Solving Linear Systems

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Introduction

Linear Algebra is the branch of Mathematics that has grown from a thorough study on solving linear systems of equations.

Here is an easy example of a linear system:

$$\begin{cases} 2x + y = 0 \\ x - y = 3 \end{cases}.$$

$$\begin{cases} 2x + y = 0 \\ x - y = 3 \end{cases} \xrightarrow[-2R_2]{} \begin{cases} 2x + y = 0 \\ -2x + 2y = -6 \end{cases} \xrightarrow[R_1 + R_2]{} \begin{cases} 2x + y = 0 \\ 0x + 3y = -6 \end{cases}$$

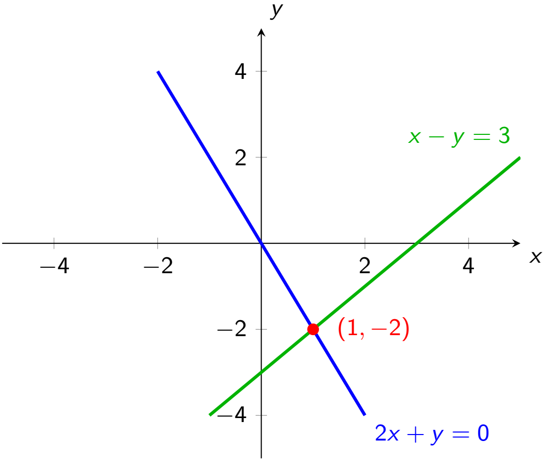
$$\longrightarrow \begin{cases} 2x + y = 0 \\ y = -2 \end{cases} \longrightarrow \begin{cases} 2x - 2 = 0 \\ y = -2 \end{cases} \longrightarrow \begin{cases} x = 1 \\ y = -2 \end{cases} \quad \text{unique solution}$$

Introduction

The two lines of the system

$$\begin{cases} 2x + y = 0 \\ x - y = 3 \end{cases}$$

intersect at the solution point.

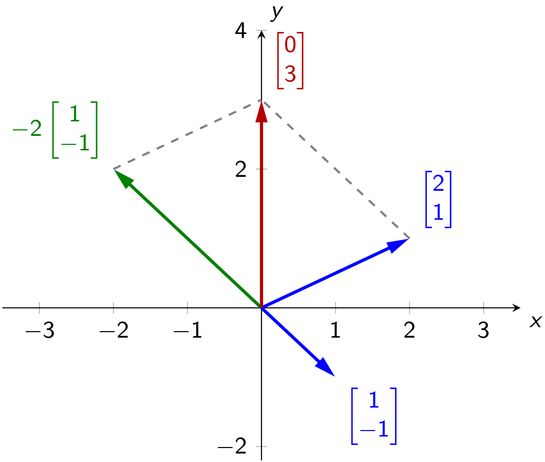


Linear systems as vector equations

On the other hand, the system is equivalent to the vector equation:

$$\begin{cases} 2x + y = 0 \\ x - y = 3 \end{cases} \longrightarrow \quad x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{1}$$

If the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent their linear combinations create any vector in the plane. So, they also create the vector in the right-hand side of (1), namely $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

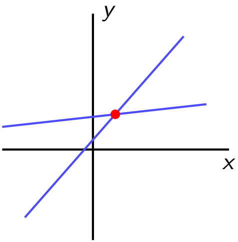


The vector $\begin{bmatrix}0\\3\end{bmatrix}$ is the linear combination of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ since there are scalars $x = 1, y = -2$ such that:

$$1 \cdot \begin{bmatrix}2\\1\end{bmatrix} + (-2) \cdot \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

Solving Linear Systems

The general form of a system with 2 equations and 2 unknowns in $\mathbb{R}^2$ is:



- The set of equations:

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are real numbers and x, y the unknowns.

- Geometrically represented by two lines in the plane.

We examine whether the system admits unique solution or not.

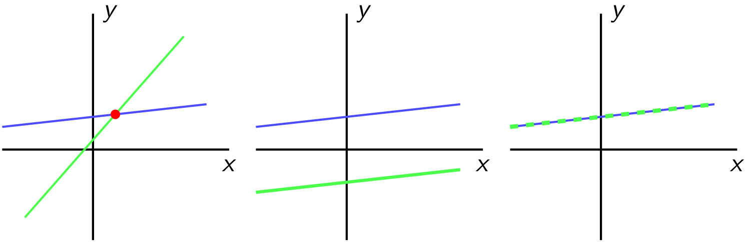
◇ By linear equations we mean equations of this form: $2x + 3y - z = -1$.

