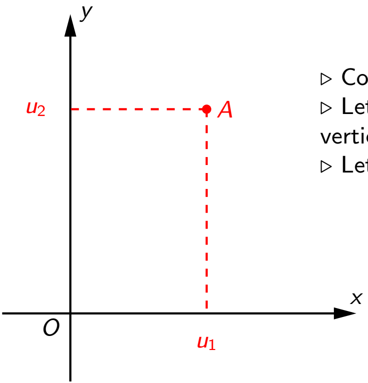
Vectors in the plane

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Vectors in the plane



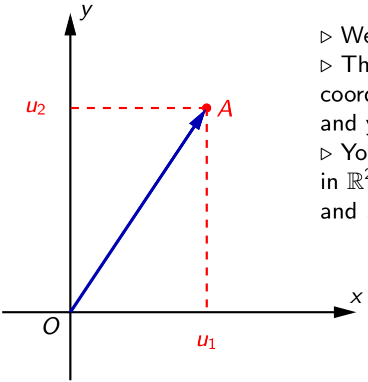
- Consider a Cartesian plane
- Let x and y denote the horizontal and vertical axis
- Let O be the origin of the axes.

Now pick up a point A in the plane.
We say that this point has coordinates the numbers u_1 and u_2.

By drawing an arrow from O to A, *you create a vector* $\mathbf{u}$:

$$\mathbf{u} = [u_1, u_2] \quad \text{or} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Vectors in the plane



▷ We denote the plane by $\mathbb{R}^2$

▷ The exponent 2 denotes the number of coordinates we have, i.e. here we have x and y.

▷ You understand that, to specify a vector in $\mathbb{R}^2$, you need to know its entries, i.e. u_1 and u_2.

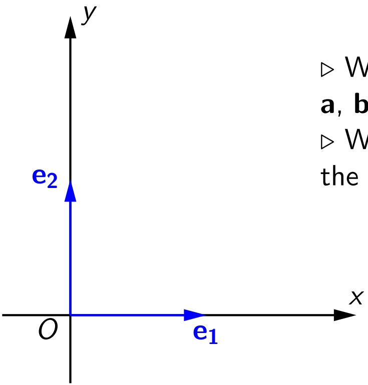
Can you draw the vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}?$$

Which is the vector from O to O?
We denote by $\mathbf{0}$ the zero-vector:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vectors in the plane



▷ We denote vectors with bold: **a**, **b**, **c** etc.

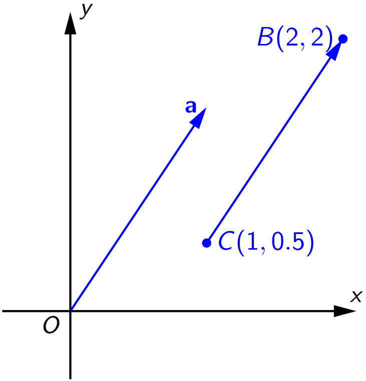
▷ We denote with $\mathbf{e_1}$ and $\mathbf{e_2}$ the standard vectors (left figure).

They are:

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Remarks

▷ Points are with brackets, i.e. $A(1, 1.5)$
▷ Vectors are with square brackets i.e. $\mathbf{a} = [1, 1.5]$



All vectors with origin other than O are transferable to their *standard position*. Example:

$$\mathbf{a} = \vec{BC} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 2-0.5 \end{bmatrix}$$

▷ The individual coordinates, 1 and 1.5, are called *components* of the vector $\mathbf{a}$.
▷ If not otherwise stated, the origin of the vector is O.

Linear Combinations

The heart of Linear Algebra is in **two operations** :

◇ *We* **add** *vectors and we* **multiply** *them by* **numbers**.

These two operations gives the **linear combination** of vectors.

