$$f(t) = \sum_{n=-\infty}^{\infty} \underbrace{c_n}_{\text{amount of}} \cdot \underbrace{e^{2\pi i n t/T}}_{\text{frequency}}$$

Tidier for engineering

$$\ldots = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \quad : \omega_n = \frac{2\pi n}{T}$$

(Similar to breaking up vectors)

over one period

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i n t/T}\, dt$$

$$= \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_n t}\, dt$$

$$: \underbrace{\Delta\omega = 2\pi/T}_{\text{diff in neighbouring frequencies}} \quad (\omega_{n+1} - \omega_n)$$

using u to avoid confusion with 2 t variables

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{i\omega_n u}\, du \; e^{i\omega_n t}$$

for simplification

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \quad : g(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-i\omega_n u}\, du$$

As $T \to \infty$, $\Delta\omega = \frac{2\pi}{T} \to 0 \ldots$

(non-periodic)

becomes continuous

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

While $g(\omega_n) \to \int_{-\infty}^{\infty} f(u) e^{i\omega u}\, du$

$$\underbrace{\ldots = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u}\, du\right]}_{\text{amount of freq. (TRANSFORM)}} e^{i\omega t}\, d\omega}_{\text{Fourier's inversion theorem}}$$

notation!

no confusion so all good to use t !!

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

FOURIER TRANSFORM OF $f(t)$

$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$

multiply together !!

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t}\, d\omega$$

INVERSE FOURIER TRANSFORM

For $f(t) = f(-t)$,

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left(\cos(-\omega t) + i \sin(-\omega t)\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(t) \left(\cos \omega t\right.}_{\text{even} = \times 2} - i \underbrace{\sin \omega t}_{\text{odd} = 0} \left.\right) dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos \omega t \, dt \leftarrow \text{even}$$

Normally $\sqrt{\frac{2}{\pi}}$ as pre-factor !! (in text books)

Hence $$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\tilde{f}(\omega) \left(\cos \omega t\right.}_{\text{even} = \times 2} + i \underbrace{\sin \omega t}_{\text{odd} = 0} \left.\right) d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega) \cos \omega t \, d\omega$$

Ambiguous so let $t \to u$ for now

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(u) \cos \omega u \, du \cos \omega t \, d\omega$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} f(u) \cos \omega u \, du \right\} \cos \omega t \, d\omega$$

FOURIER'S INVERSION THEOREM AGAIN !!

$$\tilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx \right\} e^{-ikz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz \right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du \right\} dx \;;\; u = z - x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-iku}\, e^{-ikx}\, du \right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx} \; \underbrace{\int_{-\infty}^{\infty} g(u)\, e^{iku}\, du}$$

FOURIER TRANSFORM!

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}\, \tilde{f}(k) \sqrt{2\pi}\, \tilde{g}(k)$$

$$= \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$

$$\tilde{f}(s) = \frac{s+3}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow s+3 = (s+1)A + sB$$

$\therefore$ if $s=-1$, $B=-2$ and
if $s=0$, $A=3$

$$\ldots = \frac{3}{s} - \frac{2}{s+1}$$

$$\ldots = 3\left(\frac{1}{s}\right) - 2\left(\frac{1}{s+1}\right)$$

$$\frac{1}{s+a} \to e^{-at}$$

$$\ldots = 3e^{-0t} - 2e^{-t}$$

$$\ldots = \boxed{3 - 2e^{-t}}$$

$$\mathcal{L}\left[te^{-2t}\right], \text{ let } f(t) = t$$

$$\mathcal{L}\left[e^{-2t} f(t)\right] = \tilde{f}(s+2$$

$$\mathcal{L}\left[e^{-at} f(t)\right] = \int_0^\infty \left[e^{-at} f(t)\right] e^{-st}\, dt$$

$$= \int_0^\infty e^{-at} e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \mathcal{L}(f(s+a))$$

$$= \tilde{f}(s+a)$$

THE SHIFT THEOREM

$$\frac{d^n}{ds^n} \tilde{f}(s) = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt$$

$$\ldots = \int_0^\infty \frac{d^n}{ds^n}\left(e^{-st}\right) f(t)\, dt$$

(not affecting t)

$$\ldots = \int_0^\infty (-t)^n e^{-st} f(t)\, dt$$

$$\ldots = \int_0^\infty (-1)^n t^n e^{-st} f(t)\, dt$$

$$\ldots = (-1)^n \int_0^\infty e^{-st} \left[t^n f(t)\right] dt$$

$$\ldots = (-1)^n \mathcal{L}\left[t^n f(t)\right]$$

$$\Rightarrow \mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} \tilde{f}(s)$$

$$(-1)^{-n} = (-1)^n$$