MTH 1002 Calculus

Assessments

- Exam : 60% of module mark
- WebAssign assignments: 8% of module mark
- Coursework : 7% of module mark
- Test : 25% of module mark

Reading

- Calculus, by J. Stewart
- Calculus, by M. Spivak
- any maths for physics / engineering textbook e.g. Engineering Mathematics by K. A. Stroud

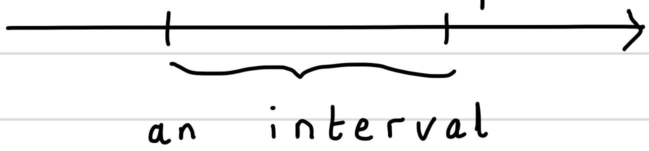
Course content

- Functions
- Complex numbers
- Limits and continuity
- Differentiation
- Curve sketching
- Taylor series
- Integration and the fundamental theorem of calculus
- Parametric equations and polar coordinates

- Multivariable calculus

Intervals

An interval is a connected portion of the real line.

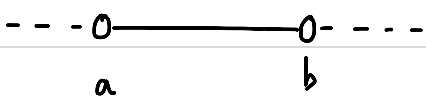


Finite intervals

Definition: A subset I of the real line is called an interval if it contains at least two numbers and every number lying between them; that is, if $x, y \in I$ and $z \in \mathbb{R}$, $x < z < y$, then $z \in I$.

- If we suppose that $a, b \in \mathbb{R}$, we can consider the following kinds of interval:

- Open interval: contains neither endpoint



$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

"such that" strict inequalities

- Closed interval: contains both endpoints

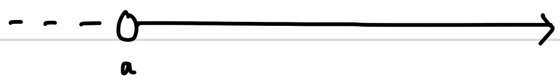
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

- <u>Half-open interval</u> : contains one endpoint

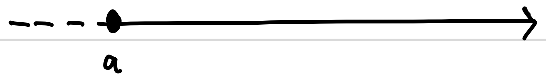
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

<u>Semi-infinite intervals</u>

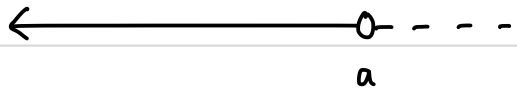
For a real value $a \in \mathbb{R}$, we have the following semi-infinite intervals:



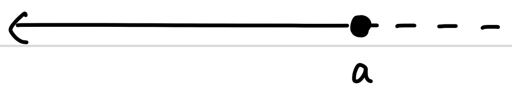
$(a, \infty) = \{x \in \mathbb{R} : x > a\}$



$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$



$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$



$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$

<u>Functions of a real variable</u>

- A function is a rule for transforming an object into another object.
e.g. the area of a circle is a function of its radius, $A = \pi r^2$.

<u>Definition</u> : A function f from a set X to a set Y is a rule that assigns a unique element $y \in Y$ to each element $x \in X$.

X is the <u>domain</u> of the function. It is the set of all values to which the function can be applied. Sometimes, the domain is specified.

- If you are asked to find the domain of a function, you will have to exclude values of x to which it cannot be applied.

<u>Example</u> The domain of the function $y = \sqrt{x}$ is $[0, \infty)$, because we cannot take the square root of a negative number.

- The <u>range</u> of the function is the set of all possible values of f as x varies throughout the domain.

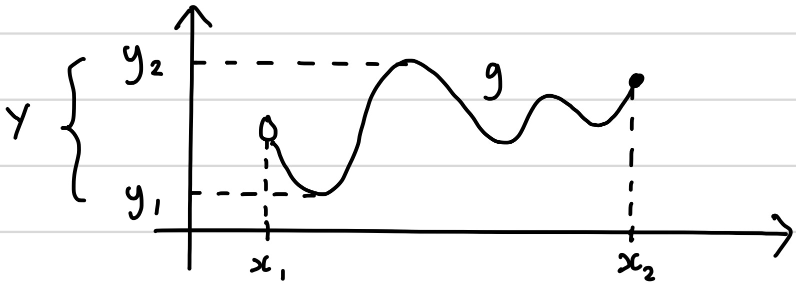
<u>Example</u>: the range of the function $y = \sqrt{x}$ is $[0, \infty)$, because we take the positive root unless otherwise specified.

<u>Example</u> What are the domain and range of the function $y = \sqrt{10 - x}$?

- Domain: $(-\infty, 10]$ (provided $x \leq 10$, we can take the square root)
- Range: $[0, \infty)$ (as for $y = \sqrt{x}$)

<u>Definition</u>: the graph of a function f is the set of all Cartesian coordinates where x is in the domain X and $y = f(x)$.

Graph of a continuous function



Graph: g

Domain:

$$X = \{x_1 < x \leq x_2\} = (x_1, x_2]$$

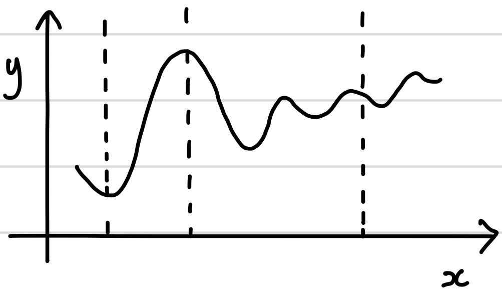
Range:

$$Y = \{y_1 \leq y \leq y_2\} = [y_1, y_2]$$

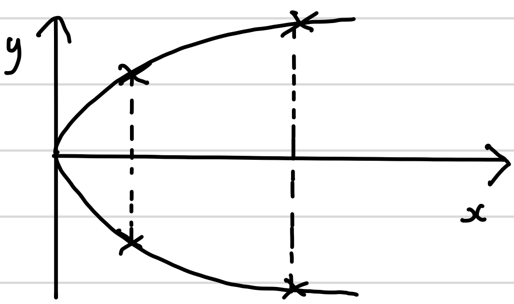
The vertical line test

Recall: a function f from a set X to a set Y is a rule that assigns a unique element $y \in Y$ to each element $x \in X$.

We can represent this graphically by the vertical line test.

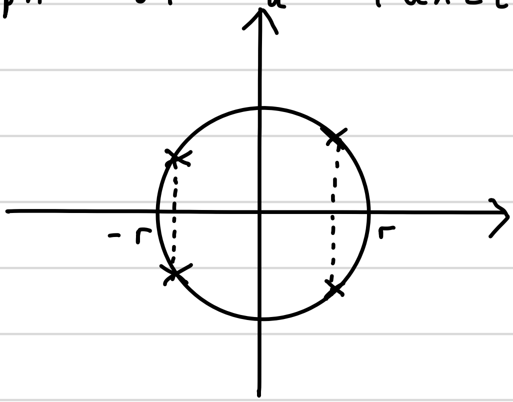


- This is the graph of a function: each line intersects the graph only once.



- This is not the graph of a function: each line intersects the graph twice.

Example A circle in the (x, y) plane is not the graph of a function:



We need to define two functions:

Equation for whole circle : $x^2 + y^2 = r^2$
Top semicircle : $y = \sqrt{r^2 - x^2}$
Bottom semicircle : $y = -\sqrt{r^2 - x^2}$

Polynomial functions

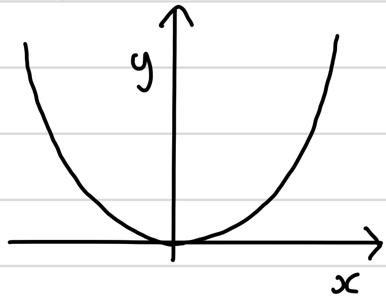
- A polynomial in x of degree n has the form

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0 = \sum_{j=0}^{n} a_j x^j$$

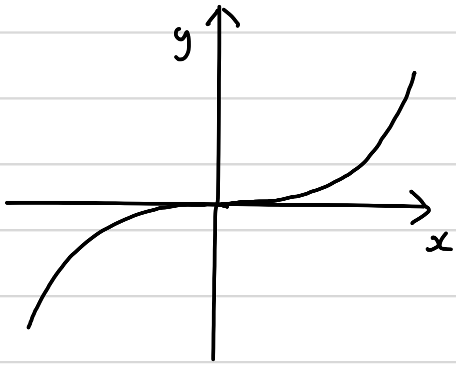
where the quantities a_j are constant coefficients and $a_n \neq 0$.
- The domain of a polynomial is all the real numbers, because
  - there are only integer powers of x, so there is no risk of taking the root of a negative number.
  - all powers of x are non-negative, so there is no risk of division by zero.

Graphs of the power x^n

- Graphs of $y = x^n$ for even $n \geq 2$ have the general form



- For odd $n \geq 3$, the graph of $y = x^n$ looks like:

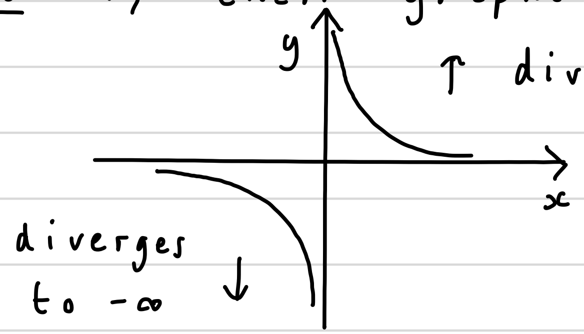


Rational functions

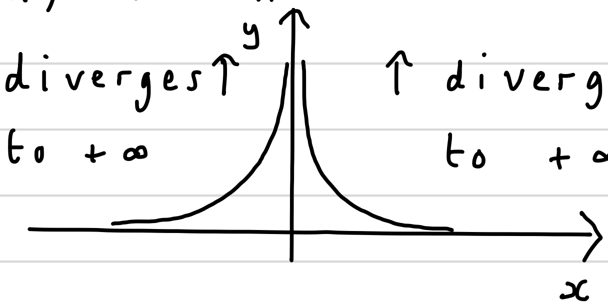
- These have the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

- Simple examples of rational functions are the functions $f(x) = 1/x^n$, where n is a positive integer

- For <u>odd</u> n, their graphs have the form



- For <u>even</u> n, we have



- These are examples of functions whose domain does not include the whole real line.
- The domain of $f(x) = 1/x^n$ for both even and odd n is $\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ ($x = 0$ must be excluded, as f is not defined here.)
- The range of $f(x) = 1/x^n$ is $\{y \in \mathbb{R} : y \neq 0\} = (-\infty, 0) \cup (0, \infty)$ for odd n, and $\{y \in \mathbb{R} : y > 0\} = (0, \infty)$ for even n.

<u>Example</u> (Exam 2020): Write down the domain and range of $f(x) = \dfrac{1}{(x-3)^4}$.

- Division by zero at $x = 3 \Rightarrow x = 3$ is excluded fron the domain.
- Domain is $\{x \in \mathbb{R} : x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

- Range: $f(x)$ can produce arbitrarily small numbers (as $x \to \pm\infty$) and arbitrarily large numbers (as $x \to 3$).
- The power is even, so these numbers are positive and the range is $(0, \infty)$.

Trigonometric functions

- Basic functions: sin and cos. These are $\pi/2$ out of phase, but are otherwise identical.
- Both have domain $(-\infty, \infty)$ and range $[-1, 1]$.
- They are defined by the power series

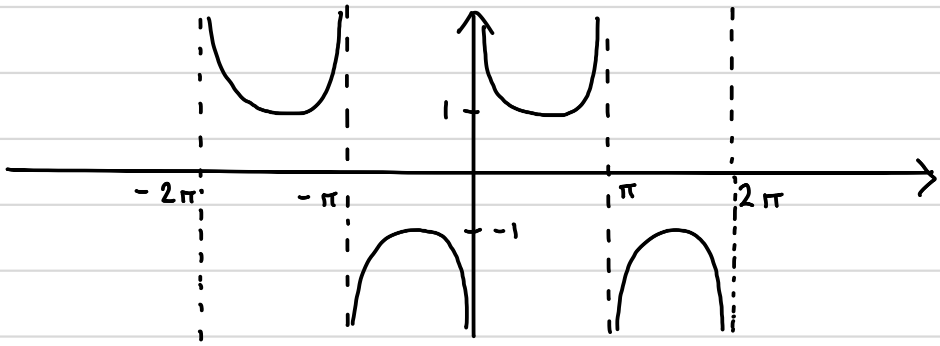
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

- Their quotient, $\tan x = \sin x / \cos x$, is undefined when $\cos x = 0$ i.e. at $x = \pi/2, 3\pi/2, 5\pi/2, \ldots,$ or $x = (n + 1/2)\pi$, $n \in \mathbb{Z}$
- Its domain is then $\{x \in \mathbb{R} : x \neq (n + \frac{1}{2})\pi, \ n \in \mathbb{Z}\}$
- Its range is all the real numbers, $y \in \mathbb{R}$.

<u>Example</u> Sketch the graph of $\operatorname{cosec} x = 1/\sin x$, and state its domain and range.

($\operatorname{cosec} x = \csc x$)



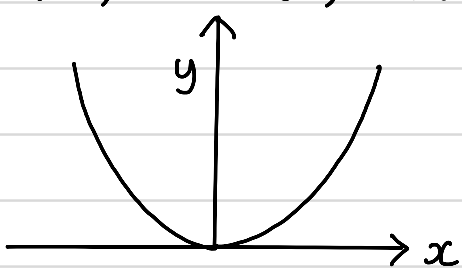
Domain : $\{x \in \mathbb{R} : x \neq n\pi,\ n \in \mathbb{Z}\}$
Range : $\{y \in \mathbb{R} : y \leq -1 \text{ or } y \geq 1\}$
or $(-\infty, -1] \cup [1, \infty)$

- The trigonometric functions are examples of periodic functions: they repeat themselves in successive intervals.

