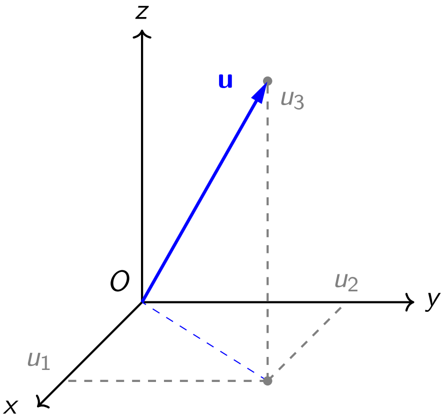
Vectors in $\mathbb{R}^3$ and beyond

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Vectors in the 3-dimensional space



▷ Consider the *xyz* coordinate system

▷ We denote the 3-dimensional vector space with $\mathbb{R}^3$

▷ Any point in $\mathbb{R}^3$ has 3 coordinates, x, y and z (or u_1, u_2, u_3 etc.)

▷ You should draw them as the left plot indicates. (In this way you keep the orientation)

Any vector $\mathbf{u}$ in $\mathbb{R}^3$ can take the following form:

$$\mathbf{u} = [u_1, u_2, u_3] \quad \text{or} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

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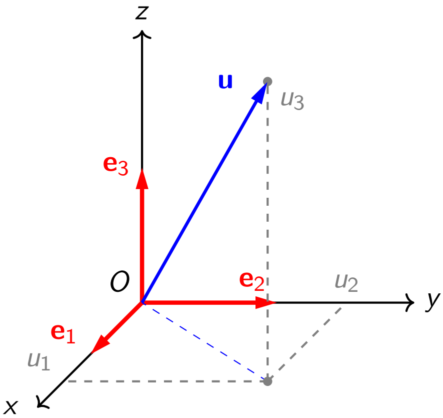
Vectors in the 3-dimensional space



▷ Let's denote by:

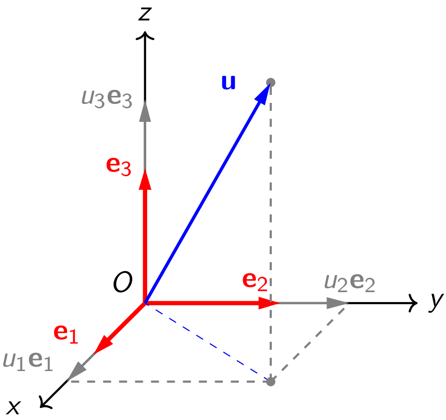
$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

▷ Any such vector $\mathbf{u}$ in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$:

$$\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$$

Since: $$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vectors in the 3-dimensional space



Left plot:

▷ Create the vectors $u_1\mathbf{e}_1$, $u_2\mathbf{e}_2$ and $u_3\mathbf{e}_3$ and sum them.

▷ $u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ is the projection of the vector $\mathbf{u}$ on the xy plane.

$$\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$$

Addition and scalar multiplication in higher dimensions

We spoke about vector addition and scalar multiplication of vectors in the plane. Here we talk about vectors in n–dimensional spaces.

- ✏ Vectors in $\mathbb{R}^n$ have n components, like: $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

- ✏ Vector Addition of 2 vectors in $\mathbb{R}^n$: $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

- ✏ Scalar Multiplication of a vector $\mathbf{u}$ in $\mathbb{R}^n$ with a real number c: $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$.

- ✏ Linear combination of k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a vector $\mathbf{v}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, where $c_1, c_2, \ldots, c_k$ are scalars.

