Solutions to Practicals

Solution to Problem 1.

$$A+B=\begin{bmatrix}1 & -1 & 1\\ 0 & 3 & 1\\ 1 & -2 & 5\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ -1 & 2 & 0\\ 3 & -2 & 3\end{bmatrix}=\begin{bmatrix}2 & -1 & 1\\ -1 & 5 & 1\\ 4 & -4 & 8\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & -1 & 1\\ 0 & 3 & 1\\ 1 & -2 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 2 & 0\\ 3 & -2 & 3\end{bmatrix}=\begin{bmatrix}5 & -4 & 3\\ 0 & 4 & 3\\ 18 & -14 & 15\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 0 & 0\\ -1 & 2 & 0\\ 3 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 0 & 3 & 1\\ 1 & -2 & 5\end{bmatrix}=\begin{bmatrix}1 & -1 & 1\\ -1 & 7 & 1\\ 6 & -15 & 16\end{bmatrix}$$

The matrices A, B do not commute since $AB \neq BA$.

Solution to Problem 2. The matrix

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

is: (a) a diagonal matrix when $b=c=0$,
(b) a symmetric matrix when $b=c$,
(c) an upper triangular matrix when $c=0$
(d) a skew-symmetric matrix when $A^T=-A$, or equivalently when

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix}=-\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

which implies that $a=-a, c=-b$ and $d=-d$. Therefore, $a=d=0$, $c=-b$, and the skew-symmetric matrix takes the form:

$$A=\begin{bmatrix}0 & b\\ -b & 0\end{bmatrix}$$

Solution to Problem 3.
✏ Recall that: *The rank of a matrix is the number of nonzero rows in the (reduced) row-echelon form.*
(a) A homogeneous system $A\mathbf{x}=\mathbf{0}$ with 3 equations and 3 unknowns admits as unique solution the zero solution, $x=0, y=0, z=0$. Its reduced row-echelon form then is:

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

and therefore the rank of the matrix A is $\text{rank}(A) = 3$.

(b) When the homogeneous system $A\mathbf{x} = \mathbf{0}$ admits infinite solutions of the form $x = 2z, y = -z$, z in $\mathbb{R}$, its reduced row-echelon form is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the rank of the matrix A is $\text{rank}(A) = 2$.

Solution to Problem 4.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1/2 \end{bmatrix} \underset{-2R_2}{\longrightarrow} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \underset{R_2 - R1}{\longrightarrow} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the row-echelon form for the matrix A, so $\text{rank}(A) = 1$.

$$A^2 = AA = \begin{bmatrix} 2 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 5 & -\frac{5}{2} \\ -\frac{5}{2} & \frac{5}{4} \end{bmatrix} \underset{-2R_2}{\longrightarrow} \begin{bmatrix} 5 & -\frac{5}{2} \\ 5 & -\frac{5}{2} \end{bmatrix} \underset{R_2 - R1}{\longrightarrow} \begin{bmatrix} 5 & -\frac{5}{2} \\ 0 & 0 \end{bmatrix}$$

that is the row-echelon form for A^2, consequently $\text{rank}(A^2) = 1$.

The matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is already in row-echelon form, so $\text{rank}(B) = 2$.

$$B^2 = BB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

that is the reduced row-echelon form for B^2, consequently $\text{rank}(B^2) = 2$.

Solution to Problem 5.

It is

$$(A - B)(A + B) = A(A + B) - B(A + B) = A^2 + AB - BA - B^2 \neq A^2 - B^2 \quad ,$$

since the matrices A and B do not necessarily commute.

Solution to Problem 6.

The vectors $\mathbf{u}, \mathbf{v}$ are two linearly independent vectors in $\mathbb{R}^2$. This means that if $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$, then the homogeneous system has a unique solution in terms of c_1 and c_2, namely the zero solution $c_1 = c_2 = 0$.

The vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$, or the homogeneous system $A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, correspond to the reduced row-echelon form:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

consequently $\operatorname{rank}(A) = 2$.