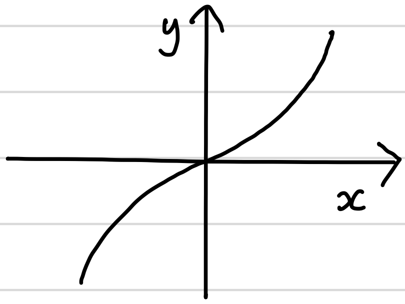
<u>Definition</u>. A function is periodic if there is a $p > 0$ such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest such number is called the period of f.

<u>Even and odd functions</u>

<u>Even function</u>: $f(-x) = f(x)$ for all x in the domain of f.



Odd function: $f(-x) = -f(x)$ for all x in the domain of f.



Even and odd functions

- To see if f is odd, even or neither, replace every instance of x with $-x$.
- If you get back the original function, f is even. If you find the negative of the original function, f is odd.
- If neither of these things is true, f is neither odd nor even.

Example $g(x) = 2x^3 + x$

$\Rightarrow$ "implies that" $g(-x) = 2(-x)^3 + (-x)$
$= -2x^3 - x = -g(x)$

Exponential functions

- An exponential function is a function of the form $y = a^x$, where a is a real constant
- The domain is the whole real line, $\mathbb{R}$, and the range is $(0, \infty)$: the graph never touches the x-axis.
- The standard exponential function is $y = e^x = \exp(x)$, where $e \approx 2.718\cdots$ (an irrational number)
- The function $\exp(x)$ is equal to its own derivative, and is defined by the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

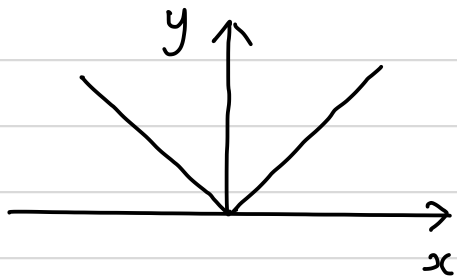
- Its inverse is the natural logarithm, $y = \ln x$, whose domain and range are $(0, \infty)$ and $\mathbb{R}$ respectively.

Piecewise-defined functions

(1) The absolute value function is given by

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Domain : $(-\infty, \infty) = \mathbb{R}$
Range : $[0, \infty)$



(2) The signum function

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

The range of the signum function is $\{-1, 0, 1\}$: the set of the three numbers -1, 0 and 1. Its domain is $\mathbb{R}$.

Hyperbolic functions

- These are even and odd combinations of e^x and e^{-x}.

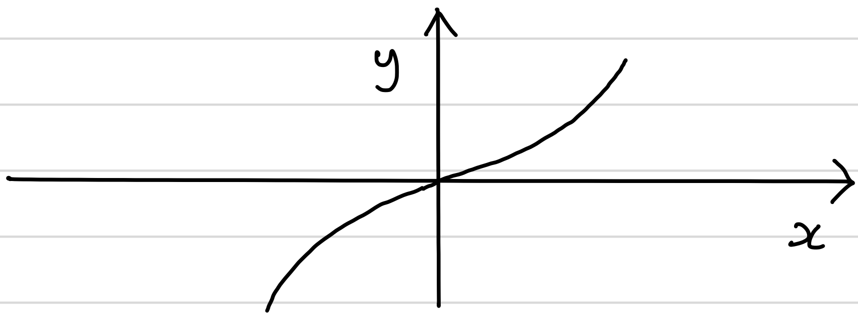
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$: hyperbolic sine

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$: hyperbolic cosine

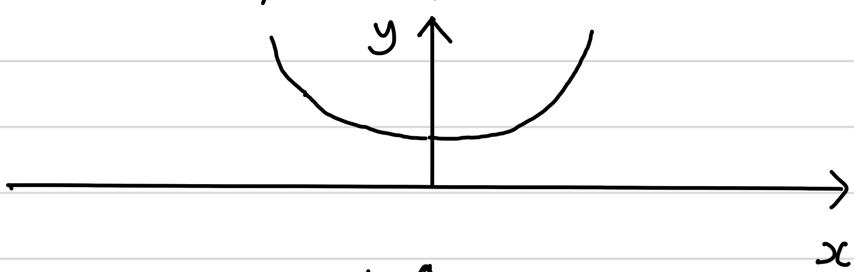
$$\tanh x = \frac{\sinh x}{\cosh x}$$: hyperbolic tangent

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

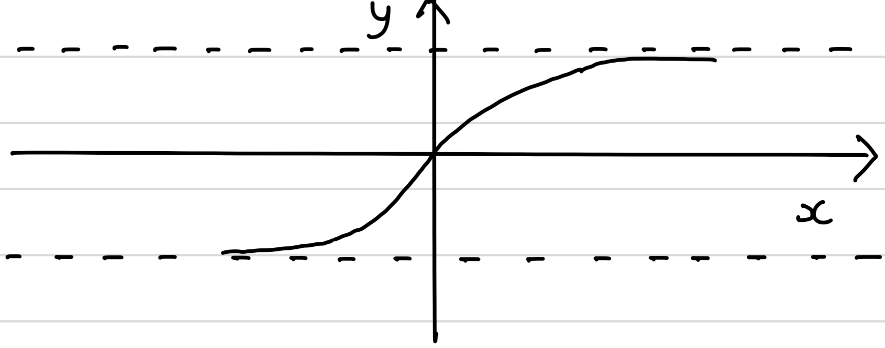
$$\coth x = \frac{\cosh x}{\sinh x}$$



$y = \sinh x$



$y = \cosh x$



$y = \tanh x$

The domains of $\sinh x$, $\cosh x$ and $\tanh x$ are all equal to $\mathbb{R}$: all the real numbers. There are no intervals where they are not defined, as none of their definitions involve prohibited operations such as division by zero.

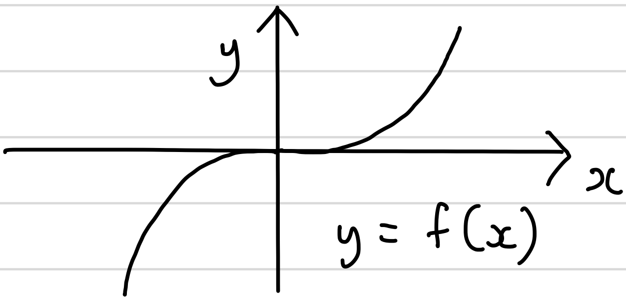
Composition of two functions

Let both f and g be functions of real variables. Their composition is a rule that assigns a value z in the range of g to each value x in the domain of f: $y = f(x)$, $z = g(y)$

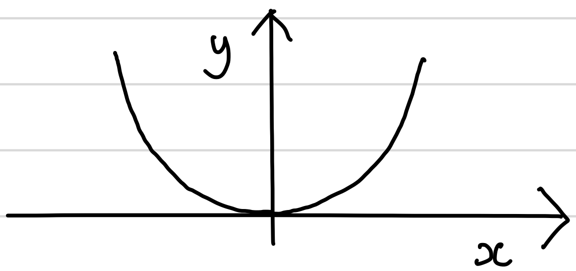
$$g \circ f(x) = g(f(x))$$

Example $f(x) = x^3$, $g(y) = |y|$

Then, $h(x) = g \circ f'(x) = |f(x)| = |x^3|$



Domain of f: $\mathbb{R}$
Range of f: $\mathbb{R}$



$$y = h(x) = g \circ f(x)$$
$$= |x^3|$$

Domain of h: $\mathbb{R}$ Range of h: $[0, \infty)$

Complex numbers

The solution of a quadratic equation $ax^2 + bx + c$ can be found using

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the term $b^2 - 4ac < 0$, we usually say that the quadratic has no (real) roots.

Example The formula above gives the roots of the quadratic equation

$$z^2 - 4z + 5 = 0 \quad \text{as} \quad \frac{2 \pm \sqrt{-4}}{2}.$$

However, if we define $i = \sqrt{-1}$, we can write these roots as

$$\frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 2 \pm \sqrt{-1} = 2 \pm i$$

This is an example of a complex number. It has a real part and an imaginary part.

More generally, we can write

$$z = a + bi, \quad \text{where } a, b \in \mathbb{R}$$

a and b are the real and imaginary parts, respectively, of z, often denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

- If $a = 0$, then z is purely imaginary.
  If $b = 0$, then z is real.
- A complex number written as the sum of a real and an imaginary term is in <u>standard form</u>.
- We denote the set of all complex numbers by $\mathbb{C}$, so that $z \in \mathbb{C}$.
- Every quadratic equation has roots in $\mathbb{C}$.

Addition and subtraction

- The real and imaginary parts are treated separately.
- Example: If $z_1 = 2 + 3i$ and $z_2 = 1 - 2i$, then

$$z_1 + z_2 = (2+1) + (3-2)i = 3 + i$$

and $$z_1 - z_2 = (2-1) + (3-(-2))i = 1 + 5i$$

Multiplication

- The essential fact to remember is that $i^2 = -1$ (since $i = \sqrt{-1}$).

Example

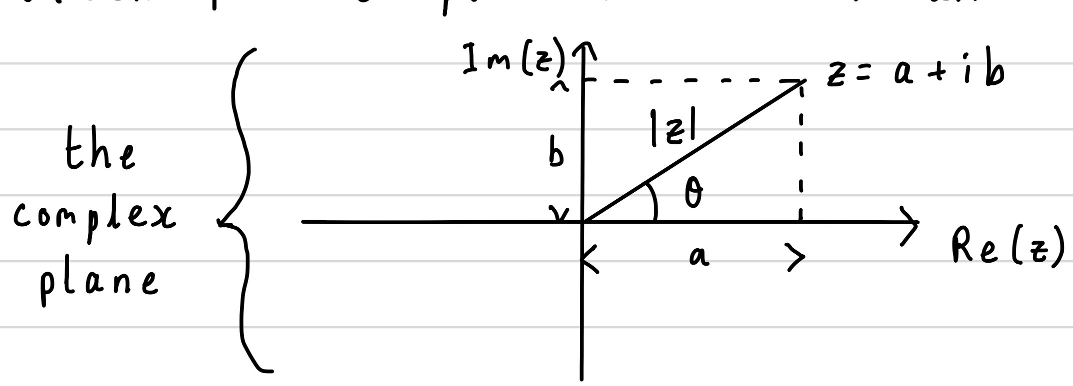
With z_1 and z_2 as above, we have

$$\begin{aligned} z_1 z_2 &= (2+3i)(1-2i) \\ &= 2 - 4i + 3i - 6(i^2) \end{aligned}$$

$$= 2 - i - 6(-1)$$
$$= 2 - i + 6$$
$$= 8 - i$$

The Argand diagram

We often plot complex numbers on an Argand diagram:



Definitions and properties

- The complex conjugate of $z = a + ib$ is $\bar{z} = a - ib$ (sometimes written $z^* = a - ib$)
- The modulus of z is $|z| = r = (a^2 + b^2)^{1/2}$
- We also have that $|z|^2 = z\bar{z}$, since

$$z\bar{z} = (a + ib)(a - ib)$$
$$= a^2 - iba + iba - (i)^2 b^2$$
$$= a^2 + b^2$$

- The argument of $z = a + ib$ is $\arg(z) \equiv \theta$.

Note: the argument is not unique. An integer multiple of 2π can be added to θ without changing z, as this just takes us back to the same point on the Argand diagram. Usually, we take $0 \leq \theta < 2\pi$.

We also have that $\theta = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

<u>Example</u> With $z = \sqrt{3} + i$, we have that $\bar{z} = \sqrt{3} - i$

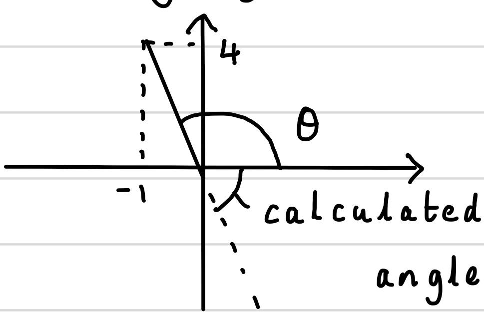
$|z| = r = [(\sqrt{3})^2 + 1^2]^{1/2} = 2$

$\arg(z) = \tan^{-1}(1/\sqrt{3}) = \pi/6$

<u>Note</u>: it is important to remember that calculating θ from $\tan^{-1}(b/a)$ can give an answer in the wrong quadrant. It can be useful to sketch an Argand diagram to make sure that you have calculated the correct angle.

<u>Example</u> With $z = -1 + 4i$, we have that $\tan^{-1}[4/(-1)] \approx -1.3258$ rad.

However, drawing an Argand diagram shows that this lies in the wrong quadrant:



We can move into the correct quadrant by adding π to our result, giving $\theta \approx 1.8158$ rad.

<u>Division</u>

The complex conjugate introduced above is important in division. For example, to calculate the quotient

$$\frac{7-4i}{4+3i}$$

we multiply the numerator + denominator by the complex conjugate of the denominator, $4-3i$.

This gives $\dfrac{(7-4i)(4-3i)}{(4+3i)(4-3i)} = \dfrac{28-37i-12}{16+9} = \dfrac{16}{25} - \dfrac{37}{25}i$

The fact that $z\bar{z} = a^2 + b^2$ has allowed us to make the denominator entirely real and write the quotient in standard form.

Polar form of complex numbers

From the Argand diagram, we can see that

$$\begin{aligned} z &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

This is the polar form of a complex number.