General Definition for Spanning $\mathbb{R}^n$

Definition Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be k vectors in $\mathbb{R}^n$. The set of all linear combinations

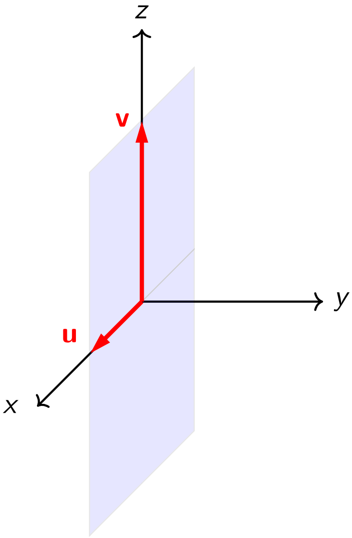
$$S = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ in } \mathbb{R}\}$$

is called *span* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by

$$span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$$

Definition Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be k vectors of $\mathbb{R}^n$. We say that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *span* $\mathbb{R}^n$ if $span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \mathbb{R}^n$.

Example 1

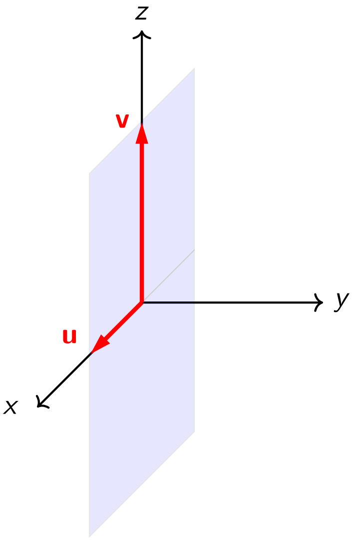


1. Compute the linear combinations of the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

and show that they span a plane in $\mathbb{R}^3$. Find a vector that is not a linear combination of $\mathbf{u}, \mathbf{v}$.

Example 1



The linear combinations of $\mathbf{u}$ and $\mathbf{v}$ are

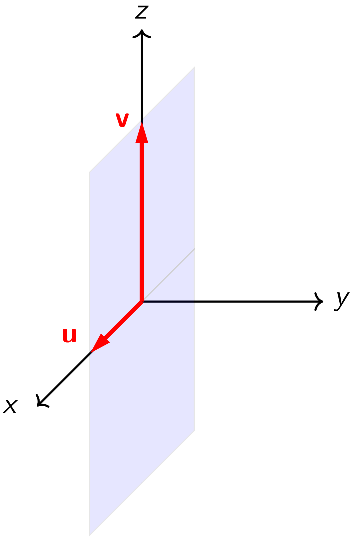
$$c\mathbf{u} + d\mathbf{v} = \begin{bmatrix} 2c \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3d \end{bmatrix} = \begin{bmatrix} 2c \\ 0 \\ 3d \end{bmatrix}$$

for any real number c and d. The span of $\mathbf{u}, \mathbf{v}$ is:

$$\begin{aligned} span(\mathbf{u}, \mathbf{v}) &= \{c\mathbf{u} + d\mathbf{v} : c, d \text{ in } \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 2c \\ 0 \\ 3d \end{bmatrix} : c, d \text{ in } \mathbb{R} \right\} \end{aligned}$$

So, this span is the xz plane in $\mathbb{R}^3$ ($y = 0$ plane), since...

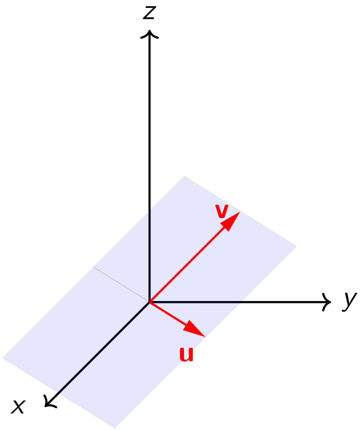
Example 1



... any vector $\mathbf{w} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$ in the xz plane can be expressed as the linear combination of $\mathbf{u}, \mathbf{v}$. Indeed, this is true for $c = x/2$, $d = y/3$.

