Ideas of mathematical proof

Slides Week 30

Arithmetic of infinite limits of sequences.
Limits of functions: arithmetic and sandwich theorems.
Infinite limits of functions. Limits as $x \to \infty$.
Continuous functions.

Infinite limits

Intuitively, $\lim_{n\to\infty} a_n = +\infty$

if a_n becomes arbitrarily large for all sufficiently large n.

Definitions

- $\lim_{n\to\infty} a_n = +\infty$ if for any M

  there exists $N(M)$ (depending on M)

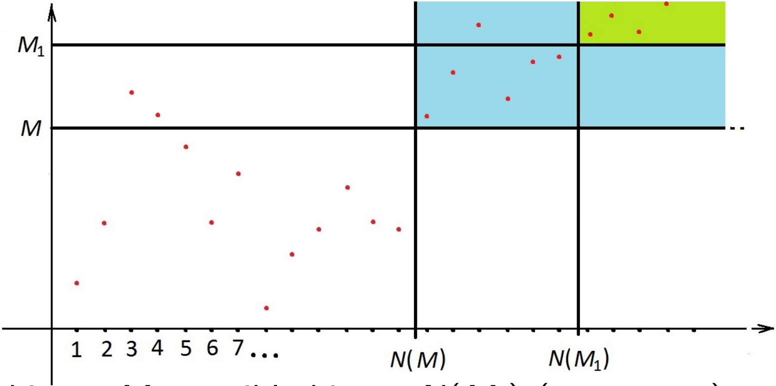
  such that $a_n > M$ for all $n > N(M)$.
- $\lim_{n\to\infty} a_n = -\infty$ if for any M

  there exists $N(M)$ (depending on M)

  such that $a_n < M$ for all $n > N(M)$.

Graph of a sequence with infinite limit

$\lim_{n\to\infty} a_n = \infty$ means: for any M there is $N(M)$ such that all points (n, a_n) of the graph on the right of $N(M)$ are in the area $(N(M), \infty) \times (M, \infty)$ (blue and green).



For bigger M_1 possibly bigger $N(M_1)$ (green area); and so on.

Example

Prove that $\lim_{n\to\infty} \log n = +\infty$ from first principles.

For any M we must find $N(M)$

such that $\log n > M$ for all $n > N(M)$.

Solve inequality for n:

using that $\log$ is an increasing function:

$\Leftrightarrow n > 10^M$.

So we put $N(M) = 10^M$,

then $\log n > M$ for all $n > N(M)$, as required.

Example

Prove that $\lim_{n\to\infty} \frac{2^n}{n^2} = +\infty$ from first principles.

For any M must find $N(M)$

such that $\frac{2^n}{n^2} > M$ for all $n > N(M)$.

Use inequality $2^n \geq n^3$ for all $n \geq 10$

(can be proved by induction, omitted here).

So for $n \geq 10$ we have $\frac{2^n}{n^2} > \frac{n^3}{n^2} = n$.

Then easy to ensure $\ldots > M$ for all $n > N(M)$:

just take $N(M) = \max\{10, M\}$.

OPTIONAL: some famous limits

$$\lim_{n\to\infty} \frac{a^n}{n^k} = \infty$$

for any constant $a > 1$ and any constant k.

E.g., $$\lim_{n\to\infty} \frac{1.00001^n}{n^{1\,000\,000}} = \infty.$$

'Any exponential is greater than any polynomial'.

Arithmetic of infinite limits of sequences

Theorem (arithmetic of infinite limits of sequences)

Suppose that $\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} b_n = +\infty$, *while* $\lim_{n\to\infty} f_n = L$ *(finite). Then*

(a) $\lim_{n\to\infty} (a_n + b_n) = +\infty;$

(b) $\lim_{n\to\infty} (a_n + f_n) = +\infty;$

(c) $\lim_{n\to\infty} (a_n \cdot b_n) = +\infty;$

CONTINUED: arithmetic of infinite limits

$\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} b_n = +\infty$,

while $\lim_{n\to\infty} f_n = L$ *(finite). Then*

(d) $\lim_{n\to\infty} \frac{1}{a_n} = 0$;

(e) *if in addition* $L > 0$, *then* $\lim_{n\to\infty} a_n f_n = +\infty$;
if in addition $L < 0$, *then* $\lim_{n\to\infty} a_n f_n = -\infty$;

(f) *if in addition* $L = 0$ *and* $f_n > 0$ *for all* n, *then* $\lim_{n\to\infty} \frac{1}{f_n} = +\infty$;

. . . .

Dots — several other combinations.

We will prove only some parts

(only those proved may appear as ‘bookwork’ questions).

But all parts can be used in examples.

Warning against using $\infty \cdot 0$, $\dfrac{1}{0}$, $\dfrac{\infty}{\infty}$, $\infty - \infty$, etc.

