MTH 3006 Methods of Mathematical Physics

Assessments

In-class test 60%
Portfolio test 40%

Course content

- Fourier and Laplace transforms
- Partial differential equations (PDEs)
- Green's functions for ODEs
- Calculus of variations
- Integral equations

Books

- Riley, Hobson and Bence: Mathematical methods for physics and engineering
- Kreyszig: Advanced engineering mathematics
- Arfken, Weber and Harris: Mathematical Methods for Physicists

Topics to revise

- Partial fractions (including the case with a quadratic factor in the denominator)
- Integration by parts
- Laws of logs and exponentials
- Ordinary differential equations
  - First order (separation of variables, homogeneous equations)
  - Second order (homogeneous equations with constant coefficients)
- Partial and total derivatives

Fourier transforms

- Fourier series: periodic function represented as sum of trigonometric functions
- Fourier transform: non-periodic function represented as an integral over trigonometric functions / complex exponentials

Recall: A function of period T may be represented as a complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T} = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \text{ with } \omega_n = \frac{2\pi n}{T}$$

- The coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-2\pi i n t/T}\, dt$$
$$= \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}\, dt,$$

where $\Delta\omega = 2\pi/T$ is the difference between neighbouring frequencies.

- Substituting this in to the series for $f(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(u)\, e^{-i\omega_n u}\, du\; e^{i\omega_n t}$$

- We rewrite this as $f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n)\, e^{i\omega_n t}$

with $g(\omega_n) = \int_{-T/2}^{T/2} f(u) e^{-i\omega_n u} \, du$

- As $T \to \infty$, $\Delta\omega = 2\pi/T$ becomes infinitesimal, and

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega) e^{i\omega t} \, d\omega$$

while $g(\omega_n) \to \int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du$

so that

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du\right] e^{i\omega t} \, d\omega}$$

This is <u>Fourier's inversion theorem</u>

- We use this to define the Fourier transform of $f(t)$ by

$$\boxed{\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt}$$

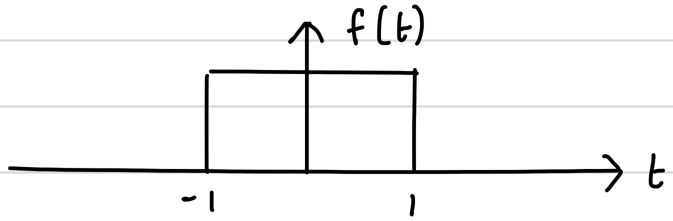
and its inverse by

$$\boxed{f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} \, d\omega}$$

Note: some books use different prefactors.

Example 1 Calculate the Fourier transform of

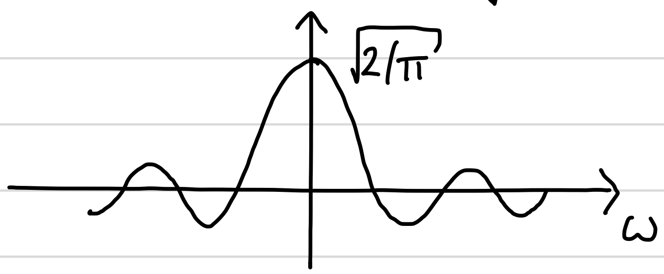
$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$



$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \, \frac{e^{-i\omega} - e^{i\omega}}{-i\omega}$$

$$= \frac{2}{\sqrt{2\pi}} \, \frac{1}{\omega} \, \frac{e^{i\omega} - e^{-i\omega}}{2i} = \sqrt{\frac{2}{\pi}} \, \frac{\sin\omega}{\omega} \quad : \text{ a sinc function}$$



Notes - this is the equivalent for non-periodic functions of calculating the coefficients in a Fourier series for periodic functions.

- roughly speaking, a localised function has a spread-out Fourier series.

Example 2 Calculate the inverse Fourier transform of the function $\tilde{f}(\omega)$ found in Example 1.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega} e^{i\omega t} \, d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega (\cos \omega t + i \sin \omega t)}{\omega} \, d\omega$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega \cos \omega t}{\omega} \, d\omega$$ since $\sin \omega / \omega$ is even.

