Classical Mechanics

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What is Classical Mechanics?

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- The physical systems governed by classical mechanics are deterministic, i.e if the present state of the system is known then all the future and past states of the system can be completely determined.

What is Classical Mechanics?

We may regard the present state of the universe as the effect of its past and the cause of its future.

"An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

— *Pierre Simon Laplace, A Philosophical Essay on Probabilities*

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Is nature predictable?

Classical Mechanics

A naive story:

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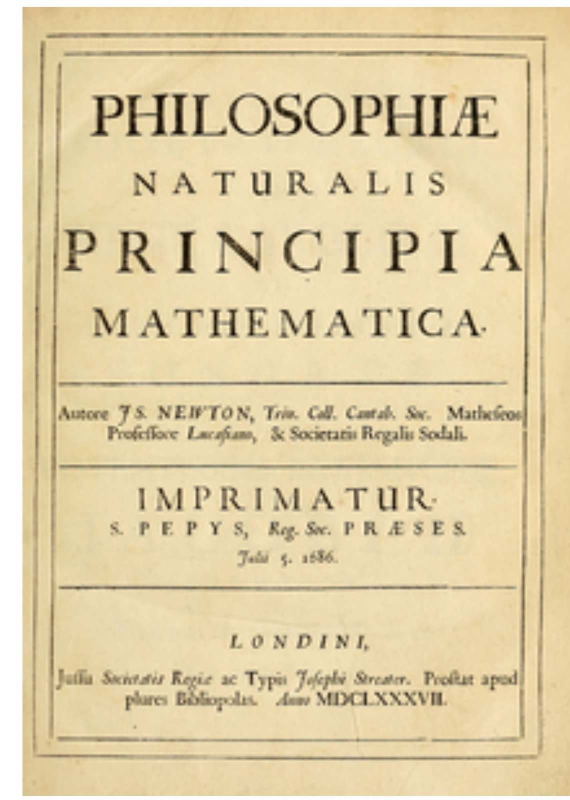

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Classical Mechanics

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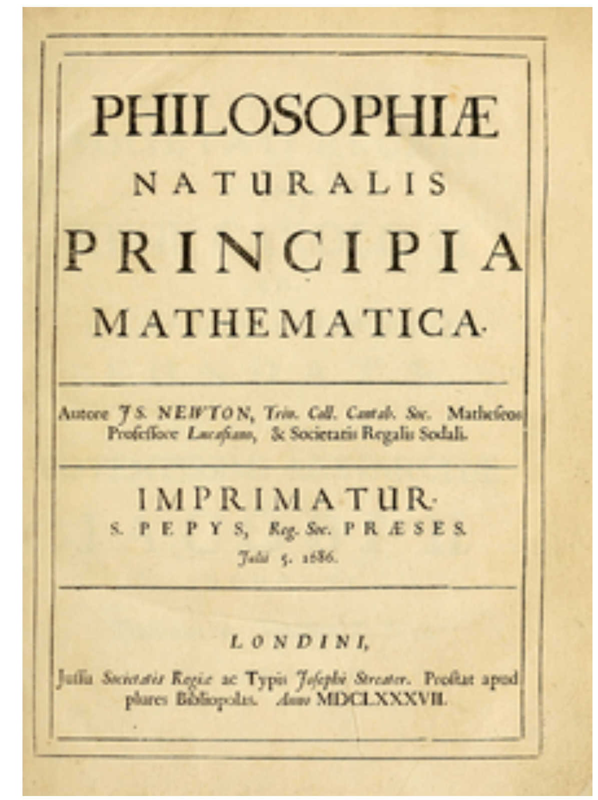

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A more accurate story:

Aristotle, Archimedes, Copernicus, Kepler, Galileo, Huygens, **Newton**, Leibniz, Euler, Lagrange, Laplace, Hamilton, Kovalevskaya, Noether, Kolmogorov,… **and many others!**

The ABCs of Classical Mechanics

Motion

- ***Motion*** is the change of the position of an object in *space* as *time* passes

- The concept of motion, which lies at the heart of classical mechanics, requires the concepts of **space** and **time**.

- In classical mechanics, space and time are ***primitive*** concepts: their meaning is considered self-evident and they do not require a definition.

Not requiring a definition of space and time does not mean that they are easy to apprehend.

The two pillars of classical mechanics

Classical mechanics is supported by two complementary sub-disciplines: *kinematics* and *dynamics*

Kinematics describes the motion of objects without considering the forces that cause them to move (*description of motion*).

Dynamics concerns the study of forces and their effects on motion (*explanation of motion*).

Introduction to Kinematics

Describing position, time and motion

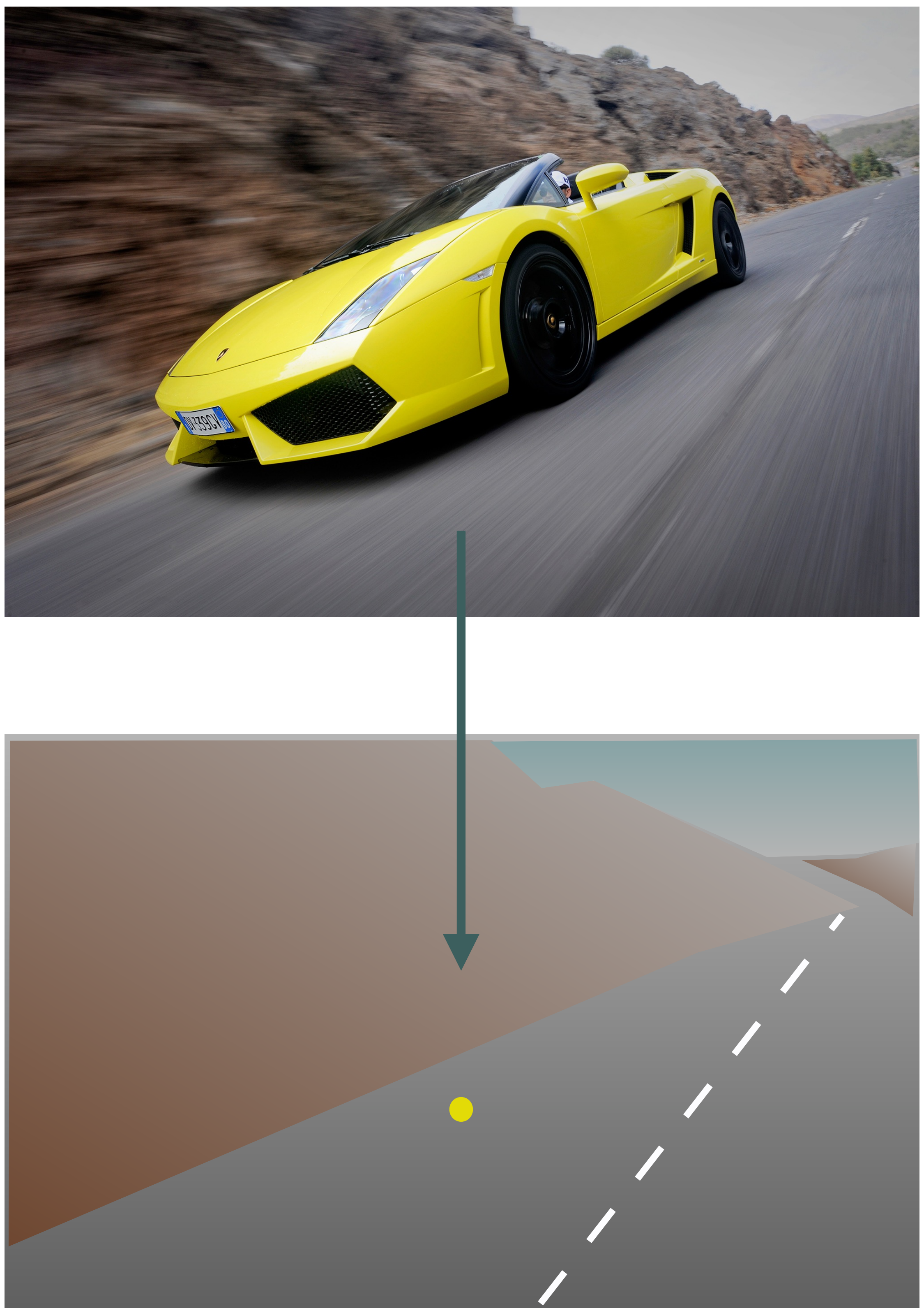
The path to abstraction

Not all details matter to characterise the motion of an “object”:

- Details **unrelated** to the description of motion like car colour, brand of the car, etc…

- Details **contributing** to motion but either too complex to be modelled or can be abstracted away for the motion under study; like pistons in the car engine, exhaust gases, shape of the car etc…

The path to abstraction



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Point Object

Trajectory

Once an object is abstracted as a point object it is often denoted by an upper case letter of the latin alphabet (A, B, C, etc…) representing the object.

For example, we could have (but it’s up to your imagination!):

- Car $\longrightarrow$ point C
- Plane $\longrightarrow$ point P
- Earth $\longrightarrow$ point E
- Sun $\longrightarrow$ point S

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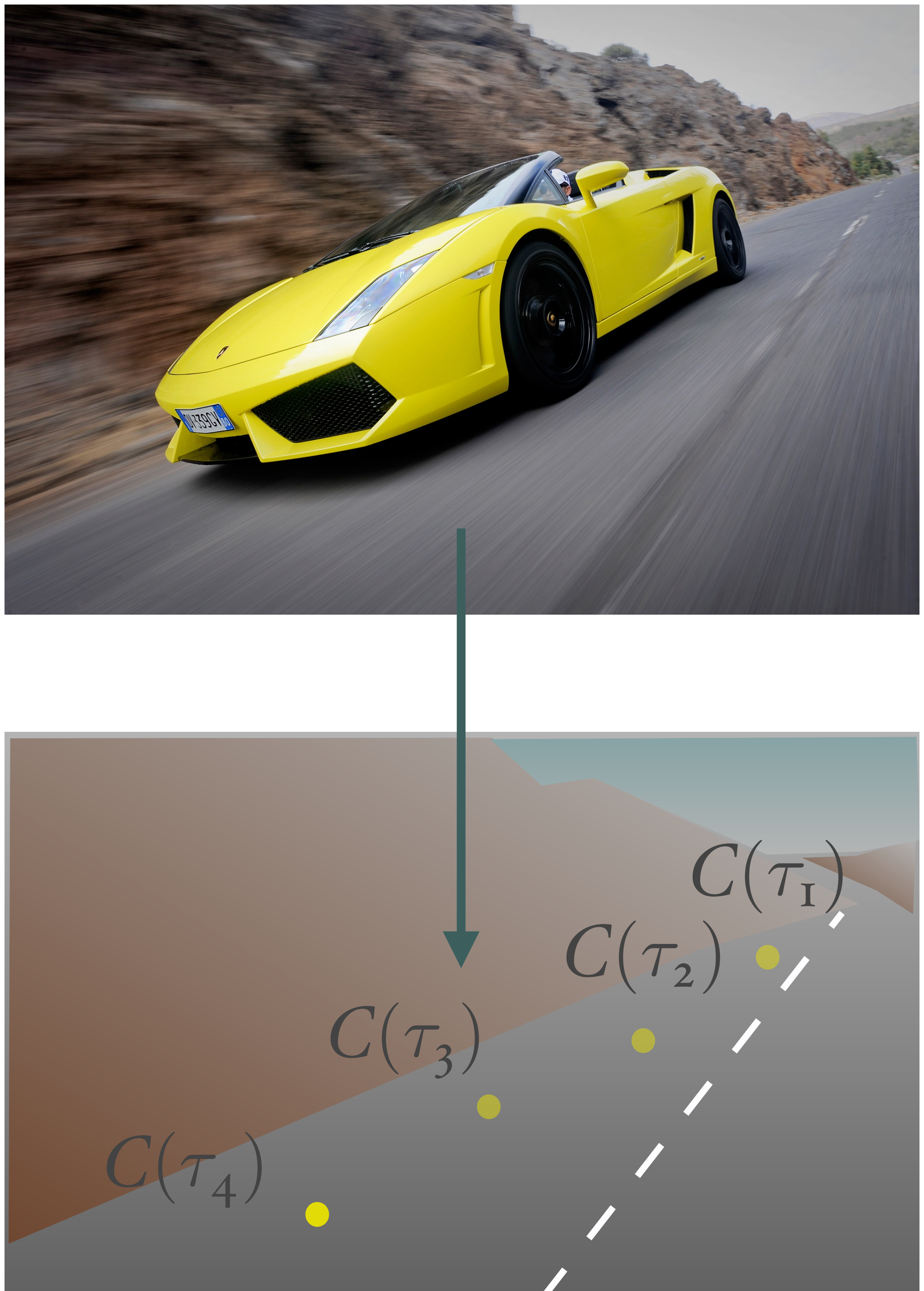
The succession of different places a point object occupies at different instants sorted in an ordered sequence $(\tau_1, \tau_2, \ldots)$ is simply denoted $(C(\tau_1), C(\tau_2), \ldots)$ and called the ***trajectory*** of the (point) object.

Trajectory



For example, in the case of the racing car, the point C representing the car occupies successively the positions

$C(\tau_1)$, $C(\tau_2)$, $C(\tau_3)$ and $C(\tau_4)$

Relative position in one dimension and time intervals

From place to (relative) position

If you were to live on an infinite straight line how would you answer a question like: “where are you?”

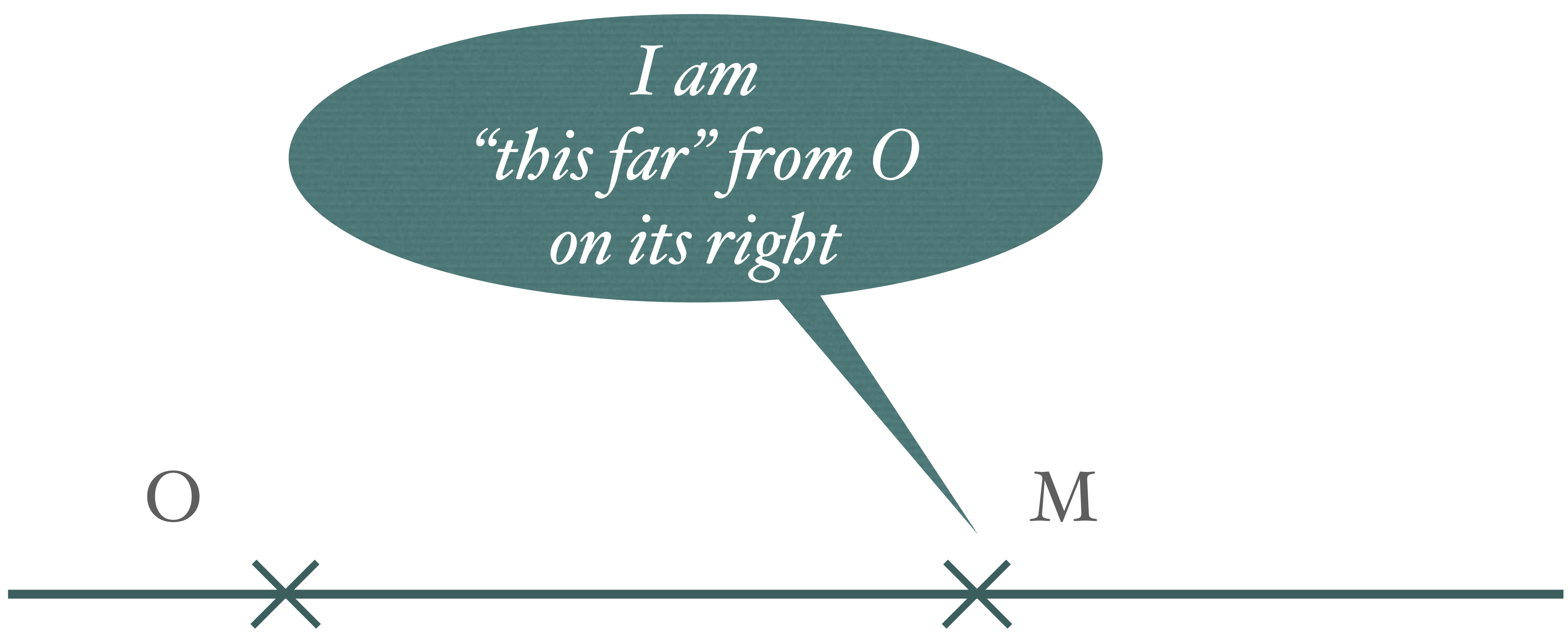
From place to (relative) position

If there is nothing around you, the most reasonable answer you can give is likely to be:

...but that would not tell much

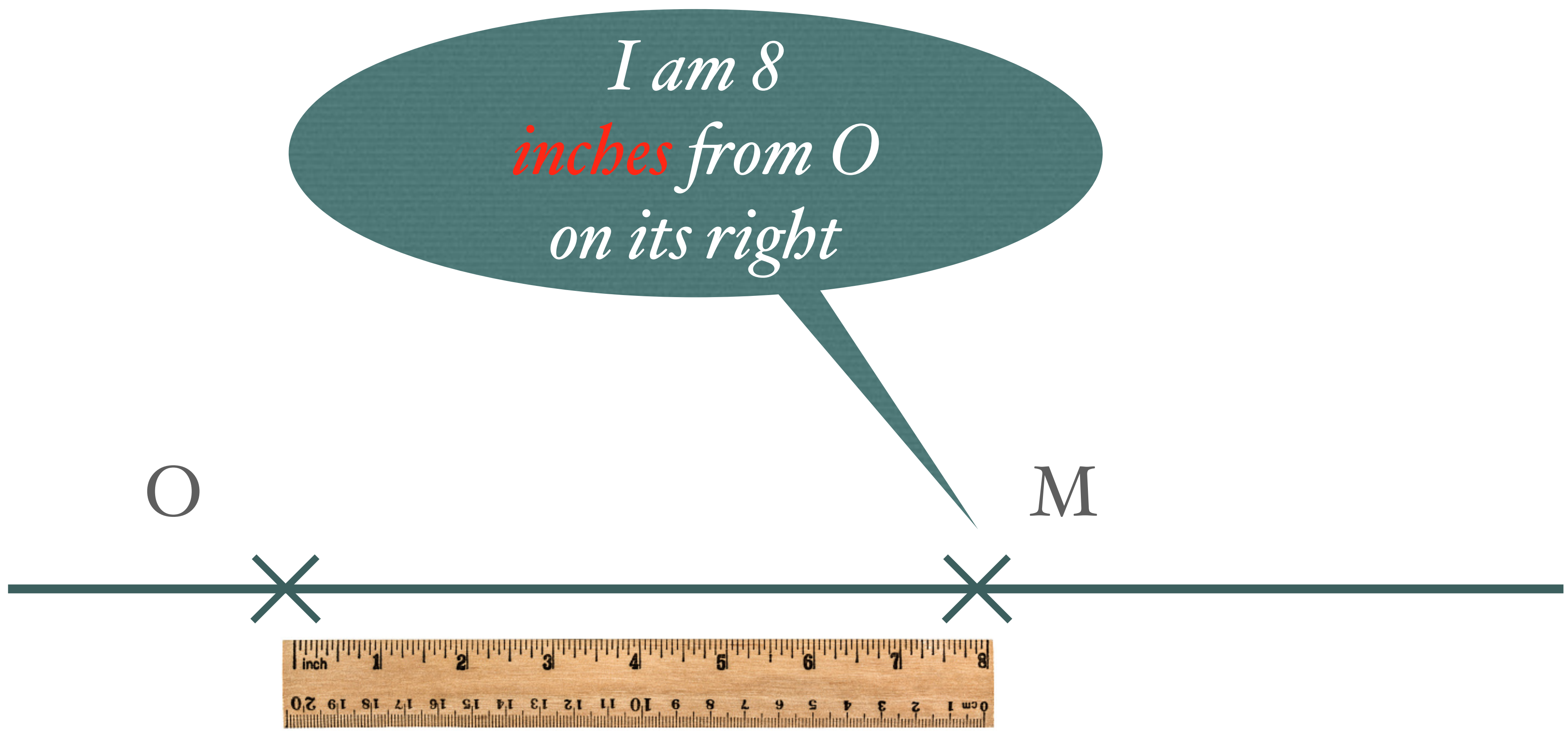
From place to (relative) position

If there is something around you, you could express your current location *relative* to another object:



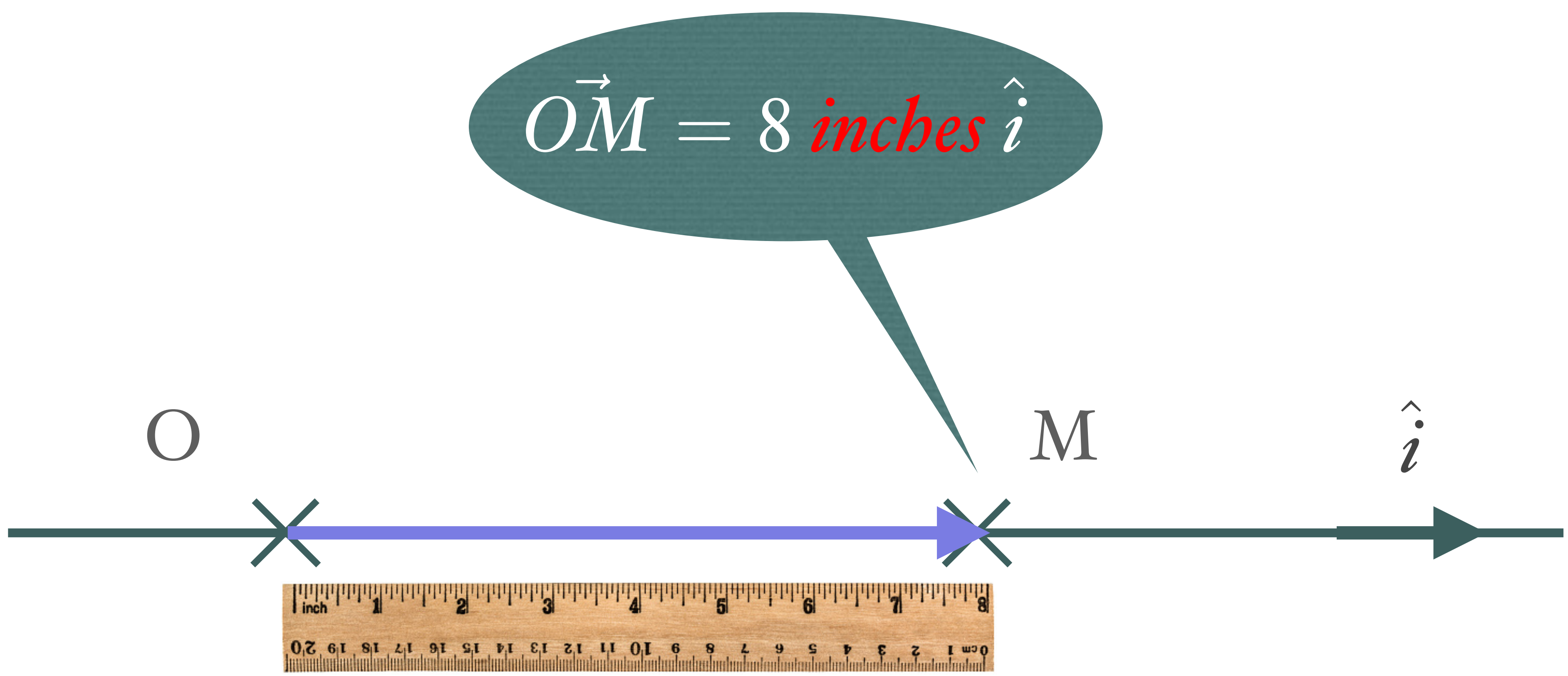
From place to (relative) position

If moreover you are given a standard of measure for lengths, i.e. a unit of length (such as inch, m, cm, feet, etc.) you could express your *relative position* more quantitatively:



From place to (relative) position

By choosing an orientation $\hat{i}$ to the line, you can state your relative position more formally as a ***position vector***:



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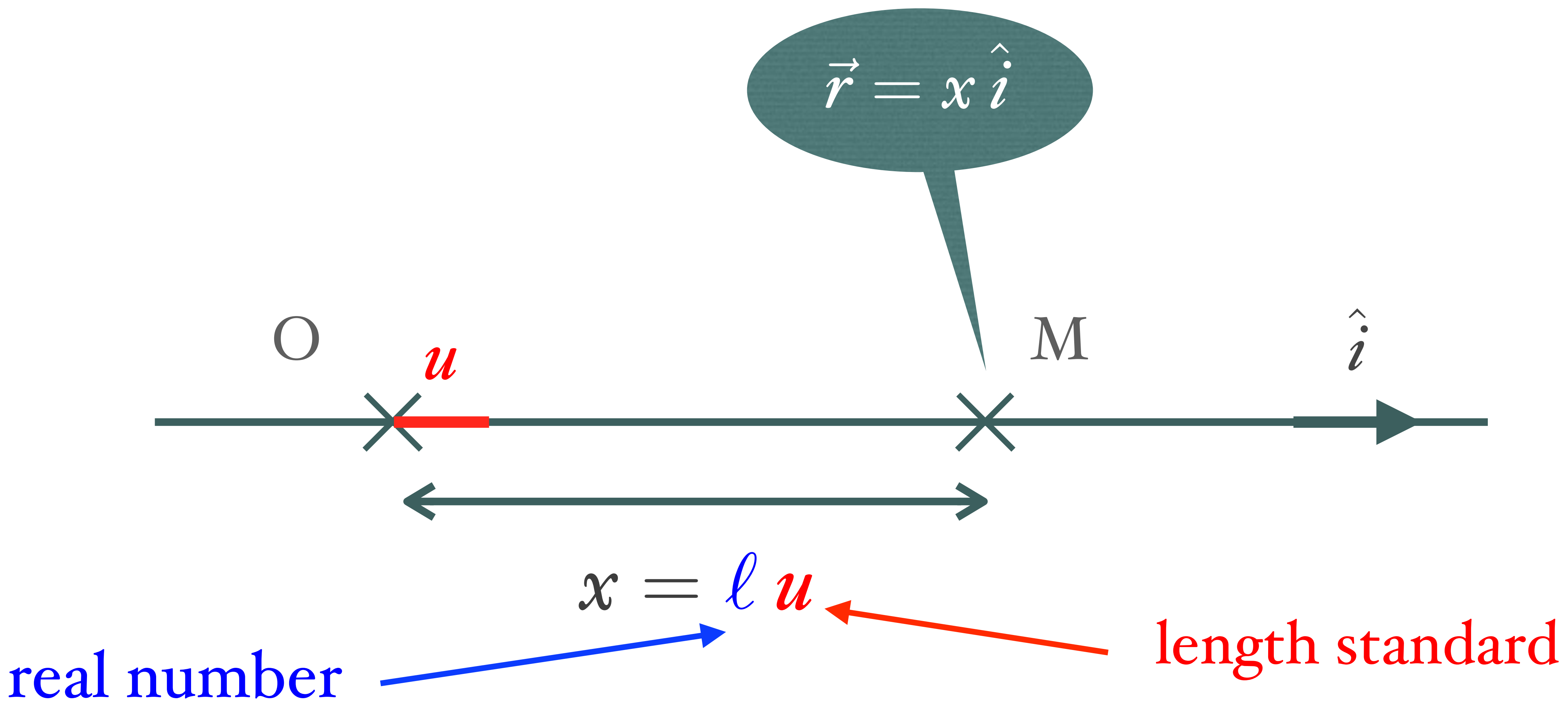

From instants to time intervals

What if you were asked the simple question “what instant is it?”
Is there an absolute way to answer to it?

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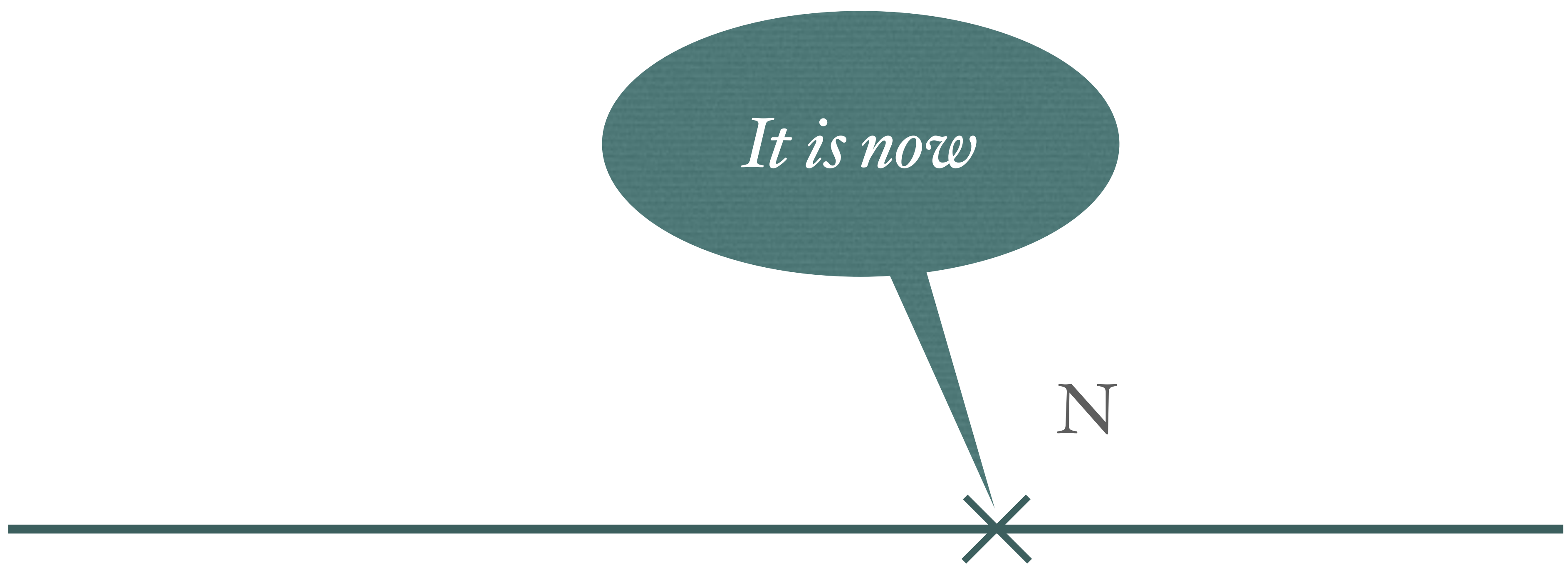
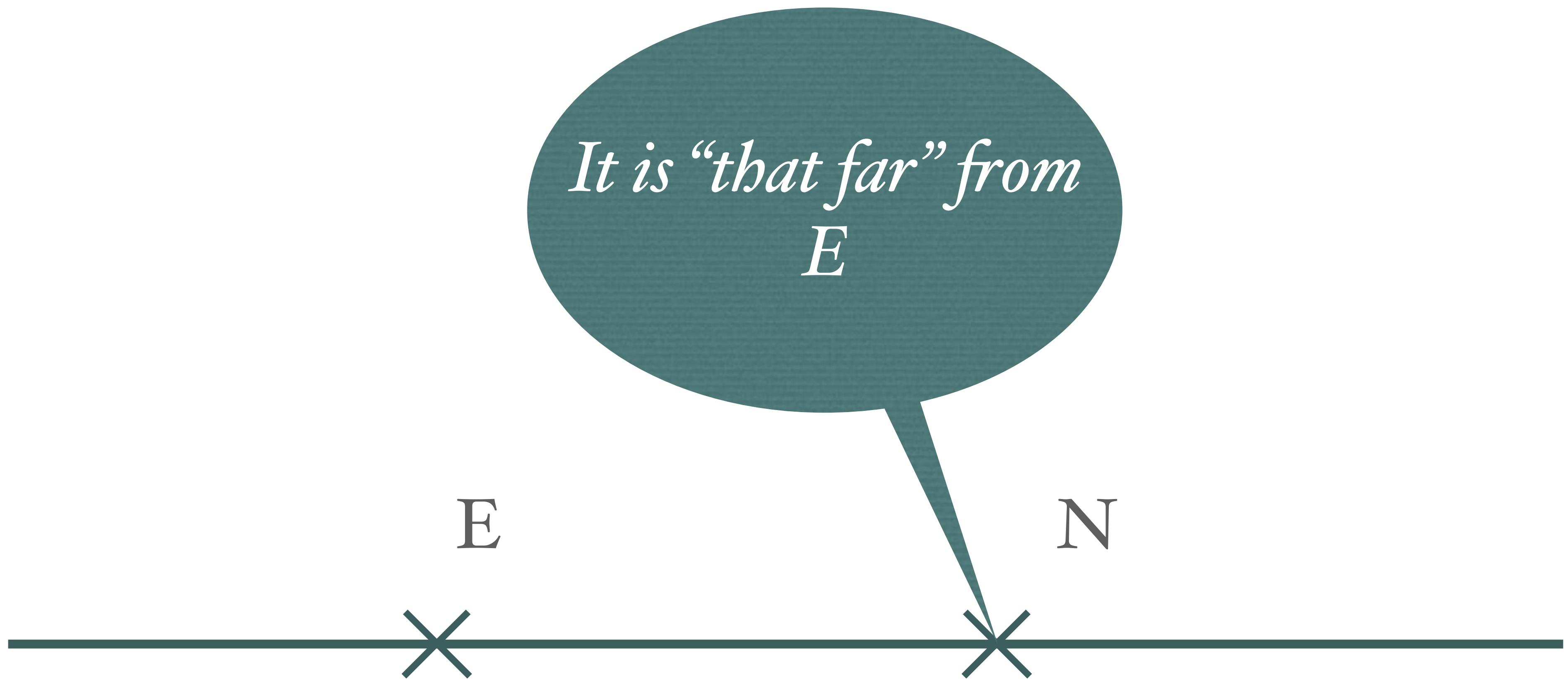


perfectly correct but sort of useless

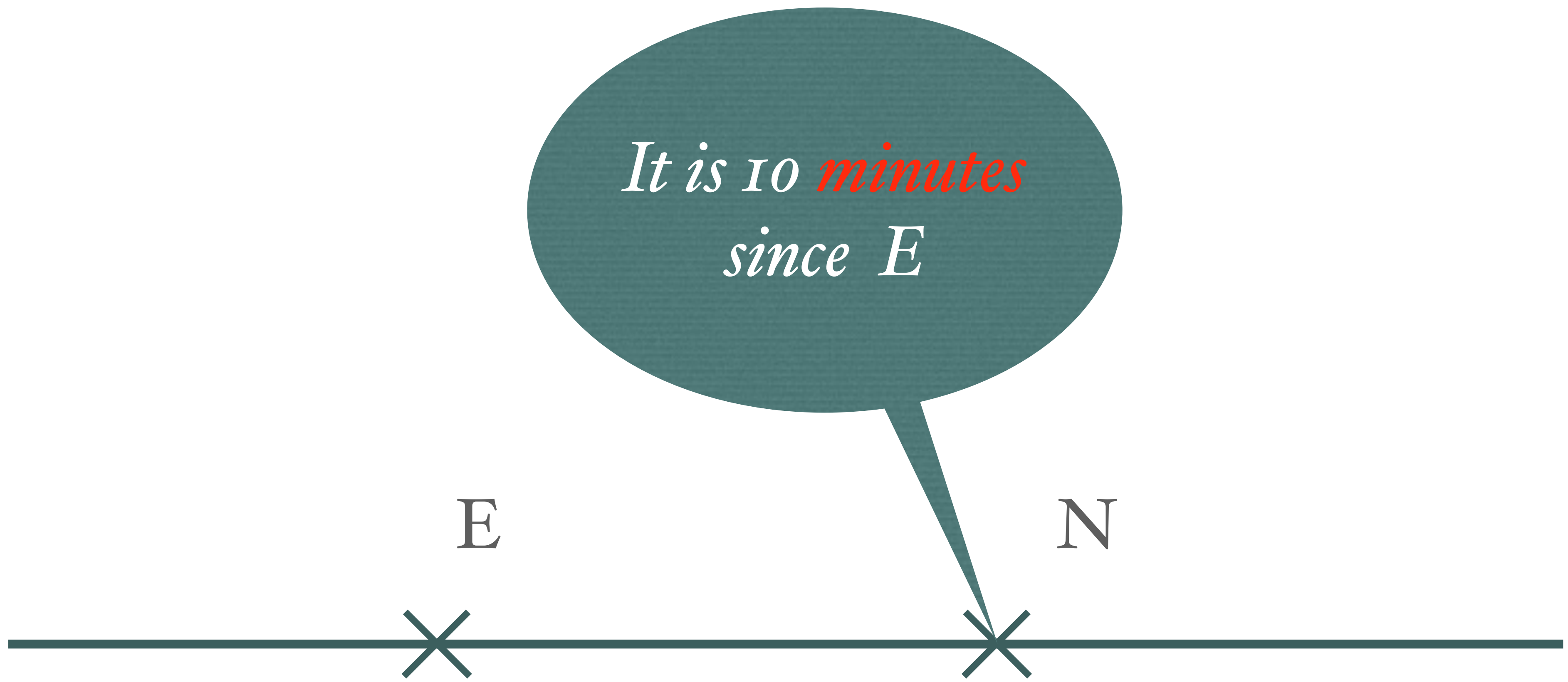
From instants to time intervals

If you have an event E of reference then you could say



From instants to time intervals

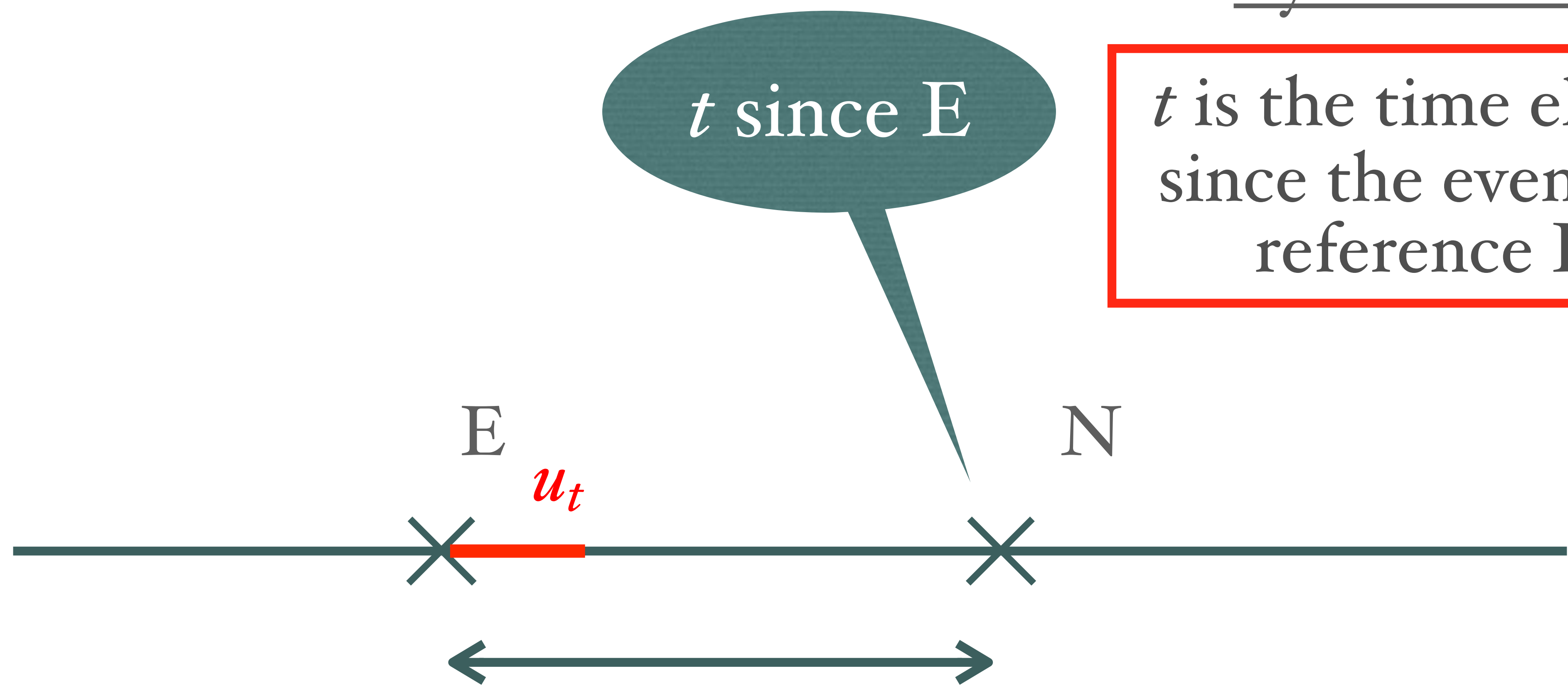
The "that far" must be measured with a standard for measuring the time elapsed between two events, i.e. a unit of time such as hours, seconds, minutes, years)



From instants to time intervals

By convention

t is the time elapsed since the event of reference E



$$t = e\, u_t$$

Real number

standard of elapsed time

Introduction to ‘physical dimension’

Physical Dimensions

We have seen that $x = \ell u$ *and* $t = e u_t$, where

- x denotes a relative position in a line and ℓ a real number related to a unit of length u (such as inch, m, km, feet)
- t denotes the time elapsed since an event of reference and e a real number related to a unit of time u_t (seconds, minutes, years)

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> In physics the word ***dimension*** denotes the physical nature of a quantity (represented by a given symbol).

1. The dimension of x, denoted $[x]$, is length and we write $[x] = L$
2. The dimension of t, denoted [t], is time and we write $[t] = T$

Similarly, we denote $[u] = L$ and $[u_t] = T$.

The real numbers ℓ and e are considered *dimensionless* and we write $[\ell] = [e] = 1$.

Dimensional Analysis

Dimensions can be treated as algebraic quantities.

Basic rules:

- Quantities can be added or subtracted only if they have the same dimensions and $[a+b]=[a]=[b]$ if $[a]=[b]$
- Numbers are dimensionless: $[a]=1$, for any number a
- The dimension of the product of two quantities is the product of their dimensions, i.e. $[ab]=[a][b]$
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Example If x denotes a relative position in $1D$ and t a time interval then

$$[\frac{2x}{t}]=\frac{[2][x]}{[t]}=\frac{1\cdot L}{T}=\frac{L}{T}$$

Notion of complete equation

A general (physics) equation reads:

Left hand side = Right hand side

<u>*Definition*</u>: An equation is said to be a ***Complete Equation*** if the dimension of its left hand side is equal to the dimension of its right hand side. The numerical values of both sides of a complete equation must be equal in all units that are multiple of the original units.

<u>Examples</u>

- The equation $x = t$ is not a complete equation
- The equation $x = (1km/h)t$ is a complete equation

Notion of complete equation

Physics theories preferentially use complete equations as they allow to relate physical concepts with each other and not just their numerical values in some units; in other words, complete equations carry **“meaning”** .

This is to be opposed to the very useful but less meaningful concept of **empirical equations** which are meant to work for practical situations but only in certain specific units.

Conversion of Units

Sometimes it is necessary to convert units from one measurement system to another or convert within a system (for example, from kilometers to meters).

- The **SI** (*International System of Units*) unit of length is the metre (m) and of time is the second (s).

Example

$$2km/h = 2\frac{1000m}{3600s} = 0.555m/s$$

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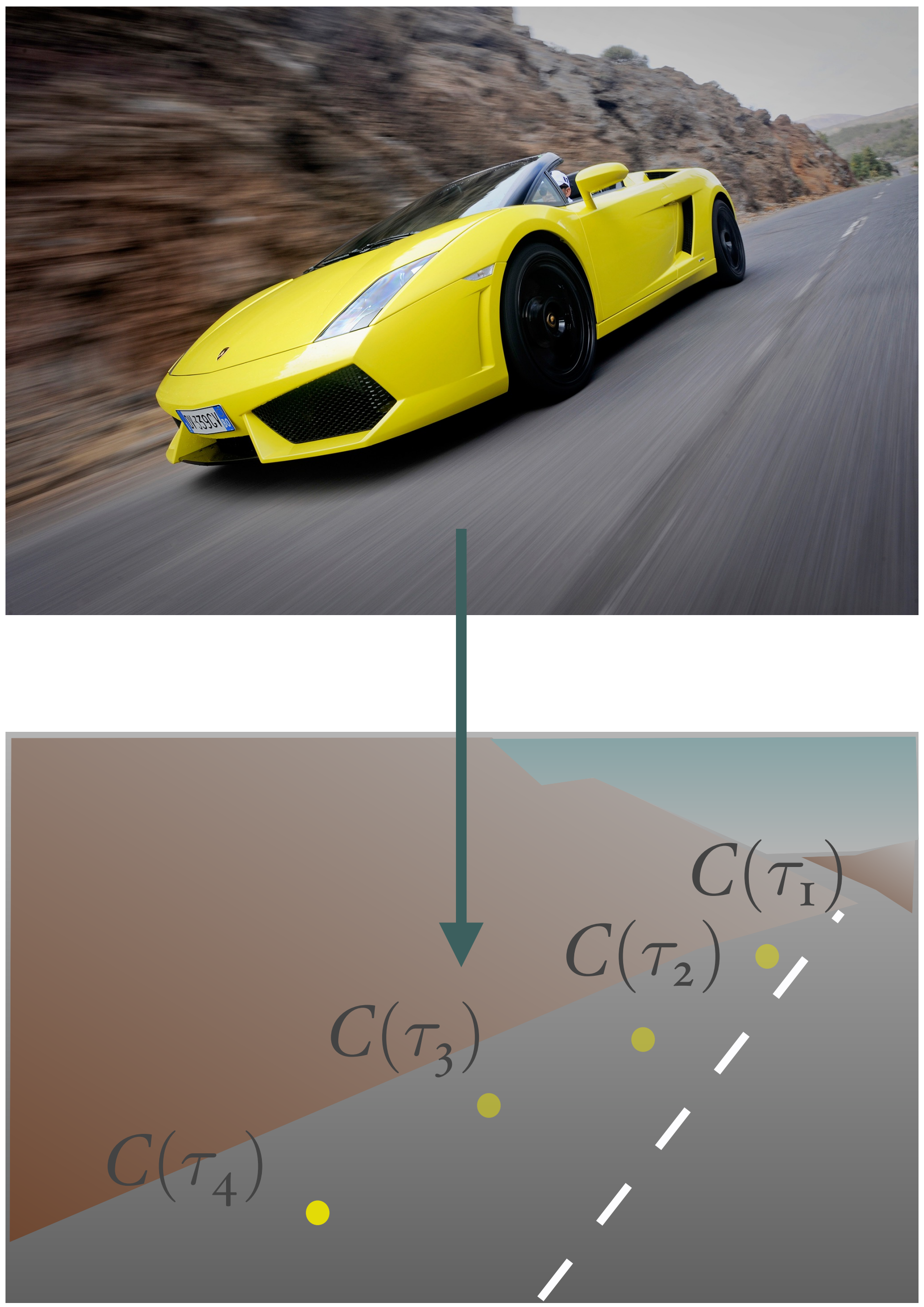
- Like dimensions, units can be treated as algebraic quantities that can cancel each other. By including the units in every step of a calculation, we can detect errors if the units for the answer turn out to be incorrect.

Kinematics in one linear dimension: tools for describing motion

From trajectory to equation

We have seen that a trajectory was a succession of places



$$C(\tau_1),\ C(\tau_2),\ C(\tau_3) \text{ and } C(\tau_4)$$

Given a point of origin they can be related to the vector positions

$$\vec{r}(\tau_1),\ \vec{r}(\tau_2),\ \vec{r}(\tau_3) \text{ and } \vec{r}(\tau_4)$$

Given a direction $\hat{i}$ in 1D, it simplifies even further into

$$x(\tau_1), x(\tau_2), x(\tau_3) \text{ and } x(\tau_4)$$

Given an instant of reference, we get

$$x(t_1), x(t_2), x(t_3) \text{ and } x(t_4)$$

From trajectory to equation

In general we can then characterise the trajectory with a mathematical *function* $x(t)$ such that when evaluated at individual time intervals it gives back the values $x(t_1), x(t_2), x(t_3)$ and $x(t_4)$

$x(t)$ is called the (explicit) ***equation of motion*** of the system and it is often assumed that this function is at least *continuous* and *differentiable* (this is where proficiency in calculus becomes useful).