A vector that is not a linear combination of $\mathbf{u}, \mathbf{v}$ is $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ since there are no c, d values such that $\mathbf{e}_2 = c\mathbf{u} + d\mathbf{v}$ $\square$

Example 2

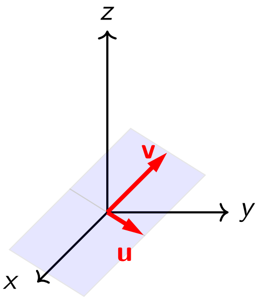


2. Compute the linear combinations of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and show that they span a plane in $\mathbb{R}^3$. Find a vector that is not a linear combination of $\mathbf{u}, \mathbf{v}$.

Example 2



The linear combinations of $\mathbf{u}$ and $\mathbf{v}$ are

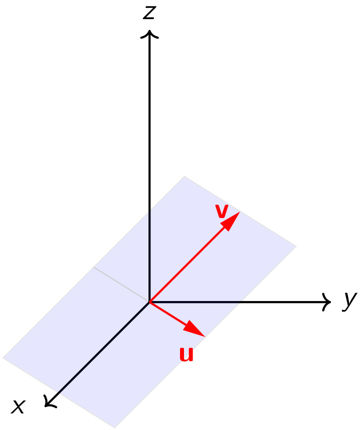
$$c\mathbf{u} + d\mathbf{v} = \begin{bmatrix} c \\ c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ d \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix}$$

for any real number c and d. The span of $\mathbf{u}, \mathbf{v}$ is:

$$\begin{aligned} span(\mathbf{u}, \mathbf{v}) &= \{c\mathbf{u} + d\mathbf{v} : c, d \text{ in } \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} c \\ c+d \\ d \end{bmatrix} : c, d \text{ in } \mathbb{R} \right\} \end{aligned}$$

in $\mathbb{R}^3$ whose second component is the sum of the first and the third ($y = x + z$).

Example 2



Any vector $\mathbf{x}$ which belongs to $span(\mathbf{u}, \mathbf{v})$ is of the form $\mathbf{x} = \begin{bmatrix} x \\ x+z \\ z \end{bmatrix}$

An example of a vector which does not belong to this plane is: $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ □

Spanning $\mathbb{R}^3$

$span(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$ since:

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Can produce any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$. In other words:

$$span(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \{x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 : x, y, z \text{ in } \mathbb{R}\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \text{ in } \mathbb{R} \right\} = \mathbb{R}^3 \; .$$

Spanning $\mathbb{R}^3$

The linear combinations of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix},$$

are:

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ -2c_3 \\ -c_1 - c_2 \end{bmatrix}$$

Any vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ when there are c_1, c_2, c_3 such that $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, which gives the system:

$$\begin{cases} c_1 = x \\ -2c_3 = y \\ -c_1 - c_2 = z \end{cases}$$

with solution $c_1 = x, c_2 = -x - z, c_3 = -y/2$, therefore:

$$span(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$$

Observations

1. One nonzero vector in $\mathbb{R}^2$ spans a line.

2. Two nonzero vectors in $\mathbb{R}^2$ which lie on the same line span that line.

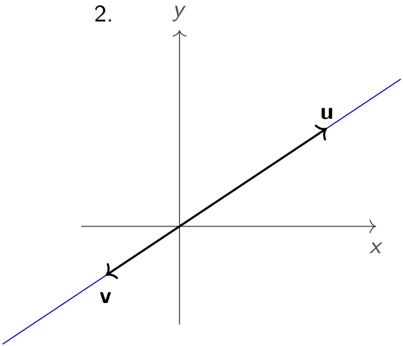
3. Two nonzero vectors in $\mathbb{R}^2$ which do not lie on the same line span $\mathbb{R}^2$.

4. Three nonzero vectors in $\mathbb{R}^3$ which lie on the same plane span a plane in $\mathbb{R}^3$.

5. Three nonzero vectors in $\mathbb{R}^3$ which do not lie on the same plane span $\mathbb{R}^3$.

Two nonzero vectors in $\mathbb{R}^2$ span ...

2.