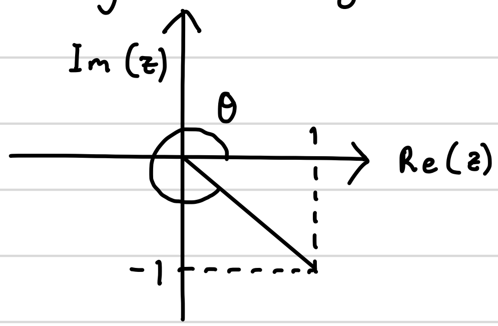
Example Write $z = 1-i$ in polar form.

1. Calculate the modulus, $|z| = r = \left[1^2 + 1^2\right]^{1/2} = \sqrt{2}$

2. Calculate $\theta = \tan^{-1}(b/a) = \tan^{-1}(-1/1)$

$$= \tan^{-1}(-1) = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

(3) Draw an Argand diagram...



... to see that $\theta = \dfrac{7\pi}{4}$

The polar form of z is then

$$z = \sqrt{2}\,[\cos(7\pi/4) + i\sin(7\pi/4)]$$

Exponential form of a complex number

Recall: the exponential, sine and cosine functions can be written as power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

If we define $e^{i\theta} = 1 + i\theta + \dfrac{(i\theta)^2}{2!} + \dfrac{(i\theta)^3}{3!} + \dfrac{(i\theta)^4}{4!} + \cdots$

then $e^{i\theta} = 1 + i\theta + \dfrac{i^2\theta^2}{2!} + \dfrac{i^3\theta^3}{3!} + \dfrac{i^4\theta^4}{4!} + \cdots$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

<u>Euler's identity.</u>

This gives us another representation of a complex number: the exponential form

$$\boxed{z = re^{i\theta}, \quad \text{with } e^{i\theta} = \cos\theta + i\sin\theta}$$

Note: $|e^{i\theta}| = \left[(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)\right]^{1/2}$

$$= \left[\cos^2\theta + \sin^2\theta\right]^{1/2} = 1$$

and $e^{i\theta}$ represents all points on a circle of unit radius in the Argand diagram.

Noting that $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta,$
we also find that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

and
$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

Similarly, $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

and
$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Note: this makes clear the connection between the trigonometric functions and the hyperbolic functions $\cosh x = (e^{x} + e^{-x})/2$ and $\sinh x = (e^{x} - e^{-x})/2$.

<u>De Moivre's theorem</u>

From $z = re^{i\theta}$ with $e^{i\theta} = \cos\theta + i\sin\theta$, it follows that
$$z^n = r^n e^{in\theta} = r^n\{\cos(n\theta) + i\sin(n\theta)\}$$

so that $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

<u>Example</u>: Use de Moivre's theorem to derive the double angle formulas.

Put $n = 2$. Then,

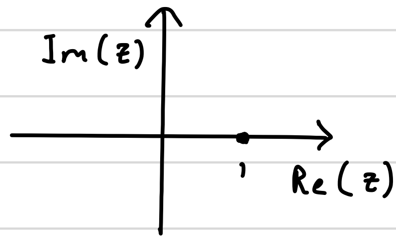
$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

Equate the real and imaginary parts:

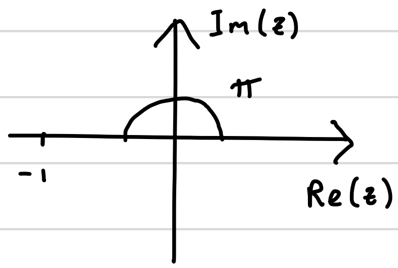
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

More examples of polar and exponential form

a) $1 = 1 + 0i$
$= 1(\cos 0 + i\sin 0)$
$= e^{i0}$



b) $-1 = 1[\cos(\pi) + i\sin(\pi)]$
$= e^{i\pi}$

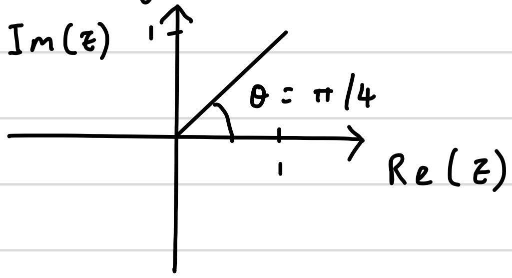


c) $z = 1 + i$

Modulus : $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Argument: $\theta = \tan^{-1}(1/1) = \tan^{-1}(1)$

This could be $\pi/4$ or $5\pi/4$. However, we see from the Argand diagram that it must be $\pi/4$:



Then, $z = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$

$= \sqrt{2}\, e^{i\pi/4}$

We can also write this as

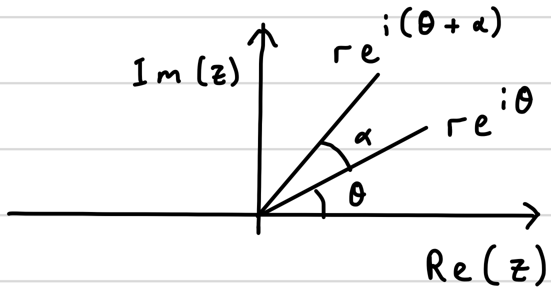
$$z = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)} \quad \text{with } k \in \mathbb{Z}$$

since adding 2π to the argument brings us back to the same point in the Argand diagram.

Rotation of a complex number

If $z = re^{i\theta}$, then $ze^{i\alpha} = re^{i\alpha}e^{i\theta} = re^{i(\alpha+\theta)}$

i.e. multiplying a complex number by $e^{i\alpha}$ rotates it by an angle α in the complex plane.



Multiplication in exponential form

Multiplication and division are neater in exponential form than in standard form. We have that, for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

We can see that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Since $|e^{i\theta}| = 1$ for all θ, we also have that $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$.

Division is similar:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$ and we see that

$|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

Roots of complex numbers

- Finding the $n^{\underline{th}}$ root of a complex number w corresponds to solving

$$z^n = w$$

- Key fact: adding an integer multiple of 2π to the argument of a complex number leaves the complex number unchanged.

Step 1: Write w in exponential form:
$w = |w| e^{i\phi}$

Step 2: Add $2k\pi$, $k \in \mathbb{Z}$, to the argument of w. This leaves w unchanged.

$$w = |w| e^{i\phi + 2k\pi i}$$

Step 3 : Write $z^n = w$, or

$$z^n = |w| e^{i\phi + 2k\pi i}$$

Step 4 : Take the n^{th} root of both sides:

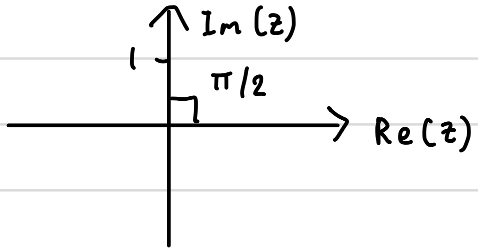
$$z = |w|^{1/n} e^{i\phi/n + 2k\pi i/n}$$

Step 5 : Let $k = 0, 1, 2, 3, \ldots, n-1$ to read off the n roots.
Note: we stop at $n-1$ as $k = n$ gives the same root as $k = 0$.

Example

$$z^3 = i$$

1 Write i in exponential form:



$$|i| = 1$$
$$\arg(i) = \pi/2$$
and $i = e^{i\pi/2}$

2 Add $2k\pi$, $k \in \mathbb{Z}$, to the argument of i:

$$e^{i\pi/2 + 2k\pi i}$$

3 Write $z^3 = i = e^{i\pi/2 + 2k\pi i}$

(4) Take the 3rd (cube) root of both sides

$$z = e^{i\pi/6 + 2k\pi i/3}$$

(5) Let $k = 0, 1, 2$:

$k = 0$: $z = e^{\pi i/6}$
$k = 1$: $z = e^{i\pi/6 + 2\pi i/3} = e^{i\pi/6 + 4\pi i/6} = e^{5\pi i/6}$
$k = 2$: $z = e^{i\pi/6 + 4\pi i/3} = e^{i\pi/6 + 8\pi i/6}$
$= e^{9\pi i/6} = e^{3\pi i/2}$

These are the three roots. When $k = 3$, $z = e^{i\pi/6 + 6\pi i/3} = e^{i\pi/6 + 2\pi i}$ $= e^{i\pi/6}$, the first root again.

Limits

- For many functions $f(x)$, the value of $f(x)$ as x approaches the value a will simply be $f(a)$.
- However, the function could be undefined at the point a.

  Example: the function

$$f(x) = \frac{\sin x}{x},$$

  used in optics and signal processing, is undefined at $x = 0$.
- In cases like this, we need the concept of a limit.

Notation and definition

Suppose $f(x)$ is defined when x is near to the number a, except possibly at a itself. If we can make $f(x)$ arbitrarily close to L by making x sufficiently close to a (on either side of it but not equal to it), then we write

$$\lim_{x \to a} f(x) = L$$

and say that the limit of $f(x)$ as x approaches a is equal to L.

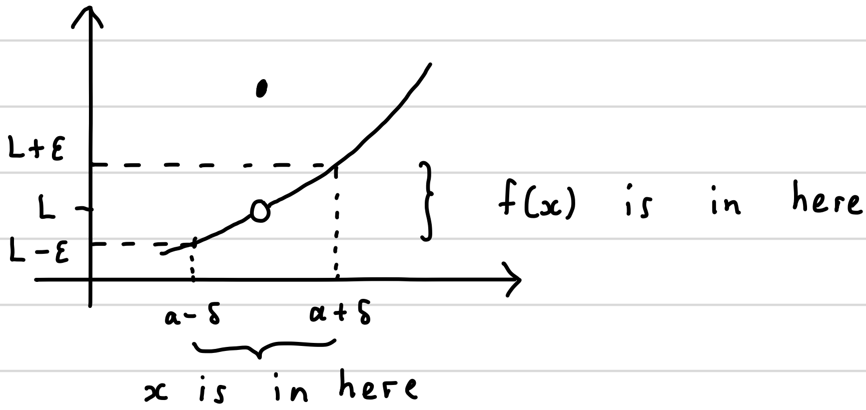
Precise definition

Suppose that $f(x)$ is defined on an open interval that contains the number a, except possibly at a itself. Then, we say that the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Note: this definition is needed in rigorous proofs: it uses variables, ε and δ, rather than statements like "arbitrarily close".



One-sided limits

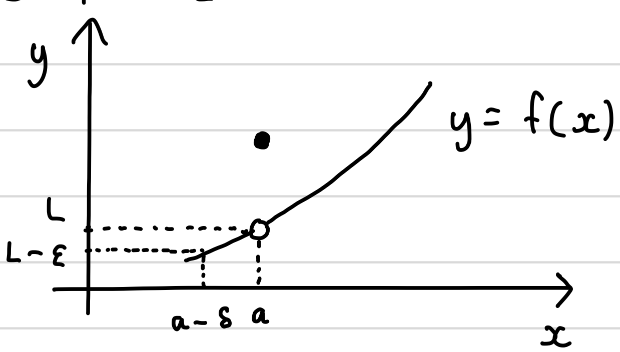
Left-sided limit

We write $\lim_{x \to a^-} f(x) = L$

and say that the limit of $f(x)$ as x approaches a from the left is equal to L if we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is less than a.

OR $\lim_{x \to a^-} f(x) = L$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$.



Right-sided limit

$$\lim_{x \to a^+} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

Note that $\lim_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Simple examples of limits

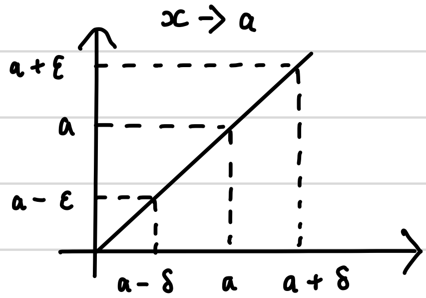
① $f(x) = A$, where A is a constant

Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = A$

② $f(x) = x$

$$\lim_{x \to a^-} f(x) = a, \quad \lim_{x \to a^+} f(x) = a$$

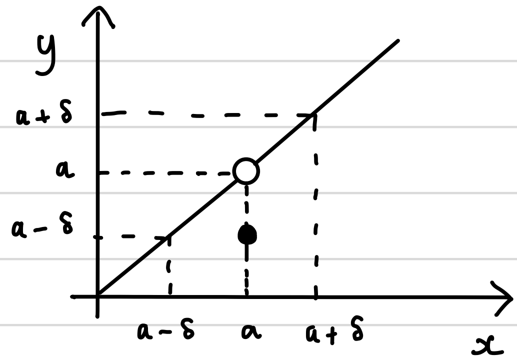
and $\lim_{x \to a} f(x) = a$



In this case, $\delta = \varepsilon$.

$\lim_{x \to a} f(x)$ versus $f(a)$

Consider the function $f(x) = \begin{cases} x & x < a \\ a/2 & x = a \\ x & x > a \end{cases}$



$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$= \lim_{x \to a} f(x) = a \neq f(a)$$

- The limit of $f(x)$ at a does not depend on the existence of $f(x)$ at a or, when $f(a)$ exists, on the value of $f(a)$.

Step functions and limits

- The signum function discussed earlier is sometimes written as $f(x) = \dfrac{x}{|x|}$.

Does this function have a limit as $x \to 0$? We see that $\lim_{x \to 0^-} f(x) = -1$ and that $\lim_{x \to 0^+} f(x) = 1$. Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, the limit $\lim_{x \to 0} f(x)$ does not exist.

However, the signum function is sometimes defined to have $f(0) = 0$.

Limits at infinity.

The limits at infinity of $f(x)$ describe its behaviour as x increases or decreases without bound.

