Algebra – Practical session 3

3.1. (a) Convert the number $(110210)_3$ (written here in base 3) to decimal notation.
(b) Convert the integer 13539 (written here in decimal notation) to base 7.

Solution: (a) One can arrange the calculation as follows:

$$
\begin{array}{c|ccccc|c}
 & 1 & 1 & 0 & 2 & 1 & 0 \\
3 & & 3 & 12 & 36 & 114 & 345 \\
\hline
 & 1 & 4 & 12 & 38 & 115 & 345
\end{array}
$$

Hence $(110210)_3 = 345$. Now convert it back to base 3 for extra practice.
(b) To convert 13539 to base 7 keep dividing as follows:

$$
\begin{aligned}
13539 &= 7 \cdot 1934 + 1 \\
1934 &= 7 \cdot 276 + 2 \\
276 &= 7 \cdot 39 + 3 \\
39 &= 7 \cdot 5 + 4 \\
5 &= 7 \cdot 0 + 5
\end{aligned}
$$

The digits in base 7 are the remainders read from the bottom up, so $(13539)_{10} = (54321)_7$. Now convert it back to decimal for extra practice.

3.2. Find a fraction m/n, with m and n integers, such that

$$\frac{m}{n} = 0.6\overline{351} = 0.6\dot{3}5\dot{1}$$

Simplify the resulting fraction using the Euclidean algorithm. (You may use a pocket calculator for this part.)

Solution: We have

$$0.6\dot{3}5\dot{1} = \frac{6351 - 6}{9990} = \frac{6345}{9990}.$$

To simplify the fraction we compute the GCD of 6345 and 9990:

$$
\begin{aligned}
9990 &= 6345 \cdot 1 + 3645 \\
6345 &= 3645 \cdot 2 - 945 \\
3645 &= 945 \cdot 4 - 135 \\
945 &= 135 \cdot 7.
\end{aligned}
$$

Dividing both numbers by their GCD 135 we find

$$0.6\dot{3}5\dot{1} = \frac{6345}{9990} = \frac{47}{74},$$

which is the simplified form of the fraction.

3.3. Recognise that the following is an infinite geometric series (with complex terms) and compute its sum:

$$1+\frac{i}{2}-\frac{1}{4}-\frac{i}{8}+\frac{1}{16}+\frac{i}{32}-\frac{1}{64}-\frac{i}{128}+\cdots$$

Draw a few partial sums in the complex plane. (That means mark the sums of the first $1, 2, 3, \ldots$ terms as points in the complex plane. If you join each partial sum to the next one by a vector, those vectors will be the individual terms that you add to make up the sum.)

Solution: The series is

$$1+\frac{i}{2}+\left(\frac{i}{2}\right)^2+\left(\frac{i}{2}\right)^3+\cdots,$$

and its sum is

$$\frac{1}{1-\frac{i}{2}}=\frac{2}{2-i}=\frac{2(2+i)}{(2-i)(2+i)}=\frac{2}{5}(2+i)=\frac{4}{5}+\frac{2}{5}i.$$

[The drawing was done on the white board in the Practical session.]

3.4. Convert the number $(21.201)_3$, written here in base 3, to decimal notation.

Solution: First multiply the given number $(21.201)_3$ by $3^3 = 27$, and get $(21201)_3$. Now convert this to decimal

$$\begin{array}{c|cccc|c} & 2 & 1 & 2 & 0 & 1 \\ 3 & & 6 & 21 & 69 & 207 \\ \hline & 2 & 7 & 23 & 69 & 208 \end{array}$$

hence $(21201)_3 = (208)_{10} = 208$, and then divide by 27. Hence we find

$$(21.201)_3 = 208/27 = 7.\dot{7}0\dot{3}.$$

3.5. Convert the decimal number 8.57 to base 6, correct to 5 digits after the point.

Solution: First separate the integer part, writing $8.57 = 8 + 0.57$. The integer part becomes $8 = (12)_6$ in base 6.

- $0.57 \cdot 6 = 3.42$, so first digit after the point will be 3;
- $0.42 \cdot 6 = 2.52$, so second digit after the point will be 2;
- $0.52 \cdot 6 = 3.12$, so third digit after the point will be 3;
- $0.12 \cdot 6 = 0.72$, so fourth digit after the point will be 0;
- $0.72 \cdot 6 = 4.32$, so fifth digit after the point will be 4.

In conclusion, we have found that $8.57 = (12.32304\cdots)_6$.

Because we know from the start how many digits we want after the point, an alternative approach would be multiplying 0.52 by 6^5, hence $0.52 \cdot 6^5 = 4432.32$, converting the integer part of this to base 6, hence $4432 = (32304)_6$, and then shifting the dot to the left by five places, finding $4432/6^5 = (0.32304)_6$, and hence $0.52 = (0.32304\cdots)_6$.

3.6. (a) Express the periodic binary number $(1.\dot{0}01\dot{1})_2$ as a fraction of integers (written as decimals).

Hint: You may adapt to binary the rule used to convert periodic decimal numbers to fractions.

(b) Write $6/5$ in binary (as a periodic binary number).

Hint: One way is converting 6 and 5 to binary, and then doing long division working in binary. Another way is expressing the fraction as a decimal first, and then converting that to binary.

Solution: (a) Separating the integer part we have $(1.\dot{0}01\dot{1})_2 = 1 + (0.\dot{0}01\dot{1})_2$, and then

$$(0.\dot{0}01\dot{1})_2 = \frac{(00011)_2}{(1111)_2} = \frac{3}{15} = \frac{1}{5},$$

and finally $(1.\dot{0}01\dot{1})_2 = 1 + 1/5 = 6/5$.

