Linear Transformations

MTH1004M Linear Algebra

UNIVERSITY OF
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Linear Transformations

Definition

A linear transformation is a transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which satisfies:

1. $T(\mathrm{u} + \mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$, for all u, v in $\mathbb{R}^n$.
2. $T(c\mathrm{u}) = cT(\mathrm{u})$, for all u in $\mathbb{R}^n$ and c in $\mathbb{R}$.

✏ The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

maps every vector in the plane to another vector in the plane.

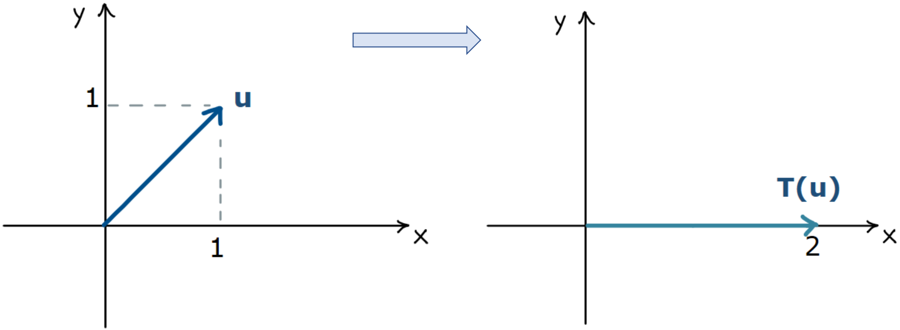
✏ The rotation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

rotates every vector in the plane counter-clockwise by the angle θ.

✏ The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$



✏ The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

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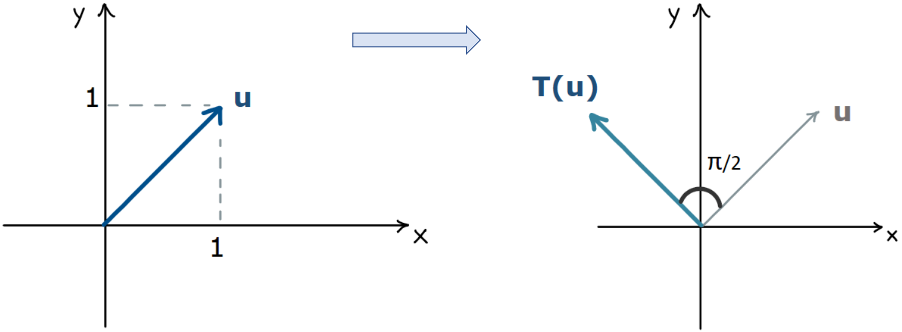


Linear Transformations

Properties

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then:

1. $T(0) = 0$
2. $T(-\mathrm{u}) = -T(\mathrm{u})$, for all u in $\mathbb{R}^n$.
3. $T(\mathrm{u} - \mathrm{v}) = T(\mathrm{u}) - T(\mathrm{v})$

✏ Consider a linear combination of the vectors u, v, namely $c_1\mathrm{u} + c_2\mathrm{v}$. When a linear transformation T acts on that vector, we have that

$$T(c_1\mathrm{u} + c_2\mathrm{v}) = c_1 T(\mathrm{u}) + c_2 T(\mathrm{v})$$

✏ The zero vector in $\mathbb{R}^n$ is always mapped to the zero vector in $\mathbb{R}^m$ $(T(0) = 0)$.

Example

Consider once again the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix}$$

To check that T is a linear transformation, we can follow the definition and define two vectors

$$\mathrm{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathrm{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

We need to show that 1. $T(\mathrm{u} + \mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ and

2. $T(c\mathrm{u}) = cT(\mathrm{u})$, where c is a scalar.

➜

Example

1. It is

$$
\begin{aligned}
T(\mathrm{u}+\mathrm{v}) &= T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2(x_1+x_2)-(y_1+y_2) \\ 3(x_1+x_2)+4(y_1+y_2) \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2x_1+2x_2-y_1-y_2 \\ 3x_1+3x_2+4y_1+4y_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ 2x_1-y_1 \\ 3x_1+4y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2-y_2 \\ 3x_2+4y_2 \end{bmatrix} = T(\mathrm{u})+T(\mathrm{v})
\end{aligned}
$$

2. Let c be a scalar. Then

$$
T(c\mathrm{u}) = T\begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 2cx_1-cy_1 \\ 3cx_1+4cy_1 \end{bmatrix} = c\begin{bmatrix} x_1 \\ 2x_1-y_1 \\ 3x_1+4y_1 \end{bmatrix} = cT(\mathrm{u})
$$

Thus, T is a linear transformation. $\square$

Cases

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then can be three cases in terms of the vector spaces dimensions:

1. $n < m$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + y \\ x - 3y \end{bmatrix}$$

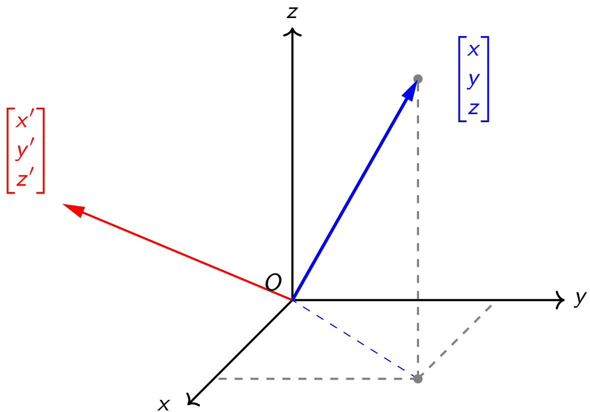
2. $n = m$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ y + z \\ x - y + 2z \end{bmatrix}$$

3. $n > m$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y - 2z \\ 2x + 3y - 4z \end{bmatrix}$$

A Linear Transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

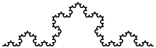
Linear Transformations

Proposition 1

Let A be an $m \times n$ matrix. Then the transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$T(\mathrm{x}) = A\mathrm{x}$$

is a linear transformation.



Proposition 2

Every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of the form

$$T(\mathrm{x}) = A\mathrm{x}$$

for some $m \times n$ matrix A.

Remarks

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad T(\mathrm{x}) = A\mathrm{x}$$

➜ We say that the $m \times n$ matrix A **acts** on a vector u and we mean that we perform the matrix multiplication Au, the result of which is another vector w in $\mathbb{R}^m$.

➜ In other words, the matrix A defines a **linear transformation** which maps any vector u in $\mathbb{R}^n$ to a some new vector w in $\mathbb{R}^m$.

➜ A matrix A defines uniquely a linear transformation and a linear transformation defines uniquely a matrix A.

Example

Show that the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

is linear.

✏ One way is to use the definition

✏ Another way is to use proposition 1:

$$A\mathrm{u} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix} = T(\mathrm{u})$$

Hence, the transformation is linear.

Example

Show that the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y + 1 \\ x - y \end{bmatrix}$$

is not a linear transformation.

✏ Here we can use the definition

$T(c\mathrm{u}) = cT(\mathrm{u})$, should be true for all scalars c.
Let $c = 0$, then

$$T(0 \cdot \mathrm{u}) = T(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot T(\mathrm{u})$$

Hence, the transformation is not a linear one.

Kernel and Range: Definitions

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

➜ The **kernel** of T, denoted $\mathrm{Ker}(T)$, is the set of all vectors in $\mathbb{R}^n$ that are mapped by T to 0 in $\mathbb{R}^m$. That is,

$$\mathrm{Ker}(T) = \{\mathsf{x} \text{ in } \mathbb{R}^n : T(\mathsf{x}) = 0\}$$

➜ The **range** of T, denoted $\mathrm{range}(T)$, is the set of all vectors in $\mathbb{R}^m$ that are images of vectors in $\mathbb{R}^n$ under T. That is,

$$\mathrm{range}(T) = \{\mathsf{y} \text{ in } \mathbb{R}^m : T(\mathsf{x}) = \mathsf{y} \text{ for some } \mathsf{x} \text{ in } \mathbb{R}^n\}$$

Kernel Example

Find the kernel of the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad\qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \end{bmatrix}$$

✏ **Remark** *The kernel of T is the Null space of its associated matrix.*

It is

$$\mathrm{Ker}(T) = \{\mathrm{x} \text{ in } \mathbb{R}^3 : T(\mathrm{x}) = 0\}$$

We search for those x in $\mathbb{R}^3$ which are mapped to the vector $0 = [0,0]^T$ in the plane.

$$\begin{aligned} T(\mathrm{x}) &= 0 \\ \begin{bmatrix} x+y \\ y+z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

which is a homogeneous system of 2 equations with 3 unknowns.

➜

By using Gauss–Jordan elimination we find:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Which yields, $x = z$ and $y = -z$, therefore all vectors in $\mathbb{R}^3$ with components $\mathrm{x} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix}$, for any real number z, give $T(\mathrm{x}) = 0$. So, the kernel of T is

$$\mathrm{Ker}(T) = \left\{ \begin{bmatrix} z \\ -z \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 : z \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

and observe that it is a vector subspace of $\mathbb{R}^3$ of dimension 1.

$$\dim(\mathrm{Ker}(T)) = 1 = nullity$$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the basis for $\mathrm{Ker}(T)$.

➜

✏ Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

be the 2×3 matrix inside the augmented matrix. This is the matrix associated with the transformation T.

