Solutions of Practical 1

1) i) Let c_1 and c_2 scalars satisfying:

$$c_1 u + c_2 v = 0$$

Then $c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or equivalently,

$$\begin{bmatrix} 3c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that implies $\begin{cases} 3c_1 = 0 \\ c_1 = c_2 \end{cases}$, hence $c_1 = c_2 = 0$.

So u and v are linearly independent.

(ii) Same with (i).

Let c_1 and c_2 be scalars such that:

$$c_1 u + c_2 v = 0$$

Then $c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 2c_1 - c_2 \\ -c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which yields $\begin{cases} 2c_1 = c_2 \\ -c_1 - c_2 = 0 \end{cases}$, so $c_1 = c_2 = 0$.

Therefore, u and v are linearly independent.

2) Let c_1, c_2 scalars satisfying:

$$c_1 u + c_2 v = 0$$

Then,

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ c_1 + c_2 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is equivalent to the system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + c_2 x = 0 \end{cases}$$

or $\begin{cases} c_2 = -c_1 \\ c_1 - c_1 x = 0 \end{cases} \rightsquigarrow \begin{cases} c_2 = -c_1 \\ c_1(1-x) = 0 \end{cases}$

i) when $x=1$, $c_1 \in \mathbb{R}$. Since c_1, c_2 are non-zero the vectors are linearly dependent.

Indeed, for $x=1$, $u=v$.

ii) When $x \neq 1$, the system becomes

$$\begin{cases} c_2 = -c_1 \\ c_1 = 0 \end{cases} \rightarrow \quad c_1 = c_2 = 0$$

and the vectors are linearly independent.

3) To solve this problem, one way is to choose any non-zero vector and search for a second vector that is not parallel to the first one. Two independent vectors in $\mathbb{R}^2$ should span $\mathbb{R}^2$.

Let $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. By choosing $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ we find the linear combinations of u and v:

$$c_1 u + c_2 v = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} .$$

Setting $c_1 u + c_2 v = 0$ yields the system $\begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{cases}$ with solution $c_1 = c_2 = 0$.

Hence the vectors are independent.

To prove that u, v span $\mathbb{R}^2$, we need to show that any vector $w = \begin{bmatrix} x \\ y \end{bmatrix}$ is a linear combination of u and v:

$$w = d_1 u + d_2 v \quad , \quad d_1, d_2 \text{ scalars} .$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ d_1 + d_2 \end{bmatrix}$. Equivalently we get: $\begin{cases} d_1 - d_2 = x \\ d_1 + d_2 = y \end{cases}$

or $\begin{cases} 2d_1 = x+y & \text{(by adding them)} \\ 2d_2 = y-x & \text{(by subtracting them)} \end{cases}$

Hence, $d_1 = \frac{x+y}{2}$, $d_2 = \frac{y-x}{2}$ and since that system

has unique solution, we conclude that there are always scalars satisfying $w = d_1 u + d_2 v$, for every w in $\mathbb{R}^2$.

So $\quad \text{span}(u,v) = \mathbb{R}^2$.

4) Let c_1, c_2 scalars satisfying:

$$c_1 u + c_2 v = 0$$

Then,

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -c_1 + \frac{c_2}{3} \\ c_1 - \frac{c_2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the system $\begin{cases} -c_1 + \frac{c_2}{3} = 0 \\ c_1 - \frac{c_2}{3} = 0 \end{cases}$

we get two identical equations, giving $c_1 = \frac{c_2}{3}$.
c_1 and c_2 can take non-zero values, so u, v are linearly dependent. In fact, it is:

$$\frac{c_2}{3} u + c_2 v = 0, \quad \text{so} \quad \frac{u}{3} + v = 0 \quad \text{(verify)}$$

5) The vector u can be written as a linear combination of v and w, if there are scalars c_1 and c_2 such that:

$$u = c_1 v + c_2 w$$

or

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_1 + 4c_2 \end{bmatrix},$$

yielding:

$$\begin{cases} c_2 = 1 \\ 2c_1 + 4c_2 = 2 \end{cases} \rightsquigarrow \begin{cases} c_2 = 1 \\ c_1 = -1 \end{cases}$$

So, $u = -v + w$