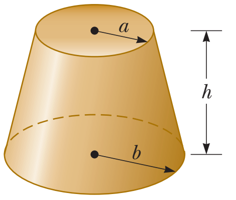
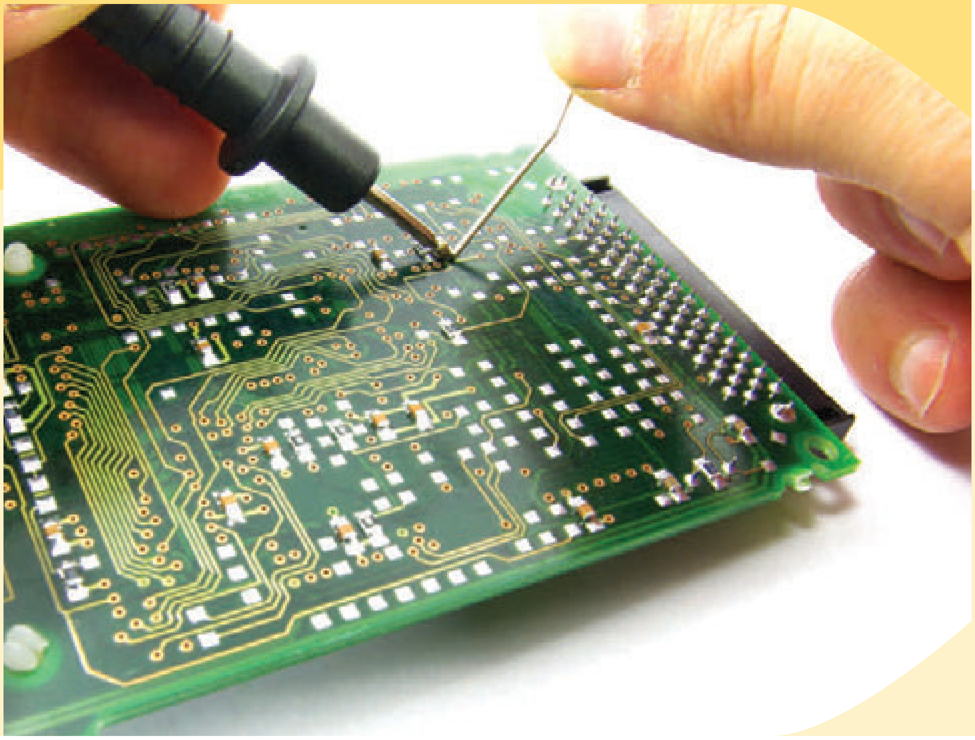


Figure P27.85

Direct-Current Circuits

CHAPTER 28



A technician repairs a connection on a circuit board from a computer. In our lives today, we use various items containing electric circuits, including many with circuit boards much smaller than the board shown in the photograph. These include handheld game players, cell phones, and digital cameras. In this chapter, we study simple types of circuits and learn how to analyze them. *(Trombax/Shutterstock.com)*

In this chapter, we analyze simple electric circuits that contain batteries, resistors, and capacitors in various combinations. Some circuits contain resistors that can be combined using simple rules. The analysis of more complicated circuits is simplified using *Kirchhoff's rules*, which follow from the laws of conservation of energy and conservation of electric charge for isolated systems. Most of the circuits analyzed are assumed to be in *steady state*, which means that currents in the circuit are constant in magnitude and direction. A current that is constant in direction is called a *direct current* (DC). We will study *alternating current* (AC), in which the current changes direction periodically, in Chapter 33. Finally, we discuss electrical circuits in the home.

28.1 Electromotive Force

In Section 27.6, we discussed a circuit in which a battery produces a current. We will generally use a battery as a source of energy for circuits in our discussion. Because the potential difference at the battery terminals is constant in a particular circuit, the current in the circuit is constant in magnitude and direction and is called **direct current.** A battery is called either a *source of electromotive force* or, more commonly, a *source of emf.* (The phrase *electromotive force* is an unfortunate historical term, describing not a force, but rather a potential difference in volts.) The **emf $\boldsymbol{\mathcal{E}}$** of a battery is **the maximum possible voltage the battery can provide between its terminals.** You can think of a source of emf as a "charge pump." When an electric potential difference exists between two points, the source moves charges "uphill" from the lower potential to the higher.

We shall generally assume the connecting wires in a circuit have no resistance. The positive terminal of a battery is at a higher potential than the negative terminal.

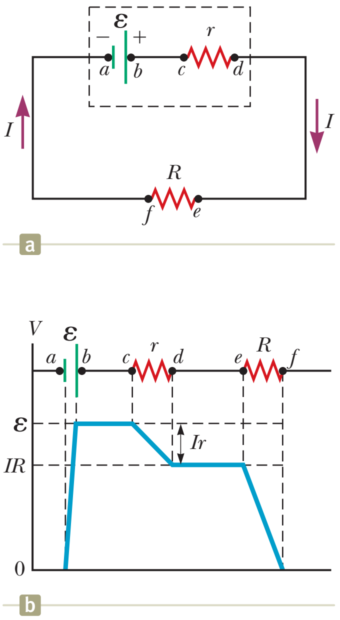


Figure 28.1 (a) Circuit diagram of a source of emf $\mathcal{E}$ (in this case, a battery), of internal resistance r, connected to an external resistor of resistance R. (b) Graphical representation showing how the electric potential changes as the circuit in (a) is traversed clockwise.

Because a real battery is made of matter, there is resistance to the flow of charge within the battery. This resistance is called **internal resistance** r. For an idealized battery with zero internal resistance, the potential difference across the battery (called its *terminal voltage*) equals its emf. For a real battery, however, the terminal voltage is *not* equal to the emf for a battery in a circuit in which there is a current. To understand why, consider the circuit diagram in Figure 28.1a. We model the battery as shown in the diagram; it is represented by the dashed rectangle containing an ideal, resistance-free emf $\mathcal{E}$ in series with an internal resistance r. A resistor of resistance R is connected across the terminals of the battery. Now imagine moving through the battery from a to d and measuring the electric potential at various locations. Passing from the negative terminal to the positive terminal, the potential increases by an amount $\mathcal{E}$. As we move through the resistance r, however, the potential *decreases* by an amount Ir, where I is the current in the circuit. Therefore, the terminal voltage of the battery $\Delta V = V_d - V_a$ is

$$\Delta V = \mathcal{E} - Ir \tag{28.1}$$

From this expression, notice that $\mathcal{E}$ is equivalent to the **open-circuit voltage,** that is, the terminal voltage when the current is zero. The emf is the voltage labeled on a battery; for example, the emf of a D cell is 1.5 V. The actual potential difference between a battery's terminals depends on the current in the battery as described by Equation 28.1. Figure 28.1b is a graphical representation of the changes in electric potential as the circuit is traversed in the clockwise direction.

Figure 28.1a shows that the terminal voltage ΔV must equal the potential difference across the external resistance R, often called the **load resistance.** The load resistor might be a simple resistive circuit element as in Figure 28.1a, or it could be the resistance of some electrical device (such as a toaster, electric heater, or lightbulb) connected to the battery (or, in the case of household devices, to the wall outlet). The resistor represents a *load* on the battery because the battery must supply energy to operate the device containing the resistance. The potential difference across the load resistance is $\Delta V = IR$. Combining this expression with Equation 28.1, we see that

$$\mathcal{E} = IR + Ir \tag{28.2}$$

Figure 28.1a shows a graphical representation of this equation. Solving for the current gives

$$I = \frac{\mathcal{E}}{R + r} \tag{28.3}$$

Equation 28.3 shows that the current in this simple circuit depends on both the load resistance R external to the battery and the internal resistance r. If R is much greater than r, as it is in many real-world circuits, we can neglect r.

Multiplying Equation 28.2 by the current I in the circuit gives

$$I\mathcal{E} = I^2R + I^2r \tag{28.4}$$

Equation 28.4 indicates that because power $P = I\,\Delta V$ (see Eq. 27.21), the total power output $I\mathcal{E}$ associated with the emf of the battery is delivered to the external load resistance in the amount I^2R and to the internal resistance in the amount I^2r.

Pitfall Prevention 28.1

What Is Constant in a Battery? It is a common misconception that a battery is a source of constant current. Equation 28.3 shows that is not true. The current in the circuit depends on the resistance R connected to the battery. It is also not true that a battery is a source of constant terminal voltage as shown by Equation 28.1. **A battery is a source of constant emf.**

Quick Quiz 28.1 To maximize the percentage of the power from the emf of a battery that is delivered to a device external to the battery, what should the internal resistance of the battery be? **(a)** It should be as low as possible. **(b)** It should be as high as possible. **(c)** The percentage does not depend on the internal resistance.

Example 28.1 Terminal Voltage of a Battery

A battery has an emf of 12.0 V and an internal resistance of 0.050 0 Ω. Its terminals are connected to a load resistance of 3.00 Ω.

28.1 continued

(A) Find the current in the circuit and the terminal voltage of the battery.

SOLUTION

Conceptualize Study Figure 28.1a, which shows a circuit consistent with the problem statement. The battery delivers energy to the load resistor.

Categorize This example involves simple calculations from this section, so we categorize it as a substitution problem.

Use Equation 28.3 to find the current in the circuit:

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0 \text{ V}}{3.00\ \Omega + 0.050\,0\ \Omega} = \boxed{3.93 \text{ A}}$$

Use Equation 28.1 to find the terminal voltage:

$$\Delta V = \mathcal{E} - Ir = 12.0 \text{ V} - (3.93 \text{ A})(0.050\,0\ \Omega) = \boxed{11.8 \text{ V}}$$

To check this result, calculate the voltage across the load resistance R:

$$\Delta V = IR = (3.93 \text{ A})(3.00\ \Omega) = 11.8 \text{ V}$$

(B) Calculate the power delivered to the load resistor, the power delivered to the internal resistance of the battery, and the power delivered by the battery.

SOLUTION

Use Equation 27.22 to find the power delivered to the load resistor:

$$P_R = I^2R = (3.93 \text{ A})^2(3.00\ \Omega) = \boxed{46.3 \text{ W}}$$

Find the power delivered to the internal resistance:

$$P_r = I^2r = (3.93 \text{ A})^2(0.050\,0\ \Omega) = \boxed{0.772 \text{ W}}$$

Find the power delivered by the battery by adding these quantities:

$$P = P_R + P_r = 46.3 \text{ W} + 0.772 \text{ W} = \boxed{47.1 \text{ W}}$$

WHAT IF? As a battery ages, its internal resistance increases. Suppose the internal resistance of this battery rises to 2.00 Ω toward the end of its useful life. How does that alter the battery's ability to deliver energy?

Answer Let's connect the same 3.00-Ω load resistor to the battery.

Find the new current in the battery:

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0 \text{ V}}{3.00\ \Omega + 2.00\ \Omega} = 2.40 \text{ A}$$

Find the new terminal voltage:

$$\Delta V = \mathcal{E} - Ir = 12.0 \text{ V} - (2.40 \text{ A})(2.00\ \Omega) = 7.2 \text{ V}$$

Find the new powers delivered to the load resistor and internal resistance:

$$P_R = I^2R = (2.40 \text{ A})^2(3.00\ \Omega) = 17.3 \text{ W}$$
$$P_r = I^2r = (2.40 \text{ A})^2(2.00\ \Omega) = 11.5 \text{ W}$$

In this situation, the terminal voltage is only 60% of the emf. Notice that 40% of the power from the battery is delivered to the internal resistance when r is 2.00 Ω. When r is 0.050 0 Ω as in part (B), this percentage is only 1.6%. Consequently, even though the emf remains fixed, the increasing internal resistance of the battery significantly reduces the battery's ability to deliver energy to an external load.

Example 28.2 Matching the Load

Find the load resistance R for which the maximum power is delivered to the load resistance in Figure 28.1a.

SOLUTION

Conceptualize Think about varying the load resistance in Figure 28.1a and the effect on the power delivered to the load resistance. When R is large, there is very little current, so the power I^2R delivered to the load resistor is small.

continued

28.2 continued

When R is small, let's say $R << r$, the current is large and the power delivered to the internal resistance is $I^2r >> I^2R$. Therefore, the power delivered to the load resistor is small compared to that delivered to the internal resistance. For some intermediate value of the resistance R, the power must maximize.

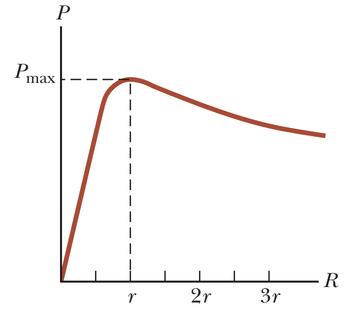


Figure 28.2 (Example 28.2) Graph of the power P delivered by a battery to a load resistor of resistance R as a function of R.

Categorize We categorize this example as an analysis problem because we must undertake a procedure to maximize the power. The circuit is the same as that in Example 28.1. The load resistance R in this case, however, is a variable.

Analyze Find the power delivered to the load resistance using Equation 27.22, with I given by Equation 28.3:

$$(1)\quad P = I^2R = \frac{\mathcal{E}^2R}{(R+r)^2}$$

Differentiate the power with respect to the load resistance R and set the derivative equal to zero to maximize the power:

$$\frac{dP}{dR} = \frac{d}{dR}\left[\frac{\mathcal{E}^2R}{(R+r)^2}\right] = \frac{d}{dR}[\mathcal{E}^2R(R+r)^{-2}] = 0$$

$$[\mathcal{E}^2(R+r)^{-2}] + [\mathcal{E}^2R(-2)(R+r)^{-3}] = 0$$

$$\frac{\mathcal{E}^2(R+r)}{(R+r)^3} - \frac{2\mathcal{E}^2R}{(R+r)^3} = \frac{\mathcal{E}^2(r-R)}{(R+r)^3} = 0$$

Solve for R:

$$R = r$$

Finalize To check this result, let's plot P versus R as in Figure 28.2. The graph shows that P reaches a maximum value at $R = r$. Equation (1) shows that this maximum value is $P_{max} = \mathcal{E}^2/4r$.

28.2 Resistors in Series and Parallel

When two or more resistors are connected together as are the incandescent lightbulbs in Figure 28.3a, they are said to be in a **series combination.** Figure 28.3b is the circuit diagram for the lightbulbs, shown as resistors, and the battery. What if you wanted to replace the series combination with a single resistor that would draw the same current from the battery? What would be its value? In a series connection, if an amount of charge Q exits resistor R_1, charge Q must also enter the second resistor R_2. Otherwise, charge would accumulate on the wire between the resistors. Therefore, the same amount of charge passes through both resistors in a given time interval and the currents are the same in both resistors:

$$I = I_1 = I_2$$

where I is the current leaving the battery, I_1 is the current in resistor R_1, and I_2 is the current in resistor R_2.

The potential difference applied across the series combination of resistors divides between the resistors. In Figure 28.3b, because the voltage drop[1] from a to b equals I_1R_1 and the voltage drop from b to c equals I_2R_2, the voltage drop from a to c is

$$\Delta V = \Delta V_1 + \Delta V_2 = I_1R_1 + I_2R_2$$

The potential difference across the battery is also applied to the **equivalent resistance** R_{eq} in Figure 28.3c:

$$\Delta V = IR_{eq}$$

[1]The term *voltage drop* is synonymous with a decrease in electric potential across a resistor. It is often used by individuals working with electric circuits.

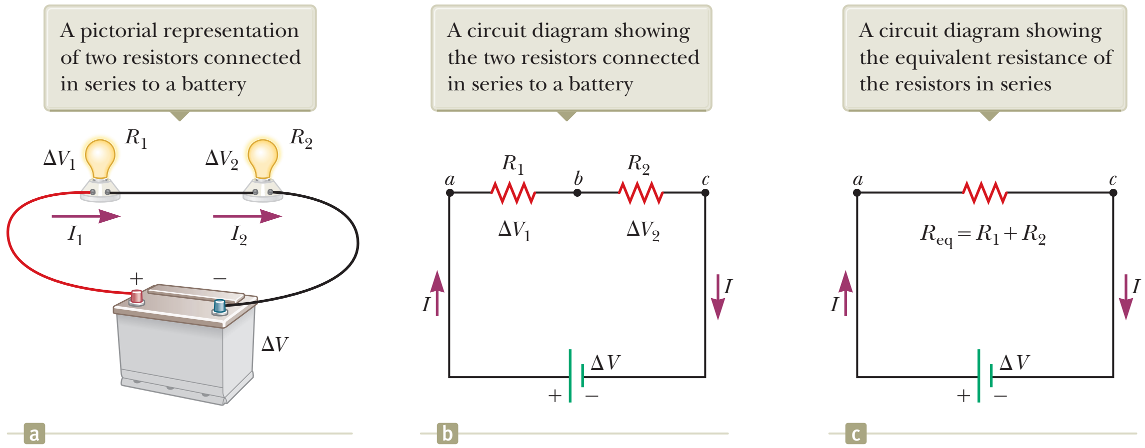


Figure 28.3 Two lightbulbs with resistances R_1 and R_2 connected in series. All three diagrams are equivalent.

where the equivalent resistance has the same effect on the circuit as the series combination because it results in the same current I in the battery. Combining these equations for ΔV gives

$$IR_{eq} = I_1R_1 + I_2R_2 \quad \rightarrow \quad R_{eq} = R_1 + R_2 \tag{28.5}$$

where we have canceled the currents I, I_1, and I_2 because they are all the same. We see that we can replace the two resistors in series with a single equivalent resistance whose value is the *sum* of the individual resistances.

The equivalent resistance of three or more resistors connected in series is

$$\boxed{R_{eq} = R_1 + R_2 + R_3 + \cdots} \tag{28.6}$$

◀ **The equivalent resistance of a series combination of resistors**

This relationship indicates that the equivalent resistance of a series combination of resistors is the numerical sum of the individual resistances and is always greater than any individual resistance.

Looking back at Equation 28.3, we see that the denominator of the right-hand side is the simple algebraic sum of the external and internal resistances. That is consistent with the internal and external resistances being in series in Figure 28.1a.

If the filament of one lightbulb in Figure 28.3 were to fail, the circuit would no longer be complete (resulting in an open-circuit condition) and the second lightbulb would also go out. This fact is a general feature of a series circuit: if one device in the series creates an open circuit, all devices are inoperative.

Quick Quiz 28.2 With the switch in the circuit of Figure 28.4a closed, there is no current in R_2 because the current has an alternate zero-resistance path through the switch. There is current in R_1, and this current is measured with the ammeter (a device for measuring current) at the bottom of the circuit. If the switch is opened (Fig. 28.4b), there is current in R_2. What happens to the reading on the ammeter when the switch is opened? **(a)** The reading goes up. **(b)** The reading goes down. **(c)** The reading does not change.

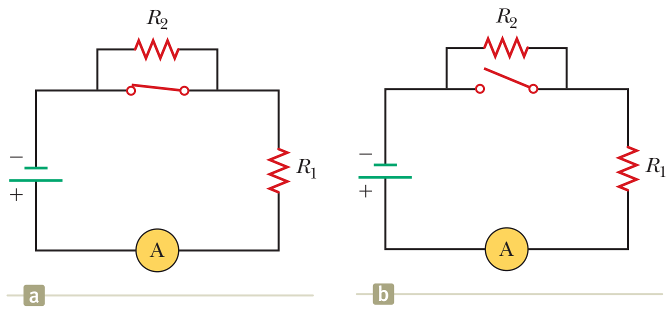


Figure 28.4 (Quick Quiz 28.2) What happens when the switch is opened?

Pitfall Prevention 28.2

Lightbulbs Don't Burn We will describe the end of the life of an incandescent lightbulb by saying *the filament fails* rather than by saying the lightbulb "burns out." The word *burn* suggests a combustion process, which is not what occurs in a lightbulb. The failure of a lightbulb results from the slow sublimation of tungsten from the very hot filament over the life of the lightbulb. The filament eventually becomes very thin because of this process. The mechanical stress from a sudden temperature increase when the lightbulb is turned on causes the thin filament to break.

Pitfall Prevention 28.3

Local and Global Changes A local change in one part of a circuit may result in a global change throughout the circuit. For example, if a single resistor is changed in a circuit containing several resistors and batteries, the currents in all resistors and batteries, the terminal voltages of all batteries, and the voltages across all resistors may change as a result.

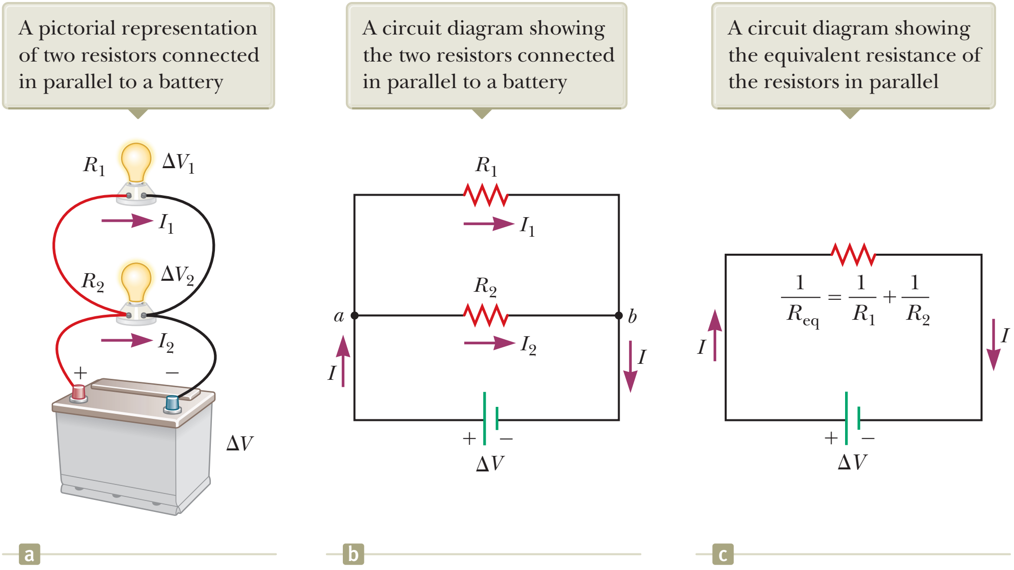


Figure 28.5 Two lightbulbs with resistances R_1 and R_2 connected in parallel. All three diagrams are equivalent.

Pitfall Prevention 28.4

Current Does Not Take the Path of Least Resistance You may have heard the phrase "current takes the path of least resistance" (or similar wording) in reference to a parallel combination of current paths such that there are two or more paths for the current to take. Such wording is incorrect. The current takes *all* paths. Those paths with lower resistance have larger currents, but even very high resistance paths carry *some* of the current. In theory, if current has a choice between a zero-resistance path and a finite resistance path, all the current takes the path of zero resistance; a path with zero resistance, however, is an idealization.

Now consider two resistors in a **parallel combination** as shown in Figure 28.5. As with the series combination, what is the value of the single resistor that could replace the combination and draw the same current from the battery? Notice that both resistors are connected directly across the terminals of the battery. Therefore, the potential differences across the resistors are the same:

$$\Delta V = \Delta V_1 = \Delta V_2$$

where ΔV is the terminal voltage of the battery.

When charges reach point a in Figure 28.5b, they split into two parts, with some going toward R_1 and the rest going toward R_2. A **junction** is any such point in a circuit where a current can split. This split results in less current in each individual resistor than the current leaving the battery. Because electric charge is conserved, the current I that enters point a must equal the total current leaving that point:

$$I = I_1 + I_2 = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2}$$

where I_1 is the current in R_1 and I_2 is the current in R_2.

The current in the **equivalent resistance** R_{eq} in Figure 28.5c is

$$I = \frac{\Delta V}{R_{eq}}$$

where the equivalent resistance has the same effect on the circuit as the two resistors in parallel; that is, the equivalent resistance draws the same current I from the battery. Combining these equations for I, we see that the equivalent resistance of two resistors in parallel is given by

$$\frac{\Delta V}{R_{eq}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{28.7}$$

where we have canceled ΔV, ΔV_1, and ΔV_2 because they are all the same.

An extension of this analysis to three or more resistors in parallel gives

The equivalent resistance of a parallel combination of resistors ▶

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \tag{28.8}$$

This expression shows that the inverse of the equivalent resistance of two or more resistors in a parallel combination is equal to the sum of the inverses of the individual resistances. Furthermore, the equivalent resistance is always less than the smallest resistance in the group.

Household circuits are always wired such that the appliances are connected in parallel. Each device operates independently of the others so that if one is switched off, the others remain on. In addition, in this type of connection, all the devices operate on the same voltage.

Let's consider two examples of practical applications of series and parallel circuits. Figure 28.6 illustrates how a three-way incandescent lightbulb is constructed to provide three levels of light intensity.[2] The socket of the lamp is equipped with a three-way switch for selecting different light intensities. The lightbulb contains two filaments. When the lamp is connected to a 120-V source, one filament receives 100 W of power and the other receives 75 W. The three light intensities are made possible by applying the 120 V to one filament alone, to the other filament alone, or to the two filaments in parallel. When switch S_1 is closed and switch S_2 is opened, current exists only in the 75-W filament. When switch S_1 is open and switch S_2 is closed, current exists only in the 100-W filament. When both switches are closed, current exists in both filaments and the total power is 175 W.

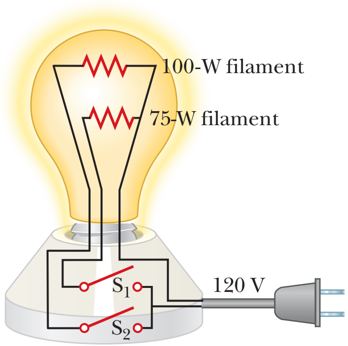


Figure 28.6 A three-way incandescent lightbulb.

If the filaments were connected in series and one of them were to break, no charges could pass through the lightbulb and it would not glow, regardless of the switch position. If, however, the filaments were connected in parallel and one of them (for example, the 75-W filament) were to break, the lightbulb would continue to glow in two of the switch positions because current exists in the other (100-W) filament.

As a second example, consider strings of incandescent lights that are used for many ornamental purposes such as decorating Christmas trees. Over the years, both parallel and series connections have been used for strings of lights. Because series-wired lightbulbs operate with less energy per bulb and at a lower temperature, they are safer than parallel-wired lightbulbs for indoor Christmas-tree use. If, however, the filament of a single lightbulb in a series-wired string were to fail (or if the lightbulb were removed from its socket), all the lights on the string would go out. The popularity of series-wired light strings diminished because troubleshooting a failed lightbulb is a tedious, time-consuming chore that involves trial-and-error substitution of a good lightbulb in each socket along the string until the defective one is found.

In a parallel-wired string, each lightbulb operates at 120 V. By design, the lightbulbs are brighter and hotter than those on a series-wired string. As a result, they are inherently more dangerous (more likely to start a fire, for instance), but if one lightbulb in a parallel-wired string fails or is removed, the rest of the lightbulbs continue to glow.

To prevent the failure of one lightbulb from causing the entire string to go out, a new design was developed for so-called miniature lights wired in series. When the filament breaks in one of these miniature lightbulbs, the break in the filament represents the largest resistance in the series, much larger than that of the intact filaments. As a result, most of the applied 120 V appears across the lightbulb with the broken filament. Inside the lightbulb, a small jumper loop covered by an insulating material is wrapped around the filament leads. When the filament fails and 120 V appears across the lightbulb, an arc burns the insulation on the jumper and connects the filament leads. This connection now completes the circuit through the lightbulb even though its filament is no longer active (Fig. 28.7, page 840).

When a lightbulb fails, the resistance across its terminals is reduced to almost zero because of the alternate jumper connection mentioned in the preceding paragraph. All the other lightbulbs not only stay on, but they glow more brightly because

[2]The three-way lightbulb and other household devices actually operate on alternating current (AC), to be introduced in Chapter 33.

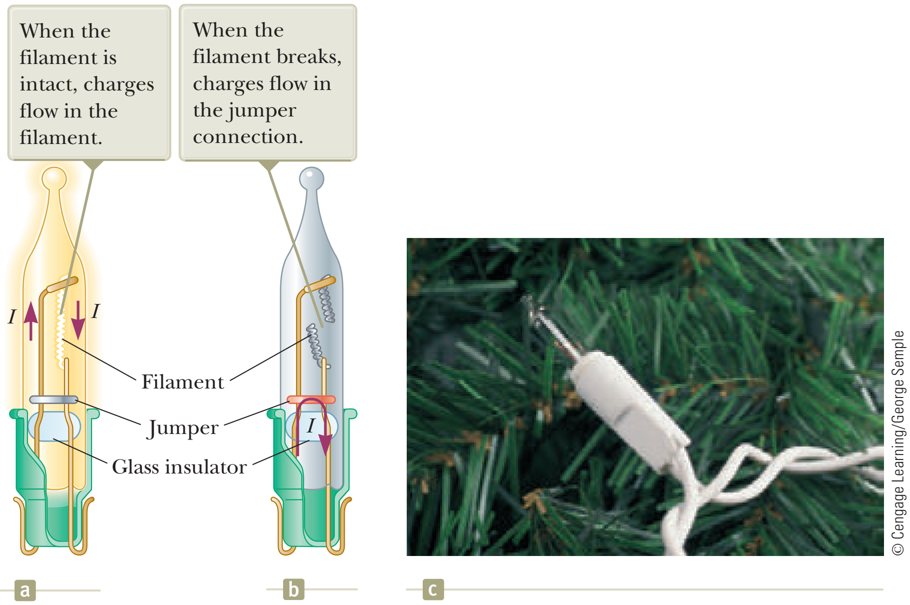


Figure 28.7 (a) Schematic diagram of a modern "miniature" incandescent holiday lightbulb, with a jumper connection to provide a current path if the filament breaks. (b) A holiday lightbulb with a broken filament. (c) A Christmas-tree lightbulb.

the total resistance of the string is reduced and consequently the current in each remaining lightbulb increases. Each lightbulb operates at a slightly higher temperature than before. As more lightbulbs fail, the current keeps rising, the filament of each remaining lightbulb operates at a higher temperature, and the lifetime of the lightbulb is reduced. For this reason, you should check for failed (nonglowing) lightbulbs in such a series-wired string and replace them as soon as possible, thereby maximizing the lifetimes of all the lightbulbs.

Quick Quiz 28.3 With the switch in the circuit of Figure 28.8a open, there is no current in R_2. There is current in R_1, however, and it is measured with the ammeter at the right side of the circuit. If the switch is closed (Fig. 28.8b), there is current in R_2. What happens to the reading on the ammeter when the switch is closed? **(a)** The reading increases. **(b)** The reading decreases. **(c)** The reading does not change.

Quick Quiz 28.4 Consider the following choices: (a) increases, (b) decreases, (c) remains the same. From these choices, choose the best answer for the following situations. **(i)** In Figure 28.3, a third resistor is added in series with the first two. What happens to the current in the battery? **(ii)** What happens to the terminal voltage of the battery? **(iii)** In Figure 28.5, a third resistor is added in parallel with the first two. What happens to the current in the battery? **(iv)** What happens to the terminal voltage of the battery?

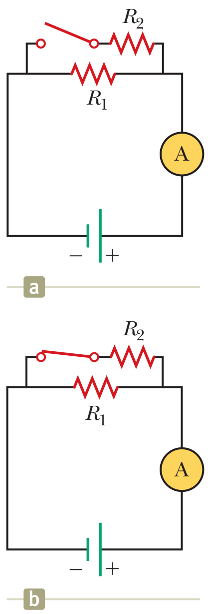


Figure 28.8 (Quick Quiz 28.3) What happens when the switch is closed?

Conceptual Example 28.3 Landscape Lights

A homeowner wishes to install low-voltage landscape lighting in his back yard. To save money, he purchases inexpensive 18-gauge cable, which has a relatively high resistance per unit length. This cable consists of two side-by-side wires separated by insulation, like the cord on an appliance. He runs a 200-foot length of this cable from the power supply to the farthest point at which he plans to position a light fixture. He attaches light fixtures across the two wires on the cable at 10-foot intervals so that the light fixtures are in parallel. Because of the cable's resistance, the brightness of the lightbulbs in the fixtures is not as desired. Which of the following problems does the homeowner have? (a) All the lightbulbs glow equally less brightly than they would if lower-resistance cable had been used. (b) The brightness of the lightbulbs decreases as you move farther from the power supply.

28.3 continued

SOLUTION

A circuit diagram for the system appears in Figure 28.9. The horizontal resistors with letter subscripts (such as R_A) represent the resistance of the wires in the cable between the light fixtures, and the vertical resistors with number subscripts (such as R_1) represent the resistance of the light fixtures themselves. Part of the terminal voltage of the power supply is dropped across resistors R_A and R_B. Therefore, the voltage across light fixture R_1 is less than the terminal voltage. There is a further voltage drop across resistors R_C and R_D. Consequently, the voltage across light fixture R_2 is smaller than that across R_1. This pattern continues down the line of light fixtures, so the correct choice is (b). Each successive light fixture has a smaller voltage across it and glows less brightly than the one before.

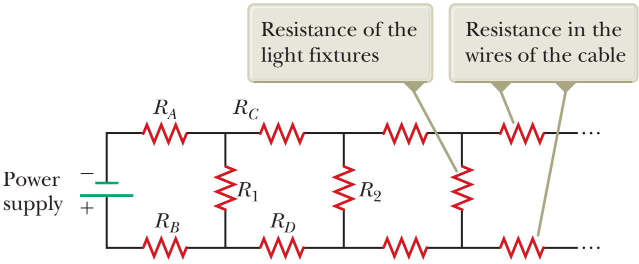


Figure 28.9 (Conceptual Example 28.3) The circuit diagram for a set of landscape light fixtures connected in parallel across the two wires of a two-wire cable.

Example 28.4 Find the Equivalent Resistance

Four resistors are connected as shown in Figure 28.10a.

(A) Find the equivalent resistance between points a and c.

SOLUTION

Conceptualize Imagine charges flowing into and through this combination from the left. All charges must pass from a to b through the first two resistors, but the charges split at b into two different paths when encountering the combination of the 6.0-Ω and the 3.0-Ω resistors.

Categorize Because of the simple nature of the combination of resistors in Figure 28.10, we categorize this example as one for which we can use the rules for series and parallel combinations of resistors.

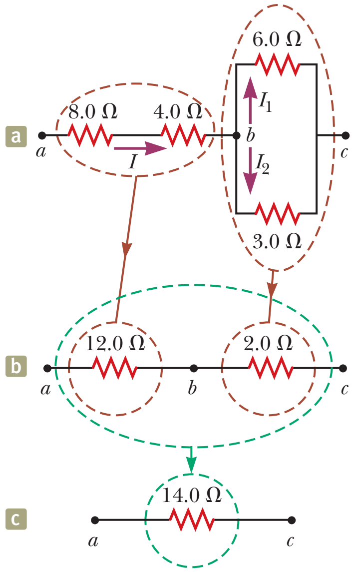


Figure 28.10 (Example 28.4) The original network of resistors is reduced to a single equivalent resistance.

Analyze The combination of resistors can be reduced in steps as shown in Figure 28.10.

Find the equivalent resistance between a and b of the 8.0-Ω and 4.0-Ω resistors, which are in series (left-hand red-brown circles):

$$R_{eq} = 8.0\ \Omega + 4.0\ \Omega = 12.0\ \Omega$$

Find the equivalent resistance between b and c of the 6.0-Ω and 3.0-Ω resistors, which are in parallel (right-hand red-brown circles):

$$\frac{1}{R_{eq}} = \frac{1}{6.0\ \Omega} + \frac{1}{3.0\ \Omega} = \frac{3}{6.0\ \Omega}$$

$$R_{eq} = \frac{6.0\ \Omega}{3} = 2.0\ \Omega$$

The circuit of equivalent resistances now looks like Figure 28.10b. The 12.0-Ω and 2.0-Ω resistors are in series (green circles). Find the equivalent resistance from a to c:

$$R_{eq} = 12.0\ \Omega + 2.0\ \Omega = \boxed{14.0\ \Omega}$$

This resistance is that of the single equivalent resistor in Figure 28.10c.

(B) What is the current in each resistor if a potential difference of 42 V is maintained between a and c?

continued

28.4 continued

SOLUTION

The currents in the 8.0-Ω and 4.0-Ω resistors are the same because they are in series. In addition, they carry the same current that would exist in the 14.0-Ω equivalent resistor subject to the 42-V potential difference.

Use Equation 27.7 ($R = \Delta V/I$) and the result from part (A) to find the current in the 8.0-Ω and 4.0-Ω resistors:

$$I = \frac{\Delta V_{ac}}{R_{eq}} = \frac{42\text{ V}}{14.0\ \Omega} = \boxed{3.0\text{ A}}$$

Set the voltages across the resistors in parallel in Figure 28.10a equal to find a relationship between the currents:

$$\Delta V_1 = \Delta V_2 \;\rightarrow\; (6.0\ \Omega)I_1 = (3.0\ \Omega)I_2 \;\rightarrow\; I_2 = 2I_1$$

Use $I_1 + I_2 = 3.0$ A to find I_1:

$$I_1 + I_2 = 3.0\text{ A} \;\rightarrow\; I_1 + 2I_1 = 3.0\text{ A} \;\rightarrow\; I_1 = \boxed{1.0\text{ A}}$$

Find I_2:

$$I_2 = 2I_1 = 2(1.0\text{ A}) = \boxed{2.0\text{ A}}$$

Finalize As a final check of our results, note that $\Delta V_{bc} = (6.0\ \Omega)I_1 = (3.0\ \Omega)I_2 = 6.0$ V and $\Delta V_{ab} = (12.0\ \Omega)I = 36$ V; therefore, $\Delta V_{ac} = \Delta V_{ab} + \Delta V_{bc} = 42$ V, as it must.

Example 28.5 Three Resistors in Parallel

Three resistors are connected in parallel as shown in Figure 28.11a. A potential difference of 18.0 V is maintained between points *a* and *b*.

(A) Calculate the equivalent resistance of the circuit.

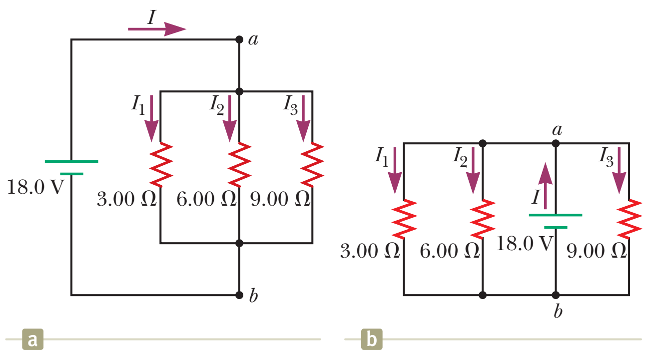


Figure 28.11 (Example 28.5) (a) Three resistors connected in parallel. The voltage across each resistor is 18.0 V. (b) Another circuit with three resistors and a battery. Is it equivalent to the circuit in (a)?

SOLUTION

Conceptualize Figure 28.11a shows that we are dealing with a simple parallel combination of three resistors. Notice that the current I splits into three currents I_1, I_2, and I_3 in the three resistors.

Categorize This problem can be solved with rules developed in this section, so we categorize it as a substitution problem. Because the three resistors are connected in parallel, we can use the rule for resistors in parallel, Equation 28.8, to evaluate the equivalent resistance.

Use Equation 28.8 to find R_{eq}:

$$\frac{1}{R_{eq}} = \frac{1}{3.00\ \Omega} + \frac{1}{6.00\ \Omega} + \frac{1}{9.00\ \Omega} = \frac{11}{18.0\ \Omega}$$

$$R_{eq} = \frac{18.0\ \Omega}{11} = \boxed{1.64\ \Omega}$$

(B) Find the current in each resistor.

SOLUTION

The potential difference across each resistor is 18.0 V. Apply the relationship $\Delta V = IR$ to find the currents:

$$I_1 = \frac{\Delta V}{R_1} = \frac{18.0\text{ V}}{3.00\ \Omega} = \boxed{6.00\text{ A}}$$

$$I_2 = \frac{\Delta V}{R_2} = \frac{18.0\text{ V}}{6.00\ \Omega} = \boxed{3.00\text{ A}}$$

$$I_3 = \frac{\Delta V}{R_3} = \frac{18.0\text{ V}}{9.00\ \Omega} = \boxed{2.00\text{ A}}$$

(C) Calculate the power delivered to each resistor and the total power delivered to the combination of resistors.

28.5 continued

SOLUTION

Apply the relationship $P = I^2R$ to each resistor using the currents calculated in part (B):

$$3.00\text{-}\Omega\text{: } P_1 = I_1{}^2R_1 = (6.00\text{ A})^2(3.00\ \Omega) = \boxed{108\text{ W}}$$

$$6.00\text{-}\Omega\text{: } P_2 = I_2{}^2R_2 = (3.00\text{ A})^2(6.00\ \Omega) = \boxed{54\text{ W}}$$

$$9.00\text{-}\Omega\text{: } P_3 = I_3{}^2R_3 = (2.00\text{ A})^2(9.00\ \Omega) = \boxed{36\text{ W}}$$

These results show that the smallest resistor receives the most power. Summing the three quantities gives a total power of $\boxed{198\text{ W}}$. We could have calculated this final result from part (A) by considering the equivalent resistance as follows: $P = (\Delta V)^2/R_{eq} = (18.0\text{ V})^2/1.64\ \Omega = 198\text{ W}$.

WHAT IF? What if the circuit were as shown in Figure 28.11b instead of as in Figure 28.11a? How would that affect the calculation?

Answer There would be no effect on the calculation. The physical placement of the battery is not important. Only the electrical arrangement is important. In Figure 28.11b, the battery still maintains a potential difference of 18.0 V between points *a* and *b*, so the two circuits in the figure are electrically identical.

28.3 Kirchhoff's Rules

As we saw in the preceding section, combinations of resistors can be simplified and analyzed using the expression $\Delta V = IR$ and the rules for series and parallel combinations of resistors. Very often, however, it is not possible to reduce a circuit to a single loop using these rules. The procedure for analyzing more complex circuits is made possible by using the following two principles, called **Kirchhoff's rules.**

1. **Junction rule.** At any junction, the sum of the currents must equal zero:

$$\sum_{\text{junction}} I = 0 \qquad \textbf{(28.9)}$$

2. **Loop rule.** The sum of the potential differences across all elements around any closed circuit loop must be zero:

$$\sum_{\text{closed loop}} \Delta V = 0 \qquad \textbf{(28.10)}$$

Kirchhoff's first rule is a statement of conservation of electric charge. All charges that enter a given point in a circuit must leave that point because charge cannot build up or disappear at a point. Currents directed into the junction are entered into the sum in the junction rule as $+I$, whereas currents directed out of a junction are entered as $-I$. Applying this rule to the junction in Figure 28.12a gives

$$I_1 - I_2 - I_3 = 0$$

Figure 28.12b represents a mechanical analog of this situation, in which water flows through a branched pipe having no leaks. Because water does not build up anywhere in the pipe, the flow rate into the pipe on the left equals the total flow rate out of the two branches on the right.

Kirchhoff's second rule follows from the law of conservation of energy for an isolated system. Let's imagine moving a charge around a closed loop of a circuit. When the charge returns to the starting point, the charge–circuit system must have the same total energy as it had before the charge was moved. The sum of the increases in energy as the charge passes through some circuit elements must equal the sum of the decreases in energy as it passes through other elements. The potential energy of the system decreases whenever the charge moves through a potential drop $-IR$ across a resistor or whenever it moves in the reverse direction through a

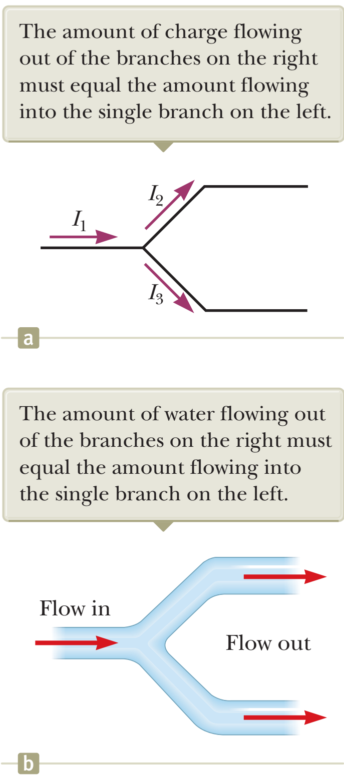


Figure 28.12 (a) Kirchhoff's junction rule. (b) A mechanical analog of the junction rule.

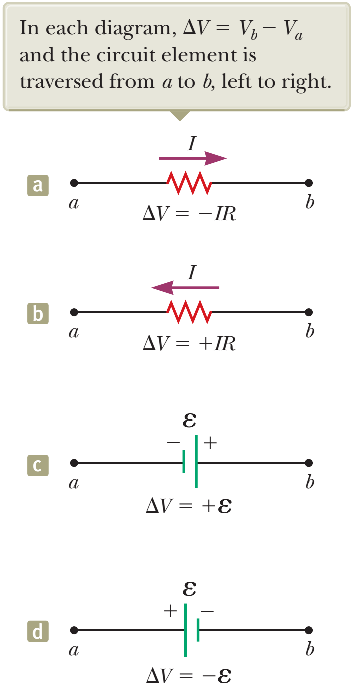


Figure 28.13 Rules for determining the signs of the potential differences across a resistor and a battery. (The battery is assumed to have no internal resistance.)

source of emf. The potential energy increases whenever the charge passes through a battery from the negative terminal to the positive terminal.

When applying Kirchhoff's second rule, imagine *traveling* around the loop and consider changes in *electric potential* rather than the changes in *potential energy* described in the preceding paragraph. Imagine traveling through the circuit elements in Figure 28.13 toward the right. The following sign conventions apply when using the second rule:

- Charges move from the high-potential end of a resistor toward the low-potential end, so if a resistor is traversed in the direction of the current, the potential difference ΔV across the resistor is $-IR$ (Fig. 28.13a).
- If a resistor is traversed in the direction *opposite* the current, the potential difference ΔV across the resistor is $+IR$ (Fig. 28.13b).
- If a source of emf (assumed to have zero internal resistance) is traversed in the direction of the emf (from negative to positive), the potential difference ΔV is $+\mathcal{E}$ (Fig. 28.13c).
- If a source of emf (assumed to have zero internal resistance) is traversed in the direction opposite the emf (from positive to negative), the potential difference ΔV is $-\mathcal{E}$ (Fig. 28.13d).

There are limits on the number of times you can usefully apply Kirchhoff's rules in analyzing a circuit. You can use the junction rule as often as you need as long as you include in it a current that has not been used in a preceding junction-rule equation. In general, the number of times you can use the junction rule is one fewer than the number of junction points in the circuit. You can apply the loop rule as often as needed as long as a new circuit element (resistor or battery) or a new current appears in each new equation. In general, to solve a particular circuit problem, the number of independent equations you need to obtain from the two rules equals the number of unknown currents.

Complex networks containing many loops and junctions generate a great number of independent linear equations and a correspondingly great number of unknowns. Such situations can be handled formally through the use of matrix algebra. Computer software can also be used to solve for the unknowns.

The following examples illustrate how to use Kirchhoff's rules. In all cases, it is assumed the circuits have reached steady-state conditions; in other words, the currents in the various branches are constant. Any capacitor acts as an open branch in a circuit; that is, the current in the branch containing the capacitor is zero under steady-state conditions.

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Gustav Kirchhoff
German Physicist (1824–1887)
Kirchhoff, a professor at Heidelberg, and Robert Bunsen invented the spectroscope and founded the science of spectroscopy, which we shall study in Chapter 42. They discovered the elements cesium and rubidium and invented astronomical spectroscopy.

Problem-Solving Strategy Kirchhoff's Rules

The following procedure is recommended for solving problems that involve circuits that cannot be reduced by the rules for combining resistors in series or parallel.

1. Conceptualize. Study the circuit diagram and make sure you recognize all elements in the circuit. Identify the polarity of each battery and try to imagine the directions in which the current would exist in the batteries.

2. Categorize. Determine whether the circuit can be reduced by means of combining series and parallel resistors. If so, use the techniques of Section 28.2. If not, apply Kirchhoff's rules according to the *Analyze* step below.

3. Analyze. Assign labels to all known quantities and symbols to all unknown quantities. You must assign *directions* to the currents in each part of the circuit. Although the assignment of current directions is arbitrary, you must adhere *rigorously* to the directions you assign when you apply Kirchhoff's rules.

Apply the junction rule (Kirchhoff's first rule) to all junctions in the circuit except one. Now apply the loop rule (Kirchhoff's second rule) to as many loops in

Problem-Solving Strategy continued

the circuit as are needed to obtain, in combination with the equations from the junction rule, as many equations as there are unknowns. To apply this rule, you must choose a direction in which to travel around the loop (either clockwise or counterclockwise) and correctly identify the change in potential as you cross each element. Be careful with signs!

Solve the equations simultaneously for the unknown quantities.

4. Finalize. Check your numerical answers for consistency. Do not be alarmed if any of the resulting currents have a negative value. That only means you have guessed the direction of that current incorrectly, but *its magnitude will be correct.*

Example 28.6 A Single-Loop Circuit

A single-loop circuit contains two resistors and two batteries as shown in Figure 28.14. (Neglect the internal resistances of the batteries.) Find the current in the circuit.

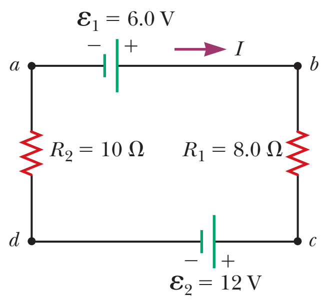

Figure 28.14 (Example 28.6) A series circuit containing two batteries and two resistors, where the polarities of the batteries are in opposition.

SOLUTION

Conceptualize Figure 28.14 shows the polarities of the batteries and a guess at the direction of the current. The 12-V battery is the stronger of the two, so the current should be counterclockwise. Therefore, we expect our guess for the direction of the current to be wrong, but we will continue and see how this incorrect guess is represented by our final answer.

Categorize We do not need Kirchhoff's rules to analyze this simple circuit, but let's use them anyway simply to see how they are applied. There are no junctions in this single-loop circuit; therefore, the current is the same in all elements.

Analyze Let's assume the current is clockwise as shown in Figure 28.14. Traversing the circuit in the clockwise direction, starting at a, we see that $a \rightarrow b$ represents a potential difference of $+\mathcal{E}_1$, $b \rightarrow c$ represents a potential difference of $-IR_1$, $c \rightarrow d$ represents a potential difference of $-\mathcal{E}_2$, and $d \rightarrow a$ represents a potential difference of $-IR_2$.

Apply Kirchhoff's loop rule to the single loop in the circuit:

$$\sum \Delta V = 0 \quad \rightarrow \quad \mathcal{E}_1 - IR_1 - \mathcal{E}_2 - IR_2 = 0$$

Solve for I and use the values given in Figure 28.14:

$$(1) \quad I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2} = \frac{6.0\text{ V} - 12\text{ V}}{8.0\ \Omega + 10\ \Omega} = \boxed{-0.33\text{ A}}$$

Finalize The negative sign for I indicates that the direction of the current is opposite the assumed direction. The emfs in the numerator subtract because the batteries in Figure 28.14 have opposite polarities. The resistances in the denominator add because the two resistors are in series.

WHAT IF? What if the polarity of the 12.0-V battery were reversed? How would that affect the circuit?

Answer Although we could repeat the Kirchhoff's rules calculation, let's instead examine Equation (1) and modify it accordingly. Because the polarities of the two batteries are now in the same direction, the signs of $\mathcal{E}_1$ and $\mathcal{E}_2$ are the same and Equation (1) becomes

$$I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{R_1 + R_2} = \frac{6.0\text{ V} + 12\text{ V}}{8.0\ \Omega + 10\ \Omega} = 1.0\text{ A}$$

Example 28.7 A Multiloop Circuit

Find the currents I_1, I_2, and I_3 in the circuit shown in Figure 28.15 on page 846.

continued

▸ 28.7 continued

SOLUTION

Conceptualize Imagine physically rearranging the circuit while keeping it electrically the same. Can you rearrange it so that it consists of simple series or parallel combinations of resistors? You should find that you cannot. (If the 10.0-V battery were removed and replaced by a wire from b to the 6.0-Ω resistor, the circuit would consist of only series and parallel combinations.)

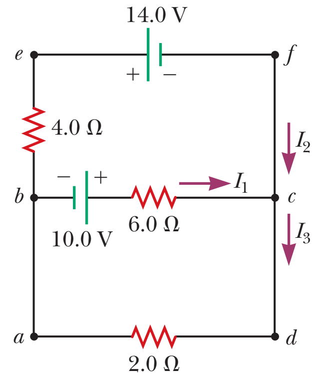

Figure 28.15 (Example 28.7) A circuit containing different branches.

Categorize We cannot simplify the circuit by the rules associated with combining resistances in series and in parallel. Therefore, this problem is one in which we must use Kirchhoff's rules.

Analyze We arbitrarily choose the directions of the currents as labeled in Figure 28.15.

Apply Kirchhoff's junction rule to junction c:

$$(1)\quad I_1 + I_2 - I_3 = 0$$

We now have one equation with three unknowns: I_1, I_2, and I_3. There are three loops in the circuit: *abcda*, *befcb*, and *aefda*. We need only two loop equations to determine the unknown currents. (The third equation would give no new information.) Let's choose to traverse these loops in the clockwise direction. Apply Kirchhoff's loop rule to loops *abcda* and *befcb*:

$$abcda\text{: } (2)\quad 10.0\text{ V} - (6.0\ \Omega)I_1 - (2.0\ \Omega)I_3 = 0$$

$$befcb\text{: } -(4.0\ \Omega)I_2 - 14.0\text{ V} + (6.0\ \Omega)I_1 - 10.0\text{ V} = 0$$

$$(3)\quad -24.0\text{ V} + (6.0\ \Omega)I_1 - (4.0\ \Omega)I_2 = 0$$

Solve Equation (1) for I_3 and substitute into Equation (2):

$$10.0\text{ V} - (6.0\ \Omega)I_1 - (2.0\ \Omega)(I_1 + I_2) = 0$$

$$(4)\quad 10.0\text{ V} - (8.0\ \Omega)I_1 - (2.0\ \Omega)I_2 = 0$$

Multiply each term in Equation (3) by 4 and each term in Equation (4) by 3:

$$(5)\quad -96.0\text{ V} + (24.0\ \Omega)I_1 - (16.0\ \Omega)I_2 = 0$$

$$(6)\quad 30.0\text{ V} - (24.0\ \Omega)I_1 - (6.0\ \Omega)I_2 = 0$$

Add Equation (6) to Equation (5) to eliminate I_1 and find I_2:

$$-66.0\text{ V} - (22.0\ \Omega)I_2 = 0$$

$$I_2 = \boxed{-3.0\text{ A}}$$

Use this value of I_2 in Equation (3) to find I_1:

$$-24.0\text{ V} + (6.0\ \Omega)I_1 - (4.0\ \Omega)(-3.0\text{ A}) = 0$$

$$-24.0\text{ V} + (6.0\ \Omega)I_1 + 12.0\text{ V} = 0$$

$$I_1 = \boxed{2.0\text{ A}}$$

Use Equation (1) to find I_3:

$$I_3 = I_1 + I_2 = 2.0\text{ A} - 3.0\text{ A} = \boxed{-1.0\text{ A}}$$

Finalize Because our values for I_2 and I_3 are negative, the directions of these currents are opposite those indicated in Figure 28.15. The numerical values for the currents are correct. Despite the incorrect direction, we *must* continue to use these negative values in subsequent calculations because our equations were established with our original choice of direction. What would have happened had we left the current directions as labeled in Figure 28.15 but traversed the loops in the opposite direction?

28.4 *RC* Circuits

So far, we have analyzed direct-current circuits in which the current is constant. In DC circuits containing capacitors, the current is always in the same direction but may vary in magnitude at different times. A circuit containing a series combination of a resistor and a capacitor is called an ***RC* circuit.**

Charging a Capacitor

Figure 28.16 shows a simple series *RC* circuit. Let's assume the capacitor in this circuit is initially uncharged. There is no current while the switch is open (Fig. 28.16a). If the switch is thrown to position *a* at $t = 0$ (Fig. 28.16b), however, charge begins to flow, setting up a current in the circuit, and the capacitor begins to charge.[3] Notice that during charging, charges do not jump across the capacitor plates because the gap between the plates represents an open circuit. Instead, charge is transferred between each plate and its connecting wires due to the electric field established in the wires by the battery until the capacitor is fully charged. As the plates are being charged, the potential difference across the capacitor increases. The value of the maximum charge on the plates depends on the voltage of the battery. Once the maximum charge is reached, the current in the circuit is zero because the potential difference across the capacitor matches that supplied by the battery.

To analyze this circuit quantitatively, let's apply Kirchhoff's loop rule to the circuit after the switch is thrown to position *a*. Traversing the loop in Figure 28.16b clockwise gives

$$\mathcal{E} - \frac{q}{C} - iR = 0 \qquad \textbf{(28.11)}$$

where q/C is the potential difference across the capacitor and iR is the potential difference across the resistor. We have used the sign conventions discussed earlier for the signs on $\mathcal{E}$ and iR. The capacitor is traversed in the direction from the positive plate to the negative plate, which represents a decrease in potential. Therefore, we use a negative sign for this potential difference in Equation 28.11. Note that lowercase q and i are *instantaneous* values that depend on time (as opposed to steady-state values) as the capacitor is being charged.

We can use Equation 28.11 to find the initial current I_i in the circuit and the maximum charge Q_{max} on the capacitor. At the instant the switch is thrown to position *a* ($t = 0$), the charge on the capacitor is zero. Equation 28.11 shows that the initial current I_i in the circuit is a maximum and is given by

$$I_i = \frac{\mathcal{E}}{R} \quad (\text{current at } t = 0) \qquad \textbf{(28.12)}$$

At this time, the potential difference from the battery terminals appears entirely across the resistor. Later, when the capacitor is charged to its maximum value Q_{max}, charges cease to flow, the current in the circuit is zero, and the potential difference from the battery terminals appears entirely across the capacitor. Substituting $i = 0$ into Equation 28.11 gives the maximum charge on the capacitor:

$$Q_{\text{max}} = C\mathcal{E} \quad (\text{maximum charge}) \qquad \textbf{(28.13)}$$

To determine analytical expressions for the time dependence of the charge and current, we must solve Equation 28.11, a single equation containing two variables q and i. The current in all parts of the series circuit must be the same. Therefore, the current in the resistance R must be the same as the current between each capacitor plate and the wire connected to it. This current is equal to the time rate of change of the charge on the capacitor plates. Therefore, we substitute $i = dq/dt$ into Equation 28.11 and rearrange the equation:

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

To find an expression for q, we solve this separable differential equation as follows. First combine the terms on the right-hand side:

$$\frac{dq}{dt} = \frac{C\mathcal{E}}{RC} - \frac{q}{RC} = -\frac{q - C\mathcal{E}}{RC}$$

[3]In previous discussions of capacitors, we assumed a steady-state situation, in which no current was present in any branch of the circuit containing a capacitor. Now we are considering the case *before* the steady-state condition is realized; in this situation, charges are moving and a current exists in the wires connected to the capacitor.

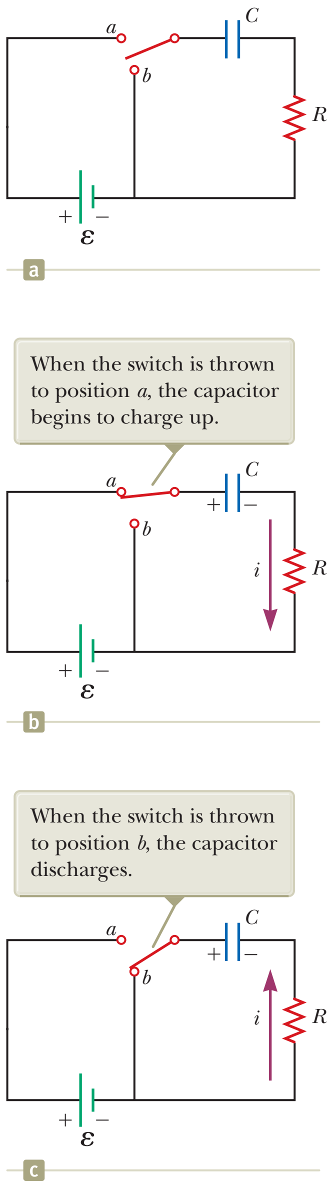

Figure 28.16 A capacitor in series with a resistor, switch, and battery.

Multiply this equation by dt and divide by $q - C\mathcal{E}$:

$$\frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}\,dt$$

Integrate this expression, using $q = 0$ at $t = 0$:

$$\int_0^q \frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

From the definition of the natural logarithm, we can write this expression as

Charge as a function of time for a capacitor being charged

$$q(t) = C\mathcal{E}(1 - e^{-t/RC}) = Q_{\max}(1 - e^{-t/RC}) \qquad \textbf{(28.14)}$$

where e is the base of the natural logarithm and we have made the substitution from Equation 28.13.

We can find an expression for the charging current by differentiating Equation 28.14 with respect to time. Using $i = dq/dt$, we find that

Current as a function of time for a capacitor being charged

$$i(t) = \frac{\mathcal{E}}{R}e^{-t/RC} \qquad \textbf{(28.15)}$$

Plots of capacitor charge and circuit current versus time are shown in Figure 28.17. Notice that the charge is zero at $t = 0$ and approaches the maximum value $C\mathcal{E}$ as $t \to \infty$. The current has its maximum value $I_i = \mathcal{E}/R$ at $t = 0$ and decays exponentially to zero as $t \to \infty$. The quantity RC, which appears in the exponents of Equations 28.14 and 28.15, is called the **time constant τ** of the circuit:

$$\tau = RC \qquad \textbf{(28.16)}$$

The time constant represents the time interval during which the current decreases to $1/e$ of its initial value; that is, after a time interval τ, the current decreases to $i = e^{-1}I_i = 0.368I_i$. After a time interval 2τ, the current decreases to $i = e^{-2}I_i = 0.135I_i$, and so forth. Likewise, in a time interval τ, the charge increases from zero to $C\mathcal{E}[1 - e^{-1}] = 0.632C\mathcal{E}$.

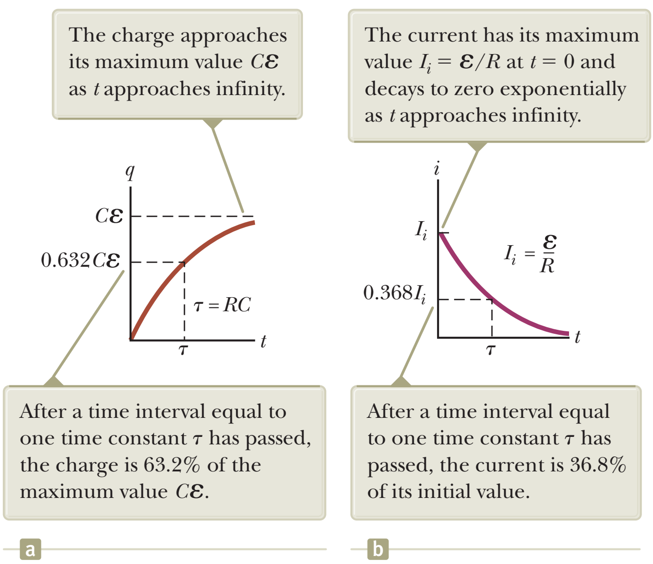

Figure 28.17 (a) Plot of capacitor charge versus time for the circuit shown in Figure 28.16b. (b) Plot of current versus time for the circuit shown in Figure 28.16b.

The following dimensional analysis shows that τ has units of time:

$$[\tau] = [RC] = \left[\left(\frac{\Delta V}{I}\right)\left(\frac{Q}{\Delta V}\right)\right] = \left[\frac{Q}{Q/\Delta t}\right] = [\Delta t] = \text{T}$$

Because $\tau = RC$ has units of time, the combination t/RC is dimensionless, as it must be to be an exponent of e in Equations 28.14 and 28.15.

The energy supplied by the battery during the time interval required to fully charge the capacitor is $Q_{\max}\boldsymbol{\mathcal{E}} = C\boldsymbol{\mathcal{E}}^2$. After the capacitor is fully charged, the energy stored in the capacitor is $\frac{1}{2}Q_{\max}\boldsymbol{\mathcal{E}} = \frac{1}{2}C\boldsymbol{\mathcal{E}}^2$, which is only half the energy output of the battery. It is left as a problem (Problem 68) to show that the remaining half of the energy supplied by the battery appears as internal energy in the resistor.

Discharging a Capacitor

Imagine that the capacitor in Figure 28.16b is completely charged. An initial potential difference Q_i/C exists across the capacitor, and there is zero potential difference across the resistor because $i = 0$. If the switch is now thrown to position b at $t = 0$ (Fig. 28.16c), the capacitor begins to discharge through the resistor. At some time t during the discharge, the current in the circuit is i and the charge on the capacitor is q. The circuit in Figure 28.16c is the same as the circuit in Figure 28.16b except for the absence of the battery. Therefore, we eliminate the emf $\boldsymbol{\mathcal{E}}$ from Equation 28.11 to obtain the appropriate loop equation for the circuit in Figure 28.16c:

$$-\frac{q}{C} - iR = 0 \qquad \textbf{(28.17)}$$

When we substitute $i = dq/dt$ into this expression, it becomes

$$-R\frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC}\,dt$$

Integrating this expression using $q = Q_i$ at $t = 0$ gives

$$\int_{Q_i}^{q}\frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q}{Q_i}\right) = -\frac{t}{RC}$$

$$q(t) = Q_i e^{-t/RC} \qquad \textbf{(28.18)}$$

◀ Charge as a function of time for a discharging capacitor

Differentiating Equation 28.18 with respect to time gives the instantaneous current as a function of time:

$$i(t) = -\frac{Q_i}{RC}e^{-t/RC} \qquad \textbf{(28.19)}$$

◀ Current as a function of time for a discharging capacitor

where $Q_i/RC = I_i$ is the initial current. The negative sign indicates that as the capacitor discharges, the current direction is opposite its direction when the capacitor was being charged. (Compare the current directions in Figs. 28.16b and 28.16c.) Both the charge on the capacitor and the current decay exponentially at a rate characterized by the time constant $\tau = RC$.

Quick Quiz 28.5 Consider the circuit in Figure 28.18 and assume the battery has no internal resistance. **(i)** Just after the switch is closed, what is the current in the battery? (a) 0 (b) $\boldsymbol{\mathcal{E}}/2R$ (c) $2\boldsymbol{\mathcal{E}}/R$ (d) $\boldsymbol{\mathcal{E}}/R$ (e) impossible to determine **(ii)** After a very long time, what is the current in the battery? Choose from the same choices.

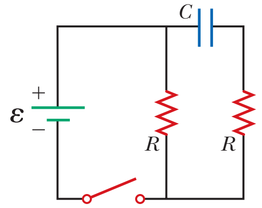

Figure 28.18 (Quick Quiz 28.5) How does the current vary after the switch is closed?

Conceptual Example 28.8 Intermittent Windshield Wipers

Many automobiles are equipped with windshield wipers that can operate intermittently during a light rainfall. How does the operation of such wipers depend on the charging and discharging of a capacitor?

SOLUTION

The wipers are part of an RC circuit whose time constant can be varied by selecting different values of R through a multiposition switch. As the voltage across the capacitor increases, the capacitor reaches a point at which it discharges and triggers the wipers. The circuit then begins another charging cycle. The time interval between the individual sweeps of the wipers is determined by the value of the time constant.

Example 28.9 Charging a Capacitor in an *RC* Circuit

An uncharged capacitor and a resistor are connected in series to a battery as shown in Figure 28.16, where $\boldsymbol{\mathcal{E}} = 12.0$ V, $C = 5.00\ \mu$F, and $R = 8.00 \times 10^5\ \Omega$. The switch is thrown to position a. Find the time constant of the circuit, the maximum charge on the capacitor, the maximum current in the circuit, and the charge and current as functions of time.

SOLUTION

Conceptualize Study Figure 28.16 and imagine throwing the switch to position a as shown in Figure 28.16b. Upon doing so, the capacitor begins to charge.

Categorize We evaluate our results using equations developed in this section, so we categorize this example as a substitution problem.

Evaluate the time constant of the circuit from Equation 28.16:

$$\tau = RC = (8.00 \times 10^5\ \Omega)(5.00 \times 10^{-6}\ \text{F}) = \boxed{4.00\ \text{s}}$$

Evaluate the maximum charge on the capacitor from Equation 28.13:

$$Q_{\text{max}} = C\boldsymbol{\mathcal{E}} = (5.00\ \mu\text{F})(12.0\ \text{V}) = \boxed{60.0\ \mu\text{C}}$$

Evaluate the maximum current in the circuit from Equation 28.12:

$$I_i = \frac{\boldsymbol{\mathcal{E}}}{R} = \frac{12.0\ \text{V}}{8.00 \times 10^5\ \Omega} = \boxed{15.0\ \mu\text{A}}$$

Use these values in Equations 28.14 and 28.15 to find the charge and current as functions of time:

$$(1)\quad q(t) = \boxed{60.0(1 - e^{-t/4.00})}$$

$$(2)\quad i(t) = \boxed{15.0e^{-t/4.00}}$$

In Equations (1) and (2), q is in microcoulombs, i is in microamperes, and t is in seconds.

Example 28.10 Discharging a Capacitor in an *RC* Circuit

Consider a capacitor of capacitance C that is being discharged through a resistor of resistance R as shown in Figure 28.16c.

(A) After how many time constants is the charge on the capacitor one-fourth its initial value?

SOLUTION

Conceptualize Study Figure 28.16 and imagine throwing the switch to position b as shown in Figure 28.16c. Upon doing so, the capacitor begins to discharge.

Categorize We categorize the example as one involving a discharging capacitor and use the appropriate equations.

28.10 continued

Analyze Substitute $q(t) = Q_i/4$ into Equation 28.18:

$$\frac{Q_i}{4} = Q_i e^{-t/RC}$$

$$\tfrac{1}{4} = e^{-t/RC}$$

Take the logarithm of both sides of the equation and solve for t:

$$-\ln 4 = -\frac{t}{RC}$$

$$t = RC\ln 4 = 1.39RC = \boxed{1.39\tau}$$

(B) The energy stored in the capacitor decreases with time as the capacitor discharges. After how many time constants is this stored energy one-fourth its initial value?

SOLUTION

Use Equations 26.11 and 28.18 to express the energy stored in the capacitor at any time t:

$$(1) \quad U(t) = \frac{q^2}{2C} = \frac{Q_i^2}{2C}e^{-2t/RC}$$

Substitute $U(t) = \frac{1}{4}(Q_i^2/2C)$ into Equation (1):

$$\tfrac{1}{4}\frac{Q_i^2}{2C} = \frac{Q_i^2}{2C}e^{-2t/RC}$$

$$\tfrac{1}{4} = e^{-2t/RC}$$

Take the logarithm of both sides of the equation and solve for t:

$$-\ln 4 = -\frac{2t}{RC}$$

$$t = \tfrac{1}{2}RC\ln 4 = 0.693RC = \boxed{0.693\tau}$$

Finalize Notice that because the energy depends on the square of the charge, the energy in the capacitor drops more rapidly than the charge on the capacitor.

WHAT IF? What if you want to describe the circuit in terms of the time interval required for the charge to fall to one-half its original value rather than by the time constant τ? That would give a parameter for the circuit called its *half-life* $t_{1/2}$. How is the half-life related to the time constant?

Answer In one half-life, the charge falls from Q_i to $Q_i/2$. Therefore, from Equation 28.18,

$$\frac{Q_i}{2} = Q_i e^{-t_{1/2}/RC} \rightarrow \tfrac{1}{2} = e^{-t_{1/2}/RC}$$

which leads to

$$t_{1/2} = 0.693\tau$$

The concept of half-life will be important to us when we study nuclear decay in Chapter 44. The radioactive decay of an unstable sample behaves in a mathematically similar manner to a discharging capacitor in an *RC* circuit.

Example 28.11 Energy Delivered to a Resistor AM

A 5.00-μF capacitor is charged to a potential difference of 800 V and then discharged through a resistor. How much energy is delivered to the resistor in the time interval required to fully discharge the capacitor?

SOLUTION

Conceptualize In Example 28.10, we considered the energy decrease in a discharging capacitor to a value of one-fourth the initial energy. In this example, the capacitor fully discharges.

Categorize We solve this example using two approaches. The first approach is to model the circuit as an *isolated system* for *energy*. Because energy in an isolated system is conserved, the initial electric potential energy U_E stored in the

continued

28.11 continued

capacitor is transformed into internal energy $E_{int} = E_R$ in the resistor. The second approach is to model the resistor as a *nonisolated system* for *energy*. Energy enters the resistor by electrical transmission from the capacitor, causing an increase in the resistor's internal energy.

Analyze We begin with the isolated system approach.

Write the appropriate reduction of the conservation of energy equation, Equation 8.2:

$$\Delta U + \Delta E_{int} = 0$$

Substitute the initial and final values of the energies:

$$(0 - U_E) + (E_{int} - 0) = 0 \rightarrow E_R = U_E$$

Use Equation 26.11 for the electric potential energy in the capacitor:

$$E_R = \tfrac{1}{2}C\boldsymbol{\mathcal{E}}^2$$

Substitute numerical values:

$$E_R = \tfrac{1}{2}(5.00 \times 10^{-6}\ \text{F})(800\ \text{V})^2 = \boxed{1.60\ \text{J}}$$

The second approach, which is more difficult but perhaps more instructive, is to note that as the capacitor discharges through the resistor, the rate at which energy is delivered to the resistor by electrical transmission is i^2R, where i is the instantaneous current given by Equation 28.19.

Evaluate the energy delivered to the resistor by integrating the power over all time because it takes an infinite time interval for the capacitor to completely discharge:

$$P = \frac{dE}{dt} \rightarrow E_R = \int_0^\infty P\,dt$$

Substitute for the power delivered to the resistor:

$$E_R = \int_0^\infty i^2 R\,dt$$

Substitute for the current from Equation 28.19:

$$E_R = \int_0^\infty \left(-\frac{Q_i}{RC}e^{-t/RC}\right)^2 R\,dt = \frac{Q_i^{\,2}}{RC^2}\int_0^\infty e^{-2t/RC}\,dt = \frac{\boldsymbol{\mathcal{E}}^2}{R}\int_0^\infty e^{-2t/RC}\,dt$$

Substitute the value of the integral, which is $RC/2$ (see Problem 44):

$$E_R = \frac{\boldsymbol{\mathcal{E}}^2}{R}\left(\frac{RC}{2}\right) = \tfrac{1}{2}C\boldsymbol{\mathcal{E}}^2$$

Finalize This result agrees with that obtained using the isolated system approach, as it must. We can use this second approach to find the total energy delivered to the resistor at *any* time after the switch is closed by simply replacing the upper limit in the integral with that specific value of t.

28.5 Household Wiring and Electrical Safety

Many considerations are important in the design of an electrical system of a home that will provide adequate electrical service for the occupants while maximizing their safety. We discuss some aspects of a home electrical system in this section.

Household Wiring

Household circuits represent a practical application of some of the ideas presented in this chapter. In our world of electrical appliances, it is useful to understand the power requirements and limitations of conventional electrical systems and the safety measures that prevent accidents.

In a conventional installation, the utility company distributes electric power to individual homes by means of a pair of wires, with each home connected in parallel to these wires. One wire is called the *live wire*[4] as illustrated in Figure 28.19, and the other is called the *neutral wire*. The neutral wire is grounded; that is, its electric potential is taken to be zero. The potential difference between the live and neutral wires is approximately 120 V. This voltage alternates in time, and the potential of the live wire oscillates relative to ground. Much of what we have learned so far for the constant-emf situation (direct current) can also be applied to the alternating current that power companies supply to businesses and households. (Alternating voltage and current are discussed in Chapter 33.)

To record a household's energy consumption, a meter is connected in series with the live wire entering the house. After the meter, the wire splits so that there are several separate circuits in parallel distributed throughout the house. Each circuit contains a circuit breaker (or, in older installations, a fuse). A circuit breaker is a special switch that opens if the current exceeds the rated value for the circuit breaker. The wire and circuit breaker for each circuit are carefully selected to meet the current requirements for that circuit. If a circuit is to carry currents as large as 30 A, a heavy wire and an appropriate circuit breaker must be selected to handle this current. A circuit used to power only lamps and small appliances often requires only 20 A. Each circuit has its own circuit breaker to provide protection for that part of the entire electrical system of the house.

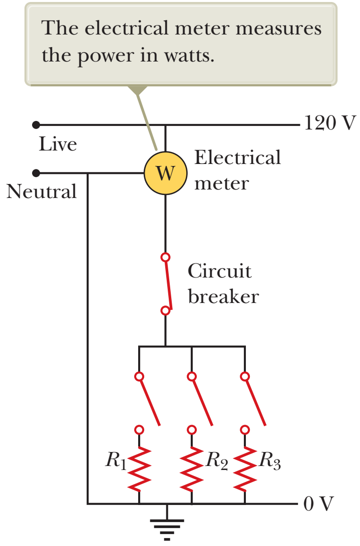

Figure 28.19 Wiring diagram for a household circuit. The resistances represent appliances or other electrical devices that operate with an applied voltage of 120 V.

As an example, consider a circuit in which a toaster oven, a microwave oven, and a coffee maker are connected (corresponding to R_1, R_2, and R_3 in Fig. 28.19). We can calculate the current in each appliance by using the expression $P = I\,\Delta V$. The toaster oven, rated at 1 000 W, draws a current of 1 000 W/120 V = 8.33 A. The microwave oven, rated at 1 300 W, draws 10.8 A, and the coffee maker, rated at 800 W, draws 6.67 A. When the three appliances are operated simultaneously, they draw a total current of 25.8 A. Therefore, the circuit must be wired to handle at least this much current. If the rating of the circuit breaker protecting the circuit is too small—say, 20 A—the breaker will be tripped when the third appliance is turned on, preventing all three appliances from operating. To avoid this situation, the toaster oven and coffee maker can be operated on one 20-A circuit and the microwave oven on a separate 20-A circuit.

Many heavy-duty appliances such as electric ranges and clothes dryers require 240 V for their operation. The power company supplies this voltage by providing a third wire that is 120 V below ground potential (Fig. 28.20). The potential difference between this live wire and the other live wire (which is 120 V above ground potential) is 240 V. An appliance that operates from a 240-V line requires half as much current compared with operating it at 120 V; therefore, smaller wires can be used in the higher-voltage circuit without overheating.

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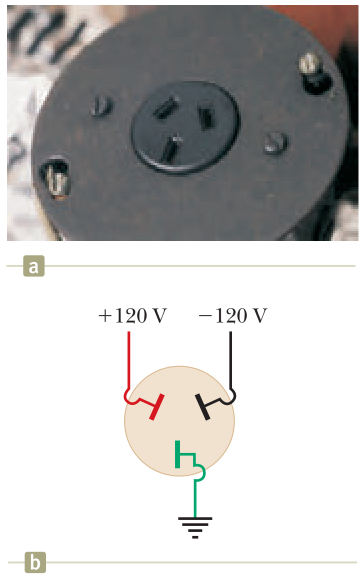


Figure 28.20 (a) An outlet for connection to a 240-V supply. (b) The connections for each of the openings in a 240-V outlet.

Electrical Safety

When the live wire of an electrical outlet is connected directly to ground, the circuit is completed and a *short-circuit condition* exists. A short circuit occurs when almost zero resistance exists between two points at different potentials, and the result is a very large current. When that happens accidentally, a properly operating circuit breaker opens the circuit and no damage is done. A person in contact with ground, however, can be electrocuted by touching the live wire of a frayed cord or other exposed conductor. An exceptionally effective (and dangerous!) ground contact is made when the person either touches a water pipe (normally at ground potential) or stands on the ground with wet feet. The latter situation represents effective ground contact because normal, nondistilled water is a conductor due to the large number of ions associated with impurities. This situation should be avoided at all cost.

[4] *Live wire* is a common expression for a conductor whose electric potential is above or below ground potential.

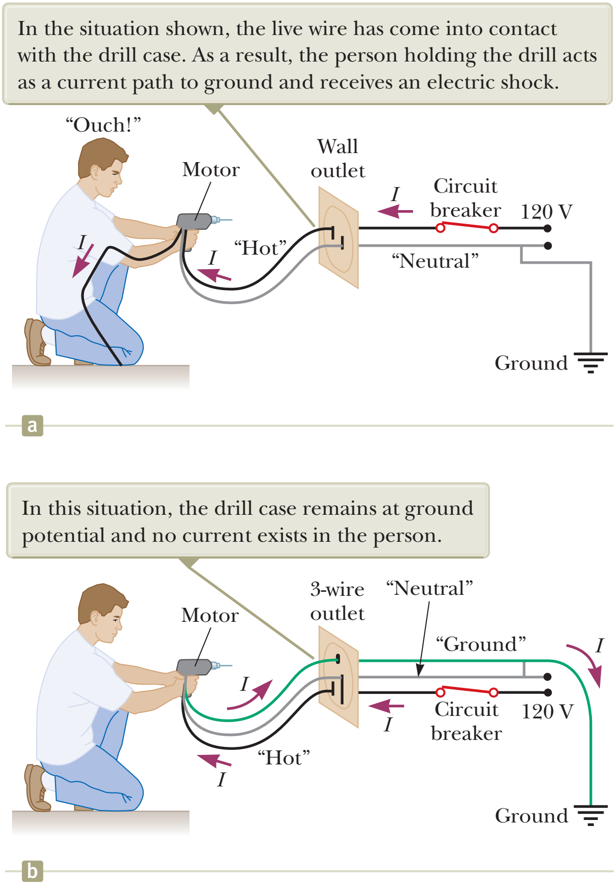


Figure 28.21 (a) A diagram of the circuit for an electric drill with only two connecting wires. The normal current path is from the live wire through the motor connections and back to ground through the neutral wire. (b) This shock can be avoided by connecting the drill case to ground through a third ground wire. The wire colors represent electrical standards in the United States: the "hot" wire is black, the ground wire is green, and the neutral wire is white (shown as gray in the figure).

Electric shock can result in fatal burns or can cause the muscles of vital organs such as the heart to malfunction. The degree of damage to the body depends on the magnitude of the current, the length of time it acts, the part of the body touched by the live wire, and the part of the body in which the current exists. Currents of 5 mA or less cause a sensation of shock, but ordinarily do little or no damage. If the current is larger than about 10 mA, the muscles contract and the person may be unable to release the live wire. If the body carries a current of about 100 mA for only a few seconds, the result can be fatal. Such a large current paralyzes the respiratory muscles and prevents breathing. In some cases, currents of approximately 1 A can produce serious (and sometimes fatal) burns. In practice, no contact with live wires is regarded as safe whenever the voltage is greater than 24 V.

Many 120-V outlets are designed to accept a three-pronged power cord. (This feature is required in all new electrical installations.) One of these prongs is the live wire at a nominal potential of 120 V. The second is the neutral wire, nominally at 0 V, which carries current to ground. Figure 28.21a shows a connection to an electric drill with only these two wires. If the live wire accidentally makes contact with the casing of the electric drill (which can occur if the wire insulation wears off), current can be carried to ground by way of the person, resulting in an electric shock. The third wire in a three-pronged power cord, the round prong, is a safety ground wire that normally carries no current. It is both grounded and connected directly to the casing of the appliance. If the live wire is accidentally shorted to the casing in this situation, most of the current takes the low-resistance path through the appliance to ground as shown in Figure 28.21b.

Special power outlets called *ground-fault circuit interrupters,* or GFCIs, are used in kitchens, bathrooms, basements, exterior outlets, and other hazardous areas of homes. These devices are designed to protect persons from electric shock by sensing small currents (< 5 mA) leaking to ground. (The principle of their operation is described in Chapter 31.) When an excessive leakage current is detected, the current is shut off in less than 1 ms.

Summary

Definition

The **emf** of a battery is equal to the voltage across its terminals when the current is zero. That is, the emf is equivalent to the **open-circuit voltage** of the battery.

Concepts and Principles

The **equivalent resistance** of a set of resistors connected in a **series combination** is

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots \quad \textbf{(28.6)}$$

The **equivalent resistance** of a set of resistors connected in a **parallel combination** is found from the relationship

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \quad \textbf{(28.8)}$$

Circuits involving more than one loop are conveniently analyzed with the use of **Kirchhoff's rules:**

1. **Junction rule.** At any junction, the sum of the currents must equal zero:

$$\sum_{\text{junction}} I = 0 \quad \textbf{(28.9)}$$

2. **Loop rule.** The sum of the potential differences across all elements around any circuit loop must be zero:

$$\sum_{\text{closed loop}} \Delta V = 0 \quad \textbf{(28.10)}$$

When a resistor is traversed in the direction of the current, the potential difference ΔV across the resistor is $-IR$. When a resistor is traversed in the direction opposite the current, $\Delta V = +IR$. When a source of emf is traversed in the direction of the emf (negative terminal to positive terminal), the potential difference is $+\mathcal{E}$. When a source of emf is traversed opposite the emf (positive to negative), the potential difference is $-\mathcal{E}$.

If a capacitor is charged with a battery through a resistor of resistance R, the charge on the capacitor and the current in the circuit vary in time according to the expressions

$$q(t) = Q_{\max}(1 - e^{-t/RC}) \quad \textbf{(28.14)}$$

$$i(t) = \frac{\mathcal{E}}{R}e^{-t/RC} \quad \textbf{(28.15)}$$

where $Q_{\max} = C\mathcal{E}$ is the maximum charge on the capacitor. The product RC is called the **time constant** τ of the circuit.

If a charged capacitor of capacitance C is discharged through a resistor of resistance R, the charge and current decrease exponentially in time according to the expressions

$$q(t) = Q_i e^{-t/RC} \quad \textbf{(28.18)}$$

$$i(t) = -\frac{Q_i}{RC}e^{-t/RC} \quad \textbf{(28.19)}$$

where Q_i is the initial charge on the capacitor and Q_i/RC is the initial current in the circuit.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Is a circuit breaker wired (a) in series with the device it is protecting, (b) in parallel, or (c) neither in series or in parallel, or (d) is it impossible to tell?
2. A battery has some internal resistance. **(i)** Can the potential difference across the terminals of the battery be equal to its emf? (a) no (b) yes, if the battery is absorbing energy by electrical transmission (c) yes, if more than one wire is connected to each terminal (d) yes, if the current in the battery is zero (e) yes, with no special condition required. **(ii)** Can the terminal voltage exceed the emf? Choose your answer from the same possibilities as in part (i).

3. The terminals of a battery are connected across two resistors in series. The resistances of the resistors are not the same. Which of the following statements are correct? Choose all that are correct. (a) The resistor with the smaller resistance carries more current than the other resistor. (b) The resistor with the larger resistance carries less current than the other resistor. (c) The current in each resistor is the same. (d) The potential difference across each resistor is the same. (e) The potential difference is greatest across the resistor closest to the positive terminal.

4. When operating on a 120-V circuit, an electric heater receives 1.30×10^3 W of power, a toaster receives 1.00×10^3 W, and an electric oven receives 1.54×10^3 W. If all three appliances are connected in parallel on a 120-V circuit and turned on, what is the total current drawn from an external source? (a) 24.0 A (b) 32.0 A (c) 40.0 A (d) 48.0 A (e) none of those answers

5. If the terminals of a battery with zero internal resistance are connected across two identical resistors in series, the total power delivered by the battery is 8.00 W. If the same battery is connected across the same resistors in parallel, what is the total power delivered by the battery? (a) 16.0 W (b) 32.0 W (c) 2.00 W (d) 4.00 W (e) none of those answers

6. Several resistors are connected in series. Which of the following statements is correct? Choose all that are correct. (a) The equivalent resistance is greater than any of the resistances in the group. (b) The equivalent resistance is less than any of the resistances in the group. (c) The equivalent resistance depends on the voltage applied across the group. (d) The equivalent resistance is equal to the sum of the resistances in the group. (e) None of those statements is correct.

7. What is the time constant of the circuit shown in Figure OQ28.7? Each of the five resistors has resistance R, and each of the five capacitors has capacitance C. The internal resistance of the battery is negligible. (a) RC (b) $5RC$ (c) $10RC$ (d) $25RC$ (e) none of those answers

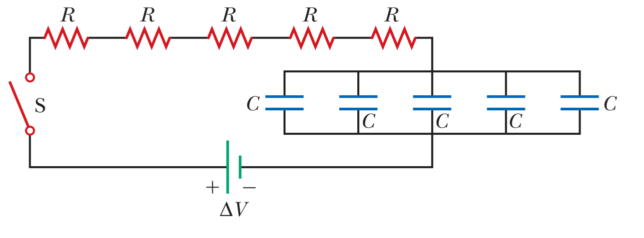


Figure OQ28.7

8. When resistors with different resistances are connected in series, which of the following must be the same for each resistor? Choose all correct answers. (a) potential difference (b) current (c) power delivered (d) charge entering each resistor in a given time interval (e) none of those answers

9. When resistors with different resistances are connected in parallel, which of the following must be the same for each resistor? Choose all correct answers. (a) potential difference (b) current (c) power delivered (d) charge entering each resistor in a given time interval (e) none of those answers

10. The terminals of a battery are connected across two resistors in parallel. The resistances of the resistors are not the same. Which of the following statements is correct? Choose all that are correct. (a) The resistor with the larger resistance carries more current than the other resistor. (b) The resistor with the larger resistance carries less current than the other resistor. (c) The potential difference across each resistor is the same. (d) The potential difference across the larger resistor is greater than the potential difference across the smaller resistor. (e) The potential difference is greater across the resistor closer to the battery.

11. Are the two headlights of a car wired (a) in series with each other, (b) in parallel, or (c) neither in series nor in parallel, or (d) is it impossible to tell?

12. In the circuit shown in Figure OQ28.12, each battery is delivering energy to the circuit by electrical transmission. All the resistors have equal resistance. **(i)** Rank the electric potentials at points a, b, c, d, and e from highest to lowest, noting any cases of equality in the ranking. **(ii)** Rank the magnitudes of the currents at the same points from greatest to least, noting any cases of equality.

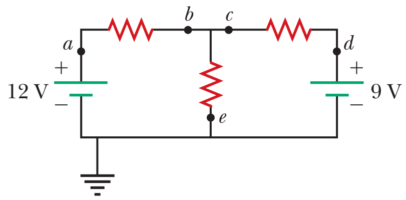


Figure OQ28.12

13. Several resistors are connected in parallel. Which of the following statements are correct? Choose all that are correct. (a) The equivalent resistance is greater than any of the resistances in the group. (b) The equivalent resistance is less than any of the resistances in the group. (c) The equivalent resistance depends on the voltage applied across the group. (d) The equivalent resistance is equal to the sum of the resistances in the group. (e) None of those statements is correct.

14. A circuit consists of three identical lamps connected to a battery as in Figure OQ28.14. The battery has some internal resistance. The switch S, originally open, is closed. **(i)** What then happens to the brightness of lamp B? (a) It increases. (b) It decreases somewhat. (c) It does not change. (d) It drops to zero. For parts (ii) to (vi), choose from the same possibilities (a) through (d). **(ii)** What happens to the brightness of lamp C? **(iii)** What happens to the current in the battery? **(iv)** What happens to the potential difference across lamp A? **(v)** What happens to the potential difference

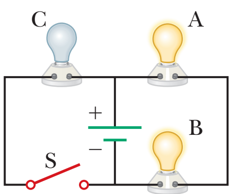


Figure OQ28.14

across lamp C? **(vi)** What happens to the total power delivered to the lamps by the battery?

15. A series circuit consists of three identical lamps connected to a battery as shown in Figure OQ28.15. The switch S, originally open, is closed. **(i)** What then happens to the brightness of lamp B? (a) It increases. (b) It decreases somewhat. (c) It does not change. (d) It drops to zero. For parts (ii) to (vi), choose from the same possibilities (a) through (d). **(ii)** What happens to the brightness of lamp C? **(iii)** What happens to the current in the battery? **(iv)** What happens to the potential difference across lamp A? **(v)** What happens to the potential difference across lamp C? **(vi)** What happens to the total power delivered to the lamps by the battery?

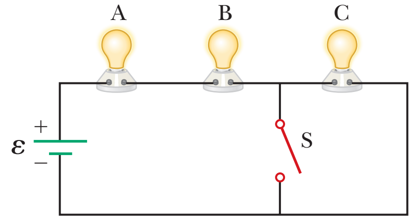


Figure OQ28.15

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. Suppose a parachutist lands on a high-voltage wire and grabs the wire as she prepares to be rescued. (a) Will she be electrocuted? (b) If the wire then breaks, should she continue to hold onto the wire as she falls to the ground? Explain.

2. A student claims that the second of two lightbulbs in series is less bright than the first because the first lightbulb uses up some of the current. How would you respond to this statement?

$\boxed{3.}$ Why is it possible for a bird to sit on a high-voltage wire without being electrocuted?

4. Given three lightbulbs and a battery, sketch as many different electric circuits as you can.

5. A ski resort consists of a few chairlifts and several interconnected downhill runs on the side of a mountain, with a lodge at the bottom. The chairlifts are analogous to batteries, and the runs are analogous to resistors. Describe how two runs can be in series. Describe how three runs can be in parallel. Sketch a junction between one chairlift and two runs. State Kirchhoff's junction rule for ski resorts. One of the skiers happens to be carrying a skydiver's altimeter. She never takes the same set of chairlifts and runs twice, but keeps passing you at the fixed location where you are working. State Kirchhoff's loop rule for ski resorts.

$\boxed{6.}$ Referring to Figure CQ28.6, describe what happens to the lightbulb after the switch is closed. Assume the capacitor has a large capacitance and is initially uncharged. Also assume the light illuminates when connected directly across the battery terminals.

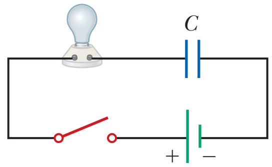


Figure CQ28.6

7. So that your grandmother can listen to *A Prairie Home Companion*, you take her bedside radio to the hospital where she is staying. You are required to have a maintenance worker test the radio for electrical safety. Finding that it develops 120 V on one of its knobs, he does not let you take it to your grandmother's room. Your grandmother complains that she has had the radio for many years and nobody has ever gotten a shock from it. You end up having to buy a new plastic radio. (a) Why is your grandmother's old radio dangerous in a hospital room? (b) Will the old radio be safe back in her bedroom?

$\boxed{8.}$ (a) What advantage does 120-V operation offer over 240 V? (b) What disadvantages does it have?

$\boxed{9.}$ Is the direction of current in a battery always from the negative terminal to the positive terminal? Explain.

10. Compare series and parallel resistors to the series and parallel rods in Figure 20.13 on page 610. How are the situations similar?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

$\boxed{1.}$ full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 28.1 Electromotive Force

$\boxed{1.}$ M A battery has an emf of 15.0 V. The terminal voltage of the battery is 11.6 V when it is delivering 20.0 W of power to an external load resistor R. (a) What is the value of R? (b) What is the internal resistance of the battery?

2. AMT Two 1.50-V batteries—with their positive terminals in the same direction—are inserted in series into a flashlight. One battery has an internal resistance of 0.255 Ω, and the other has an internal resistance of 0.153 Ω. When the switch is closed, the bulb carries a current of 600 mA. (a) What is the bulb's resistance? (b) What fraction of the chemical energy transformed appears as internal energy in the batteries?

3. W An automobile battery has an emf of 12.6 V and an internal resistance of 0.080 0 Ω. The headlights together have an equivalent resistance of 5.00 Ω (assumed constant). What is the potential difference across the headlight bulbs (a) when they are the only load on the battery and (b) when the starter motor is operated, with 35.0 A of current in the motor?

4. As in Example 28.2, consider a power supply with fixed emf $\mathcal{E}$ and internal resistance r causing current in a load resistance R. In this problem, R is fixed and r is a variable. The efficiency is defined as the energy delivered to the load divided by the energy delivered by the emf. (a) When the internal resistance is adjusted for maximum power transfer, what is the efficiency? (b) What should be the internal resistance for maximum possible efficiency? (c) When the electric company sells energy to a customer, does it have a goal of high efficiency or of maximum power transfer? Explain. (d) When a student connects a loudspeaker to an amplifier, does she most want high efficiency or high power transfer? Explain.

Section 28.2 Resistors in Series and Parallel

5. W Three 100-Ω resistors are connected as shown in Figure P28.5. The maximum power that can safely be delivered to any one resistor is 25.0 W. (a) What is the maximum potential difference that can be applied to the terminals a and b? (b) For the voltage determined in part (a), what is the power delivered to each resistor? (c) What is the total power delivered to the combination of resistors?

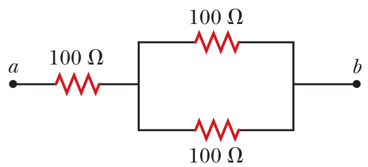


Figure P28.5

6. A lightbulb marked "75 W [at] 120 V" is screwed into a socket at one end of a long extension cord, in which each of the two conductors has resistance 0.800 Ω. The other end of the extension cord is plugged into a 120-V outlet. (a) Explain why the actual power delivered to the lightbulb cannot be 75 W in this situation. (b) Draw a circuit diagram. (c) Find the actual power delivered to the lightbulb in this circuit.

7. What is the equivalent resistance of the combination of identical resistors between points a and b in Figure P28.7?

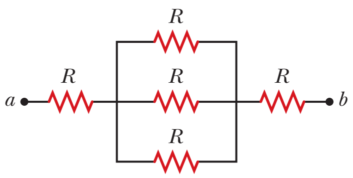


Figure P28.7

8. Consider the two circuits shown in Figure P28.8 in which the batteries are identical. The resistance of each lightbulb is R. Neglect the internal resistances of the batteries. (a) Find expressions for the currents in each lightbulb. (b) How does the brightness of B compare with that of C? Explain. (c) How does the brightness of A compare with that of B and of C? Explain.

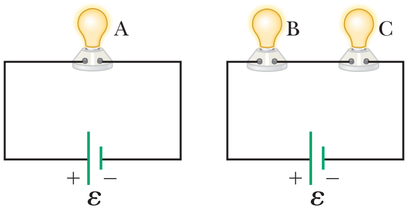


Figure P28.8

9. M Consider the circuit shown in Figure P28.9. Find (a) the current in the 20.0-Ω resistor and (b) the potential difference between points a and b.

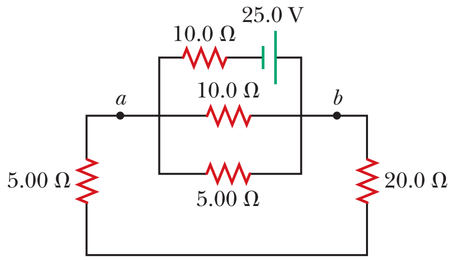


Figure P28.9

10. (a) You need a 45-Ω resistor, but the stockroom has only 20-Ω and 50-Ω resistors. How can the desired resistance be achieved under these circumstances? (b) What can you do if you need a 35-Ω resistor?

11. A battery with $\mathcal{E}$ = 6.00 V and no internal resistance supplies current to the circuit shown in Figure P28.11. When the double-throw switch S is open as shown in the figure, the current in the battery is 1.00 mA. When the switch is closed in position a, the current in the

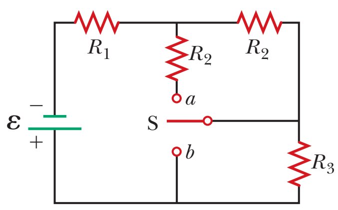


Figure P28.11
Problems 11 and 12.

battery is 1.20 mA. When the switch is closed in position b, the current in the battery is 2.00 mA. Find the resistances (a) R_1, (b) R_2, and (c) R_3.

12. A battery with emf $\mathcal{E}$ and no internal resistance supplies current to the circuit shown in Figure P28.11. When the double-throw switch S is open as shown in the figure, the current in the battery is I_0. When the switch is closed in position a, the current in the battery is I_a. When the switch is closed in position b, the current in the battery is I_b. Find the resistances (a) R_1, (b) R_2, and (c) R_3.