Example

For $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = 0$,

the sequence $a_n \cdot b_n$ may not have a limit:

e.g., let $a_n = n$ and $b_n = \dfrac{(-1)^n}{n}$;

then $a_n b_n = (-1)^n$.

Here, $a_n \to \infty$, $b_n \to 0$,

but $a_n b_n = (-1)^n$, so does not have a limit.

Example

$$a_n = \frac{(-1)^n}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

But the sequence $\dfrac{1}{a_n}$ has no limit

(not even an infinite limit),

since it has arbitrarily large absolute values

both negative and positive.

"$\infty - \infty$" limit may not exist at all,

or may be 0, or may be any other number, etc.

Example

- Let $a_n = (-1)^n + n$ and $b_n = n$;

  then $a_n \to \infty$ and $b_n \to \infty$,

  while $a_n - b_n = (-1)^n$ does not have a limit.

- Let $a_n = n$ and $b_n = n$;

  then $a_n \to \infty$ and $b_n \to \infty$,

  while $a_n - b_n = 0 \to 0$.

Example (continued)

- Let $a_n = n + 7$ and $b_n = n$;

  then $a_n \to \infty$ and $b_n \to \infty$,

  while $a_n - b_n = 7 \to 7$.

- Let $a_n = n^2$ and $b_n = n$;

  then $a_n \to \infty$ and $b_n \to \infty$,

  while $a_n - b_n = n^2 - n \to \infty$.

Bounded sequences

Before proving part (b), we prove that

a convergent sequence is bounded.

Definition

A sequence (a_n) is said to be **bounded**

if there are numbers B_1, B_2

such that $B_1 \leq a_n \leq B_2$ for all n.

Convergent is bounded

Theorem

Suppose that $\exists \lim_{n\to\infty} a_n = L$ *(finite).*

Then the sequence (a_n) *is bounded:*

there are constants B_1, B_2

such that $B_1 \le a_n \le B_2$ *for all* n.

Proof: Take $\varepsilon = 1$: there exists N_1 such that

$L - 1 < a_n < L + 1$ for all $n > N_1$.

Let $B_2 = \max\{L + 1,\ a_1, a_2, \ldots, a_{N_1}\}$;

then $a_n \le B_2$ for all n.

Convergent is bounded continued

Recall: $L - 1 < a_n < L + 1$ for all $n > N_1$.

Similarly, let $B_1 = \min\{L - 1,\ a_1, a_2, \ldots, a_{N_1}\}$;
then $B_1 \le a_n$ for all n.

Together, $B_1 \le a_n \le B_2$ for all n. $\square$

Not every bounded sequence is convergent

Remark: Every convergent sequence is bounded,

but not every bounded sequence is convergent:

e.g.: $a_n = (-1)^n$.

Proof of part (b) on arithmetic of ∞ limits

$$\lim_{n\to\infty} a_n = \infty \;\; \& \;\; \lim_{n\to\infty} f_n = L < \infty \Rightarrow \lim_{n\to\infty}(a_n + f_n) = \infty.$$

For any M need $N(M)$ s.t. $a_n + f_n > M$ for $n > N(M)$.

Use earlier theorem: convergent is bounded:

there are B_1, B_2 such that $B_1 < f_n < B_2$ for all n.

Since $\lim_{n\to\infty} a_n = +\infty$, for $M - B_1$ there is $N_1(M - B_1)$

such that $M - B_1 < a_n$ for all $n > N_1(M - B_1)$.

Choose $N(M) = N_1(M - B_1)$. Take the sum:

$M = M - B_1 + B_1 < a_n + f_n$ for all $n > N(M)$,
as required. $\square$

Proof of part (d) on arithmetic of ∞ limits

$$\lim_{n\to\infty} a_n = +\infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

For any $\varepsilon > 0$ we need to find $N(\varepsilon)$

such that $|1/a_n - 0| < \varepsilon$ for all $n > N(\varepsilon)$.

Since $\lim\limits_{n\to\infty} a_n = +\infty$, there is $N_1(1/\varepsilon)$

such that $a_n > 1/\varepsilon$ for all $n > N_1(1/\varepsilon)$.

Then $\varepsilon > 1/a_n > 0$, whence $|1/a_n - 0| < \varepsilon$.

So we can put $N(\varepsilon) = N_1(1/\varepsilon)$ to satisfy the definition:

$|1/a_n - 0| < \varepsilon$ for all $n > N(\varepsilon)$, as required. $\square$

Example

We had $\lim_{n\to\infty} \dfrac{2^n}{n^2} = +\infty$.

Now, $\lim_{n\to\infty} \dfrac{n^2}{2^n} = 0$ by Arithmetic of infinite limits.

Example

We had $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$

and <u>all terms are positive.</u>

Hence, $\lim_{n\to\infty} \sqrt{n} = +\infty$ by Arithmetic of infinite limits.

