Ideas of mathematical proof

Slides Week 21

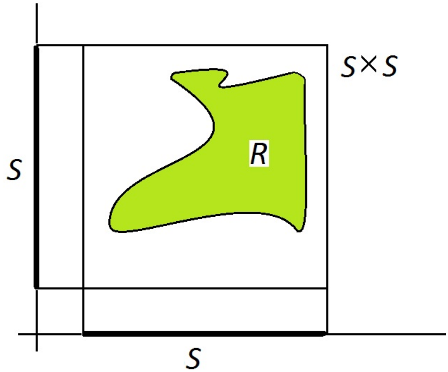
Relations. Order. Equivalence.
Mappings.

Relations

Definition

A **relation** on a set S

is any subset $R \subseteq S \times S$ of the Cartesian square.



Notation

Usually, instead of $(s, t) \in R$

we write sRt and say:

s and t are in relation R, or s, t satisfy relation R.

Order is important:

it may happen that $(s, t) \in R$ but $(t, s) \notin R$.

Some relations are written using the symbol $\sim$,

or other special symbols (like order relations $\leq$).

Examples

Example

$S =$ all people, define aRb if a shook hands with b.

Example

$S = \mathbb{R}$ and $R = \{(x, x^2) \mid x \in \mathbb{R}\}$,

that is, aRb if $b = a^2$.

If we regard the Cartesian square $S \times S$

as (x, y) coordinate plane, R is the graph of $y = x^2$.

Examples

Example

$S =$ all triangles, R is defined as follows:

aRb if a has greater area than b.

Example

$S =$ people, R is defined as follows:

$$aRb \quad \text{if} \;\; a \;\; \text{likes} \;\; b.$$

Types of relations

Definition

A relation R on a set S is said to be

- **transitive** if aRb and bRc implies aRc;
- **symmetric** if aRb implies bRa;
- **antisymmetric** if aRb and bRa implies $a = b$;
- **reflexive** if aRa for all $a \in S$.

Example

$S =$ all people, aRb if a shook hands with b. This relation: is not transitive; is symmetric; is reflexive or not depending on definition whether you can shake hands with yourself; is not antisymmetric.

Example

$S = \mathbb{R}$ and $R = \{(x, x^2) \mid x \in \mathbb{R}\}$, that is, aRb if $b = a^2$.

This relation is not transitive: $2R4$ and $4R16$, but not $2R16$ (**just one counterexample is enough!**).

It is not symmetric: $2R4$, but not $4R2$.

It is antisymmetric: if aRb and bRa, then $b = a^2$ and $a = b^2$, so both positive, and $a = b^2 = (a^2)^2 = a^4$, whence $a = 0$ or $a = 1$, and then $b = 0 = a$ or $b = 1 = a$, respectively.

It is not reflexive: $2R2$ is not true.

Example

$S = \mathbb{R}$ and R is defined as xRy if $x \leq y$.

This R transitive: $a \leq b \leq c \Rightarrow a \leq c$.

It is antisymmetric: $a \leq b$ and $b \leq a \Rightarrow a = b$.

It is reflexive: $a \leq a$.

It is not symmetric: $3R4$, but $4 \not R 3$

(that is, $4R3$ is not true).

Example

$S = \mathbb{Z}$ and aRb if $a - b$ is even. Determine if this relation is transitive, symmetric, reflexive, antisymmetric.

This R is transitive: if aRb and bRc, this means $a - b$ is even and $b - c$ is even; then the sum is also even: $a - b + b - c = a - c$, which means aRc, as required.

It is reflexive: $a - a = 0$ is even, so aRa for any $a \in \mathbb{Z}$, as required.

It is symmetric: if $a - b$ is even, then $b - a$ is even (thus, aRb implies bRa).

It is not antisymmetric: $2 - 4$ and $4 - 2$ are even, but $2 \neq 4$.

Example

Let R be a relation on $\mathbb{R}$ defined as: xRy if $x^2 \leq y^2$. Determine if this relation is transitive, symmetric, reflexive, antisymmetric.

This R is transitive: if aRb and bRc,

this means $a^2 \leq b^2 \leq c^2$, so $a^2 \leq c^2$,

which means aRc, as required.

It is reflexive: $a^2 \leq a^2$, so aRa for any $a \in \mathbb{R}$, as required.

It is not symmetric: $1^2 \leq 2^2$, but $2^2 \not\leq 1^2$.

It is not antisymmetric: $(-3)^2 \leq 3^2$ and $3^2 \leq (-3)^2$,

but $-3 \neq 3$.

Typical error: $a^2 \leq b^2$ and $b^2 \leq a^2$ does imply $a^2 = b^2$.

But does not imply $a = b$, needed for antisymmetric.

Partial order relations

Definition.

A relation R on a set S is called a (partial) **order** if R is

- transitive,
- reflexive,
- antisymmetric.

"Partial" means not every two elements of S have to be comparable.

An order R on S is **total** if for every $a, b \in S$ either aRb or bRa (that is, every two elements are comparable).

Notation

If R is an order,

then aRb is often denoted simply by $a \leq b$.

In general, $\leq$ is <u>not</u> the ordinary inequality

for numbers (but may be ordinary inequality too).