We could also have done without separating the integer part, just a bit less efficiently:

$$(1.\dot{0}01\dot{1})_2 = \frac{(10011)_2}{(1111)_2} = \frac{18}{15} = \frac{6}{5}.$$

If we prefer to not rely on a given rule but argue directly, we could note that, in binary,

$$\begin{aligned}
0.\dot{0}01\dot{1} = {} & \quad 0.0011 \\
& + 0.0000\,0011 \\
& + 0.0000\,0000\,0011 \\
& \qquad \vdots \\
= {} & 0.0011 \cdot (1 + 0.001 + 0.001^2 + \cdots) \\
= {} & \frac{0.0011}{1 - 0.0011}
\end{aligned}$$

by summing the geometric series. Of course this in binary, and in decimal it reads

$$\frac{3 \cdot 2^{-4}}{1 - 2^{-4}} = 1/5.$$

Even more directly is arguing

$$(0.\dot{0}01\dot{1})_2 \cdot 2^4 = (11.\dot{0}01\dot{1})_2 = (11)_2 + (0.\dot{0}01\dot{1})_2,$$

and so

$$(0.\dot{0}01\dot{1})_2 = 3/(2^4 - 1) = 1/5.$$

(b) In part (a) we have discovered that $(1.\dot{0}01\dot{1})_2 = 6/5$, so we have the answer, but let us pretend we do not know that and see how to proceed. One way is to divide $6 = (110)_2$ by $(101)_2$ in binary, as follows:

$$\begin{array}{r|ccccccc}
 & & & 1. & 0 & 0 & 1 & 1 \\ \hline
101 & 1 & 1 & 0 & & & & \\
 & 1 & 0 & 1 & & & & \\ \cline{2-4}
 & & & 1 & 0 & 0 & 0 & \\
 & & & & 1 & 0 & 1 & \\ \cline{4-7}
 & & & & & 1 & 1 & 0
\end{array}$$

Because we have reached a remainder, 110, which has occurred before, the calculations will repeat, and so with some care we can conclude that the digits in the quotient will repeat, namely $(1.001100110011\cdots)_2 = (1.\dot{0}01\dot{1})_2$.

Another way is using the method which we have learnt to convert a real number from decimal to an arbitrary base b:

- $6/5 = 1.2 = 1 + 0.2$, so in binary there will be a 1 before the point;
- $0.2 \cdot 2 = 0.4$, so first bit (=binary digit) after the point will be 0;
- $0.4 \cdot 2 = 0.8$, so second bit after the point will be 0;
- $0.8 \cdot 2 = 1.6$, so third bit after the point will be 1;
- $0.6 \cdot 2 = 1.2$, so fourth bit after the point will be 1;
- $0.2 \cdot 2 = 0.4$, so fifth bit after the point will be 0, but we have done exactly this calculation earlier, so from now on the bits will repeat.

Once again we conclude that $6/5 = (1.\dot{0}01\dot{1})_2$.

3.7*. Write the number π in base 3, with 10 digits after the point. Use a pocket calculator and start with typing in the approximation 3.1415926 for π.

Solution: Note that the given approximation for π is correct up to an error of less than 10^{-7}. Because this is less than 3^{-14}, we have quite a safety margin to find 10 correct digits after the point in base 3. [1] We should separately deal with the integral part 3 of π, which reads $(10)_3$ in base 3. Then we start with the fractional part 0.1415926 in our calculator, keep multiplying by three and note down each time the remainder of dividing the resulting integral part by 3. We will find that $\pi = (10.0102110122\ldots)_3$. [2]

3.8*. The positive numbers $a_1, a_2, \ldots, a_{12}$ form a geometric progression, with $a_1 = \sqrt{2}/2$ and $a_3 = \sqrt{2}$. Compute $a_1 + a_2 + a_3 + \cdots + a_{12}$, expressing the result in the simplest possible way. (Also, use the theory, do not just add together all the terms of the sum.)

Solution: Because $\sqrt{2} = a_3 = a_1 \cdot r^2 = r^2\sqrt{2}/2$ we have $r^2 = 2$. Since all terms of the progression are positive we must have $r = \sqrt{2}$. Now

$$a_1 + a_2 + a_3 + \cdots + a_{12} = a_1 \frac{r^{12} - 1}{r - 1} = \frac{\sqrt{2}}{2} \frac{(\sqrt{2})^{12} - 1}{\sqrt{2} - 1} = \frac{\sqrt{2}}{2} \frac{63}{\sqrt{2} - 1} = \frac{63}{2}(2 + \sqrt{2}).$$

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[1] Of course an error would be still possible, but unlikely: if the digits of π in base three had a sequence of zeroes from the eleventh digit after the point up to the fourteenth digit and a couple more, then decreasing from π to its approximation 3.1415926 may cause a change in the tenth digit, and possibly even in previous digits if we had more zeroes to the left!

[2] Note that

$$(10.0102110122)_3 = \frac{(100102110122)_3}{3^{10}} = \frac{185507}{3^{10}} = 3.141577334\ldots$$

As you see, because 3^{10} is less than 10^5, ten correct digits after the point in base 3 give us less than five correct decimal digits after the point. If you have a more precise calculator and feel like checking how precise it is, here is π in base three with 20 digits after the point: $\pi = (10.01021101222201021100\ldots)_3$.