NUMERICAL METHODS WEEK 10

FOURIER SERIES AND TRANSFORMS

Learning outcomes:

- Numerically calculate Fourier series of functions.
- Perform numerical Fourier transform of functions.

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FOURIER SERIES

Expanding a function in a series of basis functions, particularly periodic sin and cosines can be very useful.

- May help analysing a function.
- Can be used make a filter - removing high or low frequency noise.
- Useful solving some differential equations.
- ...

Suppose we have a periodic function - this means that

$$f(x+T)=f(x)$$

where $T=2l$ is the period.

Assume we can expand the function as a sum of (known) functions with the same periodicity

$$\phi_n(x)=\cos\left(\frac{n\pi x}{l}\right) \text{ and } \psi_n(x)=\sin\left(\frac{n\pi x}{l}\right)$$

where $n \in \mathbb{Z}$.

> Check that all the $\phi_n(x)$ and $\psi_n(x)$ have periodicity T (by considering the functions at x and $x+T$).

Note that this notation for $\phi_n(x)=\cos\left(\frac{n\pi x}{l}\right)$ and $\psi_n(x)=\sin\left(\frac{n\pi x}{l}\right)$ will be used throughout this lecture.

ORTHOGONAL FUNCTIONS

$\phi_n(x)$ and $\psi_n(x)$ have the special properties that

$$\int_{-l}^{l} \phi_n(x)\phi_m(x)\mathrm{d}x = l\delta_{nm} \text{and} \int_{-l}^{l} \psi_n(x)\psi_m(x)\mathrm{d}x = l\delta_{nm}$$

where δ_{nm} is the Kronecker delta.

$$\delta_{nm} = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

Also

$$\int_{-l}^{l} \phi_n(x)\psi_m(x)\mathrm{d}x = 0$$

for all n, m

FOURIER SERIES

Assume that we can expand our function on the interval $[-l, l]$ as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)] \end{aligned}$$

This isn't actually true for all x, but it is good enough for now.

Note this will repeat over $[l, 3l]$ and other $[il, (i+2)l]$ where $i \in \mathbb{Z}$

We can use the orthogonality of the functions to calculate the coefficients, a_m and b_m. To do this we multiply by the function whose coefficient we wish to find, and integrate over T.

For instance, to find a_m the coefficient of $\cos\left(\frac{m\pi x}{l}\right)$ we multiply by $\phi_m = \cos\left(\frac{m\pi x}{l}\right)$ and integrate over the period:

$$\begin{aligned}\int_{-l}^{l} \phi_m(x) f(x) \mathrm{d}x &= \int_{-l}^{l} \phi_m(x) \frac{a_0}{2} \mathrm{d}x + \int_{-l}^{l} \phi_m(x) \sum_{n=1}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)] \mathrm{d}x \\ &= \frac{a_0}{2} \int_{-l}^{l} \phi_m(x) \mathrm{d}x + \sum_{n=1}^{\infty} [a_n \int_{-l}^{l} \phi_m(x) \phi_n(x) \mathrm{d}x + b_n \int_{-l}^{l} \phi_m(x) \psi_n(x) \mathrm{d}x]\end{aligned}$$

Using the orthogonality relations only the term containing a_m survives, giving

$$a_m = \frac{1}{l} \int_{-l}^{l} \phi_m(x) f(x) \mathrm{d}x$$

To find b_m we multiply by ψ_m and integrate over the period:

$$\int_{-l}^{l} \psi_m(x) f(x) \mathrm{d}x = \int_{-l}^{l} \psi_m(x) \frac{a_0}{2} \mathrm{d}x + \int_{-l}^{l} \psi_m(x) \sum_{n=1}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)] \mathrm{d}x$$

$$= \frac{a_0}{2} \int_{-l}^{l} \psi_m(x) \mathrm{d}x + \sum_{n=1}^{\infty} [a_n \int_{-l}^{l} \psi_m(x) \phi_n(x) \mathrm{d}x + b_n \int_{-l}^{l} \psi_m(x) \psi_n(x) \mathrm{d}x]$$

Using the orthogonality relations only the term containing b_m survives, giving

$$b_m = \frac{1}{l} \int_{-l}^{l} \psi_m(x) f(x) \mathrm{d}x$$

- Check that $a_m = \frac{1}{l} \int_{-l}^{l} \phi_m(x) f(x) \mathrm{d}x$ and $b_m = \frac{1}{l} \int_{-l}^{l} \psi_m(x) f(x) \mathrm{d}x$

NUMERICAL FOURIER SERIES

To calculate the terms in the fourier series we need to numerically integrate our function.

