Eigenvalues and Eigenvectors

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Definition

Let A be $n \times n$ matrix. If λ is a real number and $\mathbf{x}$ non-zero vector in $\mathbb{R}^n$ satisfying:

$$A\mathbf{x} = \lambda\mathbf{x}$$

then λ is called *eigenvalue* of A and $\mathbf{x}$ is called *eigenvector* of A.

Remarks:

✏ *Eigen–* comes from German and means 'its own'.

✏ $A\mathbf{x}$ is a vector in $\mathbb{R}^n$ (matrix multiplication). Usually, the operation $A\mathbf{x}$ changes the direction of $\mathbf{x}$.

✏ $\lambda\mathbf{x}$ is a vector in $\mathbb{R}^n$ (scalar multiplication)

✏ $A\mathbf{x} = \lambda\mathbf{x}$ means that there is a special vector $\mathbf{x}$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$. $A\mathbf{x}$ only stretches or shrinks its eigenvector $\mathbf{x}$ by a factor λ, the eigenvalue.

Eigenvalues in $n \times n$ matrices

Let A be a $n \times n$ matrix, $\mathbf{x}$ a vector in $\mathbb{R}^n$ and λ a scalar.

We search for solutions in terms of $\mathbf{x}$ in the following equation:

$$A\,\mathbf{x} = \lambda \mathbf{x}$$

We bring everything to the l.h.s.

$$A\,\mathbf{x} - \lambda \mathbf{x} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

and we get a homogeneous system with n equations and n unknowns (corresponding to the eigenvector components) plus one extra unknown (corresponding to the eigenvalue λ).

✏ The system may have the zero solution (trivial case) or infinitely many solutions.

Eigenvalues in $n \times n$ matrices

A homogeneous system has infinitely many solutions when the matrix $A - \lambda I$ is singular, i.e. has zero determinant:

$$\det(A - \lambda I) = 0 \ .$$

and this is the condition to find the eigenvalues, the λ values.

✏ $\det(A - \lambda I)$ is called the characteristic polynomial of A.

Example: Eigenvalues

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

The characteristic polynomial of A is:

$$\det(A - \lambda I) \quad = \quad \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4.$$

So, the condition $\det(A - \lambda I) = 0$ is equivalent to:

$$\begin{aligned} (1-\lambda)(1-\lambda) - 4 &= 0 \\ (1-\lambda)^2 &= 4 \\ (1-\lambda) = \pm 2 \quad \text{or} \quad \lambda &= 1 \pm 2 \end{aligned}$$

It is $\det(A - \lambda I) = 0$ if and only if $\lambda = -1$ or $\lambda = 3$. So, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$ $\square$

✏ For a 2×2 matrix A, we can have up to 2 eigenvalues.

Eigenvectors in $n \times n$ matrices

✏ Eigenvectors are certain *exceptional* vectors $\mathbf{x}$ which are parallel to $A\mathbf{x}$.

✏ Knowing the eigenvalue λ_1 we search for its corresponding eigenvector $\mathbf{u}_1$. For λ_2, we search for its eigenvector $\mathbf{u}_2$ and so on.

✏ An eigenvector is any non-zero vector $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, whose components satisfy the linear system:

$$A\mathbf{u} = \lambda\mathbf{u} \text{ or } (A - \lambda I)\mathbf{u} = \mathbf{0}.$$

⋆ This is the condition to find the eigenvectors. ⋆

Example: Eigenvectors

Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

So far we know that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$.

◇ For $\lambda_1 = -1$ it is:

$$A - \lambda_1 I = A - (-1)I = A + I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

To find the eigenvector $\mathbf{u}_1$ associated with the eigenvalue $\lambda_1 = -1$, we need to solve the system $(A - \lambda_1 I)\mathbf{u}_1 = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 2y_1 \\ 2x_1 + 2y_1 \end{bmatrix}.$$

$(A - \lambda_1 I)\mathbf{u}_1 = \mathbf{0}$ is a singular system with infinitely many solutions, since it reduces to one equation with two unknowns $2x_1 + 2y_1 = 0$. So, its eigenspace is:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_1 = -x_1,\ x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = \mathit{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

Example: Eigenvectors

and the eigenvector for $\lambda_1 = -1$ is $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

◇ For $\lambda_2 = 3$ it is:

$$A - \lambda_2 I = A - 3I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} .$$

The eigenvector $\mathbf{u}_2$ associated with the eigenvalue $\lambda_2 = 3$, can be found by solving the system $(A - \lambda_2 I)\mathbf{u}_2 = \mathbf{0}$:

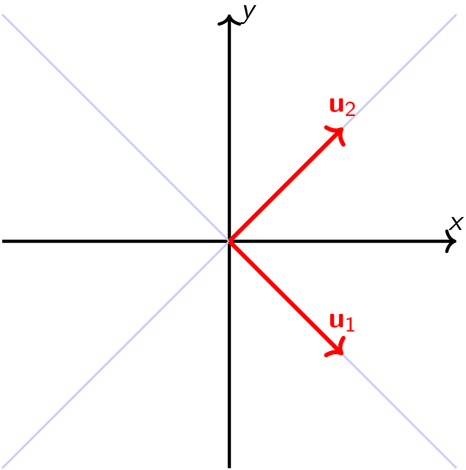
$$(A - \lambda_2 I)\mathbf{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2y_2 \\ 2x_2 - 2y_2 \end{bmatrix} .$$

So, $(A - \lambda_2 I)\mathbf{u}_2 = \mathbf{0}$ yields $-2x_2 + 2y_2 = 0$ with solution set:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_2 = x_2,\ x_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\} = \mathit{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

So, the eigenvector for $\lambda_2 = 3$ is $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvectors



The eigenvectors are
$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and
$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

They are linearly independent.

The lines correspond to the eigenspaces V_{λ_1} and V_{λ_2}.

Eigenvectors in 2×2 matrices

Condition to find the eigenvalues: $\det(A - \lambda I) = 0$

Let the matrix A have 2 real eigenvalues λ_1 and λ_2.

Process of finding the eigenvectors:

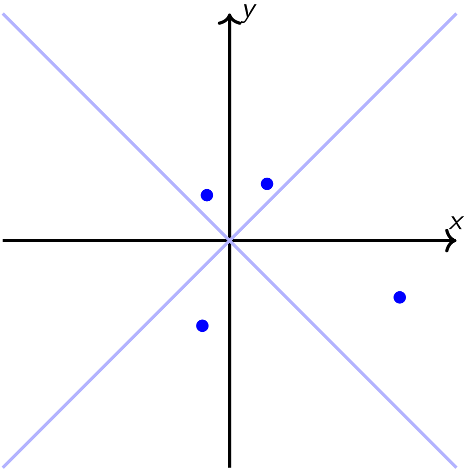
◇ First step: We input the value λ_1 in the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

◇ Second step: We get the system $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$, which we solve in terms of the components of $\mathbf{x}$.

◇ Third step: We denote its set of solutions by V_{λ_1}. V_{λ_1} is called *eigenspace* of the eigenvalue λ_1 and it is spanned by the eigenvector $\mathbf{x}$.

◇ Fourth step: Repeat for the λ_2 eigenvalue to compute its associate eigenvector.

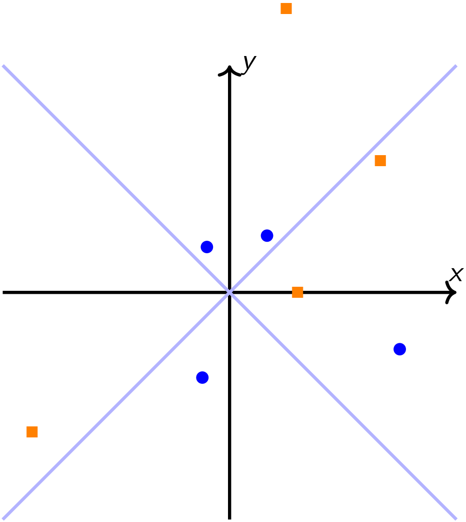
Eigenvectors: An experiment



The experiment is the following:

Choose any points (or vectors) outside the lines.

Eigenvectors: An experiment



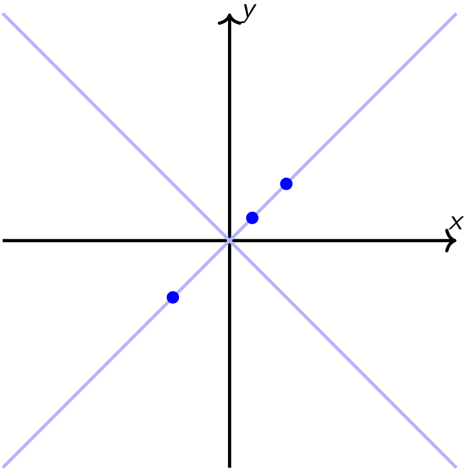
Input: $\mathbf{x} = \bullet$
Output: $A\mathbf{x} = \blacksquare$

When the matrix A acts on any point (or vector) $\mathbf{x}$ outside the line ...

$A\mathbf{x}$ will also be outside the line!

New points will be scatted on the plane and never fall on the lines.

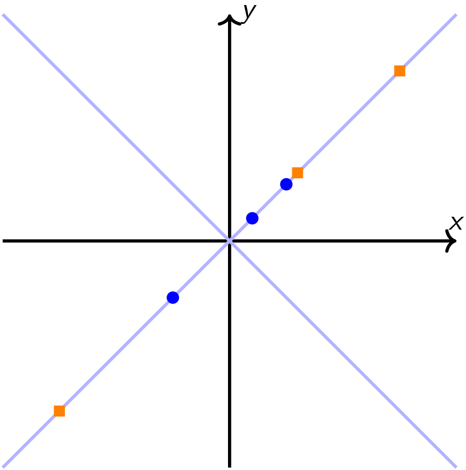
Eigenvectors: A second experiment



The second experiment is the following:

Choose the points/vectors on the lines.