Sometimes other symbols are used, to distinguish different orders:

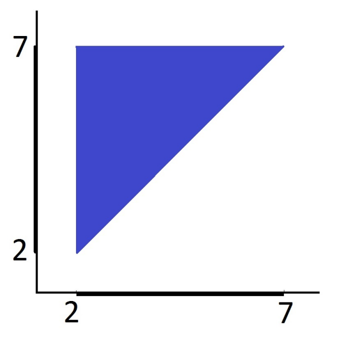
$\succ$, $\prec$, $\succeq$, $\preceq$, etc.

Example

On $\mathbb{R}$ the ordinary inequality $\leq$ is an order relation. Total order. The same for $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, or any other subset of $\mathbb{R}$.

Example

Diagram of relation ordinary order $\leq$ between real numbers on the set $[2, 7]$:



Proposition (Order by inclusion)

Let A be a set. Then the power set $\mathscr{P}(A)$ (of all subsets of A) is **ordered by inclusion:** BRC if $B \subseteq C$.

Usually denoted by $\subseteq$: instead of BRC we write $B \subseteq C$.

Proof:

It is transitive: $B \subseteq C$ and $C \subseteq D$ implies $B \subseteq D$.

It is reflexive: $B \subseteq B$ for any $B \in \mathscr{P}(A)$.

It is antisymmetric: if $B \subseteq C$ and $C \subseteq B$, then $B = C$.

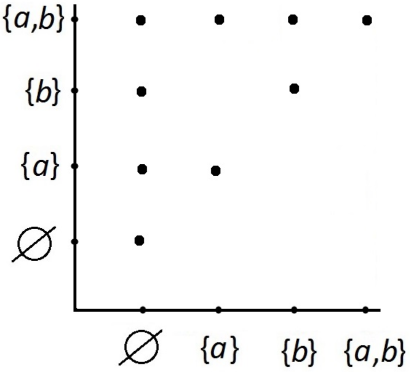
Thus, it is an order. $\square$

Not a total order: may be sets $B \not\subseteq C$ and $C \not\subseteq B$.

Example

Depict the relation $\subseteq$ on the set $\mathscr{P}(\{a, b\})$ by a diagram, as a subset of the Cartesian square.

$$\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$



Proposition (Order by divisibility)

Let $S = \mathbb{N}$ and let aRb if a divides b without remainder (which means $b = ak$ for $k \in \mathbb{Z}$). Prove that this is an order. (Usually denoted by $a \mid b$, read "a divides b".)

Proof: Transitive: $a \mid b$ and $b \mid c$

means $b = ak$ and $c = bl$ for $k, l \in \mathbb{Z}$.

Then $c = (ak)l = a(kl)$, where $kl \in \mathbb{Z}$,
so $a \mid c$, as required.

Reflexive: $a \mid a$, since $a = a \cdot 1$.

Antisymmetric: if $a \mid b$ and $b \mid a$, since both are positive,

then $a \leq b$ and $b \leq a$, so $a = b$, as required.

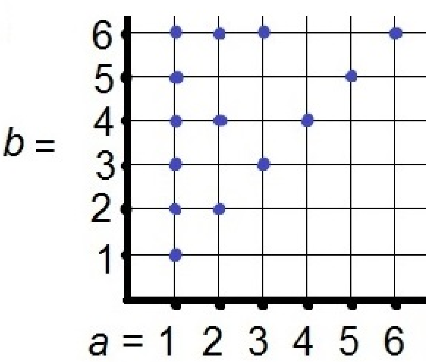
Thus, this is an order relation. □

Example

Consider $S = \{1, 2, 3, 4, 5, 6\}$ ordered by divisibility:

$$a \leq b \quad \text{if} \quad a \mid b.$$

Depict this order on the diagram as a subset of $S \times S$.



Order-related definitions

Definitions. Let $\leq$ be an order relation on a set S (here $\leq$ is not necessarily inequality for numbers!), then various self-explanatory terms are used.

Let $T \subseteq S$:

- the **greatest element** of T

  is $t_0 \in T$ such that $t \leq t_0$ for all $t \in T$ (such t_0 may not exist!);

- an **upper bound** for T is any $s \in S$ such that $t \leq s$ for all $t \in T$ (such s is usually not unique or may not exist);

Order-related definitions

- the smallest upper bound for T,
  also called **supremum**, denoted $\sup T$
  (may not exist);
- the **smallest element** of T is $t_0 \in T$ such that $t_0 \leq t$ for all $t \in T$ (such t_0 may not exist!);
- a **lower bound** for T is any $s \in S$
  such that $s \leq t$ for all $t \in T$
  (such s is not unique or may not exist!);
- the greatest lower bound for T,
  also called **infimum**, denoted $\inf T$;
- etc.

Example

Let $S = \mathbb{R}$ (with respect to ordinary inequality order) and $T = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

T has greatest element 1.

Number 4.5 is an upper bound, so is 9, or 1002, ...

$1 = \sup T$ (least upper bound = supremum).

Has no least element.

-12 is a lower bound, so is -888, etc.

$0 = \inf T$ (greatest lower bound = infimum).

Clearly, in general:

Facts:

For a subset T of an ordered set S:

If the greatest element of T exists, then $= \sup T$.

If the least element of T exists, then $= \inf T$.

If $t_0 = \sup T \in T$, then t_0 is the greatest element of T.