Tricky example

Example

Is there a limit $\lim(\sqrt{n+1} - \sqrt{n})$?

Both $\to +\infty$,

so Arithmetic Theorem cannot be applied.

Instead, some preparation first will help.

Example

Is there a limit $\lim(\sqrt{n+1}-\sqrt{n})$?

$$\sqrt{n+1}-\sqrt{n}=\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}=\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

On the right, both $\sqrt{n+1}\to+\infty$ and $\sqrt{n}\to+\infty$.

Apply Arithmetic of infinite limits: sum $\to+\infty$,

then ratio $\to 0$. Thus, $\exists\ \lim(\sqrt{n+1}-\sqrt{n})=0$.

Limits of functions

Informally: $\lim_{x \to a} f(x) = L$

if $f(x)$ "approaches" L "arbitrarily closely"

for x "sufficiently close to a" ($\underline{\text{but } x \neq a}$).

Definition of a limit of a function

Definition

A function f has a finite limit L at point a,

denoted $\lim_{x \to a} f(x) = L$

(other notation: $f(x) \to L$ as $x \to a$)

if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$

such that $|f(x) - L| < \varepsilon$

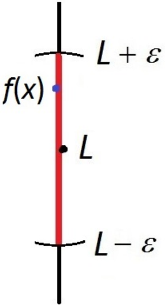
for any x such that $0 < |x - a| < \delta(\varepsilon)$.

Picture for a limit of a function

$$\lim_{x \to a} f(x) = L \quad \text{(other notation: } f(x) \to L \text{ as } x \to a)$$

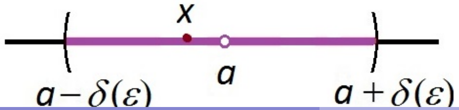
if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$

such that



$$|f(x) - L| < \varepsilon$$

for any x such that $0 < |x - a| < \delta(\varepsilon)$.



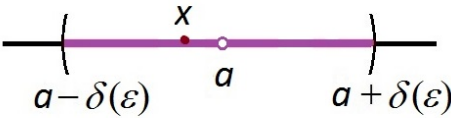
Logical quantifiers for lim $f(x)$

.....In other words: $\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0$

such that $L - \varepsilon < f(x) < L + \varepsilon$

whenever $a - \delta(\varepsilon) < x < a + \delta(\varepsilon)$ and $x \neq a$;

that is, when $x \in (a - \delta(\varepsilon),\, a) \cup (a,\, a + \delta(\varepsilon))$.



The set $(a - \delta,\, a) \cup (a,\, a + \delta)$ is often called

the **punctured δ-neighbourhood of a**.

$\forall \varepsilon > 0\, \exists \delta(\varepsilon) > 0\; \big(0 < |x - a| < \delta(\varepsilon) \Rightarrow |L - f(x)| < \varepsilon\big)$.

Why not for $x = a$?

Recall: $\lim_{x \to a} f(x) = L$ if for any $\varepsilon > 0$

there is $\delta(\varepsilon) > 0$ such that $\lim L - \varepsilon < f(x) < L + \varepsilon$

whenever $a - \delta(\varepsilon) < x < a + \delta(\varepsilon)$ and $\underline{x \neq a}$.

Deliberately <u>no requirement on $f(x)$ for $x = a$</u>.

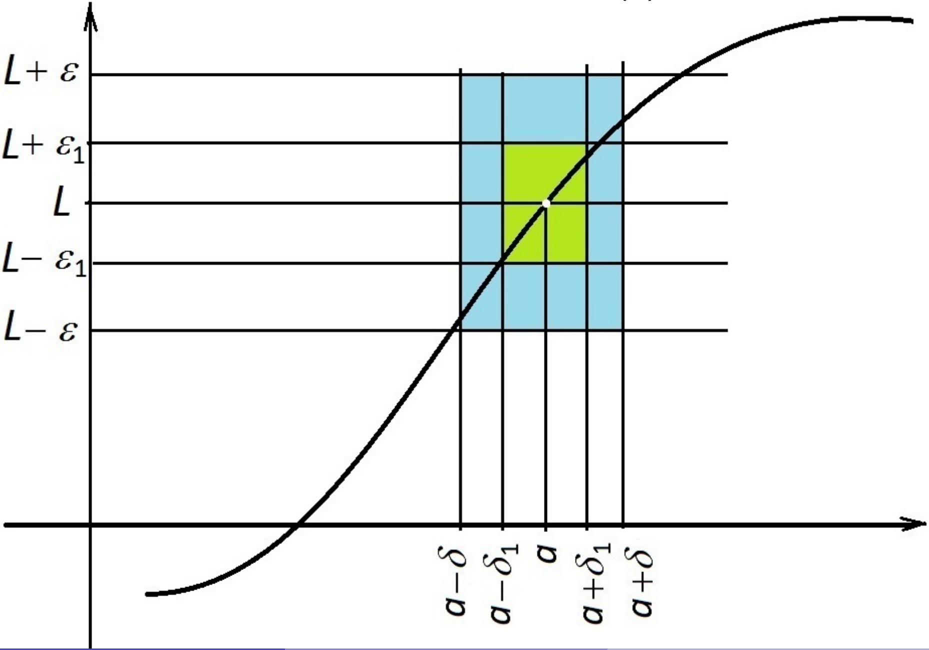
Function is allowed to be not defined at $x = a$:

important for derivative: $g'(a) = \lim_{x \to a} \dfrac{g(x) - g(a)}{x - a}$

so this is $\lim_{x \to a} f(x)$ where $f(x) = \dfrac{g(x) - g(a)}{x - a}$,

and $f(x)$ is not defined at $x = a$.

Geometrically: For $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$.......



Limits from 1st principles

There are rules for deriving limits from known limits,

but there must be some basic limits, from 1st principles.

Example

Prove from 1st principles: $\lim_{x \to a} x = a$.

Given any $\varepsilon > 0$ need $\delta(\varepsilon)$

such that $|x - a| < \varepsilon$ when $0 \neq |x - a| < \delta(\varepsilon)$.

Easy: we can put $\delta(\varepsilon) = \varepsilon$.

More from 1st principles

Example

Prove from 1st principles: $\lim_{x\to 4}\sqrt{x} = 2$.

Remark: If we know that a function is continuous,

then $\lim_{x\to a} f(x) = f(a)$,

as this is definition of continuous!

So, 'easy': $\lim\limits_{x\to 4}\sqrt{x} = \sqrt{4} = 2$.

But to prove that $\sqrt{x}$ is continuous,

we need these limits, from 1st principles.

Remark on continuous functions

If it is already known that $f(x)$ is continuous at $x = a$,

then it is OK to write straight away $\lim_{x \to a} f(x) = f(a)$.

But this only works if $f(x)$ is continuous at $x = a$.

There can still be a limit if it is not, like

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

has a limit at $x = 2$ not equal to $f(2)$,

or $f(x) = x \sin(1/x)$ has a limit at $x = 0$,

although $f(x)$ is undefined at $x = 0$.

More from 1st principles

Example

Prove from 1st principles: $\lim\limits_{x \to 4} \sqrt{x} = 2.$

Proof. For any $\varepsilon > 0$ need to find $\delta(\varepsilon) > 0$

such that $|\sqrt{x} - 2| < \varepsilon$ whenever $0 < |x - 4| < \delta(\varepsilon)$.

Solve inequality for x.

Takes different forms for $x \geq 4$ and $x \leq 4$.

<u>In the area $x \geq 4$:</u> $\sqrt{x} - 2 < \varepsilon \Leftrightarrow \sqrt{x} < \varepsilon + 2$

$\Leftrightarrow x < 4 + 4\varepsilon + \varepsilon^2$.

$\lim_{x \to 4} \sqrt{x} = 2$ continued

Recall: need $|\sqrt{x} - 2| < \varepsilon$

<u>In the area $x \leq 4$:</u> $2 - \sqrt{x} < \varepsilon \Leftrightarrow 2 - \varepsilon < \sqrt{x}$

Can assume $\varepsilon < 2$: if satisfied with small ε,
then also for bigger ε with the same δ.

Then $2 - \varepsilon$ is positive and we can square the inequality:
$\Leftrightarrow x > 4 - 4\varepsilon + \varepsilon^2$.

For $\varepsilon < 2$ we can put $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2 > 0$.

Then for $|x - 4| < \delta(\varepsilon)$: both $x < 4 + 4\varepsilon + \varepsilon^2$

and $x > 4 - 4\varepsilon + \varepsilon^2$, so $|\sqrt{x} - 2| < \varepsilon$ (in both areas),

as required by the definition. $\square$

Example

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Not continuous, even undefined at $x = 0$,

so cannot just take its value at $x = 0$.

l'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

This is not OK, because proving $(\sin x)' = \cos x$ requires this very limit!

(**Proof later, without derivative.**)

Theorem (on arithmetic of finite limits of functions)

Suppose that $\lim_{x \to a} f(x) = L$ *(finite)*

and $\lim_{x \to a} g(x) = M$ *(finite). Then*

(a) $\lim_{x \to a}(f(x) + g(x)) = L + M;$

(b) $\lim_{x \to a}(f(x) \cdot g(x)) = L \cdot M;$

in particular, $\lim_{x \to a} kf(x) = kL$ *for a constant* k*;*

(c) *if in addition* $M \neq 0$ *and* $g(x) \neq 0$,

then $\lim_{x \to a} f(x)/g(x) = L/M.$

Not all parts proved in these lectures: only (a) for sum.

But all parts can be used in examples when reducing limits to previously known limits

(unless the question specifies “from 1st principles”...).