◇ An equation of the form $2x^2 + 3y^2 - z = -1$ is non-linear.

Lines and Planes

- The equation $ax + by = c$ is a **line** in $\mathbb{R}^2$.
- The equation $ax + by + cz = d$ is a **plane** in $\mathbb{R}^3$.
- The system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ contains two lines in the plane,

which may *intersect* or may *not intersect*:



Unique solution *No solution* *Infinitely many solutions*

Example I

Solve the system: $\begin{cases} 2x - y = 3 \\ x + 3y = 5 \end{cases}$.

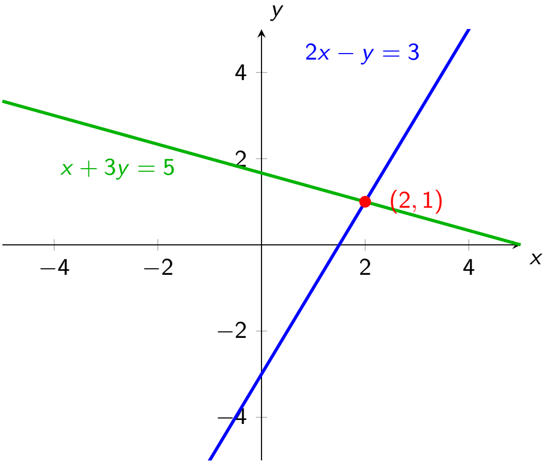
A strategy is to eliminate y from the first equation and x from the second. We are allowed to use scalar multiplication, add the equations and exchange their ordering. Let's eliminate x from the second equation:

$$\begin{cases} 2x - y = 3 \\ \\ -2(x+3y) = -10 \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ \\ -2x - 6y = -10 \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ \\ 0x - 7y = -7 \end{cases}$$

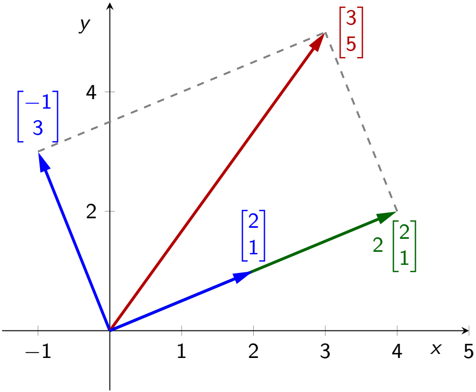
$$\longrightarrow \begin{cases} 2x - y = 3 \\ \\ y = 1 \end{cases} \longrightarrow \begin{cases} 2x - 1 = 3 \\ \\ y = 1 \end{cases} \longrightarrow \begin{cases} x = 2 \\ \\ y = 1 \end{cases} \quad \text{unique solution !}$$

Example I

The two lines $\begin{cases} 2x - y = 3 \\ x + 3y = 5 \end{cases}$ intersect at the solution point $x = 2, y = 1$.



Does $\begin{bmatrix}3\\5\end{bmatrix}$ belong to *span*$\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right)$?



Since $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}$ are linearly independent, they also produce the vector $\begin{bmatrix}3\\5\end{bmatrix}$:

$$2 \cdot \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

Example II

Solve the system: $\begin{cases} 2x - y = 3 \\ x - \frac{1}{2}y = 5 \end{cases}$.

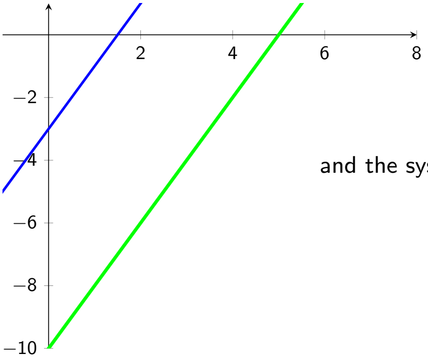
Here is an example with non-unique solution. Following the same procedure, we get:

$$\begin{cases} 2x - y = 3 \\ -2(x - \frac{1}{2})y = -10 \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ -2x + y = -10 \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ 0x + 0y = -7 \end{cases}$$

The second equation $0x + 0y = -7$ has no solution, therefore the system admits no solution.

Example II

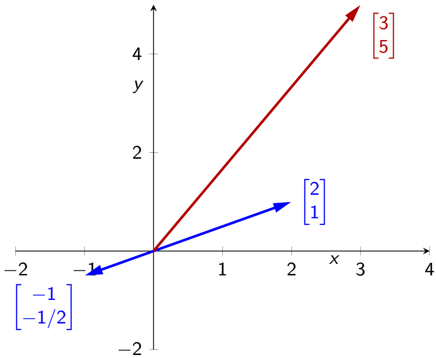
The two lines $\begin{cases} 2x - y = 3 \\ x - \frac{1}{2}y = 5 \end{cases}$ do not intersect.



and the system admits no solution.

