Properties of Fourier transforms

Note We can write the Fourier transform of $f(t)$ as $\tilde{f}(\omega)$ or $\mathcal{F}[f(t)]$.

① **Scaling** $\mathcal{F}[f(at)] = \frac{1}{|a|}\tilde{f}(\omega/a)$ $(a \neq 0)$

Proof $$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt$$

Put $t' = at$. Then, if $a > 0$,

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')\, e^{-i\omega t'/a}\, \frac{dt'}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')\, e^{-i(\omega/a)t'}\, dt' = \frac{1}{|a|}\tilde{f}(\omega/a)$$

(recalling that $a > 0$)

- When $a < 0$, we again put $t' = at$, and find

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t')\, e^{-i\omega t'/a}\, \frac{dt'}{a}$$

$$= \frac{1}{-a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')\, e^{-i(\omega/a)t'}\, dt' = \frac{1}{|a|}\tilde{f}(\omega/a)$$

since $|a| = -a$ when $a < 0$.

(2) Differentiation

$$\mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega)$$

Proof

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{-i\omega t} f(t) \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega e^{-i\omega t} f(t) \, dt$$

$$= i\omega \tilde{f}(\omega)$$

Note: we have used the fact that $f(t) \to 0$ as $t \to \pm\infty$ (this must be true for the Fourier transform to exist)

- Applying this result again, we find

$$\mathcal{F}[f''(t)] = i\omega \mathcal{F}[f'(t)] = -\omega^2 \tilde{f}(\omega)$$

Note: this is often used to transform differential equations into algebraic equations.

(3) Translation $\quad \mathcal{F}[f(t+a)] = e^{ia\omega} \tilde{f}(\omega)$

(4) Exponential multiplication $\quad \mathcal{F}[e^{\alpha t} f(t)] = \tilde{f}(\omega + i\alpha)$

Convolution

- Imagine that we are using a measuring instrument to find a function $f(x)$ e.g. the intensity as a function of position in a microscopy experiment.
- The instrument has a resolution function $g(z-x)$: the probability density that the reading will be shifted to $(z, z+dz)$ from its true value in $(x, x+dx)$.
- Integrating over all values of x that could lead to a reading in $(z, z+dz)$, we find that the observed function is

$$\boxed{h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx}$$

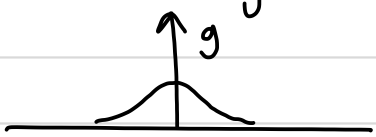
This is the convolution of f and g.

- Perfect resolution is given by the delta function

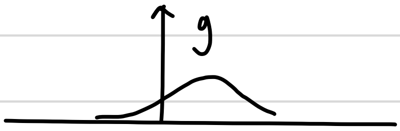
$$h(z) = \int_{-\infty}^{\infty} f(x)\, \delta(z-x)\, dx$$
$$= f(z)$$

observed function = true function

- Real resolution functions might be symmetric ("blurring")



or asymmetric (systematic error)



- The convolution of f and g is sometimes written $f * g$, and we have that $f * g = g * f$.

<u>Example</u> Find the convolution of e^{-x^2} with the sum of delta functions $\delta(x-a) + \delta(x-b)$.

$$f * g = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} [\delta(x-a) + \delta(x-b)] \exp(-(z-x)^2)\, dx$$

$$= \exp(-(z-a)^2) + \exp(-(z-b)^2)$$

- Taking the convolution of any function with a delta function leaves a copy of that function at the position of the delta function.

The convolution theorem

- The Fourier transform of the convolution is

$$\tilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx \right\} e^{-ikz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz \right\} dx$$

- Put $u = z - x$ in the second integral:

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du \right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \int_{-\infty}^{\infty} g(u)\, e^{-iku}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}\, \tilde{f}(k) \sqrt{2\pi}\, \tilde{g}(k) = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$

so that $\boxed{\mathcal{F}[f(x) * g(x)] = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)}$

This is the <u>convolution theorem</u>.

