Solutions to Practical 7

Solution to Problem 1. It is

$$B - \lambda I = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial of B is the determinant

$$\begin{aligned} \det(B - \lambda I) &= (1-\lambda)(4-\lambda) + 2 = 4 - \lambda - 4\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6 \\ &= \lambda^2 - (2+3)\lambda + 2 \cdot 3 = (\lambda - 2)(\lambda - 3) \end{aligned}$$

(this was a little trick for finding the roots, you can try to find them as usual...)
So, $\det(B - \lambda I) = 0$ implies that $(\lambda - 2)(\lambda - 3) = 0$ and so B has two real eigenvalues, $\lambda_1 = 2, \lambda_2 = 3$.

The eigenvector $\mathbf{x}_1$ associated with the eigenvalue $\lambda_1 = 2$, can be found by solving the system

$$(B - \lambda_1 I)\mathbf{x}_1 = \mathbf{0}$$

$$(B - \lambda_1 I)\mathbf{x}_1 = (B - 2I)\mathbf{x}_1 = \begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 - y_1 \\ 2x_1 + 2y_1 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} -x_1 - y_1 \\ 2x_1 + 2y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the system reduces to one equation $x_1 + y_1 = 0$. So, its eigenspace is:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_1 = -x_1,\ x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = \mathit{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

The first eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The eigenvector $\mathbf{x}_2$ associated with the eigenvalue $\lambda_2 = 3$, can be found by solving the system

$$(B - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$$

$$(B - \lambda_2 I)\mathbf{x}_2 = (B - 3I)\mathbf{x}_2 = \begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2x_2 - y_2 \\ 2x_2 + y_2 \end{bmatrix}.$$

Setting

$$\begin{bmatrix} -2x_2 - y_2 \\ 2x_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the system reduces to one equation $2x_2 + y_2 = 0$. So, its eigenspace is:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_2 = -2x_2,\ x_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_2 \\ -2x_2 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

The second eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Determinant: $\det B = 4 - (-2) = 6$ and $\lambda_1\lambda_2 = 6$.

Solution to Problem 2. The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

has as characteristic polynomial the determinant

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} a - \lambda & b \\ b & a - \lambda \end{bmatrix} = (a - \lambda)^2 - b^2$$

Then $\det(A - \lambda I) = 0$ implies that $(a - \lambda)^2 - b^2 = 0$ or $(a - \lambda)^2 = b^2$ so, $a - \lambda = \pm b$ and A has two real eigenvalues, $\lambda_1 = a - b, \lambda_2 = a + b$.

Solution to Problem 3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It is

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 0 & 1 \\ 0 & 1 - \lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

The characteristic polynomial of A is the determinant

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 - \lambda \\ 1 & 1 \end{vmatrix} \\ &= (1 - \lambda)\left(-(1 - \lambda)\lambda - 1\right) - (1 - \lambda) \\ &= (1 - \lambda)\left(-(1 - \lambda)\lambda - 1 - 1\right) \\ &= (1 - \lambda)\left(\lambda^2 - \lambda - 2\right) \\ &= (1 - \lambda)\left(\lambda^2 - (2 - 1)\lambda + 2(-1)\right) \\ &= (1 - \lambda)(\lambda - 2)(\lambda + 1) \end{aligned}$$

So, A has three real eigenvalues, $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$.

- The eigenvector $\mathbf{x}_1$ associated with the eigenvalue $\lambda_1 = 1$, can be found by solving the system

$$(A - \lambda_1 I)\mathbf{x}_1 = \mathbf{0}$$

$$(A - \lambda_1 I)\mathbf{x}_1 = (A - I)\mathbf{x}_1 = \begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 1 \\ 1 & 1 & 0-1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_1 \\ x_1 + y_1 - z_1 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} z_1 \\ z_1 \\ x_1 + y_1 - z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the system reduces to two equations $z_1 = 0$ and $x_1 + y_1 = 0$. So, its eigenspace is:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ in } \mathbb{R}^3 : z_1 = 0, y_1 = -x_1,\ x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = span \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

The first eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

- The eigenvector $\mathbf{x}_2$ associated with the eigenvalue $\lambda_2 = 2$, can be found by solving the system

$$(A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = (A - 2I)\mathbf{x}_2 = \begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} -x_2 + z_2 \\ -y_2 + z_2 \\ x_2 + y_2 - 2z_2 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} -x_2 + z_2 \\ -y_2 + z_2 \\ x_2 + y_2 - 2z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using Gauss–Jordan elimination we find:

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_1 + R_2 + R_3]{} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which yields that $-x_2 + z_2 = 0$ and $-y_2 + z_2 = 0$. So, the eigenspace is:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ in } \mathbb{R}^3 : x_2 = z_2, y_2 = z_2,\ z_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} z_2 \\ z_2 \\ z_2 \end{bmatrix} : z_2 \text{ in } \mathbb{R} \right\} = span \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

and the second eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- The eigenvector $\mathbf{x}_3$ associated with the eigenvalue $\lambda_3 = -1$, can be found by solving the system

$$(A - \lambda_3 I)\mathbf{x}_3 = \mathbf{0}$$

$$(A - \lambda_3 I)\mathbf{x}_3 = (A + I)\mathbf{x}_3 = \begin{bmatrix} 1+1 & 0 & 1 \\ 0 & 1+1 & 1 \\ 1 & 1 & 0+1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2x_3 + z_3 \\ 2y_3 + z_3 \\ x_3 + y_3 + z_3 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} 2x_3 + z_3 \\ 2y_3 + z_3 \\ x_3 + y_3 + z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using Gauss–Jordan elimination we find:

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1/2 \\ R_2/2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \longrightarrow \\ -R_1 - R_2 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which yields that $x_3 + 1/2z_3 = 0$ and $y_3 + 1/2z_3 = 0$. So, the eigenspace is:

$$V_{\lambda_3} = \left\{ \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \text{ in } \mathbb{R}^3 : x_3 = -1/2z_3, y_3 = -1/2z_3,\ z_3 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} -1/2z_3 \\ -1/2z_3 \\ z_3 \end{bmatrix} : z_3 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right)$$

The third eigenvector is

$$\mathbf{x}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

So, we produce the matrix P by introducing to its columns the components of the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$P = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix}$$

and we get:

$$P = \begin{bmatrix} 1 & 1 & -1/2 \\ -1 & 1 & -1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

with inverse

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

We can now diagonalise the matrix A. It is:

$$D = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1/2 \\ -1 & 1 & -1/2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution to Problem 4. Let

$$B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

We find the eigenvalues of B. It is

$$B - \lambda I = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}$$

The characteristic polynomial of B is the determinant

$$\det(B - \lambda I) \quad = \quad -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = \lambda^2 - (2-1)\lambda + 2(-1) = (\lambda - 2)(\lambda + 1)$$

So, $\det(B - \lambda I) = 0$ implies that $(\lambda - 2)(\lambda + 1) = 0$ and so B has two real eigenvalues, $\lambda_1 = 2, \lambda_2 = -1$.

Since the matrix B has two real **non-equal** eigenvalues, B is diagonalisable. Its diagonal form will have in the diagonal the eigenvalues:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} .$$

Diagonalisable means that there exist a matrix P, such that

$$B = PDP^{-1}$$

To find the matrix P, we need to find the eigevectors associated with $\lambda_1 = 2, \lambda_2 = -1$.

- The eigenvector $\mathbf{x}_1$ associated with the eigenvalue $\lambda_1 = 2$, can be found by solving the system

$$(B - \lambda_1 I)\mathbf{x}_1 = \mathbf{0}$$

$$(B - \lambda_1 I)\mathbf{x}_1 = (B - 2I)\mathbf{x}_1 = \begin{bmatrix} 0-2 & 2 \\ 1 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -2x_1 + 2y_1 \\ x_1 - y_1 \end{bmatrix} .$$

Consequently,

$$\begin{bmatrix} -2x_1 + 2y_1 \\ x_1 - y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the system reduces to one equation $x_1 - y_1 = 0$. So, its eigenspace is:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_1 = x_1,\ x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

The first eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- The eigenvector $\mathbf{x}_2$ associated with the eigenvalue $\lambda_2 = -1$, can be found by solving the system

$$(B - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$$

$$(B - \lambda_2 I)\mathbf{x}_2 = (B + I)\mathbf{x}_2 = \begin{bmatrix} 0+1 & 2 \\ 1 & 1+1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 + 2y_2 \\ x_2 + 2y_2 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} x_2 + 2y_2 \\ x_2 + 2y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the system reduces to one equation $x_2 + 2y_2 = 0$. So, its eigenspace is:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ in } \mathbb{R}^2 : x_2 = -2y_2,\ y_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} -2y_2 \\ y_2 \end{bmatrix} : y_2 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

The second eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So, the matrix P is

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

It is then:

$$\begin{aligned} B^n &= (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})(PDP^{-1}) \\ &= PDP^{-1}PDP^{-1}\ldots PDP^{-1}PDP^{-1} = P\underbrace{DD\ldots D}_{n}P^{-1} \\ &= PD^nP^{-1} = P\begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^{n+1} - 2(-1)^n \\ 2^n - (-1)^n & 2^{n+1} + (-1)^n \end{bmatrix} \end{aligned}$$

Solution to Problem 5.

(a) Here it is:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - (-1) = (1-\lambda)^2 + 1\,.$$

The condition $\det(A-\lambda I)=0$ implies that $(1-\lambda)^2+1=0$, so $(\lambda-1)^2=-1$. Consequently, we have that $\lambda_1-1=i$ and $\lambda_2-1=-i$, or $\lambda_{1,2}=1\pm i$. Since the matrix A has two complex conjugate eigenvalues, it cannot be diagonalised over $\mathbb{R}$.

(b) We know that the matrix

$$B=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$$

has two equal eigenvalues $\lambda_{1,2}=\lambda=2$.

• The algebraic multiplicity of λ is 2, because we have two equal eigenvalues.

We follow the standard procedure to find the eigenspace of $\lambda=2$. We solve the system

$$(B-2I)\mathbf{x}=\mathbf{0}$$

$$(\begin{bmatrix}2-2 & -1\\ 0 & 2-2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-y\\ 0\end{bmatrix}.$$

Consequently,

$$\begin{bmatrix}-y\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

yields that $y=0$. So, its eigenspace is:

$$V_\lambda=\Big\{\begin{bmatrix}x\\ y\end{bmatrix}\text{ in }\mathbb{R}^2: y=0,\ x\text{ in }\mathbb{R}\Big\}=\Big\{\begin{bmatrix}x\\ 0\end{bmatrix}: x\text{ in }\mathbb{R}\Big\}=span\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right)$$

· The geometric multiplicity of λ is 1 because $\dim V_\lambda=1$.

Since the geometric multiplicity of λ is less than its algebraic multiplicity, the matrix B is non-diagonalisable