If $s_0 = \inf T \in T$, then s_0 is the least element of T.

Example

With ordinary order, $\mathbb{N}$ has no upper bounds,

but has $\inf = 1 =$ least element.

Example

With ordinary order, $\inf([3,5)) = 3 =$ least element;

$\sup([3,5)) = 5$, has no greatest element.

Example

Consider $\mathbb{N}$ ordered by divisibility: aRb if $a \mid b$.

Let $T = \{3, 5, 6, 7\}$.
Determine $\inf T$, $\sup T$, greatest, least elements, if any.
Is 8 an upper bound for T?

There is no greatest element in T.

Number 8 is <u>not</u> an upper bound for T

(in fact, 8 is not comparable with any element of T).

The supremum of T is actually 210,

the least common multiple.

$\inf T = 1$. No least element in T.

Example

Consider $A = \mathscr{P}(\{1,2,3,4,5\})$

$=$ the set of all subsets of $\{1,2,3,4,5\}$

ordered by inclusion.

Find $\inf B$ and $\sup B$ (if exist), where $B \subset A$ is given as

$B = \{\{1,2,4\},\ \{2,3,5\},\ \{1,2,4\}\}$.

$\inf B$ $=$ biggest set contained in all three elements of B: actually $\{2\}$.

$\sup B$ $=$ smallest set containing all three elements of B: actually, $\{1,2,3,4,5\}$.

Example

More generally: consider $\mathscr{P}(S)$
$=$ the set of all subsets of a set S ordered by inclusion.

Let $B \subseteq \mathscr{P}(S)$ be a subset of $\mathscr{P}(S)$.

What is $\inf B$?
$=$ biggest set contained in all elements of B:

actually, $\inf B = \bigcap_{b \in B} b$.

What is $\sup B$?
$=$ smallest set containing all elements of B:

actually, $\sup B = \bigcup_{b \in B} b$.

Lexicographical order

Definition

Order $A \times B$ (or $A_1 \times A_2 \times A_3 \times \cdots$)

as in a dictionary: compare first elements,

only if equal, compare second elements, etc.

Example

On $\mathbb{R} \times \mathbb{R}$ define $(a, b) \leq_L (c, d)$

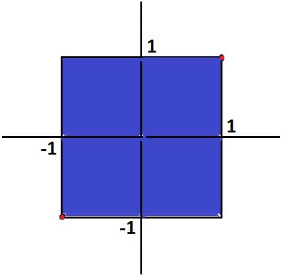
if either $a < c$ (while b and d can be any),

or $a = c$ and $b \leq d$.

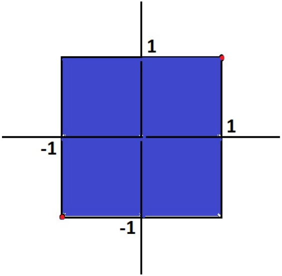
E.g.: $(2, 3) > (1, 500)$, $(2, 2) < (2, 4)$, etc.

Example

Find the supremum (least upper bound) with respect to $\leq_L$ of the set $T \subseteq \mathbb{R} \times \mathbb{R}$ consisting of all points in the square with vertices $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1)$.



T



T

Any point on the right is greater, no matter at which height the points are.

Hence all points on the right side are greater than all other points of the square.

For points on this right side, the first coordinate is the same, so we look at the second coordinate. Hence $(1,1)$ is the greatest point of T with respect to $\leq_L$.
Then $(1,1) = \sup T$ automatically.

Equivalence relations

Definition.

A relation R on a set S is called

an **equivalence** if R is

- transitive,
- symmetric,
- reflexive.

Usually, an equivalence is denoted by $\sim$,

that is, we write $a \sim b$ instead of aRb

(should be clear from the context, which equivalence relation).

'Trivial' example of equivalence is equality $=$:

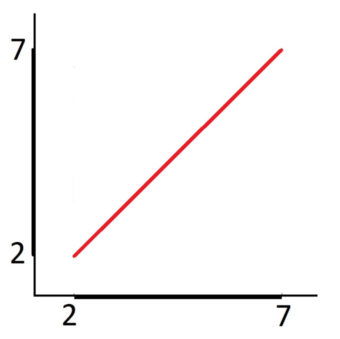
on any set, transitive: $a = b$ and $b = c \Rightarrow a = c$;

reflexive: $a = a$; symmetric: $a = b \Rightarrow b = a$.

Example

Relation "$=$" as a subset of $S \times S$, for $S = [2; 7]$:

all points (a, a) form the 'diagonal':



Example

Let $S = \mathbb{R}$ and aRb if $a^2 = b^2$.

Prove that R is an equivalence.

Transitive: aRb and bRc means $a^2 = b^2$ and $b^2 = c^2$, whence $a^2 = c^2$, which means aRc as required.

Symmetric: $a^2 = b^2$ implies $b^2 = a^2$.

Reflexive: $a^2 = a^2$.

Example

Let $\sim$ be a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ defined as

$a \sim b$ if and only if $a - b$ is divisible by 3.

Prove that $\sim$ is an equivalence relation and depict this relation on the diagram as a subset of $A \times A$.