Example If $f(x) = 1/x$, then $\lim_{x \to \infty} f(x) = 0$ and

$$\lim_{x \to -\infty} f(x) = 0.$$

Infinite limits

Some functions become infinite as x tends to a finite value

If $f(x) = \frac{1}{x}$, then $\lim_{x \to 0^-} f(x) = -\infty$

and $\lim_{x \to 0^+} f(x) = \infty$

Rules for limits

Let $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = G$

Then, $\lim_{x \to a} (k f(x)) = kF$

$$\lim_{x \to a} (f(x) \pm g(x)) = F \pm G$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \qquad (G \neq 0)$$

$$\lim_{x \to a} f(x)\, g(x) = FG$$

If $f(x) \leqslant g(x)$ on an interval containing a, then $\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$.

If $P(x)$ and $Q(x)$ are polynomials, then

$$\lim_{x \to a} P(x) = P(a), \quad \lim_{x \to a} Q(x) = Q(a)$$

and $$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \qquad Q(a) \neq 0$$

Evaluating limits

Direct substitution

Example Evaluate $\lim_{x \to -1} \frac{x^2 - 3x + 2}{x - 2}$

Substituting $x = -1$ into the above expression yields $\frac{6}{-3} = -2$.

Indeterminate forms

What happens if we try to calculate

$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$ by direct substitution?

We get $\frac{4 - 6 + 2}{2 - 2} = \frac{0}{0}$: an indeterminate form

However, we can factorise the numerator to get

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x - 2}$$

$$= \lim_{x \to 2} x - 1 = 1$$

<u>Note</u>: the simplification $\dfrac{x^2 - 3x + 2}{x - 2} = x - 1$ is not valid when $x = 2$, since it involves division by zero at this point. However, it is valid arbitrarily close to $x = 2$, allowing us to calculate the above limit.

L'Hôpital's rule

Other indeterminate limits can be evaluated using <u>l'Hôpital's rule</u>.

Suppose that f and g are differentiable functions, and that $g'(x) \neq 0$ on an open interval that contains a. Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

OR $$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then, $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

<u>Example</u>

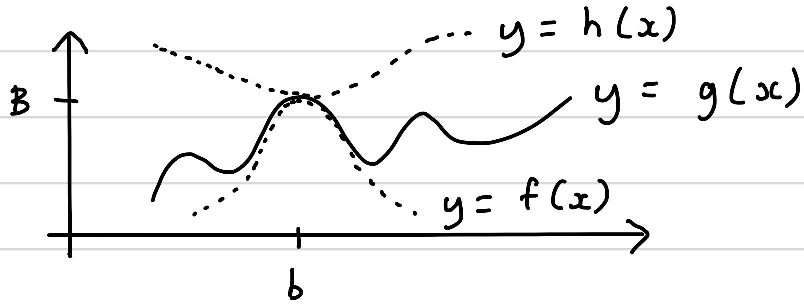
$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

The sandwich theorem (or squeeze theorem)

Let $f(x) \leq g(x) \leq h(x)$ in $(a, b) \cup (b, c)$

If $\lim_{x \to b} f(x) = \lim_{x \to b} h(x) = B$, then

$$\lim_{x \to b} g(x) = B$$



Example

Evaluate $\lim_{x \to 0^+} x \sin(1/x)$

We know that the value of the sine function always lies between -1 and 1 inclusive, so that

$$-1 \leq \sin(1/x) \leq 1$$

Since $x > 0$, we may multiply this inequality through by x to get

$$-x \leq x \sin(1/x) \leq x$$

This is in the form $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-x) = 0$

and $\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} (x) = 0$,

we also have that $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x \sin(1/x)$

$$= 0$$

by the sandwich theorem.

<u>Limits involving rational functions as $x \to \infty$</u>

<u>Example</u> If we try to calculate

$\lim_{x \to \infty} \dfrac{x - 8x^4}{7x^4 + 5x^3 + 2000x^2 - 6}$ by direct substitution, we arrive at the indeterminate form ∞ / ∞.

- To evaluate this, we divide both the numerator and denominator by the highest power of x (the leading-order term) and then take the limit.
- We have

$$\lim_{x \to \infty} \frac{x - 8x^4}{7x^4 + 5x^3 + 2000x^2 - 6}$$

$$= \lim_{x \to \infty} \frac{x/x^4 - 8x^4/x^4}{7x^4/x^4 + 5x^3/x^4 + 2000x^2/x^4 - 6/x^4}$$

$$= \lim_{x \to \infty} \frac{1/x^3 - 8}{7 + 5/x + 2000/x^2 - 6/x^4}$$

$$= \frac{0 - 8}{7 + 0 - 0 - 0} = \frac{-8}{7}$$

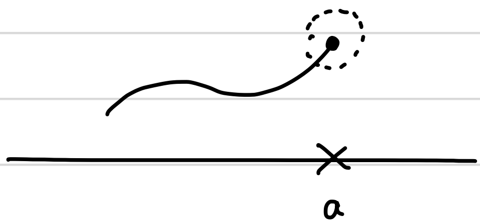
Continuity - informal definition

The graph of a continuous function can be drawn without removing your pen from the paper.

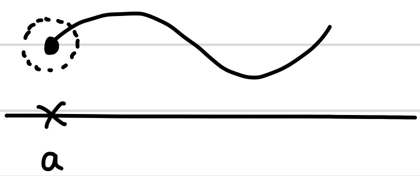
Continuity at a point - formal definition

Let f be defined on an interval that includes the point a.

f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$.



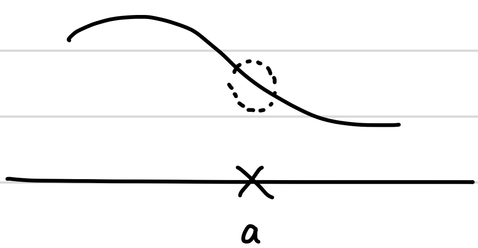
f is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$



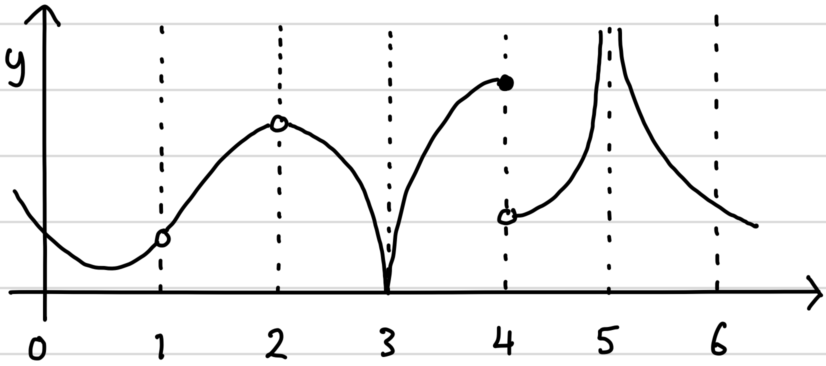
f is continuous at a if $\lim_{x \to a} f(x) = f(a)$

i.e. if $\lim_{x \to a^-} f(x) = f(a)$

and $\lim_{x \to a^+} f(x) = f(a)$



Example The function with the graph



$y = f(x)$

is continuous from the left at $x = 3$, $x = 4$ and $x = 6$, and continuous from the right at $x = 3$ and $x = 6$. This means that it is continuous at $x = 3$ and $x = 6$.

Types of discontinuity

- The discontinuities at $x = 1$ and $x = 2$ are removable discontinuities: the limits of f at $x \to 1$ and $x \to 2$ exist, but are not equal to the values of f at the respective points. They are called removable because they can be removed by redefining the function at a single point.
- The discontinuity at $x = 4$ is a jump discontinuity.
- The discontinuity at $x = 5$ is an infinite discontinuity.

Testing for continuity

1. Check whether $f(x)$ is defined at $x = a$.
2. Check whether $\lim_{x \to a} f(x)$ exists.
3. Check whether $\lim_{x \to a} f(x) = f(a)$.

Examples

Determine whether the following functions are continuous at $x = 2$

(a) $$f(x) = \frac{x^2 - 4}{x - 2}$$

This function is undefined at $x = 2$, so cannot be continuous there

(b) $$g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$$

Now, $g(x)$ is defined at $x = 2$. We further see that

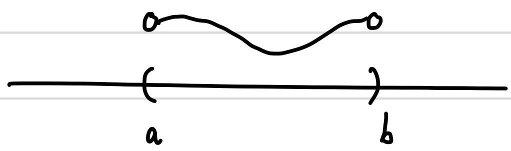
$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$

$$= \lim_{x \to 2} x + 2 = 4.$$

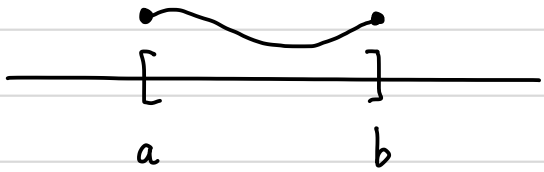
$\Rightarrow$ $g(x)$ is continuous at $x = 2$. This means that the discontinuity in $f(x)$ was removable, and was removed in $g(x)$ by defining the function separately at $x = 2$.

Continuity on intervals

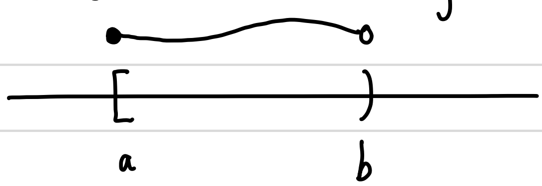
- A function f is continuous on an open interval (a, b) if it is continuous at every point in the interval.



- A function f is continuous on a closed interval $[a, b]$ if it is
  - continuous on (a, b)
  - continuous from the right at a
  - continuous from the left at b.



- A function f is continuous on a half-open interval $[a, b)$ if it is
  - continuous on (a, b), and
  - continuous from the right at a.



Continuity of combinations of functions

Suppose that the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and that c is a real constant. Then,

1. $cf(x)$ is continuous on $[a, b]$
2. $f(x) \pm g(x)$ is continuous on $[a, b]$
3. fg is continuous on $[a, b]$.
4. $\frac{f}{g}$ is continuous on $[a, b]$ provided that $g(x) \neq 0 \quad \forall x \in [a, b]$
5. Suppose that g is continuous at c and that f is continuous at $g(c)$. Then, the composition $f(g(x))$ is continuous at c.

Proof: Let $y = g(x)$. Since $g(x)$ is continuous at c, $y \to g(c)$ as $x \to c$.

Then, $\lim\limits_{x \to c} f(g(x)) = \lim\limits_{y \to g(c)} f(y)$.

Since f is continuous at $g(c)$, $\lim\limits_{y \to g(c)} f(y) = f(g(c))$.

We then have $\lim\limits_{x \to c} f(g(x)) = f(g(c))$

and the composition is continuous at c.

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions

Examples

(1) $g(x) = \dfrac{x^3 - 2x + 1}{x - 7}$ is a rational function

$\Rightarrow$ it is continuous on its domain $\{ x \in \mathbb{R} : x \neq 7 \}$.

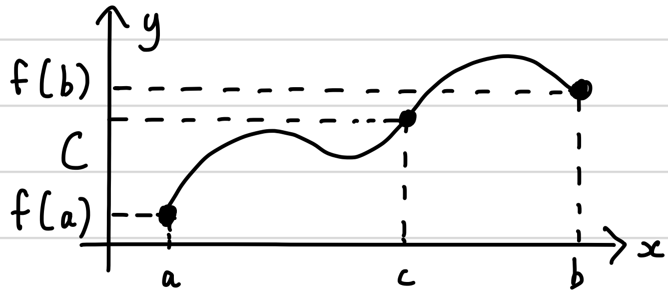
(2) $h(x) = \sqrt{x} + \dfrac{1}{x - 1}$

We can write $h(x) = G(x) + H(x)$, with $G(x) = \sqrt{x}$ and $H(x) = \dfrac{1}{x - 1}$. Then $G(x)$ is continuous on its domain, $[0, \infty)$.

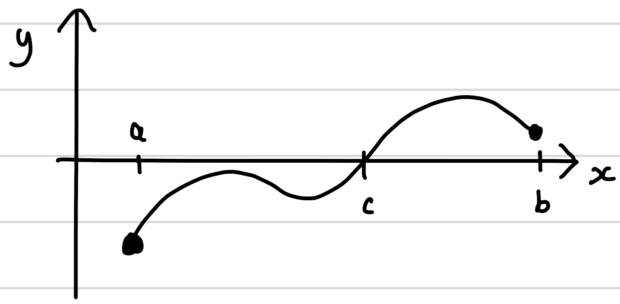
$H(x)$ is also continuous on its domain; i.e., everywhere, except at $x = 1$. So, $h(x)$ is continuous on the intervals $[0, 1)$ and $(1, \infty)$.

Intermediate value theorem

If f is a real-valued continuous function on $[a, b]$, then, for every $f(a) \leqslant C \leqslant f(b)$, or $f(b) \leqslant C \leqslant f(a)$, there exists at least one $c \in [a, b]$ such that $f(c) = C$.



Special case: $C = 0$. This can be used for root finding:



If f is continuous on $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one solution on (a, b).

Example

Show that the equation $17x^7 - 19x^5 - 1$ has a solution between -1 and 0.

Let $f(x) = 17x^7 - 19x^5 - 1$.

Now $f(-1) = 17(-1)^7 - 19(-1)^5 - 1$

$= -17 + 19 - 1$

$= 1$

and $f(0) = 17(0)^7 - 19(0)^5 - 1$

$= -1$

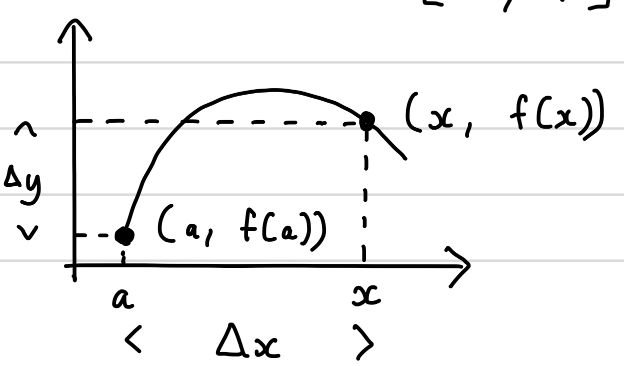
Therefore, since $f(-1)$ and $f(0)$ have opposite signs and $f(x)$ is a polynomial (a continuous function), $f(x) = 0$ has a solution between -1 and 0.