13. M (a) Find the equivalent resistance between points a and b in Figure P28.13. (b) Calculate the current in each resistor if a potential difference of 34.0 V is applied between points a and b.

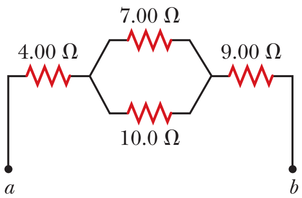


Figure P28.13

14. (a) When the switch S in the circuit of Figure P28.14 is closed, will the equivalent resistance between points a and b increase or decrease? State your reasoning. (b) Assume the equivalent resistance drops by 50.0% when the switch is closed. Determine the value of R.

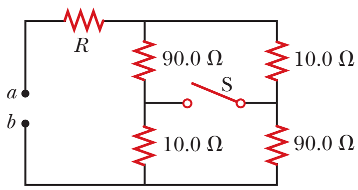


Figure P28.14

15. Two resistors connected in series have an equivalent resistance of 690 Ω. When they are connected in parallel, their equivalent resistance is 150 Ω. Find the resistance of each resistor.

16. Four resistors are connected to a battery as shown in Figure P28.16. (a) Determine the potential difference across each resistor in terms of $\mathcal{E}$. (b) Determine the current in each resistor in terms of I. (c) **What If?** If R_3 is increased, explain what happens to the current in each of the resistors. (d) In the limit that $R_3 \to \infty$, what are the new values of the current in each resistor in terms of I, the original current in the battery?

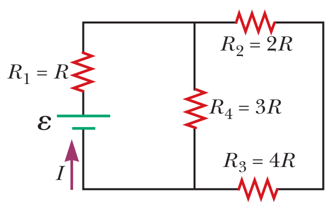


Figure P28.16

17. Consider the combination of resistors shown in Figure P28.17. (a) Find the equivalent resistance between points a and b. (b) If a voltage of 35.0 V is applied between points a and b, find the current in each resistor.

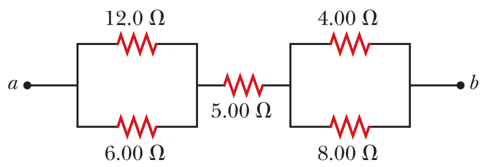


Figure P28.17

18. For the purpose of measuring the electric resistance of shoes through the body of the wearer standing on a metal ground plate, the American National Standards Institute (ANSI) specifies the circuit shown in Figure P28.18. The potential difference ΔV across the 1.00-MΩ resistor is measured with an ideal voltmeter. (a) Show that the resistance of the footwear is

$$R_{\text{shoes}} = \frac{50.0\text{ V} - \Delta V}{\Delta V}$$

(b) In a medical test, a current through the human body should not exceed 150 μA. Can the current delivered by the ANSI-specified circuit exceed 150 μA? To decide, consider a person standing barefoot on the ground plate.

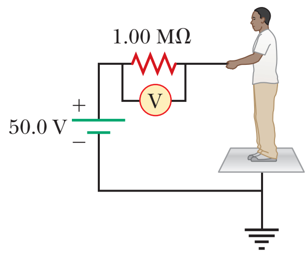


Figure P28.18

19. W Calculate the power delivered to each resistor in the circuit shown in Figure P28.19.

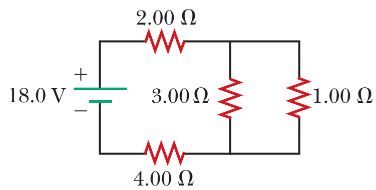


Figure P28.19

20. *Why is the following situation impossible?* A technician is testing a circuit that contains a resistance R. He realizes that a better design for the circuit would include a resistance $\frac{7}{3}R$ rather than R. He has three additional resistors, each with resistance R. By combining these additional resistors in a certain combination that is then placed in series with the original resistor, he achieves the desired resistance.

21. Consider the circuit shown in Figure P28.21 on page 860. (a) Find the voltage across the 3.00-Ω resistor. (b) Find the current in the 3.00-Ω resistor.

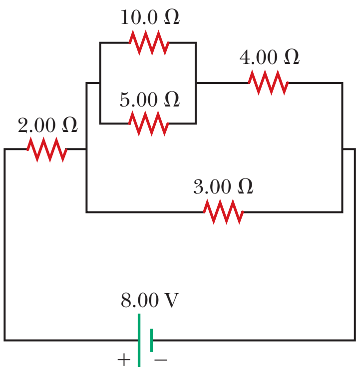


Figure P28.21

Section 28.3 Kirchhoff's Rules

22. In Figure P28.22, show how to add just enough ammeters to measure every different current. Show how to add just enough voltmeters to measure the potential difference across each resistor and across each battery.

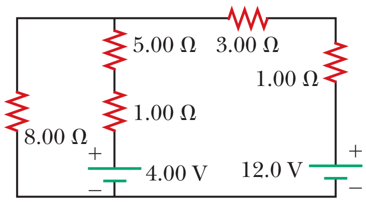


Figure P28.22 Problems 22 and 23.

23. M The circuit shown in Figure P28.22 is connected for 2.00 min. (a) Determine the current in each branch of the circuit. (b) Find the energy delivered by each battery. (c) Find the energy delivered to each resistor. (d) Identify the type of energy storage transformation that occurs in the operation of the circuit. (e) Find the total amount of energy transformed into internal energy in the resistors.

24. For the circuit shown in Figure P28.24, calculate (a) the current in the 2.00-Ω resistor and (b) the potential difference between points a and b.

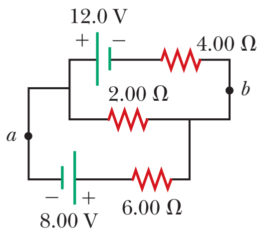


Figure P28.24

25. M What are the expected readings of (a) the ideal ammeter and (b) the ideal voltmeter in Figure P28.25?

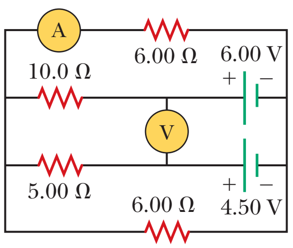


Figure P28.25

26. The following equations describe an electric circuit:

$$-I_1\,(220\ \Omega) + 5.80\ \text{V} - I_2\,(370\ \Omega) = 0$$
$$+I_2\,(370\ \Omega) + I_3\,(150\ \Omega) - 3.10\ \text{V} = 0$$
$$I_1 + I_3 - I_2 = 0$$

(a) Draw a diagram of the circuit. (b) Calculate the unknowns and identify the physical meaning of each unknown.

27. Taking $R = 1.00\ \text{k}\Omega$ and $\boldsymbol{\varepsilon} = 250$ V in Figure P28.27, determine the direction and magnitude of the current in the horizontal wire between a and e.

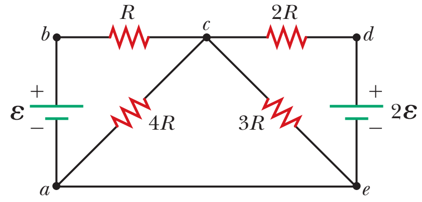


Figure P28.27

28. Jumper cables are connected from a fresh battery in one car to charge a dead battery in another car. Figure P28.28 shows the circuit diagram for this situation. While the cables are connected, the ignition switch of the car with the dead battery is closed and the starter is activated to start the engine. Determine the current in (a) the starter and (b) the dead battery. (c) Is the dead battery being charged while the starter is operating?

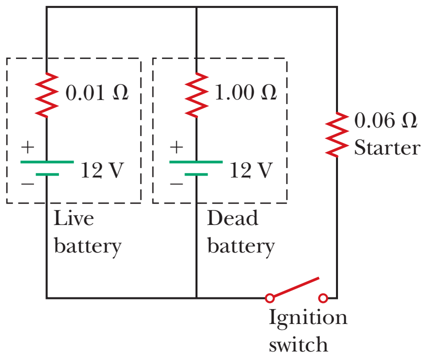


Figure P28.28

29. W The ammeter shown in Figure P28.29 reads 2.00 A. Find (a) I_1, (b) I_2, and (c) $\boldsymbol{\varepsilon}$.

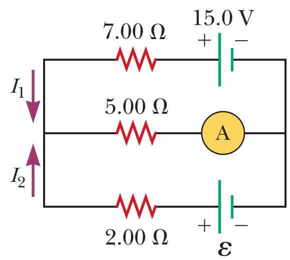


Figure P28.29

30. W In the circuit of Figure P28.30, determine (a) the current in each resistor and (b) the potential difference across the 200-Ω resistor.

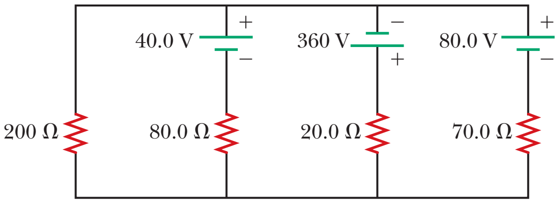


Figure P28.30

31. Using Kirchhoff's rules, (a) find the current in each resistor shown in Figure P28.31 and (b) find the potential difference between points c and f.

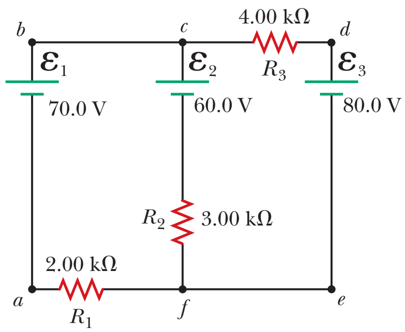


Figure P28.31

32. In the circuit of Figure P28.32, the current $I_1 = 3.00$ A and the values of $\mathcal{E}$ for the ideal battery and R are unknown. What are the currents (a) I_2 and (b) I_3? (c) Can you find the values of $\mathcal{E}$ and R? If so, find their values. If not, explain.

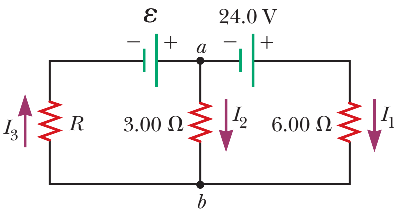


Figure P28.32

33. In Figure P28.33, find (a) the current in each resistor and (b) the power delivered to each resistor.

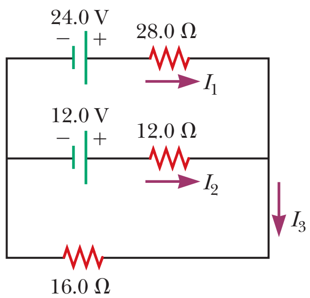


Figure P28.33

34. For the circuit shown in Figure P28.34, we wish to find the currents I_1, I_2, and I_3. Use Kirchhoff's rules to obtain equations for (a) the upper loop, (b) the lower loop, and (c) the junction on the left side. In each case, suppress units for clarity and simplify, combining the terms. (d) Solve the junction equation for I_3. (e) Using the equation found in part (d), eliminate I_3 from the equation found in part (b). (f) Solve the equations found in parts (a) and (e) simultaneously for the two unknowns I_1 and I_2. (g) Substitute the answers found in part (f) into the junction equation found in part (d), solving for I_3. (h) What is the significance of the negative answer for I_2?

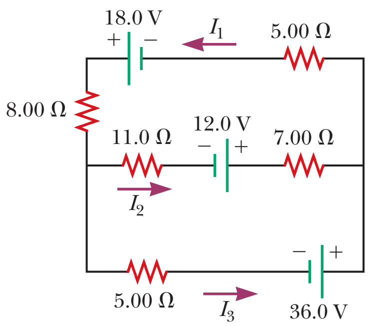


Figure P28.34

35. Find the potential difference across each resistor in Figure P28.35.

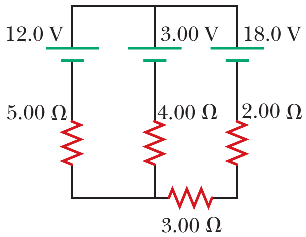


Figure P28.35

36. (a) Can the circuit shown in Figure P28.36 be reduced to a single resistor connected to a battery? Explain. Calculate the currents (b) I_1, (c) I_2, and (d) I_3.

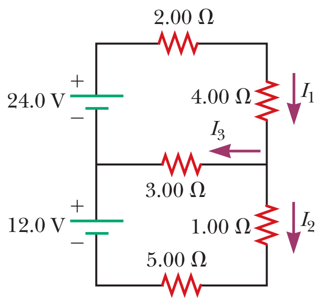


Figure P28.36

Section 28.4 *RC* Circuits

37. An uncharged capacitor and a resistor are connected in series to a source of emf. If $\mathcal{E} = 9.00$ V, $C = 20.0\ \mu$F, and $R = 100\ \Omega$, find (a) the time constant of the circuit, (b) the maximum charge on the capacitor, and (c) the charge on the capacitor at a time equal to one time constant after the battery is connected.

38. Consider a series RC circuit as in Figure P28.38 for which $R = 1.00\ \text{M}\Omega$, $C = 5.00\ \mu\text{F}$, and $\boldsymbol{\varepsilon} = 30.0$ V. Find (a) the time constant of the circuit and (b) the maximum charge on the capacitor after the switch is thrown closed. (c) Find the current in the resistor 10.0 s after the switch is closed.

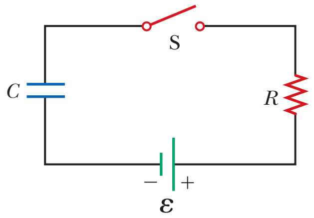


Figure P28.38 Problems 38, 67, and 68.

39. A 2.00-nF capacitor with an initial charge of 5.10 μC is discharged through a 1.30-kΩ resistor. (a) Calculate the current in the resistor 9.00 μs after the resistor is connected across the terminals of the capacitor. (b) What charge remains on the capacitor after 8.00 μs? (c) What is the maximum current in the resistor?

40. A 10.0-μF capacitor is charged by a 10.0-V battery through a resistance R. The capacitor reaches a potential difference of 4.00 V in a time interval of 3.00 s after charging begins. Find R.

41. In the circuit of Figure P28.41, the switch S has been open for a long time. It is then suddenly closed. Take $\boldsymbol{\varepsilon} = 10.0$ V, $R_1 = 50.0\ \text{k}\Omega$, $R_2 = 100\ \text{k}\Omega$, and $C = 10.0\ \mu\text{F}$. Determine the time constant (a) before the switch is closed and (b) after the switch is closed. (c) Let the switch be closed at $t = 0$. Determine the current in the switch as a function of time.

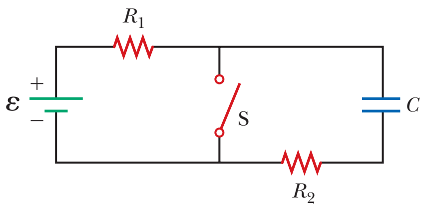


Figure P28.41 Problems 41 and 42.

42. In the circuit of Figure P28.41, the switch S has been open for a long time. It is then suddenly closed. Determine the time constant (a) before the switch is closed and (b) after the switch is closed. (c) Let the switch be closed at $t = 0$. Determine the current in the switch as a function of time.

43. The circuit in Figure P28.43 has been connected for a long time. (a) What is the potential difference across the capacitor? (b) If the battery is disconnected from the circuit, over what time interval does the capacitor discharge to one-tenth its initial voltage?

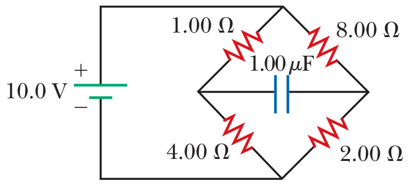


Figure P28.43

44. Show that the integral $\int_0^\infty e^{-2t/RC}\,dt$ in Example 28.11 has the value $\frac{1}{2}RC$.

45. A charged capacitor is connected to a resistor and switch as in Figure P28.45. The circuit has a time constant of 1.50 s. Soon after the switch is closed, the charge on the capacitor is 75.0% of its initial charge. (a) Find the time interval required for the capacitor to reach this charge. (b) If $R = 250\ \text{k}\Omega$, what is the value of C?

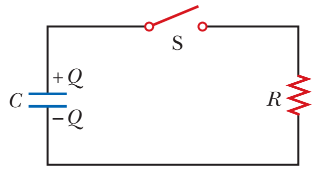


Figure P28.45

Section 28.5 Household Wiring and Electrical Safety

46. An electric heater is rated at 1.50×10^3 W, a toaster at 750 W, and an electric grill at 1.00×10^3 W. The three appliances are connected to a common 120-V household circuit. (a) How much current does each draw? (b) If the circuit is protected with a 25.0-A circuit breaker, will the circuit breaker be tripped in this situation? Explain your answer.

47. A heating element in a stove is designed to receive 3 000 W when connected to 240 V. (a) Assuming the resistance is constant, calculate the current in the heating element if it is connected to 120 V. (b) Calculate the power it receives at that voltage.

48. Turn on your desk lamp. Pick up the cord, with your thumb and index finger spanning the width of the cord. (a) Compute an order-of-magnitude estimate for the current in your hand. Assume the conductor inside the lamp cord next to your thumb is at potential $\sim 10^2$ V at a typical instant and the conductor next to your index finger is at ground potential (0 V). The resistance of your hand depends strongly on the thickness and the moisture content of the outer layers of your skin. Assume the resistance of your hand between fingertip and thumb tip is $\sim 10^4\ \Omega$. You may model the cord as having rubber insulation. State the other quantities you measure or estimate and their values. Explain your reasoning. (b) Suppose your body is isolated from any other charges or currents. In order-of-magnitude terms, estimate the potential difference between your thumb where it contacts the cord and your finger where it touches the cord.

Additional Problems

49. Assume you have a battery of emf $\mathcal{E}$ and three identical lightbulbs, each having constant resistance R. What is the total power delivered by the battery if the lightbulbs are connected (a) in series and (b) in parallel? (c) For which connection will the lightbulbs shine the brightest?

50. Find the equivalent resistance between points a and b in Figure P28.50.

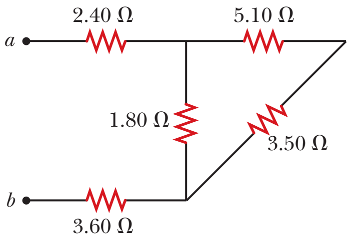


Figure P28.50

51. Four 1.50-V AA batteries in series are used to power a small radio. If the batteries can move a charge of 240 C, how long will they last if the radio has a resistance of 200 Ω?

52. Four resistors are connected in parallel across a 9.20-V battery. They carry currents of 150 mA, 45.0 mA, 14.0 mA, and 4.00 mA. If the resistor with the largest resistance is replaced with one having twice the resistance, (a) what is the ratio of the new current in the battery to the original current? (b) **What If?** If instead the resistor with the smallest resistance is replaced with one having twice the resistance, what is the ratio of the new total current to the original current? (c) On a February night, energy leaves a house by several energy leaks, including 1.50×10^3 W by conduction through the ceiling, 450 W by infiltration (airflow) around the windows, 140 W by conduction through the basement wall above the foundation sill, and 40.0 W by conduction through the plywood door to the attic. To produce the biggest saving in heating bills, which one of these energy transfers should be reduced first? Explain how you decide. Clifford Swartz suggested the idea for this problem.

53. The circuit in Figure P28.53 has been connected for several seconds. Find the current (a) in the 4.00-V battery, (b) in the 3.00-Ω resistor, (c) in the 8.00-V battery, and (d) in the 3.00-V battery. (e) Find the charge on the capacitor.

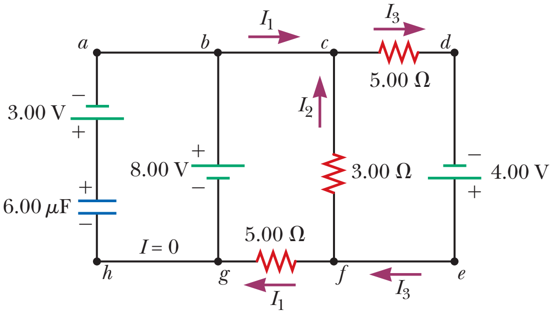


Figure P28.53

54. The circuit in Figure P28.54a consists of three resistors and one battery with no internal resistance. (a) Find the current in the 5.00-Ω resistor. (b) Find the power delivered to the 5.00-Ω resistor. (c) In each of the circuits in Figures P28.54b, P28.54c, and P28.54d, an additional 15.0-V battery has been inserted into the circuit. Which diagram or diagrams represent a circuit that requires the use of Kirchhoff's rules to find the currents? Explain why. (d) In which of these three new circuits is the smallest amount of power delivered to the 10.0-Ω resistor? (You need not calculate the power in each circuit if you explain your answer.)

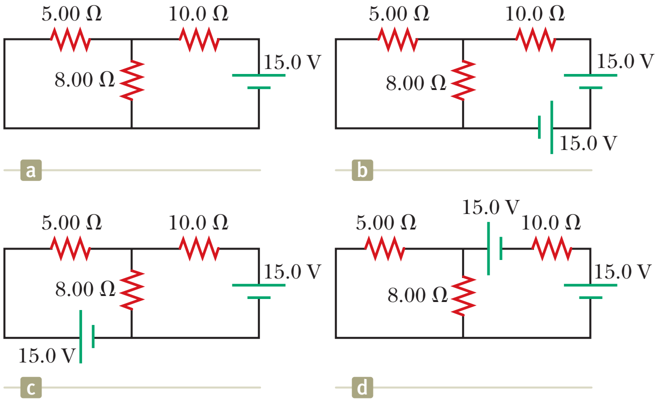


Figure P28.54

55. For the circuit shown in Figure P28.55, the ideal voltmeter reads 6.00 V and the ideal ammeter reads 3.00 mA. Find (a) the value of R, (b) the emf of the battery, and (c) the voltage across the 3.00-kΩ resistor.

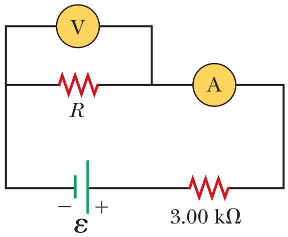


Figure P28.55

56. The resistance between terminals a and b in Figure P28.56 is 75.0 Ω. If the resistors labeled R have the same value, determine R.

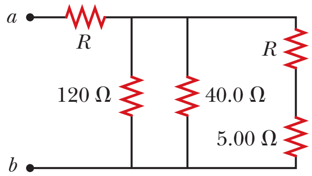


Figure P28.56

57. (a) Calculate the potential difference between points *a* and *b* in Figure P28.57 and (b) identify which point is at the higher potential.

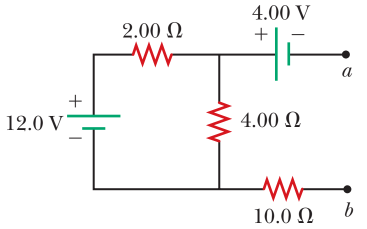


Figure P28.57

58. *Why is the following situation impossible?* A battery has an emf of $\mathcal{E}$ = 9.20 V and an internal resistance of r = 1.20 Ω. A resistance R is connected across the battery and extracts from it a power of P = 21.2 W.

59. M A rechargeable battery has an emf of 13.2 V and an internal resistance of 0.850 Ω. It is charged by a 14.7-V power supply for a time interval of 1.80 h. After charging, the battery returns to its original state as it delivers a constant current to a load resistor over 7.30 h. Find the efficiency of the battery as an energy storage device. (The efficiency here is defined as the energy delivered to the load during discharge divided by the energy delivered by the 14.7-V power supply during the charging process.)

60. Find (a) the equivalent resistance of the circuit in Figure P28.60, (b) the potential difference across each resistor, (c) each current indicated in Figure P28.60, and (d) the power delivered to each resistor.

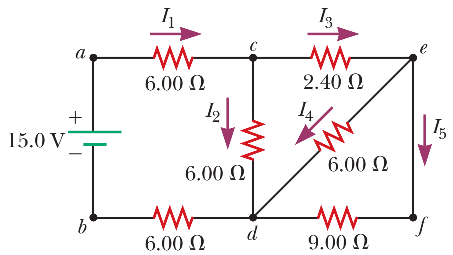


Figure P28.60

61. When two unknown resistors are connected in series with a battery, the battery delivers 225 W and carries a total current of 5.00 A. For the same total current, 50.0 W is delivered when the resistors are connected in parallel. Determine the value of each resistor.

62. When two unknown resistors are connected in series with a battery, the battery delivers total power P_s and carries a total current of I. For the same total current, a total power P_p is delivered when the resistors are connected in parallel. Determine the value of each resistor.

63. The pair of capacitors in Figure P28.63 are fully charged by a 12.0-V battery. The battery is disconnected, and the switch is then closed. After 1.00 ms has elapsed, (a) how much charge remains on the 3.00-μF capacitor? (b) How much charge remains on the 2.00-μF capacitor? (c) What is the current in the resistor at this time?

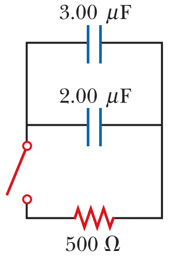


Figure P28.63

64. A power supply has an open-circuit voltage of 40.0 V and an internal resistance of 2.00 Ω. It is used to charge two storage batteries connected in series, each having an emf of 6.00 V and internal resistance of 0.300 Ω. If the charging current is to be 4.00 A, (a) what additional resistance should be added in series? At what rate does the internal energy increase in (b) the supply, (c) in the batteries, and (d) in the added series resistance? (e) At what rate does the chemical energy increase in the batteries?

65. The circuit in Figure P28.65 contains two resistors, R_1 = 2.00 kΩ and R_2 = 3.00 kΩ, and two capacitors, C_1 = 2.00 μF and C_2 = 3.00 μF, connected to a battery with emf $\mathcal{E}$ = 120 V. If there are no charges on the capacitors before switch S is closed, determine the charges on capacitors (a) C_1 and (b) C_2 as functions of time, after the switch is closed.

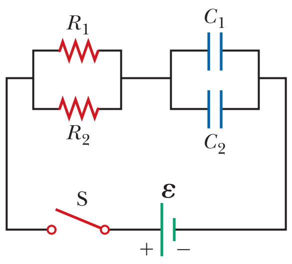


Figure P28.65

66. Two resistors R_1 and R_2 are in parallel with each other. Together they carry total current I. (a) Determine the current in each resistor. (b) Prove that this division of the total current I between the two resistors results in less power delivered to the combination than any other division. It is a general principle that *current in a direct current circuit distributes itself so that the total power delivered to the circuit is a minimum.*

67. AMT M The values of the components in a simple series RC circuit containing a switch (Fig. P28.38) are C = 1.00 μF, R = 2.00 × 10^6 Ω, and $\mathcal{E}$ = 10.0 V. At the instant 10.0 s after the switch is closed, calculate (a) the charge on the capacitor, (b) the current in the resistor, (c) the rate at which energy is being stored in the capacitor, and (d) the rate at which energy is being delivered by the battery.

68. A battery is used to charge a capacitor through a resistor as shown in Figure P28.38. Show that half the energy supplied by the battery appears as internal energy in the resistor and half is stored in the capacitor.

69. A young man owns a canister vacuum cleaner marked "535 W [at] 120 V" and a Volkswagen Beetle, which he wishes to clean. He parks the car in his apartment parking lot and uses an inexpensive extension cord 15.0 m long to plug in the vacuum cleaner. You may assume the cleaner has constant resistance. (a) If the resistance of each of the two conductors in the extension cord is 0.900 Ω, what is the actual power delivered to the cleaner? (b) If instead the power is to be at least 525 W, what must be the diameter of each of two identical copper conductors in the cord he buys? (c) Repeat part (b) assuming the power is to be at least 532 W.

70. (a) Determine the equilibrium charge on the capacitor in the circuit of Figure P28.70 as a function of R. (b) Evaluate the charge when $R = 10.0\ \Omega$. (c) Can the charge on the capacitor be zero? If so, for what value of R? (d) What is the maximum possible magnitude of the charge on the capacitor? For what value of R is it achieved? (e) Is it experimentally meaningful to take $R = \infty$? Explain your answer. If so, what charge magnitude does it imply?

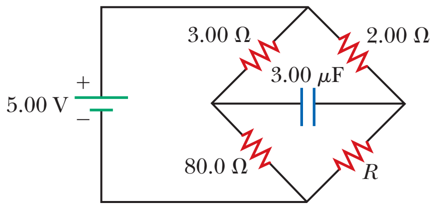


Figure P28.70

71. Switch S shown in Figure P28.71 has been closed for a long time, and the electric circuit carries a constant current. Take $C_1 = 3.00\ \mu F$, $C_2 = 6.00\ \mu F$, $R_1 = 4.00\ k\Omega$, and $R_2 = 7.00\ k\Omega$. The power delivered to R_2 is 2.40 W. (a) Find the charge on C_1. (b) Now the switch is opened. After many milliseconds, by how much has the charge on C_2 changed?

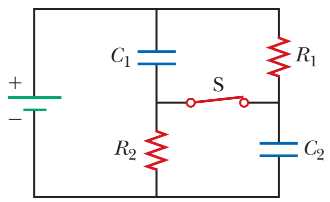


Figure P28.71

72. M Three identical 60.0-W, 120-V lightbulbs are connected across a 120-V power source as shown in Figure P28.72. Assuming the resistance of each lightbulb is constant (even though in reality the resistance might increase markedly with current), find (a) the total power supplied by the power source and (b) the potential difference across each lightbulb.

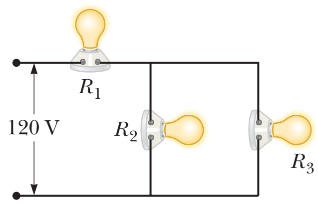


Figure P28.72

73. A regular tetrahedron is a pyramid with a triangular base and triangular sides as shown in Figure P28.73. Imagine the six straight lines in Figure P28.73 are each 10.0-Ω resistors, with junctions at the four vertices. A 12.0-V battery is connected to any two of the vertices. Find (a) the equivalent resistance of the tetrahedron between these vertices and (b) the current in the battery.

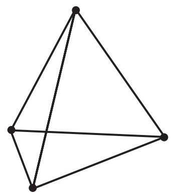


Figure P28.73

74. An ideal voltmeter connected across a certain fresh 9-V battery reads 9.30 V, and an ideal ammeter briefly connected across the same battery reads 3.70 A. We say the battery has an open-circuit voltage of 9.30 V and a short-circuit current of 3.70 A. Model the battery as a source of emf $\boldsymbol{\mathcal{E}}$ in series with an internal resistance r as in Figure 28.1a. Determine both (a) $\boldsymbol{\mathcal{E}}$ and (b) r. An experimenter connects two of these identical batteries together as shown in Figure P28.74. Find (c) the open-circuit voltage and (d) the short-circuit current of the pair of connected batteries. (e) The experimenter connects a 12.0-Ω resistor between the exposed terminals of the connected batteries. Find the current in the resistor. (f) Find the power delivered to the resistor. (g) The experimenter connects a second identical resistor in parallel with the first. Find the power delivered to each resistor. (h) Because the same pair of batteries is connected across both resistors as was connected across the single resistor, why is the power in part (g) not the same as that in part (f)?

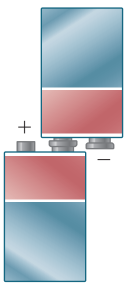


Figure P28.74

75. In Figure P28.75 on page 866, suppose the switch has been closed for a time interval sufficiently long for the capacitor to become fully charged. Find (a) the steady-state current in each resistor and (b) the charge Q_{max} on the capacitor. (c) The switch is now opened at $t = 0$. Write an equation for the current in R_2 as a function of time and (d) find the time interval required for the charge on the capacitor to fall to one-fifth its initial value.

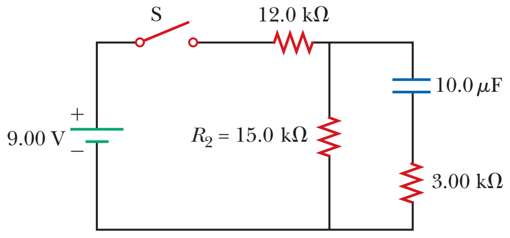


Figure P28.75

76. Figure P28.76 shows a circuit model for the transmission of an electrical signal such as cable TV to a large number of subscribers. Each subscriber connects a load resistance R_L between the transmission line and the ground. The ground is assumed to be at zero potential and able to carry any current between any ground connections with negligible resistance. The resistance of the transmission line between the connection points of different subscribers is modeled as the constant resistance R_T. Show that the equivalent resistance across the signal source is

$$R_{eq} = \tfrac{1}{2}\left[(4R_TR_L + R_T^{\,2})^{1/2} + R_T\right]$$

Suggestion: Because the number of subscribers is large, the equivalent resistance would not change noticeably if the first subscriber canceled the service. Consequently, the equivalent resistance of the section of the circuit to the right of the first load resistor is nearly equal to R_{eq}.

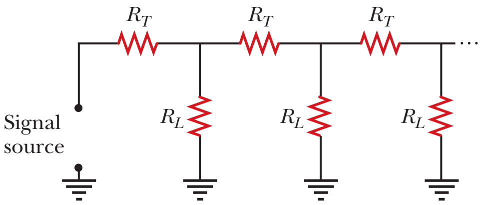


Figure P28.76

77. The student engineer of a campus radio station wishes to verify the effectiveness of the lightning rod on the antenna mast (Fig. P28.77). The unknown resistance R_x is between points C and E. Point E is a true ground, but it is inaccessible for direct measurement because this stratum is several meters below the Earth's surface. Two identical rods are driven into the ground at A and B, introducing an unknown resistance R_y. The procedure is as follows. Measure resistance R_1 between points A and B, then connect A and B with a heavy conducting wire and measure resistance R_2 between points A and C. (a) Derive an equation for R_x in terms of the observable resistances, R_1 and R_2. (b) A satisfactory ground resistance would be $R_x < 2.00\ \Omega$. Is the grounding of the station adequate if measurements give $R_1 = 13.0\ \Omega$ and $R_2 = 6.00\ \Omega$? Explain.

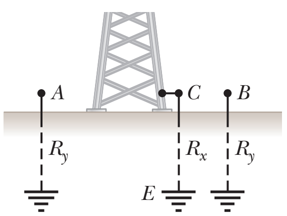


Figure P28.77

78. The circuit shown in Figure P28.78 is set up in the laboratory to measure an unknown capacitance C in series with a resistance $R = 10.0\ \text{M}\Omega$ powered by a battery whose emf is 6.19 V. The data given in the table are the measured voltages across the capacitor as a function of time, where $t = 0$ represents the instant at which the switch is thrown to position b. (a) Construct a graph of $\ln(\mathcal{E}/\Delta v)$ versus t and perform a linear least-squares fit to the data. (b) From the slope of your graph, obtain a value for the time constant of the circuit and a value for the capacitance.

Δv (V)	t (s)	$\ln(\mathcal{E}/\Delta v)$
6.19	0	
5.55	4.87	
4.93	11.1	
4.34	19.4	
3.72	30.8	
3.09	46.6	
2.47	67.3	
1.83	102.2	

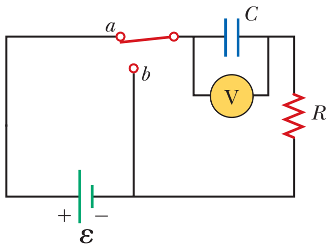


Figure P28.78

79. An electric teakettle has a multiposition switch and two heating coils. When only one coil is switched on, the well-insulated kettle brings a full pot of water to a boil over the time interval Δt. When only the other coil is switched on, it takes a time interval of $2\ \Delta t$ to boil the same amount of water. Find the time interval required to boil the same amount of water if both coils are switched on (a) in a parallel connection and (b) in a series connection.

80. A voltage ΔV is applied to a series configuration of n resistors, each of resistance R. The circuit components are reconnected in a parallel configuration, and voltage ΔV is again applied. Show that the power delivered to the series configuration is $1/n^2$ times the power delivered to the parallel configuration.

81. In places such as hospital operating rooms or factories for electronic circuit boards, electric sparks must be avoided. A person standing on a grounded floor and touching nothing else can typically have a body capacitance of 150 pF, in parallel with a foot capacitance of 80.0 pF produced by the dielectric soles of his or her shoes. The person acquires static electric charge from interactions with his or her surroundings. The static charge flows to ground through the equivalent resistance of the two shoe soles in parallel with each other. A pair of rubber-soled street shoes can present an equivalent resistance of 5.00×10^3 MΩ. A pair of shoes with special static-dissipative soles can have an equivalent resistance of 1.00 MΩ. Consider the person's body and shoes as forming an *RC* circuit with the ground. (a) How long does it take the rubber-soled shoes to reduce a person's potential from 3.00×10^3 V to 100 V? (b) How long does it take the static-dissipative shoes to do the same thing?

Challenge Problems

82. The switch in Figure P28.82a closes when $\Delta V_c > \frac{2}{3}\Delta V$ and opens when $\Delta V_c < \frac{1}{3}\Delta V$. The ideal voltmeter reads a potential difference as plotted in Figure P28.82b. What is the period T of the waveform in terms of R_1, R_2, and C?

83. The resistor R in Figure P28.83 receives 20.0 W of power. Determine the value of R.

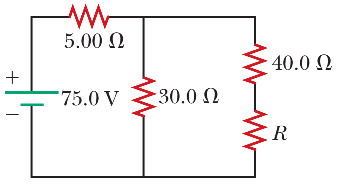


Figure P28.83

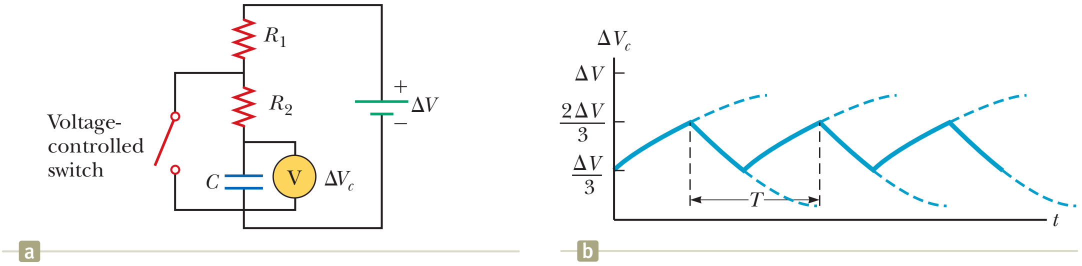
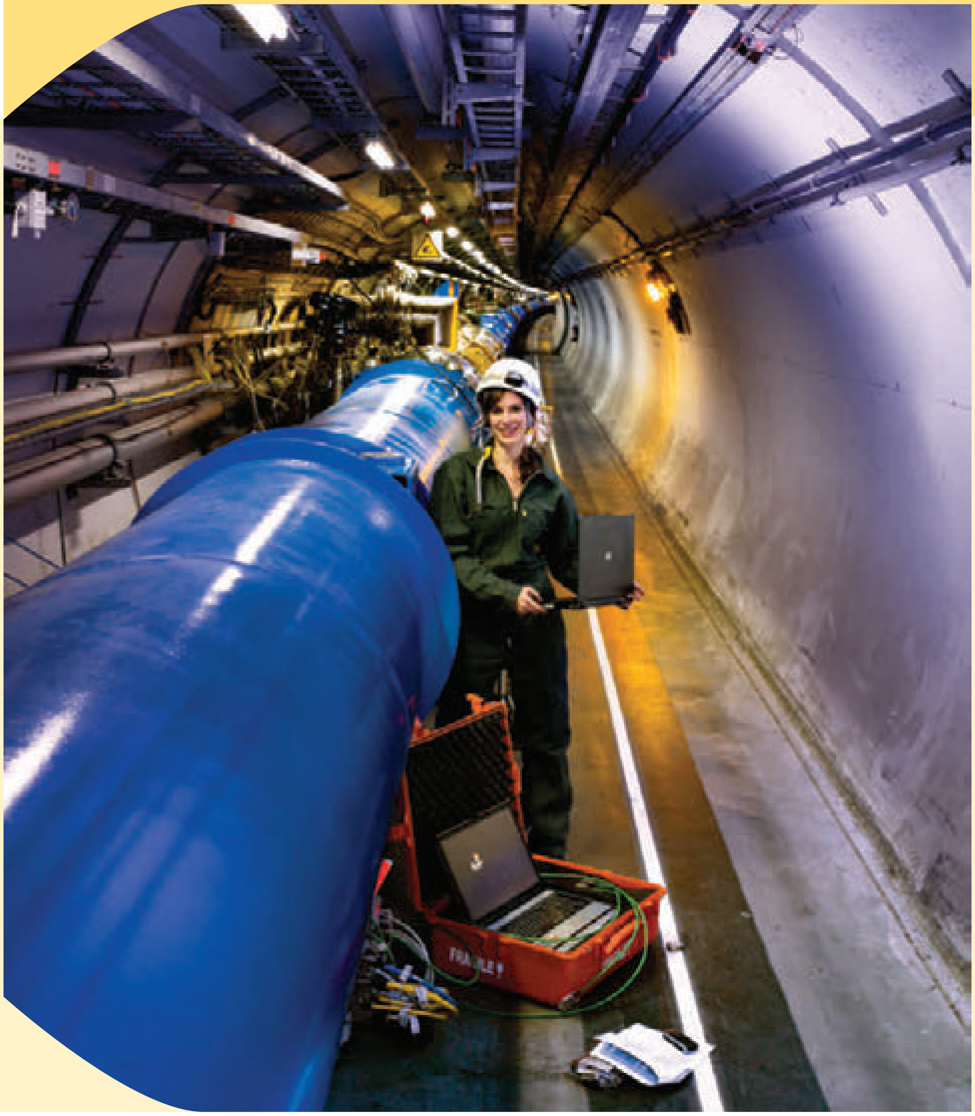


Figure P28.82

CHAPTER

29

Magnetic Fields



An engineer performs a test on the electronics associated with one of the superconducting magnets in the Large Hadron Collider at the European Laboratory for Particle Physics, operated by the European Organization for Nuclear Research (CERN). The magnets are used to control the motion of charged particles in the accelerator. We will study the effects of magnetic fields on moving charged particles in this chapter. *(CERN)*

Many historians of science believe that the compass, which uses a magnetic needle, was used in China as early as the 13th century BC, its invention being of Arabic or Indian origin. The early Greeks knew about magnetism as early as 800 BC. They discovered that the stone magnetite (Fe_3O_4) attracts pieces of iron. Legend ascribes the name *magnetite* to the shepherd Magnes, the nails of whose shoes and the tip of whose staff stuck fast to chunks of magnetite while he pastured his flocks.

In 1269, Pierre de Maricourt of France found that the directions of a needle near a spherical natural magnet formed lines that encircled the sphere and passed through two points diametrically opposite each other, which he called the *poles* of the magnet. Subsequent experiments showed that every magnet, regardless of its shape, has two poles, called *north* (N) and *south* (S) poles, that exert forces on other magnetic poles similar to the way electric charges exert forces on one another. That is, like poles (N–N or S–S) repel each other, and opposite poles (N–S) attract each other.

The poles received their names because of the way a magnet, such as that in a compass, behaves in the presence of the Earth's magnetic field. If a bar magnet is suspended from its midpoint and can swing freely in a horizontal plane, it will rotate until its north pole points to the Earth's geographic North Pole and its south pole points to the Earth's geographic South Pole.[1]

In 1600, William Gilbert (1540–1603) extended de Maricourt's experiments to a variety of materials. He knew that a compass needle orients in preferred directions, so he suggested that the Earth itself is a large, permanent magnet. In 1750, experimenters used a torsion balance to show that magnetic poles exert attractive or repulsive forces on each other and that these forces vary as the inverse square of the distance between interacting poles. Although the force between two magnetic poles is otherwise similar to the force between two electric charges, electric charges can be isolated (witness the electron and proton), whereas a single magnetic pole has never been isolated. That is, magnetic poles are always found in pairs. All attempts thus far to detect an isolated magnetic pole have been unsuccessful. No matter how many times a permanent magnet is cut in two, each piece always has a north and a south pole.[2]

The relationship between magnetism and electricity was discovered in 1819 when, during a lecture demonstration, Hans Christian Oersted found that an electric current in a wire deflected a nearby compass needle.[3] In the 1820s, further connections between electricity and magnetism were demonstrated independently by Faraday and Joseph Henry (1797–1878). They showed that an electric current can be produced in a circuit either by moving a magnet near the circuit or by changing the current in a nearby circuit. These observations demonstrate that a changing magnetic field creates an electric field. Years later, theoretical work by Maxwell showed that the reverse is also true: a changing electric field creates a magnetic field.

This chapter examines the forces that act on moving charges and on current-carrying wires in the presence of a magnetic field. The source of the magnetic field is described in Chapter 30.

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Hans Christian Oersted
Danish Physicist and Chemist (1777–1851)
Oersted is best known for observing that a compass needle deflects when placed near a wire carrying a current. This important discovery was the first evidence of the connection between electric and magnetic phenomena. Oersted was also the first to prepare pure aluminum.

29.1 Analysis Model: Particle in a Field (Magnetic)

In our study of electricity, we described the interactions between charged objects in terms of electric fields. Recall that an electric field surrounds any electric charge. In addition to containing an electric field, the region of space surrounding any *moving* electric charge also contains a **magnetic field.** A magnetic field also surrounds a magnetic substance making up a permanent magnet.

Historically, the symbol $\vec{\mathbf{B}}$ has been used to represent a magnetic field, and we use this notation in this book. The direction of the magnetic field $\vec{\mathbf{B}}$ at any location is the direction in which a compass needle points at that location. As with the electric field, we can represent the magnetic field by means of drawings with *magnetic field lines*.

Figure 29.1 shows how the magnetic field lines of a bar magnet can be traced with the aid of a compass. Notice that the magnetic field lines outside the magnet

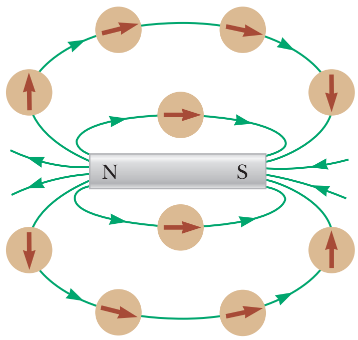


Figure 29.1 Compass needles can be used to trace the magnetic field lines in the region outside a bar magnet.

[1]The Earth's geographic North Pole is magnetically a south pole, whereas the Earth's geographic South Pole is magnetically a north pole. Because *opposite* magnetic poles attract each other, the pole on a magnet that is attracted to the Earth's geographic North Pole is the magnet's *north* pole and the pole attracted to the Earth's geographic South Pole is the magnet's *south* pole.

[2]There is some theoretical basis for speculating that magnetic *monopoles*—isolated north or south poles—may exist in nature, and attempts to detect them are an active experimental field of investigation.

[3]The same discovery was reported in 1802 by an Italian jurist, Gian Domenico Romagnosi, but was overlooked, probably because it was published in an obscure journal.

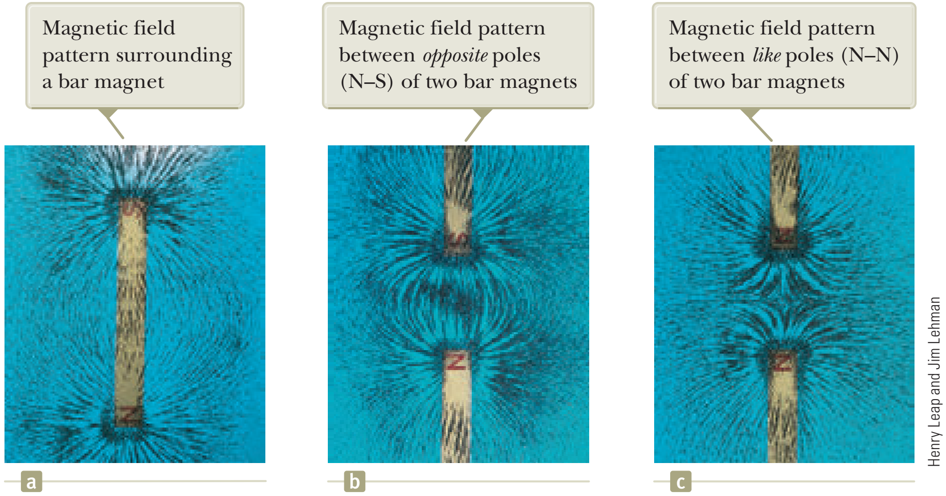


Figure 29.2 Magnetic field patterns can be displayed with iron filings sprinkled on paper near magnets.

point away from the north pole and toward the south pole. One can display magnetic field patterns of a bar magnet using small iron filings as shown in Figure 29.2.

When we speak of a compass magnet having a north pole and a south pole, it is more proper to say that it has a "north-seeking" pole and a "south-seeking" pole. This wording means that the north-seeking pole points to the north geographic pole of the Earth, whereas the south-seeking pole points to the south geographic pole. Because the north pole of a magnet is attracted toward the north geographic pole of the Earth, the Earth's south magnetic pole is located near the north geographic pole and the Earth's north magnetic pole is located near the south geographic pole. In fact, the configuration of the Earth's magnetic field, pictured in Figure 29.3, is very much like the one that would be achieved by burying a gigantic bar magnet deep in the Earth's interior. If a compass needle is supported by bearings that allow it to rotate in the vertical plane as well as in the horizontal plane, the needle is horizontal with respect to the Earth's surface only near the equator. As the compass is moved northward, the needle rotates so that it points more and more toward the Earth's surface. Finally, at a point near Hudson Bay in Canada, the north pole of the needle points directly downward. This site, first found in 1832, is considered to be the location of the south magnetic pole of the Earth. It is approximately 1 300 mi from the Earth's geographic

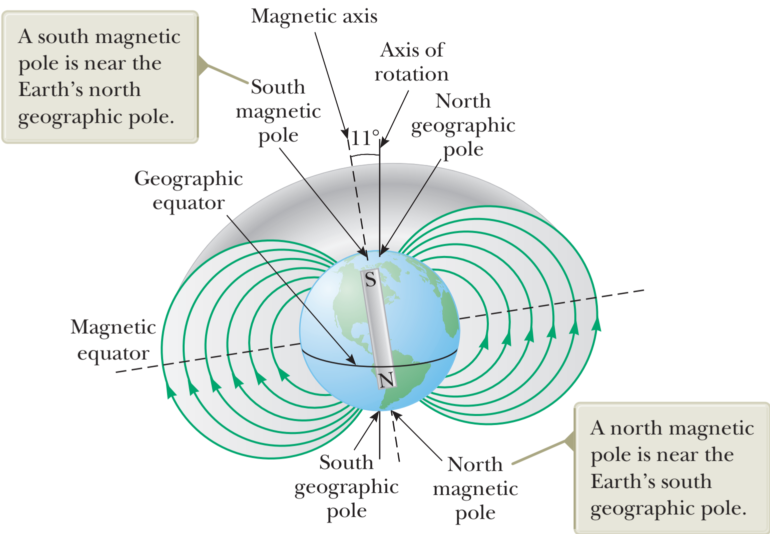


Figure 29.3 The Earth's magnetic field lines.

North Pole, and its exact position varies slowly with time. Similarly, the north magnetic pole of the Earth is about 1 200 mi away from the Earth's geographic South Pole.

Although the Earth's magnetic field pattern is similar to the one that would be set up by a bar magnet deep within the Earth, it is easy to understand why the source of this magnetic field cannot be large masses of permanently magnetized material. The Earth does have large deposits of iron ore deep beneath its surface, but the high temperatures in the Earth's core prevent the iron from retaining any permanent magnetization. Scientists consider it more likely that the source of the Earth's magnetic field is convection currents in the Earth's core. Charged ions or electrons circulating in the liquid interior could produce a magnetic field just like a current loop does, as we shall see in Chapter 30. There is also strong evidence that the magnitude of a planet's magnetic field is related to the planet's rate of rotation. For example, Jupiter rotates faster than the Earth, and space probes indicate that Jupiter's magnetic field is stronger than the Earth's. Venus, on the other hand, rotates more slowly than the Earth, and its magnetic field is found to be weaker. Investigation into the cause of the Earth's magnetism is ongoing.

The direction of the Earth's magnetic field has reversed several times during the last million years. Evidence for this reversal is provided by basalt, a type of rock that contains iron. Basalt forms from material spewed forth by volcanic activity on the ocean floor. As the lava cools, it solidifies and retains a picture of the Earth's magnetic field direction. The rocks are dated by other means to provide a time line for these periodic reversals of the magnetic field.

We can quantify the magnetic field $\vec{\mathbf{B}}$ by using our model of a particle in a field, like the model discussed for gravity in Chapter 13 and for electricity in Chapter 23. The existence of a magnetic field at some point in space can be determined by measuring the **magnetic force** $\vec{\mathbf{F}}_B$ exerted on an appropriate test particle placed at that point. This process is the same one we followed in defining the electric field in Chapter 23. If we perform such an experiment by placing a particle with charge q in the magnetic field, we find the following results that are similar to those for experiments on electric forces:

- The magnetic force is proportional to the charge q of the particle.
- The magnetic force on a negative charge is directed opposite to the force on a positive charge moving in the same direction.
- The magnetic force is proportional to the magnitude of the magnetic field vector $\vec{\mathbf{B}}$.

We also find the following results, which are *totally different* from those for experiments on electric forces:

- The magnetic force is proportional to the speed v of the particle.
- If the velocity vector makes an angle θ with the magnetic field, the magnitude of the magnetic force is proportional to sin θ.
- When a charged particle moves *parallel* to the magnetic field vector, the magnetic force on the charge is zero.
- When a charged particle moves in a direction *not* parallel to the magnetic field vector, the magnetic force acts in a direction perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$; that is, the magnetic force is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.

These results show that the magnetic force on a particle is more complicated than the electric force. The magnetic force is distinctive because it depends on the velocity of the particle and because its direction is perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. Figure 29.4 (page 872) shows the details of the direction of the magnetic force on a charged

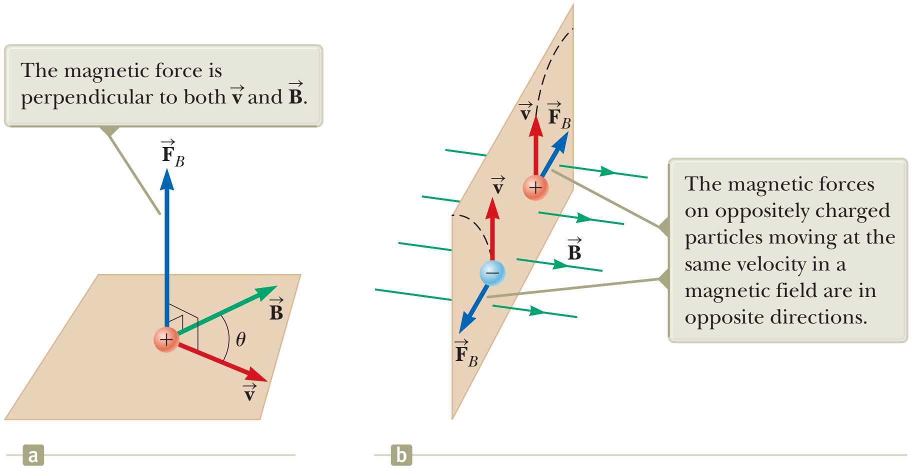


Figure 29.4 (a) The direction of the magnetic force $\vec{F}_B$ acting on a charged particle moving with a velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$. (b) Magnetic forces on positive and negative charges. The dashed lines show the paths of the particles, which are investigated in Section 29.2.

particle. Despite this complicated behavior, these observations can be summarized in a compact way by writing the magnetic force in the form

Vector expression for the magnetic force on a charged particle moving in a magnetic field

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{29.1}$$

which by definition of the cross product (see Section 11.1) is perpendicular to both $\vec{v}$ and $\vec{B}$. We can regard this equation as an operational definition of the magnetic field at some point in space. That is, the magnetic field is defined in terms of the force acting on a moving charged particle. Equation 29.1 is the mathematical representation of the magnetic version of the **particle in a field** analysis model.

Figure 29.5 reviews two right-hand rules for determining the direction of the cross product $\vec{v} \times \vec{B}$ and determining the direction of $\vec{F}_B$. The rule in Figure 29.5a depends on our right-hand rule for the cross product in Figure 11.2. Point the four fingers of your right hand along the direction of $\vec{v}$ with the palm facing $\vec{B}$ and curl them toward $\vec{B}$. Your extended thumb, which is at a right angle to your fingers, points in the direction of $\vec{v} \times \vec{B}$. Because $\vec{F}_B = q\vec{v} \times \vec{B}$, $\vec{F}_B$ is in the direction of your thumb if q is positive and is opposite the direction of your thumb if q is negative. (If you need more help understanding the cross product, you should review Section 11.1, including Fig. 11.2.)

An alternative rule is shown in Figure 29.5b. Here the thumb points in the direction of $\vec{v}$ and the extended fingers in the direction of $\vec{B}$. Now, the force $\vec{F}_B$ on a positive charge extends outward from the palm. The advantage of this rule is that the force on the charge is in the direction you would push on something with your

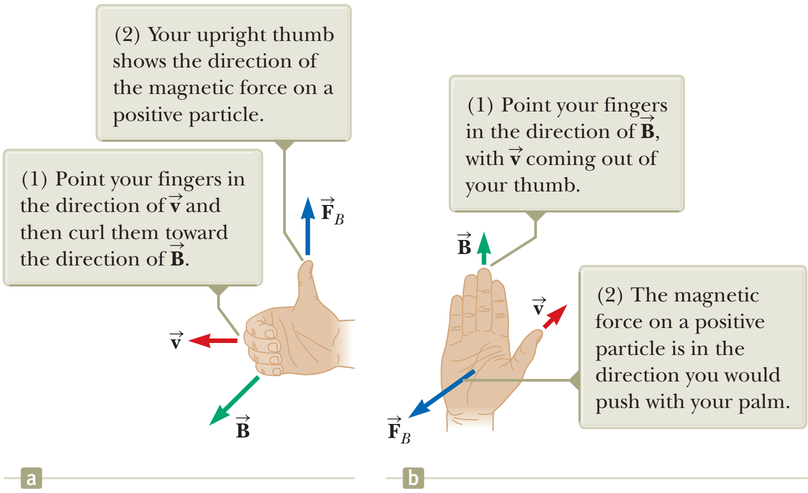


Figure 29.5 Two right-hand rules for determining the direction of the magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$ acting on a particle with charge q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$. (a) In this rule, the magnetic force is in the direction in which your thumb points. (b) In this rule, the magnetic force is in the direction of your palm, as if you are pushing the particle with your hand.

hand: outward from your palm. The force on a negative charge is in the opposite direction. You can use either of these two right-hand rules.

The magnitude of the magnetic force on a charged particle is

$$F_B = |q|vB\sin\theta \tag{29.2}$$

◀ **Magnitude of the magnetic force on a charged particle moving in a magnetic field**

where θ is the smaller angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. From this expression, we see that F_B is zero when $\vec{\mathbf{v}}$ is parallel or antiparallel to $\vec{\mathbf{B}}$ ($\theta = 0$ or $180°$) and maximum when $\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{B}}$ ($\theta = 90°$).

Let's compare the important differences between the electric and magnetic versions of the particle in a field model:

- The electric force vector is along the direction of the electric field, whereas the magnetic force vector is perpendicular to the magnetic field.
- The electric force acts on a charged particle regardless of whether the particle is moving, whereas the magnetic force acts on a charged particle only when the particle is in motion.
- The electric force does work in displacing a charged particle, whereas the magnetic force associated with a steady magnetic field does no work when a particle is displaced because the force is perpendicular to the displacement of its point of application.

From the last statement and on the basis of the work–kinetic energy theorem, we conclude that the kinetic energy of a charged particle moving through a magnetic field cannot be altered by the magnetic field alone. The field can alter the direction of the velocity vector, but it cannot change the speed or kinetic energy of the particle.

From Equation 29.2, we see that the SI unit of magnetic field is the newton per coulomb-meter per second, which is called the **tesla** (T):

$$1\ \mathrm{T} = 1\ \frac{\mathrm{N}}{\mathrm{C\cdot m/s}}$$

◀ **The tesla**

Because a coulomb per second is defined to be an ampere,

$$1\ \mathrm{T} = 1\ \frac{\mathrm{N}}{\mathrm{A\cdot m}}$$

A non-SI magnetic-field unit in common use, called the *gauss* (G), is related to the tesla through the conversion $1\ \mathrm{T} = 10^4\ \mathrm{G}$. Table 29.1 shows some typical values of magnetic fields.

Quick Quiz 29.1 An electron moves in the plane of this paper toward the top of the page. A magnetic field is also in the plane of the page and directed toward the right. What is the direction of the magnetic force on the electron? **(a)** toward the top of the page **(b)** toward the bottom of the page **(c)** toward the left edge of the page **(d)** toward the right edge of the page **(e)** upward out of the page **(f)** downward into the page

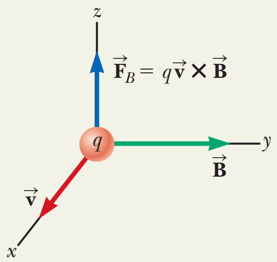
Table 29.1 Some Approximate Magnetic Field Magnitudes

Source of Field	Field Magnitude (T)
Strong superconducting laboratory magnet	30
Strong conventional laboratory magnet	2
Medical MRI unit	1.5
Bar magnet	10^{-2}
Surface of the Sun	10^{-2}
Surface of the Earth	0.5×10^{-4}
Inside human brain (due to nerve impulses)	10^{-13}

Analysis Model **Particle in a Field (Magnetic)**

Imagine some source (which we will investigate later) establishes a **magnetic field** $\vec{\mathbf{B}}$ throughout space. Now imagine a particle with charge q is placed in that field. The particle interacts with the magnetic field so that the particle experiences a magnetic force given by

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad (29.1)$$



Examples:

- an ion moves in a circular path in the magnetic field of a mass spectrometer (Section 29.3)
- a coil in a motor rotates in response to the magnetic field in the motor (Chapter 31)
- a magnetic field is used to separate particles emitted by radioactive sources (Chapter 44)
- in a bubble chamber, particles created in collisions follow curved paths in a magnetic field, allowing the particles to be identified (Chapter 46)

Example 29.1 **An Electron Moving in a Magnetic Field** AM

An electron in an old-style television picture tube moves toward the front of the tube with a speed of 8.0×10^6 m/s along the x axis (Fig. 29.6). Surrounding the neck of the tube are coils of wire that create a magnetic field of magnitude 0.025 T, directed at an angle of 60° to the x axis and lying in the xy plane. Calculate the magnetic force on the electron.

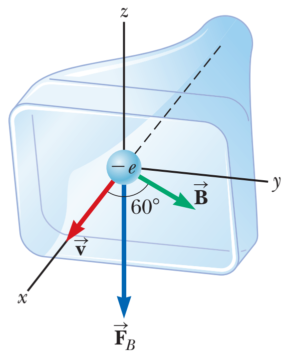


Figure 29.6 (Example 29.1) The magnetic force $\vec{\mathbf{F}}_B$ acting on the electron is in the negative z direction when $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ lie in the xy plane.

SOLUTION

Conceptualize Recall that the magnetic force on a charged particle is perpendicular to the plane formed by the velocity and magnetic field vectors. Use one of the right-hand rules in Figure 29.5 to convince yourself that the direction of the force on the electron is downward in Figure 29.6.

Categorize We evaluate the magnetic force using the *magnetic* version of the *particle in a field* model.

Analyze Use Equation 29.2 to find the magnitude of the magnetic force:

$$F_B = |q|vB \sin\theta$$
$$= (1.6 \times 10^{-19}\ \mathrm{C})(8.0 \times 10^6\ \mathrm{m/s})(0.025\ \mathrm{T})(\sin 60°)$$
$$= \boxed{2.8 \times 10^{-14}\ \mathrm{N}}$$

Finalize For practice using the vector product, evaluate this force in vector notation using Equation 29.1. The magnitude of the magnetic force may seem small to you, but remember that it is acting on a very small particle, the electron. To convince yourself that this is a substantial force for an electron, calculate the initial acceleration of the electron due to this force.

29.2 Motion of a Charged Particle in a Uniform Magnetic Field

Before we continue our discussion, some explanation of the notation used in this book is in order. To indicate the direction of $\vec{\mathbf{B}}$ in illustrations, we sometimes present perspective views such as those in Figure 29.6. If $\vec{\mathbf{B}}$ lies in the plane of the page or is present in a perspective drawing, we use green vectors or green field lines with arrowheads. In nonperspective illustrations, we depict a magnetic field perpendicular to and directed out of the page with a series of green dots, which represent the tips of arrows coming toward you (see Fig. 29.7a). In this case, the field is labeled

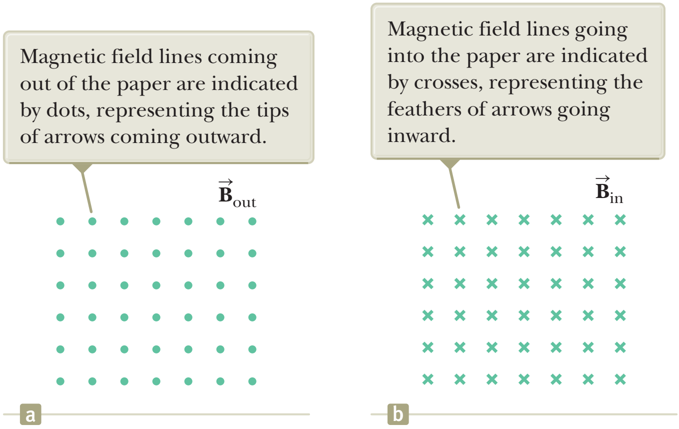


Figure 29.7 Representations of magnetic field lines perpendicular to the page.

$\vec{B}_{out}$. If $\vec{B}$ is directed perpendicularly into the page, we use green crosses, which represent the feathered tails of arrows fired away from you, as in Figure 29.7b. In this case, the field is labeled $\vec{B}_{in}$, where the subscript "in" indicates "into the page." The same notation with crosses and dots is also used for other quantities that might be perpendicular to the page such as forces and current directions.

In Section 29.1, we found that the magnetic force acting on a charged particle moving in a magnetic field is perpendicular to the particle's velocity and consequently the work done by the magnetic force on the particle is zero. Now consider the special case of a positively charged particle moving in a uniform magnetic field with the initial velocity vector of the particle perpendicular to the field. Let's assume the direction of the magnetic field is into the page as in Figure 29.8. The particle in a field model tells us that the magnetic force on the particle is perpendicular to both the magnetic field lines and the velocity of the particle. The fact that there is a force on the particle tells us to apply the particle under a net force model to the particle. As the particle changes the direction of its velocity in response to the magnetic force, the magnetic force remains perpendicular to the velocity. As we found in Section 6.1, if the force is always perpendicular to the velocity, the path of the particle is a circle! Figure 29.8 shows the particle moving in a circle in a plane perpendicular to the magnetic field. Although magnetism and magnetic forces may be new and unfamiliar to you now, we see a magnetic effect that results in something with which we are familiar: the particle in uniform circular motion model!

The particle moves in a circle because the magnetic force $\vec{F}_B$ is perpendicular to $\vec{v}$ and $\vec{B}$ and has a constant magnitude qvB. As Figure 29.8 illustrates, the

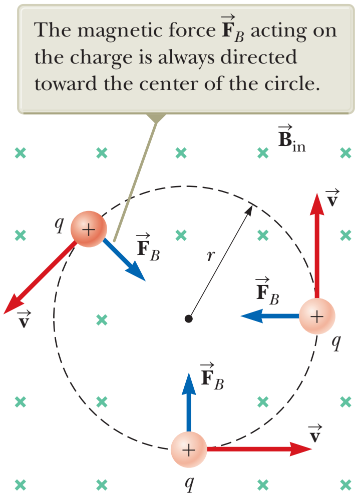


Figure 29.8 When the velocity of a charged particle is perpendicular to a uniform magnetic field, the particle moves in a circular path in a plane perpendicular to $\vec{B}$.

rotation is counterclockwise for a positive charge in a magnetic field directed into the page. If q were negative, the rotation would be clockwise. We use the particle under a net force model to write Newton's second law for the particle:

$$\sum F = F_B = ma$$

Because the particle moves in a circle, we also model it as a particle in uniform circular motion and we replace the acceleration with centripetal acceleration:

$$F_B = qvB = \frac{mv^2}{r}$$

This expression leads to the following equation for the radius of the circular path:

$$r = \frac{mv}{qB} \tag{29.3}$$

That is, the radius of the path is proportional to the linear momentum mv of the particle and inversely proportional to the magnitude of the charge on the particle and to the magnitude of the magnetic field. The angular speed of the particle (from Eq. 10.10) is

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{29.4}$$

The period of the motion (the time interval the particle requires to complete one revolution) is equal to the circumference of the circle divided by the speed of the particle:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \tag{29.5}$$

These results show that the angular speed of the particle and the period of the circular motion do not depend on the speed of the particle or on the radius of the orbit. The angular speed ω is often referred to as the **cyclotron frequency** because charged particles circulate at this angular frequency in the type of accelerator called a *cyclotron,* which is discussed in Section 29.3.

If a charged particle moves in a uniform magnetic field with its velocity at some arbitrary angle with respect to $\vec{\mathbf{B}}$, its path is a helix. For example, if the field is directed in the x direction as shown in Figure 29.9, there is no component of force in the x direction. As a result, $a_x = 0$, and the x component of velocity remains constant. The charged particle is a particle in equilibrium in this direction. The magnetic force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ causes the components v_y and v_z to change in time, however, and the resulting motion is a helix whose axis is parallel to the magnetic field. The projection of the path onto the yz plane (viewed along the x axis) is a circle. (The projections of the path onto the xy and xz planes are sinusoids!) Equations 29.3 to 29.5 still apply provided v is replaced by $v_\perp = \sqrt{v_y^2 + v_z^2}$.

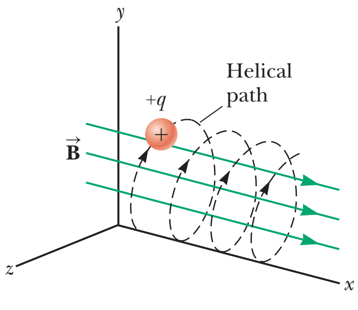


Figure 29.9 A charged particle having a velocity vector that has a component parallel to a uniform magnetic field moves in a helical path.

Quick Quiz 29.2 A charged particle is moving perpendicular to a magnetic field in a circle with a radius r. **(i)** An identical particle enters the field, with $\vec{\mathbf{v}}$ perpendicular to $\vec{\mathbf{B}}$, but with a higher speed than the first particle. Compared with the radius of the circle for the first particle, is the radius of the circular path for the second particle (a) smaller, (b) larger, or (c) equal in size? **(ii)** The magnitude of the magnetic field is increased. From the same choices, compare the radius of the new circular path of the first particle with the radius of its initial path.

Example 29.2 A Proton Moving Perpendicular to a Uniform Magnetic Field AM

A proton is moving in a circular orbit of radius 14 cm in a uniform 0.35-T magnetic field perpendicular to the velocity of the proton. Find the speed of the proton.

SOLUTION

Conceptualize From our discussion in this section, we know the proton follows a circular path when moving perpendicular to a uniform magnetic field. In Chapter 39, we will learn that the highest possible speed for a particle is the speed of light, 3.00×10^8 m/s, so the speed of the particle in this problem must come out to be smaller than that value.

Categorize The proton is described by both the *particle in a field* model and the *particle in uniform circular motion* model. These models led to Equation 29.3.

Analyze

Solve Equation 29.3 for the speed of the particle:

$$v = \frac{qBr}{m_p}$$

Substitute numerical values:

$$v = \frac{(1.60 \times 10^{-19}\ \mathrm{C})(0.35\ \mathrm{T})(0.14\ \mathrm{m})}{1.67 \times 10^{-27}\ \mathrm{kg}}$$

$$= \boxed{4.7 \times 10^6\ \mathrm{m/s}}$$

Finalize The speed is indeed smaller than the speed of light, as required.

WHAT IF? What if an electron, rather than a proton, moves in a direction perpendicular to the same magnetic field with this same speed? Will the radius of its orbit be different?

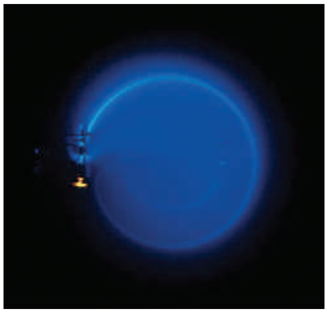
Answer An electron has a much smaller mass than a proton, so the magnetic force should be able to change its velocity much more easily than that for the proton. Therefore, we expect the radius to be smaller. Equation 29.3 shows that r is proportional to m with q, B, and v the same for the electron as for the proton. Consequently, the radius will be smaller by the same factor as the ratio of masses m_e/m_p.

Example 29.3 Bending an Electron Beam AM

In an experiment designed to measure the magnitude of a uniform magnetic field, electrons are accelerated from rest through a potential difference of 350 V and then enter a uniform magnetic field that is perpendicular to the velocity vector of the electrons. The electrons travel along a curved path because of the magnetic force exerted on them, and the radius of the path is measured to be 7.5 cm. (Such a curved beam of electrons is shown in Fig. 29.10.)

(A) What is the magnitude of the magnetic field?

continued



Henry Leap and Jim Lehman

Figure 29.10 (Example 29.3) The bending of an electron beam in a magnetic field.

29.3 continued

SOLUTION

Conceptualize This example involves electrons accelerating from rest due to an electric force and then moving in a circular path due to a magnetic force. With the help of Figures 29.8 and 29.10, visualize the circular motion of the electrons.

Categorize Equation 29.3 shows that we need the speed v of the electron to find the magnetic field magnitude, and v is not given. Consequently, we must find the speed of the electron based on the potential difference through which it is accelerated. To do so, we categorize the first part of the problem by modeling an electron and the electric field as an *isolated system* in terms of *energy*. Once the electron enters the magnetic field, we categorize the second part of the problem as one involving a *particle in a field* and a *particle in uniform circular motion*, as we have done in this section.

Analyze Write the appropriate reduction of the conservation of energy equation, Equation 8.2, for the electron–electric field system:

$$\Delta K + \Delta U = 0$$

Substitute the appropriate initial and final energies:

$$\left(\tfrac{1}{2}m_e v^2 - 0\right) + (q\,\Delta V) = 0$$

Solve for the speed of the electron:

$$v = \sqrt{\frac{-2q\,\Delta V}{m_e}}$$

Substitute numerical values:

$$v = \sqrt{\frac{-2(-1.60 \times 10^{-19}\ \text{C})(350\ \text{V})}{9.11 \times 10^{-31}\ \text{kg}}} = 1.11 \times 10^{7}\ \text{m/s}$$

Now imagine the electron entering the magnetic field with this speed. Solve Equation 29.3 for the magnitude of the magnetic field:

$$B = \frac{m_e v}{er}$$

Substitute numerical values:

$$B = \frac{(9.11 \times 10^{-31}\ \text{kg})(1.11 \times 10^{7}\ \text{m/s})}{(1.60 \times 10^{-19}\ \text{C})(0.075\ \text{m})} = \boxed{8.4 \times 10^{-4}\ \text{T}}$$

(B) What is the angular speed of the electrons?

SOLUTION

Use Equation 10.10:

$$\omega = \frac{v}{r} = \frac{1.11 \times 10^{7}\ \text{m/s}}{0.075\ \text{m}} = \boxed{1.5 \times 10^{8}\ \text{rad/s}}$$

Finalize The angular speed can be represented as $\omega = (1.5 \times 10^8\ \text{rad/s})(1\ \text{rev}/2\pi\ \text{rad}) = 2.4 \times 10^7$ rev/s. The electrons travel around the circle 24 million times per second! This answer is consistent with the very high speed found in part (A).

WHAT IF? What if a sudden voltage surge causes the accelerating voltage to increase to 400 V? How does that affect the angular speed of the electrons, assuming the magnetic field remains constant?

Answer The increase in accelerating voltage ΔV causes the electrons to enter the magnetic field with a higher speed v. This higher speed causes them to travel in a circle with a larger radius r. The angular speed is the ratio of v to r. Both v and r increase by the same factor, so the effects cancel and the angular speed remains the same. Equation 29.4 is an expression for the cyclotron frequency, which is the same as the angular speed of the electrons. The cyclotron frequency depends only on the charge q, the magnetic field B, and the mass m_e, none of which have changed. Therefore, the voltage surge has no effect on the angular speed. (In reality, however, the voltage surge may also increase the magnetic field if the magnetic field is powered by the same source as the accelerating voltage. In that case, the angular speed increases according to Eq. 29.4.)

When charged particles move in a nonuniform magnetic field, the motion is complex. For example, in a magnetic field that is strong at the ends and weak in the middle such as that shown in Figure 29.11, the particles can oscillate between two positions. A charged particle starting at one end spirals along the field lines until it reaches the other end, where it reverses its path and spirals back. This configura-

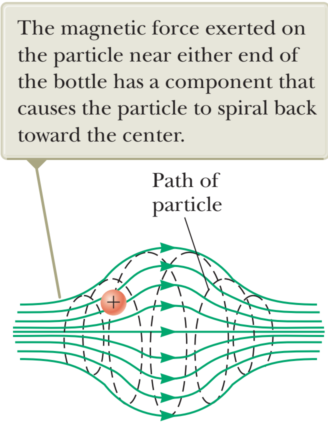


Figure 29.11 A charged particle moving in a nonuniform magnetic field (a magnetic bottle) spirals about the field and oscillates between the endpoints.

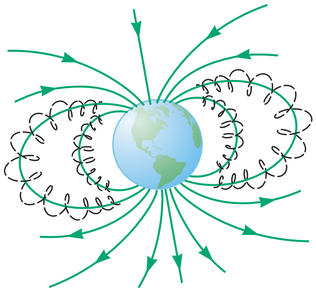


Figure 29.12 The Van Allen belts are made up of charged particles trapped by the Earth's nonuniform magnetic field. The magnetic field lines are in green, and the particle paths are dashed black lines.

tion is known as a *magnetic bottle* because charged particles can be trapped within it. The magnetic bottle has been used to confine a *plasma,* a gas consisting of ions and electrons. Such a plasma-confinement scheme could fulfill a crucial role in the control of nuclear fusion, a process that could supply us in the future with an almost endless source of energy. Unfortunately, the magnetic bottle has its problems. If a large number of particles are trapped, collisions between them cause the particles to eventually leak from the system.

The Van Allen radiation belts consist of charged particles (mostly electrons and protons) surrounding the Earth in doughnut-shaped regions (Fig. 29.12). The particles, trapped by the Earth's nonuniform magnetic field, spiral around the field lines from pole to pole, covering the distance in only a few seconds. These particles originate mainly from the Sun, but some come from stars and other heavenly objects. For this reason, the particles are called *cosmic rays*. Most cosmic rays are deflected by the Earth's magnetic field and never reach the atmosphere. Some of the particles become trapped, however, and it is these particles that make up the Van Allen belts. When the particles are located over the poles, they sometimes collide with atoms in the atmosphere, causing the atoms to emit visible light. Such collisions are the origin of the beautiful aurora borealis, or northern lights, in the northern hemisphere and the aurora australis in the southern hemisphere. Auroras are usually confined to the polar regions because the Van Allen belts are nearest the Earth's surface there. Occasionally, though, solar activity causes larger numbers of charged particles to enter the belts and significantly distort the normal magnetic field lines associated with the Earth. In these situations, an aurora can sometimes be seen at lower latitudes.

29.3 Applications Involving Charged Particles Moving in a Magnetic Field

A charge moving with a velocity $\vec{\mathbf{v}}$ in the presence of both an electric field $\vec{\mathbf{E}}$ and a magnetic field $\vec{\mathbf{B}}$ is described by two particle in a field models. It experiences both an electric force $q\vec{\mathbf{E}}$ and a magnetic force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. The total force (called the Lorentz force) acting on the charge is

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{29.6}$$

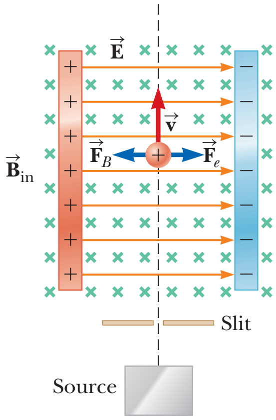


Figure 29.13 A velocity selector. When a positively charged particle is moving with velocity $\vec{\mathbf{v}}$ in the presence of a magnetic field directed into the page and an electric field directed to the right, it experiences an electric force $q\vec{\mathbf{E}}$ to the right and a magnetic force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ to the left.

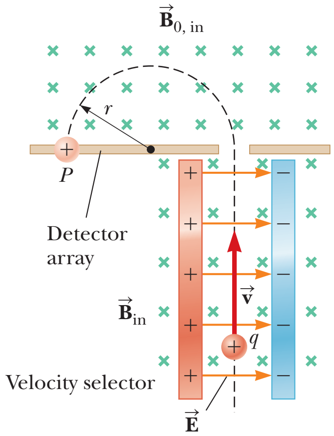


Figure 29.14 A mass spectrometer. Positively charged particles are sent first through a velocity selector and then into a region where the magnetic field $\vec{\mathbf{B}}_0$ causes the particles to move in a semicircular path and strike a detector array at P.

Velocity Selector

In many experiments involving moving charged particles, it is important that all particles move with essentially the same velocity, which can be achieved by applying a combination of an electric field and a magnetic field oriented as shown in Figure 29.13. A uniform electric field is directed to the right (in the plane of the page in Fig. 29.13), and a uniform magnetic field is applied in the direction perpendicular to the electric field (into the page in Fig. 29.13). If q is positive and the velocity $\vec{\mathbf{v}}$ is upward, the magnetic force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ is to the left and the electric force $q\vec{\mathbf{E}}$ is to the right. When the magnitudes of the two fields are chosen so that $qE = qvB$, the forces cancel. The charged particle is modeled as a particle in equilibrium and moves in a straight vertical line through the region of the fields. From the expression $qE = qvB$, we find that

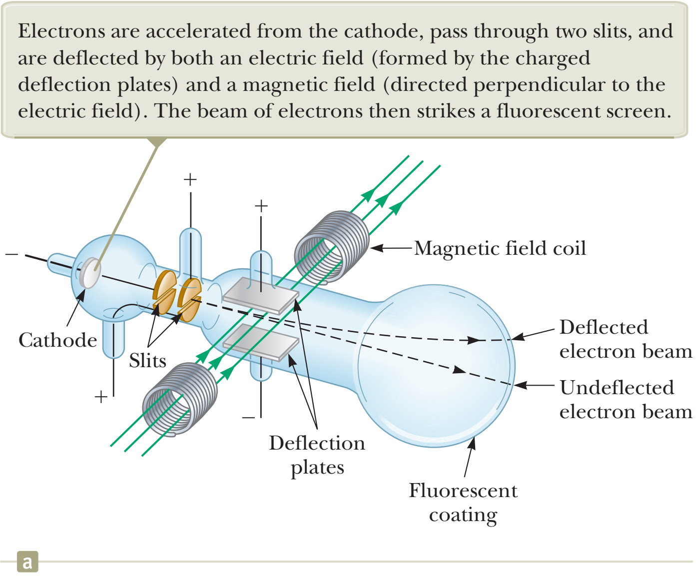
$$v = \frac{E}{B} \tag{29.7}$$

Only those particles having this speed pass undeflected through the mutually perpendicular electric and magnetic fields. The magnetic force exerted on particles moving at speeds greater than that is stronger than the electric force, and the particles are deflected to the left. Those moving at slower speeds are deflected to the right.

The Mass Spectrometer

A **mass spectrometer** separates ions according to their mass-to-charge ratio. In one version of this device, known as the *Bainbridge mass spectrometer,* a beam of ions first passes through a velocity selector and then enters a second uniform magnetic field $\vec{\mathbf{B}}_0$ that has the same direction as the magnetic field in the selector (Fig. 29.14). Upon entering the second magnetic field, the ions are described by the particle in uniform circular motion model. They move in a semicircle of radius r before striking a detector array at P. If the ions are positively charged, the beam deflects to the left as Figure 29.14 shows. If the ions are negatively charged, the beam deflects to the right. From Equation 29.3, we can express the ratio m/q as

$$\frac{m}{q} = \frac{rB_0}{v}$$



Lucent Technologies Bell Laboratory, courtesy AIP Emilio Segre Visual Archives

Figure 29.15 (a) Thomson's apparatus for measuring e/m_e. (b) J. J. Thomson *(left)* in the Cavendish Laboratory, University of Cambridge. The man on the right, Frank Baldwin Jewett, is a distant relative of John W. Jewett, Jr., coauthor of this text.

Using Equation 29.7 gives

$$\frac{m}{q} = \frac{rB_0B}{E} \qquad \textbf{(29.8)}$$

Therefore, we can determine m/q by measuring the radius of curvature and knowing the field magnitudes B, B_0, and E. In practice, one usually measures the masses of various isotopes of a given ion, with the ions all carrying the same charge q. In this way, the mass ratios can be determined even if q is unknown.

A variation of this technique was used by J. J. Thomson (1856–1940) in 1897 to measure the ratio e/m_e for electrons. Figure 29.15a shows the basic apparatus he used. Electrons are accelerated from the cathode and pass through two slits. They then drift into a region of perpendicular electric and magnetic fields. The magnitudes of the two fields are first adjusted to produce an undeflected beam. When the magnetic field is turned off, the electric field produces a measurable beam deflection that is recorded on the fluorescent screen. From the size of the deflection and the measured values of E and B, the charge-to-mass ratio can be determined. The results of this crucial experiment represent the discovery of the electron as a fundamental particle of nature.

The Cyclotron

A **cyclotron** is a device that can accelerate charged particles to very high speeds. The energetic particles produced are used to bombard atomic nuclei and thereby produce nuclear reactions of interest to researchers. A number of hospitals use cyclotron facilities to produce radioactive substances for diagnosis and treatment.

Both electric and magnetic forces play key roles in the operation of a cyclotron, a schematic drawing of which is shown in Figure 29.16a (page 882). The charges move inside two semicircular containers D_1 and D_2, referred to as *dees* because of their shape like the letter D. A high-frequency alternating potential difference is applied to the dees, and a uniform magnetic field is directed perpendicular to them. A positive ion released at P near the center of the magnet in one dee moves in a semicircular path (indicated by the dashed black line in the drawing) and arrives back at the gap in a time interval $T/2$, where T is the time interval needed to make one complete trip around the two dees, given by Equation 29.5. The frequency

Pitfall Prevention 29.1

The Cyclotron Is Not the Only Type of Particle Accelerator The cyclotron is important historically because it was the first particle accelerator to produce particles with very high speeds. Cyclotrons still play important roles in medical applications and some research activities. Many other research activities make use of a different type of accelerator called a *synchrotron*.

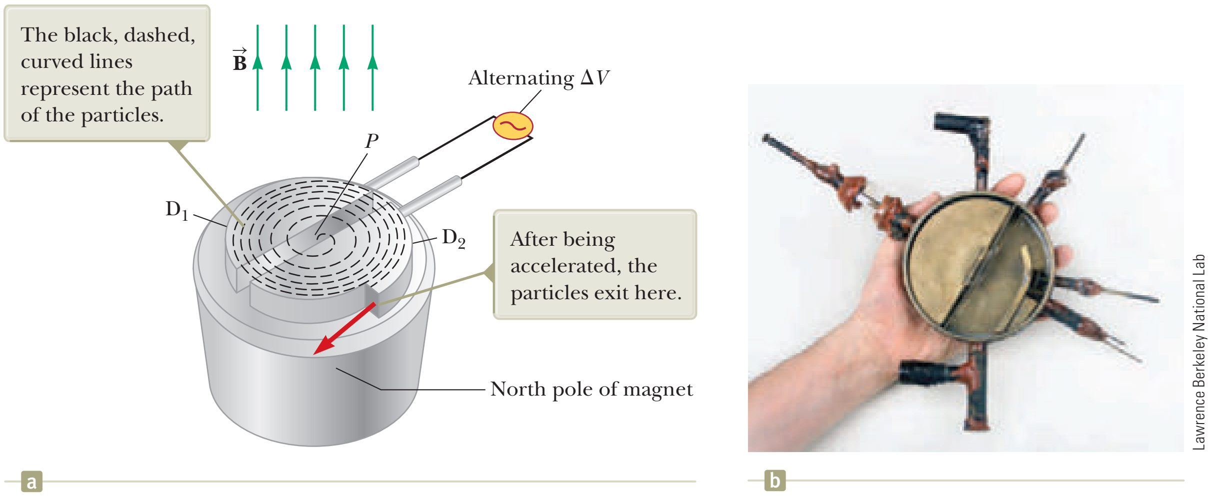


Figure 29.16 (a) A cyclotron consists of an ion source at P, two dees D_1 and D_2 across which an alternating potential difference is applied, and a uniform magnetic field. (The south pole of the magnet is not shown.) (b) The first cyclotron, invented by E. O. Lawrence and M. S. Livingston in 1934.

of the applied potential difference is adjusted so that the polarity of the dees is reversed in the same time interval during which the ion travels around one dee. If the applied potential difference is adjusted such that D_1 is at a lower electric potential than D_2 by an amount ΔV, the ion accelerates across the gap to D_1 and its kinetic energy increases by an amount $q\,\Delta V$. It then moves around D_1 in a semicircular path of greater radius (because its speed has increased). After a time interval $T/2$, it again arrives at the gap between the dees. By this time, the polarity across the dees has again been reversed and the ion is given another "kick" across the gap. The motion continues so that for each half-circle trip around one dee, the ion gains additional kinetic energy equal to $q\,\Delta V$. When the radius of its path is nearly that of the dees, the energetic ion leaves the system through the exit slit. The cyclotron's operation depends on T being independent of the speed of the ion and of the radius of the circular path (Eq. 29.5).

We can obtain an expression for the kinetic energy of the ion when it exits the cyclotron in terms of the radius R of the dees. From Equation 29.3, we know that $v = qBR/m$. Hence, the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m} \qquad \textbf{(29.9)}$$

When the energy of the ions in a cyclotron exceeds about 20 MeV, relativistic effects come into play. (Such effects are discussed in Chapter 39.) Observations show that T increases and the moving ions do not remain in phase with the applied potential difference. Some accelerators overcome this problem by modifying the period of the applied potential difference so that it remains in phase with the moving ions.

29.4 Magnetic Force Acting on a Current-Carrying Conductor

If a magnetic force is exerted on a single charged particle when the particle moves through a magnetic field, it should not surprise you that a current-carrying wire also experiences a force when placed in a magnetic field. The current is a collection of many charged particles in motion; hence, the resultant force exerted by the field on the wire is the vector sum of the individual forces exerted on all the charged particles making up the current. The force exerted on the particles is transmitted to the wire when the particles collide with the atoms making up the wire.

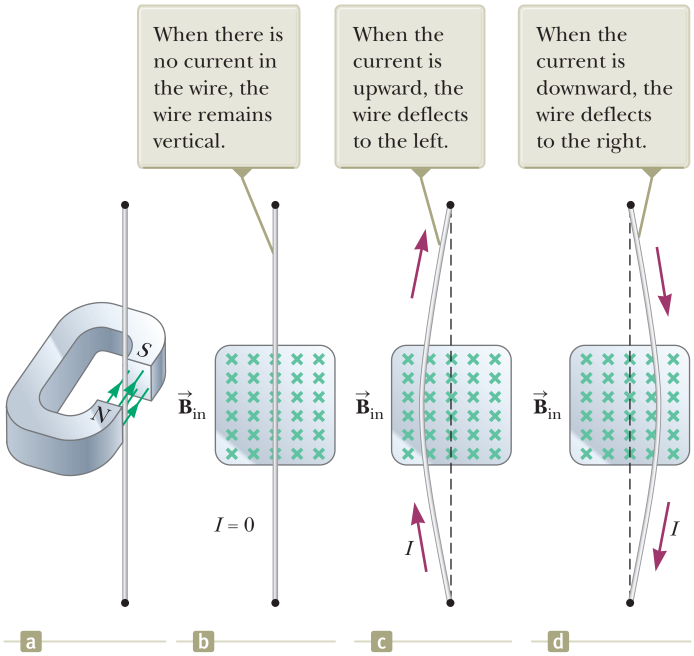


Figure 29.17 (a) A wire suspended vertically between the poles of a magnet. (b)–(d) The setup shown in (a) as seen looking at the south pole of the magnet so that the magnetic field (green crosses) is directed into the page.

One can demonstrate the magnetic force acting on a current-carrying conductor by hanging a wire between the poles of a magnet as shown in Figure 29.17a. For ease in visualization, part of the horseshoe magnet in part (a) is removed to show the end face of the south pole in parts (b) through (d) of Figure 29.17. The magnetic field is directed into the page and covers the region within the shaded squares. When the current in the wire is zero, the wire remains vertical as in Figure 29.17b. When the wire carries a current directed upward as in Figure 29.17c, however, the wire deflects to the left. If the current is reversed as in Figure 29.17d, the wire deflects to the right.

Let's quantify this discussion by considering a straight segment of wire of length L and cross-sectional area A carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$ as in Figure 29.18. According to the magnetic version of the particle in a field model, the magnetic force exerted on a charge q moving with a drift velocity $\vec{\mathbf{v}}_d$ is $q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$. To find the total force acting on the wire, we multiply the force $q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$ exerted on one charge by the number of charges in the segment. Because the volume of the segment is AL, the number of charges in the segment is nAL, where n is the number of mobile charge carriers per unit volume. Hence, the total magnetic force on the segment of wire of length L is

$$\vec{\mathbf{F}}_B = (q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}})nAL$$

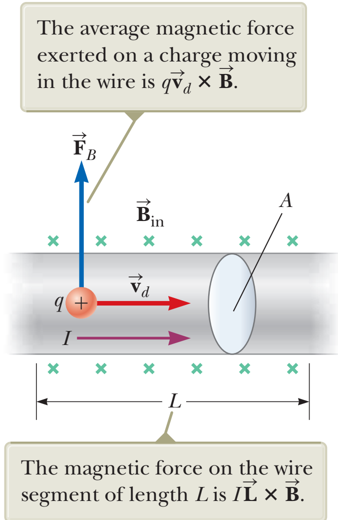


Figure 29.18 A segment of a current-carrying wire in a magnetic field $\vec{\mathbf{B}}$.

We can write this expression in a more convenient form by noting that, from Equation 27.4, the current in the wire is $I = nqv_dA$. Therefore,

$$\boxed{\vec{\mathbf{F}}_B = I\vec{\mathbf{L}} \times \vec{\mathbf{B}}} \tag{29.10}$$

◀ **Force on a segment of current-carrying wire in a uniform magnetic field**

where $\vec{\mathbf{L}}$ is a vector that points in the direction of the current I and has a magnitude equal to the length L of the segment. This expression applies only to a straight segment of wire in a uniform magnetic field.

Now consider an arbitrarily shaped wire segment of uniform cross section in a magnetic field as shown in Figure 29.19 (page 884). It follows from Equation 29.10 that the magnetic force exerted on a small segment of vector length $d\vec{\mathbf{s}}$ in the presence of a field $\vec{\mathbf{B}}$ is

$$d\vec{\mathbf{F}}_B = I\,d\vec{\mathbf{s}} \times \vec{\mathbf{B}} \tag{29.11}$$

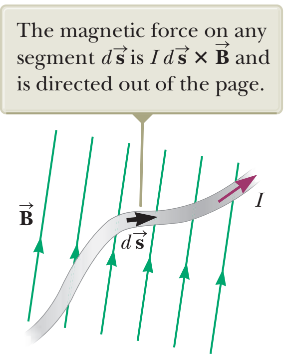


Figure 29.19 A wire segment of arbitrary shape carrying a current I in a magnetic field $\vec{\mathbf{B}}$ experiences a magnetic force.

where $d\vec{\mathbf{F}}_B$ is directed out of the page for the directions of $\vec{\mathbf{B}}$ and $d\vec{\mathbf{s}}$ in Figure 29.19. Equation 29.11 can be considered as an alternative definition of $\vec{\mathbf{B}}$. That is, we can define the magnetic field $\vec{\mathbf{B}}$ in terms of a measurable force exerted on a current element, where the force is a maximum when $\vec{\mathbf{B}}$ is perpendicular to the element and zero when $\vec{\mathbf{B}}$ is parallel to the element.

To calculate the total force $\vec{\mathbf{F}}_B$ acting on the wire shown in Figure 29.19, we integrate Equation 29.11 over the length of the wire:

$$\vec{\mathbf{F}}_B = I\int_a^b d\vec{\mathbf{s}} \times \vec{\mathbf{B}} \qquad \textbf{(29.12)}$$

where a and b represent the endpoints of the wire. When this integration is carried out, the magnitude of the magnetic field and the direction the field makes with the vector $d\vec{\mathbf{s}}$ may differ at different points.

Quick Quiz 29.3 A wire carries current in the plane of this paper toward the top of the page. The wire experiences a magnetic force toward the right edge of the page. Is the direction of the magnetic field causing this force **(a)** in the plane of the page and toward the left edge, **(b)** in the plane of the page and toward the bottom edge, **(c)** upward out of the page, or **(d)** downward into the page?

Example 29.4 Force on a Semicircular Conductor

A wire bent into a semicircle of radius R forms a closed circuit and carries a current I. The wire lies in the xy plane, and a uniform magnetic field is directed along the positive y axis as in Figure 29.20. Find the magnitude and direction of the magnetic force acting on the straight portion of the wire and on the curved portion.

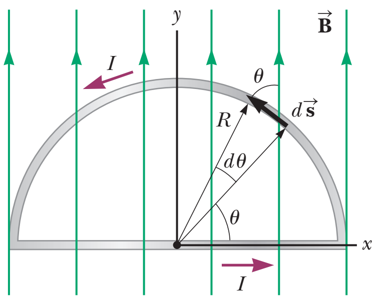


Figure 29.20 (Example 29.4) The magnetic force on the straight portion of the loop is directed out of the page, and the magnetic force on the curved portion is directed into the page.

SOLUTION

Conceptualize Using the right-hand rule for cross products, we see that the force $\vec{\mathbf{F}}_1$ on the straight portion of the wire is out of the page and the force $\vec{\mathbf{F}}_2$ on the curved portion is into the page. Is $\vec{\mathbf{F}}_2$ larger in magnitude than $\vec{\mathbf{F}}_1$ because the length of the curved portion is longer than that of the straight portion?

Categorize Because we are dealing with a current-carrying wire in a magnetic field rather than a single charged particle, we must use Equation 29.12 to find the total force on each portion of the wire.

Analyze Notice that $d\vec{\mathbf{s}}$ is perpendicular to $\vec{\mathbf{B}}$ everywhere on the straight portion of the wire. Use Equation 29.12 to find the force on this portion:

$$\vec{\mathbf{F}}_1 = I\int_a^b d\vec{\mathbf{s}} \times \vec{\mathbf{B}} = I\int_{-R}^{R} B\,dx\,\hat{\mathbf{k}} = \boxed{2IRB\,\hat{\mathbf{k}}}$$

29.4 continued

To find the magnetic force on the curved part, first write an expression for the magnetic force $d\vec{\mathbf{F}}_2$ on the element $d\vec{\mathbf{s}}$ in Figure 29.20:

$$(1)\quad d\vec{\mathbf{F}}_2 = I\,d\vec{\mathbf{s}} \times \vec{\mathbf{B}} = -IB\sin\theta\,ds\,\hat{\mathbf{k}}$$

From the geometry in Figure 29.20, write an expression for ds:

$$(2)\quad ds = R\,d\theta$$

Substitute Equation (2) into Equation (1) and integrate over the angle θ from 0 to π:

$$\vec{\mathbf{F}}_2 = -\int_0^{\pi} IRB\sin\theta\,d\theta\,\hat{\mathbf{k}} = -IRB\int_0^{\pi}\sin\theta\,d\theta\,\hat{\mathbf{k}} = -IRB\left[-\cos\theta\right]_0^{\pi}\hat{\mathbf{k}}$$

$$= IRB(\cos\pi - \cos 0)\hat{\mathbf{k}} = IRB(-1-1)\hat{\mathbf{k}} = \boxed{-2IRB\,\hat{\mathbf{k}}}$$

Finalize Two very important general statements follow from this example. First, the force on the curved portion is the same in magnitude as the force on a straight wire between the same two points. In general, the magnetic force on a curved current-carrying wire in a uniform magnetic field is equal to that on a straight wire connecting the endpoints and carrying the same current. Furthermore, $\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = 0$ is also a general result: the net magnetic force acting on any closed current loop in a uniform magnetic field is zero.

