Matrix Operations and Rank

MTH1004M Linear Algebra

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Matrices: Introduction

Recall from the previous week that a linear system of 3 equations with 3 unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

is denoted by the augmented matrix:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

where every row of this matrix represents an equation of the above system.

Under the 'augmented matrix notation' *is hidden* the following matrix operation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Matrices: Introduction

We re-write the linear system using 'matrix multiplication'.
Multiply each row of the matrix with the vector whose components are the unknowns, same as we did in the dot product of two vectors.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix}$$

◇ Recall that the dot product is:

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1x + b_1y + c_1z \end{bmatrix}$$

So, the linear system:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

is equivalent to:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A Linear System in Matrix Notation

Consider a linear system with n equations and m unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = b_2 \\ \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m = b_n \end{cases}$$

Using matrix notation it takes the following form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}}_{A \text{ matrix}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\mathbf{x} \text{ vector}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\mathbf{b} \text{ vector}}$$

Or simply,

$$\boxed{A \cdot \mathbf{x} = \mathbf{b}}$$

The Matrix

◇ A matrix is a rectangular array of numbers called the entries or elements.
◇ For a matrix with n rows and m columns we say that it is an $n \times m$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}$$

◇ Each element of this matrix is denoted by a_{ij}, which identifies that the element is at the i–th row and j–th column. The row number i goes from 1 to n and j goes from 1 to m.

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} \qquad \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 3 \\ -2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

2×3 matrix 3×2 matrix 3×3 matrix

◇ An $n \times n$ matrix is called a *square matrix*.
◇ All elements a_{ii} (i–th row and i–th column) in a square matrix are called *diagonal elements*.

Matrix Addition

Matrices are added like vectors, we sum elements which are at the same position.

⋄ For two $n \times m$ matrices A and B, we add each a_{ij} element of A to the element b_{ij} of B, so $A + B$ is:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2m} + b_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & & a_{nm} + b_{nm} \end{bmatrix}$$

To add the matrices A and B, both they have to be of the 'same size'.

If A is an $n \times m$ matrix, then $A + B$ is meaningful if also B is $n \times m$.

Examples

Find $A+B$, when:

$$i)\ A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$ii)\ A = \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$iii)\ A = \begin{bmatrix} 7 & 2 & -3 \\ -2 & -4 & 5 \\ 7 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 & 3 \\ -2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Examples

$A + B$ is :

$$i)\ A + B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

ii) It is not possible to perform the sum in this case. The matrices do not match.

$$iii)\ A + B = \begin{bmatrix} 7 & 2 & -3 \\ -2 & -4 & 5 \\ 7 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 3 \\ -2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 0 \\ -4 & -3 & 7 \\ 10 & 1 & 2 \end{bmatrix}$$

Scalar Multiplication

Derives from the addition properties for matrices.

$$k \begin{bmatrix} a_{11} & a_{12} & & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & & ka_{1m} \\ ka_{21} & ka_{22} & & ka_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1} & ka_{n2} & & ka_{nm} \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 6 \\ 0 & -8 & -4 \end{bmatrix}$$

Addition Properties

- Associativity: $(A+B)+C = A+(B+C)$
- Distributivity: $k(A+B) = kA + kB$
- Commutativity: $A+B = B+A$
- $A+O = A$, where O is the zero matrix (all of its entries are zero).
- $0A = O$.

⋆ Prove them as an exercise. ⋆

Hint: Show that the ij–entries on both sides (left and right) are equal.

Matrix Multiplication

◇ Multiplication of matrices is NOT multiplication of ‘element by element’!

◇ Multiplication of two matrices A and B is possible when the number of columns in the A matrix is equal to the number of rows in the B matrix.

◇ Multiplication of two matrices A and B is possible when their dimensions are:

$$[A_{n\times m}] \cdot [B_{m\times k}] = [C_{n\times k}]$$

The outcome is an $n \times k$ matrix C, where each of elements c_{ij} is the **dot product of the i–th row of matrix A with the j–th column of matrix B**:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj}$$

$$\begin{bmatrix} * & & & \\ a_{i1} & a_{i2} & \ldots & a_{im} \\ * & & & \\ * & & & \\ * & & & \end{bmatrix} \cdot \begin{bmatrix} * & b_{1j} & * & * & * & * \\ & b_{2j} & & & & \\ & \vdots & & & & \\ & b_{mj} & & & & \end{bmatrix} = \begin{bmatrix} & * & & & \\ * & c_{ij} & * & * & * \\ & * & & & \\ & * & & & \\ & * & & & \end{bmatrix}$$

A is n by m matrix B is m by k matrix C is n by k matrix.