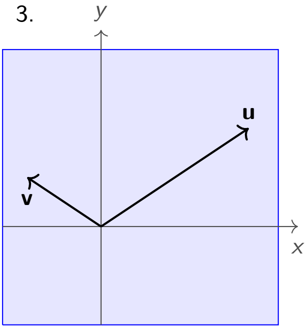


... **a line**, if they both lie on the same line, which implies that they are dependent:

$$\mathbf{u} = c\mathbf{v} \ ,$$

for some c value.

3.

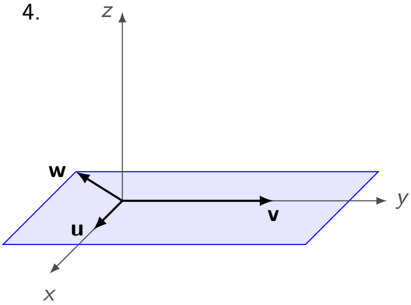


... **the plane** $\mathbb{R}^2$, if they do not lie on the same line, which implies that there is **no** real number c such that:

$$\mathbf{u} = c\mathbf{v}$$

Three nonzero vectors in $\mathbb{R}^3$ span ...

4.

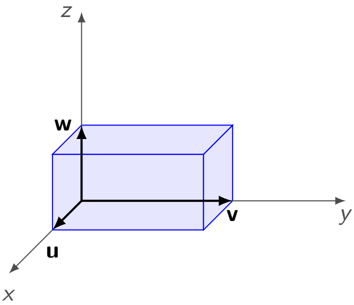


... **a plane** in $\mathbb{R}^3$, if they all lie on the same plane, which implies that they are dependent:

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$$

for some reals c_1, c_2.

5.



... **the space** $\mathbb{R}^3$, if they do not lie on the same plane, which implies that there are **no** real numbers c_1 and c_2 such that:

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$$

Linear Independence

Definition The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called *linearly independent* if the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

implies that all scalars are zero $c_1 = c_2 = \ldots = c_n = 0$

- It means that linearly independent vectors cannot be related.
- The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called *linearly dependent* if there are scalars $c_1, c_2, \ldots, c_n$ satisfying the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$, which are not all zero.
- Geometrically, the vectors are linearly dependent if at least one of them belongs to the space which is spanned by the rest vectors. Algebraically, it means that at least one of them can be expressed as the linear combination of the rest. It is like we have at least one additional vector not providing any new information.

Example

Determine whether the following vectors are linearly independent:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \ \textit{and } \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

We follow the standard methodology by solving the vector equation:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

in terms of c_1, c_2 and c_3. Their linear combinations are:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 + c_2 \\ -c_2 + 3c_3 \end{bmatrix}.$$

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0} \rightarrow \begin{cases} c_1 = 0 \\ -c_1 + c_2 = 0 \\ -c_2 + 3c_3 = 0 \end{cases} \rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

So the vectors are linearly independent.

Example

Determine whether the following vectors are linearly independent:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \ \textit{and} \ \mathbf{u}_3 = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

We follow the standard methodology by solving the vector equation:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

in terms of c_1, c_2 and c_3. Their linear combinations are:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 + c_2 - 3c_3 \\ -c_2 + 3c_3 \end{bmatrix}.$$

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0} \to \begin{cases} c_1 = 0 \\ -c_1 + c_2 - 3c_3 = 0 \\ -c_2 + 3c_3 = 0 \end{cases} \to \begin{cases} c_1 = 0 \\ c_2 - 3c_3 = 0 \\ -c_2 + 3c_3 = 0 \end{cases}$$

So, $c_2 = 3c_3$, where c_3 is any real number, therefore the vectors are linearly dependent.

Example

Find the span of the following vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \ \textit{and } \mathbf{u}_3 = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

In the previous example we found that for $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ yields that $c_1 = 0, c_2 = 3c_3$, so the vector equation becomes $0\mathbf{u}_1 + 3c_3\mathbf{u}_2 + c_3\mathbf{u}_3 = 0$, for any real number c_3, or simply:

$$3\mathbf{u}_2 + \mathbf{u}_3 = 0$$

...