▷ Vector Addition:

$$\text{If } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{w} = \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \end{bmatrix}.$$

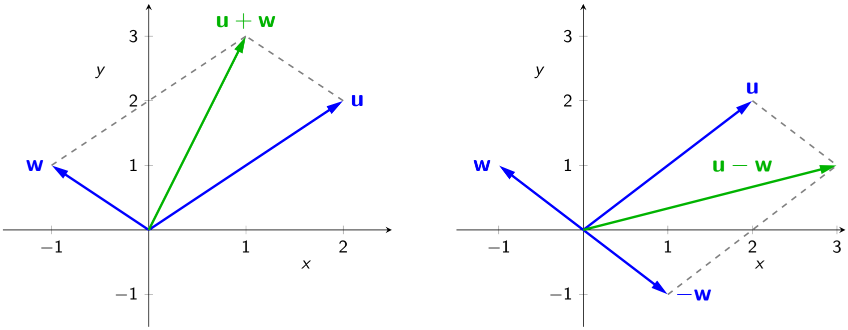
We add the first component of $\mathbf{u}$ to the first component of $\mathbf{w}$, i.e. $(u_1 + w_1)$ and the second component of $\mathbf{u}$ to the second component of $\mathbf{w}$, i.e. $(u_2 + w_2)$ to get the vector $\mathbf{u} + \mathbf{w}$

▷ Scalar Multiplication:

$$\text{If } c \text{ is a real number and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ then } c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

We multiply each component of the vector $\mathbf{u}$ with the real number c and this is how we get the vector $c\mathbf{u}$.

Vector Addition

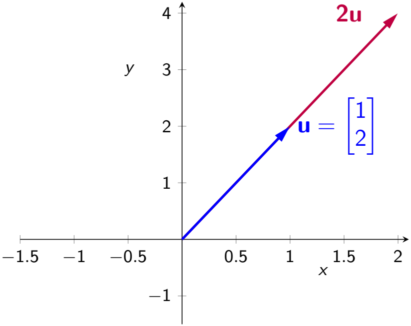


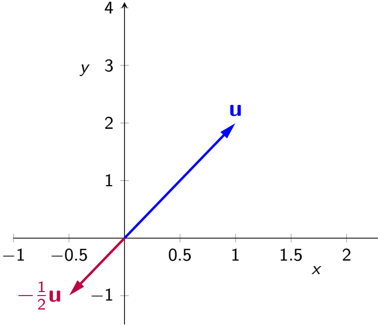
Here it is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\mathbf{u} + \mathbf{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

To calculate $\mathbf{u} - \mathbf{w}$ we find the vector $-\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and then we add the vectors $\mathbf{u}$ and $-\mathbf{w}$, as before. So,
$\mathbf{u} - \mathbf{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Scalar Multiplication





We calculate the scalar multiplication of the number $c = 2$ with the vector $\mathbf{u}$:

$$2\mathbf{u} = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}.$$

Similarly, the vector $-\frac{1}{2}\mathbf{u}$ is:

$$-\frac{1}{2}\mathbf{u} = -\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1/2\\-1\end{bmatrix}.$$

Linear Combination of Vectors

◇ This is a key concept of Linear Algebra

Definition

Let c and d be real numbers. Let $\mathbf{u}$ and $\mathbf{w}$ be vectors in the plane ($\mathbb{R}^2$). The sum of $c\mathbf{u}$ and $d\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{w}$.

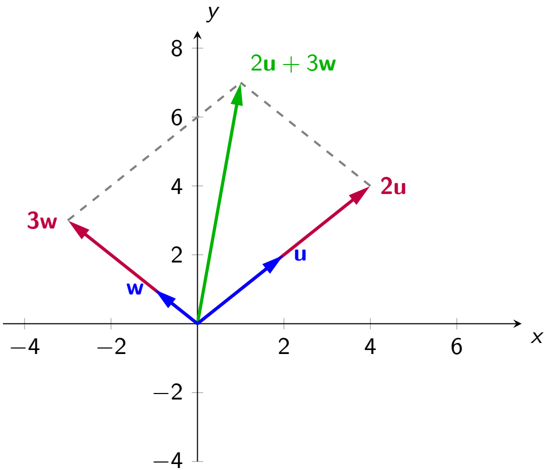
Examples

We calculated $\mathbf{u} + \mathbf{w}$ and $\mathbf{u} - \mathbf{w}$. These were both linear combinations of the vectors $\mathbf{u}$ and $\mathbf{w}$. What are the real numbers c and d here?