Examples

What is the dimension of $A \cdot B$? Calculate $A \cdot B$, when:

$$i)\ A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \end{bmatrix}$$

$$ii)\ A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & 3 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \\ 5 & -2 \end{bmatrix}$$

$$iii)\ A = \begin{bmatrix} 3 & 2 & -3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 5 \end{bmatrix}$$

$$iv)\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Examples

$$i)\ A \cdot B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 4 + (-1)(-2) + 3 \cdot 7 & 2(-5) + (-1) \cdot 4 + 3 \cdot (-1) \\ 0 \cdot 4 + (-4)(-2) + (-2) \cdot 7 & 0 \cdot 5 + (-4) \cdot 4 + (-2)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & -17 \\ -6 & -14 \end{bmatrix}$$

It is $[A_{2\times3}] \cdot [B_{3\times2}] = [C_{2\times2}]$, so $A \cdot B$ is a 2×2 matrix.

$$ii)\quad A \cdot B = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & 4 \\ 7 & -1 \\ 5 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} (-1) \cdot 4 + 2 \cdot (-2) + 0 \cdot 7 + 0 \cdot 5 & (-1)(-5) + 2 \cdot 4 + 0 \cdot (-1) + 0 \cdot (-2) \\ 1 \cdot 4 + 3 \cdot (-2) + 1 \cdot 7 + (-1) \cdot 5 & 1 \cdot (-5) + 3 \cdot 4 + 1 \cdot (-1) + (-1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 13 \\ 0 & 8 \end{bmatrix}$$ and it is $[A_{2\times4}] \cdot [B_{4\times2}] = [C_{2\times2}]$

Examples

$$iii)\quad A\cdot B=\begin{bmatrix}3 & 2 & -3 & 0\end{bmatrix}\begin{bmatrix}4\\-2\\3\\5\end{bmatrix}=3\cdot 4+2\cdot(-2)+(-3)\cdot 3+0\cdot 5=0$$

It is $[A_{1\times 4}]\cdot[B_{4\times 1}]=[C_{1\times 1}]$, so $A\cdot B$ is a number.

$$iv)\quad A\cdot B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}b & a\\ d & c\end{bmatrix}=\begin{bmatrix}a\cdot b+b\cdot d & a^2+b\cdot c\\ c\cdot b+d^2 & c\cdot a+d\cdot c\end{bmatrix}$$

It is $[A_{2\times 2}]\cdot[B_{2\times 2}]=[C_{2\times 2}]$, so $A\cdot B$ is a 2×2 matrix.

Multiplication Properties

▷ Associativity: $(AB)C = A(BC)$

▷ Distributivity: $C(A+B) = CA + CB$

▷ Distributivity: $(A+B)C = AC + BC$

Question: Which are the possible dimensions for A, B and C in each of the above properties?

⋆ Commutativity does not hold in matrix multiplication (generally)! ⋆

▷ Commutativity: $AB \neq BA$

◇ Consider that A is $n \times m$, B is $m \times k$, and that $n \neq k$. Then obviously the multiplication BA is not possible.

◇ When A, B are both square matrices, the operations AB and BA are possible, however still it is $AB \neq BA$.

$AB \neq BA$

Let $$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 8 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 10 & -1 \end{bmatrix}$$

Special Matrices

Identity Matrix: I_n or I is the square matrix with all diagonal elements equal to 1 and all the rest equal to 0.

$$I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

Property: $I \cdot A = A \cdot I = A$

Diagonal Matrix: is the square matrix with all off-diagonal elements equal to 0.

$$D = \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix}$$

Question

Is $AD = DA$ true for any $n \times n$ diagonal matrix D and any $n \times n$ matrix A?

It is $AD =$

$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & & d_n a_{1n} \\ d_1 a_{21} & d_2 a_{22} & & d_n a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 a_{n1} & d_2 a_{n2} & & d_n a_{nn} \end{bmatrix}$$

However, $DA =$

$$\begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & & d_2 a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_n a_{n1} & d_n a_{n2} & & d_n a_{nn} \end{bmatrix} \neq AD$$

Matrix Powers

Let A be an $n \times n$ square matrix. Matrix powers of A are:

$$A^2 = A \cdot A, \quad A^3 = A^2 \cdot A, \quad \ldots, \quad A^m = A^{m-1} \cdot A \quad \text{and} \quad A^0 = I \ .$$

So

$$\boxed{A^m = \underbrace{A \cdot A \ldots \cdot A}_{m \ \ times}}$$

◇ The Matrix Powers follow the same rules as numbers:

$$(A^p)^q = (A^q)^p = A^{pq} \quad .$$

In particular, it is: $I^m = I$ and $O^m = O$, where I is the identity matrix and O is the zero matrix, respectively.