Example

...

So, $span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = span(\mathbf{u}_1, \mathbf{u}_2, -3\mathbf{u}_2) = span(\mathbf{u}_1, \mathbf{u}_2)$ and:

$$\begin{aligned}
span(\mathbf{u}_1, \mathbf{u}_2) &= \left\{ c_1\mathbf{u}_1 + c_2\mathbf{u}_2 : c_1, c_2 \text{ in } \mathbb{R} \right\} \\
&= \left\{ c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : c_1, c_2 \text{ in } \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} c_1 \\ -c_1 + c_2 \\ -c_2 \end{bmatrix} : c_1, c_2 \text{ in } \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} x \\ -x - z \\ z \end{bmatrix} : x, z \text{ in } \mathbb{R} \right\}
\end{aligned}$$

Length and Angle: The Dot Product

We define the multiplication of two vectors $\mathbf{u} \cdot \mathbf{v}$, which is *NOT* a vector but **a real number**. This number is the sum of the separate component products $u_1 v_1$, $u_2 v_2$ etc.

Definition

The dot product or inner product of the vectors:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in $\mathbb{R}^n$ is the number $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

Length and Angle: The Dot Product

We avoid writing $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ while we will be writing $\begin{bmatrix} u_1, & u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Example: Find the dot product of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}.$$

Their dot product is:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = 1 \cdot 2 + (-1) \cdot 0 + 2 \cdot (-3) = -4 \quad \square$$

Properties of the Dot Product

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$
- $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the Dot Product

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

  The dot product $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$ is equal to the dot product $\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + \ldots + v_n u_n$, which means that the order of $\mathbf{u}$ and $\mathbf{v}$ makes no difference.

- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$
- $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the Dot Product

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- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

It is: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = u_1(v_1 + w_1) + u_2(v_2 + w_2) + \ldots + u_n(v_n + w_n) =$
$u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + \ldots + u_n v_n + u_n w_n =$
$u_1 v_1 + u_2 v_2 + \ldots + u_n v_n + u_1 w_1 + u_2 w_2 + \ldots + u_n w_n = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$
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Properties of the Dot Product

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$

  It is: $\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + u_2 u_2 + \ldots + u_n u_n = u_1^2 + u_2^2 + \ldots + u_n^2$, which is always non-negative, i.e. positive or zero.

- $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Length

Example: Find the product $\mathbf{u} \cdot \mathbf{u}$*, when* $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = (-1)^2 + 0 + (-1)^2 = 2 \quad \square.$$

Definition: The length $\| \mathbf{u} \|$ of a vector $\mathbf{u}$ is the square root of $\mathbf{u} \cdot \mathbf{u}$:

$$\| \mathbf{u} \| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Definition: A vector of length 1 is called a unit vector.

Length

Example: Prove that the vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, *and* $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *are unit vectors.*

It is enough to prove that $\| \mathbf{e}_i \|$, for $i = 1, 2, 3$. For $\mathbf{e}_1$ it is $\| \mathbf{e}_1 \| = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1} = \sqrt{1^2 + 0^2 + 0^2} = 1$. The same holds for the other two vectors $\mathbf{e}_2, \mathbf{e}_3$ $\quad \square$.

Example: Normalise the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

▷ To normalise a vector, we first **find its length**. So, here it is $\| \mathbf{u} \| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$.

▷ Now, we **divide** the vector $\mathbf{u}$ with its length: $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \quad \square$.

Length and Dot Product

Prove that $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector, for any vector $\mathbf{u}$ in $\mathbb{R}^n$.