Eigenvectors: A second experiment



Input: $\mathbf{x} = \bullet$
Output: $A\mathbf{x} = \blacksquare$

$A\mathbf{x}$ is a point/vector which is on the line and is λ times the initial vector.

- If $|\lambda| > 1$, then the new vector will expand.
- If $|\lambda| < 1$, the new vector will shrink.
- If $\lambda = 1$, the new vector will stay the same.
- If $\lambda = -1$, the new vector will be reflected.

Example

Determine the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}.$$

Eigenvalues

The characteristic polynomial of A is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} = (4-\lambda)(-3-\lambda) - 2(-5) \\ &= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1). \end{aligned}$$

It is $\det(A - \lambda I) = 0$ if and only if $\lambda = 2$ or $\lambda = -1$.

$\diamond$ So, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$.

Eigenvector $\mathbf{u}_1$

Let $\mathbf{u}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ be an eigenvector associated with $\lambda_1 = 2$. Since

$$\begin{aligned}(A - \lambda_1 I)\mathbf{u}_1 &= (A - 2I)\mathbf{u}_1 = \left(\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ &= \begin{bmatrix} 4-2 & -5 \\ 2 & -3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 - 5y_1 \\ 2x_1 - 5y_1 \end{bmatrix}\end{aligned}$$

the equation

$$(A - \lambda_1 I)\mathbf{u}_1 = (A - 2I)\mathbf{u}_1 = \mathbf{0}$$

becomes:

$$\begin{bmatrix} 2x_1 - 5y_1 \\ 2x_1 - 5y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{yielding} \quad 2x_1 - 5y_1 = 0$$

or $x_1 = \frac{5}{2} y_1$. The eigenvector $\mathbf{u}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is any vector of the form $\begin{bmatrix} \frac{5}{2} y_1 \\ y_1 \end{bmatrix}$ where y_1 is any non-zero real number. Let's choose $y_1 = 2$ and hence we get

$$\mathbf{u}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad .$$

Eigenvector $\mathbf{u}_2$

Let $\mathbf{u}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be an eigenvector associated with $\lambda_2 = -1$. Then $(A - \lambda_2 I)\mathbf{u}_2 = \mathbf{0}$.

$$\begin{aligned} (A - \lambda_2 I)\mathbf{u}_2 &= (A + I)\mathbf{u}_2 = \left(\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5x_2 - 5y_2 \\ 2x_2 - 2y_2 \end{bmatrix}. \end{aligned}$$

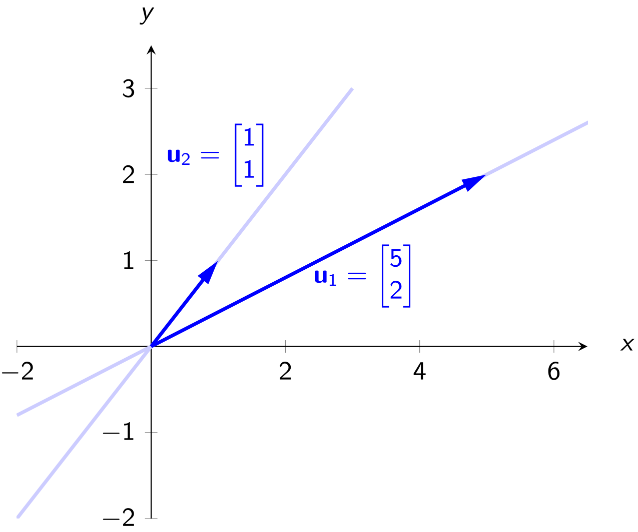
So we have

$$(A - \lambda_2 I)\mathbf{u}_2 = (A + I)\mathbf{u}_2 = \mathbf{0} \quad \text{or} \quad \begin{bmatrix} 5x_2 - 5y_2 \\ 2x_2 - 2y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This singular system reduces to one single equation $5x_2 - 5y_2 = 0$ or $y_2 = x_2$. Hence, the eigenvector $\mathbf{u}_2$ is any vector of the form $\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$, where x_2 is any non-zero real number. Setting $x_2 = 1$ we get

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad .$$

Eigenvectors



Remarks and Properties

Consider an $n \times n$ matrix A.

✏ The characteristic polynomial is an n–degree polynomial in terms of λ, which is obtained by expanding the determinant $\det(A - \lambda I)$.

✏ The characteristic polynomial $\det(A - \lambda I)$ not always has roots real numbers. When it has, the following property holds true:

✏ The determinant of the matrix A is equal to the product of its eigenvalues:

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$