Diagonalisation of Matrices

MTH1004M Linear Algebra

UNIVERSITY OF
LINCOLN

What is diagonalisation?

Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has 2 real eigenvalues λ_1, λ_2. Is A related to the diagonal matrix?

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad ?$$

The answer is yes! We are going to see why and how.

Example

We have found (in the last slides) that the matrix

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

✏ has the eigenvalue $\lambda_1 = 2$ with eigenvector $\mathrm{u}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

✏ and $\lambda_2 = -1$ with $\mathrm{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Let's construct a matrix with columns the eigenvectors of A:

$$P = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} .$$

Its inverse P^{-1} is:

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$$

➜

Example

We Perform the operation $P^{-1}AP$:

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D \quad ,$$

and we observe that the output matrix D is not only diagonal but contains the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ <u>in the same order</u> as the order of the eigenvector columns in P.

The Diagonal Form of a Matrix

So, observe that, given:

- a 2×2 matrix A which has real eigenvalues λ_1, λ_2,
- two linearly independent eigenvectors $\mathrm{u}_1, \mathrm{u}_2$,
- and a diagonal matrix D with entries λ_1, λ_2

A and D are related by a matrix P whose columns are formed by the eigenvectors $\mathrm{u}_1, \mathrm{u}_2$:

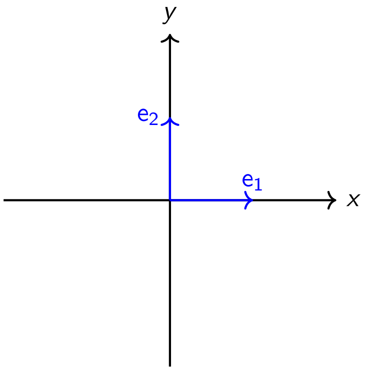
$$P = \begin{bmatrix} | & | \\ \mathrm{u}_1 & \mathrm{u}_2 \\ | & | \end{bmatrix} \quad .$$

P is called the *eigenvector matrix* and it holds that:

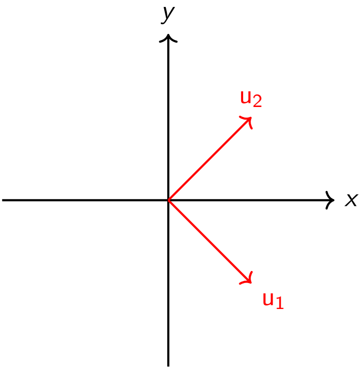
$$D = P^{-1}AP \qquad \text{or} \qquad AP = PD \; .$$

Remark: The eigenvalues will come up in D in the same order as their corresponding eigenvectors in P.

What this means geometrically?



Geometrically is means that, given a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ which is naturally expressed in $\mathbb{R}^2$ with basis the standard orthonormal vectors e_1, e_2



... when expressed in another basis of $\mathbb{R}^2$, the basis formed by the eigenvectors u_1 and u_2, then A is viewed as the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

Definitions

- Let A, B be $n \times n$ matrices. A is called **similar** to B if there is an invertible $n \times n$ matrix P, such that $P^{-1}AP = B$.

- An $n \times n$ matrix A is called **diagonalisable** if it is similar to the diagonal matrix D whose entries are the n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Example

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Verify that the matrix A with eigenvectors $\mathrm{u}_1 = [1, 1, 1]^T$, $\mathrm{u}_2 = [1, 1, -2]^T$ and $\mathrm{u}_3 = [-1, 1, 0]^T$ is similar to the diagonal matrix D.

It is always easier to prove that $AP = PD$, which requires two matrix multiplications, than finding the inverse of the matrix P. The matrix P is:

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} .$$

$$\text{Then} \quad AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\text{and} \quad PD = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \quad \square$$

How to prove that a matrix A is diagonalisable

➜ If P is the matrix with columns the eigenvectors of A.

➜ P is invertible if $\det P \neq 0$.

➜ If P is invertible, then its columns, the eigenvectors are linearly independent.

So, to prove that a matrix A is diagonalisable, it is easier to use the following Theorem.

Theorem

Let A be an $n \times n$ matrix. Then A is **diagonalisable** if and only if A has n linearly independent eigenvectors.

Example: A non-diagonalisable matrix

Show that the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is non-diagonalisable.

