Ideas of mathematical proof

Slides Week 29

Optional remarks on mathematical logic.
Limits of sequences: definition, uniqueness.
Arithmetic of finite limits of sequences.
Sandwich Theorem for sequences.
Limits of monotonic sequences.

Optional remarks: Mathematical Logic

Mathematical Logic analyses whole mathematical theories:

$$\begin{array}{c} \text{THEOREMS} \\ \uparrow \\ \text{by logical rules of inference} \\ \uparrow \\ \text{AXIOMS} \end{array}$$

The whole theory viewed as a tree, with roots = axioms, branching into theorems.

Role of mathematical logic in mathematics

Nowadays, there are, roughly, three aspects of logic within mathematics.

1. Logical foundations of mathematics, logic as a basis for the whole of mathematics. (...Never ending process, but most mathematicians are satisfied with the foundations that exist since about 100 years ago.)
2. Mathematical logic as one of mathematical disciplines, with its axioms, theory, results, journals, conferences, etc. (Other branches of mathematics often can largely ignore it, as often happens between specialized branches of mathematics.)

Role of mathematical logic in mathematics

3. Logic essentially used in other branches of mathematics. (Like continuum hypothesis, decidablity of theories, local theorems, model theory tools, etc.)

Optional remarks

Historically, Euclid's geometry was axiomatic 2000 years ago.

More difficult for 'real' mathematics. But possible in principle, based on sets and logical rules.

Logical foundations were created about 150–200 years ago (including by George Boole in Lincoln!).

Response to a crisis in foundations of mathematics.

Optional remarks

Even with sets, one must be careful:

Russell's paradox: Consider the set S of all sets A such that $A \notin A$.

Is $S \in S$?

If yes, then must have $S \notin S$ by def. of S.

If not, then must have $S \in S$ by def. of S.

Contradiction! in each case.

Optional remarks

The same **Russell's paradox** without sets:

call a phrase "self-describing"

if it applies to itself.

E.g. "involves more than 5 letters" is self-describing.

Is "non-self-describing" self-describing or not?

If yes, then it is not, if not, then it is.

Contradiction!

Optional remarks

There are some 'schools', branches of mathematical logic, or of the whole mathematics, with different approaches to foundations of mathematics: intuitionism, constructivism.

E.g. in some extreme branch of constructivism, no infinite sets are allowed!

Most of 'normal' mathematics is based

on set theory and so-called

Zermelo–Frenkel system of axioms,

usually also including Axiom of Choice.

Optional remarks

Apart from sorting out foundations of mathematics, mathematical logic also gave rise to algorithmic approaches to theories.

Decidable or undecidable theories: By definition, a theory is decidable if there is an algorithm, fixed set of rules, to decide if an arbitrary given formula is true in this theory.

Optional remarks

Example of a decidable theory: The so-called first-order theory of algebraically closed fields of a given characteristic, established by Alfred Tarski in 1949.

Example of an undecidable theory: The so-called first-order theory of groups, established by Tarski in 1953. Remarkably, not only the general theory of groups is undecidable, but also several more specific theories, for example the theory of finite groups (as established by Mal'cev 1961).

'Silver lining' of this 'negative' result is that at any moment in time, there will always be open problems in group theory! **:-)**

Optional remarks

'Substantial' results obtained by means of mathematical logic:

As already mentioned:

Optional remark: Continuum Hypothesis.

We know $\aleph_0 \lneqq \mathbf{c}$. But is there anything in between?

Continuum Hypothesis stated that there was no intermediate cardinal:

if $\aleph_0 \leq |A| \leq \mathbf{c}$, then either $|A| = \aleph_0$ or $|A| = \mathbf{c}$.

In other words, any subset of $\mathbb{R}$ is either countable, or of cardinality $\mathbf{c}$.

Almost 100 years remained a major open problem.

Answered only in the 1960s: Paul Cohen proved that this cannot be proved, and the negation also cannot be proved.

Optional remarks

Finally, mathematical logic became also an ‘applied’ discipline, as it nowadays provides a language for modern computer science, including development of AI.

Parts of computer hardware that perform logical operations AND, OR, NOT are called ‘Boolean gates’, in honour of George Boole, pioneer of mathematical logic.

4. Limits of sequences and functions

Limits of sequences

Recall notation: sequence $a_1, a_2, a_3, \ldots$

short notation: $(a_n)_{n \in \mathbb{N}}$, even shorter (a_n).

In this chapter – sequences of real numbers.

Recall: often defined by a formula:

e.g. $a_n = \dfrac{(-1)^n}{n}$, or $a_n = \dfrac{\sqrt{n} + n}{2n - 1}$.

Can also be defined recursively:

e.g. $a_1 = 1$, $a_2 = 2$ and $a_{k+2} = 2a_k - 3a_{k+1}$.

Limits of sequences informally

Recall: a sequence $(a_n)_{n\in\mathbb{N}}$ has a limit L,

denoted: $a_n \to L$ as $n \to \infty$,

or (which is the same): $\lim\limits_{n\to\infty} a_n = L$.