Limit of a sum

Proof of part (a) for sum. For any $\varepsilon > 0$ need $\delta(\varepsilon)$ such that $L + M - \varepsilon < f(x) + g(x) < L + M + \varepsilon$ when $0 < |x - a| < \delta(\varepsilon)$.

Using $\lim_{x \to a} f(x) = L$, for $\varepsilon/2$ find δ_1 such that $L - \varepsilon/2 < f(x) < L + \varepsilon/2$ when $0 < |x - a| < \delta_1$.

Using $\lim_{x \to a} g(x) = M$, for $\varepsilon/2$ find δ_2 such that $M - \varepsilon/2 < g(x) < M + \varepsilon/2$ when $0 < |x - a| < \delta_2$.

Limit of a sum continued

Put $\delta(\varepsilon) = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - a| < \delta(\varepsilon)$

<u>both</u> $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$,

so both $L - \varepsilon/2 < f(x) < L + \varepsilon/2$

and $M - \varepsilon/2 < g(x) < M + \varepsilon/2$.

Take the sum: $L + M - \varepsilon < f(x) + g(x) < L + M + \varepsilon$

when $0 < |x - a| < \delta(\varepsilon)$, as required. $\square$

Example

Use Arithm. of limits to show limit exists and find it:

$$\lim_{x\to 3} \frac{x^2 - 2x}{2x+5}$$

$$= \frac{\lim_{x\to 3} x^2 - \lim_{x\to 3} 2x}{\lim_{x\to 3} 2x + 5} = \frac{\left(\lim_{x\to 3} x\right)^2 - 2\lim_{x\to 3} x}{2\lim_{x\to 3} x + 5}$$

by Arithmetic of limits, <u>since limits on the right exist</u>

$$= \frac{3^2 - 2\cdot 3}{2\cdot 3 + 5} = \frac{9-6}{6+5} = \frac{3}{11},$$

since, e.g., $\lim_{x\to 3} x^2 = \lim_{x\to 3} x \cdot \lim_{x\to 3} x =$ known $= 3\cdot 3.$

Sandwich theorem for functions

Theorem (Sandwich theorem for functions)

Suppose that $\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$.

If $f(x) \le h(x) \le g(x)$ *for all* x

in some punctured δ_0*-neighbourhood of* a

(for some $\delta_0 > 0$*),*

then $\exists \lim_{x \to a} h(x) = L$.

Note: all three "as $x \to a$", for the same a.

Proof of Sandwich theorem for functions

Proof. For any $\varepsilon > 0$ need $\delta(\varepsilon) > 0$

such that $L - \varepsilon < h(x) < L + \varepsilon$

whenever $0 < |x - a| < \delta(\varepsilon)$.

Using $\lim_{x \to a} f(x) = L$, for this ε we find $\delta_1 > 0$

such that $L - \varepsilon < f(x) < L + \varepsilon$ when $0 < |x - a| < \delta_1$.

Using $\lim_{x \to a} g(x) = L$, for the same ε we find $\delta_2 > 0$

such that $L - \varepsilon < g(x) < L + \varepsilon$ when $0 < |x - a| < \delta_2$.

Proof of Sandwich theorem continued

Put $\delta(\varepsilon) = \min\{\delta_0, \delta_1, \delta_2\}$.

Then for $0 < |x - a| < \delta(\varepsilon)$ both $0 < |x - a| < \delta_0$,

and $0 < |x - a| < \delta_1$, and $0 < |x - a| < \delta_2$.

Hence all these ineq's hold: $f(x) \leq h(x) \leq g(x)$,

and $L - \varepsilon < f(x)$, and $g(x) < L + \varepsilon$

(only need 'halves' for $f(x)$ and $g(x)$).

Then $L - \varepsilon < f(x) \leq h(x) \leq g(x) < L + \varepsilon$,

so $L - \varepsilon < h(x) < L + \varepsilon$ when $0 < |x - a| < \delta(\varepsilon)$,

as required. $\square$

Example

Prove $\lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)\right) = 0.$

Note: not continuous (undefined) at 0, cannot take $f(0)$.

We have $-1 \leq \sin(1/x) \leq 1$.

Hence $-x^2 \;\leq\; x^2\sin(1/x) \;\leq\; x^2$.

By Arithmetic and known limits: $x^2 \to 0$ as $x \to 0$,

and $-x^2 \to -0 = 0$ as $x \to 0$.