A has two eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$. The eigenvectors $\mathrm{u}_1, \mathrm{u}_2$ are found by setting:

$$A\mathrm{u}_1 = \lambda_1 \mathrm{u}_1 \quad \text{or} \quad (A - \lambda_1 I)\mathrm{u}_1 = 0 \quad \text{which is} \quad (A - I)\mathrm{u}_1 = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and yields the system

$$\begin{cases} 0x_1 + 1y_1 = 0 \\ 0x_1 + 0y_1 = 0 \end{cases}$$

with null space the eigenspace:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_1 = 0, x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = \mathit{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Example: A non-diagonalisable matrix

Hence,

$$\mathrm{u}_1 = [1, 0]^T.$$

is the eigenvector associated with λ_1

Following the same process for u_2 we find that:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_2 = 0, x_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_2 \\ 0 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\} = span \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

So, $\mathrm{u}_2 = \mathrm{u}_1$.

Since $\mathrm{u}_1 = \mathrm{u}_2$ A is non-diagonalisable.

✏ **Remark:** When it is not obvious that the eigenvectors are linearly dependent, you can show that the matrix P is *non-invertible*, by showing that $\det P \neq 0$.

Example: A diagonalisable matrix

Let

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

with real eigenvalues $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = 5$. Show that A is diagonalisable.

We first find the eigenvectors $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ of the matrix A. For $\lambda_1 = 3$, the equation $(A - \lambda I)\mathrm{x} = 0$ becomes:

$$\left(\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and is equivalent to the following system:

$$\left\{ \begin{array}{c} x_1 + y_1 - z_1 = 0 \\ 2x_1 + 2y_1 - 2z_1 = 0 \\ x_1 + y_1 - z_1 = 0 \end{array} \right. .$$

This system is reduced to a single equation $x_1 + y_1 - z_1 = 0$. The eigenspace is:

$$\begin{aligned} V_{\lambda_1} &= \left\{ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ in } \mathbb{R}^3 : z_1 = x_1 + y_1 \right\} = \left\{ \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix} : x_1, y_1 \text{ in } \mathbb{R} \right\} \\ &= span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \end{aligned}$$

so it yields that

$$\mathrm{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathrm{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and that $\dim V_{\lambda_1} = 2$ (Check that they are linearly independent!).

For the last eigenvector associated to $\lambda_3 = 5$, we have $(A - \lambda_3 I)\mathrm{u}_3 = 0$, or:

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

➜

that is equivalent to the system:

$$\left\{\begin{array}{c} -x_3 + y_3 - z_3 = 0 \\ 2x_3 - 2z_3 = 0 \\ x_3 + y_3 - 3z_3 = 0 \end{array}\right. .$$

The above system is reduced to the following:

$$\left\{\begin{array}{c} -2x_3 + y_3 = 0 \\ z_3 = x_3 \end{array}\right.$$

with eigenspace

$$\begin{aligned} V_{\lambda_3} &= \left\{ \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \text{ in } \mathbb{R}^3 : y_3 = -2x_3, z_3 = x_3 \right\} = \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} : x_3 \text{ in } \mathbb{R} \right\} \\ &= span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Hence, the eigenvector is $\mathrm{u}_3 = [1, -2, 1]^T$.
➜

So, we produce the matrix P by introducing to its columns the components of the eigenvectors $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$:

$$P = \begin{bmatrix} | & | & | \\ \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ | & | & | \end{bmatrix}$$

and we get:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

So, now, is the matrix A diagonalisable? Since

$$\det P = 1 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (1+2) + (-1) = 2 \neq 0$$

the answer is yes, the matrix A is diagonalisable, which also implies that $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ are linearly independent. $\square$

Definitions and Remarks

✏ We call geometric multiplicity of an eigenvalue λ the dimension of the space V_λ spanned by the eigenvectors.

✏ If an $n \times n$ matrix A has:

➜ n real eigenvalues where $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$
then it yields that their corresponding eigenvectors $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ are linearly independent, and hence, the matrix A is diagonalisable.

➜ 2 equal eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with linearly independent eigenvectors
we say that the *algebraic multiplicity* of λ is 2. The geometric multiplicity of λ is 2.

➜ 2 equal eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with linearly dependent eigenvectors
then the geometric multiplicity of λ is 1, while the algebraic multiplicity of λ is 2.

Remark *Diagonalisation* is checked on the eigenvectors. *Invertibility* is checked on the eigenvalues.