Notes

- This result is the original function $f(t)$ written as an integral.
- This is the equivalent for a non-periodic function of writing a periodic function as a sum over sines and cosines.
- Since (from the definition) $f(0) = 1$, we have that

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega}{\omega} \, d\omega$$

or $$\int_{0}^{\infty} \frac{\sin \omega}{\omega} \, d\omega = \frac{\pi}{2}$$

Fourier cosine and sine transforms

- We can define Fourier cosine and sine transforms for general even and odd functions.

- For an even function, $f(t) = f(-t)$, and

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,(\cos \omega t - i \sin \omega t)\, dt = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos \omega t\, dt$$

- Noting that $\tilde{f}(\omega)$ is an even function of ω, we can write the inverse Fourier transform as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)(\cos \omega t + i \sin \omega t)\, d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega) \cos \omega t\, d\omega = \frac{2}{\pi} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} f(u) \cos \omega u\, du \right\} \cos \omega t\, d\omega$$

- We can then define the <u>Fourier cosine transform</u> and its inverse as

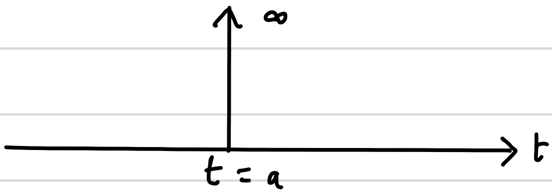
$$\boxed{\tilde{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos \omega t\, dt}$$

$$\boxed{f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \tilde{f}_c(\omega) \cos \omega t\, d\omega}$$

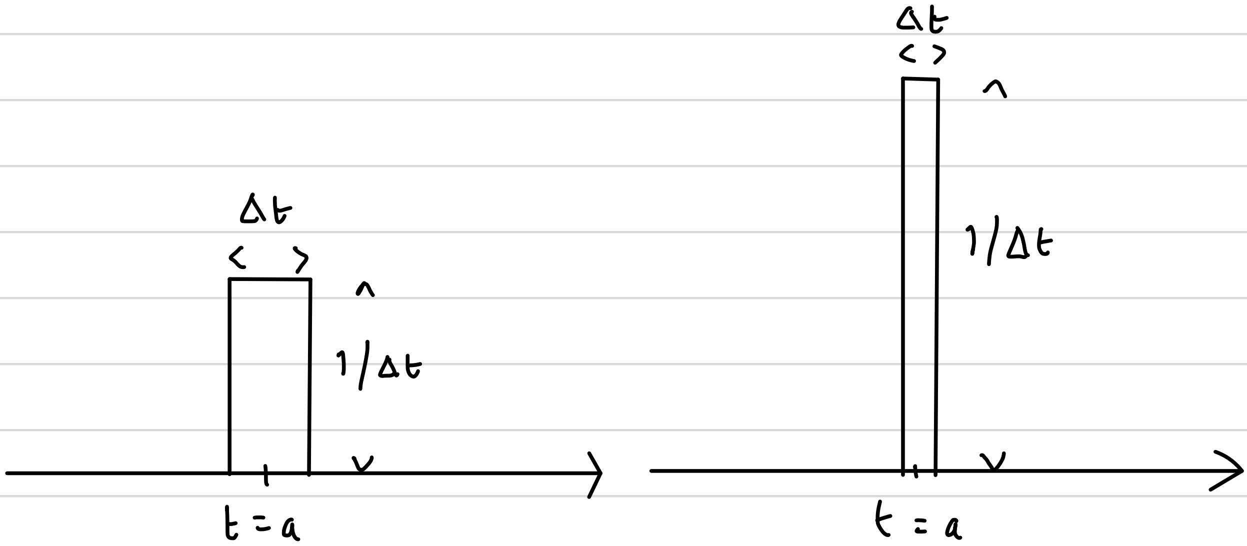
- A <u>sine transform</u> for odd functions can be defined similarly.

