Determinants and Applications to Linear Systems

MTH1004M Linear Algebra

Why Determinants?

In the previous lecture, we saw that, a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its determinant $\det A = ad - bc$ is non-zero. Its inverse then is:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad .$$

The same condition holds for larger matrices:

Theorem An $n \times n$ matrix A is invertible if and only if its determinant $\det A$ is non-zero.

Main uses of determinants

✏ Is a test for invertibility:
- ◇ If $\det A = 0$ then A is singular.
- ◇ If $\det A \neq 0$ then A is invertible.

✏ Are used to determine the so-called **eigenvalues** of the matrix (next lectures).

✏ $|\det A|$ equals the volume of a parallelepiped, whose edges come from the columns of matrix A.

✏ When A is invertible, determinants provide a fast way to compute the unique solution of the system $A\mathrm{x} = \mathrm{b}$. This method is called Cramer's rule.

The determinant of a 3×3 matrix

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} .$$

The determinant of the 3×3 matrix A is the scalar:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinants using cofactor expansions

We denote by $A_{i,j}$ the sub-matrix of A, obtained by deleting the *ith* row and the *jth* column. The determinant of the sub-matrix $A_{i,j}$ is denoted by $\det A_{i,j}$ and is called (i,j)–minor of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \qquad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

This is the cofactor expansion along the first row.

Example

Compute the determinant of the matrix:

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

It is:

$$\begin{aligned} \det A &= \begin{vmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{vmatrix} = 5 \det A_{11} - (-3) \det A_{12} + 2 \det A_{13} \\ &= 5 \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \end{aligned}$$

$$= 5(0 \cdot 3 - 2 \cdot (-1)) + 3(1 \cdot 3 - 2 \cdot 2) + 2(1 \cdot (-1) - 0 \cdot 2) = 5(2) + 3(-1) + 2(-1) = 5$$

$\square$

Definition of the determinant

Let A be an $n \times n$ matrix, where $n \geq 2$. Then the determinant of A is the scalar

$$\begin{aligned} \det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

The (i,j) cofactor of A is any term of the form

$$(-1)^{i+j} \det A_{ij}$$

Signs - Expanding any row

Expanding an $n \times n$ matrix A, the sign rule is the following:

$$\begin{bmatrix} + & - & + & - & & \dots \\ - & + & - & & \dots & \\ + & - & + & - & & \dots \\ - & + & - & & \dots & \\ & \vdots & & & & \end{bmatrix}$$

◇ Expanding the first row:
$\det A = +a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \dots$

◇ Expanding the second row:
$\det A = -a_{21} \det A_{21} + a_{22} \det A_{22} - a_{23} \det A_{23} + \dots$

◇ Expanding the second column:
$\det A = -a_{12} \det A_{12} + a_{22} \det A_{22} - a_{32} \det A_{32} + \dots$

◇ Expanding the i–th row:
$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + (-1)^{i+3} a_{i3} \det A_{i3} + \dots$

Advice: Always compute the row or column which has the most zeros!

Example

Compute the determinant of the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -2 & 0 & 1 & -3 \\ 0 & 0 & 1 & -1 \\ 2 & 8 & -3 & 6 \end{bmatrix}$$

Expanding the first row means adding 4 terms of the form $\pm a_{1i} \det A_{1i}$, where $\det A_{1i}$ are the determinants of the 3×3 sub-matrices. It is much easier to compute the third row, which has two zeros.

$$\begin{aligned}
\det A &= 0 \cdot \det A_{31} - 0 \cdot \det A_{32} + 1 \cdot \det A_{33} - (-1) \cdot \det A_{34} \\
&= \det A_{33} + \det A_{34} \\
&= \begin{vmatrix} 1 & 3 & 7 \\ -2 & 0 & -3 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 5 \\ -2 & 0 & 1 \\ 2 & 8 & -3 \end{vmatrix} \\
&= -(-2) \begin{vmatrix} 3 & 7 \\ 8 & 6 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} - (-2) \begin{vmatrix} 3 & 5 \\ 8 & -3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} \\
&= 2(3 \cdot 6 - 8 \cdot 7) + 3(1 \cdot 8 - 2 \cdot 3) + 2(3(-3) - 8 \cdot 5) - (1 \cdot 8 - 2 \cdot 3) \\
&= -170 \quad \square
\end{aligned}$$