Transitive: $a \sim b$ and $b \sim c \Rightarrow a \sim c$:

$a - b = 3k$ and $b - c = 3l \Rightarrow$
$(a - b) + (b - c) = 3k + 3l; \quad a - c = 3(k + l).$

Symmetric: $a \sim b \Rightarrow b \sim a$:

$a - b = 3k \Rightarrow b - a = 3(-k).$

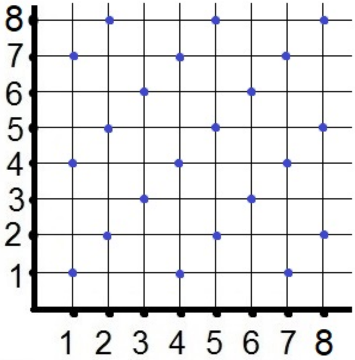
Reflexive: $a \sim a$: $\qquad a - a = 3 \cdot 0.$

Example (continued)

Let $\sim$ be a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ defined as

$a \sim b$ if and only if $a - b$ is divisible by 3.

...depict this relation as a subset of $A \times A$.



Equivalence classes

Definition

Let $\sim$ be an equivalence on a set S.

The **equivalence class** of $s_0 \in S$ is

$[s_0] := \{s \in S \mid s \sim s_0\}$

$=$ the set of all elements equivalent to s_0.

Example

Let $S = \mathbb{Z}$ and let $a \sim b$ if $a - b$ is even.

(Checked above that this is an equivalence.)

What is the equivalence class [3]? of [6]?

(with respect to this equivalence $\sim$).

We have $[3] =$ all odd numbers. Also $= [7]$, etc.

$[6] =$ all even numbers, $= [2] = [8] = [0]$, etc.

Example

Let $S = \mathbb{R}$ and let $a \sim b$ if $a^2 = b^2$.

(Checked above that this is an equivalence.)

What is the equivalence class [3]?
(with respect to this equivalence $\sim$)

We have $[3] = \{3, -3\}$.

Example

Let $S =$ the players in the Premier football league, and let $a \sim b$ if a and b are in the same team.

It is easy to show that this is an equivalence.

For a player x, then $[x] =$ the team in which x plays.

Partition into equivalence classes

Theorem

If $\sim$ is an equivalence relation on a set S,

then any two equivalence classes with respect to $\sim$

either coincide or are disjoint: for any $a, b \in S$,

either $[a] = [b]$, or $[a] \cap [b] = \varnothing$.

Proof:

If $[a] \cap [b] = \varnothing$, then there is nothing to prove.

So we can assume that $[a] \cap [b] \neq \varnothing$

($=$it remains to consider the case where $[a] \cap [b] \neq \varnothing$),

and then we need to prove $[a] = [b]$.

..... $[a] \cap [b] \neq \varnothing$ and we need to prove $[a] = [b]$.

First we prove that $a \sim b$. Let $u \in [a] \cap [b]$, which exists because $[a] \cap [b] \neq \varnothing$ in the case under consideration. So $u \in [a]$ and $u \in [b]$, that is, $u \sim a$ and $u \sim b$. Since $\sim$ is symmetric, we also have $a \sim u$; together with $u \sim b$, by transitivity $a \sim b$, as claimed.

Finally prove $[a] = [b]$. First, $[a] \subseteq [b]$: for x on the l.h.s, $x \sim a$, but $a \sim b$, hence $x \sim b$ by transitivity, that is, $x \in [b]$, as required.

The same type of argument for reverse $[a] \supseteq [b]$: for y on the r.h.s, $y \sim b$, but $a \sim b$ and $b \sim a$ by symmetric, hence $y \sim a$ by transitivity, that is, $y \in [a]$, as required.

Both $[a] \subseteq [b]$ and $[a] \supseteq [b]$ proved, so $[a] = [b]$. $\square$

Corollary

If $\sim$ is an equivalence on a set S,
then S is the <u>*disjoint union*</u>
of the equivalence classes with respect to $\sim$.
In other words, every element of S is in one of the classes, and different classes have empty intersection.

Also called a **partition** of the set S.

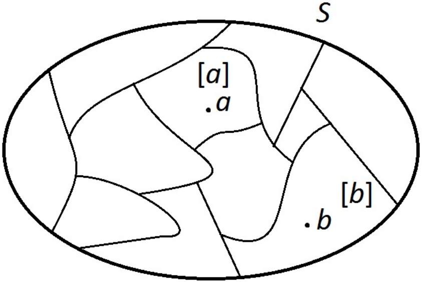
In symbolic form: $S = \dot{\bigcup}_{s \in S} [s]$

and if $[s] \neq [t]$, then $[s] \cap [t] = \varnothing$.

Disjoint = the theorem above. It remains to show that every $s \in S$ is in one of the classes. But $s \sim s$ by reflexivity, so $s \in [s]$.

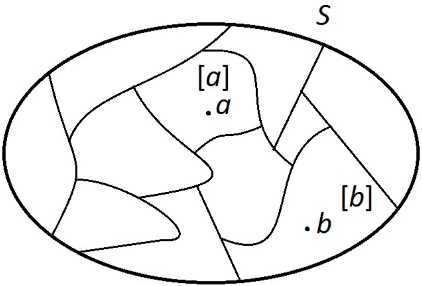
Partition = into equivalence classes

This can be visualized as 'tiling' of S into equivalence classes:



Converse: if S is a disjoint union of some subsets, then we define an equivalence by $a \sim b$ if a, b are in the same subset. Easy to show that this is an equivalence (transitive, symmetric, and reflexive), and those subsets are precisely the equivalence classes.

