Ideas of mathematical proof. Practical class week 23

Note that solutions must be by the methods specified in the problems – even if you can see another solution, even if an easier one.

5.1. Prove that the following sets have the same cardinality by producing explicitly a bijection between the sets and describing this bijection

(1) by a formula for $|(-\infty, 0]| = |[3, +\infty)|$;

(2) by representing the set on the right as a sequence: $|\mathbb{N}| = |\{a, b, c\} \cup \mathbb{N}|$;

(3) by any method: $|\{a, b\} \times \mathbb{Z}| = |\mathbb{N}|$;

(4) by using a sequence of points: $|[0, 1]| = |(0, 1)|$;

(5) geometrically: $|(0, 1)| = |(0, \infty)|$ [*Hint*: for example, first construct a bijection of $(0, 1)$ onto a quarter of a circle (without endpoints), and then a bijection of this quarter of a circle onto $(0, \infty)$ similarly to the stereographic projection in the lectures.]

5.2. A theorem in the lectures says that if $f : B \to A$ is an injective mapping and A is countable, then B is also countable.

Use this theorem to prove that $|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = \aleph_0$. [*Hint*: one method is to use products of powers of three fixed primes as images of triples of positive integers.]

5.3. Let S be the set of all sequences of the form $(a_1, a_2, a_3, \dots)$, where every a_i is either A or B. Use Cantor's "diagonal method" to prove that the set S is not countable.

5.4. Recall that $\{0, 1\} \times \{0, 1\} \times \{0, 1\}$ consists of triples of 1s and 0s, and $\mathscr{P}(\{a, b, c\})$ is the set of all subsets of $\{a, b, c\}$. Consider the mapping

$$f \,:\, \mathscr{P}(\{a, b, c\}) \longrightarrow \{0, 1\} \times \{0, 1\} \times \{0, 1\}$$

in which a subset $X \subseteq \{a, b, c\}$ is mapped to a string of length 3 consisting of 0s and 1s depending on which of the elements a, b, c belong to X (with 1 indicating yes, and 0 no). Write the images $f(\varnothing)$, $f(\{a, c\})$, $f(\{b\})$. Is f a bijection?

5.5. Use the theorem mentioned in Q5.2 to show that any set consisting of pairwise disjoint intervals of the real line (each of length > 0) is countable. [*Hint*: use the fact that $|\mathbb{Q}| = \aleph_0$.]