(This is called bracketing the root, and is an important first step when solving equations on a computer.)

Differentiation

- A basic quantity in differential calculus is the difference quotient

If $y = f(x)$ is a continuous function on the interval $[a, x]$, then the average rate of change of y with respect to x on the interval $[a, x]$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(a)}{x - a}$



- The average rate of change does not tell us anything about the details of what happened between the two endpoints.
- In contrast, the <u>derivative</u> tells us the rate of change at a point.

<u>Differentiability of functions at a point</u>

<u>Definition: differentiability</u>

Suppose that $f(x)$ is defined on an interval (a, b) containing the point x_0 (i.e. $a < x_0 < b$). Then, $f(x)$ is differentiable at x_0 if and only if the following limit exists:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- If this limit exists, it defines the derivative of f with respect to x at the point x_0, written $f'(x_0)$.
- The derivative at a point gives the gradient of the tangent at that point.
- It gives the instantaneous rate of change of $f(x)$ with respect to x at $x = x_0$.

<u>Derivative at a general point (from first principles)</u>

<u>Definition</u>: If $f(x)$ is differentiable at every

point in its domain, then it is a differentiable function. From first principles, the derivative of the function with respect to x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- If $y = f(x)$, we write $f'(x) = dy/dx$.

<u>Examples</u>

- If $f(x) = x^n$, $f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + h^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + {}^nC_2 x^{n-2}h + \dots + h^{n-1}$$

$$= nx^{n-1}$$

- This result can be used to derive other standard results:

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\sin x$$

- If $f(x) = 1/x$, $f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right) = \lim_{h\to 0} \frac{1}{h}\left(-\frac{h}{x(x+h)}\right)$$

$$= \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$$

- If $f(x) = \sqrt{x}$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Continuity and differentiability

Theorem If a function is differentiable at x_0, then it is continuous at x_0.

To prove continuity at x_0, we need to show that $\lim_{x \to x_0} f(x) = f(x_0)$, i.e. that

$$\lim_{h \to 0} f(x_0 + h) = f(x_0)$$

Proof Suppose that the function $f(x)$ is differentiable on an open interval containing the point x_0. We then know that the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. We now consider the new limit

$$\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] \times h$$

and write it in two ways.

Firstly, we write $\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] \times \lim_{h \to 0} h$

(by the product rule for limits)

$$= f'(x_0) \lim_{h \to 0} h = 0$$

since we are given that $f'(x_0)$ exists (and does not diverge, become indeterminate etc.)

Secondly, $\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] h$

$$= \lim_{h \to 0} [f(x_0 + h) - f(x_0)]$$

Equating the two results:

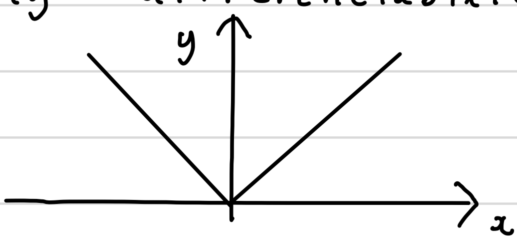
$$\lim_{h \to 0} [f(x_0 + h) - f(x_0)] = 0$$

and $\lim_{h \to 0} f(x_0 + h) = f(x_0)$

so that the function is continuous at x_0.

<u>Continuous non-differentiable functions</u>

- Differentiability implies continuity, but continuity does not imply differentiability.



$$y = f(x) = |x|$$

The absolute value function is continuous at $x = 0$. However, if we try to evaluate its derivative here, we find that

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

Since the left- and right-sided limits are different, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist, and $f(x)$ is not differentiable at $x = 0$.

Rules for differentiation

Suppose that $f(x)$ and $g(x)$ are differentiable functions on an open domain. Then,

1. $\frac{d}{dx}[c\,f(x)] = cf'(x)$, where c is a constant

2. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

3. $\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$

   (the product rule)

Proof $\quad \frac{d}{dx}[f(x)\,g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}\, g(x+h) + \lim_{h \to 0} f(x) \frac{[g(x+h) - g(x)]}{h}$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

Example If $y = xe^{x}$, then

$$\frac{dy}{dx} = \frac{d}{dx}(x)\, e^{x} + x \frac{d}{dx} e^{x}$$

$$= e^{x} + xe^{x}$$

4 $$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$$

Proof (not valid for all functions)

$$\frac{d}{dx}[f(g(x))] = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right] \times \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

(product rule for limits)

$= \quad f'(g(x))\, g'(x)$

(for a fuller explanation, see J. Stewart, Calculus)

<u>Alternative notation</u>: if y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

<u>Example</u> Differentiate $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$

(the normal distribution)

Here, $u(x) = -x^2/2$ and $g(u) = \frac{1}{\sqrt{2\pi}} \exp(u)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{\sqrt{2\pi}}\, e^{u} \,.\, (-x) = \frac{-x}{\sqrt{2\pi}}\, e^{-x^2/2}$$

⑤ $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \text{for } g(x) \neq 0$$

(the quotient rule)

<u>Example</u>: Differentiate $y = \frac{x^2 + x - 2}{x^3 + 6}$

$$\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$= \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

Logarithmic differentiation

- This is a technique for differentiating complicated products or quotients of functions
- It is related to implicit differentiation.

- Recall: to find dy/dx when (for example) $x^2 + y^2 = 25$, we differentiate both sides to find

$$2x + 2y\frac{dy}{dx} = 0$$

(where we have used the chain rule on the second term with y as the inner function)

and $$\frac{dy}{dx} = \frac{-x}{y}$$

- In logarithmic differentiation, we begin by writing $y = f(x)$, where $f(x)$ is the function to be differentiated. We then
  1. Take natural logarithms of both sides, and simplify $\ln(f(x))$ using the laws of logarithms:

(i) $\ln(ab) = \ln a + \ln b$
(ii) $\ln(a/b) = \ln a - \ln b$
(iii) $\ln(a^r) = r \ln a$

2. Differentiate both sides of the equation with respect to x.
3. Solve the resulting equation for dy/dx.

<u>Example</u> Differentiate $f(x) = \dfrac{x^2 \sin x}{\cos 2x}$

Write $y = \dfrac{x^2 \sin x}{\cos 2x}$, and

1. Take logarithms of both sides:

$$\ln y = \ln(x^2) + \ln(\sin x) - \ln(\cos 2x)$$
$$= 2 \ln x + \ln(\sin x) - \ln(\cos 2x)$$

2. Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{\cos 2x}(-2\sin 2x)$$

3. Solve for dy/dx (and simplify):

$$\frac{dy}{dx} = y\left[\frac{2}{x} + \cot x + 2\tan 2x\right]$$
$$= \frac{x^2 \sin x}{\cos 2x}\left[\frac{2}{x} + \cot x + 2\tan 2x\right]$$

where we have remembered the original formula for y.

Notation for higher-order derivatives

- We tend to use a number in brackets rather than repeated primes:

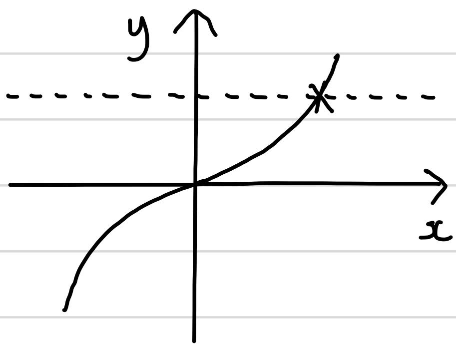
e.g. $$\frac{d^4 y}{dx^4} = f^{(4)}(x) = f^{(iv)}(x)$$

Inverse functions

- A function $f(x)$ takes an input, x, to give an output, y.
- Its inverse, $f^{-1}(y)$, tells us the value of x we should input to the function to get a given output, y.
- We can only define the inverse of one-to-one functions, which never give the same output for different values of x; that is, $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

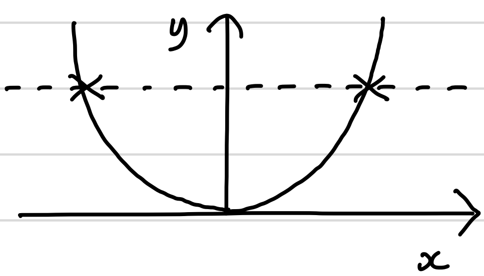
- Any horizontal line drawn through the graph of a one-to-one function will intercept it only once.

Examples



$f(x) = x^3$
$x \in \mathbb{R}$

one-to-one



$f(x) = x^2$
$x \in \mathbb{R}$, not one-to-one.

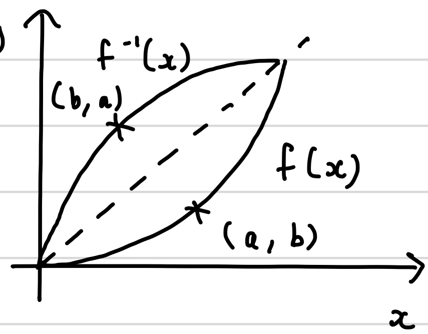
Definition Let the function f be one-to-one with domain X and range Y. Then, its inverse function has domain Y and range X

and is defined by

$$f^{-1}(y) = x \quad \text{for any } y \in Y.$$

Graph of the inverse

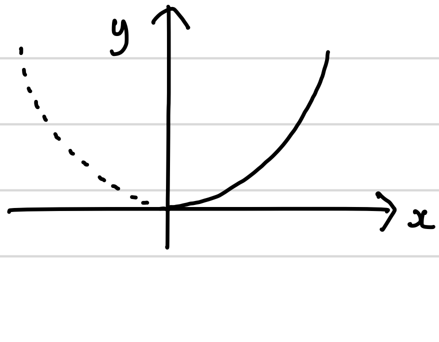
- If $f(a) = b$, then $f^{-1}(b) = a$.
- This means that, if (a, b) is on the graph of $f(x)$, then (b, a) is on the graph of $f^{-1}(x)$.



- Then, the graph of f^{-1} is found by reflecting the graph of f about $y = x$.

- Functions that are not one-to-one can be made so by restricting their domain.

e.g. $f(x) = x^2$ can be made one-to-one by restricting its domain to $[0, \infty)$.



Calculating the inverse

① Write $y = f(x)$.

② Solve this equation for x in terms of y.

③ If required, interchange x and y to express f^{-1} as a function of x: $y = f^{-1}(x)$.

Inverse hyperbolic functions

Given that $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, find $\sinh^{-1} x$.

Write $y = \sinh x$

$$y = \frac{e^{x} - e^{-x}}{2}$$

$$2y = e^{x} - e^{-x}$$

and $e^{x} - 2y - e^{-x} = 0$

so $e^{2x} - 2ye^{x} - 1 = 0$

This is a quadratic in e^{x}, with solution

$$e^{x} = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$$

so that $x = \ln\left(y \pm \sqrt{y^2 + 1}\right)$.

Since $\sqrt{y^2+1} > y$, and we cannot take the logarithm of a negative number, we must take the positive root, so that

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

and $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

Maximum and minimum values of functions

- We distinguish absolute, or global, maxima and minima from local maxima and minima.

Definition Let c be a number in the domain D of a function f. Then, $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

- Local maxima and minima are defined on small intervals rather than the whole domain.

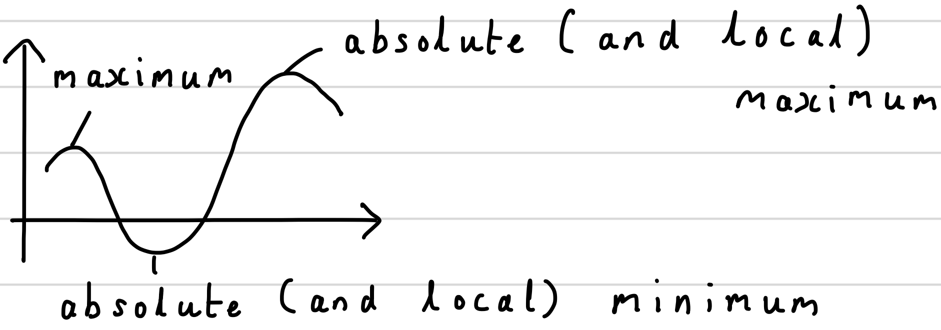
Definition The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ when x is near c.

- <u>local minimum value</u> of f if $f(c) \le f(x)$ when x is near c.

<u>Note</u>: every absolute maximum (or minimum) also satisfies the definition of a local maximum (or minimum).

<u>Example</u>

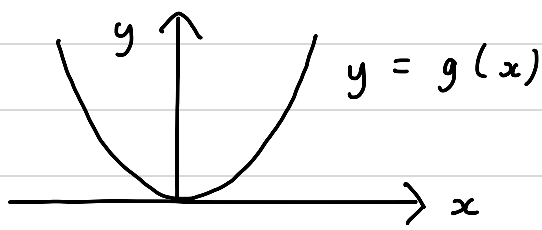


<u>Further examples</u>

① $f(x) = \sin x$ takes on its absolute (and local) maximum and minimum values infinitely many times, at intervals of 2π.