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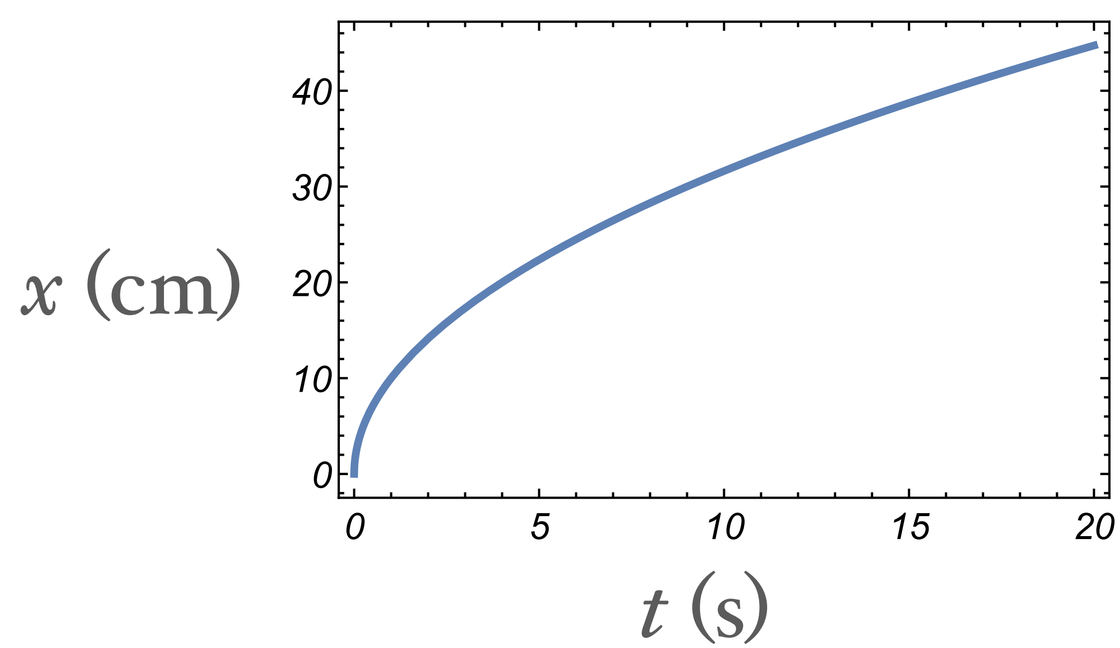
Example 1

Equation of motion

$$x(t) = 10\ cm \cdot s^{-1/2}\sqrt{t}$$

Graph

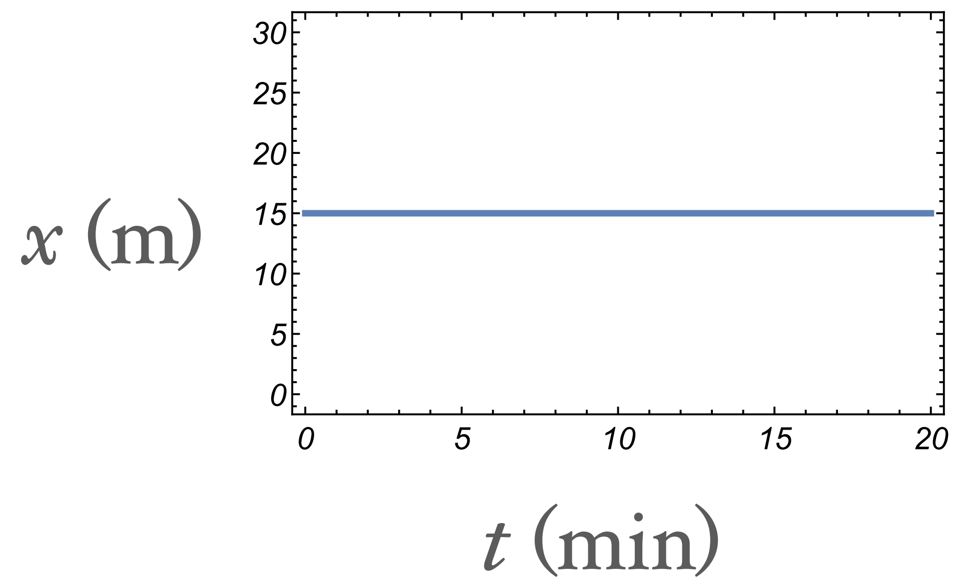


From trajectory to equation

<u>Example 2</u>

$x(t) = 15\ \mathbf{m}$

Graph

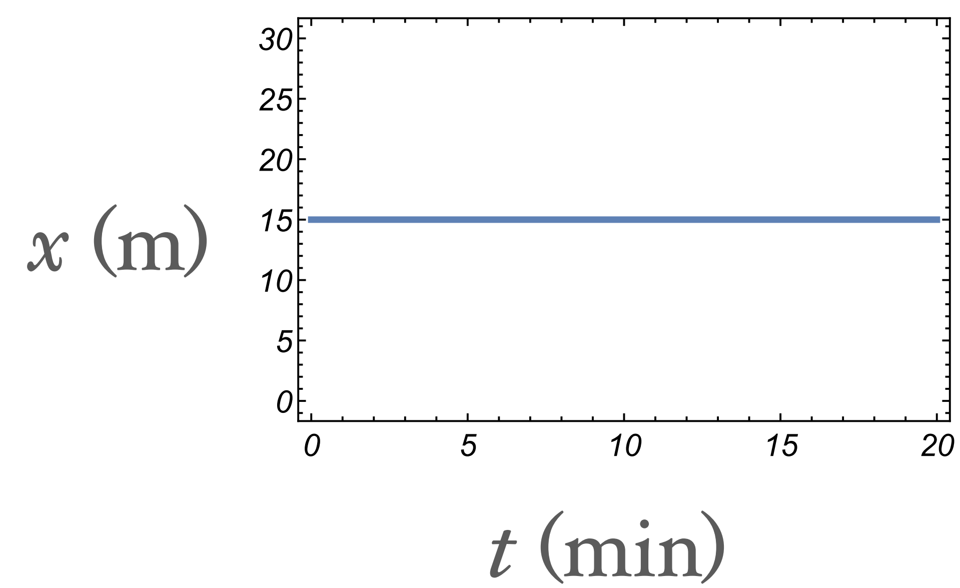


From trajectory to equation

Example 2

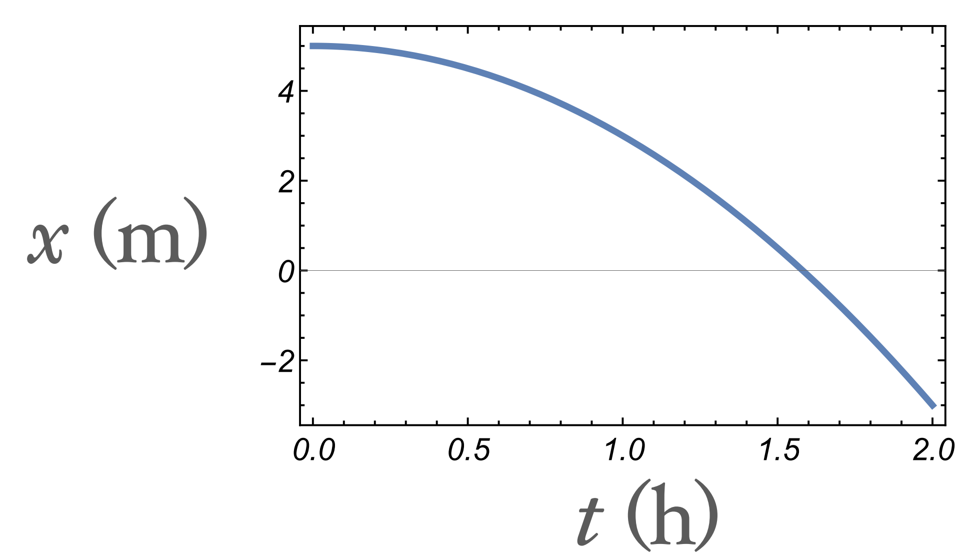
$$x(t) = \color{blue}{15}\ \color{red}{\mathbf{m}}$$

Graph



Example 3

$$x(t) = (\color{blue}{5}\ \color{red}{\mathrm{m}}) - (\color{blue}{2}\ \color{red}{\mathrm{m.h^{-2}}})t^2$$



Velocity

Average velocity and speed

- The ***average velocity*** $\upsilon_{x,avg}$ of an object between two times t_1 and t_2 is defined as the average rate of change of $x(t)$:

$$\upsilon_{x,avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

- The ***average speed*** v_{avg} of an object is defined as **the total distance** d traveled divided by the total time interval $t_2 - t_1$ required to travel that distance:

$$v_{avg} = \frac{d}{t_2 - t_1}$$

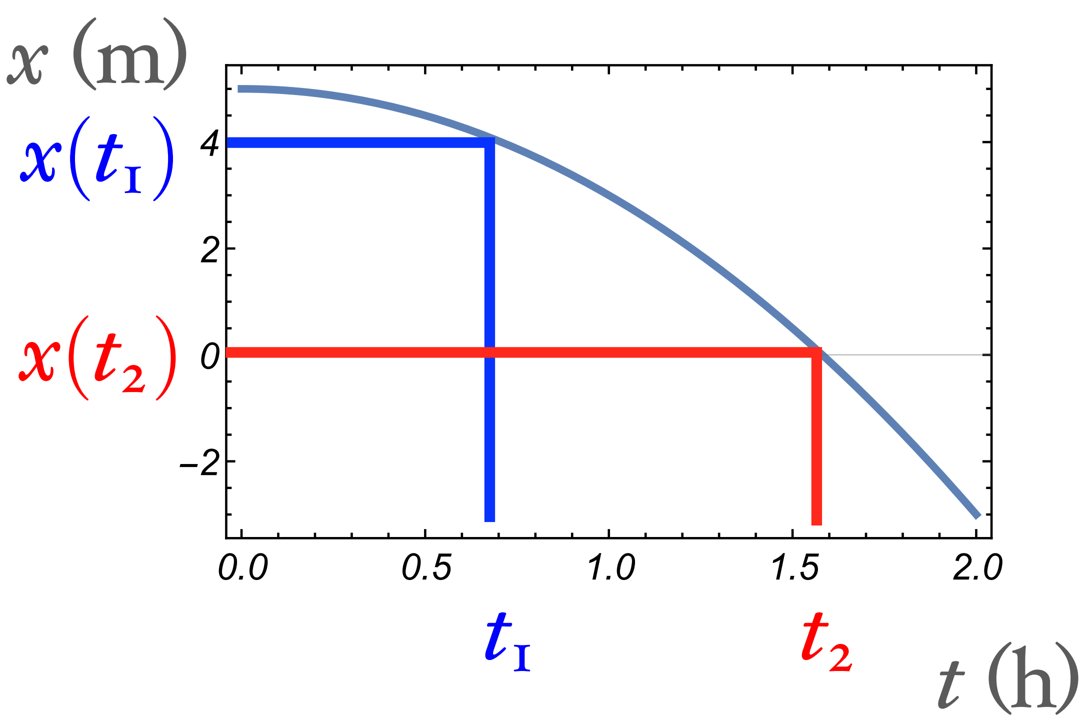
The average velocity and speed have dimensions of length divided by time (L/T) and meters per second in SI units.

Average velocity and speed

Example

The position of a particle moving on x axis varies with time according to the expression $x(t) = (5\ \text{m}) - (2\ \text{m.h}^{-2})t^2$. Find the average velocity and speed between $t_1 = 0.7\text{h}$ and $t_2 = 1.55\text{h}$.

$$v_{x,avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \text{ and } v_{avg} = \frac{d}{t_2 - t_1} \quad \text{with } x(t_1) = 4.02m,\ x(t_2) = 0.195m$$



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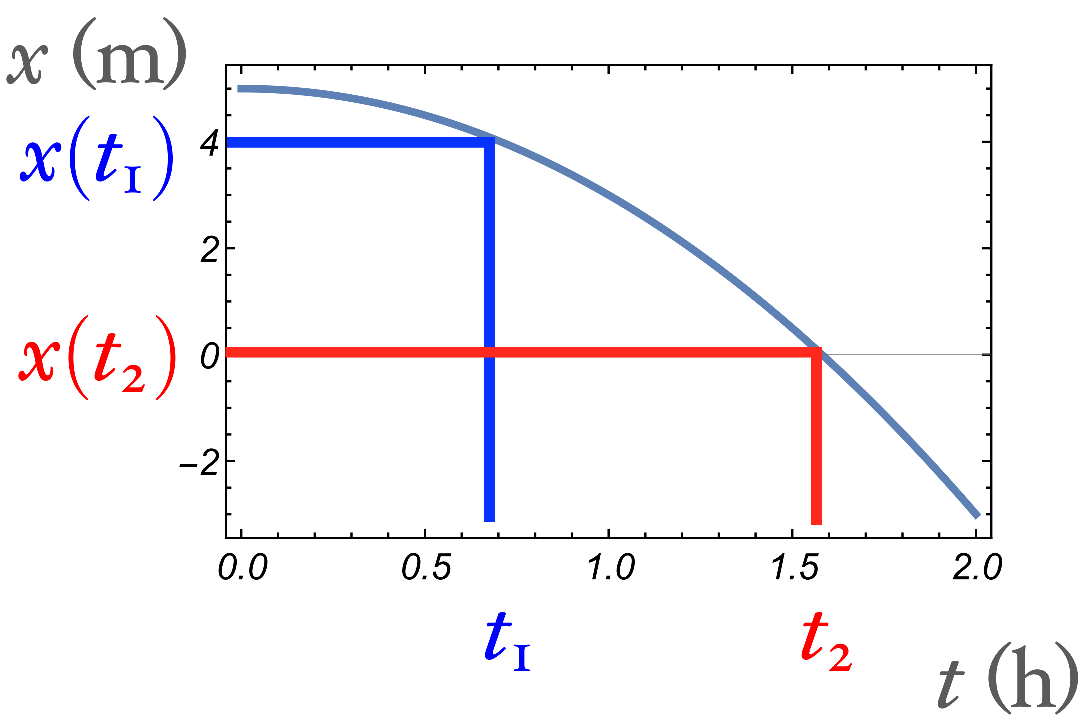


$$v_{x,avg} = \frac{(0.195 - 4.02)m}{(1.55 - 0.7)h} = -4.5m/h$$

$$v_{avg} = \frac{(|0.195 - 4.02|)m}{(1.55 - 0.7)h} = 4.5m/h$$

Reminder on calculus: definitions

We consider below functions of a single variable that take a real number as an argument and return a real number. We assume the concept of limit to be known.

Definition 1: a function $f(x)$ in $\mathbb{R}$ is said to be ***continuous*** on an interval $I = [x_{min}, x_{max}]$ if for any a in I it satisfies $\lim_{x \to a} f(x) = f(a)$

Definition 2: a continuous function is said to be ***differentiable*** at point a if the limit $\lim_{h \to 0} [f(a+h) - f(a)]/h$ exists

Definition 3: if a function f is differentiable at point a then one defines $f'(a)$ the ***derivative*** of f at point a as:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Reminder on calculus: derivatives

Functions	Derivatives
Cx^{α}	$\alpha C x^{\alpha - 1}$
Ce^{x}	Ce^{x}
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$

The “dot” notation

In mechanics, time derivatives play a central role both in the description of motion and in its prediction. On another hand modern mechanics also uses spatial derivatives for another purpose; it is thus usual to use a dedicated notation to talk about time derivatives: that’s the “dot” notation.

Let us consider a function $f(t)$ continuous and differentiable.

We then denote $\dot{f}(t)$ its time derivative at time t such that

$$\boxed{\dot{f}(t) \equiv \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \frac{df(t)}{dt}}$$

multiple time derivatives are simply denoted by adding dots

for example $$\frac{d^3 f(t)}{dt^3} = \dddot{f}(t)$$

Velocity and speed

Sometimes, being able to determine the average velocity between two times that are apart is not indicative enough, we would like some quantity associated to a given time

Definition: the ***instantaneous velocity*** at time t_1 is the limit of the average velocity as t_2 tends to t_1:

$$v_x(t_1) = \lim_{t_2 \to t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{dx(t_1)}{dt}$$

In 'dot' notation: $v_x(t_1) = \dot{x}(t_1)$

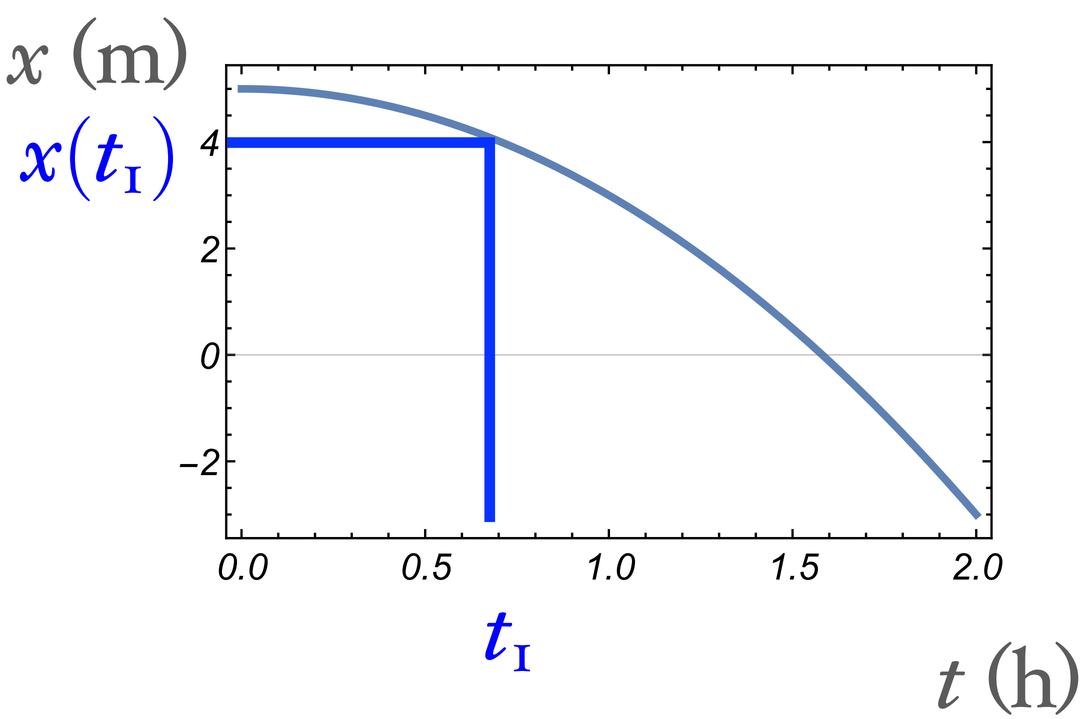
Definition: We call the ***instantaneous speed*** at time t_1, the magnitude of the instantaneous velocity at that time:

$$v(t_1) = |v_x(t_1)|$$

Velocity and speed

Example

The position of a particle moving on x axis varies with time according to the expression $x(t) = (5\ \mathrm{m}) - (2\ \mathrm{m.h^{-2}})t^2$. Find the instantaneous velocity and speed at $t_1 = 0.7\mathrm{h}$.



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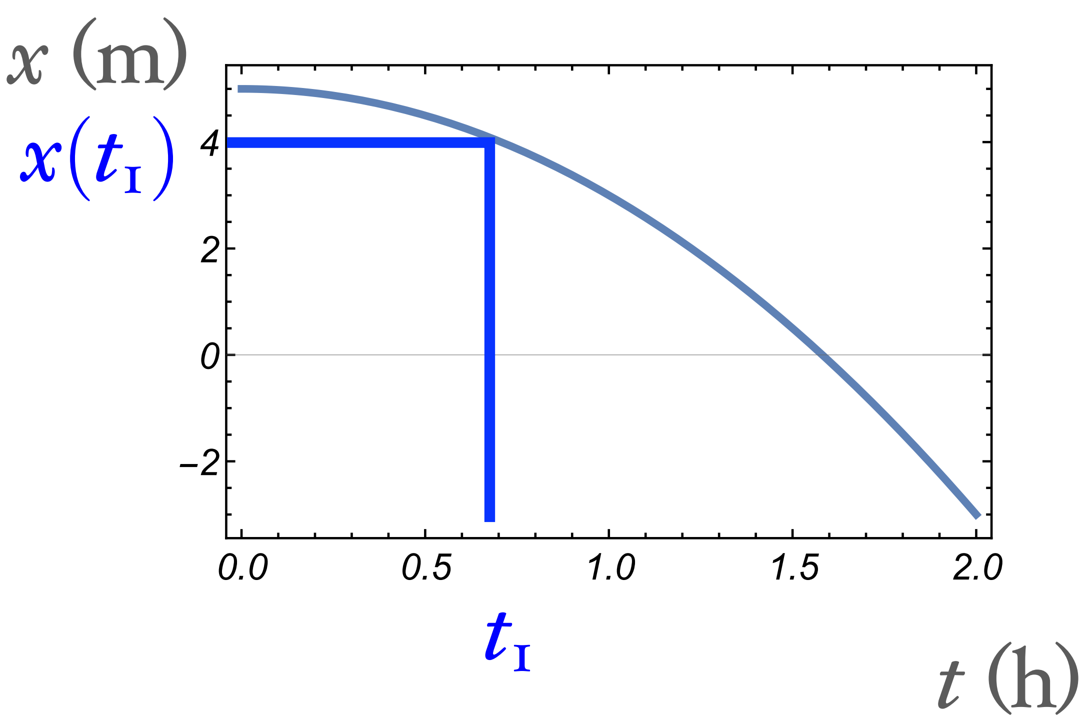
$$v_x(t) = \frac{dx(t)}{dt} = (-4\ m \cdot h^{-2})t$$



$$v_x(0.7h) = -\,2.8\ m/h$$

$$v(0.7h) = |-2.8|\ m/h = 2.8\ m/h$$

Acceleration

Acceleration

The velocity informs about the rate of change of the relative position. But an object initially at rest does not get to a finite velocity in the blink of an eye; it needs to accelerate...

- We define the ***average acceleration*** $a_{x,avg}$ of an object between two times t_1 and t_2 as the rate of change of $v_x(t)$

$$a_{x,avg} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1}$$

- We define the ***instantaneous acceleration*** at t_1 as being the limit of the average acceleration as the time t_2 tends to t_1

$$a_x(t_1) = \lim_{t_2 \to t_1} \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} = \dot{v}_x(t_1) = \ddot{x}(t_1)$$

Acceleration

Example

The position of a particle moving on x axis varies with time according to the expression $x(t) = 10\ m \cdot s^{-1/2} \sqrt{t}$.

i) Express the velocity and the acceleration as functions of time and evaluate the acceleration at $t_1 = 5s$.

ii) Find the average acceleration between $t_1 = 5s$ and $t_2 = 15s$.

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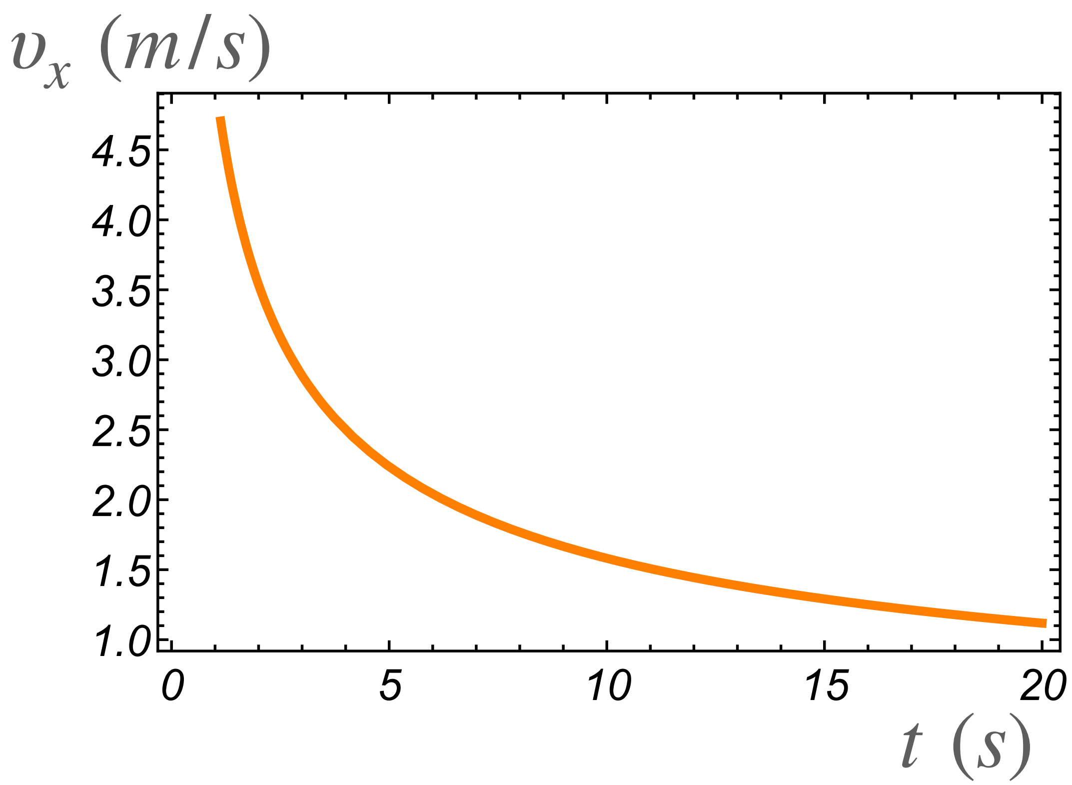


$$v_x(t) = \dot{x}(t) = \frac{(5\ m \cdot s^{-1/2})}{\sqrt{t}}$$

$$a_x(t) = \dot{v}_x(t) = (-\frac{5}{2}\ m \cdot s^{-1/2})t^{-3/2}$$

$$a_x(t_1) = (-\frac{5}{2}\ m \cdot s^{-1/2})(5s)^{-3/2} = -\frac{1}{2\sqrt{5}}\ m \cdot s^{-2}$$

Acceleration

Example

The position of a particle moving on x axis varies with time according to the expression $x(t) = 10\ m \cdot s^{-1/2}\sqrt{t}$.

ii) Find the average acceleration between $t_1 = 5s$ and $t_2 = 15s$.

$$v_x(t) = \dot{x}(t) = \frac{(5\ m \cdot s^{-1/2})}{\sqrt{t}}.\ \text{So,}$$

$$v_x(t_1) = v_x(5s) = \sqrt{5}\ m/s,\ \text{and}\ v_x(t_2) = v_x(15s) = \sqrt{\frac{5}{3}}\ m/s\,.$$

$$a_{x,avg} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} = \frac{(\sqrt{\frac{5}{3}} - \sqrt{5})\ m/s}{(15-5)s} = -\,0.0945\ m \cdot s^{-2}$$

Physical dimensions of velocity and acceleration

Both the velocity and the acceleration concepts are defined from the notions of distance and time interval: they are called ***derived quantities***. That is because they can be defined from the more primitive (undefinable) concepts that are space and time.

A natural consequence is that the dimensions (and units) of derived quantities can be expressed in terms of the dimensions of the primitive quantities they are defined from.

$$\boxed{\begin{aligned}[v_x] &= [\dot{x}] = L \cdot T^{-1} \\ [a_x] &= [\dot{v}_x] = [\ddot{x}] = L \cdot T^{-2}\end{aligned}}$$

Reminder on calculus: primitive functions

Let us consider a continuous function $f(x)$ in $\mathbb{R}$. We call a ***primitive function*** F of f a function whose derivative is f

i.e. such that $$\boxed{F'(x) = f(x)}$$

- The primitive function of a function is not unique. If $F(x)$ is a primitive function of $f(x)$ then $F(x) + c$, for a constant number c, is a primitive function of $f(x)$ as well.

To express all the primitive functions of $f(x)$, we write:

$$\int f(x)dx = F(x) + c, \text{ for a constant } c$$

(indefinite integral)

Reminder on calculus: primitive functions

Functions	Primitives
Cx^{α}	$\frac{C}{\alpha+1}x^{\alpha+1}$
$Ce^{\alpha x}$	$\frac{C}{\alpha}e^{\alpha x}$
$\frac{C}{x}$	$C\ln x$
$\cos \alpha x$	$\frac{1}{\alpha}\sin \alpha x$
$\sin \alpha x$	$-\frac{1}{\alpha}\cos \alpha x$
$\cosh \alpha x$	$\frac{1}{\alpha}\sinh \alpha x$
$\sinh \alpha x$	$\frac{1}{\alpha}\cosh \alpha x$

Reminder on calculus: primitive functions

Example

Determine all the solutions of the equation $\dfrac{df(x)}{dx} = x^3 + 1.$

Reminder on calculus: primitive functions

Example

Determine all the solutions of the equation $\dfrac{df(x)}{dx} = x^3 + 1.$

$$f(x) = \int (x^3 + 1)dx = \frac{x^4}{4} + x + c,$$

where c is an arbitrary constant.

Particle under constant velocity

Constant velocity

The simplest case of all, is that of a particle moving with constant velocity, i.e.

$$\boxed{v_x(t) = constant = v_x}$$

The position of the particle $x(t)$ satisfies the equation $\dfrac{dx(t)}{dt} = v_x$

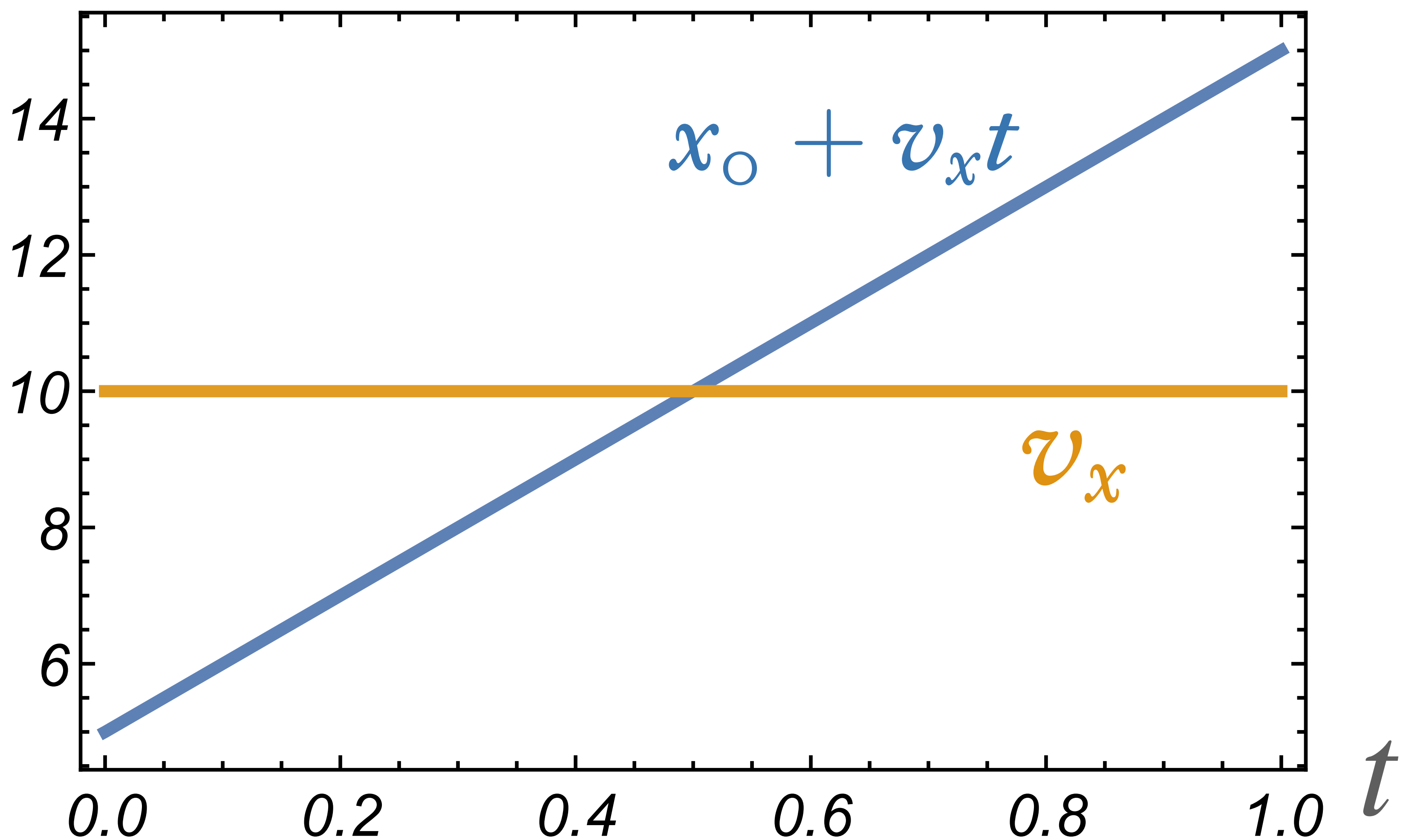
which implies that $x(t) = v_x t + c$, for a constant $c \in \mathbb{R}$.

If the initial position at $t = 0$ is $x(0) = x_0$, we derive that $c = x_0$. Hence, the position of the particle is expressed as

$$\boxed{x(t) = v_x t + x_0}$$

Constant velocity

Graphical representation



Particle under constant acceleration

Constant acceleration

The second simplest case of kinematics is that of constant acceleration. It reads:

$$\boxed{a_x(t) = constant = a_x}$$

The velocity $v_x(t)$ satisfies the equation $\dfrac{dv_x(t)}{dt} = a_x$.

So, $v_x(t) = a_x t + c_1$, for a constant $c_1 \in \mathbb{R}$.

If the initial velocity at $t = 0$ is $v_x(0) = v_0$, we derive that $c_1 = v_0$.
Hence, the velocity of the particle is expressed as

$$\boxed{v_x(t) = v_0 + a_x t}$$

Constant acceleration

We also know that $\dfrac{dx(t)}{dt} = v_x(t)$. So, for $v_x(t) = v_0 + a_x t$, we get $\dfrac{dx(t)}{dt} = v_0 + a_x t$ and by integrating $x(t) = v_0 t + a_x \dfrac{t^2}{2} + c_2$, for a constant $c_2 \in \mathbb{R}$.

If the initial position at $t = 0$ is $x(0) = x_0$, we derive that $c_2 = x_0$. Hence, the position of the particle is expressed as

$$\boxed{x(t) = x_0 + v_0 t + a_x \frac{t^2}{2}}$$

Constant acceleration

Finally, the equations that describe the motion (*equations of motion*) of a particle moving with constant acceleration a_x are:

$$x(t) = x_0 + v_0 t + a_x \frac{t^2}{2}$$

$$v_x(t) = v_0 + a_x t$$

By combining these two equations we can write

$$\boxed{x(t) = x_0 + \frac{1}{2}(v_0 + v_x(t))t}$$

Constant acceleration

Example

A racing car goes from 0 to 60 mph in 3 seconds. Assuming it has constant acceleration, what is the distance it has traveled in that time in metres?

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$$x(t) = x_0 + \frac{1}{2}(v_0 + v_x(t))t \quad \text{or} \quad x(t) - x_0 = \frac{1}{2}(v_0 + v_x(t))t.$$

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For $t = 3s = \dfrac{3}{3600}h$, we have

Constant acceleration

<u>Example</u>

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$$x(t) = x_0 + \frac{1}{2}(v_0 + v_x(t))t \quad \text{or} \quad x(t) - x_0 = \frac{1}{2}(v_0 + v_x(t))t.$$

For $t = 3s = \frac{3}{3600}h$, we have

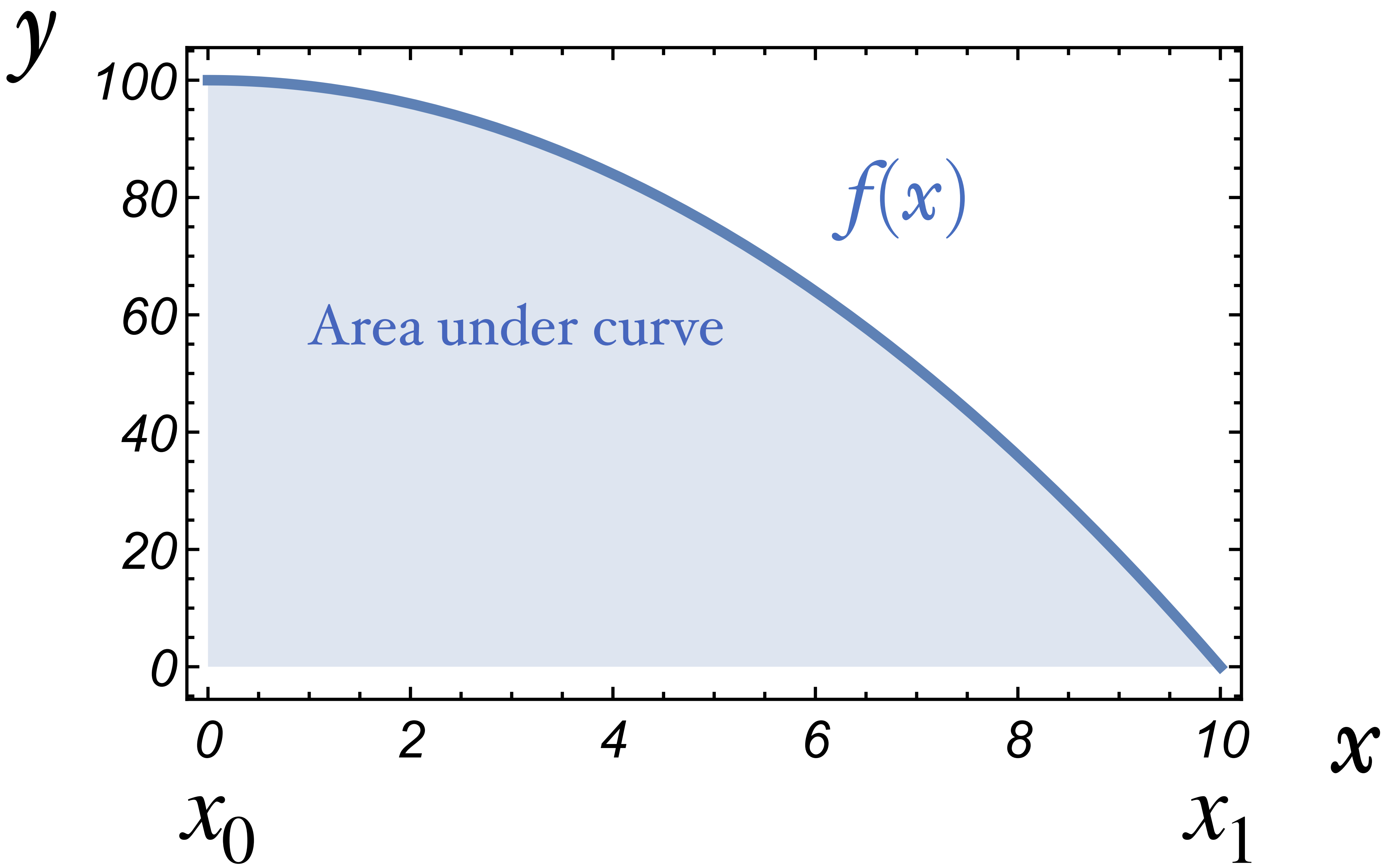
$v_x(t) = 60 \cdot 1609\ m \cdot h^{-1} = 96540\ m \cdot h^{-1}$ and $v_0 = 0$. So,

$$x(t) - x_0 = \frac{1}{2}(v_0 + v_x(t))t = \frac{1}{2}(96540m \cdot h^{-1})(\frac{3}{3600}h) = 40.225m$$

Additional notions of calculus

Riemann integral

Look at the area under a curve



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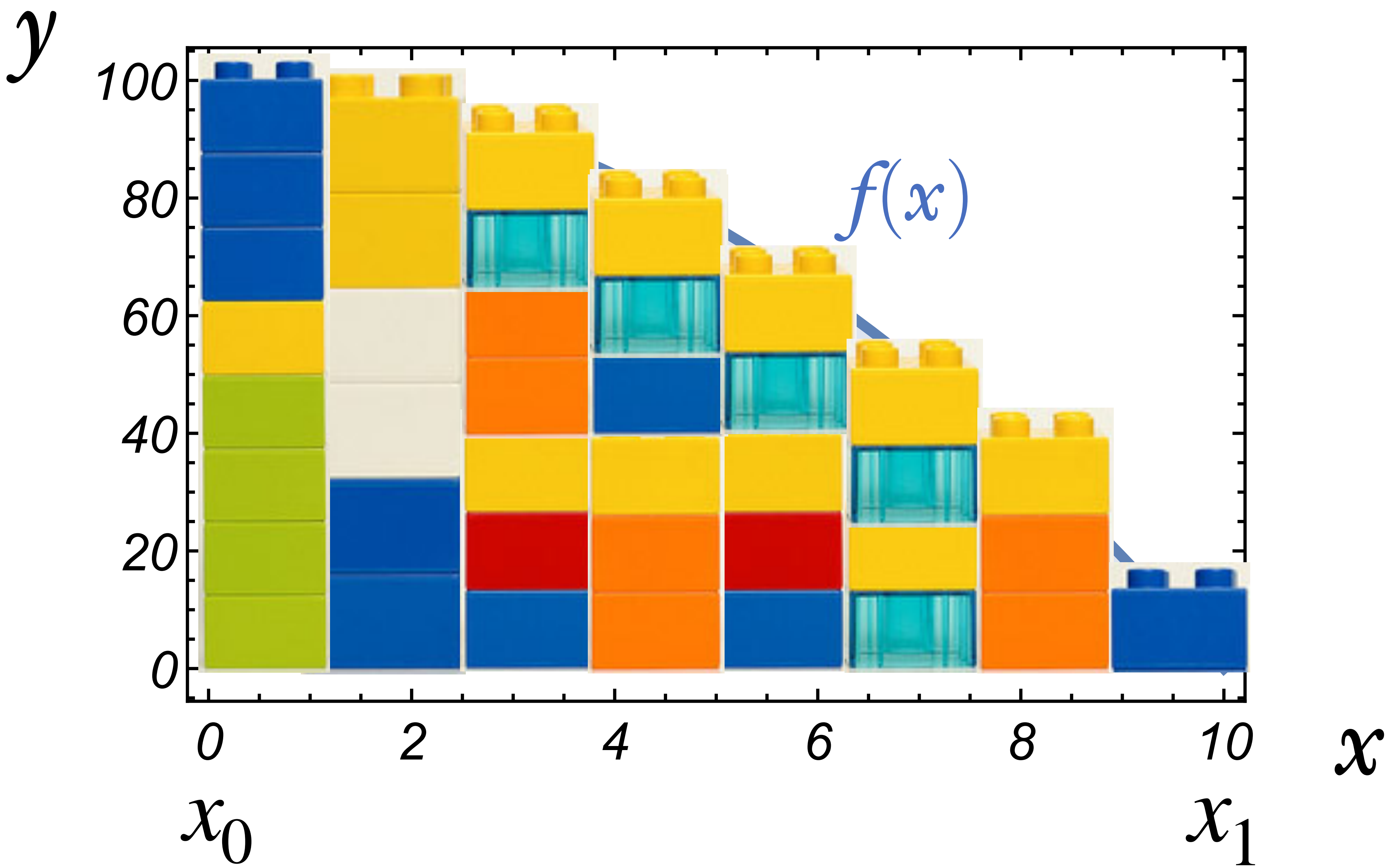
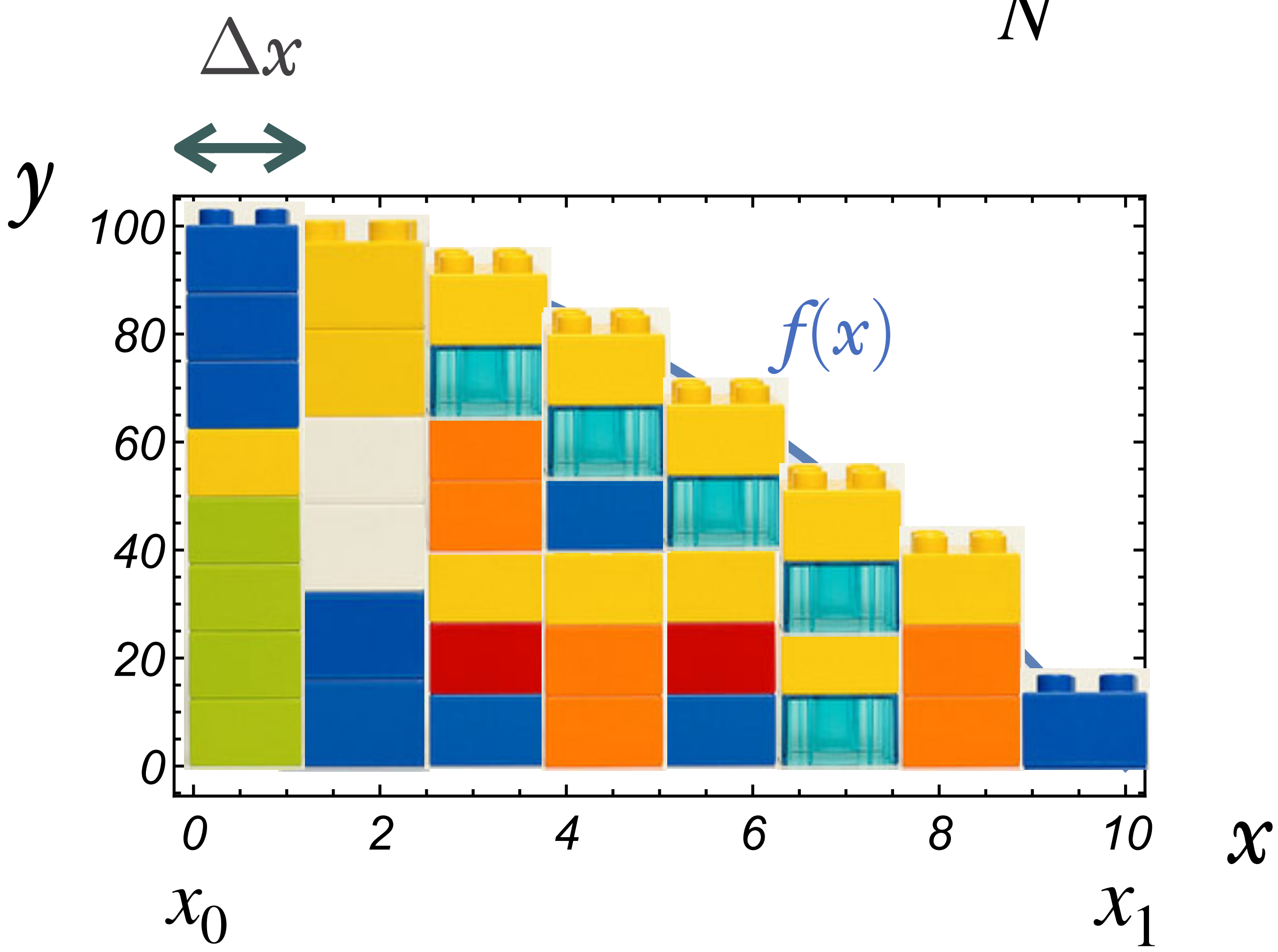


Look at the area under a curve

$$\Delta x = \frac{x_1 - x_0}{N}, \text{ where } N \text{ is the number of rectangles}$$



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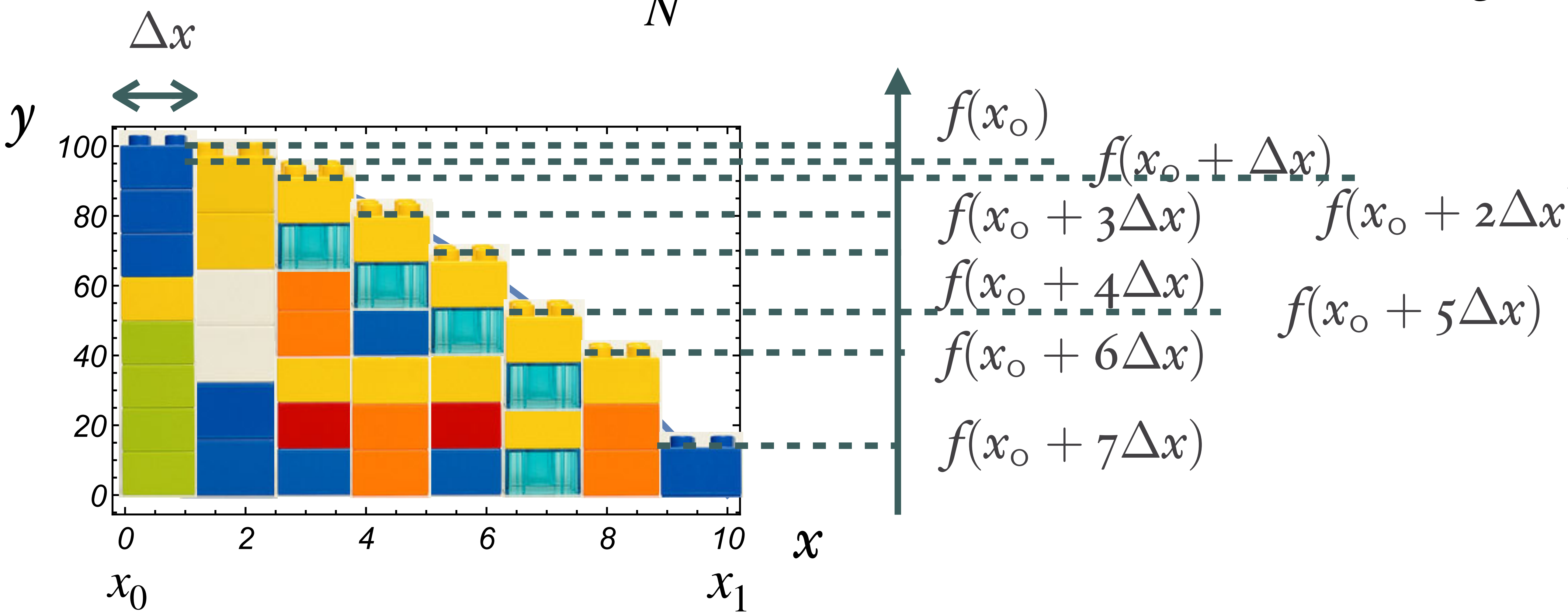


Area of the $k + 1$ rectangle (height x width): $f(x_0 + k\Delta x)\Delta x$.