29.5 Torque on a Current Loop in a Uniform Magnetic Field

In Section 29.4, we showed how a magnetic force is exerted on a current-carrying conductor placed in a magnetic field. With that as a starting point, we now show that a torque is exerted on a current loop placed in a magnetic field.

Consider a rectangular loop carrying a current I in the presence of a uniform magnetic field directed parallel to the plane of the loop as shown in Figure 29.21a. No magnetic forces act on sides ① and ③ because these wires are parallel to the field; hence, $\vec{\mathbf{L}} \times \vec{\mathbf{B}} = 0$ for these sides. Magnetic forces do, however, act on sides ② and ④ because these sides are oriented perpendicular to the field. The magnitude of these forces is, from Equation 29.10,

$$F_2 = F_4 = IaB$$

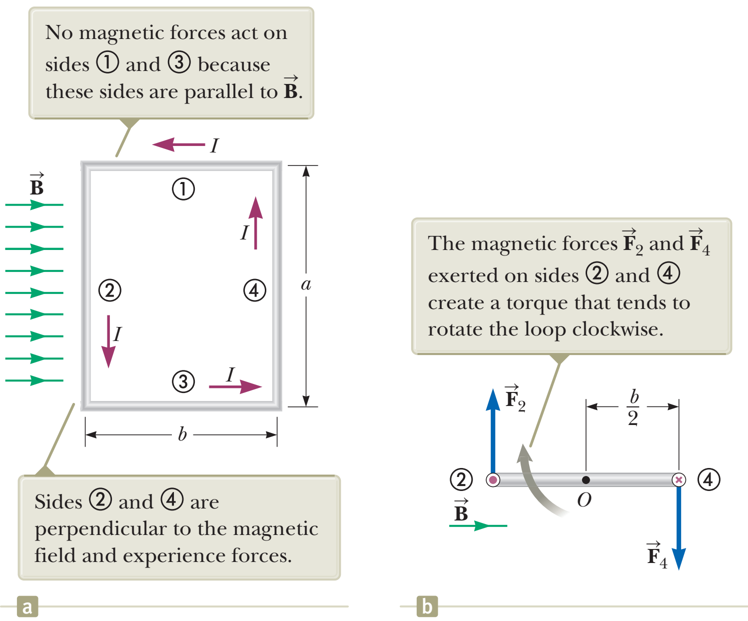


Figure 29.21 (a) Overhead view of a rectangular current loop in a uniform magnetic field. (b) Edge view of the loop sighting down sides ② and ④. The purple dot in the left circle represents current in wire ② coming toward you; the purple cross in the right circle represents current in wire ④ moving away from you.

The direction of $\vec{\mathbf{F}}_2$, the magnetic force exerted on wire ②, is out of the page in the view shown in Figure 29.20a and that of $\vec{\mathbf{F}}_4$, the magnetic force exerted on wire ④, is into the page in the same view. If we view the loop from side ③ and sight along sides ② and ④, we see the view shown in Figure 29.21b, and the two magnetic forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ are directed as shown. Notice that the two forces point in opposite directions but are *not* directed along the same line of action. If the loop is pivoted so that it can rotate about point O, these two forces produce about O a torque that rotates the loop clockwise. The magnitude of this torque $\tau_{\max}$ is

$$\tau_{\max} = F_2 \frac{b}{2} + F_4 \frac{b}{2} = (IaB)\frac{b}{2} + (IaB)\frac{b}{2} = IabB$$

where the moment arm about O is $b/2$ for each force. Because the area enclosed by the loop is $A = ab$, we can express the maximum torque as

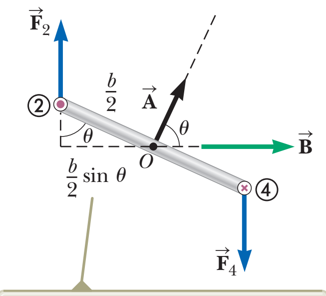
$$\tau_{\max} = IAB \tag{29.13}$$

This maximum-torque result is valid only when the magnetic field is parallel to the plane of the loop. The sense of the rotation is clockwise when viewed from side ③ as indicated in Figure 29.21b. If the current direction were reversed, the force directions would also reverse and the rotational tendency would be counterclockwise.

Now suppose the uniform magnetic field makes an angle $\theta < 90°$ with a line perpendicular to the plane of the loop as in Figure 29.22. For convenience, let's assume $\vec{\mathbf{B}}$ is perpendicular to sides ② and ④. In this case, the magnetic forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_3$ exerted on sides ① and ③ cancel each other and produce no torque because they act along the same line. The magnetic forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ acting on sides ② and ④, however, produce a torque about *any point.* Referring to the edge view shown in Figure 29.22, we see that the moment arm of $\vec{\mathbf{F}}_2$ about the point O is equal to $(b/2) \sin\theta$. Likewise, the moment arm of $\vec{\mathbf{F}}_4$ about O is also equal to $(b/2) \sin\theta$. Because $F_2 = F_4 = IaB$, the magnitude of the net torque about O is

$$\begin{aligned}\tau &= F_2 \frac{b}{2} \sin\theta + F_4 \frac{b}{2} \sin\theta \\ &= IaB\left(\frac{b}{2}\sin\theta\right) + IaB\left(\frac{b}{2}\sin\theta\right) = IabB\sin\theta \\ &= IAB\sin\theta\end{aligned}$$

where $A = ab$ is the area of the loop. This result shows that the torque has its maximum value IAB when the field is perpendicular to the normal to the plane of the loop ($\theta = 90°$) as discussed with regard to Figure 29.21 and is zero when the field is parallel to the normal to the plane of the loop ($\theta = 0$).



When the normal to the loop makes an angle θ with the magnetic field, the moment arm for the torque is $(b/2) \sin\theta$.

Figure 29.22 An edge view of the loop in Figure 29.21 with the normal to the loop at an angle θ with respect to the magnetic field.

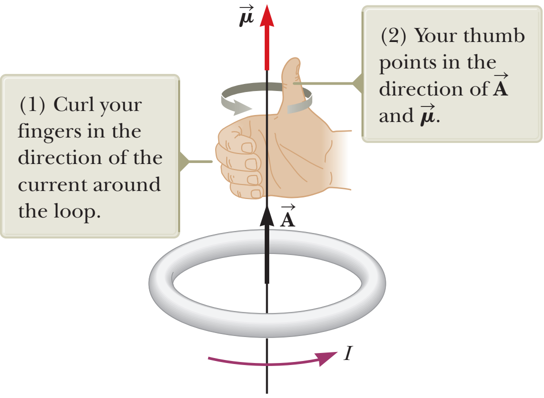


Figure 29.23 Right-hand rule for determining the direction of the vector $\vec{\mathbf{A}}$ for a current loop. The direction of the magnetic moment $\vec{\boldsymbol{\mu}}$ is the same as the direction of $\vec{\mathbf{A}}$.

A convenient vector expression for the torque exerted on a loop placed in a uniform magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\boldsymbol{\tau}} = I\vec{\mathbf{A}} \times \vec{\mathbf{B}} \tag{29.14}$$

◀ **Torque on a current loop in a magnetic field**

where $\vec{\mathbf{A}}$, the vector shown in Figure 29.22, is perpendicular to the plane of the loop and has a magnitude equal to the area of the loop. To determine the direction of $\vec{\mathbf{A}}$, use the right-hand rule described in Figure 29.23. When you curl the fingers of your right hand in the direction of the current in the loop, your thumb points in the direction of $\vec{\mathbf{A}}$. Figure 29.22 shows that the loop tends to rotate in the direction of decreasing values of θ (that is, such that the area vector $\vec{\mathbf{A}}$ rotates toward the direction of the magnetic field).

The product $I\vec{\mathbf{A}}$ is defined to be the **magnetic dipole moment** $\vec{\boldsymbol{\mu}}$ (often simply called the "magnetic moment") of the loop:

$$\vec{\boldsymbol{\mu}} \equiv I\vec{\mathbf{A}} \tag{29.15}$$

◀ **Magnetic dipole moment of a current loop**

The SI unit of magnetic dipole moment is the ampere-meter^2 ($A \cdot m^2$). If a coil of wire contains N loops of the same area, the magnetic moment of the coil is

$$\vec{\boldsymbol{\mu}}_{\text{coil}} = NI\vec{\mathbf{A}} \tag{29.16}$$

Using Equation 29.15, we can express the torque exerted on a current-carrying loop in a magnetic field $\vec{\mathbf{B}}$ as

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}} \tag{29.17}$$

◀ **Torque on a magnetic moment in a magnetic field**

This result is analogous to Equation 26.18, $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$, for the torque exerted on an electric dipole in the presence of an electric field $\vec{\mathbf{E}}$, where $\vec{\mathbf{p}}$ is the electric dipole moment.

Although we obtained the torque for a particular orientation of $\vec{\mathbf{B}}$ with respect to the loop, the equation $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$ is valid for any orientation. Furthermore, although we derived the torque expression for a rectangular loop, the result is valid for a loop of any shape. The torque on an N-turn coil is given by Equation 29.17 by using Equation 29.16 for the magnetic moment.

In Section 26.6, we found that the potential energy of a system of an electric dipole in an electric field is given by $U_E = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}$. This energy depends on the orientation of the dipole in the electric field. Likewise, the potential energy of a system of a magnetic dipole in a magnetic field depends on the orientation of the dipole in the magnetic field and is given by

$$U_B = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}} \tag{29.18}$$

◀ **Potential energy of a system of a magnetic moment in a magnetic field**

This expression shows that the system has its lowest energy $U_{\min} = -\mu B$ when $\vec{\boldsymbol{\mu}}$ points in the same direction as $\vec{\mathbf{B}}$. The system has its highest energy $U_{\max} = +\mu B$ when $\vec{\boldsymbol{\mu}}$ points in the direction opposite $\vec{\mathbf{B}}$.

Imagine the loop in Figure 29.22 is pivoted at point O on sides ① and ③, so that it is free to rotate. If the loop carries current and the magnetic field is turned on, the loop is modeled as a rigid object under a net torque, with the torque given by Equation 29.17. The torque on the current loop causes the loop to rotate; this effect is exploited practically in a **motor.** Energy enters the motor by electrical transmission, and the rotating coil can do work on some device external to the motor. For example, the motor in a car's electrical window system does work on the windows, applying a force on them and moving them up or down through some displacement. We will discuss motors in more detail in Section 31.5.

Quick Quiz 29.4 **(i)** Rank the magnitudes of the torques acting on the rectangular loops (a), (b), and (c) shown edge-on in Figure 29.24 from highest to lowest. All loops are identical and carry the same current. **(ii)** Rank the magnitudes of the net forces acting on the rectangular loops shown in Figure 29.24 from highest to lowest.

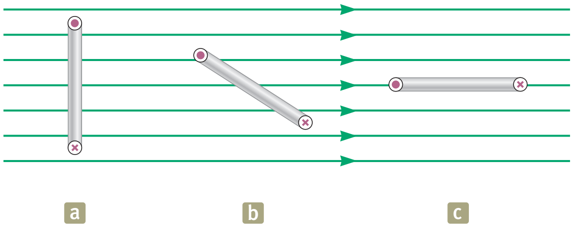


Figure 29.24 (Quick Quiz 29.4) Which current loop (seen edge-on) experiences the greatest torque, (a), (b), or (c)? Which experiences the greatest net force?

Example 29.5 The Magnetic Dipole Moment of a Coil

A rectangular coil of dimensions 5.40 cm × 8.50 cm consists of 25 turns of wire and carries a current of 15.0 mA. A 0.350-T magnetic field is applied parallel to the plane of the coil.

(A) Calculate the magnitude of the magnetic dipole moment of the coil.

SOLUTION

Conceptualize The magnetic moment of the coil is independent of any magnetic field in which the loop resides, so it depends only on the geometry of the loop and the current it carries.

Categorize We evaluate quantities based on equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 29.16 to calculate the magnetic moment associated with a coil consisting of N turns:

$$\mu_{\text{coil}} = NIA = (25)(15.0 \times 10^{-3}\text{ A})(0.054\,0\text{ m})(0.085\,0\text{ m})$$

$$= \boxed{1.72 \times 10^{-3}\text{ A} \cdot \text{m}^2}$$

(B) What is the magnitude of the torque acting on the loop?

SOLUTION

Use Equation 29.17, noting that $\vec{\mathbf{B}}$ is perpendicular to $\vec{\boldsymbol{\mu}}_{\text{coil}}$:

$$\tau = \mu_{\text{coil}} B = (1.72 \times 10^{-3}\text{ A} \cdot \text{m}^2)(0.350\text{ T})$$

$$= \boxed{6.02 \times 10^{-4}\text{ N} \cdot \text{m}}$$

Example 29.6 Rotating a Coil

Consider the loop of wire in Figure 29.25a. Imagine it is pivoted along side , which is parallel to the axis and fastened so that side remains fixed and the rest of the loop hangs vertically in the gravitational field of the Earth but can rotate around side (Fig. 29.25b). The mass of the loop is 50.0 g, and the sides are of lengths 0.200 m and 0.100 m. The loop carries a current of 3.50 A and is immersed in a vertical uniform magnetic field of magnitude 0.010 0 T in the positive direction (Fig. 29.25c). What angle does the plane of the loop make with the vertical?

SOLU ION

Conceptualize In the edge view of Figure 29.25b, notice that the magnetic moment of the loop is to the left. Therefore, when the loop is in the magnetic field, the magnetic torque on the loop causes it to rotate in a clockwise direction around side which we choose as the rotation axis. Imagine the loop making this clockwise rotation so that the plane of the loop is at some angle to the vertical as in Figure 29.25c. The gravitational force on the loop exerts a torque that would cause a rotation in the counterclockwise direction if the magnetic field were turned off.

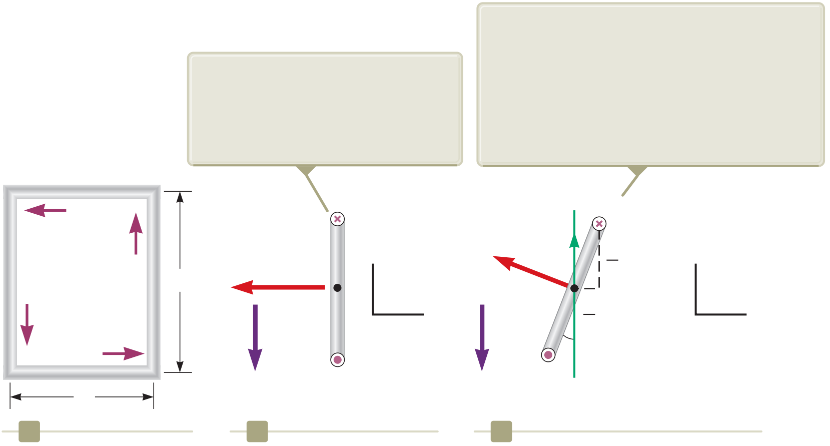


Figure 29.25 (Example 29.6) (a) The dimensions of a rectangular current loop. (b) Edge view of the loop sighting down sides and . (c) An edge view of the loop in (b) rotated through an angle with respect to the horizontal when it is placed in a magnetic field.

Categorize At some angle of the loop, the two torques described in the Conceptualize step are equal in magnitude and the loop is at rest. We therefore model the loop as a *rigid object in equilibrium.*

Analyze Evaluate the magnetic torque on the loop about side from Equation 29.17:

$$= -\mu \sin \quad 90° - \theta \quad = -IAB \cos \quad \mathbf{k} = -IabB \cos \quad \mathbf{k}$$

Evaluate the gravitational torque on the loop, noting that the gravitational force can be modeled to act at the center of the loop:

$$mg - \sin \quad \mathbf{k}$$

From the rigid body in equilibrium model, add the torques and set the net torque equal to zero:

$$= -IabB \cos \qquad mg - \sin \quad \mathbf{k}$$

Solve for

$$IabB \cos \theta = mg - \sin \quad \tan \qquad \theta = \frac{IaB}{mg}$$

$$\theta = \tan \quad \frac{IaB}{mg}$$

Substitute numerical values:

$$\theta = \tan \quad \frac{3.50\text{ A})(0.200\text{ m})(0.010\text{ 0 T}}{0.050\text{ 0 kg})(9.80\text{ m}} \qquad 1.64$$

Finalize The angle is relatively small, so the loop still hangs almost vertically. If the current or the magnetic field is increased, however, the angle increases as the magnetic torque becomes stronger.

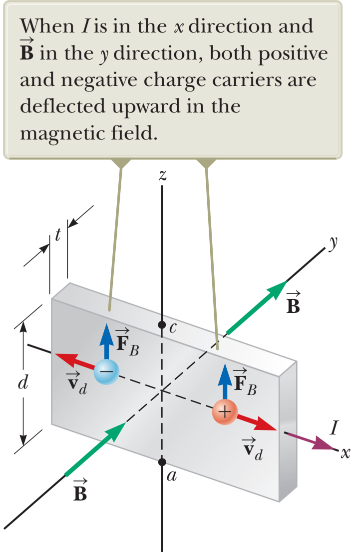


Figure 29.26 To observe the Hall effect, a magnetic field is applied to a current-carrying conductor. The Hall voltage is measured between points a and c.

29.6 The Hall Effect

When a current-carrying conductor is placed in a magnetic field, a potential difference is generated in a direction perpendicular to both the current and the magnetic field. This phenomenon, first observed by Edwin Hall (1855–1938) in 1879, is known as the *Hall effect*. The arrangement for observing the Hall effect consists of a flat conductor carrying a current I in the x direction as shown in Figure 29.26. A uniform magnetic field $\vec{\mathbf{B}}$ is applied in the y direction. If the charge carriers are electrons moving in the negative x direction with a drift velocity $\vec{\mathbf{v}}_d$, they experience an upward magnetic force $\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$, are deflected upward, and accumulate at the upper edge of the flat conductor, leaving an excess of positive charge at the lower edge (Fig. 29.27a). This accumulation of charge at the edges establishes an electric field in the conductor and increases until the electric force on carriers remaining in the bulk of the conductor balances the magnetic force acting on the carriers. The electrons can now be described by the particle in equilibrium model, and they are no longer deflected upward. A sensitive voltmeter connected across the sample as shown in Figure 29.27 can measure the potential difference, known as the **Hall voltage** ΔV_H, generated across the conductor.

If the charge carriers are positive and hence move in the positive x direction (for rightward current) as shown in Figures 29.26 and 29.27b, they also experience an upward magnetic force $q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$, which produces a buildup of positive charge on the upper edge and leaves an excess of negative charge on the lower edge. Hence, the sign of the Hall voltage generated in the sample is opposite the sign of the Hall voltage resulting from the deflection of electrons. The sign of the charge carriers can therefore be determined from measuring the polarity of the Hall voltage.

In deriving an expression for the Hall voltage, first note that the magnetic force exerted on the carriers has magnitude qv_dB. In equilibrium, this force is balanced by the electric force qE_H, where E_H is the magnitude of the electric field due to the charge separation (sometimes referred to as the *Hall field*). Therefore,

$$qv_dB = qE_H$$

$$E_H = v_dB$$

If d is the width of the conductor, the Hall voltage is

$$\Delta V_H = E_H d = v_d B d \tag{29.19}$$

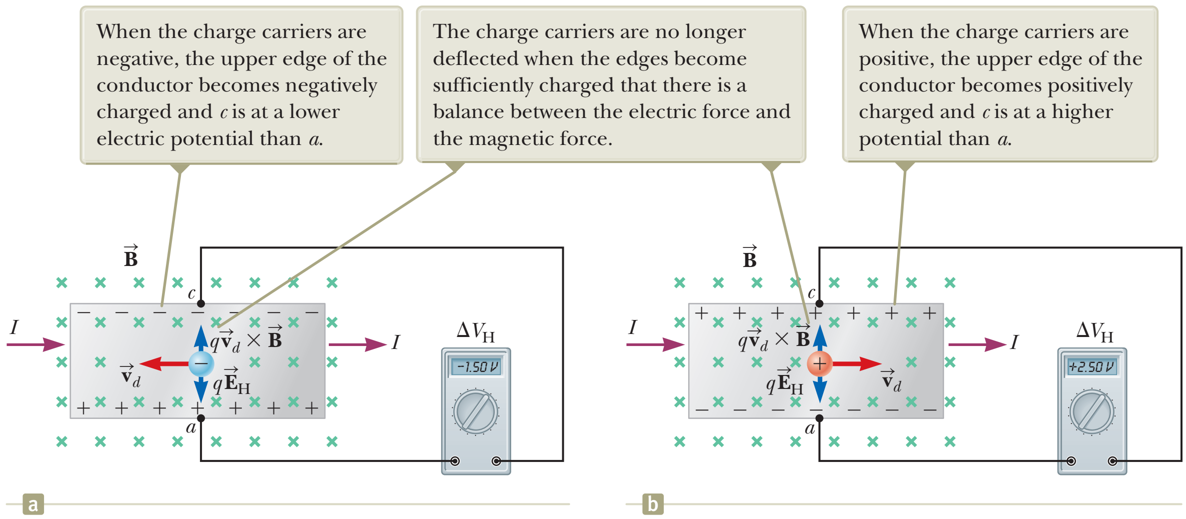


Figure 29.27 The sign of the Hall voltage depends on the sign of the charge carriers.

Therefore, the measured Hall voltage gives a value for the drift speed of the charge carriers if d and B are known.

We can obtain the charge-carrier density n by measuring the current in the sample. From Equation 27.4, we can express the drift speed as

$$v_d = \frac{I}{nqA} \tag{29.20}$$

where A is the cross-sectional area of the conductor. Substituting Equation 29.20 into Equation 29.19 gives

$$\Delta V_H = \frac{IBd}{nqA} \tag{29.21}$$

Because $A = td$, where t is the thickness of the conductor, we can also express Equation 29.21 as

$$\Delta V_H = \frac{IB}{nqt} = \frac{R_H IB}{t} \tag{29.22}$$

◀ The Hall voltage

where $R_H = 1/nq$ is called the **Hall coefficient.** This relationship shows that a properly calibrated conductor can be used to measure the magnitude of an unknown magnetic field.

Because all quantities in Equation 29.22 other than nq can be measured, a value for the Hall coefficient is readily obtainable. The sign and magnitude of R_H give the sign of the charge carriers and their number density. In most metals, the charge carriers are electrons and the charge-carrier density determined from Hall-effect measurements is in good agreement with calculated values for such metals as lithium (Li), sodium (Na), copper (Cu), and silver (Ag), whose atoms each give up one electron to act as a current carrier. In this case, n is approximately equal to the number of conducting electrons per unit volume. This classical model, however, is not valid for metals such as iron (Fe), bismuth (Bi), and cadmium (Cd) or for semiconductors. These discrepancies can be explained only by using a model based on the quantum nature of solids.

Example 29.7 The Hall Effect for Copper

A rectangular copper strip 1.5 cm wide and 0.10 cm thick carries a current of 5.0 A. Find the Hall voltage for a 1.2-T magnetic field applied in a direction perpendicular to the strip.

SOLUTION

Conceptualize Study Figures 29.26 and 29.27 carefully and make sure you understand that a Hall voltage is developed between the top and bottom edges of the strip.

Categorize We evaluate the Hall voltage using an equation developed in this section, so we categorize this example as a substitution problem.

Assuming one electron per atom is available for conduction, find the charge-carrier density in terms of the molar mass M and density ρ of copper:

$$n = \frac{N_A}{V} = \frac{N_A \rho}{M}$$

Substitute this result into Equation 29.22:

$$\Delta V_H = \frac{IB}{nqt} = \frac{MIB}{N_A \rho q t}$$

Substitute numerical values:

$$\Delta V_H = \frac{(0.063\,5\text{ kg/mol})(5.0\text{ A})(1.2\text{ T})}{(6.02 \times 10^{23}\text{ mol}^{-1})(8\,920\text{ kg/m}^3)(1.60 \times 10^{-19}\text{ C})(0.001\,0\text{ m})}$$

$$= \boxed{0.44\ \mu\text{V}}$$

continued

29.7 continued

Such an extremely small Hall voltage is expected in good conductors. (Notice that the width of the conductor is not needed in this calculation.)

WHAT IF? What if the strip has the same dimensions but is made of a semiconductor? Will the Hall voltage be smaller or larger?

Answer In semiconductors, n is much smaller than it is in metals that contribute one electron per atom to the current; hence, the Hall voltage is usually larger because it varies as the inverse of n. Currents on the order of 0.1 mA are generally used for such materials. Consider a piece of silicon that has the same dimensions as the copper strip in this example and whose value for n is 1.0×10^{20} electrons/m^3. Taking $B = 1.2$ T and $I = 0.10$ mA, we find that $\Delta V_H = 7.5$ mV. A potential difference of this magnitude is readily measured.

Summary

Definition

The **magnetic dipole moment** $\vec{\boldsymbol{\mu}}$ of a loop carrying a current I is

$$\vec{\boldsymbol{\mu}} \equiv I\vec{\mathbf{A}} \qquad (29.15)$$

where the area vector $\vec{\mathbf{A}}$ is perpendicular to the plane of the loop and $|\vec{\mathbf{A}}|$ is equal to the area of the loop. The SI unit of $\vec{\boldsymbol{\mu}}$ is $\mathrm{A \cdot m^2}$.

Concepts and Principles

If a charged particle moves in a uniform magnetic field so that its initial velocity is perpendicular to the field, the particle moves in a circle, the plane of which is perpendicular to the magnetic field. The radius of the circular path is

$$r = \frac{mv}{qB} \qquad (29.3)$$

where m is the mass of the particle and q is its charge. The angular speed of the charged particle is

$$\omega = \frac{qB}{m} \qquad (29.4)$$

If a straight conductor of length L carries a current I, the force exerted on that conductor when it is placed in a uniform magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{F}}_B = I\vec{\mathbf{L}} \times \vec{\mathbf{B}} \qquad (29.10)$$

where the direction of $\vec{\mathbf{L}}$ is in the direction of the current and $|\vec{\mathbf{L}}| = L$.

If an arbitrarily shaped wire carrying a current I is placed in a magnetic field, the magnetic force exerted on a very small segment $d\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{F}}_B = I\, d\vec{\mathbf{s}} \times \vec{\mathbf{B}} \qquad (29.11)$$

To determine the total magnetic force on the wire, one must integrate Equation 29.11 over the wire, keeping in mind that both $\vec{\mathbf{B}}$ and $d\vec{\mathbf{s}}$ may vary at each point.

The torque $\vec{\boldsymbol{\tau}}$ on a current loop placed in a uniform magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}} \qquad (29.17)$$

The potential energy of the system of a magnetic dipole in a magnetic field is

$$U_B = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}} \qquad (29.18)$$

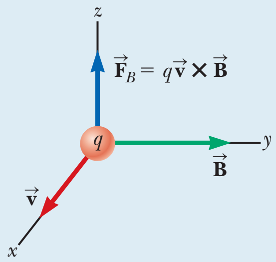
Analysis Models for Problem Solving

Particle in a Field (Magnetic) A source (to be discussed in Chapter 30) establishes a **magnetic field $\vec{\mathbf{B}}$** throughout space. When a particle with charge q and moving with velocity $\vec{\mathbf{v}}$ is placed in that field, it experiences a magnetic force given by

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \quad \textbf{(29.1)}$$

The direction of this magnetic force is perpendicular both to the velocity of the particle and to the magnetic field. The magnitude of this force is

$$F_B = |q|vB\sin\theta \quad \textbf{(29.2)}$$



where θ is the smaller angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. The SI unit of $\vec{\mathbf{B}}$ is the **tesla** (T), where 1 T = 1 N/A · m.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

Objective Questions 3, 4, and 6 in Chapter 11 can be assigned with this chapter as review for the vector product.

1. A spatially uniform magnetic field cannot exert a magnetic force on a particle in which of the following circumstances? There may be more than one correct statement. (a) The particle is charged. (b) The particle moves perpendicular to the magnetic field. (c) The particle moves parallel to the magnetic field. (d) The magnitude of the magnetic field changes with time. (e) The particle is at rest.

2. Rank the magnitudes of the forces exerted on the following particles from largest to smallest. In your ranking, display any cases of equality. (a) an electron moving at 1 Mm/s perpendicular to a 1-mT magnetic field (b) an electron moving at 1 Mm/s parallel to a 1-mT magnetic field (c) an electron moving at 2 Mm/s perpendicular to a 1-mT magnetic field (d) a proton moving at 1 Mm/s perpendicular to a 1-mT magnetic field (e) a proton moving at 1 Mm/s at a 45° angle to a 1-mT magnetic field

3. A particle with electric charge is fired into a region of space where the electric field is zero. It moves in a straight line. Can you conclude that the magnetic field in that region is zero? (a) Yes, you can. (b) No; the field might be perpendicular to the particle's velocity. (c) No; the field might be parallel to the particle's velocity. (d) No; the particle might need to have charge of the opposite sign to have a force exerted on it. (e) No; an observation of an object with *electric* charge gives no information about a *magnetic* field.

4. A proton moving horizontally enters a region where a uniform magnetic field is directed perpendicular to the proton's velocity as shown in Figure OQ29.4. After the proton enters the field, does it (a) deflect downward, with its speed remaining constant; (b) deflect upward, moving in a semicircular path with constant speed, and exit the field moving to the left; (c) continue to move in the horizontal direction with constant velocity; (d) move in a circular orbit and become trapped by the field; or (e) deflect out of the plane of the paper?

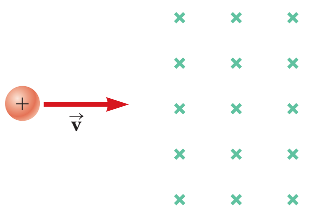


Figure OQ29.4

5. At a certain instant, a proton is moving in the positive x direction through a magnetic field in the negative z direction. What is the direction of the magnetic force exerted on the proton? (a) positive z direction (b) negative z direction (c) positive y direction (d) negative y direction (e) The force is zero.

6. A thin copper rod 1.00 m long has a mass of 50.0 g. What is the minimum current in the rod that would allow it to levitate above the ground in a magnetic field of magnitude 0.100 T? (a) 1.20 A (b) 2.40 A (c) 4.90 A (d) 9.80 A (e) none of those answers

7. Electron A is fired horizontally with speed 1.00 Mm/s into a region where a vertical magnetic field exists. Electron B is fired along the same path with speed 2.00 Mm/s. **(i)** Which electron has a larger magnetic force exerted on it? (a) A does. (b) B does. (c) The forces have the same nonzero magnitude. (d) The forces are both zero. **(ii)** Which electron has a path that curves more sharply? (a) A does. (b) B does. (c) The particles follow the same curved path. (d) The particles continue to go straight.

8. Classify each of the following statements as a characteristic (a) of electric forces only, (b) of magnetic forces only, (c) of both electric and magnetic forces, or (d) of neither electric nor magnetic forces. **(i)** The force is proportional to the magnitude of the field exerting it. **(ii)** The force is proportional to the magnitude of the charge of the object on which the force is exerted. **(iii)** The force exerted on a negatively charged object is opposite in direction to the force on a positive charge. **(iv)** The force exerted on a stationary charged object is nonzero. **(v)** The force exerted on a moving charged object is zero. **(vi)** The force exerted on a charged object is proportional to its speed. **(vii)** The force exerted on a charged object cannot alter the object's speed. **(viii)** The magnitude of the force depends on the charged object's direction of motion.

9. An electron moves horizontally across the Earth's equator at a speed of 2.50×10^6 m/s and in a direction 35.0° N of E. At this point, the Earth's magnetic field has a direction due north, is parallel to the surface, and has a value of 3.00×10^{-5} T. What is the force acting on the electron due to its interaction with the Earth's magnetic field? (a) 6.88×10^{-18} N due west (b) 6.88×10^{-18} N toward the Earth's surface (c) 9.83×10^{-18} N toward the Earth's surface (d) 9.83×10^{-18} N away from the Earth's surface (e) 4.00×10^{-18} N away from the Earth's surface

10. A charged particle is traveling through a uniform magnetic field. Which of the following statements are true of the magnetic field? There may be more than one correct statement. (a) It exerts a force on the particle parallel to the field. (b) It exerts a force on the particle along the direction of its motion. (c) It increases the kinetic energy of the particle. (d) It exerts a force that is perpendicular to the direction of motion. (e) It does not change the magnitude of the momentum of the particle.

11. In the velocity selector shown in Figure 29.13, electrons with speed $v = E/B$ follow a straight path. Electrons moving significantly faster than this speed through the same selector will move along what kind of path? (a) a circle (b) a parabola (c) a straight line (d) a more complicated trajectory

12. Answer each question yes or no. Assume the motions and currents mentioned are along the x axis and fields are in the y direction. (a) Does an electric field exert a force on a stationary charged object? (b) Does a magnetic field do so? (c) Does an electric field exert a force on a moving charged object? (d) Does a magnetic field do so? (e) Does an electric field exert a force on a straight current-carrying wire? (f) Does a magnetic field do so? (g) Does an electric field exert a force on a beam of moving electrons? (h) Does a magnetic field do so?

13. A magnetic field exerts a torque on each of the current-carrying single loops of wire shown in Figure OQ29.13. The loops lie in the xy plane, each carrying the same magnitude current, and the uniform magnetic field points in the positive x direction. Rank the loops by the magnitude of the torque exerted on them by the field from largest to smallest.

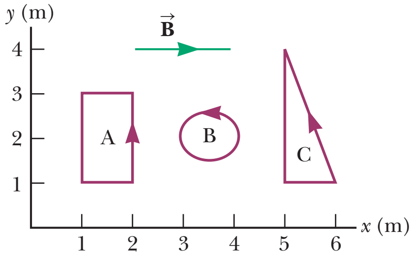


Figure OQ29.13

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Can a constant magnetic field set into motion an electron initially at rest? Explain your answer.

2. Explain why it is not possible to determine the charge and the mass of a charged particle separately by measuring accelerations produced by electric and magnetic forces on the particle.

3. Is it possible to orient a current loop in a uniform magnetic field such that the loop does not tend to rotate? Explain.

4. How can the motion of a moving charged particle be used to distinguish between a magnetic field and an electric field? Give a specific example to justify your argument.

5. How can a current loop be used to determine the presence of a magnetic field in a given region of space?

6. Charged particles from outer space, called cosmic rays, strike the Earth more frequently near the poles than near the equator. Why?

7. Two charged particles are projected in the same direction into a magnetic field perpendicular to their velocities. If the particles are deflected in opposite directions, what can you say about them?

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 29.1 Analysis Model: Particle in a Field (Magnetic)

Problems 1–4, 6–7, and 10 in Chapter 11 can be assigned with this section as review for the vector product.

1. At the equator, near the surface of the Earth, the magnetic field is approximately 50.0 μT northward, and the electric field is about 100 N/C downward in fair weather. Find the gravitational, electric, and magnetic forces on an electron in this environment, assuming that the electron has an instantaneous velocity of 6.00×10^6 m/s directed to the east.

2. W Determine the initial direction of the deflection of charged particles as they enter the magnetic fields shown in Figure P29.2.

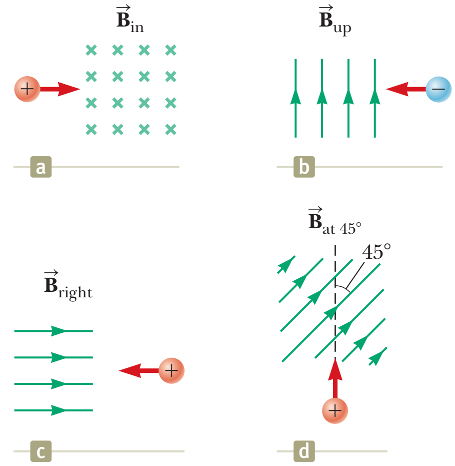


Figure P29.2

3. Find the direction of the magnetic field acting on a positively charged particle moving in the various situations shown in Figure P29.3 if the direction of the magnetic force acting on it is as indicated.

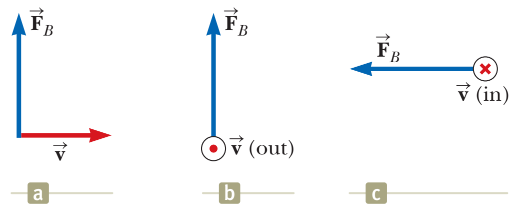


Figure P29.3

4. Consider an electron near the Earth's equator. In which direction does it tend to deflect if its velocity is (a) directed downward? (b) Directed northward? (c) Directed westward? (d) Directed southeastward?

5. A proton is projected into a magnetic field that is directed along the positive x axis. Find the direction of the magnetic force exerted on the proton for each of the following directions of the proton's velocity: (a) the positive y direction, (b) the negative y direction, (c) the positive x direction.

6. M A proton moving at 4.00×10^6 m/s through a magnetic field of magnitude 1.70 T experiences a magnetic force of magnitude 8.20×10^{-13} N. What is the angle between the proton's velocity and the field?

7. W An electron is accelerated through 2.40×10^3 V from rest and then enters a uniform 1.70-T magnetic field. What are (a) the maximum and (b) the minimum values of the magnetic force this particle experiences?

8. W A proton moves with a velocity of $\vec{v} = (2\hat{i} - 4\hat{j} + \hat{k})$ m/s in a region in which the magnetic field is $\vec{B} = (\hat{i} + 2\hat{j} - \hat{k})$T. What is the magnitude of the magnetic force this particle experiences?

9. AMT A proton travels with a speed of 5.02×10^6 m/s in a direction that makes an angle of 60.0° with the direction of a magnetic field of magnitude 0.180 T in the positive x direction. What are the magnitudes of (a) the magnetic force on the proton and (b) the proton's acceleration?

10. A laboratory electromagnet produces a magnetic field of magnitude 1.50 T. A proton moves through this field with a speed of 6.00×10^6 m/s. (a) Find the magnitude of the maximum magnetic force that could be exerted on the proton. (b) What is the magnitude of the maximum acceleration of the proton? (c) Would the field exert the same magnetic force on an electron moving through the field with the same speed? (d) Would the electron experience the same acceleration? Explain.

11. M A proton moves perpendicular to a uniform magnetic field $\vec{B}$ at a speed of 1.00×10^7 m/s and experiences an acceleration of 2.00×10^{13} m/s^2 in the positive x direction when its velocity is in the positive z direction. Determine the magnitude and direction of the field.

12. **Review.** A charged particle of mass 1.50 g is moving at a speed of 1.50×10^4 m/s. Suddenly, a uniform magnetic field of magnitude 0.150 mT in a direction perpendicular to the particle's velocity is turned on and then turned off in a time interval of 1.00 s. During this time interval, the magnitude and direction of the velocity of the particle undergo a negligible change, but the particle moves by a distance of 0.150 m in a direction perpendicular to the velocity. Find the charge on the particle.

Section 29.2 Motion of a Charged Particle in a Uniform Magnetic Field

13. An electron moves in a circular path perpendicular to a uniform magnetic field with a magnitude of 2.00 mT. If the speed of the electron is 1.50×10^7 m/s, determine (a) the radius of the circular path and (b) the time interval required to complete one revolution.

14. An accelerating voltage of 2.50×10^3 V is applied to an electron gun, producing a beam of electrons originally traveling horizontally north in vacuum toward the center of a viewing screen 35.0 cm away. What are (a) the magnitude and (b) the direction of the deflection on the screen caused by the Earth's gravitational field? What are (c) the magnitude and (d) the direction of the deflection on the screen caused by the vertical component of the Earth's magnetic field, taken as 20.0 μT down? (e) Does an electron in this vertical magnetic field move as a projectile, with constant vector acceleration perpendicular to a constant northward component of velocity? (f) Is it a good approximation to assume it has this projectile motion? Explain.

15. A proton (charge $+e$, mass m_p), a deuteron (charge $+e$, mass $2m_p$), and an alpha particle (charge $+2e$, mass $4m_p$) are accelerated from rest through a common potential difference ΔV. Each of the particles enters a uniform magnetic field $\vec{\mathbf{B}}$, with its velocity in a direction perpendicular to $\vec{\mathbf{B}}$. The proton moves in a circular path of radius r_p. In terms of r_p, determine (a) the radius r_d of the circular orbit for the deuteron and (b) the radius r_α for the alpha particle.

16. A particle with charge q and kinetic energy K travels in a uniform magnetic field of magnitude B. If the particle moves in a circular path of radius R, find expressions for (a) its speed and (b) its mass.

17. AMT **Review.** One electron collides elastically with a second electron initially at rest. After the collision, the radii of their trajectories are 1.00 cm and 2.40 cm. The trajectories are perpendicular to a uniform magnetic field of magnitude 0.044 0 T. Determine the energy (in keV) of the incident electron.

18. **Review.** One electron collides elastically with a second electron initially at rest. After the collision, the radii of their trajectories are r_1 and r_2. The trajectories are perpendicular to a uniform magnetic field of magnitude B. Determine the energy of the incident electron.

19. **Review.** An electron moves in a circular path perpendicular to a constant magnetic field of magnitude 1.00 mT. The angular momentum of the electron about the center of the circle is 4.00×10^{-25} kg $\cdot$ m^2/s. Determine (a) the radius of the circular path and (b) the speed of the electron.

20. **Review.** A 30.0-g metal ball having net charge $Q = 5.00$ μC is thrown out of a window horizontally north at a speed $v = 20.0$ m/s. The window is at a height $h = 20.0$ m above the ground. A uniform, horizontal magnetic field of magnitude $B = 0.010\,0$ T is perpendicular to the plane of the ball's trajectory and directed toward the west. (a) Assuming the ball follows the same trajectory as it would in the absence of the magnetic field, find the magnetic force acting on the ball just before it hits the ground. (b) Based on the result of part (a), is it justified for three-significant-digit precision to assume the trajectory is unaffected by the magnetic field? Explain.

21. M A cosmic-ray proton in interstellar space has an energy of 10.0 MeV and executes a circular orbit having a radius equal to that of Mercury's orbit around the Sun (5.80×10^{10} m). What is the magnetic field in that region of space?

22. Assume the region to the right of a certain plane contains a uniform magnetic field of magnitude 1.00 mT and the field is zero in the region to the left of the plane as shown in Figure P29.22. An electron, originally traveling perpendicular to the boundary plane, passes into the region of the field. (a) Determine the time interval required for the electron to leave the "field-filled" region, noting that the electron's path is a semicircle. (b) Assuming the maximum depth of penetration into the field is 2.00 cm, find the kinetic energy of the electron.

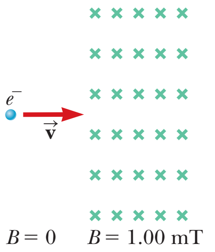


Figure P29.22

23. A singly charged ion of mass m is accelerated from rest by a potential difference ΔV. It is then deflected by a uniform magnetic field (perpendicular to the ion's velocity) into a semicircle of radius R. Now a doubly charged ion of mass m' is accelerated through the same potential difference and deflected by the same magnetic field into a semicircle of radius $R' = 2R$. What is the ratio of the masses of the ions?

Section 29.3 Applications Involving Charged Particles Moving in a Magnetic Field

24. M A cyclotron designed to accelerate protons has a magnetic field of magnitude 0.450 T over a region of radius 1.20 m. What are (a) the cyclotron frequency and (b) the maximum speed acquired by the protons?

25. W Consider the mass spectrometer shown schematically in Figure 29.14. The magnitude of the electric field between the plates of the velocity selector is 2.50×10^3 V/m, and the magnetic field in both the velocity selector and the deflection chamber has a magnitude of 0.035 0 T. Calculate the radius of the path for a singly charged ion having a mass $m = 2.18 \times 10^{-26}$ kg.

26. Singly charged uranium-238 ions are accelerated through a potential difference of 2.00 kV and enter a uniform magnetic field of magnitude 1.20 T directed perpendicular to their velocities. (a) Determine the radius of their circular path. (b) Repeat this calculation for uranium-235 ions. (c) **What If?** How does the ratio of these path radii depend on the accelerating voltage? (d) On the magnitude of the magnetic field?

27. A cyclotron (Fig. 29.16) designed to accelerate protons has an outer radius of 0.350 m. The protons are emitted nearly at rest from a source at the center and are accelerated through 600 V each time they cross the gap between the dees. The dees are between the poles of an electromagnet where the field is 0.800 T. (a) Find the cyclotron frequency for the protons in this cyclotron. Find (b) the speed at which protons exit the cyclotron and (c) their maximum kinetic energy. (d) How many revolutions does a proton make in the cyclotron? (e) For what time interval does the proton accelerate?

28. A particle in the cyclotron shown in Figure 29.16a gains energy $q\,\Delta V$ from the alternating power supply each time it passes from one dee to the other. The time interval for each full orbit is

$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$$

so the particle's average rate of increase in energy is

$$\frac{2q\,\Delta V}{T} = \frac{q^2 B\,\Delta V}{\pi m}$$

Notice that this power input is constant in time. On the other hand, the rate of increase in the radius r of its path is *not* constant. (a) Show that the rate of increase in the radius r of the particle's path is given by

$$\frac{dr}{dt} = \frac{1}{r}\frac{\Delta V}{\pi B}$$

(b) Describe how the path of the particles in Figure 29.16a is consistent with the result of part (a). (c) At what rate is the radial position of the protons in a cyclotron increasing immediately before the protons leave the cyclotron? Assume the cyclotron has an outer radius of 0.350 m, an accelerating voltage of $\Delta V =$ 600 V, and a magnetic field of magnitude 0.800 T. (d) By how much does the radius of the protons' path increase during their last full revolution?

29. W A velocity selector consists of electric and magnetic fields described by the expressions $\vec{\mathbf{E}} = E\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B\hat{\mathbf{j}}$, with $B = 15.0$ mT. Find the value of E such that a 750-eV electron moving in the negative x direction is undeflected.

30. In his experiments on "cathode rays" during which he discovered the electron, J. J. Thomson showed that the same beam deflections resulted with tubes having cathodes made of *different* materials and containing *various* gases before evacuation. (a) Are these observations important? Explain your answer. (b) When he applied various potential differences to the deflection plates and turned on the magnetic coils, alone or in combination with the deflection plates, Thomson observed that the fluorescent screen continued to show a *single small* glowing patch. Argue whether his observation is important. (c) Do calculations to show that the charge-to-mass ratio Thomson obtained was huge compared with that of any macroscopic object or of any ionized atom or molecule. How can one make sense of this comparison? (d) Could Thomson observe any deflection of the beam due to gravitation? Do a calculation to argue for your answer. *Note:* To obtain a visibly glowing patch on the fluorescent screen, the potential difference between the slits and the cathode must be 100 V or more.

31. The picture tube in an old black-and-white television uses magnetic deflection coils rather than electric deflection plates. Suppose an electron beam is accelerated through a 50.0-kV potential difference and then through a region of uniform magnetic field 1.00 cm wide. The screen is located 10.0 cm from the center of the coils and is 50.0 cm wide. When the field is turned off, the electron beam hits the center of the screen. Ignoring relativistic corrections, what field magnitude is necessary to deflect the beam to the side of the screen?

Section 29.4 Magnetic Force Acting on a Current-Carrying Conductor

32. A straight wire carrying a 3.00-A current is placed in a uniform magnetic field of magnitude 0.280 T directed perpendicular to the wire. (a) Find the magnitude of the magnetic force on a section of the wire having a length of 14.0 cm. (b) Explain why you can't determine the direction of the magnetic force from the information given in the problem.

33. A conductor carrying a current $I = 15.0$ A is directed along the positive x axis and perpendicular to a uniform magnetic field. A magnetic force per unit length of 0.120 N/m acts on the conductor in the negative y direction. Determine (a) the magnitude and (b) the direction of the magnetic field in the region through which the current passes.

34. W A wire 2.80 m in length carries a current of 5.00 A in a region where a uniform magnetic field has a magnitude of 0.390 T. Calculate the magnitude of the magnetic force on the wire assuming the angle between the magnetic field and the current is (a) 60.0°, (b) 90.0°, and (c) 120°.

35. W A wire carries a steady current of 2.40 A. A straight section of the wire is 0.750 m long and lies along the x axis within a uniform magnetic field, $\vec{\mathbf{B}} = 1.60\hat{\mathbf{k}}$ T. If the current is in the positive x direction, what is the magnetic force on the section of wire?

36. *Why is the following situation impossible?* Imagine a copper wire with radius 1.00 mm encircling the Earth at its magnetic equator, where the field direction is horizontal. A power supply delivers 100 MW to the wire to maintain a current in it, in a direction such that the magnetic force from the Earth's magnetic field is upward. Due to this force, the wire is levitated immediately above the ground.

37. AMT W **Review.** A rod of mass 0.720 kg and radius 6.00 cm rests on two parallel rails (Fig. P29.37) that are $d =$ 12.0 cm apart and $L = 45.0$ cm long. The rod carries a

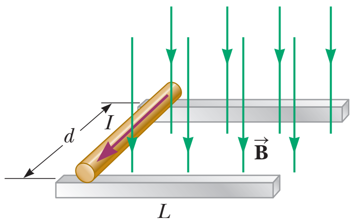


Figure P29.37 Problems 37 and 38.

current of $I = 48.0$ A in the direction shown and rolls along the rails without slipping. A uniform magnetic field of magnitude 0.240 T is directed perpendicular to the rod and the rails. If it starts from rest, what is the speed of the rod as it leaves the rails?

38. **Review.** A rod of mass m and radius R rests on two parallel rails (Fig. P29.37) that are a distance d apart and have a length L. The rod carries a current I in the direction shown and rolls along the rails without slipping. A uniform magnetic field B is directed perpendicular to the rod and the rails. If it starts from rest, what is the speed of the rod as it leaves the rails?

39. M A wire having a mass per unit length of 0.500 g/cm carries a 2.00-A current horizontally to the south. What are (a) the direction and (b) the magnitude of the minimum magnetic field needed to lift this wire vertically upward?

40. Consider the system pictured in Figure P29.40. A 15.0-cm horizontal wire of mass 15.0 g is placed between two thin, vertical conductors, and a uniform magnetic field acts perpendicular to the page. The wire is free to move vertically without friction on the two vertical conductors. When a 5.00-A current is directed as shown in the figure, the horizontal wire moves upward at constant velocity in the presence of gravity. (a) What forces act on the horizontal wire, and (b) under what condition is the wire able to move upward at constant velocity? (c) Find the magnitude and direction of the minimum magnetic field required to move the wire at constant speed. (d) What happens if the magnetic field exceeds this minimum value?

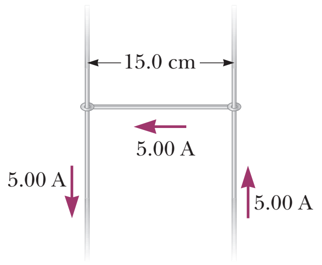


Figure P29.40

41. A horizontal power line of length 58.0 m carries a current of 2.20 kA northward as shown in Figure P29.41. The Earth's magnetic field at this location has a magnitude of 5.00×10^{-5} T. The field at this location is directed toward the north at an angle 65.0° below the power line. Find (a) the magnitude and (b) the direction of the magnetic force on the power line.

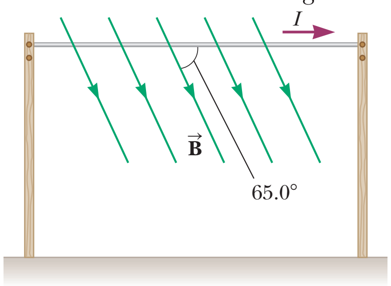


Figure P29.41

42. A strong magnet is placed under a horizontal conducting ring of radius r that carries current I as shown in Figure P29.42. If the magnetic field $\vec{\mathbf{B}}$ makes an angle θ with the vertical at the ring's location, what are (a) the magnitude and (b) the direction of the resultant magnetic force on the ring?

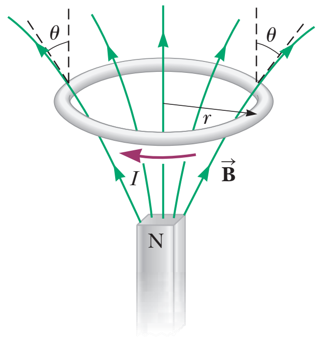


Figure P29.42

43. Assume the Earth's magnetic field is 52.0 μT northward at 60.0° below the horizontal in Atlanta, Georgia. A tube in a neon sign stretches between two diagonally opposite corners of a shop window—which lies in a north–south vertical plane—and carries current 35.0 mA. The current enters the tube at the bottom south corner of the shop's window. It exits at the opposite corner, which is 1.40 m farther north and 0.850 m higher up. Between these two points, the glowing tube spells out DONUTS. Determine the total vector magnetic force on the tube. *Hint:* You may use the first "important general statement" presented in the Finalize section of Example 29.4.

44. In Figure P29.44, the cube is 40.0 cm on each edge. Four straight segments of wire—*ab*, *bc*, *cd*, and *da*—form a closed loop that carries a current $I = 5.00$ A in the direction shown. A uniform magnetic field of magnitude $B = 0.020\,0$ T is in the positive y direction. Determine the magnetic force vector on (a) *ab*, (b) *bc*, (c) *cd*, and (d) *da*. (e) Explain how you could find the force exerted on the fourth of these segments from the forces on the other three, without further calculation involving the magnetic field.

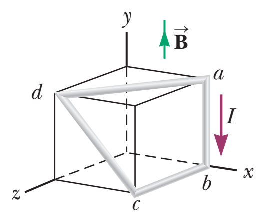


Figure P29.44

Section 29.5 Torque on a Current Loop in a Uniform Magnetic Field

45. A typical magnitude of the external magnetic field in a cardiac catheter ablation procedure using remote magnetic navigation is $B = 0.080$ T. Suppose that the permanent magnet in the catheter used in the procedure is inside the left atrium of the heart and subject to this external magnetic field. The permanent magnet has a magnetic moment of $0.10\ \text{A} \cdot \text{m}^2$. The orientation of the permanent magnet is 30° from the direction of the external magnetic field lines. (a) What is the magnitude of the torque on the tip of the catheter containing this permanent magnet? (b) What is the potential energy of the system consisting of the permanent magnet in the catheter and the magnetic field provided by the external magnets?

46. A 50.0-turn circular coil of radius 5.00 cm can be oriented in any direction in a uniform magnetic field having a magnitude of 0.500 T. If the coil carries a current of 25.0 mA, find the magnitude of the maximum possible torque exerted on the coil.

47. A magnetized sewing needle has a magnetic moment of $9.70\ \text{mA} \cdot \text{m}^2$. At its location, the Earth's magnetic field is 55.0 μT northward at 48.0° below the horizontal. Identify the orientations of the needle that represent (a) the minimum potential energy and (b) the maximum potential energy of the needle–field system. (c) How much work must be done on the system to move the needle from the minimum to the maximum potential energy orientation?

48. W A current of 17.0 mA is maintained in a single circular loop of 2.00 m circumference. A magnetic field of 0.800 T is directed parallel to the plane of the loop. (a) Calculate the magnetic moment of the loop. (b) What is the magnitude of the torque exerted by the magnetic field on the loop?

49. M An eight-turn coil encloses an elliptical area having a major axis of 40.0 cm and a minor axis of 30.0 cm (Fig. P29.49). The coil lies in the plane of the page and has a 6.00-A current flowing clockwise around it. If the coil is in a uniform magnetic field of 2.00×10^{-4} T directed toward the left of the page, what is the magnitude of the torque on the coil? *Hint:* The area of an ellipse is $A = \pi ab$, where a and b are, respectively, the semimajor and semiminor axes of the ellipse.

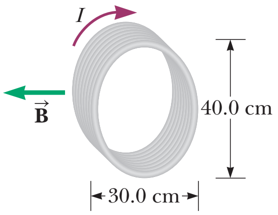


Figure P29.49

50. The rotor in a certain electric motor is a flat, rectangular coil with 80 turns of wire and dimensions 2.50 cm by 4.00 cm. The rotor rotates in a uniform magnetic field of 0.800 T. When the plane of the rotor is perpendicular to the direction of the magnetic field, the rotor carries a current of 10.0 mA. In this orientation, the magnetic moment of the rotor is directed opposite the magnetic field. The rotor then turns through one-half revolution. This process is repeated to cause the rotor to turn steadily at an angular speed of 3.60×10^3 rev/min. (a) Find the maximum torque acting on the rotor. (b) Find the peak power output of the motor. (c) Determine the amount of work performed by the magnetic field on the rotor in every full revolution. (d) What is the average power of the motor?

51. M A rectangular coil consists of $N = 100$ closely wrapped turns and has dimensions $a = 0.400$ m and $b = 0.300$ m. The coil is hinged along the y axis, and its plane makes an angle $\theta = 30.0°$ with the x axis (Fig. P29.51). (a) What is the magnitude of the torque exerted on the coil by a uniform magnetic field $B = 0.800$ T directed in the positive x direction when the current is $I = 1.20$ A in the direction shown? (b) What is the expected direction of rotation of the coil?

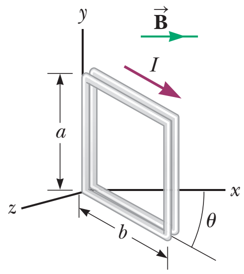


Figure P29.51

52. GP A rectangular loop of wire has dimensions 0.500 m by 0.300 m. The loop is pivoted at the x axis and lies in the xy plane as shown in Figure P29.52. A uniform magnetic field of magnitude 1.50 T is directed at an angle of 40.0° with respect to the y axis with field lines parallel to the yz plane. The loop carries a current of 0.900 A in the direction shown. (Ignore gravitation.) We wish to evaluate the torque on the current loop. (a) What is the direction of the magnetic force exerted on wire segment ab? (b) What is the direction of the torque associated with this force about an axis through the origin? (c) What is the direction of the magnetic force exerted on segment cd? (d) What is the direction of the torque associated with this force about an axis through the origin? (e) Can the forces examined in parts (a) and (c) combine to cause the loop to rotate around the x axis? (f) Can they affect the motion of the loop in any way? Explain. (g) What is the direction of the magnetic force exerted on segment bc? (h) What is the direction of the torque associated with this force about an axis through the origin? (i) What is the torque on segment ad about an axis through the origin? (j) From the point of view of Figure P29.52, once the loop is released from rest at

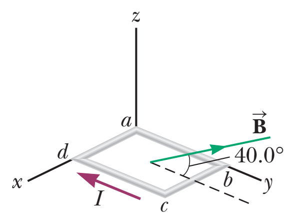


Figure P29.52

the position shown, will it rotate clockwise or counterclockwise around the x axis? (k) Compute the magnitude of the magnetic moment of the loop. (l) What is the angle between the magnetic moment vector and the magnetic field? (m) Compute the torque on the loop using the results to parts (k) and (l).

53. W A wire is formed into a circle having a diameter of 10.0 cm and is placed in a uniform magnetic field of 3.00 mT. The wire carries a current of 5.00 A. Find (a) the maximum torque on the wire and (b) the range of potential energies of the wire–field system for different orientations of the circle.

Section 29.6 The Hall Effect

54. A Hall-effect probe operates with a 120-mA current. When the probe is placed in a uniform magnetic field of magnitude 0.080 0 T, it produces a Hall voltage of 0.700 μV. (a) When it is used to measure an unknown magnetic field, the Hall voltage is 0.330 μV. What is the magnitude of the unknown field? (b) The thickness of the probe in the direction of $\vec{\mathbf{B}}$ is 2.00 mm. Find the density of the charge carriers, each of which has charge of magnitude e.

55. M In an experiment designed to measure the Earth's magnetic field using the Hall effect, a copper bar 0.500 cm thick is positioned along an east–west direction. Assume $n = 8.46 \times 10^{28}$ electrons/m^3 and the plane of the bar is rotated to be perpendicular to the direction of $\vec{\mathbf{B}}$. If a current of 8.00 A in the conductor results in a Hall voltage of 5.10×10^{-12} V, what is the magnitude of the Earth's magnetic field at this location?

Additional Problems

56. Carbon-14 and carbon-12 ions (each with charge of magnitude e) are accelerated in a cyclotron. If the cyclotron has a magnetic field of magnitude 2.40 T, what is the difference in cyclotron frequencies for the two ions?

57. In Niels Bohr's 1913 model of the hydrogen atom, the single electron is in a circular orbit of radius 5.29×10^{-11} m and its speed is 2.19×10^6 m/s. (a) What is the magnitude of the magnetic moment due to the electron's motion? (b) If the electron moves in a horizontal circle, counterclockwise as seen from above, what is the direction of this magnetic moment vector?

58. Heart–lung machines and artificial kidney machines employ electromagnetic blood pumps. The blood is confined to an electrically insulating tube, cylindrical in practice but represented here for simplicity as a rectangle of interior width w and height h. Figure P29.58 shows a rectangular section of blood within the tube. Two electrodes fit into the top and the bottom of the tube. The potential difference between them establishes an electric current through the blood, with current density J over the section of length L shown in Figure P29.58. A perpendicular magnetic field exists in the same region. (a) Explain why this arrangement produces on the liquid a force that is directed along the length of the pipe. (b) Show that the section of liquid in the magnetic field experiences a pressure increase JLB. (c) After the blood leaves the pump, is it charged? (d) Is it carrying current? (e) Is it magnetized? (The same electromagnetic pump can be used for any fluid that conducts electricity, such as liquid sodium in a nuclear reactor.)

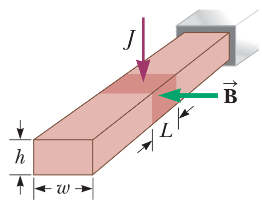


Figure P29.58

59. M A particle with positive charge $q = 3.20 \times 10^{-19}$ C moves with a velocity $\vec{\mathbf{v}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ m/s through a region where both a uniform magnetic field and a uniform electric field exist. (a) Calculate the total force on the moving particle (in unit-vector notation), taking $\vec{\mathbf{B}} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$ T and $\vec{\mathbf{E}} = (4\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ V/m. (b) What angle does the force vector make with the positive x axis?

60. Figure 29.11 shows a charged particle traveling in a nonuniform magnetic field forming a magnetic bottle. (a) Explain why the positively charged particle in the figure must be moving clockwise when viewed from the right of the figure. The particle travels along a helix whose radius decreases and whose pitch decreases as the particle moves into a stronger magnetic field. If the particle is moving to the right along the x axis, its velocity in this direction will be reduced to zero and it will be reflected from the right-hand side of the bottle, acting as a "magnetic mirror." The particle ends up bouncing back and forth between the ends of the bottle. (b) Explain qualitatively why the axial velocity is reduced to zero as the particle moves into the region of strong magnetic field at the end of the bottle. (c) Explain why the tangential velocity increases as the particle approaches the end of the bottle. (d) Explain why the orbiting particle has a magnetic dipole moment.

61. AMT **Review.** The upper portion of the circuit in Figure P29.61 is fixed. The horizontal wire at the bottom has a mass of 10.0 g and is 5.00 cm long. This wire hangs in the gravitational field of the Earth from identical light springs connected to the upper portion of the circuit. The springs stretch 0.500 cm under the weight of the

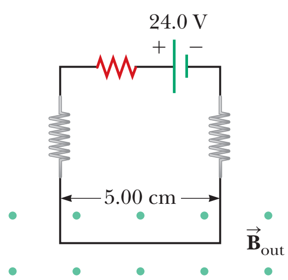


Figure P29.61

wire, and the circuit has a total resistance of 12.0 Ω. When a magnetic field is turned on, directed out of the page, the springs stretch an additional 0.300 cm. Only the horizontal wire at the bottom of the circuit is in the magnetic field. What is the magnitude of the magnetic field?

62. Within a cylindrical region of space of radius 100 Mm, a magnetic field is uniform with a magnitude 25.0 μT and oriented parallel to the axis of the cylinder. The magnetic field is zero outside this cylinder. A cosmic-ray proton traveling at one-tenth the speed of light is heading directly toward the center of the cylinder, moving perpendicular to the cylinder's axis. (a) Find the radius of curvature of the path the proton follows when it enters the region of the field. (b) Explain whether the proton will arrive at the center of the cylinder.

63. **Review.** A proton is at rest at the plane boundary of a region containing a uniform magnetic field B (Fig. P29.63). An alpha particle moving horizontally makes a head-on elastic collision with the proton. Immediately after the collision, both particles enter the magnetic field, moving perpendicular to the direction of the field. The radius of the proton's trajectory is R. The mass of the alpha particle is four times that of the proton, and its charge is twice that of the proton. Find the radius of the alpha particle's trajectory.

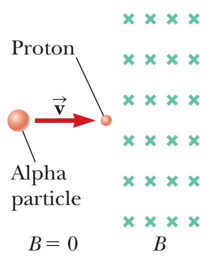


Figure P29.63

64. (a) A proton moving with velocity $\vec{v} = v_i\hat{i}$ experiences a magnetic force $\vec{F} = F_i\hat{j}$. Explain what you can and cannot infer about $\vec{B}$ from this information. (b) **What If?** In terms of F_i, what would be the force on a proton in the same field moving with velocity $\vec{v} = -v_i\hat{i}$? (c) What would be the force on an electron in the same field moving with velocity $\vec{v} = -v_i\hat{i}$?

65. AMT **Review.** A 0.200-kg metal rod carrying a current of 10.0 A glides on two horizontal rails 0.500 m apart. If the coefficient of kinetic friction between the rod and rails is 0.100, what vertical magnetic field is required to keep the rod moving at a constant speed?

66. **Review.** A metal rod of mass m carrying a current I glides on two horizontal rails a distance d apart. If the coefficient of kinetic friction between the rod and rails is μ, what vertical magnetic field is required to keep the rod moving at a constant speed?

67. A proton having an initial velocity of $20.0\hat{i}$ Mm/s enters a uniform magnetic field of magnitude 0.300 T with a direction perpendicular to the proton's velocity. It leaves the field-filled region with velocity $-20.0\hat{j}$ Mm/s. Determine (a) the direction of the magnetic field, (b) the radius of curvature of the proton's path while in the field, (c) the distance the proton traveled in the field, and (d) the time interval during which the proton is in the field.

68. Model the electric motor in a handheld electric mixer as a single flat, compact, circular coil carrying electric current in a region where a magnetic field is produced by an external permanent magnet. You need consider only one instant in the operation of the motor. (We will consider motors again in Chapter 31.) Make order-of-magnitude estimates of (a) the magnetic field, (b) the torque on the coil, (c) the current in the coil, (d) the coil's area, and (e) the number of turns in the coil. The input power to the motor is electric, given by $P = I\Delta V$, and the useful output power is mechanical, $P = \tau\omega$.

69. AMT A nonconducting sphere has mass 80.0 g and radius 20.0 cm. A flat, compact coil of wire with five turns is wrapped tightly around it, with each turn concentric with the sphere. The sphere is placed on an inclined plane that slopes downward to the left (Fig. P29.69), making an angle θ with the horizontal so that the coil is parallel to the inclined plane. A uniform magnetic field of 0.350 T vertically upward exists in the region of the sphere. (a) What current in the coil will enable the sphere to rest in equilibrium on the inclined plane? (b) Show that the result does not depend on the value of θ.

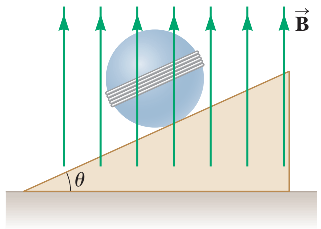


Figure P29.69

70. *Why is the following situation impossible?* Figure P29.70 shows an experimental technique for altering the direction of travel for a charged particle. A particle of charge $q = 1.00\ \mu$C and mass $m = 2.00 \times 10^{-13}$ kg enters the bottom of the region of uniform magnetic field at speed $v = 2.00 \times 10^5$ m/s, with a velocity vector

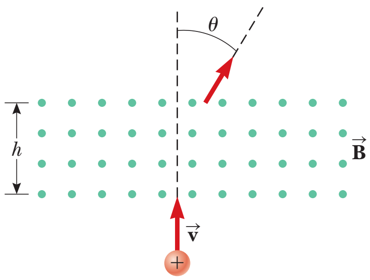


Figure P29.70

perpendicular to the field lines. The magnetic force on the particle causes its direction of travel to change so that it leaves the region of the magnetic field at the top traveling at an angle from its original direction. The magnetic field has magnitude $B = 0.400$ T and is directed out of the page. The length h of the magnetic field region is 0.110 m. An experimenter performs the technique and measures the angle θ at which the particles exit the top of the field. She finds that the angles of deviation are exactly as predicted.

71. Figure P29.71 shows a schematic representation of an apparatus that can be used to measure magnetic fields. A rectangular coil of wire contains N turns and has a width w. The coil is attached to one arm of a balance and is suspended between the poles of a magnet. The magnetic field is uniform and perpendicular to the plane of the coil. The system is first balanced when the current in the coil is zero. When the switch is closed and the coil carries a current I, a mass m must be added to the right side to balance the system. (a) Find an expression for the magnitude of the magnetic field. (b) Why is the result independent of the vertical dimensions of the coil? (c) Suppose the coil has 50 turns and a width of 5.00 cm. When the switch is closed, the coil carries a current of 0.300 A, and a mass of 20.0 g must be added to the right side to balance the system. What is the magnitude of the magnetic field?

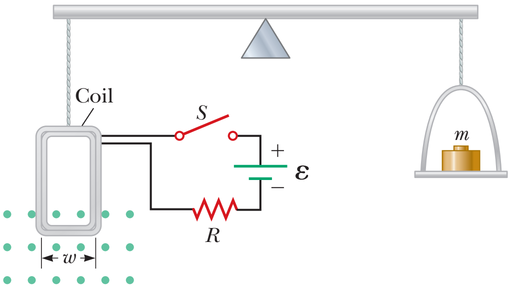


Figure P29.71

72. A heart surgeon monitors the flow rate of blood through an artery using an electromagnetic flowmeter (Fig. P29.72). Electrodes A and B make contact with the outer surface of the blood vessel, which has a diameter of 3.00 mm. (a) For a magnetic field magnitude of 0.040 0 T, an emf of 160 μV appears between the electrodes. Calculate the speed of the blood. (b) Explain why electrode A has to be positive as shown. (c) Does the sign of the emf depend on whether the mobile ions in the blood are predominantly positively or negatively charged? Explain.

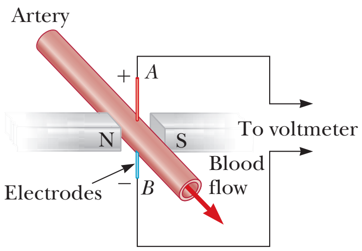


Figure P29.72

73. A uniform magnetic field of magnitude 0.150 T is directed along the positive x axis. A positron moving at a speed of 5.00×10^6 m/s enters the field along a direction that makes an angle of $\theta = 85.0°$ with the x axis (Fig. P29.73). The motion of the particle is expected to be a helix as described in Section 29.2. Calculate (a) the pitch p and (b) the radius r of the trajectory as defined in Figure P29.73.

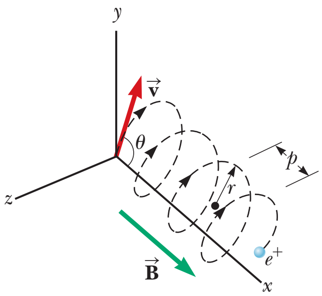


Figure P29.73

74. **Review.** (a) Show that a magnetic dipole in a uniform magnetic field, displaced from its equilibrium orientation and released, can oscillate as a torsional pendulum (Section 15.5) in simple harmonic motion. (b) Is this statement true for all angular displacements, for all displacements less than 180°, or only for small angular displacements? Explain. (c) Assume the dipole is a compass needle—a light bar magnet—with a magnetic moment of magnitude μ. It has moment of inertia I about its center, where it is mounted on a frictionless, vertical axle, and it is placed in a horizontal magnetic field of magnitude B. Determine its frequency of oscillation. (d) Explain how the compass needle can be conveniently used as an indicator of the magnitude of the external magnetic field. (e) If its frequency is 0.680 Hz in the Earth's local field, with a horizontal component of 39.2 μT, what is the magnitude of a field parallel to the needle in which its frequency of oscillation is 4.90 Hz?

75. The accompanying table shows measurements of the Hall voltage and corresponding magnetic field for a probe used to measure magnetic fields. (a) Plot these data and deduce a relationship between the two variables. (b) If the measurements were taken with a current of 0.200 A and the sample is made from a material having a charge-carrier density of 1.00×10^{26} carriers/m^3, what is the thickness of the sample?

ΔV_H (μV)	B (T)
0	0.00
11	0.10
19	0.20
28	0.30
42	0.40
50	0.50
61	0.60
68	0.70
79	0.80
90	0.90
102	1.00

76. A metal rod having a mass per unit length λ carries a current I. The rod hangs from two wires in a uniform vertical magnetic field as shown in Figure P29.76. The wires make an angle θ with the vertical when in equilibrium. Determine the magnitude of the magnetic field.

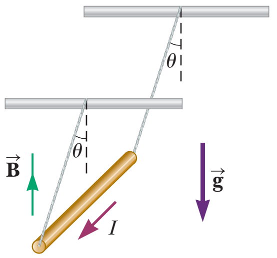


Figure P29.76

Challenge Problems

77. Consider an electron orbiting a proton and maintained in a fixed circular path of radius $R = 5.29 \times 10^{-11}$ m by the Coulomb force. Treat the orbiting particle as a current loop. Calculate the resulting torque when the electron–proton system is placed in a magnetic field of 0.400 T directed perpendicular to the magnetic moment of the loop.

78. Protons having a kinetic energy of 5.00 MeV (1 eV = 1.60×10^{-19} J) are moving in the positive x direction and enter a magnetic field $\vec{\mathbf{B}} = 0.050\,0\hat{\mathbf{k}}$ T directed out of the plane of the page and extending from $x = 0$ to $x = 1.00$ m as shown in Figure P29.78. (a) Ignoring relativistic effects, find the angle α between the initial velocity vector of the proton beam and the velocity vector after the beam emerges from the field. (b) Calculate the y component of the protons' momenta as they leave the magnetic field.

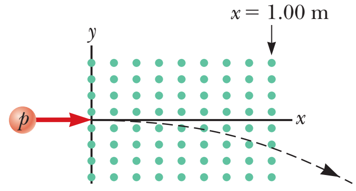


Figure P29.78

79. Review. A wire having a linear mass density of 1.00 g/cm is placed on a horizontal surface that has a coefficient of kinetic friction of 0.200. The wire carries a current of 1.50 A toward the east and slides horizontally to the north at constant velocity. What are (a) the magnitude and (b) the direction of the smallest magnetic field that enables the wire to move in this fashion?

80. A proton moving in the plane of the page has a kinetic energy of 6.00 MeV. A magnetic field of magnitude $B = 1.00$ T is directed into the page. The proton enters the magnetic field with its velocity vector at an angle $\theta = 45.0°$ to the linear boundary of the field as shown in Figure P29.80. (a) Find x, the distance from the point of entry to where the proton will leave the field. (b) Determine θ, the angle between the boundary and the proton's velocity vector as it leaves the field.

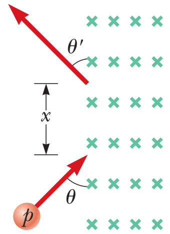


Figure P29.80

CHAPTER

30

Sources of the Magnetic Field

A cardiac catheterization laboratory stands ready to receive a patient suffering from atrial fibrillation. The large white objects on either side of the operating table are strong magnets that place the patient in a magnetic field. The electrophysiologist performing a catheter ablation procedure sits at a computer in the room to the left. With guidance from the magnetic field, he or she uses a joystick and other controls to thread the magnetically sensitive tip of a cardiac catheter through blood vessels and into the chambers of the heart. *(©Courtesy of Stereotaxis, Inc.)*

In Chapter 29, we discussed the magnetic force exerted on a charged particle moving in a magnetic field. To complete the description of the magnetic interaction, this chapter explores the origin of the magnetic field, moving charges. We begin by showing how to use the law of Biot and Savart to calculate the magnetic field produced at some point in space by a small current element. This formalism is then used to calculate the total magnetic field due to various current distributions. Next, we show how to determine the force between two current-carrying conductors, leading to the definition of the ampere. We also introduce Ampère's law, which is useful in calculating the magnetic field of a highly symmetric configuration carrying a steady current.

This chapter is also concerned with the complex processes that occur in magnetic materials. All magnetic effects in matter can be explained on the basis of atomic magnetic moments, which arise both from the orbital motion of electrons and from an intrinsic property of electrons known as spin.

30.1 The Biot–Savart Law

Shortly after Oersted's discovery in 1819 that a compass needle is deflected by a current-carrying conductor, Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841) performed quantitative experiments on the force exerted by an electric current on a nearby magnet. From their experimental results, Biot and Savart arrived at a mathematical expression that gives the magnetic field at some point in space in terms of the current that produces the field. That expression is based on the following experimental observations for the magnetic field $d\vec{\mathbf{B}}$ at a point P associated with a length element $d\vec{\mathbf{s}}$ of a wire carrying a steady current I (Fig. 30.1):

- The vector $d\vec{\mathbf{B}}$ is perpendicular both to $d\vec{\mathbf{s}}$ (which points in the direction of the current) and to the unit vector $\hat{\mathbf{r}}$ directed from $d\vec{\mathbf{s}}$ toward P.
- The magnitude of $d\vec{\mathbf{B}}$ is inversely proportional to r^2, where r is the distance from $d\vec{\mathbf{s}}$ to P.
- The magnitude of $d\vec{\mathbf{B}}$ is proportional to the current I and to the magnitude ds of the length element $d\vec{\mathbf{s}}$.
- The magnitude of $d\vec{\mathbf{B}}$ is proportional to $\sin\theta$, where θ is the angle between the vectors $d\vec{\mathbf{s}}$ and $\hat{\mathbf{r}}$.

Pitfall Prevention 30.1

The Biot–Savart Law The magnetic field described by the Biot–Savart law is the field *due to* a given current-carrying conductor. Do not confuse this field with any *external* field that may be applied to the conductor from some other source.

These observations are summarized in the mathematical expression known today as the **Biot–Savart law:**

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{30.1}$$

◀ Biot–Savart law

where μ_0 is a constant called the **permeability of free space:**

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\cdot m/A} \tag{30.2}$$

◀ Permeability of free space

Notice that the field $d\vec{\mathbf{B}}$ in Equation 30.1 is the field created at a point by the current in only a small length element $d\vec{\mathbf{s}}$ of the conductor. To find the *total* magnetic field $\vec{\mathbf{B}}$ created at some point by a current of finite size, we must sum up contributions from all current elements $I\,d\vec{\mathbf{s}}$ that make up the current. That is, we must evaluate $\vec{\mathbf{B}}$ by integrating Equation 30.1:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{30.3}$$

where the integral is taken over the entire current distribution. This expression must be handled with special care because the integrand is a cross product and therefore a vector quantity. We shall see one case of such an integration in Example 30.1.

Although the Biot–Savart law was discussed for a current-carrying wire, it is also valid for a current consisting of charges flowing through space such as the particle beam in an accelerator. In that case, $d\vec{\mathbf{s}}$ represents the length of a small segment of space in which the charges flow.

Interesting similarities and differences exist between Equation 30.1 for the magnetic field due to a current element and Equation 23.9 for the electric field due to a point charge. The magnitude of the magnetic field varies as the inverse square of the distance from the source, as does the electric field due to a point charge. The directions of the two fields are quite different, however. The electric field created by a point charge is radial, but the magnetic field created by a current element is perpendicular to both the length element $d\vec{\mathbf{s}}$ and the unit vector $\hat{\mathbf{r}}$ as described by the cross product in Equation 30.1. Hence, if the conductor lies in the plane of the page as shown in Figure 30.1, $d\vec{\mathbf{B}}$ points out of the page at P and into the page at P'.

Another difference between electric and magnetic fields is related to the source of the field. An electric field is established by an isolated electric charge. The Biot–Savart law gives the magnetic field of an isolated current element at some point, but such an isolated current element cannot exist the way an isolated electric charge can. A current element *must* be part of an extended current distribution because a complete circuit is needed for charges to flow. Therefore,

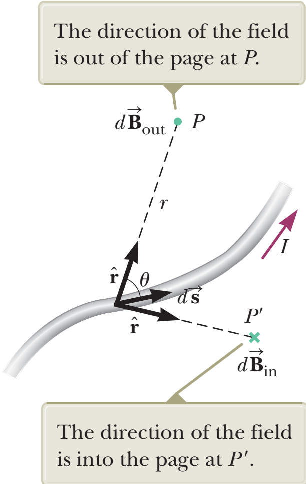


Figure 30.1 The magnetic field $d\vec{\mathbf{B}}$ at a point due to the current I through a length element $d\vec{\mathbf{s}}$ is given by the Biot–Savart law.

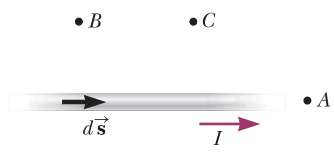


Figure 30.2 (Quick Quiz 30.1) Where is the magnetic field due to the current element the greatest?

the Biot–Savart law (Eq. 30.1) is only the first step in a calculation of a magnetic field; it must be followed by an integration over the current distribution as in Equation 30.3.

Quick Quiz 30.1 Consider the magnetic field due to the current in the wire shown in Figure 30.2. Rank the points A, B, and C in terms of magnitude of the magnetic field that is due to the current in just the length element $d\vec{\mathbf{s}}$ shown from greatest to least.

Example 30.1 Magnetic Field Surrounding a Thin, Straight Conductor

Consider a thin, straight wire of finite length carrying a constant current I and placed along the x axis as shown in Figure 30.3. Determine the magnitude and direction of the magnetic field at point P due to this current.

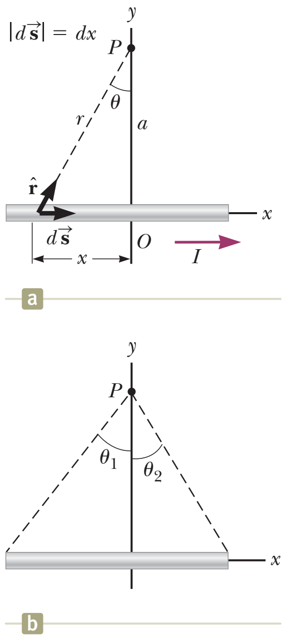


Figure 30.3 (Example 30.1) (a) A thin, straight wire carrying a current I. (b) The angles θ_1 and θ_2 used for determining the net field.

SOLUTION

Conceptualize From the Biot–Savart law, we expect that the magnitude of the field is proportional to the current in the wire and decreases as the distance a from the wire to point P increases. We also expect the field to depend on the angles θ_1 and θ_2 in Figure 30.3b. We place the origin at O and let point P be along the positive y axis, with $\hat{\mathbf{k}}$ being a unit vector pointing out of the page.

Categorize We are asked to find the magnetic field due to a simple current distribution, so this example is a typical problem for which the Biot–Savart law is appropriate. We must find the field contribution from a small element of current and then integrate over the current distribution.

Analyze Let's start by considering a length element $d\vec{\mathbf{s}}$ located a distance r from P. The direction of the magnetic field at point P due to the current in this element is out of the page because $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ is out of the page. In fact, because *all* the current elements $I\,d\vec{\mathbf{s}}$ lie in the plane of the page, they all produce a magnetic field directed out of the page at point P. Therefore, the direction of the magnetic field at point P is out of the page and we need only find the magnitude of the field.

Evaluate the cross product in the Biot–Savart law:

$$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = |d\vec{\mathbf{s}} \times \hat{\mathbf{r}}|\hat{\mathbf{k}} = \left[dx \sin\left(\frac{\pi}{2} - \theta\right)\right]\hat{\mathbf{k}} = (dx\cos\theta)\hat{\mathbf{k}}$$

Substitute into Equation 30.1:

$$(1)\quad d\vec{\mathbf{B}} = (dB)\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi}\frac{dx\cos\theta}{r^2}\,\hat{\mathbf{k}}$$

From the geometry in Figure 30.3a, express r in terms of θ:

$$(2)\quad r = \frac{a}{\cos\theta}$$

Notice that $\tan\theta = -x/a$ from the right triangle in Figure 30.3a (the negative sign is necessary because $d\vec{\mathbf{s}}$ is located at a negative value of x) and solve for x:

$$x = -a\tan\theta$$

Find the differential dx:

$$(3)\quad dx = -a\sec^2\theta\,d\theta = -\frac{a\,d\theta}{\cos^2\theta}$$

Substitute Equations (2) and (3) into the expression for the z component of the field from Equation (1):

$$(4)\quad dB = -\frac{\mu_0 I}{4\pi}\left(\frac{a\,d\theta}{\cos^2\theta}\right)\left(\frac{\cos^2\theta}{a^2}\right)\cos\theta = -\frac{\mu_0 I}{4\pi a}\cos\theta\,d\theta$$

30.1 continued

Integrate Equation (4) over all length elements on the wire, where the subtending angles range from θ_1 to θ_2 as defined in Figure 30.3b:

$$B = -\frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \boxed{\frac{\mu_0 I}{4\pi a}(\sin\theta_1 - \sin\theta_2)} \qquad \textbf{(30.4)}$$

Finalize We can use this result to find the magnitude of the magnetic field of *any* straight current-carrying wire if we know the geometry and hence the angles θ_1 and θ_2. Consider the special case of an infinitely long, straight wire. If the wire in Figure 30.3b becomes infinitely long, we see that $\theta_1 = \pi/2$ and $\theta_2 = -\pi/2$ for length elements ranging between positions $x = -\infty$ and $x = +\infty$. Because $(\sin\theta_1 - \sin\theta_2) = [\sin\pi/2 - \sin(-\pi/2)] = 2$, Equation 30.4 becomes

$$B = \frac{\mu_0 I}{2\pi a} \qquad \textbf{(30.5)}$$

Equations 30.4 and 30.5 both show that the magnitude of the magnetic field is proportional to the current and decreases with increasing distance from the wire, as expected. Equation 30.5 has the same mathematical form as the expression for the magnitude of the electric field due to a long charged wire (see Eq. 24.7).

Example 30.2 Magnetic Field Due to a Curved Wire Segment

Calculate the magnetic field at point O for the current-carrying wire segment shown in Figure 30.4. The wire consists of two straight portions and a circular arc of radius a, which subtends an angle θ.

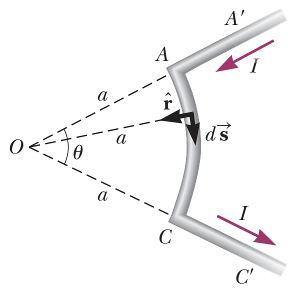


Figure 30.4 (Example 30.2) The length of the curved segment AC is s.

SOLUTION

Conceptualize The magnetic field at O due to the current in the straight segments AA' and CC' is zero because $d\vec{\mathbf{s}}$ is parallel to $\hat{\mathbf{r}}$ along these paths, which means that $d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = 0$ for these paths. Therefore, we expect the magnetic field at O to be due only to the current in the curved portion of the wire.

Categorize Because we can ignore segments AA' and CC', this example is categorized as an application of the Biot–Savart law to the curved wire segment AC.

Analyze Each length element $d\vec{\mathbf{s}}$ along path AC is at the same distance a from O, and the current in each contributes a field element $d\vec{\mathbf{B}}$ directed into the page at O. Furthermore, at every point on AC, $d\vec{\mathbf{s}}$ is perpendicular to $\hat{\mathbf{r}}$; hence, $|d\vec{\mathbf{s}} \times \hat{\mathbf{r}}| = ds$.

From Equation 30.1, find the magnitude of the field at O due to the current in an element of length ds:

$$dB = \frac{\mu_0}{4\pi}\frac{I\,ds}{a^2}$$

Integrate this expression over the curved path AC, noting that I and a are constants:

$$B = \frac{\mu_0 I}{4\pi a^2}\int ds = \frac{\mu_0 I}{4\pi a^2}s$$

From the geometry, note that $s = a\theta$ and substitute:

$$B = \frac{\mu_0 I}{4\pi a^2}(a\theta) = \boxed{\frac{\mu_0 I}{4\pi a}\theta} \qquad \textbf{(30.6)}$$

Finalize Equation 30.6 gives the magnitude of the magnetic field at O. The direction of $\vec{\mathbf{B}}$ is into the page at O because $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ is into the page for every length element.

WHAT IF? What if you were asked to find the magnetic field at the center of a circular wire loop of radius R that carries a current I? Can this question be answered at this point in our understanding of the source of magnetic fields?

continued

30.2 continued

Answer Yes, it can. The straight wires in Figure 30.4 do not contribute to the magnetic field. The only contribution is from the curved segment. As the angle θ increases, the curved segment becomes a full circle when $\theta = 2\pi$. Therefore, you can find the magnetic field at the center of a wire loop by letting $\theta = 2\pi$ in Equation 30.6:

$$B = \frac{\mu_0 I}{4\pi a} 2\pi = \frac{\mu_0 I}{2a}$$

This result is a limiting case of a more general result discussed in Example 30.3.

Example 30.3 Magnetic Field on the Axis of a Circular Current Loop

Consider a circular wire loop of radius a located in the yz plane and carrying a steady current I as in Figure 30.5. Calculate the magnetic field at an axial point P a distance x from the center of the loop.

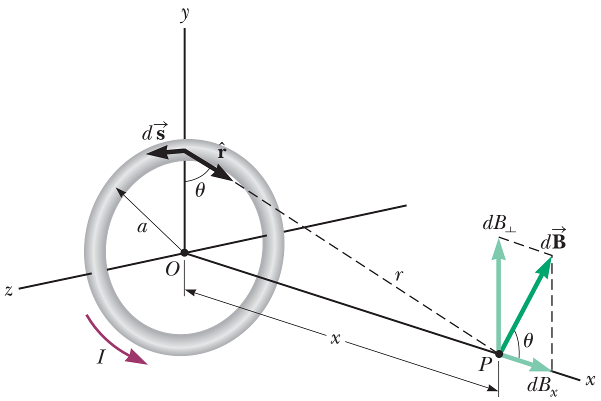


Figure 30.5 (Example 30.3) Geometry for calculating the magnetic field at a point P lying on the axis of a current loop. By symmetry, the total field $\vec{\mathbf{B}}$ is along this axis.

SOLUTION

Conceptualize Compare this problem to Example 23.8 for the electric field due to a ring of charge. Figure 30.5 shows the magnetic field contribution $d\vec{\mathbf{B}}$ at P due to a single current element at the top of the ring. This field vector can be resolved into components dB_x parallel to the axis of the ring and $dB_\perp$ perpendicular to the axis. Think about the magnetic field contributions from a current element at the bottom of the loop. Because of the symmetry of the situation, the perpendicular components of the field due to elements at the top and bottom of the ring cancel. This cancellation occurs for all pairs of segments around the ring, so we can ignore the perpendicular component of the field and focus solely on the parallel components, which simply add.

Categorize We are asked to find the magnetic field due to a simple current distribution, so this example is a typical problem for which the Biot–Savart law is appropriate.

Analyze In this situation, every length element $d\vec{\mathbf{s}}$ is perpendicular to the vector $\hat{\mathbf{r}}$ at the location of the element. Therefore, for any element, $|d\vec{\mathbf{s}} \times \hat{\mathbf{r}}| = (ds)(1) \sin 90° = ds$. Furthermore, all length elements around the loop are at the same distance r from P, where $r^2 = a^2 + x^2$.

Use Equation 30.1 to find the magnitude of $d\vec{\mathbf{B}}$ due to the current in any length element $d\vec{\mathbf{s}}$:

$$dB = \frac{\mu_0 I}{4\pi} \frac{|d\vec{\mathbf{s}} \times \hat{\mathbf{r}}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{ds}{(a^2 + x^2)}$$

Find the x component of the field element:

$$dB_x = \frac{\mu_0 I}{4\pi} \frac{ds}{(a^2 + x^2)} \cos\theta$$

Integrate over the entire loop:

$$B_x = \oint dB_x = \frac{\mu_0 I}{4\pi} \oint \frac{ds \cos\theta}{a^2 + x^2}$$

From the geometry, evaluate $\cos\theta$:

$$\cos\theta = \frac{a}{(a^2 + x^2)^{1/2}}$$

Substitute this expression for $\cos\theta$ into the integral and note that x and a are both constant:

$$B_x = \frac{\mu_0 I}{4\pi} \oint \frac{ds}{a^2 + x^2} \left[\frac{a}{(a^2 + x^2)^{1/2}}\right] = \frac{\mu_0 I}{4\pi} \frac{a}{(a^2 + x^2)^{3/2}} \oint ds$$

Integrate around the loop:

$$B_x = \frac{\mu_0 I}{4\pi} \frac{a}{(a^2 + x^2)^{3/2}} (2\pi a) = \boxed{\frac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}}} \quad (30.7)$$

30.3 continued

Finalize To find the magnetic field at the center of the loop, set $x = 0$ in Equation 30.7. At this special point,

$$B = \frac{\mu_0 I}{2a} \quad (\text{at } x = 0) \tag{30.8}$$

which is consistent with the result of the **What If?** feature of Example 30.2.

The pattern of magnetic field lines for a circular current loop is shown in Figure 30.6a. For clarity, the lines are drawn for only the plane that contains the axis of the loop. The field-line pattern is axially symmetric and looks like the pattern around a bar magnet, which is shown in Figure 30.6b.

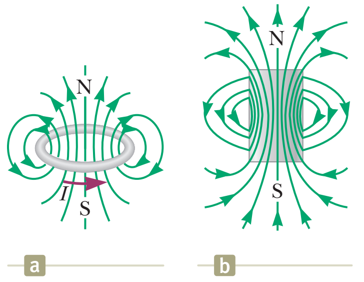


Figure 30.6 (Example 30.3) (a) Magnetic field lines surrounding a current loop. (b) Magnetic field lines surrounding a bar magnet. Notice the similarity between this line pattern and that of a current loop.

WHAT IF? What if we consider points on the x axis very far from the loop? How does the magnetic field behave at these distant points?

Answer In this case, in which $x >> a$, we can neglect the term a^2 in the denominator of Equation 30.7 and obtain

$$B \approx \frac{\mu_0 I a^2}{2x^3} \quad (\text{for } x >> a) \tag{30.9}$$

The magnitude of the magnetic moment μ of the loop is defined as the product of current and loop area (see Eq. 29.15): $\mu = I(\pi a^2)$ for our circular loop. We can express Equation 30.9 as

$$B \approx \frac{\mu_0}{2\pi} \frac{\mu}{x^3} \tag{30.10}$$

This result is similar in form to the expression for the electric field due to an electric dipole, $E = k_e(p/y^3)$ (see Example 23.6), where $p = 2aq$ is the electric dipole moment as defined in Equation 26.16.

30.2 The Magnetic Force Between Two Parallel Conductors

In Chapter 29, we described the magnetic force that acts on a current-carrying conductor placed in an external magnetic field. Because a current in a conductor sets up its own magnetic field, it is easy to understand that two current-carrying conductors exert magnetic forces on each other. One wire establishes the magnetic field and the other wire is modeled as a collection of particles in a magnetic field. Such forces between wires can be used as the basis for defining the ampere and the coulomb.

Consider two long, straight, parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction as in Figure 30.7. Let's determine the force exerted on one wire due to the magnetic field set up by the other wire. Wire 2, which carries a current I_2 and is identified arbitrarily as the source wire, creates a magnetic field $\vec{\mathbf{B}}_2$ at the location of wire 1, the test wire. The magnitude of this magnetic field is the same at all points on wire 1. The direction of $\vec{\mathbf{B}}_2$ is perpendicular to wire 1 as shown in Figure 30.7. According to Equation 29.10, the magnetic force on a length ℓ of wire 1 is $\vec{\mathbf{F}}_1 = I_1 \vec{\ell} \times \vec{\mathbf{B}}_2$. Because $\vec{\ell}$ is perpendicular to $\vec{\mathbf{B}}_2$ in this situation, the magnitude of $\vec{\mathbf{F}}_1$ is $F_1 = I_1 \ell B_2$. Because the magnitude of $\vec{\mathbf{B}}_2$ is given by Equation 30.5,

$$F_1 = I_1 \ell B_2 = I_1 \ell \left(\frac{\mu_0 I_2}{2\pi a} \right) = \frac{\mu_0 I_1 I_2}{2\pi a} \ell \tag{30.11}$$

The direction of $\vec{\mathbf{F}}_1$ is toward wire 2 because $\vec{\ell} \times \vec{\mathbf{B}}_2$ is in that direction. When the field set up at wire 2 by wire 1 is calculated, the force $\vec{\mathbf{F}}_2$ acting on wire 2 is found to be equal in magnitude and opposite in direction to $\vec{\mathbf{F}}_1$, which is what we expect because Newton's third law must be obeyed. When the currents are in opposite directions (that is, when one of the currents is reversed in Fig. 30.7), the forces

The field $\vec{\mathbf{B}}_2$ due to the current in wire 2 exerts a magnetic force of magnitude $F_1 = I_1 \ell B_2$ on wire 1.

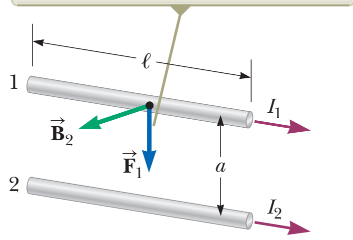


Figure 30.7 Two parallel wires that each carry a steady current exert a magnetic force on each other. The force is attractive if the currents are parallel (as shown) and repulsive if the currents are antiparallel.

are reversed and the wires repel each other. Hence, parallel conductors carrying currents in the *same* direction *attract* each other, and parallel conductors carrying currents in *opposite* directions *repel* each other.

Because the magnitudes of the forces are the same on both wires, we denote the magnitude of the magnetic force between the wires as simply F_B. We can rewrite this magnitude in terms of the force per unit length:

$$\frac{F_B}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{30.12}$$

The force between two parallel wires is used to define the **ampere** as follows:

Definition of the ampere

When the magnitude of the force per unit length between two long, parallel wires that carry identical currents and are separated by 1 m is 2×10^{-7} N/m, the current in each wire is defined to be 1 A.

The value 2×10^{-7} N/m is obtained from Equation 30.12 with $I_1 = I_2 = 1$ A and $a = 1$ m. Because this definition is based on a force, a mechanical measurement can be used to standardize the ampere. For instance, the National Institute of Standards and Technology uses an instrument called a *current balance* for primary current measurements. The results are then used to standardize other, more conventional instruments such as ammeters.

The SI unit of charge, the **coulomb,** is defined in terms of the ampere: When a conductor carries a steady current of 1 A, the quantity of charge that flows through a cross section of the conductor in 1 s is 1 C.

In deriving Equations 30.11 and 30.12, we assumed both wires are long compared with their separation distance. In fact, only one wire needs to be long. The equations accurately describe the forces exerted on each other by a long wire and a straight, parallel wire of limited length ℓ.

Quick Quiz 30.2 A loose spiral spring carrying no current is hung from a ceiling. When a switch is thrown so that a current exists in the spring, do the coils **(a)** move closer together, **(b)** move farther apart, or **(c)** not move at all?

Example 30.4 Suspending a Wire AM

Two infinitely long, parallel wires are lying on the ground a distance $a = 1.00$ cm apart as shown in Figure 30.8a. A third wire, of length $L = 10.0$ m and mass 400 g, carries a current of $I_1 = 100$ A and is levitated above the first two wires, at a horizontal position midway between them. The infinitely long wires carry equal currents I_2 in the same direction, but in the direction opposite that in the levitated wire. What current must the infinitely long wires carry so that the three wires form an equilateral triangle?