Example II

In terms of Linear Algebra you can easily spot whether the system admits a solution or not.



The vectors $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\-1/2\end{bmatrix}$ are linearly *dependent*, so they span a line which excludes the vector $\begin{bmatrix}3\\5\end{bmatrix}$.

Example III

Solve the system: $\begin{cases} 2x - y = 3 \\ -x + \frac{1}{2}y = -\frac{3}{2} \end{cases}$.

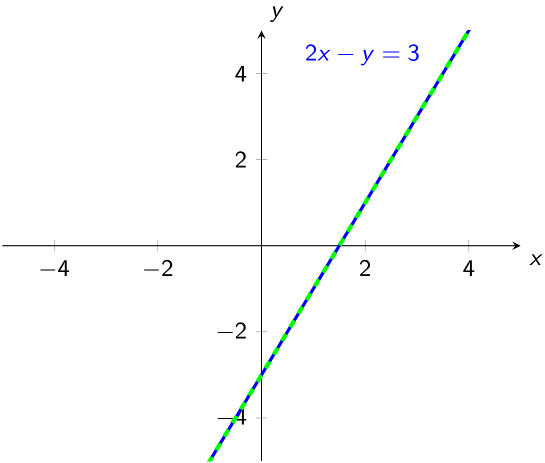
$$\begin{cases} 2x - y = 3 \\ \\ 2(-x + \frac{1}{2}y) = -2\frac{3}{2} \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ \\ -2x + y = -3 \end{cases} \longrightarrow \begin{cases} 2x - y = 3 \\ \\ 0x + 0y = 0 \end{cases}$$

The second equation $0x + 0y = 0$ admits infinitely many solutions. Only the first equation provides information; the solution set for this system:

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : y = 2x - 3,\ x \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ 2x - 3 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$$

Example III

The two lines $\begin{cases} 2x - y = 3 \\ -x + \frac{1}{2}y = -\frac{3}{2} \end{cases}$ are essentially one line.



Solving Linear Systems with Elimination Methods

Solve the following system:

$$\begin{cases} x - 2y - 3z = 2 \\ x + 2y + 2z = 5 \\ -x - y - z = 1 \end{cases}$$

Ideally, we would like to eliminate y, z from the first equation and x, z from the second and x, y from the third. The strategy is, first we try to eliminate x from the last two equations and then y from the third equation. This procedure will reduce the system to an upper triangular form.

We can start from the last equation R_3: x can be eliminated from R_3 if we add to it the second R_2:

$$\begin{cases} x - 2y - 3z = 2 \\ x + 2y + 2z = 5 \\ y + z = 6 \end{cases}$$

Step 1

We can eliminate x from the second equation by subtracting R_1 from R_2:

$$\begin{cases} x - 2y - 3z = 2 \\ \quad\;\; 4y + 5z = 3 \\ \qquad\;\; y + z = 6 \end{cases}$$

Now we can eliminate y from the third equation by adding $-R_2/4$ to R_3. The last equation then becomes $-1/4z = 6 - 3/4$ or simply $z = -21$:

$$\begin{cases} x - 2y - 3z = 2 \\ \quad\;\; 4y + 5z = 3 \\ \qquad\qquad z = -21 \end{cases}$$

Our system has been reduced to **an upper triangular system**!

Step 2

The next step is **back-substitution**: We start from below and substitute z in the second and, y and z in the first one. So:

$$\begin{cases} x - 2y - 3z = 2 \\ 4y + 5(-21) = 3 \\ z = -21 \end{cases}$$

or

$$\begin{cases} x - 2(27) - 3(-21) = 2 \\ y = 27 \\ z = -21 \end{cases}$$

or

$$\begin{cases} x = -7 \\ y = 27 \\ z = -21 \end{cases}$$

which is the *unique solution* of the system.

Elimination Procedure

In our computations we used the following steps

For the system:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

◇ The elimination procedure produces an **upper triangular system**.

$$\begin{aligned} \alpha_1 x + \beta_1 y + \gamma_1 z &= \delta_1 \\ \beta_2 y + \gamma_2 z &= \delta_2 \\ \gamma_3 z &= \delta_3 \end{aligned}$$

◇ Then the system is solved from the bottom upwards and this procedure is called **back substitution**.