✏ The Kernel of the transformation T is a vector subspace of $\mathbb{R}^3$

✏ $\mathrm{Ker}(T) = \mathrm{Null}(A)$

✏ **Nullity** is also the dimension of $\mathrm{Ker}(T)$.

Kernel and Range: Theorem

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

(i) The kernel of T is a subspace of $\mathbb{R}^n$

(ii) The range of T is a subspace of $\mathbb{R}^m$

✏ The range of T is equal to the span of the columns of the matrix A. In other words the range of T is the column space of the matrix A

$$\mathrm{range}(T) = \mathrm{col}(A)$$

✏ The dimension of the subspace $\mathrm{range}(T)$ is equal to the dimension of the subspace $\mathrm{col}(A)$. So,

$$\dim[\mathrm{range}(T)] = \mathrm{rank}(A)$$

Range Example

Find the range of the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad\qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \end{bmatrix}$$

and its basis. What is the rank of T?

✏ **Remark** *The range of T is the Column space of its associated matrix.*

Now let's find the range of T.

$$\mathrm{range}(T) = \{\mathsf{y} \text{ in } \mathbb{R}^2 : T(\mathsf{x}) = \mathsf{y} \text{ for some } \mathsf{x} \text{ in } \mathbb{R}^3\}$$

The definition says that we search for those 'values' of T which derive by the vectors x in $\mathbb{R}^3$. (In literature you will find the range also called as **image** of a transformation and denoted as $\mathrm{Im}(T)$.)

It is easier to find $\mathrm{col}(A)$ than to follow the definition above.

Range Example

It is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\mathrm{col}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent and form a basis for $\mathrm{range}(T)$.

The rank of T is 2.

One-to-One and Onto Linear Transformations

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

Definition

T is called *one-to-one* if T maps distinct vectors in $\mathbb{R}^n$ to distinct vectors in $\mathbb{R}^m$:

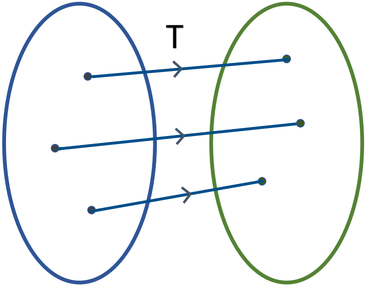
$$T(\mathrm{u}) = T(\mathrm{v}) \text{ implies that } \mathrm{u} = \mathrm{v}, \text{ for all u and v in } \mathbb{R}^n$$

Definition

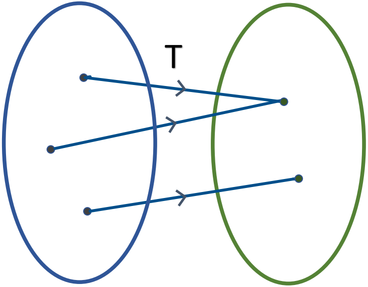
If $\mathrm{range}(T) = \mathbb{R}^m$, then T is called *onto*.

In other words, for all w in $\mathbb{R}^m$, there is at least one u in $\mathbb{R}^n$ such that $\mathrm{w} = T(\mathrm{u})$

One-to-one transformations

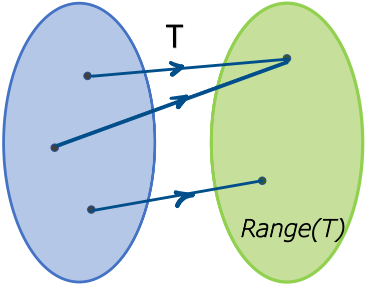


I. T is one-to-one

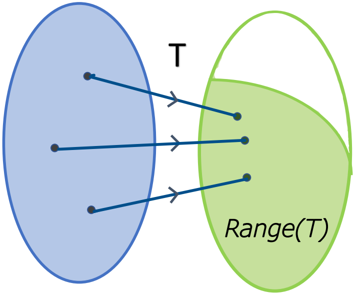


II. T is *not* one-to-one

Onto transformations



I. T is onto



II. T is *not* onto

Quiz

Which of the following linear transformations are one-to-one? Which are onto?

1. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + y \\ x - 3y \end{bmatrix}$

2. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -y \\ x \end{bmatrix}$

3. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 3y \end{bmatrix}$

Transformation 1

$$1. \quad T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad\qquad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + y \\ x - 3y \end{bmatrix}$$

✏ Is it one-to-one?

Let $T(\mathrm{u}) = T(\mathrm{v})$, where $\mathrm{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. It yields that

$$\begin{bmatrix} 2x_1 - y_1 \\ x_1 + y_1 \\ x_1 - 3y_1 \end{bmatrix} = \begin{bmatrix} 2x_2 - y_2 \\ x_2 + y_2 \\ x_2 - 3y_2 \end{bmatrix}$$

hence we get that $x_1 = x_2$ and $y_1 = y_2$, so $\mathrm{u} = \mathrm{v}$. We conclude that T is one-to-one.