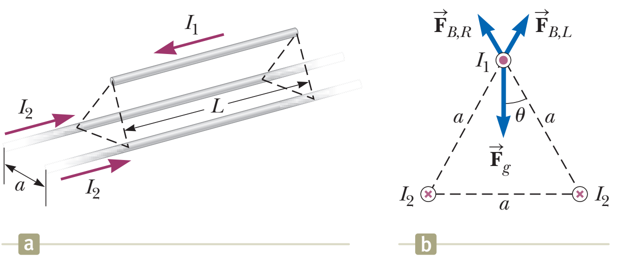


Figure 30.8 (Example 30.4) (a) Two current-carrying wires lie on the ground and suspend a third wire in the air by magnetic forces. (b) End view. In the situation described in the example, the three wires form an equilateral triangle. The two magnetic forces on the levitated wire are $\vec{F}_{B,L}$, the force due to the left-hand wire on the ground, and $\vec{F}_{B,R}$, the force due to the right-hand wire. The gravitational force $\vec{F}_g$ on the levitated wire is also shown.

SOLUTION

Conceptualize Because the current in the short wire is opposite those in the long wires, the short wire is repelled from both of the others. Imagine the currents in the long wires in Figure 30.8a are increased. The repulsive force becomes stronger, and the levitated wire rises to the point at which the wire is once again levitated in equilibrium at a higher position. Figure 30.8b shows the desired situation with the three wires forming an equilateral triangle.

Categorize Because the levitated wire is subject to forces but does not accelerate, it is modeled as a *particle in equilibrium*.

30.4 continued

Analyze The horizontal components of the magnetic forces on the levitated wire cancel. The vertical components are both positive and add together. Choose the z axis to be upward through the top wire in Figure 30.8b and in the plane of the page.

Find the total magnetic force in the upward direction on the levitated wire:

$$\vec{\mathbf{F}}_B = 2\left(\frac{\mu_0 I_1 I_2}{2\pi a}\ell\right)\cos\theta\,\hat{\mathbf{k}} = \frac{\mu_0 I_1 I_2}{\pi a}\ell\cos\theta\,\hat{\mathbf{k}}$$

Find the gravitational force on the levitated wire:

$$\vec{\mathbf{F}}_g = -mg\hat{\mathbf{k}}$$

Apply the particle in equilibrium model by adding the forces and setting the net force equal to zero:

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_B + \vec{\mathbf{F}}_g = \frac{\mu_0 I_1 I_2}{\pi a}\ell\cos\theta\,\hat{\mathbf{k}} - mg\hat{\mathbf{k}} = 0$$

Solve for the current in the wires on the ground:

$$I_2 = \frac{mg\pi a}{\mu_0 I_1 \ell\cos\theta}$$

Substitute numerical values:

$$I_2 = \frac{(0.400\text{ kg})(9.80\text{ m/s}^2)\pi(0.010\,0\text{ m})}{(4\pi\times10^{-7}\text{ T}\cdot\text{m/A})(100\text{ A})(10.0\text{ m})\cos 30.0^\circ}$$

$$= \boxed{113\text{ A}}$$

Finalize The currents in all wires are on the order of 10^2 A. Such large currents would require specialized equipment. Therefore, this situation would be difficult to establish in practice. Is the equilibrium of wire 1 stable or unstable?

30.3 Ampère's Law

Looking back, we can see that the result of Example 30.1 is important because a current in the form of a long, straight wire occurs often. Figure 30.9 is a perspective view of the magnetic field surrounding a long, straight, current-carrying wire. Because of the wire's symmetry, the magnetic field lines are circles concentric with the wire and lie in planes perpendicular to the wire. The magnitude of $\vec{\mathbf{B}}$ is constant on any circle of radius a and is given by Equation 30.5. A convenient rule for determining the direction of $\vec{\mathbf{B}}$ is to grasp the wire with the right hand, positioning the thumb along the direction of the current. The four fingers wrap in the direction of the magnetic field.

Figure 30.9 also shows that the magnetic field line has no beginning and no end. Rather, it forms a closed loop. That is a major difference between magnetic field lines and electric field lines, which begin on positive charges and end on negative charges. We will explore this feature of magnetic field lines further in Section 30.5.

Oersted's 1819 discovery about deflected compass needles demonstrates that a current-carrying conductor produces a magnetic field. Figure 30.10a (page 912) shows how this effect can be demonstrated in the classroom. Several compass needles are placed in a horizontal plane near a long, vertical wire. When no current is present in the wire, all the needles point in the same direction (that of the horizontal component of the Earth's magnetic field) as expected. When the wire carries a strong, steady current, the needles all deflect in a direction tangent to the circle as in Figure 30.10b. These observations demonstrate that the direction of the magnetic field produced by the current in the wire is consistent with the right-hand rule described in Figure 30.9. When the current is reversed, the needles in Figure 30.10b also reverse.

Now let's evaluate the product $\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ for a small length element $d\vec{\mathbf{s}}$ on the circular path defined by the compass needles and sum the products for all elements

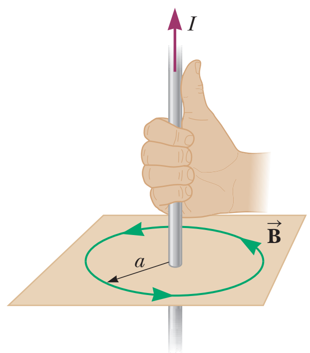


Figure 30.9 The right-hand rule for determining the direction of the magnetic field surrounding a long, straight wire carrying a current. Notice that the magnetic field lines form circles around the wire.

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Andre-Marie Ampère
French Physicist (1775–1836)
Ampère is credited with the discovery of electromagnetism, which is the relationship between electric currents and magnetic fields. Ampère's genius, particularly in mathematics, became evident by the time he was 12 years old; his personal life, however, was filled with tragedy. His father, a wealthy city official, was guillotined during the French Revolution, and his wife died young, in 1803. Ampère died at the age of 61 of pneumonia.

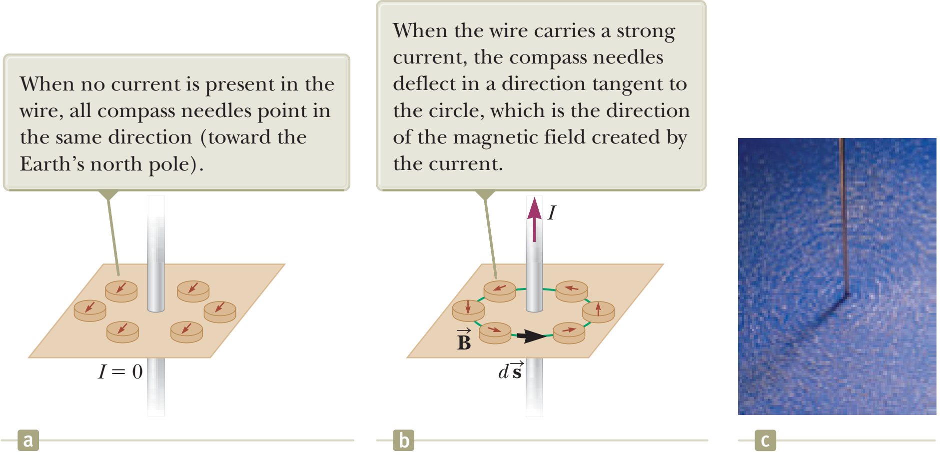

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Figure 30.10 (a) and (b) Compasses show the effects of the current in a nearby wire. (c) Circular magnetic field lines surrounding a current-carrying conductor, displayed with iron filings.

over the closed circular path.[1] Along this path, the vectors $d\vec{\mathbf{s}}$ and $\vec{\mathbf{B}}$ are parallel at each point (see Fig. 30.10b), so $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\,ds$. Furthermore, the magnitude of $\vec{\mathbf{B}}$ is constant on this circle and is given by Equation 30.5. Therefore, the sum of the products $B\,ds$ over the closed path, which is equivalent to the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$, is

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I$$

where $\oint ds = 2\pi r$ is the circumference of the circular path of radius r. Although this result was calculated for the special case of a circular path surrounding a wire of infinite length, it holds for a closed path of *any* shape (an *amperian loop*) surrounding a current that exists in an unbroken circuit. The general case, known as **Ampère's law,** can be stated as follows:

Pitfall Prevention 30.2
Avoiding Problems with Signs When using Ampère's law, apply the following right-hand rule. Point your thumb in the direction of the current through the amperian loop. Your curled fingers then point in the direction that you should integrate when traversing the loop to avoid having to define the current as negative.

Ampère's law ▶

The line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed path equals $\mu_0 I$, where I is the total steady current passing through any surface bounded by the closed path:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I \qquad \textbf{(30.13)}$$

Ampère's law describes the creation of magnetic fields by all continuous current configurations, but at our mathematical level it is useful only for calculating the magnetic field of current configurations having a high degree of symmetry. Its use is similar to that of Gauss's law in calculating electric fields for highly symmetric charge distributions.

Quick Quiz 30.3 Rank the magnitudes of $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for the closed paths a through d in Figure 30.11 from greatest to least.

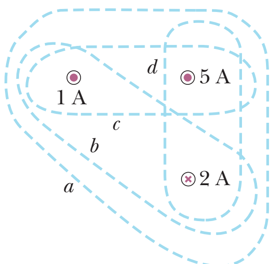


Figure 30.11 (Quick Quiz 30.3) Four closed paths around three current-carrying wires.

[1] You may wonder why we would choose to evaluate this scalar product. The origin of Ampère's law is in 19th-century science, in which a "magnetic charge" (the supposed analog to an isolated electric charge) was imagined to be moved around a circular field line. The work done on the charge was related to $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$, just as the work done moving an electric charge in an electric field is related to $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$. Therefore, Ampère's law, a valid and useful principle, arose from an erroneous and abandoned work calculation!

Quick Quiz 30.4 Rank the magnitudes of $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for the closed paths a through d in Figure 30.12 from greatest to least.

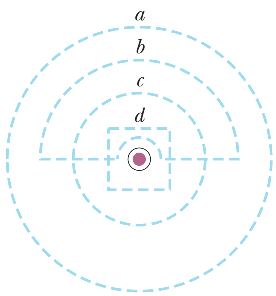


Figure 30.12 (Quick Quiz 30.4) Several closed paths near a single current-carrying wire.

Example 30.5 The Magnetic Field Created by a Long Current-Carrying Wire

A long, straight wire of radius R carries a steady current I that is uniformly distributed through the cross section of the wire (Fig. 30.13). Calculate the magnetic field a distance r from the center of the wire in the regions $r \geq R$ and $r < R$.

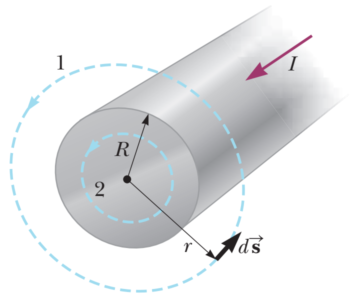


Figure 30.13 (Example 30.5) A long, straight wire of radius R carrying a steady current I uniformly distributed across the cross section of the wire. The magnetic field at any point can be calculated from Ampère's law using a circular path of radius r, concentric with the wire.

SOLUTION

Conceptualize Study Figure 30.13 to understand the structure of the wire and the current in the wire. The current creates magnetic fields everywhere, both inside and outside the wire. Based on our discussions about long, straight wires, we expect the magnetic field lines to be circles centered on the central axis of the wire.

Categorize Because the wire has a high degree of symmetry, we categorize this example as an Ampère's law problem. For the $r \geq R$ case, we should arrive at the same result as was obtained in Example 30.1, where we applied the Biot–Savart law to the same situation.

Analyze For the magnetic field exterior to the wire, let us choose for our path of integration circle 1 in Figure 30.13. From symmetry, $\vec{\mathbf{B}}$ must be constant in magnitude and parallel to $d\vec{\mathbf{s}}$ at every point on this circle.

Note that the total current passing through the plane of the circle is I and apply Ampère's law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 I$$

Solve for B:

$$B = \boxed{\frac{\mu_0 I}{2\pi r}} \quad (\text{for } r \geq R) \tag{30.14}$$

Now consider the interior of the wire, where $r < R$. Here the current I' passing through the plane of circle 2 is less than the total current I.

Set the ratio of the current I' enclosed by circle 2 to the entire current I equal to the ratio of the area πr^2 enclosed by circle 2 to the cross-sectional area πR^2 of the wire:

$$\frac{I'}{I} = \frac{\pi r^2}{\pi R^2}$$

Solve for I':

$$I' = \frac{r^2}{R^2} I$$

Apply Ampère's law to circle 2:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r) = \mu_0 I' = \mu_0\left(\frac{r^2}{R^2} I\right)$$

Solve for B:

$$B = \boxed{\left(\frac{\mu_0 I}{2\pi R^2}\right) r} \quad (\text{for } r < R) \tag{30.15}$$

continued

30.5 continued

Finalize The magnetic field exterior to the wire is identical in form to Equation 30.5. As is often the case in highly symmetric situations, it is much easier to use Ampère's law than the Biot–Savart law (Example 30.1). The magnetic field interior to the wire is similar in form to the expression for the electric field inside a uniformly charged sphere (see Example 24.3). The magnitude of the magnetic field versus r for this configuration is plotted in Figure 30.14. Inside the wire, $B \rightarrow 0$ as $r \rightarrow 0$. Furthermore, Equations 30.14 and 30.15 give the same value of the magnetic field at $r = R$, demonstrating that the magnetic field is continuous at the surface of the wire.

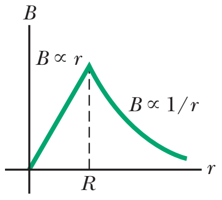


Figure 30.14 (Example 30.5) Magnitude of the magnetic field versus r for the wire shown in Figure 30.13. The field is proportional to r inside the wire and varies as $1/r$ outside the wire.

Example 30.6 The Magnetic Field Created by a Toroid

A device called a *toroid* (Fig. 30.15) is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring (a *torus*) made of a nonconducting material. For a toroid having N closely spaced turns of wire, calculate the magnetic field in the region occupied by the torus, a distance r from the center.

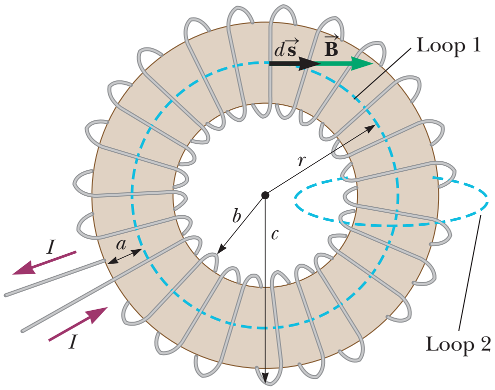


Figure 30.15 (Example 30.6) A toroid consisting of many turns of wire. If the turns are closely spaced, the magnetic field in the interior of the toroid is tangent to the dashed circle (loop 1) and varies as $1/r$. The dimension a is the cross-sectional radius of the torus. The field outside the toroid is very small and can be described by using the amperian loop (loop 2) at the right side, perpendicular to the page.

SOLUTION

Conceptualize Study Figure 30.15 carefully to understand how the wire is wrapped around the torus. The torus could be a solid material or it could be air, with a stiff wire wrapped into the shape shown in Figure 30.15 to form an empty toroid. Imagine each turn of the wire to be a circular loop as in Example 30.3. The magnetic field at the center of the loop is perpendicular to the plane of the loop. Therefore, the magnetic field lines of the collection of loops will form circles within the toroid such as suggested by loop 1 in Figure 30.15.

Categorize Because the toroid has a high degree of symmetry, we categorize this example as an Ampère's law problem.

Analyze Consider the circular amperian loop (loop 1) of radius r in the plane of Figure 30.15. By symmetry, the magnitude of the field is constant on this circle and tangent to it, so $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\,ds$. Furthermore, the wire passes through the loop N times, so the total current through the loop is NI.

Apply Ampère's law to loop 1:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 NI$$

Solve for B:

$$B = \boxed{\frac{\mu_0 NI}{2\pi r}} \qquad \textbf{(30.16)}$$

Finalize This result shows that B varies as $1/r$ and hence is *nonuniform* in the region occupied by the torus. If, however, r is very large compared with the cross-sectional radius a of the torus, the field is approximately uniform inside the torus.

For an ideal toroid, in which the turns are closely spaced, the external magnetic field is close to zero, but it is not exactly zero. In Figure 30.15, imagine the radius r of amperian loop 1 to be either smaller than b or larger than c. In either case, the loop encloses zero net current, so $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0$. You might think this result proves that $\vec{\mathbf{B}} = 0$, but it does not. Consider the amperian loop (loop 2) on the right side of the toroid in Figure 30.15. The plane of this loop is perpendicular to the page, and the toroid passes through the loop. As charges enter the toroid as indicated by the current directions in Figure 30.15,

30.6 continued

they work their way counterclockwise around the toroid. Therefore, there is a counterclockwise current around the toroid, so that a current passes through amperian loop 2! This current is small, but not zero. As a result, the toroid acts as a current loop and produces a weak external field of the form shown in Figure 30.6. The reason $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0$ for amperian loop 1 of radius $r < b$ or $r > c$ is that the field lines are perpendicular to $d\vec{\mathbf{s}}$, *not* because $\vec{\mathbf{B}} = 0$.

30.4 The Magnetic Field of a Solenoid

A **solenoid** is a long wire wound in the form of a helix. With this configuration, a reasonably uniform magnetic field can be produced in the space surrounded by the turns of wire—which we shall call the *interior* of the solenoid—when the solenoid carries a current. When the turns are closely spaced, each can be approximated as a circular loop; the net magnetic field is the vector sum of the fields resulting from all the turns.

Figure 30.16 shows the magnetic field lines surrounding a loosely wound solenoid. The field lines in the interior are nearly parallel to one another, are uniformly distributed, and are close together, indicating that the field in this space is strong and almost uniform.

If the turns are closely spaced and the solenoid is of finite length, the external magnetic field lines are as shown in Figure 30.17a. This field line distribution is similar to that surrounding a bar magnet (Fig. 30.17b). Hence, one end of the solenoid behaves like the north pole of a magnet and the opposite end behaves like the south pole. As the length of the solenoid increases, the interior field becomes more uniform and the exterior field becomes weaker. An *ideal solenoid* is approached when the turns are closely spaced and the length is much greater than the radius of the turns. Figure 30.18 (page 916) shows a longitudinal cross section of part of such a solenoid carrying a current I. In this case, the external field is close to zero and the interior field is uniform over a great volume.

Consider the amperian loop (loop 1) perpendicular to the page in Figure 30.18 (page 916), surrounding the ideal solenoid. This loop encloses a small

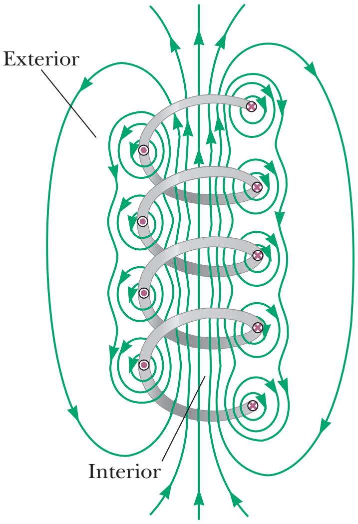


Figure 30.16 The magnetic field lines for a loosely wound solenoid.

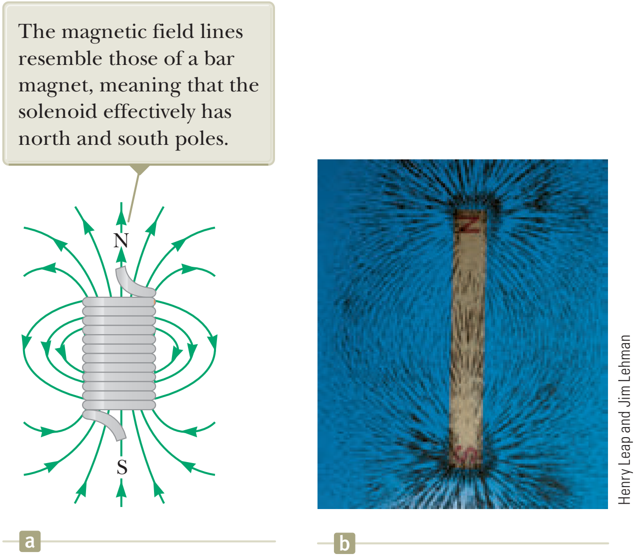


Figure 30.17 (a) Magnetic field lines for a tightly wound solenoid of finite length, carrying a steady current. The field in the interior space is strong and nearly uniform. (b) The magnetic field pattern of a bar magnet, displayed with small iron filings on a sheet of paper.

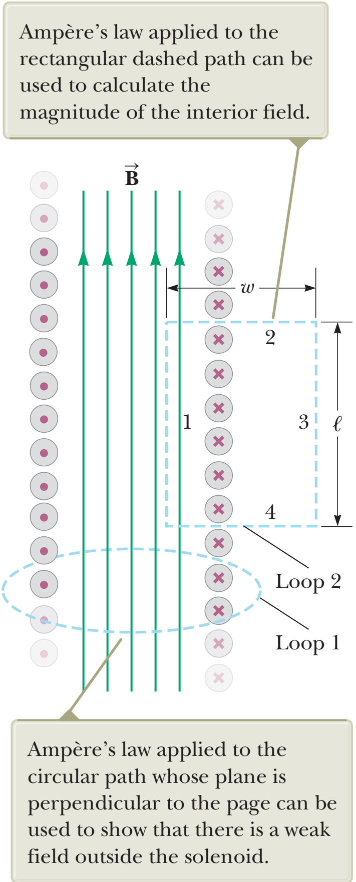


Figure 30.18 Cross-sectional view of an ideal solenoid, where the interior magnetic field is uniform and the exterior field is close to zero.

current as the charges in the wire move coil by coil along the length of the solenoid. Therefore, there is a nonzero magnetic field outside the solenoid. It is a weak field, with circular field lines, like those due to a line of current as in Figure 30.9. For an ideal solenoid, this weak field is the only field external to the solenoid.

We can use Ampère's law to obtain a quantitative expression for the interior magnetic field in an ideal solenoid. Because the solenoid is ideal, $\vec{\mathbf{B}}$ in the interior space is uniform and parallel to the axis and the magnetic field lines in the exterior space form circles around the solenoid. The planes of these circles are perpendicular to the page. Consider the rectangular path (loop 2) of length ℓ and width w shown in Figure 30.18. Let's apply Ampère's law to this path by evaluating the integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ over each side of the rectangle. The contribution along side 3 is zero because the external magnetic field lines are perpendicular to the path in this region. The contributions from sides 2 and 4 are both zero, again because $\vec{\mathbf{B}}$ is perpendicular to $d\vec{\mathbf{s}}$ along these paths, both inside and outside the solenoid. Side 1 gives a contribution to the integral because along this path $\vec{\mathbf{B}}$ is uniform and parallel to $d\vec{\mathbf{s}}$. The integral over the closed rectangular path is therefore

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_{\text{path 1}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \int_{\text{path 1}} ds = B\ell$$

The right side of Ampère's law involves the total current I through the area bounded by the path of integration. In this case, the total current through the rectangular path equals the current through each turn multiplied by the number of turns. If N is the number of turns in the length ℓ, the total current through the rectangle is NI. Therefore, Ampère's law applied to this path gives

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\ell = \mu_0 NI$$

Magnetic field inside a solenoid ▶

$$\boxed{B = \mu_0 \frac{N}{\ell} I = \mu_0 n I} \qquad \textbf{(30.17)}$$

where $n = N/\ell$ is the number of turns per unit length.

We also could obtain this result by reconsidering the magnetic field of a toroid (see Example 30.6). If the radius r of the torus in Figure 30.15 containing N turns is much greater than the toroid's cross-sectional radius a, a short section of the toroid approximates a solenoid for which $n = N/2\pi r$. In this limit, Equation 30.16 agrees with Equation 30.17.

Equation 30.17 is valid only for points near the center (that is, far from the ends) of a very long solenoid. As you might expect, the field near each end is smaller than the value given by Equation 30.17. As the length of a solenoid increases, the magnitude of the field at the end approaches half the magnitude at the center (see Problem 69).

Quick Quiz 30.5 Consider a solenoid that is very long compared with its radius. Of the following choices, what is the most effective way to increase the magnetic field in the interior of the solenoid? **(a)** double its length, keeping the number of turns per unit length constant **(b)** reduce its radius by half, keeping the number of turns per unit length constant **(c)** overwrap the entire solenoid with an additional layer of current-carrying wire

30.5 Gauss's Law in Magnetism

The flux associated with a magnetic field is defined in a manner similar to that used to define electric flux (see Eq. 24.3). Consider an element of area dA on an arbitrarily shaped surface as shown in Figure 30.19. If the magnetic field at this element is $\vec{\mathbf{B}}$, the magnetic flux through the element is $\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$, where $d\vec{\mathbf{A}}$ is a vector that is perpendicular to the surface and has a magnitude equal to the area dA. Therefore, the total magnetic flux Φ_B through the surface is

$$\Phi_B \equiv \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \qquad (30.18)$$

◀ **Definition of magnetic flux**

Consider the special case of a plane of area A in a uniform field $\vec{\mathbf{B}}$ that makes an angle θ with $d\vec{\mathbf{A}}$. The magnetic flux through the plane in this case is

$$\Phi_B = BA \cos \theta \qquad (30.19)$$

If the magnetic field is parallel to the plane as in Figure 30.20a, then $\theta = 90°$ and the flux through the plane is zero. If the field is perpendicular to the plane as in Figure 30.20b, then $\theta = 0$ and the flux through the plane is BA (the maximum value).

The unit of magnetic flux is $\text{T} \cdot \text{m}^2$, which is defined as a *weber* (Wb); $1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2$.

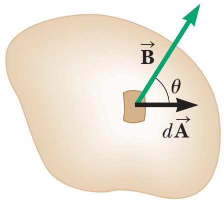


Figure 30.19 The magnetic flux through an area element dA is $\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = B\,dA\cos\theta$, where $d\vec{\mathbf{A}}$ is a vector perpendicular to the surface.

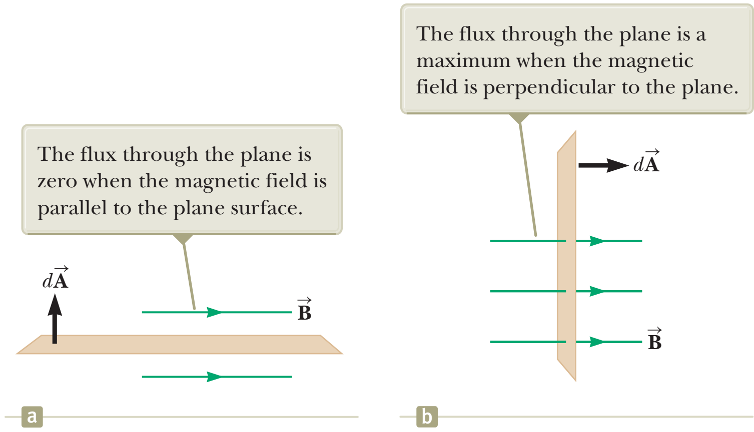


Figure 30.20 Magnetic flux through a plane lying in a magnetic field.

Example 30.7 Magnetic Flux Through a Rectangular Loop

A rectangular loop of width a and length b is located near a long wire carrying a current I (Fig. 30.21). The distance between the wire and the closest side of the loop is c. The wire is parallel to the long side of the loop. Find the total magnetic flux through the loop due to the current in the wire.

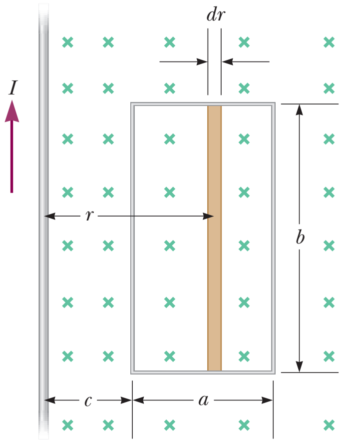


Figure 30.21 (Example 30.7) The magnetic field due to the wire carrying a current I is not uniform over the rectangular loop.

SOLUTION

Conceptualize As we saw in Section 30.3, the magnetic field lines due to the wire will be circles, many of which will pass through the rectangular loop. We know that the magnetic field is a function of distance r from a long wire. Therefore, the magnetic field varies over the area of the rectangular loop.

Categorize Because the magnetic field varies over the area of the loop, we must integrate over this area to find the total flux. That identifies this as an analysis problem.

Analyze Noting that $\vec{\mathbf{B}}$ is parallel to $d\vec{\mathbf{A}}$ at any point within the loop, find the magnetic flux through the rectangular area using Equation 30.18 and incorporate Equation 30.14 for the magnetic field:

$$\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \int B\,dA = \int \frac{\mu_0 I}{2\pi r}\,dA$$

continued

30.7 continued

Express the area element (the tan strip in Fig. 30.21) as $dA = b\,dr$ and substitute:

$$\Phi_B = \int \frac{\mu_0 I}{2\pi r}\, b\, dr = \frac{\mu_0 I b}{2\pi} \int \frac{dr}{r}$$

Integrate from $r = c$ to $r = a + c$:

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \int_c^{a+c} \frac{dr}{r} = \frac{\mu_0 I b}{2\pi} \ln r \Big|_c^{a+c}$$

$$= \frac{\mu_0 I b}{2\pi} \ln\left(\frac{a+c}{c}\right) = \boxed{\frac{\mu_0 I b}{2\pi} \ln\left(1 + \frac{a}{c}\right)}$$

Finalize Notice how the flux depends on the size of the loop. Increasing either a or b increases the flux as expected. If c becomes large such that the loop is very far from the wire, the flux approaches zero, also as expected. If c goes to zero, the flux becomes infinite. In principle, this infinite value occurs because the field becomes infinite at $r = 0$ (assuming an infinitesimally thin wire). That will not happen in reality because the thickness of the wire prevents the left edge of the loop from reaching $r = 0$.

In Chapter 24, we found that the electric flux through a closed surface surrounding a net charge is proportional to that charge (Gauss's law). In other words, the number of electric field lines leaving the surface depends only on the net charge within it. This behavior exists because electric field lines originate and terminate on electric charges.

The situation is quite different for magnetic fields, which are continuous and form closed loops. In other words, as illustrated by the magnetic field lines of a current in Figure 30.9 and of a bar magnet in Figure 30.22, magnetic field lines do not begin or end at any point. For any closed surface such as the one outlined by the dashed line in Figure 30.22, the number of lines entering the surface equals the number leaving the surface; therefore, the net magnetic flux is zero. In contrast, for a closed surface surrounding one charge of an electric dipole (Fig. 30.23), the net electric flux is not zero.

Gauss's law in magnetism states that

Gauss's law in magnetism ▶

the net magnetic flux through any closed surface is always zero:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \qquad \textbf{(30.20)}$$

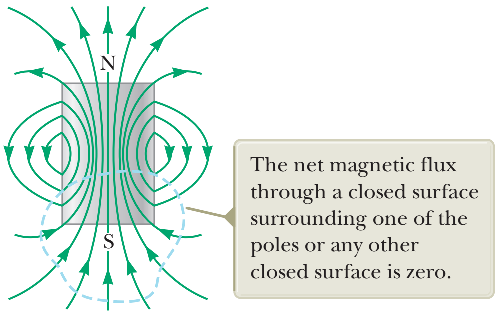


Figure 30.22 The magnetic field lines of a bar magnet form closed loops. (The dashed line represents the intersection of a closed surface with the page.)

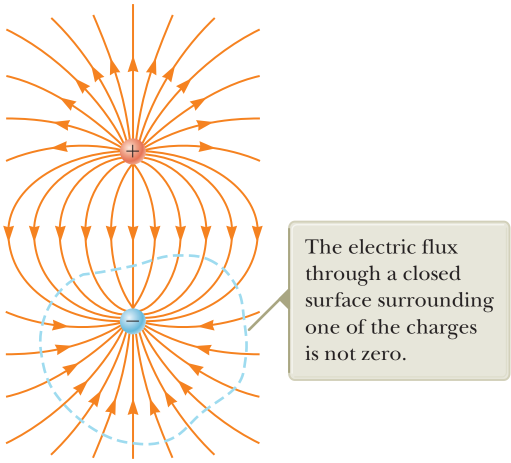


Figure 30.23 The electric field lines surrounding an electric dipole begin on the positive charge and terminate on the negative charge.

This statement represents that isolated magnetic poles (monopoles) have never been detected and perhaps do not exist. Nonetheless, scientists continue the search because certain theories that are otherwise successful in explaining fundamental physical behavior suggest the possible existence of magnetic monopoles.

30.6 Magnetism in Matter

The magnetic field produced by a current in a coil of wire gives us a hint as to what causes certain materials to exhibit strong magnetic properties. Earlier we found that a solenoid like the one shown in Figure 30.17a has a north pole and a south pole. In general, *any* current loop has a magnetic field and therefore has a magnetic dipole moment, including the atomic-level current loops described in some models of the atom.

The Magnetic Moments of Atoms

Let's begin our discussion with a classical model of the atom in which electrons move in circular orbits around the much more massive nucleus. In this model, an orbiting electron constitutes a tiny current loop (because it is a moving charge), and the magnetic moment of the electron is associated with this orbital motion. Although this model has many deficiencies, some of its predictions are in good agreement with the correct theory, which is expressed in terms of quantum physics.

In our classical model, we assume an electron is a particle in uniform circular motion: it moves with constant speed v in a circular orbit of radius r about the nucleus as in Figure 30.24. The current I associated with this orbiting electron is its charge e divided by its period T. Using Equation 4.15 from the particle in uniform circular motion model, $T = 2\pi r/v$, gives

$$I = \frac{e}{T} = \frac{ev}{2\pi r}$$

The magnitude of the magnetic moment associated with this current loop is given by $\mu = IA$, where $A = \pi r^2$ is the area enclosed by the orbit. Therefore,

$$\mu = IA = \left(\frac{ev}{2\pi r}\right)\pi r^2 = \tfrac{1}{2}evr \tag{30.21}$$

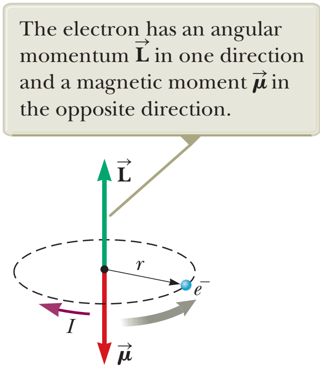


Figure 30.24 An electron moving in the direction of the gray arrow in a circular orbit of radius r. Because the electron carries a negative charge, the direction of the current due to its motion about the nucleus is opposite the direction of that motion.

Because the magnitude of the orbital angular momentum of the electron is given by $L = m_e vr$ (Eq. 11.12 with $\phi = 90°$), the magnetic moment can be written as

$$\mu = \left(\frac{e}{2m_e}\right)L \tag{30.22}$$

◀ **Orbital magnetic moment**

This result demonstrates that the magnetic moment of the electron is proportional to its orbital angular momentum. Because the electron is negatively charged, the vectors $\vec{\boldsymbol{\mu}}$ and $\vec{\mathbf{L}}$ point in *opposite* directions. Both vectors are perpendicular to the plane of the orbit as indicated in Figure 30.24.

A fundamental outcome of quantum physics is that orbital angular momentum is quantized and is equal to multiples of $\hbar = h/2\pi = 1.05 \times 10^{-34}\ \text{J}\cdot\text{s}$, where h is Planck's constant (see Chapter 40). The smallest nonzero value of the electron's magnetic moment resulting from its orbital motion is

$$\mu = \sqrt{2}\,\frac{e}{2m_e}\,\hbar \tag{30.23}$$

We shall see in Chapter 42 how expressions such as Equation 30.23 arise.

Because all substances contain electrons, you may wonder why most substances are not magnetic. The main reason is that, in most substances, the magnetic

Pitfall Prevention 30.3

The Electron Does Not Spin The electron is *not* physically spinning. It has an intrinsic angular momentum *as if it were spinning,* but the notion of rotation for a point particle is meaningless. Rotation applies only to a *rigid object,* with an extent in space, as in Chapter 10. Spin angular momentum is actually a relativistic effect.

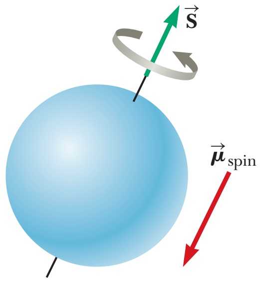


Figure 30.25 Classical model of a spinning electron. We can adopt this model to remind ourselves that electrons have an intrinsic angular momentum. The model should not be pushed too far, however; it gives an incorrect magnitude for the magnetic moment, incorrect quantum numbers, and too many degrees of freedom.

Table 30.1 Magnetic Moments of Some Atoms and Ions

Atom or Ion	Magnetic Moment (10^{-24} J/T)
H	9.27
He	0
Ne	0
Ce^{3+}	19.8
Yb^{3+}	37.1

moment of one electron in an atom is canceled by that of another electron orbiting in the opposite direction. The net result is that, for most materials, the magnetic effect produced by the orbital motion of the electrons is either zero or very small.

In addition to its orbital magnetic moment, an electron (as well as protons, neutrons, and other particles) has an intrinsic property called **spin** that also contributes to its magnetic moment. Classically, the electron might be viewed as spinning about its axis as shown in Figure 30.25, but you should be very careful with the classical interpretation. The magnitude of the angular momentum $\vec{S}$ associated with spin is on the same order of magnitude as the magnitude of the angular momentum $\vec{L}$ due to the orbital motion. The magnitude of the spin angular momentum of an electron predicted by quantum theory is

$$S = \frac{\sqrt{3}}{2}\hbar$$

The magnetic moment characteristically associated with the spin of an electron has the value

$$\mu_{spin} = \frac{e\hbar}{2m_e} \tag{30.24}$$

This combination of constants is called the **Bohr magneton** μ_B:

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \mathrm{J/T} \tag{30.25}$$

Therefore, atomic magnetic moments can be expressed as multiples of the Bohr magneton. (Note that $1\ \mathrm{J/T} = 1\ \mathrm{A \cdot m^2}$.)

In atoms containing many electrons, the electrons usually pair up with their spins opposite each other; therefore, the spin magnetic moments cancel. Atoms containing an odd number of electrons, however, must have at least one unpaired electron and therefore some spin magnetic moment. The total magnetic moment of an atom is the vector sum of the orbital and spin magnetic moments, and a few examples are given in Table 30.1. Notice that helium and neon have zero moments because their individual spin and orbital moments cancel.

The nucleus of an atom also has a magnetic moment associated with its constituent protons and neutrons. The magnetic moment of a proton or neutron, however, is much smaller than that of an electron and can usually be neglected. We can understand this smaller value by inspecting Equation 30.25 and replacing the mass of the electron with the mass of a proton or a neutron. Because the masses of the proton and neutron are much greater than that of the electron, their magnetic moments are on the order of 10^3 times smaller than that of the electron.

Ferromagnetism

A small number of crystalline substances exhibit strong magnetic effects called **ferromagnetism.** Some examples of ferromagnetic substances are iron, cobalt, nickel, gadolinium, and dysprosium. These substances contain permanent atomic magnetic moments that tend to align parallel to each other even in a weak external magnetic field. Once the moments are aligned, the substance remains magnetized after the external field is removed. This permanent alignment is due to a strong coupling between neighboring moments, a coupling that can be understood only in quantum-mechanical terms.

All ferromagnetic materials are made up of microscopic regions called **domains,** regions within which all magnetic moments are aligned. These domains have volumes of about 10^{-12} to 10^{-8} m^3 and contain 10^{17} to 10^{21} atoms. The boundaries between the various domains having different orientations are called **domain walls.** In an unmagnetized sample, the magnetic moments in the domains are randomly oriented so that the net magnetic moment is zero as in Figure 30.26a. When the sample is placed in an external magnetic field $\vec{\mathbf{B}}$, the size of those domains with magnetic moments aligned with the field grows, which results in a magnetized sample as in Figure 30.26b. As the external field becomes very strong as in Figure 30.26c, the domains in which the magnetic moments are not aligned with the field become very small. When the external field is removed, the sample may retain a net magnetization in the direction of the original field. At ordinary temperatures, thermal agitation is not sufficient to disrupt this preferred orientation of magnetic moments.

When the temperature of a ferromagnetic substance reaches or exceeds a critical temperature called the **Curie temperature,** the substance loses its residual magnetization. Below the Curie temperature, the magnetic moments are aligned and the substance is ferromagnetic. Above the Curie temperature, the thermal agitation is great enough to cause a random orientation of the moments and the substance becomes paramagnetic. Curie temperatures for several ferromagnetic substances are given in Table 30.2.

Paramagnetism

Paramagnetic substances have a weak magnetism resulting from the presence of atoms (or ions) that have permanent magnetic moments. These moments interact only weakly with one another and are randomly oriented in the absence of an external magnetic field. When a paramagnetic substance is placed in an external magnetic field, its atomic moments tend to line up with the field. This alignment process, however, must compete with thermal motion, which tends to randomize the magnetic moment orientations.

Diamagnetism

When an external magnetic field is applied to a diamagnetic substance, a weak magnetic moment is induced in the direction opposite the applied field, causing diamagnetic substances to be weakly repelled by a magnet. Although diamagnetism is present in all matter, its effects are much smaller than those of paramagnetism or ferromagnetism and are evident only when those other effects do not exist.

We can attain some understanding of diamagnetism by considering a classical model of two atomic electrons orbiting the nucleus in opposite directions but with the same speed. The electrons remain in their circular orbits because of the attractive electrostatic force exerted by the positively charged nucleus. Because the magnetic moments of the two electrons are equal in magnitude and opposite in direction, they cancel each other and the magnetic moment of the atom is zero. When an external magnetic field is applied, the electrons experience an additional magnetic force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. This added magnetic force combines with the electrostatic force to increase the orbital speed of the electron whose magnetic moment is antiparallel to the field and to decrease the speed of the electron whose magnetic moment is parallel to the field. As a result, the two magnetic moments of the electrons no longer cancel and the substance acquires a net magnetic moment that is opposite the applied field.

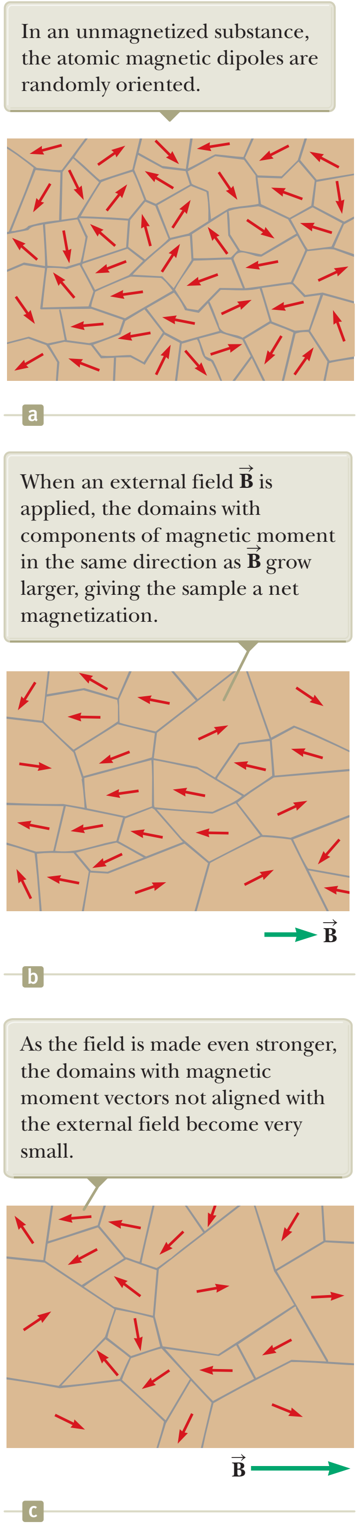
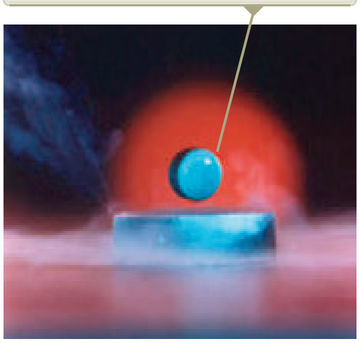


Figure 30.26 Orientation of magnetic dipoles before and after a magnetic field is applied to a ferromagnetic substance.

Table 30.2 Curie Temperatures for Several Ferromagnetic Substances

Substance	T_{Curie} (K)
Iron	1 043
Cobalt	1 394
Nickel	631
Gadolinium	317
Fe_2O_3	893



Courtesy Argonne National Laboratory

Figure 30.27 An illustration of the Meissner effect, shown by this magnet suspended above a cooled ceramic superconductor disk, has become our most visual image of high-temperature superconductivity. Superconductivity is the loss of all resistance to electrical current and is a key to more-efficient energy use.

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Courtesy of Dr. Andre Geim, Manchester University

(*Left*) Paramagnetism. (*Right*) Diamagnetism: a frog is levitated in a 16-T magnetic field at the Nijmegen High Field Magnet Laboratory in the Netherlands.

As you recall from Chapter 27, a superconductor is a substance in which the electrical resistance is zero below some critical temperature. Certain types of superconductors also exhibit perfect diamagnetism in the superconducting state. As a result, an applied magnetic field is expelled by the superconductor so that the field is zero in its interior. This phenomenon is known as the **Meissner effect.** If a permanent magnet is brought near a superconductor, the two objects repel each other. This repulsion is illustrated in Figure 30.27, which shows a small permanent magnet levitated above a superconductor maintained at 77 K.

Summary

Definition

The **magnetic flux** Φ_B through a surface is defined by the surface integral

$$\Phi_B \equiv \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \tag{30.18}$$

Concepts and Principles

The **Biot–Savart law** says that the magnetic field $d\vec{\mathbf{B}}$ at a point P due to a length element $d\vec{\mathbf{s}}$ that carries a steady current I is

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{30.1}$$

where μ_0 is the **permeability of free space,** r is the distance from the element to the point P, and $\hat{\mathbf{r}}$ is a unit vector pointing from $d\vec{\mathbf{s}}$ toward point P. We find the total field at P by integrating this expression over the entire current distribution.

The magnetic force per unit length between two parallel wires separated by a distance a and carrying currents I_1 and I_2 has a magnitude

$$\frac{F_B}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{30.12}$$

The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions.

Ampère's law says that the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed path equals $\mu_0 I$, where I is the total steady current through any surface bounded by the closed path:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I \quad \textbf{(30.13)}$$

The magnitude of the magnetic field at a distance r from a long, straight wire carrying an electric current I is

$$B = \frac{\mu_0 I}{2\pi r} \quad \textbf{(30.14)}$$

The field lines are circles concentric with the wire.

The magnitudes of the fields inside a toroid and solenoid are

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(toroid)} \quad \textbf{(30.16)}$$

$$B = \mu_0 \frac{N}{\ell} I = \mu_0 n I \quad \text{(solenoid)} \quad \textbf{(30.17)}$$

where N is the total number of turns.

Gauss's law of magnetism states that the net magnetic flux through any closed surface is zero:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \quad \textbf{(30.20)}$$

Substances can be classified into one of three categories that describe their magnetic behavior. **Diamagnetic** substances are those in which the magnetic moment is weak and opposite the applied magnetic field. **Paramagnetic** substances are those in which the magnetic moment is weak and in the same direction as the applied magnetic field. In **ferromagnetic** substances, interactions between atoms cause magnetic moments to align and create a strong magnetization that remains after the external field is removed.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. **(i)** What happens to the magnitude of the magnetic field inside a long solenoid if the current is doubled? (a) It becomes four times larger. (b) It becomes twice as large. (c) It is unchanged. (d) It becomes one-half as large. (e) It becomes one-fourth as large. **(ii)** What happens to the field if instead the length of the solenoid is doubled, with the number of turns remaining the same? Choose from the same possibilities as in part (i). **(iii)** What happens to the field if the number of turns is doubled, with the length remaining the same? Choose from the same possibilities as in part (i). **(iv)** What happens to the field if the radius is doubled? Choose from the same possibilities as in part (i).

2. In Figure 30.7, assume $I_1 = 2.00$ A and $I_2 = 6.00$ A. What is the relationship between the magnitude F_1 of the force exerted on wire 1 and the magnitude F_2 of the force exerted on wire 2? (a) $F_1 = 6F_2$ (b) $F_1 = 3F_2$ (c) $F_1 = F_2$ (d) $F_1 = \frac{1}{3}F_2$ (e) $F_1 = \frac{1}{6}F_2$

3. Answer each question yes or no. (a) Is it possible for each of three stationary charged particles to exert a force of attraction on the other two? (b) Is it possible for each of three stationary charged particles to repel both of the other particles? (c) Is it possible for each of three current-carrying metal wires to attract the other two wires? (d) Is it possible for each of three current-carrying metal wires to repel the other two wires? André-Marie Ampère's experiments on electromagnetism are models of logical precision and included observation of the phenomena referred to in this question.

4. Two long, parallel wires each carry the same current I in the same direction (Fig. OQ30.4). Is the total magnetic field at the point P midway between the wires (a) zero, (b) directed into the page, (c) directed out of the page, (d) directed to the left, or (e) directed to the right?

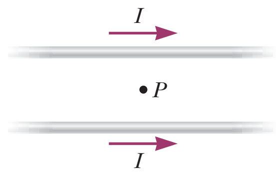

Figure OQ30.4

5. Two long, straight wires cross each other at a right angle, and each carries the same current I (Fig. OQ30.5). Which of the following statements is true regarding the total magnetic field due to the two wires at the various points in the figure? More than one statement may be correct. (a) The field is strongest at points B and D. (b) The field is strongest at points A and C. (c) The field is out of the page at point B and

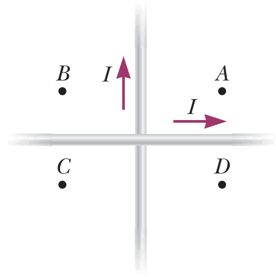

Figure OQ30.5

into the page at point D. (d) The field is out of the page at point C and out of the page at point D. (e) The field has the same magnitude at all four points.

6. A long, vertical, metallic wire carries downward electric current. **(i)** What is the direction of the magnetic field it creates at a point 2 cm horizontally east of the center of the wire? (a) north (b) south (c) east (d) west (e) up **(ii)** What would be the direction of the field if the current consisted of positive charges moving downward instead of electrons moving upward? Choose from the same possibilities as in part (i).

7. Suppose you are facing a tall makeup mirror on a vertical wall. Fluorescent tubes framing the mirror carry a clockwise electric current. **(i)** What is the direction of the magnetic field created by that current at the center of the mirror? (a) left (b) right (c) horizontally toward you (d) horizontally away from you (e) no direction because the field has zero magnitude **(ii)** What is the direction of the field the current creates at a point on the wall outside the frame to the right? Choose from the same possibilities as in part (i).

8. A long, straight wire carries a current I (Fig. OQ30.8). Which of the following statements is true regarding the magnetic field due to the wire? More than one statement may be correct. (a) The magnitude is proportional to I/r, and the direction is out of the page at P. (b) The magnitude is proportional to I/r^2, and the direction is out of the page at P. (c) The magnitude is proportional to I/r, and the direction is into the page at P. (d) The magnitude is proportional to I/r^2, and the direction is into the page at P. (e) The magnitude is proportional to I, but does not depend on r.

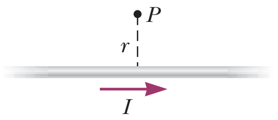

Figure OQ30.8

9. Two long, parallel wires carry currents of 20.0 A and 10.0 A in opposite directions (Fig. OQ30.9). Which of the following statements is true? More than one statement may be correct. (a) In region I, the magnetic field is into the page and is never zero. (b) In region II, the field is into the page and can be zero. (c) In region III, it is possible for the field to be zero. (d) In region I, the magnetic field is out of the page and is never zero. (e) There are no points where the field is zero.

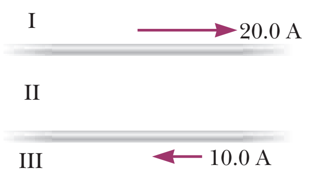

Figure OQ30.9 Objective Questions 9 and 10.

10. Consider the two parallel wires carrying currents in opposite directions in Figure OQ30.9. Due to the magnetic interaction between the wires, does the lower wire experience a magnetic force that is (a) upward, (b) downward, (c) to the left, (d) to the right, or (e) into the paper?

11. What creates a magnetic field? More than one answer may be correct. (a) a stationary object with electric charge (b) a moving object with electric charge (c) a stationary conductor carrying electric current (d) a difference in electric potential (e) a charged capacitor disconnected from a battery and at rest *Note:* In Chapter 34, we will see that a changing electric field also creates a magnetic field.

12. A long solenoid with closely spaced turns carries electric current. Does each turn of wire exert (a) an attractive force on the next adjacent turn, (b) a repulsive force on the next adjacent turn, (c) zero force on the next adjacent turn, or (d) either an attractive or a repulsive force on the next turn, depending on the direction of current in the solenoid?

13. A uniform magnetic field is directed along the x axis. For what orientation of a flat, rectangular coil is the flux through the rectangle a maximum? (a) It is a maximum in the xy plane. (b) It is a maximum in the xz plane. (c) It is a maximum in the yz plane. (d) The flux has the same nonzero value for all these orientations. (e) The flux is zero in all cases.

14. Rank the magnitudes of the following magnetic fields from largest to smallest, noting any cases of equality. (a) the field 2 cm away from a long, straight wire carrying a current of 3 A (b) the field at the center of a flat, compact, circular coil, 2 cm in radius, with 10 turns, carrying a current of 0.3 A (c) the field at the center of a solenoid 2 cm in radius and 200 cm long, with 1 000 turns, carrying a current of 0.3 A (d) the field at the center of a long, straight, metal bar, 2 cm in radius, carrying a current of 300 A (e) a field of 1 mT

15. Solenoid A has length L and N turns, solenoid B has length $2L$ and N turns, and solenoid C has length $L/2$ and $2N$ turns. If each solenoid carries the same current, rank the magnitudes of the magnetic fields in the centers of the solenoids from largest to smallest.

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Is the magnetic field created by a current loop uniform? Explain.

2. One pole of a magnet attracts a nail. Will the other pole of the magnet attract the nail? Explain. Also explain how a magnet sticks to a refrigerator door.

3. Compare Ampère's law with the Biot–Savart law. Which is more generally useful for calculating $\vec{\mathbf{B}}$ for a current-carrying conductor?

4. A hollow copper tube carries a current along its length. Why is $B = 0$ inside the tube? Is B nonzero outside the tube?

5. Imagine you have a compass whose needle can rotate vertically as well as horizontally. Which way would the compass needle point if you were at the Earth's north magnetic pole?

6. Is Ampère's law valid for all closed paths surrounding a conductor? Why is it not useful for calculating $\vec{\mathbf{B}}$ for all such paths?

7. A magnet attracts a piece of iron. The iron can then attract another piece of iron. On the basis of domain alignment, explain what happens in each piece of iron.

8. Why does hitting a magnet with a hammer cause the magnetism to be reduced?

9. The quantity $\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ in Ampère's law is called *magnetic circulation*. Figures 30.10 and 30.13 show paths around which the magnetic circulation is evaluated. Each of these paths encloses an area. What is the magnetic flux through each area? Explain your answer.

10. Figure CQ30.10 shows four permanent magnets, each having a hole through its center. Notice that the blue and yellow magnets are levitated above the red ones. (a) How does this levitation occur? (b) What purpose do the rods serve? (c) What can you say about the poles of the magnets from this observation? (d) If the blue magnet were inverted, what do you suppose would happen?

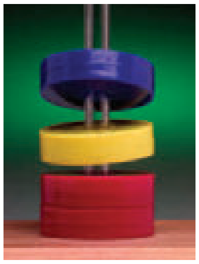

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Figure CQ30.10

$\boxed{\textbf{11.}}$ Explain why two parallel wires carrying currents in opposite directions repel each other.

12. Consider a magnetic field that is uniform in direction throughout a certain volume. (a) Can the field be uniform in magnitude? (b) Must it be uniform in magnitude? Give evidence for your answers.

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

$\boxed{\textbf{1.}}$ full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 30.1 The Biot–Savart Law

1. Review. In studies of the possibility of migrating birds using the Earth's magnetic field for navigation, birds have been fitted with coils as "caps" and "collars" as shown in Figure P30.1. (a) If the identical coils have radii of 1.20 cm and are 2.20 cm apart, with 50 turns of wire apiece, what current should they both carry to produce a magnetic field of 4.50×10^{-5} T halfway between them? (b) If the resistance of each coil is 210 Ω, what voltage should the battery supplying each coil have? (c) What power is delivered to each coil?

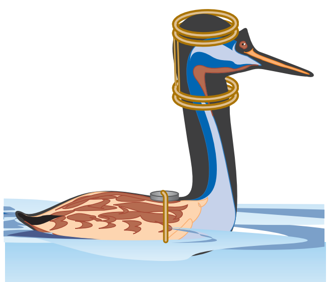


Figure P30.1

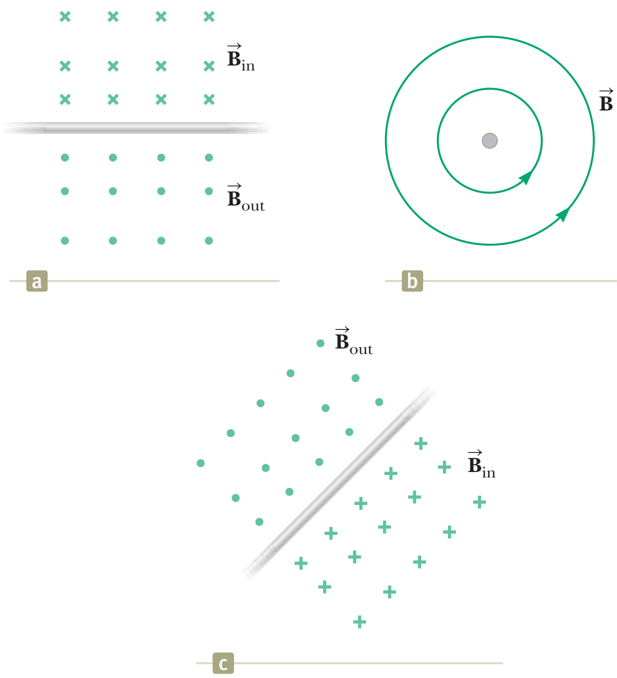


Figure P30.2

2. In each of parts (a) through (c) of Figure P30.2, find the direction of the current in the wire that would produce a magnetic field directed as shown.

3. W Calculate the magnitude of the magnetic field at a point 25.0 cm from a long, thin conductor carrying a current of 2.00 A.

4. In 1962, measurements of the magnetic field of a large tornado were made at the Geophysical Observatory in Tulsa, Oklahoma. If the magnitude of the tornado's field was $B = 1.50 \times 10^{-8}$ T pointing north when the tornado was 9.00 km east of the observatory, what current was carried up or down the funnel of the tornado? Model the vortex as a long, straight wire carrying a current.

5. M (a) A conducting loop in the shape of a square of edge length $\ell = 0.400$ m carries a current $I = 10.0$ A as shown in Figure P30.5. Calculate the magnitude and direction of the magnetic field at the center of the square. (b) **What If?** If this conductor is reshaped to form a circular loop and carries the same current, what is the value of the magnetic field at the center?

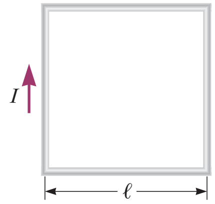


Figure P30.5

6. W In Niels Bohr's 1913 model of the hydrogen atom, an electron circles the proton at a distance of 5.29×10^{-11} m with a speed of 2.19×10^6 m/s. Compute the magnitude of the magnetic field this motion produces at the location of the proton.

7. A conductor consists of a circular loop of radius $R = 15.0$ cm and two long, straight sections as shown in Figure P30.7. The wire lies in the plane of the paper and carries a current $I = 1.00$ A. Find the magnetic field at the center of the loop.

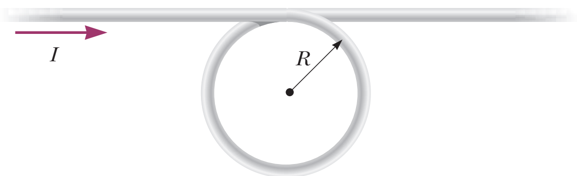


Figure P30.7 Problems 7 and 8.

8. A conductor consists of a circular loop of radius R and two long, straight sections as shown in Figure P30.7. The wire lies in the plane of the paper and carries a current I. (a) What is the direction of the magnetic field at the center of the loop? (b) Find an expression for the magnitude of the magnetic field at the center of the loop.

9. Two long, straight, parallel wires carry currents that are directed perpendicular to the page as shown in Figure P30.9. Wire 1 carries a current I_1 into the page (in the negative z direction) and passes through the x axis at $x = +a$. Wire 2 passes through the x axis at $x = -2a$ and carries an unknown current I_2. The total magnetic field at the origin due to the current-carrying wires has the magnitude $2\mu_0 I_1/(2\pi a)$. The current I_2 can have either of two possible values. (a) Find the value of I_2 with the smaller magnitude, stating it in terms of I_1 and giving its direction. (b) Find the other possible value of I_2.

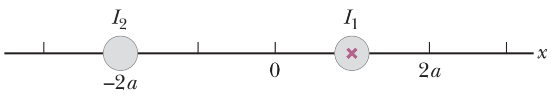


Figure P30.9

10. An infinitely long wire carrying a current I is bent at a right angle as shown in Figure P30.10. Determine the magnetic field at point P, located a distance x from the corner of the wire.

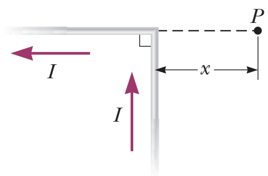


Figure P30.10

11. A long, straight wire carries a current I. A right-angle bend is made in the middle of the wire. The bend forms an arc of a circle of radius r as shown in Figure P30.11. Determine the magnetic field at point P, the center of the arc.

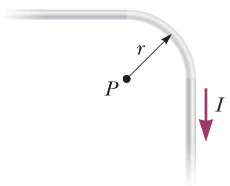


Figure P30.11

12. Consider a flat, circular current loop of radius R carrying a current I. Choose the x axis to be along the axis of the loop, with the origin at the loop's center. Plot a graph of the ratio of the magnitude of the magnetic field at coordinate x to that at the origin for $x = 0$ to $x = 5R$. It may be helpful to use a programmable calculator or a computer to solve this problem.

13. A current path shaped as shown in Figure P30.13 produces a magnetic field at P, the center of the arc. If the arc subtends an angle of $\theta = 30.0°$ and the radius of the arc is 0.600 m, what are the magnitude and

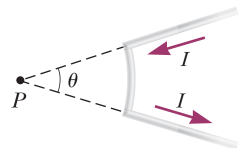


Figure P30.13

direction of the field produced at P if the current is 3.00 A?

14. AMT M One long wire carries current 30.0 A to the left along the x axis. A second long wire carries current 50.0 A to the right along the line (y = 0.280 m, z = 0). (a) Where in the plane of the two wires is the total magnetic field equal to zero? (b) A particle with a charge of $-2.00\ \mu$C is moving with a velocity of $150\hat{\mathbf{i}}$ Mm/s along the line (y = 0.100 m, z = 0). Calculate the vector magnetic force acting on the particle. (c) **What If?** A uniform electric field is applied to allow this particle to pass through this region undeflected. Calculate the required vector electric field.

15. Three long, parallel conductors each carry a current of I = 2.00 A. Figure P30.15 is an end view of the conductors, with each current coming out of the page. Taking a = 1.00 cm, determine the magnitude and direction of the magnetic field at (a) point A, (b) point B, and (c) point C.

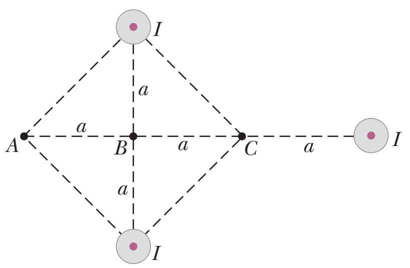


Figure P30.15

16. In a long, straight, vertical lightning stroke, electrons move downward and positive ions move upward and constitute a current of magnitude 20.0 kA. At a location 50.0 m east of the middle of the stroke, a free electron drifts through the air toward the west with a speed of 300 m/s. (a) Make a sketch showing the various vectors involved. Ignore the effect of the Earth's magnetic field. (b) Find the vector force the lightning stroke exerts on the electron. (c) Find the radius of the electron's path. (d) Is it a good approximation to model the electron as moving in a uniform field? Explain your answer. (e) If it does not collide with any obstacles, how many revolutions will the electron complete during the 60.0-μs duration of the lightning stroke?

17. Determine the magnetic field (in terms of I, a, and d) at the origin due to the current loop in Figure P30.17. The loop extends to infinity above the figure.

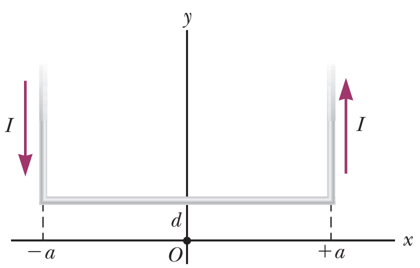


Figure P30.17

18. A wire carrying a current I is bent into the shape of an equilateral triangle of side L. (a) Find the magnitude of the magnetic field at the center of the triangle. (b) At a point halfway between the center and any vertex, is the field stronger or weaker than at the center? Give a qualitative argument for your answer.

19. The two wires shown in Figure P30.19 are separated by d = 10.0 cm and carry currents of I = 5.00 A in opposite directions. Find the magnitude and direction of the net magnetic field (a) at a point midway between the wires; (b) at point P_1, 10.0 cm to the right of the wire on the right; and (c) at point P_2, $2d$ = 20.0 cm to the left of the wire on the left.

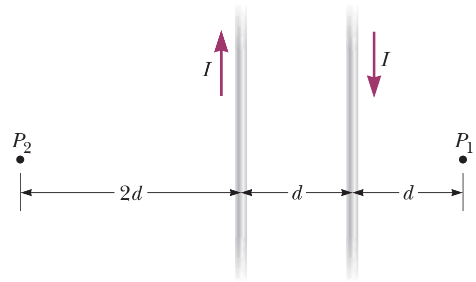


Figure P30.19

20. Two long, parallel wires carry currents of I_1 = 3.00 A and I_2 = 5.00 A in the directions indicated in Figure P30.20. (a) Find the magnitude and direction of the magnetic field at a point midway between the wires. (b) Find the magnitude and direction of the magnetic field at point P, located d = 20.0 cm above the wire carrying the 5.00-A current.

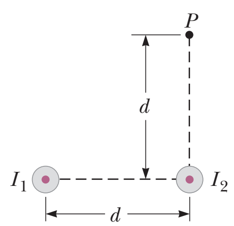


Figure P30.20

Section 30.2 The Magnetic Force Between Two Parallel Conductors

21. W Two long, parallel conductors, separated by 10.0 cm, carry currents in the same direction. The first wire carries a current I_1 = 5.00 A, and the second carries I_2 = 8.00 A. (a) What is the magnitude of the magnetic field created by I_1 at the location of I_2? (b) What is the force per unit length exerted by I_1 on I_2? (c) What is the magnitude of the magnetic field created by I_2 at the location of I_1? (d) What is the force per length exerted by I_2 on I_1?

22. Two parallel wires separated by 4.00 cm repel each other with a force per unit length of 2.00×10^{-4} N/m. The current in one wire is 5.00 A. (a) Find the current in the other wire. (b) Are the currents in the same direction or in opposite directions? (c) What would happen if the direction of one current were reversed and doubled?

23. Two parallel wires are separated by 6.00 cm, each carrying 3.00 A of current in the same direction. (a) What is the magnitude of the force per unit length between the wires? (b) Is the force attractive or repulsive?

24. Two long wires hang vertically. Wire 1 carries an upward current of 1.50 A. Wire 2, 20.0 cm to the right of wire 1, carries a downward current of 4.00 A. A third wire, wire 3, is to be hung vertically and located such that when it carries a certain current, each wire experiences no net force. (a) Is this situation possible? Is it possible in more than one way? Describe (b) the position of wire 3 and (c) the magnitude and direction of the current in wire 3.

25. M In Figure P30.25, the current in the long, straight wire is $I_1 = 5.00$ A and the wire lies in the plane of the rectangular loop, which carries a current $I_2 = 10.0$ A. The dimensions in the figure are $c = 0.100$ m, $a = 0.150$ m, and $\ell = 0.450$ m. Find the magnitude and direction of the net force exerted on the loop by the magnetic field created by the wire.

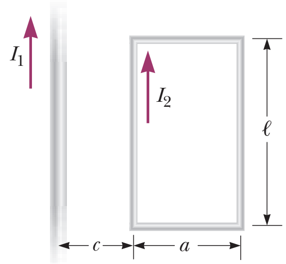


Figure P30.25 Problems 25 and 26.

26. In Figure P30.25, the current in the long, straight wire is I_1 and the wire lies in the plane of a rectangular loop, which carries a current I_2. The loop is of length ℓ and width a. Its left end is a distance c from the wire. Find the magnitude and direction of the net force exerted on the loop by the magnetic field created by the wire.

27. Two long, parallel wires are attracted to each other by a force per unit length of 320 μN/m. One wire carries a current of 20.0 A to the right and is located along the line $y = 0.500$ m. The second wire lies along the x axis. Determine the value of y for the line in the plane of the two wires along which the total magnetic field is zero.

28. *Why is the following situation impossible?* Two parallel copper conductors each have length $\ell = 0.500$ m and radius $r = 250$ μm. They carry currents $I = 10.0$ A in opposite directions and repel each other with a magnetic force $F_B = 1.00$ N.

29. AMT The unit of magnetic flux is named for Wilhelm Weber. A practical-size unit of magnetic field is named for Johann Karl Friedrich Gauss. Along with their individual accomplishments, Weber and Gauss built a telegraph in 1833 that consisted of a battery and switch, at one end of a transmission line 3 km long, operating an electromagnet at the other end. Suppose their transmission line was as diagrammed in Figure P30.29. Two long, parallel wires, each having a mass per unit length of 40.0 g/m, are supported in a horizontal plane by strings $\ell = 6.00$ cm long. When both wires carry the same current I, the wires repel each other so that the angle between the supporting strings is $\theta = 16.0°$. (a) Are the currents in the same direction or in opposite directions? (b) Find the magnitude of the current. (c) If this transmission line were taken to Mars, would the current required to separate the wires by the same angle be larger or smaller than that required on the Earth? Why?

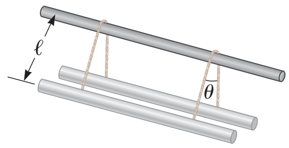


Figure P30.29

Section 30.3 Ampère's Law

30. Niobium metal becomes a superconductor when cooled below 9 K. Its superconductivity is destroyed when the surface magnetic field exceeds 0.100 T. In the absence of any external magnetic field, determine the maximum current a 2.00-mm-diameter niobium wire can carry and remain superconducting.

31. W Figure P30.31 is a cross-sectional view of a coaxial cable. The center conductor is surrounded by a rubber layer, an outer conductor, and another rubber layer. In a particular application, the current in the inner conductor is $I_1 = 1.00$ A out of the page and the current in the outer conductor is $I_2 = 3.00$ A into the page. Assuming the distance $d = 1.00$ mm, determine the magnitude and direction of the magnetic field at (a) point a and (b) point b.

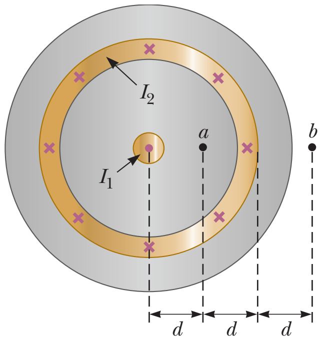


Figure P30.31

32. W The magnetic coils of a tokamak fusion reactor are in the shape of a toroid having an inner radius of 0.700 m and an outer radius of 1.30 m. The toroid has 900 turns of large-diameter wire, each of which carries a current of 14.0 kA. Find the magnitude of the magnetic field inside the toroid along (a) the inner radius and (b) the outer radius.

33. A long, straight wire lies on a horizontal table and carries a current of 1.20 μA. In a vacuum, a proton moves parallel to the wire (opposite the current) with a constant speed of 2.30×10^4 m/s at a distance d above the wire. Ignoring the magnetic field due to the Earth, determine the value of d.

34. An infinite sheet of current lying in the yz plane carries a surface current of linear density J_s. The current is in the positive z direction, and J_s represents the current per unit length measured along the y axis. Figure P30.34 is an edge view of the sheet. Prove that the magnetic field near the sheet is parallel to the sheet and perpendicular to the current direction, with magnitude $\mu_0 J_s/2$.

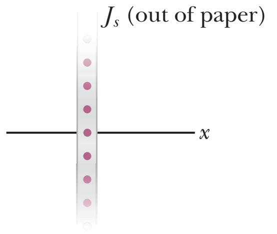


Figure P30.34

35. W The magnetic field 40.0 cm away from a long, straight wire carrying current 2.00 A is 1.00 μT. (a) At what distance is it 0.100 μT? (b) **What If?** At one instant, the two conductors in a long household extension cord carry equal 2.00-A currents in opposite directions. The two wires are 3.00 mm apart. Find the magnetic field 40.0 cm away from the middle of the straight cord, in the plane of the two wires. (c) At what distance is it one-tenth as large? (d) The center wire in a coaxial cable carries current 2.00 A in one direction, and the sheath around it carries current 2.00 A in the opposite direction. What magnetic field does the cable create at points outside the cable?

36. A packed bundle of 100 long, straight, insulated wires forms a cylinder of radius $R = 0.500$ cm. If each wire carries 2.00 A, what are (a) the magnitude and (b) the direction of the magnetic force per unit length acting on a wire located 0.200 cm from the center of the bundle? (c) **What If?** Would a wire on the outer edge of the bundle experience a force greater or smaller than the value calculated in parts (a) and (b)? Give a qualitative argument for your answer.

37. The magnetic field created by a large current passing through plasma (ionized gas) can force current-carrying particles together. This *pinch effect* has been used in designing fusion reactors. It can be demonstrated by making an empty aluminum can carry a large current parallel to its axis. Let R represent the radius of the can and I the current, uniformly distributed over the can's curved wall. Determine the magnetic field (a) just inside the wall and (b) just outside. (c) Determine the pressure on the wall.

38. A long, cylindrical conductor of radius R carries a current I as shown in Figure P30.38. The current density J, however, is not uniform over the cross section of the conductor but rather is a function of the radius according to $J = br$, where b is a constant. Find an expression for the magnetic field magnitude B (a) at a distance $r_1 < R$ and (b) at a distance $r_2 > R$, measured from the center of the conductor.

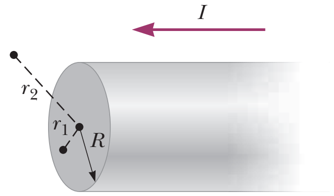


Figure P30.38

39. M Four long, parallel conductors carry equal currents of $I = 5.00$ A. Figure P30.39 is an end view of the conductors. The current direction is into the page at points A and B and out of the page at points C and D. Calculate (a) the magnitude and (b) the direction of the magnetic field at point P, located at the center of the square of edge length $\ell = 0.200$ m.

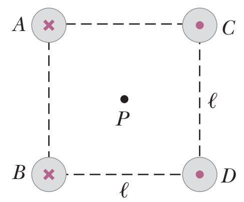


Figure P30.39

Section 30.4 The Magnetic Field of a Solenoid

40. A certain superconducting magnet in the form of a solenoid of length 0.500 m can generate a magnetic field of 9.00 T in its core when its coils carry a current of 75.0 A. Find the number of turns in the solenoid.

41. M A long solenoid that has 1 000 turns uniformly distributed over a length of 0.400 m produces a magnetic field of magnitude 1.00×10^{-4} T at its center. What current is required in the windings for that to occur?

42. You are given a certain volume of copper from which you can make copper wire. To insulate the wire, you can have as much enamel as you like. You will use the wire to make a tightly wound solenoid 20 cm long having the greatest possible magnetic field at the center and using a power supply that can deliver a current of 5 A. The solenoid can be wrapped with wire in one or more layers. (a) Should you make the wire long and thin or shorter and thick? Explain. (b) Should you make the radius of the solenoid small or large? Explain.

43. W A single-turn square loop of wire, 2.00 cm on each edge, carries a clockwise current of 0.200 A. The loop is inside a solenoid, with the plane of the loop perpendicular to the magnetic field of the solenoid. The solenoid has

30.0 turns/cm and carries a clockwise current of 15.0 A. Find (a) the force on each side of the loop and (b) the torque acting on the loop.

44. A solenoid 10.0 cm in diameter and 75.0 cm long is made from copper wire of diameter 0.100 cm, with very thin insulation. The wire is wound onto a cardboard tube in a single layer, with adjacent turns touching each other. What power must be delivered to the solenoid if it is to produce a field of 8.00 mT at its center?

45. It is desired to construct a solenoid that will have a resistance of 5.00 Ω (at 20.0°C) and produce a magnetic field of 4.00×10^{-2} T at its center when it carries a current of 4.00 A. The solenoid is to be constructed from copper wire having a diameter of 0.500 mm. If the radius of the solenoid is to be 1.00 cm, determine (a) the number of turns of wire needed and (b) the required length of the solenoid.

Section 30.5 Gauss's Law in Magnetism

46. Consider the hemispherical closed surface in Figure P30.46. The hemisphere is in a uniform magnetic field that makes an angle θ with the vertical. Calculate the magnetic flux through (a) the flat surface S_1 and (b) the hemispherical surface S_2.

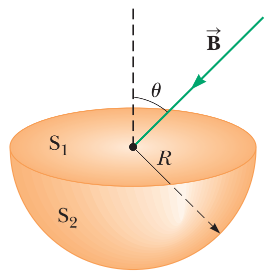


Figure P30.46

47. M A cube of edge length $\ell = 2.50$ cm is positioned as shown in Figure P30.47. A uniform magnetic field given by $\vec{\mathbf{B}} = (5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ T exists throughout the region. (a) Calculate the magnetic flux through the shaded face. (b) What is the total flux through the six faces?

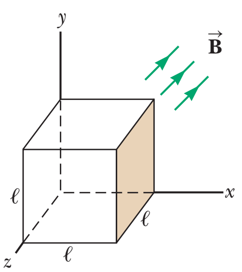


Figure P30.47

48. W A solenoid of radius $r = 1.25$ cm and length $\ell = 30.0$ cm has 300 turns and carries 12.0 A. (a) Calculate the flux through the surface of a disk-shaped area of radius $R = 5.00$ cm that is positioned perpendicular to and centered on the axis of the solenoid as shown in Figure P30.48a. (b) Figure P30.48b shows an enlarged end view of the same solenoid. Calculate the flux through the tan area, which is an annulus with an inner radius of $a = 0.400$ cm and an outer radius of $b = 0.800$ cm.

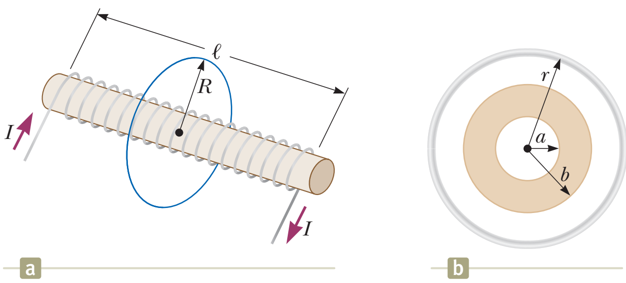


Figure P30.48

Section 30.6 Magnetism in Matter

49. M The magnetic moment of the Earth is approximately 8.00×10^{22} A·m². Imagine that the planetary magnetic field were caused by the complete magnetization of a huge iron deposit with density 7 900 kg/m³ and approximately 8.50×10^{28} iron atoms/m³. (a) How many unpaired electrons, each with a magnetic moment of 9.27×10^{-24} A · m², would participate? (b) At two unpaired electrons per iron atom, how many kilograms of iron would be present in the deposit?

50. At *saturation,* when nearly all the atoms have their magnetic moments aligned, the magnetic field is equal to the permeability constant μ_0 multiplied by the magnetic moment per unit volume. In a sample of iron, where the number density of atoms is approximately 8.50×10^{28} atoms/m³, the magnetic field can reach 2.00 T. If each electron contributes a magnetic moment of 9.27×10^{-24} A·m² (1 Bohr magneton), how many electrons per atom contribute to the saturated field of iron?

Additional Problems

51. A 30.0-turn solenoid of length 6.00 cm produces a magnetic field of magnitude 2.00 mT at its center. Find the current in the solenoid.

52. M A wire carries a 7.00-A current along the x axis, and another wire carries a 6.00-A current along the y axis, as shown in Figure P30.52. What is the magnetic field at point P, located at $x = 4.00$ m, $y = 3.00$ m?

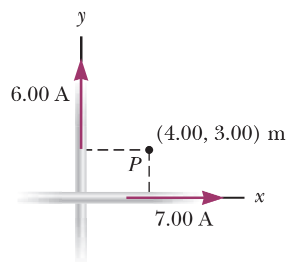


Figure P30.52

53. Suppose you install a compass on the center of a car's dashboard. (a) Assuming the dashboard is made mostly of plastic, compute an order-of-magnitude estimate for the magnetic field at this location produced by the current when you switch on the car's headlights. (b) How does this estimate compare with the Earth's magnetic field?

54. *Why is the following situation impossible?* The magnitude of the Earth's magnetic field at either pole is approximately 7.00×10^{-5} T. Suppose the field fades away to zero before its next reversal. Several scientists propose plans for artificially generating a replacement magnetic field to assist with devices that depend on the presence of the field. The plan that is selected is to lay a copper wire around the equator and supply it with a current that would generate a magnetic field of magnitude 7.00×10^{-5} T at the poles. (Ignore magnetization of any materials inside the Earth.) The plan is implemented and is highly successful.

55. M A nonconducting ring of radius 10.0 cm is uniformly charged with a total positive charge 10.0 μC. The ring rotates at a constant angular speed 20.0 rad/s about an axis through its center, perpendicular to the plane of the ring. What is the magnitude of the magnetic field on the axis of the ring 5.00 cm from its center?

56. A nonconducting ring of radius R is uniformly charged with a total positive charge q. The ring rotates at a constant angular speed ω about an axis through its center, perpendicular to the plane of the ring. What is the magnitude of the magnetic field on the axis of the ring a distance $\frac{1}{2}R$ from its center?

57. A very long, thin strip of metal of width w carries a current I along its length as shown in Figure P30.57. The current is distributed uniformly across the width of the strip. Find the magnetic field at point P in the diagram. Point P is in the plane of the strip at distance b away from its edge.

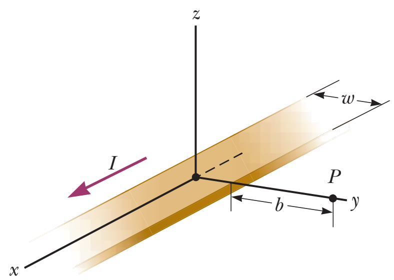


Figure P30.57

58. A circular coil of five turns and a diameter of 30.0 cm is oriented in a vertical plane with its axis perpendicular to the horizontal component of the Earth's magnetic field. A horizontal compass placed at the coil's center is made to deflect 45.0° from magnetic north by a current of 0.600 A in the coil. (a) What is the horizontal component of the Earth's magnetic field? (b) The current in the coil is switched off. A "dip needle" is a magnetic compass mounted so that it can rotate in a vertical north–south plane. At this location, a dip needle makes an angle of 13.0° from the vertical. What is the total magnitude of the Earth's magnetic field at this location?

59. A very large parallel-plate capacitor has uniform charge per unit area $+\sigma$ on the upper plate and $-\sigma$ on the lower plate. The plates are horizontal, and both move horizontally with speed v to the right. (a) What is the magnetic field between the plates? (b) What is the magnetic field just above or just below the plates? (c) What are the magnitude and direction of the magnetic force per unit area on the upper plate? (d) At what extrapolated speed v will the magnetic force on a plate balance the electric force on the plate? *Suggestion:* Use Ampere's law and choose a path that closes between the plates of the capacitor.

60. Two circular coils of radius R, each with N turns, are perpendicular to a common axis. The coil centers are a distance R apart. Each coil carries a steady current I in the same direction as shown in Figure P30.60. (a) Show that the magnetic field on the axis at a distance x from the center of one coil is

$$B = \frac{N\mu_0 IR^2}{2}\left[\frac{1}{(R^2 + x^2)^{3/2}} + \frac{1}{(2R^2 + x^2 - 2Rx)^{3/2}}\right]$$

(b) Show that dB/dx and d^2B/dx^2 are both zero at the point midway between the coils. We may then conclude that the magnetic field in the region midway between the coils is uniform. Coils in this configuration are called *Helmholtz coils.*

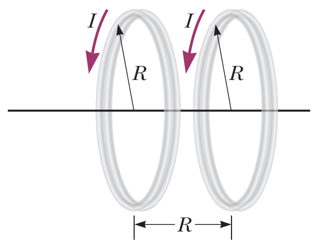


Figure P30.60 Problems 60 and 61.

61. Two identical, flat, circular coils of wire each have 100 turns and radius $R = 0.500$ m. The coils are arranged as a set of Helmholtz coils so that the separation distance between the coils is equal to the radius of the coils (see Fig. P30.60). Each coil carries current $I = 10.0$ A. Determine the magnitude of the magnetic field at a point on the common axis of the coils and halfway between them.

62. AMT Two circular loops are parallel, coaxial, and almost in contact, with their centers 1.00 mm apart (Fig. P30.62, page 932). Each loop is 10.0 cm in radius. The top loop carries a clockwise current of $I = 140$ A. The bottom loop carries a counterclockwise current of $I = 140$ A. (a) Calculate the magnetic force exerted by the bottom loop on the top loop. (b) Suppose a student thinks the first step in solving part (a) is to use Equation 30.7 to find the magnetic field created by one of the loops.

How would you argue for or against this idea? (c) The upper loop has a mass of 0.021 0 kg. Calculate its acceleration, assuming the only forces acting on it are the force in part (a) and the gravitational force.

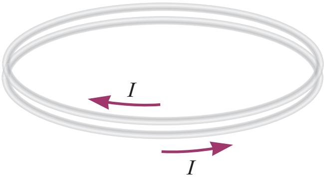


Figure P30.62

63. Two long, straight wires cross each other perpendicularly as shown in Figure P30.63. The wires are thin so that they are effectively in the same plane but do not touch. Find the magnetic field at a point 30.0 cm above the point of intersection of the wires along the z axis; that is, 30.0 cm out of the page, toward you.

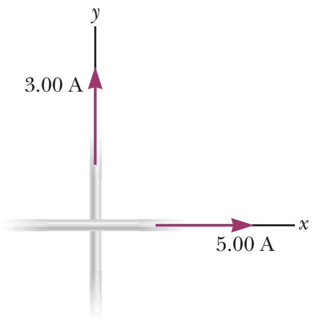


Figure P30.63

64. Two coplanar and concentric circular loops of wire carry currents of $I_1 = 5.00$ A and $I_2 = 3.00$ A in opposite directions as in Figure P30.64. If $r_1 = 12.0$ cm and $r_2 = 9.00$ cm, what are (a) the magnitude and (b) the direction of the net magnetic field at the center of the two loops? (c) Let r_1 remain fixed at 12.0 cm and let r_2 be a variable. Determine the value of r_2 such that the net field at the center of the loops is zero.

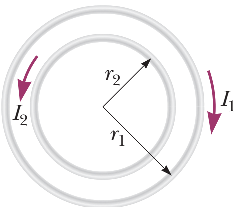


Figure P30.64

65. As seen in previous chapters, any object with electric charge, stationary or moving, other than the charged object that created the field, experiences a force in an electric field. Also, any object with electric charge, stationary or moving, can create an electric field (Chapter 23). Similarly, an electric current or a moving electric charge, other than the current or charge that created the field, experiences a force in a magnetic field (Chapter 29), and an electric current creates a magnetic field (Section 30.1). (a) To understand how a moving charge can also create a magnetic field, consider a particle with charge q moving with velocity $\vec{\mathbf{v}}$. Define the position vector $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$ leading from the particle to some location. Show that the magnetic field at that location is

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

(b) Find the magnitude of the magnetic field 1.00 mm to the side of a proton moving at 2.00×10^7 m/s. (c) Find the magnetic force on a second proton at this point, moving with the same speed in the opposite direction. (d) Find the electric force on the second proton.

66. AMT GP **Review.** Rail guns have been suggested for launching projectiles into space without chemical rockets. A tabletop model rail gun (Fig. P30.66) consists of two long, parallel, horizontal rails $\ell = 3.50$ cm apart, bridged by a bar of mass $m = 3.00$ g that is free to slide without friction. The rails and bar have low electric resistance, and the current is limited to a constant $I = 24.0$ A by a power supply that is far to the left of the figure, so it has no magnetic effect on the bar. Figure P30.66 shows the bar at rest at the midpoint of the rails at the moment the current is established. We wish to find the speed with which the bar leaves the rails after being released from the midpoint of the rails. (a) Find the magnitude of the magnetic field at a distance of 1.75 cm from a single long wire carrying a current of 2.40 A. (b) For purposes of evaluating the magnetic field, model the rails as infinitely long. Using the result of part (a), find the magnitude and direction of the magnetic field at the midpoint of the bar. (c) Argue that this value of the field will be the same at all positions of the bar to the right of the midpoint of the rails. At other points along the bar, the field is in the same direction as at the midpoint, but is larger in magnitude. Assume the average effective magnetic field along the bar is five times larger than the field at the midpoint. With this assumption, find (d) the magnitude and (e) the direction of the force on the bar. (f) Is the bar properly modeled as a particle under constant acceleration? (g) Find the velocity of the bar after it has traveled a distance $d = 130$ cm to the end of the rails.

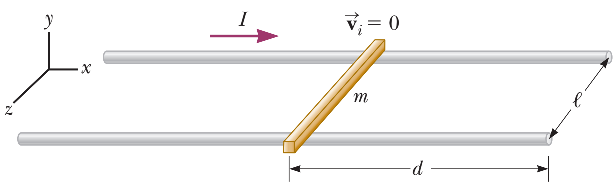


Figure P30.66

67. Fifty turns of insulated wire 0.100 cm in diameter are tightly wound to form a flat spiral. The spiral fills a disk surrounding a circle of radius 5.00 cm and extending to a radius 10.00 cm at the outer edge. Assume the wire carries a current I at the center of its cross section. Approximate each turn of wire as a circle. Then a loop of current exists at radius 5.05 cm, another at 5.15 cm, and so on. Numerically calculate the magnetic field at the center of the coil.

68. An infinitely long, straight wire carrying a current I_1 is partially surrounded by a loop as shown in Figure P30.68. The loop has a length L and radius R, and it carries a current I_2. The axis of the loop coincides with the wire. Calculate the magnetic force exerted on the loop.

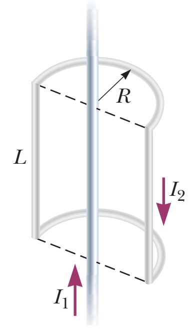


Figure P30.68

Challenge Problems

69. Consider a solenoid of length ℓ and radius a containing N closely spaced turns and carrying a steady current I. (a) In terms of these parameters, find the magnetic field at a point along the axis as a function of position x from the end of the solenoid. (b) Show that as ℓ becomes very long, B approaches $\mu_0 NI/2\ell$ at each end of the solenoid.

70. We have seen that a long solenoid produces a uniform magnetic field directed along the axis of a cylindrical region. To produce a uniform magnetic field directed parallel to a *diameter* of a cylindrical region, however, one can use the *saddle coils* illustrated in Figure P30.70. The loops are wrapped over a long, somewhat flattened tube. Figure P30.70a shows one wrapping of wire around the tube. This wrapping is continued in this manner until the visible side has many long sections of wire carrying current to the left in Figure P30.70a and the back side has many lengths carrying current to the right. The end view of the tube in Figure P30.70b shows these wires and the currents they carry. By wrapping the wires carefully, the distribution of wires can take the shape suggested in the end view such that the overall current distribution is approximately the superposition of two overlapping, circular cylinders of radius R (shown by the dashed lines) with uniformly distributed current, one toward you and one away from you. The current density J is the same for each cylinder. The center of one cylinder is described by a position vector $\vec{\mathbf{d}}$ relative to the center of the other cylinder. Prove that the magnetic field inside the hollow tube is $\mu_0 Jd/2$ downward. *Suggestion:* The use of vector methods simplifies the calculation.

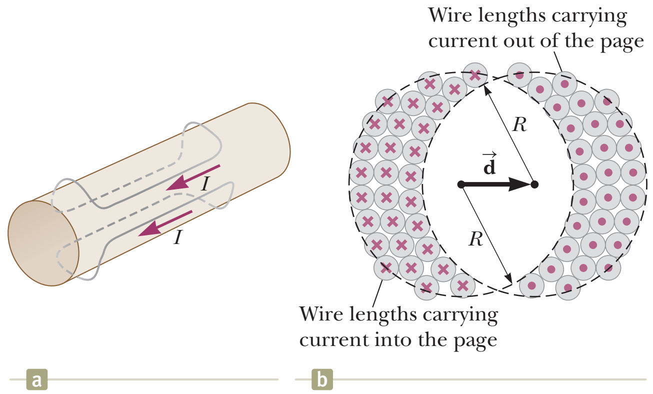


Figure P30.70

71. A thin copper bar of length $\ell = 10.0$ cm is supported horizontally by two (nonmagnetic) contacts at its ends. The bar carries a current of $I_1 = 100$ A in the negative x direction as shown in Figure P30.71. At a distance $h = 0.500$ cm below one end of the bar, a long, straight wire carries a current of $I_2 = 200$ A in the positive z direction. Determine the magnetic force exerted on the bar.

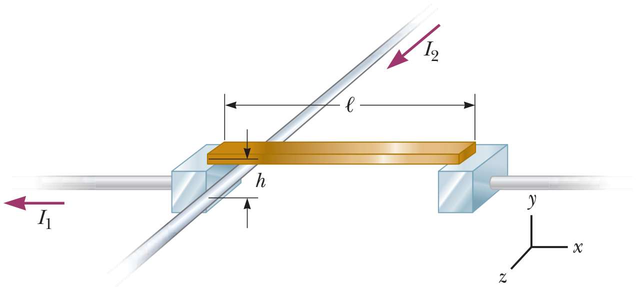


Figure P30.71

72. In Figure P30.72, both currents in the infinitely long wires are 8.00 A in the negative x direction. The wires are separated by the distance $2a = 6.00$ cm. (a) Sketch the magnetic field pattern in the yz plane. (b) What is the value of the magnetic field at the origin? (c) At ($y = 0$, $z \to \infty$)? (d) Find the magnetic field at points along the z axis as a function of z. (e) At what distance d along the positive z axis is the magnetic field a maximum? (f) What is this maximum value?

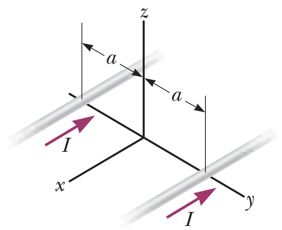


Figure P30.72

73. A wire carrying a current I is bent into the shape of an exponential spiral, $r = e^{\theta}$, from $\theta = 0$ to $\theta = 2\pi$ as suggested in Figure P30.73 (page 934). To complete a loop, the ends of the spiral are connected by a straight wire along the x axis. (a) The angle β between a radial line and its tangent line at any point on a curve $r = f(\theta)$ is related to the function by

$$\tan \beta = \frac{r}{dr/d\theta}$$

Use this fact to show that $\beta = \pi/4$. (b) Find the magnetic field at the origin.

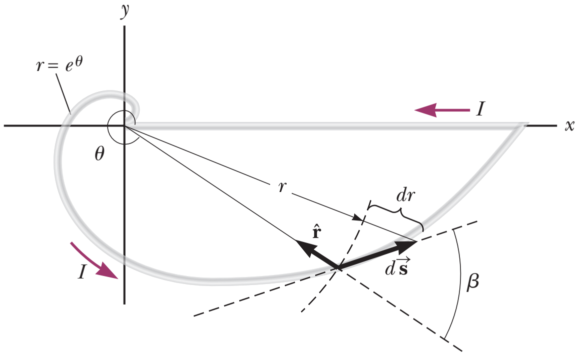


Figure P30.73

74. A sphere of radius R has a uniform volume charge density ρ. When the sphere rotates as a rigid object with angular speed ω about an axis through its center (Fig. P30.74), determine (a) the magnetic field at the center of the sphere and (b) the magnetic moment of the sphere.

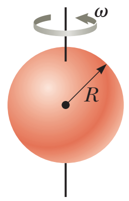


Figure P30.74

75. A long, cylindrical conductor of radius a has two cylindrical cavities each of diameter a through its entire length as shown in the end view of Figure P30.75. A current I is directed out of the page and is uniform through a cross section of the conducting material. Find the magnitude and direction of the magnetic field in terms of μ_0, I, r, and a at (a) point P_1 and (b) point P_2.

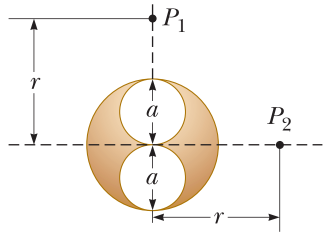


Figure P30.75

76. A wire is formed into the shape of a square of edge length L (Fig. P30.76). Show that when the current in the loop is I, the magnetic field at point P a distance x from the center of the square along its axis is

$$B = \frac{\mu_0 I L^2}{2\pi(x^2 + L^2/4)\sqrt{x^2 + L^2/2}}$$

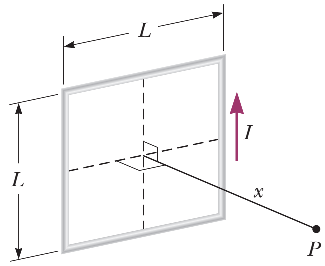


Figure P30.76

77. The magnitude of the force on a magnetic dipole $\vec{\mu}$ aligned with a nonuniform magnetic field in the positive x direction is $F_x = |\vec{\mu}|\,dB/dx$. Suppose two flat loops of wire each have radius R and carry a current I. (a) The loops are parallel to each other and share the same axis. They are separated by a variable distance $x >> R$. Show that the magnetic force between them varies as $1/x^4$. (b) Find the magnitude of this force, taking $I = 10.0$ A, $R = 0.500$ cm, and $x = 5.00$ cm.

Faraday's Law

CHAPTER 31

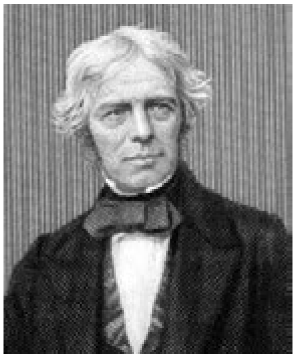
An artist's impression of the Skerries SeaGen Array, a tidal energy generator under development near the island of Anglesey, North Wales. When it is brought online, it will offer 10.5 MW of power from generators turned by tidal streams. The image shows the underwater blades that are driven by the tidal currents. The second blade system has been raised from the water for servicing. We will study generators in this chapter. *(Marine Current Turbines TM Ltd.)*

So far, our studies in electricity and magnetism have focused on the electric fields produced by stationary charges and the magnetic fields produced by moving charges. This chapter explores the effects produced by magnetic fields that vary in time.

Experiments conducted by Michael Faraday in England in 1831 and independently by Joseph Henry in the United States that same year showed that an emf can be induced in a circuit by a changing magnetic field. The results of these experiments led to a very basic and important law of electromagnetism known as *Faraday's law of induction.* An emf (and therefore a current as well) can be induced in various processes that involve a change in a magnetic flux.