By Sandwich theorem middle term has the same limit:

$\exists \lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)\right) = 0.$

Limit $\lim_{x \to 0} \dfrac{\sin x}{x}$

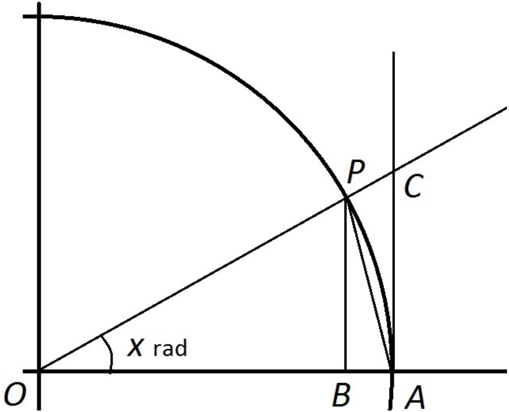
Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

We use Sandwich theorem, and continuity of $\cos x$

(which we assume here;

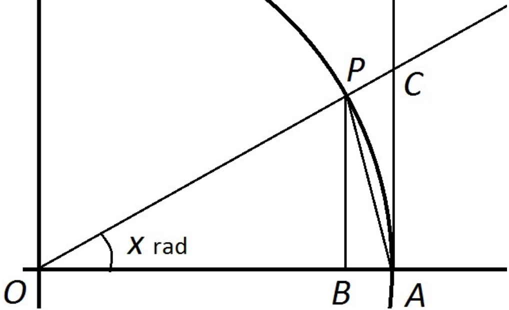
not difficult to prove from 1st principles).



$\angle AOP = x$ radians, $\quad OA = 1$, $\quad$ then $PB = \sin x$,

where PB is perpendicular to OA.

Next: CA is perpendicular to OA.



By similar triangles: $\tan x = \dfrac{PB}{OB} = \dfrac{CA}{OA} = \dfrac{CA}{1} = CA$,

so, $CA = \tan x$. Since $\triangle OAP \subseteq$ sector $OAP \subseteq \triangle OAC$,

areas: $\dfrac{OA \cdot PB}{2} \leq \dfrac{1 \cdot x}{2} \leq \dfrac{OA \cdot CA}{2} = \dfrac{1 \cdot \tan x}{2}$,

times 2: $1 \cdot \sin x \leq 1 \cdot x \leq 1 \cdot \tan x = \dfrac{\sin x}{\cos x}$, the same: $\sin x \leq x \leq \dfrac{\sin x}{\cos x}$.

From the left inequality: $\dfrac{\sin x}{x} \leq 1$.

From the right inequality: $\cos x \leq \dfrac{\sin x}{x}$.

Thus: $\cos x \leq \dfrac{\sin x}{x} \leq 1$.

Since $\cos x$ is continuous, we have $\lim\limits_{x \to 0} \cos x = \cos 0 = 1$,

and of course, $\lim\limits_{x \to 0} 1 = 1$.

Then by the Sandwich theorem the middle term also converges to the same limit: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, as required.

Above we only considered $0 < x < \pi/2$;

but this is sufficient, because $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$

and the Sandwich theorem only requires the inequality

in some punctured δ_0-neighbourhood.

Here can take $\delta_0 = \pi/2$:

on $\quad (-\pi/2, 0) \cup (0, \pi/2)$. $\square$

Infinite limits of functions

Definition

A function $f(x)$ has limit $+\infty$ at point a,

denoted $\lim_{x \to a} f(x) = +\infty$

(other notation: $f(x) \to +\infty$ as $x \to a$)

if for any M there is $\delta = \delta(M) > 0$

such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

Geometrically: vertical asymptote $x = a$

Again: $f(x)$ may not be defined at $x = a$.

Example

From 1st principles: $\displaystyle \lim_{x\to 2} \frac{1}{(x-2)^2} = +\infty.$

Proof. For any M we need $\delta(M) > 0$

such that $\displaystyle \frac{1}{(x-2)^2} > M$ when $0 < |x-2| < \delta(M)$.

Solve inequality for x.

Can assume $M > 0$, as automatically true for $M \leq 0$.

Then divide by M and multiply by $(x-2)^2$:

$$\cdots \Leftrightarrow \frac{1}{M} > (x-2)^2 \;\Leftrightarrow\; \frac{1}{\sqrt{M}} > |x-2|.$$

Just put $\delta(M) = 1/\sqrt{M}$; then inequality holds, as req.

Theorem (on arithmetic of infinite limits of functions)

Suppose that $\lim_{x\to a} f(x) = +\infty$ *and* $\lim_{x\to a} g(x) = +\infty$, *while* $\lim_{x\to a} h(x) = L$ *(finite). Then*

(a) $\lim_{x\to a}(f(x) + g(x)) = +\infty;$

(b) *if* $L > 0$, *then* $\lim_{x\to a}(f(x)\cdot h(x)) = +\infty;$

(c) *if in addition* $L = 0$ *and* $h(x) > 0$, *then* $\lim_{x\to a} 1/h(x) = +\infty;$

....

We assume the theorem without proof, can use in examples/problems

(unless specified to prove from 1st principles, by verifying the definition).

Examples show that 'limits' of type

$$+\infty + (-\infty), \quad 0 \cdot \infty, \ \text{etc.},$$

may have various values, often do not exist at all.