◇ $(kA)^m = k^m A^m$. Why?

Question

If D is an $n \times n$ diagonal matrix, then are $D^2, D^3, \ldots, D^m$ also diagonal?

$$D^2 = \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & & 0 \\ 0 & d_2^2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} d_1^2 & 0 & & 0 \\ 0 & d_2^2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n^2 \end{bmatrix} \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix} = \begin{bmatrix} d_1^3 & 0 & & 0 \\ 0 & d_2^3 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n^3 \end{bmatrix}$$

By induction can be proved that also D^m is diagonal and its elements are d_i^m.

Exercise

Find a, b, c such that the following matrices commute $AB = BA$:

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and for those values find A^m.

It is

$$AB = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ b & b+c \end{bmatrix}$$

while also

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a+b & c \\ b & c \end{bmatrix}$$

So $AB = BA$ implies $b = 0$ and $a = c$, therefore

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI \quad .$$

Now, for $b = 0$ and that $a = c$ is $A^m = (aI)^m = a^m I^m = a^m I$. $\square$

Exercise

(i) Expand the expressions: $(A+B)^2$ *and* $(A+B)^3$.

(ii) $(AB)^2 = A^2B^2$. *True or false?*

(i) It is $(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2$ and careful, $(A+B)^2 \neq A^2 + 2AB + B^2$

In the same way, $(A+B)^3 = (A+B)(A+B)^2 = (A+B)(A^2+AB+BA+B^2)$

$$= A(A^2 + AB + BA + B^2) + B(A^2 + AB + BA + B^2)$$
$$= (A^3 + A^2B + ABA + AB^2) + (BA^2 + BAB + B^2A + B^3)$$
$$= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3$$

(ii) $(AB)^2 = (AB)(AB) = ABAB$ and $(AB)^2 \neq A^2B^2$!
Because we have associativity and we remove all parentheses, but we don't have commutativity.

Remark

The matrix equation $A^2 = O$ does not imply $A = O$. For example, evaluate A^2 when A is the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It is $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$, even though $A \neq O$.

Transpose of a Matrix

... is when the rows of a matrix A become the columns of the other matrix.

For an $n \times k$ matrix $A = (a_{ij})$ its transpose is a $k \times n$ matrix, denoted by A^T, with elements $A^T = (a_{ji})$

Example:

$$\text{If} \quad A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$$

Properties of the Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A \cdot B)^T = B^T \cdot A^T$
- $(\lambda A)^T = \lambda A^T$, λ in $\mathbb{R}$
- If D is a diagonal matrix then $D^T = D$

Properties of the Matrix Transpose

Show that $(A \cdot B)^T = B^T \cdot A^T$

Proof:
Let's denote the i–th row of a matrix A by $\mathrm{row}_i(A)$, its j–th column by $\mathrm{col}_j(A)$, and the same for the matrix B. Then, the i, j element of $(AB)^T$ is:

$$
\begin{aligned}
\left[(AB)^T\right]_{ij} &= (AB)_{ji} \\
&= \mathrm{row}_j(A) \cdot \mathrm{col}_i(B) \\
&= \mathrm{col}_j\left(A^T\right) \cdot \mathrm{row}_i\left(B^T\right) \\
&= \mathrm{row}_i\left(B^T\right) \cdot \mathrm{col}_j\left(A^T\right) = \left[B^T A^T\right]_{ij}
\end{aligned}
$$

(since the dot product is commutative) is equal to the i, j element of $B^T A^T$ for any i and j.

Exercise

Find the transpose of the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \\ 1 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & 2 \\ 3 & 2 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad M^2 \quad .$$

Is $(M^2)^T = (M^T)^2$ *?*

It is $A^T = \begin{bmatrix} 4 & 2 & 1 \\ -1 & 3 & 1/2 \end{bmatrix}$, $B^T = [1 \ -3 \ 5]$, $C^T = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & 2 \\ 3 & 2 & -2 \end{bmatrix} = C$

Notice that $C^T = C$.

It is $M^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ so $(M^2)^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ while ,

$$(M^T)^2 = M^T M^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{so} \quad (M^2)^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Therefore, it holds $(M^2)^T = (M^T)^2$.