31.1 Faraday's Law of Induction

To see how an emf can be induced by a changing magnetic field, consider the experimental results obtained when a loop of wire is connected to a sensitive ammeter as illustrated in Figure 31.1 (page 936). When a magnet is moved toward the loop, the reading on the ammeter changes from zero to a nonzero value, arbitrarily shown as negative in Figure 31.1a. When the magnet is brought to rest and held stationary relative to the loop (Fig. 31.1b), a reading of zero is observed. When the magnet is moved away from the loop, the reading on the ammeter changes to a positive value as shown in Figure 31.1c. Finally, when the magnet is held stationary and the loop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Michael Faraday
British Physicist and Chemist (1791–1867)
Faraday is often regarded as the greatest experimental scientist of the 1800s. His many contributions to the study of electricity include the invention of the electric motor, electric generator, and transformer as well as the discovery of electromagnetic induction and the laws of electrolysis. Greatly influenced by religion, he refused to work on the development of poison gas for the British military.

is moved either toward or away from it, the reading changes from zero. From these observations, we conclude that the loop detects that the magnet is moving relative to it and we relate this detection to a change in magnetic field. Therefore, it seems that a relationship exists between a current and a changing magnetic field.

These results are quite remarkable because a current is set up even though no batteries are present in the circuit! We call such a current an *induced current* and say that it is produced by an *induced emf.*

Now let's describe an experiment conducted by Faraday and illustrated in Figure 31.2. A primary coil is wrapped around an iron ring and connected to a switch and a battery. A current in the coil produces a magnetic field when the switch is closed. A secondary coil also is wrapped around the ring and is connected to a sensitive ammeter. No battery is present in the secondary circuit, and the secondary coil is not electrically connected to the primary coil. Any current detected in the secondary circuit must be induced by some external agent.

Initially, you might guess that no current is ever detected in the secondary circuit. Something quite amazing happens when the switch in the primary circuit is either opened or thrown closed, however. At the instant the switch is closed, the ammeter reading changes from zero momentarily and then returns to zero. At the instant the switch is opened, the ammeter changes to a reading with the opposite sign and again returns to zero. Finally, the ammeter reads zero when there is either a steady current or no current in the primary circuit. To understand what happens in this experiment, note that when the switch is closed, the current in the primary circuit produces a magnetic field that penetrates the secondary circuit. Furthermore, when the switch is thrown closed, the magnetic field produced by the current in the primary circuit changes from zero to some value over some finite time, and this changing field induces a current in the secondary circuit. Notice that no current is induced in the secondary coil even when a steady current exists in the primary coil. It is a *change* in the current in the primary coil that induces a current in the secondary coil, not just the *existence* of a current.

As a result of these observations, Faraday concluded that an electric current can be induced in a loop by a changing magnetic field. The induced current exists only while the magnetic field through the loop is changing. Once the magnetic field reaches a steady value, the current in the loop disappears. In effect, the loop behaves as though a source of emf were connected to it for a short time. It is customary to say that an induced emf is produced in the loop by the changing magnetic field.

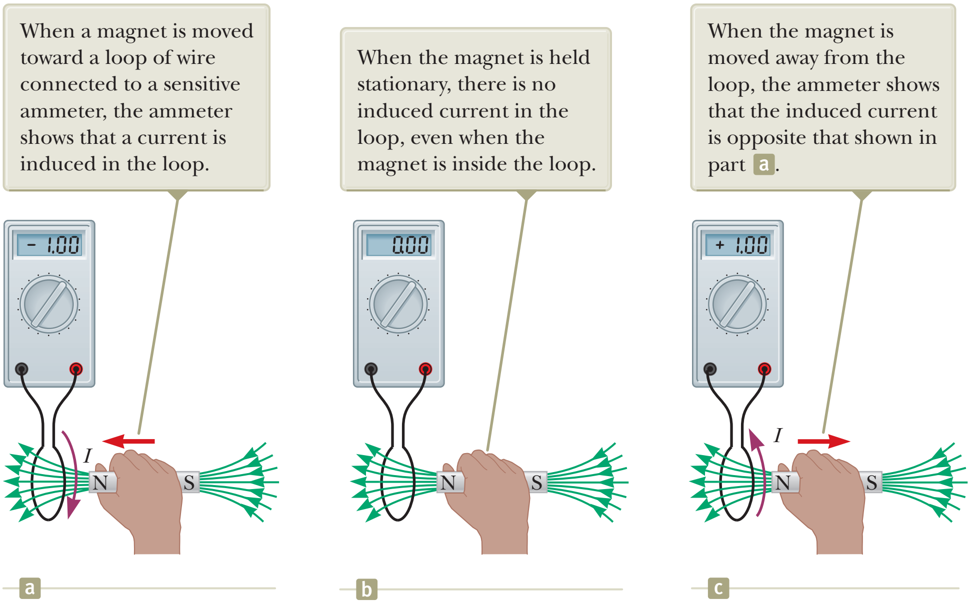

Figure 31.1 A simple experiment showing that a current is induced in a loop when a magnet is moved toward or away from the loop.

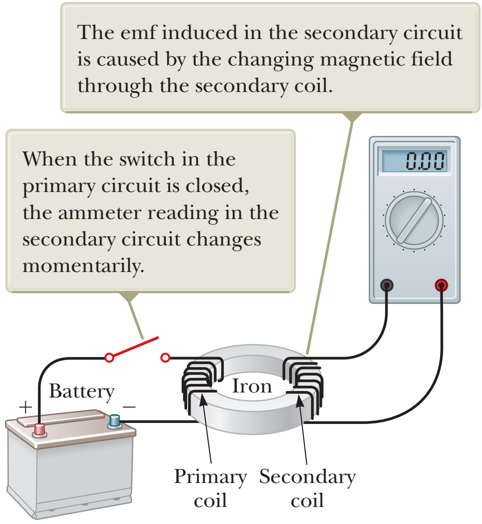

Figure 31.2 Faraday's experiment.

The experiments shown in Figures 31.1 and 31.2 have one thing in common: in each case, an emf is induced in a loop when the magnetic flux through the loop changes with time. In general, this emf is directly proportional to the time rate of change of the magnetic flux through the loop. This statement can be written mathematically as **Faraday's law of induction:**

$$\boldsymbol{\mathcal{E}} = -\frac{d\Phi_B}{dt} \tag{31.1}$$

◀ **Faraday's law of induction**

where $\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$ is the magnetic flux through the loop. (See Section 30.5.)

If a coil consists of N loops with the same area and Φ_B is the magnetic flux through one loop, an emf is induced in every loop. The loops are in series, so their emfs add; therefore, the total induced emf in the coil is given by

$$\boldsymbol{\mathcal{E}} = -N\frac{d\Phi_B}{dt} \tag{31.2}$$

The negative sign in Equations 31.1 and 31.2 is of important physical significance and will be discussed in Section 31.3.

Suppose a loop enclosing an area A lies in a uniform magnetic field $\vec{\mathbf{B}}$ as in Figure 31.3. The magnetic flux through the loop is equal to $BA\cos\theta$, where θ is the angle between the magnetic field and the normal to the loop; hence, the induced emf can be expressed as

$$\boldsymbol{\mathcal{E}} = -\frac{d}{dt}(BA\cos\theta) \tag{31.3}$$

From this expression, we see that an emf can be induced in the circuit in several ways:

- The magnitude of $\vec{\mathbf{B}}$ can change with time.
- The area enclosed by the loop can change with time.
- The angle θ between $\vec{\mathbf{B}}$ and the normal to the loop can change with time.
- Any combination of the above can occur.

Quick Quiz 31.1 A circular loop of wire is held in a uniform magnetic field, with the plane of the loop perpendicular to the field lines. Which of the following will *not* cause a current to be induced in the loop? **(a)** crushing the loop **(b)** rotating the loop about an axis perpendicular to the field lines **(c)** keeping the orientation of the loop fixed and moving it along the field lines **(d)** pulling the loop out of the field

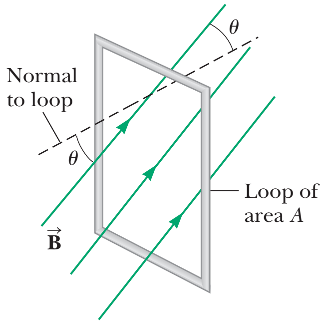

Figure 31.3 A conducting loop that encloses an area A in the presence of a uniform magnetic field $\vec{\mathbf{B}}$. The angle between $\vec{\mathbf{B}}$ and the normal to the loop is θ.

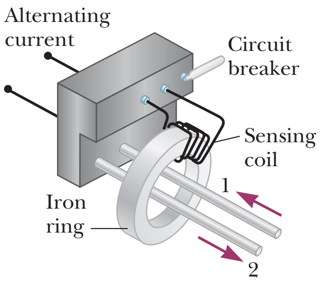

Figure 31.4 Essential components of a ground fault circuit interrupter.

Some Applications of Faraday's Law

The ground fault circuit interrupter (GFCI) is an interesting safety device that protects users of electrical appliances against electric shock. Its operation makes use of Faraday's law. In the GFCI shown in Figure 31.4, wire 1 leads from the wall outlet to the appliance to be protected and wire 2 leads from the appliance back to the wall outlet. An iron ring surrounds the two wires, and a sensing coil is wrapped around part of the ring. Because the currents in the wires are in opposite directions and of equal magnitude, there is zero net current flowing through the ring and the net magnetic flux through the sensing coil is zero. Now suppose the return current in wire 2 changes so that the two currents are not equal in magnitude. (That can happen if, for example, the appliance becomes wet, enabling current to leak to ground.) Then the net current through the ring is not zero and the magnetic flux through the sensing coil is no longer zero. Because household current is alternating (meaning that its direction keeps reversing), the magnetic flux through the sensing coil changes with time, inducing an emf in the coil. This induced emf is used to trigger a circuit breaker, which stops the current before it is able to reach a harmful level.

Another interesting application of Faraday's law is the production of sound in an electric guitar. The coil in this case, called the *pickup coil,* is placed near the vibrating guitar string, which is made of a metal that can be magnetized. A permanent magnet inside the coil magnetizes the portion of the string nearest the coil (Fig. 31.5a). When the string vibrates at some frequency, its magnetized segment produces a changing magnetic flux through the coil. The changing flux induces an emf in the coil that is fed to an amplifier. The output of the amplifier is sent to the loudspeakers, which produce the sound waves we hear.

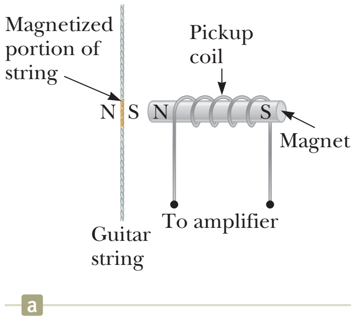

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Figure 31.5 (a) In an electric guitar, a vibrating magnetized string induces an emf in a pickup coil. (b) The pickups (the circles beneath the metallic strings) of this electric guitar detect the vibrations of the strings and send this information through an amplifier and into speakers. (A switch on the guitar allows the musician to select which set of six pickups is used.)

Example 31.1 Inducing an emf in a Coil

A coil consists of 200 turns of wire. Each turn is a square of side d = 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly from 0 to 0.50 T in 0.80 s, what is the magnitude of the induced emf in the coil while the field is changing?

SOLUTION

Conceptualize From the description in the problem, imagine magnetic field lines passing through the coil. Because the magnetic field is changing in magnitude, an emf is induced in the coil.

Categorize We will evaluate the emf using Faraday's law from this section, so we categorize this example as a substitution problem.

31.1 continued

Evaluate Equation 31.2 for the situation described here, noting that the magnetic field changes linearly with time:

$$|\mathcal{E}| = N\frac{\Delta\Phi_B}{\Delta t} = N\frac{\Delta(BA)}{\Delta t} = NA\frac{\Delta B}{\Delta t} = Nd^2\frac{B_f - B_i}{\Delta t}$$

Substitute numerical values:

$$|\mathcal{E}| = (200)(0.18 \text{ m})^2\frac{(0.50 \text{ T} - 0)}{0.80 \text{ s}} = \boxed{4.0 \text{ V}}$$

WHAT IF? What if you were asked to find the magnitude of the induced current in the coil while the field is changing? Can you answer that question?

Answer If the ends of the coil are not connected to a circuit, the answer to this question is easy: the current is zero! (Charges move within the wire of the coil, but they cannot move into or out of the ends of the coil.) For a steady current to exist, the ends of the coil must be connected to an external circuit. Let's assume the coil is connected to a circuit and the total resistance of the coil and the circuit is 2.0 Ω. Then, the magnitude of the induced current in the coil is

$$I = \frac{|\mathcal{E}|}{R} = \frac{4.0 \text{ V}}{2.0 \ \Omega} = 2.0 \text{ A}$$

Example 31.2 An Exponentially Decaying Magnetic Field

A loop of wire enclosing an area A is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of $\vec{\mathbf{B}}$ varies in time according to the expression $B = B_{\max}e^{-at}$, where a is some constant. That is, at $t = 0$, the field is $B_{\max}$, and for $t > 0$, the field decreases exponentially (Fig. 31.6). Find the induced emf in the loop as a function of time.

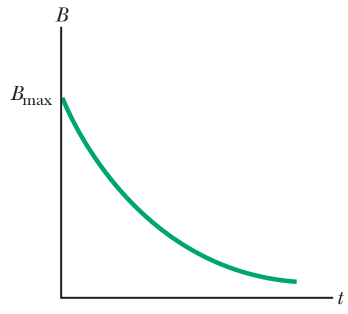

Figure 31.6 (Example 31.2) Exponential decrease in the magnitude of the magnetic field through a loop with time. The induced emf and induced current in a conducting path attached to the loop vary with time in the same way.

SOLUTION

Conceptualize The physical situation is similar to that in Example 31.1 except for two things: there is only one loop, and the field varies exponentially with time rather than linearly.

Categorize We will evaluate the emf using Faraday's law from this section, so we categorize this example as a substitution problem.

Evaluate Equation 31.1 for the situation described here:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(AB_{\max}e^{-at}) = -AB_{\max}\frac{d}{dt}e^{-at} = \boxed{aAB_{\max}e^{-at}}$$

This expression indicates that the induced emf decays exponentially in time. The maximum emf occurs at $t = 0$, where $\mathcal{E}_{\max} = aAB_{\max}$. The plot of $\mathcal{E}$ versus t is similar to the B-versus-t curve shown in Figure 31.6.

31.2 Motional emf

In Examples 31.1 and 31.2, we considered cases in which an emf is induced in a stationary circuit placed in a magnetic field when the field changes with time. In this section, we describe **motional emf,** the emf induced in a conductor moving through a constant magnetic field.

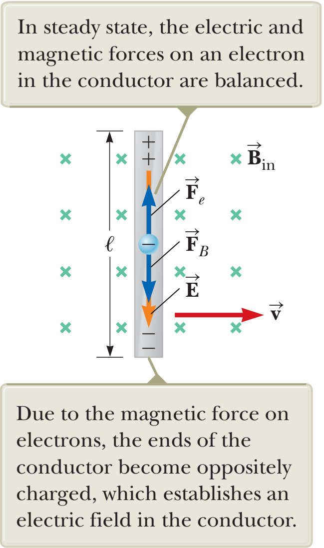

Figure 31.7 A straight electrical conductor of length ℓ moving with a velocity $\vec{v}$ through a uniform magnetic field $\vec{B}$ directed perpendicular to $\vec{v}$.

The straight conductor of length ℓ shown in Figure 31.7 is moving through a uniform magnetic field directed into the page. For simplicity, let's assume the conductor is moving in a direction perpendicular to the field with constant velocity under the influence of some external agent. From the magnetic version of the particle in a field model, the electrons in the conductor experience a force $\vec{F}_B = q\vec{v} \times \vec{B}$ (Eq. 29.1) that is directed along the length ℓ, perpendicular to both $\vec{v}$ and $\vec{B}$. Under the influence of this force, the electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. As a result of this charge separation, an electric field $\vec{E}$ is produced inside the conductor. Therefore, the electrons are also described by the electric version of the particle in a field model. The charges accumulate at both ends until the downward magnetic force qvB on charges remaining in the conductor is balanced by the upward electric force qE. The electrons are then described by the particle in equilibrium model. The condition for equilibrium requires that the forces on the electrons balance:

$$qE = qvB \quad \text{or} \quad E = vB$$

The magnitude of the electric field produced in the conductor is related to the potential difference across the ends of the conductor according to the relationship $\Delta V = E\ell$ (Eq. 25.6). Therefore, for the equilibrium condition,

$$\Delta V = E\ell = B\ell v \tag{31.4}$$

where the upper end of the conductor in Figure 31.7 is at a higher electric potential than the lower end. Therefore, a potential difference is maintained between the ends of the conductor as long as the conductor continues to move through the uniform magnetic field. If the direction of the motion is reversed, the polarity of the potential difference is also reversed.

A more interesting situation occurs when the moving conductor is part of a closed conducting path. This situation is particularly useful for illustrating how a changing magnetic flux causes an induced current in a closed circuit. Consider a circuit consisting of a conducting bar of length ℓ sliding along two fixed, parallel conducting rails as shown in Figure 31.8a. For simplicity, let's assume the bar has zero resistance and the stationary part of the circuit has a resistance R. A uniform and constant magnetic field $\vec{B}$ is applied perpendicular to the plane of the circuit. As the bar is pulled to the right with a velocity $\vec{v}$ under the influence of an applied force $\vec{F}_{app}$, free charges in the bar are moving particles in a magnetic field that experience a magnetic force directed along the length of the bar. This force sets up an induced current because the charges are free to move in the closed conducting path. In this case, the rate of change of magnetic flux through the circuit and the corresponding

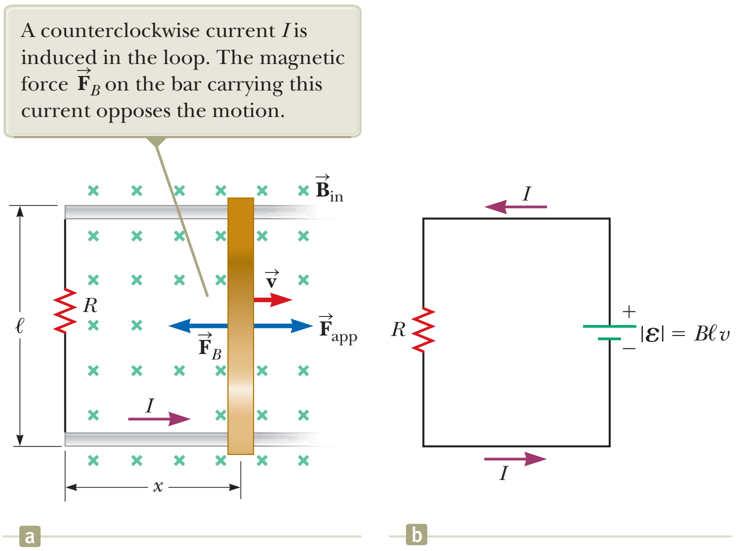

Figure 31.8 (a) A conducting bar sliding with a velocity $\vec{v}$ along two conducting rails under the action of an applied force $\vec{F}_{app}$. (b) The equivalent circuit diagram for the setup shown in (a).

induced motional emf across the moving bar are proportional to the change in area of the circuit.

Because the area enclosed by the circuit at any instant is ℓx, where x is the position of the bar, the magnetic flux through that area is

$$\Phi_B = B\ell x$$

Using Faraday's law and noting that x changes with time at a rate $dx/dt = v$, we find that the induced motional emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\ell x) = -B\ell\frac{dx}{dt}$$

$$\mathcal{E} = -B\ell v \tag{31.5}$$

◀ **Motional emf**

Because the resistance of the circuit is R, the magnitude of the induced current is

$$I = \frac{|\mathcal{E}|}{R} = \frac{B\ell v}{R} \tag{31.6}$$

The equivalent circuit diagram for this example is shown in Figure 31.8b.

Let's examine the system using energy considerations. Because no battery is in the circuit, you might wonder about the origin of the induced current and the energy delivered to the resistor. We can understand the source of this current and energy by noting that the applied force does work on the conducting bar. Therefore, we model the circuit as a nonisolated system. The movement of the bar through the field causes charges to move along the bar with some average drift velocity; hence, a current is established. The change in energy in the system during some time interval must be equal to the transfer of energy into the system by work, consistent with the general principle of conservation of energy described by Equation 8.2. The appropriate reduction of Equation 8.2 is $W = \Delta E_{\text{int}}$, because the input energy appears as internal energy in the resistor.

Let's verify this equality mathematically. As the bar moves through the uniform magnetic field $\vec{\mathbf{B}}$, it experiences a magnetic force $\vec{\mathbf{F}}_B$ of magnitude $I\ell B$ (see Section 29.4). Because the bar moves with constant velocity, it is modeled as a particle in equilibrium and the magnetic force must be equal in magnitude and opposite in direction to the applied force, or to the left in Figure 31.8a. (If $\vec{\mathbf{F}}_B$ acted in the direction of motion, it would cause the bar to accelerate, violating the principle of conservation of energy.) Using Equation 31.6 and $F_{\text{app}} = F_B = I\ell B$, the power delivered by the applied force is

$$P = F_{\text{app}}v = (I\ell B)v = \frac{B^2\ell^2v^2}{R} = \frac{\mathcal{E}^2}{R} \tag{31.7}$$

From Equation 27.22, we see that this power input is equal to the rate at which energy is delivered to the resistor, consistent with the principle of conservation of energy.

Quick Quiz 31.2 In Figure 31.8a, a given applied force of magnitude F_{app} results in a constant speed v and a power input P. Imagine that the force is increased so that the constant speed of the bar is doubled to $2v$. Under these conditions, what are the new force and the new power input? **(a)** $2F$ and $2P$ **(b)** $4F$ and $2P$ **(c)** $2F$ and $4P$ **(d)** $4F$ and $4P$

Example 31.3 Magnetic Force Acting on a Sliding Bar AM

The conducting bar illustrated in Figure 31.9 (page 942) moves on two frictionless, parallel rails in the presence of a uniform magnetic field directed into the page. The bar has mass m, and its length is ℓ. The bar is given an initial velocity $\vec{\mathbf{v}}_i$ to the right and is released at $t = 0$.

continued

31.3 continued

(A) Using Newton's laws, find the velocity of the bar as a function of time.

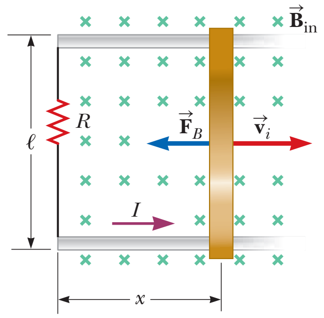

Figure 31.9 (Example 31.3) A conducting bar of length ℓ on two fixed conducting rails is given an initial velocity $\vec{\mathbf{v}}_i$ to the right.

SOLUTION

Conceptualize As the bar slides to the right in Figure 31.9, a counterclockwise current is established in the circuit consisting of the bar, the rails, and the resistor. The upward current in the bar results in a magnetic force to the left on the bar as shown in the figure. Therefore, the bar must slow down, so our mathematical solution should demonstrate that.

Categorize The text already categorizes this problem as one that uses Newton's laws. We model the bar as a *particle under a net force.*

Analyze From Equation 29.10, the magnetic force is $F_B = -I\ell B$, where the negative sign indicates that the force is to the left. The magnetic force is the *only* horizontal force acting on the bar.

Using the particle under a net force model, apply Newton's second law to the bar in the horizontal direction:

$$F_x = ma \rightarrow -I\ell B = m\frac{dv}{dt}$$

Substitute $I = B\ell v/R$ from Equation 31.6:

$$m\frac{dv}{dt} = -\frac{B^2\ell^2}{R}v$$

Rearrange the equation so that all occurrences of the variable v are on the left and those of t are on the right:

$$\frac{dv}{v} = -\left(\frac{B^2\ell^2}{mR}\right)dt$$

Integrate this equation using the initial condition that $v = v_i$ at $t = 0$ and noting that $(B^2\ell^2/mR)$ is a constant:

$$\int_{v_i}^{v}\frac{dv}{v} = -\frac{B^2\ell^2}{mR}\int_0^t dt$$

$$\ln\left(\frac{v}{v_i}\right) = -\left(\frac{B^2\ell^2}{mR}\right)t$$

Define the constant $\tau = mR/B^2\ell^2$ and solve for the velocity:

$$(1)\quad v = \boxed{v_i e^{-t/\tau}}$$

Finalize This expression for v indicates that the velocity of the bar decreases with time under the action of the magnetic force as expected from our conceptualization of the problem.

(B) Show that the same result is found by using an energy approach.

SOLUTION

Categorize The text of this part of the problem tells us to use an energy approach for the same situation. We model the entire circuit in Figure 31.9 as an *isolated system.*

Analyze Consider the sliding bar as one system component possessing kinetic energy, which decreases because energy is transferring *out* of the bar by electrical transmission through the rails. The resistor is another system component possessing internal energy, which rises because energy is transferring *into* the resistor. Because energy is not leaving the system, the rate of energy transfer out of the bar equals the rate of energy transfer into the resistor.

Equate the power entering the resistor to that leaving the bar:

$$P_{\text{resistor}} = -P_{\text{bar}}$$

Substitute for the electrical power delivered to the resistor and the time rate of change of kinetic energy for the bar:

$$I^2R = -\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)$$

Use Equation 31.6 for the current and carry out the derivative:

$$\frac{B^2\ell^2v^2}{R} = -mv\frac{dv}{dt}$$

31.3 continued

Rearrange terms: $$\frac{dv}{v} = -\left(\frac{B^2\ell^2}{mR}\right)dt$$

Finalize This result is the same expression to be integrated that we found in part (A).

WHAT IF? Suppose you wished to increase the distance through which the bar moves between the time it is initially projected and the time it essentially comes to rest. You can do so by changing one of three variables—v_i, R, or B—by a factor of 2 or $\frac{1}{2}$. Which variable should you change to maximize the distance, and would you double it or halve it?

Answer Increasing v_i would make the bar move farther. Increasing R would decrease the current and therefore the magnetic force, making the bar move farther. Decreasing B would decrease the magnetic force and make the bar move farther. Which method is most effective, though?

Use Equation (1) to find the distance the bar moves by integration:

$$v = \frac{dx}{dt} = v_i e^{-t/\tau}$$

$$x = \int_0^\infty v_i e^{-t/\tau}\,dt = -v_i\tau e^{-t/\tau}\Big|_0^\infty$$

$$= -v_i\tau(0-1) = v_i\tau = v_i\left(\frac{mR}{B^2\ell^2}\right)$$

This expression shows that doubling v_i or R will double the distance. Changing B by a factor of $\frac{1}{2}$, however, causes the distance to be four times as great!

Example 31.4 Motional emf Induced in a Rotating Bar

A conducting bar of length ℓ rotates with a constant angular speed ω about a pivot at one end. A uniform magnetic field $\vec{\mathbf{B}}$ is directed perpendicular to the plane of rotation as shown in Figure 31.10. Find the motional emf induced between the ends of the bar.

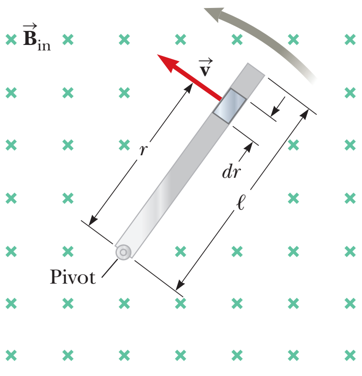

Figure 31.10 (Example 31.4) A conducting bar rotating around a pivot at one end in a uniform magnetic field that is perpendicular to the plane of rotation. A motional emf is induced between the ends of the bar.

SOLUTION

Conceptualize The rotating bar is different in nature from the sliding bar in Figure 31.8. Consider a small segment of the bar, however. It is a short length of conductor moving in a magnetic field and has an emf generated in it like the sliding bar. By thinking of each small segment as a source of emf, we see that all segments are in series and the emfs add.

Categorize Based on the conceptualization of the problem, we approach this example as we did in the discussion leading to Equation 31.5, with the added feature that the short segments of the bar are traveling in circular paths.

Analyze Evaluate the magnitude of the emf induced in a segment of the bar of length dr having a velocity $\vec{\mathbf{v}}$ from Equation 31.5:

$$d\mathcal{E} = Bv\,dr$$

Find the total emf between the ends of the bar by adding the emfs induced across all segments:

$$\mathcal{E} = \int Bv\,dr$$

The tangential speed v of an element is related to the angular speed ω through the relationship $v = r\omega$ (Eq. 10.10); use that fact and integrate:

$$\mathcal{E} = B\int v\,dr = B\omega\int_0^\ell r\,dr = \boxed{\tfrac{1}{2}B\omega\ell^2}$$

continued

▶ **31.4** continued

Finalize In Equation 31.5 for a sliding bar, we can increase $\mathcal{E}$ by increasing B, ℓ, or v. Increasing any one of these variables by a given factor increases $\mathcal{E}$ by the same factor. Therefore, you would choose whichever of these three variables is most convenient to increase. For the rotating rod, however, there is an advantage to increasing the length of the rod to raise the emf because ℓ is squared. Doubling the length gives four times the emf, whereas doubling the angular speed only doubles the emf.

WHAT IF? Suppose, after reading through this example, you come up with a brilliant idea. A Ferris wheel has radial metallic spokes between the hub and the circular rim. These spokes move in the magnetic field of the Earth, so each spoke acts like the bar in Figure 31.10. You plan to use the emf generated by the rotation of the Ferris wheel to power the lightbulbs on the wheel. Will this idea work?

Answer Let's estimate the emf that is generated in this situation. We know the magnitude of the magnetic field of the Earth from Table 29.1: $B = 0.5 \times 10^{-4}$ T. A typical spoke on a Ferris wheel might have a length on the order of 10 m. Suppose the period of rotation is on the order of 10 s.

Determine the angular speed of the spoke:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{10\ \text{s}} = 0.63\ \text{s}^{-1} \sim 1\ \text{s}^{-1}$$

Assume the magnetic field lines of the Earth are horizontal at the location of the Ferris wheel and perpendicular to the spokes. Find the emf generated:

$$\mathcal{E} = \tfrac{1}{2}B\omega\ell^2 = \tfrac{1}{2}(0.5 \times 10^{-4}\ \text{T})(1\ \text{s}^{-1})(10\ \text{m})^2$$
$$= 2.5 \times 10^{-3}\ \text{V} \sim 1\ \text{mV}$$

This value is a tiny emf, far smaller than that required to operate lightbulbs.

An additional difficulty is related to energy. Even assuming you could find lightbulbs that operate using a potential difference on the order of millivolts, a spoke must be part of a circuit to provide a voltage to the lightbulbs. Consequently, the spoke must carry a current. Because this current-carrying spoke is in a magnetic field, a magnetic force is exerted on the spoke in the direction opposite its direction of motion. As a result, the motor of the Ferris wheel must supply more energy to perform work against this magnetic drag force. The motor must ultimately provide the energy that is operating the lightbulbs, and you have not gained anything for free!

31.3 Lenz's Law

Faraday's law (Eq. 31.1) indicates that the induced emf and the change in flux have opposite algebraic signs. This feature has a very real physical interpretation that has come to be known as **Lenz's law:**[1]

Lenz's law ▶

> The induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop.

That is, the induced current tends to keep the original magnetic flux through the loop from changing. We shall show that this law is a consequence of the law of conservation of energy.

To understand Lenz's law, let's return to the example of a bar moving to the right on two parallel rails in the presence of a uniform magnetic field (the *external* magnetic field, shown by the green crosses in Fig. 31.11a). As the bar moves to the right, the magnetic flux through the area enclosed by the circuit increases with time because the area increases. Lenz's law states that the induced current must be directed so that the magnetic field it produces opposes the change in the external magnetic flux. Because the magnetic flux due to an external field directed into the page is increasing, the induced current—if it is to oppose this change—must

[1]Developed by German physicist Heinrich Lenz (1804–1865).

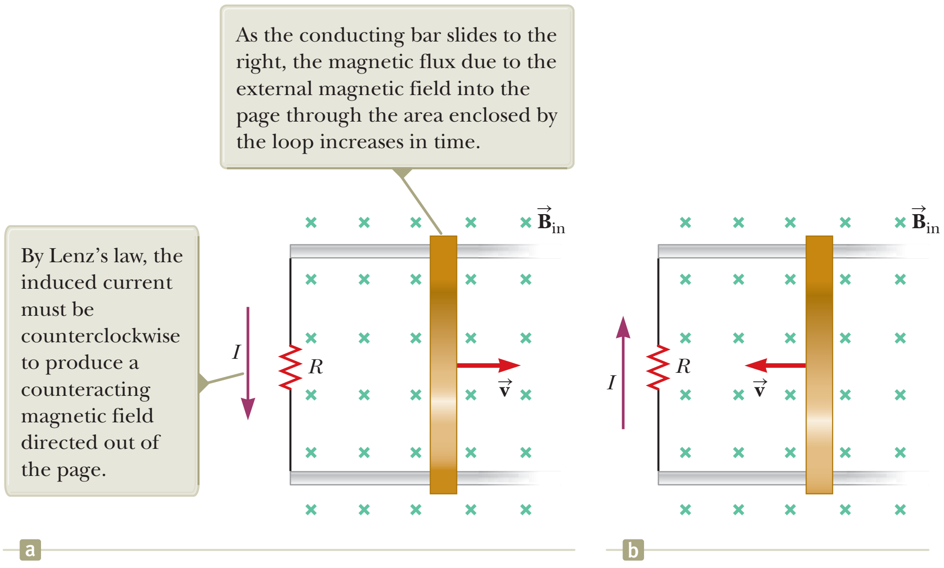

Figure 31.11 (a) Lenz's law can be used to determine the direction of the induced current. (b) When the bar moves to the left, the induced current must be clockwise. Why?

produce a field directed out of the page. Hence, the induced current must be directed counterclockwise when the bar moves to the right. (Use the right-hand rule to verify this direction.) If the bar is moving to the left as in Figure 31.11b, the external magnetic flux through the area enclosed by the loop decreases with time. Because the field is directed into the page, the direction of the induced current must be clockwise if it is to produce a field that also is directed into the page. In either case, the induced current attempts to maintain the original flux through the area enclosed by the current loop.

Let's examine this situation using energy considerations. Suppose the bar is given a slight push to the right. In the preceding analysis, we found that this motion sets up a counterclockwise current in the loop. What happens if we assume the current is clockwise such that the direction of the magnetic force exerted on the bar is to the right? This force would accelerate the rod and increase its velocity, which in turn would cause the area enclosed by the loop to increase more rapidly. The result would be an increase in the induced current, which would cause an increase in the force, which would produce an increase in the current, and so on. In effect, the system would acquire energy with no input of energy. This behavior is clearly inconsistent with all experience and violates the law of conservation of energy. Therefore, the current must be counterclockwise.

Quick Quiz 31.3 Figure 31.12 shows a circular loop of wire falling toward a wire carrying a current to the left. What is the direction of the induced current in the loop of wire? **(a)** clockwise **(b)** counterclockwise **(c)** zero **(d)** impossible to determine

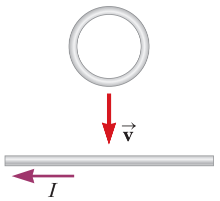

Figure 31.12 (Quick Quiz 31.3)

Conceptual Example 31.5 Application of Lenz's Law

A magnet is placed near a metal loop as shown in Figure 31.13a (page 946).

(A) Find the direction of the induced current in the loop when the magnet is pushed toward the loop.

SOLUTION

As the magnet moves to the right toward the loop, the external magnetic flux through the loop increases with time. To counteract this increase in flux due to a field toward the right, the induced current produces its own magnetic field to the left as illustrated in Figure 31.13b; hence, the induced current is in the direction shown. Knowing that like

continued

31.5 continued

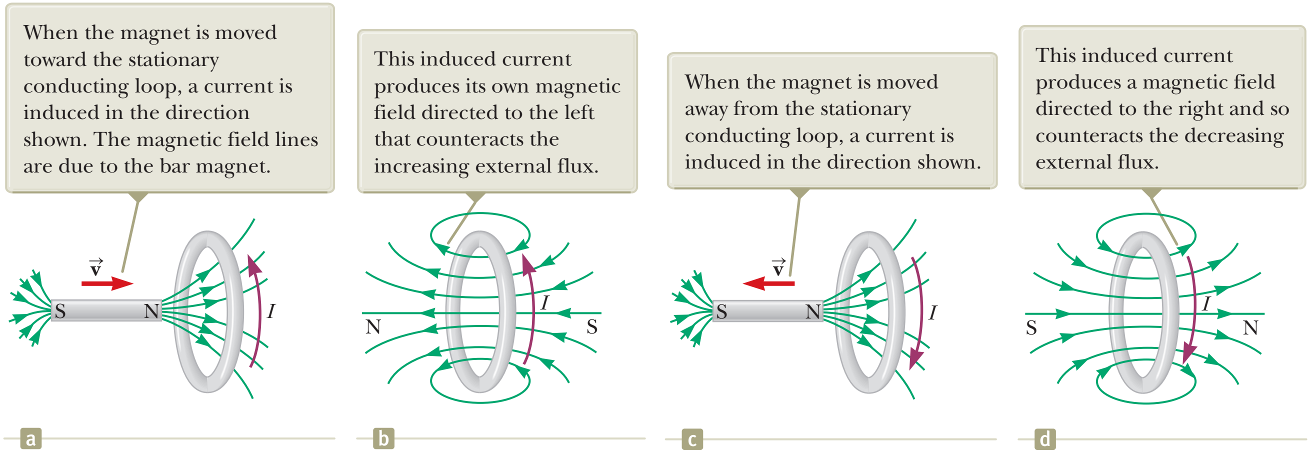

Figure 31.13 (Conceptual Example 31.5) A moving bar magnet induces a current in a conducting loop.

magnetic poles repel each other, we conclude that the left face of the current loop acts like a north pole and the right face acts like a south pole.

(B) Find the direction of the induced current in the loop when the magnet is pulled away from the loop.

SOLUTION

If the magnet moves to the left as in Figure 31.13c, its flux through the area enclosed by the loop decreases in time. Now the induced current in the loop is in the direction shown in Figure 31.13d because this current direction produces a magnetic field in the same direction as the external field. In this case, the left face of the loop is a south pole and the right face is a north pole.

Conceptual Example 31.6 A Loop Moving Through a Magnetic Field

A rectangular metallic loop of dimensions ℓ and w and resistance R moves with constant speed v to the right as in Figure 31.14a. The loop passes through a uniform magnetic field $\vec{\mathbf{B}}$ directed into the page and extending a distance $3w$ along the x axis. Define x as the position of the right side of the loop along the x axis.

(A) Plot the magnetic flux through the area enclosed by the loop as a function of x.

SOLUTION

Figure 31.14b shows the flux through the area enclosed by the loop as a function of x. Before the loop enters the field, the flux through the loop is zero. As the loop enters the field, the flux increases linearly with position until the left edge of the loop is just inside the field. Finally, the flux through the loop decreases linearly to zero as the loop leaves the field.

(B) Plot the induced motional emf in the loop as a function of x.

SOLUTION

Before the loop enters the field, no motional emf is induced in it because no field is present (Fig. 31.14c). As the right side of the loop enters the field, the magnetic flux directed into the page increases. Hence, according to Lenz's law, the induced current is counterclockwise because it must produce its own magnetic field directed out of the page. The motional emf $-B\ell v$ (from Eq. 31.5) arises from the magnetic force experienced by charges in the right side of the loop. When the loop is entirely in the field, the change in magnetic flux through the loop is zero; hence, the motional emf vanishes. That happens because once the left side of the loop enters the field, the motional emf induced in it

31.6 continued

cancels the motional emf present in the right side of the loop. As the right side of the loop leaves the field, the flux through the loop begins to decrease, a clockwise current is induced, and the induced emf is $B\ell v$. As soon as the left side leaves the field, the emf decreases to zero.

(C) Plot the external applied force necessary to counter the magnetic force and keep v constant as a function of x.

SOLUTION

The external force that must be applied to the loop to maintain this motion is plotted in Figure 31.14d. Before the loop enters the field, no magnetic force acts on it; hence, the applied force must be zero if v is constant. When the right side of the loop enters the field, the applied force necessary to maintain constant speed must be equal in magnitude and opposite in direction to the magnetic force exerted on that side, so that the loop is a particle in equilibrium. When the loop is entirely in the field, the flux through the loop is not changing with time. Hence, the net emf induced in the loop is zero and the current also is zero. Therefore, no external force is needed to maintain the motion. Finally, as the right side leaves the field, the applied force must be equal in magnitude and opposite in direction to the magnetic force acting on the left side of the loop.

From this analysis, we conclude that power is supplied only when the loop is either entering or leaving the field. Furthermore, this example shows that the motional emf induced in the loop can be zero even when there is motion through the field! A motional emf is induced *only* when the magnetic flux through the loop *changes in time*.

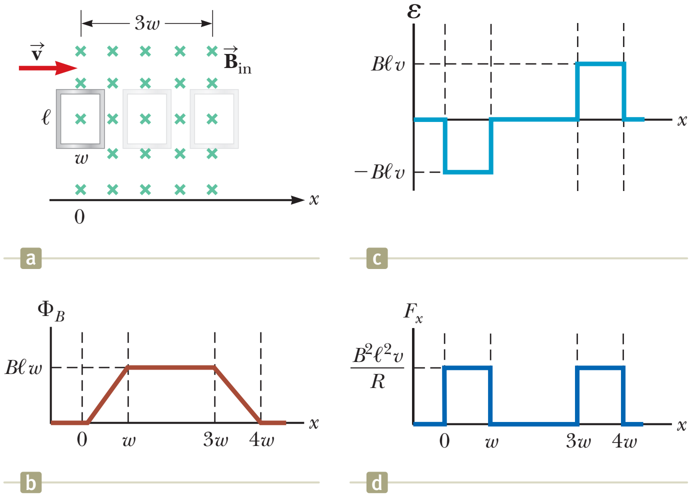

Figure 31.14 (Conceptual Example 31.6) (a) A conducting rectangular loop of width w and length ℓ moving with a velocity $\vec{\mathbf{v}}$ through a uniform magnetic field extending a distance $3w$. (b) Magnetic flux through the area enclosed by the loop as a function of loop position. (c) Induced emf as a function of loop position. (d) Applied force required for constant velocity as a function of loop position.

31.4 Induced emf and Electric Fields

We have seen that a changing magnetic flux induces an emf and a current in a conducting loop. In our study of electricity, we related a current to an electric field that applies electric forces on charged particles. In the same way, we can relate an induced current in a conducting loop to an electric field by claiming that an electric field is created in the conductor as a result of the changing magnetic flux.

We also noted in our study of electricity that the existence of an electric field is independent of the presence of any test charges. This independence suggests that even in the absence of a conducting loop, a changing magnetic field generates an electric field in empty space.

This induced electric field is *nonconservative,* unlike the electrostatic field produced by stationary charges. To illustrate this point, consider a conducting loop of radius r situated in a uniform magnetic field that is perpendicular to the plane of the loop as in Figure 31.15. If the magnetic field changes with time, an emf $\mathcal{E} = -d\Phi_B/dt$ is, according to Faraday's law (Eq. 31.1), induced in the loop. The induction of a current in the loop implies the presence of an induced electric field $\vec{\mathbf{E}}$, which must be tangent to the loop because that is the direction in which the charges in the wire move in response to the electric force. The work done by the electric field in moving a charge q once around the loop is equal to $q\mathcal{E}$. Because the electric force acting on the charge is $q\vec{\mathbf{E}}$, the work done by the electric field in

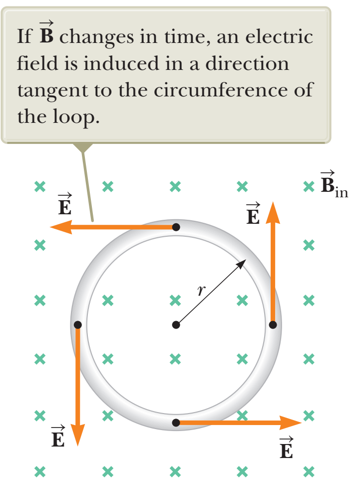

Figure 31.15 A conducting loop of radius r in a uniform magnetic field perpendicular to the plane of the loop.

Pitfall Prevention 31.1

Induced Electric Fields The changing magnetic field does *not* need to exist at the location of the induced electric field. In Figure 31.15, even a loop outside the region of magnetic field experiences an induced electric field.

moving the charge once around the loop is $qE(2\pi r)$, where $2\pi r$ is the circumference of the loop. These two expressions for the work done must be equal; therefore,

$$q\mathcal{E} = qE(2\pi r)$$

$$E = \frac{\mathcal{E}}{2\pi r}$$

Using this result along with Equation 31.1 and that $\Phi_B = BA = B\pi r^2$ for a circular loop, the induced electric field can be expressed as

$$E = -\frac{1}{2\pi r}\frac{d\Phi_B}{dt} = -\frac{r}{2}\frac{dB}{dt} \qquad (31.8)$$

If the time variation of the magnetic field is specified, the induced electric field can be calculated from Equation 31.8.

The emf for any closed path can be expressed as the line integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ over that path: $\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$. In more general cases, E may not be constant and the path may not be a circle. Hence, Faraday's law of induction, $\mathcal{E} = -d\Phi_B/dt$, can be written in the general form

Faraday's law in general form ▶

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \qquad (31.9)$$

The induced electric field $\vec{\mathbf{E}}$ in Equation 31.9 is a nonconservative field that is generated by a changing magnetic field. The field $\vec{\mathbf{E}}$ that satisfies Equation 31.9 cannot possibly be an electrostatic field because were the field electrostatic and hence conservative, the line integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ over a closed loop would be zero (Section 25.1), which would be in contradiction to Equation 31.9.

Example 31.7 Electric Field Induced by a Changing Magnetic Field in a Solenoid

A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{max} \cos \omega t$, where I_{max} is the maximum current and ω is the angular frequency of the alternating current source (Fig. 31.16).

(A) Determine the magnitude of the induced electric field outside the solenoid at a distance $r > R$ from its long central axis.

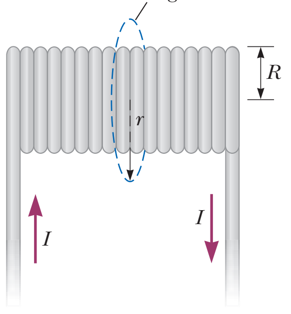

Figure 31.16 (Example 31.7) A long solenoid carrying a time-varying current given by $I = I_{max} \cos \omega t$. An electric field is induced both inside and outside the solenoid.

SOLUTION

Conceptualize Figure 31.16 shows the physical situation. As the current in the coil changes, imagine a changing magnetic field at all points in space as well as an induced electric field.

Categorize In this analysis problem, because the current varies in time, the magnetic field is changing, leading to an induced electric field as opposed to the electrostatic electric fields due to stationary electric charges.

Analyze First consider an external point and take the path for the line integral to be a circle of radius r centered on the solenoid as illustrated in Figure 31.16.

Evaluate the right side of Equation 31.9, noting that the magnetic field $\vec{\mathbf{B}}$ inside the solenoid is perpendicular to the circle bounded by the path of integration:

$$(1) \quad -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\pi R^2) = -\pi R^2 \frac{dB}{dt}$$

Evaluate the magnetic field inside the solenoid from Equation 30.17:

$$(2) \quad B = \mu_0 n I = \mu_0 n I_{max} \cos \omega t$$

31.7 continued

Substitute Equation (2) into Equation (1): (3) $-\frac{d\Phi_B}{dt} = -\pi R^2 \mu_0 n I_{\max} \frac{d}{dt}(\cos \omega t) = \pi R^2 \mu_0 n I_{\max} \omega \sin \omega t$

Evaluate the left side of Equation 31.9, noting that the magnitude of $\vec{\mathbf{E}}$ is constant on the path of integration and $\vec{\mathbf{E}}$ is tangent to it: (4) $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E(2\pi r)$

Substitute Equations (3) and (4) into Equation 31.9: $E(2\pi r) = \pi R^2 \mu_0 n I_{\max} \omega \sin \omega t$

Solve for the magnitude of the electric field: $E = \boxed{\frac{\mu_0 n I_{\max} \omega R^2}{2r} \sin \omega t}$ (for $r > R$)

Finalize This result shows that the amplitude of the electric field outside the solenoid falls off as $1/r$ and varies sinusoidally with time. It is proportional to the current I as well as to the frequency ω, consistent with the fact that a larger value of ω means more change in magnetic flux per unit time. As we will learn in Chapter 34, the time-varying electric field creates an additional contribution to the magnetic field. The magnetic field can be somewhat stronger than we first stated, both inside and outside the solenoid. The correction to the magnetic field is small if the angular frequency ω is small. At high frequencies, however, a new phenomenon can dominate: The electric and magnetic fields, each re-creating the other, constitute an electromagnetic wave radiated by the solenoid as we will study in Chapter 34.

(B) What is the magnitude of the induced electric field inside the solenoid, a distance r from its axis?

SOLUTION

Analyze For an interior point ($r < R$), the magnetic flux through an integration loop is given by $\Phi_B = B\pi r^2$.

Evaluate the right side of Equation 31.9: (5) $-\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\pi r^2) = -\pi r^2 \frac{dB}{dt}$

Substitute Equation (2) into Equation (5): (6) $-\frac{d\Phi_B}{dt} = -\pi r^2 \mu_0 n I_{\max} \frac{d}{dt}(\cos \omega t) = \pi r^2 \mu_0 n I_{\max} \omega \sin \omega t$

Substitute Equations (4) and (6) into Equation 31.9: $E(2\pi r) = \pi r^2 \mu_0 n I_{\max} \omega \sin \omega t$

Solve for the magnitude of the electric field: $E = \boxed{\frac{\mu_0 n I_{\max} \omega}{2} r \sin \omega t}$ (for $r < R$)

Finalize This result shows that the amplitude of the electric field induced inside the solenoid by the changing magnetic flux through the solenoid increases linearly with r and varies sinusoidally with time. As with the field outside the solenoid, the field inside is proportional to the current I and the frequency ω.

31.5 Generators and Motors

Electric generators are devices that take in energy by work and transfer it out by electrical transmission. To understand how they operate, let us consider the **alternating-current (AC) generator.** In its simplest form, it consists of a loop of wire rotated by some external means in a magnetic field (Fig. 31.17a, page 950).

In commercial power plants, the energy required to rotate the loop can be derived from a variety of sources. For example, in a hydroelectric plant, falling water directed against the blades of a turbine produces the rotary motion; in a coal-fired plant, the energy released by burning coal is used to convert water to steam, and this steam is directed against the turbine blades.

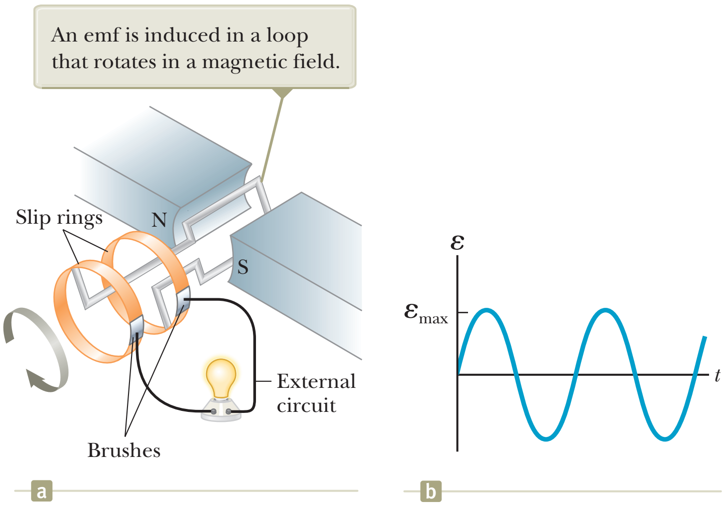

Figure 31.17 (a) Schematic diagram of an AC generator. (b) The alternating emf induced in the loop plotted as a function of time.

As a loop rotates in a magnetic field, the magnetic flux through the area enclosed by the loop changes with time, and this change induces an emf and a current in the loop according to Faraday's law. The ends of the loop are connected to slip rings that rotate with the loop. Connections from these slip rings, which act as output terminals of the generator, to the external circuit are made by stationary metallic brushes in contact with the slip rings.

Instead of a single turn, suppose a coil with N turns (a more practical situation), with the same area A, rotates in a magnetic field with a constant angular speed ω. If θ is the angle between the magnetic field and the normal to the plane of the coil as in Figure 31.18, the magnetic flux through the coil at any time t is

$$\Phi_B = BA\cos\theta = BA\cos\omega t$$

where we have used the relationship $\theta = \omega t$ between angular position and angular speed (see Eq. 10.3). (We have set the clock so that $t = 0$ when $\theta = 0$.) Hence, the induced emf in the coil is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -NBA\frac{d}{dt}(\cos\omega t) = NBA\omega\sin\omega t \tag{31.10}$$

This result shows that the emf varies sinusoidally with time as plotted in Figure 31.17b. Equation 31.10 shows that the maximum emf has the value

$$\mathcal{E}_{\max} = NBA\omega \tag{31.11}$$

which occurs when $\omega t = 90°$ or $270°$. In other words, $\mathcal{E} = \mathcal{E}_{\max}$ when the magnetic field is in the plane of the coil and the time rate of change of flux is a maximum. Furthermore, the emf is zero when $\omega t = 0$ or $180°$, that is, when $\vec{\mathbf{B}}$ is perpendicular to the plane of the coil and the time rate of change of flux is zero.

The frequency for commercial generators in the United States and Canada is 60 Hz, whereas in some European countries it is 50 Hz. (Recall that $\omega = 2\pi f$, where f is the frequency in hertz.)

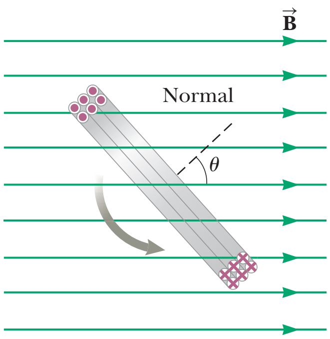

Figure 31.18 A cutaway view of a loop enclosing an area A and containing N turns, rotating with constant angular speed ω in a magnetic field. The emf induced in the loop varies sinusoidally in time.

Quick Quiz 31.4 In an AC generator, a coil with N turns of wire spins in a magnetic field. Of the following choices, which does *not* cause an increase in the emf generated in the coil? **(a)** replacing the coil wire with one of lower resistance **(b)** spinning the coil faster **(c)** increasing the magnetic field **(d)** increasing the number of turns of wire on the coil

Example 31.8 emf Induced in a Generator

The coil in an AC generator consists of 8 turns of wire, each of area $A = 0.090\,0\ \text{m}^2$, and the total resistance of the wire is $12.0\ \Omega$. The coil rotates in a 0.500-T magnetic field at a constant frequency of 60.0 Hz.

(A) Find the maximum induced emf in the coil.

SOLUTION

Conceptualize Study Figure 31.17 to make sure you understand the operation of an AC generator.

Categorize We evaluate parameters using equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 31.11 to find the maximum induced emf:

$$\mathcal{E}_{\text{max}} = NBA\omega = NBA(2\pi f)$$

Substitute numerical values:

$$\mathcal{E}_{\text{max}} = 8(0.500\ \text{T})(0.090\,0\ \text{m}^2)(2\pi)(60.0\ \text{Hz}) = \boxed{136\ \text{V}}$$

(B) What is the maximum induced current in the coil when the output terminals are connected to a low-resistance conductor?

SOLUTION

Use Equation 27.7 and the result to part (A):

$$I_{\text{max}} = \frac{\mathcal{E}_{\text{max}}}{R} = \frac{136\ \text{V}}{12.0\ \Omega} = \boxed{11.3\ \text{A}}$$

The **direct-current (DC) generator** is illustrated in Figure 31.19a. Such generators are used, for instance, in older cars to charge the storage batteries. The components are essentially the same as those of the AC generator except that the contacts to the rotating coil are made using a split ring called a *commutator.*

In this configuration, the output voltage always has the same polarity and pulsates with time as shown in Figure 31.19b. We can understand why by noting that the contacts to the split ring reverse their roles every half cycle. At the same time, the polarity of the induced emf reverses; hence, the polarity of the split ring (which is the same as the polarity of the output voltage) remains the same.

A pulsating DC current is not suitable for most applications. To obtain a steadier DC current, commercial DC generators use many coils and commutators distributed so that the sinusoidal pulses from the various coils are out of phase. When these pulses are superimposed, the DC output is almost free of fluctuations.

A **motor** is a device into which energy is transferred by electrical transmission while energy is transferred out by work. A motor is essentially a generator operating

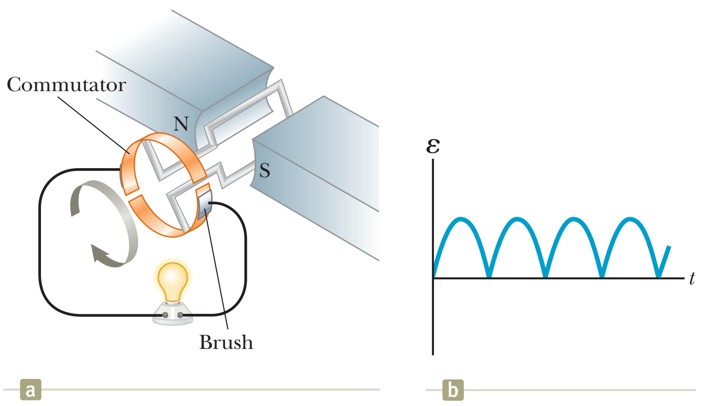

Figure 31.19 (a) Schematic diagram of a DC generator. (b) The magnitude of the emf varies in time, but the polarity never changes.

in reverse. Instead of generating a current by rotating a coil, a current is supplied to the coil by a battery, and the torque acting on the current-carrying coil (Section 29.5) causes it to rotate.

Useful mechanical work can be done by attaching the rotating coil of a motor to some external device. As the coil rotates in a magnetic field, however, the changing magnetic flux induces an emf in the coil; consistent with Lenz's law, this induced emf always acts to reduce the current in the coil. The back emf increases in magnitude as the rotational speed of the coil increases. (The phrase *back emf* is used to indicate an emf that tends to reduce the supplied current.) Because the voltage available to supply current equals the difference between the supply voltage and the back emf, the current in the rotating coil is limited by the back emf.

When a motor is turned on, there is initially no back emf, and the current is very large because it is limited only by the resistance of the coil. As the coil begins to rotate, the induced back emf opposes the applied voltage and the current in the coil decreases. If the mechanical load increases, the motor slows down, which causes the back emf to decrease. This reduction in the back emf increases the current in the coil and therefore also increases the power needed from the external voltage source. For this reason, the power requirements for running a motor are greater for heavy loads than for light ones. If the motor is allowed to run under no mechanical load, the back emf reduces the current to a value just large enough to overcome energy losses due to internal energy and friction. If a very heavy load jams the motor so that it cannot rotate, the lack of a back emf can lead to dangerously high current in the motor's wire. This dangerous situation is explored in the What If? section of Example 31.9.

John W. Jewett, Jr.

Figure 31.20 The engine compartment of a Toyota Prius, a hybrid vehicle.

A modern application of motors in automobiles is seen in the development of *hybrid drive systems*. In these automobiles, a gasoline engine and an electric motor are combined to increase the fuel economy of the vehicle and reduce its emissions. Figure 31.20 shows the engine compartment of a Toyota Prius, one of the hybrids available in the United States. In this automobile, power to the wheels can come from either the gasoline engine or the electric motor. In normal driving, the electric motor accelerates the vehicle from rest until it is moving at a speed of about 15 mi/h (24 km/h). During this acceleration period, the engine is not running, so gasoline is not used and there is no emission. At higher speeds, the motor and engine work together so that the engine always operates at or near its most efficient speed. The result is a significantly higher gasoline mileage than that obtained by a traditional gasoline-powered automobile. When a hybrid vehicle brakes, the motor acts as a generator and returns some of the vehicle's kinetic energy back to the battery as stored energy. In a normal vehicle, this kinetic energy is not recovered because it is transformed to internal energy in the brakes and roadway.

Example 31.9 The Induced Current in a Motor

A motor contains a coil with a total resistance of 10 Ω and is supplied by a voltage of 120 V. When the motor is running at its maximum speed, the back emf is 70 V.

(A) Find the current in the coil at the instant the motor is turned on.

SOLUTION

Conceptualize Think about the motor just after it is turned on. It has not yet moved, so there is no back emf generated. As a result, the current in the motor is high. After the motor begins to turn, a back emf is generated and the current decreases.

Categorize We need to combine our new understanding about motors with the relationship between current, voltage, and resistance in this substitution problem.

31.9 continued

Evaluate the current in the coil from Equation 27.7 with no back emf generated:

$$I = \frac{\mathcal{E}}{R} = \frac{120\ \text{V}}{10\ \Omega} = \boxed{12\ \text{A}}$$

(B) Find the current in the coil when the motor has reached maximum speed.

SOLUTION

Evaluate the current in the coil with the maximum back emf generated:

$$I = \frac{\mathcal{E} - \mathcal{E}_{\text{back}}}{R} = \frac{120\ \text{V} - 70\ \text{V}}{10\ \Omega} = \frac{50\ \text{V}}{10\ \Omega} = \boxed{5.0\ \text{A}}$$

The current drawn by the motor when operating at its maximum speed is significantly less than that drawn before it begins to turn.

WHAT IF? Suppose this motor is in a circular saw. When you are operating the saw, the blade becomes jammed in a piece of wood and the motor cannot turn. By what percentage does the power input to the motor increase when it is jammed?

Answer You may have everyday experiences with motors becoming warm when they are prevented from turning. That is due to the increased power input to the motor. The higher rate of energy transfer results in an increase in the internal energy of the coil, an undesirable effect.

Set up the ratio of power input to the motor when jammed, using the current calculated in part (A), to that when it is not jammed, part (B):

$$\frac{P_{\text{jammed}}}{P_{\text{not jammed}}} = \frac{I_A^2 R}{I_B^2 R} = \frac{I_A^2}{I_B^2}$$

Substitute numerical values:

$$\frac{P_{\text{jammed}}}{P_{\text{not jammed}}} = \frac{(12\ \text{A})^2}{(5.0\ \text{A})^2} = 5.76$$

That represents a 476% increase in the input power! Such a high power input can cause the coil to become so hot that it is damaged.

31.6 Eddy Currents

As we have seen, an emf and a current are induced in a circuit by a changing magnetic flux. In the same manner, circulating currents called **eddy currents** are induced in bulk pieces of metal moving through a magnetic field. This phenomenon can be demonstrated by allowing a flat copper or aluminum plate attached at the end of a rigid bar to swing back and forth through a magnetic field (Fig. 31.21). As the plate enters the field, the changing magnetic flux induces an emf in the plate, which in turn causes the free electrons in the plate to move, producing the swirling eddy currents. According to Lenz's law, the direction of the eddy currents is such that they create magnetic fields that oppose the change that causes the currents. For this reason, the eddy currents must produce effective magnetic poles on the plate, which are repelled by the poles of the magnet; this situation gives rise to a repulsive force that opposes the motion of the plate. (If the opposite were true, the plate would accelerate and its energy would increase after each swing, in violation of the law of conservation of energy.)

As indicated in Figure 31.22a (page 954), with $\vec{\mathbf{B}}$ directed into the page, the induced eddy current is counterclockwise as the swinging plate enters the field at position 1 because the flux due to the external magnetic field into the page through the plate is increasing. Hence, by Lenz's law, the induced current must provide its own magnetic field out of the page. The opposite is true as the plate

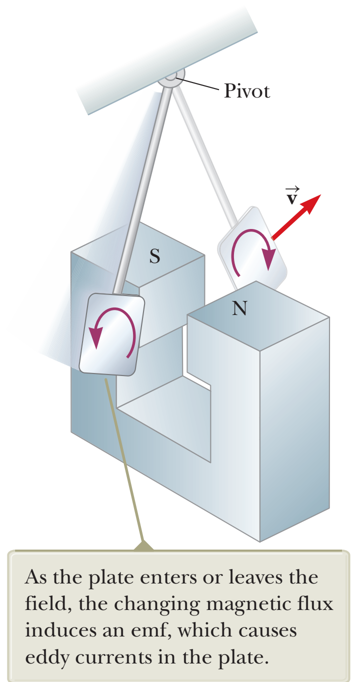

Figure 31.21 Formation of eddy currents in a conducting plate moving through a magnetic field.

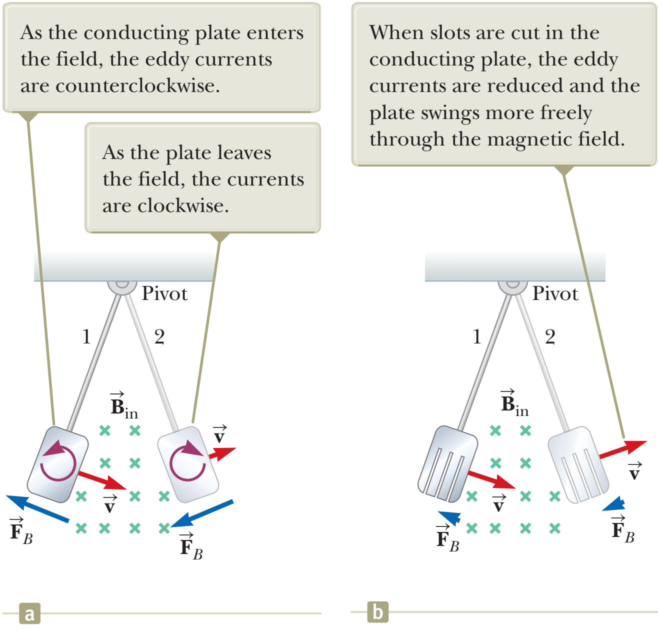

Figure 31.22 When a conducting plate swings through a magnetic field, eddy currents are induced and the magnetic force $\vec{F}_B$ on the plate opposes its velocity, causing it to eventually come to rest.

leaves the field at position 2, where the current is clockwise. Because the induced eddy current always produces a magnetic retarding force $\vec{F}_B$ when the plate enters or leaves the field, the swinging plate eventually comes to rest.

If slots are cut in the plate as shown in Figure 31.22b, the eddy currents and the corresponding retarding force are greatly reduced. We can understand this reduction in force by realizing that the cuts in the plate prevent the formation of any large current loops.

The braking systems on many subway and rapid-transit cars make use of electromagnetic induction and eddy currents. An electromagnet attached to the train is positioned near the steel rails. (An electromagnet is essentially a solenoid with an iron core.) The braking action occurs when a large current is passed through the electromagnet. The relative motion of the magnet and rails induces eddy currents in the rails, and the direction of these currents produces a drag force on the moving train. Because the eddy currents decrease steadily in magnitude as the train slows down, the braking effect is quite smooth. As a safety measure, some power tools use eddy currents to stop rapidly spinning blades once the device is turned off.

Eddy currents are often undesirable because they represent a transformation of mechanical energy to internal energy. To reduce this energy loss, conducting parts are often laminated; that is, they are built up in thin layers separated by a nonconducting material such as lacquer or a metal oxide. This layered structure prevents large current loops and effectively confines the currents to small loops in individual layers. Such a laminated structure is used in transformer cores (see Section 33.8) and motors to minimize eddy currents and thereby increase the efficiency of these devices.

Quick Quiz 31.5 In an equal-arm balance from the early 20th century (Fig. 31.23), an aluminum sheet hangs from one of the arms and passes between the poles of a magnet, causing the oscillations of the balance to decay rapidly. In the absence of such magnetic braking, the oscillation might continue for a long time, and the experimenter would have to wait to take a reading. Why do the oscillations decay? **(a)** because the aluminum sheet is attracted to the magnet

(b) because currents in the aluminum sheet set up a magnetic field that opposes the oscillations **(c)** because aluminum is paramagnetic

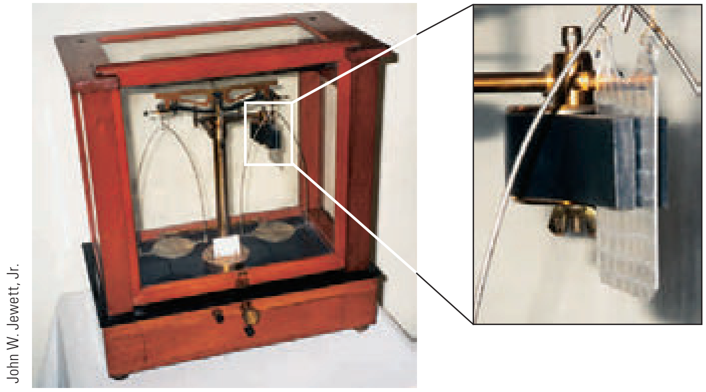

Figure 31.23 (Quick Quiz 31.5) In an old-fashioned equal-arm balance, an aluminum sheet hangs between the poles of a magnet.

Summary

Concepts and Principles

Faraday's law of induction states that the emf induced in a loop is directly proportional to the time rate of change of magnetic flux through the loop, or

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad (31.1)$$

where $\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$ is the magnetic flux through the loop.

When a conducting bar of length ℓ moves at a velocity $\vec{\mathbf{v}}$ through a magnetic field $\vec{\mathbf{B}}$, where $\vec{\mathbf{B}}$ is perpendicular to the bar and to $\vec{\mathbf{v}}$, the **motional emf** induced in the bar is

$$\mathcal{E} = -B\ell v \quad (31.5)$$

Lenz's law states that the induced current and induced emf in a conductor are in such a direction as to set up a magnetic field that opposes the change that produced them.

A general form of **Faraday's law of induction** is

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \quad (31.9)$$

where $\vec{\mathbf{E}}$ is the nonconservative electric field that is produced by the changing magnetic flux.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Figure OQ31.1 is a graph of the magnetic flux through a certain coil of wire as a function of time during an interval while the radius of the coil is increased, the coil is rotated through 1.5 revolutions, and the external source of the magnetic field is turned off, in that order. Rank the emf induced in the coil at the instants marked A through E from the largest positive value to the largest-magnitude negative value. In your ranking,

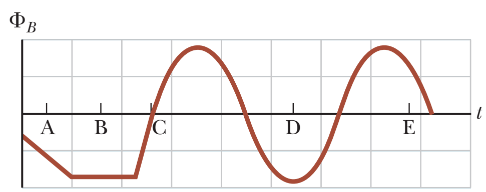

Figure OQ31.1

note any cases of equality and also any instants when the emf is zero.

2. A flat coil of wire is placed in a uniform magnetic field that is in the y direction. **(i)** The magnetic flux through the coil is a maximum if the plane of the coil is where? More than one answer may be correct. (a) in the xy plane (b) in the yz plane (c) in the xz plane (d) in any orientation, because it is a constant **(ii)** For what orientation is the flux zero? Choose from the same possibilities as in part (i).

3. A rectangular conducting loop is placed near a long wire carrying a current I as shown in Figure OQ31.3. If I decreases in time, what can be said of the current induced in the loop? (a) The direction of the current depends on the size of the loop. (b) The current is clockwise. (c) The current is counterclockwise. (d) The current is zero. (e) Nothing can be said about the current in the loop without more information.

Figure OQ31.3

4. A circular loop of wire with a radius of 4.0 cm is in a uniform magnetic field of magnitude 0.060 T. The plane of the loop is perpendicular to the direction of the magnetic field. In a time interval of 0.50 s, the magnetic field changes to the opposite direction with a magnitude of 0.040 T. What is the magnitude of the average emf induced in the loop? (a) 0.20 V (b) 0.025 V (c) 5.0 mV (d) 1.0 mV (e) 0.20 mV

5. A square, flat loop of wire is pulled at constant velocity through a region of uniform magnetic field directed perpendicular to the plane of the loop as shown in Figure OQ31.5. Which of the following statements are correct? More than one statement may be correct. (a) Current is induced in the loop in the clockwise direction. (b) Current is induced in the loop in the counterclockwise direction. (c) No current is induced in the loop. (d) Charge separation occurs in the loop, with the top edge positive. (e) Charge separation occurs in the loop, with the top edge negative.

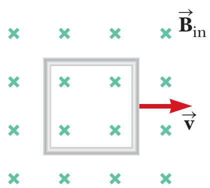

Figure OQ31.5

6. The bar in Figure OQ31.6 moves on rails to the right with a velocity $\vec{v}$, and a uniform, constant magnetic field is directed out of the page. Which of the following statements are correct? More than one statement may be correct. (a) The induced current in the loop is zero. (b) The induced current in the loop is clockwise. (c) The induced current in the loop is counterclockwise. (d) An external force is required to keep the bar moving at constant speed. (e) No force is required to keep the bar moving at constant speed.

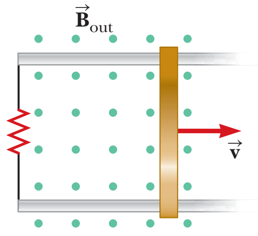

Figure OQ31.6

7. A bar magnet is held in a vertical orientation above a loop of wire that lies in the horizontal plane as shown in Figure OQ31.7. The south end of the magnet is toward the loop. After the magnet is dropped, what is true of the induced current in the loop as viewed from above? (a) It is clockwise as the magnet falls toward the loop. (b) It is counterclockwise as the magnet falls toward the loop. (c) It is clockwise after the magnet has moved through the loop and moves away from it. (d) It is always clockwise. (e) It is first counterclockwise as the magnet approaches the loop and then clockwise after it has passed through the loop.

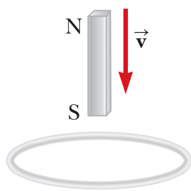

Figure OQ31.7

8. What happens to the amplitude of the induced emf when the rate of rotation of a generator coil is doubled? (a) It becomes four times larger. (b) It becomes two times larger. (c) It is unchanged. (d) It becomes one-half as large. (e) It becomes one-fourth as large.

9. Two coils are placed near each other as shown in Figure OQ31.9. The coil on the left is connected to a battery and a switch, and the coil on the right is connected to a resistor. What is the direction of the cur-

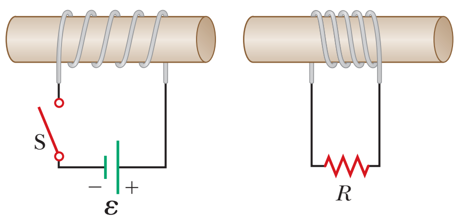

Figure OQ31.9

rent in the resistor **(i)** at an instant immediately after the switch is thrown closed, **(ii)** after the switch has been closed for several seconds, and **(iii)** at an instant after the switch has then been thrown open? Choose each answer from the possibilities (a) left, (b) right, or (c) the current is zero.

10. A circuit consists of a conducting movable bar and a lightbulb connected to two conducting rails as shown in Figure OQ31.10. An external magnetic field is directed perpendicular to the plane of the circuit. Which of the following actions will make the bulb light up? More than one statement may be correct. (a) The bar is moved to the left. (b) The bar is moved to the right. (c) The magnitude of the magnetic field is increased. (d) The magnitude of the magnetic field is decreased. (e) The bar is lifted off the rails.

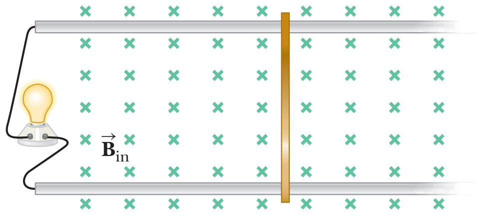

Figure OQ31.10

11. Two rectangular loops of wire lie in the same plane as shown in Figure OQ31.11. If the current I in the outer loop is counterclockwise and increases with time, what is true of the current induced in the inner loop? More than one statement may be correct. (a) It is zero. (b) It is clockwise. (c) It is counterclockwise. (d) Its magnitude depends on the dimensions of the loops. (e) Its direction depends on the dimensions of the loops.

Figure OQ31.11

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. In Section 7.7, we defined conservative and nonconservative forces. In Chapter 23, we stated that an electric charge creates an electric field that produces a conservative force. Argue now that induction creates an electric field that produces a nonconservative force.

2. A spacecraft orbiting the Earth has a coil of wire in it. An astronaut measures a small current in the coil, although there is no battery connected to it and there are no magnets in the spacecraft. What is causing the current?

3. In a hydroelectric dam, how is energy produced that is then transferred out by electrical transmission? That is, how is the energy of motion of the water converted to energy that is transmitted by AC electricity?

4. A bar magnet is dropped toward a conducting ring lying on the floor. As the magnet falls toward the ring, does it move as a freely falling object? Explain.

5. A circular loop of wire is located in a uniform and constant magnetic field. Describe how an emf can be induced in the loop in this situation.

6. A piece of aluminum is dropped vertically downward between the poles of an electromagnet. Does the magnetic field affect the velocity of the aluminum?

7. What is the difference between magnetic flux and magnetic field?

8. When the switch in Figure CQ31.8a is closed, a current is set up in the coil and the metal ring springs upward (Fig. CQ31.8b). Explain this behavior.

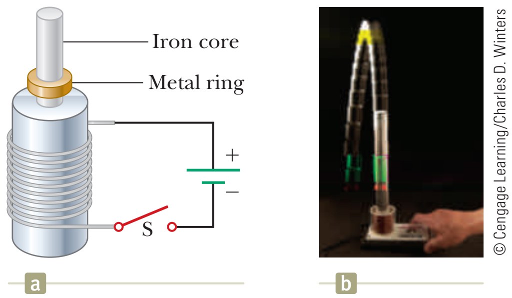

Figure CQ31.8 Conceptual Questions 8 and 9.

9. Assume the battery in Figure CQ31.8a is replaced by an AC source and the switch is held closed. If held down, the metal ring on top of the solenoid becomes hot. Why?

10. A loop of wire is moving near a long, straight wire carrying a constant current I as shown in Figure CQ31.10. (a) Determine the direction of the induced current in the loop as it moves away from the wire. (b) What would be the direction of the induced current in the loop if it were moving toward the wire?

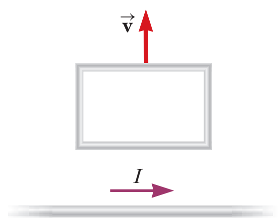

Figure CQ31.10

Problems

The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 31.1 Faraday's Law of Induction

1. A flat loop of wire consisting of a single turn of cross-sectional area 8.00 cm^2 is perpendicular to a magnetic field that increases uniformly in magnitude from 0.500 T to 2.50 T in 1.00 s. What is the resulting induced current if the loop has a resistance of 2.00 Ω?

2. An instrument based on induced emf has been used to measure projectile speeds up to 6 km/s. A small magnet is imbedded in the projectile as shown in Figure P31.2. The projectile passes through two coils separated by a distance d. As the projectile passes through each coil, a pulse of emf is induced in the coil. The time interval between pulses can be measured accurately with an oscilloscope, and thus the speed can be determined. (a) Sketch a graph of ΔV versus t for the arrangement shown. Consider a current that flows counterclockwise as viewed from the starting point of the projectile as positive. On your graph, indicate which pulse is from coil 1 and which is from coil 2. (b) If the pulse separation is 2.40 ms and $d = 1.50$ m, what is the projectile speed?

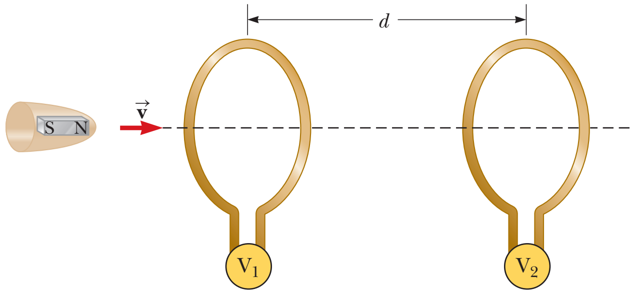

Figure P31.2

3. Transcranial magnetic stimulation (TMS) is a noninvasive technique used to stimulate regions of the human brain (Figure P31.3). In TMS, a small coil is placed on the scalp and a brief burst of current in the coil produces a rapidly changing magnetic field inside the brain. The induced emf can stimulate neuronal activity. (a) One such device generates an upward magnetic field within the brain that rises from zero to 1.50 T in 120 ms. Determine the induced emf around a horizontal circle of tissue of radius 1.60 mm. (b) **What If?** The field next changes to 0.500 T downward in 80.0 ms. How does the emf induced in this process compare with that in part (a)?

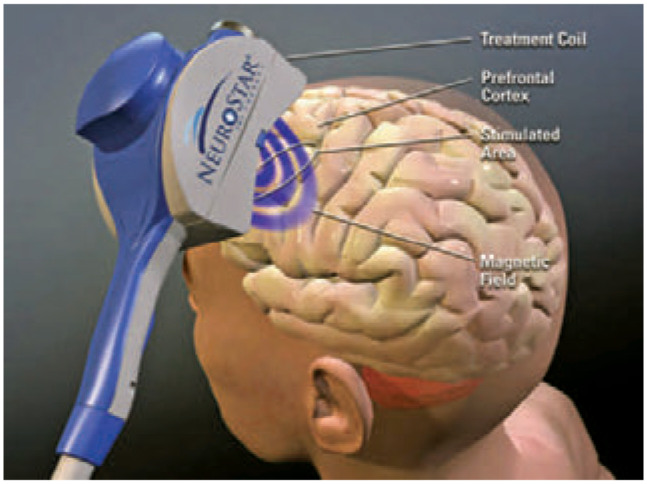

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Figure P31.3 Problems 3 and 51. The magnetic coil of a Neurostar TMS apparatus is held near the head of a patient.

4. W A 25-turn circular coil of wire has diameter 1.00 m. It is placed with its axis along the direction of the Earth's magnetic field of 50.0 μT and then in 0.200 s is flipped 180°. An average emf of what magnitude is generated in the coil?

5. The flexible loop in Figure P31.5 has a radius of 12.0 cm and is in a magnetic field of magnitude 0.150 T. The loop is grasped at points A and B and stretched until its area is nearly zero. If it takes 0.200 s to close the loop, what is the magnitude of the average induced emf in it during this time interval?

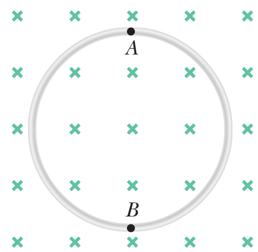

Figure P31.5 Problems 5 and 6.

6. A circular loop of wire of radius 12.0 cm is placed in a magnetic field directed perpendicular to the plane of the loop as in Figure P31.5. If the field decreases at the rate of 0.050 0 T/s in some time interval, find the magnitude of the emf induced in the loop during this interval.

7. To monitor the breathing of a hospital patient, a thin belt is girded around the patient's chest. The belt is a 200-turn coil. When the patient inhales, the area encircled by the coil increases by 39.0 cm^2. The magnitude of the Earth's magnetic field is 50.0 μT and makes an angle of 28.0° with the plane of the coil. Assuming a patient takes 1.80 s to inhale, find the average induced emf in the coil during this time interval.

8. W A strong electromagnet produces a uniform magnetic field of 1.60 T over a cross-sectional area of 0.200 m^2. A coil having 200 turns and a total resistance of 20.0 Ω is placed around the electromagnet. The current in the electromagnet is then smoothly reduced until it reaches zero in 20.0 ms. What is the current induced in the coil?

9. W A 30-turn circular coil of radius 4.00 cm and resistance 1.00 Ω is placed in a magnetic field directed perpendicular to the plane of the coil. The magnitude of the magnetic field varies in time according to the expression $B = 0.010\,0t + 0.040\,0t^2$, where B is in teslas and t is in seconds. Calculate the induced emf in the coil at $t = 5.00$ s.

10. Scientific work is currently under way to determine whether weak oscillating magnetic fields can affect human health. For example, one study found that drivers of trains had a higher incidence of blood cancer than other railway workers, possibly due to long exposure to mechanical devices in the train engine cab. Consider a magnetic field of magnitude 1.00×10^{-3} T, oscillating sinusoidally at 60.0 Hz. If the diameter of a red blood cell is 8.00 μm, determine the maximum emf that can be generated around the perimeter of a cell in this field.

11. M An aluminum ring of radius $r_1 = 5.00$ cm and resistance 3.00×10^{-4} Ω is placed around one end of a long air-core solenoid with 1 000 turns per meter and radius $r_2 = 3.00$ cm as shown in Figure P31.11. Assume the axial component of the field produced by the solenoid is one-half as strong over the area of the end of the solenoid as at the center of the solenoid. Also assume the solenoid produces negligible field outside its cross-sectional area. The current in the solenoid is increasing at a rate of 270 A/s. (a) What is the induced current in the ring? At the center of the ring, what are (b) the magnitude and (c) the direction of the magnetic field produced by the induced current in the ring?

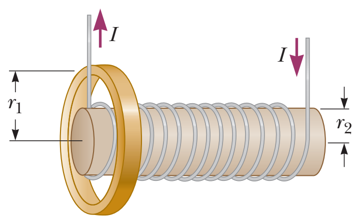

Figure P31.11 Problems 11 and 12.

12. An aluminum ring of radius r_1 and resistance R is placed around one end of a long air-core solenoid with n turns per meter and smaller radius r_2 as shown in Figure P31.11. Assume the axial component of the field produced by the solenoid over the area of the end of the solenoid is one-half as strong as at the center of the solenoid. Also assume the solenoid produces negligible field outside its cross-sectional area. The current in the solenoid is increasing at a rate of $\Delta I/\Delta t$. (a) What is the induced current in the ring? (b) At the center of the ring, what is the magnetic field produced by the induced current in the ring? (c) What is the direction of this field?

13. W A loop of wire in the shape of a rectangle of width w and length L and a long, straight wire carrying a current I lie on a tabletop as shown in Figure P31.13. (a) Determine the magnetic flux through the loop due to the current I. (b) Suppose the current is changing with time according to $I = a + bt$, where a and b are constants. Determine the emf that is induced in the loop if $b = 10.0$ A/s, $h = 1.00$ cm, $w = 10.0$ cm, and $L = 1.00$ m. (c) What is the direction of the induced current in the rectangle?

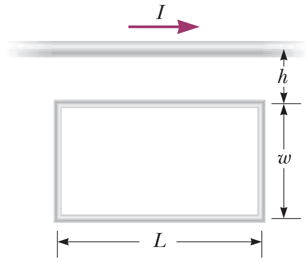

Figure P31.13

14. W A coil of 15 turns and radius 10.0 cm surrounds a long solenoid of radius 2.00 cm and 1.00×10^3 turns/meter (Fig. P31.14). The current in the solenoid changes as $I = 5.00 \sin 120t$, where I is in amperes and t is in seconds. Find the induced emf in the 15-turn coil as a function of time.

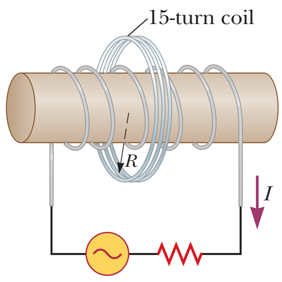

Figure P31.14

15. A square, single-turn wire loop $\ell = 1.00$ cm on a side is placed inside a solenoid that has a circular cross section of radius $r = 3.00$ cm as shown in the end view of Figure P31.15 (page 960). The solenoid is 20.0 cm long and wound with 100 turns of wire. (a) If the current in the solenoid is 3.00 A, what is the magnetic flux through the square loop? (b) If the current in the solenoid is reduced to zero in 3.00 s, what is the magnitude of the average induced emf in the square loop?

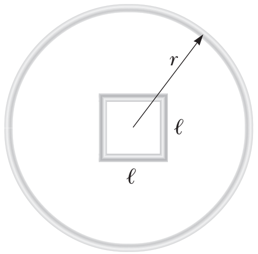

Figure P31.15

16. M A long solenoid has n = 400 turns per meter and carries a current given by $I = 30.0(1 - e^{-1.60t})$, where I is in amperes and t is in seconds. Inside the solenoid and coaxial with it is a coil that has a radius of R = 6.00 cm and consists of a total of N = 250 turns of fine wire (Fig. P31.16). What emf is induced in the coil by the changing current?

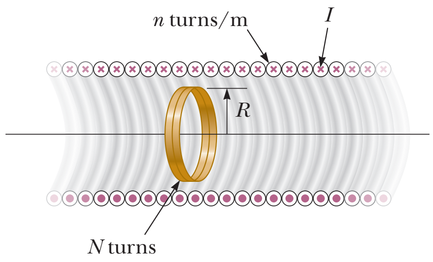

Figure P31.16

17. A coil formed by wrapping 50 turns of wire in the shape of a square is positioned in a magnetic field so that the normal to the plane of the coil makes an angle of 30.0° with the direction of the field. When the magnetic field is increased uniformly from 200 μT to 600 μT in 0.400 s, an emf of magnitude 80.0 mV is induced in the coil. What is the total length of the wire in the coil?

18. When a wire carries an AC current with a known frequency, you can use a *Rogowski coil* to determine the amplitude I_{max} of the current without disconnecting the wire to shunt the current through a meter. The Rogowski coil, shown in Figure P31.18, simply clips around the wire. It consists of a toroidal conductor wrapped around a circular return cord. Let n represent the number of turns in the toroid per unit distance along it. Let A represent the cross-sectional area of the toroid. Let $I(t) = I_{max} \sin \omega t$ represent the current to be measured. (a) Show that the amplitude of the emf induced in the Rogowski coil is $\mathcal{E}_{max} = \mu_0 n A \omega I_{max}$. (b) Explain why the wire carrying the unknown current need not be at the center of the Rogowski coil and why the coil will not respond to nearby currents that it does not enclose.

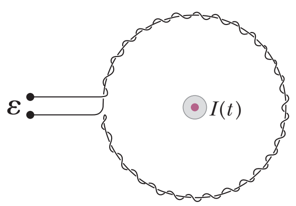

Figure P31.18

19. A toroid having a rectangular cross section (a = 2.00 cm by b = 3.00 cm) and inner radius R = 4.00 cm consists of N = 500 turns of wire that carry a sinusoidal current $I = I_{max} \sin \omega t$, with I_{max} = 50.0 A and a frequency $f = \omega/2\pi$ = 60.0 Hz. A coil that consists of N' = 20 turns of wire is wrapped around one section of the toroid as shown in Figure P31.19. Determine the emf induced in the coil as a function of time.

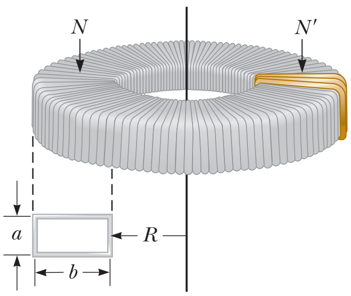

Figure P31.19

20. A piece of insulated wire is shaped into a figure eight as shown in Figure P31.20. For simplicity, model the two halves of the figure eight as circles. The radius of the upper circle is 5.00 cm and that of the lower circle is 9.00 cm. The wire has a uniform resistance per unit length of 3.00 Ω/m. A uniform magnetic field is applied perpendicular to the plane of the two circles, in the direction shown. The magnetic field is increasing at a constant rate of 2.00 T/s. Find (a) the magnitude and (b) the direction of the induced current in the wire.

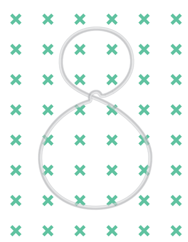

Figure P31.20

Section 31.2 Motional emf

Section 31.3 Lenz's Law

Problem 72 in Chapter 29 can be assigned with this section.

21. A helicopter (Fig. P31.21) has blades of length 3.00 m, extending out from a central hub and rotating at 2.00 rev/s. If the vertical component of the Earth's magnetic field is 50.0 μT, what is the emf induced between the blade tip and the center hub?

Sascha Hahn/Shutterstock.com

Figure P31.21

22. W Use Lenz's law to answer the following questions concerning the direction of induced currents. Express your answers in terms of the letter labels a and b in each part of Figure P31.22. (a) What is the direction of the induced current in the resistor R in Figure P31.22a when the bar magnet is moved to the left? (b) What is the direction of the current induced in the resistor R immediately after the switch S in Figure P31.22b is closed? (c) What is the direction of the induced current in the resistor R when the current I in Figure P31.22c decreases rapidly to zero?

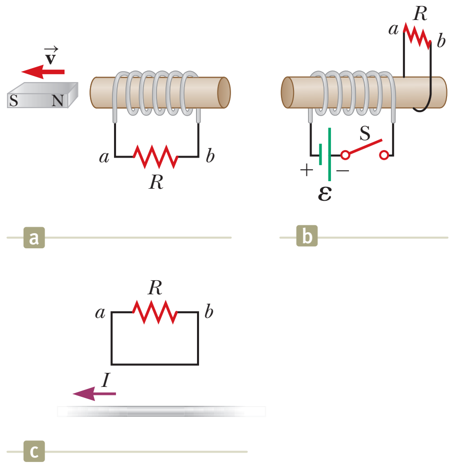

Figure P31.22

23. A truck is carrying a steel beam of length 15.0 m on a freeway. An accident causes the beam to be dumped off the truck and slide horizontally along the ground at a speed of 25.0 m/s. The velocity of the center of mass of the beam is northward while the length of the beam maintains an east–west orientation. The vertical component of the Earth's magnetic field at this location has a magnitude of 35.0 μT. What is the magnitude of the induced emf between the ends of the beam?

24. A small airplane with a wingspan of 14.0 m is flying due north at a speed of 70.0 m/s over a region where the vertical component of the Earth's magnetic field is 1.20 μT downward. (a) What potential difference is developed between the airplane's wingtips? (b) Which wingtip is at higher potential? (c) **What If?** How would the answers to parts (a) and (b) change if the plane turned to fly due east? (d) Can this emf be used to power a lightbulb in the passenger compartment? Explain your answer.

25. A 2.00-m length of wire is held in an east–west direction and moves horizontally to the north with a speed of 0.500 m/s. The Earth's magnetic field in this region is of magnitude 50.0 μT and is directed northward and 53.0° below the horizontal. (a) Calculate the magnitude of the induced emf between the ends of the wire and (b) determine which end is positive.

26. Consider the arrangement shown in Figure P31.26. Assume that $R = 6.00\ \Omega$, $\ell = 1.20$ m, and a uniform 2.50-T magnetic field is directed into the page. At what speed should the bar be moved to produce a current of 0.500 A in the resistor?

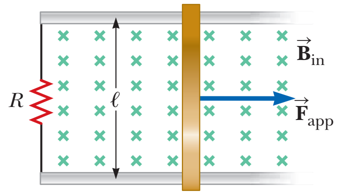

Figure P31.26 Problems 26 through 29.

27. M Figure P31.26 shows a top view of a bar that can slide on two frictionless rails. The resistor is $R = 6.00\ \Omega$, and a 2.50-T magnetic field is directed perpendicularly downward, into the paper. Let $\ell = 1.20$ m. (a) Calculate the applied force required to move the bar to the right at a constant speed of 2.00 m/s. (b) At what rate is energy delivered to the resistor?

28. A metal rod of mass m slides without friction along two parallel horizontal rails, separated by a distance ℓ and connected by a resistor R, as shown in Figure P31.26. A uniform vertical magnetic field of magnitude B is applied perpendicular to the plane of the paper. The applied force shown in the figure acts only for a moment, to give the rod a speed v. In terms of m, ℓ, R, B, and v, find the distance the rod will then slide as it coasts to a stop.

29. A conducting rod of length ℓ moves on two horizontal, frictionless rails as shown in Figure P31.26. If a constant force of 1.00 N moves the bar at 2.00 m/s through a magnetic field $\vec{\mathbf{B}}$ that is directed into the page, (a) what is the current through the 8.00-Ω resistor R? (b) What is the rate at which energy is delivered to the resistor? (c) What is the mechanical power delivered by the force $\vec{\mathbf{F}}_{\text{app}}$?

30. *Why is the following situation impossible?* An automobile has a vertical radio antenna of length $\ell = 1.20$ m. The automobile travels on a curvy, horizontal road where the Earth's magnetic field has a magnitude of $B = 50.0\ \mu$T and is directed toward the north and downward at an angle of $\theta = 65.0°$ below the horizontal. The motional emf developed between the top and bottom of the antenna varies with the speed and direction of the automobile's travel and has a maximum value of 4.50 mV.

31. AMT **Review.** Figure P31.31 shows a bar of mass $m = 0.200$ kg that can slide without friction on a pair of rails separated by a distance $\ell = 1.20$ m and located on an inclined plane that makes an angle $\theta = 25.0°$ with respect to the ground. The resistance of the resistor is $R = 1.00\ \Omega$ and a uniform magnetic field of magnitude $B = 0.500$ T is directed downward, perpendicular to the ground, over the entire region through which the bar moves. With what constant speed v does the bar slide along the rails?

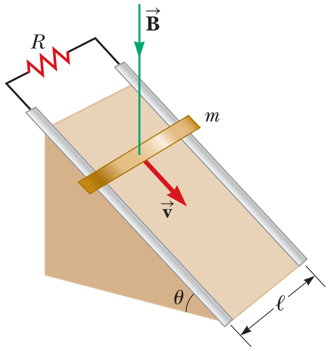

Figure P31.31 Problems 31 and 32.

32. **Review.** Figure P31.31 shows a bar of mass m that can slide without friction on a pair of rails separated by a distance ℓ and located on an inclined plane that makes an angle θ with respect to the ground. The resistance of the resistor is R, and a uniform magnetic field of magnitude B is directed downward, perpendicular to the ground, over the entire region through which the bar moves. With what constant speed v does the bar slide along the rails?

33. M The *homopolar generator,* also called the *Faraday disk,* is a low-voltage, high-current electric generator. It consists of a rotating conducting disk with one stationary brush (a sliding electrical contact) at its axle and another at a point on its circumference as shown in Figure P31.33. A uniform magnetic field is applied perpendicular to the plane of the disk. Assume the field is 0.900 T, the angular speed is 3.20×10^3 rev/min, and the radius of the disk is 0.400 m. Find the emf generated between the brushes. When superconducting coils are used to produce a large magnetic field, a homopolar generator can have a power output of several megawatts. Such a generator is useful, for example, in purifying metals by electrolysis. If a voltage is applied to the output terminals of the generator, it runs in reverse as a *homopolar motor* capable of providing great torque, useful in ship propulsion.

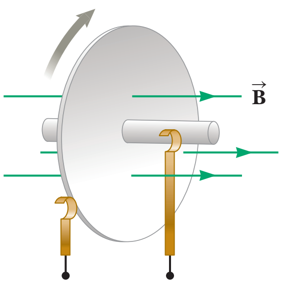

Figure P31.33

34. A conducting bar of length ℓ moves to the right on two frictionless rails as shown in Figure P31.34. A uniform magnetic field directed into the page has a magnitude of 0.300 T. Assume $R = 9.00\ \Omega$ and $\ell = 0.350$ m. (a) At what constant speed should the bar move to produce an 8.50-mA current in the resistor? (b) What is the direction of the induced current? (c) At what rate is energy delivered to the resistor? (d) Explain the origin of the energy being delivered to the resistor.

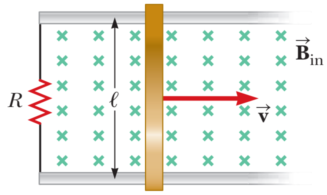

Figure P31.34

35. AMT **Review.** After removing one string while restringing his acoustic guitar, a student is distracted by a video game. His experimentalist roommate notices his inattention and attaches one end of the string, of linear density $\mu = 3.00 \times 10^{-3}$ kg/m, to a rigid support. The other end passes over a pulley, a distance $\ell = 64.0$ cm from the fixed end, and an object of mass $m = 27.2$ kg is attached to the hanging end of the string. The roommate places a magnet across the string as shown in Figure P31.35. The magnet does not touch the string, but produces a uniform field of 4.50 mT over a 2.00-cm length of the string and negligible field elsewhere. Strumming the string sets it vibrating vertically at its fundamental (lowest) frequency. The section of the string in the magnetic field moves perpendicular to the field with a uniform amplitude of 1.50 cm. Find (a) the frequency and (b) the amplitude of the emf induced between the ends of the string.

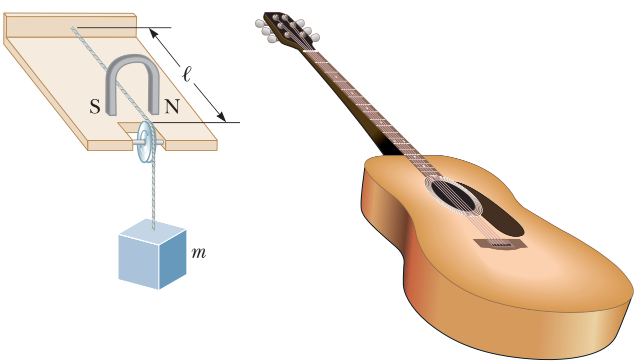

Figure P31.35

36. A rectangular coil with resistance R has N turns, each of length ℓ and width w as shown in Figure P31.36. The coil moves into a uniform magnetic field $\vec{\mathbf{B}}$ with constant velocity $\vec{\mathbf{v}}$. What are the magnitude and direction of the total magnetic force on the coil (a) as it enters the magnetic field, (b) as it moves within the field, and (c) as it leaves the field?