We set $$A_N[f](x_0, x_1) = \sum_{k=0}^{N-1} f(x_0 + k\Delta x)\Delta x \qquad \textit{(Riemann sum)}$$

Riemann integral and the fundamental theorem of calculus

Let f be a continuous function on $[x_0, x_1]$ and F a primitive function of f in $[x_0, x_1]$. Then

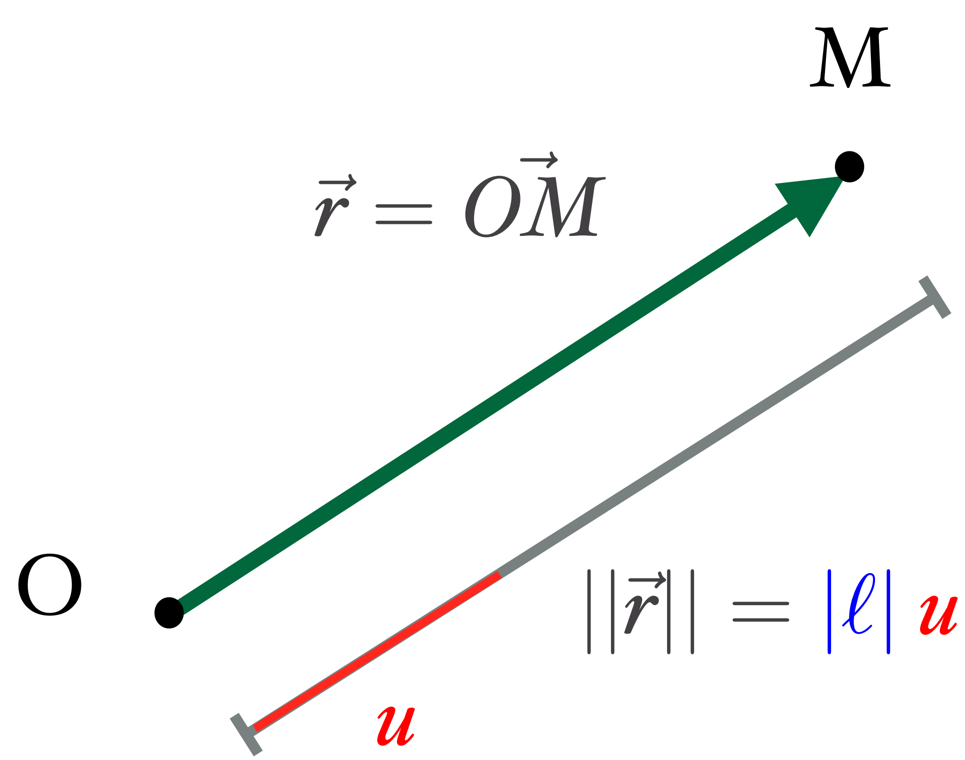
$$\int_{x_0}^{x_1} f(x)dx = \lim_{N\to\infty} A_N[f](x_0, x_1)$$ *(Riemann or definite integral)*

$$\boxed{\int_{x_0}^{x_1} f(x)dx = F(x_1) - F(x_0)}$$

Introduction to kinematics in 2D

Representation of a point in 2D

In 2D the position of a point M relative to a reference point O is identified by a position vector $\vec{r} = \overrightarrow{OM}$.

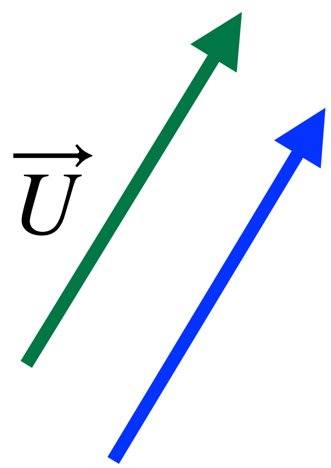


The position vector has a ***magnitude*** or ***norm*** $||\vec{r}||$ associated to a positive number $|\ell|$ related to some length unit u = cm, inches, feet, etc...

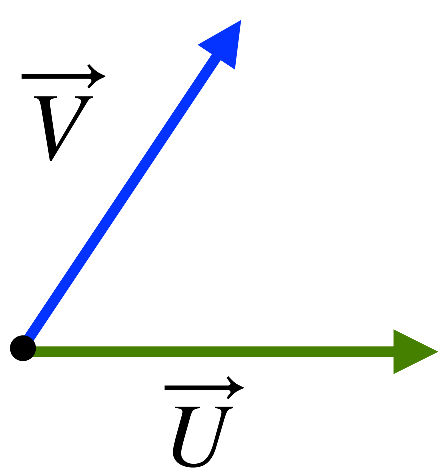
Contrary to 1D not all position vectors are proportional to each others!

A short reminder on vector algebra

☆ $\overrightarrow{U} = \overrightarrow{V}$ if and only if $\|\overrightarrow{U}\| = \|\overrightarrow{V}\|$ and $\overrightarrow{U}, \overrightarrow{V}$ point in the same direction along parallel lines.



$$\Rightarrow \quad \overrightarrow{U} = \overrightarrow{V}$$

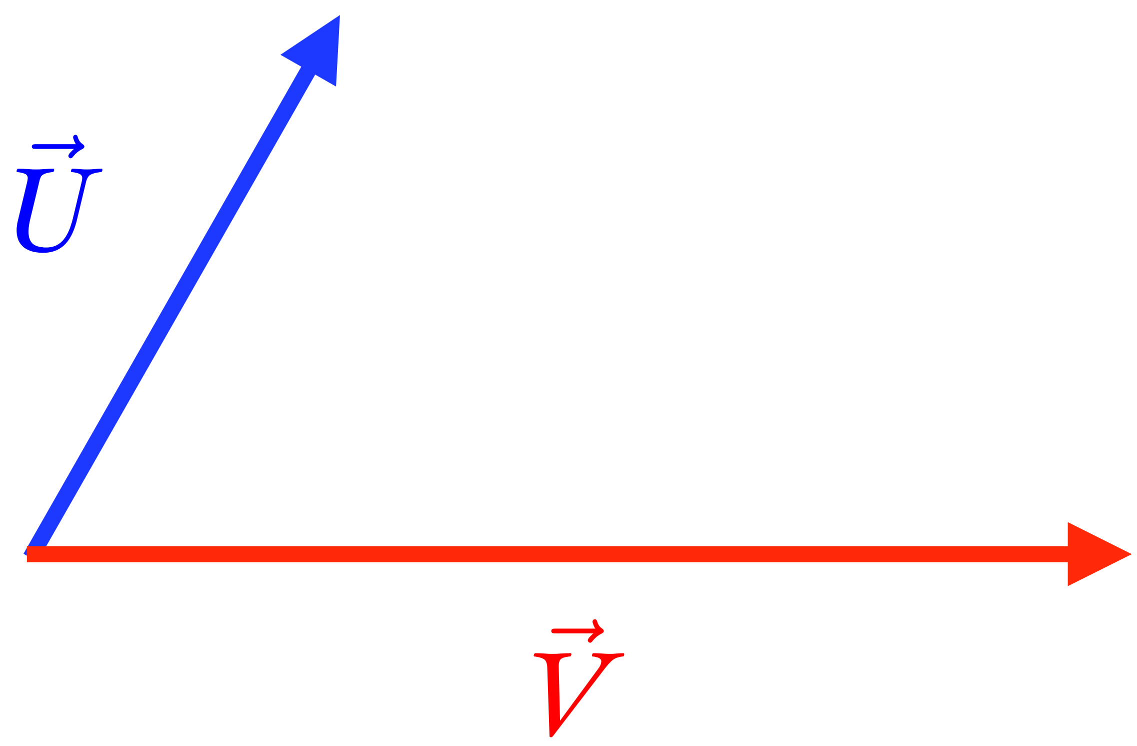


$$\Rightarrow \quad \overrightarrow{U} \neq \overrightarrow{V}$$

A short reminder on vector algebra

Addition

$\star\ \vec{U} + \vec{V}$



A short reminder on vector algebra

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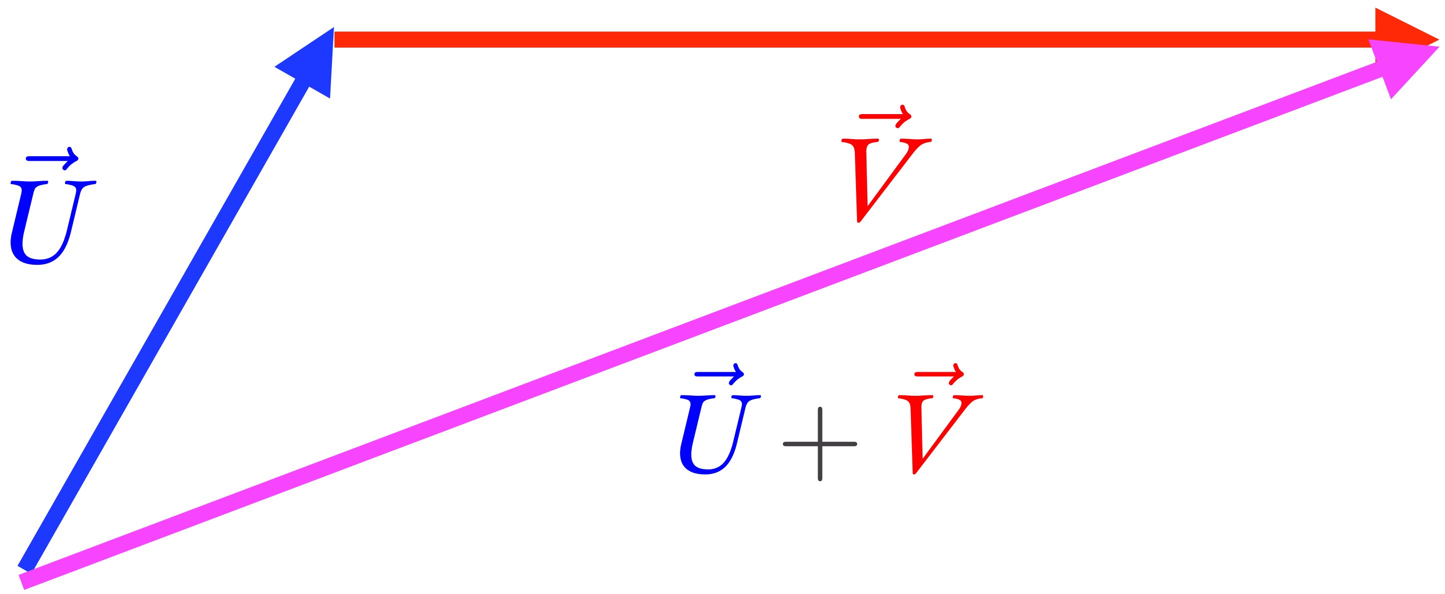
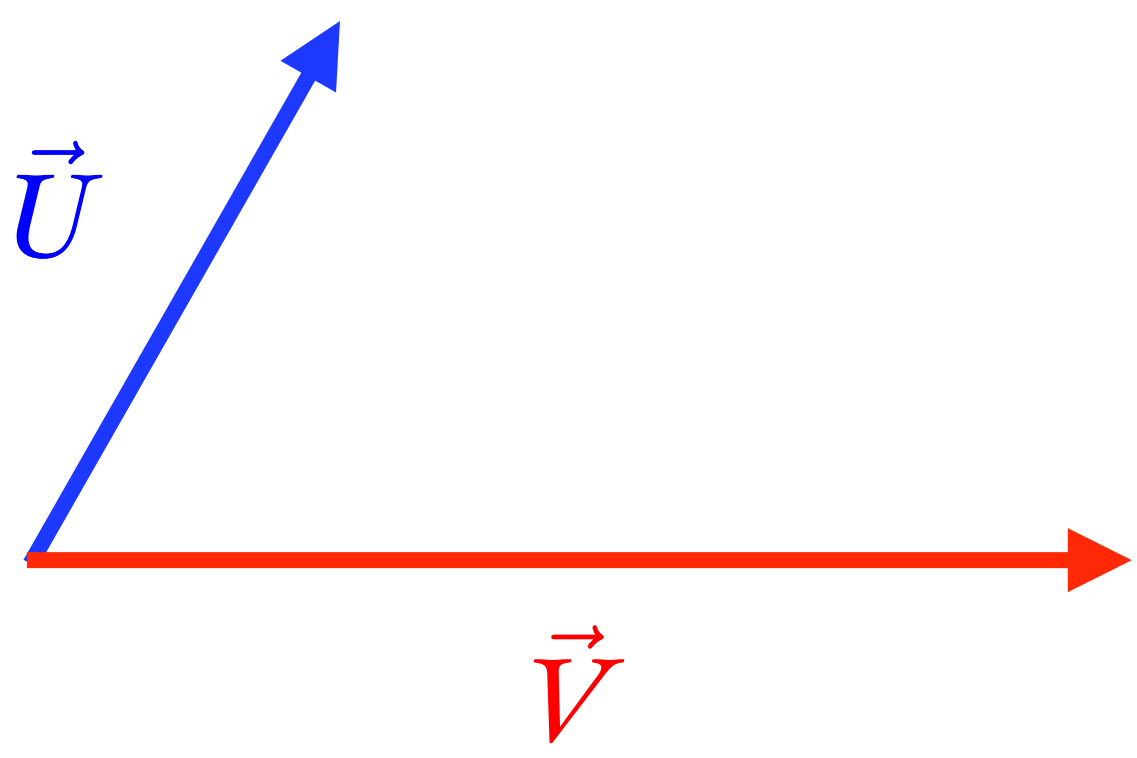


A short reminder on vector algebra

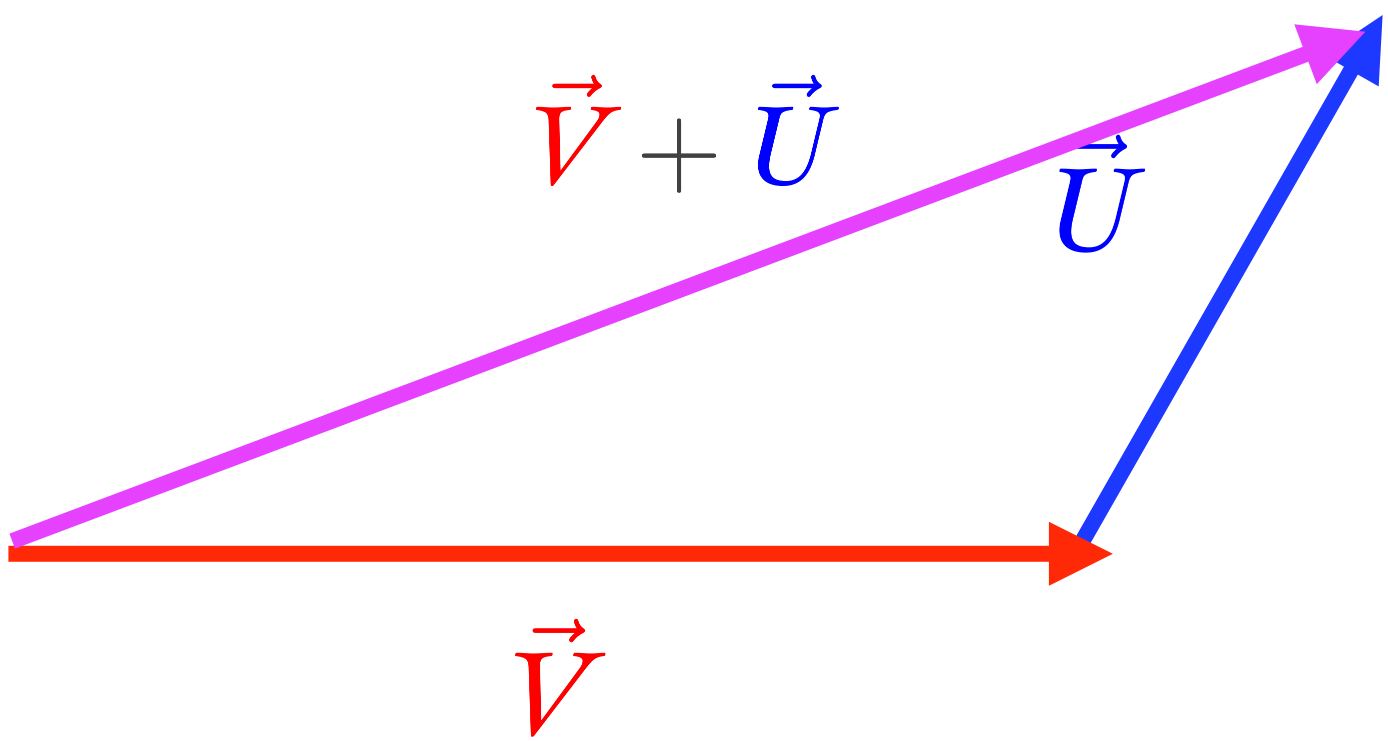
Addition



- $\star\ \vec{U} + \vec{V}$
- $\star\ \vec{V} + \vec{U}$

A short reminder on vector algebra

Addition



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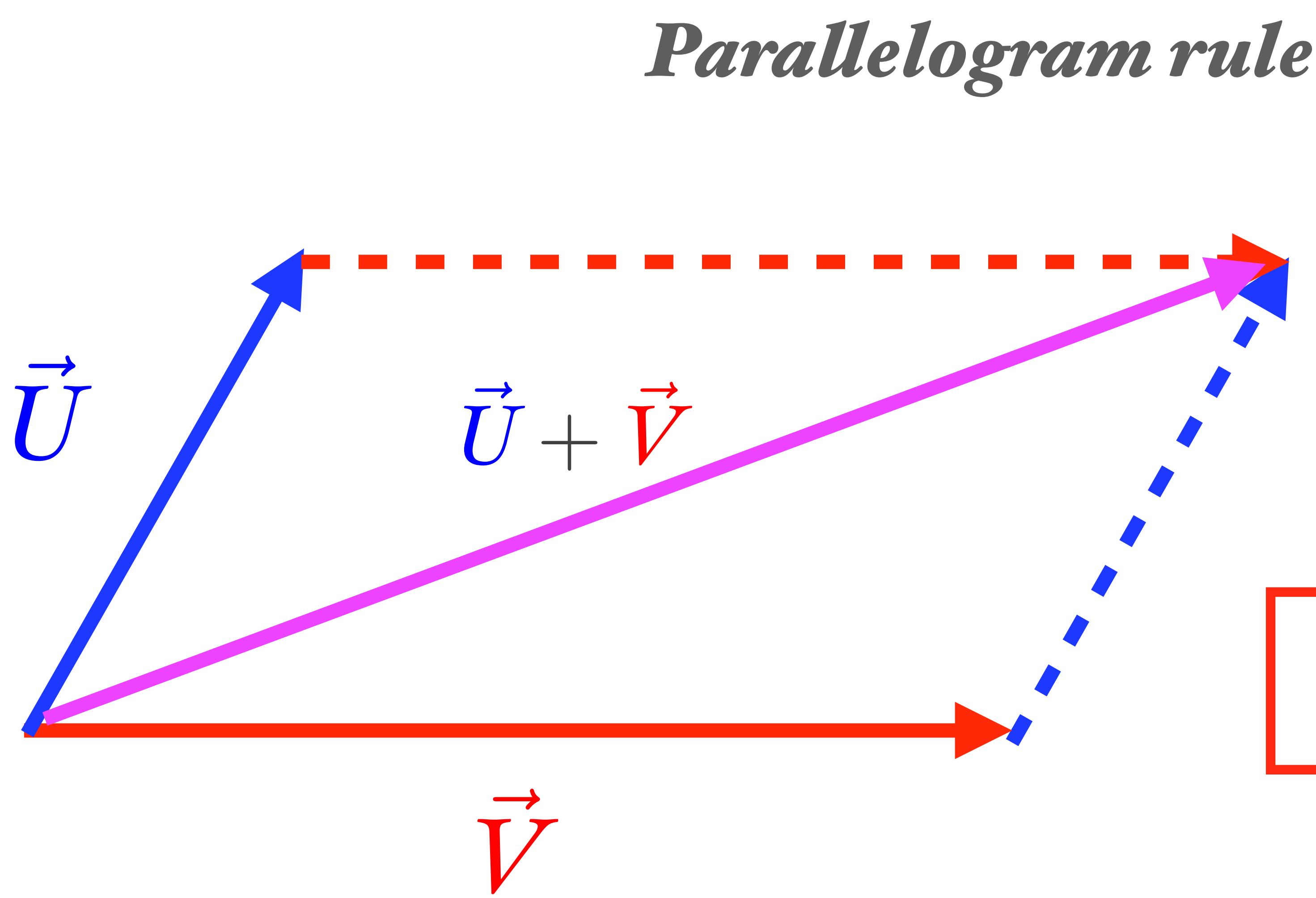
A short reminder on vector algebra

Addition



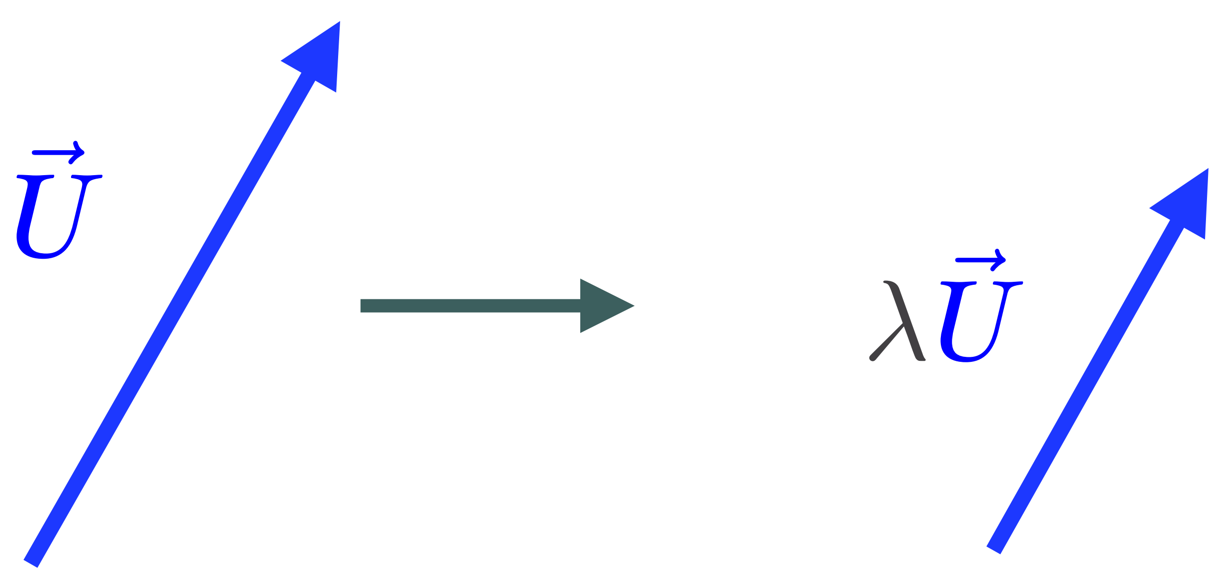
$\star\ \vec{U} + \vec{V}$

$\star\ \vec{V} + \vec{U}$

$$\boxed{\vec{U} + \vec{V} = \vec{V} + \vec{U}}$$

A short reminder on vector algebra

Multiplication by a real number



* Keep the orientation
* $||\lambda\vec{U}|| \equiv |\lambda| \times ||\vec{U}||$
* Same direction if

$$\lambda > 0$$

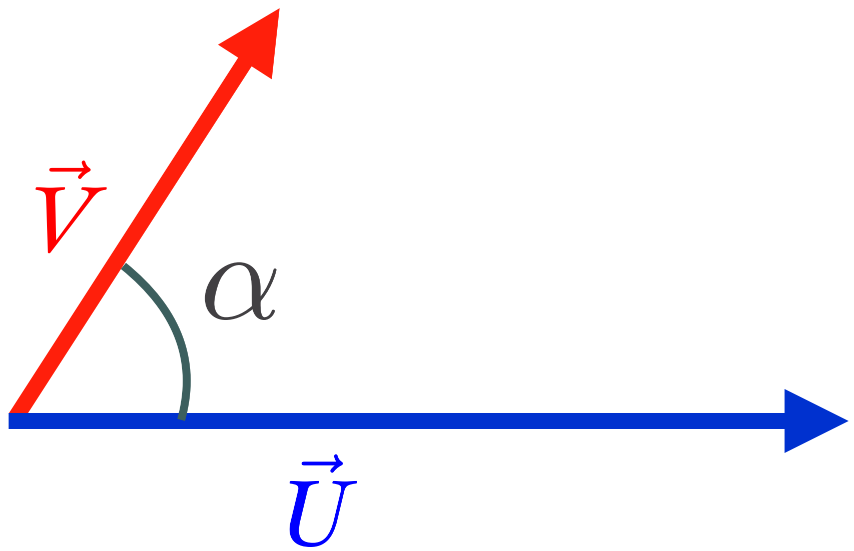
* Opposite direction if

$$\lambda < 0$$

A short reminder on vector algebra

Scalar product

$$\vec{U} \cdot \vec{V} = ||\vec{U}|| \times ||\vec{V}|| \cos\alpha \quad (\textit{scalar quantity})$$



$$\boxed{\vec{V} \cdot \vec{U} = \vec{U} \cdot \vec{V}}$$

$$\boxed{||\vec{U}|| = \sqrt{\vec{U} \cdot \vec{U}}}$$

A short reminder on vector algebra

Scalar product

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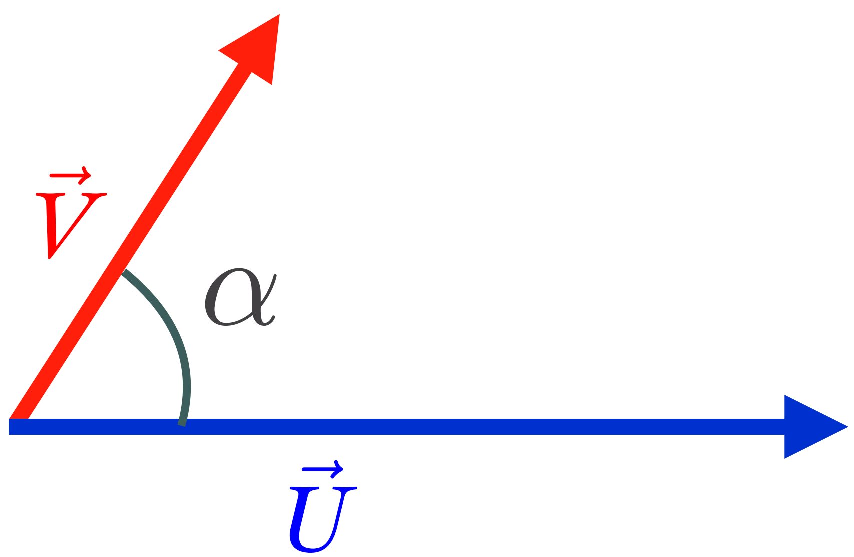


$$\boxed{\vec{V} \cdot \vec{U} = \vec{U} \cdot \vec{V}}$$

$$\boxed{||\vec{U}|| = \sqrt{\vec{U} \cdot \vec{U}}}$$

- Two vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are said to be ***orthogonal*** or perpendicular if $\overrightarrow{U} \cdot \overrightarrow{V} = 0$, i.e. $\alpha = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$.
- Two vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are said to be ***collinear*** or parallel if $\overrightarrow{U} \cdot \overrightarrow{V} = \pm \|\overrightarrow{U}\| \|\overrightarrow{V}\|$, i.e. $\alpha = n\pi,\ n \in \mathbb{Z}$.

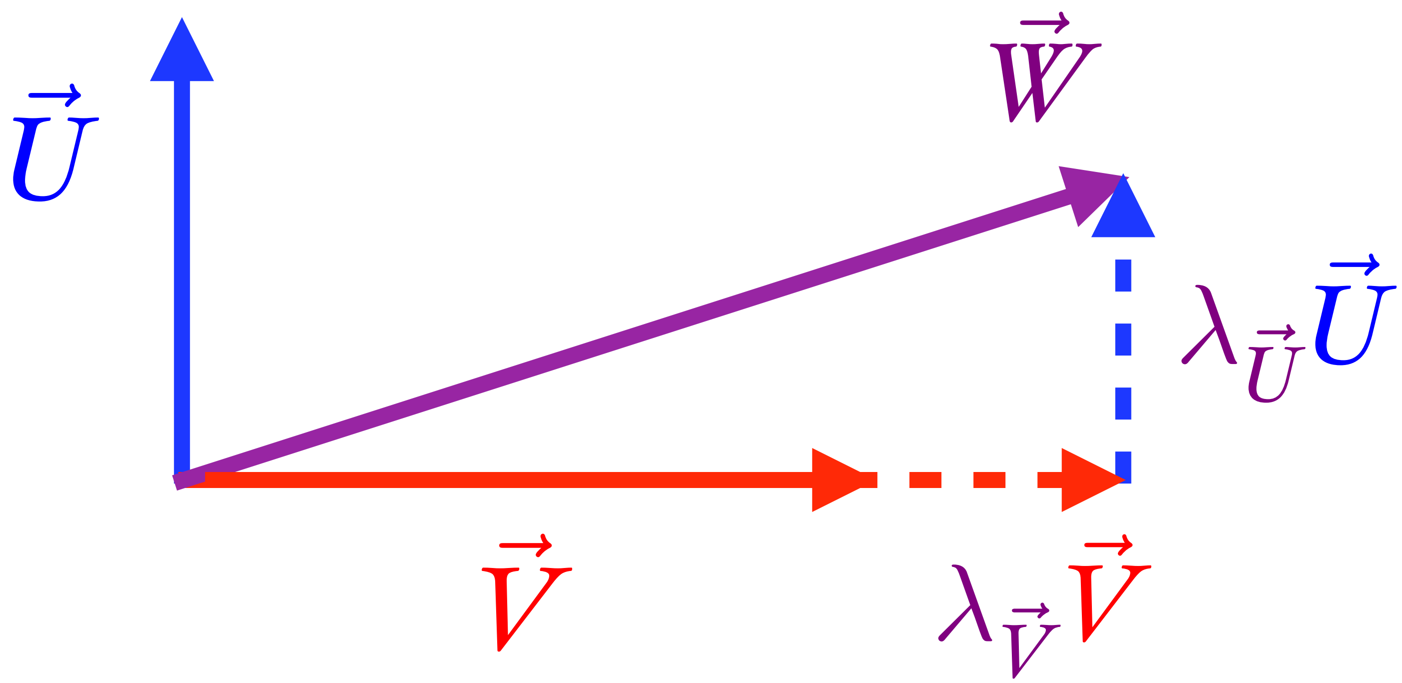
A short reminder on vector algebra

Basis and components

By the parallelogram rule, any vector in the plane can be written as the sum of two non-parallel vectors. It is in particular true with perpendicular vectors:

$$\vec{W} = \lambda_{\vec{U}}\vec{U} + \lambda_{\vec{V}}\vec{V}$$

$$\vec{W} = \lambda_{\vec{U}}||\vec{U}||\frac{\vec{U}}{||\vec{U}||} + \lambda_{\vec{V}}||\vec{V}||\frac{\vec{V}}{||\vec{V}||}$$



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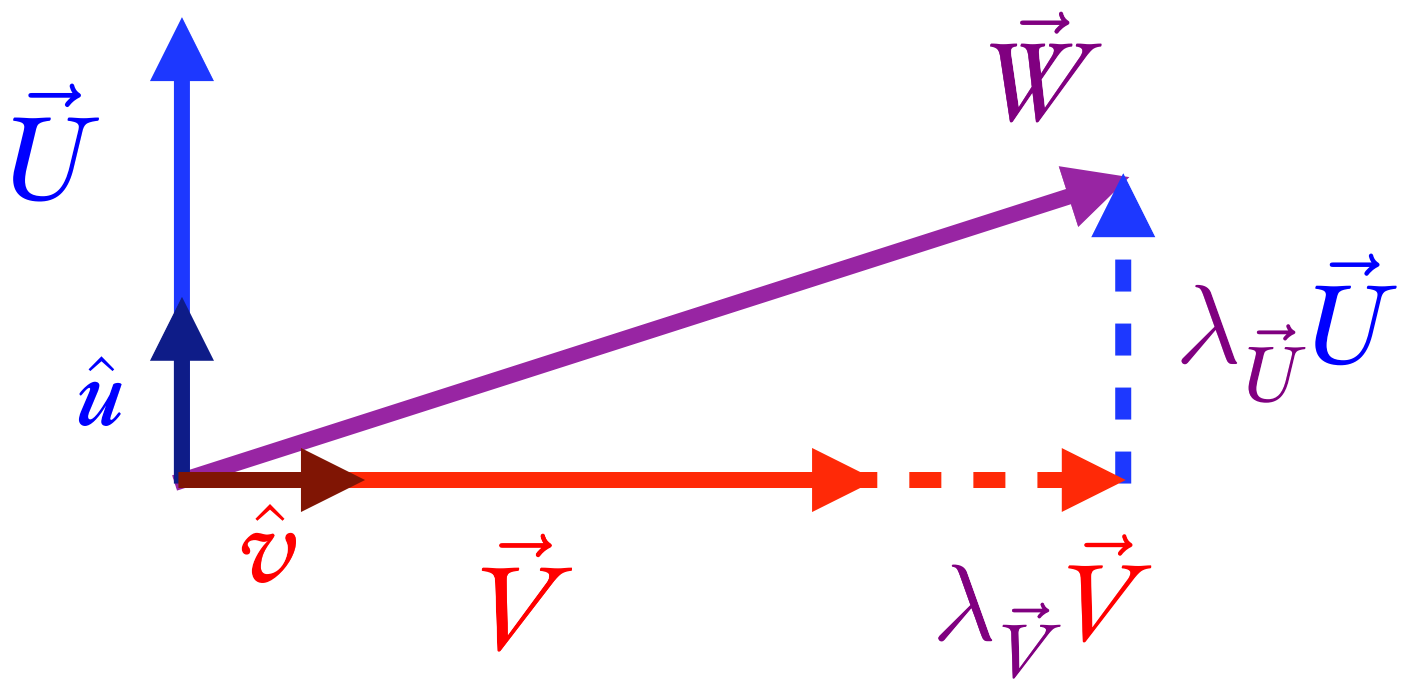

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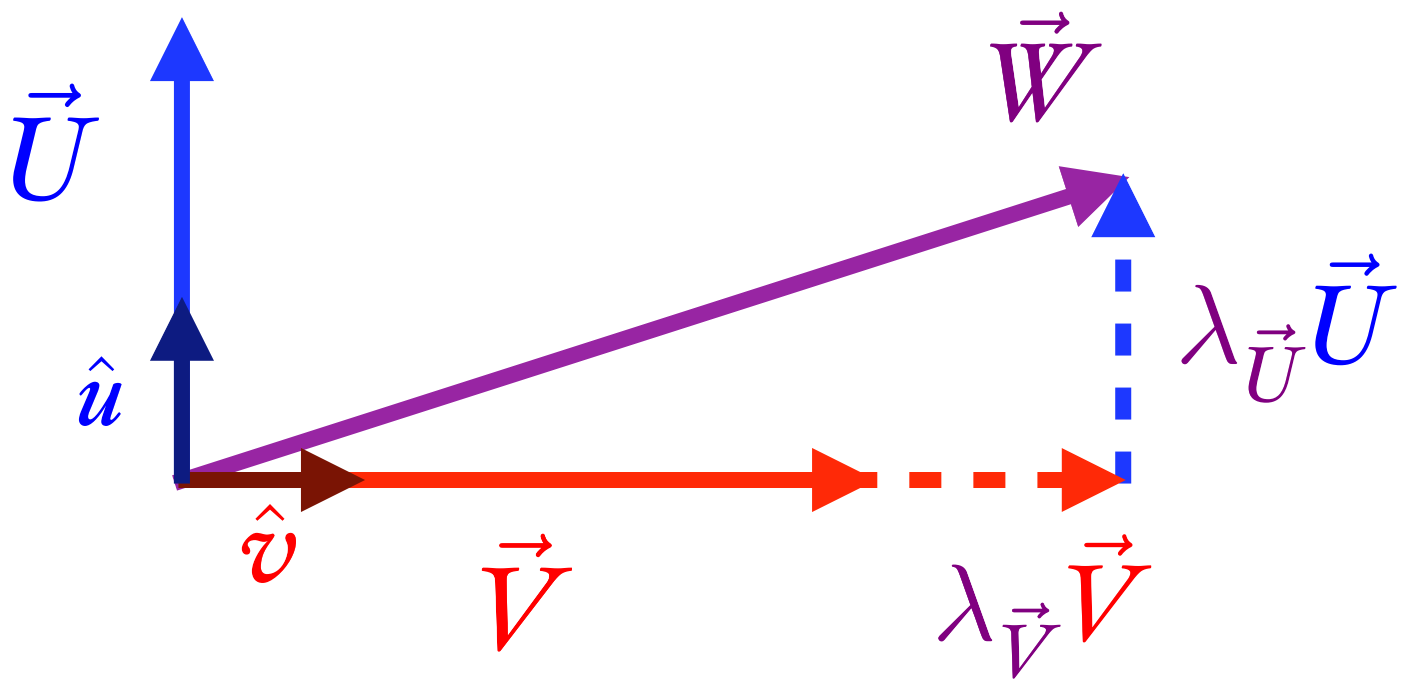


$$\vec{W} = \lambda_{\vec{U}}\vec{U} + \lambda_{\vec{V}}\vec{V}$$

$$\vec{W} = \underbrace{\lambda_{\vec{U}}||\vec{U}||}_{W_{\hat{u}}}\underbrace{\frac{\vec{U}}{||\vec{U}||}}_{\hat{u}} + \underbrace{\lambda_{\vec{V}}||\vec{V}||}_{W_{\hat{v}}}\underbrace{\frac{\vec{V}}{||\vec{V}||}}_{\hat{v}}$$

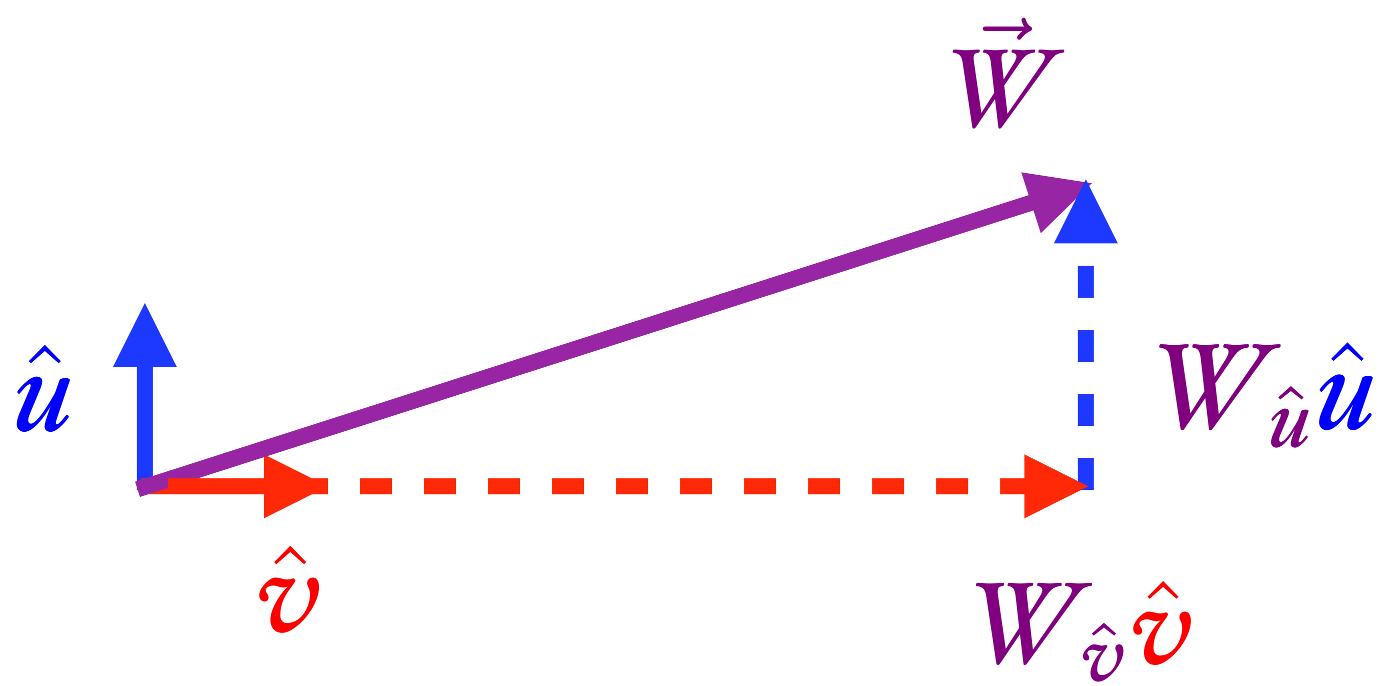
$$\boxed{\vec{W} = W_{\hat{u}}\hat{u} + W_{\hat{v}}\hat{v}}$$

A short reminder on vector algebra

Basis and components

$$\boxed{\vec{W} = W_{\hat{u}}\,\hat{u} + W_{\hat{v}}\,\hat{v}}$$

- The pair $(\hat{u}, \hat{v})$ with $\hat{u} \perp \hat{v}$ and $\|\hat{u}\| = \|\hat{v}\| = 1$, is called an ***orthonormal basis***.
- The numbers $W_{\hat{u}}$ and $W_{\hat{v}}$ are called the ***components*** of $\vec{W}$.



A short reminder on vector algebra

Vector algebra in terms of components

All the vector operations can be implemented with components

- If $\vec{W} = W_{\hat{u}}\,\hat{u} + W_{\hat{v}}\,\hat{v}$ and $\vec{Z} = Z_{\hat{u}}\,\hat{u} + Z_{\hat{v}}\,\hat{v}$

Addition: $$\vec{W} + \vec{Z} = (Z_{\hat{u}} + W_{\hat{u}})\,\hat{u} + (W_{\hat{v}} + Z_{\hat{v}})\,\hat{v}$$

Multiplication by a number: $$\lambda \vec{W} = \lambda W_{\hat{u}}\,\hat{u} + \lambda W_{\hat{v}}\,\hat{v}$$

Scalar product: $$\vec{W} \cdot \vec{Z} = W_{\hat{u}} Z_{\hat{u}} + W_{\hat{v}} Z_{\hat{v}}$$

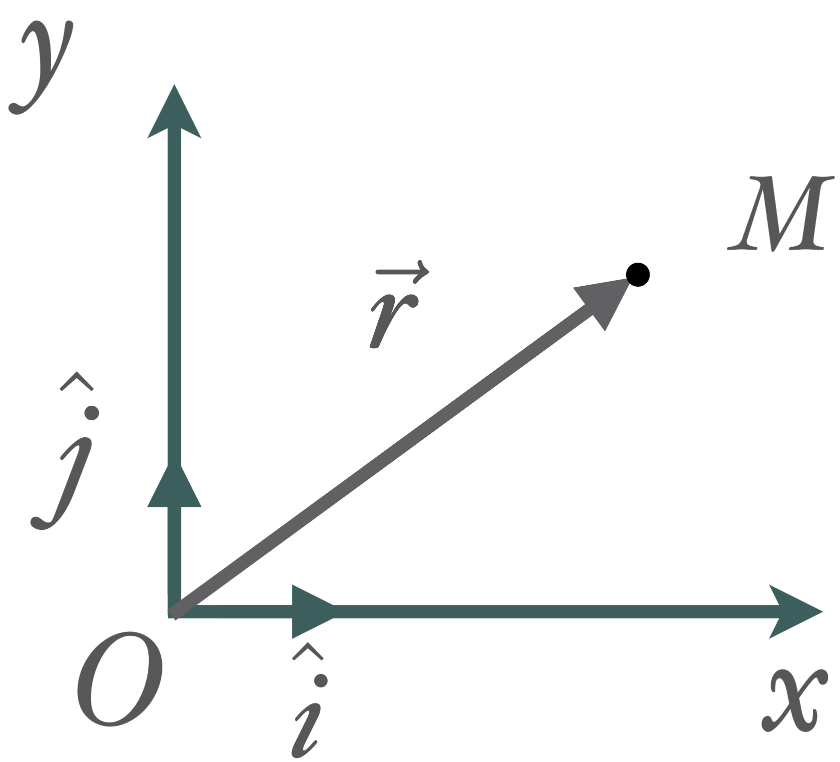
Norm: $$||\vec{W}|| = \sqrt{\vec{W} \cdot \vec{W}} = \sqrt{W_{\hat{u}}{}^2 + W_{\hat{v}}{}^2}$$

Kinematics in 2D

Relative position

Given a ***frame*** consisting of an orthonormal basis $(\hat{i}, \hat{j})$ and an origin O, the position of a point M relative to the origin is characterised by its position vector $\vec{r} = \vec{OM}$ in this frame

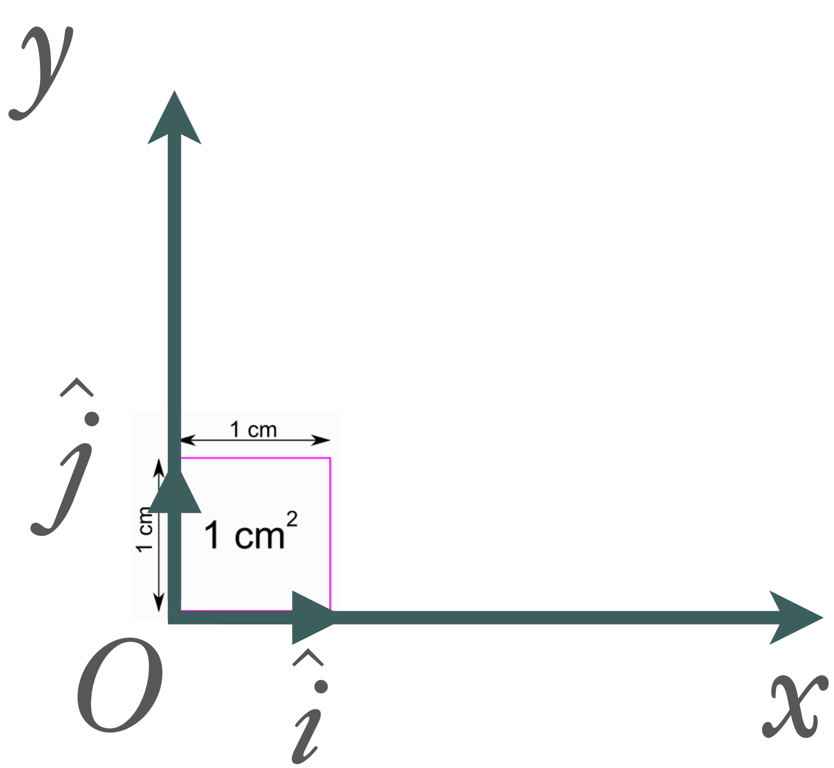
$$\boxed{\vec{r} = x\,\hat{i} + y\,\hat{j}}$$



The components x and y of $\vec{r}$ in the frame $(O, \hat{i}, \hat{j})$ are called the (cartesian) ***coordinates*** of M

Relative position

Example 1



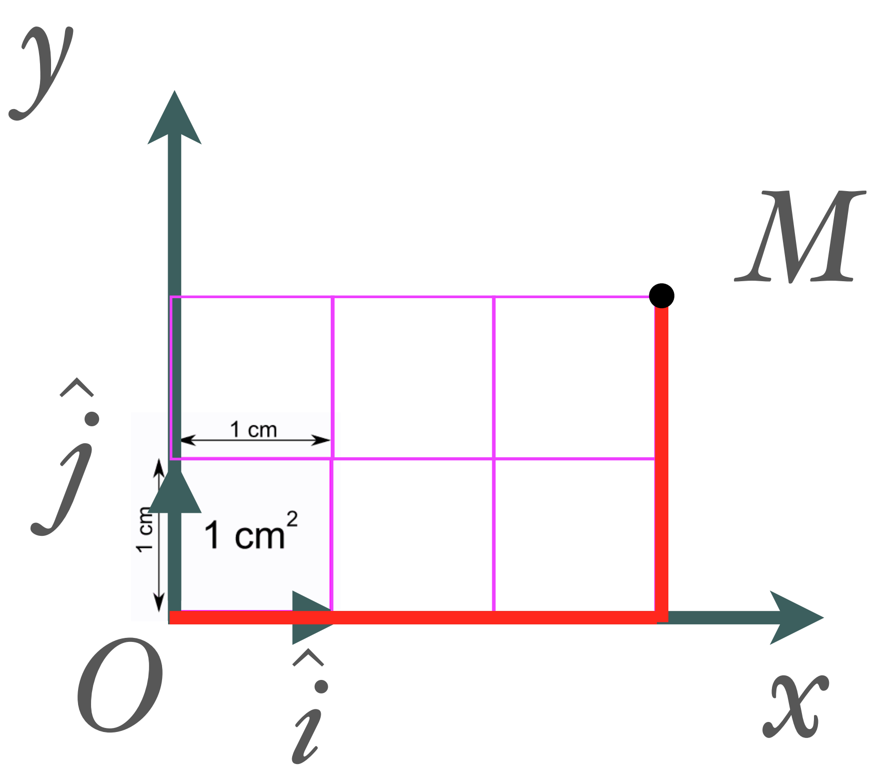
Given the position vector below

$$\vec{r} = (3\ \mathrm{cm})\,\hat{i} + (2\ \mathrm{cm})\,\hat{j}$$

find the position of the corresponding point M on the graph.