Examples

1. Compute the determinant of the upper triangular matrix:

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

We will expand the last row:

$$\begin{aligned} \det A &= 0 \cdot \det A_{31} - 0 \cdot \det A_{32} + 1 \cdot \det A_{33} = \det A_{33} = \begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} \\ &= 1 \cdot 1 - 0 \cdot a = 1 \end{aligned}$$

2. Compute the determinant of the upper triangular matrix:

$$A = \begin{bmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

We will expand the last row:

$$\det A = 0 \cdot \det A_{31} - 0 \cdot \det A_{32} + \lambda_3 \cdot \det A_{33} = \lambda_3 \det A_{33} =$$

Examples

$$
\begin{aligned}
&= \lambda_3 \begin{vmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{vmatrix} \\
&= \lambda_3(\lambda_1\lambda_2 - 0 \cdot a) = \lambda_1\lambda_2\lambda_3 \quad \square
\end{aligned}
$$

3. Compute the determinant of the diagonal matrix:

$$
A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

We will expand the last row:

$$
\begin{aligned}
\det A &= 0 \cdot \det A_{31} - 0 \cdot \det A_{32} + \lambda_3 \det A_{33} = \lambda_3 \det A_{33} = \lambda_3 \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \\
&= \lambda_3(\lambda_1 \cdot \lambda_2 - 0) = \lambda_1\lambda_2\lambda_3 \quad \square
\end{aligned}
$$

Properties

- $\det I = 1$
- A row as the sum of two vectors:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

- Scalar multiplication of a row:

$$\begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} = \lambda \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- Interchanging two rows:

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- Two identical rows:

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

- If the matrix has a zero row or a zero column, then $\det A = 0$.

Properties

If A and B are $n \times n$ matrices and A is invertible, then:

- $\det(AB) = \det A \det B$
- $\det(A^{-1}) = \frac{1}{\det A}$

From the first property yields that $\det(AA^{-1}) = \det A \det A^{-1}$. Since A is invertible, $AA^{-1} = I$, which implies that $\det(AA^{-1}) = \det I = 1$, therefore $\det A \det A^{-1} = 1$.

- $\det(A^T) = \det A$
- $\det(\lambda A) = \lambda^n \det A$
- $\det(A^n) = (\det A)^n$

From the first property yields that
$\det(A^n) = \det(A \cdot \ldots \cdot A) = \det A \ldots \det A = (\det A)^n$.

Generalisations

✏ The determinant of any diagonal matrix is equal to the product of its diagonal elements.

$$\begin{vmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & d_n \end{vmatrix} = d_1 d_2 \ldots d_n$$

✏ The determinant of any upper triangular matrix is equal to the product of its diagonal elements.

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & a_{nn} \end{vmatrix} = a_{11} a_{22} \ldots a_{nn}$$

Volume and Determinants

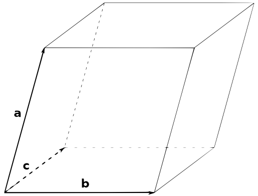
Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then the volume of the parallelepiped in $\mathbb{R}^3$ whose edges are the column vectors of the matrix A, is equal to the absolute value of the determinant of A

Let $\mathrm{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\mathrm{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ be the column vectors in $\mathbb{R}^3$ of the matrix A, then

$$|\det A| = \text{Volume}$$



$\star$ This property is generalised to $n \times n$ matrices $\star$

Examples

Find the area formed by the two vectors $\mathrm{a} = [1, 3]^T$, $\mathrm{b} = [-2, 5]^T$ *in* $\mathbb{R}^2$.