Example

$$f(x) = \frac{1}{(x-2)^2} + \sin\left(\frac{1}{x-2}\right), \quad g(x) = \frac{-1}{(x-2)^2}.$$

Claim: $\lim\limits_{x\to 2} f(x) = +\infty, \quad \lim\limits_{x\to 2} g(x) = -\infty,$

but $\lim\limits_{x\to 2} f(x) + g(x)$ does not exist.

For $\lim\limits_{x\to 2} f(x) = +\infty$: for any M need $\delta(M)$

such that $f(x) > M$ whenever $0 < |x-2| < \delta(M)$.

Since $|\sin| < 1$, enough to have $\dfrac{1}{(x-2)^2} > M + 1$.

Can assume $M > 1$, solve $\dfrac{1}{(x-2)^2} > M + 1$

$$\Leftrightarrow \frac{1}{M+1} > (x-2)^2 \Leftrightarrow |x-2| < \frac{1}{\sqrt{M+1}}.$$

Thus, we can put $\delta(M) = \dfrac{1}{\sqrt{M+1}}$.

For g: using $\displaystyle\lim_{x\to 2} \frac{1}{(x-2)^2} = +\infty$ proved before,

by Arithmetic of limits, $\displaystyle\lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$.

The sum $f(x) + g(x) = \sin\left(\frac{1}{x-2}\right)$

has no limit as $x \to 2$,

since in any δ-neighbourhood of 2

there are points with values $+1$ and -1:

$\frac{1}{x-2} = n\pi + \frac{\pi}{2}$ for n even and odd

$\Leftrightarrow x = 2 + \frac{1}{n\pi + \frac{\pi}{2}}$,

arbitrarily close to 2 for large enough n.

Example

$$f(x) = \frac{1}{(x-2)^2} \quad \text{and} \quad g(x) = \sin\left(\frac{1}{x-2}\right)(x-2)^2.$$

Then $\lim_{x\to 2} f(x) = +\infty$, $\qquad \lim_{x\to 2} g(x) = 0$

but $\lim_{x\to 2} f(x)g(x)$ does not exist.

We already know $\lim_{x\to 2} f(x) = +\infty$.

For $g(x)$, we have

$$-(x-2)^2 \leq \sin\left(\frac{1}{x-2}\right)(x-2)^2 \leq (x-2)^2.$$

Since $(x-2)^2$ is continuous, we have

$$\lim_{x\to 2}(x-2)^2 = 0 = \lim_{x\to 2}(-(x-2)^2).$$

Hence, $\lim\limits_{x\to 2} g(x) = 0$ by Sandwich Theorem.

For the product: $f(x)g(x) = \sin\left(\frac{1}{x-2}\right)$, which has no limit as $x \to 2$, as we saw above.

Limits of functions as $x \to \infty$

Definition

$$\lim_{x \to +\infty} f(x) = L \text{ (finite)},$$

other notation: $f(x) \to L$ as $x \to +\infty$,

if for any $\varepsilon > 0$ there is $N(\varepsilon)$

such that $|f(x) - L| < \varepsilon$ whenever $x > N(\varepsilon)$.

(Geometrically: $y = L$ is a **horizontal asymptote** of the graph $y = f(x)$ as $x \to +\infty$.)

Example

Prove from 1st principles: $\lim_{x \to +\infty} \dfrac{1}{x} = 0.$

Proof. For any $\varepsilon > 0$ we need $N(\varepsilon)$

such that $|1/x - 0| < \varepsilon$ for all $x > N(\varepsilon)$.

Solving the inequality: $\Leftrightarrow 1/|x| < \varepsilon \Leftrightarrow 1/\varepsilon < |x|$.

Thus we can put $N(\varepsilon) = 1/\varepsilon$:

then $x > 1/\varepsilon > 0 \Rightarrow |x| > 1/\varepsilon$, as required.

Theorems on arithmetic of limits as $x \to \infty$......

Sandwich theorem ....

Infinite limits as $x \to \infty$, with similar definitions and properties.........

Similar definition for limits of functions as $x \to -\infty$.....

Negation of existence of a limit

$\lim_{x \to a} f(x) = L$ means

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in (a-\delta, a) \cup (a, a+\delta) \; |f(x) - L| < \varepsilon$$

Negation: L is not a limit:

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \in (a-\delta, a) \cup (a, a+\delta) \; |f(x) - L| \geq \varepsilon$$

Negation: there is no (finite) limit at all:

$$\forall L \; \exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \in (a-\delta, a) \cup (a, a+\delta) \; |f(x) - L| \geq \varepsilon$$

Or one can use proof by contradiction.

Example

Prove that $\sin\frac{1}{x}$ has no limit as $x \to 0$.

