Practical 5

Problem 1. Find the inverse of the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

using the standard formula for the inverse of 2×2 matrices:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem 2. Find the inverse of the matrix

$$B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

using Gauss–Jordan elimination and its rank.

Problem 3. Prove that:
(a) If A is invertible and $AB = O$, then $B = O$.
(b) If A is invertible and $AB = AC$, then $B = C$.
(c) If A satisfies the matrix equation $A^2 - 2A + I = O$, then A is invertible and find its inverse.
(d) If A satisfies the matrix equation $A^3 + \alpha A^2 + \beta A - I = O$, then A is invertible and find its inverse.

Problem 4. (a) Compute the determinant of the following matrices:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 4 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(b) Find the determinant of the matrix A^n.

Problem 5. Using the determinant properties (and not cofactor expansions), show that:

$$\det\left(P^{-1}AP\right) = \det A$$

$$\det\left(P^{T}AP\right) = \det A(\det P)^2$$

(c) If $\det A = 1$ and $ABA^T = B^2$, then $\det B = 0$ or 1.

Problem 6. Use Cramer's rule to solve the following linear systems:

$$\begin{cases} 2x + y - z = 1 \\ -x + 2y + z = 1 \\ x - y + 2z = 1 \end{cases}, \quad \begin{cases} x + ay + az = 0 \\ x - y + az = 0 \\ x + y + z = 0 \end{cases}$$

For which a values the second system has unique solution?