The Dirac δ-function

- Can be visualised as a sharp spike



coming from the limit of a sequence of 'top-hat functions':



- As Δt shrinks, the function becomes concentrated at one point, so

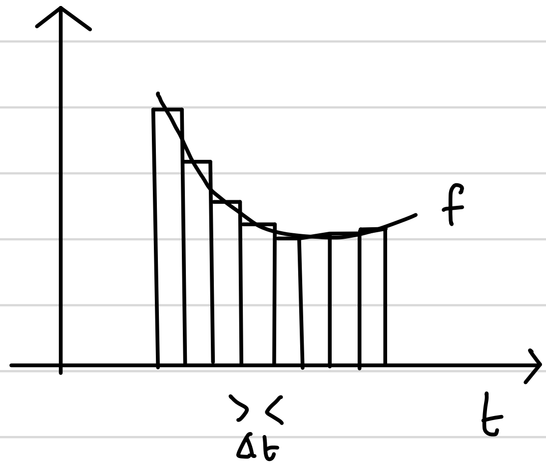
$$\delta(t-a) = 0 \qquad (t \neq a)$$

- However, the area under the graph remains constant: $\Delta t\,(1/\Delta t) = 1$, and

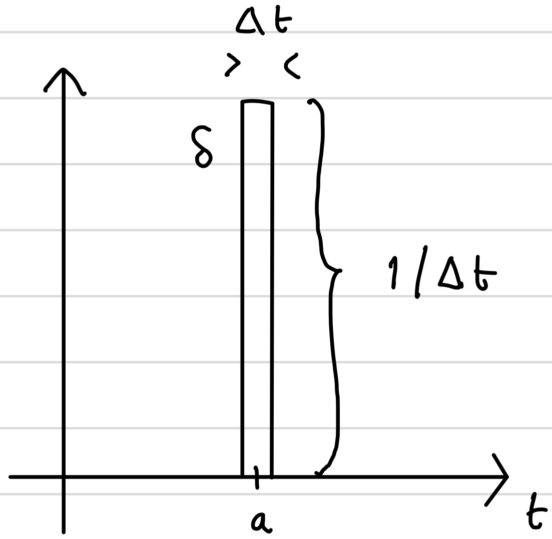
$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$$

Integrals involving the delta function

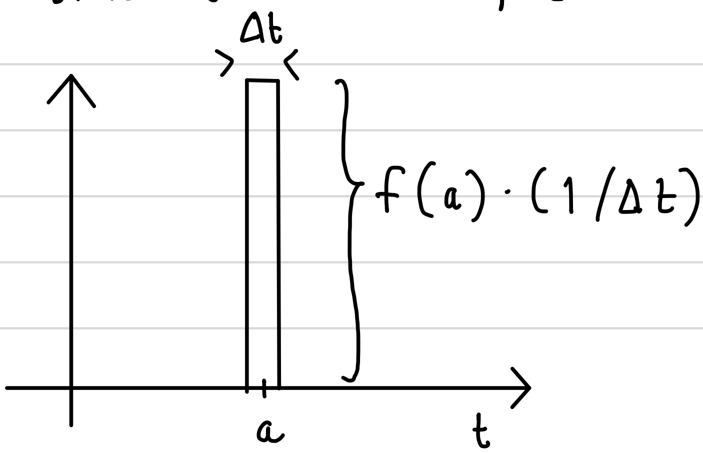
- The integral of a function $f(t)$ is defined via a sum of areas of rectangles



- We now calculate the integral of $[f(t) \times \delta(t-a)]$



- In the integral of $f \cdot \delta$, all contributions from either side of the spike at $t = a$ are zero.



- The one remaining contribution is

$$\lim_{\Delta t \to 0} \underbrace{f(a)(1/\Delta t)}_{\text{height of rectangle}} \cdot \underbrace{\Delta t}_{\text{width of rectangle}} = f(a)$$

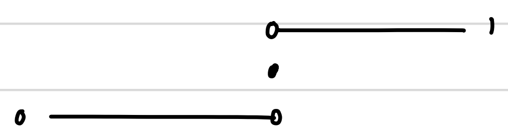
so that

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)}$$

<u>The Heaviside step function</u>

- Defined as $H(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$

Usually, we also set $H(0) = 1/2$.



- The step function and the delta function are related by

$$H'(t) = \delta(t)$$