② $g(x) = x^2$

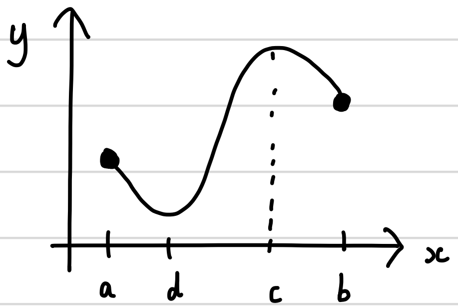
$g(0)$ is the absolute (and local) minimum value. There is <u>no</u> absolute maximum.

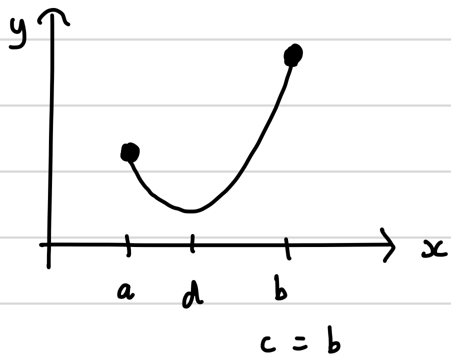


<u>Note</u>: maximum and minimum values are often called <u>extrema</u>, or <u>extreme values</u>.

The extreme value theorem

If $f(x)$ is continuous on a closed interval $[a, b]$, then it has an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



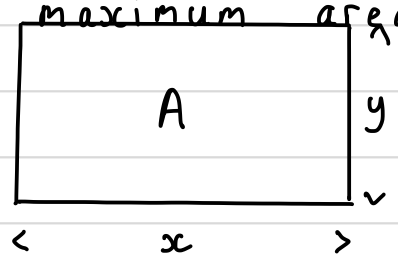


- Extreme values can occur at the endpoints of the interval, as well as at peaks and troughs.

Fermat's theorem

If $f(x)$ has a local extremum (i.e. maximum or minimum value) at c, and $f'(c)$ exists, then $f'(c) = 0$.

Example Show that, of all rectangles with the same perimeter, the square has maximum area.



Area $A = xy$
Perimeter $p = 2x + 2y$
The perimeter is fixed

to a value p: this is called a <u>constraint</u>. We want to find the maximum value of A for a given p.

- Firstly, we use the equation for p to write y in terms of x: $y = \dfrac{p - 2x}{2}$

- We then substitute this in the expression for A to find that

$$A = x\left(\frac{p - 2x}{2}\right) = \frac{px}{2} - x^2$$

- We now find dA/dx and set it to 0, so that $\frac{p}{2} - 2x = 0$, and

$$x = \frac{p}{4}, \quad y = \frac{p - 2(p/4)}{2} = \frac{p}{4}$$

and we have a square.

- We can confirm that this is a maximum by noting that $\dfrac{d^2A}{dx^2} = -2 < 0$

- This is an example of an <u>optimisation problem</u> with a constraint.

To find the absolute maximum and minimum values of a continuous function on a closed interval $[a, b]$, we

(1) Find the values of f at the stationary points.

(2) Find the values of f at a and b.

(3) The absolute maximum and minimum values are the largest and smallest values respectively from steps (1) and (2).

<u>Example</u> Find the absolute maximum and minimum values of $f(x) = 3x^2 - 12x + 5$ on $[0, 3]$, giving a justification of your answer.

- The function is continuous on the given interval, and its absolute maximum and minimum will each occur at a stationary point or an endpoint of the interval.

(1) $f'(x) = 6x - 12$
$f'(x) = 0$ at $x = 2$, and
$f(2) = 3 \times 2^2 - 12 \times 2 + 5 = -7$

(2) At the endpoints, we have that $f(0) = 5$ and $f(3) = 3 \times 3^2 - 12 \times 3 + 5 = -4$.

(3) The absolute maximum is 5 and the absolute minimum is -7.

<u>Critical points</u>

<u>Definition</u> A critical point of a function

is any value c in its domain where $f'(c) = 0$ or $f'(c)$ does not exist.

Example : Find the critical points of $f(x) = x^{2/3}(1-x)$. We have that $f(x) = x^{2/3} - x^{5/3}$, so that

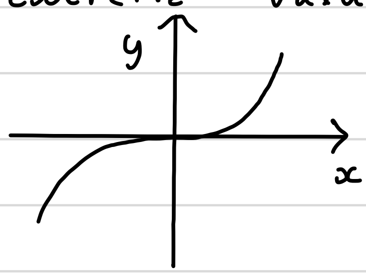
$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2-5x}{3x^{1/3}}$$

$f'(x)$ is 0 at $x = 2/5$ and does not exist at $x = 0$. These are the two critical points of $f(x)$.

Notes

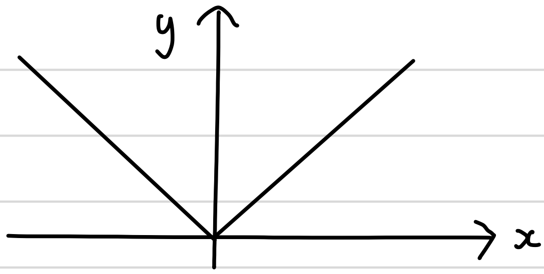
- If $f'(c) = 0$, there is not necessarily a maximum or minimum value of f at c.

Example : If $f(x) = x^3$, then $f'(0) = 0$, but there is no extreme value.



- There may be an extremum even when $f'(x)$ does not exist

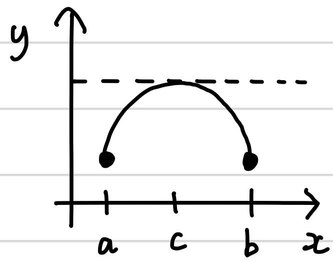
Example



$f(x) = |x|$

Rolle's theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there must be at least one number c in (a, b) such that $f'(c) = 0$.



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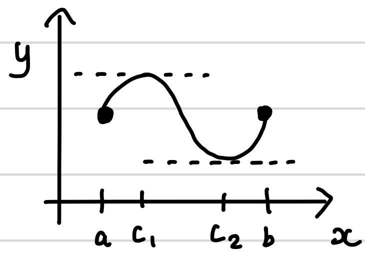

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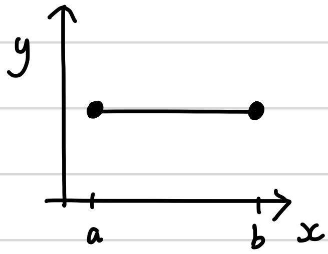


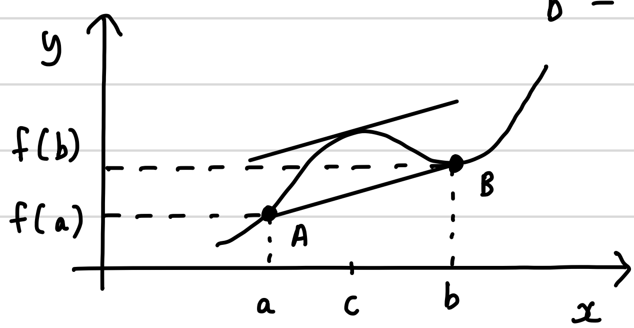
3

Case 1 : one point where $f' = 0$.
" 2 : two points where $f' = 0$.
" 3 : $f' = 0$ everywhere in the interval.

Mean value theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$y = f(x)$

To understand the mean value theorem, note that the equation of the line AB is

$$g(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a}$$

The difference between the curve and the line is

$$h(x) = f(x) - g(x) = f(x) - f(a) - (x-a)\frac{f(b)-f(a)}{b-a}$$

Since the curve and the line intersect at A and B, $h(x) = 0$ at both of these points. We can then apply Rolle's theorem, which tells us that $h'(x) = 0$ for at least one point c between A and B. Differentiating $h(x)$, we find that

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Since $h'(c) = 0$, we have that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$: the mean value theorem

<u>Intervals of increase and decrease</u>

<u>Definition</u> Suppose that f is continuous on an interval I. Then, f is an increasing

function if, for every pair $x_1, x_2 \in I$, where $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. Conversely, f is a decreasing function if, for every pair $x_1, x_2 \in I$, where $x_1 < x_2$, we have that $f(x_1) > f(x_2)$.

Then, for differentiable functions,

1. If $f'(x) > 0$ for all $x \in I$, then f is increasing on the interval I.
2. If $f'(x) < 0$ for all $x \in I$, then f is decreasing on the interval I.

<u>Proof of 1</u> (2 is proved similarly)

The function is differentiable on (x_1, x_2), so we can apply the mean value theorem, which tells us that there is a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We know that $f'(c) > 0$ and that $x_2 - x_1 > 0$. It then follows that $f(x_2) - f(x_1) > 0$, so that $f(x_2) > f(x_1)$.

Since this argument can be applied to any pair $x_1, x_2 \in I : x_1 < x_2$, we have shown that f is increasing on the interval I.

To find the intervals of increase or decrease for a function $f(x)$ on its domain D, we

1. Find the critical points of f (and other points where f' does not exist, e.g, $x = 0$ for $f(x) = 1/x$)
2. Divide D into sub-intervals with endpoints at these points.
3. Check the sign of f' within each subinterval.
4. If $f' > 0$, then f is increasing on that interval. If $f' < 0$, then f is decreasing.

<u>Example</u> Determine the intervals of increase and decrease for $f(x) = x^3 - 3x^2 + 2$, $x \in \mathbb{R}$.

1. Find the critical points of f
$f'(x) = 3x^2 - 6x = 3x(x-2)$
This exists for all $x \in \mathbb{R}$, and $f'(x) = 0$ at $x = 0$ and $x = 2$.

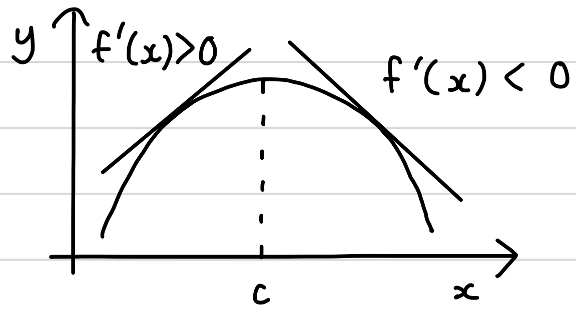
2. We divide the domain, $\mathbb{R}$, into three subintervals: $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.
3. We now choose a point in each interval, and check the sign of f' at that point.
$f'(-1) = 9 > 0$, $f'(1) = -3 < 0$ and $f'(3) = 9 > 0$.
4. So, f is increasing on $(-\infty, 0)$
decreasing on $(0, 2)$
and increasing on $(2, \infty)$.

Classifying critical points as local maxima or minima

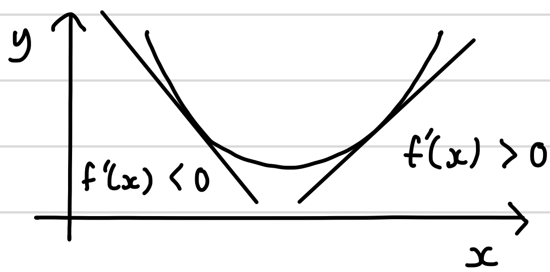
Using the first derivative

Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Then,

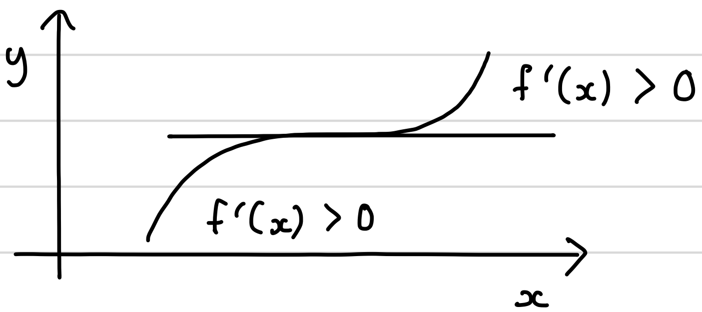
① if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has a local maximum at $x = c$.



② if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b), then f has a local minimum at $x = c$.



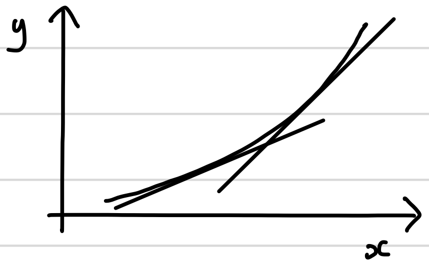
③ if $f'(x) > 0$ on (a, c) and $f'(x) > 0$ on (c, b), or $f'(x) < 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has neither a local maximum nor a local minimum at $x = c$.

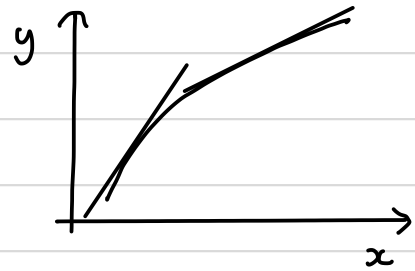


Concavity

Definition Suppose that the function f is differentiable on $x \in (a, b)$. Then,

1. if $f'(x)$ is increasing on (a, b), the graph is concave upwards on (a, b).
2. if $f'(x)$ is decreasing on (a, b), the graph is concave downwards on (a, b).





Concave upwards: curve lies above its tangent at all points in the interval

Concave downwards: curve below tangent.

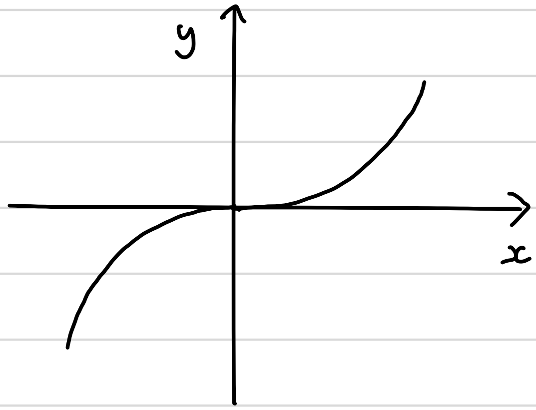
Testing for concavity

Suppose that the function $f(x)$ is twice differentiable on $x \in (a, b)$. Then,

1. if $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is concave upwards on (a, b).