Linear combinations of $\mathbf{u}$ and $\mathbf{w}$

$$2\mathbf{u} + 3\mathbf{w}$$
$$\mathbf{u} - 1/2\mathbf{w}$$
$$-2\mathbf{u} + 4\mathbf{w}$$
$$-1\mathbf{u} + 1\mathbf{w}$$
$$0\mathbf{u} + \mathbf{w}$$

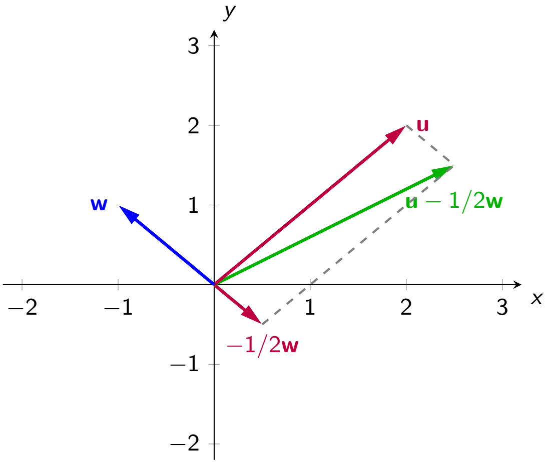
Example: Calculate and Draw $2\mathbf{u} + 3\mathbf{w}$



Here it is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$2\mathbf{u} + 3\mathbf{w} = 2\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

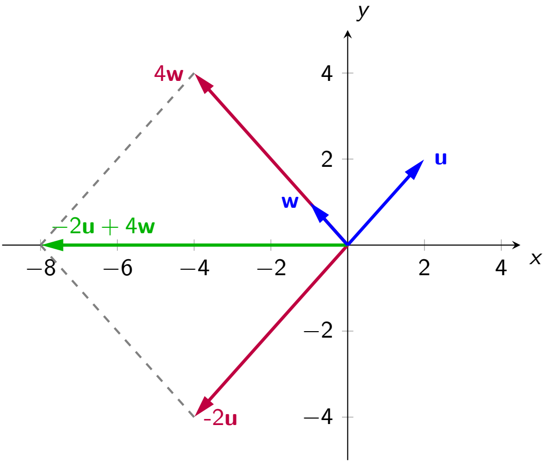
Example: Calculate and Draw $\mathbf{u} - 1/2\mathbf{w}$



Here it is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\mathbf{u} - \tfrac{1}{2}\mathbf{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}.$$

Example: Calculate and Draw $-2\mathbf{u} + 4\mathbf{w}$



It is
$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and
$\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$-2\mathbf{u} + 4\mathbf{w} = -2\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}.$$

Properties

◇ *Let that c, d be real numbers and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ vectors in the plane.*

▷ *The zero vector $\mathbf{0}$:*

$0\mathbf{u} = \mathbf{0}$

$0\mathbf{u} + \mathbf{w} = \mathbf{w}$

$c\mathbf{u} + 0\mathbf{w} = c\mathbf{u}$

$c\mathbf{0} = \mathbf{0}$

▷ *Commutativity:* $\mathbf{u} + \mathbf{w} = \mathbf{w} + \mathbf{u}$

In other words, the order of vector addition makes no difference.

Proof:

$$\mathbf{u} + \mathbf{w} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \end{bmatrix} = \begin{bmatrix} w_1 + u_1 \\ w_2 + u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{w} + \mathbf{u}$$

Example: Let $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Show that $\mathbf{u} + \mathbf{w} = \mathbf{w} + \mathbf{u}$.

It is, $\mathbf{u} + \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

while also $\mathbf{w} + \mathbf{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Properties

▷ *Associativity:* $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Example: Let $\mathbf{u}=\begin{bmatrix}-2\\1\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}0\\-1\end{bmatrix}$ *and* $\mathbf{w}=\begin{bmatrix}3\\-2\end{bmatrix}$. *Show that* $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

It is, $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-2\\1\end{bmatrix}+\begin{bmatrix}0\\-1\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}$.

So, then $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\begin{bmatrix}-2\\0\end{bmatrix}+\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$.

On the other hand, it is $\mathbf{v}+\mathbf{w}=\begin{bmatrix}0\\-1\end{bmatrix}+\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}3\\-3\end{bmatrix}$,

so, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}-2\\1\end{bmatrix}+\begin{bmatrix}3\\-3\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$ $\square$

Properties

▷ *Distributivity (I):* $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Example: Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$*. Show that* $3(\mathbf{u}+\mathbf{v}) = 3\mathbf{u} + 3\mathbf{v}$*.*

It is, $3(\mathbf{u}+\mathbf{v}) = 3(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix}) = 3\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ -12 \end{bmatrix}$,

while it is $3\mathbf{u} + 3\mathbf{v} = 3\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -12 \end{bmatrix} = \begin{bmatrix} 9 \\ -12 \end{bmatrix}$ $\square$

Properties

- *Distributivity (II):* $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- *Other properties:*

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$
$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$
$$1\mathbf{u} = \mathbf{u}$$

Example: Simplify the vector expression

$$3(\mathbf{u} - 4\mathbf{v}) + 2((2+1)\mathbf{w} + 3\mathbf{v}) + 0(1\mathbf{u} + 5\mathbf{w}).$$

Alternatively one can ask, which is the simplest linear combination of the above expression?

