Vector Spaces & Subspaces, Bases & Dimension

MTH1004M Linear Algebra

UNIVERSITY OF LINCOLN

Examples of Vector Spaces

A vector space is a set V together with two operations:
vector addition and **scalar multiplication**
(definition in the next slide)

Some basic examples of vector spaces are:

✏ Real numbers: $\mathbb{R}$

✏ The plane: $\mathbb{R}^2$

✏ The space: $\mathbb{R}^3$

✏ The complex plane: $\mathbb{C}$

✏ The set of matrices: $M_{2x2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, \ldots, a_{22} \in \mathbb{R} \right\}$

Definition of Vector Spaces

A vector space is a set V together with two operations (**vector addition** and **scalar multiplication**) that satisfy:

1. For each $\mathbf{x}$ and $\mathbf{y}$ in V, $\mathbf{x}+\mathbf{y}$ is in V (closure under addition)
2. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$, for all $\mathbf{x},\mathbf{y}$ in V (commutativity)
3. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$, for all $\mathbf{x},\mathbf{y},\mathbf{z}$ in V (associativity)
4. There exists an element, called the zero vector $\mathbf{0}$, such that $\mathbf{0}+\mathbf{x}=\mathbf{x}$
5. For each $\mathbf{x}$ in V there exists an element $-\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$
6. For each scalar c and $\mathbf{x}$ in V, $c\mathbf{x}$ is in V (closure under scalar multiplication)
7. $c(\mathbf{x}+\mathbf{y})=c\mathbf{x}+c\mathbf{y}$, for all scalars c and $\mathbf{x},\mathbf{y}$ in V (distributivity)
8. $(c+d)\mathbf{x}=c\mathbf{x}+d\mathbf{x}$, for all scalars c,d and $\mathbf{x}$ in V (distributivity)
9. $(cd)\mathbf{x}=c(d\mathbf{x})$, for all scalars c,d and $\mathbf{x}$ in V
10. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}$ in V

Examples of Vector Spaces

✏ $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ are vector spaces, since they satisfy all the axioms in the definition of vector spaces.

✏ We write $\mathbb{R}^n$ as a set of vectors:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \ldots, x_n \text{ in } \mathbb{R} \right\}$$

✏ The complex plane $\mathbb{C}$ is also a vector space, since its elements satisfy all axioms. Each element $\mathbf{z} = x + iy$ is associated to a vector $\mathbf{z} = [x, y]^T$.

Spaces and Subspaces

Definition

Let S be a subset of a vector space V. We say that S is a vector subspace S of V if:

1. The zero vector $\mathbf{0}$ is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S. (closure under addition)
3. If $\mathbf{u}$ is in S and c is a scalar, then $c\mathbf{u}$ is in S. (closure under scalar multiplication)

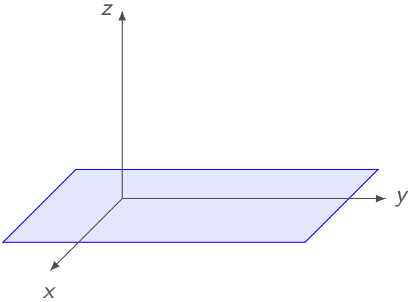
✏ Any vector subspace is also a vector space itself.

Spaces and Subspaces

Examples:

1. Any line in the plane passing through the origin is a subspace of $\mathbb{R}^2$.
2. Any line in the plane passing through the origin is a subspace of $\mathbb{R}^n$, for any n.
3. The $x-y$ plane is a subspace of $\mathbb{R}^3$.
4. $y-z$ plane is also a subspace of $\mathbb{R}^3$.
5. Any plane passing through the origin is a subspace of $\mathbb{R}^3$.
6. The zero vector $\mathbf{0}$ is subspace of $\mathbb{R}^n$, for any n.