Remarks

- ✏ *Vectors* are $n \times 1$ matrices.
- ✏ The *dot product* of two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

can be represented by matrix multiplication in the following way:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \mathbf{u}^T \mathbf{v}$$

Special Matrices

- *Symmetric Matrix* is a square matrix whose transpose is the same matrix $A^T = A$. For those matrices holds that $a_{ij} = a_{ji}$ for all i, j.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad \text{then also} \quad A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

- *Triangular Matrix* is a square matrix with zero elements below or above the diagonal. If the zero elements are below the diagonal is called upper triangular matrix and if the zero elements are above the diagonal is called lower triangular:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Upper triangular matrix — Lower triangular matrix

Examples

▷ Symmetric matrices:

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = A, \quad B^T = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix} = B, \quad C^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C$$

▷ Let A be an $n \times n$ matrix. Show that $B = A + A^T$ is always a symmetric matrix

It is $B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$, so B is symmetric.

Trace of a Matrix

The trace of a square $n \times n$ matrix A is the sum of the diagonal elements

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

and is denoted by $\mathrm{tr}(A)$.

Examples:

1. let $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 4 & 4 \\ 1 & 5 & -2 \end{bmatrix}$. The it is $\mathrm{tr}(A) = 1 + 4 - 2 = 3$.

2. Let $B = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$, then $\mathrm{tr}(B) = 0 + 0 = 0$.

3. For $D = \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{bmatrix}$ we have that $\mathrm{tr}(D) = d_1 + d_2 + \ldots + d_n$
$= \sum_{i=1}^{n} d_i$.

Remarks

❖ The matrices:

$$A = \begin{bmatrix} d_1 & a_1 & a_2 \\ b_1 & d_2 & a_3 \\ b_2 & b_3 & d_3 \end{bmatrix}, \ B = \begin{bmatrix} d_1 & c_1 & c_2 \\ 0 & d_2 & c_3 \\ 0 & 0 & d_3 \end{bmatrix}, \ D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

have the same trace, namely $\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(D) = d_1 + d_2 + d_3$.

❖ The $n \times n$ zero matrix O_n and the $n \times n$ identity matrix I_n have traces:

$$\mathrm{tr}(O_n) = \underbrace{0 + 0 + \ldots + 0}_{n \quad times} = 0$$

$$\mathrm{tr}(I_n) = \underbrace{1 + 1 + \ldots + 1}_{n \quad times} = n$$

Properties of the Trace

Let A and B be $n \times n$ square matrices.

- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

$\mathrm{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \ldots + (a_{nn} + b_{nn}) =$
$(a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) = \mathrm{tr}(A) + \mathrm{tr}(B)$

- $\mathrm{tr}(kA) = k\mathrm{tr}(A)$

$\mathrm{tr}(kA) = ka_{11} + ka_{22} + \ldots + ka_{nn} = k(a_{11} + a_{22} + \ldots + a_{nn}) = k\mathrm{tr}(A)$

- $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

- But $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$ and $tr(A^n) \neq \mathrm{tr}(A)^n$!

Example

Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *and that* $\mathrm{tr}(A^2) \neq \mathrm{tr}(A)^2$

It is

$$AB = \begin{bmatrix} -4 & 13 \\ -1 & 12 \end{bmatrix}, \text{ and } BA = \begin{bmatrix} 7 & 14 \\ 3 & 1 \end{bmatrix}$$

where $\mathrm{tr}(AB) = -4 + 12 = 8$ and $\mathrm{tr}(BA) = 7 + 1$.

While

$$A^2 = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 20 \\ 8 & 19 \end{bmatrix}$$

where $\mathrm{tr}(A^2) = 30 \neq 16 = \mathrm{tr}(A)^2 \quad \square$

Rank of a Matrix

Definition The rank of a matrix A is the number of nonzero rows in the (reduced) row-echelon form. We denote by $\mathrm{rank}(A)$.

✏ [Or equivalently] The rank of a matrix is the number of pivots in the (reduced) row-echelon form.

Theorem Elementary row operations do not change the rank of a matrix.

Rank of a Matrix: Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank} = 3 \qquad \text{rank} = 2 \qquad \text{rank} = 2 \qquad \text{rank} = 3 \qquad \text{rank} = 1$$

Rank and Transpose

Theorem The rank of a matrix is equal to the rank of its transpose.

$$\mathrm{rank} A^T = \mathrm{rank} A$$

Remark: Any square matrix B has the same row echelon form with its transpose

Example

Find the rank of the following upper and lower triangular matrices

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a^2+1 & 0 & 0 \\ -1 & b^2+1 & 0 \\ 2 & -1 & c^2+1 \end{bmatrix}$$

The matrix A is in a row echelon form, so following the definition of the rank

$$\mathrm{rank} A = 3$$

The matrix B is not in row echelon form. However, the matrix B^T becomes an upper triangular matrix which is in row echelon form. Since all diagonal elements are non-zero, B^T has 3 nonzero rows and from theorem,

$$\mathrm{rank} B^T = \mathrm{rank} B = 3$$