Task 1

We denote the trigonometric functions as:

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \psi_m(x) = \sin\left(\frac{n\pi x}{L}\right)$$

where $n \in \mathbb{Z}$. These functions are **orthogonal**, meaning that

$$\int_{-L}^{L} \phi_n(x)\phi_m(x)dx = L\delta_{nm} \quad \text{and} \quad \int_{-L}^{L} \psi_n(x)\psi_m(x)dx = L\delta_{nm}$$

where δ_{nm} is the Kronecker delta. We also have that

$$\int_{-L}^{L} \phi_n(x)\psi_m(x)dx = 0$$

for all n and m.

We assume that any periodic function $f(x)$ can be written in terms of our periodic functions $\phi_n(x)$ and $\psi_m(x)$ over an interval of length $2L$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\phi_n(x) + b_n\psi_n(x)\right)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} \phi_n(x)f(x)dx, \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} \psi_n(x)f(x)dx.$$

Discretised, we can find a_n and b_n by numerically integrating as follows:

$$a_0 = \frac{1}{L}\sum_{i=0}^{N-1} f(x_i)\Delta x, \quad a_k = \frac{1}{L}\sum_{i=0}^{N-1} \phi_k(x_i)f(x_i)\Delta x, \quad \text{and} \quad b_k = \frac{1}{L}\sum_{i=0}^{N-1} \psi_k(x_i)f(x_i)\Delta x$$

where Δx will be the difference between each x value, $\Delta x = \dfrac{2L}{N}$. Each summand can be quickly found using array operations from the `NumPy` library. Treating the arrays of values $\phi_k(x)$, $\psi_k(x)$, and $f(x)$ as vectors, each summand is simply the dot product (`np.dot`) of two given vectors.

i) Write functions to calculate a_n and b_n up to $n = 50$ for a general function $f(x)$.

ii) Calculate the Fourier series of $f(x) = \cos(x)$ in the interval $[-\pi, \pi]$.

iii) Calculate the Fourier series of $f(x) = \cos(x) + \cos(2x)$ in the interval $[-\pi, \pi]$.

Task 2

This method can be applied to non-periodic functions, but it will only reconstruct a given function in a limited interval.

1. Calculate the Fourier series of $f(x) = x^2$ in the interval $[-\pi, \pi)$.
2. Calculate the Fourier series of $f(x) = x$ in the interval $[-\pi, \pi)$.
    - What does the series converge to at $x = -\pi$?

If there is something up with this worksheet, please contact me (Ewan Dalgliesh - 26192682) and I will endeavour to fix any mistakes! (Apologies in advance, of course)