Meaning: "a_n approaches L as n goes to ∞",

but what does this mean exactly?

Better: "a_n becomes as close to L as required

for all sufficiently large n".

This is still imprecise: what is "as close as required"?

what is "for all n sufficiently large"?

Historic remark

Calculus first appeared in 16–17 centuries

with imprecise definitions of limits, derivatives, integrals.

So-called infinitesimals were used

— "infinitely small quantities".

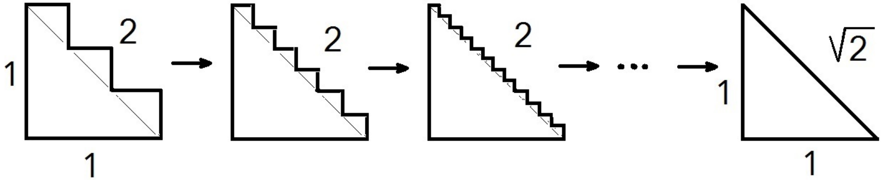
Mathematicians relied on their intuition to avoid errors

(not always successfully).

A simple example of improper use of limits:

Example of improper use of limits

Making zigzag 'steps' ever smaller, we make the zigzag line approach in the limit the hypotenuse.



At every step the length of the zigzag line is 2:
the sum of all horizontal pieces is 1,
and the sum of all vertical pieces is 1.

Then 'surely' the length of the limit – the hypotenuse – must be the limit of these lengths: $\lim 2 = 2$.

We obtained $2 = \sqrt{2}$....

Historic remark on rigorous definitions

It was only in the 19th century
that precise definitions of limits, derivatives, integrals
were given and rigorous foundations of calculus
were established, as we know today.

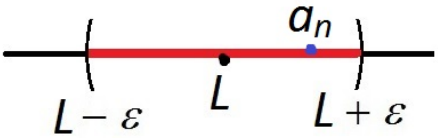
Measure of "closeness":

a_n is "ε-close" to L if $a_n \in (L - \varepsilon,\ L + \varepsilon)$,

the same as $L - \varepsilon < a_n < L + \varepsilon$,

the same as $|a_n - L| < \varepsilon$.

Graphically:



"for all n large enough" means

for all $n > N$ for some $N = N(\varepsilon)$.

Definition of a finite limit of a sequence

Definition

A sequence (a_n) has a **finite limit** $\lim\limits_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ (depending on ε) such that $|a_n - L| < \varepsilon$ for all $n > N(\varepsilon)$.

<u>Using logical expressions:</u> $\lim\limits_{n\to\infty} a_n = L$ if

$$\forall\, \varepsilon > 0\ \exists\, N(\varepsilon)\ \forall\, n > N(\varepsilon)\ \ |a_n - L| < \varepsilon.$$

Remarks on notation

1. Alternative notation for $\lim_{n \to \infty} a_n = L$

   is often used:

   $a_n \to L$ as $n \to \infty$.

2. Note that ∞ is not a number,

   just a symbol used in this notation,

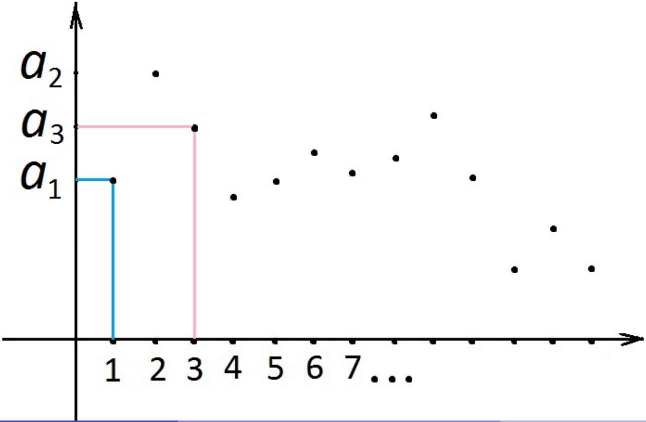
   the precise meaning is in that definition.

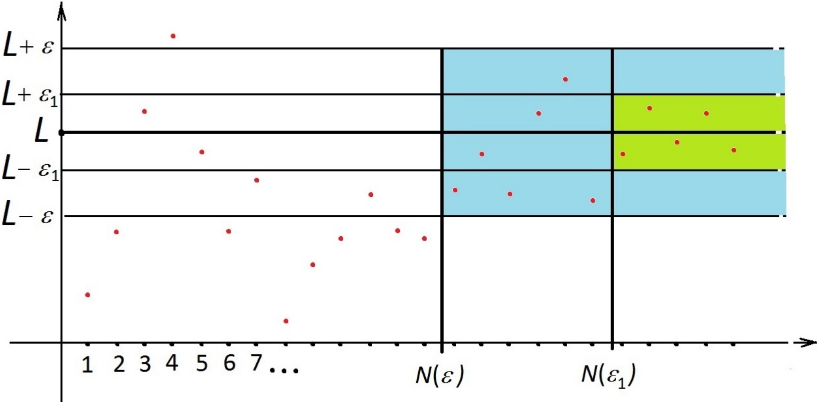
Graphical interpretation of a finite limit

Recall: a sequence (a_n) can be regarded

as a mapping (function) $f : \mathbb{N} \to \mathbb{R}$

with domain positive integers, so that $a_n = f(n)$.