Relative position

Example 1



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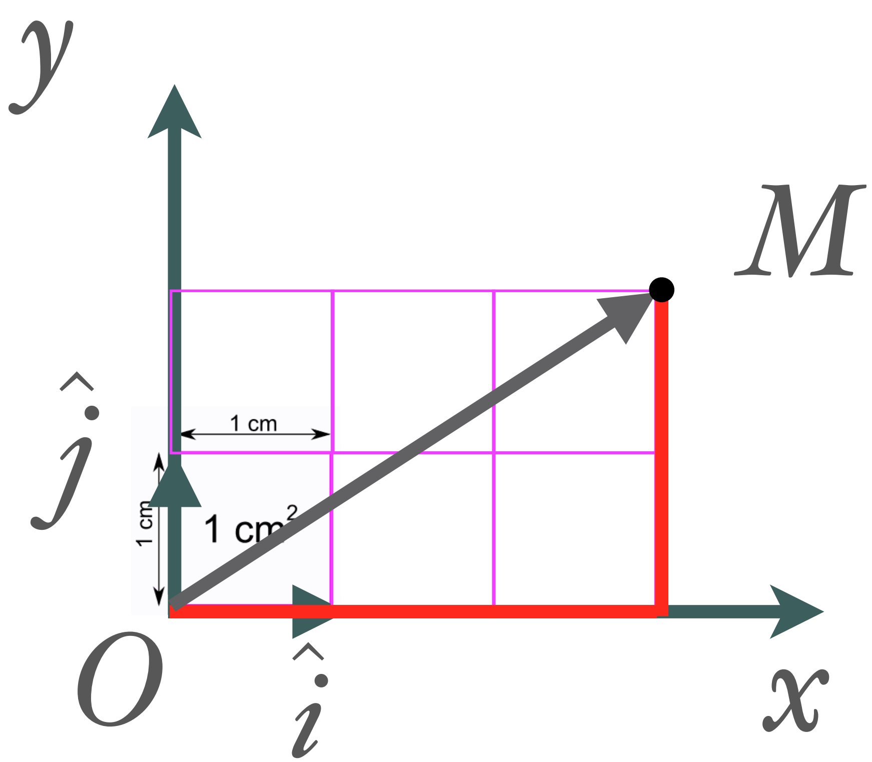

Given the position vector below

$$\vec{r} = (3\ \mathrm{cm})\,\hat{i} + (2\ \mathrm{cm})\,\hat{j}$$

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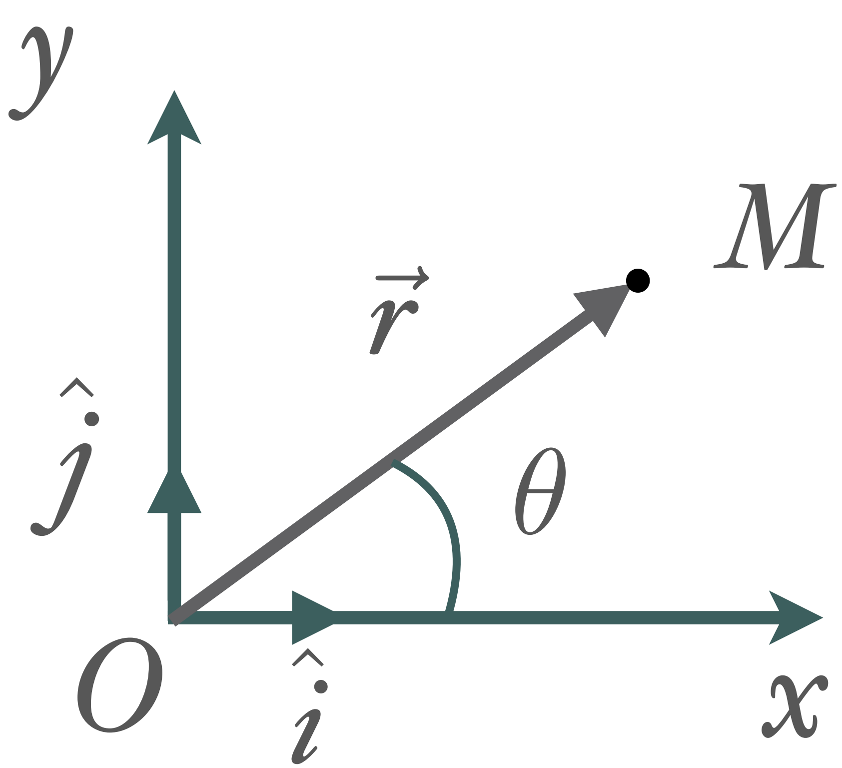
Relative position

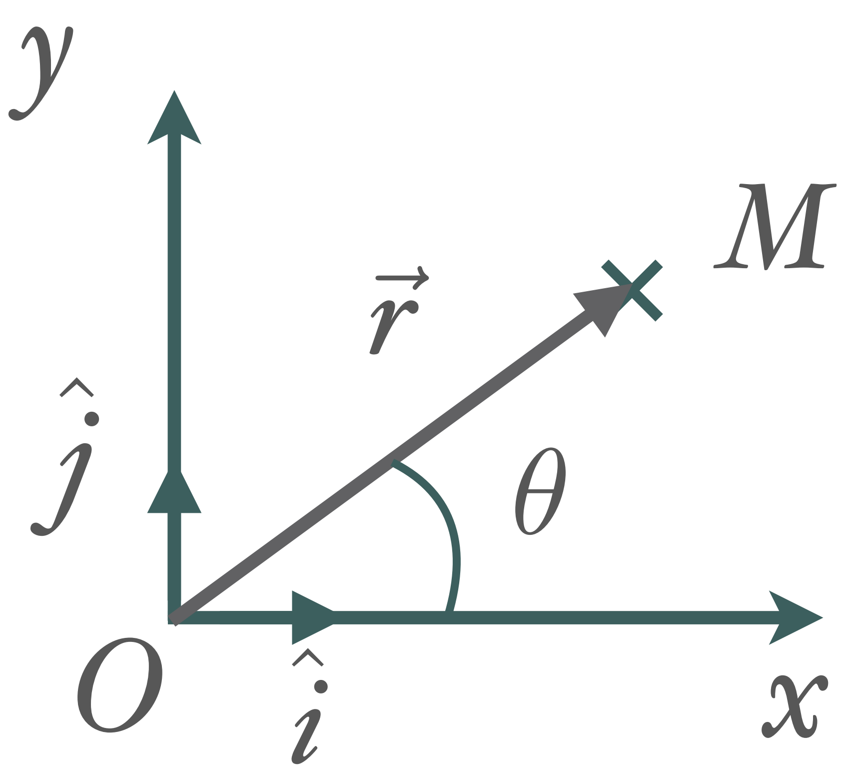
Example 2



Given that $||\vec{r}|| = 4$ cm and $\theta = \pi/4$ rad, find the coordinates of M in the frame $(O, \hat{i}, \hat{j})$.

Relative position

Example 2



Given that $||\vec{r}|| = 4$ cm and $\theta = \pi/4$ rad, find the coordinates of M in the frame $(O, \hat{i}, \hat{j})$.

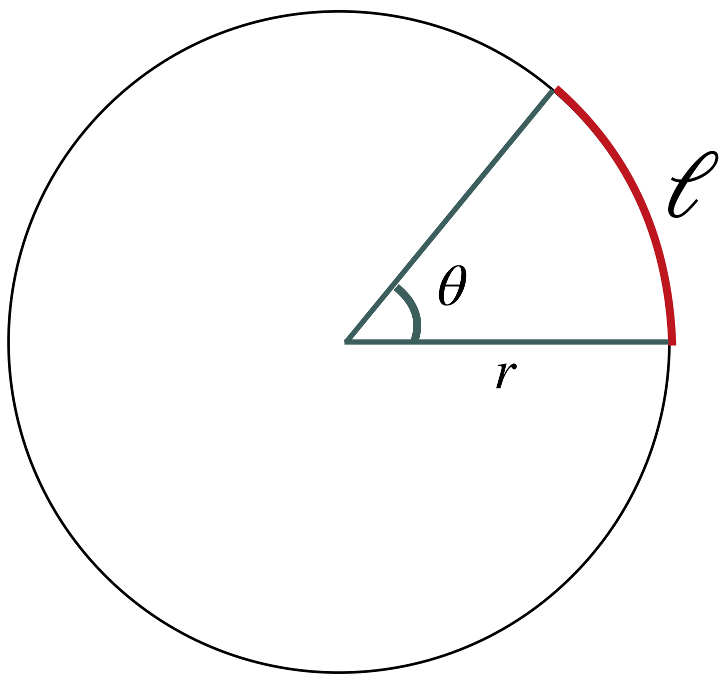
Answer: the coordinates are the components of $\vec{r} = x\hat{i} + y\hat{j}$, with

$$x = \vec{r} \cdot \hat{i} = ||\vec{r}|| \cos\theta = 2\sqrt{2} \text{ cm}$$

$$y = \vec{r} \cdot \hat{j} = ||\vec{r}|| \sin\theta = 2\sqrt{2} \text{ cm}$$

A quick word on the dimension of an angle

The arc length ℓ of a circle is $\ell = r\theta$, where r is the radius of the circle and θ the corresponding angle in **radians**.



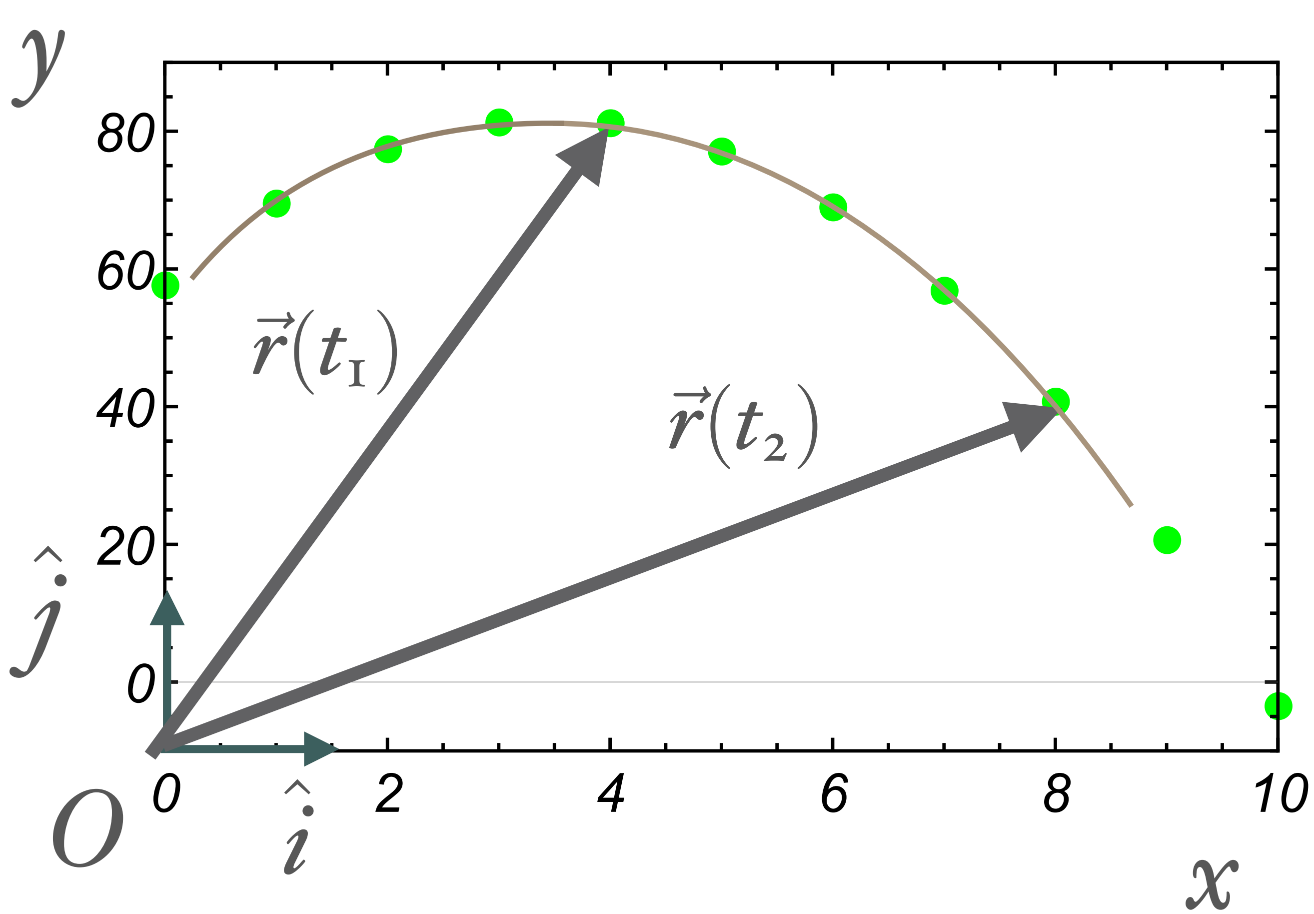
$$\theta = \frac{\ell}{r}. \text{ So, } [\theta] = \frac{[\ell]}{[r]} = \frac{L}{L} = 1,$$

i.e., the angle θ is dimensionless.

Hence, we will consider any angle θ dimensionless and correspondingly $[cos\theta] = [\sin\theta] = 1$.

Average velocity

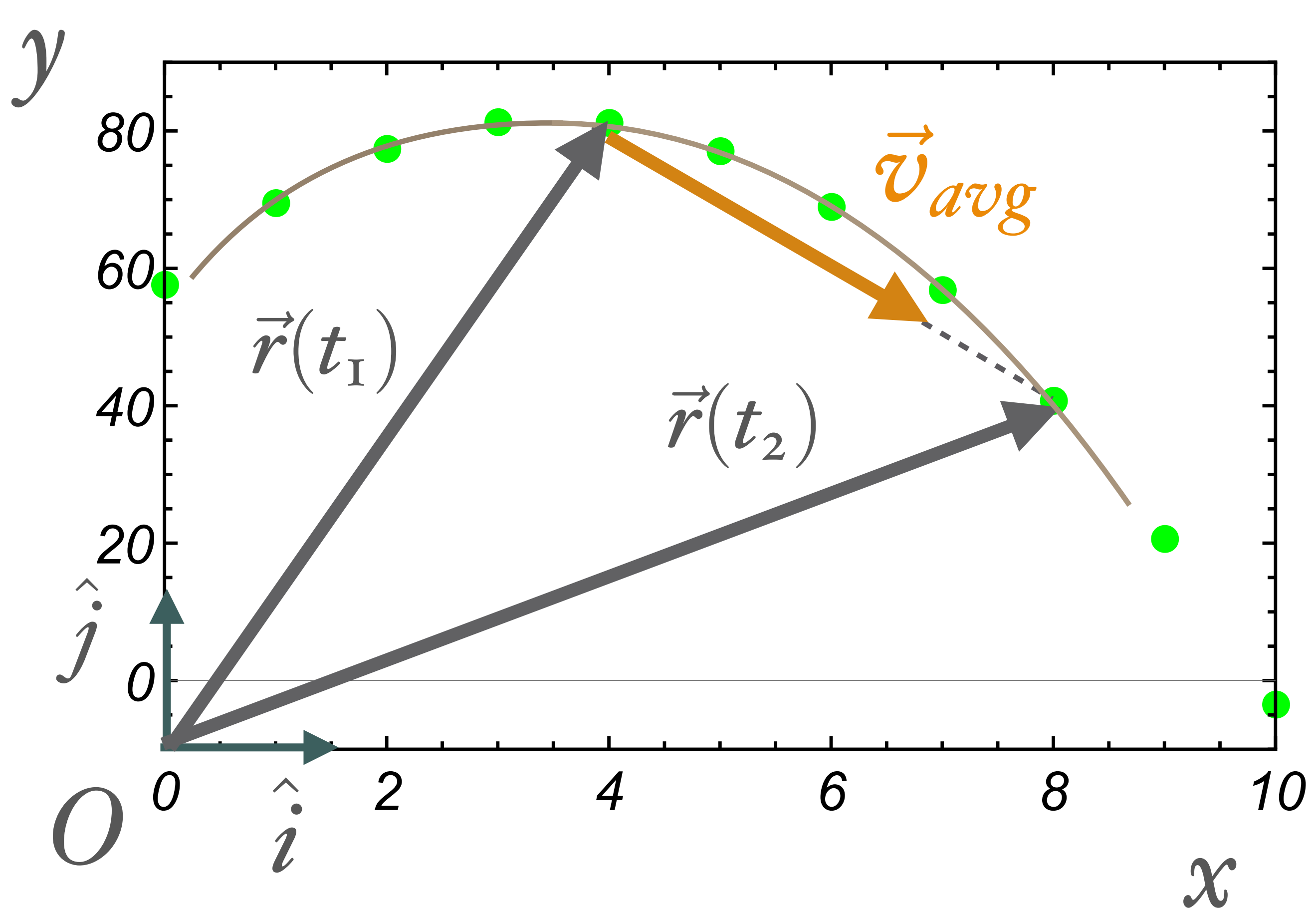
The trajectory of a point object can be represented by its vector position as a function of time $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$.



Average velocity

The trajectory of a point object can be represented by its vector position as a function of time $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$.

The average velocity of point M between t_1 and t_2 is the vector:



$$\vec{v}_{avg} \equiv \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

with components:

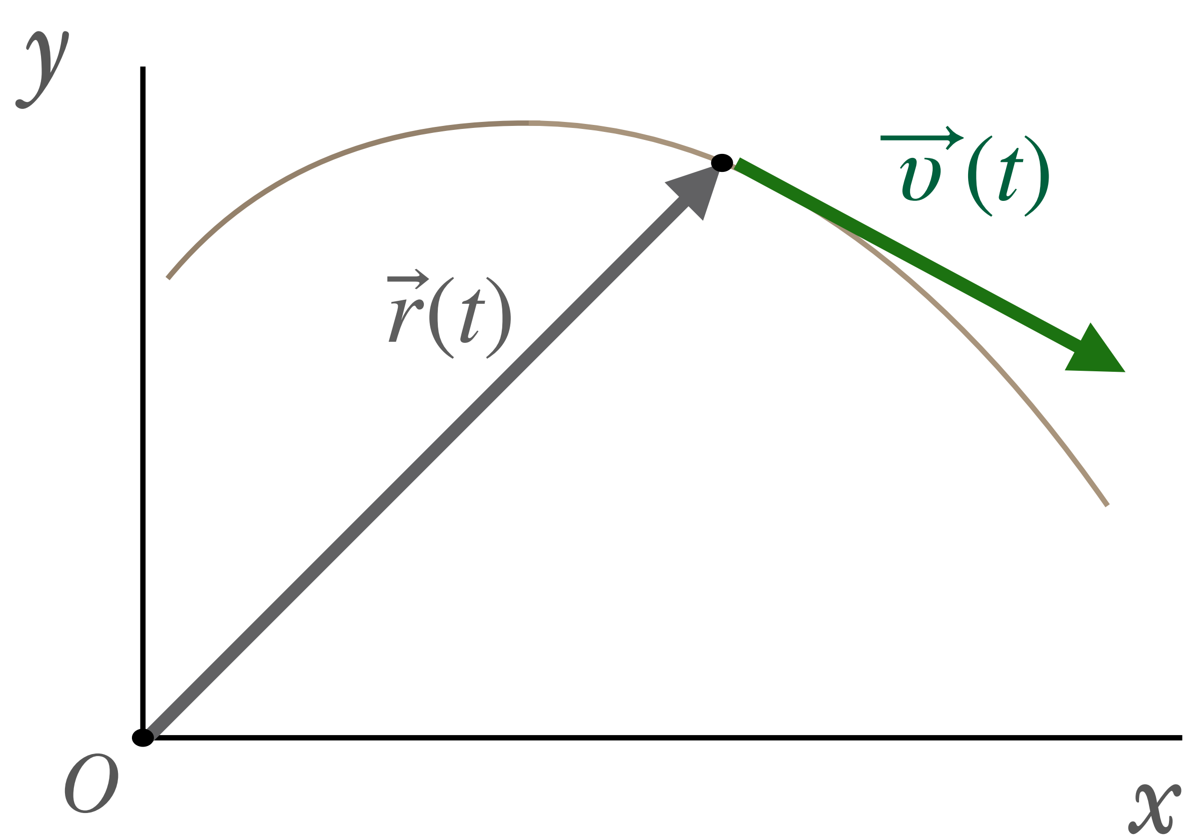
$$v_{x,avg} \equiv \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$v_{y,avg} \equiv \frac{y(t_2) - y(t_1)}{t_2 - t_1}$$

Instantaneous velocity

The instantaneous velocity in 2D is defined as the vector

$$\boxed{\vec{v}(t) \equiv \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \dot{\vec{r}}(t)}$$



$$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$$

with components:

$$v_x(t) = \dot{x}(t)$$

$$v_y(t) = \dot{y}(t)$$

Instantaneous velocity

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous velocity at time t .

Instantaneous velocity

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We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous velocity at time t.

Solution

$$\overrightarrow{v}(t) = \dot{\vec{r}}(t) = 0 \cdot\ \hat{i} - (20\ m/s^2)\ t\ \hat{j} = -\ (20\ m/s^2)\ t\ \hat{j}.$$

Acceleration

We consider a point object moving with velocity

$$v(t) = v_x(t)\hat{i} + v_y(t)\hat{j}$$

Average acceleration between t_1 and t_2:

$$\vec{a}_{avg} = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1}\hat{i} + \frac{v_y(t_2) - v_y(t_1)}{t_2 - t_1}\hat{j}$$

Instantaneous acceleration at t:

$$\vec{a}(t) = \lim_{h \to 0} \frac{\vec{v}(t+h) - \vec{v}(t)}{h} = \dot{\vec{v}}(t) = \dot{v}_x(t)\hat{i} + \dot{v}_y(t)\hat{j}$$

Instantaneous acceleration

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous acceleration at time t .

Solution

$$\overrightarrow{v}(t) = \dot{\vec{r}}(t) = 0 \cdot\ \hat{i} - (20\ m/s^2)\ t\ \hat{j} = -(20\ m/s^2)\ t\ \hat{j}.$$

Instantaneous acceleration

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}.$
Find the instantaneous acceleration at time t.

Solution

$$\overrightarrow{v}(t) = \dot{\vec{r}}(t) = 0 \cdot\ \hat{i} - (20\ m/s^2)\ t\ \hat{j} = -\ (20\ m/s^2)\ t\ \hat{j}.$$

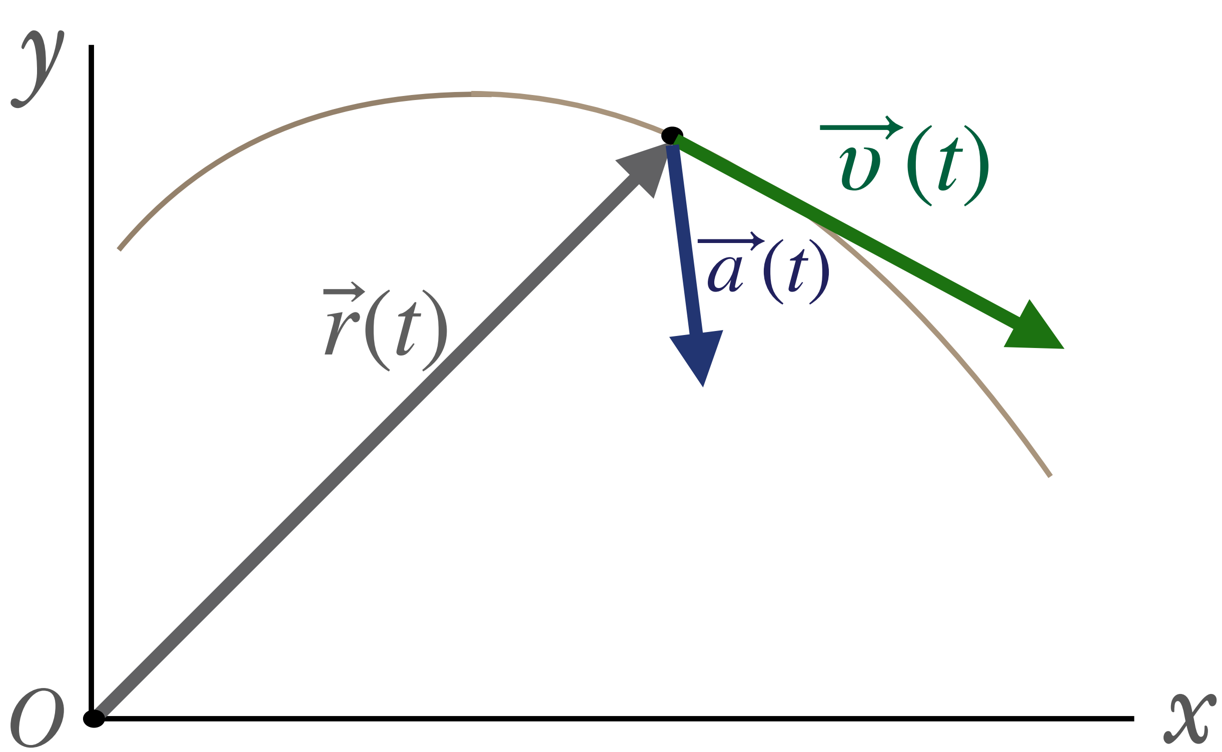
$$\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = -\ (20\ m/s^2)\ \hat{j}.$$

Summary: position, velocity and acceleration

Position $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Velocity $\overrightarrow{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$

Acceleration $\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$



Uniformly accelerated motion in 2D

The motion of a point object is said to be ***uniformly accelerated*** if at all time during the motion:

$$\boxed{\vec{a}(t) = \vec{a}}$$

where $\overrightarrow{a} = a_x\hat{i} + a_y\hat{j}$ is a constant vector.

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where $\overrightarrow{a} = a_x\hat{i} + a_y\hat{j}$ is a constant vector.

For $\overrightarrow{a}(t) = a_x(t)\hat{i} + a_y(t)\hat{j}$ we derive

$$\boxed{\begin{aligned} a_x(t) &= a_x \\ a_y(t) &= a_y \end{aligned}}$$

i.e. both components of the acceleration are constants.

Uniformly accelerated motion in 2D

In the case of uniformly accelerated motion we have:

$$a_x(t) = a_x \qquad\qquad \longleftrightarrow \qquad\qquad \ddot{x}(t) = a_x$$

$$a_y(t) = a_y \qquad\qquad\qquad\qquad\qquad \ddot{y}(t) = a_y$$

What happens in x-direction is independent of the y-direction

We literally just have to solve twice a 1D problem!

Uniformly accelerated motion in 2D

$$a_x(t) = a_x \qquad \ddot{x}(t) = a_x$$

$$a_y(t) = a_y \qquad \ddot{y}(t) = a_y$$

vectors / components	$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$	$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$
x	$v_x(t) = v_{x,0} + a_x\,t$	$x(t) = x_0 + v_{x,0}\,t + \frac{1}{2}a_x\,t^2$
y	$v_y(t) = v_{y,0} + a_y\,t$	$y(t) = y_0 + v_{y,0}\,t + \frac{1}{2}a_y\,t^2$

The relativity of motion

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

* Although they are running Alice and Julie are not moving with respect to each other

* They nevertheless move with respect to the stop sign

Relative motion in equation

* How could we express the fact that Alice and Julie do not move relative to each other?

  Introduce the point objects A for Alice and J for Julie. It then follows that

$$\boxed{\overrightarrow{\mathrm{AJ}} = \textit{constant} \text{ or } \dot{\overrightarrow{\mathrm{AJ}}} = 0}$$

Relative motion in equation

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Introduce the point objects A for Alice and J for Julie. It then follows that

$$\boxed{\overrightarrow{\mathrm{AJ}} = \textit{constant} \text{ or } \dot{\overrightarrow{\mathrm{AJ}}} = 0}$$

* How could we express the fact that Alice and Julie move however with respect to the stop sign?

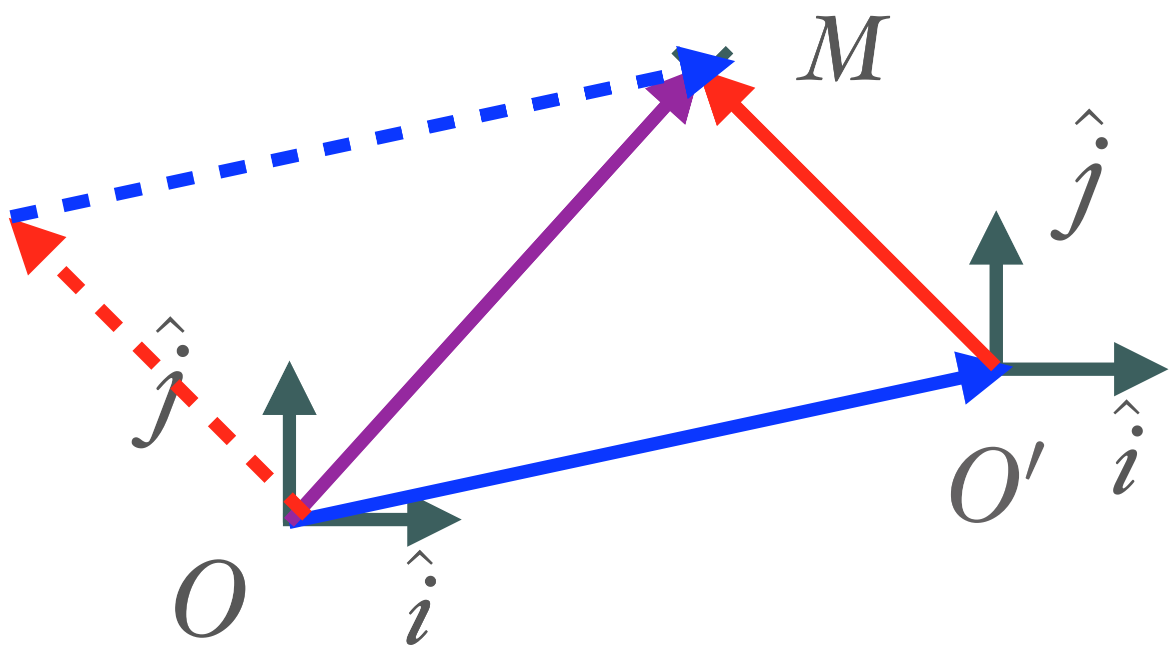
Introduce the point object S for the stop sign. It then follows that

$$\boxed{\dot{\overrightarrow{\mathrm{SA}}} \neq 0 \text{ and } \dot{\overrightarrow{\mathrm{SJ}}} \neq 0}$$

Relative motion for frames in translation

We consider two frames $\mathcal{F} = (O,\, \hat{i},\, \hat{j})$ and $\mathcal{F}\,' = (O',\, \hat{i},\, \hat{j})$ and the point object M

$$\boxed{\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}}$$



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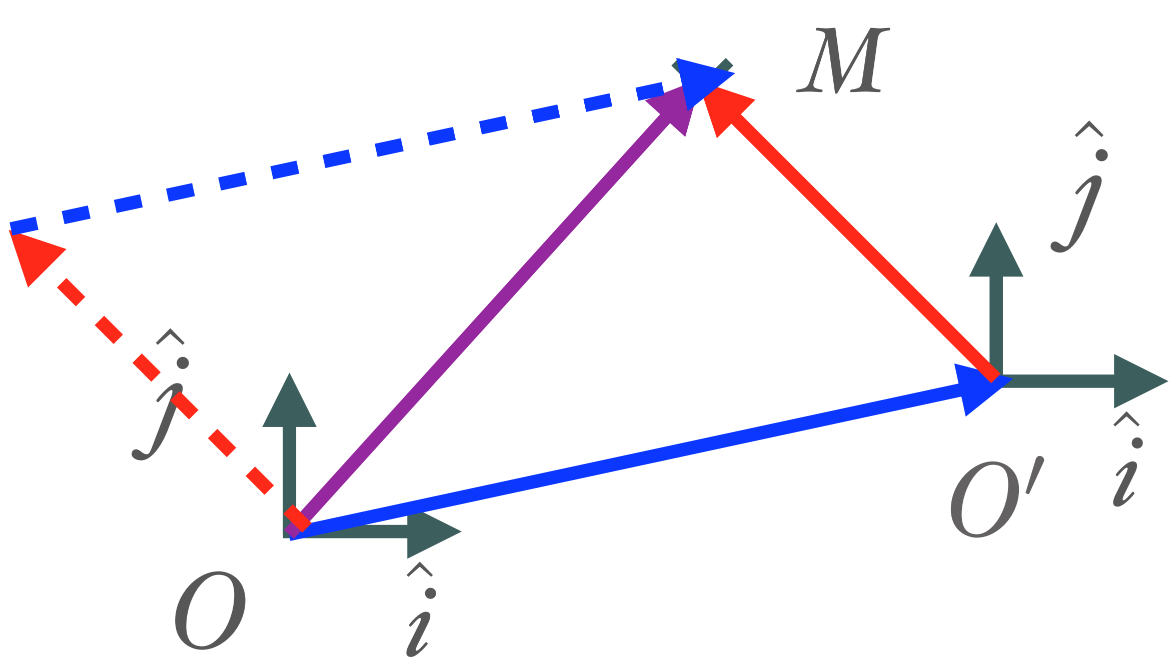

$$\boxed{\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}}$$

The velocity and the acceleration of a point M relative to O is defined as

$$\vec{v}(M|O) \equiv \dot{\overrightarrow{OM}}$$
$$\vec{a}(M|O) \equiv \dot{\vec{v}}(M|O) = \ddot{\overrightarrow{OM}}$$

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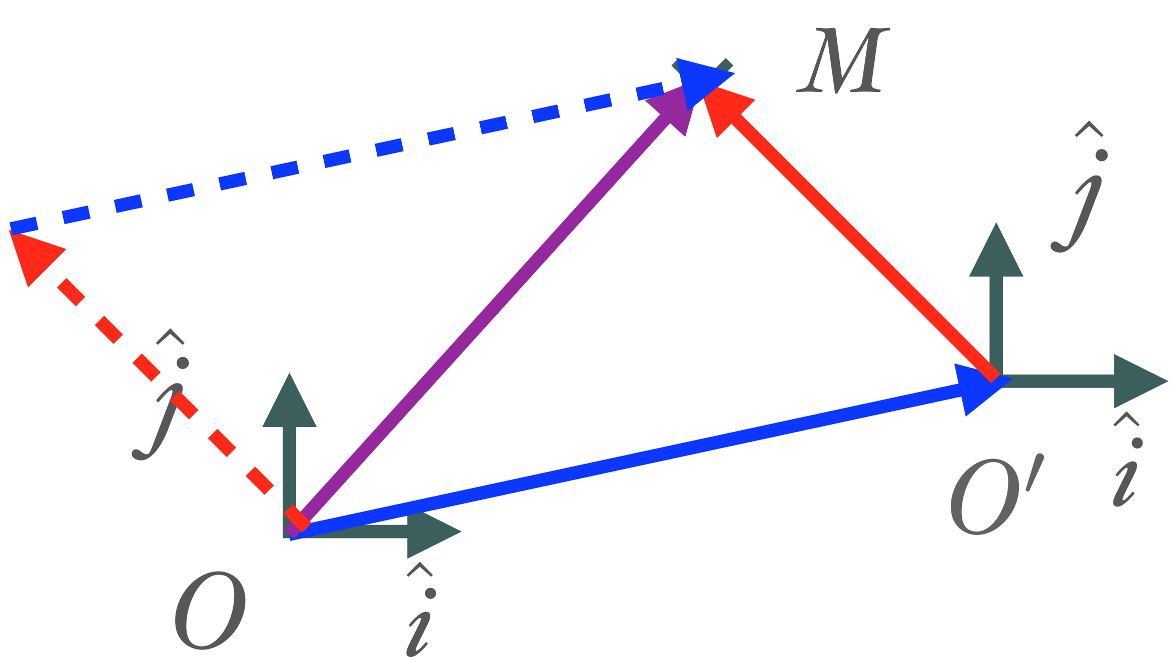
We consider two frames $\mathcal{F} = (O, \hat{i}, \hat{j})$ and $\mathcal{F}' = (O', \hat{i}, \hat{j})$ and the point object M



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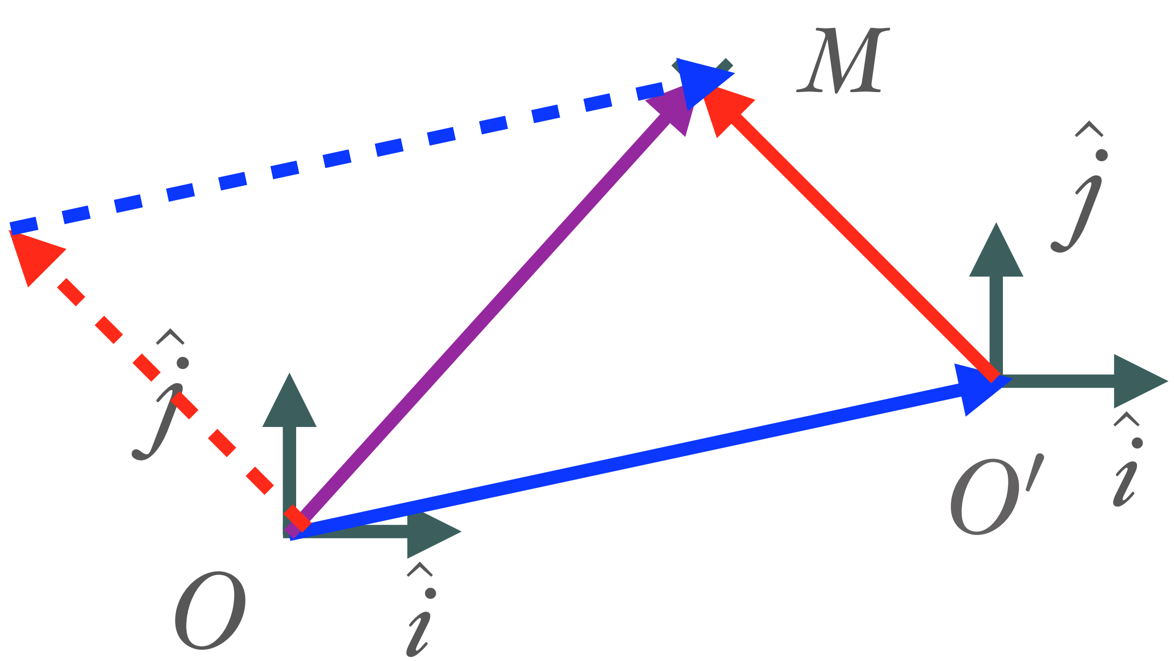
$$\dot{\overrightarrow{OM}} = \dot{\overrightarrow{OO'}} + \dot{\overrightarrow{O'M}} \longrightarrow \overrightarrow{v}(M|O) = \overrightarrow{v}(O'|O) + \overrightarrow{v}(M|O')$$

$$\ddot{\overrightarrow{OM}} = \ddot{\overrightarrow{OO'}} + \ddot{\overrightarrow{O'M}} \longrightarrow \overrightarrow{a}(M|O) = \overrightarrow{a}(O'|O) + \overrightarrow{a}(M|O')$$

Relative motion for frames in translation



Law of composition of velocities $\overrightarrow{v}(M\,|\,O) = \overrightarrow{v}(O'\,|\,O) + \overrightarrow{v}(M\,|\,O')$

Law of composition of accelerations $\overrightarrow{a}(M\,|\,O) = \overrightarrow{a}(O'\,|\,O) + \overrightarrow{a}(M\,|\,O')$

Relative motion

Example

Julie (left) and Alice (right) running

Let us suppose that the velocity of Julie relative to the stop is

$$\vec{v}(\mathrm{J}|\mathrm{S}) = (7\ \mathrm{km}\cdot\mathrm{h}^{-1})\ \hat{\mathrm{i}}$$

and that the velocity of Alice relative to Julie is zero.

Find the velocity and the acceleration of Alice relative to the stop.

Relative motion

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Julie (left) and Alice (right) running

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and that the velocity of Alice relative to Julie is zero.

Find the velocity and the acceleration of Alice relative to the stop.

Solution

$$\vec{v}(\mathrm{A}|\mathrm{S}) = \vec{v}(\mathrm{A}|\mathrm{J}) + \vec{v}(\mathrm{J}|\mathrm{S}) = (7\ \mathrm{km} \cdot \mathrm{h}^{-1})\ \hat{i}$$

$$\vec{a}(\mathrm{A}\,|\,S) = \dot{\vec{v}}(\mathrm{A}\,|\,S) = 0$$

The Laws of Motion

The three laws of Newton

- **Law 1**: *A body continues in its state of rest, or in uniform motion in a straight line (motion with constant velocity), unless acted upon by a force.*

- **Law 2**: *The acceleration produced when a force acts is directly proportional to the force and takes place in the direction in which the force acts.*

i

- **Law 3**: *To every action there is an equal and opposite reaction: whenever one object exerts a force on another object, the second object exerts an equal in magnitude and opposite in direction force on the first.*

Newton's First Law

the law of inertia

Deciphering Newton's 1st law

Newton rightly pointed out that if **both** the full relativity of motion **and** his first law are true at the same time, then contradictions arise.

Deciphering Newton’s 1st law

Take for example two free falling objects: an apple (A) and a stone (S) which are dropped from the top of the Tower of Pisa with zero initial velocity.

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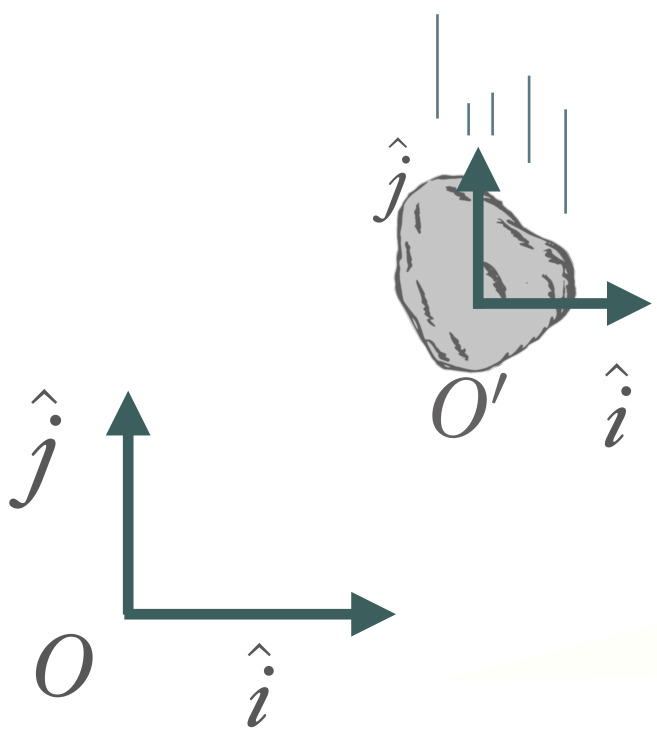

We can consider a frame of references $F(O, \hat{i}, \hat{j})$ on the ground and a second one $F(O', \hat{i}, \hat{j})$ attached to the stone.

Deciphering Newton's 1st law

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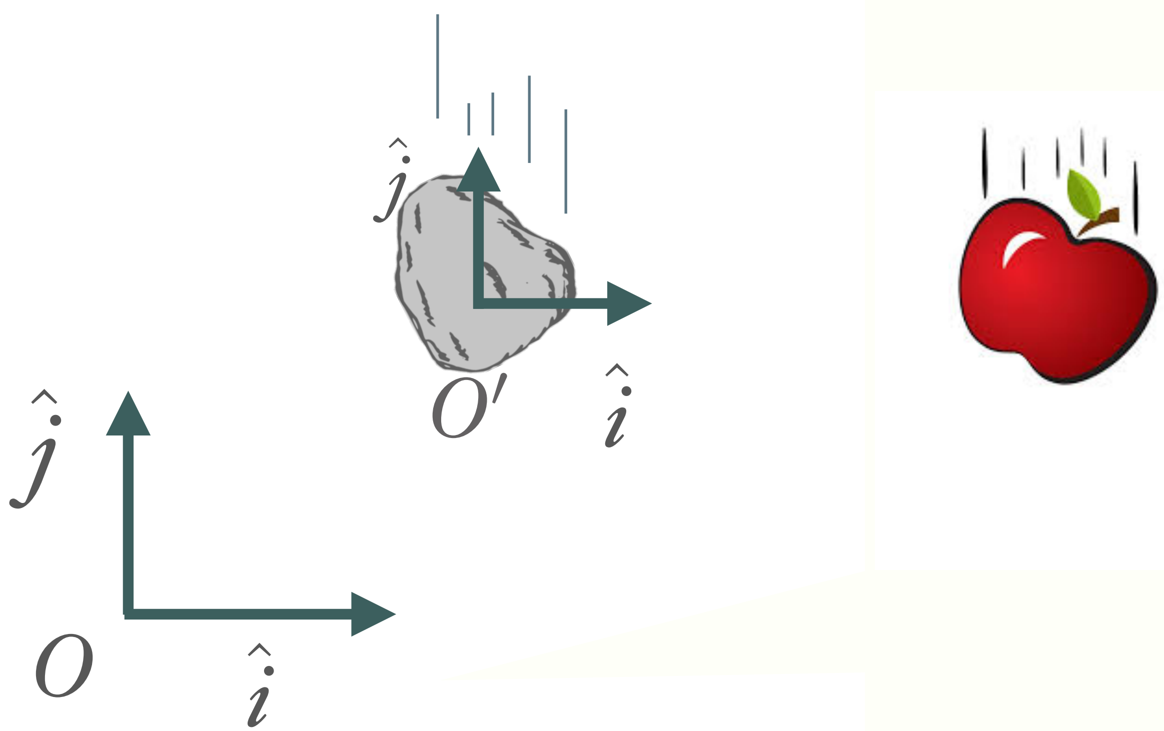

We can consider a frame of references $F(O, \hat{i}, \hat{j})$ on the ground and a second one $F(O', \hat{i}, \hat{j})$ attached to the stone.

Galileo tells us that $\overrightarrow{v}(A\,|\,O) = \overrightarrow{v}(S\,|\,O)$. Also by the law of composition of velocities $\overrightarrow{v}(A\,|\,O) = \overrightarrow{v}(S\,|\,O) + \overrightarrow{v}(A\,|\,O')$, hence

$\overrightarrow{v}(A\,|\,O') = 0$, i.e. the apple remains at rest with respect to the frame of reference of the stone.

Deciphering Newton's 1st law

Newton rightly pointed out that if **both** the full relativity of motion **and** his first law are true at the same time, then contradictions arise.

Any motion, even under impressed forces, can look like a uniform motion in a special chosen frame of reference.

This fact seems to contradict the fist law of motion!

Deciphering Newton's 1st law

Newton's first law holds only with respect to a certain very special class of frames of reference which are called ***inertial frames of reference*** or ***Galilean frames of reference***.

An inertial frame of reference is, by definition, one in which isolated objects, not subject to forces, move at constant velocity, i.e. a frame of reference in which Newton's first law holds.

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An inertial frame of reference is, by definition, one in which isolated objects, not subject to forces, move at constant velocity, i.e. a frame of reference in which Newton's first law holds.

By the law of composition of velocities any frame in uniform translation (moving with constant velocity and not rotating) with respect to an inertial frame of reference will also witness a uniform motion in absence of external forces, hence it constitutes an inertial frame of reference as well.

Deciphering Newton's 1st law

Reformulation of the first law:

There exists a class of reference frames, called inertial or Galilean frames, such that for any Galilean frame F and any moving point object M the following holds:

If $\vec{a}(M|F) = 0$, then no net motive force is acting on M and conversely, if no net motive force is acting on M then $\vec{a}(M|F) = 0$.

Deciphering Newton's 1st law

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The special principle of relativity

The laws of mechanics are the same with respect to all Galilean frames of reference.

Deciphering Newton's 1st law

Example of application of the fist law:

We consider a Galilean frame $(O,\ \hat{i},\ \hat{j})$ in which we observe the motion of a point object with the following position vector

$$\vec{r} = (3\ \mathrm{cm})\,\hat{i} + (-87\ \mathrm{cm}\cdot\mathrm{s}^{-1})\,t\,\hat{j}$$

Question: Is there any net force acting on this point object?

Deciphering Newton's 1st law

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Answer: We answer this question in two steps. First we calculate the velocity vector and then we apply the first law as expressed in the previous slide.

Deciphering Newton's 1st law

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$$\vec{v} = (-87\ \mathrm{cm}\cdot\mathrm{s}^{-1})\ \hat{j}$$

The velocity vector is ***constant***, therefore, by the 1st law of Newton, there is ***no net force*** acting on the point object.

Newton's Third Law

the law of action-reaction

Deciphering Newton's 3rd law

To every action there is an equal and opposite reaction: whenever one object exerts a force on another object, the second object exerts an equal in magnitude and opposite in direction force on the first.

Deciphering Newton's 3rd law

To every action there is an equal and opposite reaction: whenever one object exerts a force on another object, the second object exerts an equal in magnitude and opposite in direction force on the first.

Forces are ***vector quantities*** with a magnitude and a direction. They are often denoted $\overrightarrow{F}$ (not be confused with reference frames!!!)

Let us consider 2 bodies/objects 1 and 2. Being in mutual interaction means that they exert forces on each other. We denote $\overrightarrow{F}_{1\to 2}$ the force of object 1 on object 2 and $\overrightarrow{F}_{2\to 1}$ the force of object 2 on object 1.

Deciphering Newton's 3rd law

To every action there is an equal and opposite reaction: whenever one object exerts a force on another object, the second object exerts an equal in magnitude and opposite in direction force on the first.

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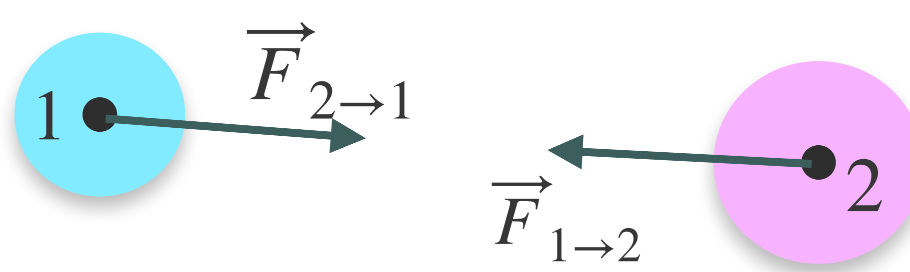

Deciphering Newton's 3rd law

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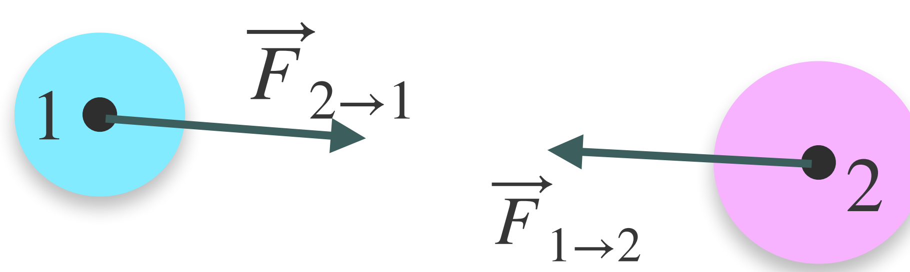

Newton's 3rd law: $\overrightarrow{F}_{1\to 2} = -\overrightarrow{F}_{2\to 1}$

Newton's Second Law

the fundamental law of dynamics

Deciphering Newton’s 2nd law

- ***The acceleration produced when a force acts is directly proportional to the force and takes place in the direction in which the force acts.***

Deciphering Newton's 2nd law

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Reformulation of the 2nd law

1. Let $\overrightarrow{a}(t)$ be the acceleration of a point object M as observed in a Galilean frame $(O, \hat{i} \,.\, \hat{j})$
2. Let $\overrightarrow{F}$ be the force vector characterising the force impressed on M

   Then

Deciphering Newton's 2nd law

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Deciphering Newton's 2nd law

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Then

Inertial mass

$$\boxed{m\overrightarrow{a}(t) = \overrightarrow{F}}$$

Deciphering Newton's 2nd law

Force and mass

The concept of the force action that we have introduced is new and was not present in kinematics.