➜

Transformation 1

✏ Is T onto?

To be onto, every vector in $\mathbb{R}^3$ should come from a vector in the domain of T. Let's see whether there is a vector u that is mapped in e_3 under the transformation T. It should be:

$$\begin{aligned} T(\mathrm{u}) &= \mathrm{e}_3 \\ \begin{bmatrix} 2x - y \\ x + y \\ x - 3y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

which cannot be true, since the first two equations give $x = y = 0$ and the last $0 = 1$. Hence, we conclude that T is not onto.

Transformation 2

$$2. \quad T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad\qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -y \\ x \end{bmatrix}$$

✏ Is it one-to-one?

Let $T(\mathrm{u}) = T(\mathrm{v})$, where $\mathrm{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. It yields that

$$\begin{bmatrix} 3z_1 \\ -y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3z_2 \\ -y_2 \\ x_2 \end{bmatrix}$$

hence we get that $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$, so $\mathrm{u} = \mathrm{v}$. We conclude that T is one-to-one.

➜

Transformation 2

✏ Is T onto?

Let's see whether there a vector $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ mapped to the vector $v = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ under the transformation T. It should be:

$$\begin{aligned} T(u) &= v \\ \begin{bmatrix} 3z \\ -y \\ x \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \end{aligned}$$

The system admits the unique solution

$$x = z_0, y = -y_0, z = x_0/3$$

Hence, T is onto.

Transformation 3

$$3. \quad T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad\qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+3y \end{bmatrix}$$

✏ Is it one-to-one?

Let $T(\mathrm{u}) = T(\mathrm{v})$, where $\mathrm{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. It yields that

$$\begin{bmatrix} x_1 + y_1 \\ 2x_1 + 3y_1 \end{bmatrix} = \begin{bmatrix} x_2 + y_2 \\ 2x_2 + 3y_2 \end{bmatrix}$$

so we get that $x_1 = x_2$, $y_1 = y_2$. However, we do not get $z_1 = z_2$, so we cannot conclude that $\mathrm{u} = \mathrm{v}$. Hence T is not one-to-one.

For example, the vectors $[1, 1, 0]^T$ and $[1, 1, 1]^T$ are mapped to the same vector $[2, 5]^T$.

➜

Transformation 3

✏ Is T onto?

Let's see whether there is a vector $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ mapped to the vector $v = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ under the transformation T. It should be:

$$\begin{aligned} T(u) &= v \\ \begin{bmatrix} x + y \\ 2x + 3y \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \end{aligned}$$

The system admits infinitely many solutions:

$$x = 2x_0 - y_0/2, y = (y_0 - 2x_0)/2, z \text{ in } \mathbb{R}$$

Hence, T is onto.

Theorems

Theorem

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one-to-one if and only if $\mathrm{Ker}(T) = \{0\}$.

Theorem

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is one-to-one if and only if it is onto.

Rank Theorem

Theorem (Revision)

Let A be an $m \times n$ matrix. Then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n \ .$$

✏ Rank Theorem translated for transformations:
Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

$$dim[\mathrm{range}(T)] + dim[\mathrm{Ker}(T)] = n \ .$$

✏ Example

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad\qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+3y \end{bmatrix}$$

$$dim[\mathrm{range}(T)] = 2, \quad dim[\mathrm{Ker}(T)] = 1 \ .$$

Isomorphism

Definition

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called an isomorphism if it is one-to-one and onto.

✏ An isomorphism is always a linear transformation from a vector space to a vector space of the **same dimension**.

✏ The matrix associated to an isomorphism $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a $n \times n$ matrix A, which is **invertible** ($\det A \neq 0$).

✏ Example

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -y \\ x \end{bmatrix}$$

$$dim[\mathrm{range}(T)] = 3, \quad dim[\mathrm{Ker}(T)] = 0\ .$$

Isomorphism example

Is the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ y + z \\ x - y + 2z \end{bmatrix}$$

an isomorphism?

✏ There are many different ways to determine whether this is an isomorphism. One way is to show that $\mathrm{Ker}(T) = \{0\}$. If this is true, then T is one-to-one, and by the theorem is also onto, hence an isomorphism.

✏ We can check the determinant of the associated matrix A:

$$\det A = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 4 \neq 0$$

Isomorphism example

Hence, A is invertible with $\operatorname{rank}(A) = 3$, so

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 3$$

yields that $\operatorname{nullity}(A) = 0$ and therefore $\operatorname{Null}(A) = \operatorname{Ker}(T) = \{0\}$.