(2) if $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is concave downwards on (a, b).

Example



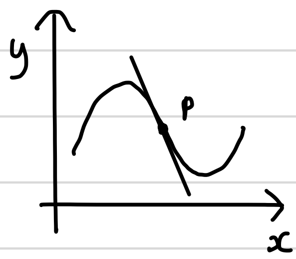
If $f(x) = x^3$, then $f'(x) = 3x^2$
and $f''(x) = 6x$

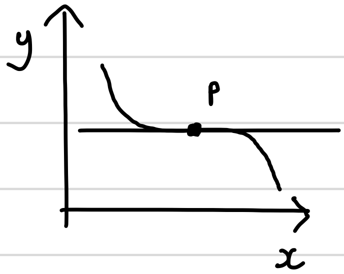
- For $x < 0$, $f''(x) < 0$ and f is concave downwards.
- For $x > 0$, $f''(x) > 0$ and f is concave upwards.

Points of inflection

Definition: Suppose that the function $f(x)$ is continuous on an interval I. If there

is a point $P \in I$ where the graph changes from concave upwards to concave downwards (or vice-versa), then P is an inflection point of f.





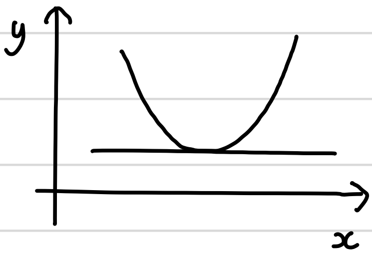
horizontal tangent

At a point of inflection, the curve crosses its tangent.

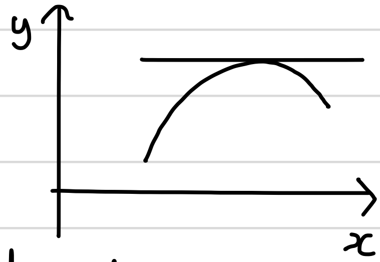
Second derivative test for maxima and minima

Suppose that the function f is continuous near $x = c$.
Then,

1. if $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



$f'(c) = 0$
$f''(c) > 0$



$f'(c) = 0$
$f''(c) < 0$

local min, concave up

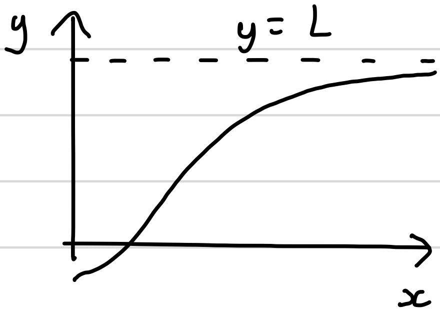
local max, concave down

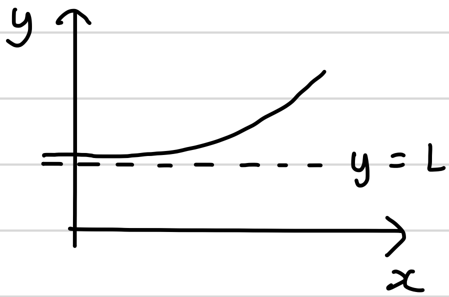
Note: this test fails when $f''(c) = 0$, or does not exist.

Example: $f(x) = x^4$ has a local minimum at $x = 0$, but $f''(x) = 0$ here.

Asymptotes

Definition Suppose that either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. Then, the line $y = L$ is a horizontal asymptote to the graph of $y = f(x)$.

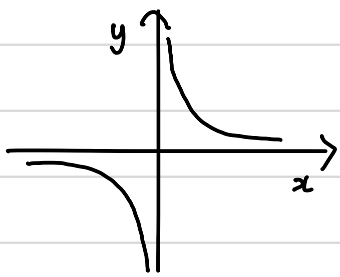




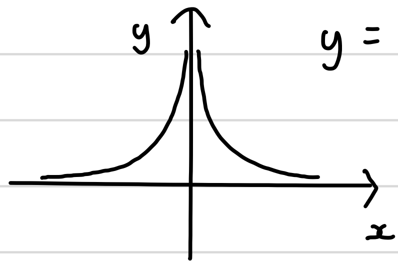
$\lim_{x \to \infty} f(x) = L$ $\qquad$ $\lim_{x \to -\infty} f(x) = L$

Definition The line $x = a$ is a vertical asymptote to the graph of $y = f(x)$ if at least one of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ is equal to ∞ or $-\infty$

Examples

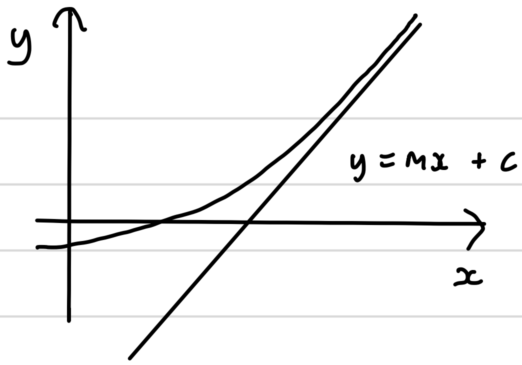


$y = \frac{1}{x}$



$y = 1/x^2$

Definition If $\lim_{x \to \infty} [f(x) - (mx + c)] = 0$, for $m \neq 0$, then the line $y = mx + c$ is a slant asymptote of the curve.



This occurs for rational functions when the degree of the numerator is one higher than the degree of the denominator.

Checklist for curve sketching.

1. Domain
2. Intercepts with axes
3. Symmetry and periodicity
4. Asymptotes : vertical, horizontal, slant
5. Intervals of increase and decrease
6. Local maxima and minima
7. Concavity and points of inflection.

Example Sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$

(1) Domain : $\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$

$$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

(2) Intercepts : $y = 0$ when $x = 0$.

(3) $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 1} = \dfrac{2x^2}{x^2 - 1} = f(x)$

and the function is even.

(4) $$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

and $y = 2$ is a (double) horizontal asymptote.

The denominator is 0 when $x = \pm 1$, so we have that

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty, \quad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

and we have vertical asymptotes at $x = \pm 1$.

(5) $$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 . 2x}{(x^2 - 1)^2}$$

$$= \frac{-4x}{(x^2 - 1)^2}$$

So, $f'(x) > 0$ when $x < 0$ (with $x \neq -1$)
and $f'(x) < 0$ when $x > 0$ (with $x \neq 1$).
Then $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

(6) $f' = 0$ at $x = 0$ and does not exist at $x = \pm 1$. We have already dealt with the vertical asymptotes at $x = \pm 1$. We know that $(0, 0)$ is a local maximum, since $f'(x)$ changes from positive to negative here.

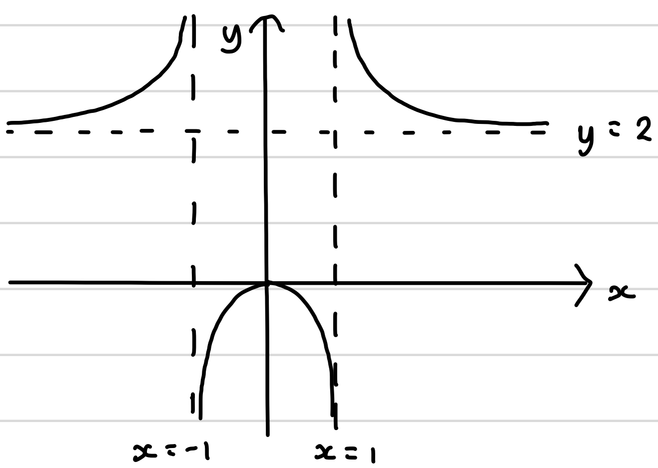
(7) $$f''(x) = \frac{(x^2-1)^2(-4) + 4x.2(x^2-1)2x}{(x^2-1)^4}$$

$$= \frac{-4(x^2-1) + 16x^2}{(x^2-1)^3}$$

$$= \frac{12x^2+4}{(x^2-1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have that $f''(x) > 0$ when $x^2 > 1$ and that $f''(x) < 0$ when $x^2 < 1$. This means that the curve is concave upwards on $(-\infty, -1)$ and $(1, \infty)$ and concave downwards on $(-1, 1)$.

Putting all this together, we find that



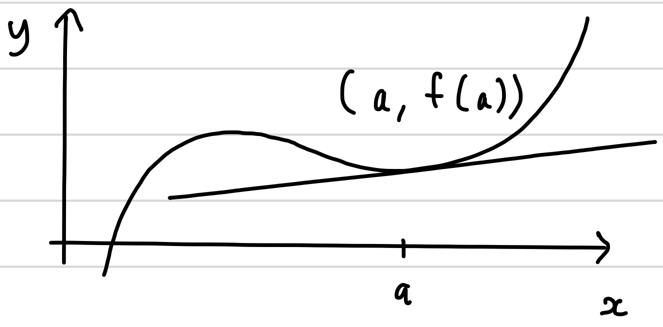
Taylor polynomials and Taylor series

- We can use polynomials to approximate complicated functions.
- A systematic way of doing this is to use Taylor polynomials.
- We build up to this by looking at linear and quadratic approximations.

Linearisation

- Imagine we have a function, $f(x)$, that can be differentiated as many times as we like.
- Near the point $(a, f(a))$, we can approximate the function by its tangent line,

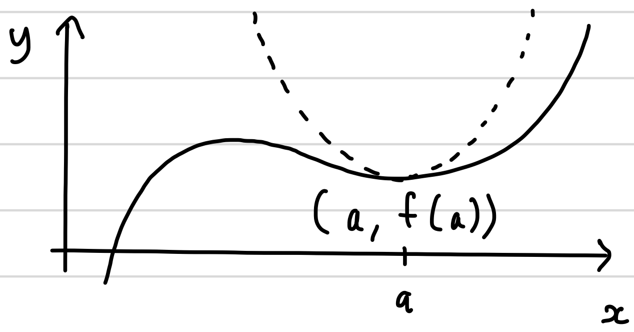
$$y = f(a) + f'(a)(x-a)$$



- The closer we get to $x = a$, the better the approximation becomes.

Quadratic approximations

- It is often possible to improve the approximation by using a quadratic.



- This is given by

$$y = P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Why does this work?

① When $x = a$, $P_2(a) = f(a)$, so P_2 and the function match at $x = a$.

② $P_2'(x) = f'(a) + f''(a)(x-a)$

Putting $x = a$, we see that $P_2'(a) = f'(a)$, and the derivatives match at $x = a$.

③ $P_2''(x) = f''(a)$, so the second derivatives match at $x = a$.

- Our approximating polynomial therefore reproduces some important features of $f(x)$.

- Note that $P_2'''(x) = 0$ for all x, so $P_2(x)$ contains no information about the third (and higher) derivatives of $f(x)$.

Higher-degree approximations

We can improve the approximation further by including more terms:

$$P_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

or, in sigma notation,

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- This is the N^{th} order Taylor polynomial of $f(x)$ at $x = a$
- All derivatives of $P_N(x)$, up to and including the N^{th} derivative, match those of $f(x)$, that is,

$$P_N^{(n)}(a) = f^{(n)}(a) \quad \text{for } n = 0, 1, 2, \dots, N$$

but all higher derivatives of P_N must be zero everywhere.

- The function P_N includes the information about f that comes from its derivatives up to order N at $x = a$.

<u>Example</u>: Find the third-order Taylor polynomial approximation to $f(x) = e^x$, valid near $a = 0$.

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!}$$

All derivatives of e^x with respect to x are e^x, so

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = e^0 = 1$$

We then have that $P_3(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

<u>The error term</u>

- We would now like to quantify the error made when approximating a function by a Taylor polynomial.
- To begin, we define the N^{th}-order error term as

$$R_N(x) = f(x) - P_N(x)$$

Taylor's theorem then states that the N^{th} order error term about $x = a$ is

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

where c is a number that lies between x and a.

Note

- Taylor's theorem is an extension of the mean value theorem, and is exactly the same if we set $N = 0$ in the above formula. In this case, $R_0(x) = f'(c)(x-a)$, so that $f(x) - f(a) = f'(c)(x-a)$. Rearranging this, we find that

$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

for some c between x and a, which is the mean value theorem.

Example We found earlier that the third-order Taylor approximation to e^x around $a = 0$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Taylor's theorem states that the error term is given by

$$R_3(x) = \frac{f^{(4)}(c)\, x^4}{4!}$$

where c is between 0 and x. Again, any derivative of e^x with respect to x is e^x, and $4! = 24$, so,

$$R_3(x) = \frac{e^c x^4}{24}$$

Taylor's theorem does not give us the value of c. However, we can calculate the maximum error for a given value of x. For example, if we use the polynomial given above to estimate $e^{-1/10}$ by expanding about $a=0$, we find that

$$e^{-1/10} = 1 - \left(\frac{1}{10}\right) + \frac{(1/100)}{2} - \frac{(1/1000)}{6} + \frac{e^c}{24}\left(\frac{1}{10000}\right)$$

$$= \frac{5429}{6000} + \frac{e^c}{240000}$$

We know that $-1/10 < c < 0$, so the maximum error term is $\dfrac{e^0}{240000} = \dfrac{1}{240000}$.

<u>Note</u> : the key when using the above technique to estimate the value of a function at a given point is to start at a nearby point where the value of $f(x)$ is known. In the above example, we knew that $e^0 = 1$.