Assuming that we have a series of n evenly spaced $\{x, f(x)\}$ values we can simply calculate the integrals using the trapezium rule as

$$a_m = \frac{1}{l} \sum_{i=0}^{n-1} \phi_m(x_i) f(x_i) \Delta x$$

This discretization is very similar to what we did in previous weeks - we discretize the period $2l$ along the x axis with n points in each period. The $n+1$th point is a replica of 0th. We don't want to include the $n+1$ th point at $2l$ because it is the same as the first one. Δx will be the period of the function divided by the number of points, $\Delta x = \frac{2l}{n}$.

> Suppose we want to sample a function with period 2π with n = 4 points in each period. This means the 5th point will be at 2π the beginning of the next period. The spacing between the points will be the period divided by the 4 spaces between the 5 points. So $\Delta x = 2\pi/4 = \pi/2$.

> Note that there is a maximum frequency of wave that can be represented. Don't go above $n/2$ terms in the Fourier series otherwise you will get 'aliasing' (https://en.wikipedia.org/wiki/Aliasing).

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- Write code to perform these integrations. Check the orthogonality conditions are satisfied by your code.
- Calculate the Fourier series of the periodic function $f(x) = \cos(x)$ within $[-\pi, \pi]$, which then repeats itself every 2π
- Calculate the Fourier series of the periodic function $f(x) = \cos(x) + \cos(2x)$ within $[-\pi, \pi]$.

EXTRA THINGS TO TRY.

- Calculate the Fourier series of the periodic function $f(x) = x^2$ within $[-\pi, \pi)$, which then repeats itself every 2π
- Calculate the Fourier series of the periodic function $f(x) = x$ within $[-\pi, \pi)$, which then repeats itself every 2π
- What does the previous series converge to at $x = -\pi$

PARSEVAL'S THEOREM

I the 'power' of the function is conversed, or multiplying on both sides by $f(x)$ and integrating over the period

$$\frac{1}{l}\int_{-l}^{l} f(x)^2\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

- Check that Parseval's Theorem is satisfied for $f(x) = x^2$.
- Check that Parseval's Theorem is satisfied for $f(x) = x$ taking the periodic interval to be $[-2\pi, 0)$.

OTHER FORMS OF FOURIER SERIES

Odd or even functions can be expanded as just $\sin$ or $\cos$ series.

The expansion can be written in terms of complex exponentials instead of $\sin$ and $\cos$ terms - instead of our functions ϕ and ψ we use

$$\chi_n(x) = \exp\left(\frac{inx\pi}{l}\right)$$

and expand $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \chi_n(x)$$

This is just rewriting the series, because χ_n are just linear combinations of ϕ_n and ψ_n. The χ_n also satisfy orthogonality relations, and the coefficients can be found in the same way - multiply by the complex conjugate of the function whose coefficient you want to find and then integrate over the period.

$$c_n = \int_{-l}^{l} \chi_n^*(x) f(x) \mathrm{d}x$$

SOLVING DIFFERENTIAL EQUATIONS

Differential equations can often be transformed into algebraic form using Fourier series. Often the exponential form is most easy to use.

Take an example of the from

$$\ddot{y} + \omega_0^2 y = f(t)$$

where the function $f(t)$ is a periodic function with the period $T = 2\pi/\omega$. We can expand $f(t)$ as a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\pi t\omega/\pi} = \sum_{n=-\infty}^{\infty} f_n e^{in\omega t}$$

with

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-in\omega t)\mathrm{d}t$$

- but we can also expand y as a fourier series with the same period, T.

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{in\omega t}$$

substituting and performing the straightforward derivative of the exponential we get

$$\sum_{n=-\infty}^{\infty} \left\{\left[-(n\omega)^2 + \omega_0^2\right] y_n - f_n\right\} \exp(in\omega t) = 0$$

which implies that

$$y_n = \frac{f_n}{\omega_0^2 - (n\omega^2)}$$

From which the full solution can be found.

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{f_n}{\omega_0^2 - (n\omega^2)} \exp(in\omega t)$$

SUMMARY AND FURTHER READING

You should be reading additional material to provide a solid background to what we do in class

All the textbooks contain sections on root finding and solving non-linear equations, for instance chapters 12 and 13 of Numerical Recipes (http://www.nrbook.com/a/bookcpdf.php).