Ideas of mathematical proof. Practical Week 20

Note that solutions must be by the methods specified in the problems – even if you can see another solution, even if an easier one.

Here, "induction" means any of the induction methods: Axiom of Math. Induction, or Extended A.M.I., of Cumulative A.M.I.

2.1. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence defined recursively as $a_1 = 1$ and $a_{i+1} = 3a_i - 1$. Use induction to prove that $a_n = \dfrac{3^{n-1}+1}{2}$ for all $n \in \mathbb{N}$.

2.2. Let $(b_i)_{i\in\mathbb{N}}$ be a sequence defined recursively as $b_1 = 0,\ \ b_2 = 1$, and $b_i = 3b_{i-1} - 2b_{i-2}$ for $i \geqslant 3$. Use induction to prove that $b_n = 2^{n-1} - 1$ for all $n \in \mathbb{N}$.

2.3. Let the universal set be $\mathscr{U} = \{x \in \mathbb{Z} \mid -9 \leqslant x \leqslant 9\}$, and let A be the set of all odd integers in $\mathscr{U}$, let $B = \{x \in \mathscr{U} \mid x^2 > 25\}$, and $C = \{x \in \mathscr{U} \mid x < 0\}$. Determine each of the following sets and list their elements:
(1) $A \cap B \cap C$;
(2) $A \cap \overline{B}$;
(3) $A \cap \overline{C}$;
(4) $B \cup C$.

2.4. Use Venn diagrams to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for any sets A, B, C.

2.5. Use the properties of operations on sets to simplify the expression $B \cup \overline{(\overline{A} \cap \overline{B})}$.

2.6. Solve the (simultaneous) system of inequalities using intersection of solutions of individual inequalities and write the solution as a set:

$$\begin{cases} x^2 + x - 6 \geqslant 0 \\ x^2 - 7x + 10 \geqslant 0. \end{cases}$$

2.7. Prove 'from 1st principles' (based on the definitions) that $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.

2.8. Use mathematical induction to prove that every positive integer $n \geqslant 8$ can be represented as $n = 3a + 5b$, where a and b are non-negative integers. [*Hint*: verify induction base for $n = 8, 9, 10$, then use Cumulative AMI.]

2.9. Consider the function $f(x) = \dfrac{x}{x+1}$, and define recursively a sequence of functions

$$f_1(x) = f(x) \quad \text{and} \quad f_{i+1}(x) = f(f_i(x)).$$

Calculate $f_1(x)$, $f_2(x)$, $f_3(x)$, make a guess, and then prove your guess by induction.