Set of equivalence classes = quotient set



$$\boxed{\text{Equivalence} \longleftrightarrow \text{Partition}}$$

Importance: we then think of those classes as new entities, instead of their elements.

E.g.: partition of footballers into teams: then we speak about results between teams.

Integers mod n

Example (from Algebra)

Let n be a fixed integer and let $\sim$ be the relation on $\mathbb{Z}$ defined as $a \sim b$ if $a - b$ is divisible by n.

Prove that this is an equivalence.

Transitive: $a \sim b$ and $b \sim c$ means

$a - b = nk$ and $b - c = nl$ for $k, l \in \mathbb{Z}$. Take the sum:

$a - b + b - c = a - c = nk + nl = n(k + l)$, so $a \sim c$.

Symmetric: $a \sim b \;\Rightarrow\; a - b = nk \;\Rightarrow\; b - a = n(-k)$, that is, $b \sim a$.

Reflexive: $a - a = n \cdot 0$.

Arithmetic mod n

Hence $\mathbb{Z}$ is partitioned into equivalence classes mod n:

$a - b$ is divisible by $n \Leftrightarrow a$, b have the same remainders after division by n.

So the classes actually are $[0], [1], \ldots, [n-1]$,

where $[i]$ is the set of all integers

with remainder i after division by n,

that is, $[i] = \{i + kn \mid k \in \mathbb{Z}\}$.

In Algebra these classes are regarded as elements, can be added, multiplied, etc.

'Arithmetic modulo n': so-called ring $\mathbb{Z}/n\mathbb{Z}$ consisting of n elements.

Example

Let $\sim$ on $\mathbb{R}^2$ be defined as

$(a, b) \sim (c, d)$ if $a^2 + b^2 = c^2 + d^2$.

Prove that this is an equivalence and find the equivalence classes.

Transitive:

$(a, b) \sim (c, d)$ and $(c, d) \sim (u, v)$ means
$a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = u^2 + v^2$, then
$a^2 + b^2 = u^2 + v^2$, so $(a, b) \sim (u, v)$, as required.

Reflexive: $a^2 + b^2 = a^2 + b^2$, so $(a, b) \sim (a, b)$.

Symmetric: $a^2 + b^2 = c^2 + d^2$ implies
$c^2 + d^2 = a^2 + b^2$.

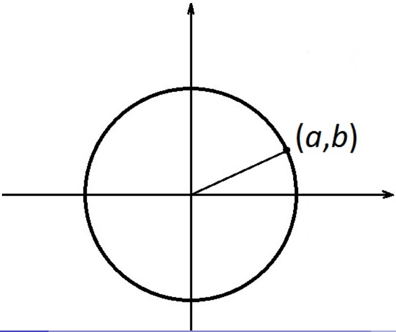
Equivalence classes are circles with centre at $(0, 0)$:

$[(a, b)] = \{(x, y) \mid (x, y) \sim (a, b)\}$, that is,

$x^2 + y^2 = a^2 + b^2 = const$, so by Pythagoras this is

the circle of radius $\sqrt{a^2 + b^2}$ with centre at $(0, 0)$.

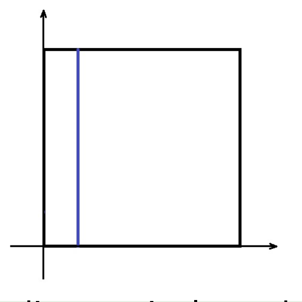
These circles clearly partition the whole plane $\mathbb{R}^2$ (together with $\{(0, 0)\}$ = circle of radius 0, the class $[(0, 0)]$).



Partition → equivalence

Example

The square $[0,1] \times [0,1]$ is partitioned into vertical lines.



Write the corresponding equivalence by a formula.

One possible answer is $(a, b) \sim (c, d)$ if $a = c$.

Then equivalence classes are precisely the vertical lines.

Example

Let R be a relation on $\mathbb{N}$ defined as aRb if $a \leq b + 1$.

Determine if it is transitive, symmetric, antisymmetric, reflexive. Is it an order or equivalence, or neither?

It is not transitive: $3R2$ and $2R1$, but $3R1$ is not true:

$3 \leq 2 + 1$ and $2 \leq 1 + 1$, but $3 \not\leq 1 + 1$.

So neither order nor equivalence.

Not symmetric: $1R5$, but $5R1$ not true.

Not antisymmetric: $1R2$ and $2R1$, but $1 \neq 2$.

Reflexive: kRk as $k \leq k + 1$.

Example

Let $\sim$ be a relation on $\mathbb{R} \times \mathbb{R}$

defined as $(a, b) \sim (c, d)$ if $ab = cd$.

Prove that $\sim$ is an equivalence.

Find the equivalence classes of $(1, 2)$ and of $(3, 0)$.

Transitive: $(a, b) \sim (c, d)$ and $(c, d) \sim (u, v)$

means $ab = cd$ and $cd = uv$

whence $ab = uv$, that is, $(a, b) \sim (u, v)$.