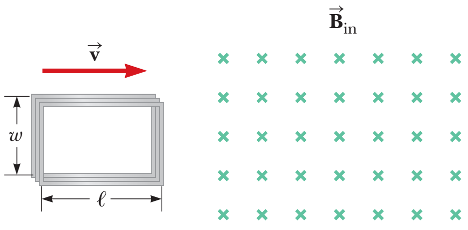

Figure P31.36

37. Two parallel rails with negligible resistance are 10.0 cm apart and are connected by a resistor of resistance $R_3 = 5.00\ \Omega$. The circuit also contains two metal rods having resistances of $R_1 = 10.0\ \Omega$ and $R_2 = 15.0\ \Omega$ sliding along the rails (Fig. P31.37). The rods are pulled away from the resistor at constant speeds of $v_1 = 4.00$ m/s and $v_2 = 2.00$ m/s, respectively. A uniform magnetic field of magnitude $B = 0.010\ 0$ T is applied perpendicular to the plane of the rails. Determine the current in R_3.

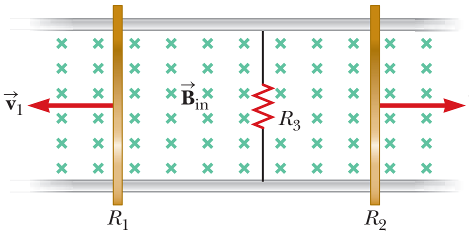

Figure P31.37

38. An astronaut is connected to her spacecraft by a 25.0-m-long tether cord as she and the spacecraft orbit the Earth in a circular path at a speed of 7.80×10^3 m/s. At one instant, the emf between the ends of a wire embedded in the cord is measured to be 1.17 V. Assume the long dimension of the cord is perpendicular to the Earth's magnetic field at that instant. Assume also the tether's center of mass moves with a velocity perpendicular to the Earth's magnetic field. (a) What is the magnitude of the Earth's field at this location? (b) Does the emf change as the system moves from one location to another? Explain. (c) Provide two conditions under which the emf would be zero even though the magnetic field is not zero.

Section 31.4 Induced emf and Electric Fields

39. Within the green dashed circle shown in Figure P31.39, the magnetic field changes with time according to the expression $B = 2.00t^3 - 4.00t^2 + 0.800$, where B is in teslas, t is in seconds, and $R = 2.50$ cm. When $t = 2.00$ s, calculate (a) the magnitude and (b) the direction of the force exerted on an electron located at point P_1, which is at a distance $r_1 = 5.00$ cm from the center of the circular field region. (c) At what instant is this force equal to zero?

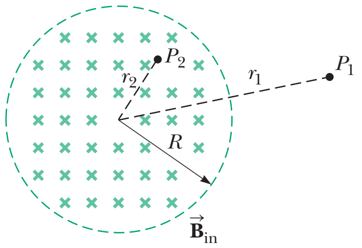

Figure P31.39 Problems 39 and 40.

40. M A magnetic field directed into the page changes with time according to $B = 0.030\ 0t^2 + 1.40$, where B is in teslas and t is in seconds. The field has a circular cross section of radius $R = 2.50$ cm (see Fig. P31.39). When $t = 3.00$ s and $r_2 = 0.020\ 0$ m, what are (a) the magnitude and (b) the direction of the electric field at point P_2?

41. A long solenoid with 1.00×10^3 turns per meter and radius 2.00 cm carries an oscillating current $I = 5.00 \sin 100\pi t$, where I is in amperes and t is in seconds. (a) What is the electric field induced at a radius $r = 1.00$ cm from the axis of the solenoid? (b) What is the direction of this electric field when the current is increasing counterclockwise in the solenoid?

Section 31.5 Generators and Motors

Problems 50 and 68 in Chapter 29 can be assigned with this section.

42. A 100-turn square coil of side 20.0 cm rotates about a vertical axis at 1.50×10^3 rev/min as indicated in Figure P31.42. The horizontal component of the Earth's magnetic field at the coil's location is equal to 2.00×10^{-5} T. (a) Calculate the maximum emf induced in the coil by this field. (b) What is the orientation of the coil with respect to the magnetic field when the maximum emf occurs?

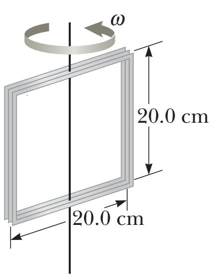

Figure P31.42

43. A generator produces 24.0 V when turning at 900 rev/min. What emf does it produce when turning at 500 rev/min?

44. Figure P31.44 (page 964) is a graph of the induced emf versus time for a coil of N turns rotating with angular speed ω in a uniform magnetic field directed perpendicular to the coil's axis of rotation. **What If?** Copy this sketch (on a larger scale) and on the same set of axes show the graph of emf versus t (a) if the number of turns in the coil is doubled, (b) if instead the angular speed is doubled, and (c) if the angular speed is doubled while the number of turns in the coil is halved.

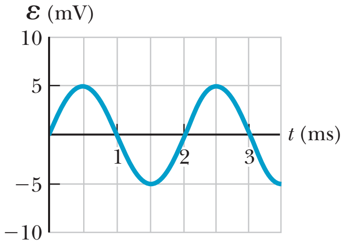

Figure P31.44

45. In a 250-turn automobile alternator, the magnetic flux in each turn is $\Phi_B = 2.50 \times 10^{-4} \cos \omega t$, where Φ_B is in webers, ω is the angular speed of the alternator, and t is in seconds. The alternator is geared to rotate three times for each engine revolution. When the engine is running at an angular speed of 1.00×10^3 rev/min, determine (a) the induced emf in the alternator as a function of time and (b) the maximum emf in the alternator.

46. In Figure P31.46, a semicircular conductor of radius $R = 0.250$ m is rotated about the axis AC at a constant rate of 120 rev/min. A uniform magnetic field of magnitude 1.30 T fills the entire region below the axis and is directed out of the page. (a) Calculate the maximum value of the emf induced between the ends of the conductor. (b) What is the value of the average induced emf for each complete rotation? (c) **What If?** How would your answers to parts (a) and (b) change if the magnetic field were allowed to extend a distance R above the axis of rotation? Sketch the emf versus time (d) when the field is as drawn in Figure P31.46 and (e) when the field is extended as described in part (c).

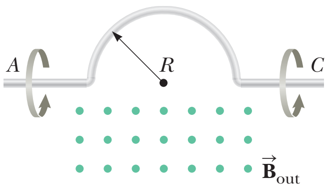

Figure P31.46

47. A long solenoid, with its axis along the x axis, consists of 200 turns per meter of wire that carries a steady current of 15.0 A. A coil is formed by wrapping 30 turns of thin wire around a circular frame that has a radius of 8.00 cm. The coil is placed inside the solenoid and mounted on an axis that is a diameter of the coil and coincides with the y axis. The coil is then rotated with an angular speed of 4.00π rad/s. The plane of the coil is in the yz plane at $t = 0$. Determine the emf generated in the coil as a function of time.

48. A motor in normal operation carries a direct current of 0.850 A when connected to a 120-V power supply. The resistance of the motor windings is 11.8 Ω. While in normal operation, (a) what is the back emf generated by the motor? (b) At what rate is internal energy produced in the windings? (c) **What If?** Suppose a malfunction stops the motor shaft from rotating. At what rate will internal energy be produced in the windings in this case? (Most motors have a thermal switch that will turn off the motor to prevent overheating when this stalling occurs.)

49. The rotating loop in an AC generator is a square 10.0 cm on each side. It is rotated at 60.0 Hz in a uniform field of 0.800 T. Calculate (a) the flux through the loop as a function of time, (b) the emf induced in the loop, (c) the current induced in the loop for a loop resistance of 1.00 Ω, (d) the power delivered to the loop, and (e) the torque that must be exerted to rotate the loop.

Section 31.6 Eddy Currents

50. Figure P31.50 represents an electromagnetic brake that uses eddy currents. An electromagnet hangs from a railroad car near one rail. To stop the car, a large current is sent through the coils of the electromagnet. The moving electromagnet induces eddy currents in the rails, whose fields oppose the change in the electromagnet's field. The magnetic fields of the eddy currents exert force on the current in the electromagnet, thereby slowing the car. The direction of the car's motion and the direction of the current in the electromagnet are shown correctly in the picture. Determine which of the eddy currents shown on the rails is correct. Explain your answer.

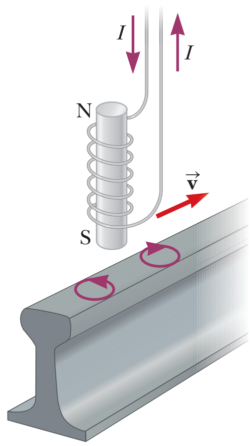

Figure P31.50

Additional Problems

51. Consider a transcranial magnetic stimulation (TMS) device (Figure P31.3) containing a coil with several turns of wire, each of radius 6.00 cm. In a circular area of the brain of radius 6.00 cm directly below and coaxial with the coil, the magnetic field changes at the rate of 1.00×10^4 T/s. Assume that this rate of change is the same everywhere inside the circular area. (a) What is the emf induced around the circumference of this circular area in the brain? (b) What electric field is induced on the circumference of this circular area?

52. Suppose you wrap wire onto the core from a roll of cellophane tape to make a coil. Describe how you can use a bar magnet to produce an induced voltage in the coil. What is the order of magnitude of the emf you generate? State the quantities you take as data and their values.

53. M A circular coil enclosing an area of 100 cm^2 is made of 200 turns of copper wire (Figure P31.53). The wire making up the coil has no resistance; the ends of the wire are connected across a 5.00-Ω resistor to form a closed circuit. Initially, a 1.10-T uniform magnetic field points perpendicularly upward through the plane of the coil. The direction of the field then reverses so that the final magnetic field has a magnitude of 1.10 T and points downward through the coil. If the time interval required for the field to reverse directions is 0.100 s, what is the average current in the coil during that time?

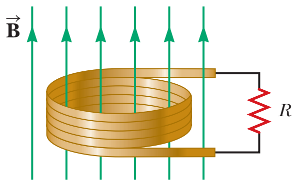

Figure P31.53

54. A circular loop of wire of resistance $R = 0.500\ \Omega$ and radius $r = 8.00$ cm is in a uniform magnetic field directed out of the page as in Figure P31.54. If a clockwise current of $I = 2.50$ mA is induced in the loop, (a) is the magnetic field increasing or decreasing in time? (b) Find the rate at which the field is changing with time.

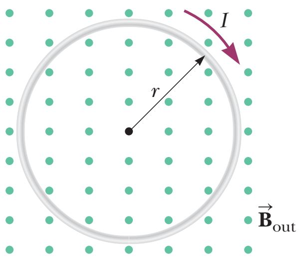

Figure P31.54

55. A rectangular loop of area $A = 0.160\ m^2$ is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of the field is allowed to vary in time according to $B = 0.350\ e^{-t/2.00}$, where B is in teslas and t is in seconds. The field has the constant value 0.350 T for $t < 0$. What is the value for $\mathcal{E}$ at $t = 4.00$ s?

56. A rectangular loop of area A is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of the field is allowed to vary in time according to $B = B_{max}e^{-t/\tau}$, where B_{max} and τ are constants. The field has the constant value B_{max} for $t < 0$. Find the emf induced in the loop as a function of time.

57. Strong magnetic fields are used in such medical procedures as magnetic resonance imaging, or MRI. A technician wearing a brass bracelet enclosing area 0.005 00 m^2 places her hand in a solenoid whose magnetic field is 5.00 T directed perpendicular to the plane of the bracelet. The electrical resistance around the bracelet's circumference is 0.020 0 Ω. An unexpected power failure causes the field to drop to 1.50 T in a time interval of 20.0 ms. Find (a) the current induced in the bracelet and (b) the power delivered to the bracelet. *Note:* As this problem implies, you should not wear any metal objects when working in regions of strong magnetic fields.

58. GP Consider the apparatus shown in Figure P31.58 in which a conducting bar can be moved along two rails connected to a lightbulb. The whole system is immersed in a magnetic field of magnitude $B = 0.400$ T perpendicular and into the page. The distance between the horizontal rails is $\ell = 0.800$ m. The resistance of the lightbulb is $R = 48.0\ \Omega$, assumed to be constant. The bar and rails have negligible resistance. The bar is moved toward the right by a constant force of magnitude $F = 0.600$ N. We wish to find the maximum power delivered to the lightbulb. (a) Find an expression for the current in the lightbulb as a function of B, ℓ, R, and v, the speed of the bar. (b) When the maximum power is delivered to the lightbulb, what analysis model properly describes the moving bar? (c) Use the analysis model in part (b) to find a numerical value for the speed v of the bar when the maximum power is being delivered to the lightbulb. (d) Find the current in the lightbulb when maximum power is being delivered to it. (e) Using $P = I^2R$, what is the maximum power delivered to the lightbulb? (f) What is the maximum mechanical input power delivered to the bar by the force F? (g) We have assumed the resistance of the lightbulb is constant. In reality, as the power delivered to the lightbulb increases, the filament temperature increases and the resistance increases. Does the speed found in part (c) change if the resistance increases and all other quantities are held constant? (h) If so, does the speed found in part (c) increase or decrease? If not, explain. (i) With the assumption that the resistance of the lightbulb increases as the current increases, does the power found in part (f) change? (j) If so, is the power found in part (f) larger or smaller? If not, explain.

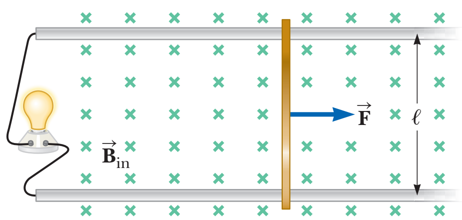

Figure P31.58

59. A guitar's steel string vibrates (see Fig. 31.5). The component of magnetic field perpendicular to the area of a pickup coil nearby is given by

$$B = 50.0 + 3.20 \sin 1\,046\pi t$$

where B is in milliteslas and t is in seconds. The circular pickup coil has 30 turns and radius 2.70 mm. Find the emf induced in the coil as a function of time.

60. *Why is the following situation impossible?* A conducting rectangular loop of mass $M = 0.100$ kg, resistance $R = 1.00\ \Omega$, and dimensions $w = 50.0$ cm by $\ell = 90.0$ cm is held with its lower edge just above a region with a uniform magnetic field of magnitude $B = 1.00$ T as shown in Figure P31.60. The loop is released from rest. Just as the top edge of the loop reaches the region containing the field, the loop moves with a speed 4.00 m/s.

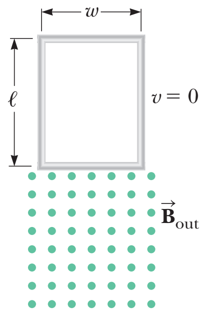

Figure P31.60

61. The circuit in Figure P31.61 is located in a magnetic field whose magnitude varies with time according to the expression $B = 1.00 \times 10^{-3}\ t$, where B is in teslas and t is in seconds. Assume the resistance per length of the wire is 0.100 Ω/m. Find the current in section PQ of length $a = 65.0$ cm.

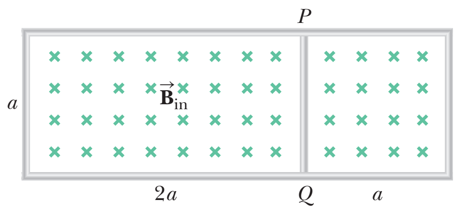

Figure P31.61

62. Magnetic field values are often determined by using a device known as a *search coil*. This technique depends on the measurement of the total charge passing through a coil in a time interval during which the magnetic flux linking the windings changes either because of the coil's motion or because of a change in the value of B. (a) Show that as the flux through the coil changes from Φ_1 to Φ_2, the charge transferred through the coil is given by $Q = N(\Phi_2 - \Phi_1)/R$, where R is the resistance of the coil and N is the number of turns. (b) As a specific example, calculate B when a total charge of 5.00×10^{-4} C passes through a 100-turn coil of resistance 200 Ω and cross-sectional area 40.0 cm^2 as it is rotated in a uniform field from a position where the plane of the coil is perpendicular to the field to a position where it is parallel to the field.

63. A conducting rod of length $\ell = 35.0$ cm is free to slide on two parallel conducting bars as shown in Figure P31.63. Two resistors $R_1 = 2.00\ \Omega$ and $R_2 = 5.00\ \Omega$ are connected across the ends of the bars to form a loop. A constant magnetic field $B = 2.50$ T is directed perpendicularly into the page. An external agent pulls the rod to the left with a constant speed of $v = 8.00$ m/s. Find (a) the currents in both resistors, (b) the total power delivered to the resistance of the circuit, and (c) the magnitude of the applied force that is needed to move the rod with this constant velocity.

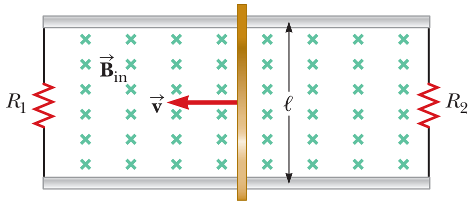

Figure P31.63

64. **Review.** A particle with a mass of 2.00×10^{-16} kg and a charge of 30.0 nC starts from rest, is accelerated through a potential difference ΔV, and is fired from a small source in a region containing a uniform, constant magnetic field of magnitude 0.600 T. The particle's velocity is perpendicular to the magnetic field lines. The circular orbit of the particle as it returns to the location of the source encloses a magnetic flux of 15.0 μWb. (a) Calculate the particle's speed. (b) Calculate the potential difference through which the particle was accelerated inside the source.

65. The plane of a square loop of wire with edge length $a = 0.200$ m is oriented vertically and along an east–west axis. The Earth's magnetic field at this point is of magnitude $B = 35.0\ \mu$T and is directed northward at 35.0° below the horizontal. The total resistance of the loop and the wires connecting it to a sensitive ammeter is 0.500 Ω. If the loop is suddenly collapsed by horizontal forces as shown in Figure P31.65, what total charge enters one terminal of the ammeter?

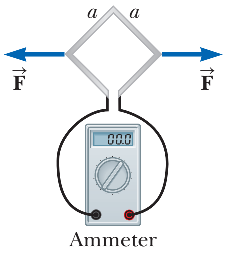

Figure P31.65

66. In Figure P31.66, the rolling axle, 1.50 m long, is pushed along horizontal rails at a constant speed $v = 3.00$ m/s. A resistor $R = 0.400\ \Omega$ is connected to the rails at points a and b, directly opposite each other.

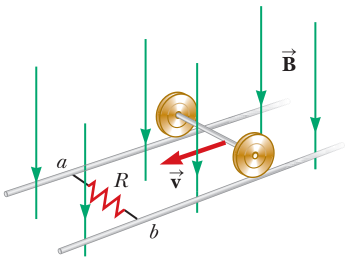

Figure P31.66

The wheels make good electrical contact with the rails, so the axle, rails, and R form a closed-loop circuit. The only significant resistance in the circuit is R. A uniform magnetic field $B = 0.080\ 0$ T is vertically downward. (a) Find the induced current I in the resistor. (b) What horizontal force F is required to keep the axle rolling at constant speed? (c) Which end of the resistor, a or b, is at the higher electric potential? (d) **What If?** After the axle rolls past the resistor, does the current in R reverse direction? Explain your answer.

67. Figure P31.67 shows a stationary conductor whose shape is similar to the letter e. The radius of its circular portion is $a = 50.0$ cm. It is placed in a constant magnetic field of 0.500 T directed out of the page. A straight conducting rod, 50.0 cm long, is pivoted about point O and rotates with a constant angular speed of 2.00 rad/s. (a) Determine the induced emf in the loop POQ. Note that the area of the loop is $\theta a^2/2$. (b) If all the conducting material has a resistance per length of 5.00 Ω/m, what is the induced current in the loop POQ at the instant 0.250 s after point P passes point Q?

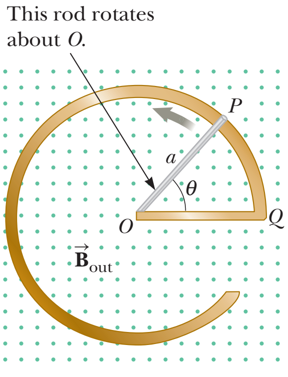

Figure P31.67

68. A conducting rod moves with a constant velocity in a direction perpendicular to a long, straight wire carrying a current I as shown in Figure P31.68. Show that the magnitude of the emf generated between the ends of the rod is

$$|\mathcal{E}| = \frac{\mu_0 v I \ell}{2\pi r}$$

In this case, note that the emf decreases with increasing r as you might expect.

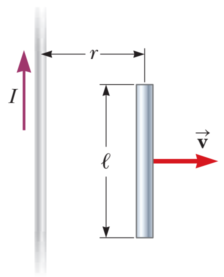

Figure P31.68

69. AMT A small, circular washer of radius $a = 0.500$ cm is held directly below a long, straight wire carrying a current of $I = 10.0$ A. The washer is located $h = 0.500$ m above the top of a table (Fig. P31.69). Assume the magnetic field is nearly constant over the area of the washer and equal to the magnetic field at the center of the washer. (a) If the washer is dropped from rest, what is the magnitude of the average induced emf in the washer over the time interval between its release and the moment it hits the tabletop? (b) What is the direction of the induced current in the washer?

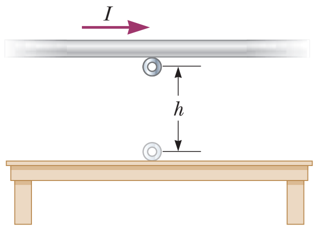

Figure P31.69

70. Figure P31.70 shows a compact, circular coil with 220 turns and radius 12.0 cm immersed in a uniform magnetic field parallel to the axis of the coil. The rate of change of the field has the constant magnitude 20.0 mT/s. (a) What additional information is necessary to determine whether the coil is carrying clockwise or counterclockwise current? (b) The coil overheats if more than 160 W of power is delivered to it. What resistance would the coil have at this critical point? (c) To run cooler, should it have lower resistance or higher resistance?

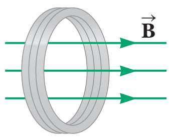

Figure P31.70

71. A rectangular coil of 60 turns, dimensions 0.100 m by 0.200 m, and total resistance 10.0 Ω rotates with angular speed 30.0 rad/s about the y axis in a region where a 1.00-T magnetic field is directed along the x axis. The time $t = 0$ is chosen to be at an instant when the plane of the coil is perpendicular to the direction of $\vec{B}$. Calculate (a) the maximum induced emf in the coil, (b) the maximum rate of change of magnetic flux through the coil, (c) the induced emf at $t = 0.050\ 0$ s, and (d) the torque exerted by the magnetic field on the coil at the instant when the emf is a maximum.

72. **Review.** In Figure P31.72, a uniform magnetic field decreases at a constant rate $dB/dt = -K$, where K is a positive constant. A circular loop of wire of radius a containing a resistance R and a capacitance C is placed

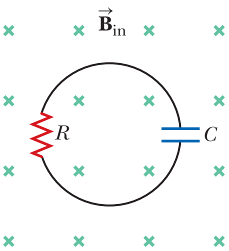

Figure P31.72

with its plane normal to the field. (a) Find the charge Q on the capacitor when it is fully charged. (b) Which plate, upper or lower, is at the higher potential? (c) Discuss the force that causes the separation of charges.

73. An N-turn square coil with side ℓ and resistance R is pulled to the right at constant speed v in the presence of a uniform magnetic field B acting perpendicular to the coil as shown in Figure P31.73. At $t = 0$, the right side of the coil has just departed the right edge of the field. At time t, the left side of the coil enters the region where $B = 0$. In terms of the quantities N, B, ℓ, v, and R, find symbolic expressions for (a) the magnitude of the induced emf in the loop during the time interval from $t = 0$ to t, (b) the magnitude of the induced current in the coil, (c) the power delivered to the coil, and (d) the force required to remove the coil from the field. (e) What is the direction of the induced current in the loop? (f) What is the direction of the magnetic force on the loop while it is being pulled out of the field?

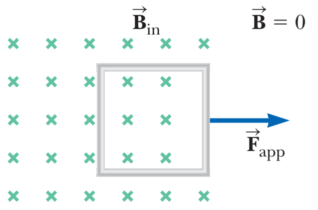

Figure P31.73

74. A conducting rod of length ℓ moves with velocity $\vec{\mathbf{v}}$ parallel to a long wire carrying a steady current I. The axis of the rod is maintained perpendicular to the wire with the near end a distance r away (Fig. P31.74). Show that the magnitude of the emf induced in the rod is

$$|\boldsymbol{\mathcal{E}}| = \frac{\mu_0 I v}{2\pi} \ln\left(1 + \frac{\ell}{r}\right)$$

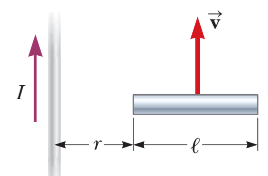

Figure P31.74

75. M The magnetic flux through a metal ring varies with time t according to $\Phi_B = at^3 - bt^2$, where Φ_B is in webers, $a = 6.00\ \text{s}^{-3}$, $b = 18.0\ \text{s}^{-2}$, and t is in seconds. The resistance of the ring is $3.00\ \Omega$. For the interval from $t = 0$ to $t = 2.00$ s, determine the maximum current induced in the ring.

76. A rectangular loop of dimensions ℓ and w moves with a constant velocity $\vec{\mathbf{v}}$ away from a long wire that carries a current I in the plane of the loop (Fig. P31.76). The total resistance of the loop is R. Derive an expression that gives the current in the loop at the instant the near side is a distance r from the wire.

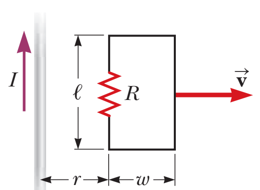

Figure P31.76

77. M A long, straight wire carries a current given by $I = I_{max} \sin(\omega t + \phi)$. The wire lies in the plane of a rectangular coil of N turns of wire as shown in Figure P31.77. The quantities I_{max}, ω, and ϕ are all constants. Assume $I_{max} = 50.0$ A, $\omega = 200\pi\ \text{s}^{-1}$, $N = 100$, $h = w = 5.00$ cm, and $L = 20.0$ cm. Determine the emf induced in the coil by the magnetic field created by the current in the straight wire.

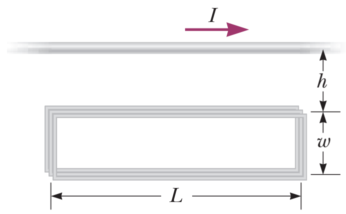

Figure P31.77

78. A thin wire $\ell = 30.0$ cm long is held parallel to and $d = 80.0$ cm above a long, thin wire carrying $I = 200$ A and fixed in position (Fig. P31.78). The 30.0-cm wire is released at the instant $t = 0$ and falls, remaining parallel to the current-carrying wire as it falls. Assume the falling wire accelerates at $9.80\ \text{m/s}^2$. (a) Derive an equation for the emf induced in it as a function of time. (b) What is the minimum value of the emf? (c) What is the maximum value? (d) What is the induced emf 0.300 s after the wire is released?

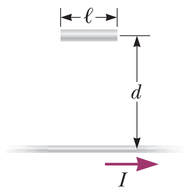

Figure P31.78

Challenge Problems

79. Two infinitely long solenoids (seen in cross section) pass through a circuit as shown in Figure P31.79. The magnitude of $\vec{\mathbf{B}}$ inside each is the same and is increasing at the rate of 100 T/s. What is the current in each resistor?

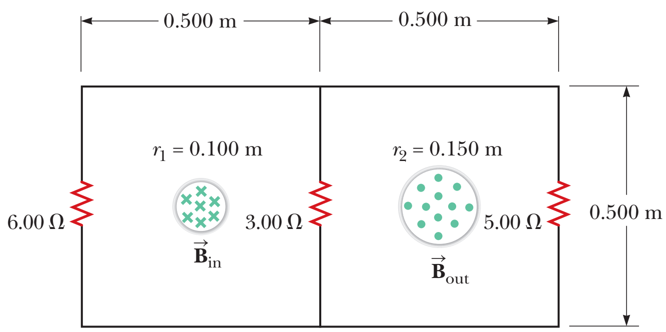

Figure P31.79

80. An *induction furnace* uses electromagnetic induction to produce eddy currents in a conductor, thereby raising the conductor's temperature. Commercial units operate at frequencies ranging from 60 Hz to about 1 MHz and deliver powers from a few watts to several megawatts. Induction heating can be used for warming a metal pan on a kitchen stove. It can be used to avoid oxidation and contamination of the metal when welding in a vacuum enclosure. To explore induction heating, consider a flat conducting disk of radius R, thickness b, and resistivity ρ. A sinusoidal magnetic field $B_{max} \cos \omega t$ is applied perpendicular to the disk. Assume the eddy currents occur in circles concentric with the disk. (a) Calculate the average power delivered to the disk. (b) **What If?** By what factor does the power change when the amplitude of the field doubles? (c) When the frequency doubles? (d) When the radius of the disk doubles?

81. A bar of mass m and resistance R slides without friction in a horizontal plane, moving on parallel rails as shown in Figure P31.81. The rails are separated by a distance d. A battery that maintains a constant emf $\mathcal{E}$ is connected between the rails, and a constant magnetic field $\vec{\mathbf{B}}$ is directed perpendicularly out of the page. Assuming the bar starts from rest at time $t = 0$, show that at time t it moves with a speed

$$v = \frac{\mathcal{E}}{Bd}\left(1 - e^{-B^2 d^2 t/mR}\right)$$

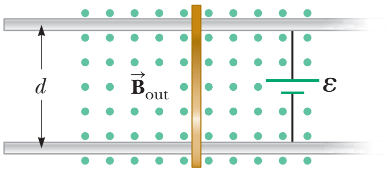

Figure P31.81

82. A *betatron* is a device that accelerates electrons to energies in the MeV range by means of electromagnetic induction. Electrons in a vacuum chamber are held in a circular orbit by a magnetic field perpendicular to the orbital plane. The magnetic field is gradually increased to induce an electric field around the orbit. (a) Show that the electric field is in the correct direction to make the electrons speed up. (b) Assume the radius of the orbit remains constant. Show that the average magnetic field over the area enclosed by the orbit must be twice as large as the magnetic field at the circle's circumference.

83. **Review.** The bar of mass m in Figure P31.83 is pulled horizontally across parallel, frictionless rails by a massless string that passes over a light, frictionless pulley and is attached to a suspended object of mass M. The uniform upward magnetic field has a magnitude B, and the distance between the rails is ℓ. The only significant electrical resistance is the load resistor R shown connecting the rails at one end. Assuming the suspended object is released with the bar at rest at $t = 0$, derive an expression that gives the bar's horizontal speed as a function of time.

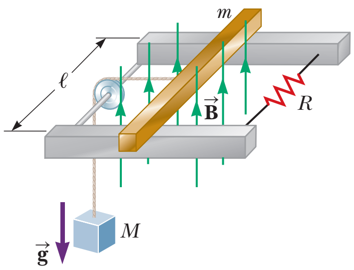

Figure P31.83

CHAPTER

32

Inductance

A treasure hunter uses a metal detector to search for buried objects at a beach. At the end of the metal detector is a coil of wire that is part of a circuit. When the coil comes near a metal object, the inductance of the coil is affected and the current in the circuit changes. This change triggers a signal in the earphones worn by the treasure hunter. We investigate inductance in this chapter. *(Andy Ryan/Stone/Getty Images)*

In Chapter 31, we saw that an emf and a current are induced in a loop of wire when the magnetic flux through the area enclosed by the loop changes with time. This phenomenon of electromagnetic induction has some practical consequences. In this chapter, we first describe an effect known as *self-induction,* in which a time-varying current in a circuit produces an induced emf opposing the emf that initially set up the time-varying current. Self-induction is the basis of the *inductor,* an electrical circuit element. We discuss the energy stored in the magnetic field of an inductor and the energy density associated with the magnetic field.

Next, we study how an emf is induced in a coil as a result of a changing magnetic flux produced by a second coil, which is the basic principle of *mutual induction.* Finally, we examine the characteristics of circuits that contain inductors, resistors, and capacitors in various combinations.

32.1 Self-Induction and Inductance

In this chapter, we need to distinguish carefully between emfs and currents that are caused by physical sources such as batteries and those that are induced by changing magnetic fields. When we use a term (such as *emf* or *current*) without an adjective, we are describing the parameters associated with a physical source. We use the adjective *induced* to describe those emfs and currents caused by a changing magnetic field.

Consider a circuit consisting of a switch, a resistor, and a source of emf as shown in Figure 32.1. The circuit diagram is represented in perspective to show the orientations of some of the magnetic field lines due to the current in the circuit. When the switch is thrown to its closed position, the current does not immediately jump from zero to its maximum value $\mathcal{E}/R$. Faraday's law of electromagnetic induction (Eq. 31.1) can be used to describe this effect as follows. As the current increases with time, the magnetic field lines surrounding the wires pass through the loop represented by the circuit itself. This magnetic field passing through the loop causes a magnetic flux through the loop. This increasing flux creates an induced emf in the circuit. The direction of the induced emf is such that it would cause an induced current in the loop (if the loop did not already carry a current), which would establish a magnetic field opposing the change in the original magnetic field. Therefore, the direction of the induced emf is opposite the direction of the emf of the battery, which results in a gradual rather than instantaneous increase in the current to its final equilibrium value. Because of the direction of the induced emf, it is also called a *back emf,* similar to that in a motor as discussed in Chapter 31. This effect is called **self-induction** because the changing flux through the circuit and the resultant induced emf arise from the circuit itself. The emf $\mathcal{E}_L$ set up in this case is called a **self-induced emf.**

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Joseph Henry
American Physicist (1797–1878)
Henry became the first director of the Smithsonian Institution and first president of the Academy of Natural Science. He improved the design of the electromagnet and constructed one of the first motors. He also discovered the phenomenon of self-induction, but he failed to publish his findings. The unit of inductance, the henry, is named in his honor.

To obtain a quantitative description of self-induction, recall from Faraday's law that the induced emf is equal to the negative of the time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field, which in turn is proportional to the current in the circuit. Therefore, a self-induced emf is always proportional to the time rate of change of the current. For any loop of wire, we can write this proportionality as

$$\mathcal{E}_L = -L\frac{di}{dt} \qquad (32.1)$$

where L is a proportionality constant—called the **inductance** of the loop—that depends on the geometry of the loop and other physical characteristics. If we consider a closely spaced coil of N turns (a toroid or an ideal solenoid) carrying a current i and containing N turns, Faraday's law tells us that $\mathcal{E}_L = -N\,d\Phi_B/dt$. Combining this expression with Equation 32.1 gives

Inductance of an *N*-turn coil

$$L = \frac{N\Phi_B}{i} \qquad (32.2)$$

where it is assumed the same magnetic flux passes through each turn and L is the inductance of the entire coil.

From Equation 32.1, we can also write the inductance as the ratio

$$L = -\frac{\mathcal{E}_L}{di/dt} \qquad (32.3)$$

Recall that resistance is a measure of the opposition to current as given by Equation 27.7, $R = \Delta V/I$; in comparison, Equation 32.3, being of the same mathematical form as Equation 27.7, shows us that inductance is a measure of the opposition to a *change* in current.

The SI unit of inductance is the **henry** (H), which as we can see from Equation 32.3 is 1 volt-second per ampere: $1\text{ H} = 1\text{ V}\cdot\text{s/A}$.

As shown in Example 32.1, the inductance of a coil depends on its geometry. This dependence is analogous to the capacitance of a capacitor depending on the geometry of its plates as we found in Equation 26.3 and the resistance of a resistor depending on the length and area of the conducting material in Equation 27.10. Inductance calculations can be quite difficult to perform for complicated geometries, but the examples below involve simple situations for which inductances are easily evaluated.

After the switch is closed, the current produces a magnetic flux through the area enclosed by the loop. As the current increases toward its equilibrium value, this magnetic flux changes in time and induces an emf in the loop.

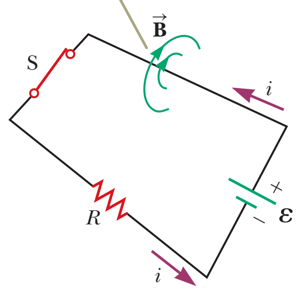


Figure 32.1 Self-induction in a simple circuit.

Quick Quiz 32.1 A coil with zero resistance has its ends labeled a and b. The potential at a is higher than at b. Which of the following could be consistent with this situation? **(a)** The current is constant and is directed from a to b. **(b)** The current is constant and is directed from b to a. **(c)** The current is increasing and is directed from a to b. **(d)** The current is decreasing and is directed from a to b. **(e)** The current is increasing and is directed from b to a. **(f)** The current is decreasing and is directed from b to a.

Example 32.1 Inductance of a Solenoid

Consider a uniformly wound solenoid having N turns and length ℓ. Assume ℓ is much longer than the radius of the windings and the core of the solenoid is air.

(A) Find the inductance of the solenoid.

SOLUTION

Conceptualize The magnetic field lines from each turn of the solenoid pass through all the turns, so an induced emf in each coil opposes changes in the current.

Categorize We categorize this example as a substitution problem. Because the solenoid is long, we can use the results for an ideal solenoid obtained in Chapter 30.

Find the magnetic flux through each turn of area A in the solenoid, using the expression for the magnetic field from Equation 30.17:

$$\Phi_B = BA = \mu_0 niA = \mu_0 \frac{N}{\ell} iA$$

Substitute this expression into Equation 32.2:

$$L = \frac{N\Phi_B}{i} = \boxed{\mu_0 \frac{N^2}{\ell} A} \qquad \textbf{(32.4)}$$

(B) Calculate the inductance of the solenoid if it contains 300 turns, its length is 25.0 cm, and its cross-sectional area is 4.00 cm^2.

SOLUTION

Substitute numerical values into Equation 32.4:

$$L = (4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}) \frac{300^2}{25.0 \times 10^{-2}\ \mathrm{m}} (4.00 \times 10^{-4}\ \mathrm{m^2})$$

$$= 1.81 \times 10^{-4}\ \mathrm{T \cdot m^2/A} = \boxed{0.181\ \mathrm{mH}}$$

(C) Calculate the self-induced emf in the solenoid if the current it carries decreases at the rate of 50.0 A/s.

SOLUTION

Substitute $di/dt = -50.0$ A/s and the answer to part (B) into Equation 32.1:

$$\mathcal{E}_L = -L\frac{di}{dt} = -(1.81 \times 10^{-4}\ \mathrm{H})(-50.0\ \mathrm{A/s})$$

$$= \boxed{9.05\ \mathrm{mV}}$$

The result for part (A) shows that L depends on geometry and is proportional to the square of the number of turns. Because $N = n\ell$, we can also express the result in the form

$$L = \mu_0 \frac{(n\ell)^2}{\ell} A = \mu_0 n^2 A\ell = \mu_0 n^2 V \qquad \textbf{(32.5)}$$

where $V = A\ell$ is the interior volume of the solenoid.

32.2 *RL* Circuits

If a circuit contains a coil such as a solenoid, the inductance of the coil prevents the current in the circuit from increasing or decreasing instantaneously. A circuit element that has a large inductance is called an **inductor** and has the circuit symbol —ᴕᴕᴕ—. We always assume the inductance of the remainder of a circuit is negligible compared with that of the inductor. Keep in mind, however, that even a circuit without a coil has some inductance that can affect the circuit's behavior.

Because the inductance of an inductor results in a back emf, an inductor in a circuit opposes changes in the current in that circuit. The inductor attempts to keep the current the same as it was before the change occurred. If the battery voltage in the circuit is increased so that the current rises, the inductor opposes this change and the rise is not instantaneous. If the battery voltage is decreased, the inductor causes a slow drop in the current rather than an immediate drop. Therefore, the inductor causes the circuit to be "sluggish" as it reacts to changes in the voltage.

Consider the circuit shown in Figure 32.2, which contains a battery of negligible internal resistance. This circuit is an ***RL* circuit** because the elements connected to the battery are a resistor and an inductor. The curved lines on switch S_2 suggest this switch can never be open; it is always set to either a or b. (If the switch is connected to neither a nor b, any current in the circuit suddenly stops.) Suppose S_2 is set to a and switch S_1 is open for $t < 0$ and then thrown closed at $t = 0$. The current in the circuit begins to increase, and a back emf (Eq. 32.1) that opposes the increasing current is induced in the inductor.

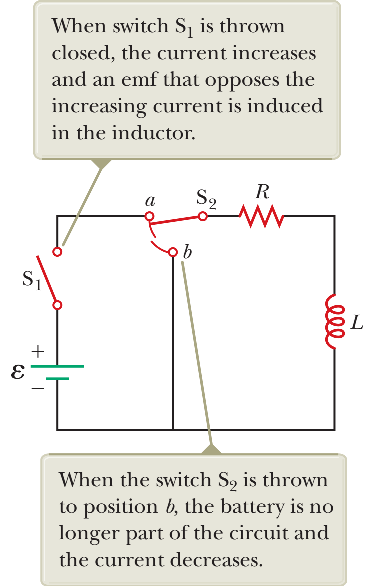


Figure 32.2 An *RL* circuit. When switch S_2 is in position a, the battery is in the circuit.

With this point in mind, let's apply Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction:

$$\mathcal{E} - iR - L\frac{di}{dt} = 0 \tag{32.6}$$

where iR is the voltage drop across the resistor. (Kirchhoff's rules were developed for circuits with steady currents, but they can also be applied to a circuit in which the current is changing if we imagine them to represent the circuit at one *instant* of time.) Now let's find a solution to this differential equation, which is similar to that for the *RC* circuit (see Section 28.4).

A mathematical solution of Equation 32.6 represents the current in the circuit as a function of time. To find this solution, we change variables for convenience, letting $x = (\mathcal{E}/R) - i$, so $dx = -di$. With these substitutions, Equation 32.6 becomes

$$x + \frac{L}{R}\frac{dx}{dt} = 0$$

Rearranging and integrating this last expression gives

$$\int_{x_0}^{x} \frac{dx}{x} = -\frac{R}{L}\int_0^t dt$$

$$\ln\frac{x}{x_0} = -\frac{R}{L}t$$

where x_0 is the value of x at time $t = 0$. Taking the antilogarithm of this result gives

$$x = x_0 e^{-Rt/L}$$

Because $i = 0$ at $t = 0$, note from the definition of x that $x_0 = \mathcal{E}/R$. Hence, this last expression is equivalent to

$$\frac{\mathcal{E}}{R} - i = \frac{\mathcal{E}}{R}e^{-Rt/L}$$

$$i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

This expression shows how the inductor affects the current. The current does not increase instantly to its final equilibrium value when the switch is closed, but instead increases according to an exponential function. If the inductance is removed from the circuit, which corresponds to letting L approach zero, the exponential term

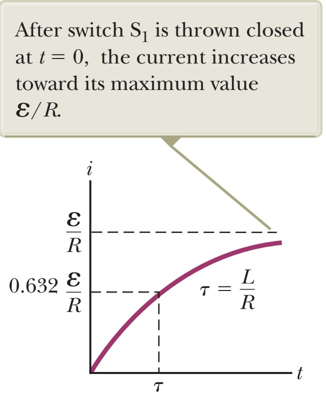


Figure 32.3 Plot of the current versus time for the RL circuit shown in Figure 32.2. The time constant τ is the time interval required for i to reach 63.2% of its maximum value.

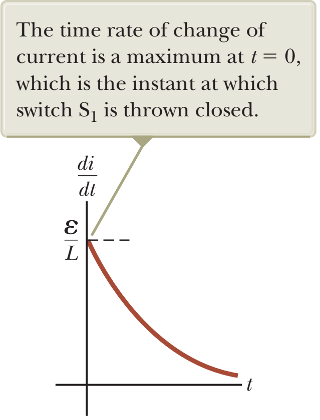


Figure 32.4 Plot of di/dt versus time for the RL circuit shown in Figure 32.2. The rate decreases exponentially with time as i increases toward its maximum value.

becomes zero and there is no time dependence of the current in this case; the current increases instantaneously to its final equilibrium value in the absence of the inductance.

We can also write this expression as

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \tag{32.7}$$

where the constant τ is the **time constant** of the RL circuit:

$$\tau = \frac{L}{R} \tag{32.8}$$

Physically, τ is the time interval required for the current in the circuit to reach $(1 - e^{-1}) = 0.632 = 63.2\%$ of its final value $\mathcal{E}/R$. The time constant is a useful parameter for comparing the time responses of various circuits.

Figure 32.3 shows a graph of the current versus time in the RL circuit. Notice that the equilibrium value of the current, which occurs as t approaches infinity, is $\mathcal{E}/R$. That can be seen by setting di/dt equal to zero in Equation 32.6 and solving for the current i. (At equilibrium, the change in the current is zero.) Therefore, the current initially increases very rapidly and then gradually approaches the equilibrium value $\mathcal{E}/R$ as t approaches infinity.

Let's also investigate the time rate of change of the current. Taking the first time derivative of Equation 32.7 gives

$$\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau} \tag{32.9}$$

This result shows that the time rate of change of the current is a maximum (equal to $\mathcal{E}/L$) at $t = 0$ and falls off exponentially to zero as t approaches infinity (Fig. 32.4).

Now consider the RL circuit in Figure 32.2 again. Suppose switch S_2 has been set at position a long enough (and switch S_1 remains closed) to allow the current to reach its equilibrium value $\mathcal{E}/R$. In this situation, the circuit is described by the outer loop in Figure 32.2. If S_2 is thrown from a to b, the circuit is now described by only the right-hand loop in Figure 32.2. Therefore, the battery has been eliminated from the circuit. Setting $\mathcal{E} = 0$ in Equation 32.6 gives

$$iR + L\frac{di}{dt} = 0$$

It is left as a problem (Problem 22) to show that the solution of this differential equation is

$$i = \frac{\mathcal{E}}{R} e^{-t/\tau} = I_i e^{-t/\tau} \tag{32.10}$$

where $\mathcal{E}$ is the emf of the battery and $I_i = \mathcal{E}/R$ is the initial current at the instant the switch is thrown to b.

If the circuit did not contain an inductor, the current would immediately decrease to zero when the battery is removed. When the inductor is present, it opposes the decrease in the current and causes the current to decrease exponentially. A graph of the current in the circuit versus time (Fig. 32.5) shows that the current is continuously decreasing with time.

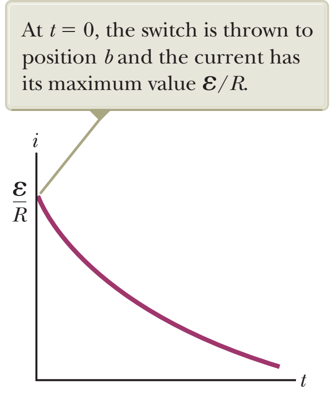


Figure 32.5 Current versus time for the right-hand loop of the circuit shown in Figure 32.2. For $t < 0$, switch S_2 is at position a.

Quick Quiz 32.2 Consider the circuit in Figure 32.2 with S_1 open and S_2 at position a. Switch S_1 is now thrown closed. **(i)** At the instant it is closed, across which circuit element is the voltage equal to the emf of the battery? (a) the resistor (b) the inductor (c) both the inductor and resistor **(ii)** After a very long time, across which circuit element is the voltage equal to the emf of the battery? Choose from among the same answers.

Example 32.2 Time Constant of an *RL* Circuit

Consider the circuit in Figure 32.2 again. Suppose the circuit elements have the following values: $\mathcal{E}$ = 12.0 V, R = 6.00 Ω, and L = 30.0 mH.

(A) Find the time constant of the circuit.

SOLUTION

Conceptualize You should understand the operation and behavior of the circuit in Figure 32.2 from the discussion in this section.

Categorize We evaluate the results using equations developed in this section, so this example is a substitution problem.

Evaluate the time constant from Equation 32.8:

$$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3}\ \text{H}}{6.00\ \Omega} = \boxed{5.00\ \text{ms}}$$

(B) Switch S_2 is at position a, and switch S_1 is thrown closed at $t = 0$. Calculate the current in the circuit at $t = 2.00$ ms.

SOLUTION

Evaluate the current at $t = 2.00$ ms from Equation 32.7:

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{12.0\ \text{V}}{6.00\ \Omega}(1 - e^{-2.00\ \text{ms}/5.00\ \text{ms}}) = 2.00\ \text{A}\ (1 - e^{-0.400})$$
$$= \boxed{0.659\ \text{A}}$$

(C) Compare the potential difference across the resistor with that across the inductor.

SOLUTION

At the instant the switch is closed, there is no current and therefore no potential difference across the resistor. At this instant, the battery voltage appears entirely across the inductor in the form of a back emf of 12.0 V as the inductor tries to maintain the zero-current condition. (The top end of the inductor in Fig. 32.2 is at a higher electric potential than the bottom end.) As time passes, the emf across the inductor decreases and the current in the resistor (and hence the voltage across it) increases as shown in Figure 32.6 (page 976). The sum of the two voltages at all times is 12.0 V.

WHAT IF? In Figure 32.6, the voltages across the resistor and inductor are equal at 3.4 ms. What if you wanted to delay the condition in which the voltages are equal to some later instant, such as $t = 10.0$ ms? Which parameter, L or R, would require the least adjustment, in terms of a percentage change, to achieve that?

continued

Answer Figure 32.6 shows that the voltages are equal when the voltage across the inductor has fallen to half its original value. Therefore, the time interval required for the voltages to become equal is the *half-life* $_{1/2}$ of the decay. We introduced the half-life in the What If? section of Example 28.10 to describe the exponential decay in circuits, where $_{1/2}$ 0.693

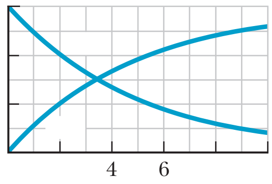


Figure 32.6 (Example 32.2) The time behavior of the voltages across the resistor and inductor in Figure 32.2 given the values provided in this example.

From the desired half-life of 10.0 ms, use the result from Example 28.10 to find the time constant of the circuit:

$$\tau = \frac{}{0.693} \quad \frac{10.0 \text{ ms}}{0.693} \quad 14.4 \text{ ms}$$

Hold fixed and find the value of that gives this time constant:

$$\tau = - \qquad - \quad \frac{30.0 \quad 10 \quad \text{H}}{14.4 \text{ ms}} \quad 2.08$$

Now hold fixed and find the appropriate value of

$$\tau = - \qquad = \tau \qquad 14.4 \text{ ms})(6.00 \qquad 86.4 \quad 10 \quad \text{H}$$

The change in corresponds to a 65% decrease compared with the initial resistance. The change in represents a 188% increase in inductance! Therefore, a much smaller percentage adjustment in can achieve the desired effect than would an adjustment in

Pitfall Prevention 32.1

Capacitors, Resistors, and Inductors Store Energy Differently
Different energy-storage mechanisms are at work in capacitors, inductors, and resistors. A charged capacitor stores energy as electrical potential energy. An inductor stores energy as what we could call magnetic potential energy when it carries current. Energy delivered to a resistor is transformed to internal energy.

32.3 Energy in a Magnetic Field

A battery in a circuit containing an inductor must provide more energy than one in a circuit without the inductor. Consider Figure 32.2 with switch S in position When switch S is thrown closed, part of the energy supplied by the battery appears as internal energy in the resistance in the circuit, and the remaining energy is stored in the magnetic field of the inductor. Multiplying each term in Equation 32.6 by and rearranging the expression gives

$$Li\frac{di}{dt} \tag{32.11}$$

Recognizing as the rate at which energy is supplied by the battery and as the rate at which energy is delivered to the resistor, we see that di) must represent the rate at which energy is being stored in the inductor. If is the energy stored in the inductor at any time, we can write the rate at which energy is stored as

$$\frac{dU}{dt} \qquad Li\frac{di}{dt}$$

To find the total energy stored in the inductor at any instant, let's rewrite this expression as $Li\,di$ and integrate:

$$dU \qquad Li\,di \qquad di$$

Energy stored in an inductor

$$-Li \tag{32.12}$$

where is constant and has been removed from the integral. Equation 32.12 represents the energy stored in the magnetic field of the inductor when the current is It is similar in form to Equation 26.11 for the energy stored in the electric field of a capacitor, – . In either case, energy is required to establish a field.

We can also determine the energy density of a magnetic field. For simplicity, consider a solenoid whose inductance is given by Equation 32.5:

The magnetic field of a solenoid is given by Equation 30.17:

$$B = \mu_0 n i$$

Substituting the expression for L and $i = B/\mu_0 n$ into Equation 32.12 gives

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}\mu_0 n^2 V\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}V \tag{32.13}$$

The magnetic energy density, or the energy stored per unit volume in the magnetic field of the inductor, is $u_B = U_B/V$, or

$$u_B = \frac{B^2}{2\mu_0} \tag{32.14}$$

◀ **Magnetic energy density**

Although this expression was derived for the special case of a solenoid, it is valid for any region of space in which a magnetic field exists. Equation 32.14 is similar in form to Equation 26.13 for the energy per unit volume stored in an electric field, $u_E = \frac{1}{2}\epsilon_0 E^2$. In both cases, the energy density is proportional to the square of the field magnitude.

Quick Quiz 32.3 You are performing an experiment that requires the highest-possible magnetic energy density in the interior of a very long current-carrying solenoid. Which of the following adjustments increases the energy density? (More than one choice may be correct.) **(a)** increasing the number of turns per unit length on the solenoid **(b)** increasing the cross-sectional area of the solenoid **(c)** increasing only the length of the solenoid while keeping the number of turns per unit length fixed **(d)** increasing the current in the solenoid

Example 32.3 What Happens to the Energy in the Inductor? AM

Consider once again the RL circuit shown in Figure 32.2, with switch S_2 at position a and the current having reached its steady-state value. When S_2 is thrown to position b, the current in the right-hand loop decays exponentially with time according to the expression $i = I_i e^{-t/\tau}$, where $I_i = \mathcal{E}/R$ is the initial current in the circuit and $\tau = L/R$ is the time constant. Show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

SOLUTION

Conceptualize Before S_2 is thrown to b, energy is being delivered at a constant rate to the resistor from the battery and energy is stored in the magnetic field of the inductor. After $t = 0$, when S_2 is thrown to b, the battery can no longer provide energy and energy is delivered to the resistor only from the inductor.

Categorize We model the right-hand loop of the circuit as an *isolated system* so that energy is transferred between components of the system but does not leave the system.

Analyze We begin by evaluating the energy delivered to the resistor, which appears as internal energy in the resistor.

Begin with Equation 27.22 and recognize that the rate of change of internal energy in the resistor is the power delivered to the resistor:

$$\frac{dE_{\text{int}}}{dt} = P = i^2 R$$

Substitute the current given by Equation 32.10 into this equation:

$$\frac{dE_{\text{int}}}{dt} = i^2 R = (I_i e^{-Rt/L})^2 R = I_i^2 R e^{-2Rt/L}$$

Solve for dE_{int} and integrate this expression over the limits $t = 0$ to $t \to \infty$:

$$E_{\text{int}} = \int_0^\infty I_i^2 R e^{-2Rt/L}\,dt = I_i^2 R \int_0^\infty e^{-2Rt/L}\,dt$$

The value of the definite integral can be shown to be $L/2R$ (see Problem 36). Use this result to evaluate E_{int}:

$$E_{\text{int}} = I_i^2 R\left(\frac{L}{2R}\right) = \boxed{\tfrac{1}{2}LI_i^2}$$

continued

32.3 continued

Finalize This result is equal to the initial energy stored in the magnetic field of the inductor, given by Equation 32.12, as we set out to prove.

Example 32.4 The Coaxial Cable

Coaxial cables are often used to connect electrical devices, such as your video system, and in receiving signals in television cable systems. Model a long coaxial cable as a thin, cylindrical conducting shell of radius b concentric with a solid cylinder of radius a as in Figure 32.7. The conductors carry the same current I in opposite directions. Calculate the inductance L of a length ℓ of this cable.

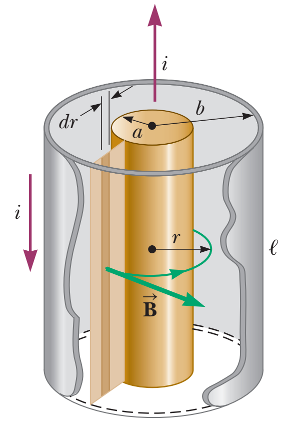


Figure 32.7 (Example 32.4) Section of a long coaxial cable. The inner and outer conductors carry equal currents in opposite directions.

SOLUTION

Conceptualize Consider Figure 32.7. Although we do not have a visible coil in this geometry, imagine a thin, radial slice of the coaxial cable such as the light gold rectangle in Figure 32.7. If the inner and outer conductors are connected at the ends of the cable (above and below the figure), this slice represents one large conducting loop. The current in the loop sets up a magnetic field between the inner and outer conductors that passes through this loop. If the current changes, the magnetic field changes and the induced emf opposes the original change in the current in the conductors.

Categorize We categorize this situation as one in which we must return to the fundamental definition of inductance, Equation 32.2.

Analyze We must find the magnetic flux through the light gold rectangle in Figure 32.7. Ampère's law (see Section 30.3) tells us that the magnetic field in the region between the conductors is due to the inner conductor alone and that its magnitude is $B = \mu_0 i/2\pi r$, where r is measured from the common center of the cylinders. A sample circular field line is shown in Figure 32.7, along with a field vector tangent to the field line. The magnetic field is zero outside the outer shell because the net current passing through the area enclosed by a circular path surrounding the cable is zero; hence, from Ampère's law, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0$.

The magnetic field is perpendicular to the light gold rectangle of length ℓ and width $b - a$, the cross section of interest. Because the magnetic field varies with radial position across this rectangle, we must use calculus to find the total magnetic flux.

Divide the light gold rectangle into strips of width dr such as the darker strip in Figure 32.7. Evaluate the magnetic flux through such a strip:

$$d\Phi_B = B\,dA = B\ell\,dr$$

Substitute for the magnetic field and integrate over the entire light gold rectangle:

$$\Phi_B = \int_a^b \frac{\mu_0 i}{2\pi r}\ell\,dr = \frac{\mu_0 i\ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 i\ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Use Equation 32.2 to find the inductance of the cable:

$$L = \frac{\Phi_B}{i} = \boxed{\frac{\mu_0\ell}{2\pi}\ln\left(\frac{b}{a}\right)}$$

Finalize The inductance depends only on geometric factors related to the cable. It increases if ℓ increases, if b increases, or if a decreases. This result is consistent with our conceptualization: any of these changes increases the size of the loop represented by our radial slice and through which the magnetic field passes, increasing the inductance.

32.4 Mutual Inductance

Very often, the magnetic flux through the area enclosed by a circuit varies with time because of time-varying currents in nearby circuits. This condition induces an emf through a process known as *mutual induction,* so named because it depends on the interaction of two circuits.

Consider the two closely wound coils of wire shown in cross-sectional view in Figure 32.8. The current i_1 in coil 1, which has N_1 turns, creates a magnetic field. Some of the magnetic field lines pass through coil 2, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through coil 2 is represented by Φ_{12}. In analogy to Equation 32.2, we can identify the **mutual inductance** M_{12} of coil 2 with respect to coil 1:

$$M_{12} = \frac{N_2\Phi_{12}}{i_1} \tag{32.15}$$

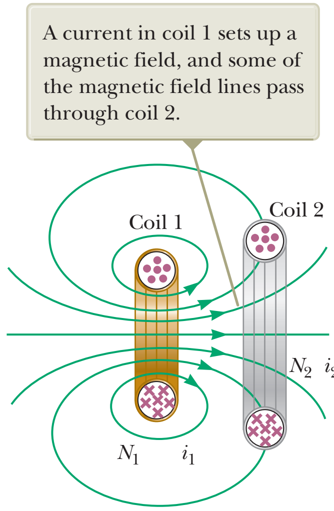


Figure 32.8 A cross-sectional view of two adjacent coils.

Mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other. As the circuit separation distance increases, the mutual inductance decreases because the flux linking the circuits decreases.

If the current i_1 varies with time, we see from Faraday's law and Equation 32.15 that the emf induced by coil 1 in coil 2 is

$$\boldsymbol{\mathcal{E}}_2 = -N_2\frac{d\Phi_{12}}{dt} = -N_2\frac{d}{dt}\left(\frac{M_{12}i_1}{N_2}\right) = -M_{12}\frac{di_1}{dt} \tag{32.16}$$

In the preceding discussion, it was assumed the current is in coil 1. Let's also imagine a current i_2 in coil 2. The preceding discussion can be repeated to show that there is a mutual inductance M_{21}. If the current i_2 varies with time, the emf induced by coil 2 in coil 1 is

$$\boldsymbol{\mathcal{E}}_1 = -M_{21}\frac{di_2}{dt} \tag{32.17}$$

In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing. Although the proportionality constants M_{12} and M_{21} have been treated separately, it can be shown that they are equal. Therefore, with $M_{12} = M_{21} = M$, Equations 32.16 and 32.17 become

$$\boldsymbol{\mathcal{E}}_2 = -M\frac{di_1}{dt} \quad \text{and} \quad \boldsymbol{\mathcal{E}}_1 = -M\frac{di_2}{dt}$$

These two equations are similar in form to Equation 32.1 for the self-induced emf $\boldsymbol{\mathcal{E}} = -L\,(di/dt)$. The unit of mutual inductance is the henry.

Quick Quiz 32.4 In Figure 32.8, coil 1 is moved closer to coil 2, with the orientation of both coils remaining fixed. Because of this movement, the mutual induction of the two coils **(a)** increases, **(b)** decreases, or **(c)** is unaffected.

Example 32.5 "Wireless" Battery Charger

An electric toothbrush has a base designed to hold the toothbrush handle when not in use. As shown in Figure 32.9a, the handle has a cylindrical hole that fits loosely over a matching cylinder on the base. When the handle is placed on the base, a changing current in a solenoid inside the base cylinder induces a current in a coil inside the handle. This induced current charges the battery in the handle.

We can model the base as a solenoid of length ℓ with N_B turns (Fig. 32.9b), carrying a current i, and having a cross-sectional area A. The handle coil contains N_H turns and completely surrounds the base coil. Find the mutual inductance of the system.

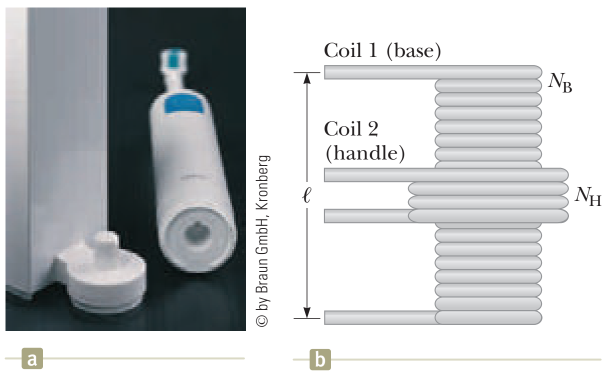


Figure 32.9 (Example 32.5) (a) This electric toothbrush uses the mutual induction of solenoids as part of its battery-charging system. (b) A coil of N_H turns wrapped around the center of a solenoid of N_B turns.

continued

32.5 continued

SOLUTION

Conceptualize Be sure you can identify the two coils in the situation and understand that a changing current in one coil induces a current in the second coil.

Categorize We will determine the result using concepts discussed in this section, so we categorize this example as a substitution problem.

Use Equation 30.17 to express the magnetic field in the interior of the base solenoid:

$$B = \mu_0 \frac{N_B}{\ell} i$$

Find the mutual inductance, noting that the magnetic flux Φ_{BH} through the handle's coil caused by the magnetic field of the base coil is BA:

$$M = \frac{N_H \Phi_{BH}}{i} = \frac{N_H BA}{i} = \boxed{\mu_0 \frac{N_B N_H}{\ell} A}$$

Wireless charging is used in a number of other "cordless" devices. One significant example is the inductive charging used by some manufacturers of electric cars that avoids direct metal-to-metal contact between the car and the charging apparatus.

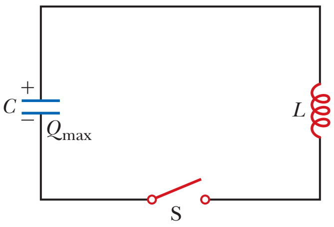


Figure 32.10 A simple *LC* circuit. The capacitor has an initial charge Q_{max}, and the switch is open for $t < 0$ and then closed at $t = 0$.

32.5 Oscillations in an *LC* Circuit

When a capacitor is connected to an inductor as illustrated in Figure 32.10, the combination is an ***LC* circuit.** If the capacitor is initially charged and the switch is then closed, both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy. In the following analysis, the resistance in the circuit is neglected. We also assume an idealized situation in which energy is not radiated away from the circuit. This radiation mechanism is discussed in Chapter 34.

Assume the capacitor has an initial charge Q_{max} (the maximum charge) and the switch is open for $t < 0$ and then closed at $t = 0$. Let's investigate what happens from an energy viewpoint.

When the capacitor is fully charged, the energy U in the circuit is stored in the capacitor's electric field and is equal to $Q_{max}^2/2C$ (Eq. 26.11). At this time, the current in the circuit is zero; therefore, no energy is stored in the inductor. After the switch is closed, the rate at which charges leave or enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. After the switch is closed and the capacitor begins to discharge, the energy stored in its electric field decreases. The capacitor's discharge represents a current in the circuit, and some energy is now stored in the magnetic field of the inductor. Therefore, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value and all the energy in the circuit is stored in the inductor. The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity. This process is followed by another discharge until the circuit returns to its original state of maximum charge Q_{max} and the plate polarity shown in Figure 32.10. The energy continues to oscillate between inductor and capacitor.

The oscillations of the *LC* circuit are an electromagnetic analog to the mechanical oscillations of the particle in simple harmonic motion studied in Chapter 15. Much of what was discussed there is applicable to *LC* oscillations. For example, we investigated the effect of driving a mechanical oscillator with an external force,

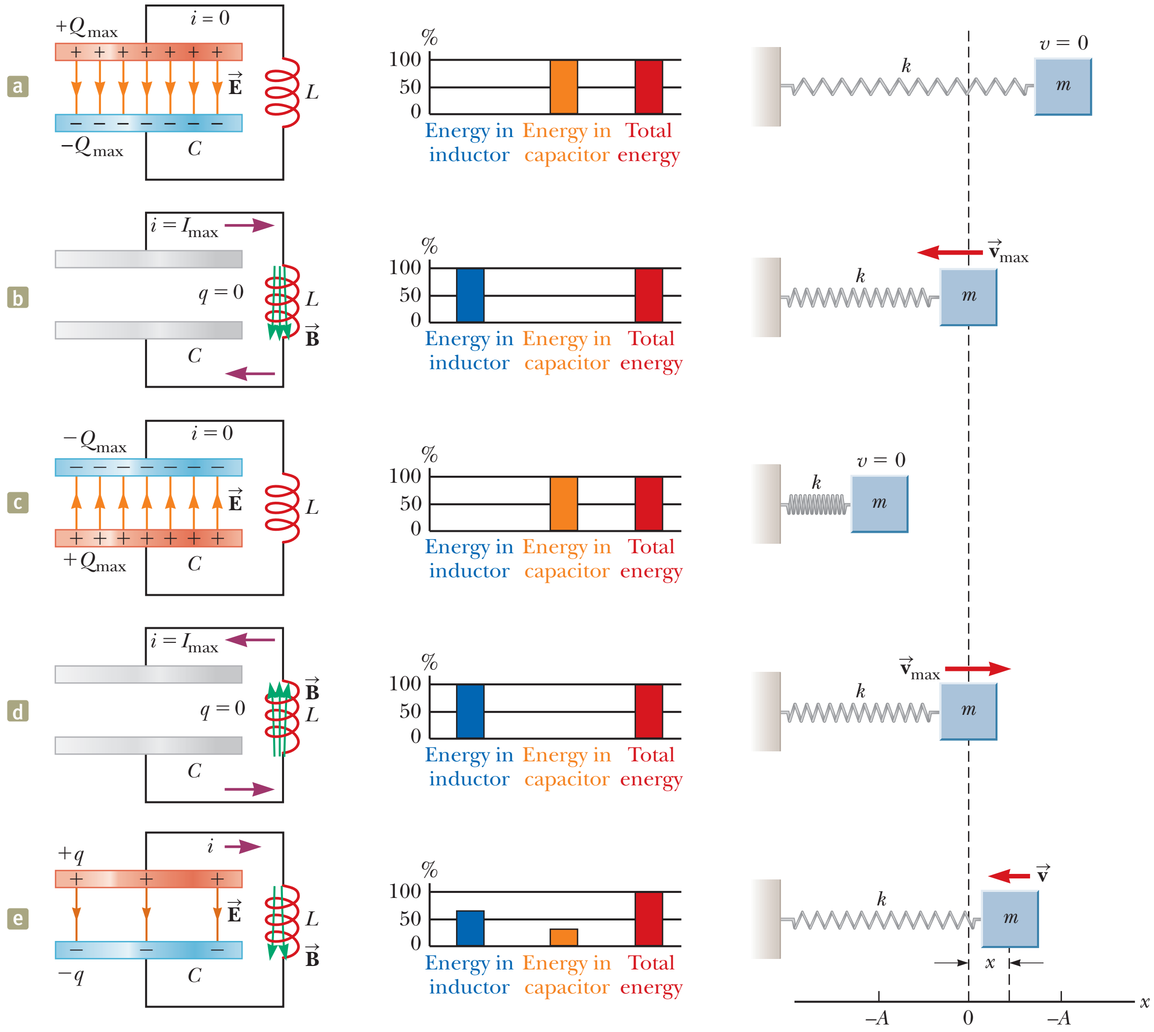


Figure 32.11 Energy transfer in a resistanceless, nonradiating *LC* circuit. The capacitor has a charge Q_{max} at $t = 0$, the instant at which the switch in Figure 32.10 is closed. The mechanical analog of this circuit is the particle in simple harmonic motion, represented by the block–spring system at the right of the figure. (a)–(d) At these special instants, all of the energy in the circuit resides in one of the circuit elements. (e) At an arbitrary instant, the energy is split between the capacitor and the inductor.

which leads to the phenomenon of *resonance*. The same phenomenon is observed in the *LC* circuit. (See Section 33.7.)

A representation of the energy transfer in an *LC* circuit is shown in Figure 32.11. As mentioned, the behavior of the circuit is analogous to that of the particle in simple harmonic motion studied in Chapter 15. For example, consider the block–spring system shown in Figure 15.10. The oscillations of this system are shown at the right of Figure 32.11. The potential energy $\frac{1}{2}kx^2$ stored in the stretched spring is analogous to the potential energy $Q_{max}^2/2C$ stored in the capacitor in Figure 32.11. The kinetic energy $\frac{1}{2}mv^2$ of the moving block is analogous to the magnetic energy $\frac{1}{2}Li^2$ stored in the inductor, which requires the presence of moving charges. In Figure 32.11a, all the energy is stored as electric potential energy in the capacitor at $t = 0$ (because $i = 0$), just as all the energy in a block–spring system is initially stored as potential energy in the spring if it is stretched and released at $t = 0$. In Figure 32.11b, all the energy is stored as magnetic energy $\frac{1}{2}LI_{max}^2$ in the inductor, where I_{max} is the maximum current. Figures 32.11c and 32.11d show subsequent quarter-cycle situations in which the energy is all electric or all magnetic. At intermediate points, part of the energy is electric and part is magnetic.

Consider some arbitrary time t after the switch is closed so that the capacitor has a charge $q < Q_{\max}$ and the current is $i < I_{\max}$. At this time, both circuit elements store energy, as shown in Figure 32.11e, but the sum of the two energies must equal the total initial energy U stored in the fully charged capacitor at $t = 0$:

Total energy stored in an LC circuit ▶

$$U = U_E + U_B = \frac{q^2}{2C} + \tfrac{1}{2}Li^2 = \frac{Q_{\max}^2}{2C} \qquad \textbf{(32.18)}$$

Because we have assumed the circuit resistance to be zero and we ignore electromagnetic radiation, no energy is transformed to internal energy and none is transferred out of the system of the circuit. Therefore, with these assumptions, the system of the circuit is isolated: *the total energy of the system must remain constant in time.* We describe the constant energy of the system mathematically by setting $dU/dt = 0$. Therefore, by differentiating Equation 32.18 with respect to time while noting that q and i vary with time gives

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \tfrac{1}{2}Li^2\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0 \qquad \textbf{(32.19)}$$

We can reduce this result to a differential equation in one variable by remembering that the current in the circuit is equal to the rate at which the charge on the capacitor changes: $i = dq/dt$. It then follows that $di/dt = d^2q/dt^2$. Substitution of these relationships into Equation 32.19 gives

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\frac{d^2q}{dt^2} = -\frac{1}{LC}q \qquad \textbf{(32.20)}$$

Let's solve for q by noting that this expression is of the same form as the analogous Equations 15.3 and 15.5 for a particle in simple harmonic motion:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

where k is the spring constant, m is the mass of the block, and $\omega = \sqrt{k/m}$. The solution of this mechanical equation has the general form (Eq. 15.6):

$$x = A\cos(\omega t + \phi)$$

where A is the amplitude of the simple harmonic motion (the maximum value of x), ω is the angular frequency of this motion, and ϕ is the phase constant; the values of A and ϕ depend on the initial conditions. Because Equation 32.20 is of the same mathematical form as the differential equation of the simple harmonic oscillator, it has the solution

Charge as a function of time for an ideal LC circuit ▶

$$q = Q_{\max}\cos(\omega t + \phi) \qquad \textbf{(32.21)}$$

where $Q_{\max}$ is the maximum charge of the capacitor and the angular frequency ω is

Angular frequency of oscillation in an LC circuit ▶

$$\omega = \frac{1}{\sqrt{LC}} \qquad \textbf{(32.22)}$$

Note that the angular frequency of the oscillations depends solely on the inductance and capacitance of the circuit. Equation 32.22 gives the *natural frequency* of oscillation of the LC circuit.

Because q varies sinusoidally with time, the current in the circuit also varies sinusoidally. We can show that by differentiating Equation 32.21 with respect to time:

Current as a function of time for an ideal LC current ▶

$$i = \frac{dq}{dt} = -\omega Q_{\max}\sin(\omega t + \phi) \qquad \textbf{(32.23)}$$

To determine the value of the phase angle ϕ, let's examine the initial conditions, which in our situation require that at $t = 0$, $i = 0$, and $q = Q_{max}$. Setting $i = 0$ at $t = 0$ in Equation 32.23 gives

$$0 = -\omega Q_{max} \sin \phi$$

which shows that $\phi = 0$. This value for ϕ also is consistent with Equation 32.21 and the condition that $q = Q_{max}$ at $t = 0$. Therefore, in our case, the expressions for q and i are

$$q = Q_{max} \cos \omega t \quad \textbf{(32.24)}$$

$$i = -\omega Q_{max} \sin \omega t = -I_{max} \sin \omega t \quad \textbf{(32.25)}$$

Graphs of q versus t and i versus t are shown in Figure 32.12. The charge on the capacitor oscillates between the extreme values Q_{max} and $-Q_{max}$, and the current oscillates between I_{max} and $-I_{max}$. Furthermore, the current is 90° out of phase with the charge. That is, when the charge is a maximum, the current is zero, and when the charge is zero, the current has its maximum value.

Let's return to the energy discussion of the LC circuit. Substituting Equations 32.24 and 32.25 in Equation 32.18, we find that the total energy is

$$U = U_E + U_B = \frac{Q_{max}^2}{2C} \cos^2 \omega t + \tfrac{1}{2} L I_{max}^2 \sin^2 \omega t \quad \textbf{(32.26)}$$

This expression contains all the features described qualitatively at the beginning of this section. It shows that the energy of the LC circuit continuously oscillates between energy stored in the capacitor's electric field and energy stored in the inductor's magnetic field. When the energy stored in the capacitor has its maximum value $Q_{max}^2/2C$, the energy stored in the inductor is zero. When the energy stored in the inductor has its maximum value $\frac{1}{2}LI_{max}^2$, the energy stored in the capacitor is zero.

Plots of the time variations of U_E and U_B are shown in Figure 32.13. The sum $U_E + U_B$ is a constant and is equal to the total energy $Q_{max}^2/2C$, or $\frac{1}{2}LI_{max}^2$. Analytical verification is straightforward. The amplitudes of the two graphs in Figure 32.13 must be equal because the maximum energy stored in the capacitor (when $I = 0$) must equal the maximum energy stored in the inductor (when $q = 0$). This equality is expressed mathematically as

$$\frac{Q_{max}^2}{2C} = \frac{LI_{max}^2}{2}$$

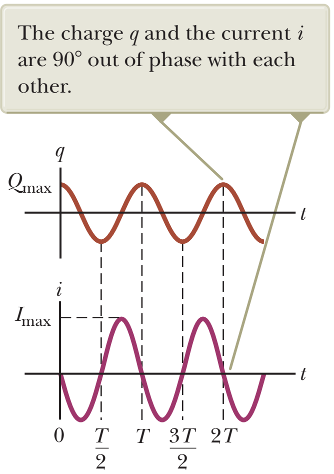


Figure 32.12 Graphs of charge versus time and current versus time for a resistanceless, nonradiating LC circuit.

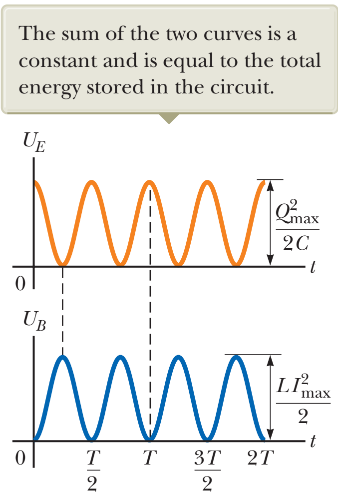


Figure 32.13 Plots of U_E versus t and U_B versus t for a resistanceless, nonradiating LC circuit.

Using this expression in Equation 32.26 for the total energy gives

$$U = \frac{Q_{\max}^2}{2C}(\cos^2 \omega t + \sin^2 \omega t) = \frac{Q_{\max}^2}{2C} \tag{32.27}$$

because $\cos^2 \omega t + \sin^2 \omega t = 1$.

In our idealized situation, the oscillations in the circuit persist indefinitely; the total energy U of the circuit, however, remains constant only if energy transfers and transformations are neglected. In actual circuits, there is always some resistance and some energy is therefore transformed to internal energy. We mentioned at the beginning of this section that we are also ignoring radiation from the circuit. In reality, radiation is inevitable in this type of circuit, and the total energy in the circuit continuously decreases as a result of this process.

Quick Quiz 32.5 **(i)** At an instant of time during the oscillations of an LC circuit, the current is at its maximum value. At this instant, what happens to the voltage across the capacitor? (a) It is different from that across the inductor. (b) It is zero. (c) It has its maximum value. (d) It is impossible to determine. **(ii)** Now consider an instant when the current is momentarily zero. From the same choices, describe the magnitude of the voltage across the capacitor at this instant.

Example 32.6 Oscillations in an *LC* Circuit

In Figure 32.14, the battery has an emf of 12.0 V, the inductance is 2.81 mH, and the capacitance is 9.00 pF. The switch has been set to position a for a long time so that the capacitor is charged. The switch is then thrown to position b, removing the battery from the circuit and connecting the capacitor directly across the inductor.

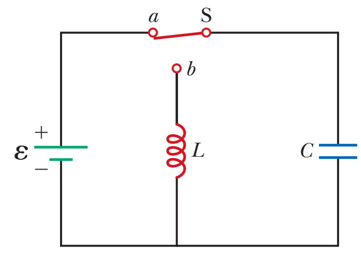


Figure 32.14 (Example 32.6) First the capacitor is fully charged with the switch set to position a. Then the switch is thrown to position b, and the battery is no longer in the circuit.

(A) Find the frequency of oscillation of the circuit.

SOLUTION

Conceptualize When the switch is thrown to position b, the active part of the circuit is the right-hand loop, which is an LC circuit.

Categorize We use equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 32.22 to find the frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Substitute numerical values:

$$f = \frac{1}{2\pi[(2.81 \times 10^{-3}\text{ H})(9.00 \times 10^{-12}\text{ F})]^{1/2}} = \boxed{1.00 \times 10^6\text{ Hz}}$$

(B) What are the maximum values of charge on the capacitor and current in the circuit?

SOLUTION

Find the initial charge on the capacitor, which equals the maximum charge:

$$Q_{\max} = C\,\Delta V = (9.00 \times 10^{-12}\text{ F})(12.0\text{ V}) = \boxed{1.08 \times 10^{-10}\text{ C}}$$

Use Equation 32.25 to find the maximum current from the maximum charge:

$$I_{\max} = \omega Q_{\max} = 2\pi f Q_{\max} = (2\pi \times 10^6\text{ s}^{-1})(1.08 \times 10^{-10}\text{ C})$$

$$= \boxed{6.79 \times 10^{-4}\text{ A}}$$

32.6 The *RLC* Circuit

Let's now turn our attention to a more realistic circuit consisting of a resistor, an inductor, and a capacitor connected in series as shown in Figure 32.15. We assume the resistance of the resistor represents all the resistance in the circuit. Suppose the switch is at position a so that the capacitor has an initial charge $Q_{\max}$. The switch is now thrown to position b. At this instant, the total energy stored in the capacitor and inductor is $Q_{\max}^2/2C$. This total energy, however, is no longer constant as it was in the LC circuit because the resistor causes transformation to internal energy. (We continue to ignore electromagnetic radiation from the circuit in this discussion.) Because the rate of energy transformation to internal energy within a resistor is i^2R,

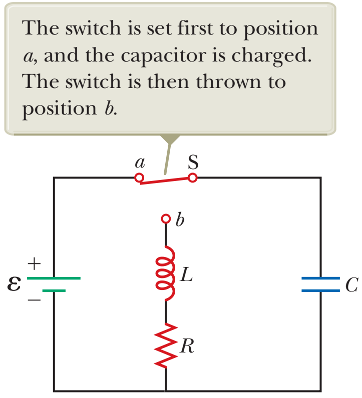


Figure 32.15 A series *RLC* circuit.

$$\frac{dU}{dt} = -i^2R$$

where the negative sign signifies that the energy U of the circuit is decreasing in time. Substituting $U = U_E + U_B$ gives

$$\frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2R \tag{32.28}$$

To convert this equation into a form that allows us to compare the electrical oscillations with their mechanical analog, we first use $i = dq/dt$ and move all terms to the left-hand side to obtain

$$Li\frac{d^2q}{dt^2} + i^2R + \frac{q}{C}i = 0$$

Now divide through by i:

$$L\frac{d^2q}{dt^2} + iR + \frac{q}{C} = 0$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{32.29}$$

The *RLC* circuit is analogous to the damped harmonic oscillator discussed in Section 15.6 and illustrated in Figure 15.20. The equation of motion for a damped block–spring system is, from Equation 15.31,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{32.30}$$

Comparing Equations 32.29 and 32.30, we see that q corresponds to the position x of the block at any instant, L to the mass m of the block, R to the damping coefficient b, and C to $1/k$, where k is the force constant of the spring. These and other relationships are listed in Table 32.1 on page 986.

Because the analytical solution of Equation 32.29 is cumbersome, we give only a qualitative description of the circuit behavior. In the simplest case, when $R = 0$, Equation 32.29 reduces to that of a simple LC circuit as expected, and the charge and the current oscillate sinusoidally in time. This situation is equivalent to removing all damping in the mechanical oscillator.

When R is small, a situation that is analogous to light damping in the mechanical oscillator, the solution of Equation 32.29 is

$$q = Q_{\max}e^{-Rt/2L}\cos\omega_d t \tag{32.31}$$

where ω_d, the angular frequency at which the circuit oscillates, is given by

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \tag{32.32}$$

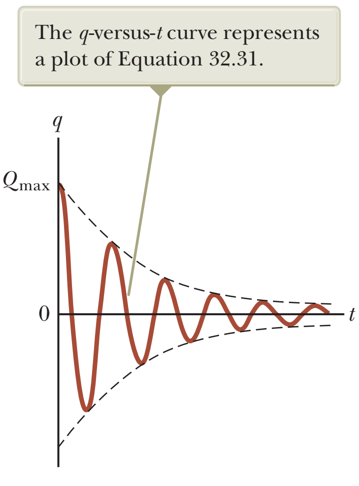
That is, the value of the charge on the capacitor undergoes a damped harmonic oscillation in analogy with a block–spring system moving in a viscous medium. Equation 32.32 shows that when $R << \sqrt{4L/C}$ (so that the second term in the

Table 32.1 **Analogies Between the *RLC* Circuit and the Particle in Simple Harmonic Motion**

***RLC* Circuit**		**One-Dimensional Particle in Simple Harmonic Motion**
Charge	$q \leftrightarrow x$	Position
Current	$i \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$i = \dfrac{dq}{dt} \leftrightarrow v_x = \dfrac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\dfrac{di}{dt} = \dfrac{d^2q}{dt^2} \leftrightarrow a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_B = \frac{1}{2}Li^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_E = \frac{1}{2}\dfrac{q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$i^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0 \leftrightarrow m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$	Damped object on a spring

brackets is much smaller than the first), the frequency ω_d of the damped oscillator is close to that of the undamped oscillator, $1/\sqrt{LC}$. Because $i = dq/dt$, it follows that the current also undergoes damped harmonic oscillation. A plot of the charge versus time for the damped oscillator is shown in Figure 32.16a, and an oscilloscope trace for a real *RLC* circuit is shown in Figure 32.16b. The maximum value of q decreases after each oscillation, just as the amplitude of a damped block–spring system decreases in time.

For larger values of R, the oscillations damp out more rapidly; in fact, there exists a critical resistance value $R_c = \sqrt{4L/C}$ above which no oscillations occur. A system with $R = R_c$ is said to be *critically damped*. When R exceeds R_c, the system is said to be *overdamped*.



a

b

iStockphoto.com/A_Carina

Figure 32.16 (a) Charge versus time for a damped *RLC* circuit. The charge decays in this way when $R < \sqrt{4L/C}$. (b) Oscilloscope pattern showing the decay in the oscillations of an *RLC* circuit.

Summary

Concepts and Principles

When the current in a loop of wire changes with time, an emf is induced in the loop according to Faraday's law. The **self-induced emf** is

$$\mathcal{E}_L = -L\frac{di}{dt} \quad \textbf{(32.1)}$$

where L is the **inductance** of the loop. Inductance is a measure of how much opposition a loop offers to a change in the current in the loop. Inductance has the SI unit of **henry** (H), where 1 H = 1 V · s/A.

The inductance of any coil is

$$L = \frac{N\Phi_B}{i} \quad \textbf{(32.2)}$$

where N is the total number of turns and Φ_B is the magnetic flux through the coil. The inductance of a device depends on its geometry. For example, the inductance of an air-core solenoid is

$$L = \mu_0\frac{N^2}{\ell}A \quad \textbf{(32.4)}$$

where ℓ is the length of the solenoid and A is the cross-sectional area.

If a resistor and inductor are connected in series to a battery of emf $\mathcal{E}$ at time $t = 0$, the current in the circuit varies in time according to the expression

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \quad \textbf{(32.7)}$$

where $\tau = L/R$ is the **time constant** of the RL circuit. If we replace the battery in the circuit by a resistanceless wire, the current decays exponentially with time according to the expression

$$i = \frac{\mathcal{E}}{R}e^{-t/\tau} \quad \textbf{(32.10)}$$

where $\mathcal{E}/R$ is the initial current in the circuit.

The energy stored in the magnetic field of an inductor carrying a current i is

$$U_B = \tfrac{1}{2}Li^2 \quad \textbf{(32.12)}$$

This energy is the magnetic counterpart to the energy stored in the electric field of a charged capacitor.

The energy density at a point where the magnetic field is B is

$$u_B = \frac{B^2}{2\mu_0} \quad \textbf{(32.14)}$$

The **mutual inductance** of a system of two coils is

$$M_{12} = \frac{N_2\Phi_{12}}{i_1} = M_{21} = \frac{N_1\Phi_{21}}{i_2} = M \quad \textbf{(32.15)}$$

This mutual inductance allows us to relate the induced emf in a coil to the changing source current in a nearby coil using the relationships

$$\mathcal{E}_2 = -M_{12}\frac{di_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M_{21}\frac{di_2}{dt} \quad \textbf{(32.16, 32.17)}$$

In an LC circuit that has zero resistance and does not radiate electromagnetically (an idealization), the values of the charge on the capacitor and the current in the circuit vary sinusoidally in time at an angular frequency given by

$$\omega = \frac{1}{\sqrt{LC}} \quad \textbf{(32.22)}$$

The energy in an LC circuit continuously transfers between energy stored in the capacitor and energy stored in the inductor.

In an RLC circuit with small resistance, the charge on the capacitor varies with time according to

$$q = Q_{\max}e^{-Rt/2L}\cos\omega_d t \quad \textbf{(32.31)}$$

where

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \quad \textbf{(32.32)}$$

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. The centers of two circular loops are separated by a fixed distance. **(i)** For what relative orientation of the loops is their mutual inductance a maximum? (a) coaxial and lying in parallel planes (b) lying in the same plane (c) lying in perpendicular planes, with the center of one on the axis of the other (d) The orientation makes no difference. **(ii)** For what relative orientation is their mutual inductance a minimum? Choose from the same possibilities as in part (i).

2. A long, fine wire is wound into a coil with inductance 5 mH. The coil is connected across the terminals of a battery, and the current is measured a few seconds after the connection is made. The wire is unwound and wound again into a different coil with $L = 10$ mH. This second coil is connected across the same battery, and the current is measured in the same way. Compared with the current in the first coil, is the current in the second coil (a) four times as large, (b) twice as large, (c) unchanged, (d) half as large, or (e) one-fourth as large?

3. A solenoidal inductor for a printed circuit board is being redesigned. To save weight, the number of turns is reduced by one-half, with the geometric dimensions kept the same. By how much must the current change if the energy stored in the inductor is to remain the same? (a) It must be four times larger. (b) It must be two times larger. (c) It should be left the same. (d) It should be one-half as large. (e) No change in the current can compensate for the reduction in the number of turns.

4. In Figure OQ32.4, the switch is left in position *a* for a long time interval and is then quickly thrown to position *b*. Rank the magnitudes of the voltages across the four circuit elements a short time thereafter from the largest to the smallest.

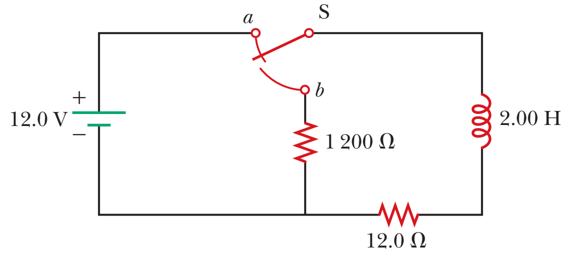


Figure OQ32.4

5. Two solenoids, A and B, are wound using equal lengths of the same kind of wire. The length of the axis of each solenoid is large compared with its diameter. The axial length of A is twice as large as that of B, and A has twice as many turns as B. What is the ratio of the inductance of solenoid A to that of solenoid B? (a) 4 (b) 2 (c) 1 (d) $\frac{1}{2}$ (e) $\frac{1}{4}$

6. If the current in an inductor is doubled, by what factor is the stored energy multiplied? (a) 4 (b) 2 (c) 1 (d) $\frac{1}{2}$ (e) $\frac{1}{4}$

7. Initially, an inductor with no resistance carries a constant current. Then the current is brought to a new constant value twice as large. *After* this change, when the current is constant at its higher value, what has happened to the emf in the inductor? (a) It is larger than before the change by a factor of 4. (b) It is larger by a factor of 2. (c) It has the same nonzero value. (d) It continues to be zero. (e) It has decreased.

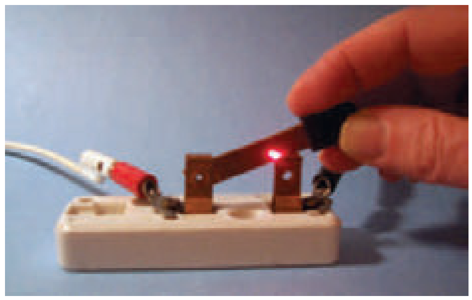
Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. Consider this thesis: "Joseph Henry, America's first professional physicist, caused a basic change in the human view of the Universe when he discovered self-induction during a school vacation at the Albany Academy about 1830. Before that time, one could think of the Universe as composed of only one thing: matter. The energy that temporarily maintains the current after a battery is removed from a coil, on the other hand, is not energy that belongs to any chunk of matter. It is energy in the massless magnetic field surrounding the coil. With Henry's discovery, Nature forced us to admit that the Universe consists of fields as well as matter." (a) Argue for or against the statement. (b) In your view, what makes up the Universe?

2. (a) What parameters affect the inductance of a coil? (b) Does the inductance of a coil depend on the current in the coil?

3. A switch controls the current in a circuit that has a large inductance. The electric arc at the switch (Fig. CQ32.3) can melt and oxidize the contact surfaces, resulting in high resistivity of the contacts and eventual destruction of the switch. Is a spark more likely to be produced at the switch when the switch is being closed, when it is being opened, or does it not matter?



Alexandra Héder

Figure CQ32.3

4. Consider the four circuits shown in Figure CQ32.4, each consisting of a battery, a switch, a lightbulb, a resistor, and either a capacitor or an inductor. Assume the capacitor has a large capacitance and the inductor has a large inductance but no resistance. The lightbulb has high efficiency, glowing whenever it carries electric current. **(i)** Describe what the lightbulb does in each of circuits (a) through (d) after the switch is thrown closed. **(ii)** Describe what the lightbulb does in each of circuits (a) through (d) when, having been closed for a long time interval, the switch is opened.

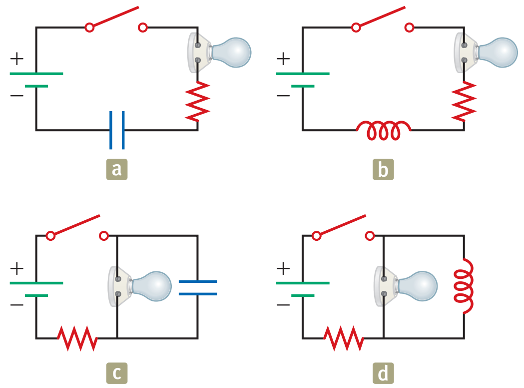


Figure CQ32.4

5. The current in a circuit containing a coil, a resistor, and a battery has reached a constant value. (a) Does the coil have an inductance? (b) Does the coil affect the value of the current?

6. (a) Can an object exert a force on itself? (b) When a coil induces an emf in itself, does it exert a force on itself?

7. The open switch in Figure CQ32.7 is thrown closed at $t = 0$. Before the switch is closed, the capacitor is uncharged and all currents are zero. Determine the currents in L, C, and R, the emf across L, and the potential differences across C and R (a) at the instant after the switch is closed and (b) long after it is closed.

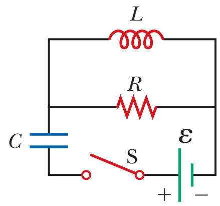


Figure CQ32.7

8. After the switch is closed in the LC circuit shown in Figure CQ32.8, the charge on the capacitor is sometimes zero, but at such instants the current in the circuit is not zero. How is this behavior possible?

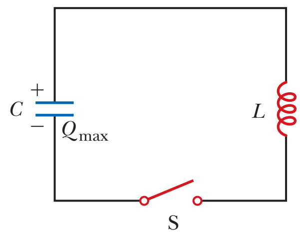


Figure CQ32.8 Conceptual Question 8 and Problems 52, 54, and 55.

9. How can you tell whether an RLC circuit is overdamped or underdamped?

10. Discuss the similarities between the energy stored in the electric field of a charged capacitor and the energy stored in the magnetic field of a current-carrying coil.

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 32.1 Self-Induction and Inductance

1. A coil has an inductance of 3.00 mH, and the current in it changes from 0.200 A to 1.50 A in a time interval of 0.200 s. Find the magnitude of the average induced emf in the coil during this time interval.

2. A coiled telephone cord forms a spiral with 70.0 turns, a diameter of 1.30 cm, and an unstretched length of 60.0 cm. Determine the inductance of one conductor in the unstretched cord.

3. A 2.00-H inductor carries a steady current of 0.500 A. When the switch in the circuit is opened, the current is effectively zero after 10.0 ms. What is the average induced emf in the inductor during this time interval?

4. M A solenoid of radius 2.50 cm has 400 turns and a length of 20.0 cm. Find (a) its inductance and (b) the rate at which current must change through it to produce an emf of 75.0 mV.

5. An emf of 24.0 mV is induced in a 500-turn coil when the current is changing at the rate of 10.0 A/s. What is the magnetic flux through each turn of the coil at an instant when the current is 4.00 A?

6. A 40.0-mA current is carried by a uniformly wound air-core solenoid with 450 turns, a 15.0-mm diameter, and 12.0-cm length. Compute (a) the magnetic field inside the solenoid, (b) the magnetic flux through each turn, and (c) the inductance of the solenoid. (d) **What If?** If the current were different, which of these quantities would change?

7. The current in a coil changes from 3.50 A to 2.00 A in the same direction in 0.500 s. If the average emf induced in the coil is 12.0 mV, what is the inductance of the coil?

8. A technician wraps wire around a tube of length 36.0 cm having a diameter of 8.00 cm. When the windings are evenly spread over the full length of the tube, the result is a solenoid containing 580 turns of wire. (a) Find the inductance of this solenoid. (b) If the current in this solenoid increases at the rate of 4.00 A/s, find the self-induced emf in the solenoid.

9. The current in a 90.0-mH inductor changes with time as $i = 1.00t^2 - 6.00t$, where i is in amperes and t is in seconds. Find the magnitude of the induced emf at (a) $t = 1.00$ s and (b) $t = 4.00$ s. (c) At what time is the emf zero?

10. An inductor in the form of a solenoid contains 420 turns and is 16.0 cm in length. A uniform rate of decrease of current through the inductor of 0.421 A/s induces an emf of 175 μV. What is the radius of the solenoid?

11. A self-induced emf in a solenoid of inductance L changes in time as $\mathcal{E} = \mathcal{E}_0 e^{-kt}$. Assuming the charge is finite, find the total charge that passes a point in the wire of the solenoid.

12. A toroid has a major radius R and a minor radius r and is tightly wound with N turns of wire on a hollow cardboard torus. Figure P32.12 shows half of this toroid, allowing us to see its cross section. If $R >> r$, the magnetic field in the region enclosed by the wire is essentially the same as the magnetic field of a solenoid that has been bent into a large circle of radius R. Modeling the field as the uniform field of a long solenoid, show that the inductance of such a toroid is approximately

$$L \approx \tfrac{1}{2}\mu_0 N^2 \frac{r^2}{R}$$

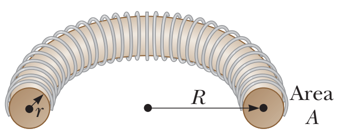


Figure P32.12

13. A 10.0-mH inductor carries a current $i = I_{\max} \sin \omega t$, with $I_{\max} = 5.00$ A and $f = \omega/2\pi = 60.0$ Hz. What is the self-induced emf as a function of time?

14. The current in a 4.00 mH-inductor varies in time as shown in Figure P32.14. Construct a graph of the self-induced emf across the inductor over the time interval $t = 0$ to $t = 12.0$ ms.

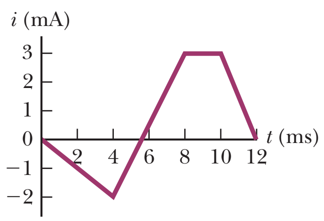


Figure P32.14

Section 32.2 *RL* Circuits

15. A 510-turn solenoid has a radius of 8.00 mm and an overall length of 14.0 cm. (a) What is its inductance? (b) If the solenoid is connected in series with a 2.50-Ω resistor and a battery, what is the time constant of the circuit?