Yet, in Newtonian mechanics forces exist, we can talk about them, about their action on objects...but without being able to clearly define them; a bit like space and time in fact.

Alternatively, the 2nd law enables the force concept to be derived from space, time and yet another concept the ***inertial mass*** (or just mass).

Deciphering Newton’s 2nd law

- The (inertial) **mass** of an object is a (scalar) quantity associated with the amount of material that is present in the object. It is that property of an object that specifies how much resistance an object exhibits to changes in its velocity

- Mass is an inherent property of an object and is independent of the object’s surroundings and of the method used to measure it.

- In mechanics, we consider the mass as a primitive concept akin to space and time.

Physical dimensions of mass and force

The inertial mass is a primitive concept like space and time and must be measured with respect to a mass standard (unit) u_m .

Any mass can then be expressed as $m = ru_m$, where r is a positive number and u_m a unit of mass.

$$\boxed{\text{The SI unit of mass is the kilogram (kg)}}$$

- **Dimension:** $[m] = [u_m] = M$

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The three fundamental quantities in mechanics are length, mass and time. All other quantities can be expressed in terms of these three.

Physical dimensions of mass and force

The force concept then derives from the inertial mass, space and time concepts by application of the second law: $\overrightarrow{F} = m\overrightarrow{a}$

- Dimension: $[\overrightarrow{F}] = [m\overrightarrow{a}] = [m][\overrightarrow{a}] = M \cdot L \cdot T^{-2}$

The SI unit of force is the Newton (N)

$$1\ \mathrm{N} = 1\ \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$$

Newton's 2nd law

<u>Example</u>

A 3kg object is moving with velocity

$$\overrightarrow{v}(t) = (2t \ ms^{-2})\hat{i} + (5t \ ms^{-2})\hat{j}.$$

Find the resultant force (*net force*) acting on the object.

Newton's 2nd law

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$$\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = (2\hat{i} + 5\hat{j})m/s^2\,.$$

$$\overrightarrow{F} = m\overrightarrow{a} = (3kg)(2\hat{i} + 5\hat{j})m/s^2 = (6\hat{i} + 15\hat{j})N$$

Newton's 2nd law

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The **net force** is the vector sum of forces acting on a particle or object. The net force is a single force that replaces the effect of the original forces on the particle's motion.

Introducing the notion of weight

We call ***weight*** and denote $\overrightarrow{W}$, the ***force impressed*** on a body pulling it towards the ground on Earth.

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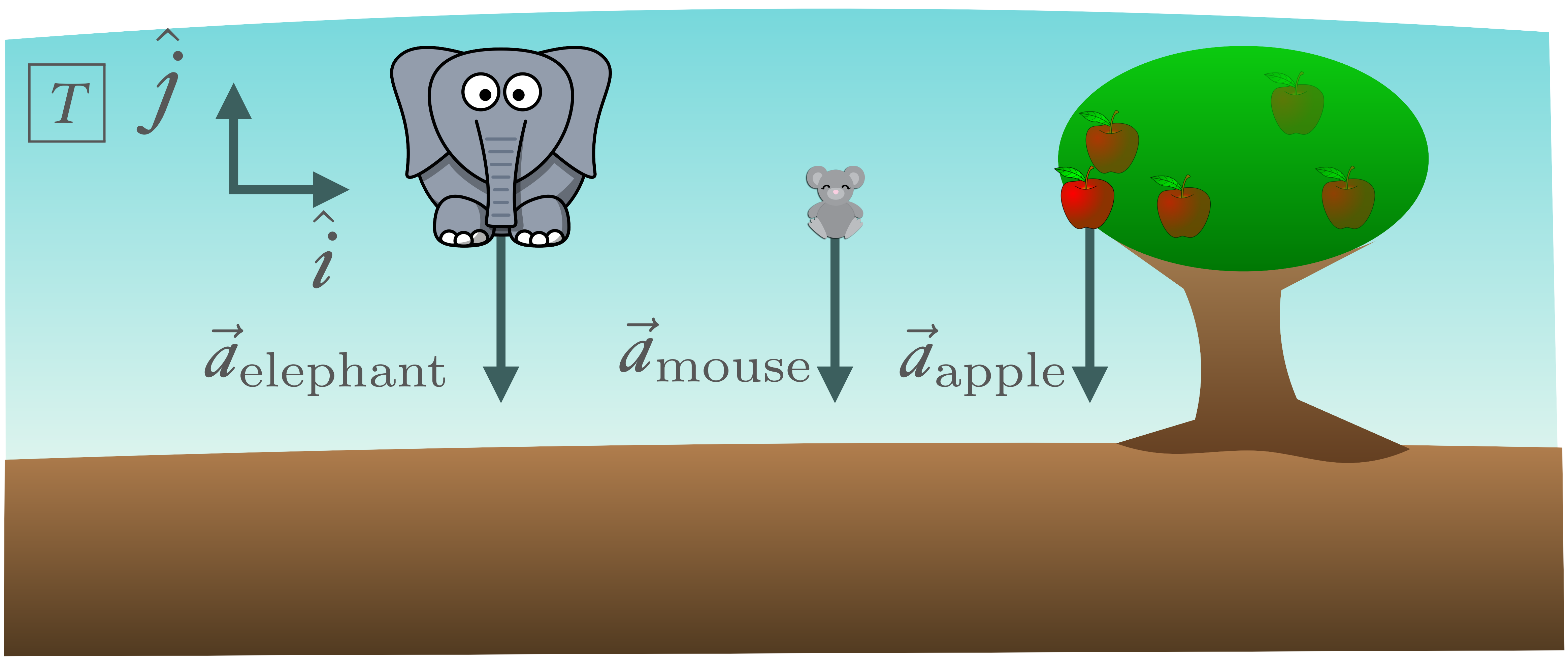
- According to Galileo: "*Provided air resistance can be neglected, all bodies let go from the same place **fall on Earth with the same acceleration** as observed from a terrestrial frame T*"



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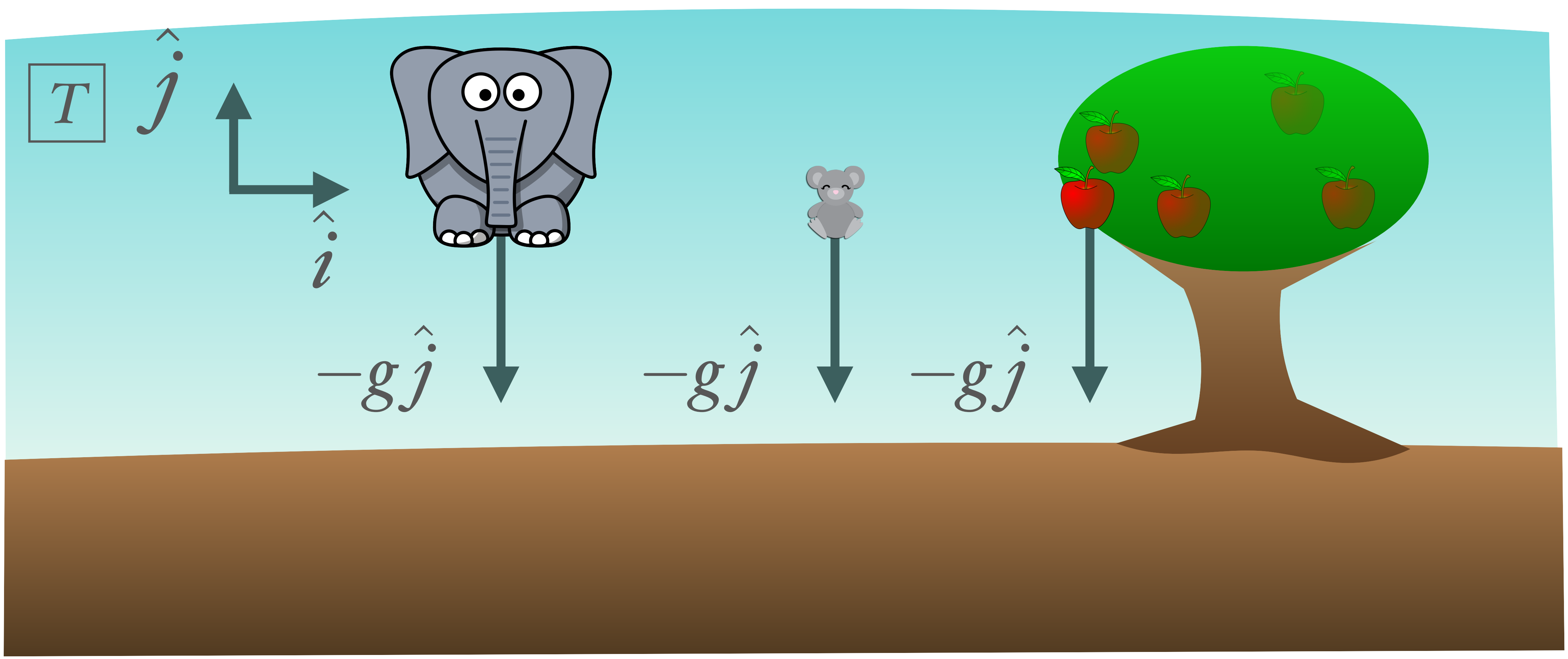

Quantifying the weight

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$$\overrightarrow{a} = \overrightarrow{a}_{elephant} = \overrightarrow{a}_{mouse} = \overrightarrow{a}_{apple} = -g\hat{j} \qquad (g \approx 9.81 \mathrm{ms}^{-2})$$

Quantifying the weight

We call ***weight*** and denote $\overrightarrow{W}$, the ***force impressed*** on a body pulling it towards the ground on Earth.

- According to Galileo: "*Provided air resistance can be neglected, all bodies let go from the same place* ***fall on Earth with the same acceleration*** *as observed from a terrestrial frame T*"

$$\overrightarrow{a} = \overrightarrow{a}_{elephant} = \overrightarrow{a}_{mouse} = \overrightarrow{a}_{apple} = -g\hat{j} \qquad (g \approx 9.81 \mathrm{ms}^{-2})$$

- According to Newton's 2nd law, if T is a Galilean frame then

$$\overrightarrow{W} = m\overrightarrow{a}$$

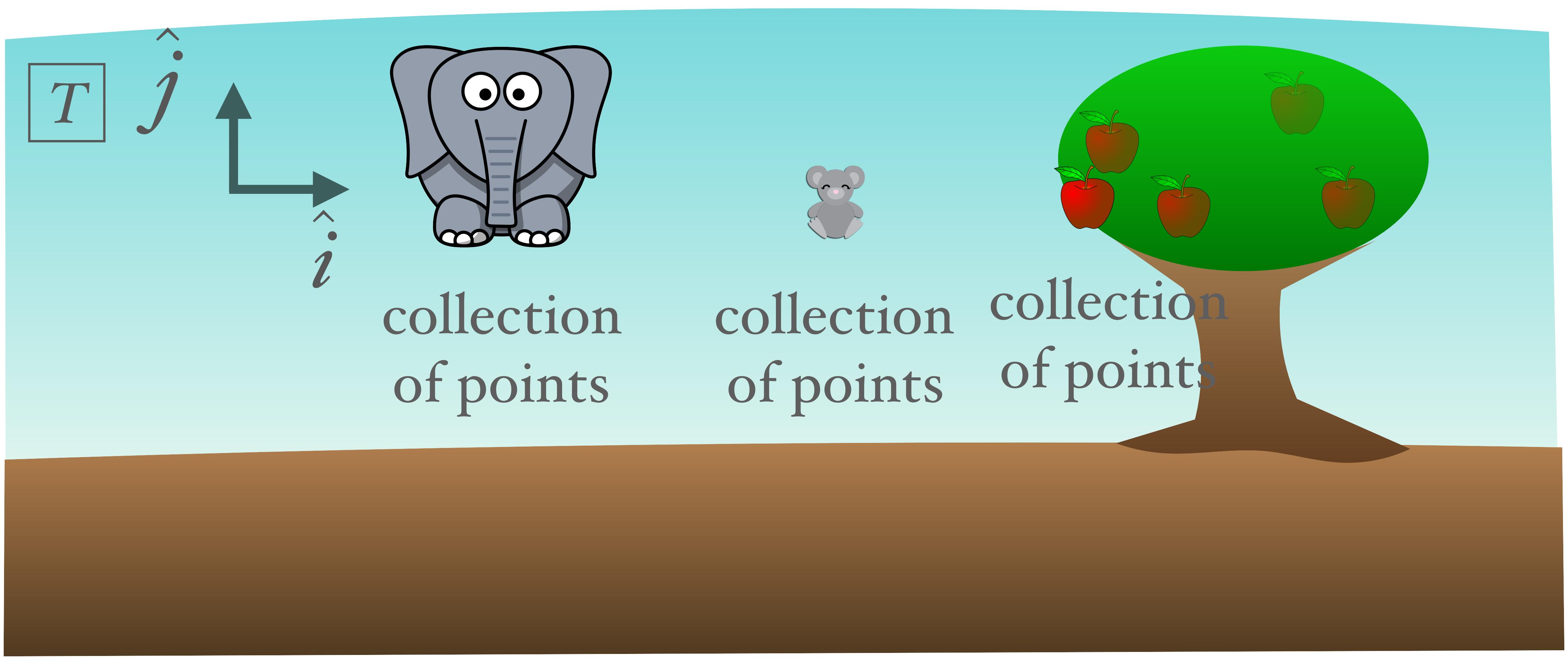
- By comparison with Galileo's proposal we get

$$\boxed{\overrightarrow{W} = -mg\hat{j}}$$

Why the concept of point object works

System comprising multiple parts

The laws of Newton concern the motion of point objects on which forces are acting. In reality, however a real object is not a point but an extended object; in other words a ***collection of points***



System comprising multiple parts

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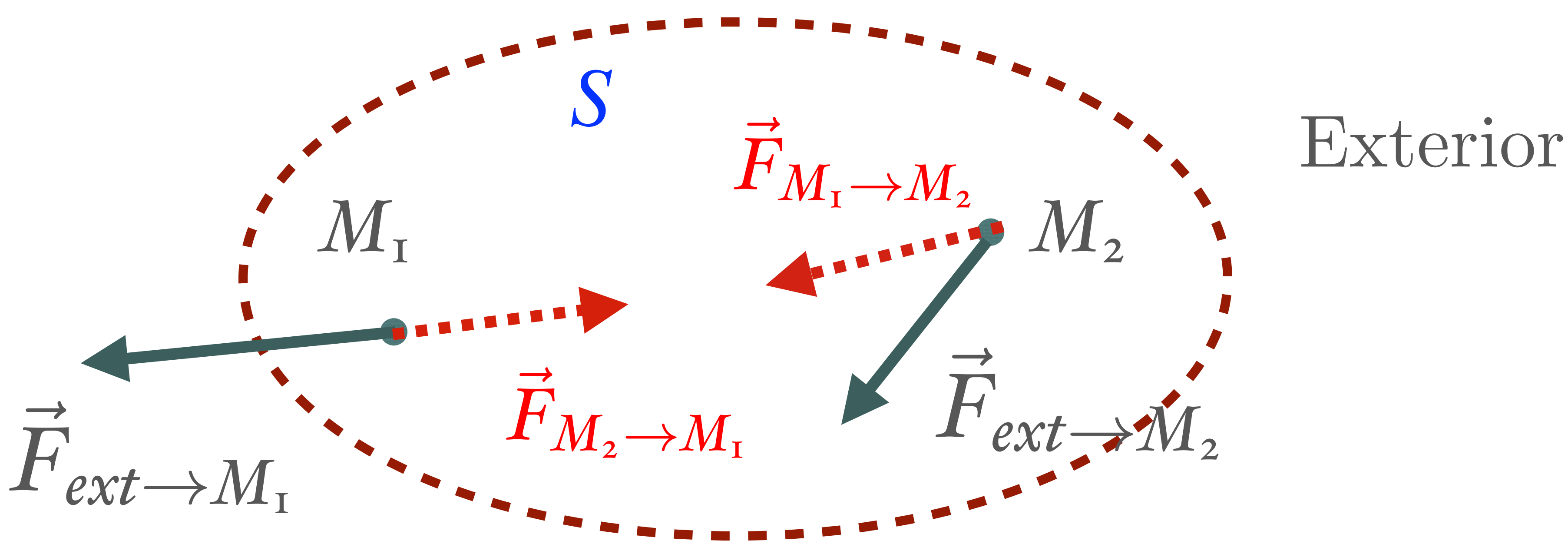
We consider a system S comprising two point objects M_1 and M_2 with respective masses m_1 and m_2 each subject to external forces $\overrightarrow{F}_{ext\to M_1}$ and $\overrightarrow{F}_{ext\to M_2}$ and moreover having a mutual interaction.



System comprising multiple parts

If the dynamics of the system S is observed from a Galilean frame of reference, then Newton's 2nd law applies for each part of the system:

$$m_1\overrightarrow{a_1} = \overrightarrow{F}_{ext \to M_1} + \overrightarrow{F}_{M_2 \to M_1}$$

$$m_2\overrightarrow{a_2} = \overrightarrow{F}_{ext \to M_2} + \overrightarrow{F}_{M_1 \to M_2}$$

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$$\overrightarrow{F}_{M_2\to M_1} + \overrightarrow{F}_{M_1\to M_2} = \vec{0}$$

Newton's 3rd law

System comprising multiple parts

If the dynamics of the system S is observed from a Galilean frame of reference, then Newton's 2nd law applies for each part of the system:

$$? = \quad m_1\overrightarrow{a_1} = \overrightarrow{F}_{ext\to M_1} + \overrightarrow{F}_{M_2\to M_1}$$

$$+$$

$$m_2\overrightarrow{a_2} = \overrightarrow{F}_{ext\to M_2} + \overrightarrow{F}_{M_1\to M_2} \quad = \vec{0}$$

Newton's 3rd law

Reminder on the centre of mass

The centre of mass (CM) of a system S of N points $M_1, \ldots, M_N$ with masses $m_1, \ldots, m_N$ is the geometric point M_{cm} of mass $m_s = m_1 + m_2 + \ldots + m_N$, whose position vector relative to an origin O is given by:

$$\boxed{\vec{r}_{cm} = \frac{1}{m_s} \sum_{i=1}^{N} m_i \vec{r}_i}$$

Reminder on the centre of mass

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$$\boxed{\vec{r}_{cm} = \frac{1}{m_s}\sum_{i=1}^{N} m_i \vec{r}_i}$$

Example: For N = 2, $\vec{r}_{cm} = \dfrac{1}{m_1 + m_2}(m_1\vec{r}_1 + m_2\vec{r}_2)$

with acceleration: $\overrightarrow{a}_{cm} = \ddot{\vec{r}}_{cm} = \dfrac{1}{m_1 + m_2}(m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2)$

$$\longrightarrow \quad \boxed{(m_1 + m_2)\overrightarrow{a}_{cm} = (m_1\overrightarrow{a}_1 + m_2\overrightarrow{a}_2)}$$

System comprising multiple parts

If the dynamics of the system S is observed from a Galilean frame of reference, then Newton's 2nd law applies for each part of the system:

$$m_S\,\vec{a}_{cm} = \begin{array}{c} m_1\vec{a}_1 = \vec{F}_{ext\to M_1} + \vec{F}_{M_2\to M_1} \\ + \\ m_2\vec{a}_2 = \vec{F}_{ext\to M_2} + \vec{F}_{M_1\to M_2} \end{array} = \vec{0}$$

Newton's 3rd law

$$m_S\,\vec{a}_{cm} = \vec{F}_{ext\to M_1} + \vec{F}_{ext\to M_2}$$

System comprising multiple parts

For any extended system S, the dynamics of its centre of mass, as seen in a Galilean frame, is entirely specified by the following equation of motion:

$$\boxed{m_S\,\vec{a}_{cm} = \sum \vec{F}_{ext \to S}}$$

Where the $m_s = m_1 + \ldots + m_n$ is the total mass and $\overrightarrow{a}_{cm}$ the acceleration of the centre of mass.

System comprising multiple parts

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Where the $m_s = m_1 + \ldots + m_n$ is the total mass and $\overrightarrow{a}_{cm}$ the acceleration of the centre of mass.

The system S could be stretched or rotating so that the above equation does not fully characterise the dynamics of S.

However, the centre of mass of S is always a good geometrical point to characterise the translation of S via the above equation.

Practical classification of mechanical forces

- ***Forces at a distance (related to "fundamental forces")***
  - Act at any distance from the object they act upon
  - The net point of application of the force on the object is usually the centre of mass
  - Examples: weight, gravitational force, electric force, magnetic force.
  - Can often be determined by formulae or calculations

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* ***Contact forces***
  - A force that acts only when there is physical contact between two bodies (act solely at the point of contact)
  - Are usually unknown and need supplementary informations to be determined
  - Examples: applied force, reaction force, friction, tension, air resistance etc.

Vertical free fall

A ball of mass m is dropped with initial zero velocity from the top of Lincoln's cathedral whose position vector is $\vec{r}_\circ = (83\ \mathrm{m})\,\hat{j}$ in a frame $(O,\ \hat{i},\ \hat{j})$ assumed Galilean.

<u>Question</u>: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

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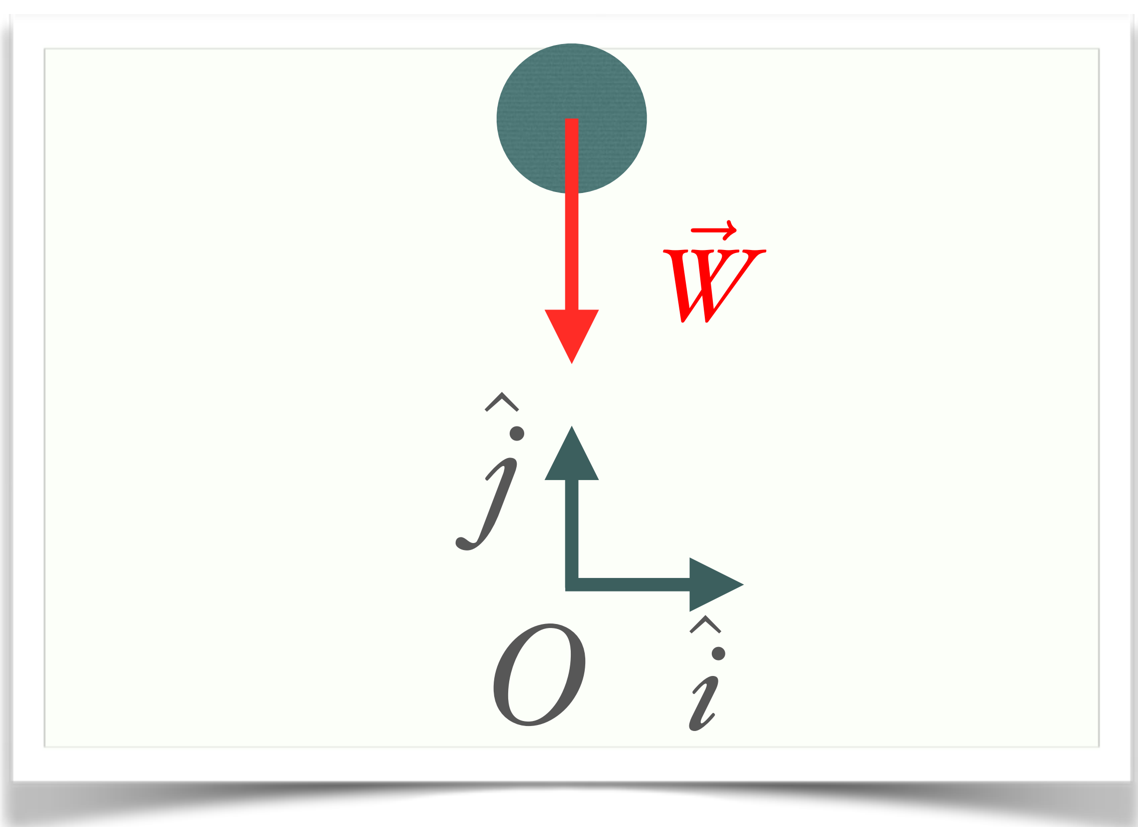
Question: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

Answer: we shall answer this question in multiple steps.

1st step: make a diagram with all forces



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Question: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

Answer: we shall answer this question in multiple steps.

2nd step: list all the forces acting on the ball

Contact forces	Forces at a distance
None	Weight: $\vec{W} = -m\,g\,\hat{j}$

Vertical free fall

A ball of mass m is dropped with initial zero velocity from the top of Lincoln's cathedral whose position vector is $\vec{r}_{\circ} = (83\ \mathrm{m})\,\hat{j}$ in a frame $(O,\ \hat{i},\ \hat{j})$ assumed Galilean.

<u>Question</u>: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

<u>Answer</u>: we shall answer this question in multiple steps.

3rd step: use Newton's second law

The frame is Galilean therefore Newton's 2nd law applies

$$m\,\vec{a} = \vec{W} = -m\,g\,\hat{j}$$

4th step: get the acceleration vector and components

$$\vec{a} = -g\,\hat{j}\ .\ So,\ a_x = 0,\ a_y = -\,g\ .$$

Vertical free fall

A ball of mass m is dropped with initial zero velocity from the top of Lincoln's cathedral whose position vector is $\vec{r}_\circ = (83\ \mathrm{m})\,\hat{j}$ in a frame $(O,\ \hat{i},\ \hat{j})$ assumed Galilean.

Question: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

Answer: we shall answer this question in multiple steps.

5th step: integrate to get the velocity components and vector

$$\dot{v}_x = a_x = 0,\ so\ v_x(t) = v_x(0) = 0$$

$$\dot{v}_y = a_y = -g,\ so\ v_y(t) = v_y(0) - gt = -gt$$

$$\boxed{\vec{v} = -g\,t\,\hat{j}}$$

Vertical free fall

A ball of mass m is dropped with initial zero velocity from the top of Lincoln's cathedral whose position vector is $\vec{r}_\circ = (83\ \mathrm{m})\,\hat{j}$ in a frame $(O,\ \hat{i},\ \hat{j})$ assumed Galilean.

<u>Question</u>: assuming the air resistance can be neglected, determine the acceleration, velocity and position vectors of the ball.

<u>Answer</u>: we shall answer this question in multiple steps.

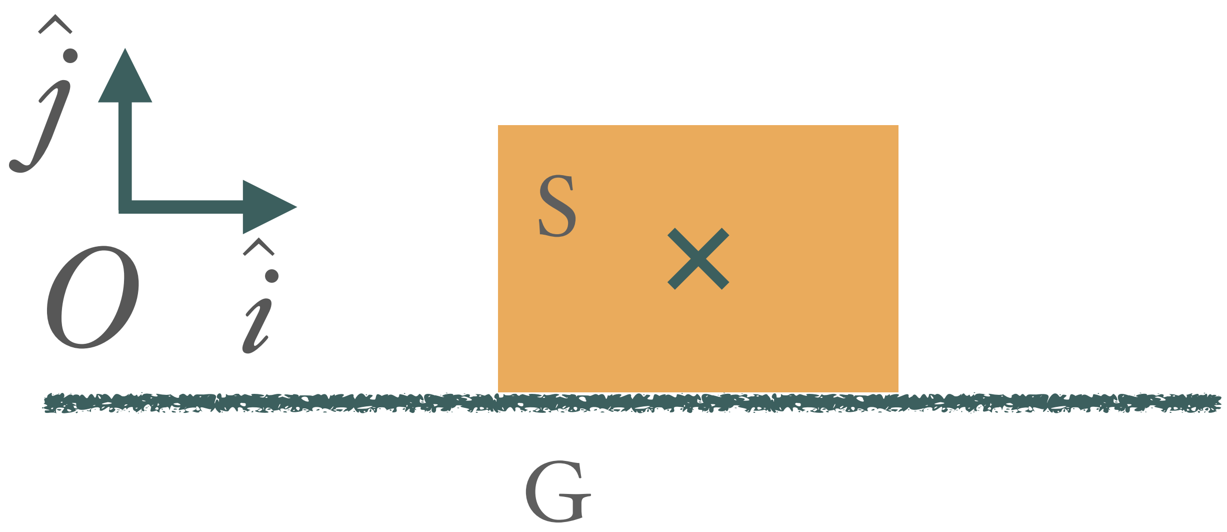
6***th step***: integrate to get the position vector

$$\dot{x}(t) = v_x(t) = 0,\ so\ x(t) = x(0) = 0$$

$$\dot{y}(t) = v_y(t) = -gt,\ so\ v_y(t) = y(0) - \frac{1}{2}gt^2 = 83m - \frac{1}{2}(9.8\ m\cdot s^{-2})t^2$$

$$\boxed{\vec{r}(t) = [(83\ \mathrm{m}) - (4.9\ \mathrm{m}\cdot\mathrm{s}^{-2})\,t^2]\ \hat{j}}$$

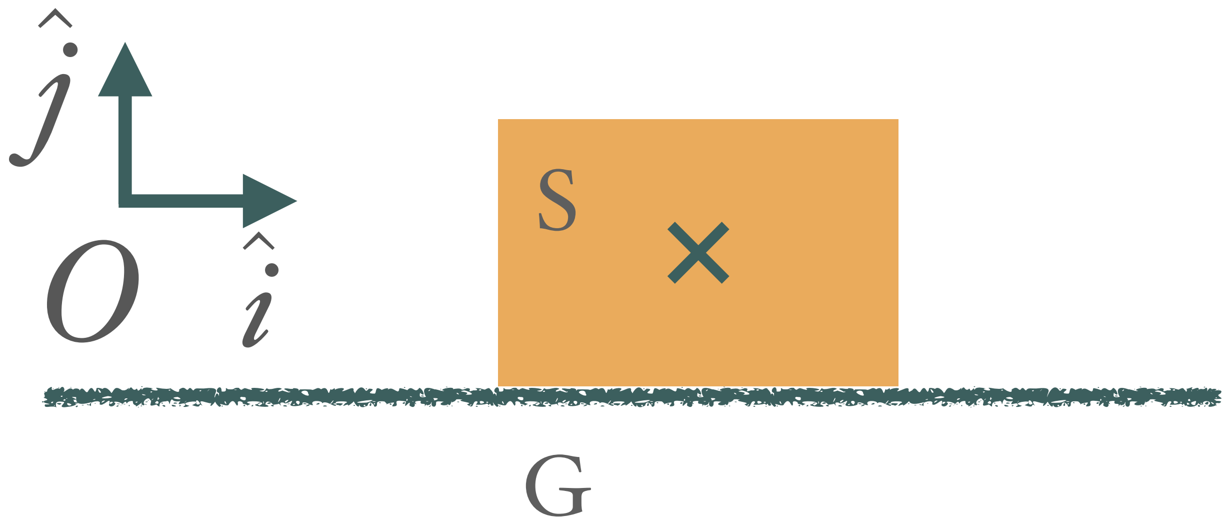
First encounter with contact forces



A block S of mass m is resting without motion on the ground G in the frame $(O, \hat{i}, \hat{j})$ considered Galilean.

<u>Question</u>: Determine the force exerted by the block on the ground.

First encounter with contact forces

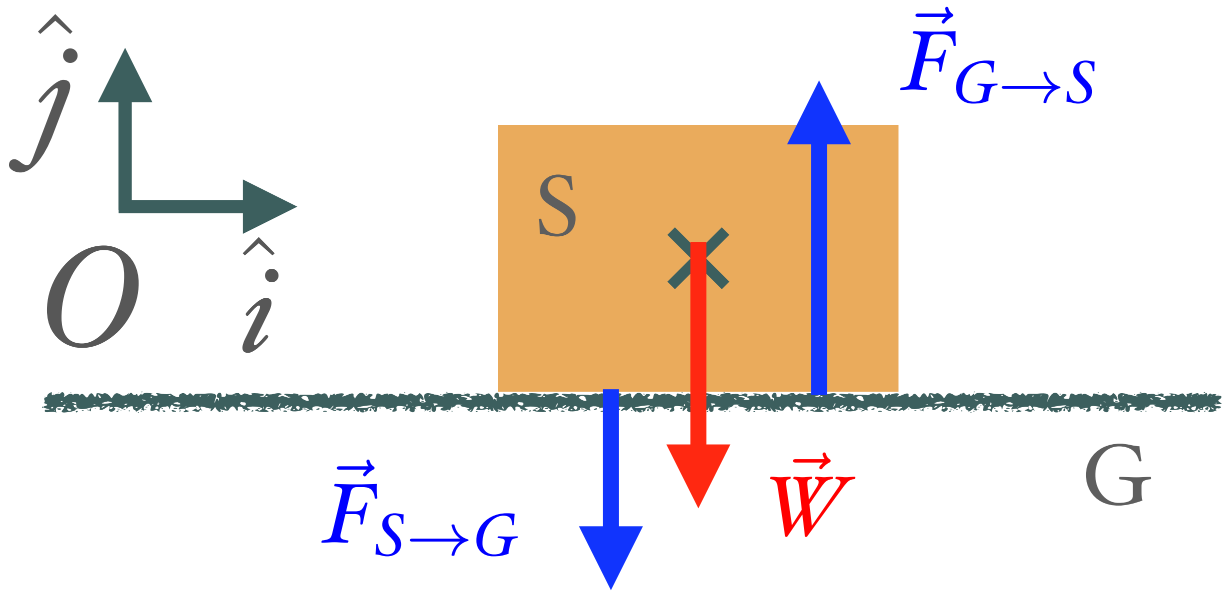


A block S of mass m is resting without motion on the ground G in the frame $(O, \hat{i}, \hat{j})$ considered Galilean.

Question: Determine the force exerted by the block on the ground.

Answer: Again we proceed in multiple steps

First encounter with contact forces



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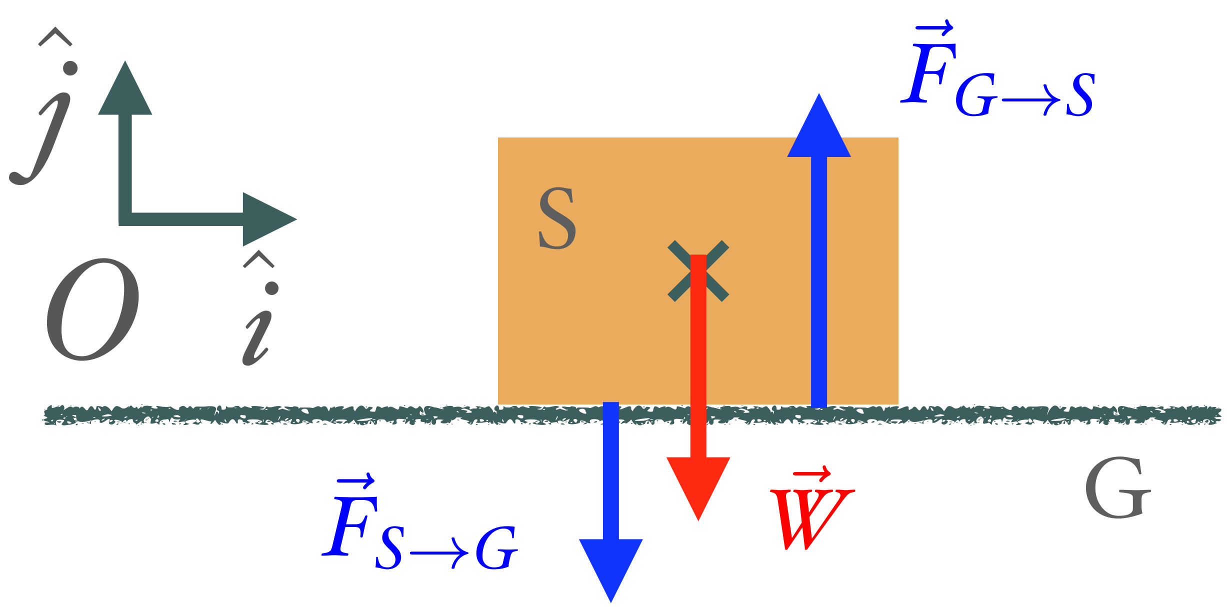
Question: Determine the force exerted by the block on the ground.

Answer: Again we proceed in multiple steps

1st step: make a diagram

First encounter with contact forces



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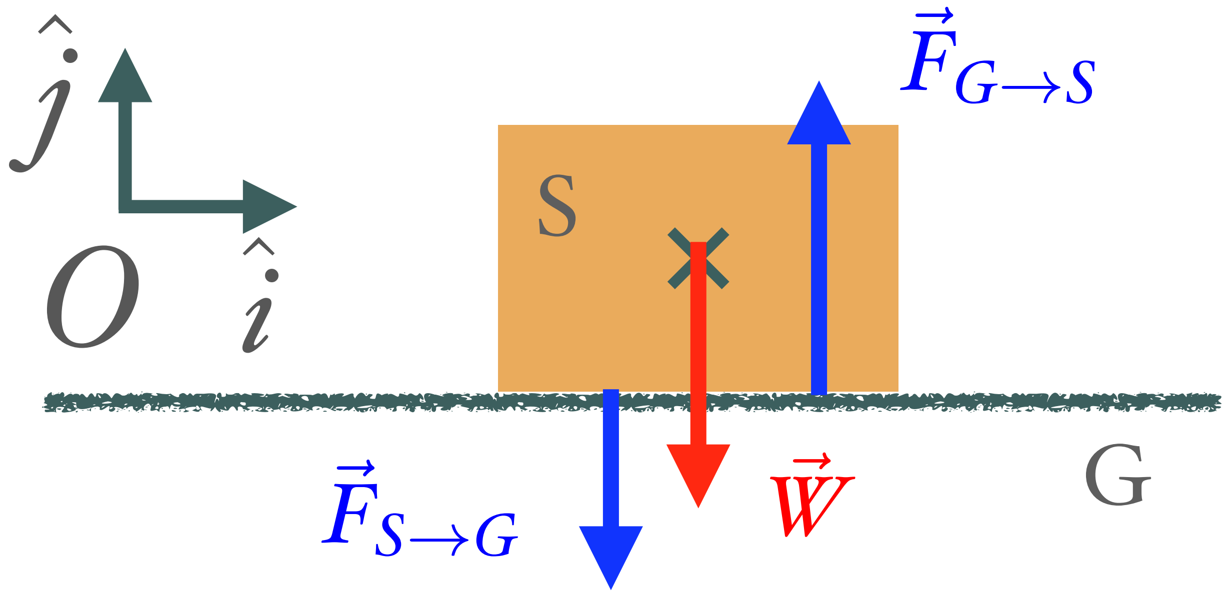
Question: Determine the force exerted by the block on the ground.

Answer: Again we proceed in multiple steps

2nd step: list forces acting on S

Contact forces	Forces at a distance
Ground reaction :$\vec{F}_{G\to S}$	Weight: $\vec{W} = -m\,g\,\hat{j}$

First encounter with contact forces



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Question: Determine the force exerted by the block on the ground.

Answer: Again we proceed in multiple steps

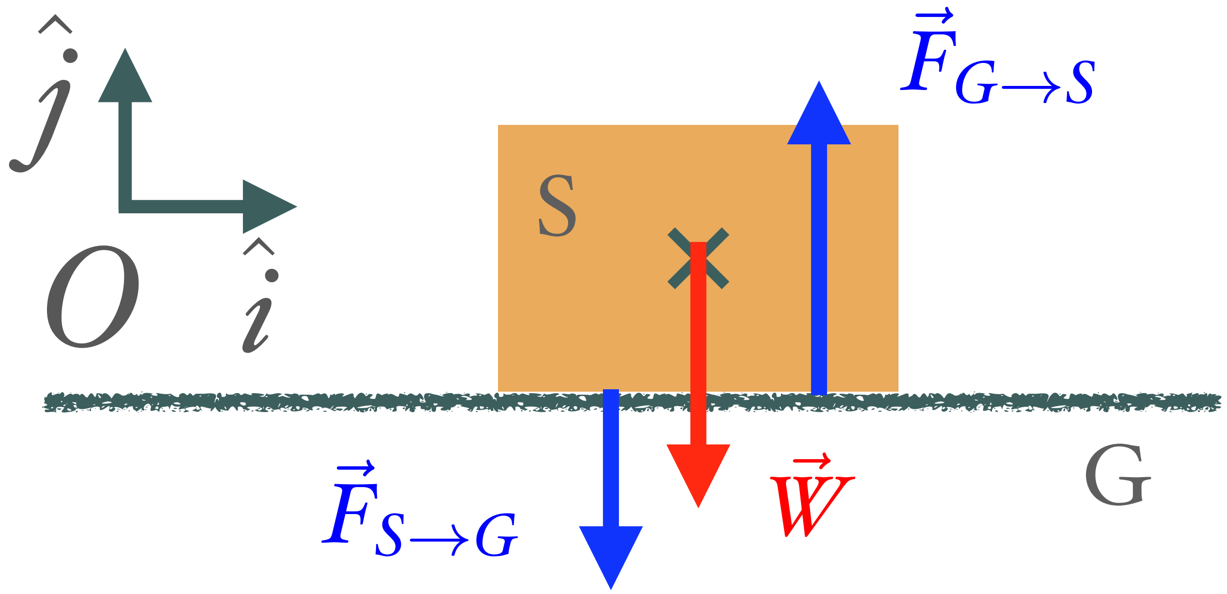
3rd step: apply Newton's 2nd law

The frame is Galilean therefore Newton's 2nd law applies

$$m\,\vec{a} = \vec{W} + \vec{F}_{G\to S}$$

$$\vec{a} = \vec{0} \implies \vec{F}_{G\to S} = -\vec{W}$$

First encounter with contact forces



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Question: Determine the force exerted by the block on the ground.

Answer: Again we proceed in multiple steps

4th step: apply Newton's 3rd law

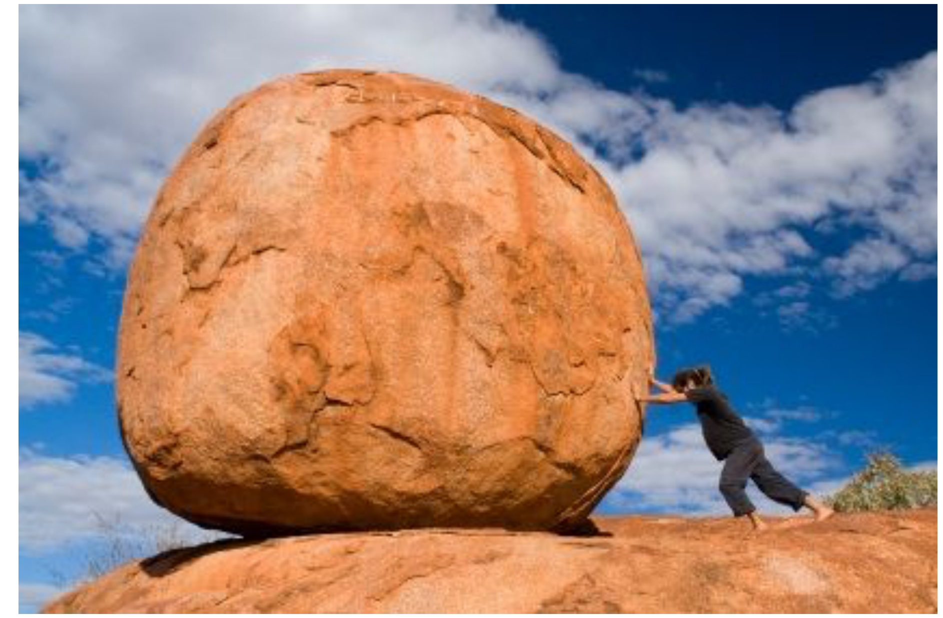
$$\vec{F}_{S\to G} = -\vec{F}_{G\to S} \implies \boxed{\vec{F}_{S\to G} = \vec{W}}$$

Friction forces

Solid friction force

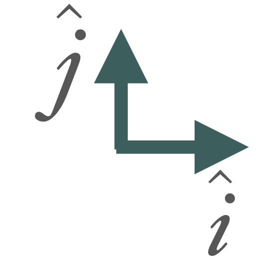
Solid friction

Can we make a 1000 tonnes boulder move by simply applying a reasonable force?



Let's try to figure it out!!

Solid friction



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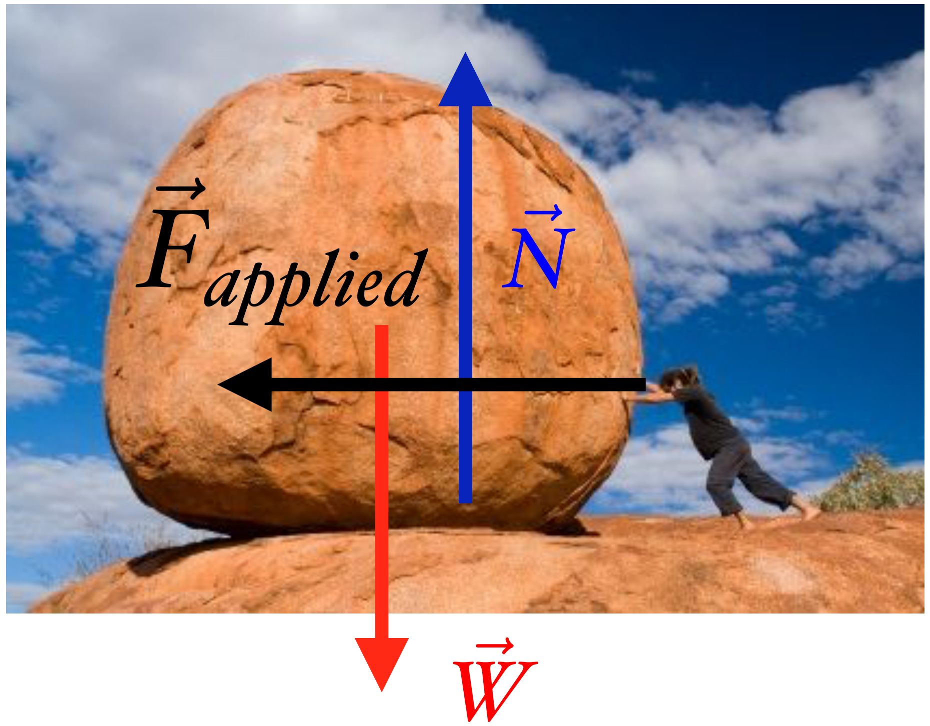
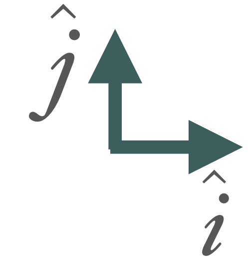


Let's try to figure it out!!

1st step: make a diagram with all the forces acting on the boulder

Solid friction



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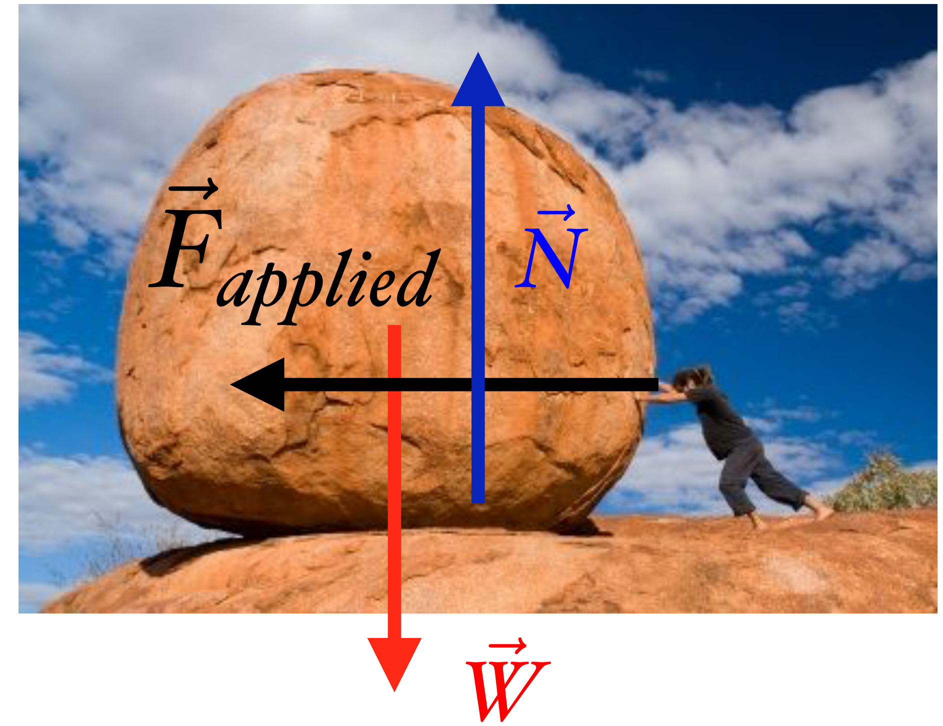
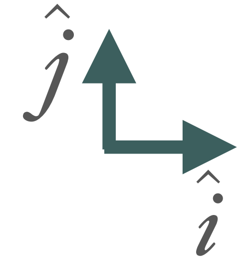


Let's try to figure it out!!