Then $\lim_{n\to\infty} a_n = L$ means that for any $\varepsilon > 0$ there is $N(\varepsilon)$ such that all points (n, a_n) of the graph with $n > N(\varepsilon)$, that is, to the right of $N(\varepsilon)$, are in the strip $(N(\varepsilon), \infty) \times (L - \varepsilon,\ L + \varepsilon)$ (blue and green).

For a smaller ε_1 possibly bigger $N(\varepsilon_1)$ (green strip)....

Limits from 1st principles

Example

Prove 'from first principles' that $\lim_{n\to\infty} 1/n = 0$.

Given any $\varepsilon > 0$ we need to find $N(\varepsilon)$
such that $|1/n - 0| < \varepsilon$ for all $n > N(\varepsilon)$.

So we solve the inequality with respect to n:

$|1/n - 0| < \varepsilon \;\Leftrightarrow\; 1/n < \varepsilon$;

here $\varepsilon > 0$ and $n > 0$, so we can divide by ε and multiply by n: ...$\Leftrightarrow 1/\varepsilon < n$. This tells us for which n it holds.

So we can put $N(\varepsilon) = 1/\varepsilon$, then for $n > N(\varepsilon)$ all hold,

so $|1/n - 0| < \varepsilon$ for all $n > N(\varepsilon)$, as required.

Example

Prove from 'first principles' that $\lim_{n\to\infty} \frac{n^2-1}{n^2} = 1$.

Given any $\varepsilon > 0$, need to find $N(\varepsilon)$ such that $\left|\frac{n^2-1}{n^2} - 1\right| < \varepsilon$ for all $n > N(\varepsilon)$.

Solve the inequality with respect to n:

$$\frac{|n^2 - 1 - n^2|}{n^2} < \varepsilon \;\Leftrightarrow\; \frac{1}{n^2} < \varepsilon \;\Leftrightarrow\; \frac{1}{\varepsilon} < n^2 \;\Leftrightarrow\; \frac{1}{\sqrt{\varepsilon}} < n.$$

Thus, we can put $N(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$, and then we have

$\left|\frac{n^2-1}{n^2} - 1\right| < \varepsilon$ for any $n > N(\varepsilon)$, as required.

Example

Prove 'from first principles' that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

Given any $\varepsilon > 0$, need to find $N(\varepsilon)$ such that $|\frac{1}{\sqrt{n}} - 0| < \varepsilon$ for all $n > N(\varepsilon)$.

Solve the inequality with respect to n: $\ldots \Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon.$

We can square: $\ldots \Leftrightarrow 1/n < \varepsilon^2$ since both sides > 0.

We can divide by ε^2 and multiply by n: $\ldots \Leftrightarrow 1/\varepsilon^2 < n$.

Thus, we can put $N(\varepsilon) = 1/\varepsilon^2$ and then we have $|\frac{1}{\sqrt{n}} - 0| < \varepsilon$ for all $n > N(\varepsilon)$, as required.

Convergent and non-convergent sequences

We are now dealing only with **finite limits**;

infinite limits will be defined a bit later.

Definition

A sequences that has a finite limit

is said to be **convergent**.

Of course, not every sequence has a limit: e.g.

$a_n = (-1)^n$ (so it is -1 for odd n, and 1 for n even).

How to prove rigorously, based on definition,

that this (a_n) has no limit?

Triangle inequality

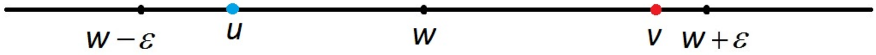
... is often useful (also true for complex numbers):

Fact

If $|u - w| < \varepsilon$ and $|v - w| < \varepsilon$, then $|u - v| < 2\varepsilon$.

Proof: $|u - w| < \varepsilon$ and $|v - w| < \varepsilon$ mean that

both u and v are in the interval $(w - \varepsilon, w + \varepsilon)$:



which has length 2ε,

so the distance $|u - v|$ between u, v is at most 2ε. $\square$

Example

Prove that $a_n = (-1)^n$ is not convergent.

Proof by contradiction: suppose the opposite,

$\exists \lim_{n\to\infty} a_n = L$.

Choose $\varepsilon = 1/2$. By def'n there must exist N

such that $|a_n - L| < 1/2$ for all $n > N$.

Choose some $n_0 > N$. Then both $|a_{n_0} - L| < 1/2$

and $|a_{n_0+1} - L| < 1/2$.

Hence $|a_{n_0} - a_{n_0+1}| < 2 \cdot 1/2 = 1$ by triangle inequality.