16. A 12.0-V battery is connected into a series circuit containing a 10.0-Ω resistor and a 2.00-H inductor. In what time interval will the current reach (a) 50.0% and (b) 90.0% of its final value?

17. A series RL circuit with $L = 3.00$ H and a series RC circuit with $C = 3.00\ \mu$F have equal time constants. If the two circuits contain the same resistance R, (a) what is the value of R? (b) What is the time constant?

18. In the circuit diagrammed in Figure P32.18, take $\mathcal{E} = 12.0$ V and $R = 24.0\ \Omega$. Assume the switch is open for $t < 0$ and is closed at $t = 0$. On a single set of axes, sketch graphs of the current in the circuit as a function of time for $t \geq 0$, assuming (a) the inductance in the circuit is essentially zero, (b) the inductance has an intermediate value, and (c) the inductance has a very large value. Label the initial and final values of the current.

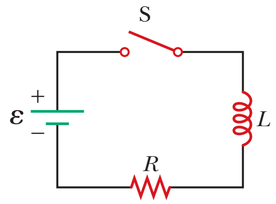


Figure P32.18

Problems 18, 20, 23, 24, and 27.

19. Consider the circuit shown in Figure P32.19. (a) When the switch is in position a, for what value of R will the circuit have a time constant of 15.0 μs? (b) What is the current in the inductor at the instant the switch is thrown to position b?

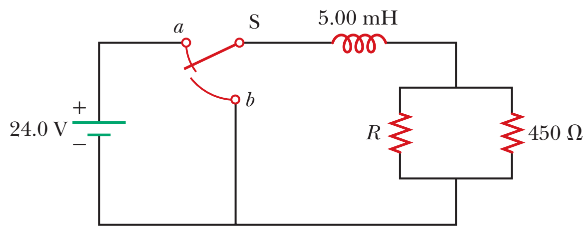


Figure P32.19

20. When the switch in Figure P32.18 is closed, the current takes 3.00 ms to reach 98.0% of its final value. If $R = 10.0\ \Omega$, what is the inductance?

21. A circuit consists of a coil, a switch, and a battery, all in series. The internal resistance of the battery is negligible compared with that of the coil. The switch is originally open. It is thrown closed, and after a time interval Δt, the current in the circuit reaches 80.0%

of its final value. The switch then remains closed for a time interval much longer than Δt. The wires connected to the terminals of the battery are then short-circuited with another wire and removed from the battery, so that the current is uninterrupted. (a) At an instant that is a time interval Δt after the short circuit, the current is what percentage of its maximum value? (b) At the moment $2\Delta t$ after the coil is short-circuited, the current in the coil is what percentage of its maximum value?

22. Show that $i = I_i e^{-t/\tau}$ is a solution of the differential equation

$$iR + L\frac{di}{dt} = 0$$

where I_i is the current at $t = 0$ and $\tau = L/R$.

23. W In the circuit shown in Figure P32.18, let $L = 7.00$ H, $R = 9.00\ \Omega$, and $\boldsymbol{\mathcal{E}} = 120$ V. What is the self-induced emf 0.200 s after the switch is closed?

24. Consider the circuit in Figure P32.18, taking $\boldsymbol{\mathcal{E}} = 6.00$ V, $L = 8.00$ mH, and $R = 4.00\ \Omega$. (a) What is the inductive time constant of the circuit? (b) Calculate the current in the circuit 250 μs after the switch is closed. (c) What is the value of the final steady-state current? (d) After what time interval does the current reach 80.0% of its maximum value?

25. The switch in Figure P32.25 is open for $t < 0$ and is then thrown closed at time $t = 0$. Assume $R = 4.00\ \Omega$, $L = 1.00$ H, and $\boldsymbol{\mathcal{E}} = 10.0$ V. Find (a) the current in the inductor and (b) the current in the switch as functions of time thereafter.

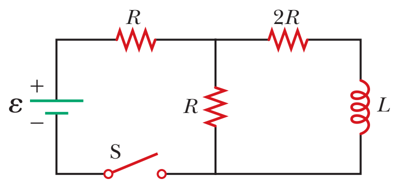


Figure P32.25 Problems 25, 26, and 64.

26. The switch in Figure P32.25 is open for $t < 0$ and is then thrown closed at time $t = 0$. Find (a) the current in the inductor and (b) the current in the switch as functions of time thereafter.

27. For the *RL* circuit shown in Figure P32.18, let the inductance be 3.00 H, the resistance 8.00 Ω, and the battery emf 36.0 V. (a) Calculate $\Delta V_R/\boldsymbol{\mathcal{E}}_L$, that is, the ratio of the potential difference across the resistor to the emf across the inductor when the current is 2.00 A. (b) Calculate the emf across the inductor when the current is 4.50 A.

28. Consider the current pulse $i(t)$ shown in Figure P32.28a. The current begins at zero, becomes 10.0 A between $t = 0$ and $t = 200\ \mu$s, and then is zero once again. This pulse is applied to the input of the partial circuit shown in Figure P32.28b. Determine the current in the inductor as a function of time.

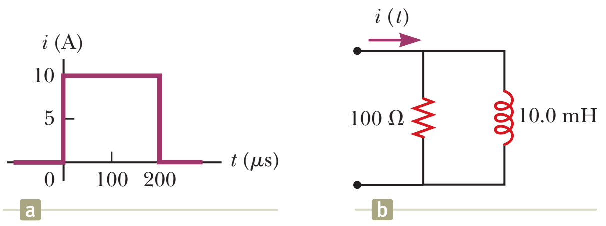


Figure P32.28

29. An inductor that has an inductance of 15.0 H and a resistance of 30.0 Ω is connected across a 100-V battery. What is the rate of increase of the current (a) at $t = 0$ and (b) at $t = 1.50$ s?

30. Two ideal inductors, L_1 and L_2, have *zero* internal resistance and are far apart, so their magnetic fields do not influence each other. (a) Assuming these inductors are connected in series, show that they are equivalent to a single ideal inductor having $L_{eq} = L_1 + L_2$. (b) Assuming these same two inductors are connected in parallel, show that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$. (c) **What If?** Now consider two inductors L_1 and L_2 that have *nonzero* internal resistances R_1 and R_2, respectively. Assume they are still far apart, so their mutual inductance is zero, and assume they are connected in series. Show that they are equivalent to a single inductor having $L_{eq} = L_1 + L_2$ and $R_{eq} = R_1 + R_2$. (d) If these same inductors are now connected in parallel, is it necessarily true that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$ and $1/R_{eq} = 1/R_1 + 1/R_2$? Explain your answer.

31. M A 140-mH inductor and a 4.90-Ω resistor are connected with a switch to a 6.00-V battery as shown in Figure P32.31. (a) After the switch is first thrown to *a* (connecting the battery), what time interval elapses before the current reaches 220 mA? (b) What is the current in the inductor 10.0 s after the switch is closed? (c) Now the switch is quickly thrown from *a* to *b*. What time interval elapses before the current in the inductor falls to 160 mA?

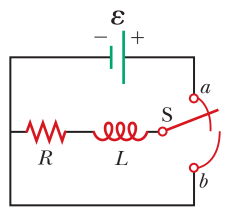


Figure P32.31

Section 32.3 Energy in a Magnetic Field

32. Calculate the energy associated with the magnetic field of a 200-turn solenoid in which a current of 1.75 A produces a magnetic flux of 3.70×10^{-4} T $\cdot$ m^2 in each turn.

33. M An air-core solenoid with 68 turns is 8.00 cm long and has a diameter of 1.20 cm. When the solenoid carries a current of 0.770 A, how much energy is stored in its magnetic field?

34. W A 10.0-V battery, a 5.00-Ω resistor, and a 10.0-H inductor are connected in series. After the current in the circuit has reached its maximum value, calculate (a) the power being supplied by the battery, (b) the power being delivered to the resistor, (c) the power being delivered to the inductor, and (d) the energy stored in the magnetic field of the inductor.

35. On a clear day at a certain location, a 100-V/m vertical electric field exists near the Earth's surface. At the same place, the Earth's magnetic field has a magnitude of 0.500×10^{-4} T. Compute the energy densities of (a) the electric field and (b) the magnetic field.

36. Complete the calculation in Example 32.3 by proving that

$$\int_0^\infty e^{-2Rt/L}\, dt = \frac{L}{2R}$$

37. M A 24.0-V battery is connected in series with a resistor and an inductor, with $R = 8.00\ \Omega$ and $L = 4.00$ H, respectively. Find the energy stored in the inductor (a) when the current reaches its maximum value and (b) at an instant that is a time interval of one time constant after the switch is closed.

38. A flat coil of wire has an inductance of 40.0 mH and a resistance of 5.00 Ω. It is connected to a 22.0-V battery at the instant $t = 0$. Consider the moment when the current is 3.00 A. (a) At what rate is energy being delivered by the battery? (b) What is the power being delivered to the resistance of the coil? (c) At what rate is energy being stored in the magnetic field of the coil? (d) What is the relationship among these three power values? (e) Is the relationship described in part (d) true at other instants as well? (f) Explain the relationship at the moment immediately after $t = 0$ and at a moment several seconds later.

39. The magnetic field inside a superconducting solenoid is 4.50 T. The solenoid has an inner diameter of 6.20 cm and a length of 26.0 cm. Determine (a) the magnetic energy density in the field and (b) the energy stored in the magnetic field within the solenoid.

Section 32.4 Mutual Inductance

40. An emf of 96.0 mV is induced in the windings of a coil when the current in a nearby coil is increasing at the rate of 1.20 A/s. What is the mutual inductance of the two coils?

41. Two coils, held in fixed positions, have a mutual inductance of 100 μH. What is the peak emf in one coil when the current in the other coil is $i(t) = 10.0 \sin (1.00 \times 10^3 t)$, where i is in amperes and t is in seconds?

42. Two coils are close to each other. The first coil carries a current given by $i(t) = 5.00\, e^{-0.025\,0t} \sin 120\pi t$, where i is in amperes and t is in seconds. At $t = 0.800$ s, the emf measured across the second coil is -3.20 V. What is the mutual inductance of the coils?

43. M Two solenoids A and B, spaced close to each other and sharing the same cylindrical axis, have 400 and 700 turns, respectively. A current of 3.50 A in solenoid A produces an average flux of 300 μWb through each turn of A and a flux of 90.0 μWb through each turn of B. (a) Calculate the mutual inductance of the two solenoids. (b) What is the inductance of A? (c) What emf is induced in B when the current in A changes at the rate of 0.500 A/s?

44. Solenoid S_1 has N_1 turns, radius R_1, and length ℓ. It is so long that its magnetic field is uniform nearly everywhere inside it and is nearly zero outside. Solenoid S_2 has N_2 turns, radius $R_2 < R_1$, and the same length as S_1. It lies inside S_1, with their axes parallel. (a) Assume S_1 carries variable current i. Compute the mutual inductance characterizing the emf induced in S_2. (b) Now assume S_2 carries current i. Compute the mutual inductance to which the emf in S_1 is proportional. (c) State how the results of parts (a) and (b) compare with each other.

45. On a printed circuit board, a relatively long, straight conductor and a conducting rectangular loop lie in the same plane as shown in Figure P32.45. Taking $h = 0.400$ mm, $w = 1.30$ mm, and $\ell = 2.70$ mm, find their mutual inductance.

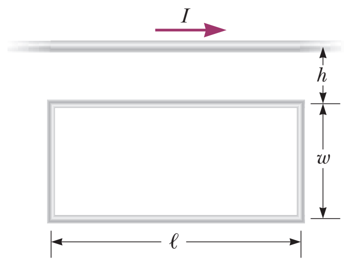


Figure P32.45

46. Two single-turn circular loops of wire have radii R and r, with $R >> r$. The loops lie in the same plane and are concentric. (a) Show that the mutual inductance of the pair is approximately $M = \mu_0 \pi r^2/2R$. (b) Evaluate M for $r = 2.00$ cm and $R = 20.0$ cm.

Section 32.5 Oscillations in an *LC* Circuit

47. In the circuit of Figure P32.47, the battery emf is 50.0 V, the resistance is 250 Ω, and the capacitance is 0.500 μF. The switch S is closed for a long time

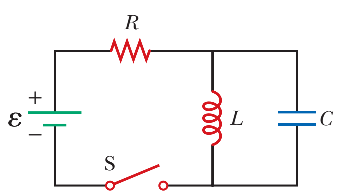


Figure P32.47

interval, and zero potential difference is measured across the capacitor. After the switch is opened, the potential difference across the capacitor reaches a maximum value of 150 V. What is the value of the inductance?

48. A 1.05-μH inductor is connected in series with a variable capacitor in the tuning section of a shortwave radio set. What capacitance tunes the circuit to the signal from a transmitter broadcasting at 6.30 MHz?

49. A 1.00-μF capacitor is charged by a 40.0-V power supply. The fully charged capacitor is then discharged through a 10.0-mH inductor. Find the maximum current in the resulting oscillations.

50. Calculate the inductance of an LC circuit that oscillates at 120 Hz when the capacitance is 8.00 μF.

51. An LC circuit consists of a 20.0-mH inductor and a 0.500-μF capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

52. *Why is the following situation impossible?* The LC circuit shown in Figure CQ32.8 has L = 30.0 mH and C = 50.0 μF. The capacitor has an initial charge of 200 μC. The switch is closed, and the circuit undergoes undamped LC oscillations. At periodic instants, the energies stored by the capacitor and the inductor are equal, with each of the two components storing 250 μJ.

53. AMT The switch in Figure P32.53 is connected to position a for a long time interval. At $t = 0$, the switch is thrown to position b. After this time, what are (a) the frequency of oscillation of the LC circuit, (b) the maximum charge that appears on the capacitor, (c) the maximum current in the inductor, and (d) the total energy the circuit possesses at t = 3.00 s?

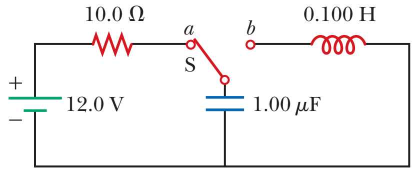


Figure P32.53

54. M An LC circuit like that in Figure CQ32.8 consists of a 3.30-H inductor and an 840-pF capacitor that initially carries a 105-μC charge. The switch is open for $t < 0$ and is then thrown closed at $t = 0$. Compute the following quantities at t = 2.00 ms: (a) the energy stored in the capacitor, (b) the energy stored in the inductor, and (c) the total energy in the circuit.

55. AMT An LC circuit like the one in Figure CQ32.8 contains an 82.0-mH inductor and a 17.0-μF capacitor that initially carries a 180-μC charge. The switch is open for $t < 0$ and is then thrown closed at $t = 0$. (a) Find the frequency (in hertz) of the resulting oscillations. At t = 1.00 ms, find (b) the charge on the capacitor and (c) the current in the circuit.

Section 32.6 The *RLC* Circuit

56. Show that Equation 32.28 in the text is Kirchhoff's loop rule as applied to the circuit in Figure P32.56 with the switch thrown to position b.

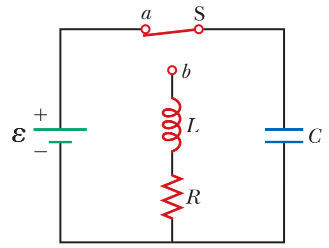


Figure P32.56 Problems 56 and 57.

57. In Figure P32.56, let R = 7.60 Ω, L = 2.20 mH, and C = 1.80 μF. (a) Calculate the frequency of the damped oscillation of the circuit when the switch is thrown to position b. (b) What is the critical resistance for damped oscillations?

58. M Consider an LC circuit in which L = 500 mH and C = 0.100 μF. (a) What is the resonance frequency ω_0? (b) If a resistance of 1.00 kΩ is introduced into this circuit, what is the frequency of the damped oscillations? (c) By what percentage does the frequency of the damped oscillations differ from the resonance frequency?

59. Electrical oscillations are initiated in a series circuit containing a capacitance C, inductance L, and resistance R. (a) If $R << \sqrt{4L/C}$ (weak damping), what time interval elapses before the amplitude of the current oscillation falls to 50.0% of its initial value? (b) Over what time interval does the energy decrease to 50.0% of its initial value?

Additional Problems

60. **Review.** This problem extends the reasoning of Section 26.4, Problem 38 in Chapter 26, Problem 34 in Chapter 30, and Section 32.3. (a) Consider a capacitor with vacuum between its large, closely spaced, oppositely charged parallel plates. Show that the force on one plate can be accounted for by thinking of the electric field between the plates as exerting a "negative pressure" equal to the energy density of the electric field. (b) Consider two infinite plane sheets carrying electric currents in opposite directions with equal linear current densities J_s. Calculate the force per area acting on one sheet due to the magnetic field, of magnitude $\mu_0 J_s/2$, created by the other sheet. (c) Calculate the net magnetic field between the sheets and the field outside of the volume between them. (d) Calculate the energy density in the magnetic field between the sheets. (e) Show that the force on one sheet can be accounted for by thinking of the magnetic field between the sheets as exerting a positive pressure equal to its energy density. This result for magnetic pressure applies to all current configurations, not only to sheets of current.

61. A 1.00-mH inductor and a 1.00-μF capacitor are connected in series. The current in the circuit increases linearly in time as $i = 20.0t$, where i is in amperes and t is in seconds. The capacitor initially has no charge. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

62. An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increases linearly in time as described by $i = Kt$, where K is a constant. The capacitor is initially uncharged. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

63. A capacitor in a series LC circuit has an initial charge Q and is being discharged. When the charge on the capacitor is $Q/2$, find the flux through each of the N turns in the coil of the inductor in terms of Q, N, L, and C.

64. AMT In the circuit diagrammed in Figure P32.25, assume the switch has been closed for a long time interval and is opened at $t = 0$. Also assume $R = 4.00\ \Omega$, $L = 1.00$ H, and $\mathcal{E} = 10.0$ V. (a) Before the switch is opened, does the inductor behave as an open circuit, a short circuit, a resistor of some particular resistance, or none of those choices? (b) What current does the inductor carry? (c) How much energy is stored in the inductor for $t < 0$? (d) After the switch is opened, what happens to the energy previously stored in the inductor? (e) Sketch a graph of the current in the inductor for $t \geq 0$. Label the initial and final values and the time constant.

65. When the current in the portion of the circuit shown in Figure P32.65 is 2.00 A and increases at a rate of 0.500 A/s, the measured voltage is $\Delta V_{ab} = 9.00$ V. When the current is 2.00 A and decreases at the rate of 0.500 A/s, the measured voltage is $\Delta V_{ab} = 5.00$ V. Calculate the values of (a) L and (b) R.

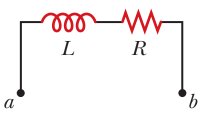


Figure P32.65

66. At the moment $t = 0$, a 24.0-V battery is connected to a 5.00-mH coil and a 6.00-Ω resistor. (a) Immediately thereafter, how does the potential difference across the resistor compare to the emf across the coil? (b) Answer the same question about the circuit several seconds later. (c) Is there an instant at which these two voltages are equal in magnitude? If so, when? Is there more than one such instant? (d) After a 4.00-A current is established in the resistor and coil, the battery is suddenly replaced by a short circuit. Answer parts (a), (b), and (c) again with reference to this new circuit.

67. (a) A flat, circular coil does not actually produce a uniform magnetic field in the area it encloses. Nevertheless, estimate the inductance of a flat, compact, circular coil with radius R and N turns by assuming the field at its center is uniform over its area. (b) A circuit on a laboratory table consists of a 1.50-volt battery, a 270-Ω resistor, a switch, and three 30.0-cm-long patch cords connecting them. Suppose the circuit is arranged to be circular. Think of it as a flat coil with one turn. Compute the order of magnitude of its inductance and (c) of the time constant describing how fast the current increases when you close the switch.

68. *Why is the following situation impossible?* You are working on an experiment involving a series circuit consisting of a charged 500-μF capacitor, a 32.0-mH inductor, and a resistor R. You discharge the capacitor through the inductor and resistor and observe the decaying oscillations of the current in the circuit. When the resistance R is 8.00 Ω, the decay in the oscillations is too slow for your experimental design. To make the decay faster, you double the resistance. As a result, you generate decaying oscillations of the current that are perfect for your needs.

69. A time-varying current i is sent through a 50.0-mH inductor from a source as shown in Figure P32.69a. The current is constant at $i = -1.00$ mA until $t = 0$ and then varies with time afterward as shown in Figure P32.69b. Make a graph of the emf across the inductor as a function of time.

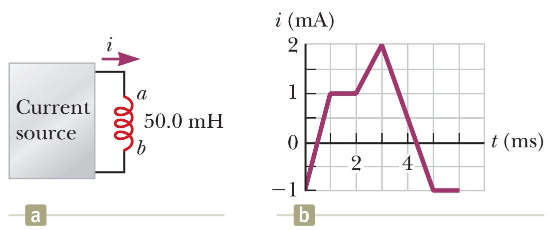


Figure P32.69

70. GP At $t = 0$, the open switch in Figure P32.70 is thrown closed. We wish to find a symbolic expression for the current in the inductor for time $t > 0$. Let this current be called i and choose it to be downward in the inductor in Figure P32.70. Identify i_1 as the current to the right through R_1 and i_2 as the current downward through R_2. (a) Use Kirchhoff's junction rule to find

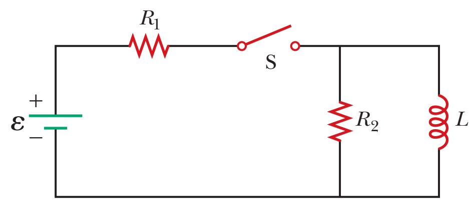


Figure P32.70

a relation among the three currents. (b) Use Kirchhoff's loop rule around the left loop to find another relationship. (c) Use Kirchhoff's loop rule around the outer loop to find a third relationship. (d) Eliminate i_1 and i_2 among the three equations to find an equation involving only the current i. (e) Compare the equation in part (d) with Equation 32.6 in the text. Use this comparison to rewrite Equation 32.7 in the text for the situation in this problem and show that

$$i(t) = \frac{\boldsymbol{\mathcal{E}}}{R_1}\left[1 - e^{-(R'/L)t}\right]$$

where $R' = R_1R_2/(R_1 + R_2)$.

71. The toroid in Figure P32.71 consists of N turns and has a rectangular cross section. Its inner and outer radii are a and b, respectively. The figure shows half of the toroid to allow us to see its cross-section. Compute the inductance of a 500-turn toroid for which $a = 10.0$ cm, $b = 12.0$ cm, and $h = 1.00$ cm.

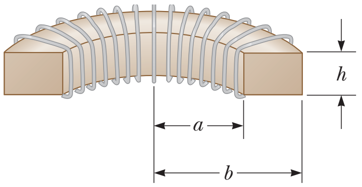


Figure P32.71 Problems 71 and 72.

72. The toroid in Figure P32.71 consists of N turns and has a rectangular cross section. Its inner and outer radii are a and b, respectively. Find the inductance of the toroid.

> Problems 73 through 76 apply ideas from this and earlier chapters to some properties of superconductors, which were introduced in Section 27.5.

73. Review. A novel method of storing energy has been proposed. A huge underground superconducting coil, 1.00 km in diameter, would be fabricated. It would carry a maximum current of 50.0 kA through each winding of a 150-turn Nb_3Sn solenoid. (a) If the inductance of this huge coil were 50.0 H, what would be the total energy stored? (b) What would be the compressive force per unit length acting between two adjacent windings 0.250 m apart?

74. Review. In an experiment carried out by S. C. Collins between 1955 and 1958, a current was maintained in a superconducting lead ring for 2.50 yr with no observed loss, even though there was no energy input. If the inductance of the ring were 3.14×10^{-8} H and the sensitivity of the experiment were 1 part in 10^9, what was the maximum resistance of the ring? *Suggestion:* Treat the ring as an *RL* circuit carrying decaying current and recall that the approximation $e^{-x} \approx 1 - x$ is valid for small x.

75. Review. The use of superconductors has been proposed for power transmission lines. A single coaxial cable (Fig. P32.75) could carry a power of 1.00×10^3 MW (the output of a large power plant) at 200 kV, DC, over a distance of 1.00×10^3 km without loss. An inner wire of radius $a = 2.00$ cm, made from the superconductor Nb_3Sn, carries the current I in one direction. A surrounding superconducting cylinder of radius $b = 5.00$ cm would carry the return current I. In such a system, what is the magnetic field (a) at the surface of the inner conductor and (b) at the inner surface of the outer conductor? (c) How much energy would be stored in the magnetic field in the space between the conductors in a 1.00×10^3 km superconducting line? (d) What is the pressure exerted on the outer conductor due to the current in the inner conductor?

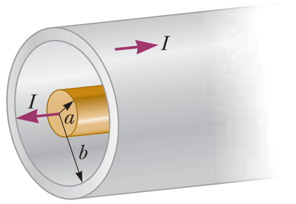


Figure P32.75

76. Review. A fundamental property of a type I superconducting material is *perfect diamagnetism,* or demonstration of the *Meissner effect,* illustrated in Figure 30.27 in Section 30.6 and described as follows. If a sample of superconducting material is placed into an externally produced magnetic field or is cooled to become superconducting while it is in a magnetic field, electric currents appear on the surface of the sample. The currents have precisely the strength and orientation required to make the total magnetic field be zero throughout the interior of the sample. This problem will help you understand the magnetic force that can then act on the sample. Compare this problem with Problem 65 in Chapter 26, pertaining to the force attracting a perfect dielectric into a strong electric field.

A vertical solenoid with a length of 120 cm and a diameter of 2.50 cm consists of 1 400 turns of copper wire carrying a counterclockwise current (when viewed from above) of 2.00 A as shown in Figure P32.76a (page 996). (a) Find the magnetic field in the vacuum inside the solenoid. (b) Find the energy density of the magnetic field. Now a superconducting bar 2.20 cm in diameter is inserted partway into the solenoid. Its upper end is far outside the solenoid, where the magnetic field is negligible. The lower end of the bar is deep inside the solenoid. (c) Explain how you identify the direction required for the current on the curved surface of the bar so that the total magnetic field is zero within the bar. The field created by the supercurrents is sketched in Figure P32.76b, and the total field is sketched in Figure P32.76c. (d) The field of the solenoid exerts a force on the current in the superconductor. Explain how you determine the direction of the force on the bar. (e) Noting that the units J/m^3 of energy density are the same as the units N/m^2 of pressure, calculate the magnitude of the force by multiplying the energy density of the solenoid field times the area of the bottom end of the superconducting bar.

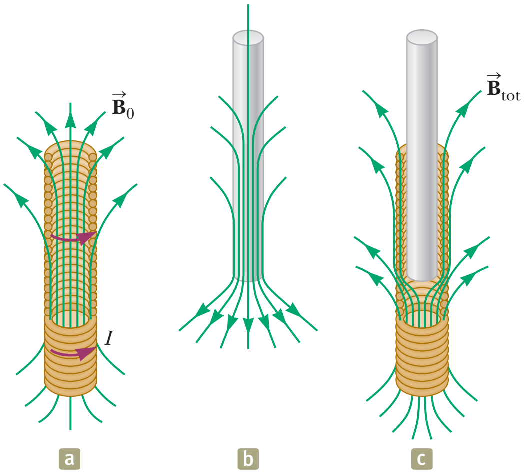


Figure P32.76

77. A wire of nonmagnetic material, with radius R, carries current uniformly distributed over its cross section. The total current carried by the wire is I. Show that the magnetic energy per unit length inside the wire is $\mu_0 I^2/16\pi$.

Challenge Problems

78. In earlier times when many households received nondigital television signals from an antenna, the lead-in wires from the antenna were often constructed in the form of two parallel wires (Fig. P32.78). The two wires carry currents of equal magnitude in opposite directions. The center-to-center separation of the wires is w, and a is their radius. Assume w is large enough compared with a that the wires carry the current uniformly distributed over their surfaces and negligible magnetic field exists inside the wires. (a) Why does this configuration of conductors have an inductance? (b) What constitutes the flux loop for this configuration? (c) Show that the inductance of a length x of this type of lead-in is

$$L = \frac{\mu_0 x}{\pi} \ln\left(\frac{w - a}{a}\right)$$

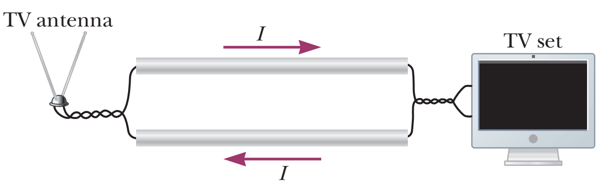


Figure P32.78

79. Assume the magnitude of the magnetic field outside a sphere of radius R is $B = B_0(R/r)^2$, where B_0 is a constant. (a) Determine the total energy stored in the magnetic field outside the sphere. (b) Evaluate your result from part (a) for $B_0 = 5.00 \times 10^{-5}$ T and $R = 6.00 \times 10^6$ m, values appropriate for the Earth's magnetic field.

80. In Figure P32.80, the battery has emf $\boldsymbol{\varepsilon} = 18.0$ V and the other circuit elements have values $L = 0.400$ H, $R_1 = 2.00$ kΩ, and $R_2 = 6.00$ kΩ. The switch is closed for $t < 0$, and steady-state conditions are established. The switch is then opened at $t = 0$. (a) Find the emf across L immediately after $t = 0$. (b) Which end of the coil, a or b, is at the higher potential? (c) Make graphs of the currents in R_1 and in R_2 as a function of time, treating the steady-state directions as positive. Show values before and after $t = 0$. (d) At what moment after $t = 0$ does the current in R_2 have the value 2.00 mA?

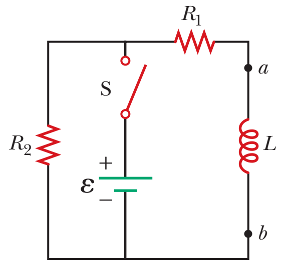


Figure P32.80

81. To prevent damage from arcing in an electric motor, a discharge resistor is sometimes placed in parallel with the armature. If the motor is suddenly unplugged while running, this resistor limits the voltage that appears across the armature coils. Consider a 12.0-V DC motor with an armature that has a resistance of 7.50 Ω and an inductance of 450 mH. Assume the magnitude of the self-induced emf in the armature coils is 10.0 V when the motor is running at normal speed. (The equivalent circuit for the armature is shown in Fig. P32.81.) Calculate the maximum resistance R that limits the voltage across the armature to 80.0 V when the motor is unplugged.

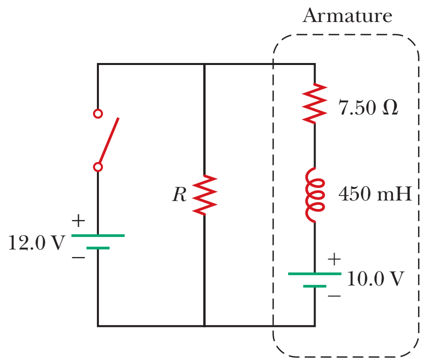


Figure P32.81

82. One application of an RL circuit is the generation of time-varying high voltage from a low-voltage source as shown in Figure P32.82. (a) What is the current in the circuit a long time after the switch has been in position a? (b) Now the switch is thrown quickly from a to b. Compute the initial voltage across each resistor and across the inductor. (c) How much time elapses before the voltage across the inductor drops to 12.0 V?

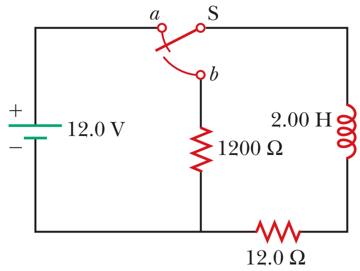


Figure P32.82

83. Two inductors having inductances L_1 and L_2 are connected in parallel as shown in Figure P32.83a. The mutual inductance between the two inductors is M. Determine the equivalent inductance L_{eq} for the system (Fig. P32.83b).

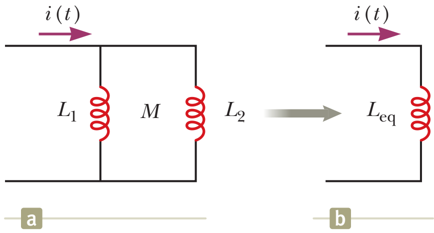


Figure P32.83

CHAPTER

33

Alternating-Current Circuits

These large transformers are used to increase the voltage at a power plant for distribution of energy by electrical transmission to the power grid. Voltages can be changed relatively easily because power is distributed by alternating current rather than direct current. *(©Lester Lefkowitz/Getty Images)*

In this chapter, we describe alternating-current (AC) circuits. Every time you turn on a television set, a computer, or any of a multitude of other electrical appliances in a home, you are calling on alternating currents to provide the power to operate them. We begin our study by investigating the characteristics of simple series circuits that contain resistors, inductors, and capacitors and that are driven by a sinusoidal voltage. The primary aim of this chapter can be summarized as follows: if an AC source applies an alternating voltage to a series circuit containing resistors, inductors, and capacitors, we want to know the amplitude and time characteristics of the alternating current. We conclude this chapter with two sections concerning transformers, power transmission, and electrical filters.

33.1 AC Sources

An AC circuit consists of circuit elements and a power source that provides an alternating voltage Δv. This time-varying voltage from the source is described by

$$\Delta v = \Delta V_{\max} \sin \omega t$$

where $\Delta V_{\max}$ is the maximum output voltage of the source, or the **voltage amplitude.** There are various possibilities for AC sources, including generators as discussed in Section 31.5 and electrical oscillators. In a home, each electrical outlet serves as an AC source. Because the output voltage of an AC source varies sinusoidally with time, the voltage is positive during one half of the cycle and negative during the other half as in Figure 33.1. Likewise, the current in any circuit driven by an AC source is an alternating current that also varies sinusoidally with time.

From Equation 15.12, the angular frequency of the AC voltage is

$$\omega = 2\pi f = \frac{2\pi}{T}$$

where f is the frequency of the source and T is the period. The source determines the frequency of the current in any circuit connected to it. Commercial electric-power plants in the United States use a frequency of 60.0 Hz, which corresponds to an angular frequency of 377 rad/s.

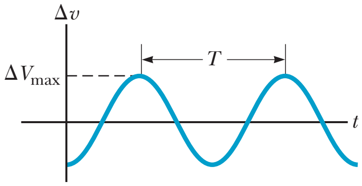


Figure 33.1 The voltage supplied by an AC source is sinusoidal with a period T.

33.2 Resistors in an AC Circuit

Consider a simple AC circuit consisting of a resistor and an AC source  as shown in Figure 33.2. At any instant, the algebraic sum of the voltages around a closed loop in a circuit must be zero (Kirchhoff's loop rule). Therefore, $\Delta v + \Delta v_R = 0$ or, using Equation 27.7 for the voltage across the resistor,

$$\Delta v - i_R R = 0$$

If we rearrange this expression and substitute $\Delta V_{max} \sin \omega t$ for Δv, the instantaneous current in the resistor is

$$i_R = \frac{\Delta v}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin \omega t \quad \textbf{(33.1)}$$

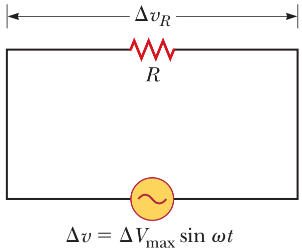


Figure 33.2 A circuit consisting of a resistor of resistance R connected to an AC source, designated by the symbol .

where I_{max} is the maximum current:

$$I_{max} = \frac{\Delta V_{max}}{R} \quad \textbf{(33.2)}$$

◀ **Maximum current in a resistor**

Equation 33.1 shows that the instantaneous voltage across the resistor is

$$\Delta v_R = i_R R = I_{max} R \sin \omega t \quad \textbf{(33.3)}$$

◀ **Voltage across a resistor**

A plot of voltage and current versus time for this circuit is shown in Figure 33.3a on page 1000. At point *a*, the current has a maximum value in one direction, arbitrarily called the positive direction. Between points *a* and *b*, the current is decreasing in magnitude but is still in the positive direction. At point *b*, the current is momentarily zero; it then begins to increase in the negative direction between points *b* and *c*. At point *c*, the current has reached its maximum value in the negative direction.

The current and voltage are in step with each other because they vary identically with time. Because i_R and Δv_R both vary as $\sin \omega t$ and reach their maximum values at the same time as shown in Figure 33.3a, they are said to be **in phase,** similar to the way two waves can be in phase as discussed in our study of wave motion in Chapter 18. Therefore, for a sinusoidal applied voltage, the current in a resistor is always in phase with the voltage across the resistor. For resistors in AC circuits, there are no new concepts to learn. Resistors behave essentially the same way in both DC and AC circuits. That, however, is not the case for capacitors and inductors.

To simplify our analysis of circuits containing two or more elements, we use a graphical representation called a *phasor diagram.* A **phasor** is a vector whose length is proportional to the maximum value of the variable it represents (ΔV_{max} for voltage and I_{max} for current in this discussion). The phasor rotates counterclockwise at an angular speed equal to the angular frequency associated with the variable. The

Pitfall Prevention 33.1

Time-Varying Values We continue to use lowercase symbols Δv and i to indicate the instantaneous values of time-varying voltages and currents. We will add a subscript to indicate the appropriate circuit element. Capital letters represent fixed values of voltage and current such as ΔV_{max} and I_{max}.

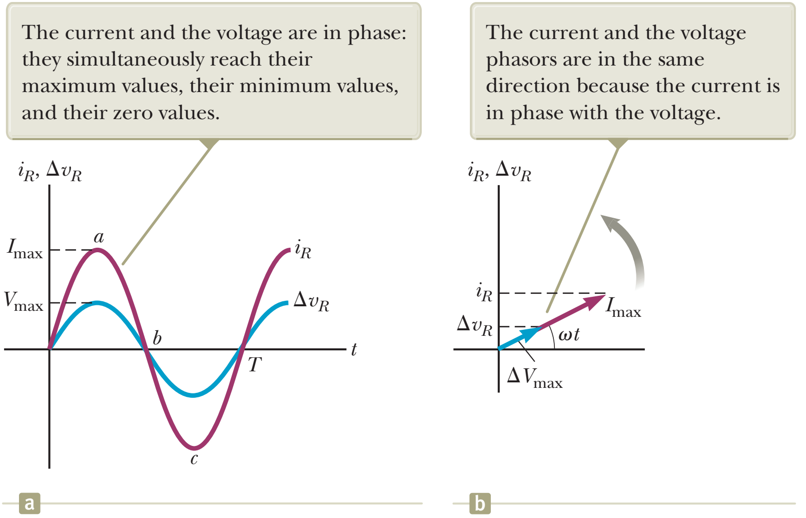


Figure 33.3 (a) Plots of the instantaneous current i_R and instantaneous voltage Δv_R across a resistor as functions of time. At time $t = T$, one cycle of the time-varying voltage and current has been completed. (b) Phasor diagram for the resistive circuit showing that the current is in phase with the voltage.

Pitfall Prevention 33.2

A Phasor Is Like a Graph An alternating voltage can be presented in different representations. One graphical representation is shown in Figure 33.1 in which the voltage is drawn in rectangular coordinates, with voltage on the vertical axis and time on the horizontal axis. Figure 33.3b shows another graphical representation. The phase space in which the phasor is drawn is similar to polar coordinate graph paper. The radial coordinate represents the amplitude of the voltage. The angular coordinate is the phase angle. The vertical-axis coordinate of the tip of the phasor represents the instantaneous value of the voltage. The horizontal coordinate represents nothing at all. As shown in Figure 33.3b, alternating currents can also be represented by phasors.

To help with this discussion of phasors, review Section 15.4, where we represented the simple harmonic motion of a real object by the projection of an imaginary object's uniform circular motion onto a coordinate axis. A phasor is a direct analog to this representation.

projection of the phasor onto the vertical axis represents the instantaneous value of the quantity it represents.

Figure 33.3b shows voltage and current phasors for the circuit of Figure 33.2 at some instant of time. The projections of the phasor arrows onto the vertical axis are determined by a sine function of the angle of the phasor with respect to the horizontal axis. For example, the projection of the current phasor in Figure 33.3b is $I_{max} \sin \omega t$. Notice that this expression is the same as Equation 33.1. Therefore, the projections of phasors represent current values that vary sinusoidally in time. We can do the same with time-varying voltages. The advantage of this approach is that the phase relationships among currents and voltages can be represented as vector additions of phasors using the vector addition techniques discussed in Chapter 3.

In the case of the single-loop resistive circuit of Figure 33.2, the current and voltage phasors are in the same direction in Figure 33.3b because i_R and Δv_R are in phase. The current and voltage in circuits containing capacitors and inductors have different phase relationships.

Quick Quiz 33.1 Consider the voltage phasor in Figure 33.4, shown at three instants of time. **(i)** Choose the part of the figure, (a), (b), or (c), that represents the instant of time at which the instantaneous value of the voltage has the largest magnitude. **(ii)** Choose the part of the figure that represents the instant of time at which the instantaneous value of the voltage has the smallest magnitude.

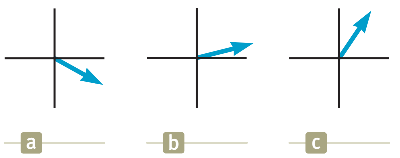


Figure 33.4 (Quick Quiz 33.1) A voltage phasor is shown at three instants of time, (a), (b), and (c).

For the simple resistive circuit in Figure 33.2, notice that the average value of the current over one cycle is zero. That is, the current is maintained in the positive direction for the same amount of time and at the same magnitude as it is maintained in the negative direction. The direction of the current, however, has no effect on the behavior of the resistor. We can understand this concept by realizing that collisions between electrons and the fixed atoms of the resistor result in an

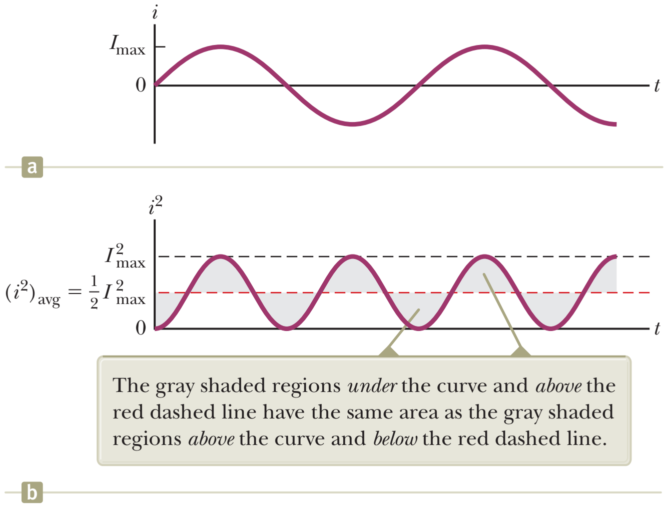


Figure 33.5 (a) Graph of the current in a resistor as a function of time. (b) Graph of the current squared in a resistor as a function of time, showing that the red dashed line is the average of $I^2_{max}\sin^2\omega t$. In general, the average value of $\sin^2\omega t$ or $\cos^2\omega t$ over one cycle is $\frac{1}{2}$.

increase in the resistor's temperature. Although this temperature increase depends on the magnitude of the current, it is independent of the current's direction.

We can make this discussion quantitative by recalling that the rate at which energy is delivered to a resistor is the power $P = i^2R$, where i is the instantaneous current in the resistor. Because this rate is proportional to the square of the current, it makes no difference whether the current is direct or alternating, that is, whether the sign associated with the current is positive or negative. The temperature increase produced by an alternating current having a maximum value I_{max}, however, is not the same as that produced by a direct current equal to I_{max} because the alternating current has this maximum value for only an instant during each cycle (Fig. 33.5a). What is of importance in an AC circuit is an average value of current, referred to as the **rms current.** As we learned in Section 21.1, the notation *rms* stands for *root-mean-square,* which in this case means the square root of the mean (average) value of the square of the current: $I_{rms} = \sqrt{(i^2)_{avg}}$. Because i^2 varies as $\sin^2\omega t$ and because the average value of i^2 is $\frac{1}{2}I^2_{max}$ (see Fig. 33.5b), the rms current is

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max} \quad (33.4)$$

◀ **rms current**

This equation states that an alternating current whose maximum value is 2.00 A delivers to a resistor the same power as a direct current that has a value of (0.707)(2.00 A) = 1.41 A. The average power delivered to a resistor that carries an alternating current is

$$P_{avg} = I^2_{rms}R$$

◀ **Average power delivered to a resistor**

Alternating voltage is also best discussed in terms of rms voltage, and the relationship is identical to that for current:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707\,\Delta V_{max} \quad (33.5)$$

◀ **rms voltage**

When we speak of measuring a 120-V alternating voltage from an electrical outlet, we are referring to an rms voltage of 120 V. A calculation using Equation 33.5 shows that such an alternating voltage has a maximum value of about 170 V. One reason rms values are often used when discussing alternating currents and voltages is that AC ammeters and voltmeters are designed to read rms values. Furthermore, with rms values, many of the equations we use have the same form as their direct-current counterparts.

Example 33.1 What Is the rms Current?

The voltage output of an AC source is given by the expression $\Delta v = 200 \sin \omega t$, where Δv is in volts. Find the rms current in the circuit when this source is connected to a 100-Ω resistor.

SOLUTION

Conceptualize Figure 33.2 shows the physical situation for this problem.

Categorize We evaluate the current with an equation developed in this section, so we categorize this example as a substitution problem.

Combine Equations 33.2 and 33.4 to find the rms current:

$$I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} = \frac{\Delta V_{\text{max}}}{\sqrt{2}\,R}$$

Comparing the expression for voltage output with the general form $\Delta v = \Delta V_{\text{max}} \sin \omega t$ shows that $\Delta V_{\text{max}} = 200$ V. Substitute numerical values:

$$I_{\text{rms}} = \frac{200\text{ V}}{\sqrt{2}\,(100\ \Omega)} = \boxed{1.41\text{ A}}$$

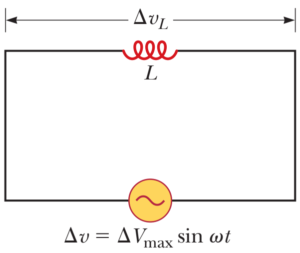


Figure 33.6 A circuit consisting of an inductor of inductance L connected to an AC source.

33.3 Inductors in an AC Circuit

Now consider an AC circuit consisting only of an inductor connected to the terminals of an AC source as shown in Figure 33.6. Because $\Delta v_L = -L(di_L/dt)$ is the self-induced instantaneous voltage across the inductor (see Eq. 32.1), Kirchhoff's loop rule applied to this circuit gives $\Delta v + \Delta v_L = 0$, or

$$\Delta v - L\frac{di_L}{dt} = 0$$

Substituting $\Delta V_{\text{max}} \sin \omega t$ for Δv and rearranging gives

$$\Delta v = L\frac{di_L}{dt} = \Delta V_{\text{max}} \sin \omega t \qquad \textbf{(33.6)}$$

Solving this equation for di_L gives

$$di_L = \frac{\Delta V_{\text{max}}}{L} \sin \omega t\, dt$$

Integrating this expression[1] gives the instantaneous current i_L in the inductor as a function of time:

$$i_L = \frac{\Delta V_{\text{max}}}{L}\int \sin \omega t\, dt = -\frac{\Delta V_{\text{max}}}{\omega L}\cos \omega t \qquad \textbf{(33.7)}$$

Using the trigonometric identity $\cos \omega t = -\sin(\omega t - \pi/2)$, we can express Equation 33.7 as

Current in an inductor ▶

$$i_L = \frac{\Delta V_{\text{max}}}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) \qquad \textbf{(33.8)}$$

Comparing this result with Equation 33.6 shows that the instantaneous current i_L in the inductor and the instantaneous voltage Δv_L across the inductor are *out* of phase by $\pi/2$ rad $= 90°$.

A plot of voltage and current versus time is shown in Figure 33.7a. When the current i_L in the inductor is a maximum (point b in Fig. 33.7a), it is momentarily

[1]We neglect the constant of integration here because it depends on the initial conditions, which are not important for this situation.

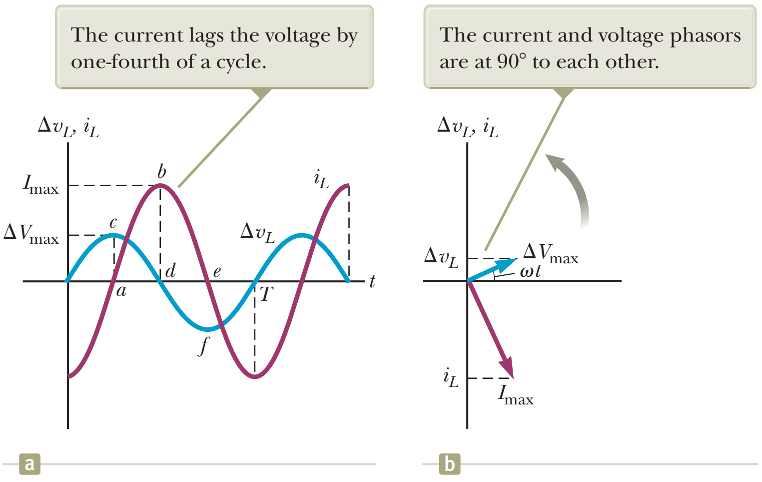


Figure 33.7 (a) Plots of the instantaneous current i_L and instantaneous voltage Δv_L across an inductor as functions of time. (b) Phasor diagram for the inductive circuit.

not changing, so the voltage across the inductor is zero (point d). At points such as a and e, the current is zero and the rate of change of current is at a maximum. Therefore, the voltage across the inductor is also at a maximum (points c and f). Notice that the voltage reaches its maximum value one-quarter of a period before the current reaches its maximum value. Therefore, for a sinusoidal applied voltage, the current in an inductor always *lags* behind the voltage across the inductor by 90° (one-quarter cycle in time).

As with the relationship between current and voltage for a resistor, we can represent this relationship for an inductor with a phasor diagram as in Figure 33.7b. The phasors are at 90° to each other, representing the 90° phase difference between current and voltage.

Equation 33.7 shows that the current in an inductive circuit reaches its maximum value when $\cos \omega t = \pm 1$:

$$I_{max} = \frac{\Delta V_{max}}{\omega L} \qquad (33.9)$$

◀ **Maximum current in an inductor**

This expression is similar to the relationship between current, voltage, and resistance in a DC circuit, $I = \Delta V/R$ (Eq. 27.7). Because I_{max} has units of amperes and ΔV_{max} has units of volts, ωL must have units of ohms. Therefore, ωL has the same units as resistance and is related to current and voltage in the same way as resistance. It must behave in a manner similar to resistance in the sense that it represents opposition to the flow of charge. Because ωL depends on the applied frequency ω, the inductor *reacts* differently, in terms of offering opposition to current, for different frequencies. For this reason, we define ωL as the **inductive reactance** X_L:

$$X_L \equiv \omega L \qquad (33.10)$$

◀ **Inductive reactance**

Therefore, we can write Equation 33.9 as

$$I_{max} = \frac{\Delta V_{max}}{X_L} \qquad (33.11)$$

The expression for the rms current in an inductor is similar to Equation 33.11, with I_{max} replaced by I_{rms} and ΔV_{max} replaced by ΔV_{rms}.

Equation 33.10 indicates that, for a given applied voltage, the inductive reactance increases as the frequency increases. This conclusion is consistent with Faraday's law: the greater the rate of change of current in the inductor, the larger the back emf. The larger back emf translates to an increase in the reactance and a decrease in the current.

Using Equations 33.6 and 33.11, we find that the instantaneous voltage across the inductor is

Voltage across an inductor ▶

$$\Delta v_L = -L\frac{di_L}{dt} = -\Delta V_{\max}\sin\omega t = -I_{\max}X_L\sin\omega t \tag{33.12}$$

Quick Quiz 33.2 Consider the AC circuit in Figure 33.8. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest? **(a)** It glows brightest at high frequencies. **(b)** It glows brightest at low frequencies. **(c)** The brightness is the same at all frequencies.

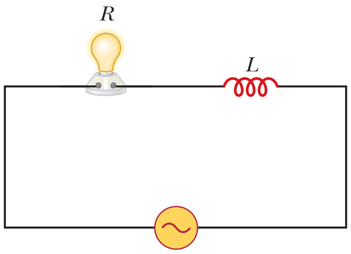


Figure 33.8 (Quick Quiz 33.2) At what frequencies does the lightbulb glow the brightest?

Example 33.2 A Purely Inductive AC Circuit

In a purely inductive AC circuit, $L = 25.0$ mH and the rms voltage is 150 V. Calculate the inductive reactance and rms current in the circuit if the frequency is 60.0 Hz.

SOLUTION

Conceptualize Figure 33.6 shows the physical situation for this problem. Keep in mind that inductive reactance increases with increasing frequency of the applied voltage.

Categorize We determine the reactance and the current from equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 33.10 to find the inductive reactance:

$$X_L = \omega L = 2\pi f L = 2\pi(60.0\text{ Hz})(25.0\times10^{-3}\text{ H})$$
$$= \boxed{9.42\ \Omega}$$

Use an rms version of Equation 33.11 to find the rms current:

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{X_L} = \frac{150\text{ V}}{9.42\ \Omega} = \boxed{15.9\text{ A}}$$

WHAT IF? If the frequency increases to 6.00 kHz, what happens to the rms current in the circuit?

Answer If the frequency increases, the inductive reactance also increases because the current is changing at a higher rate. The increase in inductive reactance results in a lower current.

Let's calculate the new inductive reactance and the new rms current:

$$X_L = 2\pi(6.00\times10^{3}\text{ Hz})(25.0\times10^{-3}\text{ H}) = 942\ \Omega$$

$$I_{\text{rms}} = \frac{150\text{ V}}{942\ \Omega} = 0.159\text{ A}$$

33.4 Capacitors in an AC Circuit

Figure 33.9 shows an AC circuit consisting of a capacitor connected across the terminals of an AC source. Kirchhoff's loop rule applied to this circuit gives $\Delta v + \Delta v_C = 0$, or

$$\Delta v - \frac{q}{C} = 0 \tag{33.13}$$

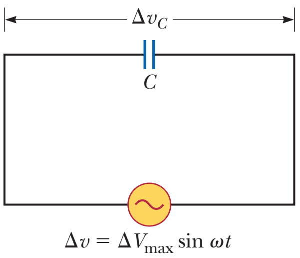


Figure 33.9 A circuit consisting of a capacitor of capacitance C connected to an AC source.

Substituting $\Delta V_{max} \sin \omega t$ for Δv and rearranging gives

$$q = C \Delta V_{max} \sin \omega t \qquad \textbf{(33.14)}$$

where q is the instantaneous charge on the capacitor. Differentiating Equation 33.14 with respect to time gives the instantaneous current in the circuit:

$$i_C = \frac{dq}{dt} = \omega C \Delta V_{max} \cos \omega t \qquad \textbf{(33.15)}$$

Using the trigonometric identity

$$\cos \omega t = \sin \left(\omega t + \frac{\pi}{2} \right)$$

we can express Equation 33.15 in the alternative form

$$i_C = \omega C \Delta V_{max} \sin \left(\omega t + \frac{\pi}{2} \right) \qquad \textbf{(33.16)}$$

◀ **Current in a capacitor**

Comparing this expression with $\Delta v = \Delta V_{max} \sin \omega t$ shows that the current is $\pi/2$ rad $= 90°$ out of phase with the voltage across the capacitor. A plot of current and voltage versus time (Fig. 33.10a) shows that the current reaches its maximum value one-quarter of a cycle sooner than the voltage reaches its maximum value.

Consider a point such as b in Figure 33.10a where the current is zero at this instant. That occurs when the capacitor reaches its maximum charge so that the voltage across the capacitor is a maximum (point d). At points such as a and e, the current is a maximum, which occurs at those instants when the charge on the capacitor reaches zero and the capacitor begins to recharge with the opposite polarity. When the charge is zero, the voltage across the capacitor is zero (points c and f).

As with inductors, we can represent the current and voltage for a capacitor on a phasor diagram. The phasor diagram in Figure 33.10b shows that for a sinusoidally applied voltage, the current always *leads* the voltage across a capacitor by 90°.

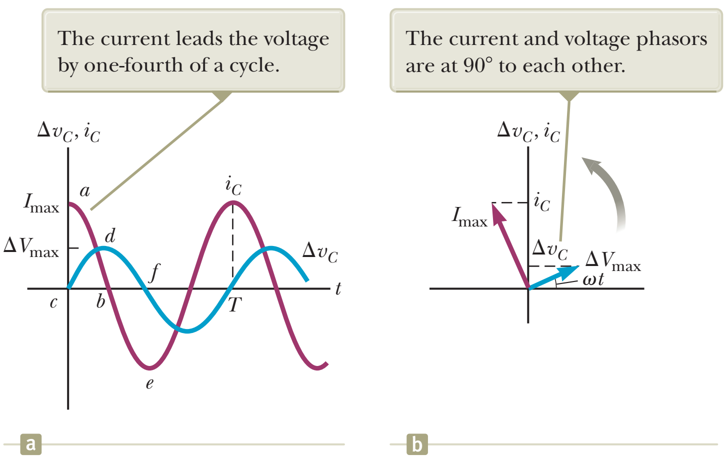


Figure 33.10 (a) Plots of the instantaneous current i_C and instantaneous voltage Δv_C across a capacitor as functions of time. (b) Phasor diagram for the capacitive circuit.

Equation 33.15 shows that the current in the circuit reaches its maximum value when $\cos \omega t = \pm 1$:

$$I_{\max} = \omega C \Delta V_{\max} = \frac{\Delta V_{\max}}{(1/\omega C)} \tag{33.17}$$

As in the case with inductors, this looks like Equation 27.7, so the denominator plays the role of resistance, with units of ohms. We give the combination $1/\omega C$ the symbol X_C, and because this function varies with frequency, we define it as the **capacitive reactance:**

Capacitive reactance ▶

$$X_C \equiv \frac{1}{\omega C} \tag{33.18}$$

We can now write Equation 33.17 as

Maximum current in a capacitor ▶

$$I_{\max} = \frac{\Delta V_{\max}}{X_C} \tag{33.19}$$

The rms current is given by an expression similar to Equation 33.19, with $I_{\max}$ replaced by I_{rms} and $\Delta V_{\max}$ replaced by ΔV_{rms}.

Using Equation 33.19, we can express the instantaneous voltage across the capacitor as

Voltage across a capacitor ▶

$$\Delta v_C = \Delta V_{\max} \sin \omega t = I_{\max} X_C \sin \omega t \tag{33.20}$$

Equations 33.18 and 33.19 indicate that as the frequency of the voltage source increases, the capacitive reactance decreases and the maximum current therefore increases. The frequency of the current is determined by the frequency of the voltage source driving the circuit. As the frequency approaches zero, the capacitive reactance approaches infinity and the current therefore approaches zero. This conclusion makes sense because the circuit approaches direct current conditions as ω approaches zero and the capacitor represents an open circuit.

Quick Quiz 33.3 Consider the AC circuit in Figure 33.11. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest? **(a)** It glows brightest at high frequencies. **(b)** It glows brightest at low frequencies. **(c)** The brightness is the same at all frequencies.

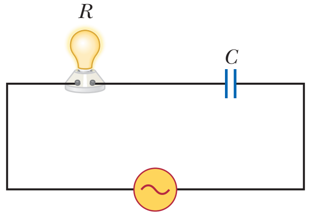


Figure 33.11 (Quick Quiz 33.3)

Quick Quiz 33.4 Consider the AC circuit in Figure 33.12. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest? **(a)** It glows brightest at high frequencies. **(b)** It glows brightest at low frequencies. **(c)** The brightness is the same at all frequencies.

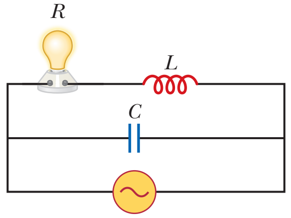


Figure 33.12 (Quick Quiz 33.4)

Example 33.3 A Purely Capacitive AC Circuit

An 8.00-μF capacitor is connected to the terminals of a 60.0-Hz AC source whose rms voltage is 150 V. Find the capacitive reactance and the rms current in the circuit.

SOLUTION

Conceptualize Figure 33.9 shows the physical situation for this problem. Keep in mind that capacitive reactance decreases with increasing frequency of the applied voltage.

Categorize We determine the reactance and the current from equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 33.18 to find the capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(60.0\ \text{Hz})(8.00 \times 10^{-6}\ \text{F})} = \boxed{332\ \Omega}$$

Use an rms version of Equation 33.19 to find the rms current:

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{X_C} = \frac{150\ \text{V}}{332\ \Omega} = \boxed{0.452\ \text{A}}$$

WHAT IF? What if the frequency is doubled? What happens to the rms current in the circuit?

Answer If the frequency increases, the capacitive reactance decreases, which is just the opposite from the case of an inductor. The decrease in capacitive reactance results in an increase in the current.

Let's calculate the new capacitive reactance and the new rms current:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi(120\ \text{Hz})(8.00 \times 10^{-6}\ \text{F})} = 166\ \Omega$$

$$I_{\text{rms}} = \frac{150\ \text{V}}{166\ \Omega} = 0.904\ \text{A}$$

33.5 The *RLC* Series Circuit

In the previous sections, we considered individual circuit elements connected to an AC source. Figure 33.13a shows a circuit that contains a combination of circuit elements: a resistor, an inductor, and a capacitor connected in series across an alternating-voltage source. If the applied voltage varies sinusoidally with time, the instantaneous applied voltage is

$$\Delta v = \Delta V_{\max} \sin \omega t$$

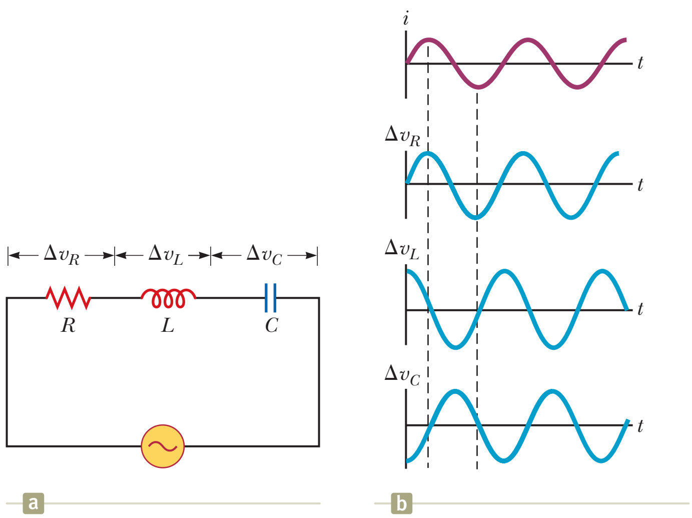


Figure 33.13 (a) A series circuit consisting of a resistor, an inductor, and a capacitor connected to an AC source. (b) Phase relationships between the current and the voltages in the individual circuit elements if they were connected alone to the AC source.

Figure 33.13b shows the voltage versus time across each element in the circuit and its phase relationships to the current if it were connected individually to the AC source, as discussed in Sections 33.2–33.4.

When the circuit elements are all connected together to the AC source, as in Figure 33.13a, the current in the circuit is given by

$$i = I_{\max} \sin (\omega t - \phi)$$

where ϕ is some **phase angle** between the current and the applied voltage. Based on our discussions of phase in Sections 33.3 and 33.4, we expect that the current will generally not be in phase with the voltage in an *RLC* circuit.

Because the circuit elements in Figure 33.13a are in series, the current everywhere in the circuit must be the same at any instant. That is, the current at all points in a series AC circuit has the same amplitude and phase. Based on the preceding sections, we know that the voltage across each element has a different amplitude and phase. In particular, the voltage across the resistor is in phase with the current, the voltage across the inductor leads the current by 90°, and the voltage across the capacitor lags behind the current by 90°. Using these phase relationships, we can express the instantaneous voltages across the three circuit elements as

$$\Delta v_R = I_{\max} R \sin \omega t = \Delta V_R \sin \omega t \tag{33.21}$$

$$\Delta v_L = I_{\max} X_L \sin \left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t \tag{33.22}$$

$$\Delta v_C = I_{\max} X_C \sin \left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t \tag{33.23}$$

The sum of these three voltages must equal the instantaneous voltage from the AC source, but it is important to recognize that because the three voltages have different phase relationships with the current, they cannot be added directly. Figure 33.14 represents the phasors at an instant at which the current in all three elements is momentarily zero. The zero current is represented by the current phasor along the horizontal axis in each part of the figure. Next the voltage phasor is drawn at the appropriate phase angle to the current for each element.

Because phasors are rotating vectors, the voltage phasors in Figure 33.14 can be combined using vector addition as in Figure 33.15. In Figure 33.15a, the voltage phasors in Figure 33.14 are combined on the same coordinate axes. Figure 33.15b shows the vector addition of the voltage phasors. The voltage phasors ΔV_L and ΔV_C are in *opposite* directions along the same line, so we can construct the difference phasor $\Delta V_L - \Delta V_C$, which is perpendicular to the phasor ΔV_R. This diagram shows that the vector sum of the voltage amplitudes ΔV_R, ΔV_L, and ΔV_C equals a phasor whose length is the maximum applied voltage $\Delta V_{\max}$ and which makes an angle ϕ with the current phasor $I_{\max}$. From the right triangle in Figure 33.15b, we see that

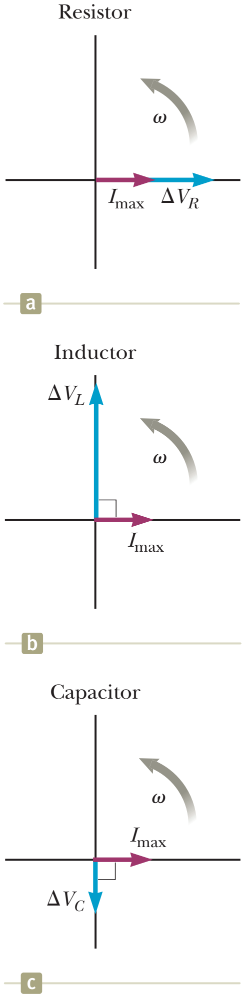


Figure 33.14 Phase relationships between the voltage and current phasors for (a) a resistor, (b) an inductor, and (c) a capacitor connected in series.

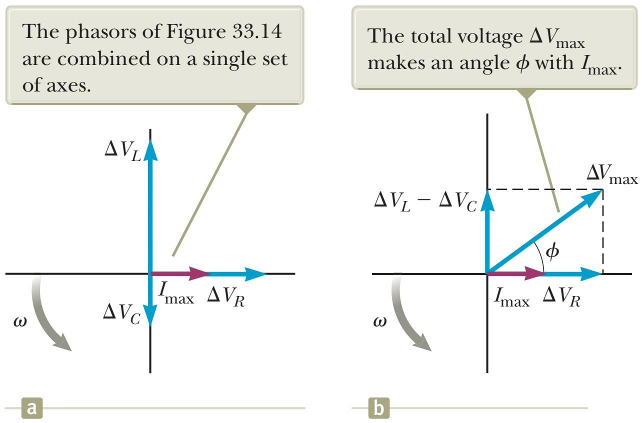


Figure 33.15 (a) Phasor diagram for the series *RLC* circuit shown in Figure 33.13a. (b) The inductance and capacitance phasors are added together and then added vectorially to the resistance phasor.

$$\Delta V_{max} = \sqrt{\Delta V_R^2 + (\Delta V_L - \Delta V_C)^2} = \sqrt{(I_{max}R)^2 + (I_{max}X_L - I_{max}X_C)^2}$$

$$\Delta V_{max} = I_{max}\sqrt{R^2 + (X_L - X_C)^2}$$

Therefore, we can express the maximum current as

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{33.24}$$

Maximum current in an *RLC* circuit

Once again, this expression has the same mathematical form as Equation 27.7. The denominator of the fraction plays the role of resistance and is called the **impedance** Z of the circuit:

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \tag{33.25}$$

Impedance

where impedance also has units of ohms. Therefore, Equation 33.24 can be written in the form

$$I_{max} = \frac{\Delta V_{max}}{Z} \tag{33.26}$$

Equation 33.26 is the AC equivalent of Equation 27.7. Note that the impedance and therefore the current in an AC circuit depend on the resistance, the inductance, the capacitance, and the frequency (because the reactances are frequency dependent).

From the right triangle in the phasor diagram in Figure 33.15b, the phase angle ϕ between the current and the voltage is found as follows:

$$\phi = \tan^{-1}\left(\frac{\Delta V_L - \Delta V_C}{\Delta V_R}\right) = \tan^{-1}\left(\frac{I_{max}X_L - I_{max}X_C}{I_{max}R}\right)$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \tag{33.27}$$

Phase angle

When $X_L > X_C$ (which occurs at high frequencies), the phase angle is positive, signifying that the current lags the applied voltage as in Figure 33.15b. We describe this situation by saying that the circuit is *more inductive than capacitive.* When $X_L < X_C$, the phase angle is negative, signifying that the current leads the applied voltage, and the circuit is *more capacitive than inductive.* When $X_L = X_C$, the phase angle is zero and the circuit is *purely resistive.*

Quick Quiz 33.5 Label each part of Figure 33.16, (a), (b), and (c), as representing $X_L > X_C$, $X_L = X_C$, or $X_L < X_C$.

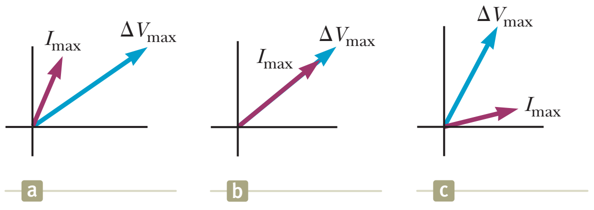


Figure 33.16 (Quick Quiz 33.5) Match the phasor diagrams to the relationships between the reactances.

Example 33.4 Analyzing a Series *RLC* Circuit

A series *RLC* circuit has $R = 425\ \Omega$, $L = 1.25$ H, and $C = 3.50\ \mu$F. It is connected to an AC source with $f = 60.0$ Hz and $\Delta V_{max} = 150$ V.

(A) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit.

continued

▸ 33.4 continued

SOLUTION

Conceptualize The circuit of interest in this example is shown in Figure 33.13a. The current in the combination of the resistor, inductor, and capacitor oscillates at a particular phase angle with respect to the applied voltage.

Categorize The circuit is a simple series *RLC* circuit, so we can use the approach discussed in this section.

Analyze Find the angular frequency:

$$\omega = 2\pi f = 2\pi(60.0\text{ Hz}) = 377\text{ s}^{-1}$$

Use Equation 33.10 to find the inductive reactance:

$$X_L = \omega L = (377\text{ s}^{-1})(1.25\text{ H}) = \boxed{471\ \Omega}$$

Use Equation 33.18 to find the capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{(377\text{ s}^{-1})(3.50\times10^{-6}\text{ F})} = \boxed{758\ \Omega}$$

Use Equation 33.25 to find the impedance:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(425\ \Omega)^2 + (471\ \Omega - 758\ \Omega)^2} = \boxed{513\ \Omega}$$

(B) Find the maximum current in the circuit.

SOLUTION

Use Equation 33.26 to find the maximum current:

$$I_{\text{max}} = \frac{\Delta V_{\text{max}}}{Z} = \frac{150\text{ V}}{513\ \Omega} = \boxed{0.293\text{ A}}$$

(C) Find the phase angle between the current and voltage.

SOLUTION

Use Equation 33.27 to calculate the phase angle:

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{471\ \Omega - 758\ \Omega}{425\ \Omega}\right) = \boxed{-34.0^\circ}$$

(D) Find the maximum voltage across each element.

SOLUTION

Use Equations 33.2, 33.11, and 33.19 to calculate the maximum voltages:

$$\Delta V_R = I_{\text{max}}R = (0.293\text{ A})(425\ \Omega) = \boxed{124\text{ V}}$$

$$\Delta V_L = I_{\text{max}}X_L = (0.293\text{ A})(471\ \Omega) = \boxed{138\text{ V}}$$

$$\Delta V_C = I_{\text{max}}X_C = (0.293\text{ A})(758\ \Omega) = \boxed{222\text{ V}}$$

(E) What replacement value of *L* should an engineer analyzing the circuit choose such that the current leads the applied voltage by 30.0° rather than 34.0°? All other values in the circuit stay the same.

SOLUTION

Solve Equation 33.27 for the inductive reactance:

$$X_L = X_C + R\tan\phi$$

Substitute Equations 33.10 and 33.18 into this expression:

$$\omega L = \frac{1}{\omega C} + R\tan\phi$$

Solve for *L*:

$$L = \frac{1}{\omega}\left(\frac{1}{\omega C} + R\tan\phi\right)$$

Substitute the given values:

$$L = \frac{1}{(377\text{ s}^{-1})}\left[\frac{1}{(377\text{ s}^{-1})(3.50\times10^{-6}\text{ F})} + (425\ \Omega)\tan(-30.0^\circ)\right]$$

$$L = \boxed{1.36\text{ H}}$$

Finalize Because the capacitive reactance is larger than the inductive reactance, the circuit is more capacitive than inductive. In this case, the phase angle ϕ is negative, so the current leads the applied voltage.

33.4 continued

Using Equations 33.21, 33.22, and 33.23, the instantaneous voltages across the three elements are

$$\Delta v_R = (124\text{ V})\sin 377t$$

$$\Delta v_L = (138\text{ V})\cos 377t$$

$$\Delta v_C = (-222\text{ V})\cos 377t$$

WHAT IF? What if you added up the maximum voltages across the three circuit elements? Is that a physically meaningful quantity?

Answer The sum of the maximum voltages across the elements is $\Delta V_R + \Delta V_L + \Delta V_C = 484$ V. This sum is much greater than the maximum voltage of the source, 150 V. The sum of the maximum voltages is a meaningless quantity because when sinusoidally varying quantities are added, *both their amplitudes and their phases* must be taken into account. The maximum voltages across the various elements occur at different times. Therefore, the voltages must be added in a way that takes account of the different phases as shown in Figure 33.15.

33.6 Power in an AC Circuit

Now let's take an energy approach to analyzing AC circuits and consider the transfer of energy from the AC source to the circuit. The power delivered by a battery to an external DC circuit is equal to the product of the current and the terminal voltage of the battery. Likewise, the instantaneous power delivered by an AC source to a circuit is the product of the current and the applied voltage. For the *RLC* circuit shown in Figure 33.13a, we can express the instantaneous power P as

$$P = i\,\Delta v = I_{\max}\sin(\omega t - \phi)\,\Delta V_{\max}\sin\omega t$$

$$P = I_{\max}\,\Delta V_{\max}\sin\omega t\sin(\omega t - \phi) \qquad \textbf{(33.28)}$$

This result is a complicated function of time and is therefore not very useful from a practical viewpoint. What is generally of interest is the average power over one or more cycles. Such an average can be computed by first using the trigonometric identity $\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$. Substituting this identity into Equation 33.28 gives

$$P = I_{\max}\,\Delta V_{\max}\sin^2\omega t\cos\phi - I_{\max}\,\Delta V_{\max}\sin\omega t\cos\omega t\sin\phi \qquad \textbf{(33.29)}$$

Let's now take the time average of P over one or more cycles, noting that $I_{\max}$, $\Delta V_{\max}$, ϕ, and ω are all constants. The time average of the first term on the right of the equal sign in Equation 33.29 involves the average value of $\sin^2\omega t$, which is $\frac{1}{2}$. The time average of the second term on the right of the equal sign is identically zero because $\sin\omega t\cos\omega t = \frac{1}{2}\sin 2\omega t$, and the average value of $\sin 2\omega t$ is zero. Therefore, we can express the **average power** P_{avg} as

$$P_{\text{avg}} = \tfrac{1}{2}I_{\max}\,\Delta V_{\max}\cos\phi \qquad \textbf{(33.30)}$$

It is convenient to express the average power in terms of the rms current and rms voltage defined by Equations 33.4 and 33.5:

$$P_{\text{avg}} = I_{\text{rms}}\,\Delta V_{\text{rms}}\cos\phi \qquad \textbf{(33.31)}$$

Average power delivered to an *RLC* circuit

where the quantity $\cos\phi$ is called the **power factor.** Figure 33.15b shows that the maximum voltage across the resistor is given by $\Delta V_R = \Delta V_{\max}\cos\phi = I_{\max}R$. Therefore, $\cos\phi = I_{\max}R/\Delta V_{\max} = R/Z$, and we can express P_{avg} as

$$P_{\text{avg}} = I_{\text{rms}}\,\Delta V_{\text{rms}}\cos\phi = I_{\text{rms}}\,\Delta V_{\text{rms}}\left(\frac{R}{Z}\right) = I_{\text{rms}}\left(\frac{\Delta V_{\text{rms}}}{Z}\right)R$$

Recognizing that $\Delta V_{\text{rms}}/Z = I_{\text{rms}}$ gives

$$\boxed{P_{\text{avg}} = I_{\text{rms}}^2 R} \tag{33.32}$$

The average power delivered by the source is converted to internal energy in the resistor, just as in the case of a DC circuit. When the load is purely resistive, $\phi = 0$, $\cos\phi = 1$, and, from Equation 33.31, we see that

$$P_{\text{avg}} = I_{\text{rms}}\,\Delta V_{\text{rms}}$$

Note that no power losses are associated with pure capacitors and pure inductors in an AC circuit. To see why that is true, let's first analyze the power in an AC circuit containing only a source and a capacitor. When the current begins to increase in one direction in an AC circuit, charge begins to accumulate on the capacitor and a voltage appears across it. When this voltage reaches its maximum value, the energy stored in the capacitor as electric potential energy is $\frac{1}{2}C(\Delta V_{\text{max}})^2$. This energy storage, however, is only momentary. The capacitor is charged and discharged twice during each cycle: charge is delivered to the capacitor during two quarters of the cycle and is returned to the voltage source during the remaining two quarters. Therefore, the average power supplied by the source is zero. In other words, no power losses occur in a capacitor in an AC circuit.

Now consider the case of an inductor. When the current in an inductor reaches its maximum value, the energy stored in the inductor is a maximum and is given by $\frac{1}{2}LI_{\text{max}}^2$. When the current begins to decrease in the circuit, this stored energy in the inductor returns to the source as the inductor attempts to maintain the current in the circuit.

Equation 33.31 shows that the power delivered by an AC source to any circuit depends on the phase, a result that has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, technicians introduce capacitance in the circuits to shift the phase.

Quick Quiz 33.6 An AC source drives an *RLC* circuit with a fixed voltage amplitude. If the driving frequency is ω_1, the circuit is more capacitive than inductive and the phase angle is $-10°$. If the driving frequency is ω_2, the circuit is more inductive than capacitive and the phase angle is $+10°$. At what frequency is the largest amount of power delivered to the circuit? **(a)** It is largest at ω_1. **(b)** It is largest at ω_2. **(c)** The same amount of power is delivered at both frequencies.

Example 33.5 Average Power in an *RLC* Series Circuit

Calculate the average power delivered to the series *RLC* circuit described in Example 33.4.

SOLUTION

Conceptualize Consider the circuit in Figure 33.13a and imagine energy being delivered to the circuit by the AC source. Review Example 33.4 for other details about this circuit.

Categorize We find the result by using equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 33.5 and the maximum voltage from Example 33.4 to find the rms voltage from the source:

$$\Delta V_{\text{rms}} = \frac{\Delta V_{\text{max}}}{\sqrt{2}} = \frac{150\text{ V}}{\sqrt{2}} = 106\text{ V}$$

Similarly, find the rms current in the circuit:

$$I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} = \frac{0.293\text{ A}}{\sqrt{2}} = 0.207\text{ A}$$

33.5 continued

Use Equation 33.31 to find the power delivered by the source:

$$P_{avg} = I_{rms} V_{rms} \cos \phi = (0.207 \text{ A})(106 \text{ V}) \cos (-34.0°)$$
$$= \boxed{18.2 \text{ W}}$$

33.7 Resonance in a Series *RLC* Circuit

We investigated resonance in mechanical oscillating systems in Chapter 15. As shown in Chapter 32, a series *RLC* circuit is an electrical oscillating system. Such a circuit is said to be **in resonance** when the driving frequency is such that the rms current has its maximum value. In general, the rms current can be written

$$I_{rms} = \frac{\Delta V_{rms}}{Z} \quad \textbf{(33.33)}$$

where Z is the impedance. Substituting the expression for Z from Equation 33.25 into Equation 33.33 gives

$$I_{rms} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \quad \textbf{(33.34)}$$

Because the impedance depends on the frequency of the source, the current in the *RLC* circuit also depends on the frequency. The angular frequency ω_0 at which $X_L - X_C = 0$ is called the **resonance frequency** of the circuit. To find ω_0, we set $X_L = X_C$, which gives $\omega_0 L = 1/\omega_0 C$, or

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \textbf{(33.35)}$$

◀ Resonance frequency

This frequency also corresponds to the natural frequency of oscillation of an *LC* circuit (see Section 32.5). Therefore, the rms current in a series *RLC* circuit has its maximum value when the frequency of the applied voltage matches the natural oscillator frequency, which depends only on L and C. Furthermore, at the resonance frequency, the current is in phase with the applied voltage.

Quick Quiz 33.7 What is the impedance of a series *RLC* circuit at resonance? **(a)** larger than R **(b)** less than R **(c)** equal to R **(d)** impossible to determine

A plot of rms current versus angular frequency for a series *RLC* circuit is shown in Figure 33.17a on page 1014. The data assume a constant $\Delta V_{rms} = 5.0$ mV, $L = 5.0\ \mu$H, and $C = 2.0$ nF. The three curves correspond to three values of R. In each case, the rms current has its maximum value at the resonance frequency ω_0. Furthermore, the curves become narrower and taller as the resistance decreases.

Equation 33.34 shows that when $R = 0$, the current becomes infinite at resonance. Real circuits, however, always have some resistance, which limits the value of the current to some finite value.

We can also calculate the average power as a function of frequency for a series *RLC* circuit. Using Equations 33.32, 33.33, and 33.25 gives

$$P_{avg} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R = \frac{(\Delta V_{rms})^2 R}{R^2 + (X_L - X_C)^2} \quad \textbf{(33.36)}$$

Because $X_L = \omega L$, $X_C = 1/\omega C$, and $\omega_0{}^2 = 1/LC$, the term $(X_L - X_C)^2$ can be expressed as

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = \frac{L^2}{\omega^2}(\omega^2 - \omega_0^2)^2$$

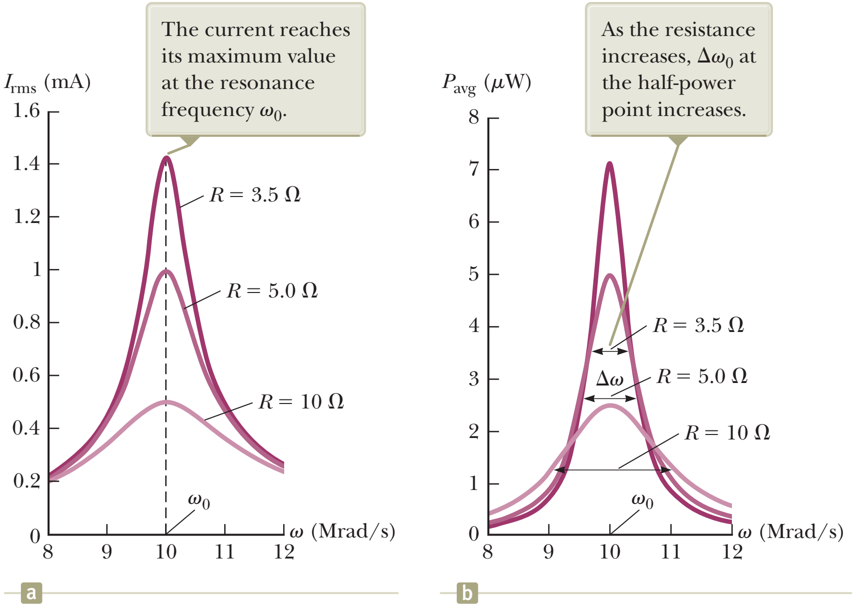


Figure 33.17 (a) The rms current versus frequency for a series *RLC* circuit for three values of *R*. (b) Average power delivered to the circuit versus frequency for the series *RLC* circuit for three values of *R*.

Using this result in Equation 33.36 gives

Average power as a function of frequency in an *RLC* circuit

$$P_{avg} = \frac{(\Delta V_{rms})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2 - \omega_0^{\,2})^2} \quad \textbf{(33.37)}$$

Equation 33.37 shows that at resonance, when $\omega = \omega_0$, the average power is a maximum and has the value $(\Delta V_{rms})^2/R$. Figure 33.17b is a plot of average power versus frequency for three values of R in a series *RLC* circuit. As the resistance is made smaller, the curve becomes sharper in the vicinity of the resonance frequency. This curve sharpness is usually described by a dimensionless parameter known as the **quality factor,**[2] denoted by Q:

Quality factor

$$Q = \frac{\omega_0}{\Delta\omega}$$

where $\Delta\omega$ is the width of the curve measured between the two values of ω for which P_{avg} has one-half its maximum value, called the *half-power points* (see Fig. 33.17b.) It is left as a problem (Problem 76) to show that the width at the half-power points has the value $\Delta\omega = R/L$ so that

$$Q = \frac{\omega_0 L}{R} \quad \textbf{(33.38)}$$

A radio's receiving circuit is an important application of a resonant circuit. The radio is tuned to a particular station (which transmits an electromagnetic wave or signal of a specific frequency) by varying a capacitor, which changes the receiving circuit's resonance frequency. When the circuit is driven by the electromagnetic oscillations a radio signal produces in an antenna, the tuner circuit responds with a large amplitude of electrical oscillation only for the station frequency that matches the resonance frequency. Therefore, only the signal from one radio station is passed on to the amplifier and loudspeakers even though signals from all stations are driving the circuit at the same time. Because many signals are often present over a range of frequencies, it is important to design a high-Q circuit to eliminate unwanted signals. In this manner, stations whose frequencies are near but not equal to the resonance frequency have a response at the receiver that is negligibly small relative to the signal that matches the resonance frequency.

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[2]The quality factor is also defined as the ratio $2\pi E/\Delta E$, where E is the energy stored in the oscillating system and ΔE is the energy decrease per cycle of oscillation due to the resistance.

Example 33.6 **A Resonating Series *RLC* Circuit**

Consider a series *RLC* circuit for which $R = 150\ \Omega$, $L = 20.0$ mH, $\Delta V_{\text{rms}} = 20.0$ V, and $\omega = 5\ 000\ \text{s}^{-1}$. Determine the value of the capacitance for which the current is a maximum.

SOLUTION

Conceptualize Consider the circuit in Figure 33.13a and imagine varying the frequency of the AC source. The current in the circuit has its maximum value at the resonance frequency ω_0.

Categorize We find the result by using equations developed in this section, so we categorize this example as a substitution problem.

Use Equation 33.35 to solve for the required capacitance in terms of the resonance frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}} \rightarrow C = \frac{1}{\omega_0^2 L}$$

Substitute numerical values:

$$C = \frac{1}{(5.00 \times 10^3\ \text{s}^{-1})^2(20.0 \times 10^{-3}\ \text{H})} = \boxed{2.00\ \mu\text{F}}$$

33.8 The Transformer and Power Transmission

As discussed in Section 27.6, it is economical to use a high voltage and a low current to minimize the I^2R loss in transmission lines when electric power is transmitted over great distances. Consequently, 350-kV lines are common, and in many areas, even higher-voltage (765-kV) lines are used. At the receiving end of such lines, the consumer requires power at a low voltage (for safety and for efficiency in design). In practice, the voltage is decreased to approximately 20 000 V at a distribution substation, then to 4 000 V for delivery to residential areas, and finally to 120 V and 240 V at the customer's site. Therefore, a device is needed that can change the alternating voltage and current without causing appreciable changes in the power delivered. The AC transformer is that device.

In its simplest form, the **AC transformer** consists of two coils of wire wound around a core of iron as illustrated in Figure 33.18. (Compare this arrangement to Faraday's experiment in Figure 31.2.) The coil on the left, which is connected to the input alternating-voltage source and has N_1 turns, is called the *primary winding* (or the *primary*). The coil on the right, consisting of N_2 turns and connected to a load resistor R_L, is called the *secondary winding* (or the *secondary*). The purposes of the iron core are to increase the magnetic flux through the coil and to provide a medium in which nearly all the magnetic field lines through one coil pass through the other coil. Eddy-current losses are reduced by using a laminated core. Transformation of energy to internal energy in the finite resistance of the coil wires is usually quite small. Typical transformers have power efficiencies from 90% to

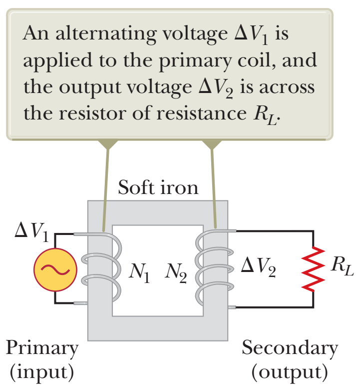


Figure 33.18 An ideal transformer consists of two coils wound on the same iron core.

99%. In the discussion that follows, let's assume we are working with an *ideal transformer,* one in which the energy losses in the windings and core are zero.

Faraday's law states that the voltage Δv_1 across the primary is

$$\Delta v_1 = -N_1 \frac{d\Phi_B}{dt} \quad \textbf{(33.39)}$$

where Φ_B is the magnetic flux through each turn. If we assume all magnetic field lines remain within the iron core, the flux through each turn of the primary equals the flux through each turn of the secondary. Hence, the voltage across the secondary is

$$\Delta v_2 = -N_2 \frac{d\Phi_B}{dt} \quad \textbf{(33.40)}$$

Solving Equation 33.39 for $d\Phi_B/dt$ and substituting the result into Equation 33.40 gives

$$\Delta v_2 = \frac{N_2}{N_1} \Delta v_1 \quad \textbf{(33.41)}$$

When $N_2 > N_1$, the output voltage Δv_2 exceeds the input voltage Δv_1. This configuration is referred to as a *step-up transformer.* When $N_2 < N_1$, the output voltage is less than the input voltage, and we have a *step-down transformer.* A circuit diagram for a transformer connected to a load resistance is shown in Figure 33.19.

When a current I_1 exists in the primary circuit, a current I_2 is induced in the secondary. (In this discussion, uppercase I and ΔV refer to rms values.) If the load in the secondary circuit is a pure resistance, the induced current is in phase with the induced voltage. The power supplied to the secondary circuit must be provided by the AC source connected to the primary circuit. In an ideal transformer where there are no losses, the power $I_1 \Delta V_1$ supplied by the source is equal to the power $I_2 \Delta V_2$ in the secondary circuit. That is,

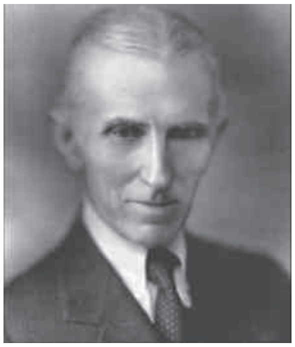
$$I_1 \Delta V_1 = I_2 \Delta V_2 \quad \textbf{(33.42)}$$

The value of the load resistance R_L determines the value of the secondary current because $I_2 = \Delta V_2/R_L$. Furthermore, the current in the primary is $I_1 = \Delta V_1/R_{eq}$, where

$$R_{eq} = \left(\frac{N_1}{N_2}\right)^2 R_L \quad \textbf{(33.43)}$$

is the equivalent resistance of the load resistance when viewed from the primary side. We see from this analysis that a transformer may be used to match resistances between the primary circuit and the load. In this manner, maximum power transfer can be achieved between a given power source and the load resistance. For example, a transformer connected between the 1-kΩ output of an audio amplifier and an 8-Ω speaker ensures that as much of the audio signal as possible is transferred into the speaker. In stereo terminology, this process is called *impedance matching.*

To operate properly, many common household electronic devices require low voltages. A small transformer that plugs directly into the wall like the one illustrated in Figure 33.20 can provide the proper voltage. The photograph shows the two windings wrapped around a common iron core that is found inside all these



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Nikola Tesla
American Physicist (1856–1943)
Tesla was born in Croatia, but he spent most of his professional life as an inventor in the United States. He was a key figure in the development of alternating-current electricity, high-voltage transformers, and the transport of electrical power using AC transmission lines. Tesla's viewpoint was at odds with the ideas of Thomas Edison, who committed himself to the use of direct current in power transmission. Tesla's AC approach won out.

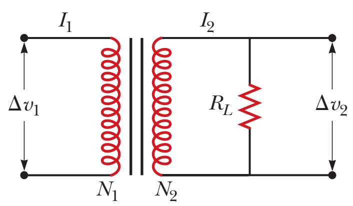

Figure 33.19 Circuit diagram for a transformer.

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This transformer is smaller than the one in the opening photograph of this chapter. In addition, it is a step-down transformer. It drops the voltage from 4 000 V to 240 V for delivery to a group of residences.

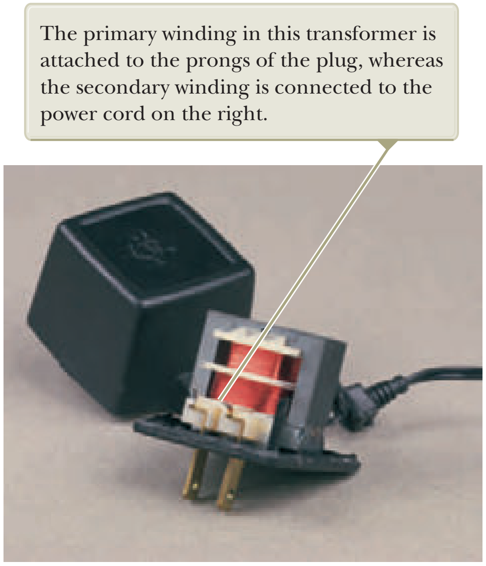

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Figure 33.20 Electronic devices are often powered by AC adaptors containing transformers such as this one. These adaptors alter the AC voltage. In many applications, the adaptors also convert alternating current to direct current.

little "black boxes." This particular transformer converts the 120-V AC in the wall socket to 12.5-V AC. (Can you determine the ratio of the numbers of turns in the two coils?) Some black boxes also make use of diodes to convert the alternating current to direct current. (See Section 33.9.)

Example 33.7 The Economics of AC Power

An electricity-generating station needs to deliver energy at a rate of 20 MW to a city 1.0 km away. A common voltage for commercial power generators is 22 kV, but a step-up transformer is used to boost the voltage to 230 kV before transmission.

(A) If the resistance of the wires is 2.0 Ω and the energy costs are about 11¢/kWh, estimate the cost of the energy converted to internal energy in the wires during one day.

SOLUTION

Conceptualize The resistance of the wires is in series with the resistance representing the load (homes and businesses). Therefore, there is a voltage drop in the wires, which means that some of the transmitted energy is converted to internal energy in the wires and never reaches the load.

Categorize This problem involves finding the power delivered to a resistive load in an AC circuit. Let's ignore any capacitive or inductive characteristics of the load and set the power factor equal to 1.

Analyze Calculate I_{rms} in the wires from Equation 33.31:

$$I_{\text{rms}} = \frac{P_{\text{avg}}}{\Delta V_{\text{rms}}} = \frac{20 \times 10^6 \text{ W}}{230 \times 10^3 \text{ V}} = 87 \text{ A}$$

Determine the rate at which energy is delivered to the resistance in the wires from Equation 33.32:

$$P_{\text{wires}} = I_{\text{rms}}^2 R = (87 \text{ A})^2 (2.0 \ \Omega) = 15 \text{ kW}$$

Calculate the energy T_{ET} delivered to the wires over the course of a day:

$$T_{\text{ET}} = P_{\text{wires}} \Delta t = (15 \text{ kW})(24 \text{ h}) = \boxed{363 \text{ kWh}}$$

Find the cost of this energy at a rate of 11¢/kWh:

$$\text{Cost} = (363 \text{ kWh})(\$0.11/\text{kWh}) = \boxed{\$40}$$

(B) Repeat the calculation for the situation in which the power plant delivers the energy at its original voltage of 22 kV.

continued

▶ **33.7** continued

SOLUTION

Calculate I_{rms} in the wires from Equation 33.31:

$$I_{\text{rms}} = \frac{P_{\text{avg}}}{\Delta V_{\text{rms}}} = \frac{20 \times 10^6\ \text{W}}{22 \times 10^3\ \text{V}} = 909\ \text{A}$$

From Equation 33.32, determine the rate at which energy is delivered to the resistance in the wires:

$$P_{\text{wires}} = I_{\text{rms}}^2 R = (909\ \text{A})^2(2.0\ \Omega) = 1.7 \times 10^3\ \text{kW}$$

Calculate the energy delivered to the wires over the course of a day:

$$T_{\text{ET}} = P_{\text{wires}}\,\Delta t = (1.7 \times 10^3\ \text{kW})(24\ \text{h}) = 4.0 \times 10^4\ \text{kWh}$$

Find the cost of this energy at a rate of 11¢/kWh:

$$\text{Cost} = (4.0 \times 10^4\ \text{kWh})(\$0.11/\text{kWh}) = \boxed{\$4.4 \times 10^3}$$

Finalize Notice the tremendous savings that are possible through the use of transformers and high-voltage transmission lines. Such savings in combination with the efficiency of using alternating current to operate motors led to the universal adoption of alternating current instead of direct current for commercial power grids.

33.9 Rectifiers and Filters

Portable electronic devices such as radios and laptop computers are often powered by direct current supplied by batteries. Many devices come with AC–DC converters such as that shown in Figure 33.20. Such a converter contains a transformer that steps the voltage down from 120 V to, typically, 6 V or 9 V and a circuit that converts alternating current to direct current. The AC–DC converting process is called **rectification,** and the converting device is called a **rectifier.**

The most important element in a rectifier circuit is a **diode,** a circuit element that conducts current in one direction but not the other. Most diodes used in modern electronics are semiconductor devices. The circuit symbol for a diode is , where the arrow indicates the direction of the current in the diode. A diode has low resistance to current in one direction (the direction of the arrow) and high resistance to current in the opposite direction. To understand how a diode rectifies a current, consider Figure 33.21a, which shows a diode and a resistor connected to the secondary of a transformer. The transformer reduces the voltage from 120-V AC to the lower voltage that is needed for the device having a resistance R (the load

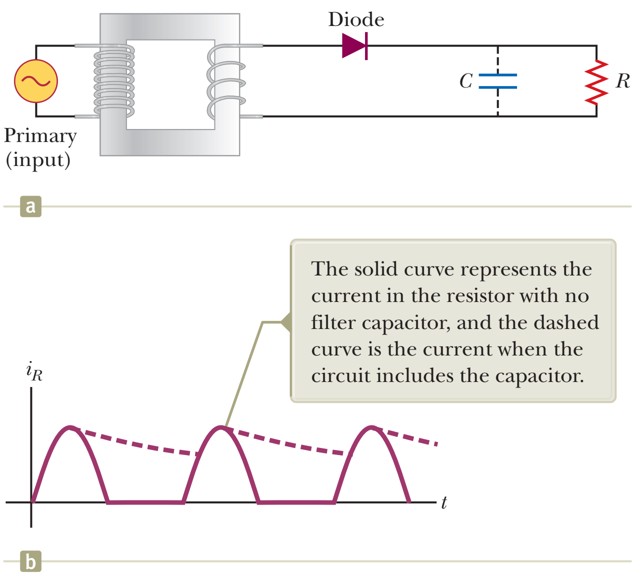


Figure 33.21 (a) A half-wave rectifier with an optional filter capacitor. (b) Current versus time in the resistor.

resistance). Because the diode conducts current in only one direction, the alternating current in the load resistor is reduced to the form shown by the solid curve in Figure 33.21b. The diode conducts current only when the side of the symbol containing the arrowhead has a positive potential relative to the other side. In this situation, the diode acts as a *half-wave rectifier* because current is present in the circuit only during half of each cycle.

