Note: inverting a Laplace transform is often very hard. We usually use a table of transforms and hope to find the relevant function / transform pair.

Inverting Laplace transforms

Example Using the facts that
$\mathcal{L}[1] = 1/s$ and $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$,

where a is a constant, find $f(t)$ if

$$\tilde{f}(s) = \frac{s+3}{s(s+1)}.$$

- To bring $\tilde{f}(s)$ into a form where we can use the known transforms, use partial fractions.

$$\tilde{f}(s) = \frac{s+3}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow s+3 = A(s+1) + Bs$$
$$= (A+B)s + A$$

Equating coefficients:
$A = 3$ and $A + B = 1 \Rightarrow B = 1 - A = -2$

$$\Rightarrow \quad \tilde{f}(s) = \frac{3}{s} - \frac{2}{s+1}$$

Then, the inverse Laplace transform of $3/s$ is given by

$$\mathcal{L}^{-1}[3/s] = 3\mathcal{L}^{-1}[1/s] = 3$$

and that of $2/(s+1)$ by

$$\mathcal{L}^{-1}[2/(s+1)] = 2\mathcal{L}^{-1}[1/(s+1)] = 2e^{-t}$$

so that $f(t) = 3 - 2e^{-t}$.

Properties of Laplace transforms

① The shift theorem

$$\mathcal{L}[e^{-at} f(t)] = \tilde{f}(s+a)$$

Proof $\mathcal{L}[e^{-at} f(t)] = \int_0^\infty e^{-at} e^{-st} f(t)\,dt$

$$= \underbrace{\int_0^\infty e^{-(s+a)t} f(t)\,dt}_{\text{definition of the Laplace transform with } s \text{ replaced by } s+a} = \tilde{f}(s+a)$$

Example Since $\mathcal{L}[\sin \omega t] = \dfrac{\omega}{s^2+\omega^2}$,

the Laplace transform of $\exp(-\gamma t)\sin \omega t$ (a damped oscillation) is

$$\mathcal{L}[e^{-\gamma t}\sin\omega t] = \frac{\omega}{(s+\gamma)^2+\omega^2}$$

(2) <u>Scaling</u> $\quad \mathcal{L}[f(at)] = \frac{1}{a}\tilde{f}(s/a)$

proved by a change of variable as for Fourier transforms.

(3) $$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\tilde{f}(s)$$

<u>Proof</u>

$$\frac{d^n}{ds^n}\tilde{f}(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d^n}{ds^n}(e^{-st})f(t)\,dt$$

$$= \int_0^\infty (-1)^n t^n e^{-st} f(t)\,dt = (-1)^n \int_0^\infty e^{-st}\left[t^n f(t)\right]dt$$

$$= (-1)^n \mathcal{L}[t^n f(t)] \Rightarrow \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\tilde{f}(s)$$

<u>Example</u> Given that $\mathcal{L}[1] = 1/s$, calculate the Laplace transform of $f(t) = t^n$, $n \in \mathbb{Z}_+$.

$$\mathcal{L}[t^n . 1] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[1] = (-1)^n \frac{d^n}{ds^n}\frac{1}{s}$$

$$= (-1)^n(-1)^n \frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

④ The convolution theorem

$$\boxed{\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\,du\right] = \tilde{f}(s)\,\tilde{g}(s)}$$

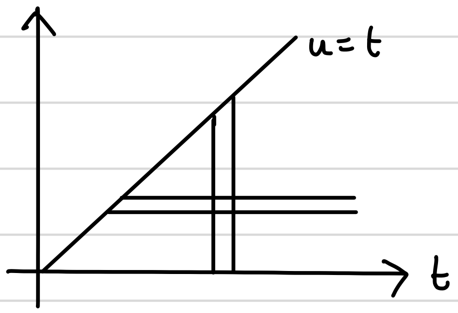
- The integral $\int_0^t f(t-u)\,g(u)\,du$ is the convolution of f and g, denoted by $f * g$.

- Note the difference in integration limits from the Fourier transform case.

Proof

- By definition,

$$\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\,du\right] = \int_0^\infty e^{-st} \int_0^t f(t-u)\,g(u)\,du\,dt$$



- The region of integration is an infinite wedge

- The inner integral over u from $u=0$ to $u=t$ is performed first. It runs over the vertical strip shown above.
- The outer integral over t sums the integrals over the vertical strips to give the integral over the whole infinite wedge.

- We can also integrate over horizontal strips running from $t = u$ to ∞ and then integrate over these from $u = 0$ to ∞. This gives

$$\int_0^\infty g(u) \int_{t=u}^\infty e^{-ts} f(t-u)\, dt\, du$$

- The inner integral is performed at a fixed value of u, so we can write $t' = t - u$, so that

$$\int_0^\infty g(u) \int_{t'=0}^\infty e^{-s(u+t')} f(t')\, dt'\, du$$

$$= \int_0^\infty g(u)\, e^{-su}\, du \int_0^\infty f(t')\, e^{-st'}\, dt'$$

- Then,

$$\mathcal{L}\left[\int_0^t f(t-u)\, g(u)\, du\right] = \int_0^\infty f(t')\, e^{-st'}\, dt' \int_0^\infty g(u)\, e^{-su}\, du$$

$$= \tilde{f}(s)\, \tilde{g}(s)$$

<u>Example</u> Given that

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L}[e^{at}] = \frac{1}{s-a},$$

calculate $\mathcal{L}\left[\int_0^t e^{au} \cos(b(t-u))\, du\right]$.