But in fact $|a_{n_0} - a_{n_0+1}| = 2$, a contradiction,

so the assumption is false, so there is no limit.

Connection with logical notation

A sequence (a_n) has a finite limit L if

$$\forall\, \varepsilon > 0\ \exists\, N(\varepsilon)\ \forall\, n > N(\varepsilon)\ |a_n - L| < \varepsilon.$$

"L is not a limit" is the negation of "L is a limit":

$$\neg \left(\forall\, \varepsilon > 0\ \exists\, N(\varepsilon)\ \forall\, n > N(\varepsilon)\ |a_n - L| < \varepsilon\right)$$

$$\equiv\ \exists\, \varepsilon > 0\ \forall\, N(\varepsilon)\ \exists\, n > N(\varepsilon)\ |a_n - L| \geq \varepsilon.$$

Connection with logical notation

A sequence (a_n) has a finite limit (is convergent) if

$$\exists\, L\, \forall\, \varepsilon > 0\ \exists\, N(\varepsilon)\ \forall\, n > N(\varepsilon)\ |a_n - L| < \varepsilon.$$

Is not convergent is the negation:

$$\neg\left(\exists\, L\, \forall\, \varepsilon > 0\ \exists\, N(\varepsilon)\ \forall\, n > N(\varepsilon)\ |a_n - L| < \varepsilon\right)$$

$$\equiv\ \forall\, L\, \exists\, \varepsilon > 0\ \forall\, N(\varepsilon)\ \exists\, n > N(\varepsilon)\ |a_n - L| \geq \varepsilon$$

In the above example, $a_n = (-1)^n$ is not convergent, for any L we found $\varepsilon = 1/2$ such that negation is true.

Reducing to known limits

In practice limits are rarely computed from first principles.

There are properties, rules, theorems

that reduce complicated limits to simpler ones.

Example. $\lim_{n\to\infty} \frac{n^2+n}{2n^2-1} = \frac{1}{2}$,

because $\frac{n^2+n}{2n^2-1} = \frac{n^2/n^2+n/n^2}{2n^2/n^2-1/n^2} = \frac{1+1/n}{2-1/n^2}$

and then we use properties and known limits.

Rigorous foundations for limits

But there must be some ‘beginnings’,

basic limits to which to reduce,

and those rules (like limit of sum, of product, etc)

must be proved.

One simple example was above: $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

Another is a **constant sequence**:

if $a_n = C = \text{const}$ for all $n \in \mathbb{N}$,

then $\lim\limits_{n\to\infty} a_n = C$ (in the definition

that difference $|a_n - C|$ is always $= 0 < \varepsilon$).

Theorem (on uniqueness of a limit)

If a sequence (a_n) is convergent (to a finite limit), then the limit $\lim\limits_{n\to\infty} a_n$ is unique.

Proof: by contradiction: suppose that

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} a_n = M \quad \text{and} \quad L \neq M.$$

Choose $\varepsilon = |L - M|/3$.

By definition of $\lim\limits_{n\to\infty} a_n = L$

there is $N_1(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > N_1(\varepsilon)$.

By definition of $\lim\limits_{n\to\infty} a_n = M$

there is $N_2(\varepsilon)$ such that $|a_n - M| < \varepsilon$ for all $n > N_2(\varepsilon)$.

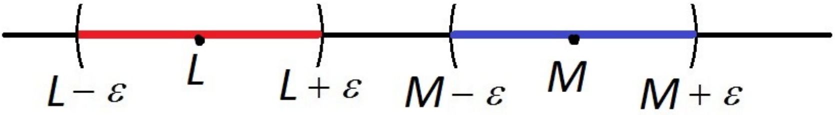
Choose $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$,
then both inequalities hold:

$$|a_n - L| < \varepsilon \qquad \text{and} \qquad |a_n - M| < \varepsilon.$$

This means that a_n is

both in $(L - \varepsilon, L + \varepsilon)$ and in $(M - \varepsilon, M + \varepsilon)$.

But these intervals are disjoint, since $\varepsilon = |L - M|/3$:



This is a contradiction. So the theorem is proved. $\square$

Limits ‘in practice’ computed by reduction

... to known limits. Recall: In practice limits are not computed from 1st principles.

There are properties, rules, theorems
that reduce complicated limits to simpler ones.

E.g.: $$\lim_{n\to\infty} \frac{n^2+n}{2n^2-1} = \frac{1}{2},$$

because $$\frac{n^2+n}{2n^2-1} = \frac{n^2/n^2+n/n^2}{2n^2/n^2-1/n^2}n = \frac{1+1/n}{2-1/n^2};$$

we now can use theorems on arithmetic of limits
and known limits: as $n \to \infty$,

$$\frac{n^2+n}{2n^2-1} = \frac{1+1/n}{2-1/n^2} \to \frac{1+0}{2-0\cdot 0} = \frac{1}{2}\,.$$

But <u>there must be some ‘beginnings’</u>,

basic limits to which to reduce,

and those rules (like limit of sum, of product, etc.)

must be proved.