When a capacitor is added to the circuit as shown by the dashed lines and the capacitor symbol in Figure 33.21a, the circuit is a simple DC power supply. The time variation of the current in the load resistor (the dashed curve in Fig. 33.21b) is close to being zero, as determined by the RC time constant of the circuit. As the current in the circuit begins to rise at $t = 0$ in Figure 33.21b, the capacitor charges up. When the current begins to fall, however, the capacitor discharges through the resistor, so the current in the resistor does not fall as quickly as the current from the transformer.

The RC circuit in Figure 33.21a is one example of a **filter circuit,** which is used to smooth out or eliminate a time-varying signal. For example, radios are usually powered by a 60-Hz alternating voltage. After rectification, the voltage still contains a small AC component at 60 Hz (sometimes called *ripple*), which must be filtered. By "filtered," we mean that the 60-Hz ripple must be reduced to a value much less than that of the audio signal to be amplified because without filtering, the resulting audio signal includes an annoying hum at 60 Hz.

We can also design filters that respond differently to different frequencies. Consider the simple series RC circuit shown in Figure 33.22a. The input voltage is across the series combination of the two elements. The output is the voltage across the resistor. A plot of the ratio of the output voltage to the input voltage as a function of the logarithm of angular frequency (see Fig. 33.22b) shows that at low frequencies, ΔV_{out} is much smaller than ΔV_{in}, whereas at high frequencies, the two voltages are equal. Because the circuit preferentially passes signals of higher frequency while blocking low-frequency signals, the circuit is called an ***RC* high-pass filter.** (See Problem 54 for an analysis of this filter.)

Physically, a high-pass filter works because a capacitor "blocks out" direct current and AC current at low frequencies. At low frequencies, the capacitive reactance is large and much of the applied voltage appears across the capacitor rather than across the output resistor. As the frequency increases, the capacitive reactance drops and more of the applied voltage appears across the resistor.

Now consider the circuit shown in Figure 33.23a on page 1020, where we have interchanged the resistor and capacitor and where the output voltage is taken across the capacitor. At low frequencies, the reactance of the capacitor and the voltage across the capacitor is high. As the frequency increases, the voltage across the capacitor drops. Therefore, this filter is an ***RC* low-pass filter.** The ratio of output voltage to input voltage (see Problem 56), plotted as a function of the logarithm of ω in Figure 33.23b, shows this behavior.

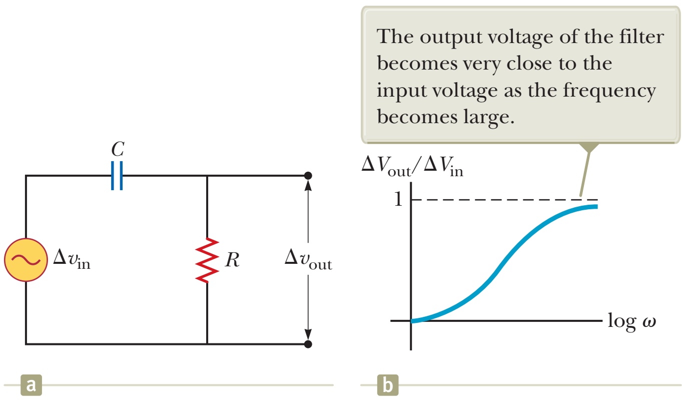


Figure 33.22 (a) A simple RC high-pass filter. (b) Ratio of output voltage to input voltage for an RC high-pass filter as a function of the angular frequency of the AC source.

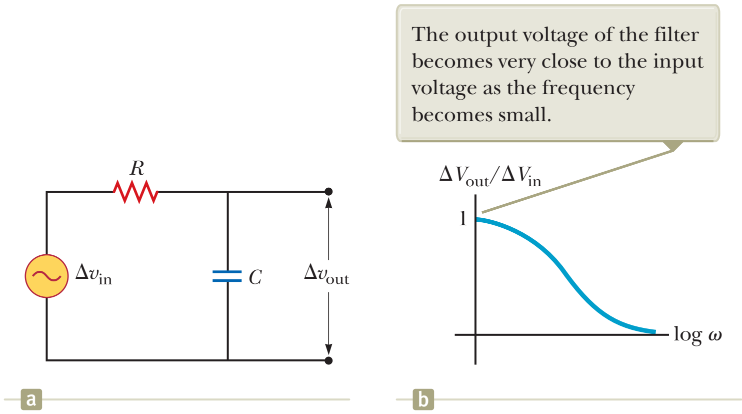


Figure 33.23 (a) A simple *RC* low-pass filter. (b) Ratio of output voltage to input voltage for an *RC* low-pass filter as a function of the angular frequency of the AC source.

You may be familiar with crossover networks, which are an important part of the speaker systems for high-quality audio systems. These networks use low-pass filters to direct low frequencies to a special type of speaker, the "woofer," which is designed to reproduce the low notes accurately. The high frequencies are sent by a high-pass filter to the "tweeter" speaker.

Summary

Definitions

In AC circuits that contain inductors and capacitors, it is useful to define the **inductive reactance** X_L and the **capacitive reactance** X_C as

$$X_L \equiv \omega L \qquad (33.10)$$

$$X_C \equiv \frac{1}{\omega C} \qquad (33.18)$$

where ω is the angular frequency of the AC source. The SI unit of reactance is the ohm.

The **impedance** Z of an *RLC* series AC circuit is

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \qquad (33.25)$$

This expression illustrates that we cannot simply add the resistance and reactances in a circuit. We must account for the applied voltage and current being out of phase, with the **phase angle** ϕ between the current and voltage being

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \qquad (33.27)$$

The sign of ϕ can be positive or negative, depending on whether X_L is greater or less than X_C. The phase angle is zero when $X_L = X_C$.

Concepts and Principles

The **rms current** and **rms voltage** in an AC circuit in which the voltages and current vary sinusoidally are given by

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max} \qquad (33.4)$$

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707 \Delta V_{max} \qquad (33.5)$$

where I_{max} and ΔV_{max} are the maximum values.

If an AC circuit consists of a source and a resistor, the current is in phase with the voltage. That is, the current and voltage reach their maximum values at the same time.

If an AC circuit consists of a source and an inductor, the current lags the voltage by 90°. That is, the voltage reaches its maximum value one-quarter of a period before the current reaches its maximum value.

If an AC circuit consists of a source and a capacitor, the current leads the voltage by 90°. That is, the current reaches its maximum value one-quarter of a period before the voltage reaches its maximum value.

The **average power** delivered by the source in an *RLC* circuit is

$$P_{avg} = I_{rms}\,\Delta V_{rms}\cos\phi \quad (33.31)$$

An equivalent expression for the average power is

$$P_{avg} = I_{rms}^2 R \quad (33.32)$$

The average power delivered by the source results in increasing internal energy in the resistor. No power loss occurs in an ideal inductor or capacitor.

The rms current in a series *RLC* circuit is

$$I_{rms} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (33.34)$$

A series *RLC* circuit is in resonance when the inductive reactance equals the capacitive reactance. When this condition is met, the rms current given by Equation 33.34 has its maximum value. The **resonance frequency** ω_0 of the circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (33.35)$$

The rms current in a series *RLC* circuit has its maximum value when the frequency of the source equals ω_0, that is, when the "driving" frequency matches the resonance frequency.

AC transformers allow for easy changes in alternating voltage according to

$$\Delta v_2 = \frac{N_2}{N_1}\Delta v_1 \quad (33.41)$$

where N_1 and N_2 are the numbers of windings on the primary and secondary coils, respectively, and Δv_1 and Δv_2 are the voltages on these coils.

Objective Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. An inductor and a resistor are connected in series across an AC source as in Figure OQ33.1. Immediately after the switch is closed, which of the following statements is true? (a) The current in the circuit is $\Delta V/R$. (b) The voltage across the inductor is zero. (c) The current in the circuit is zero. (d) The voltage across the resistor is ΔV. (e) The voltage across the inductor is half its maximum value.

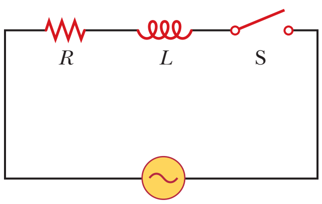


Figure OQ33.1

2. **(i)** When a particular inductor is connected to a source of sinusoidally varying emf with constant amplitude and a frequency of 60.0 Hz, the rms current is 3.00 A. What is the rms current if the source frequency is doubled? (a) 12.0 A (b) 6.00 A (c) 4.24 A (d) 3.00 A (e) 1.50 A **(ii)** Repeat part (i) assuming the load is a capacitor instead of an inductor. **(iii)** Repeat part (i) assuming the load is a resistor instead of an inductor.

3. A capacitor and a resistor are connected in series across an AC source as shown in Figure OQ33.3. After the switch is closed, which of the following statements is true? (a) The voltage across the capacitor lags the current by 90°. (b) The voltage across the resistor is out of phase with the current. (c) The voltage across the capacitor leads the current by 90°. (d) The current decreases as the frequency of the source is increased, but its peak voltage remains the same. (e) None of those statements is correct.

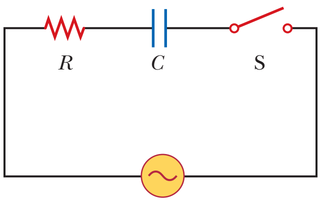


Figure OQ33.3

4. **(i)** What is the time average of the "square-wave" potential shown in Figure OQ33.4? (a) $\sqrt{2}\,\Delta V_{max}$ (b) ΔV_{max} (c) $\Delta V_{max}/\sqrt{2}$ (d) $\Delta V_{max}/2$ (e) $\Delta V_{max}/4$ **(ii)** What is the rms voltage? Choose from the same possibilities as in part (i).

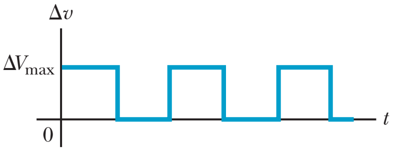


Figure OQ33.4

5. If the voltage across a circuit element has its maximum value when the current in the circuit is zero, which of the following statements *must* be true? (a) The circuit element is a resistor. (b) The circuit element is a capacitor. (c) The circuit element is an inductor. (d) The current and voltage are 90° out of phase. (e) The current and voltage are 180° out of phase.

6. A sinusoidally varying potential difference has amplitude 170 V. **(i)** What is its minimum instantaneous value? (a) 170 V (b) 120 V (c) 0 (d) −120 V (e) −170 V **(ii)** What is its average value? **(iii)** What is its rms value? Choose from the same possibilities as in part (i) in each case.

7. A series *RLC* circuit contains a 20.0-Ω resistor, a 0.750-μF capacitor, and a 120-mH inductor. **(i)** If a sinusoidally varying rms voltage of 120 V at f = 500 Hz is applied across this combination of elements, what is the rms current in the circuit? (a) 2.33 A (b) 6.00 A (c) 10.0 A (d) 17.0 A (e) none of those answers **(ii) What If?** What is the rms current in the circuit when operating at its resonance frequency? Choose from the same possibilities as in part (i).

8. A resistor, a capacitor, and an inductor are connected in series across an AC source. Which of the following statements is *false?* (a) The instantaneous voltage across the capacitor lags the current by 90°. (b) The instantaneous voltage across the inductor leads the current by 90°. (c) The instantaneous voltage across the resistor is in phase with the current. (d) The voltages across the resistor, capacitor, and inductor are not in phase. (e) The rms voltage across the combination of the three elements equals the algebraic sum of the rms voltages across each element separately.

9. Under what conditions is the impedance of a series *RLC* circuit equal to the resistance in the circuit? (a) The driving frequency is lower than the resonance frequency. (b) The driving frequency is equal to the resonance frequency. (c) The driving frequency is higher than the resonance frequency. (d) always (e) never

10. What is the phase angle in a series *RLC* circuit at resonance? (a) 180° (b) 90° (c) 0 (d) −90° (e) None of those answers is necessarily correct.

11. A circuit containing an AC source, a capacitor, an inductor, and a resistor has a high-Q resonance at 1 000 Hz. From greatest to least, rank the following contributions to the impedance of the circuit at that frequency and at lower and higher frequencies. Note any cases of equality in your ranking. (a) X_C at 500 Hz (b) X_C at 1 500 Hz (c) X_L at 500 Hz (d) X_L at 1 500 Hz (e) R at 1 000 Hz

12. A 6.00-V battery is connected across the primary coil of a transformer having 50 turns. If the secondary coil of the transformer has 100 turns, what voltage appears across the secondary? (a) 24.0 V (b) 12.0 V (c) 6.00 V (d) 3.00 V (e) none of those answers

13. Do AC ammeters and voltmeters read (a) peak-to-valley, (b) maximum, (c) rms, or (d) average values?

Conceptual Questions

$\boxed{1.}$ denotes answer available in *Student Solutions Manual/Study Guide*

1. (a) Explain how the quality factor is related to the response characteristics of a radio receiver. (b) Which variable most strongly influences the quality factor?

2. (a) Explain how the mnemonic "ELI the ICE man" can be used to recall whether current leads voltage or voltage leads current in *RLC* circuits. Note that E represents emf $\mathcal{E}$. (b) Explain how "CIVIL" works as another mnemonic device, where V represents voltage.

3. Why is the sum of the maximum voltages across each element in a series *RLC* circuit usually greater than the maximum applied voltage? Doesn't that inequality violate Kirchhoff's loop rule?

$\boxed{4.}$ (a) Does the phase angle in an *RLC* series circuit depend on frequency? (b) What is the phase angle for the circuit when the inductive reactance equals the capacitive reactance?

5. Do some research to answer these questions: Who invented the metal detector? Why? What are its limitations?

6. As shown in Figure CQ33.6, a person pulls a vacuum cleaner at speed v across a horizontal floor, exerting on it a force of magnitude F directed upward at an angle θ with the horizontal. (a) At what rate is the person doing work on the cleaner? (b) State as completely as you can the analogy between power in this situation and in an electric circuit.

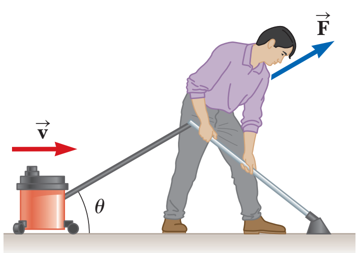


Figure CQ33.6

7. A certain power supply can be modeled as a source of emf in series with both a resistance of 10 Ω and an inductive reactance of 5 Ω. To obtain maximum power delivered to the load, it is found that the load should have a resistance of R_L = 10 Ω, an inductive reactance of zero, and a capacitive reactance of 5 Ω. (a) With this load, is the circuit in resonance? (b) With this load, what fraction of the average power put out by the source of emf is delivered to the load? (c) To increase the fraction of the power delivered to the load, how could the load be changed? You may wish to review Example 28.2 and Problem 4 in Chapter 28 on maximum power transfer in DC circuits.

$\boxed{8.}$ Will a transformer operate if a battery is used for the input voltage across the primary? Explain.

9. (a) Why does a capacitor act as a short circuit at high frequencies? (b) Why does a capacitor act as an open circuit at low frequencies?

10. An ice storm breaks a transmission line and interrupts electric power to a town. A homeowner starts a gasoline-powered 120-V generator and clips its output terminals to "hot" and "ground" terminals of the electrical panel for his house. On a power pole down the block is a transformer designed to step down the voltage for household use. It has a ratio of turns N_1/N_2 of 100 to 1. A repairman climbs the pole. What voltage will he encounter on the input side of the transformer? As this question implies, safety precautions must be taken in the use of home generators and during power failures in general.

Problems

ENHANCED WebAssign The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate; **3.** challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Section 33.1 AC Sources

Section 33.2 Resistors in an AC Circuit

1. When an AC source is connected across a 12.0-Ω resistor, the rms current in the resistor is 8.00 A. Find (a) the rms voltage across the resistor, (b) the peak voltage of the source, (c) the maximum current in the resistor, and (d) the average power delivered to the resistor.

2. (a) What is the resistance of a lightbulb that uses an average power of 75.0 W when connected to a 60.0-Hz power source having a maximum voltage of 170 V? (b) **What If?** What is the resistance of a 100-W lightbulb?

3. An AC power supply produces a maximum voltage ΔV_{max} = 100 V. This power supply is connected to a resistor R = 24.0 Ω, and the current and resistor voltage are measured with an ideal AC ammeter and voltmeter as shown in Figure P33.3. An ideal ammeter has zero resistance, and an ideal voltmeter has infinite resistance. What is the reading on (a) the ammeter and (b) the voltmeter?

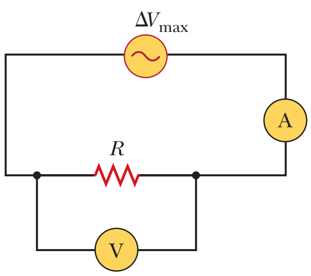


Figure P33.3

4. A certain lightbulb is rated at 60.0 W when operating at an rms voltage of 120 V. (a) What is the peak voltage applied across the bulb? (b) What is the resistance of the bulb? (c) Does a 100-W bulb have greater or less resistance than a 60.0-W bulb? Explain.

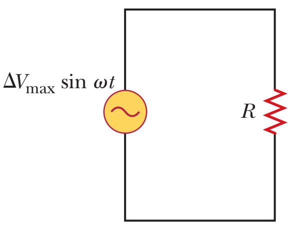


Figure P33.5 Problems 5 and 6.

5. The current in the circuit shown in Figure P33.5 equals 60.0% of the peak current at t = 7.00 ms. What is the lowest source frequency that gives this current?

6. In the AC circuit shown in Figure P33.5, R = 70.0 Ω and the output voltage of the AC source is $\Delta V_{max} \sin \omega t$. (a) If $\Delta V_R = 0.250\ \Delta V_{max}$ for the first time at t = 0.010 0 s, what is the angular frequency of the source? (b) What is the next value of t for which $\Delta V_R = 0.250\ \Delta V_{max}$?

7. An audio amplifier, represented by the AC source and resistor in Figure P33.7, delivers to the speaker alternating voltage at audio frequencies. If the source voltage has an amplitude of 15.0 V, R = 8.20 Ω, and the speaker is equivalent to a resistance of 10.4 Ω, what is the time-averaged power transferred to it?

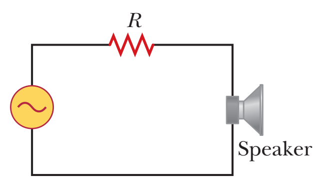


Figure P33.7

8. Figure P33.8 shows three lightbulbs connected to a 120-V AC (rms) household supply voltage. Bulbs 1 and 2 have a power rating of 150 W, and bulb 3 has a 100-W rating. Find (a) the rms current in each bulb and (b) the resistance of each bulb. (c) What is the total resistance of the combination of the three lightbulbs?

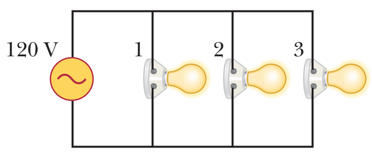


Figure P33.8

Section 33.3 Inductors in an AC Circuit

9. An inductor has a 54.0-Ω reactance when connected to a 60.0–Hz source. The inductor is removed and then connected to a 50.0-Hz source that produces a 100-V rms voltage. What is the maximum current in the inductor?

10. M In a purely inductive AC circuit as shown in Figure P33.10 (page 1024), ΔV_{max} = 100 V. (a) The maximum current is 7.50 A at 50.0 Hz. Calculate the inductance L. (b) **What If?** At what angular frequency ω is the maximum current 2.50 A?

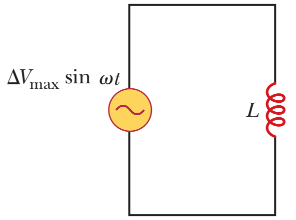


Figure P33.10 Problems 10 and 11.

11. W For the circuit shown in Figure P33.10, $\Delta V_{max} = 80.0$ V, $\omega = 65.0\pi$ rad/s, and $L = 70.0$ mH. Calculate the current in the inductor at $t = 15.5$ ms.

12. An inductor is connected to an AC power supply having a maximum output voltage of 4.00 V at a frequency of 300 Hz. What inductance is needed to keep the rms current less than 2.00 mA?

13. An AC source has an output rms voltage of 78.0 V at a frequency of 80.0 Hz. If the source is connected across a 25.0-mH inductor, what are (a) the inductive reactance of the circuit, (b) the rms current in the circuit, and (c) the maximum current in the circuit?

14. W A 20.0-mH inductor is connected to a North American electrical outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Assuming the energy stored in the inductor is zero at $t = 0$, determine the energy stored at $t = \frac{1}{180}$ s.

15. Review. Determine the maximum magnetic flux through an inductor connected to a North American electrical outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz).

16. The output voltage of an AC source is given by $\Delta v = 120 \sin 30.0\pi t$, where Δv is in volts and t is in seconds. The source is connected across a 0.500-H inductor. Find (a) the frequency of the source, (b) the rms voltage across the inductor, (c) the inductive reactance of the circuit, (d) the rms current in the inductor, and (e) the maximum current in the inductor.

Section 33.4 Capacitors in an AC Circuit

17. A 1.00-mF capacitor is connected to a North American electrical outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Assuming the energy stored in the capacitor is zero at $t = 0$, determine the magnitude of the current in the wires at $t = \frac{1}{180}$ s.

18. An AC source with an output rms voltage of 36.0 V at a frequency of 60.0 Hz is connected across a 12.0-μF capacitor. Find (a) the capacitive reactance, (b) the rms current, and (c) the maximum current in the circuit. (d) Does the capacitor have its maximum charge when the current has its maximum value? Explain.

19. (a) For what frequencies does a 22.0-μF capacitor have a reactance below 175 Ω? (b) **What If?** What is the reactance of a 44.0-μF capacitor over this same frequency range?

20. A source delivers an AC voltage of the form $\Delta v = 98.0 \sin 80\pi t$, where Δv is in volts and t is in seconds, to a capacitor. The maximum current in the circuit is 0.500 A. Find (a) the rms voltage of the source, (b) the frequency of the source, and (c) the value of the capacitance.

21. M What maximum current is delivered by an AC source with $\Delta V_{max} = 48.0$ V and $f = 90.0$ Hz when connected across a 3.70-μF capacitor?

22. A capacitor C is connected to a power supply that operates at a frequency f and produces an rms voltage ΔV. What is the maximum charge that appears on either capacitor plate?

23. W What is the maximum current in a 2.20-μF capacitor when it is connected across (a) a North American electrical outlet having $\Delta V_{rms} = 120$ V and $f = 60.0$ Hz and (b) a European electrical outlet having $\Delta V_{rms} = 240$ V and $f = 50.0$ Hz?

Section 33.5 The *RLC* Series Circuit

24. W An AC source with $\Delta V_{max} = 150$ V and $f = 50.0$ Hz is connected between points a and d in Figure P33.24. Calculate the maximum voltages between (a) points a and b, (b) points b and c, (c) points c and d, and (d) points b and d.

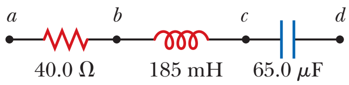


Figure P33.24 Problems 24 and 81.

25. In addition to phasor diagrams showing voltages such as in Figure 33.15, we can draw phasor diagrams with resistance and reactances. The resultant of adding the phasors is the impedance. Draw to scale a phasor diagram showing Z, X_L, X_C, and ϕ for an AC series circuit for which $R = 300$ Ω, $C = 11.0$ μF, $L = 0.200$ H, and $f = 500/\pi$ Hz.

26. W A sinusoidal voltage $\Delta v = 40.0 \sin 100t$, where Δv is in volts and t is in seconds, is applied to a series *RLC* circuit with $L = 160$ mH, $C = 99.0$ μF, and $R = 68.0$ Ω. (a) What is the impedance of the circuit? (b) What is the maximum current? Determine the numerical values for (c) ω and (d) ϕ in the equation $i = I_{max} \sin(\omega t - \phi)$.

27. W A series AC circuit contains a resistor, an inductor of 150 mH, a capacitor of 5.00 μF, and a source with $\Delta V_{max} = 240$ V operating at 50.0 Hz. The maximum current in the circuit is 100 mA. Calculate (a) the inductive reactance, (b) the capacitive reactance, (c) the impedance, (d) the resistance in the circuit, and (e) the phase angle between the current and the source voltage.

28. At what frequency does the inductive reactance of a 57.0-μH inductor equal the capacitive reactance of a 57.0-μF capacitor?

29. An *RLC* circuit consists of a 150-Ω resistor, a 21.0-μF capacitor, and a 460-mH inductor connected in series with a 120-V, 60.0-Hz power supply. (a) What is the phase angle between the current and the applied voltage? (b) Which reaches its maximum earlier, the current or the voltage?

30. Draw phasors to scale for the following voltages in SI units: (a) 25.0 sin ωt at $\omega t = 90.0°$, (b) 30.0 sin ωt at $\omega t = 60.0°$, and (c) 18.0 sin ωt at $\omega t = 300°$.

31. M An inductor ($L = 400$ mH), a capacitor ($C = 4.43\ \mu$F), and a resistor ($R = 500\ \Omega$) are connected in series. A 50.0-Hz AC source produces a peak current of 250 mA in the circuit. (a) Calculate the required peak voltage ΔV_{max}. (b) Determine the phase angle by which the current leads or lags the applied voltage.

32. A 60.0-Ω resistor is connected in series with a 30.0-μF capacitor and a source whose maximum voltage is 120 V, operating at 60.0 Hz. Find (a) the capacitive reactance of the circuit, (b) the impedance of the circuit, and (c) the maximum current in the circuit. (d) Does the voltage lead or lag the current? (e) How will adding an inductor in series with the existing resistor and capacitor affect the current? Explain.

33. **Review.** In an *RLC* series circuit that includes a source of alternating current operating at fixed frequency and voltage, the resistance R is equal to the inductive reactance. If the plate separation of the parallel-plate capacitor is reduced to one-half its original value, the current in the circuit doubles. Find the initial capacitive reactance in terms of R.

Section 33.6 Power in an AC Circuit

34. *Why is the following situation impossible?* A series circuit consists of an ideal AC source (no inductance or capacitance in the source itself) with an rms voltage of ΔV at a frequency f and a magnetic buzzer with a resistance R and an inductance L. By carefully adjusting the inductance L of the circuit, a power factor of exactly 1.00 is attained.

35. W A series *RLC* circuit has a resistance of 45.0 Ω and an impedance of 75.0 Ω. What average power is delivered to this circuit when $\Delta V_{rms} = 210$ V?

36. W An AC voltage of the form $\Delta v = 100 \sin 1\,000t$, where Δv is in volts and t is in seconds, is applied to a series *RLC* circuit. Assume the resistance is 400 Ω, the capacitance is 5.00 μF, and the inductance is 0.500 H. Find the average power delivered to the circuit.

37. A series *RLC* circuit has a resistance of 22.0 Ω and an impedance of 80.0 Ω. If the rms voltage applied to the circuit is 160 V, what average power is delivered to the circuit?

38. An AC voltage of the form $\Delta v = 90.0 \sin 350t$, where Δv is in volts and t is in seconds, is applied to a series *RLC* circuit. If $R = 50.0\ \Omega$, $C = 25.0\ \mu$F, and $L = 0.200$ H, find (a) the impedance of the circuit, (b) the rms current in the circuit, and (c) the average power delivered to the circuit.

39. W In a certain series *RLC* circuit, $I_{rms} = 9.00$ A, $\Delta V_{rms} = 180$ V, and the current leads the voltage by 37.0°. (a) What is the total resistance of the circuit? (b) Calculate the reactance of the circuit ($X_L - X_C$).

40. W Suppose you manage a factory that uses many electric motors. The motors create a large inductive load to the electric power line as well as a resistive load. The electric company builds an extra-heavy distribution line to supply you with two components of current: one that is 90° out of phase with the voltage and another that is in phase with the voltage. The electric company charges you an extra fee for "reactive volt-amps" in addition to the amount you pay for the energy you use. You can avoid the extra fee by installing a capacitor between the power line and your factory. The following problem models this solution.

In an *RL* circuit, a 120-V (rms), 60.0-Hz source is in series with a 25.0-mH inductor and a 20.0-Ω resistor. What are (a) the rms current and (b) the power factor? (c) What capacitor must be added in series to make the power factor equal to 1? (d) To what value can the supply voltage be reduced if the power supplied is to be the same as before the capacitor was installed?

41. A diode is a device that allows current to be carried in only one direction (the direction indicated by the arrowhead in its circuit symbol). Find the average power delivered to the diode circuit of Figure P33.41 in terms of ΔV_{rms} and R.

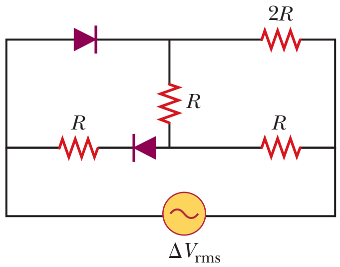


Figure P33.41

Section 33.7 Resonance in a Series *RLC* Circuit

42. W A series *RLC* circuit has components with the following values: $L = 20.0$ mH, $C = 100$ nF, $R = 20.0\ \Omega$, and $\Delta V_{max} = 100$ V, with $\Delta v = \Delta V_{max} \sin \omega t$. Find (a) the resonant frequency of the circuit, (b) the amplitude of the current at the resonant frequency, (c) the Q of the circuit, and (d) the amplitude of the voltage across the inductor at resonance.

43. AMT M An *RLC* circuit is used in a radio to tune into an FM station broadcasting at $f = 99.7$ MHz. The resistance in the circuit is $R = 12.0\ \Omega$, and the inductance is $L = 1.40\ \mu$H. What capacitance should be used?

44. The *LC* circuit of a radar transmitter oscillates at 9.00 GHz. (a) What inductance is required for the circuit to resonate at this frequency if its capacitance is 2.00 pF? (b) What is the inductive reactance of the circuit at this frequency?

45. A 10.0-Ω resistor, 10.0-mH inductor, and 100-μF capacitor are connected in series to a 50.0-V (rms) source having variable frequency. If the operating frequency is twice the resonance frequency, find the energy delivered to the circuit during one period.

46. A resistor R, inductor L, and capacitor C are connected in series to an AC source of rms voltage ΔV and variable frequency. If the operating frequency is twice the resonance frequency, find the energy delivered to the circuit during one period.

47. **Review.** A radar transmitter contains an LC circuit oscillating at 1.00×10^{10} Hz. (a) For a one-turn loop having an inductance of 400 pH to resonate at this frequency, what capacitance is required in series with the loop? (b) The capacitor has square, parallel plates separated by 1.00 mm of air. What should the edge length of the plates be? (c) What is the common reactance of the loop and capacitor at resonance?

Section 33.8 The Transformer and Power Transmission

48. W A step-down transformer is used for recharging the batteries of portable electronic devices. The turns ratio N_2/N_1 for a particular transformer used in a DVD player is 1:13. When used with 120-V (rms) household service, the transformer draws an rms current of 20.0 mA from the house outlet. Find (a) the rms output voltage of the transformer and (b) the power delivered to the DVD player.

49. M The primary coil of a transformer has $N_1 = 350$ turns, and the secondary coil has $N_2 = 2\,000$ turns. If the input voltage across the primary coil is $\Delta v = 170 \cos \omega t$, where Δv is in volts and t is in seconds, what rms voltage is developed across the secondary coil?

50. AMT A transmission line that has a resistance per unit length of 4.50×10^{-4} Ω/m is to be used to transmit 5.00 MW across 400 mi (6.44×10^5 m). The output voltage of the source is 4.50 kV. (a) What is the line loss if a transformer is used to step up the voltage to 500 kV? (b) What fraction of the input power is lost to the line under these circumstances? (c) **What If?** What difficulties would be encountered in attempting to transmit the 5.00 MW at the source voltage of 4.50 kV?

51. In the transformer shown in Figure P33.51, the load resistance R_L is 50.0 Ω. The turns ratio N_1/N_2 is 2.50, and the rms source voltage is $\Delta V_s = 80.0$ V. If a voltmeter across the load resistance measures an rms voltage of 25.0 V, what is the source resistance R_s?

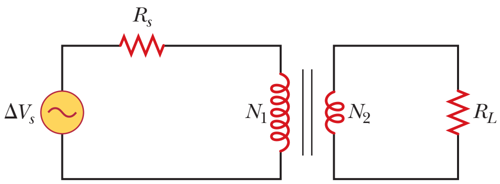


Figure P33.51

52. A person is working near the secondary of a transformer as shown in Figure P33.52. The primary voltage is 120 V at 60.0 Hz. The secondary voltage is 5 000 V. The capacitance C_s, which is the stray capacitance between the hand and the secondary winding, is 20.0 pF. Assuming the person has a body resistance to ground of $R_b = 50.0$ kΩ, determine the rms voltage across the body. *Suggestion:* Model the secondary of the transformer as an AC source.

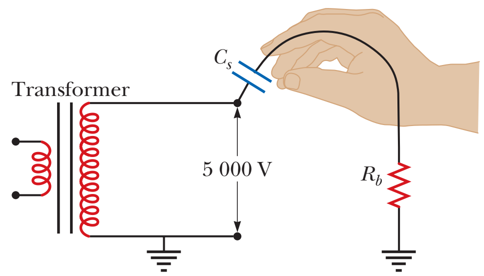


Figure P33.52

Section 33.9 Rectifiers and Filters

53. M The RC high-pass filter shown in Figure P33.53 has a resistance $R = 0.500$ Ω and a capacitance $C = 613$ μF. What is the ratio of the amplitude of the output voltage to that of the input voltage for this filter for a source frequency of 600 Hz?

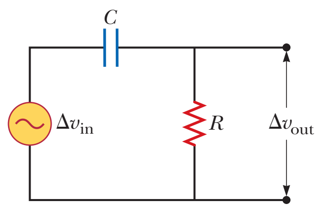


Figure P33.53
Problems 53 and 54.

54. Consider the RC high-pass filter circuit shown in Figure P33.53. (a) Find an expression for the ratio of the amplitude of the output voltage to that of the input voltage in terms of R, C, and the AC source frequency ω. (b) What value does this ratio approach as the frequency decreases toward zero? (c) What value does this ratio approach as the frequency increases without limit?

55. One particular plug-in power supply for a radio looks similar to the one shown in Figure 33.20 and is marked with the following information: Input 120 V AC 8 W Output 9 V DC 300 mA. Assume these values are accurate to two digits. (a) Find the energy efficiency of the device when the radio is operating. (b) At what rate is energy wasted in the device when the radio is operating? (c) Suppose the input power to the transformer is 8.00 W when the radio is switched off and energy costs \$0.110/kWh from the electric company. Find the cost of having six such transformers around the house, each plugged in for 31 days.

56. Consider the filter circuit shown in Figure P33.56. (a) Show that the ratio of the amplitude of the output voltage to that of the input voltage is

$$\frac{\Delta V_{out}}{\Delta V_{in}} = \frac{1/\omega C}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}}$$

(b) What value does this ratio approach as the frequency decreases toward zero? (c) What value does this ratio approach as the frequency increases without limit? (d) At what frequency is the ratio equal to one-half?

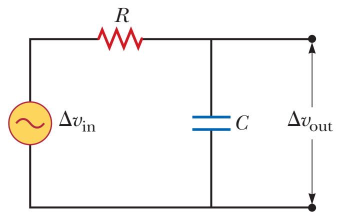


Figure P33.56

Additional Problems

57. A step-up transformer is designed to have an output voltage of 2 200 V (rms) when the primary is connected across a 110-V (rms) source. (a) If the primary winding has exactly 80 turns, how many turns are required on the secondary? (b) If a load resistor across the secondary draws a current of 1.50 A, what is the current in the primary, assuming ideal conditions? (c) **What If?** If the transformer actually has an efficiency of 95.0%, what is the current in the primary when the secondary current is 1.20 A?

58. *Why is the following situation impossible?* An *RLC* circuit is used in a radio to tune into a North American AM commercial radio station. The values of the circuit components are $R = 15.0\ \Omega$, $L = 2.80\ \mu$H, and $C = 0.910$ pF.

59. **Review.** The voltage phasor diagram for a certain series *RLC* circuit is shown in Figure P33.59. The resistance of the circuit is 75.0 Ω, and the frequency is 60.0 Hz. Find (a) the maximum voltage ΔV_{max}, (b) the phase angle ϕ, (c) the maximum current, (d) the impedance, (e) the capacitance and (f) the inductance of the circuit, and (g) the average power delivered to the circuit.

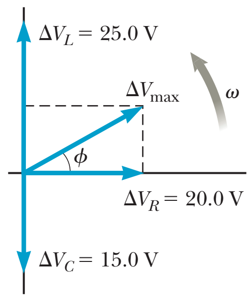


Figure P33.59

60. Consider a series *RLC* circuit having the parameters $R = 200\ \Omega$, $L = 663$ mH, and $C = 26.5\ \mu$F. The applied voltage has an amplitude of 50.0 V and a frequency of 60.0 Hz. Find (a) the current I_{max} and its phase relative to the applied voltage Δv, (b) the maximum voltage ΔV_R across the resistor and its phase relative to the current, (c) the maximum voltage ΔV_C across the capacitor and its phase relative to the current, and (d) the maximum voltage ΔV_L across the inductor and its phase relative to the current.

61. Energy is to be transmitted over a pair of copper wires in a transmission line at the rate of 20.0 kW with only a 1.00% loss over a distance of 18.0 km at potential difference $\Delta V_{rms} = 1.50 \times 10^3$ V between the wires. Assuming the current density is uniform in the conductors, what is the diameter required for each of the two wires?

62. Energy is to be transmitted over a pair of copper wires in a transmission line at a rate P with only a fractional loss f over a distance ℓ at potential difference ΔV_{rms} between the wires. Assuming the current density is uniform in the conductors, what is the diameter required for each of the two wires?

63. A 400-Ω resistor, an inductor, and a capacitor are in series with an AC source. The reactance of the inductor is 700 Ω, and the circuit impedance is 760 Ω. (a) What are the possible values of the reactance of the capacitor? (b) If you find that the power delivered to the circuit decreases as you raise the frequency, what is the capacitive reactance in the original circuit? (c) Repeat part (a) assuming the resistance is 200 Ω instead of 400 Ω and the circuit impedance continues to be 760 Ω.

64. Show that the rms value for the sawtooth voltage shown in Figure P33.64 is $\Delta V_{max}/\sqrt{3}$.

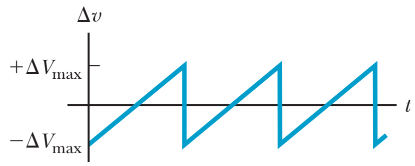


Figure P33.64

65. A transformer may be used to provide maximum power transfer between two AC circuits that have different impedances Z_1 and Z_2. This process is called *impedance matching.* (a) Show that the ratio of turns N_1/N_2 for this transformer is

$$\frac{N_1}{N_2} = \sqrt{\frac{Z_1}{Z_2}}$$

(b) Suppose you want to use a transformer as an impedance-matching device between an audio amplifier that has an output impedance of 8.00 kΩ and a speaker that has an input impedance of 8.00 Ω. What should your N_1/N_2 ratio be?

66. A capacitor, a coil, and two resistors of equal resistance are arranged in an AC circuit as shown in Figure P33.66 (page 1028). An AC source provides an emf of $\Delta V_{rms} = 20.0$ V at a frequency of 60.0 Hz. When the double-throw switch S is open as shown in the figure, the rms current is 183 mA. When the switch is closed in position *a*, the rms current is 298 mA. When the switch is closed in position *b*, the rms current is 137 mA. Determine the values of (a) R, (b) C, and (c) L. (d) Is more than one set of values possible? Explain.

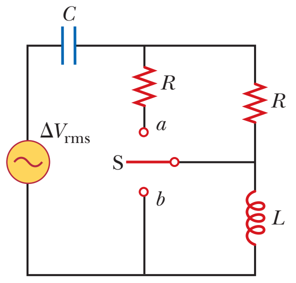


Figure P33.66

67. W Marie Cornu, a physicist at the Polytechnic Institute in Paris, invented phasors in about 1880. This problem helps you see their general utility in representing oscillations. Two mechanical vibrations are represented by the expressions

$$y_1 = 12.0 \sin 4.50t$$

and

$$y_2 = 12.0 \sin (4.50t + 70.0°)$$

where y_1 and y_2 are in centimeters and t is in seconds. Find the amplitude and phase constant of the sum of these functions (a) by using a trigonometric identity (as from Appendix B) and (b) by representing the oscillations as phasors. (c) State the result of comparing the answers to parts (a) and (b). (d) Phasors make it equally easy to add traveling waves. Find the amplitude and phase constant of the sum of the three waves represented by

$$y_1 = 12.0 \sin (15.0x - 4.50t + 70.0°)$$

$$y_2 = 15.5 \sin (15.0x - 4.50t - 80.0°)$$

$$y_3 = 17.0 \sin (15.0x - 4.50t + 160°)$$

where x, y_1, y_2, and y_3 are in centimeters and t is in seconds.

68. A series RLC circuit has resonance angular frequency 2.00×10^3 rad/s. When it is operating at some input frequency, $X_L = 12.0\ \Omega$ and $X_C = 8.00\ \Omega$. (a) Is this input frequency higher than, lower than, or the same as the resonance frequency? Explain how you can tell. (b) Explain whether it is possible to determine the values of both L and C. (c) If it is possible, find L and C. If it is not possible, give a compact expression for the condition that L and C must satisfy.

69. AMT **Review.** One insulated conductor from a household extension cord has a mass per length of 19.0 g/m. A section of this conductor is held under tension between two clamps. A subsection is located in a magnetic field of magnitude 15.3 mT directed perpendicular to the length of the cord. When the cord carries an AC current of 9.00 A at a frequency of 60.0 Hz, it vibrates in resonance in its simplest standing-wave vibration mode. (a) Determine the relationship that must be satisfied between the separation d of the clamps and the tension T in the cord. (b) Determine one possible combination of values for these variables.

70. (a) Sketch a graph of the phase angle for an RLC series circuit as a function of angular frequency from zero to a frequency much higher than the resonance frequency. (b) Identify the value of ϕ at the resonance angular frequency ω_0. (c) Prove that the slope of the graph of ϕ versus ω at the resonance point is $2Q/\omega_0$.

71. In Figure P33.71, find the rms current delivered by the 45.0-V (rms) power supply when (a) the frequency is very large and (b) the frequency is very small.

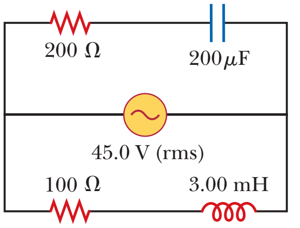


Figure P33.71

72. **Review.** In the circuit shown in Figure P33.72, assume all parameters except C are given. Find (a) the current in the circuit as a function of time and (b) the power delivered to the circuit. (c) Find the current as a function of time after *only* switch 1 is opened. (d) After switch 2 is *also* opened, the current and voltage are in phase. Find the capacitance C. Find (e) the impedance of the circuit when both switches are open, (f) the maximum energy stored in the capacitor during oscillations, and (g) the maximum energy stored in the inductor during oscillations. (h) Now the frequency of the voltage source is doubled. Find the phase difference between the current and the voltage. (i) Find the frequency that makes the inductive reactance one-half the capacitive reactance.

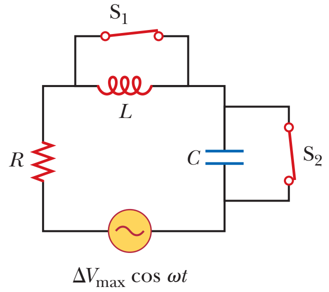


Figure P33.72

73. GP A series RLC circuit contains the following components: $R = 150\ \Omega$, $L = 0.250$ H, $C = 2.00\ \mu$F, and a source with $\Delta V_{\max} = 210$ V operating at 50.0 Hz. Our goal is to find the phase angle, the power factor, and the power input for this circuit. (a) Find the inductive reactance in the circuit. (b) Find the capacitive reactance in the circuit. (c) Find the impedance in the circuit. (d) Calculate the maximum current in the circuit. (e) Determine the phase angle between the current and source voltage. (f) Find the power factor for the circuit. (g) Find the power input to the circuit.

74. A series *RLC* circuit is operating at 2.00×10^3 Hz. At this frequency, $X_L = X_C = 1\ 884\ \Omega$. The resistance of the circuit is 40.0 Ω. (a) Prepare a table showing the values of X_L, X_C, and Z for $f = 300$, 600, 800, 1.00×10^3, 1.50×10^3, 2.00×10^3, 3.00×10^3, 4.00×10^3, 6.00×10^3, and 1.00×10^4 Hz. (b) Plot on the same set of axes X_L, X_C, and Z as a function of $\ln f$.

75. M A series *RLC* circuit consists of an 8.00-Ω resistor, a 5.00-μF capacitor, and a 50.0-mH inductor. A variable-frequency source applies an emf of 400 V (rms) across the combination. Assuming the frequency is equal to one-half the resonance frequency, determine the power delivered to the circuit.

76. A series *RLC* circuit in which $R = 1.00\ \Omega$, $L = 1.00$ mH, and $C = 1.00$ nF is connected to an AC source delivering 1.00 V (rms). (a) Make a precise graph of the power delivered to the circuit as a function of the frequency and (b) verify that the full width of the resonance peak at half-maximum is $R/2\pi L$.

Challenge Problems

77. The resistor in Figure P33.77 represents the midrange speaker in a three-speaker system. Assume its resistance to be constant at 8.00 Ω. The source represents an audio amplifier producing signals of uniform amplitude $\Delta V_{max} = 10.0$ V at all audio frequencies. The inductor and capacitor are to function as a band-pass filter with $\Delta V_{out}/\Delta V_{in} = \frac{1}{2}$ at 200 Hz and at 4.00×10^3 Hz. Determine the required values of (a) L and (b) C. Find (c) the maximum value of the ratio $\Delta V_{out}/\Delta V_{in}$; (d) the frequency f_0 at which the ratio has its maximum value; (e) the phase shift between Δv_{in} and Δv_{out} at 200 Hz, at f_0, and at 4.00×10^3 Hz; and (f) the average power transferred to the speaker at 200 Hz, at f_0, and at 4.00×10^3 Hz. (g) Treating the filter as a resonant circuit, find its quality factor.

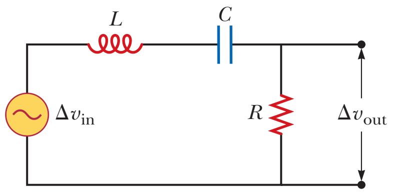


Figure P33.77

78. An 80.0-Ω resistor and a 200-mH inductor are connected in *parallel* across a 100-V (rms), 60.0-Hz source. (a) What is the rms current in the resistor? (b) By what angle does the total current lead or lag behind the voltage?

79. A voltage $\Delta v = 100 \sin \omega t$, where Δv is in volts and t is in seconds, is applied across a series combination of a 2.00-H inductor, a 10.0-μF capacitor, and a 10.0-Ω resistor. (a) Determine the angular frequency ω_0 at which the power delivered to the resistor is a maximum. (b) Calculate the average power delivered at that frequency. (c) Determine the two angular frequencies ω_1 and ω_2 at which the power is one-half the maximum value. *Note:* The Q of the circuit is $\omega_0/(\omega_2 - \omega_1)$.

80. Figure P33.80a shows a parallel *RLC* circuit. The instantaneous voltages (and rms voltages) across each of the three circuit elements are the same, and each is in phase with the current in the resistor. The currents in C and L lead or lag the current in the resistor as shown in the current phasor diagram, Figure P33.80b. (a) Show that the rms current delivered by the source is

$$I_{rms} = \Delta V_{rms}\left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{1/2}$$

(b) Show that the phase angle ϕ between ΔV_{rms} and I_{rms} is given by

$$\tan \phi = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$

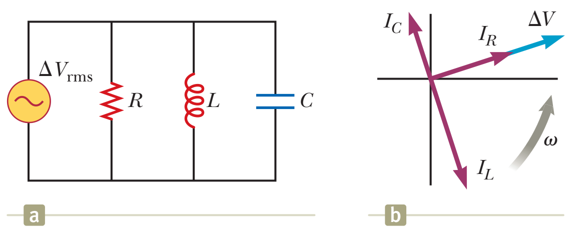
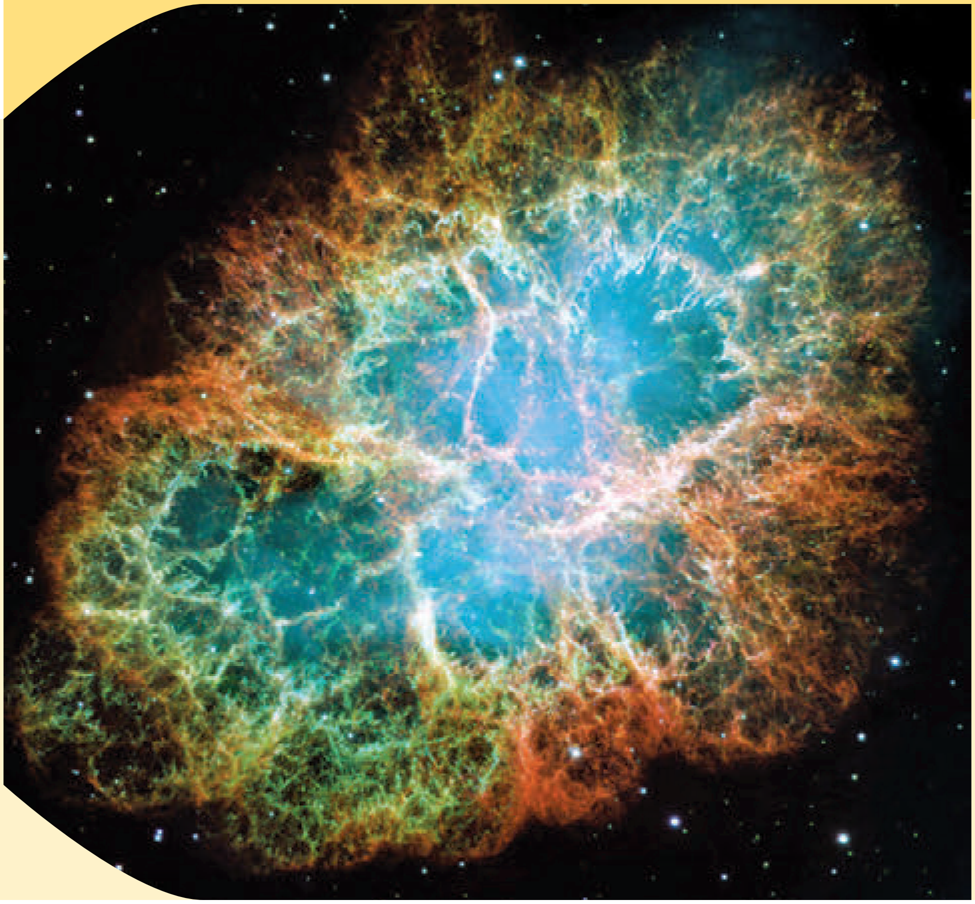


Figure P33.80

81. An AC source with $\Delta V_{rms} = 120$ V is connected between points a and d in Figure P33.24. At what frequency will it deliver a power of 250 W? Explain your answer.

CHAPTER

34 Electromagnetic Waves



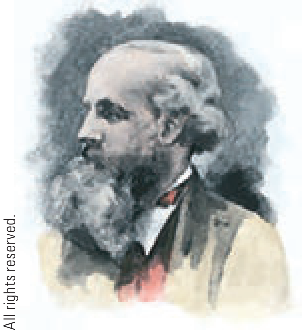
This image of the Crab Nebula taken with visible light shows a variety of colors, with each color representing a different wavelength of visible light. *(NASA, ESA, J. Hester, A. Loll (ASU))*

The waves described in Chapters 16, 17, and 18 are mechanical waves. By definition, the propagation of mechanical disturbances—such as sound waves, water waves, and waves on a string—requires the presence of a medium. This chapter is concerned with the properties of electromagnetic waves, which (unlike mechanical waves) can propagate through empty space.

We begin by considering Maxwell's contributions in modifying Ampère's law, which we studied in Chapter 30. We then discuss Maxwell's equations, which form the theoretical basis of all electromagnetic phenomena. These equations predict the existence of electromagnetic waves that propagate through space at the speed of light c according to the traveling wave analysis model. Heinrich Hertz confirmed Maxwell's prediction when he generated and detected electromagnetic waves in 1887. That discovery has led to many practical communication systems, including radio, television, cell phone systems, wireless Internet connectivity, and optoelectronics.

Next, we learn how electromagnetic waves are generated by oscillating electric charges. The waves radiated from the oscillating charges can be detected at great distances. Furthermore, because electromagnetic waves carry energy (T_{ER} in Eq. 8.2) and momentum, they can exert pressure on a surface. The chapter concludes with a description of the various frequency ranges in the electromagnetic spectrum.

34.1 Displacement Current and the General Form of Ampère's Law



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James Clerk Maxwell
Scottish Theoretical Physicist (1831–1879)
Maxwell developed the electromagnetic theory of light and the kinetic theory of gases, and explained the nature of Saturn's rings and color vision. Maxwell's successful interpretation of the electromagnetic field resulted in the field equations that bear his name. Formidable mathematical ability combined with great insight enabled him to lead the way in the study of electromagnetism and kinetic theory. He died of cancer before he was 50.

In Chapter 30, we discussed using Ampère's law (Eq. 30.13) to analyze the magnetic fields created by currents:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I$$

In this equation, the line integral is over any closed path through which conduction current passes, where conduction current is defined by the expression $I = dq/dt$. (In this section, we use the term *conduction current* to refer to the current carried by charge carriers in the wire to distinguish it from a new type of current we shall introduce shortly.) We now show that Ampère's law in this form is valid only if any electric fields present are constant in time. James Clerk Maxwell recognized this limitation and modified Ampère's law to include time-varying electric fields.

Consider a capacitor being charged as illustrated in Figure 34.1. When a conduction current is present, the charge on the positive plate changes, but no conduction current exists in the gap between the plates because there are no charge carriers in the gap. Now consider the two surfaces S_1 and S_2 in Figure 34.1, bounded by the same path P. Ampère's law states that $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around this path must equal $\mu_0 I$, where I is the total current through *any* surface bounded by the path P.

When the path P is considered to be the boundary of S_1, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I$ because the conduction current I passes through S_1. When the path is considered to be the boundary of S_2, however, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0$ because no conduction current passes through S_2. Therefore, we have a contradictory situation that arises from the discontinuity of the current! Maxwell solved this problem by postulating an additional term on the right side of Ampère's law, which includes a factor called the **displacement current** I_d defined as[1]

$$I_d \equiv \epsilon_0 \frac{d\Phi_E}{dt} \tag{34.1}$$

◀ **Displacement current**

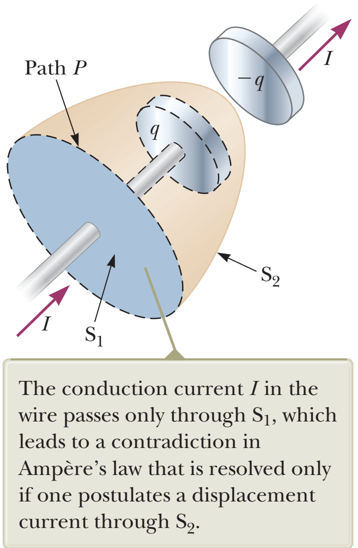

Figure 34.1 Two surfaces S_1 and S_2 near the plate of a capacitor are bounded by the same path P.

[1] *Displacement* in this context does not have the meaning it does in Chapter 2. Despite the inaccurate implications, the word is historically entrenched in the language of physics, so we continue to use it.

where ϵ_0 is the permittivity of free space (see Section 23.3) and $\Phi_E \equiv \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ is the electric flux (see Eq. 24.3) through the surface bounded by the path of integration.

As the capacitor is being charged (or discharged), the changing electric field between the plates may be considered equivalent to a current that acts as a continuation of the conduction current in the wire. When the expression for the displacement current given by Equation 34.1 is added to the conduction current on the right side of Ampère's law, the difficulty represented in Figure 34.1 is resolved. No matter which surface bounded by the path P is chosen, either a conduction current or a displacement current passes through it. With this new term I_d, we can express the general form of Ampère's law (sometimes called the **Ampère–Maxwell law**) as

Ampère–Maxwell law ▶

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0(I + I_d) = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt} \qquad (34.2)$$

We can understand the meaning of this expression by referring to Figure 34.2. The electric flux through surface S is $\Phi_E = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA$, where A is the area of the capacitor plates and E is the magnitude of the uniform electric field between the plates. If q is the charge on the plates at any instant, then $E = q/(\epsilon_0 A)$ (see Section 26.2). Therefore, the electric flux through S is

$$\Phi_E = EA = \frac{q}{\epsilon_0}$$

Hence, the displacement current through S is

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} = \frac{dq}{dt} \qquad (34.3)$$

That is, the displacement current I_d through S is precisely equal to the conduction current I in the wires connected to the capacitor!

By considering surface S, we can identify the displacement current as the source of the magnetic field on the surface boundary. The displacement current has its physical origin in the time-varying electric field. The central point of this formalism is that magnetic fields are produced *both* by conduction currents *and* by time-varying electric fields. This result was a remarkable example of theoretical work by Maxwell, and it contributed to major advances in the understanding of electromagnetism.

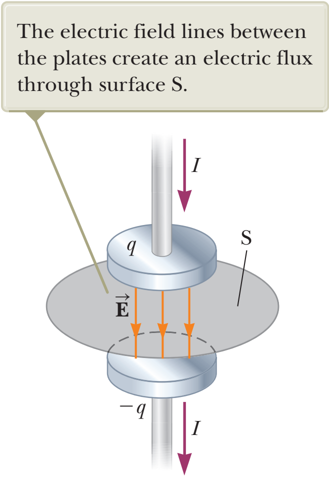

Figure 34.2 When a conduction current exists in the wires, a changing electric field $\vec{\mathbf{E}}$ exists between the plates of the capacitor.

Quick Quiz 34.1 In an *RC* circuit, the capacitor begins to discharge. **(i)** During the discharge, in the region of space between the plates of the capacitor, is there (a) conduction current but no displacement current, (b) displacement current but no conduction current, (c) both conduction and displacement current, or (d) no current of any type? **(ii)** In the same region of space, is there (a) an electric field but no magnetic field, (b) a magnetic field but no electric field, (c) both electric and magnetic fields, or (d) no fields of any type?

Example 34.1 Displacement Current in a Capacitor

A sinusoidally varying voltage is applied across a capacitor as shown in Figure 34.3. The capacitance is $C = 8.00\ \mu\text{F}$, the frequency of the applied voltage is $f = 3.00$ kHz, and the voltage amplitude is $\Delta V_{\max} = 30.0$ V. Find the displacement current in the capacitor.

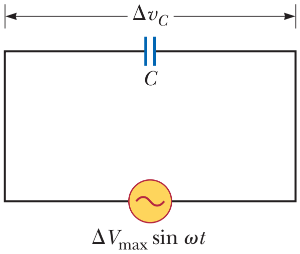

Figure 34.3 (Example 34.1)

SOLUTION

Conceptualize Figure 34.3 represents the circuit diagram for this situation. Figure 34.2 shows a close-up of the capacitor and the electric field between the plates.

Categorize We determine results using equations discussed in this section, so we categorize this example as a substitution problem.

34.1 continued

Evaluate the angular frequency of the source from Equation 15.12:

$$\omega = 2\pi f = 2\pi(3.00 \times 10^3 \text{ Hz}) = 1.88 \times 10^4 \text{ s}^{-1}$$

Use Equation 33.20 to express the potential difference in volts across the capacitor as a function of time in seconds:

$$\Delta v_C = \Delta V_{\max} \sin \omega t = 30.0 \sin (1.88 \times 10^4\, t)$$

Use Equation 34.3 to find the displacement current in amperes as a function of time. Note that the charge on the capacitor is $q = C\,\Delta v_C$:

$$i_d = \frac{dq}{dt} = \frac{d}{dt}(C\,\Delta v_C) = C\frac{d}{dt}(\Delta V_{\max} \sin \omega t)$$

$$= \omega C\,\Delta V_{\max} \cos \omega t$$

Substitute numerical values:

$$i_d = (1.88 \times 10^4 \text{ s}^{-1})(8.00 \times 10^{-6} \text{ C})(30.0 \text{ V}) \cos (1.88 \times 10^4\, t)$$

$$= \boxed{4.51 \cos (1.88 \times 10^4\, t)}$$

34.2 Maxwell's Equations and Hertz's Discoveries

We now present four equations that are regarded as the basis of all electrical and magnetic phenomena. These equations, developed by Maxwell, are as fundamental to electromagnetic phenomena as Newton's laws are to mechanical phenomena. In fact, the theory that Maxwell developed was more far-reaching than even he imagined because it turned out to be in agreement with the special theory of relativity, as Einstein showed in 1905.

Maxwell's equations represent the laws of electricity and magnetism that we have already discussed, but they have additional important consequences. For simplicity, we present **Maxwell's equations** as applied to free space, that is, in the absence of any dielectric or magnetic material. The four equations are

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q}{\epsilon_0} \quad (34.4)$$

◀ Gauss's law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \quad (34.5)$$

◀ Gauss's law in magnetism

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \quad (34.6)$$

◀ Faraday's law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad (34.7)$$

◀ Ampère–Maxwell law

Equation 34.4 is Gauss's law: the total electric flux through any closed surface equals the net charge inside that surface divided by ϵ_0. This law relates an electric field to the charge distribution that creates it.

Equation 34.5 is Gauss's law in magnetism, and it states that the net magnetic flux through a closed surface is zero. That is, the number of magnetic field lines that enter a closed volume must equal the number that leave that volume, which implies that magnetic field lines cannot begin or end at any point. If they did, it would mean that isolated magnetic monopoles existed at those points. That isolated magnetic monopoles have not been observed in nature can be taken as a confirmation of Equation 34.5.

Equation 34.6 is Faraday's law of induction, which describes the creation of an electric field by a changing magnetic flux. This law states that the emf, which is the line integral of the electric field around any closed path, equals the rate of change of magnetic flux through any surface bounded by that path. One consequence of Faraday's law is the current induced in a conducting loop placed in a time-varying magnetic field.

Equation 34.7 is the Ampère–Maxwell law, discussed in Section 34.1, and it describes the creation of a magnetic field by a changing electric field and by electric current: the line integral of the magnetic field around any closed path is the sum of μ_0 multiplied by the net current through that path and $\epsilon_0\mu_0$ multiplied by the rate of change of electric flux through any surface bounded by that path.

Once the electric and magnetic fields are known at some point in space, the force acting on a particle of charge q can be calculated from the electric and magnetic versions of the particle in a field model:

Lorentz force law ▶

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \quad \textbf{(34.8)}$$

This relationship is called the **Lorentz force law.** (We saw this relationship earlier as Eq. 29.6.) Maxwell's equations, together with this force law, completely describe all classical electromagnetic interactions in a vacuum.

Notice the symmetry of Maxwell's equations. Equations 34.4 and 34.5 are symmetric, apart from the absence of the term for magnetic monopoles in Equation 34.5. Furthermore, Equations 34.6 and 34.7 are symmetric in that the line integrals of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ around a closed path are related to the rate of change of magnetic flux and electric flux, respectively. Maxwell's equations are of fundamental importance not only to electromagnetism, but to all science. Hertz once wrote, "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we put into them."

In the next section, we show that Equations 34.6 and 34.7 can be combined to obtain a wave equation for both the electric field and the magnetic field. In empty space, where $q = 0$ and $I = 0$, the solution to these two equations shows that the speed at which electromagnetic waves travel equals the measured speed of light. This result led Maxwell to predict that light waves are a form of electromagnetic radiation.

Hertz performed experiments that verified Maxwell's prediction. The experimental apparatus Hertz used to generate and detect electromagnetic waves is shown schematically in Figure 34.4. An induction coil is connected to a transmitter made up of two spherical electrodes separated by a narrow gap. The coil provides short voltage surges to the electrodes, making one positive and the other negative. A spark is generated between the spheres when the electric field near either electrode surpasses the dielectric strength for air (3×10^6 V/m; see Table 26.1). Free electrons in a strong electric field are accelerated and gain enough energy to ionize any molecules they strike. This ionization provides more electrons, which can accelerate and cause further ionizations. As the air in the gap is ionized, it becomes a much better conductor and the discharge between the electrodes exhibits an oscillatory behavior at a very high frequency. From an electric-circuit viewpoint, this experimental apparatus is equivalent to an LC circuit in which the inductance is that of the coil and the capacitance is due to the spherical electrodes.

Because L and C are small in Hertz's apparatus, the frequency of oscillation is high, on the order of 100 MHz. (Recall from Eq. 32.22 that $\omega = 1/\sqrt{LC}$ for an LC circuit.) Electromagnetic waves are radiated at this frequency as a result of the oscillation of free charges in the transmitter circuit. Hertz was able to detect these waves using a single loop of wire with its own spark gap (the receiver). Such a receiver loop, placed several meters from the transmitter, has its own effective inductance, capacitance, and natural frequency of oscillation. In Hertz's experiment, sparks were induced across the gap of the receiving electrodes when the receiver's frequency was adjusted to match that of the transmitter. In this way,

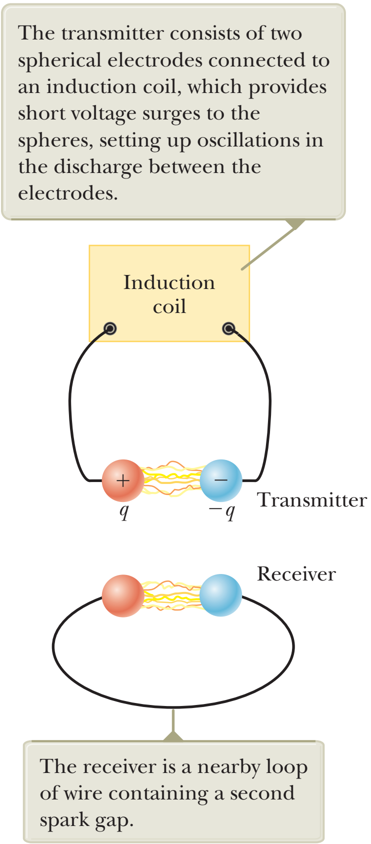

Figure 34.4 Schematic diagram of Hertz's apparatus for generating and detecting electromagnetic waves.

Hertz demonstrated that the oscillating current induced in the receiver was produced by electromagnetic waves radiated by the transmitter. His experiment is analogous to the mechanical phenomenon in which a tuning fork responds to acoustic vibrations from an identical tuning fork that is oscillating nearby.

In addition, Hertz showed in a series of experiments that the radiation generated by his spark-gap device exhibited the wave properties of interference, diffraction, reflection, refraction, and polarization, which are all properties exhibited by light as we shall see in Part 5. Therefore, it became evident that the radio-frequency waves Hertz was generating had properties similar to those of light waves and that they differed only in frequency and wavelength. Perhaps his most convincing experiment was the measurement of the speed of this radiation. Waves of known frequency were reflected from a metal sheet and created a standing-wave interference pattern whose nodal points could be detected. The measured distance between the nodal points enabled determination of the wavelength λ. Using the relationship $v = \lambda f$ (Eq. 16.12) from the traveling wave model, Hertz found that v was close to 3×10^8 m/s, the known speed c of visible light.

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Heinrich Rudolf Hertz
German Physicist (1857–1894)
Hertz made his most important discovery of electromagnetic waves in 1887. After finding that the speed of an electromagnetic wave was the same as that of light, Hertz showed that electromagnetic waves, like light waves, could be reflected, refracted, and diffracted. The hertz, equal to one complete vibration or cycle per second, is named after him.

34.3 Plane Electromagnetic Waves

The properties of electromagnetic waves can be deduced from Maxwell's equations. One approach to deriving these properties is to solve the second-order differential equation obtained from Maxwell's third and fourth equations. A rigorous mathematical treatment of that sort is beyond the scope of this text. To circumvent this problem, let's assume the vectors for the electric field and magnetic field in an electromagnetic wave have a specific space–time behavior that is simple but consistent with Maxwell equations.

To understand the prediction of electromagnetic waves more fully, let's focus our attention on an electromagnetic wave that travels in the x direction (the *direction of propagation*). For this wave, the electric field $\vec{\mathbf{E}}$ is in the y direction and the magnetic field $\vec{\mathbf{B}}$ is in the z direction as shown in Figure 34.5. Such waves, in which the electric and magnetic fields are restricted to being parallel to a pair of perpendicular axes, are said to be **linearly polarized waves.** Furthermore, let's assume the field magnitudes E and B depend on x and t only, not on the y or z coordinate.

Let's also imagine that the source of the electromagnetic waves is such that a wave radiated from *any* position in the yz plane (not only from the origin as might be suggested by Fig. 34.5) propagates in the x direction and all such waves are emitted in phase. If we define a **ray** as the line along which the wave travels, all rays for these waves are parallel. This entire collection of waves is often called a **plane wave.** A surface connecting points of equal phase on all waves is a geometric plane called a **wave front,** introduced in Chapter 17. In comparison, a point source of radiation sends waves out radially in all directions. A surface connecting points of equal phase for this situation is a sphere, so this wave is called a **spherical wave.**

Pitfall Prevention 34.1
What Is "a" Wave? What do we mean by a *single* wave? The word *wave* represents both the emission from a *single point* ("wave radiated from *any* position in the yz plane" in the text) and the collection of waves from *all points* on the source ("**plane wave**" in the text). You should be able to use this term in both ways and understand its meaning from the context.

To generate the prediction of plane electromagnetic waves, we start with Faraday's law, Equation 34.6:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

To apply this equation to the wave in Figure 34.5, consider a rectangle of width dx and height ℓ lying in the xy plane as shown in Figure 34.6 (page 1036). Let's first evaluate the line integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ around this rectangle in the counterclockwise direction at an instant of time when the wave is passing through the rectangle. The contributions from the top and bottom of the rectangle are zero because $\vec{\mathbf{E}}$ is perpendicular to $d\vec{\mathbf{s}}$ for these paths. We can express the electric field on the right side of the rectangle as

$$E(x + dx) \approx E(x) + \left.\frac{dE}{dx}\right|_{t \text{ constant}} dx = E(x) + \frac{\partial E}{\partial x}\, dx$$

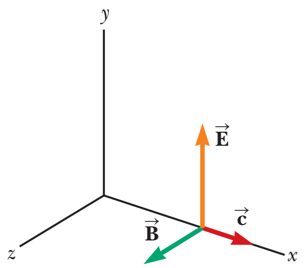

Figure 34.5 Electric and magnetic fields of an electromagnetic wave traveling at velocity $\vec{\mathbf{c}}$ in the positive x direction. The field vectors are shown at one instant of time and at one position in space. These fields depend on x and t.

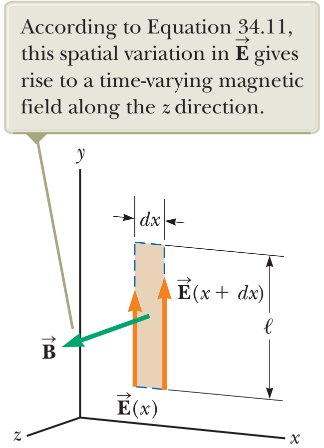

Figure 34.6 At an instant when a plane wave moving in the positive x direction passes through a rectangular path of width dx lying in the xy plane, the electric field in the y direction varies from $\vec{\mathbf{E}}(x)$ to $\vec{\mathbf{E}}(x + dx)$.

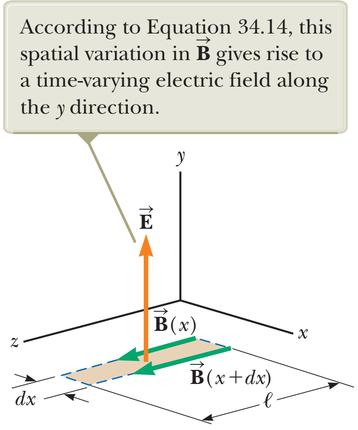

Figure 34.7 At an instant when a plane wave passes through a rectangular path of width dx lying in the xz plane, the magnetic field in the z direction varies from $\vec{\mathbf{B}}(x)$ to $\vec{\mathbf{B}}(x + dx)$.

where $E(x)$ is the field on the left side of the rectangle at this instant.[2] Therefore, the line integral over this rectangle is approximately

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = [E(x + dx)]\ell - [E(x)]\ell \approx \ell\left(\frac{\partial E}{\partial x}\right) dx \tag{34.9}$$

Because the magnetic field is in the z direction, the magnetic flux through the rectangle of area $\ell\, dx$ is approximately $\Phi_B = B\ell\, dx$ (assuming dx is very small compared with the wavelength of the wave). Taking the time derivative of the magnetic flux gives

$$\frac{d\Phi_B}{dt} = \ell\, dx \left.\frac{dB}{dt}\right|_{x \text{ constant}} = \ell\, dx \frac{\partial B}{\partial t} \tag{34.10}$$

Substituting Equations 34.9 and 34.10 into Equation 34.6 gives

$$\ell\left(\frac{\partial E}{\partial x}\right) dx = -\ell\, dx \frac{\partial B}{\partial t}$$

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{34.11}$$

In a similar manner, we can derive a second equation by starting with Maxwell's fourth equation in empty space (Eq. 34.7). In this case, the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ is evaluated around a rectangle lying in the xz plane and having width dx and length ℓ as in Figure 34.7. Noting that the magnitude of the magnetic field changes from $B(x)$ to $B(x + dx)$ over the width dx and that the direction for taking the line integral is counterclockwise when viewed from above in Figure 34.7, the line integral over this rectangle is found to be approximately

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = [B(x)]\ell - [B(x + dx)]\ell \approx -\ell\left(\frac{\partial B}{\partial x}\right) dx \tag{34.12}$$

[2]Because dE/dx in this equation is expressed as the change in E with x at a given instant t, dE/dx is equivalent to the partial derivative $\partial E/\partial x$. Likewise, dB/dt means the change in B with time at a particular position x; therefore, in Equation 34.10, we can replace dB/dt with $\partial B/\partial t$.

The electric flux through the rectangle is $\Phi_E = E\ell\,dx$, which, when differentiated with respect to time, gives

$$\frac{\partial \Phi_E}{\partial t} = \ell\ dx\,\frac{\partial E}{\partial t} \tag{34.13}$$

Substituting Equations 34.12 and 34.13 into Equation 34.7 gives

$$-\ell\left(\frac{\partial B}{\partial x}\right) dx = \mu_0\,\epsilon_0\,\ell\ dx\left(\frac{\partial E}{\partial t}\right)$$

$$\frac{\partial B}{\partial x} = -\mu_0\epsilon_0\,\frac{\partial E}{\partial t} \tag{34.14}$$

Taking the derivative of Equation 34.11 with respect to x and combining the result with Equation 34.14 gives

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right) = -\frac{\partial}{\partial t}\left(-\mu_0\epsilon_0\,\frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0\epsilon_0\,\frac{\partial^2 E}{\partial t^2} \tag{34.15}$$

In the same manner, taking the derivative of Equation 34.14 with respect to x and combining it with Equation 34.11 gives

$$\frac{\partial^2 B}{\partial x^2} = \mu_0\epsilon_0\,\frac{\partial^2 B}{\partial t^2} \tag{34.16}$$

Equations 34.15 and 34.16 both have the form of the linear wave equation[3] with the wave speed v replaced by c, where

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{34.17}$$

Speed of electromagnetic waves

Let's evaluate this speed numerically:

$$c = \frac{1}{\sqrt{(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})(8.854\ 19 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})}}$$
$$= 2.997\ 92 \times 10^8\ \mathrm{m/s}$$

Because this speed is precisely the same as the speed of light in empty space, we are led to believe (correctly) that light is an electromagnetic wave.

The simplest solution to Equations 34.15 and 34.16 is a sinusoidal wave for which the field magnitudes E and B vary with x and t according to the expressions

$$E = E_{\max}\cos(kx - \omega t) \tag{34.18}$$

$$B = B_{\max}\cos(kx - \omega t) \tag{34.19}$$

Sinusoidal electric and magnetic fields

where $E_{\max}$ and $B_{\max}$ are the maximum values of the fields. The angular wave number is $k = 2\pi/\lambda$, where λ is the wavelength. The angular frequency is $\omega = 2\pi f$, where f is the wave frequency. According to the traveling wave model, the ratio ω/k equals the speed of an electromagnetic wave, c:

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$$

[3]The linear wave equation is of the form $(\partial^2 y/\partial x^2) = (1/v^2)(\partial^2 y/\partial t^2)$, where v is the speed of the wave and y is the wave function. The linear wave equation was introduced as Equation 16.27, and we suggest you review Section 16.6.

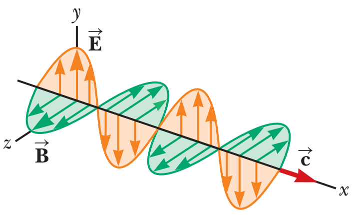

Figure 34.8 A sinusoidal electromagnetic wave moves in the positive x direction with a speed c.

where we have used Equation 16.12, $v = c = \lambda f$, which relates the speed, frequency, and wavelength of a sinusoidal wave. Therefore, for electromagnetic waves, the wavelength and frequency of these waves are related by

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{f} \tag{34.20}$$

Figure 34.8 is a pictorial representation, at one instant, of a sinusoidal, linearly polarized electromagnetic wave moving in the positive x direction.

We can generate other mathematical representations of the traveling wave model for electromagnetic waves. Taking partial derivatives of Equations 34.18 (with respect to x) and 34.19 (with respect to t) gives

$$\frac{\partial E}{\partial x} = -kE_{\max} \sin(kx - \omega t)$$

$$\frac{\partial B}{\partial t} = \omega B_{\max} \sin(kx - \omega t)$$

Substituting these results into Equation 34.11 shows that, at any instant,

$$kE_{\max} = \omega B_{\max}$$

$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c$$

Using these results together with Equations 34.18 and 34.19 gives

$$\boxed{\frac{E_{\max}}{B_{\max}} = \frac{E}{B} = c} \tag{34.21}$$

That is, at every instant, the ratio of the magnitude of the electric field to the magnitude of the magnetic field in an electromagnetic wave equals the speed of light.

Finally, note that electromagnetic waves obey the superposition principle as described in the waves in interference analysis model (which we discussed in Section 18.1 with respect to mechanical waves) because the differential equations involving E and B are linear equations. For example, we can add two waves with the same frequency and polarization simply by adding the magnitudes of the two electric fields algebraically.

Pitfall Prevention 34.2

$\vec{E}$ Stronger Than $\vec{B}$? Because the value of c is so large, some students incorrectly interpret Equation 34.21 as meaning that the electric field is much stronger than the magnetic field. Electric and magnetic fields are measured in different units, however, so they cannot be directly compared. In Section 34.4, we find that the electric and magnetic fields contribute equally to the wave's energy.

Quick Quiz 34.2 What is the phase difference between the sinusoidal oscillations of the electric and magnetic fields in Figure 34.8? **(a)** 180° **(b)** 90° **(c)** 0 **(d)** impossible to determine

Quick Quiz 34.3 An electromagnetic wave propagates in the negative y direction. The electric field at a point in space is momentarily oriented in the positive x direction. In which direction is the magnetic field at that point momentarily oriented? **(a)** the negative x direction **(b)** the positive y direction **(c)** the positive z direction **(d)** the negative z direction

Example 34.2 An Electromagnetic Wave AM

A sinusoidal electromagnetic wave of frequency 40.0 MHz travels in free space in the x direction as in Figure 34.9.

(A) Determine the wavelength and period of the wave.

▶ 34.2 continued

SOLUTION

Conceptualize Imagine the wave in Figure 34.9 moving to the right along the x axis, with the electric and magnetic fields oscillating in phase.

Categorize We use the mathematical representation of the *traveling wave* model for electromagnetic waves.

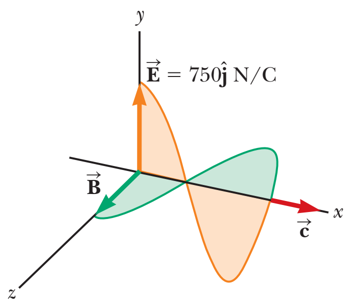

Figure 34.9 (Example 34.2) At some instant, a plane electromagnetic wave moving in the x direction has a maximum electric field of 750 N/C in the positive y direction.

Analyze

Use Equation 34.20 to find the wavelength of the wave:

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{40.0 \times 10^6 \text{ Hz}} = \boxed{7.50 \text{ m}}$$

Find the period T of the wave as the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{40.0 \times 10^6 \text{ Hz}} = \boxed{2.50 \times 10^{-8} \text{ s}}$$

(B) At some point and at some instant, the electric field has its maximum value of 750 N/C and is directed along the y axis. Calculate the magnitude and direction of the magnetic field at this position and time.

SOLUTION

Use Equation 34.21 to find the magnitude of the magnetic field:

$$B_{\max} = \frac{E_{\max}}{c} = \frac{750 \text{ N/C}}{3.00 \times 10^8 \text{ m/s}} = \boxed{2.50 \times 10^{-6} \text{ T}}$$

Because $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ must be perpendicular to each other and perpendicular to the direction of wave propagation (x in this case), we conclude that $\vec{\mathbf{B}}$ is in the z direction.

Finalize Notice that the wavelength is several meters. This is relatively long for an electromagnetic wave. As we will see in Section 34.7, this wave belongs to the radio range of frequencies.

34.4 Energy Carried by Electromagnetic Waves

In our discussion of the nonisolated system model for energy in Section 8.1, we identified electromagnetic radiation as one method of energy transfer across the boundary of a system. The amount of energy transferred by electromagnetic waves is symbolized as T_{ER} in Equation 8.2. The rate of transfer of energy by an electromagnetic wave is described by a vector $\vec{\mathbf{S}}$, called the **Poynting vector,** which is defined by the expression

$$\boxed{\vec{\mathbf{S}} \equiv \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}} \quad \textbf{(34.22)}$$

◀ Poynting vector

The magnitude of the Poynting vector represents the rate at which energy passes through a unit surface area perpendicular to the direction of wave propagation. Therefore, the magnitude of $\vec{\mathbf{S}}$ represents *power per unit area.* The direction of the vector is along the direction of wave propagation (Fig. 34.10, page 1040). The SI units of $\vec{\mathbf{S}}$ are J/s · m² = W/m².

As an example, let's evaluate the magnitude of $\vec{\mathbf{S}}$ for a plane electromagnetic wave where $|\vec{\mathbf{E}} \times \vec{\mathbf{B}}| = EB$. In this case,

$$S = \frac{EB}{\mu_0} \quad \textbf{(34.23)}$$

Pitfall Prevention 34.3

An Instantaneous Value The Poynting vector given by Equation 34.22 is time dependent. Its magnitude varies in time, reaching a maximum value at the same instant the magnitudes of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ do. The *average* rate of energy transfer is given by Equation 34.24 on the next page.

Pitfall Prevention 34.4

Irradiance In this discussion, intensity is defined in the same way as in Chapter 17 (as power per unit area). In the optics industry, however, power per unit area is called the *irradiance.* Radiant intensity is defined as the power in watts per solid angle (measured in steradians).

Because $B = E/c$, we can also express this result as

$$S = \frac{E^2}{\mu_0 c} = \frac{cB^2}{\mu_0}$$

These equations for S apply at any instant of time and represent the *instantaneous* rate at which energy is passing through a unit area in terms of the instantaneous values of E and B.

What is of greater interest for a sinusoidal plane electromagnetic wave is the time average of S over one or more cycles, which is called the *wave intensity I.* (We discussed the intensity of sound waves in Chapter 17.) When this average is taken, we obtain an expression involving the time average of $\cos^2(kx - \omega t)$, which equals $\frac{1}{2}$. Hence, the average value of S (in other words, the intensity of the wave) is

Wave intensity ▶

$$I = S_{\text{avg}} = \frac{E_{\max} B_{\max}}{2\mu_0} = \frac{E_{\max}^2}{2\mu_0 c} = \frac{cB_{\max}^2}{2\mu_0} \quad (34.24)$$

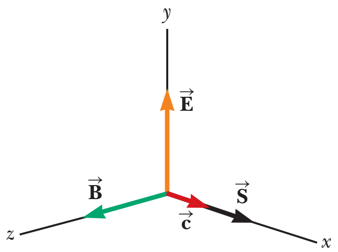

Figure 34.10 The Poynting vector $\vec{\mathbf{S}}$ for a plane electromagnetic wave is along the direction of wave propagation.

Recall that the energy per unit volume associated with an electric field, which is the instantaneous energy density u_E, is given by Equation 26.13:

$$u_E = \tfrac{1}{2}\epsilon_0 E^2$$

Also recall that the instantaneous energy density u_B associated with a magnetic field is given by Equation 32.14:

$$u_B = \frac{B^2}{2\mu_0}$$

Because E and B vary with time for an electromagnetic wave, the energy densities also vary with time. Using the relationships $B = E/c$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, the expression for u_B becomes

$$u_B = \frac{(E/c)^2}{2\mu_0} = \frac{\mu_0\epsilon_0}{2\mu_0}E^2 = \tfrac{1}{2}\epsilon_0 E^2$$

Comparing this result with the expression for u_E, we see that

$$u_B = u_E = \tfrac{1}{2}\epsilon_0 E^2 = \frac{B^2}{2\mu_0}$$

That is, the instantaneous energy density associated with the magnetic field of an electromagnetic wave equals the instantaneous energy density associated with the electric field. Hence, in a given volume, the energy is equally shared by the two fields.

The **total instantaneous energy density** u is equal to the sum of the energy densities associated with the electric and magnetic fields:

Total instantaneous energy density of an electromagnetic wave ▶

$$u = u_E + u_B = \epsilon_0 E^2 = \frac{B^2}{\mu_0}$$

When this total instantaneous energy density is averaged over one or more cycles of an electromagnetic wave, we again obtain a factor of $\frac{1}{2}$. Hence, for any electromagnetic wave, the total average energy per unit volume is

Average energy density of an electromagnetic wave ▶

$$u_{\text{avg}} = \epsilon_0 (E^2)_{\text{avg}} = \tfrac{1}{2}\epsilon_0 E_{\max}^2 = \frac{B_{\max}^2}{2\mu_0} \quad (34.25)$$

Comparing this result with Equation 34.24 for the average value of S, we see that

$$I = S_{\text{avg}} = cu_{\text{avg}} \quad (34.26)$$

In other words, the intensity of an electromagnetic wave equals the average energy density multiplied by the speed of light.

The Sun delivers about 10^3 W/m^2 of energy to the Earth's surface via electromagnetic radiation. Let's calculate the total power that is incident on the roof of a home. The roof's dimensions are 8.00 m × 20.0 m. We assume the average magnitude of the Poynting vector for solar radiation at the surface of the Earth is $S_{avg} = 1\,000$ W/m^2. This average value represents the power per unit area, or the light intensity. Assuming the radiation is incident normal to the roof, we obtain

$$P_{avg} = S_{avg}A = (1\,000 \text{ W/m}^2)(8.00 \text{ m} \times 20.0 \text{ m}) = 1.60 \times 10^5 \text{ W}$$

This power is large compared with the power requirements of a typical home. If this power could be absorbed and made available to electrical devices, it would provide more than enough energy for the average home. Solar energy is not easily harnessed, however, and the prospects for large-scale conversion are not as bright as may appear from this calculation. For example, the efficiency of conversion from solar energy is typically 12–18% for photovoltaic cells, reducing the available power by an order of magnitude. Other considerations reduce the power even further. Depending on location, the radiation is most likely not incident normal to the roof and, even if it is, this situation exists for only a short time near the middle of the day. No energy is available for about half of each day during the nighttime hours, and cloudy days further reduce the available energy. Finally, while energy is arriving at a large rate during the middle of the day, some of it must be stored for later use, requiring batteries or other storage devices. All in all, complete solar operation of homes is not currently cost effective for most homes.

Example 34.3 Fields on the Page

Estimate the maximum magnitudes of the electric and magnetic fields of the light that is incident on this page because of the visible light coming from your desk lamp. Treat the lightbulb as a point source of electromagnetic radiation that is 5% efficient at transforming energy coming in by electrical transmission to energy leaving by visible light.

SOLUTION

Conceptualize The filament in your lightbulb emits electromagnetic radiation. The brighter the light, the larger the magnitudes of the electric and magnetic fields.

Categorize Because the lightbulb is to be treated as a point source, it emits equally in all directions, so the outgoing electromagnetic radiation can be modeled as a spherical wave.

Analyze Recall from Equation 17.13 that the wave intensity I a distance r from a point source is $I = P_{avg}/4\pi r^2$, where P_{avg} is the average power output of the source and $4\pi r^2$ is the area of a sphere of radius r centered on the source.

Set this expression for I equal to the intensity of an electromagnetic wave given by Equation 34.24:

$$I = \frac{P_{avg}}{4\pi r^2} = \frac{E_{max}^2}{2\mu_0 c}$$

Solve for the electric field magnitude:

$$E_{max} = \sqrt{\frac{\mu_0 c P_{avg}}{2\pi r^2}}$$

Let's make some assumptions about numbers to enter in this equation. The visible light output of a 60-W lightbulb operating at 5% efficiency is approximately 3.0 W by visible light. (The remaining energy transfers out of the lightbulb by thermal conduction and invisible radiation.) A reasonable distance from the lightbulb to the page might be 0.30 m.

Substitute these values:

$$E_{max} = \sqrt{\frac{(4\pi \times 10^{-7} \text{ T}\cdot\text{m/A})(3.00 \times 10^8 \text{ m/s})(3.0 \text{ W})}{2\pi(0.30 \text{ m})^2}}$$

$$= \boxed{45 \text{ V/m}}$$

continued

34.3 continued

Use Equation 34.21 to find the magnetic field magnitude:

$$B_{max} = \frac{E_{max}}{c} = \frac{45\ \text{V/m}}{3.00 \times 10^8\ \text{m/s}} = \boxed{1.5 \times 10^{-7}\ \text{T}}$$

Finalize This value of the magnetic field magnitude is two orders of magnitude smaller than the Earth's magnetic field.

34.5 Momentum and Radiation Pressure

Electromagnetic waves transport linear momentum as well as energy. As this momentum is absorbed by some surface, pressure is exerted on the surface. Therefore, the surface is a nonisolated system for momentum. In this discussion, let's assume the electromagnetic wave strikes the surface at normal incidence and transports a total energy T_{ER} to the surface in a time interval Δt. Maxwell showed that if the surface absorbs all the incident energy T_{ER} in this time interval (as does a black body, introduced in Section 20.7), the total momentum $\vec{\mathbf{p}}$ transported to the surface has a magnitude

Momentum transported to a perfectly absorbing surface

$$p = \frac{T_{ER}}{c} \quad \text{(complete absorption)} \qquad \textbf{(34.27)}$$

The pressure P exerted on the surface is defined as force per unit area F/A, which when combined with Newton's second law gives

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt}$$

Substituting Equation 34.27 into this expression for pressure P gives

$$P = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{T_{ER}}{c}\right) = \frac{1}{c}\frac{(dT_{ER}/dt)}{A}$$

We recognize $(dT_{ER}/dt)/A$ as the rate at which energy is arriving at the surface per unit area, which is the magnitude of the Poynting vector. Therefore, the radiation pressure P exerted on the perfectly absorbing surface is

Radiation pressure exerted on a perfectly absorbing surface

$$P = \frac{S}{c} \quad \text{(complete absorption)} \qquad \textbf{(34.28)}$$

Pitfall Prevention 34.5

So Many p's We have p for momentum and P for pressure, and they are both related to P for power! Be sure to keep all these symbols straight.

If the surface is a perfect reflector (such as a mirror) and incidence is normal, the momentum transported to the surface in a time interval Δt is twice that given by Equation 34.27. That is, the momentum transferred to the surface by the incoming light is $p = T_{ER}/c$ and that transferred by the reflected light is also $p = T_{ER}/c$. Therefore,

$$p = \frac{2T_{ER}}{c} \quad \text{(complete reflection)} \qquad \textbf{(34.29)}$$

The radiation pressure exerted on a perfectly reflecting surface for normal incidence of the wave is

Radiation pressure exerted on a perfectly reflecting surface

$$P = \frac{2S}{c} \quad \text{(complete reflection)} \qquad \textbf{(34.30)}$$

The pressure on a surface having a reflectivity somewhere between these two extremes has a value between S/c and $2S/c$, depending on the properties of the surface.

Although radiation pressures are very small (about 5×10^{-6} N/m^2 for direct sunlight), *solar sailing* is a low-cost means of sending spacecraft to the planets. Large sheets experience radiation pressure from sunlight and are used in much the way canvas sheets are used on earthbound sailboats. In 2010, the Japan Aerospace Exploration Agency (JAXA) launched the first spacecraft to use solar sailing as its primary propulsion, *IKAROS* (Interplanetary Kite-craft Accelerated by Radiation of the Sun). Successful testing of this spacecraft would lead to a larger effort to send a spacecraft to Jupiter by radiation pressure later in the present decade.

Quick Quiz 34.4 To maximize the radiation pressure on the sails of a spacecraft using solar sailing, should the sheets be **(a)** very black to absorb as much sunlight as possible or **(b)** very shiny to reflect as much sunlight as possible?

Conceptual Example 34.4 Sweeping the Solar System

A great amount of dust exists in interplanetary space. Although in theory these dust particles can vary in size from molecular size to a much larger size, very little of the dust in our solar system is smaller than about 0.2 μm. Why?

SOLUTION

The dust particles are subject to two significant forces: the gravitational force that draws them toward the Sun and the radiation-pressure force that pushes them away from the Sun. The gravitational force is proportional to the cube of the radius of a spherical dust particle because it is proportional to the mass and therefore to the volume $4\pi r^3/3$ of the particle. The radiation pressure is proportional to the square of the radius because it depends on the planar cross section of the particle. For large particles, the gravitational force is greater than the force from radiation pressure. For particles having radii less than about 0.2 μm, the radiation-pressure force is greater than the gravitational force. As a result, these particles are swept out of our solar system by sunlight.

Example 34.5 Pressure from a Laser Pointer

When giving presentations, many people use a laser pointer to direct the attention of the audience to information on a screen. If a 3.0-mW pointer creates a spot on a screen that is 2.0 mm in diameter, determine the radiation pressure on a screen that reflects 70% of the light that strikes it. The power 3.0 mW is a time-averaged value.

SOLUTION

Conceptualize Imagine the waves striking the screen and exerting a radiation pressure on it. The pressure should not be very large.

Categorize This problem involves a calculation of radiation pressure using an approach like that leading to Equation 34.28 or Equation 34.30, but it is complicated by the 70% reflection.

Analyze We begin by determining the magnitude of the beam's Poynting vector.

Divide the time-averaged power delivered via the electromagnetic wave by the cross-sectional area of the beam:

$$S_{avg} = \frac{(Power)_{avg}}{A} = \frac{(Power)_{avg}}{\pi r^2} = \frac{3.0 \times 10^{-3}\text{ W}}{\pi\left(\dfrac{2.0 \times 10^{-3}\text{ m}}{2}\right)^2} = 955\text{ W/m}^2$$

Now let's determine the radiation pressure from the laser beam. Equation 34.30 indicates that a completely reflected beam would apply an average pressure of $P_{avg} = 2S_{avg}/c$. We can model the actual reflection as follows. Imagine that the surface absorbs the beam, resulting in pressure $P_{avg} = S_{avg}/c$. Then the surface emits the beam, resulting in additional pressure $P_{avg} = S_{avg}/c$. If the surface emits only a fraction f of the beam (so that f is the amount of the incident beam reflected), the pressure due to the emitted beam is $P_{avg} = fS_{avg}/c$.

Use this model to find the total pressure on the surface due to absorption and re-emission (reflection):

$$P_{avg} = \frac{S_{avg}}{c} + f\,\frac{S_{avg}}{c} = (1 + f)\,\frac{S_{avg}}{c}$$

continued

▶ **34.5 continued**

Evaluate this pressure for a beam that is 70% reflected:

$$P_{\text{avg}} = (1 + 0.70)\frac{955\ \text{W/m}^2}{3.0 \times 10^8\ \text{m/s}} = \boxed{5.4 \times 10^{-6}\ \text{N/m}^2}$$

Finalize The pressure has an extremely small value, as expected. (Recall from Section 14.2 that atmospheric pressure is approximately 10^5 N/m^2.) Consider the magnitude of the Poynting vector, $S_{\text{avg}} = 955$ W/m^2. It is about the same as the intensity of sunlight at the Earth's surface. For this reason, it is not safe to shine the beam of a laser pointer into a person's eyes, which may be more dangerous than looking directly at the Sun.

WHAT IF? What if the laser pointer is moved twice as far away from the screen? Does that affect the radiation pressure on the screen?

Answer Because a laser beam is popularly represented as a beam of light with constant cross section, you might think that the intensity of radiation, and therefore the radiation pressure, is independent of distance from the screen. A laser beam, however, does not have a constant cross section at all distances from the source; rather, there is a small but measurable divergence of the beam. If the laser is moved farther away from the screen, the area of illumination on the screen increases, decreasing the intensity. In turn, the radiation pressure is reduced.

In addition, the doubled distance from the screen results in more loss of energy from the beam due to scattering from air molecules and dust particles as the light travels from the laser to the screen. This energy loss further reduces the radiation pressure on the screen.

34.6 Production of Electromagnetic Waves by an Antenna

The electric field lines resemble those of an electric dipole (shown in Fig. 23.20).

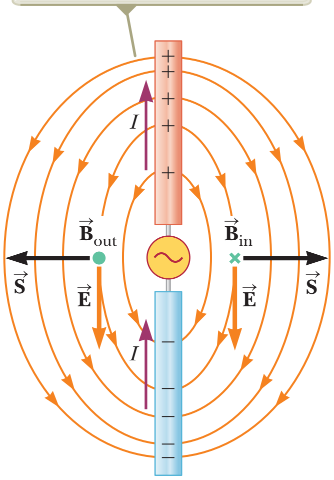

Figure 34.11 A half-wave antenna consists of two metal rods connected to an alternating voltage source. This diagram shows $\vec{E}$ and $\vec{B}$ at an arbitrary instant when the current is upward.

Stationary charges and steady currents cannot produce electromagnetic waves. If the current in a wire changes with time, however, the wire emits electromagnetic waves. The fundamental mechanism responsible for this radiation is the acceleration of a charged particle. **Whenever a charged particle accelerates, energy is transferred away from the particle by electromagnetic radiation.**

Let's consider the production of electromagnetic waves by a *half-wave antenna.* In this arrangement, two conducting rods are connected to a source of alternating voltage (such as an *LC* oscillator) as shown in Figure 34.11. The length of each rod is equal to one-quarter the wavelength of the radiation emitted when the oscillator operates at frequency *f*. The oscillator forces charges to accelerate back and forth between the two rods. Figure 34.11 shows the configuration of the electric and magnetic fields at some instant when the current is upward. The separation of charges in the upper and lower portions of the antenna make the electric field lines resemble those of an electric dipole. (As a result, this type of antenna is sometimes called a *dipole antenna.*) Because these charges are continuously oscillating between the two rods, the antenna can be approximated by an oscillating electric dipole. The current representing the movement of charges between the ends of the antenna produces magnetic field lines forming concentric circles around the antenna that are perpendicular to the electric field lines at all points. The magnetic field is zero at all points along the axis of the antenna. Furthermore, $\vec{E}$ and $\vec{B}$ are 90° out of phase in time; for example, the current is zero when the charges at the outer ends of the rods are at a maximum.

At the two points where the magnetic field is shown in Figure 34.11, the Poynting vector $\vec{S}$ is directed radially outward, indicating that energy is flowing away from the antenna at this instant. At later times, the fields and the Poynting vector reverse direction as the current alternates. Because $\vec{E}$ and $\vec{B}$ are 90° out of phase at points near the dipole, the net energy flow is zero. From this fact, you might conclude (incorrectly) that no energy is radiated by the dipole.

Energy is indeed radiated, however. Because the dipole fields fall off as $1/r^3$ (as shown in Example 23.6 for the electric field of a static dipole), they are negligible at great distances from the antenna. At these great distances, something else causes