Elimination Procedure

$\diamond$ If the above system admits unique solution then back substitution will result to:

$$\begin{aligned} 1x + 0y + 0z &= k_1 \\ 1y + 0z &= k_2 \\ 1z &= k_3 \end{aligned}$$

which implies the system's solution is $(x, y, z) = (k_1, k_2, k_3)$.

Now we will use a little different notation to solve linear systems...

Elimination in Matrix Notation: The Gauss–Jordan method

We repeat these steps using matrix notation. For the system:

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

◇ We construct a matrix whose entries are all coefficients:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Every row of this matrix represents an equation of the above system. (Or every column of this matrix represents the coefficients of x, y and z, respectively.)

◇ We proceed exactly as before, by performing elementary row operations such as scalar multiplication and addition of rows in order to reduce the matrix in the so-called *echelon form*

$$\left[\begin{array}{ccc|c} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ 0 & \beta_2 & \gamma_2 & \delta_2 \\ 0 & 0 & \gamma_3 & \delta_3 \end{array}\right]$$

Elimination in Matrix Notation: The Gauss–Jordan method

$\diamond$ Using back substitution we solve the system. If the system admits unique solution, then the procedure stops when the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right]$$

and the last column indicates that the solution is $(x, y, z) = (k_1, k_2, k_3)$.

Remarks:
$\diamond$ The strategy is to create a leading entry in a column and then use it to create zeros below it.
$\diamond$ Although not strictly necessary, it is often convenient to make each leading entry a 1.

Elementary Row Operations

The following elementary row operations can be performed on a matrix:

1. Interchange two rows.

2. Multiply a row by a nonzero constant [scalar multiplication].

3. Add a multiple of a row to another row [linear combination of rows].

The process of applying elementary row operations to bring a matrix into row echelon form, called row reduction, is used to reduce a matrix to echelon form.

Row Echelon Form

DEFINITION The pivot of a row is the first non–zero entry (reading from left to right).

DEFINITION A matrix is in a row echelon form if:

(a) Any rows consisting entirely of zeros are at the bottom.

(b) In each non-zero row, the pivot is in a column to the left of any pivots below it.

✏ All entries below a pivot are zero!

✏ A row echelon matrix resembles a staircase. Some 3×3 examples are the following:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Form

$$
\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\quad
\begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}
$$

Reduced Row Echelon Form

The row echelon form simplifies the linear system and makes it easier to see the solution. However, the augmented matrix can simplify even further. It can take a *reduced row echelon form*, that is the simplest form for an augmented matrix.

DEFINITION A matrix is reduced row echelon form (RREF) if
(a) it is in row echelon form (REF)
(b) all pivots are 1
(c) in any column with a pivot all other entries are zero.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

Reduced Row Echelon Form

Row Echelon Form of the matrices

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

and their Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 29 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Elimination in Matrix Notation

Solve the following system:

$$\begin{cases} x + 2y - z = 1 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$

We first construct the *augmented matrix*:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{-4R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_3/4} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example

Solve the following system:

$$\begin{cases} 3y + 2z = 0 \\ x + y + z = 1 \\ 3x + 2y + z = -2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 3 & 2 & 1 & -2 \end{array}\right] \xrightarrow[-3R_1 + R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & -2 & -5 \end{array}\right]$$

$$\xrightarrow[R_2 + 3R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -4 & -15 \end{array}\right] \xrightarrow[R_3/(-4)]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 15/4 \end{array}\right] \xrightarrow{-2R_3 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & -15/2 \\ 0 & 0 & 1 & 15/4 \end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & 15/4 \end{array}\right] \xrightarrow{R_1 - R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & 15/4 \end{array}\right]$$

Example

Solve the system with 3 equations and 4 unknowns:

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 2 \\ 2x_1 + 4x_2 - 4x_3 + 6x_4 = 10 \\ 3x_1 + 6x_2 - 6x_3 + 9x_4 = 13 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 3 & 6 & -6 & 9 & 13 \end{array}\right] \xrightarrow[-3R_1 + R_3]{} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{array}\right]$$

$$\xrightarrow[-R_2 + R_3]{} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[-R_2 + R_3]{}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

which implies that

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$$

so the systems admits no solution.

