Lecture One (29th September 2025)

Fully understand everything noted down in the topics to revise pre-lecture notes!! Teaching hours are now 36 h

What's the difference between a Fourier series and Fourier transform?
Fourier series: periodic function (period T or L) represented as a sum of trigonometric functions (like sin/cos).
Fourier transform: non-periodic function (infinite period) represented as an integral over trigonometric functions

A function of period T may be represented as a complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} \underbrace{c_n}_{\text{amount of frequency}} e^{2\pi i n t/T}$$

Tidier for ease

$$\ldots = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \quad \omega_n = \frac{2\pi n}{T}$$

(similar to breaking up vectors)

over one period

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i n t/T}\, dt$$

$$= \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_n t}\, dt$$

$$\therefore \underbrace{\Delta\omega = 2\pi/T}_{\text{diff in neighbouring frequencies}} \quad (\omega_{n+1} - \omega_n)$$

Using u to avoid confusion with 2 t variables

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{i\omega_n u}\, du \; e^{i\omega_n t}$$

for simplification

$$\ldots = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \; : \; g(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-i\omega_n u}\, du$$

As $T \to \infty$, $\Delta\omega = \frac{2\pi}{T} \to 0 \ldots$
(non-periodic)

becomes continuous

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_n) e^{i\omega_n t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

$$\text{While } g(\omega_n) \to \int_{-\infty}^{\infty} f(u) e^{i\omega u}\, du$$

$$\underbrace{\ldots = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u}\, du\right]}_{\text{amount of freq. (TRANSFORM)}} e^{i\omega t}\, d\omega}_{\text{Fourier's inversion theorem}}$$

notation!

no confusion so all good to use t !!

$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

FOURIER TRANSFORM OF f(t)

multiply together !!

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

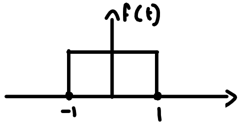
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{F}(\omega) e^{i\omega t}\, d\omega$$

INVERSE FOURIER TRANSFORM

Example 1

Calculate the Fourier Transform of…

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$



TOPHAT FUNCTION

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t} \, dt,$$ by subbing in $f(t)$ / narrowing limits by def.

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega} - e^{i\omega}}{-i\omega}$$

re-arranging to find sin

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{\omega} \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$

Sinc function

(damped oscillation)

Signal processing (turning off/on) Very Common!!

↳ l'hôpital's rule

often used to find $\frac{0}{0}$ (indeterminate form) at $\omega = 0$

NOTE: Basically finding co-effs in Fourier transform

hence