$$f(t) = \sum_{n=-\infty}^{\infty} \underbrace{c_n}_{\text{amount}} \cdot \underbrace{e^{2\pi i n t/T}}_{\text{of frequency}}$$

Tidier for engineering

$$\ldots = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \; : \; \omega_n = \frac{2\pi n}{T}$$

(Similar to breaking up vectors)

over one period

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i n t/T} \, dt$$

$$= \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_n t} \, dt$$

$$: \underbrace{\Delta\omega = 2\pi/T}_{\text{diff in neighbouring frequencies}} \quad (\omega_{n+1} - \omega_n)$$

Using u to avoid confusion with 2 t variables

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{i\omega_n u} \, du \; e^{i\omega_n t}$$

for simplification

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \; : \; g(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-i\omega_n u} \, du$$

As $T \to \infty$, $\Delta\omega = \frac{2\pi}{T} \to 0 \ldots$
(non-periodic)

becomes continuous

$$\hookrightarrow \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega) e^{i\omega t} \, d\omega$$

While $g(\omega_n) \to \displaystyle\int_{-\infty}^{\infty} f(u) e^{i\omega u} \, du$

$$\ldots = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du\right]}_{\text{amount of freq. (TRANSFORM)}} e^{i\omega t} \, d\omega}_{\text{Fourier's inversion theorem}}$$

notation!

no confusion so all good to use t !!

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

FOURIER TRANSFORM OF $F(t)$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

multiply together !!

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} \, d\omega$$

INVERSE FOURIER TRANSFORM

For $f(t) = f(-t)$,

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left(\cos(-\omega t) + i\sin(-\omega t)\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(t)\left(\cos \omega t\right.}_{\text{even} = \times 2} - \underbrace{i \sin \omega t}_{\text{odd} = 0}\left.\right) dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos \omega t \, dt \leftarrow \underline{\text{even}}$$

Normally $\sqrt{\frac{2}{\pi}}$ as pre-factor !! (in text books)

Hence $$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\tilde{f}(\omega)\left(\cos \omega t\right.}_{\text{even} = \times 2} + \underbrace{i \sin \omega t}_{\text{odd} = 0}\left.\right) d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega) \cos \omega t \, d\omega$$

Ambiguous so let $t \to u$ for now

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(u) \cos \omega u \, du \cos \omega t \, d\omega$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} f(u) \cos \omega u \, du \right\} \cos \omega t \, d\omega$$

FOURIER'S INVERSION THEOREM AGAIN !!

$$\tilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx \right\} e^{-ikz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz \right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du \right\} dx \; ; \; u = z - x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-iku}\, e^{-ikx}\, du \right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx}\; \underbrace{\int_{-\infty}^{\infty} g(u)\, e^{iku}\, du}$$

FOURIER TRANSFORM!

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}\, \tilde{f}(k)\, \sqrt{2\pi}\, \tilde{g}(k)$$

$$= \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$