Already proved the uniqueness of a limit (if it exists).

Theorem (on arithmetics of finite limits of sequences)

Suppose that $\exists \lim_{n\to\infty} a_n = L$ *(finite) and* $\exists \lim_{n\to\infty} b_n = M$ *(finite). Then*

(a) $\exists \lim_{n\to\infty}(a_n + b_n) = L + M;$

(b) $\exists \lim_{n\to\infty}(a_n \cdot b_n) = L \cdot M;$

in particular, $\lim_{n\to\infty} ka_n = kL$ *for any constant* k;

(c) *if in addition* $M \neq 0$ *and* $b_n \neq 0$ *for all* n, *then*

$$\exists \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}.$$

Not all parts are proved in these lectures

(and only parts proved may appear as ‘bookwork’ questions in exam).

But all parts can be used in examples when computing limits by reduction to previously known limits.

The rules work only if the limits exist

Important to observe the conditions under which the rules work:

e.g.: $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

is true <u>only when both limits on the right exist!</u>

If the limits on the right do not exist, and then the formula is useless:

e.g.: $\lim_{n\to\infty}(\sqrt{n} - \sqrt{n+2}) = ?$

Product, sum, quotient rules only work as stated:

"limits" like $\frac{0}{0}$, $\frac{\infty}{\infty}$ can be 1, or not 1, or not exist, etc.

Limit of a sum (=proof of part (a))

(a) $\exists \lim_{n\to\infty} a_n = L$ and $\exists \lim_{n\to\infty} b_n = M$
$\Rightarrow \exists \lim_{n\to\infty}(a_n + b_n) = L + M$.

For any $\varepsilon > 0$ we need to find $N(\varepsilon)$ such that

$(L+M)-\varepsilon < a_n + b_n < (L+M)+\varepsilon$ for all $n > N(\varepsilon)$.

For $\varepsilon/2$, by definition of $\lim_{n\to\infty} a_n = L$, there is $N_1(\varepsilon/2)$

such that $L - \varepsilon/2 < a_n < L + \varepsilon/2$ for all $n > N_1(\varepsilon/2)$.

For $\varepsilon/2$, by definition of $\lim_{n\to\infty} b_n = M$, there is $N_2(\varepsilon/2)$

such that $M - \varepsilon/2 < b_n < M + \varepsilon/2$ for all $n > N_2(\varepsilon/2)$.

Limit of a sum continued

Choose $N(\varepsilon) = \max\{N_1(\varepsilon/2), N_2(\varepsilon/2)\}$,

then for any $n > N(\varepsilon)$ both inequalities hold:

$L - \varepsilon/2 < a_n < L + \varepsilon/2$ and

$M - \varepsilon/2 < b_n < M + \varepsilon/2$.

Take the sum:

$$(L + M) - \varepsilon < a_n + b_n < (L + M) + \varepsilon$$

for all $n > N(\varepsilon)$, as required. $\square$

Parts (b) and (c) without proof

Part (a) for sums is proved.

We assume parts (b) and (c) on products and quotients without proof in these lectures.

We can use the theorem on arithmetic, together with some basic limits we already proved.

Example of using Arithmetics of limits

Example

Use the arithmetics of limits to

prove that the limit exists and compute the limit

$$\lim_{n\to\infty} \frac{2n^3+4n^2+3}{5n^3-3n^2+2n-2}.$$

Solution. Divide both numerator and denominator by n^3:

$$\frac{2n^3+4n^2+3}{5n^3-3n^2+2n-2} = \frac{2+\frac{4}{n}+\frac{3}{n^3}}{5-\frac{3}{n}+\frac{2}{n^2}-\frac{2}{n^3}}.$$

Example continued

Proved earlier: $\exists \lim_{n\to\infty} \frac{1}{n} = 0$.

By the arithmetic theorem for products,

$$\exists \lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = 0 \cdot 0 = 0$$

and

$$\exists \lim_{n\to\infty} \frac{1}{n^3} = \lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{1}{n}\right) = 0 \cdot 0 = 0.$$

Example continued

By the arithmetic theorem for sums,

and for "times constant":

$$\exists \lim_{n\to\infty}\left(2+\frac{4}{n}+\frac{3}{n^3}\right)=2+4\cdot 0+3\cdot 0=2$$

and

$$\exists \lim_{n\to\infty}\left(5-\frac{3}{n}+\frac{2}{n^2}-\frac{2}{n^3}\right)=5-3\cdot 0+2\cdot 0-2\cdot 0=5.$$

Example continued

Finally, by the arithmetic theorem for quotients,

$$\exists \lim_{n\to\infty} \frac{2n^3+4n^2+3}{5n^3-3n^2+2n-2}$$

$$= \frac{\lim_{n\to\infty}(2+\frac{4}{n}+\frac{3}{n^3})}{\lim_{n\to\infty}(5-\frac{3}{n}+\frac{2}{n^2}-\frac{2}{n^3})} = \frac{2}{5}.$$

$\square$

Note: we always remind ourselves that the limit exists: formulae for arithmetics <u>hold only when the limits exist</u>.

