MATH 314 Fall 2011
Section 001

Practice Midterm 2

This midterm covers:	
Chapter 4	4.1, 4.2, 4.3, 4.4, 4.6
Chapter 5	5.1

Things you need to know:

(a) Concepts:

- linear independence, spanning set, basis
- characteristic equation, eigenvalue, eigenvector, eigenspace
- orthogonal vectors, orthogonal basis, orthonormal basis
- coordinates with respect to a given basis

(b) Procedures:

- computing determinants: using the quick formulas (2x2 and 3x3 only), by expansion along a row or column, by row operations
- properties of determinants
- computing cross products
- finding eigenvalues, eigenvectors, eigenspaces, multiplicities
- diagonalizing a matrix (finding P and D)
- computing coordinates with respect to an orthogonal basis
- computing orthogonal projections

No notes, books or calculators are to be used during the test.
The midterm will be in 106 Avery Hall from 6-8 pm on Thursday 11/03.

1. Compute the following:

(a) $\begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} =$

(b) $\begin{vmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} =$

(c) $\begin{vmatrix} a & b & 0 & \dots & 0 \\ c & d & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} =$

(all entries off the diagonal are 0 and all entries on the diagonal are 1 except in the leftmost 2×2 block)

(d) Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, compute:

(a) $det(A), det(5A), det(A^{-1}), det(A^{10})$

(b) all the minors of A

(c) all the cofactors of A

(d) the adjoint matrix of A

(e) the inverse matrix A^{-1} using the adjoint formula

2. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the characteristic equation of A.

(b) Find the eigenvalues of A.

(c) Find the eigenspaces of A.

(d) Find for each eigenvalue the algebraic and geometric multiplicity. Is A diagonalizable?

3. Let A be an unknown 3x3 matrix that has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding eigenspaces

$$E_{\lambda_1} = Span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}, E_{\lambda_2} = Span\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$$

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(a) What are the algebraic and geometric multiplicities of λ_1, λ_2? Explain.

(b) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(c) Compute A^{100}.

(d) Find a *probability* vector $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{x}}$.

(e) Can A be a stochastic matrix?

4. (a) Give an example of three linearly independent vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^3$ such that $\vec{\mathbf{u}}$ is orthogonal to $\vec{\mathbf{v}}$, $\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{w}}$, but $\vec{\mathbf{w}}$ is not orthogonal to $\vec{\mathbf{u}}$. You will use these vectors for the rest of the problem.

(b) Let $S = Span\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$. Compute $proj_S(\vec{\mathbf{w}})$ and $perp_S(\vec{\mathbf{w}})$.

(c) Find an orthogonal basis $\mathcal{B}$ of $\mathbb{R}^3$ that contains the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

(d) Find the coordinates of $\vec{\mathbf{w}}$ with respect to the basis $\mathcal{B}$ described above.

5. The purpose of this problem is to prove that eigenspaces are subspaces.

(a) Let A be an $n \times n$ matrix. Define what an eigenvalue λ of A is and what an eigenvector $\vec{\mathbf{v}}$ corresponding to the eigenvalue λ is.

(b) Let $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ be eigenvectors corresponding to the *same* eigenvalue λ. Prove that $\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$ is an eigenvector corresponding to the eigenvalue λ as well.

(c) Let $\vec{\mathbf{v}}$ be an eigenvectors corresponding to the eigenvalue λ and let c be a scalar. Prove that $c\vec{\mathbf{v}}$ is an eigenvector corresponding to the eigenvalue λ.

(d) Define the eigenspace E_λ and explain how the facts above prove it is a subspace of $\mathbb{R}^n$.