Ideas of mathematical proof. Practical class week 29

Note that solutions must be by the methods specified in the problems – even if you can see another solution, even if an easier one.

Reminder: Definition. A sequence (a_n) has a finite limit $\lim\limits_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ (depending on ε) such that $|a_n - L| < \varepsilon$ for all $n > N(\varepsilon)$.
Another notation: $a_n \to L$ as $n \to \infty$.

8.1. Prove from first principles, by verifying the definition, that

(a) $\displaystyle \lim_{n\to\infty} \frac{2n-20}{n+5} = 2.$

(Note that there is no need to simplify the expression for $N(\varepsilon)$.)

(b) $\displaystyle \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0.$

8.2. Based on the definition of a finite limit, prove that the sequence $a_n = \sqrt{n}$ has no (finite) limit.

8.3. Prove by contrapositive that if $5n + 6$ is even for $n \in \mathbb{N}$, then n is even.

8.4. By considering all possible cases of remainders after division of k by 3, prove that $k(k+1)(2k+1)$ is divisible by 3 for any $k \in \mathbb{N}$.

8.5. Prove that if 120 points are chosen in a square 10×10 cm, then there must exist two of these points at distance less than 1.5 cm from each other.

8.6. Let $\mathscr{U} = \mathbb{N}$. Determine which of these statements are true:

(a) $\forall n_0 \, \forall n \, ((n > n_0) \Rightarrow (\frac{1}{n} < 0.003))$;

(b) $\exists n_0 \, \forall n \, ((n > n_0) \Rightarrow (\frac{1}{n} < 0.003))$;

(c) $\exists n_0 \, \exists n \, ((n < n_0) \wedge (\frac{1}{n} < 0.003))$.

8.7. Use the formula in the lectures for $a^{2k+1} + 1$ to prove that $8^n + 1$ cannot be a prime number for any $n \in \mathbb{N}$.