Remarks on level of detail

Here we look at rigorous definitions and proofs for possibly familiar examples.

Therefore worked in every detail, with references to the theorems and properties that are used.

This level of detail may be unnecessary in ‘everyday life’, in other modules like Calculus, or Probability, or Differential Equations.

But it is important to remember the conditions under which these manipulations are allowed.

We had the Arithmetic Theorem, which makes it easy to find/prove limits.

But some limits must be ‘from 1st principles’, by verifying the definition.

Example

Prove from 1st principles that $\lim_{n\to\infty} \dfrac{1}{\log(n+1)} = 0.$

Given any $\varepsilon > 0$ we need to find $N(\varepsilon)$

such that $\left|\dfrac{1}{\log(n+1)} - 0\right| < \varepsilon$ for all $n > N(\varepsilon)$.

So we solve the inequality for n:

$$\ldots \Leftrightarrow \frac{1}{\log(n+1)} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \log(n+1).$$

Use that log is increasing: $\ldots \Leftrightarrow 10^{1/\varepsilon} < n+1$

$\Leftrightarrow 10^{1/\varepsilon} - 1 < n$. So we can put $N(\varepsilon) = 10^{1/\varepsilon} - 1$, then

for all $n > N(\varepsilon)$ all the above inequalities hold, as req.

Example

Prove from 1st principles: $\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2}$.

Given any $\varepsilon > 0$ we need to find $N(\varepsilon)$ such that $\left|\frac{3n-1}{2n+1} - \frac{3}{2}\right| < \varepsilon$ for all $n > N(\varepsilon)$.

Solve the inequality for n: $\Leftrightarrow \left|\frac{6n-2-6n-3}{4n+2}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{-5}{4n+2}\right| < \varepsilon \Leftrightarrow \frac{5}{4n+2} < \varepsilon \Leftrightarrow \frac{5}{\varepsilon} < 4n+2$

$\Leftrightarrow \frac{5}{\varepsilon} - 2 < 4n \Leftrightarrow \frac{5}{4\varepsilon} - \frac{1}{2} < n$. So put $N(\varepsilon) = \frac{5}{4\varepsilon} - \frac{1}{2}$,

then for all $n > N(\varepsilon)$ all inequalities hold, as req.

Example

Prove from 1st principles that $\lim_{n\to\infty} \dfrac{200}{\sqrt[3]{n} - 9.5} = 0.$

Given any $\varepsilon > 0$ we need to find $N(\varepsilon)$
such that $\left| \frac{200}{\sqrt[3]{n}-9.5} - 0 \right| < \varepsilon$ for all $n > N(\varepsilon)$.

So we solve the inequality for n: for big $n > 1000$

$\Leftrightarrow \frac{200}{\sqrt[3]{n}-9.5} < \varepsilon \Leftrightarrow \frac{200}{\varepsilon} < \sqrt[3]{n} - 9.5 \Leftrightarrow \frac{200}{\varepsilon} + 9.5 < \sqrt[3]{n}.$

Take the cubes: $\Leftrightarrow \left(\frac{200}{\varepsilon} + 9.5\right)^3 < n$. So we can put

$N(\varepsilon) = \max\{1000, \left(\frac{200}{\varepsilon} + 9.5\right)^3\}$, then for all $n > N(\varepsilon)$

all the above inequalities hold, as req.

Sandwich theorem for sequences

Theorem (Sandwich theorem for sequences)

Suppose that (a_n) *and* (c_n) *are two convergent sequences with equal (finite) limits:* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$.

If (b_n) *is another sequence such that*

$$a_n \leq b_n \leq c_n \qquad \text{for all} \quad n,$$

then it also converges to the same limit: $\exists \lim_{n\to\infty} b_n = L$.

Sandwich theorem for sequences

Theorem (Sandwich theorem for sequences)

If $\begin{array}{ccccc} a_n & \le & b_n & \le & c_n, \\ \downarrow & & & & \downarrow \\ L & & & & L \end{array}$ *then* $\begin{array}{ccccc} a_n & \le & b_n & \le & c_n \\ \downarrow & & \downarrow & & \downarrow \\ L & & L & & L \end{array}$

Proof. Let $\varepsilon > 0$ be any positive number.

By definition of $\lim\limits_{n\to\infty} a_n = L$ there is $N_1(\varepsilon)$

such that $L - \varepsilon < a_n < L + \varepsilon$ for all $n > N_1(\varepsilon)$.

Similarly, for this ε by definition of $\lim\limits_{n\to\infty} c_n = L$

there is $N_2(\varepsilon)$ such that

$L - \varepsilon < c_n < L + \varepsilon$ for all $n > N_2(\varepsilon)$.

