Task 1

To solve a second order ODE of the form

$$\frac{d^2y}{dx^2} = g(x),$$

construct the following n x n matrix derived from finite differences:

$$M = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & -2 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

where each diagonal element is

$$M_{i,i} = -2$$

and the elements either side horizontally are

$$M_{i,i-1} = M_{i,i+1} = 1$$

except for the first and last rows. Then construct the array of known values

$$\mathbf{b} = \begin{bmatrix} -2y_L \\ g(x_1)\Delta x^2 \\ \vdots \\ g(x_{n-2})\Delta x^2 \\ -2y_R \end{bmatrix}$$

and set up the array of the values you want to find,

$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}.$$

Using your Gauss / Gauss-Jordan elimination code from previous weeks, you can then solve the matrix equation $M\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

i) Solve the differential equation for $g(x) = x$ subject to $y_L = 0.2$ and $y_R = 1.5$ in the interval $[0, 1]$ for

   a) $n = 10$

   b) $n = 100$

   c) $n = 500$

ii) Solve the differential equation for $g(x_{n/2}) = \dfrac{1}{\Delta x}$ and $g(x_i) = 0$ for all other x (like a discrete Dirac Delta function) subject to $y_L = 0$ and $y_R = 0$ in the interval $[-5, 5]$ for

a) $n = 10$

b) $n = 100$

c) $n = 500$

Task 2

When dealing with a second order ODE of the form

$$\frac{d^2y}{dx^2} = g(x),$$

we use the finite difference approximation

$$\frac{d^2y}{dx^2}(x_i) \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2}$$

to create the matrix M. If our ODE is instead of the form

$$\frac{d^2y}{dx^2} + ky = c,$$

we must derive a slightly different formula to construct M. We can substitute the approximation of the second derivative straight into our ODE:

$$\frac{d^2y}{dx^2}(x_i) + ky(x_i) = g(x_i) \quad \rightsquigarrow \quad \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} + ky_i = g(x_i)$$

$$\Rightarrow y_{i-1} - 2y_i + y_{i+1} + ky_i\Delta x^2 = g(x_i)\Delta x^2$$
$$\Rightarrow y_{i-1} + (k\Delta x^2 - 2)y_i + y_{i+1} = g(x_i)\Delta x^2$$

Use the above approximation to determine what values to populate your matrix M with. You can use a similar method to determine the form of M when your ODE includes $\dfrac{dy}{dx}$.

i) Solve the differential equation $\dfrac{d^2y}{dx^2} - 3y = -1$ for, $y_L = 5$, $y_R = 0$ in the interval $[-3, 3]$ for $n = 500$.

ii) Generalise your code slightly to be able to solve an ODE of the form $\dfrac{d^2y}{dx^2} + ky = c$.

Online Test

Solve the question in the week 9 online test on BlackBoard.

If there is something up with this worksheet, please contact me (Ewan Dalgliesh - 26192682) and I will endeavour to fix any mistakes! (Apologies in advance, of course)