Proof by contradiction: suppose there is a limit L.
Then for $\varepsilon = 0.3$ there is $\delta > 0$ such that
$x \in (-\delta, 0) \cup (0, \delta) \Rightarrow |\sin\frac{1}{x} - L| < 0.3$. Then

$|\sin\frac{1}{x_1} - \sin\frac{1}{x_2}| \le |\sin\frac{1}{x_1} - L| + |L - \sin\frac{1}{x_2}| < 0.3 + 0.3 = 0.6$

for any $x_1, x_2 \in (-\delta, 0) \cup (0, \delta)$.
But whatever δ, there are points $0 \neq x_{1,2} \in (-\delta, \delta)$
where $\sin\frac{1}{x}$ has values $+1$ and -1. Namely $\frac{1}{x} = n\pi + \frac{\pi}{2}$
for n even and odd, $x = \frac{1}{n\pi + \frac{\pi}{2}}$, when n is large enough.
Then $|\sin\frac{1}{x_1} - \sin\frac{1}{x_2}| = 2$ for such $x_{1,2}$ for $n_{1,2}$ even, odd.
Contradiction: hence there is no limit.

Continuous functions

Informally: $f(x)$ is continuous if you can

'draw the graph without taking the pen off the paper'.

Definition

$f(x)$ is continuous at a point $x = a$

if $\exists \lim_{x \to a} f(x)$ and it is $= f(a)$.

$f(x)$ is continuous everywhere (or on an interval)

if it is continuous at every point (of this interval).

Expanding definition of continuous

Using the definition of limit:

Definition

$f(x)$ is continuous at a point $x = a$

if $\forall \ \varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0$ such that

$|f(x) - f(a)| < \varepsilon$ when $|x - a| < \delta(\varepsilon)$.

Verifying from 1st principles similar to finding those limits

(solving inequality $|f(x) - f(a)| < \varepsilon$ for $x \ldots$)

Examples of continuous functions

$f(x) = x, \;\; \sin x, \;\; 2^x, \ldots$ are continuous everywhere;

$\dfrac{1}{x}$ is continuous on $(-\infty, 0)$ and on $(0, +\infty)$.

Some must be basic;

then Arithmetic, Sandwich theorems give more:

e.g.: $f(x) = 3x^3 - 2x + \cos x$ is continuous.

Arithmetic of continuous functions

Arithmetic for limits $\Rightarrow$ Arithmetic of continuous functions:

Theorem (Arithmetic of continuous functions)

Suppose $f(x)$ *and* $g(x)$ *are continuous at* $x = a$.

Then

(a) $f(x) + g(x)$ *is continuous at* $x = a$;

(b) $f(x) \cdot g(x)$ *is continuous at* $x = a$;

(c) *if in addition* $g(x) \neq 0$, *then* $f(x)/g(x)$ *is continuous at* $x = a$.

Proof for sum of continuous

Sum of continuous is continuous

If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

Proof: We have $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

Then by Arithmetic for limits

$$\exists \lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a),$$

$=$ value of $f(x) + g(x)$ at $x = a$, as required. $\square$

Definition of continuity via sequences

Definition

$f(x)$ is continuous at $x = a$

if for <u>every</u> sequence a_i such that $\exists\ \lim_{i\to\infty} a_i = a$,

we have $\lim_{i\to\infty} f(a_i) = f(a)$.

Theorem (assumed without proof)

The two definitions above are equivalent.

Useful for limits of sequences:

combining known limits and known continuous functions.

Example

We know $\lim_{n\to\infty} (1/n) = 0$ and assume $\cos x$ is continuous.

Then $\lim_{n\to\infty} \cos(1/n) = \cos 0 = 1$.

Example

Assume that $\log x$ and $\sqrt{x}$ are continuous.

Then

$$\lim_{n\to\infty} \sqrt{(\log(1+1/n)+2} = \sqrt{(\log(1+0)+2} = \sqrt{2}.$$

Theorem (composite of continuous is continuous)

If $f(x)$ is continuous at $x = a$,
and $g(x)$ is continuous at $x = f(a)$,
then $g \circ f$ is continuous at $x = a$.

Proof. Use Def-2: if a sequence $a_i \to a$ as $i \to \infty$,

then $f(a_i) \to f(a)$ as $i \to \infty$,

since f is continuous at $x = a$.

Then $g(f(a_i)) \to g(f(a))$ as $i \to \infty$,

since g is continuous at $x = f(a)$.

This means that $a_i \to a \;\Rightarrow\; (g \circ f)(a_i) \to (g \circ f)(a)$,

as required for $g \circ f$ to be continuous at $x = a$. $\square$

We assume many functions as known to be continuous:

$f(x) = x$, $\sin x$, 2^x, $\ldots$ continuous everywhere;

$\dfrac{1}{x}$ continuous on $(-\infty, 0)$ and on $(0, +\infty)$, $\ldots$

Example of application of Theorem above:

$\sin(1/x)$ is continuous on $(0, +\infty)$.

(But $\lim\limits_{x\to 0} \sin(1/x)$ does not exist!)