Sandwich theorem for sequences cont'd

Choose $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$;

then for any $n > N(\varepsilon)$ both inequalities hold:

$$L - \varepsilon < a_n < L + \varepsilon \qquad \text{and} \qquad L - \varepsilon < c_n < L + \varepsilon.$$

Then $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, so

$L - \varepsilon < b_n < L + \varepsilon$ for all $n > N(\varepsilon)$,

as required in the definition of $\lim\limits_{n\to\infty} b_n = L$.

Theorem is proved. $\square$

Example

Prove that $\lim_{n\to\infty} \frac{\sin n}{n} = 0.$

Proof. We have $-1 \le \sin n \le 1$, so

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

We know already: $\exists \lim_{n\to\infty} \frac{1}{n} = 0,$

then by arithmetic of limits: $\lim_{n\to\infty} \frac{-1}{n} = -0 = 0.$

Hence, by Sandwich Theorem, also $\exists \lim_{n\to\infty} \frac{\sin n}{n} = 0.$

Example

Prove that $\displaystyle\lim_{n\to\infty} \frac{\sqrt{\log n}}{n} = 0.$

Proof. We know $\log n \le n$, so

$$0 \le \frac{\sqrt{\log n}}{n} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

We had $\dfrac{1}{\sqrt{n}} \to 0$; also $0 \to 0$.

Hence, $\dfrac{\sqrt{\log n}}{n} \to 0$ by the Sandwich Theorem.

Bounded sequences

Definitions

A sequence (a_n) is **bounded above**

if there is a constant B such that

$a_n \leq B$ for all $n \in \mathbb{N}$.

A sequence (a_n) is **bounded below**

if there is a constant A such that

$a_n \geq A$ for all $n \in \mathbb{N}$.

Monotonic sequences

Definitions

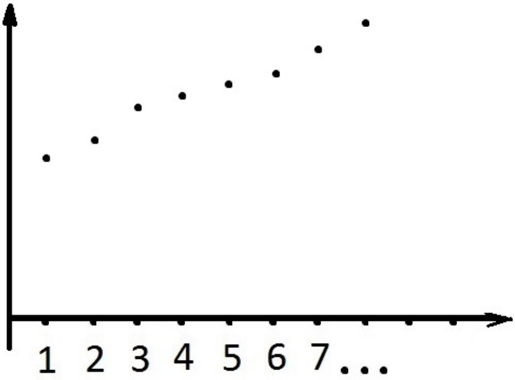
A sequence (a_n) is **monotonically increasing** if $a_k \leq a_{k+1}$ for all $k \in \mathbb{N}$.

A sequence (a_n) is **monotonically decreasing** if $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$.

(If strict inequalities — monotonically **strictly** increasing, or decreasing.)

Graph of a monotonic sequence

On the graph: increasing



Limits of monotonic sequences

Theorem

If a sequence (a_n) *is monotonically increasing and bounded above, then it has a limit* $\lim_{n\to\infty} a_n$.

In fact, $\lim = \sup\{a_n \mid n \in \mathbb{N}\}$ is the supremum (lowest upper bound) of its terms.

Theorem

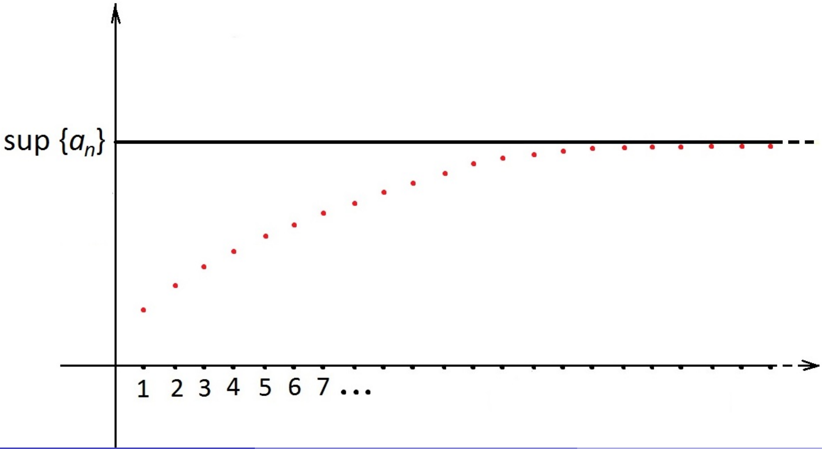
If a sequence (a_n) *is monotonically decreasing and bounded below, then it has a limit* $\lim_{n\to\infty} a_n$.

In fact, $\lim = \inf\{a_n \mid n \in \mathbb{N}\}$ is the infimum (greatest lower bound) of its terms.

Graph of a bounded monotonic sequence

We omit the proofs of these theorems
(although not really difficult).

Picture for monotonically increasing and bounded above:



Example

The sequence $a_n = 1 - \dfrac{1}{n}$

is bounded above: $a_n \leq 1$

and monotonically increasing,

since ever smaller quantity is subtracted.

Hence must have a limit by the theorem.

