Ideas of mathematical proof. Practical class week 21

3.1. Use a diagram to prove the property $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

3.2. Let $A = \{1, 2, \ldots, 100\}$ be the set of all integers from 1 to 100. Let R be a relation on $\mathscr{P}(A)$ (the set of all subsets of A) defined as BRC if $|B| \leqslant |C|$. Determine if R is (1) transitive, (2) reflexive, (3) symmetric, (4) antisymmetric, in each case giving a proof if yes, or a counterexample if not. Hence state whether R is an order, or an equivalence, or neither.

3.3. On the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, consider the relation R defined as aRb if $a \mid b$ (that is, a divides b). Draw the diagram of R as a subset of $S \times S$.

3.4. Let $\sim$ be a relation on the set $\mathbb{R} \times \mathbb{R}$ defined by $(x, y) \sim (a, b)$ if $y - |x| = b - |a|$.

(1) Show that $\sim$ is an equivalence.

(2) What is the equivalence class of $(2, -3)$ with respect to this equivalence $\sim$?

3.5. Consider the order relation on $\mathbb{N}$ defined by divisibility: $a \mid b$ (when $b = ak$ for $k \in \mathbb{N}$). What are the infimum and supremum of the set $S = \{8, 12, 36\}$? Does this set have the greatest element? the smallest element?

3.6. Consider the lexicographical order $\leqslant_L$ on $\mathbb{R} \times \mathbb{R}$, which means that by definition $(a, b) \leqslant_L (c, d)$ if either $a < c$ (while b and d can be any), or $a = c$ and $b \leqslant d$.

(a) Find the supremum (least upper bound), if it exists, with respect to $\leqslant_L$ of the set $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$ (the disc of radius 1 with centre at the origin, including the unit circle).

(b) Find the supremum (least upper bound), if it exists, with respect to $\leqslant_L$ of the set $D_0 = \{(x, y) \mid x^2 + y^2 < 1\}$ (the disc of radius 1 with centre at the origin, without the boundary unit circle).

3.7. On the set $S = \mathbb{N} \times \mathbb{N}$, consider the relation $\sim$ defined as $(m_1, n_1) \sim (m_2, n_2)$ if $m_1 \cdot n_2 = m_2 \cdot n_1$. Show that $\sim$ is an equivalence. Describe the equivalence class of $(1, 2)$ with respect to this equivalence $\sim$.