$$f(t) = \sum_{n=-\infty}^{\infty} \underbrace{c_n}_{\text{amount of}} \cdot \underbrace{e^{2\pi i n t/T}}_{\text{frequency}}$$

Tidier for engineering

$$\dots = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \;:\; \omega_n = \frac{2\pi n}{T}$$

(Similar to breaking up vectors)

over one period

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i n t/T} \, dt$$

$$= \frac{\Delta\omega}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_n t} \, dt$$

$$: \underbrace{\Delta\omega = 2\pi/T}_{\text{diff in neighbouring frequencies}} \quad (\omega_{n+1} - \omega_n)$$

Using u to avoid confusion with 2 t variables

$$\dots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{i\omega_n u} \, du \; e^{i\omega_n t}$$

for simplification

$$\dots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \;:\; g(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-i\omega_n u} \, du$$

As $T \to \infty$, $\Delta\omega = \frac{2\pi}{T} \to 0 \dots$

(non-periodic)

becomes continuous

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} \, d\omega$$

While $g(\omega_n) \to \int_{-\infty}^{\infty} f(u) e^{i\omega u} \, du$

$$\dots = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du \right]}_{\text{amount of freq. (TRANSFORM)}} e^{i\omega t} \, d\omega$$

Fourier's inversion theorem

notation!

no confusion so all good to use t !!

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

FOURIER TRANSFORM OF $f(t)$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

multiply together !!

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} \, d\omega$$

INVERSE FOURIER TRANSFORM

For $f(t) = f(-t)$,

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left(\cos(-\omega t) + i \sin(-\omega t)\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(t) \left(\cos \omega t\right.}_{\text{even} = \times 2} - i \underbrace{\left.\sin \omega t\right)}_{\text{odd} = 0} dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos \omega t\, dt \longleftarrow \text{even}$$

Normally $\sqrt{\frac{2}{\pi}}$ as pre-factor !! (in text books)

Hence $$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\tilde{f}(\omega) \left(\cos \omega t\right.}_{\text{even} = \times 2} + i \underbrace{\left.\sin \omega t\right)}_{\text{odd} = 0} d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega) \cos \omega t\, d\omega$$

Ambiguous so let $t \to u$ for now

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(u) \cos \omega u\, du \cos \omega t\, d\omega$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} f(u) \cos \omega u\, du \right\} \cos \omega t\, d\omega$$

FOURIER'S INVERSION THEOREM AGAIN !!