Lemma

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

✏ *less than* implies that the matrix is non-diagonalisable.

✏ *equal to* implies that the matrix is diagonalisable.

Application: The n–th power of a Matrix

Let A be a 2x2 diagonalisable matrix. Find A^n.

If A is diagonalisable then, there is a matrix P and a diagonal matrix D, such that:

$$A = PDP^{-1}$$

It is then:

$$\begin{aligned} A^n &= (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})(PDP^{-1}) \\ &= PDP^{-1}PDP^{-1}\dots PDP^{-1}PDP^{-1} = P\underbrace{DD\dots D}_{n}P^{-1} \\ &= PD^nP^{-1} = P\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}P^{-1}. \quad \square \end{aligned}$$

Application: The n–th power of a Matrix

$$\text{Let } A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}. \text{ Find } A^{100}.$$

Of course, we will never calculate A^{100} with A in its original form.
We first search for the eigenvalues λ.

$$\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{bmatrix} = (-3-\lambda)(3-\lambda) - 4(-2) = \lambda^2 - 1$$

The condition $\det(A - \lambda I) = 0$ implies $\lambda^2 = 1$, so there are two non-equal eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$.

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} = \text{..show that..} = PD^{100}P^{-1} = P \begin{bmatrix} (-1)^{100} & 0 \\ 0 & (1)^{100} \end{bmatrix} P^{-1} \\ &= PIP^{-1} = PP^{-1} = I. \quad \square \end{aligned}$$

Example: Symmetric Matrices

Prove that any symmetric matrix of the form:

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

where $a \neq b$, a, b in $\mathbb{R}$, is diagonalisable and find P.

Show that A has two real eigenvalues: $\lambda_1 = a + b$, $\lambda_2 = a - b$ (it one of our practical problems for this week). These eigenvalues are non-equal as long as $b \neq 0$. So there are two cases:

(i) $b \neq 0$, where a 2×2 matrix with two non-equal eigenvalues has two linearly independent eigenvectors, therefore A is diagonalisable, and

(ii) $b = 0$, where A becomes $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, so it is diagonal in its original form.

✏ Therefore, any symmetric 2×2 matrix A is diagonalisable.

Example: Symmetric Matrices

Only for case (i) $b \neq 0$ is necessary to find the eigenvectors. (Why?)

In order to find the matrix P, we need the eigenvectors $\mathrm{u}_1, \mathrm{u}_2$, which we get by replacing the eigenvalues into $(A - \lambda I)\mathrm{x} = 0$.

✧ For $\lambda_1 = a + b$ it is:

$$\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} - (a+b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where the components of the eigenvectors u_1 follow the equation $-bx_1 + by_1 = 0$, or equivalently $-x_1 + y_1 = 0$, because $b \neq 0$, so the eigenspace for $\lambda_1 = a + b$ is:

$$V_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_1 = x_1, x_1 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} : x_1 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

The eigenvector is $\mathrm{u}_1 = [1, 1]^T$.

Example: Symmetric Matrices

We follow the same process for λ_2.

✧ For $\lambda_2 = a - b$ it is:

$$(A - \lambda_2 I)\mathrm{u}_2 = 0$$

$$\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} - (a-b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b & b \\ b & b \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where the components of the eigenvectors u_2 follow the equation $bx_2 + by_2 = 0$ or $x_2 + y_2 = 0$ $(b \neq 0)$, so the eigenspace for $\lambda_2 = a - b$ is:

$$V_{\lambda_2} = \left\{ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ in } \mathbb{R}^2 : y_2 = -x_2, x_2 \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

and we get $\mathrm{u}_2 = [1, -1]^T$.

So the matrix P is $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\square$

Example: Skew–Symmetric Matrices

Is the skew-symmetric matrix

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} ,$$

with $\omega > 0$ diagonalisable?

✏ There are cases where a matrix has *no real eigenvalues*.

✏ A is a skew-symmetric matrix, which means that $A^T = -A$.

Here it is:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix} = \lambda^2 + \omega^2$$

so, the condition $\det(A - \lambda I) = 0$ implies that $\lambda^2 + \omega^2 = 0$ and consequently that $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$. Since the matrix A has two purely imaginary complex conjugate eigenvalues, it cannot be diagonalised over $\mathbb{R}$. $\square$