Algebra – Practical session 5

5.1. Simplify the fraction
$$\frac{4x^3 - 3x + 9}{2x^3 + x^2 - 3x + 6},$$
by finding the greatest common divisor of numerator and denominator by means of the Euclidean algorithm, and then dividing both numerator and denominator by it.

5.2. Find polynomials $u(x)$ and $v(x)$ such that $(2x^2 + x + 3) \cdot u(x) + (x^2 + 1) \cdot v(x) = 1$.
(Apply Euclid's algorithm to $f(x) = 2x^2 + x + 3$ and $g(x) = x^2 + 1$, check that their gcd is 1 (otherwise such $u(x)$ and $v(x)$ would not exist), and then do the extended part of the algorithm to find $u(x)$ and $v(x)$.)

5.3. Find polynomials $u(x)$ and $v(x)$ such that $(x^3 + x + 1) \cdot u(x) + (x^2 - x - 1) \cdot v(x) = 1$.

5.4. Factorise the integer $10^6 - 3^6 = 999271$ into a product of primes. (You will need to show that the factors which you find are actually primes.)

5.5. Consider the polynomial $f(x) = x^4 + 2x^3 - 6x^2 + 8x + 80$.

(a) Check that $2 + 2i$ is a root of $f(x)$. (Use Ruffini's rule.)

(b) Using the information in part (a), find the remaining complex roots of $f(x)$, and write the complete factorisation of $f(x)$ in $\mathbb{C}[x]$.

5.6. Use Ruffini's rule or other means to compute the quotient and the remainder of dividing $x^{100} + 2x^6 + 1$ by $x - 1$.
Note: You need not write down individually ALL the terms of the quotient, but you must make it clear that you have found them, and how. (For example, make appropriate use of dots in your calculation and in your final answer.)

5.7. Find the unique polynomial $f(x)$ of degree at most 3 such that
$$f(-1) = -7, \qquad f(0) = -3, \qquad f(1) = -3, \qquad f(2) = -1.$$

5.8*. Consider a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$.

(a) Show that the sum of all coefficients of even powers of x in $f(x)$ (which means the coefficients of $x^0 = 1$, x^2, x^4, etc.) equals $\frac{1}{2}\big(f(1) + f(-1)\big)$.

(b) Find a similar expression for the sum of all coefficients of odd powers of x in $f(x)$.

5.9*. Compute the sum of the coefficients of the polynomial
$$(2x - 1)(3x^2 - 2)(4x^3 - 3)(5x^4 - 4)(6x^5 - 5)$$
without bringing it to normal form (that is, without explicitly computing the product).