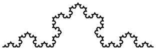
$$\begin{aligned}
&3(\mathbf{u} - 4\mathbf{v}) + 2(3\mathbf{w} + 3\mathbf{v}) + 0(1\mathbf{u} + 5\mathbf{w}) = \\
&3\mathbf{u} - 12\mathbf{v} + 6\mathbf{w} + 6\mathbf{v} + \mathbf{0} = \\
&3\mathbf{u} - 6\mathbf{v} + 6\mathbf{w}
\end{aligned}$$

Spanning of two vectors in $\mathbb{R}^2$

Definition Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two vectors in $\mathbb{R}^2$. The set of all linear combinations

$$S = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 : c_1, c_2 \text{ in } \mathbb{R}\}$$

is called *span* of $\mathbf{u}_1$ and $\mathbf{u}_2$ and is denoted by $span(\mathbf{u}_1, \mathbf{u}_2)$.



Definition If the linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ can produce any vector in the plane, then the vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ span the plane. We denote by

$$span(\mathbf{u}_1, \mathbf{u}_2) = \mathbb{R}^2$$

Examples spanning $\mathbb{R}^2$

➜*Find the span of the standard vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$

It is

$$span(\mathbf{e}_1, \mathbf{e}_2) = \{c_1\mathbf{e}_1 + c_2\mathbf{e}_2 : c_1, c_2 \text{ in } \mathbb{R}\}$$

The linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$ are:

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

So,

$$span(\mathbf{e}_1, \mathbf{e}_2) = \{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} : c_1, c_2 \text{ in } \mathbb{R}\}$$

➜*Show that the standard vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *span the plane.*

We check whether the linear combinations of $\mathbf{e}_1, \mathbf{e}_2$ can produce any vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$. Let $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = \mathbf{x}$, then $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. So, for $c_1 = x$ and $c_1 = y$ any vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be expressed as the linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$, therefore they span the plane:

$$span(\mathbf{e}_1, \mathbf{e}_2) = \mathbb{R}^2$$

Example spanning a line in $\mathbb{R}^2$

➜*Find* $span(\mathbf{e}_1, -2\mathbf{e}_1)$

The linear combinations of $\mathbf{e}_1, -2\mathbf{e}_1$ are:

$$c_1\mathbf{e}_1 + c_2(-2\mathbf{e}_1) = (c_1 - 2c_2)\mathbf{e}_1 = (c_1 - 2c_2)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}c_1 - 2c_2\\0\end{bmatrix}$$

Let $d = c_1 - 2c_2$, then:

$$span(\mathbf{e}_1, -2\mathbf{e}_1) = \{d\mathbf{e}_1 : d \text{ in } \mathbb{R}\} = \left\{ \begin{bmatrix}d\\0\end{bmatrix} : d \text{ in } \mathbb{R} \right\},$$

which is the x–axis and is a line $\mathbb{R}^2$ (whose equation is $y = 0$).

Therefore, $span(\mathbf{e}_1, -2\mathbf{e}_1) \neq \mathbb{R}^2$ because $\mathbf{e}_1$ and $-2\mathbf{e}_1$ **cannot** produce all vectors in $\mathbb{R}^2$.

Example

➜*Find span*$(\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_2)$

The linear combinations of $\mathbf{e}_1 + 3\mathbf{e}_2$ and $\mathbf{e}_2$ are:

$$c_1(\mathbf{e}_1 + 3\mathbf{e}_2) + c_2\mathbf{e}_2 = c_1\mathbf{e}_1 + (3c_1 + c_2)\mathbf{e}_2 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (3c_1 + c_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 3c_1 + c_2 \end{bmatrix}$$

So:

$$span(\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_2) = \{ \begin{bmatrix} c_1 \\ 3c_1 + c_2 \end{bmatrix} : c_1, c_2 \text{ in } \mathbb{R} \}$$

➜*Show that span*$(\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_2) = \mathbb{R}^2$.