- We also have that

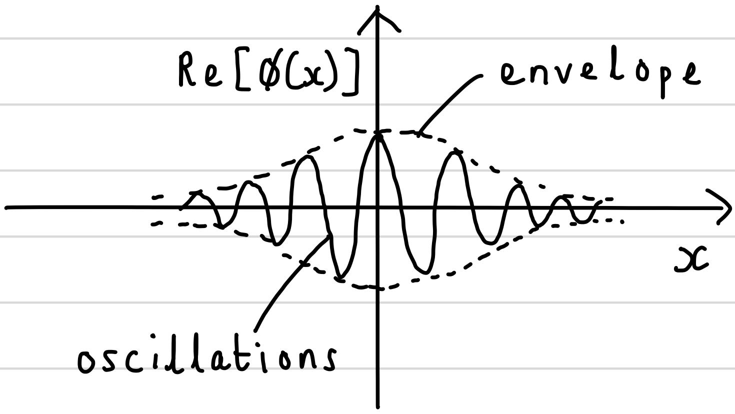
$$\boxed{\mathcal{F}[f(x)\, g(x)] = \frac{1}{\sqrt{2\pi}} \tilde{f}(k) * \tilde{g}(k)}$$

(the proof is similar)

<u>Example</u> Calculate the Fourier transform of the wave packet

$$\phi(x) = \exp(-x^2/2a^2)\, e^{iqx}$$

<u>Note</u>: the real part of this function has the form



You may use the facts that

$$\mathcal{F}[\exp(-x^2/2a^2)] = a \exp(-k^2a^2/2)$$

and $\mathcal{F}[e^{iqx}] = \sqrt{2\pi}\,\delta(k-q)$

- Second form of the convolution theorem:

$$\mathcal{F}[f(x)\,g(x)] = \frac{1}{\sqrt{2\pi}}\, \tilde{f}(k) * \tilde{g}(k)$$

$$\mathcal{F}[\exp(-x^2/2a^2)\, e^{iqx}] = \frac{1}{\sqrt{2\pi}}\, a\exp(-k^2a^2/2) * \sqrt{2\pi}\,\delta(k-q)$$

$$= a \int_{-\infty}^{\infty} [\delta(u-q) \exp(-(k-u)^2 a^2/2)]\, du$$

$= a \exp(-(k-q)^2 a^2/2)$

<u>Note</u>: we could also have found this result using the exponential multiplication property derived earlier.

Laplace transforms

- The Laplace transform of a function $f(t)$ is defined by

$$\boxed{\tilde{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

- Laplace transforms can exist when $\lim_{t\to\infty} f(t) \neq 0$, when the Fourier transform does not exist.

- They are often used when we are only interested in $t > 0$ (e.g. initial-value problems).
- We often denote the Laplace transform of $f(t)$ by $\mathcal{L}[f(t)]$.

<u>Example</u> Show that $\mathcal{L}(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}$

and $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}$.

- Writing $L_c = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$, we find

$$L_c = \int_0^\infty e^{-st} \cos \omega t \, dt$$

$$= \left[\frac{e^{-st}}{-s} \cos \omega t \right]_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt$$

$$= \frac{1}{s} - \frac{\omega}{s} L_s \qquad (1)$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \, dt$$

$$= \left[\frac{e^{-st}}{-s} \sin \omega t \right]_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt$$

$$= \frac{\omega}{s} L_c \qquad (2)$$

- Substituting (2) in (1):

$$L_c = \frac{1}{s} - \frac{\omega}{s} \left(\frac{\omega}{s} L_c \right) \Rightarrow L_c \left(1 + \frac{\omega^2}{s^2} \right) = \frac{1}{s}$$

$$\Rightarrow L_c \left(\frac{s^2 + \omega^2}{s^2} \right) = \frac{1}{s} \text{ and } L_c = \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

- Substituting ① in ②: $L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right)$

$\Rightarrow L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2} \Rightarrow L_s\left(\frac{s^2 + \omega^2}{s^2}\right) = \frac{\omega}{s^2}$

and $L_s = \mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}$