Let $\mathbf{u}$ a vector in $\mathbb{R}^n$. Then

$$\left\| \frac{1}{\| \mathbf{u} \|}\mathbf{u} \right\|^2 = \left(\frac{1}{\| \mathbf{u} \|}\mathbf{u}\right) \cdot \left(\frac{1}{\| \mathbf{u} \|}\mathbf{u}\right) = \frac{1}{\| \mathbf{u} \|^2}\mathbf{u} \cdot \mathbf{u} = \frac{1}{\| \mathbf{u} \|^2} \| \mathbf{u} \|^2 = 1$$

Remark: For any real number c and vector $\mathbf{u}$, it is $\| c\mathbf{u} \| = | c | \| \mathbf{u} \|$

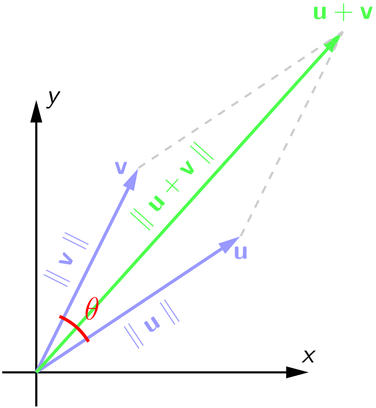
Definition: The vectors $\mathbf{u}, \mathbf{v}$ are called orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$

Example: The vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 3 + 1 \cdot 1 - 2 \cdot 2 = 4 - 4 = 0$$

Definitions and Properties

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be any vectors in* $\mathbb{R}^n$.



▷ Angle Formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

we use this formula to calculate the angle between two vectors $\mathbf{u}, \mathbf{v}$

▷ Cauchy–Schwarz Inequality

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

look at the angle formula

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

▷ Triangle Inequality

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Look at the left plot. Can you understand why the length of $\mathbf{u}+\mathbf{v}$ is less than the sum of the lengths of the vectors $\mathbf{u}$ and $\mathbf{v}$?

The Triangle Inequality

For all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$,

$$\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$$

Since both sides of the inequality are non–negative, it is enough to show that the square of $\| \mathbf{u} + \mathbf{v} \|$ is less or equal to the square of $\| \mathbf{u} \| + \| \mathbf{v} \|$. It is:

$$\begin{aligned}
\| \mathbf{u} + \mathbf{v} \|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\
&= \| \mathbf{u} \|^2 + 2\, \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2 \\
&\leq \| \mathbf{u} \|^2 + 2 |\mathbf{u} \cdot \mathbf{v}| + \| \mathbf{v} \|^2 \\
&\leq \| \mathbf{u} \|^2 + 2 \| \mathbf{u} \| \| \mathbf{v} \| + \| \mathbf{v} \|^2 \qquad \textit{Cauchy–Schwarz} \quad \textit{Inequality} \\
&\leq (\| \mathbf{u} \| + \| \mathbf{v} \|)^2
\end{aligned}$$

Example

Let $\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *two vectors in the plane:*
i) Find the length and the angle between these vectors.
ii) Verify the Cauchy–Schwarz inequality and the Triangle inequality.

i) We first compute the length for each of the vectors, so it is $\| \mathbf{u} \| = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = 2$ and $\| \mathbf{v} \| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$. To find the angle we need the dot product and the lengths of the vectors, which we have. The dot product is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 = 1 \cdot 1 + \sqrt{3} \cdot 0 = 1$, so from the equation $\mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos\theta$, we find that $\cos\theta = \frac{1}{2 \cdot 1} = \frac{1}{2}$, and therefore $\theta = \pi/3$.

ii) The Cauchy–Schwarz inequality reads $| \mathbf{u} \cdot \mathbf{v} | \leq \| \mathbf{u} \| \| \mathbf{v} \|$, where $\mathbf{u} \cdot \mathbf{v} = 1$ and $\| \mathbf{u} \| \| \mathbf{v} \| = 2 \cdot 1 = 2$.
Triangle inequality reads $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$, where $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1+1 \\ \sqrt{3}+0 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{3} \end{bmatrix}$ so $\| \mathbf{u} + \mathbf{v} \| = \sqrt{2^2 + \sqrt{3}^2} = \sqrt{7}$, whereas $\| \mathbf{u} \| + \| \mathbf{v} \| = 2 + 1 = 3$. Is it true that $\sqrt{7} < 3$?