Ideas of mathematical proof. Practical class week 30

Note that solutions must be by the methods specified in the problems – even if you can see another solution, even if an easier one.

9.1. Use previously known limits (like $1/n \to 0$) and the Arithmetic of Limits theorem to prove that the following limits exist and to find these limits.

(a) $\lim\limits_{n\to\infty} \dfrac{3n^3 - 7}{2n^3 + 2n^2 - 5n}$. (b) $\lim\limits_{n\to\infty} \dfrac{100n^2 + 200}{n^3 - 25n}$.

9.2. Use the Sandwich theorem to prove that the following limits exist and to find these limits.

(a) $\lim\limits_{n\to\infty} \dfrac{\cos(\sqrt{n})}{n}$. (b) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n} + \log n}{n^2}$.

9.3. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence defined recursively as $a_1 = 2$ and $a_{i+1} = \dfrac{a_i + 3}{2}$.

(a) Use induction to prove that $a_n \leqslant 3$ for all $n \in \mathbb{N}$.

(b) Use part (a) to prove that the sequence (a_i) is monotonically increasing.

(c) By the theorem on monotonic sequence, deduce from parts (a) and (b) that the sequence (a_i) has a finite limit.

(d) Find this limit by passing to the limit in the defining equation.

Reminder: Definition. A sequence (a_n) has positive infinite limit $\lim\limits_{n\to\infty} a_n = +\infty$ if for any number M there exists $N(M)$ (depending on M) such that $a_n > M$ for all $n > N(M)$.

9.4. Prove from first principles, by verifying the definition, that $\lim\limits_{n\to\infty} \dfrac{n^2 - 100}{n} = +\infty$.

[*Hint*: one does not have to solve that inequality for n precisely, a rough estimate is enough as long as it guarantees $a_n > M$ for all $n > N(M)$.]

9.5. Give examples of two sequences (a_n) and (b_n) such that

(1) $a_n \to +\infty$ and $b_n \to +\infty$, while $a_n/b_n \to 0$;

(2) $a_n \to +\infty$ and $b_n \to +\infty$, while $a_n/b_n \to 2$;

(3) $a_n \to +\infty$ and $b_n \to +\infty$, while a_n/b_n does not have a limit.