We create the matrix A whose columns are those two vectors:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

The determinant of the matrix A is:

$$\det A = \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = 5 - (-2 \cdot 3) = 11$$

Therefore, the area formed by a, b is $|\det A| = 11$ $\square$

Remark: Repeat the same example by forming a matrix B with rows the vectors a and b. Do you find the same area? Which determinant property guarantees that you will get the same result?

Applications to Linear Systems

This part of the lecture is about solving the system $A\,\mathrm{x} = \mathrm{b}$, without using the elimination method. It is called Cramer's rule and is applied only for linear systems with n equations and n unknowns.

The idea is that we explicitly write the solution:

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \ldots, \quad x_n = \frac{\det B_n}{\det A}$$

for this system in terms of the components $x_1, x_2, \ldots, x_n$ of x, by using the *determinants* of some matrices $B_1, B_2, \ldots, B_n$.

The matrices $B_1, B_2, \ldots, B_n$ are formed by replacing the i–the column of the matrix A with the column b. For example, in the 3×3 case:

$$B_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad B_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Notation: These matrices are denoted by B_i or by $A_i(b)$ (in your ebook).

Cramer's rule

Let A be an invertible $n \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by:

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \ldots, \quad x_n = \frac{\det B_n}{\det A}$$

Example

Solve the system using determinants:

$$\begin{cases} x + y + z = 5 \\ x - 2y - 3z = -1 \\ 2x + y - z = 3 \end{cases}$$

Let form the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$$

We compute the main determinant of the matrix A:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & -3 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5 - 5 + 5 = 5$$

We then compute the determinants:

$$\det B_1 = \begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix} = 5 \begin{vmatrix} -2 & -3 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} -1 & -3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= 25 - 10 + 5 = 20$$

Example

and

$$\begin{aligned}\det B_2 &= \begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -3 \\ 3 & -1 \end{vmatrix} - 5\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ &= 10 - 5 \cdot 5 + 5 = -10\end{aligned}$$

and finally

$$\begin{aligned}\det B_3 &= \begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \\ &= -5 - 5 + 5 \cdot 5 = 15 \quad .\end{aligned}$$

By Cramer's rule:

$$x_1 = \frac{\det B_1}{\det A} = \frac{20}{5} = 4, \;\; x_2 = \frac{\det B_2}{\det A} = \frac{-10}{5} = -2, \;\; x_3 = \frac{\det B_3}{\det A} = \frac{15}{5} = 3 \;\; \square$$

Example

Solve the system using Cramer's rule:

$$\begin{cases} x + 2y - 2z = 1 \\ 2x + 3y - 4z = 0 \\ -x - 2y = -1 \end{cases}$$

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -4 \\ -1 & -2 & 0 \end{bmatrix}$$

We compute the determinant of the matrix A:

$$\det A = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & -4 \\ -1 & -2 & 0 \end{vmatrix} = -1 \begin{vmatrix} 2 & -2 \\ 3 & -4 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = -(-2) + 2 \cdot 0 = 2$$

Example

So the matrix A is invertible and the system has unique solution. We now compute the determinants:

$$\det B_1 = \begin{vmatrix} 1 & 2 & -2 \\ 0 & 3 & -4 \\ -1 & -2 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 3(-2) + 0 = -6$$

$$\det B_2 = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -4 \\ -1 & -1 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -2(-2) = 4$$

$$\det B_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ -1 & -2 & -1 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0 + 0 = 0$$

And by Cramer's rule the solution of the system is:

$$x_1 = \frac{\det B_1}{\det A} = \frac{-6}{2} = -3, \ x_2 = \frac{\det B_2}{\det A} = \frac{4}{2} = 2, \ x_3 = \frac{\det B_3}{\det A} = \frac{0}{2} = 0 \qquad \square$$