The Inverse of a Matrix

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Inverse of a Matrix

A real number c has as inverse the number $1/c$ as long as c is non-zero. In matrices, the situation is similar.

✏ Matrices, like numbers, might be invertible or non invertible.

✏ An essential application of inverting a matrix is solving a linear system.

✏ Only square matrices can be inverted.

✏ The system of n linear equations with n unknowns

$$A\mathbf{x} = \mathbf{b}$$

has unique solution, if the matrix A is invertible. Its solution then is:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for any $\mathbf{b}$ in $\mathbb{R}^n$.

Definition An $n \times n$ square matrix A is called *invertible* if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I \ .$$

The matrix A^{-1} is called *inverse* of the matrix A.

Theorem If A is an invertible matrix then its inverse is unique.

Proof:
Suppose that A has more than one inverse, hence there exists A_1 and A_2 such that

$$A_1A = I = AA_1, \quad \text{and} \quad A_2A = I = AA_2$$

Then

$$A_1 = A_1I = A_1(AA_2) = (A_1A)A_2 = IA_2 = A_2. \quad \square$$

Properties of the Inverse

Let A and B are invertible $n \times n$ matrices, then:

- A^{-1} is invertible and its inverse is
$$(A^{-1})^{-1} = A$$
- AB is invertible and its inverse is
$$(AB)^{-1} = B^{-1}A^{-1}$$
- A^T is invertible and its inverse is
$$(A^T)^{-1} = (A^{-1})^T$$
- A^n is invertible for all $n = 0, 1, 2, \ldots$ and its inverse is
$$(A^n)^{-1} = (A^{-1})^n.$$

Most properties are shown by uniqueness of the inverse matrix theorem.

Examples

1. If A satisfies the matrix equation $3A^2 - A + I = O$, then A is invertible.

It is

$$\begin{aligned} 3A^2 - A + I &= O \\ 3A^2 - A &= -I \\ (3A - I)A &= -I \\ (I - 3A)A &= I \end{aligned}$$

which implies that

$$A^{-1}A = I \quad \text{where} \quad A^{-1} = I - 3A$$

So A is invertible, with inverse the matrix $A^{-1} = I - 3A$. $\square$

Examples

2. If A satisfies the matrix equation $A^3 - A^2 - I = O$, then A is invertible.

It is

$$\begin{aligned} A^3 - A^2 - I &= O \\ A^3 - A^2 &= I \\ A(A^2 - A) &= I \end{aligned}$$

which implies that

$$AA^{-1} = I \ \text{ where } \ A^{-1} = A^2 - A$$

So A is invertible, with inverse the matrix $A^{-1} = A^2 - A$. $\square$

Examples

1. Prove that matrix B is the inverse of the matrix A, when

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad BA = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we have that $AB = I$ and $BA = I$, therefore B is the inverse of matrix A and A is the inverse of matrix B. $\square$

2. *Show that the inverse of the identity matrix is* $I^{-1} = I$

Obviously $II = I$, so the inverse of the identity matrix is I. $\square$

Examples

3. The inverse of the diagonal matrix $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ *is* $D^{-1} = \begin{bmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{bmatrix}$

The product of diagonal matrices is straightforward, like numbers.

$$\text{It is } DD^{-1} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{bmatrix} = \begin{bmatrix} d_1/d_1 & 0 \\ 0 & d_2/d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, $D^{-1}D = I$, so the matrix D^{-1} is the inverse of D. $\square$

4. If $A^{-1} = A$*, where* A *is a non-diagonal and symmetric matrix, find* A*.*

$A = A^{-1}$ is equivalent to $AA = AA^{-1}$ or $A^2 = I$.

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Since A is non-diagonal $b \neq 0$ and ...

Examples

... the equation $A^2 = I$ becomes:

$$A^2 = \begin{bmatrix} a^2 + b^2 & ab + cb \\ ab + cb & b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So, } a^2 + b^2 = 1, ab + cb = 0, b^2 + c^2 = 1$$

From $ab + cb = 0$ we get that $c = -a$ (since $b \neq 0$). So, the matrix is

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad \text{where} \quad a^2 + b^2 = 1$$

. $\square$

Example of a non-invertible matrix

The matrix: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is non-invertible. Try to solve a system of the form $A\mathbf{x} = \mathbf{b}$, for example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and you will realise that it is not possible.

Equivalently, is like asking whether the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a linear combination of the columns of A, namely:

$$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

But the columns of the matrix A, viewed as vectors, are linearly dependent and therefore they don't span the plane. Consequently they cannot produce the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which does not lie on the line $x + y = 0$.

What is the inverse of a 2×2 matrix?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We are then searching for a matrix $X = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$AX = I \ .$$

The left-hand side is:

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} ax + by & az + bw \\ cx + dy & cz + cw \end{bmatrix}$$

so $AX = I$ is equivalent to two systems of equations:

$$\begin{cases} ax + by = 1 \\ cx + dy = 0 \end{cases} \quad \text{and} \quad \begin{cases} az + bw = 0 \\ cz + cw = 1 \end{cases}$$

to be continued...

...

The first system in terms of Gauss–Jordan elimination is written as:

$$\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right] \xrightarrow[bR_2]{} \left[\begin{array}{cc|c} a & b & 1 \\ bc & bd & 0 \end{array}\right] \xrightarrow[R_2 - dR_1]{} \left[\begin{array}{cc|c} a & b & 1 \\ bc - ad & 0 & -d \end{array}\right] \xrightarrow[-R_2]{} \left[\begin{array}{cc|c} a & b & 1 \\ ad - bc & 0 & d \end{array}\right]$$

Does the system have a unique solution or not?

Answer: It depends whether $ad - bc$ is zero or not.