- We identify $f(t-u) = \cos(b(t-u))$
and $g(u) = e^{au}$

so that $f(t) = \cos(bt)$
and $g(t) = e^{at}$

- By the convolution theorem,

$$\mathcal{L}\left[\int_0^t e^{au}\cos(b(t-u))\,du\right] = \tilde{f}(s)\tilde{g}(s) = \frac{s}{s^2+b^2}\,\frac{1}{s-a}$$

$$= \frac{s}{(s^2+b^2)(s-a)}$$

<u>Inversion using the convolution theorem</u>

- Since $\mathcal{L}\left[\int_0^t f(t-u)g(u)\,du\right] = \tilde{f}(s)\,\tilde{g}(s)$

we have that $\mathcal{L}^{-1}[\tilde{f}(s)\tilde{g}(s)] = \int_0^t f(t-u)g(u)\,du$

- If a function $\tilde{F}(s)$ is given as the product of two functions $\tilde{f}(s)$ and $\tilde{g}(s)$, the inverse transforms of which are known, we can use the above to calculate $F(t)$, the inverse transform of $\tilde{F}(s)$.

<u>Example</u> Evaluate $\mathcal{L}^{-1}\left[\dfrac{4}{s^2(s+2)^2}\right]$

- We write $\tilde{f}(s) = \dfrac{4}{s^2}$ and $\tilde{g}(s) = \dfrac{1}{(s+2)^2}$

- We saw before that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$
- Then, $\mathcal{L}^{-1}\left[\dfrac{n!}{s^{n+1}}\right] = t^n$, so $\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$

(setting $n = 1$) and $f(t) = \mathcal{L}^{-1}\left[\dfrac{4}{s^2}\right] = 4t$

- The other function, $\tilde{g}(s) = 1/(s+2)^2$, has the same form, but with s shifted to $s+2$.
- We therefore use the shift theorem

$$\mathcal{L}[e^{-at} f(t)] = \tilde{f}(s+a) \quad \text{or} \quad \mathcal{L}^{-1}[\tilde{f}(s+a)] = e^{-at} f(t)$$

to see that $g(t) = \mathcal{L}^{-1}\left[\dfrac{1}{(s+2)^2}\right] = e^{-2t} \cdot t = te^{-2t}$

- From the convolution theorem,

$$\mathcal{L}^{-1}[\tilde{f}(s)\,\tilde{g}(s)] = \int_0^t f(t-u)\, g(u)\, du$$

and $\mathcal{L}^{-1}\left[\dfrac{4}{s^2} \dfrac{1}{(s+2)^2}\right] = 4\displaystyle\int_0^t (t-u)\, u e^{-2u}\, du$

$$= 4\left\{ \cancelto{0}{\left[\frac{(t-u)\,u\,e^{-2u}}{-2}\right]_0^t} + \frac{1}{2}\int_0^t (t-2u)\, e^{-2u}\, du \right\}$$

$$= 2\left\{ \left[\frac{(t-2u)\,e^{-2u}}{-2}\right]_0^t - \int_0^t \frac{(\cancel{-2})\, e^{-2u}}{\cancel{-2}}\, du \right\}$$

$$= \cancel{2}\left\{ \frac{t}{\cancel{2}} e^{-2t} + \frac{t}{\cancel{2}} - \left[\frac{e^{-2u}}{-\cancel{2}}\right]_0^t \right\} = te^{-2t} + t + e^{-2t} - 1$$

$$= (t+1)e^{-2t} + t - 1.$$

The gamma function

- We wish to calculate the Laplace transform of $f(t) = t^p$ for $p > -1$, or

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p \, dt$$

- Putting $u = st$,

$$\mathcal{L}[t^p] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^p \frac{du}{s} = \frac{1}{s^{p+1}} \int_0^\infty e^{-u} u^p \, du$$

- The integral on the right is independent of s.
- It is a function of p called the gamma function. Usually, we define

$$\Gamma(p) = \int_0^\infty e^{-u} u^{p-1} \, du, \qquad p > 0.$$

- We can then write

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}} \qquad p > -1$$

- We saw earlier for positive integers n that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad n \in \mathbb{Z}_+$$

- This implies that $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}_+$, and the gamma function is a generalisation of the factorial to non-integer arguments.

Properties of the gamma function

(1) For every $p > 0$, $\Gamma(p+1) = p\,\Gamma(p)$

Proof: $\Gamma(p+1) = \int_0^\infty e^{-u} u^p \, du$

$$= \left[-e^{-u} u^p\right]_0^\infty - \int_0^\infty (-e^{-u})(p u^{p-1}) \, du$$

$$= p \int_0^\infty e^{-u} u^{p-1} \, du$$

$$= p\,\Gamma(p)$$

(2) $\Gamma(1) = 1$

Proof: $\int_0^\infty e^{-u} \, du = \left[-e^{-u}\right]_0^\infty = 1$