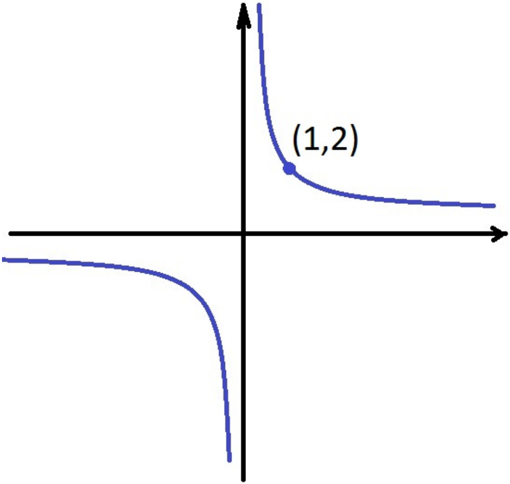
Symmetric: $ab = cd \Rightarrow cd = ab$,

that is, $(a, b) \sim (c, d) \Rightarrow (c, d) \sim (a, b)$.

Reflexive: $ab = ab$.

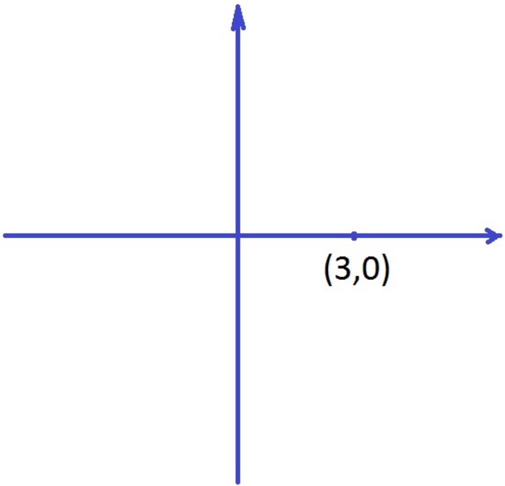
$[(1,2)] = \{(x,y) \mid xy = 1 \cdot 2 = 2\}$, that is, $y = 2/x$.

The class = the graph of $y = 2/x$:



$[(3,0)] = \{(x,y) \mid xy = 3 \cdot 0 = 0\}$, that is, $xy = 0$, which means $x = 0$ or $y = 0$.

The class = union of the two axes:



Mappings

Definition

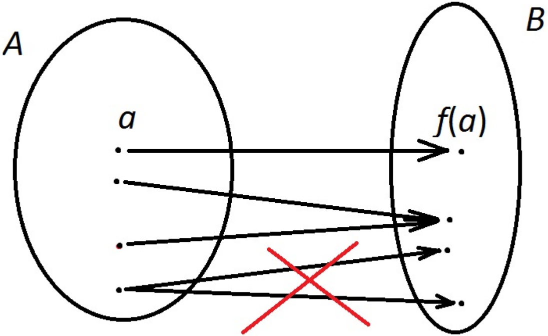
A **mapping** $f : A \to B$ from a set A
to a set B is a rule that associates with every $a \in A$
a certain unique (well-defined) element $b \in B$,
denoted $b = f(a)$ and called the **image** of a under f.

Here, when we write $f : A \to B$, it is assumed that f is defined on the whole of A, that is, the **domain** of f is A.

(In some books or articles, $f : A \to B$ only means that the domain is a subset of A.)

For $A, B \subseteq \mathbb{R}$ or $\mathbb{C}$, mappings are often called functions (in some books, any mappings are called functions).

Illustrations by diagrams



(We can draw a similar picture even if $A = B$.)
From each $a \in A$ there must be an outgoing arrow, but only one.

Two different elements can have equal images.

More terminology and notation

Given a mapping $f : A \to B$, the **image** of f is

$f(A) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$;

also called the range of f, or the image of A.

'Destination' set B is **codomain**. In general, $f(A) \subseteq B$.

For a subset $A_1 \subseteq A$, the image of A_1 is
$f(A_1) = \{b \in B \mid b = f(a) \text{ for some } a \in A_1\}$.

The image $f(a)$ of an element $a \in A$
is also called the **value** of f at a.

We say that f **maps** a to $f(a)$.

The rule defining a mapping $f : A \to B$ can be anything, it only matters that images are well-defined:

- a formula: $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$;
- a table: $f : \{a, b, c, d\} \to \{1, 2, 3\}$, where

$x =$	a	b	c	d
$f(x) =$	1	1	2	1

;

- piece-wise formulae: $f : \mathbb{R} \to \mathbb{Z}$,

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0 \end{cases}$$

- any well-defined rule: $f : T \to \mathbb{R}$, where T is the set of all triangles on a given plane, and $f(t) =$ area of the triangle t.

Equal mappings

Two mappings are equal if they have **the same domain and the same image of every element in the domain**

... does not matter if they are defined by different rules.

Also, even if they are defined by the same rule, they are not equal if the domains are different.

E.g.: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ for all x, and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin^2 x + \cos^2 x$ for all x, are equal mappings.

"Ye shall know them by their fruits"

But $u : \mathbb{Z} \to \mathbb{N}$, $u(x) = x^2$, and $v : \mathbb{N} \to \mathbb{N}$, $v(x) = x^2$ are <u>different</u>, since have different domains assigned.