If $ad - bc \neq 0$, then

$$x = d/(ad - bc)$$

Similarly, we get that $y = -c/(ad - bc)$, $z = -b/(ad - bc)$, $w = a/(ad - bc)$. So,

$$X = \boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

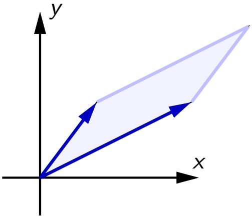
is the inverse of a matrix A and exists when $ad - bc \neq 0$. $\square$

Determinant of a 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A, denoted by $\det A$, is the number:

$$\det A = ad - bc$$

✏ A is invertible only when the determinant $\det A = ad - bc$ is nonzero.



Geometrically, the absolute value of the determinant of the matrix A is the volume of the parallelogram formed by its column vectors (or equivalently, its row vectors)

$$|\det A| = \text{Volume}$$

Linear systems with unique solution

Theorem If A is an invertible $n \times n$ matrix, then the system of linear equations given by

$$A\mathbf{x} = \mathbf{b}$$

has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for any $\mathbf{b}$ in $\mathbb{R}^n$.

✏ Gauss–Jordan elimination for a linear system with unique solution

$$[A|\,\mathbf{b}] \longrightarrow \left[I|\,A^{-1}\mathbf{b}\right]$$

✏ Gauss–Jordan elimination for a *homogeneous* linear system with unique solution

$$[A|\,\mathbf{0}] \longrightarrow [I|\,\mathbf{0}]$$

Example

Are the following matrices invertible? If yes, find their inverse:

$$A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

For the matrix A we have that $\det A = 3 \cdot 2 - (-5)(-1) = 6 - 5 = 1$, for B we have $\det B = 1(-1) - 1(-1) = 0$ and for C it is $\det C = 2 \cdot 1 - 1 \cdot 0 = 2$.

Therefore, only A and C are invertible. For those two matrices their inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

and

$$C^{-1} = \frac{1}{\det C} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \qquad \square$$

Verification

Indeed,

$$A^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

is the inverse of the matrix

$$A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

since,

$$A^{-1}A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 5(-1) & 2(-5) + 5 \cdot 2 \\ 1 \cdot 3 + 3(-1) & 1(-5) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, for the matrix C, it is:

$$C^{-1}C = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 2 + 0 \cdot 1 & 1/2 \cdot 0 + 0 \cdot 1 \\ -1/2 \cdot 2 + 1 \cdot 1 & -1/2 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

A and C are the matrices defined in the previous exercise. Use their inverses to solve the two linear systems:

$$A\mathbf{x} = \mathbf{b}, \qquad C\mathbf{x} = \mathbf{b},$$

where $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

If the matrix is invertible, then the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. We 've seen that (previous exercise) the matrix A is invertible and that $\det A = 1$. Therefore, the system's unique solution is:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot (-1) \\ 1 \cdot 1 + 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \quad .$$

Similarly for the system $C\mathbf{x} = \mathbf{b}$, its unique solution is:

$$\mathbf{x} = C^{-1}\mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 1 + 0 \cdot (-1) \\ -1/2 \cdot 1 + 1 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \end{bmatrix} \quad \square$$

The Gauss–Jordan Method for Computing the Inverse

To find the inverse of a matrix we will follow almost the same procedure for solving linear systems. We will use an augmented matrix, whose right side is the identity matrix. Following the same Gauss–Jordan elimination steps we used for solving linear systems, the identity matrix appears in the left side and the inverse of the matrix appears in the right side of the augmented matrix.

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

The matrix A | Identity matrix Identity matrix | The inverse $B = A^{-1}$

If A (left side of the augmented matrix) cannot be reduced to I,

then the matrix A is non invertible.

Remark: The Gauss–Jordan method is the easiest way to compute the inverse of a matrix.

Inverse and Rank

Proposition The $n \times n$ matrix A is invertible if and only if $\mathrm{rank}(A) = n$.

✏ When a matrix is invertible, the matrix is non-singular. It cannot, for example, have two equal rows or a zero row.

✏ What is the reduced row-echelon form of an invertible matrix?

Example

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \underset{R_3 - 3R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 10 & 0 & -3 & 1 \end{array}\right]$$

$$\underset{R_3/10}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3/10 & 1/10 \end{array}\right] \overset{R_2 + 2R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/5 & 1/5 \\ 0 & 0 & 1 & 0 & -3/10 & 1/10 \end{array}\right]$$

$$\overset{R_1 - 3R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 9/10 & -3/10 \\ 0 & 1 & 0 & 0 & 2/5 & 1/5 \\ 0 & 0 & 1 & 0 & -3/10 & 1/10 \end{array}\right]$$

The left side of the augmented matrix has been transformed into the identity matrix. Therefore, the matrix A is invertible and its inverse is the matrix:

$$A^{-1} = \begin{bmatrix} 1 & 9/10 & -3/10 \\ 0 & 2/5 & 1/5 \\ 0 & -3/10 & 1/10 \end{bmatrix} \quad \square$$

Example: A non-invertible matrix

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \underset{R_3 - 2R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\underset{R_3/5}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2/5 & 0 & 1/5 \end{array}\right] \overset{R_2 - 3R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 1 & -1 & -2/5 & 0 & 1/5 \end{array}\right]$$

$$\overset{R_2/4}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/4 & 1/4 & 0 \\ 0 & 1 & -1 & -2/5 & 0 & 1/5 \end{array}\right]$$

2 equal rows $\rightarrow$ is a sign of non-invertibility

Example: A non-invertible matrix

$$\xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 7/20 & -1/4 & 1/5 \end{array}\right]$$

Indeed, we got a zero row $\rightarrow$ it is now evident that is not possible to transform the left side of the augmented matrix into the identity matrix.
Therefore, the matrix A is non-invertible. $\square$