2nd step: list the forces acting on the boulder

Contact forces	Forces at a distance
$\vec{F}_{applied} = F_{applied}\,\hat{i}$	$\vec{W} = -m\,g\,\hat{j}, \quad g = 9.8\ \mathrm{m \cdot s^{-2}}$
$\vec{N} = N\hat{j}$	

Solid friction



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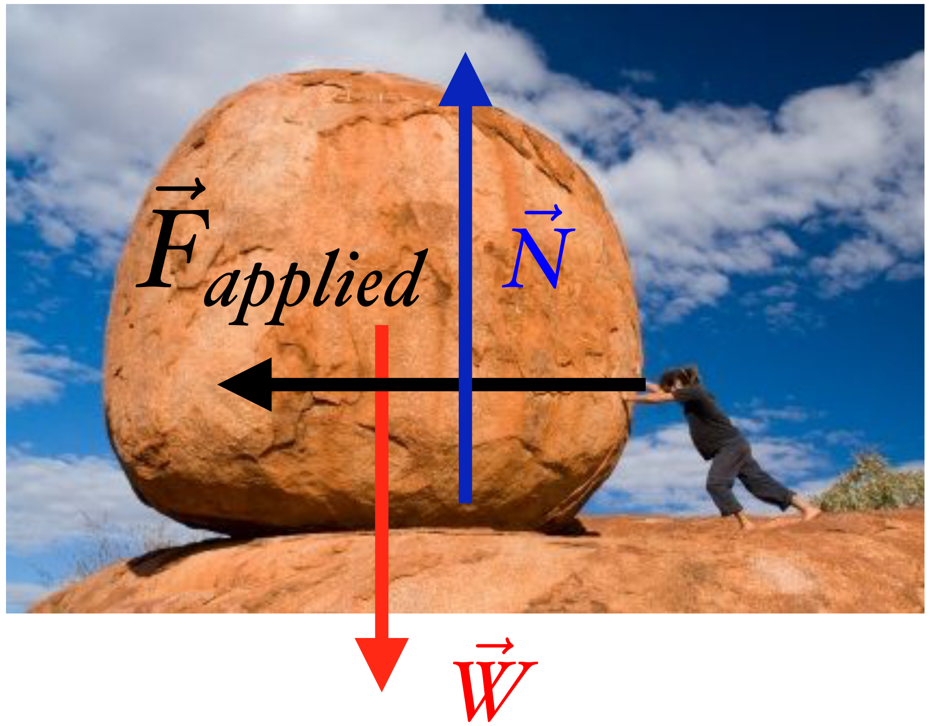


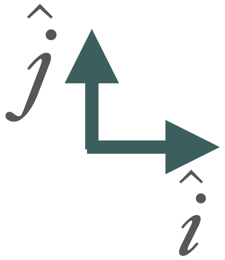
Let's try to figure it out!!

3rd step: apply Newton's 2nd law

We assume the frame of reference to be Galilean and therefore we can apply Newton's 2nd law

$$m\,\vec{a} = \vec{F}_{applied} + \vec{N} + \vec{W}$$

Solid friction



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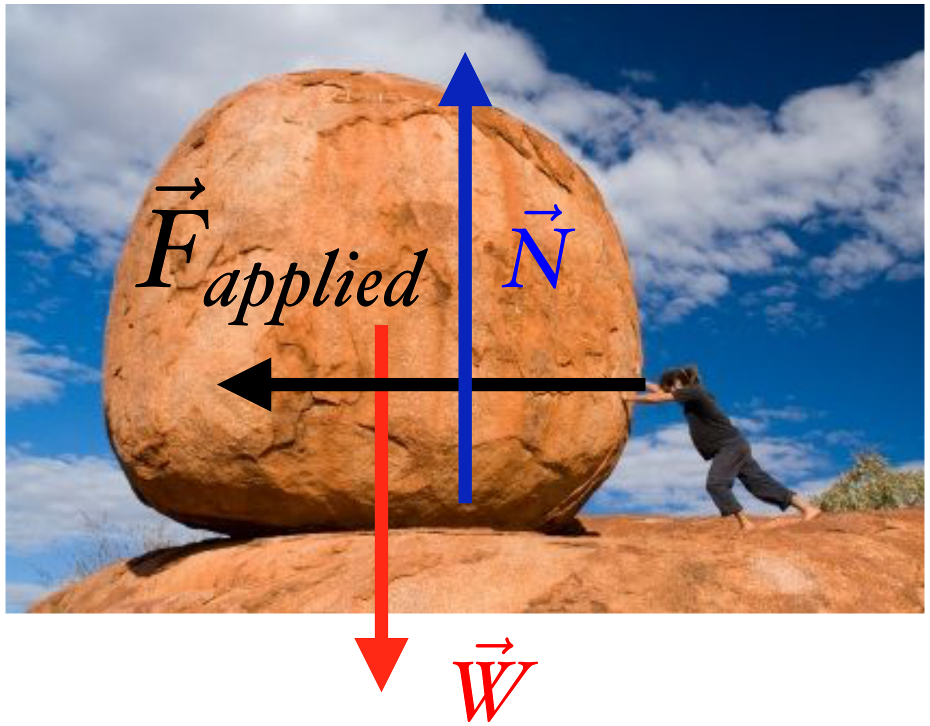


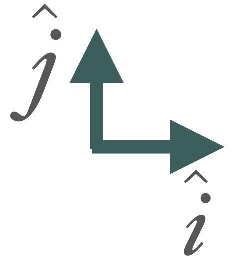
Let's try to figure it out!!

4th step: get the components

$$m\,a_x = F_{applied}$$

$$m\,a_y = N - m\,g$$

Solid friction



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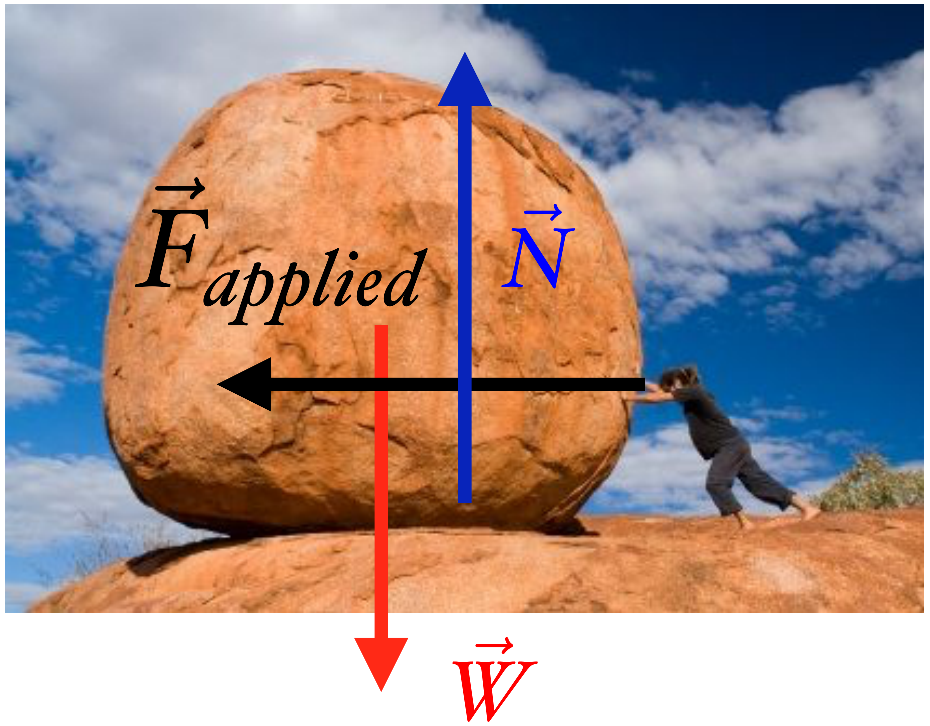


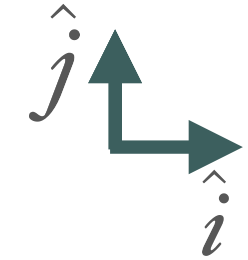
Let's try to figure it out!!

4th step: get the components

$$F_{applied} \neq 0 \implies a_x \neq 0$$

$$a_y = 0 \implies N = m\,g$$

Solid friction



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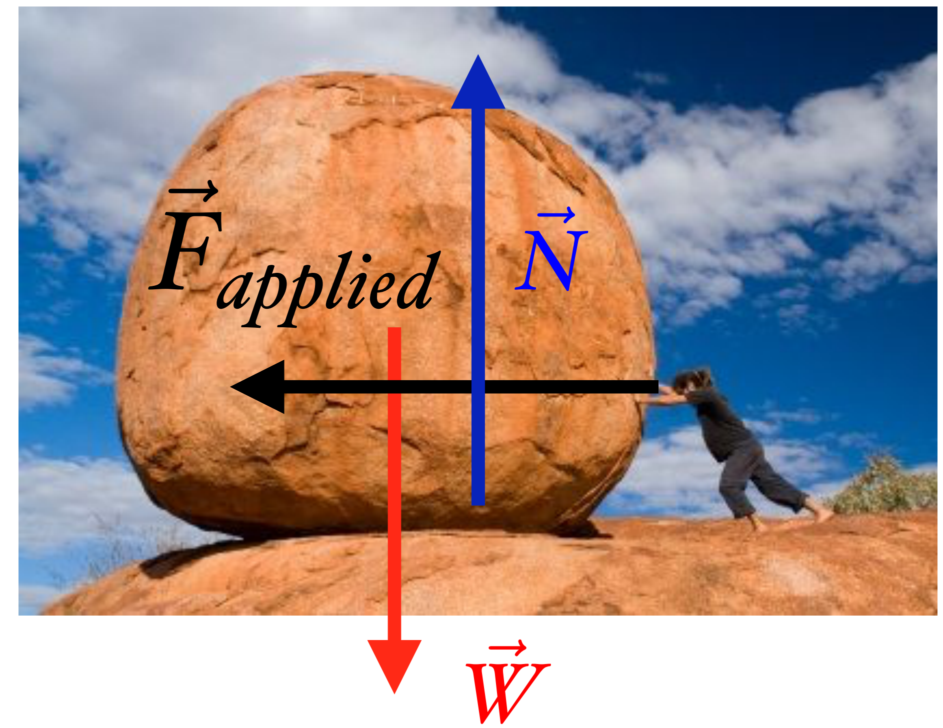

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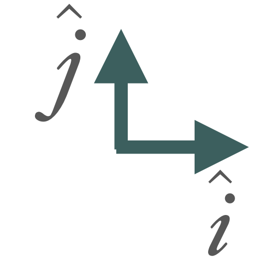
Solid friction



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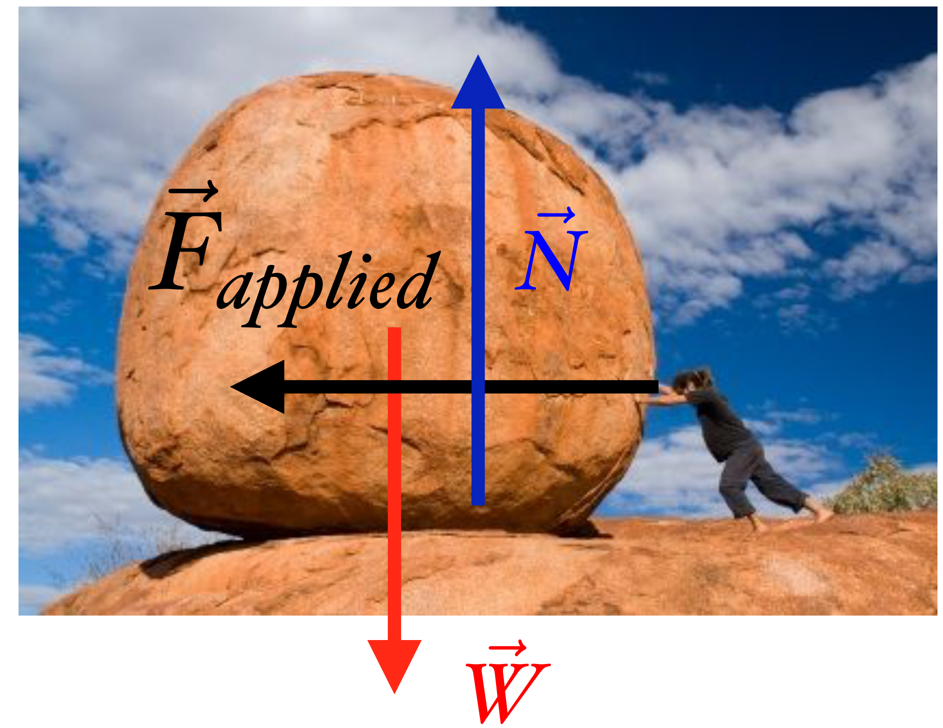


Let's try to figure it out!!

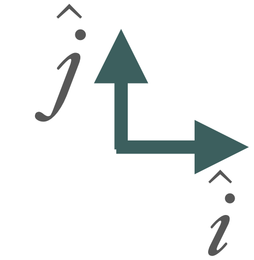
4th step: get the components

$$\boxed{F_{applied} \neq 0 \implies a_x \neq 0}$$

Really?
We must have missed something

$$a_y = 0 \implies N = m\, g$$

Solid friction



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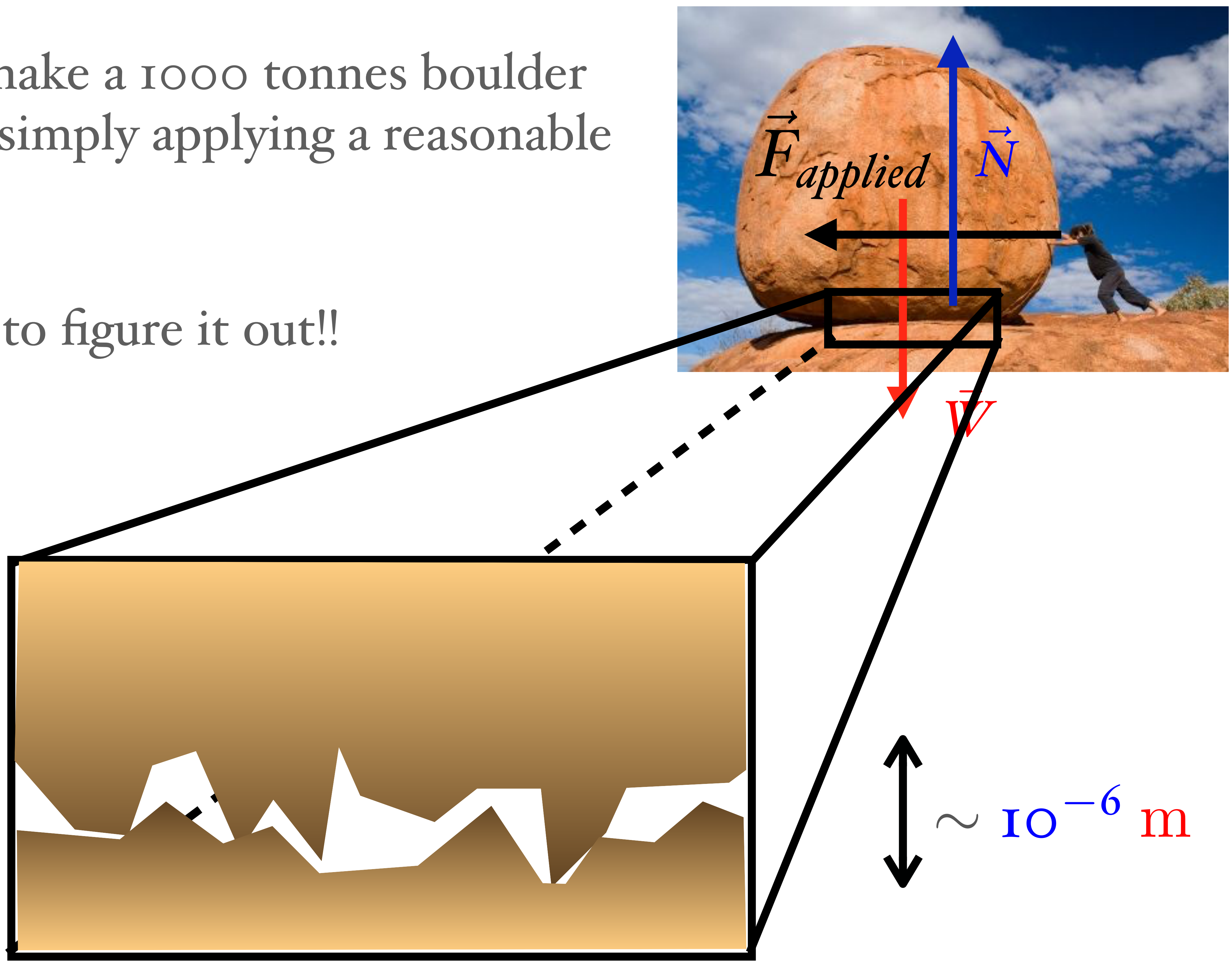
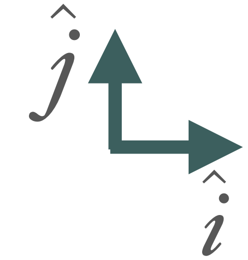
Let's try to figure it out!!



Rough surfaces

Solid friction



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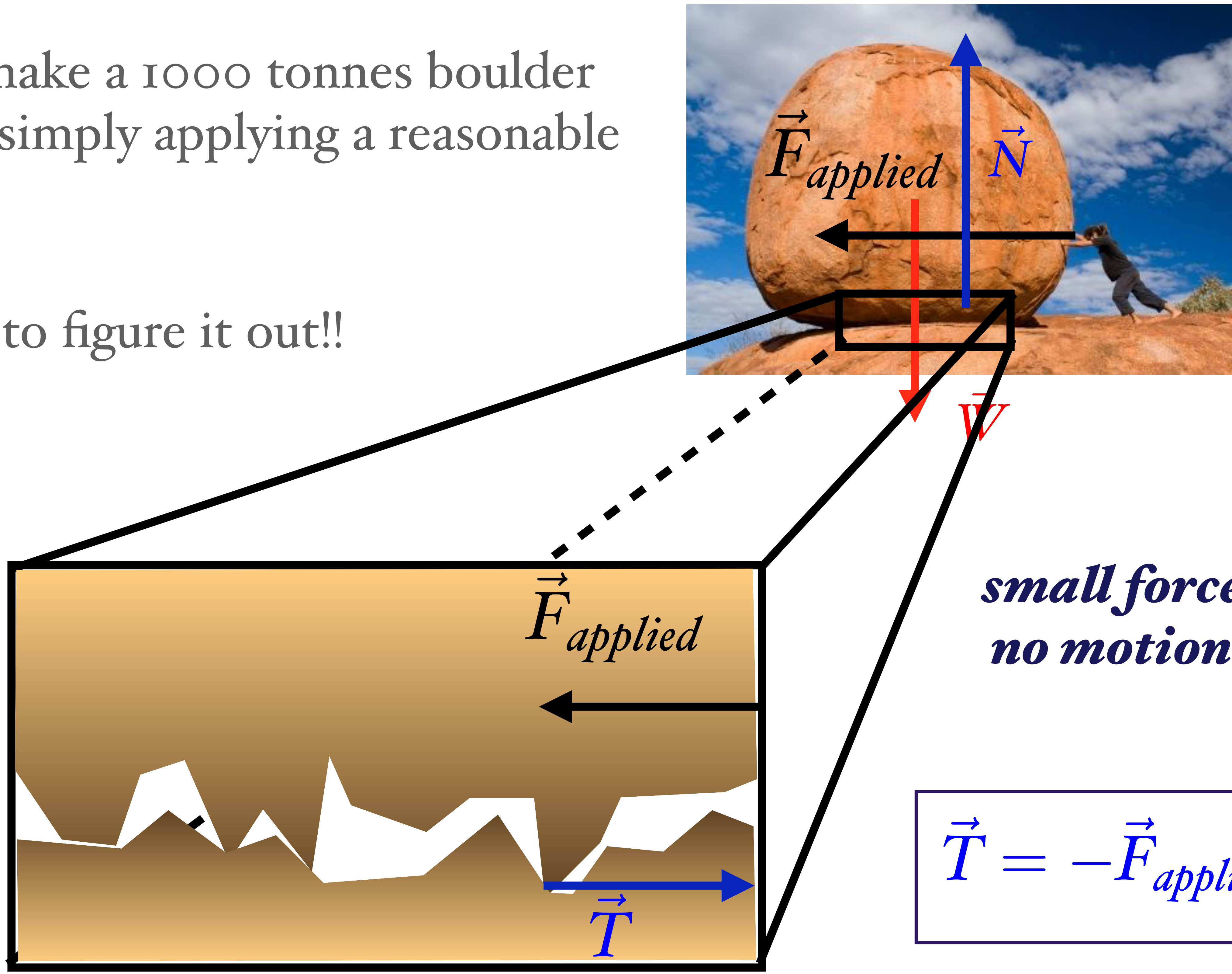


Rough surfaces

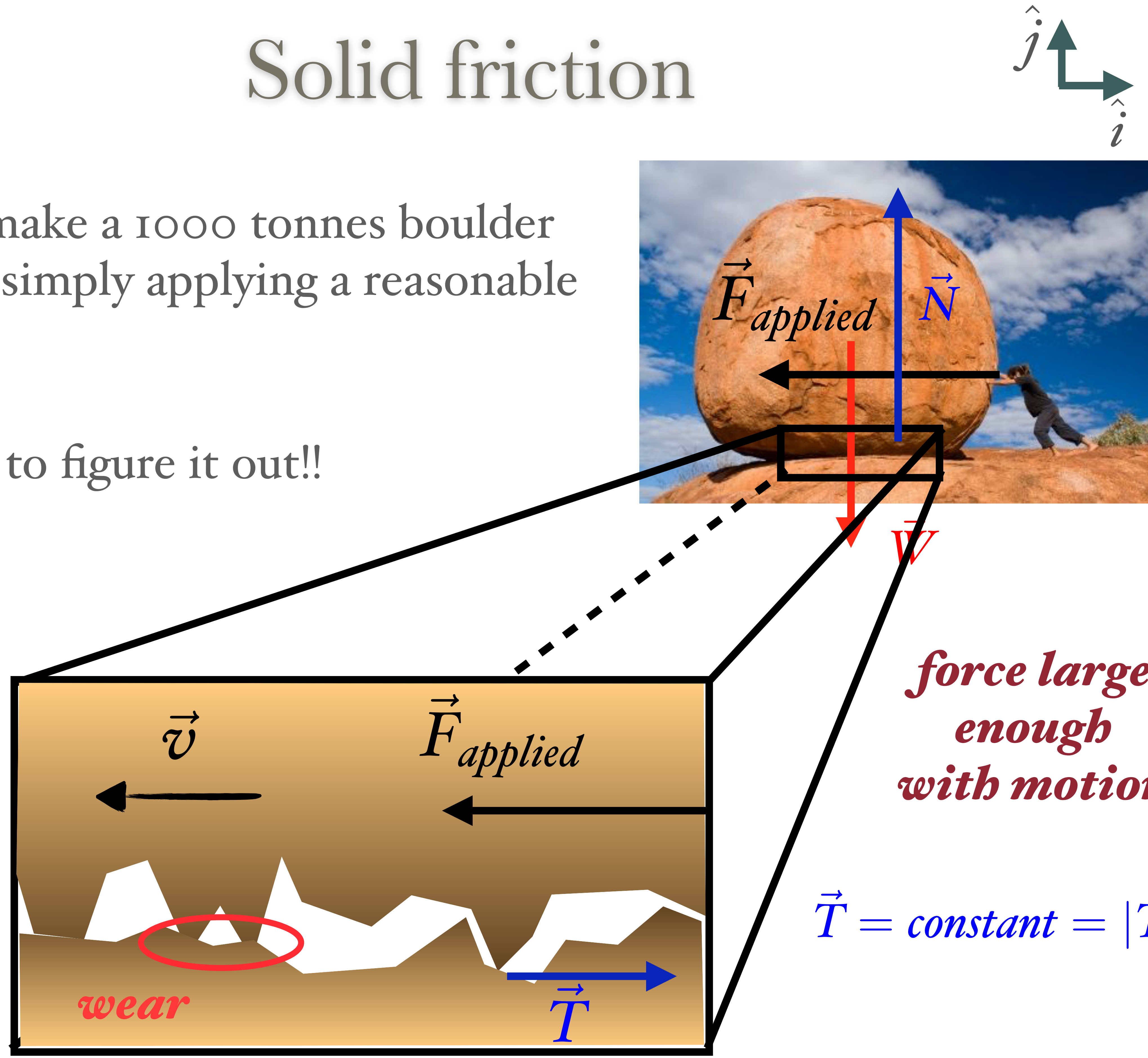
small force no motion

$$\boxed{\vec{T} = -\vec{F}_{applied}}$$

Solid friction

Can we make a 1000 tonnes boulder move by simply applying a reasonable force?

Let's try to figure it out!!



Rough surfaces

force large enough with motion

$\vec{T} = constant = |T|\,\hat{i}$

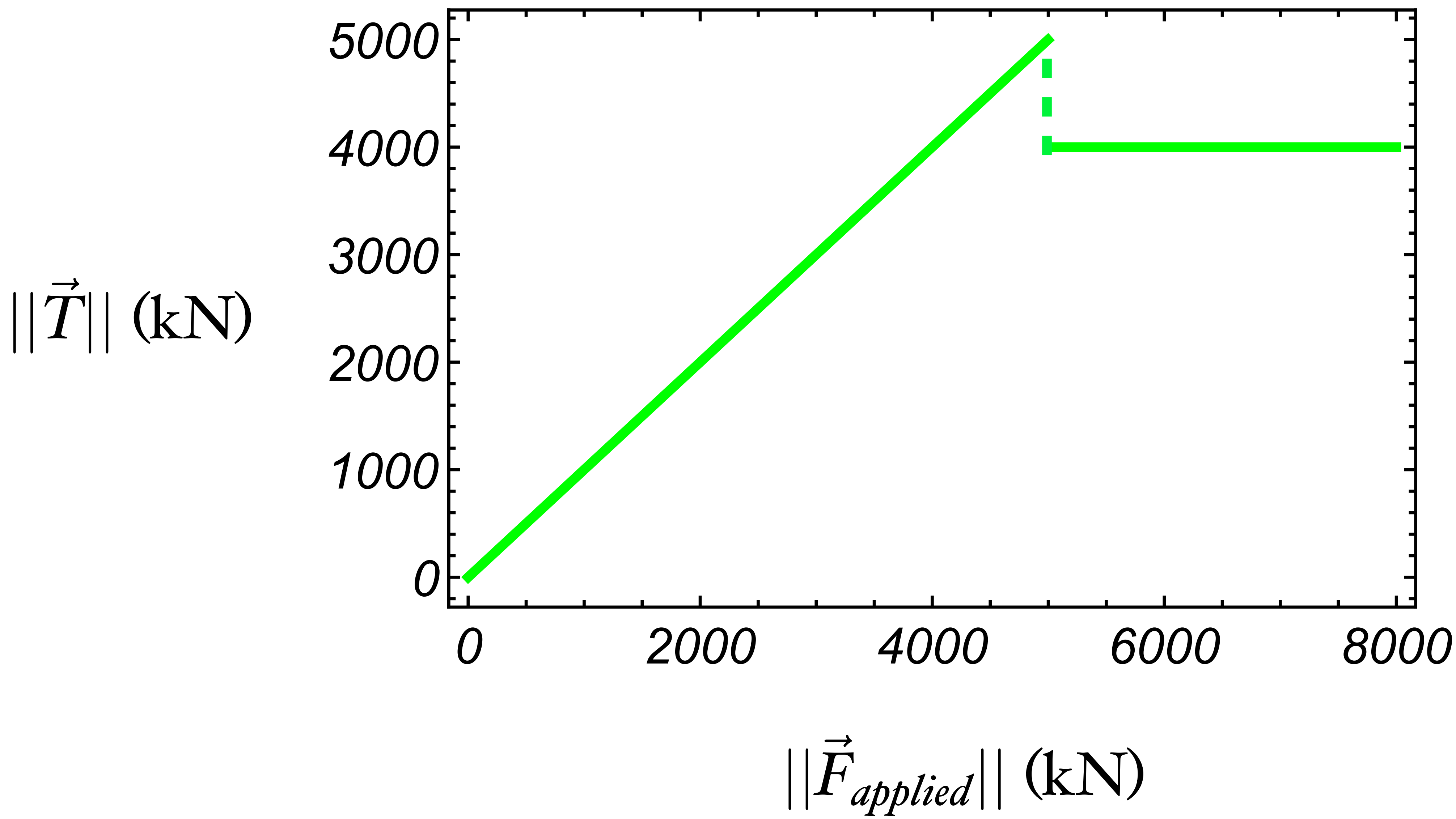
Solid friction

On rather soft grounds, the wear created by heavy objects is easily visible with the naked eye

Solid friction

Mathematical modelling

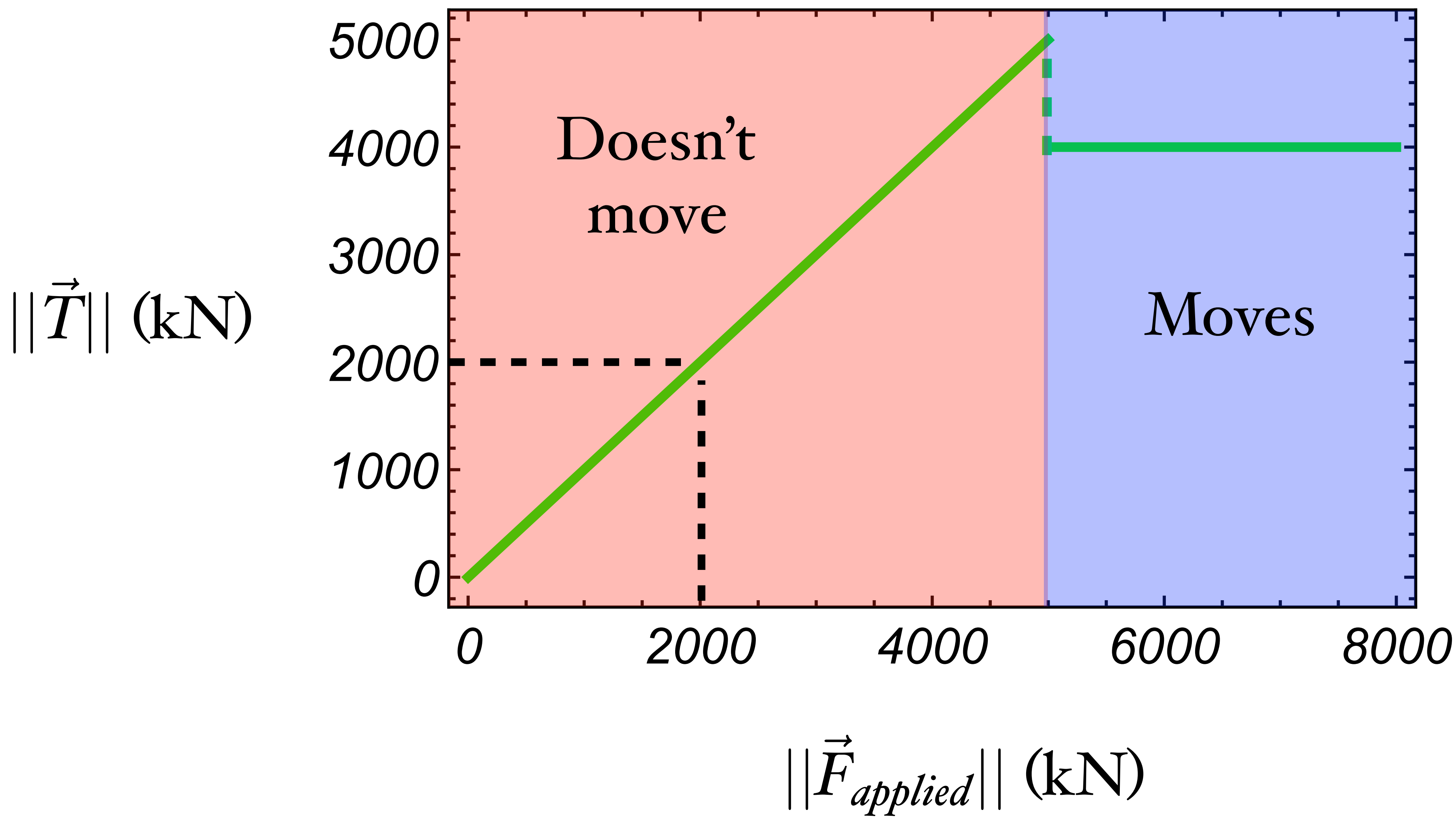
for a 1000 tonnes boulder



Solid friction

Mathematical modelling

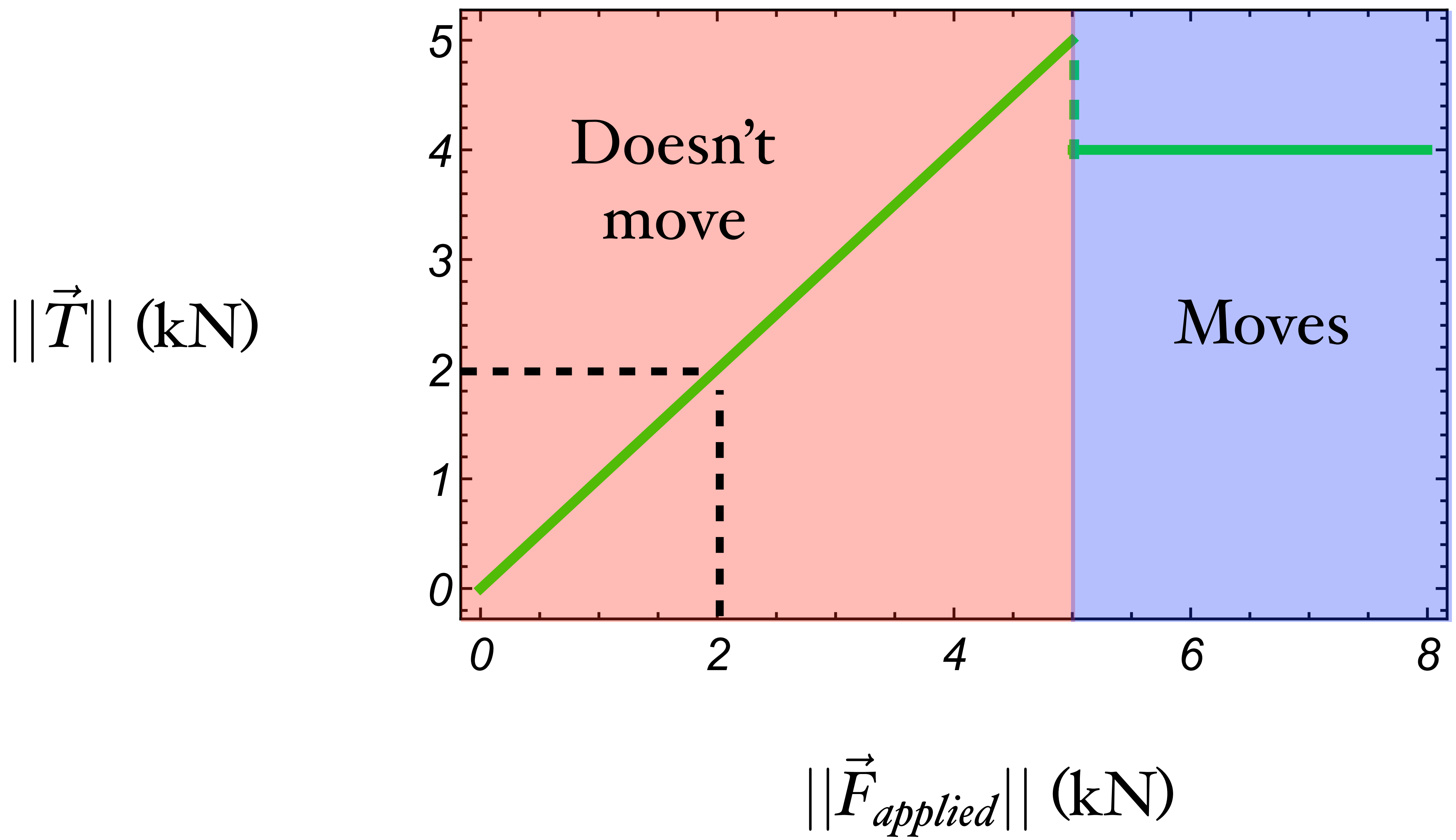
for a 1000 tonnes boulder



Solid friction

Mathematical modelling

for a 1 tonne boulder



Solid friction

Mathematical modelling

☆ If $\|\overrightarrow{F}_{applied}\| \leq \mu_s \|\overrightarrow{N}\|$ the tangential reaction $\overrightarrow{T}$ from the ground (*static friction force*) balances the applied force $\overrightarrow{F}_{applied}$,

$$\boxed{\overrightarrow{T} = -\overrightarrow{F}_{applied}}$$

and the object ***doesn't move***. The factor μ_s is called the ***static friction coefficient***.

Solid friction

Mathematical modelling

☆ If $\|\overrightarrow{F}_{applied}\| \leq \mu_s \|\overrightarrow{N}\|$ the tangential reaction $\overrightarrow{T}$ from the ground (*static friction force*) balances the applied force $\overrightarrow{F}_{applied}$,

$$\boxed{\overrightarrow{T} = -\overrightarrow{F}_{applied}}$$

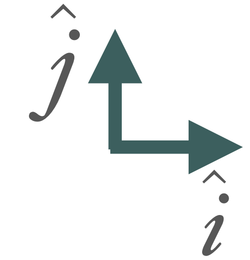
and the object ***doesn't move***. The factor μ_s is called the ***static friction coefficient***.

☆ If $\|\overrightarrow{F}_{applied}\| > \mu_s \|\overrightarrow{N}\|$ the object moves. The tangential reaction $\overrightarrow{T}$ from the ground (*kinetic friction force*) opposes the motion with constant magnitude

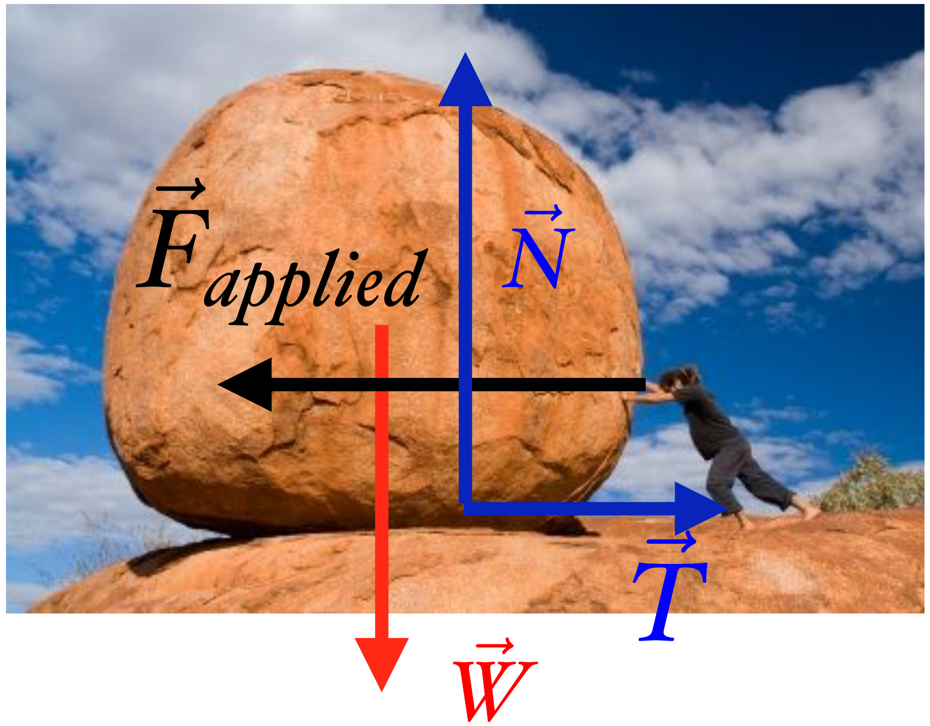
$$\boxed{\|\overrightarrow{T}\| = \mu_k \|\overrightarrow{N}\|}$$

The factor $\mu_k \leq \mu_s$ is called the ***kinetic friction coefficient***.

Solid friction



Can we make a 1000 tonnes boulder move by simply applying a reasonable force given that the static friction coefficient is $\mu_s = 0.5$?



- If $\|\overrightarrow{F}_{applied}\| \leq \mu_s \|\overrightarrow{N}\|$ the boulder doesn't move
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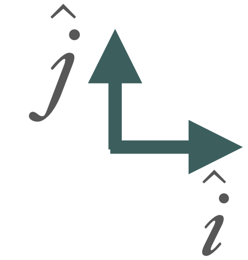

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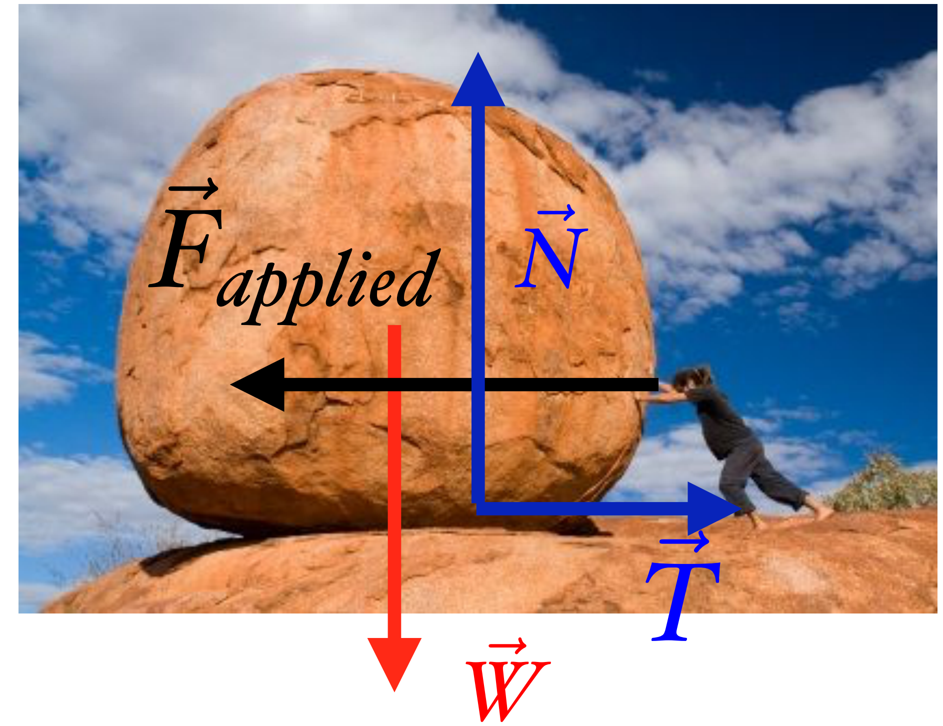


- If $||\overrightarrow{F}_{applied}|| \leq \mu_s ||\overrightarrow{N}||$ the boulder doesn't move
- If $||\overrightarrow{F}_{applied}|| > \mu_s ||\overrightarrow{N}||$ the boulder moves

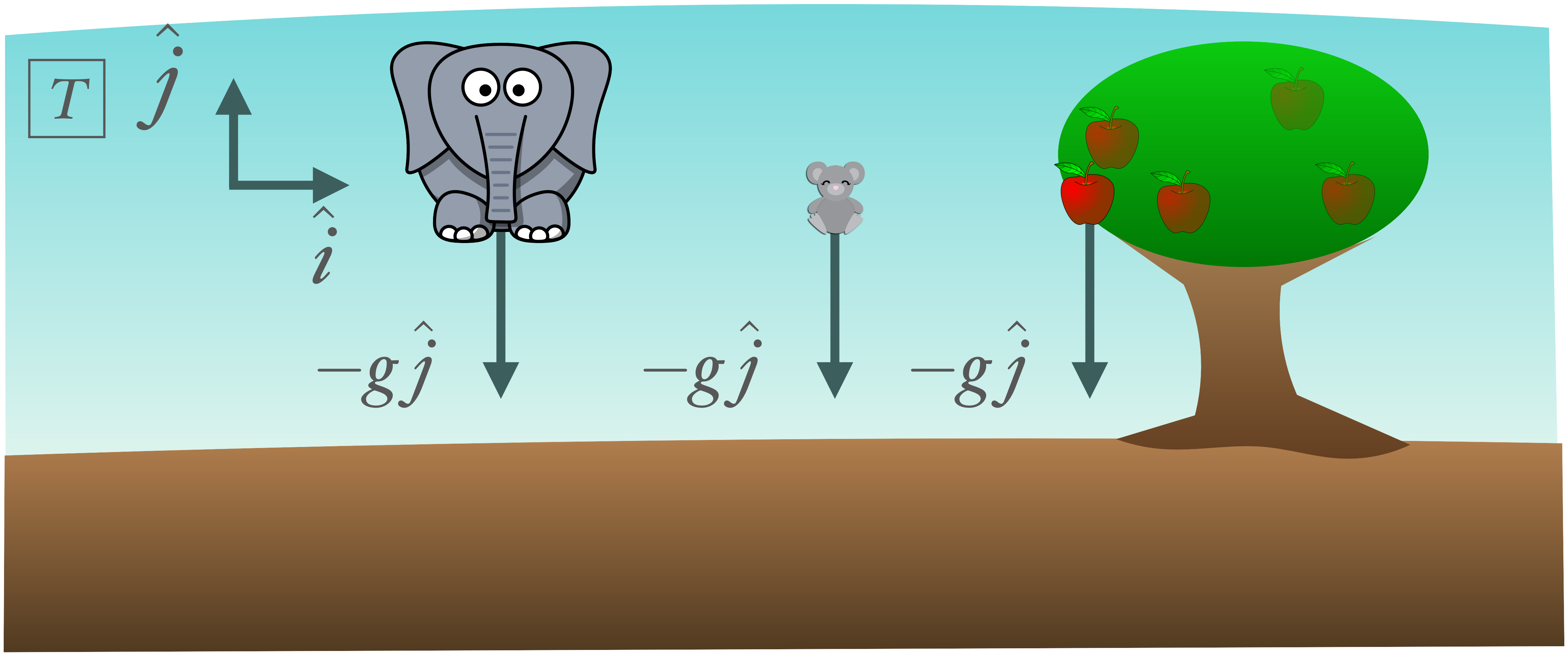
$\overrightarrow{N} = -\overrightarrow{W} = -mg\hat{j}$, where $m = 1000^2 kg$ and $g = 9.8ms^{-2}$
Hence, the boulder will move if

$$||\overrightarrow{F}_{applied}|| > \mu_s mg = 0.5(1000^2 kg)(9.8m/s^2) = 4.9(10^6)N$$

Enormous force!!!

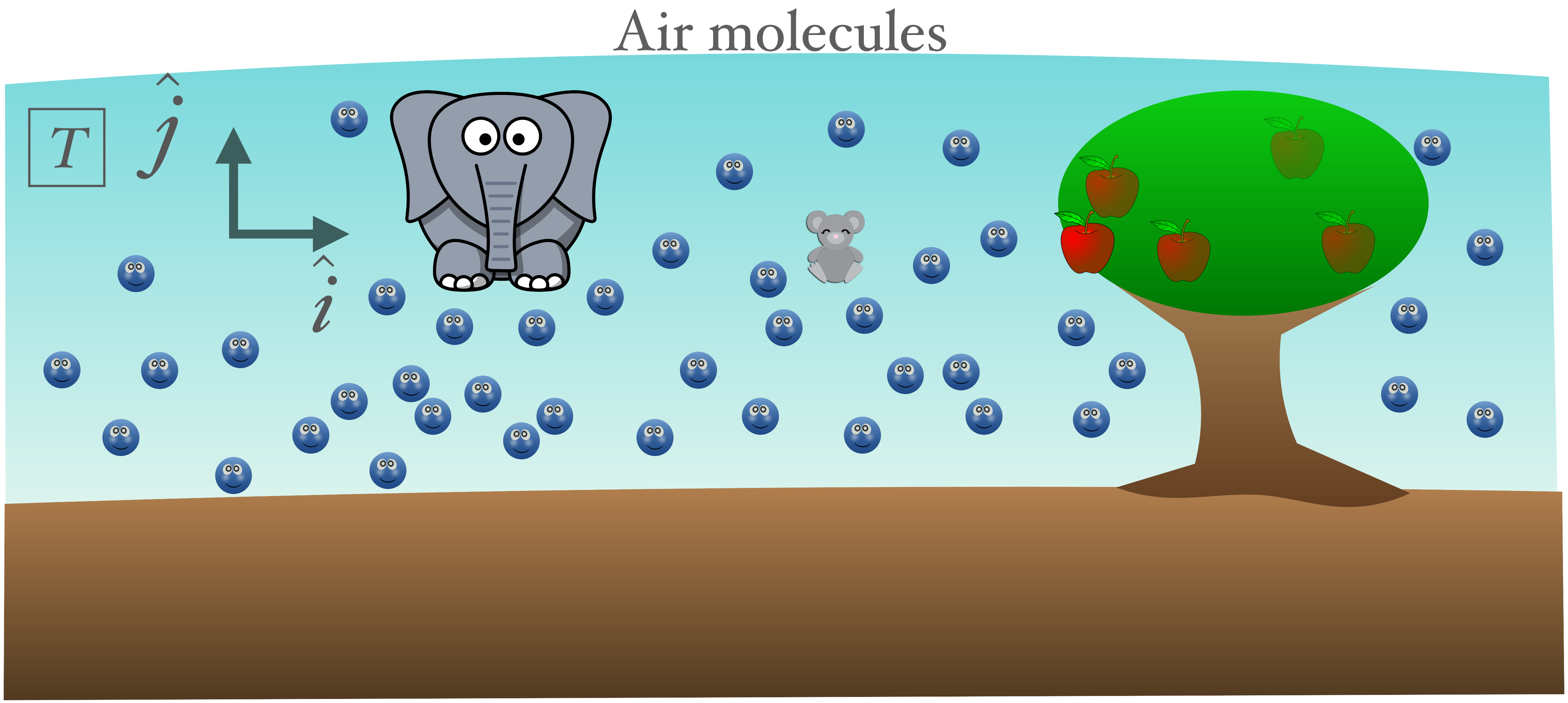
Fluid friction force

Going back to Galileo



Is it actually realistic to imagine that every object dropped from the same height will fall at the same speed on Earth in normal conditions?