Example

Let $A = \mathscr{P}(\{a, b, c\})$ (set of all subsets of $\{a, b, c\}$),

let $f : A \to A$, $f(X) = X \cap \{a\}$ (here $X \in \mathscr{P}(\{a, b, c\})$).

For example, $f(\{b, c\}) = \{b, c\} \cap \{a\} = \varnothing$,
or $f(\{a, c\}) = \{a, c\} \cap \{a\} = \{a\}$.

The image of f is $\{\varnothing, \{a\}\}$.

Example

Let $A = \mathscr{P}(\{a, b, c\})$,

let $f : A \to \mathbb{Z}$ be given by $f(X) = |X|$.

For example, $f(\{b, c\}) = 2$.

The image of f then is $\{0, 1, 2, 3\}$. (Note: $0 = f(\varnothing)$.)

Example

Find the image of $f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = \dfrac{2x}{x^2+1}$.

Simply means, for which y there is x satisfying $f(x) = y$, that is, for which y there exists a solution

of $\dfrac{2x}{x^2+1} = y$, where x is unknown.

Can multiply by denominator $\neq 0$:

$2x = yx^2 + y, \;\; yx^2 - 2x + y = 0, \;\; x_{1,2} = \dfrac{2 \pm \sqrt{4 - 4y^2}}{2y}$

if $y \neq 0$.

This solutions exist if and only if $4 - 4y^2 \geq 0$,

that is, for $y \in [-1, 1] \setminus \{0\}$.

The case $y = 0$ must be considered separately:
$\dfrac{2x}{x^2 + 1} = 0$ also has a solution $x = 0$.
Thus, the image of f is $[-1, 1]$.

Example

How many mappings are there

from a finite set A with m elements

to a finite set B with n elements?

Solution: Every element of A can be mapped (=sent) to any of n elements of B, independently. Therefore there are $\underbrace{n \cdot n \cdots n}_{m} = n^m$ possible mappings.

Rigorous definition of a mapping

The word ‘rule’ in the above definition of a mapping is rather imprecise.

A rigorous definition is in terms of sets.

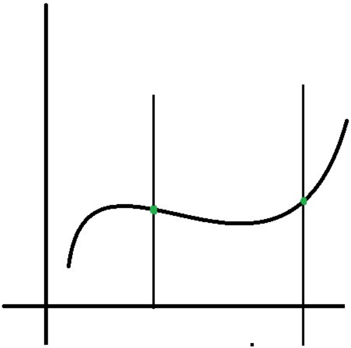
Definition

A mapping $f : A \to B$ is a subset of $A \times B$ such that for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$, then of course we write $b = f(a)$.

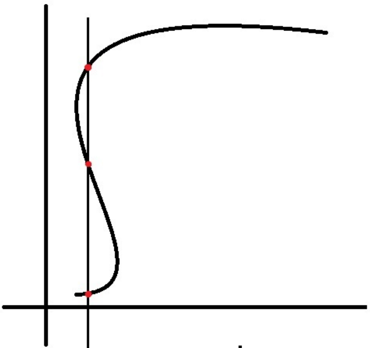
In other words, “function is its graph”.

Vertical Line Test

When $A, B \subseteq \mathbb{R}$, a set on the plane $\mathbb{R} \times \mathbb{R}$ above/below A is a mapping (a function) if it satisfies the “Vertical Line Test”: it must have exactly one intersection point with every vertical line through A.



a mapping



not a mapping

Implicitly defined mappings

Often a mapping is defined 'implicitly':

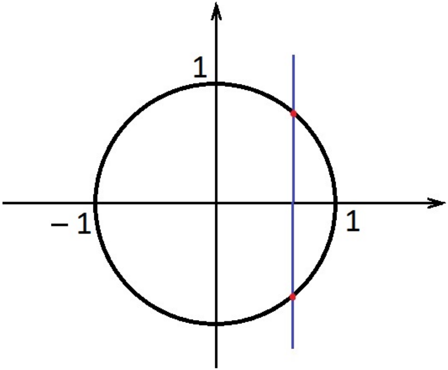
some condition defines a subset of $A \times B$,

which may, or may not, be a mapping.

Example

Is $F = \{(x, y) \mid x^2 + y^2 = 1\}$ a mapping?

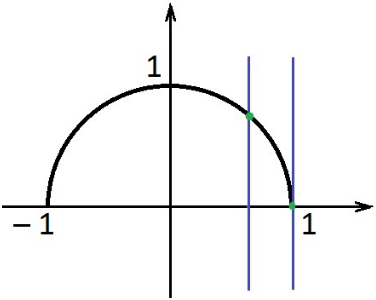
This is a circle, of radius 1 centred at $(0, 0)$.



Fails Vertical Line Test; not a mapping.

Example

Is $F_1 = \{(x,y) \mid y \geq 0 \text{ and } x^2 + y^2 = 1\}$ a mapping?



This is the upper semicircle.

Vertical Line Test OK; domain $[-1,1]$

is a mapping $F_1 : [-1,1] \to \mathbb{R}$,

actually, $y = \sqrt{1-x^2}$ or $F_1(x) = \sqrt{1-x^2}$.

Example

Is $F = \{(x, y) \in \mathbb{R}^2 \mid x + |x| = y + |y|\}$ a mapping?