Here, actually, $\lim \left(1 - \dfrac{1}{n}\right) = 1.$

Limit of a subsequence

Theorem (Limit of a subsequence)

If (a_n) is a convergent sequence, then any **subsequence** $b_k = a_{i_k}$, *where* $i_1 < i_2 < \cdots$, *is also convergent and has the same limit.*

$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, \ldots$

If $a_n \to L$, then $a_{i_k} \to L$ too.

Proof not difficult (but omitted in these lectures).

E.g.: $\dfrac{1}{n} \to 0$; then $\dfrac{1}{2^k}$ is a subsequence, also $\to 0$.

E.g.: if $\lim a_n = L$, then $\lim a_{n+1} = L = \lim a_{3n+2}$, etc.

Passing to the limit in equation

Example

Let $a_1 = 1$, and $a_{k+1} = \dfrac{a_k}{2}$.

Bounded below: all $a_i \geq 0$. Clearly decreasing.

Hence has a limit: $\lim a_n = L$. To find the limit

pass to the limit in the defining equation

using that limit of subsequence (a_{k+1}) is the same:

$L = \lim a_{k+1} = \lim \dfrac{a_k}{2}$ by arithmetic theorem

$= \dfrac{\lim a_k}{2} = \dfrac{L}{2}$. So equation: $L = \dfrac{L}{2}$; $2L = L$; $L = 0$.

Example

Let (a_i) be a sequence defined recursively:

$$a_1 = 2 \quad \text{and} \quad a_{i+1} = \frac{2a_i + 5}{3}.$$

(a) Prove that it is bounded.

(b) Prove that it is monotonically increasing.

(c) Conclude that hence it has a limit by Theorem.

(d) Find this limit.

Example continued (a)

(a) Recall: $a_1 = 2$ and $a_{i+1} = \dfrac{2a_i + 5}{3}$.

We use induction to prove that $a_n \leq 5$ for all $n \in \mathbb{N}$.

1°. For $n = 1$ we have $a_1 = 2 \leq 5$, true.

2°. Suppose that $a_k \leq 5$. Then by def.

$$a_{k+1} = \frac{2a_k + 5}{3} \leq \text{ by ind. hyp. } \leq \frac{2 \cdot 5 + 5}{3} = 5,$$

so "true for $n = k$" implies "true for $n = k + 1$".

Thus, $a_n \leq 5$ for all $n \in \mathbb{N}$ by A.M.I.

Example continued (b), (c)

(b) We use (a) to prove that (a_i) is monotonically increasing.

We need $a_k \leq a_{k+1} = \dfrac{2a_k + 5}{3} \Leftrightarrow 3a_k \leq 2a_k + 5$

$\Leftrightarrow a_k \leq 5,$ true by part (a).

(c) The sequence in bounded by (a),

and is monotonically increasing by (b),

hence by Theorem (a_i) has a finite limit $\lim\limits_{n \to \infty} a_k = L.$

Example continued (d)

(d) Once we know that the limit exists,

we can pass to the limit in the defining equation

using that the limit of a subsequence is the same:

$$L = \lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} \frac{2a_k + 5}{3} =$$

by arithmetic of limits

$$= \frac{2\lim a_k + 5}{3} = \frac{2L + 5}{3}.$$

So we solve the equation $L = \dfrac{2L+5}{3}$, whence $L = 5$.

OPTIONAL: one famous limit

More difficult:

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

can be shown to be monotonically increasing

and bounded.

Therefore it has a limit.

In fact, this limit is e, the base of natural logarithms.

Remark on limits in defining relations

RECALL: $a_1 = 2$ and $a_{i+1} = \dfrac{2a_i + 5}{3}$.

We proved that this sequence is

(a) bounded; (b) monotonically increasing;

(c) <u>hence has a finite limit</u> by that theorem;

<u>only then</u> we can find this limit $L = \lim\limits_{k\to\infty} a_k$:

$$L = \lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} \frac{2a_k + 5}{3} = \frac{2\lim a_k + 5}{3} = \frac{2L + 5}{3}.$$

So we solve the equation $L = \dfrac{2L+5}{3}$, whence $L = 5$.

Importance of existence theorems

Passing to the limit in defining equation works **only if the limit exists and is finite**.

Example

Let $a_1 = 2$ and $a_{k+1} = 2a_k$.

(...Forgot to check if there is a finite limit:)

$L = \lim a_{k+1} = \lim 2a_k = 2L; \quad L = 2L; \quad L = 0.$

But in fact, $\lim a_n = +\infty$ (since $a_n = 2^n$).

Importance of existence theorems-2

Not only with infinite limits:

Example

Let $a_1 = 3$ and $a_{k+1} = -a_k$.

(...Forgot to check if there is a finite limit:)

$$L = \lim a_{k+1} = \lim(-a_k) = -L; \quad L = -L; \quad L = 0.$$

But in fact, $\lim a_n$ does not exist!

(not even an infinite limit)

since $(a_i) = 3, -3, 3, -3, 3, -3, \ldots.$