Going back to Galileo



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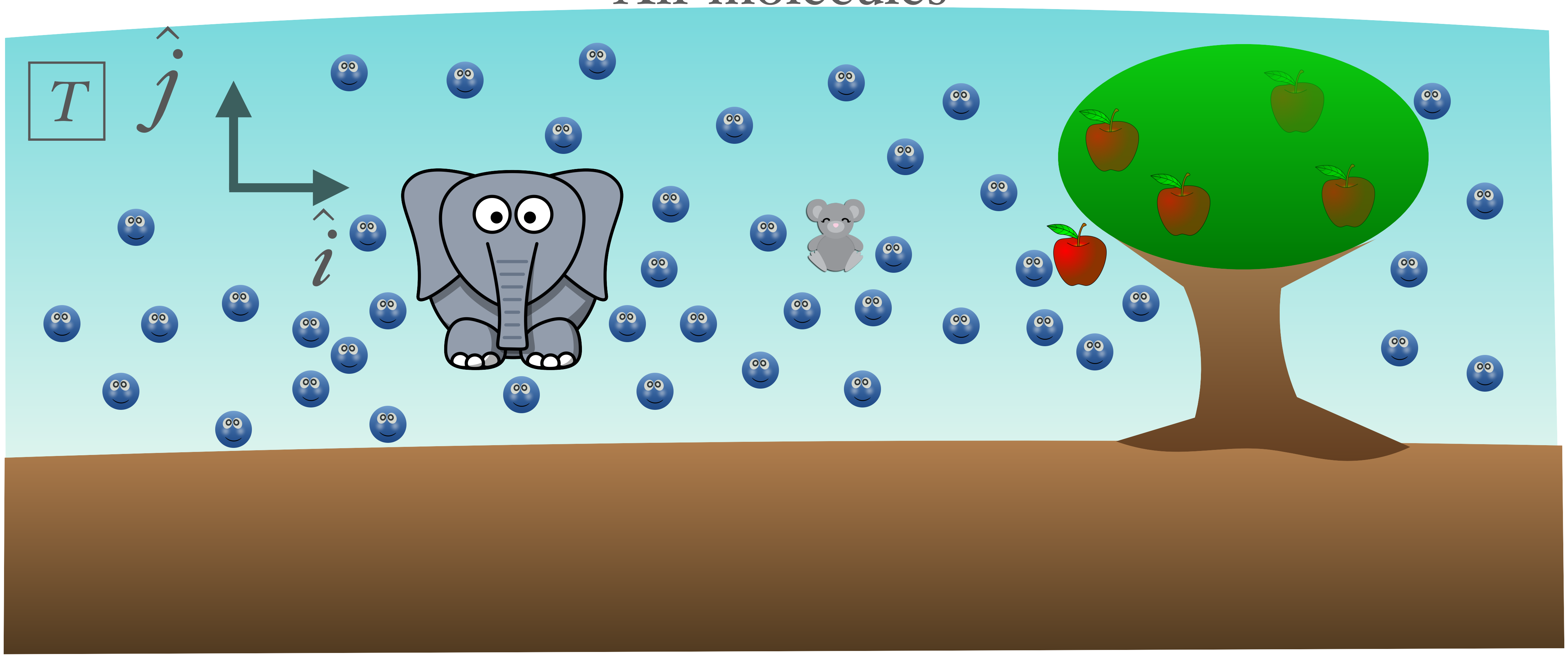
Going back to Galileo

Air molecules



Is it actually realistic to imagine that every object dropped from the same height will fall at the same speed on Earth in normal conditions?

Upon moving down through air, all bodies are subject to a ***decelerating force*** whose final effect depends on their size

Fluid friction

Mathematical modelling

Every body moving through a fluid with velocity $\vec{v}$ relative to it is subject to a force $\vec{F}_{drag}$ opposing its motion and called ***drag force***

$$\vec{F}_{drag} = -\frac{1}{2}\,\rho\,C_D(||\vec{v}||)\,A\,||\vec{v}||\vec{v}(t)$$

Fluid friction

<u>Mathematical modelling</u>

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Fluid mass density

Drag coefficient

Cross sectional area of the body

Fluid friction

<u>Mathematical modelling</u>

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Very complicated

Fluid friction

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$$\vec{F}_{drag} = -\frac{1}{2}\,\rho\, C_D(||\vec{v}||)\, A\, ||\vec{v}||\vec{v}(t)$$

In case of very small objects and/or very small velocities,

$$\frac{1}{2}||\vec{v}||\rho C_D(||\vec{v}||)A \approx constant = \gamma \quad \text{(friction coefficient)}$$

i.e. the drag force is simply proportional to the velocity:

$$\boxed{\vec{F}_{drag} = -\gamma\vec{v}(t)}$$

We will only consider this equation in this module!

Energy

Motivations

* As we have seen them, Newton's laws of motion are ***enough*** to solve any dynamical problem, provided we are given an adequate model for the forces.

* Yet many contemporaries of Newton's felt unsatisfied with his laws

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  • *The concept of force was not totally clear*
  • *Could the 2nd law emerge from a deeper principle?*

* Newton's laws are not able to capture all of our own sensations and apprehension of forces. For example:
  • *Why is it harder to climb stairs rather than walk on a flat surface if the same distance is traveled?*
  • *Is it possible to quantify the effort one needs to generate to perform a given mechanical task?*

The energy concept

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* This typical sentence implies that exerting a force on an object is ***costly***...but costly in what exactly?

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* In our society: "costly" means that whatever service or good we want, we will have to trade it against an amount of scarce resource that we call "money".

* In mechanics this "money" is called ***Energy***.

Kinetic energy

We consider a point object of mass m moving with velocity $\overrightarrow{v}(t)$

We define the ***kinetic energy*** of this point object as being:

$$\boxed{K(t) = \frac{1}{2}m\|\overrightarrow{v}\|^2}$$

If the object moves in one dimension with velocity $\overrightarrow{v}(t) = v_x(t)\hat{i}$ in a Galilean frame $(O, \hat{i})$, then

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$$K(t) = \frac{1}{2}mv_x(t)^2 = \frac{1}{2}(3kg)(6ms^{-1})^2 = 54\ kg \cdot m^2 \cdot s^{-2} = 54N \cdot m$$

How is energy spent or earned?

Let us consider a point object of mass m moving with velocity $\vec{v}(t) = v_x(t)\,\hat{i}$ in a Galilean frame $(O,\ \hat{i})$ and subject to a **constant** force $\vec{F} = F\,\hat{i}$

Since we are in a Galilean frame, Newton's 2nd law applies and, in 1D, can be directly written in component:

$$m\,a_x(t) = F \quad \Longrightarrow \; m\,\dot{v}_x(t)\cdot v_x(t) = F\cdot v_x(t)$$

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If we consider that the position of the point object at time t_A is $x(t_A) = x_A$ and at time t_b is $x(t_B) = x_B$, then we derive

$$\int_{t_A}^{t_B} m\;v_x(t)\cdot \dot{v}_x(t)dt = \int_{t_A}^{t_B} F\cdot v_x(t)dt\ ,\quad \text{which implies that}$$

$$\frac{1}{2}mv_x(t_B)^2 - \frac{1}{2}mv_x(t_A)^2 = F(x_B - x_A) \quad \text{or} \quad K_B - K_A = F(x_B - x_A)$$

How is energy spent or earned?

The equation $K_B - K_A = F(x_B - x_A)$ is a specific case (for a constant force) of what is known as the ***work-energy theorem***.

It shows that when a constant force is applied on a point object over a non-zero distance, it changes its kinetic energy.

The quantity $F(x_B - x_A)$ is called the ***work done*** by the **constant** force F on the point object from points A to B.

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Remark 2: if F and $(x_B - x_A)$ have same signs, the force facilitates the motion of the object. It can then be interpreted as a ***monetary incentive*** to earn by the object to move from A to B

Work in one dimension

Work in one dimension

The general formula for the work done by a force $\overrightarrow{F}(x) = F(x)\hat{i}$ along a 1D path $\Gamma_{A\to B}$ from point A to point B is:

$$\boxed{W(F\,|\,\Gamma_{A\to B}) = \int_{x_A}^{x_B} F(x)dx}$$

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If we set $x = x(t)$, with t_A, t_B such that $x(t_A) = x_A$ and $x(t_B) = x_B$, then

$$dx = \frac{dx(t)}{dt}dt = v_x(t)dt \ , \quad \text{hence}$$

$$\boxed{W(F\,|\,\Gamma_{A\to B}) = \int_{t_A}^{t_B} F(x(t)) \cdot v_x(t)dt}$$

Work in one dimension

Example

Let us consider a path $\Gamma_{A\to B}$ that goes from x_A to x_C and then from x_C to x_B with $x_A < x_B < x_c$ and

$$F(x) = F, \quad for\ x \in [x_A, x_B],$$

$$F(x) = 2F,\ for\ x \in [x_B, x_C],$$

where F is a constant.

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where F is a constant.

Solution

$$\mathcal{W}(F|\Gamma_{A\to B}) = \int_{x_A}^{x_B} F\,dx + \int_{x_B}^{x_C} 2F\,dx + \int_{x_C}^{x_B} 2F\,dx$$

$$\boxed{\mathcal{W}(F|\Gamma_{A\to B}) = F\cdot(x_B - x_A)}$$

Work-energy theorem in 1D

Let us consider a point object of mass m moving with velocity $\vec{v}(t) = v_x(t)\,\hat{i}$ in a Galilean frame $(O,\,\hat{i})$ and subject to a force $\overrightarrow{F}(x) = F(x)\hat{i}$, whilst moving on a path $\Gamma_{A\to B}$ from point A to B

Then, according to Newton's 2nd law the following is true:

$$ma_x(t) = F(x(t)) \implies m\,\dot{v}_x(t)\,v_x(t) = F(x(t))\,v_x(t)$$

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If we consider that $x(t_A) = x_A$ and $x(t_B) = x_B$, then

$$\int_{t_A}^{t_B} m\,v_x(t)\cdot\dot{v}_x(t)dt = \int_{t_A}^{t_B} F(x(t))v_x(t)dt\ ,$$ which implies that

$$\boxed{K_B - K_A = \mathcal{W}(F|\Gamma_{A\to B})}$$

This equation is called the ***1D work-energy theorem***

Conservative forces

Conservative forces

A 1D force $\overrightarrow{F}(x) = F(x)\hat{i}$ is said to be ***conservative*** if either of the ***equivalent*** propositions is true:

- The work done by the force on a point object moving from a point A to a point B is ***independent of the path taken***

- The work done by the force on a point object along ***any closed*** path is ***zero***

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So, $$W(F\,|\,\Gamma_{A\to B}) = \int_{x_A}^{x_B} F(x)dx = U(x_A) - U(x_B)$$

Conservation of mechanical energy

If a 1D force F is conservative, then necessarily it can be associated to a function $U(x)$ such that $F(x) = -U'(x)$

In this case the work-energy theorem reads:

$$K_B - K_A = W(F\,|\,\Gamma_{A\to B}) = U(x_A) - U(x_B)$$

Reshuffling the terms yields:

$$\boxed{K_B + U(x_B) = K_A + U(x_A)}$$

There is ***conservation*** of what is called ***mechanical energy***

Mechanical and potential energy

If the forces acting on a system are conservative, then the ***mechanical energy*** $E = K + U$ is conserved.

The quantity $U(x)$ is called the ***potential energy***.

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Physical dimension

$$[E] = [K] = [U] = [W] = M \cdot L^2 \cdot T^{-2}$$

The SI unit of energy/work is the ***joule*** (J)

$$1\,\mathrm{J} = 1\mathrm{kg}\left(\frac{m}{s}\right)^2 = 1\mathrm{N} \cdot m$$

“Deriving” Newton’s 2nd law in one dimension

Consider a point object of mass m in a Galilean frame subject to a conservative force $F(x)$ characterised by a potential energy $U(x)$

The mechanical energy is $$E = \frac{1}{2}\, m\, v_x(t)^2 + U(x)$$

Conservation of energy means that $\dot{E} = 0$

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Now, $$\dot{E} = \frac{1}{2}\, m\, \frac{d}{dt}\left(v_x(t)^2\right) + \frac{dU(x)}{dt}$$

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Now, $$\dot{E} = \frac{1}{2}\, m \boxed{\frac{d}{dt}\left(v_x(t)^2\right)} + \boxed{\frac{dU(x)}{dt}}$$

product rule

$$2\,\frac{dv_x(t)}{dt} \cdot v_x(t)$$

chain rule

$$\frac{dU(x)}{dx} \cdot \frac{dx}{dt}$$

“Deriving” Newton’s 2nd law in one dimension

Hence, $\dot{E} = 0$ implies $\left(m\dfrac{dv_x(t)}{dt} + \dfrac{dU(x)}{dx} \right) \cdot v_x(t) = 0$.

If we consider that $v_x(t) \neq 0$, we get that

$$m\frac{dv_x(t)}{dt} = -\frac{dU(x)}{dx}$$

and since $-\dfrac{dU(x)}{dx} = F(x)$, we derive $\boxed{ma_x(t) = F(x)}$

that is Newton’s 2nd law.

"Deriving" Newton's 2nd law in one dimension

Hence, $\dot{E} = 0$ implies $\left(m \dfrac{dv_x(t)}{dt} + \dfrac{dU(x)}{dx} \right) \cdot v_x(t) = 0$.

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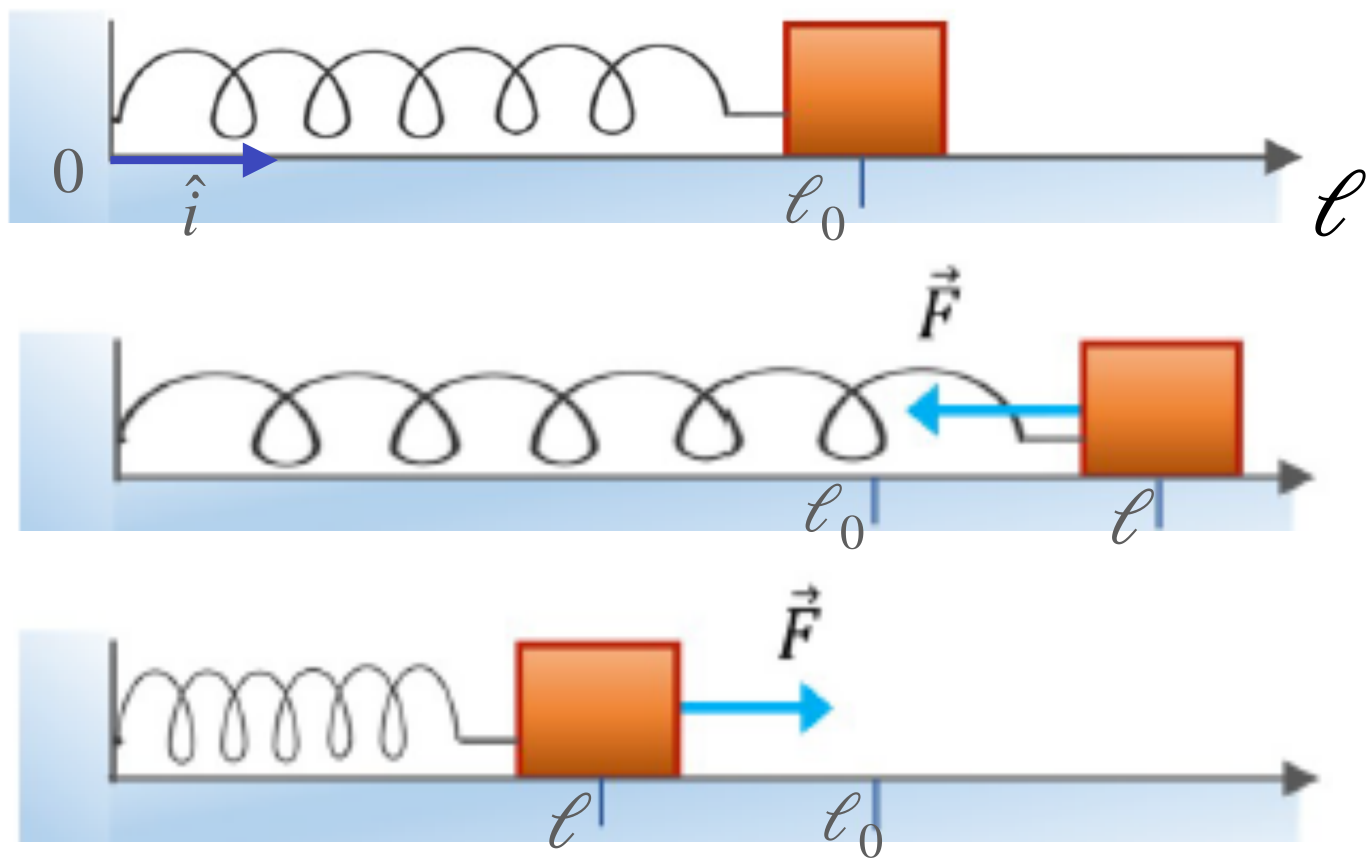
Notice that this derivation of Newton's 2nd law works only for a point object subject to a conservative force.

Forces exerted by a spring

Hooke's law

We consider an object of mass m on a frictionless horizontal surface attached to the end of a spring whose rest length is ℓ_0. When the block is displaced from its equilibrium position, the spring exerts a restoring force $\overrightarrow{F} = F(\ell)\hat{i}$.



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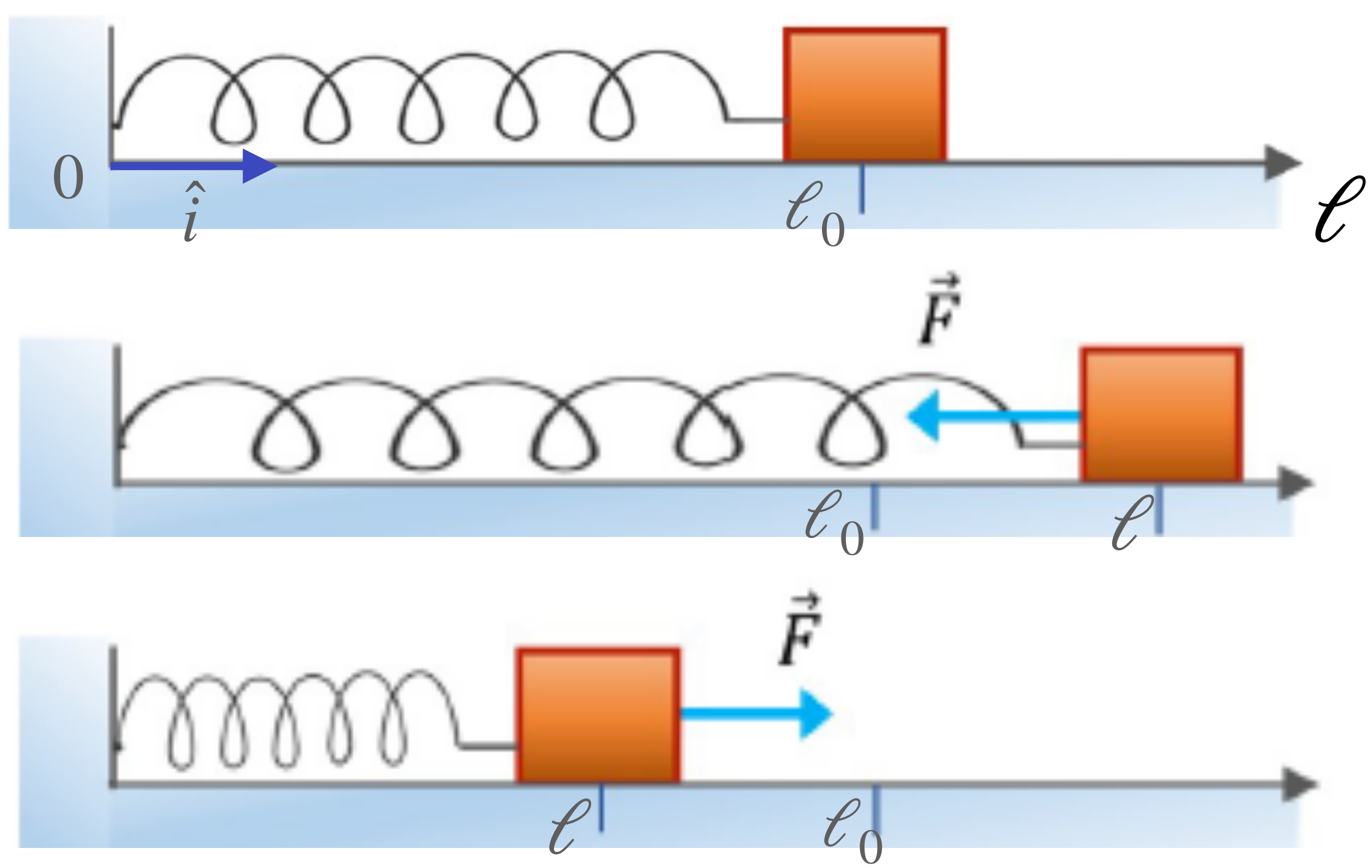

Hooke's Law: $F(\ell) = -k(\ell - \ell_0)$, for a constant $k > 0$.

The constant k (with units N/m) is a measure of the stiffness of the spring.

Hooke's law

Exercise

Show that the spring force $\overrightarrow{F} = F(\ell)\hat{\imath}$, with $F(\ell) = -k(\ell - \ell_0)$ is a conservative force with potential energy

$$U(\ell) = \frac{1}{2}k(\ell - \ell_0)^2$$

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$$U(\ell) = \frac{1}{2}k(\ell - \ell_0)^2$$

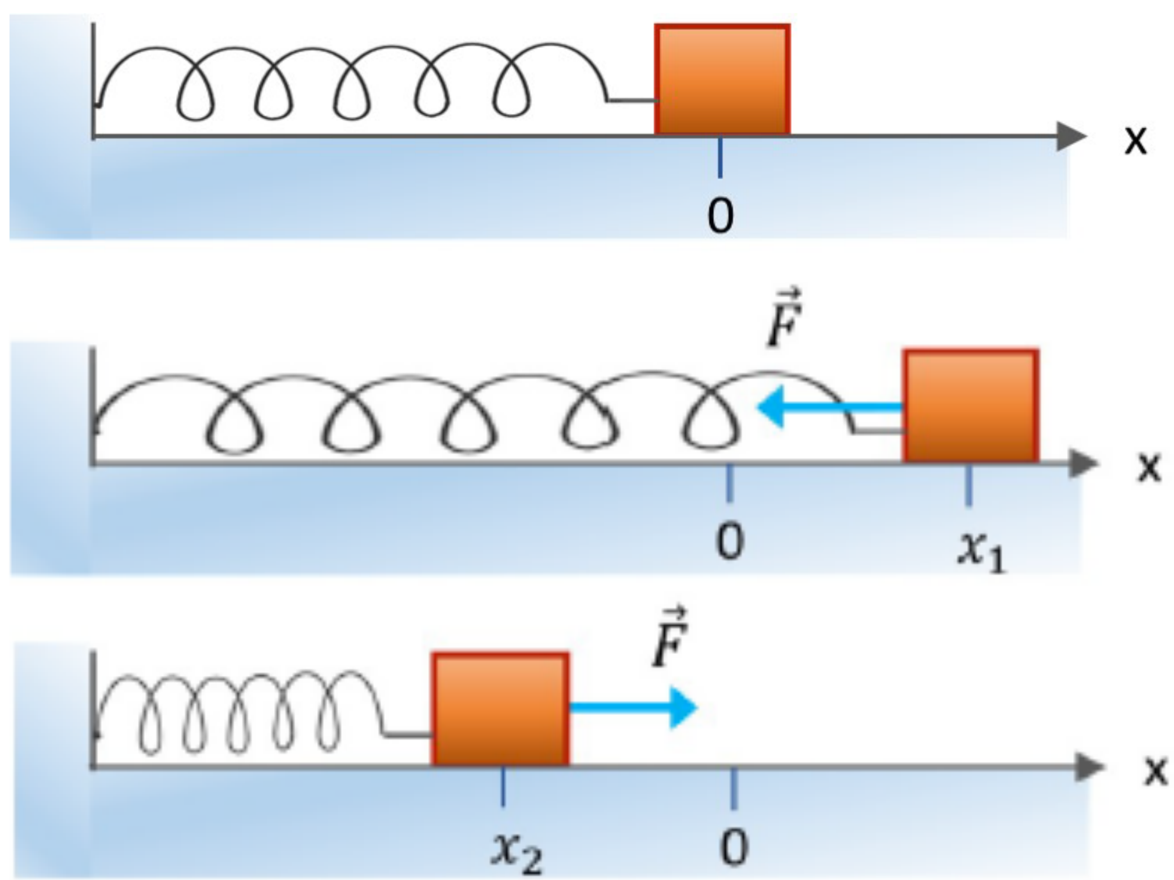
Solution

$$\frac{dU(\ell)}{d\ell} = k(\ell - \ell_0), \text{ so } F(\ell) = -\frac{dU(\ell)}{d\ell}.$$

Hooke's law

Remark

If we set the origin of the reference frame at ℓ_0 and x denotes the displaced from the equilibrium position, i.e. $x = \ell - \ell_0$,



then the spring force can be expressed as $\overrightarrow{F} = F(x)\hat{i}$, with $F(x) = -kx$.

Additional material in calculus

Scalar functions of multiple variables

A function of two variables f is a mathematical objects which takes two (real) numbers in and outputs a single real number.

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$x,\ y \mapsto f(x, y)$$

Examples

- $\star$ $f(x, y) = \sqrt{x^2 + y^2} \implies f(\sqrt{7}, 3) = 4$
- $\star$ $f(x, y) = x \cdot y \implies f(\sqrt{7}, 3) = 3\sqrt{7}$
- $\star$ $f(x, y) = \ln \dfrac{x}{y} \implies f(\sqrt{7}, 3) = \ln \dfrac{\sqrt{7}}{3} \approx -0.12$
- $\star$ $f(x, y) = 2 \cdot (x + y) \implies f(\sqrt{7}, 3) = 6 + 2\sqrt{7}$

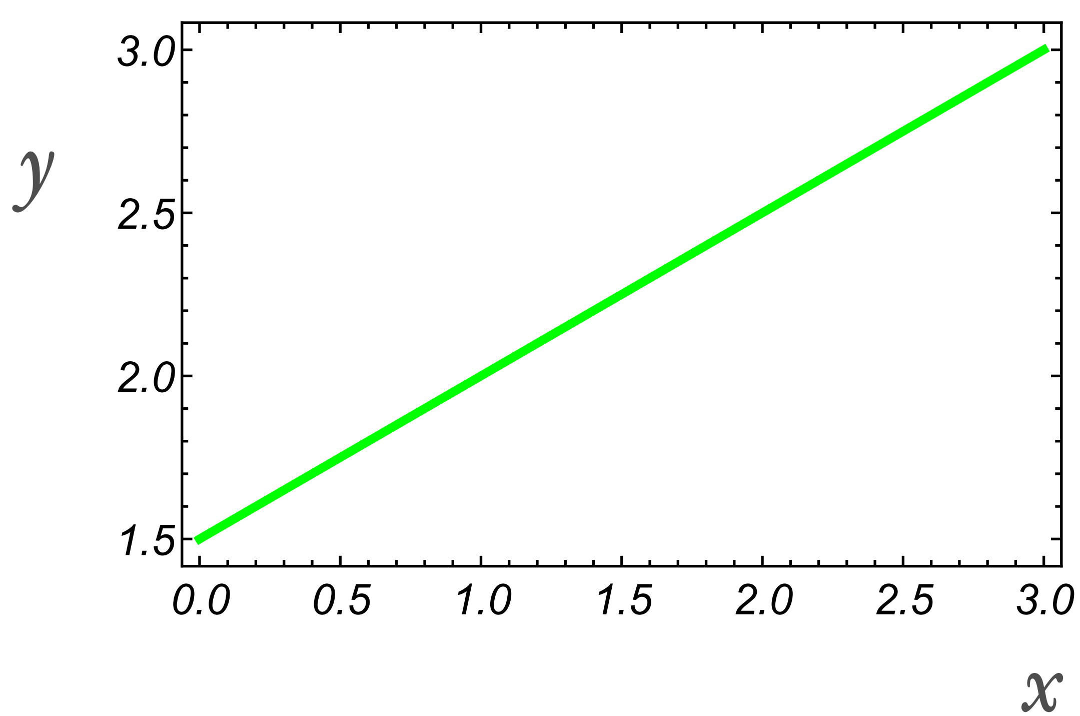
Scalar functions of multiple variables

Graphical representation

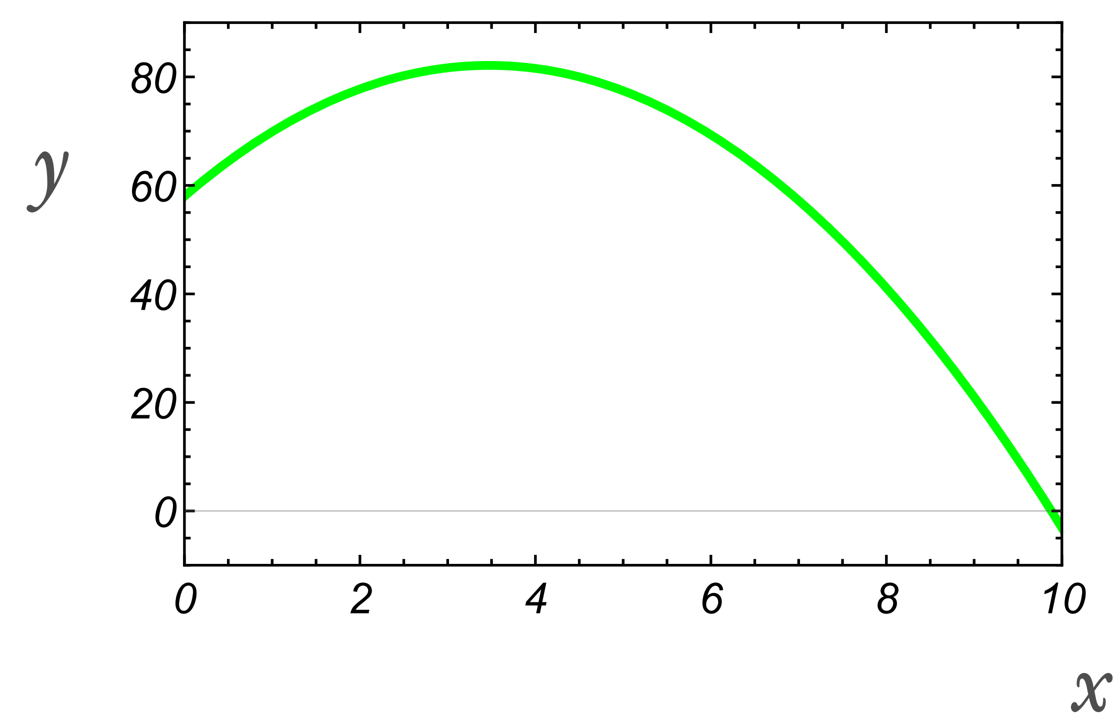
A function of a ***single*** variable can be graphically represented by ***interpreting*** the image $f(x)$ of the variable x as being the y coordinate in the (x, y) plane.

Examples

$$y = f(x) = x/2 + 1.5$$



$$y = f(x) = 58 + 13.89\,x - 2\,x^2$$



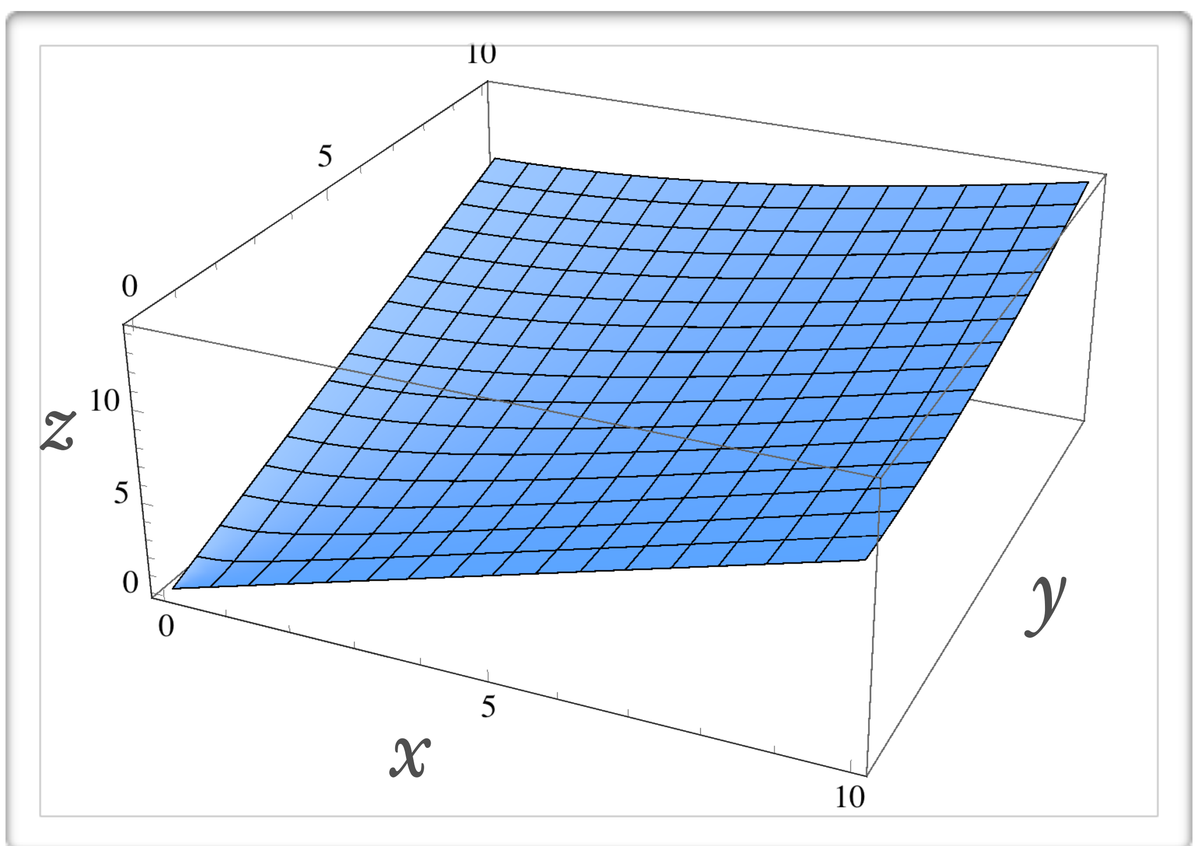
Scalar functions of multiple variables

Graphical representation

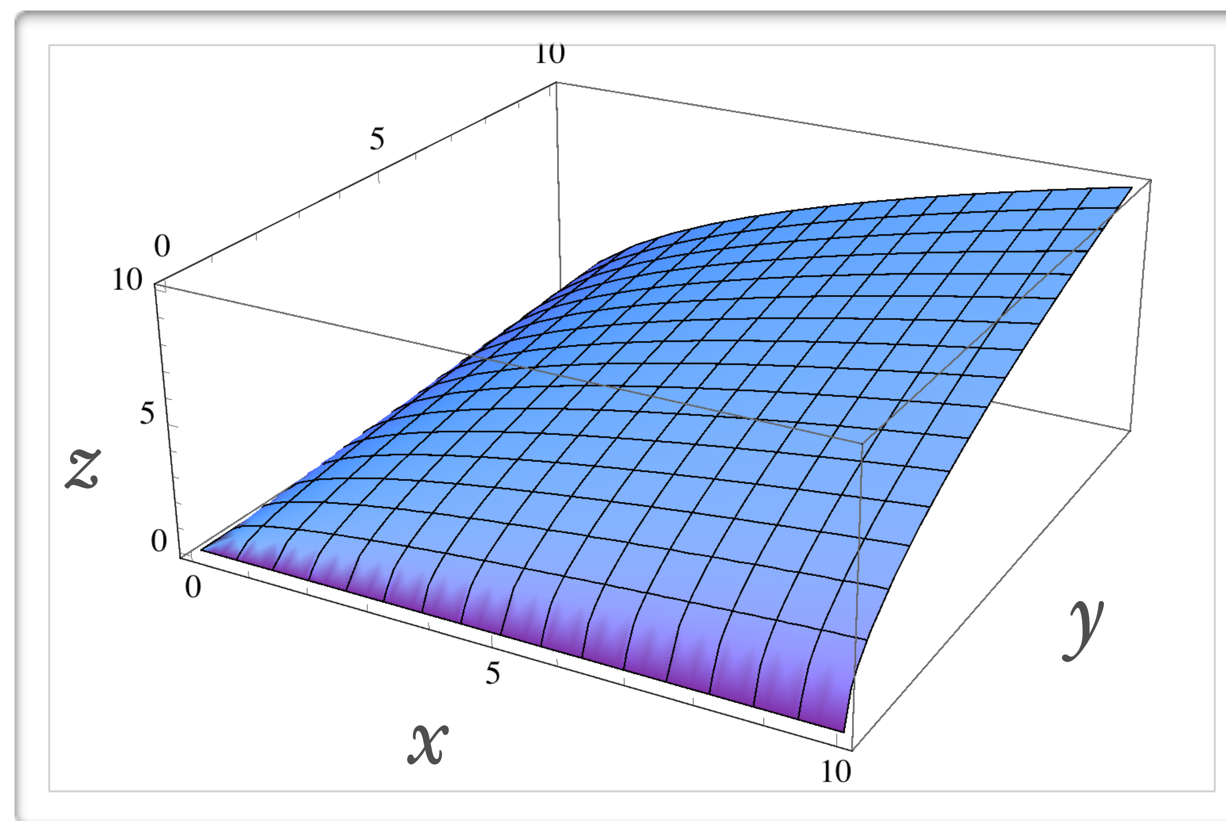
A function of a ***two*** variables can be graphically represented by ***interpreting*** the image $f(x,y)$ of the variables x and y as being the z coordinate in the 3D $(x,\ y,\ z)$ cartesian frame.

Examples

$$z = f(x,y) = \sqrt{x^2 + y^2}$$



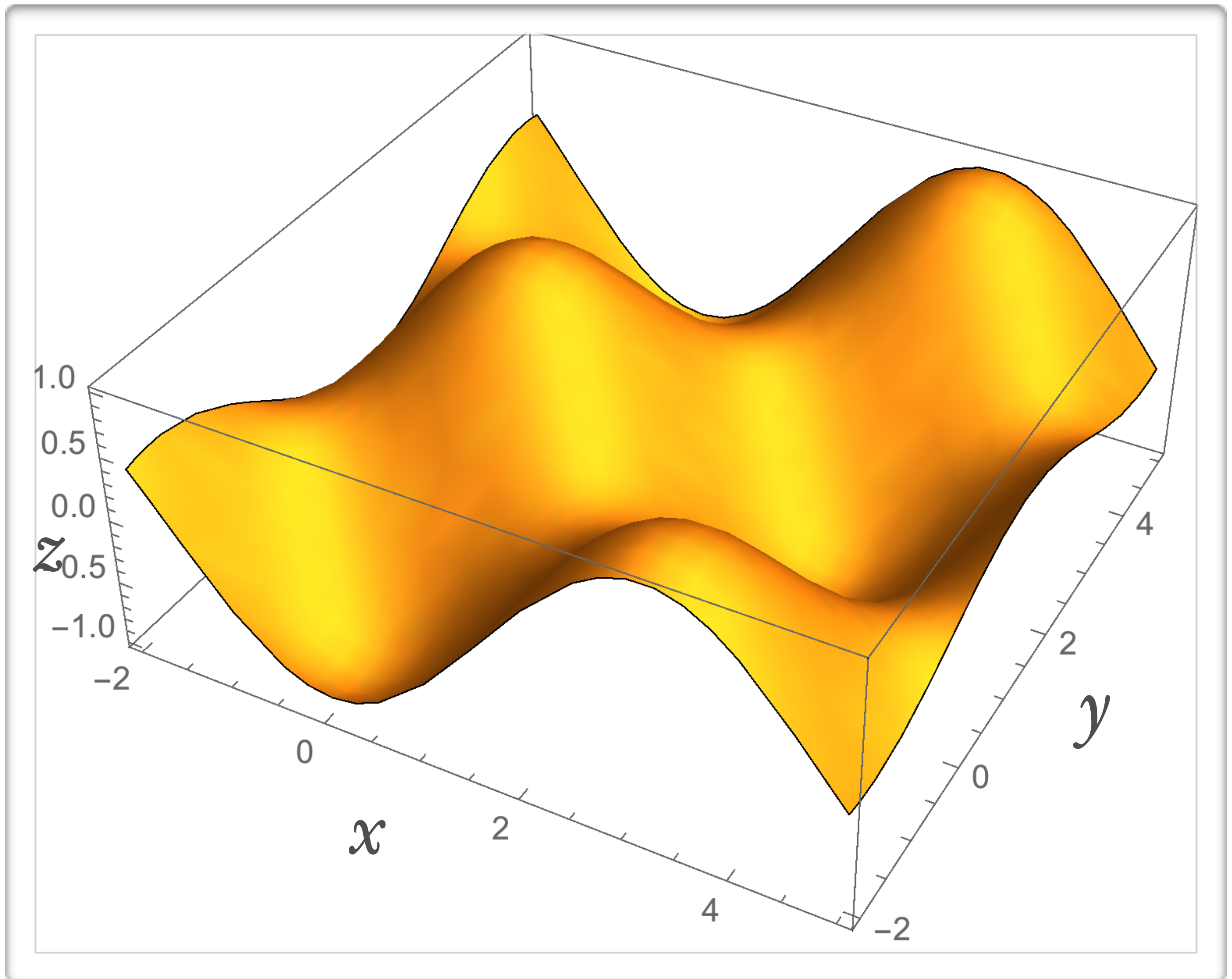
$$z = f(x,y) = x \cdot y$$



Scalar functions of multiple variables

<u>Examples</u>

$$z = f(x, y) = \cos(x) \cdot \sin(y)$$



Scalar functions of multiple variables

Partial derivatives and gradient

Let $f(x, y)$ a function of two variables.

- The partial derivative of f with respect to x, denoted as $\dfrac{\partial f}{\partial x}$, is defined as

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

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We define the ***gradient*** of f denoted ∇f as being the vector field

$$\boxed{\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\hat{i} + \frac{\partial f}{\partial y}(x, y)\hat{j}}$$

Scalar functions of multiple variables

Remark

Sometimes the partial derivative of f with respect to x at fixed y is denoted by $\left(\frac{\partial f}{\partial x}\right)_y$ and the partial derivative of f with respect to y at fixed x is denoted by $\left(\frac{\partial f}{\partial y}\right)_x$.

☆The partial derivative of a function can be seen as another function and can again be partially integrated.

Second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

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Partial derivatives and gradient

Example

Find the partial derivatives and the gradient of $f(x, y) = 2xy + y$

Scalar functions of multiple variables

Partial derivatives and gradient

Example

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$$\frac{\partial f}{\partial x}(x, y) = 2y, \;\; \frac{\partial f}{\partial y}(x, y) = 2x + 1$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\hat{\imath} + \frac{\partial f}{\partial y}(x, y)\hat{\jmath} = 2y\hat{\imath} + (2x + 1)\hat{\jmath}$$

Energy concepts in two dimensions

Kinetic energy and work in 2 dimensions

In **two *dimensions*** the kinetic energy of a point object of mass m moving with velocity $\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$ with respect to a frame $(O,\ \hat{i},\ \hat{j})$ is

$$K(t) = \frac{1}{2}m\|\overrightarrow{v}(t)\|^2 = \frac{1}{2}m(v_x(t)^2 + v_y(t)^2)$$

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The work done by a force $\overrightarrow{F}$ on a point object along a 2D path $\Gamma_{A\to B}$ with position vector $\vec{r}(t)$ and velocity $\overrightarrow{v}(t) = \dot{\vec{r}}(t)$ going from $\vec{r}(t_A)$ to $\vec{r}(t_B)$ is:

$$W(\overrightarrow{F}\,|\,\Gamma_{A\to B}) = \int_{\Gamma_{A\to B}} \overrightarrow{F}(\Gamma)\cdot d\vec{r} \equiv \int_{t_A}^{t_B} \overrightarrow{F}(\vec{r}(t))\cdot\overrightarrow{v}(t)dt$$

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The work is zero if $\overrightarrow{F} \perp \overrightarrow{v}$

Conservative forces in 2 dimensions

A 2D force $\overrightarrow{F}$ is said to be ***conservative*** if any of the ***equivalent*** propositions is true:

- $\overrightarrow{F}$ is a function of position such that there exists a function $U(x, y)$ satisfying

$$\overrightarrow{F}(x, y) = -\nabla U(x, y) = -\frac{\partial U}{\partial x}(x, y)\hat{i} - \frac{\partial U}{\partial y}(x, y)\hat{j}$$

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- The work done by $\overrightarrow{F}$ on a point object moving from a point A to a point B is ***independent of the path taken***

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- The work done by $\vec{F}$ on a point object along ***any closed*** path is ***zero***

Conservative forces in 2 dimensions

Example

We consider the force $\overrightarrow{F}(x, y) = -2y\hat{i} - (2x + 1)\hat{j}$, where x, y is given in metres and $\overrightarrow{F}$ in newtons (more formally $\overrightarrow{F}(x, y) = (-2y\ N \cdot m^{-1})\hat{i} - (2x\ N \cdot m^{-1} + 1\ N)\hat{j}$).

Show that $\overrightarrow{F}$ is conservative and evaluate the work along any path from the point A with coordinates $(x_A, y_A) = (1,0)$ to the point B with $(x_B, y_B) = (1,1)$.

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From the previous example, for $U(x,y) = 2xy + y$ we have that

$$\nabla U(x,y) = 2y\hat{\imath} + (2x+1)\hat{\jmath}, \text{ so } \overrightarrow{F} = -\nabla U(x,y),$$

which means that $\overrightarrow{F}$ is conservative.

$$W(F\,|\,\Gamma_{A\to B}) = U(x_A, y_A) - U(x_B, y_B) = 0 - (2+1)\ \mathrm{J} = -3\ \mathrm{J}\,.$$

Work-energy theorem and energy conservation

We consider point object of mass m in a Galilean frame $(O, \hat{i}, \hat{j})$ subject to a net force $\overrightarrow{F}$, moving on a path $\Gamma_{A \to B}$, from point A to B.

☆ Work-energy theorem: $K_B - K_A = W(\overrightarrow{F} \,|\, \Gamma_{A \to B})$

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- If the net force $\overrightarrow{F}$ is not conservative but can be expressed as the sum of a conservative force $\overrightarrow{F}_c$, with potential energy U, and a non-conservative force $\overrightarrow{F}_{n-c}$, i.e $\overrightarrow{F} = \overrightarrow{F}_c + \overrightarrow{F}_{n-c} = -\nabla U + \overrightarrow{F}_{n-c}$,

  then $\boxed{E_B - E_A = W(\overrightarrow{F}_{n-c} \,|\, \Gamma_{A\to B})}$ or

$$(K_B + U_B) - (K_A + U_A) = \int_{\Gamma_{A\to B}} \overrightarrow{F}_{n-c}(\Gamma)\cdot d\vec{r}$$

Linear momentum

Definition of linear momentum

We consider a point object of mass m with position vector $\vec{r}(t)$ and velocity $\vec{v}(t)$ relative to a frame $(O,\ \hat{i},\ \hat{j})$

We define the ***linear momentum*** $\vec{p}(t)$ of this point object as being

$$\boxed{\overrightarrow{p}(t) = m\overrightarrow{v}(t)}$$

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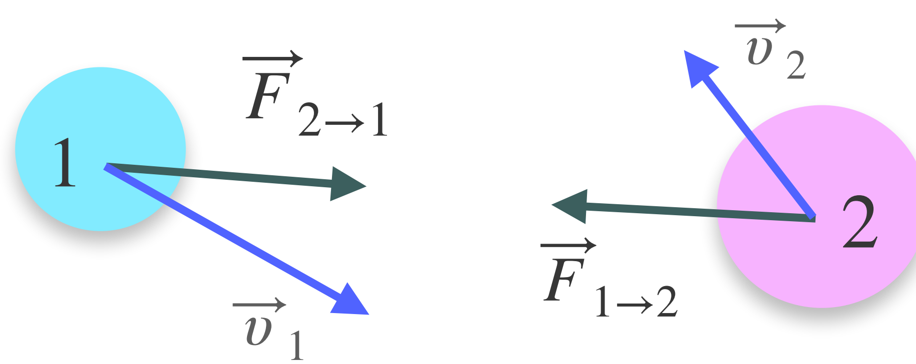
For a system comprising multiple parts, $\{(m_1,\ \vec{v}_1);\ \ldots;\ (m_N,\ \vec{v}_N)\}$ its ***total linear momentum*** reads:

$$\boxed{\vec{p}_{total}(t) \equiv \sum_{n=1}^{N} m_n\,\vec{v}_n(t)}$$

Conservation of linear momentum

We consider an isolated system (no external forces) consisting of two point objects with masses m_1, m_2 and velocities $\vec{v}_1(t)$ and $\vec{v}_2(t)$ with respect to a Galilean frame of reference.