<u>Taylor series</u>

- If the number of terms becomes infinite ($N \to \infty$), a Taylor polynomial becomes an infinite series.
- This is a convergent series : it approaches a limit.

<u>Example</u> : the Taylor series about $x = 1$ of e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^1(x-1)^n}{n!} = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

since all the derivatives $f^{(n)}(x)$ of e^x are e^x.

<u>Example (Exam 2021/22)</u>
Find the first three non-zero terms in the Taylor series of $\cos x$ around $x = \pi/2$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	0
5	$-\sin x$	-1

$$f(x) = f(\pi/2) + f'(\pi/2)(x-\pi/2)$$

$$+ \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f^{(3)}(\pi/2)}{3!}(x-\pi/2)^3$$

$$+ \frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4 + \frac{f^{(5)}(\pi/2)}{5!}(x-\pi/2)^5$$

$$+ \dots$$

$$= -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \dots$$

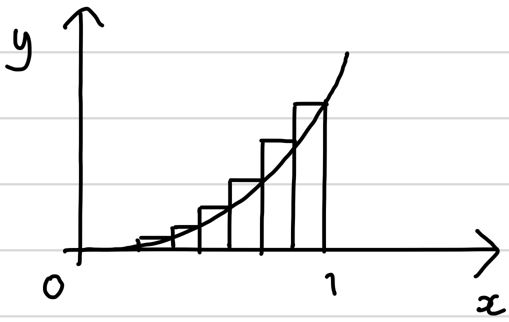
Integration

Integration from first principles

Suppose that a function is continuous and non-negative on an interval $x \in I$, and we wish to determine the area under the curve on I.

Example: What is the area under the curve $y = x^2$ on the interval $x \in [0, 1]$?

We can estimate the area using rectangles:



- Our estimate becomes more accurate as we increase the number of rectangles.
- Suppose that we have n rectangles. Since the interval is of width 1, each rectangle is of width $1/n$.

- The height of the first rectangle is $\left(\frac{1}{n}\right)^2$, that of the second is $\left(\frac{2}{n}\right)^2$, and so on.
- The sum of the areas of the rectangles is given by

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$
$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)$$

- To evaluate this sum, we need to know that the sum of the squares of the first n positive integers is given by

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- Taking the limit of an infinite number of rectangles, we find that

$$\lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n \to \infty} \frac{2 + 3/n + 1/n^2}{6} = \frac{1}{3}$$

<u>Definition</u> Suppose that $f(x)$ is a continuous function on the interval $[a, b]$. We divide the interval into n sub-intervals of equal width $\Delta x = (b-a)/n$. Let $x_0 = a$, $x_1, x_2, \dots, x_n = b$ denote the endpoints of the subintervals, and let $c_i \in [x_{i-1}, x_i]$ be sample points. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

<u>Notes</u>

- if the limit exists, we say that the function is integrable
- the limit gives the same value for all possible choices of sample points c_i.
- The sum $\sum_{i=1}^{n} f(c_i)\,\Delta x$ is called the Riemann sum
- Since the value of the limit is independent of the sample of points chosen, we can simplify the definition of the definite integral to

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $x_i = a + i\,\Delta x$.

Properties of the definite integral

① $\int_a^b c \, dx = c(b-a)$, where c is any constant

② $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

③ $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$

④ $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$

(splitting the range)

⑤ If $f(x) \geq 0$ for $a \leq x \leq b$, then

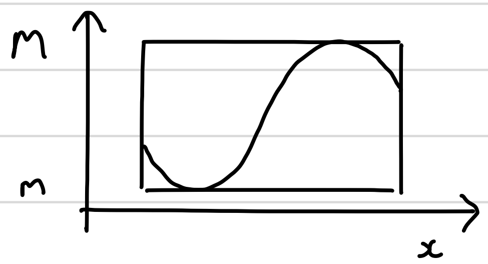
$$\int_a^b f(x) \, dx \geq 0$$

⑥ If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$$

⑦ If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)$$



<u>Note</u>: this may be used to estimate the definite integrals of complicated functions

Fundamental theorem of calculus

- This essentially states that integration and differentiation are the inverses of each other.

Part ① If f is continuous on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b$$

is continuous on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$.

Alternatively, $$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

Notes: - the choice of the lower limit is not important: it sets the constant of integration, which vanishes when we differentiate.
- this theorem allows us to view integration as solving a differential equation: we are given $F'(x) = f(x)$, and then find $F(x)$.

Antiderivatives

If F is a function whose derivative is f, so that $F'(x) = f(x)$, we say that F is an antiderivative of f.

Example: Any function $\frac{x^3}{3} + c$, for some constant c, is an antiderivative of x^2.

Fundamental theorem of calculus part 2

If f is continuous on $[a, b]$, then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Example: $\int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Indefinite integrals

- The indefinite integral $\int f(x)\,dx$ is the family of all antiderivatives of f.
  Example
  $$\int x^2\,dx = \frac{x^3}{3} + c$$

- We can find many indefinite integrals $\int f(x)\,dx$ by recalling the function $F(x)$ such that $F'(x) = f(x)$ e.g.
  $\int \cos x\,dx = \sin x + c.$

Standard cases

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1$$

$$\int x^{-1} \, dx = \ln|x| + c$$

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + c$$

$$\int \sin kx \, dx = -\frac{\cos kx}{k} + c$$

$$\int \cos kx \, dx = \frac{\sin kx}{k} + c$$

$$\int \sinh kx \, dx = \frac{\cosh kx}{k} + c$$

$$\int \cosh kx \, dx = \frac{\sinh kx}{k} + c$$

Techiques of integration   Substitution

- Using the chain rule, we can differentiate e^{x^2} with respect to x to find that

$$\frac{d}{dx} e^{x^2} = 2x \, e^{x^2}$$

- Using the results of the preceding section, we can turn this around to get

$$\int 2x \, e^{x^2} \, dx = e^{x^2} + c$$

- More generally,

If $u = g(x)$ is a differentiable function with range I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

<u>Note</u>: this can be proved by using the chain rule together with the fundamental theorem of calculus.

<u>Example</u> Evaluate $\displaystyle\int_0^{\sqrt[3]{\pi/2}} x^2 \cos(x^3)\, dx$

We have $u = x^3$, $du = 3x^2\, dx$ and $x^2\, dx = \dfrac{du}{3}$

We also have that, when $x = 0$, $u = 0$ and that, when $x = \sqrt[3]{\pi/2}$, $u = \pi/2$.

Then, $\displaystyle\int_0^{\sqrt[3]{\pi/2}} x^2 \cos(x^3)\, dx = \int_0^{\pi/2} \frac{1}{3} \cos u\, du$

$$= \frac{1}{3} \sin u \Big|_0^{\pi/2} = \frac{1}{3}\left[\sin(\pi/2) - \sin(0)\right] = \frac{1}{3}$$

<u>Example</u> <u>(Exam 2021)</u>

Evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{2 - x^2}}\, dx$.

Let $u = 2 - x^2 \Rightarrow du/dx = -2x$
and $x\, dx = -du/2$
At $x = 0$, $u = 2$; at $x = 1$, $u = 1$.

$$\int_0^1 \frac{x}{\sqrt{2-x^2}}\, dx = -\frac{1}{2}\int_2^1 \frac{du}{\sqrt{u}} = \frac{-1}{2}\int_2^1 u^{-1/2}\, du$$

$$= -\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]_2^1 = -\frac{1}{2} \cdot \left[2u^{1/2}\right]_2^1 = -\left[u^{1/2}\right]_2^1$$

$$= -\left[1 - \sqrt{2}\right] = \sqrt{2} - 1$$

Integrals involving the natural Logarithm

Example Evaluate $\int \frac{1}{ax}\, dx$, where $a \neq 0$ is a constant.

$$\int \frac{1}{ax}\, dx = \frac{1}{a}\int \frac{1}{x}\, dx = \frac{1}{a}\ln|x| + c$$

$$= \ln|x|^{1/a} + c$$

$1/a$ is taken out to the front by the rule $\int c\, f(x)\, dx = c\int f(x)\, dx$.

Combined example

Evaluate $\int \frac{4x^3 - 2}{x^4 - 2x}\, dx$.

Let $u = x^4 - 2x \Rightarrow \frac{du}{dx} = 4x^3 - 2$

Then, $\int \frac{4x^3-2}{x^4-2x}\, dx = \int \frac{du}{u} = \ln|u| + c$

$$= \ln|x^4 - 2x| + c$$

These integrals can also be evaluated by remembering the rule

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Integrals involving trigonometric identities

Example Evaluate $\int \sin^2 x \; dx$.

Identity: $\cos 2x \equiv 1 - 2\sin^2 x$
Then, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, and

$$\int \sin^2 x \; dx = \frac{1}{2}\int (1 - \cos 2x)\,dx$$
$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

Integration by parts - recursive version

- In standard integration by parts, one of the terms simplifies as it is differentiated e.g. in $\int x^2 e^x \,dx$, x^2 becomes $2x$, then 2.
- However, integration by parts can sometimes be used when this does not happen.

Example

$$I = \int e^x \sin x \; dx$$

- In the formula

$\int u\,dv = uv - \int v\,du$, we choose

$$u = e^x \qquad dv = \sin x\,dx$$
$$du = e^x\,dx \qquad v = -\cos x$$

$$I = \int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

- We now integrate $\int e^x \cos x\,dx$ by parts, using

$$U = e^x \qquad dV = \cos x\,dx$$
$$dU = e^x\,dx \qquad V = \sin x$$

so $\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$

- Substituting this in the expression for I, we find

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$
$$= e^x(\sin x - \cos x) - I$$

and the original integral has appeared on the right-hand side. We can now solve for I, finding

$$2I = e^x(\sin x - \cos x)$$

and $I = \frac{1}{2} e^x(\sin x - \cos x) + c$

where we have inserted the constant of integration, c.

Reduction formulas

- These are used to evaluate integrals, I_n, in which all or part of the function being integrated is raised to an integer power, n.
- Examples are $\int x^2 e^x \, dx$, $\int \tan^4 x \, dx$ and $\int (1 - x^3)^2 \, dx$.

- A reduction formula expresses I_n in terms of simpler integrals involving lower powers, such as I_{n-1} or I_{n-2}.
- If we can evaluate one of these simpler integrals directly, we can then use the reduction formula to gradually build up to higher powers.

<u>Example</u> Suppose that

$$I_n = \int x^n e^x \, dx$$

We then use integration by parts, with

$$u = x^n \qquad dv = e^x \, dx$$
$$du = n x^{n-1} \, dx \qquad v = e^x$$

Then, $I_n = \int x^n e^x \, dx$

$$= uv - \int v \, du$$

$$= x^n e^x - \int n x^{n-1} e^x \, dx$$
$$= x^n e^x - n \int x^{n-1} e^x \, dx$$
$$= x^n e^x - n I_{n-1}$$

We can evaluate I_0 directly:

$$I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + c_0.$$

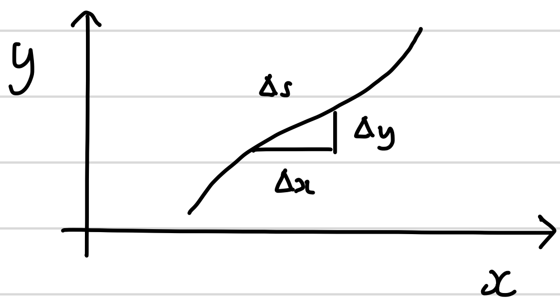
Using the reduction formula, we find that

$$I_1 = x^1 e^x - 1(e^x + c_0)$$
$$= x e^x - e^x - c_0$$

and
$$I_2 = x^2 e^x - 2(xe^x - e^x - c_0)$$
$$= x^2 e^x - 2xe^x + 2e^x + 2c_0$$
$$= e^x (x^2 - 2x + 2) + c \qquad \text{with } c = 2c_0$$

Notes:
- this process can be continued indefinitely.
- it is not necessary to keep track of the constant of integration throughout the calculation, as long as you add it in at the end.

Application of integration: curve length



$y = f(x)$

- We have a curve defined by $f(x)$.
- The distance, Δs, that corresponds to small changes Δx and Δy in x and y is given by

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Taking the limit as the number of small distances becomes infinite, we find that

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

(For a more rigorous derivation, see Stewart, Calculus)

<u>Example</u> Find the length of the curve $y = x^{3/2}$ between $x = 0$ and $x = 2$.

We have that $dy/dx = \frac{3}{2}\sqrt{x}$, so

$$s = \int_0^2 \sqrt{1 + \frac{9x}{4}}\ dx = \left[\frac{2}{3}\left(\frac{4}{9}\right)\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^2$$

$$= \frac{8}{27}\left[\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^2 = \frac{8}{27}\left[\left(\frac{11}{2}\right)^{3/2} - 1\right]$$

Parametric equations and polar coordinates

- Instead of relating x and y directly, as in $y = x^2 \sin x$, we can relate them indirectly, via a common parameter.

Example A circle of radius a centred on the origin can be described by

- the direct relation $x^2 + y^2 = a^2$
- the pair of functions $y = \pm\sqrt{a^2 - x^2}$
- the parametric equations

$$x = a \cos t \quad \text{and} \quad y = a \sin t$$

- As t increases, we gradually trace out a circle: at $t = \pi/2$, we have a quarter arc of a circle; at $t = \pi$, we have a semicircle, and so on.
- For all values of t, we have that

$$\begin{aligned} x^2 + y^2 &= a^2 \cos^2 t + a^2 \sin^2 t \\ &= a^2 (\cos^2 t + \sin^2 t) \\ &= a^2 \end{aligned}$$

and recover the original direct relation.

Derivatives of parametric equations

- The chain rule tells us that

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Rearranging this, we find that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$$

where the dot represents differentiation with respect to t.