Example: The x–y plane in $\mathbb{R}^3$



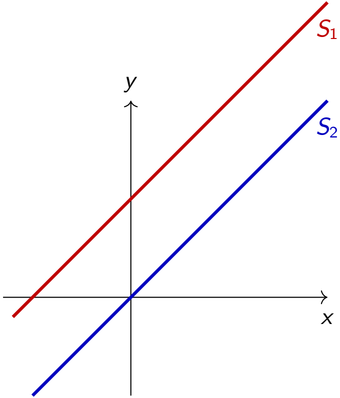
Let S be the x–y plane in $\mathbb{R}^3$. Then S is a subspace of $\mathbb{R}^3$. The set S is written as:

$$S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

Solution:

1. The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ belongs to S.
2. If $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}$ are in S, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix}$ is in S.
3. If $\mathbf{u} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is in S and c is a scalar, then $c\mathbf{u} = \begin{bmatrix} cx \\ cy \\ 0 \end{bmatrix}$ is in S.

Examples



Consider the two subsets of $\mathbb{R}^2$:

$$S_1 = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$$

While S_2 consists a vectors subspace of $\mathbb{R}^2$, because satisfies all three properties, the subset S_1 is not.

The *easiest way* to prove that S_1 is not a vectors subspace of $\mathbb{R}^2$ is proving that the zero vector does not belong in S_1.

Vector Subspaces

Proposition

Let

$$S = span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$$

be the span of k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V. Then S is a vector subspace of the vector space V.

✏ The span of a set of vectors in $V = \mathbb{R}^n$ guarantees the three properties which every vector subspace satisfies:

Property 1: the zero vector can be produced by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ (if we regard zero scalars)

Properties 2 and 3: derive from the linear combinations of those k vectors

Basis and Dimension

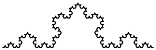
Definition

A *basis* for a subspace S of V is a set of vectors in S that:

1. spans S and

2. is linearly independent*.

* **Remark:** By ‘linearly independent set of vectors’ we mean that the vectors are linearly independent. Think of $V = \mathbb{R}^n$.



Definition

The number of vectors in a basis for a subspace S is called the *dimension* of S and is denoted by $\dim S$.

Examples of Bases

1. The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ span $\mathbb{R}^n$ and are linearly independent. They are the *standard basis* of $\mathbb{R}^n$ and $\dim(\mathbb{R}^n) = n$.

2. The two vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

span the plane and are linearly independent (can be shown). Therefore, they form a basis for $\mathbb{R}^2$.

2. The two vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

do not span the plane and are linearly dependent. So, they do not form a basis for $\mathbb{R}^2$.

Remarks

✏ Not only the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$, but also the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$.

✏ There can be infinitely many bases for $\mathbb{R}^n$.

✏ If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent vectors, then they form a basis for $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$.

✏ On the other hand, if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly dependent, they do not form a basis for $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$. A basis for $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ will be a subset of these vectors, say k vectors where $k < n$ and

$$\dim[\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)] = k \quad .$$

Example

Show that

$$S = \left\{ \begin{bmatrix} x \\ x+y \\ -y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

is the vector subspace of $\mathbb{R}^3$ *and find a basis for* S.

Property 1: The zero vector $\mathbf{0}$ belongs to S (we get the zero vector when $x = y = 0$).

Property 2: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 + y_1 \\ -y_1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} x_2 \\ x_2 + y_2 \\ -y_2 \end{bmatrix}$ two vectors in S, then

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_1 + y_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 + y_2 \\ -y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ (x_1 + x_2) + (y_1 + y_2) \\ -(y_1 + y_2) \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ -y \end{bmatrix}$$

which is in S for $x = x_1 + x_2$ and $y = y_1 + y_2$

➜

Example

Property 3: Let c a scalar and $\mathbf{x} = \begin{bmatrix} x' \\ x' + y' \\ -y' \end{bmatrix}$ in S. It is

$$c\mathbf{x} = \begin{bmatrix} cx' \\ cx' + cy' \\ -cy' \end{bmatrix} = \begin{bmatrix} x \\ x + y \\ -y \end{bmatrix}$$

which is in S for $x = cx'$ and $y = cy'$. So, S is a vector subspace of $\mathbb{R}^3$.