Let us denote $\vec{F}_{1\to2}$ the force of object 1 on object 2 and $\vec{F}_{2\to1}$ the force of object 2 on object 1.

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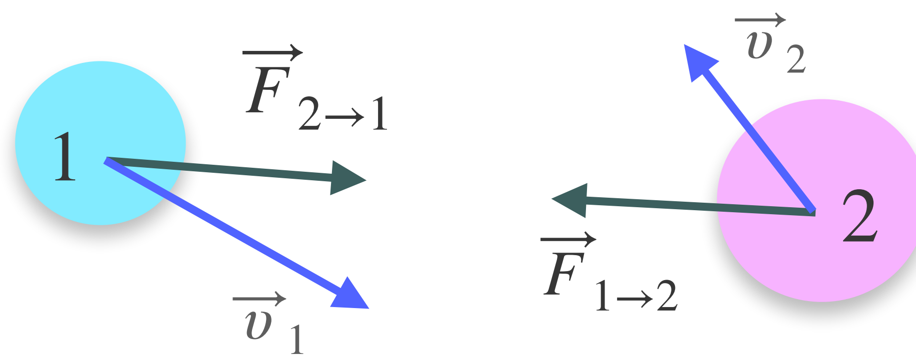

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Newton's 3rd law: $\overrightarrow{F}_{2\to 1} + \overrightarrow{F}_{1\to 2} = 0$. So,

$$m_1 \frac{d\overrightarrow{v}_1(t)}{dt} + m_2 \frac{d\overrightarrow{v}_2(t)}{dt} = 0, \text{ or } \frac{d}{dt}(m_1 \overrightarrow{v}_1(t) + m_2 \overrightarrow{v}_2(t)) = 0.$$

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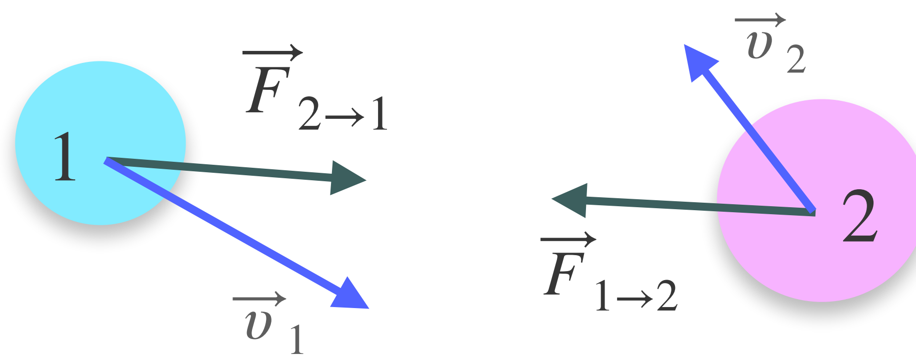

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$$\text{Hence, } \frac{d}{dt}(\vec{p}_1(t) + \vec{p}_2(t)) = 0, \text{ i.e. } \dot{\vec{p}}_{tot}(t) = 0.$$

Conservation of linear momentum

In absence of external forces in a Galilean frame, the total linear momentum of a system is ***conserved***, that means $\vec{p}_{tot}(t)$ *is constant*.

Conservation of linear momentum

Example

A shooter, initially standing in aiming position, fires his $2kg$ gun that shoots a $10g$ bullet. Assuming the frame is Galilean, determine the recoil velocity of the gun if the bullet velocity is $(470\ m/s)\hat{i}$.

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From the conservation of momentum we have that

$$m_{gun}\overrightarrow{v}'_{gun} + m_{bullet}\overrightarrow{v}'_{bullet} = m_{gun}\overrightarrow{v}_{gun} + m_{bullet}\overrightarrow{v}_{bullet}, \text{ so}$$

$$(2kg)\overrightarrow{v}'_{gun} + (0.01kg)(470\ m/s)\hat{i} = 0, \text{ or}$$

$$\overrightarrow{v}'_{gun} = -\frac{(0.01kg)(470\ m/s)}{2kg}\hat{i} = -(2.35m/s)\hat{i}$$

Elastic collision

We consider two solid objects of mass m_1 and m_2 initially moving in one dimension at velocities $\overrightarrow{v_1} = v_1\hat{i}$ and $\overrightarrow{v_2} = v_2\hat{i}$.

They then ***collide*** and go apart from each other with velocities $\overrightarrow{v_1}' = v_1'\hat{i}$ and $\overrightarrow{v_2}' = v_2'\hat{i}$.

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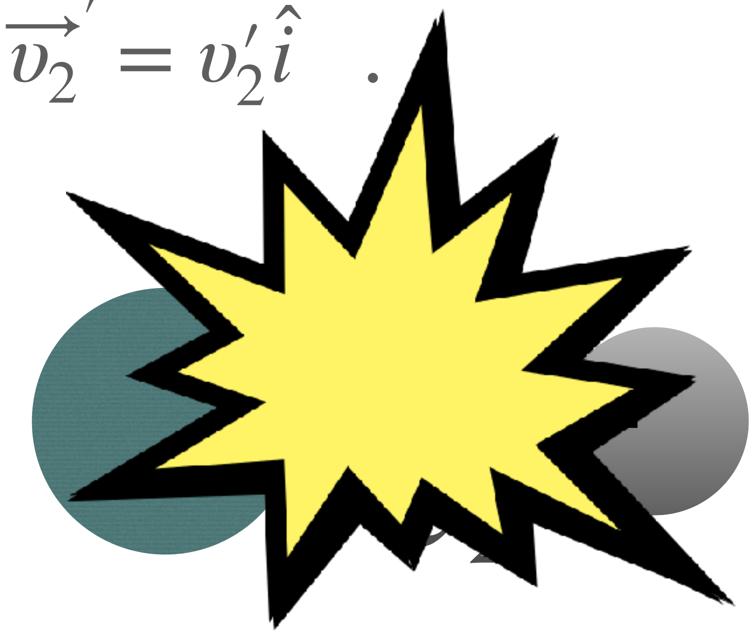

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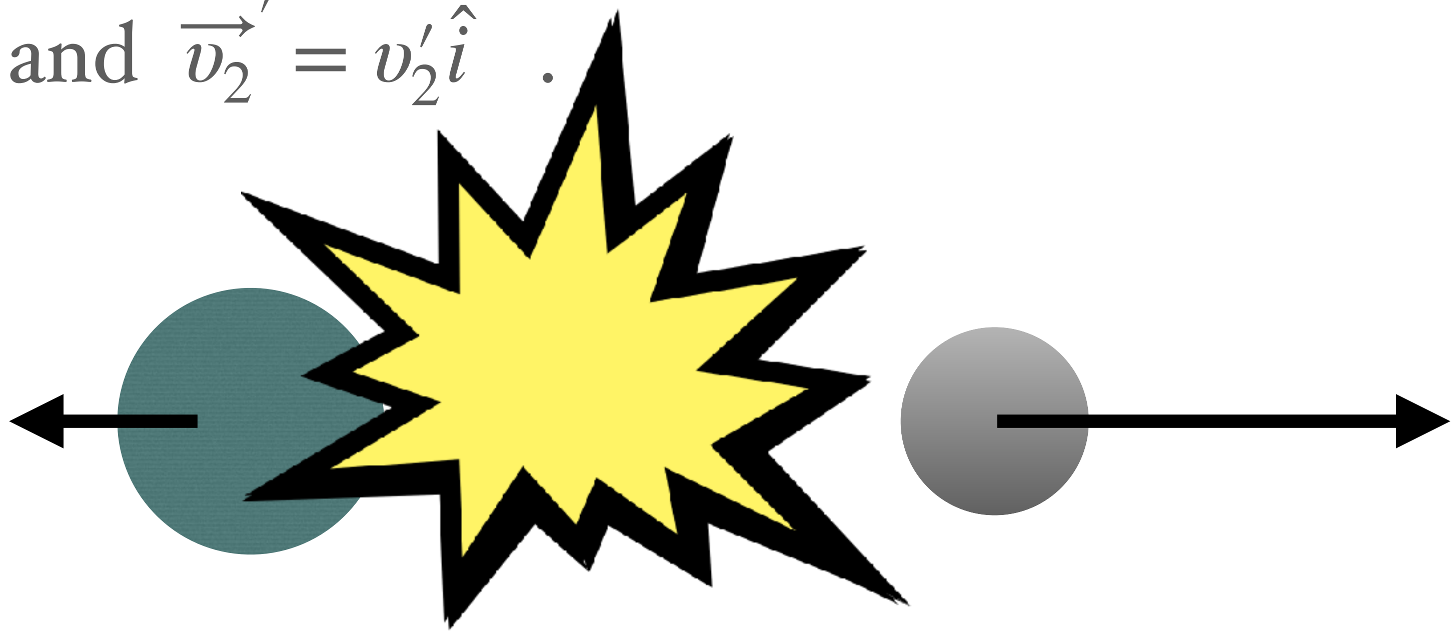

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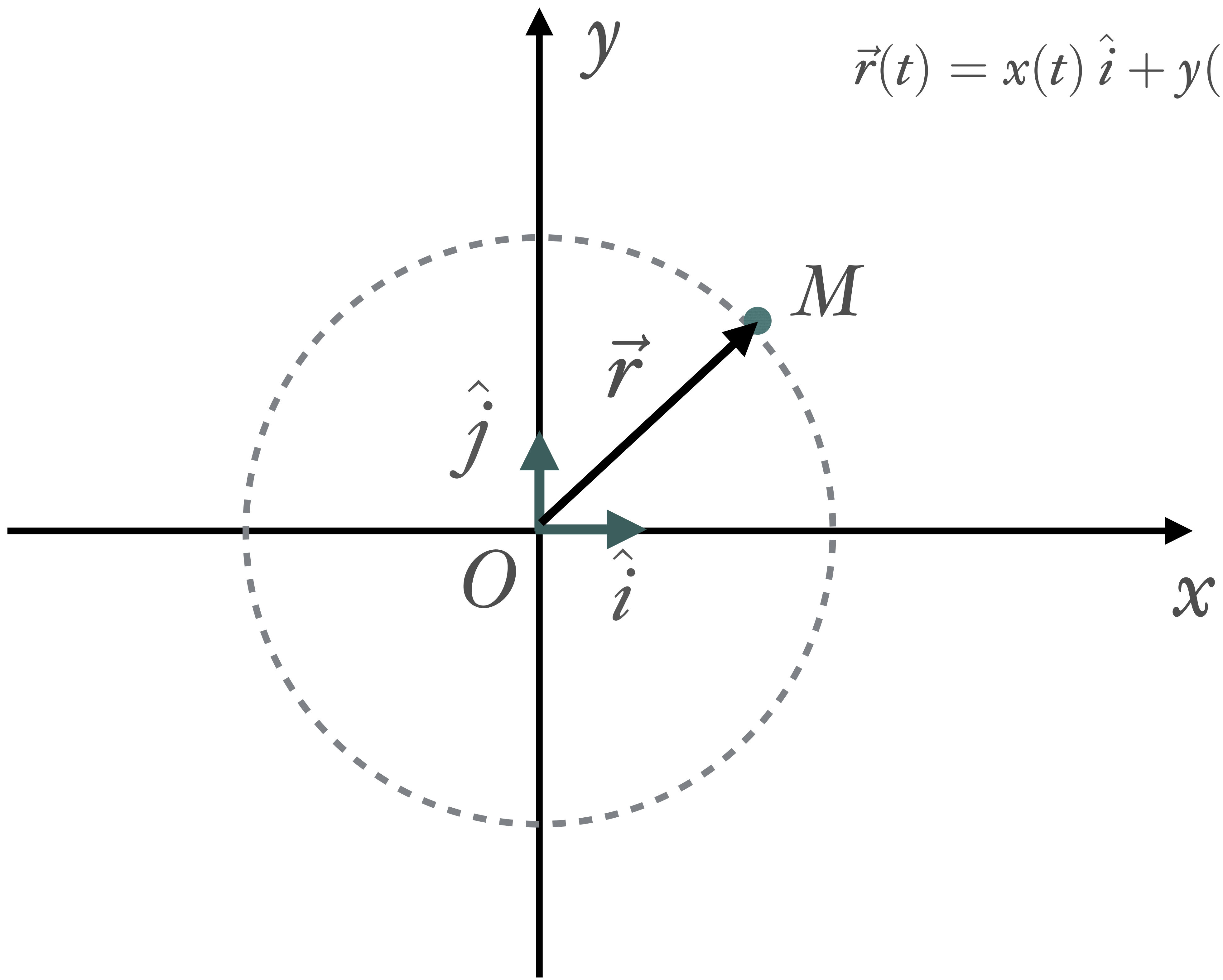
A collision process between two solid objects is said to be ***elastic*** if both the total linear momentum and the total kinetic energy are the same before and after the collision:

$$\boxed{\begin{gathered} m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \\ \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1^{'2} + \frac{1}{2}m_2v_2^{'2} \end{gathered}}$$

A short detour on circular motion

Characterising circular motion



$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

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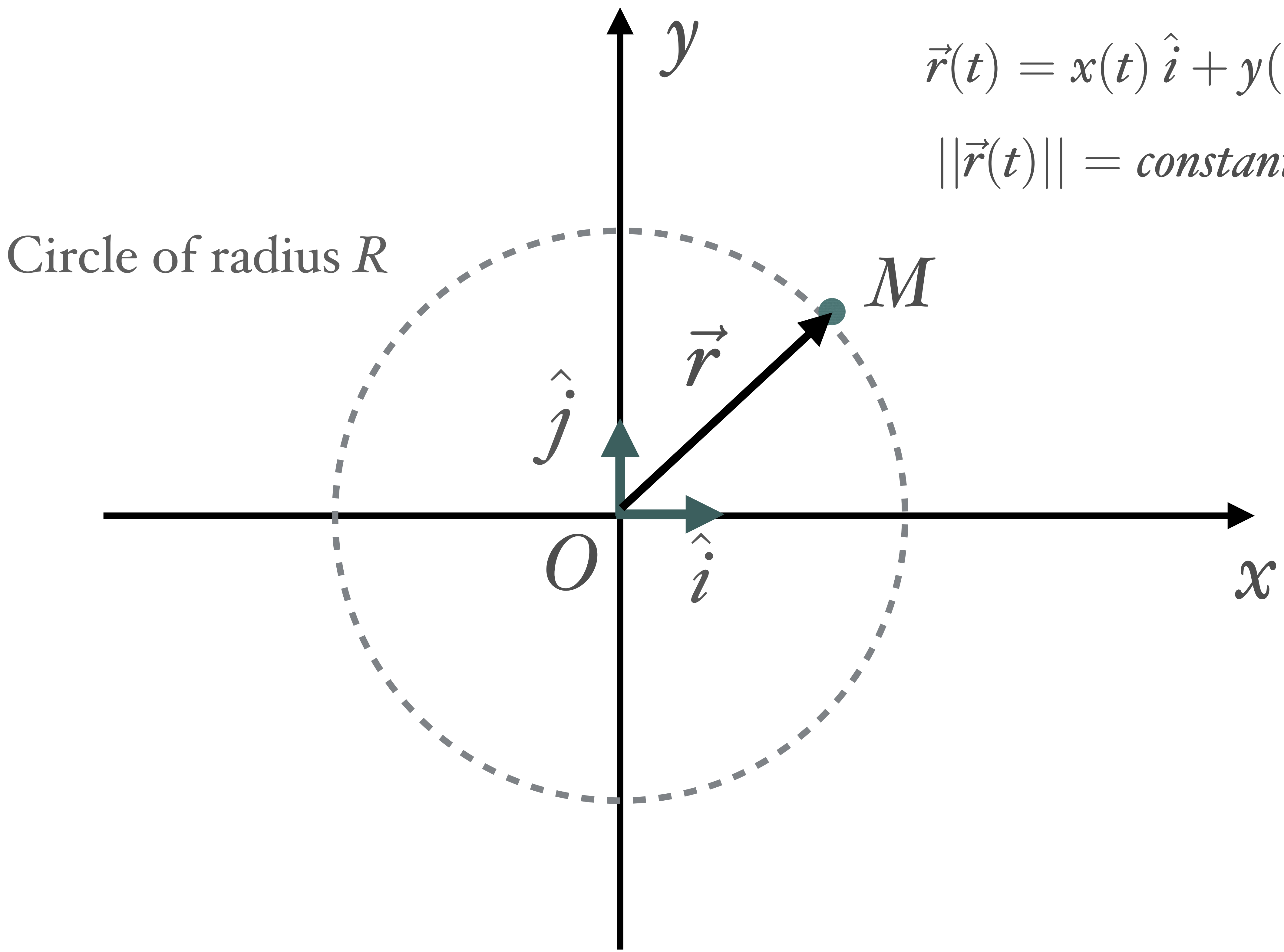

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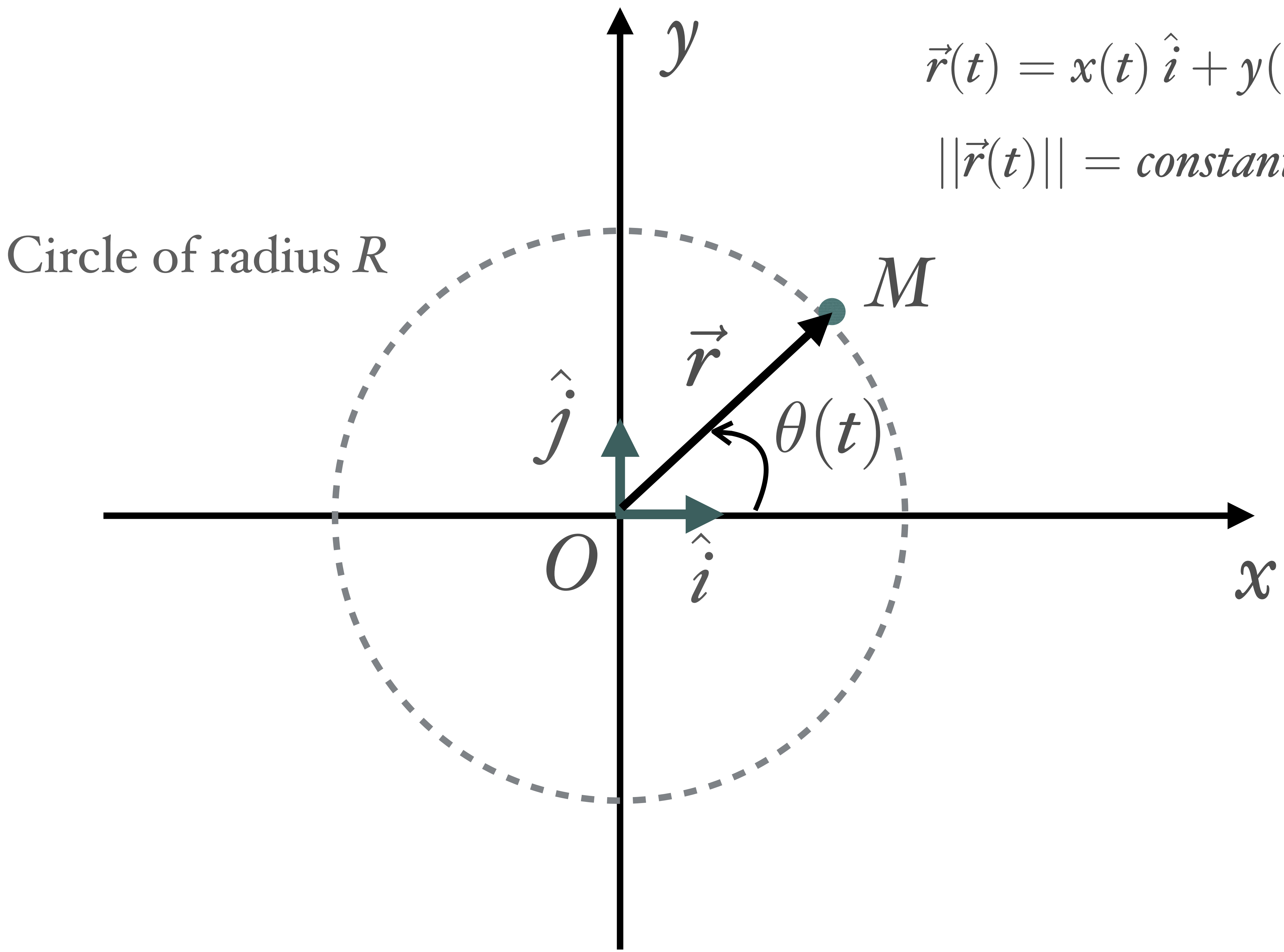

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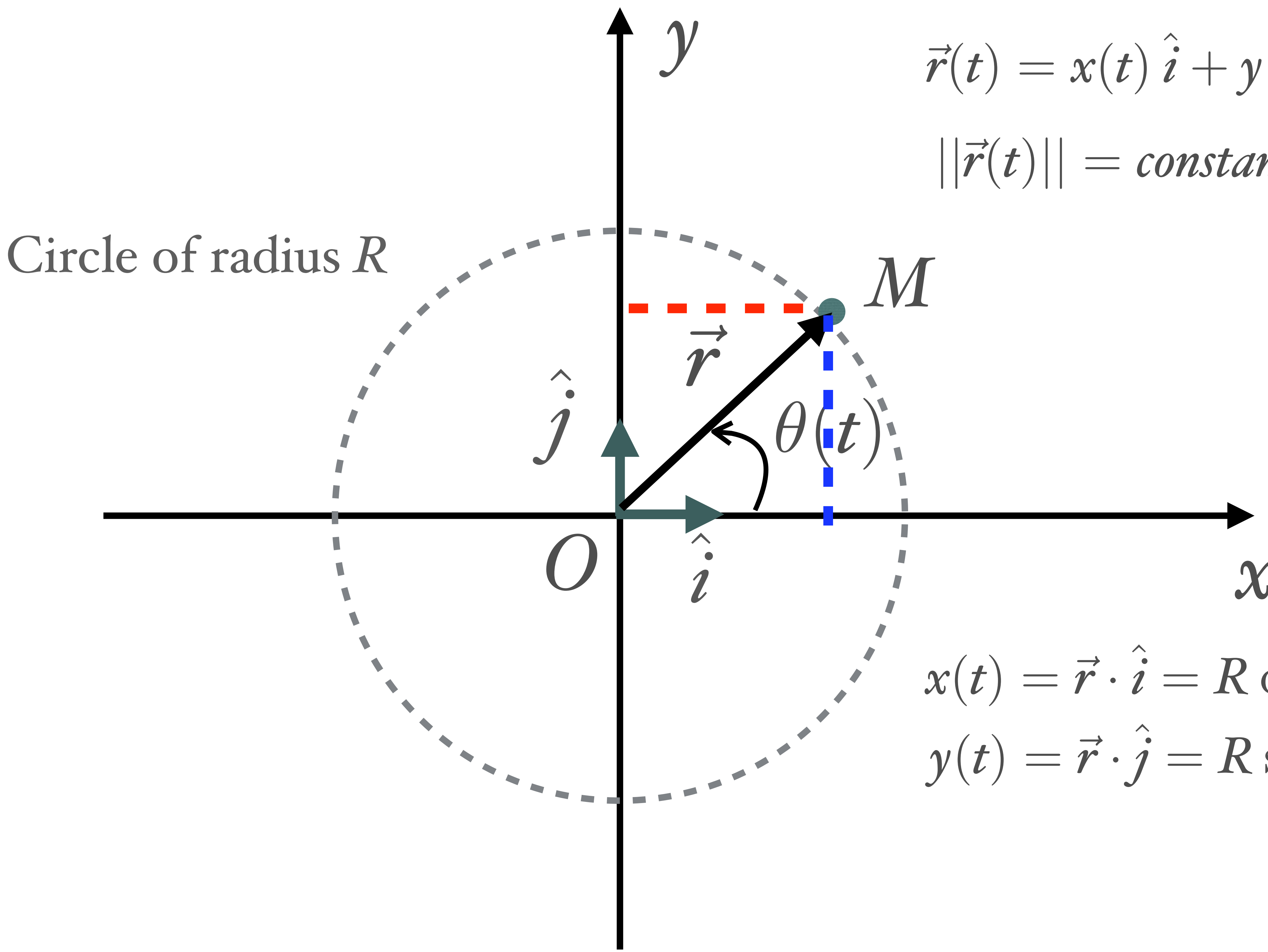

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$||\vec{r}(t)|| = constant = R$$

$$x(t) = \vec{r}\cdot\hat{i} = R\cos\theta(t)$$

$$y(t) = \vec{r}\cdot\hat{j} = R\sin\theta(t)$$

Velocity vector for circular motion

Circular motion around a circle of radius R is uniquely characterised by the angle $\theta(t)$, i.e.

$$\vec{r}(t) = R\cos\theta(t)\,\hat{\imath} + R\sin\theta(t)\,\hat{\jmath}$$

$$\vec{v}(t) \equiv \dot{\vec{r}}(t) = -R\dot{\theta}(t)\sin\theta(t)\,\hat{\imath} + R\dot{\theta}(t)\cos\theta(t)\,\hat{\jmath}$$

Angular velocity: $\boxed{\omega(t) = \dot{\theta}(t)}$

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Properties

☆ $$\|\overrightarrow{v}(t)\| = \sqrt{R^2\dot{\theta}(t)^2(sin^2\theta(t) + cos^2\theta(t))} = R\,|\,\dot{\theta}(t)\,|\text{, so}$$

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☆ $\vec{r}(t)\cdot\overrightarrow{v}(t) = 0$, so $\vec{r}(t) \perp \overrightarrow{v}(t)$

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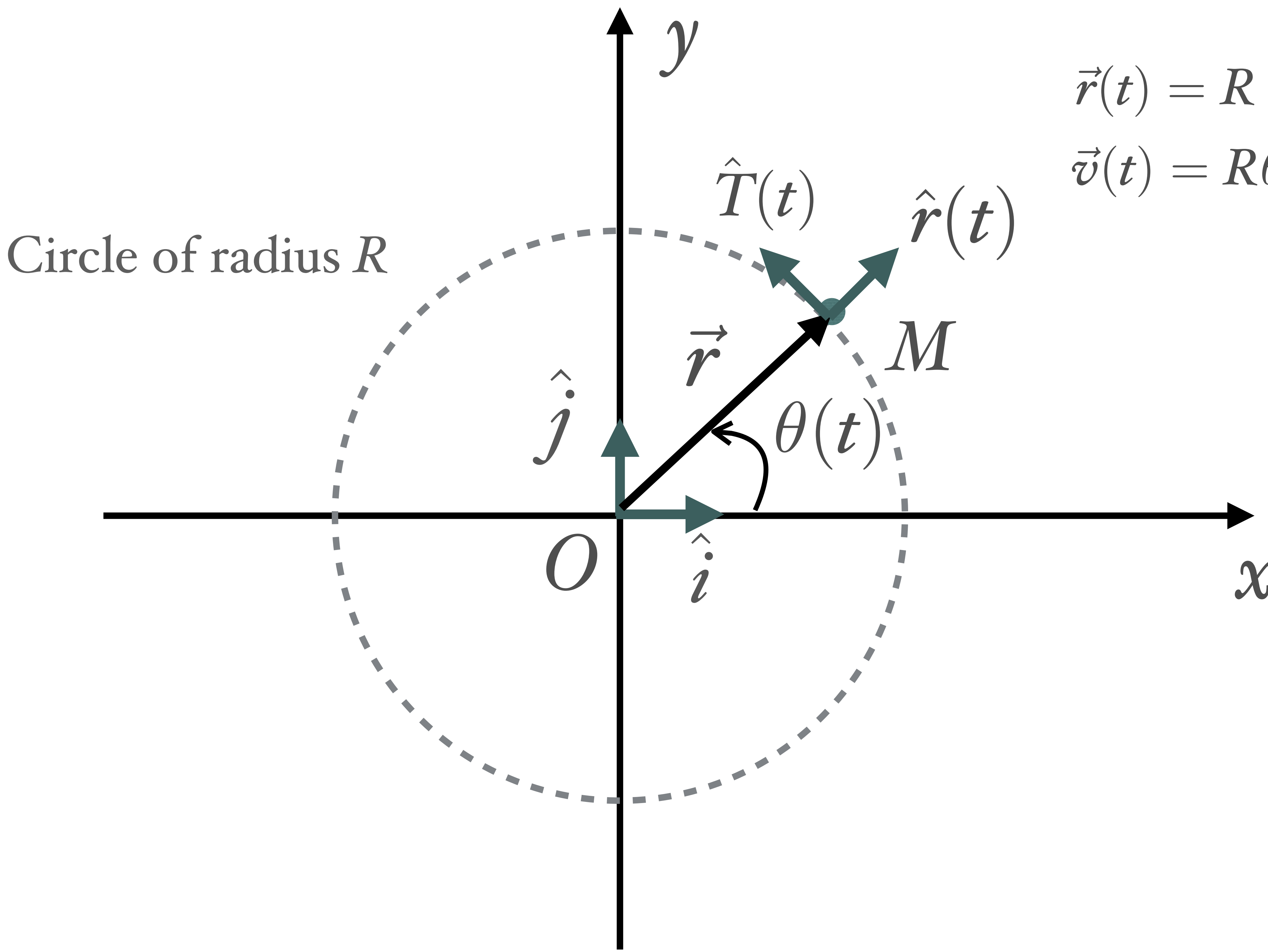
We consider the vectors

$$\hat{r}(t) = cos\theta(t)\hat{i} + sin\theta(t)\hat{j},\ \hat{T}(t) = -\ sin\theta(t)\hat{i} + cos\theta(t)\hat{j}$$

with $\|\hat{r}(t)\| = \|\hat{T}(t)\| = 1$ and $\hat{r}(t) \perp \hat{T}(t)$. Then

$$\boxed{\begin{aligned} \vec{r}(t) &= R\,\hat{r}(t) \\ \vec{v}(t) &= R\dot{\theta}(t)\,\hat{T}(t) \end{aligned}}$$

Velocity vector for circular motion



$$\vec{r}(t) = R\,\hat{r}(t)$$

$$\vec{v}(t) = R\dot{\theta}(t)\;\hat{T}(t)$$

Acceleration vector for circular motion

$$\vec{v}(t) = R\dot{\theta}(t) \cdot (-\sin\theta(t)\,\hat{i} + \cos\theta(t)\,\hat{j})$$

$$\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = R\ddot{\theta}(t)(-sin\theta(t)\hat{i} + cos\theta(t)\hat{j}) + R\dot{\theta}^2(t)(-cos\theta(t)\hat{i} - sin\theta(t)\hat{j})$$

Acceleration vector for circular motion

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and $\omega(t) = \dot{\theta}(t)$, we derive $\boxed{\vec{a}(t) = R\ddot{\theta}(t)\,\hat{T}(t) + R\omega^2(t)\,\hat{N}(t)}$

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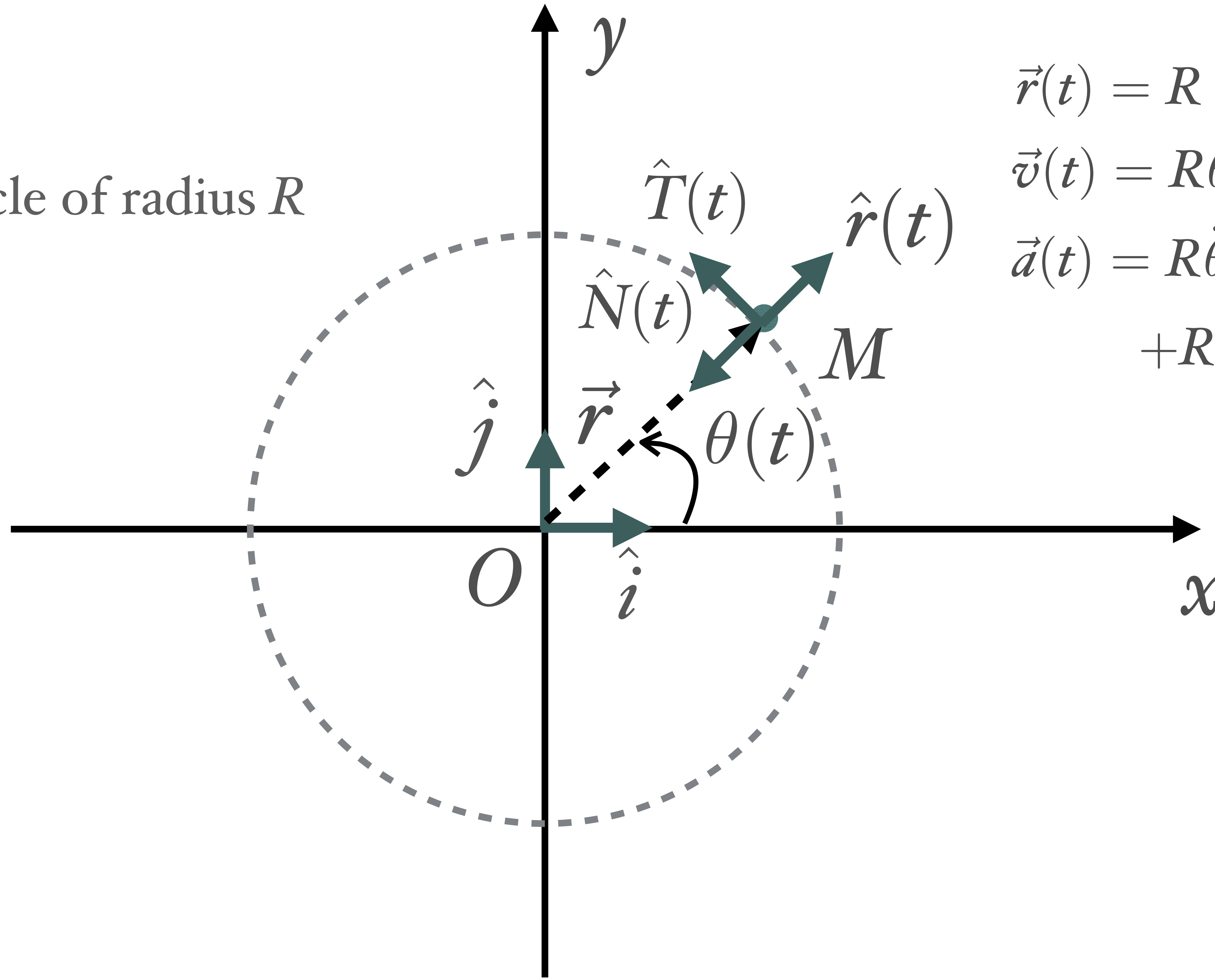
⭐Notice that even if the speed $\|\overrightarrow{v}(t)\|$ is constant the centripetal acceleration is not zero.

Kinematics for circular motion

Circle of radius R



$$\vec{r}(t) = R\,\hat{r}(t)$$

$$\vec{v}(t) = R\dot{\theta}(t)\ \hat{T}(t)$$

$$\vec{a}(t) = R\ddot{\theta}(t)\ \hat{T}(t) + R\omega(t)^2\ \hat{N}(t)$$

Kinematics for circular motion

Example

A sprinter runs at $10m/s$ on a circular racing track of radius $R = 50m$. Determine their angular velocity and centripetal acceleration.

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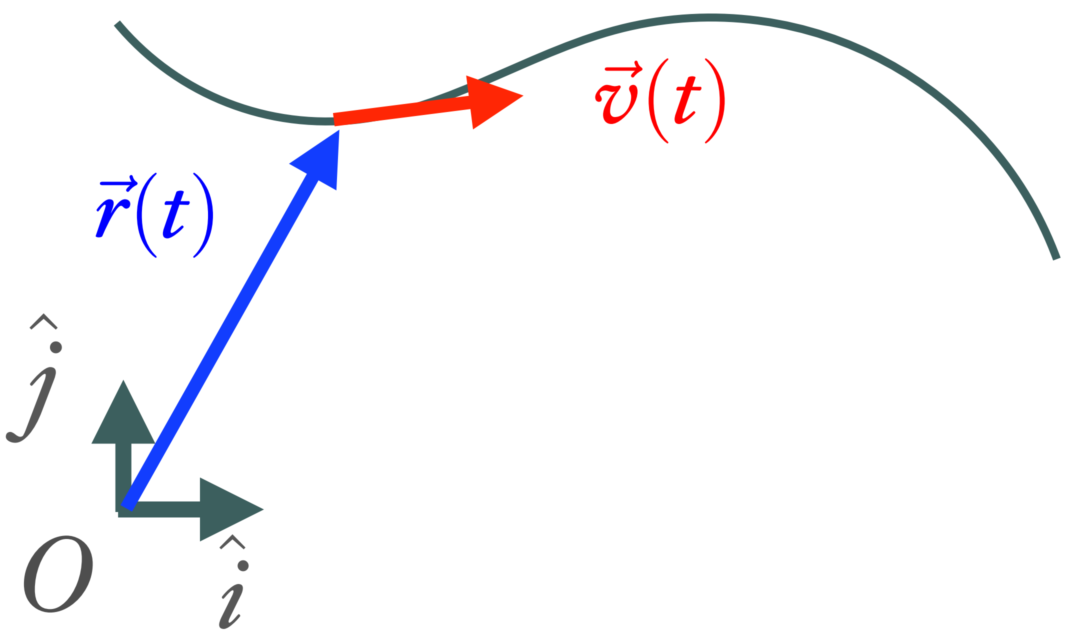
- Centripetal acceleration:

$$a_N = R\omega^2 = (50m)(0.2s^{-1})^2 = 2\ m \cdot s^2$$

A short introduction to the angular momentum

Definition in two dimensions

We consider a point object of mass m with position vector $\vec{r}(t)$ and velocity $\vec{v}(t)$ relative to a frame $(O,\ \hat{i},\ \hat{j})$



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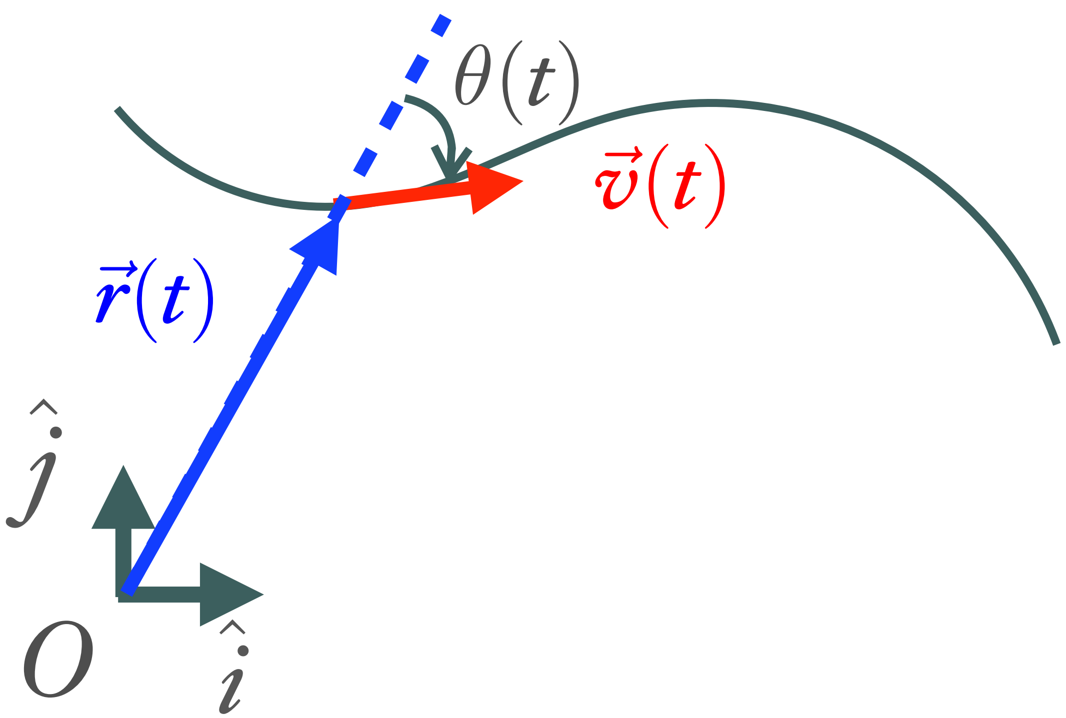

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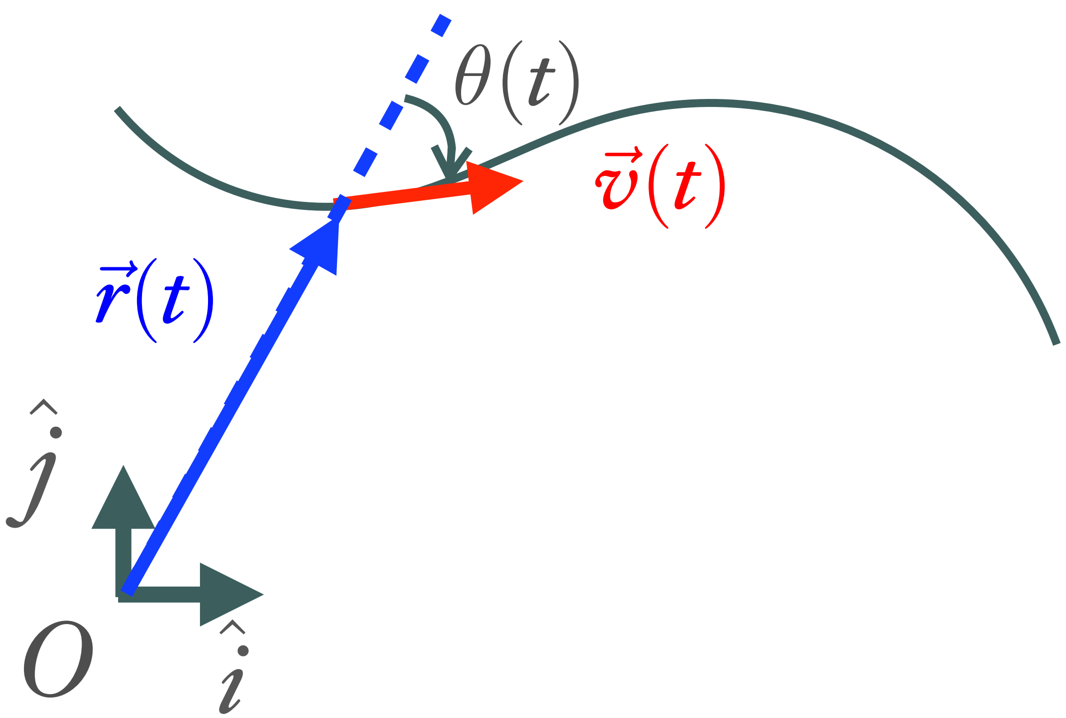


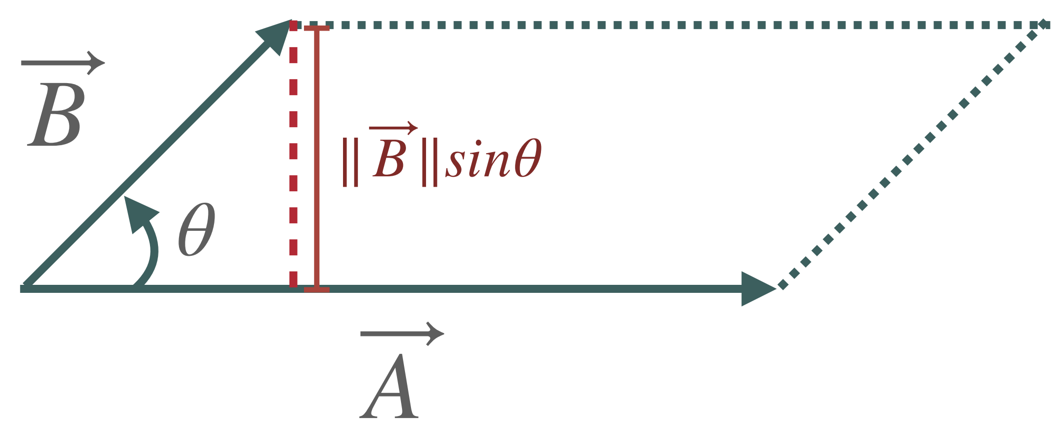
We define the ***angular momentum*** as:

$$L_O(t) \equiv m \cdot ||\vec{r}(t)|| \cdot ||\vec{v}(t)|| \cdot \sin\theta(t)$$

Definition in two dimensions

In two dimensions, the angular momentum can be written in term of a 2D ***vector product*** or ***cross product*** defined as:

$$\vec{A} \wedge \vec{B} = ||\vec{A}|| \cdot ||\vec{B}|| \cdot \sin \theta_{\vec{A}\vec{B}}$$



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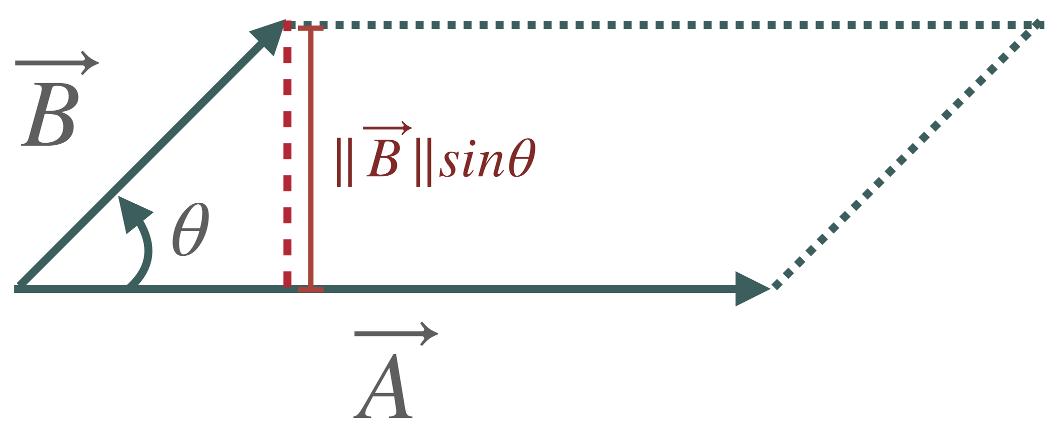

In 2D only, the cross product is a ***number*** which is positive if the angle going from $\overrightarrow{A}$ to $\overrightarrow{B}$ goes anticlockwise and negative otherwise.

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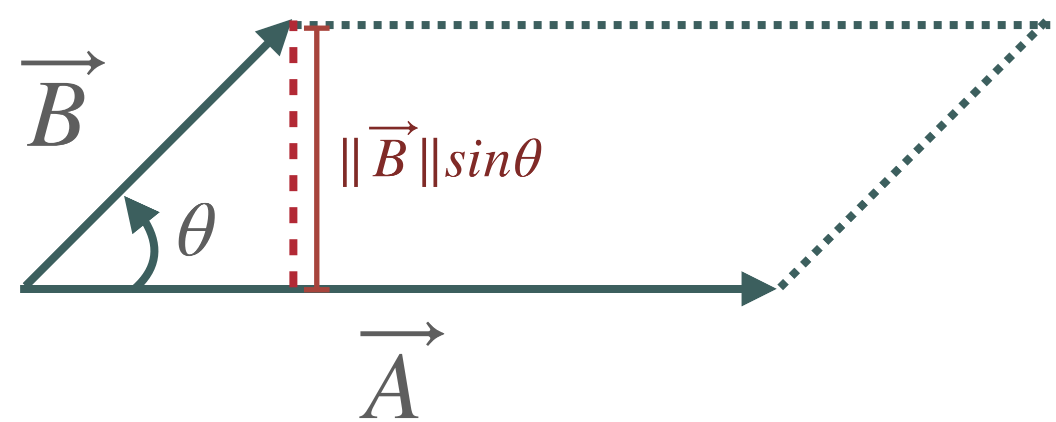

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$$\boxed{L_O(t) = \vec{r}(t) \wedge m\vec{v}(t)}$$

Time evolution of the angular momentum

We have seen that for a single point object, the angular momentum relative to point O is

$$L_O(t) = \vec{r}(t) \wedge m\vec{v}(t)$$

Then

$$\dot{L}_O(t) = \dot{\vec{r}}(t) \wedge m\vec{v}(t) + \vec{r}(t) \wedge m\dot{\vec{v}}(t)$$

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But $\vec{v}(t) \wedge \vec{v}(t) = 0$ and in a Galilean frame $m\overrightarrow{a}(t) = \overrightarrow{F}$. Hence,

$$\boxed{\dot{L}_0 = \vec{r}(t) \wedge \overrightarrow{F}}$$

Time evolution of the 2D angular momentum

The quantity $M_0 = \vec{r} \wedge \overrightarrow{F}$ is called the **moment** or **torque** of the force $\overrightarrow{F}$ with respect to point O.

As we have shown $\boxed{\dot{L}_0 = M_0}$

<u>Remark</u>:
If more than one force is acting on a point object, we can specify which force the moment refers to by adding it in parentheses $M_0(\overrightarrow{F})$.

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Conservation of angular momentum

In absence of external forces the angular momentum is conserved, $\dot{L}_0 = 0$.

In general the total angular momentum $L_{tot} = L_{O_1} + L_{O_2} + \ldots L_{O_n}$ of a system of n point objects varies according to:

$$\dot{L}_{tot} = \sum_i \mathcal{M}_O(\vec{F}^{(i)}_{ext})$$

Angular momentum

Example

A sprinter of m=60kg runs anticlockwise at $10m/s$ on a circular racing track of radius $R = 50m$. Determine their angular momentum relative to the centre of the circular track.

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$\overrightarrow{v}(t) \perp \vec{r}(t)$ and the angle from $\vec{r}(t)$ to $\overrightarrow{v}(t)$ goes anticlockwise. So, $sin\theta = 1$. Hence,

$$L_O(t) = (60kg)(50m)(10m \cdot s^{-1}) = 30000\ kg \cdot m^2 \cdot s^{-1}$$