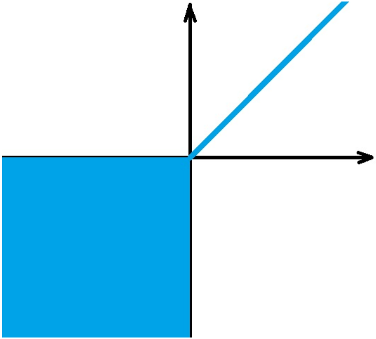
$a + |a| = 2a$ if $a \geq 0$, and $a + |a| = 0$ if $a \leq 0$.

When $x \geq 0$ and $y \geq 0$
the condition means $2x = 2y$, $x = y$.

When $x < 0$ and $y \geq 0$
the condition means $0 = 2y$, $y = 0$.

When $x \geq 0$ and $y < 0$
the condition means $2x = 0$, $x = 0$.

When $x \leq 0$ and $y \leq 0$
the condition means $0 = 0$, that is, always holds throughout this area.



Vertical Line Test not satisfied for $x \leq 0$, so not a mapping.

But 'becomes' a mapping within $(0, \infty) \times \mathbb{R}$, that is,

$F_1 = \{(x, y) \in (0, \infty) \times \mathbb{R} \mid x + |x| = y + |y|\}$

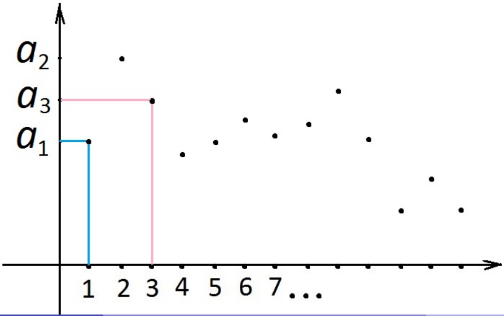
is a mapping $F_1 : (0, \infty) \to \mathbb{R}$, $F_1(x) = x$.

Sequences as mappings

Definition

A **sequence** is a mapping $f : \mathbb{N} \to A$;
traditional notation: $a_i = f(i)$; $(a_i)_{i \in \mathbb{N}}$.

Diagram of a sequence: domain is $\mathbb{N}$,
codomain here is $\mathbb{R}$ (for a sequence of real numbers).



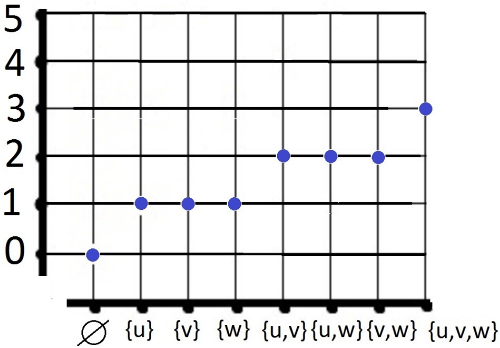
Example

Let $A = \mathscr{P}(\{u, v, w\})$ (all subsets of $\{u, v, w\}$).

Depict $f : A \to \{0, 1, 2, 3, 4, 5\}, \quad f(X) = |X|,$

as a subset of the Cartesian product.

E.g., $f(\varnothing) = 0, \quad f(\{u, v\}) = 2,$ etc.



Theorem (Image of union and intersection)

Let $f : A \to B$ *be a mapping, and* $U, V \subseteq A$.

(a) $f(U \cup V) = f(U) \cup f(V)$;

(b) $f(U \cap V) \subseteq f(U) \cap f(V)$, *and "*$\neq$*" in general.*

Proof of (a): "$\subseteq$": On the left: $f(x)$ where $x \in U$ or $x \in V$ by def. of union.
Then $f(x) \in f(U)$ or $f(x) \in f(V)$,
so $f(x) \in$ r.h.s. by def. of union.

"$\supseteq$": On the right: in $f(U)$ or in $f(V)$ by def. of union.
So it is $f(x)$ for $x \in U$ or $x \in V$,
which means $x \in U \cup V$ by def. of union,
and then $f(x) \in f(U \cup V) =$ l.h.s.

Image of intersection

Proof of (b): $f(U \cap V) \subseteq f(U) \cap f(V)$,
and "$\neq$" in general.

On the left: $f(x)$ where $x \in U$ and $x \in V$ by def. of intersection.

Then $f(x) \in f(U)$ and $f(x) \in f(V)$,
so $f(x) \in$ r.h.s. by def. of intersection.

A simple example where $f(U \cap V) \neq f(U) \cap f(V)$:

Let $f : \{a, b\} \to \{1\}$, $\quad f(a) = 1$ and $f(b) = 1$.

Let $U = \{a\}$ and $V = \{b\}$.

Then $f(U \cap V) = f(\varnothing) = \varnothing$,

while $f(U) \cap f(V) = \{1\} \cap \{1\} = \{1\}$. $\square$

Example

Another example where $f(U \cap V) \neq f(U) \cap f(V)$:

Let $f : [-2, 2] \to \mathbb{R}$ be defined as $f(x) = x^2$.

Let $U = [-2, 1]$ and $V = [0, 2]$.

We have $U \cap V = [0, 1]$, so $f(U \cap V) = [0, 1]$.

We have $f(U) = [0, 4]$ and $f(V) = [0, 4]$,

so $f(U) \cap f(V) = [0, 4]$

$\neq f(U \cap V) = [0, 1]$.