An **easier way** to prove that S is a subspace and find its basis, is to appropriately decompose the general form of its vectors, namely

$$S = \left\{ \begin{bmatrix} x \\ x + y \\ -y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

By the previous proposition, S is a vector subspace of $\mathbb{R}^3$. ➜

Example

So, we have shown that the two vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$ span the space S and remains to show that they are linearly independent. The equation

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

immediadely yields $c_1 = c_2 = 0$, therefore these vectors form a basis for S. $\square$

Theorem

Theorem

Any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Proof:
A set of n linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ when they span $\mathbb{R}^n$.

By proposition we know that $S = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a vector subspace of $\mathbb{R}^n$ and we need to show that $S = \mathbb{R}^n$.

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ do not form a basis for $\mathbb{R}^n$. There should exist m additional linearly independent vectors $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_{n+m}$ which, together with the n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$.

But this assumption would imply $\dim \mathbb{R}^n = n + m$, which is false since $\dim \mathbb{R}^n = n$. Therefore $S = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \mathbb{R}^n$.

Example

Determine whether the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \ \textit{and} \ \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

According to the Theorem, it is enough to check whether those three vectors are linearly independent. We solve the homogeneous system
$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ in terms of the scalars c_1, c_2, c_3.
by the Gauss–Jordan elimination method:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

➜

Example

➜

From the reduced row-echelon form above, we conclude that its solution is $c_1 - c_2 = 0$, $c_2 - c_3 = 0$ or $c_1 = c_2 = c_3$, which implies that the vectors are linearly dependent, and the initial vector equation becomes:

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$$

(since c_1 is a free parameter). Let $\mathbf{u}_3 = -\mathbf{u}_1 - \mathbf{u}_2$. Are the remaining two vectors $\mathbf{u}_1, \mathbf{u}_2$ linearly independent?

Example

Find a basis for the set of 2×2 *matrices*

$$M_{2x2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, \ldots, a_{22} \in \mathbb{R} \right\}$$

and its dimension.

Notice that the elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for M_{2x2}, since they are linearly independent and they span the space. It is $\dim M_{2x2} = 4$.

Example

Show that the set of 2×2 *symmetric matrices*

$$S_{2x2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}$$

is a vector subspace of M_{2x2}.

We can prove this by showing the three properties:

1. *Closure under Matrix addition:* Let A, B in S_{2x2}, then $A + B$ is also in S_{2x2}.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{12} + b_{12} & a_{22} + b_{22} \end{bmatrix} \text{ is symmetric}$$

2. *Closure under Scalar multiplication:*

Let λ be a scalar and A in S_{2x2}, then λA belongs to S_{2x2}.

3. *The zero matrix O:* belongs to S_{2x2} as well.

Challenge

Can you prove that the set of $n \times n$ *symmetric matrices*

$$S_{nxn} = \left\{ A \text{ in } M_{n \times n} : A^T = A \right\}$$

is a vector subspace of M_{nxn} *?*

Example

Find a basis for the set of 2×2 *symmetric matrices*

$$S_{2x2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}$$

and determine its dimension.

Any 2×2 symmetric matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ is the linear combination of some basic matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = a_{11} A_1 + a_{12} A_2 + a_{22} A_3.$$

Hence, we get that

$$1. \quad \mathrm{span}(A_1, A_2, A_3) = S_{2x2}.$$

(Remark: Equivalent way to prove that S_{2x2} is a vector subspace of M_{2x2}.)

➜

➜

2. But also A_1, A_2, A_3 are linearly independent elements:

Let

$$c_1 A_1 + c_2 A_2 + c_3 A_3 = O$$

then we get that

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies $c_1 = c_2 = c_3 = 0$.

So, we conclude that A_1, A_2, A_3 is a basis for S_{2x2} and $\dim S_{2x2} = 3$.

Applications to Matrices

Let A be a $n \times m$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}$$

We consider the rows or the columns of a matrix as vectors.