We check whether the linear combinations of $\mathbf{e}_1 + 3\mathbf{e}_2$ and $\mathbf{e}_2$ can produce any vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$. Setting $c_1(\mathbf{e}_1 + 3\mathbf{e}_2) + c_2\mathbf{e}_2 = \mathbf{x}$, it yields that:

$$\begin{bmatrix} c_1 \\ 3c_1 + c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

So $c_1 = x$ and $3c_1 + c_2 = y$ or better, $c_1 = x$ and $c_2 = y - 3c_1 = y - 3x$.
So, since there are $c_1 = x$ and $c_2 = y - 3x$, then any vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be expressed as the linear combination of $\mathbf{e}_1 + 3\mathbf{e}_2$ and $\mathbf{e}_2$, therefore they span the plane, namely $span(\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_2) = \mathbb{R}^2$.

Example

→*Find span*$(\mathbf{u}, \mathbf{v})$ *where* $\mathbf{u} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} -3/4 \\ -1/2 \end{bmatrix}$

The linear combinations of $\mathbf{u}, \mathbf{v}$ read:

$$c_1\mathbf{u} + c_2\mathbf{v} = c_1 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3/4 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3/2c_1 - 3/4c_2 \\ c_1 - 1/2c_2 \end{bmatrix} = \begin{bmatrix} 3/2(c_1 - 1/2c_2) \\ c_1 - 1/2c_2 \end{bmatrix}$$

Let $d = c_1 - 1/2c_2$, then

$$span(\mathbf{u}, \mathbf{v}) = \left\{ \begin{bmatrix} 3/2d \\ d \end{bmatrix} : d \text{ in } \mathbb{R} \right\} = \left\{ d \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} : d \text{ in } \mathbb{R} \right\} = span(\mathbf{u})$$

which is a line in plane $\mathbb{R}^2$. (By setting $x = 3/2d$ and $y = d$ one finds that the line is $x = 3/2y$).

So, *span*$(\mathbf{u}, \mathbf{v})$ is **not** the whole plane $\mathbb{R}^2$.

Linear Independence

Definition Two vectors $\mathbf{u}_1, \mathbf{u}_2$ are called *linearly independent* if the linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$$

implies that all scalars are zero $c_1 = c_2 = 0$.

✏ It actually means that whose vectors are not related. They are, simply, independent.

Example 1

➜*Show that the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *are linearly independent.*

The standard methodology is to start with the vector equation

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$$

and try to find possible solutions in terms of scalars c_1 and c_2. It is:

$$c_1\mathbf{u} + c_2\mathbf{v} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 \end{bmatrix}.$$

The equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ is equivalent to

$$\begin{bmatrix} c_1 + c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} 0 + c_2 = 0 \\ c_1 = 0 \end{cases}$$

So, $c_1 = c_2 = 0$ therefore we conclude that the vectors $\mathbf{u}, \mathbf{v}$ are linearly independent.

Example 2

➜*Determine whether the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *are linearly independent.*

Like in the previous example, we start with the vector equation

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$$

and solve it in terms of the scalars c_1 and c_2. We find:

$$c_1\mathbf{u} + c_2\mathbf{v} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix}.$$

The equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ gives

$$\begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{cases} 0 + c_2 = 0 \\ c_1 = 0 \end{cases}$$

Hence, $c_1 = c_2 = 0$ and we conclude that $\mathbf{u}, \mathbf{v}$ are linearly independent.

Example 3

➜*Determine whether the vectors* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *are linearly independent.*

Again, we consider the vector equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

which we solve in terms of the scalars c_1, c_2 and c_3. We find:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_2 \end{bmatrix}.$$

Then, the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ gives

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{cases} c_3 = -2c_1 \\ c_2 = c_1 \end{cases}$$

Since, c_1, c_2, c_3 can take non-zero values, we conclude that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

Key-concepts of vectors in the plane

- Vector Operations (Vector Addition, Scalar Multiplication)
- Linear Combinations of Vectors
- Spanning of Vectors:

  1. One nonzero vector in $\mathbb{R}^2$ spans a line
  2. Two nonzero vectors in $\mathbb{R}^2$ span the whole plane $\mathbb{R}^2$,
     provided that they are not collinear (do not lie on the same line)

- Linear Independence of vectors