MTH3006 Methods of Mathematical Physics **Weekly problems 2**

1. A function is defined by

$$T(x) = \begin{cases} 1 & \text{when} \quad |x| < a \\ 0 & \text{when} \quad |x| > a \end{cases}$$

where a is a real, positive constant.

(a) Sketch the graph of $T(x)$, and use your result to sketch the graph of the convolution $h(x) = T(x) * [\delta(x-b) + \delta(x+b)]$ when $b > a$.

(b) Use the convolution theorem to calculate $\tilde{h}(k)$, the Fourier transform of $h(x)$, and show that $\tilde{h}(k)$ has zeros at $k = n\pi/a$ when $n \neq 0$ and $k = [(n+1/2)\pi]/b$ for all n ($n \in \mathbb{Z}$).

2. (a) Find the Fourier transform of the function $f(t) = \exp(-|t|)$. Before starting the calculation, think carefully about which form of the transform will give the most straightforward integral.
(b) Take the inverse Fourier transform of your result from part (a) to show that

$$\frac{\pi}{2}\exp(-|t|) = \int_0^\infty \frac{\cos \omega t}{1+\omega^2}\, d\omega.$$

3. Ignoring the point $t = 0$, the Heaviside step function is defined by

$$H(t) = \begin{cases} 1 & \text{when} \quad t > 0 \\ 0 & \text{when} \quad t < 0 \end{cases}.$$

(a) Calculate the Fourier transform of $f(t) = e^{-at}H(t)$, where $a > 0$ is a real constant.
(b) Use your result, together with the convolution theorem, to evaluate the inverse Fourier transform of

$$\tilde{f}(\omega) = \frac{1}{2\pi(a+i\omega)^2}.$$

Hint: Write $\tilde{f}(\omega)$ in terms of your answer to part (a).