The matrix A has:

- n vectors in $\mathbb{R}^m$ (its rows, e.g. $[a_{11}, a_{12}, \ldots, a_{1m}]^T$)
- m vectors in $\mathbb{R}^n$ (its columns, e.g. $[a_{11}, a_{21}, \ldots, a_{n1}]^T$)

➜ Aim: Find the vector subspace of $\mathbb{R}^m$ that is spanned by the rows of the matrix.

➜ Aim: Find the vector subspace of $\mathbb{R}^n$ that is spanned by the columns of the matrix.

➜ Aim: Determine whether the rows or the columns of the matrix are linearly independent.

Subspaces associated with matrices

Definitions for an $n \times m$ matrix A:

1. The **row space** of A is the subspace $\mathrm{row}(A)$ of $\mathbb{R}^m$ spanned by the rows of A.

2. The **column space** of A is the subspace $\mathrm{col}(A)$ of $\mathbb{R}^n$ spanned by the columns of A.

3. The **null space** of A, denoted by $\mathrm{null}(A)$, is the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

Theorem
The row and column spaces of a matrix A have the same dimension.

✏ The rank of A is equal to the dimension of its row space

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A))$$

✏ and is equal to the dimension of its column space

$$\mathrm{rank}(A) = \dim(\mathrm{col}(A))$$

Example

Find the subspaces $\mathrm{row}(A)$, $\mathrm{col}(A)$, $\mathrm{null}(A)$ *for the* 3×4 *matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and determine their dimensions.

Row space: $\mathrm{row}(A) = \mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where

$$\mathbf{u}_1 = [1, 0, 0, 0], \mathbf{u}_2 = [0, 1, 0, 0], \mathbf{u}_3 = [0, 0, 1, 0] \ .$$

The vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are 3 vectors in $\mathbb{R}^4$. They are linearly independent and they span $\mathrm{row}(A)$, therefore they consist a basis for $\mathrm{row}(A)$. Moreover, it is $\dim(\mathrm{row}(A)) = 3$.

Example

Column space: $\mathrm{col}(A) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, where

$$\mathbf{v}_1 = [1, 0, 0], \mathbf{v}_2 = [0, 1, 0], \mathbf{v}_3 = [0, 0, 1], \mathbf{v}_4 = [0, 0, 0] \ .$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are 4 vectors in $\mathbb{R}^3$, which are linearly dependent since $\mathbf{v}_4$ is the zero vector.

However, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent and span $\mathrm{col}(A)$, therefore they consist a basis for $\mathrm{col}(A)$. Moreover, it is $\dim(\mathrm{col}(A)) = 3$.

Null space: $\mathrm{null}(A)$. The homogeneous linear system $A\mathbf{x} = \mathbf{0}$

$$Ax = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solutions $x = 0$, $y = 0$, $z = 0$ and w in $\mathbb{R}$.

Example

So,

$$\mathrm{null}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \text{for } x = y = z = 0,\ w \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \end{bmatrix} : w \text{ in } \mathbb{R} \right\} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

which is the w–axis, an one-dimensional subspace of $\mathbb{R}^4$

$$\dim(\mathrm{null}(A)) = 1$$

Rank: $\mathrm{rank}(A)$ is equal to the dimension of row or column spaces, so it is:

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)) = 3\ .$$

✏ Notice that $\dim(\mathrm{null}(A)) + \mathrm{rank}(A) = 4 = \dim(\mathbb{R}^4)$.

Nullity and Theorems

Definition

The nullity of a matrix A is the dimension of its null space $\mathrm{null}(A)$ and is denoted by $\mathrm{nullity}(A)$.

Theorem

Let A be an $n \times m$ matrix. Then $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^m$.

Rank Theorem

Let A be an $n \times m$ matrix. Then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = m \ .$$

The fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced echelon row form for A is I_n.

e. $\mathrm{rank}(A) = n$

f. $\mathrm{null}(A) = \{\mathbf{0}\}$ (or $\mathrm{nullity}(A) = 0$)

g. The column/row vectors of A form a basis for $\mathbb{R}^n$.

h. $\det A \neq 0$