Example

Find a value for c which gives: i) infinitely many solutions and ii) no solution to the following system:

$$\begin{cases} 3x + 2y = 10 \\ 6x + 4y = c \end{cases}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 6 & 4 & c \end{array}\right] \xrightarrow[R_2/2]{} \left[\begin{array}{cc|c} 3 & 2 & 10 \\ 3 & 2 & c/2 \end{array}\right] \xrightarrow[-R_1 + R_2]{} \left[\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & c/2 - 10 \end{array}\right].$$

The value of $c/2 - 10$ can determine whether the system admits infinitely many solutions or no solution, since the second equation has become

$$0x + 0y = c/2 - 10$$

(i) If $c/2 - 10 = 0$, so if $c = 20$, the system admits the following solutions:

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : 3x + 2y = 10 \right\} = \left\{ \begin{bmatrix} x \\ 5 - \frac{3x}{2} \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$$

(ii) $c \neq 20$ the system admits no solution.

Example

Determine whether the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *belongs to the span of the vectors:*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

The question is the same with: *Is the vector* $\mathbf{u}$ *a linear combination of* $\mathbf{v}, \mathbf{w}$*?*
So, let x, y be scalars, then:

$$x \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and we determine whether the system admits unique solution in terms of x and y...

Example

... so, we construct the augmented matrix and use Gauss–Jordan elimination:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \underset{R_3/3}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{array}\right] \underset{-R_1+R_3}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] (\textit{echelon})$$

$$\overset{R_1+R_2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

So, the system admits the unique solution $x = 3$ and $y = 2$ and the answer is yes, the vector $\mathbf{u}$ belongs to $span(\mathbf{v}, \mathbf{w})$, since it is a linear combination of $\mathbf{v}, \mathbf{w}$:

$$\mathbf{u} = 3\mathbf{v} + 2\mathbf{w} \ .$$

Homogeneous Systems

Definition A linear system is called homogeneous system, if the constant term in each equation is equal to zero.

Examples

$$\begin{cases} 2x + y - z = 0 \\ -x + 2y + z = 0 \\ x - y + 2z = 0 \end{cases}, \quad \begin{cases} 3x - 5y + 7z = 0 \\ -x + 4y - 3z = 0 \end{cases}, \quad \begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$$

Homogeneous Systems

Theorem If a homogeneous system has:

• m equations with n unknowns and $m < n$, then the system has infinitely many solutions.

• n equations with n unknowns, then has either infinitely many solutions or the zero solution.

Examples

$$\begin{cases} 2x + y - z = 0 \\ -x + 2y + z = 0 \\ x - y + 2z = 0 \end{cases} \rightarrow \text{Zero solution } x = 0, y = 0, z = 0$$

Homogeneous Systems

$$\begin{cases} 3x - 5y + 7z = 0 \\ -x + y + z = 0 \end{cases} \rightarrow \text{ Infinitely many solutions } x = 6z, y = 5z$$

We write its solutions set:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 : x = 6z, y = 5z, z \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} 6z \\ 5z \\ z \end{bmatrix} : z \text{ in } \mathbb{R} \right\}$$

$$\text{So, finally } S = span\left(\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right)$$

$$\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases} \rightarrow \text{ Infinitely many solutions } y = x$$

so the solution set is

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : y = x, x \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Solution set example 1

Find the solution set which corresponds to the following reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

We understand that the augmented matrix corresponds to a system with 2 equations and 3 unknowns:

$$\begin{cases} x - z = 0 \\ y + 2z = 0 \end{cases}$$

where the variables x, y can be easily expressed in terms of z, so $x = z, y = -2z$. The solution set then is:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 : x = z, y = -2z, z \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} : z \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

Solution set example 2

Find the solution set which corresponds to the following reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

The augmented matrix corresponds to a system with 2 equations and 4 unknowns:

$$\begin{cases} x + 2w = 0 \\ y - z = 0 \end{cases}$$

where the variables x, y can be easily expressed in terms of z, w, so $x = -2w, y = z$. The solution set then is:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ in } \mathbb{R}^4 : x = -2w, y = z, \text{ and } z, w \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} -2w \\ z \\ z \\ w \end{bmatrix} : z, w \text{ in } \mathbb{R} \right\}$$

But ...

Solution set example 2

...

$$S = \left\{ \begin{bmatrix} -2w \\ z \\ z \\ w \end{bmatrix} : z, w \text{ in } \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : z, w \text{ in } \mathbb{R} \right\}$$

Therefore, the solution set is spanned by two vectors:

$$S = span\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Example

Determine whether the following vectors are linearly independent:

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

To check linear independence we form the homogeneous linear system $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$. Then, this is equivalent to the augmented matrix form:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

Using Gauss-Jordan elimination we find that the system has the unique solution $c_1 = c_2 = c_3 = 0$ and therefore the vectors are linearly independent.