1 Motivation

We begin by recalling the definition of WLP:

Definition 1. *Let $I \subseteq S = \mathbf{K}[x_1, \ldots, x_n]$ be an ideal such that $A = S/I$ is Artinian. Then A has the* Weak Lefschetz property *(WLP) if there is an $\ell \in S_1$ such that for all m, the map μ_ℓ*

$$A_m \xrightarrow{\cdot \ell} A_{m+1}$$

is either injective or surjective.

To study the WLP one is motivated by the interest in determining the Hilbert functions of ideals generated by generic forms.
In the following we fix $I(\alpha_1, \ldots, \alpha_r) = (L_1^{\alpha_1}, \ldots, L_r^{\alpha_r})$.

2 Inverse systems

In [EI95], Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms, and ideals of fatpoints.
Let $R = \mathbf{K}[y_1, \ldots, y_n]$, $p_i = [p_{i1} : \cdots : p_{in}] \in \mathbb{P}^{n-1}$, $I(p_i) = \wp_i$. A fat point ideal is an ideal of the form

$$F = \bigcap_{i=1}^{r} \wp_i^{\alpha_i+1} \subset R.$$

Define an action of R on S by partial differentiation: $y_j = \partial/\partial x_j$. Since F is a submodule of R, it acts on S. The set of elements annihilated by the action of F is denoted F^{-1}. Let $L_{p_i} = \sum_{j=1}^{n} p_{i_j} x_j \in S$. Emsalem and Iarrobino show

Theorem 2 (Emsalem and Iarrobino, [EI95])**.** *Let F be an ideal of fatpoints:*

$$F = \wp_1^{\alpha_1+1} \cap \cdots \cap \wp_r^{\alpha_r+1} \subset R.$$

Then

$$(F^{-1})_j = \begin{cases} S_j & \text{for } j \leq \max\{\alpha_i\} \\ \\ L_{p_1}^{j-\alpha_1} S_{\alpha_1} + \cdots + L_{p_r}^{j-\alpha_n} S_{\alpha_r} & \text{for } j \geq \max\{\alpha_i + 1\} \end{cases}$$

and

$$HF(F^{-1}, j) = HF(\frac{R}{I(j-\alpha_1, \ldots j-\alpha_r)}, j).$$

We are trying to use the correspondence in the "reverse direction". Note that to obtain the Hilbert function of a fixed ideal of linear forms, it is necessary to consider an infinite family of ideals of fat points.

3 Powers of linear forms and blowing up projective space

There is a well-known correspondence between the graded pieces of an ideal of fat points $F \subseteq \mathbf{K}[x_1, \ldots, x_r]$ and the global sections of a line bundle on the variety X which is the blow up of $\mathbb{P}^{r-1}$ at the points. We briefly review this. Let $\mathcal{I}_Z(j)$ be the ideal sheaf of the fat points subscheme Z defined by F. Of course, $\dim_{\mathbf{K}} F_j = h^0(\mathbf{P}^{r-1}, \mathcal{I}_Z(j))$.
Let E_i be the class of the exceptional divisor over the point p_i, and E_0 the pullback of a hyperplane on $\mathbb{P}^{r-1}$. Define

$$D_j = jE_0 - \sum_{i=1}^{n} m_i E_i.$$

Moreover, $h^i(X, D) = h^i(\mathbf{P}^{r-1}, \mathcal{I}_Z(j))$ for all $i \geq 0$. Taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_Z(j) \longrightarrow \mathcal{O}_{\mathbf{P}^{r-1}}(j) \longrightarrow \mathcal{O}_Z(j) \longrightarrow 0$$

and using the fact that $\mathcal{O}_Z(j) \cong \mathcal{O}_Z$ and thus $h^0(Z, \mathcal{O}_Z(j)) = h^0(Z, \mathcal{O}_Z) = \sum_i \binom{n-2+m_i}{n-1}$, shows that

$$h^0(X, D) = h^0(\mathcal{I}_Z(j)) = \binom{n-1+j}{n-1} - \sum_i \binom{n-2+m_i}{n-1} + h^1(\mathcal{I}_Z(j)). \tag{3.1}$$

In the context of **Theorem 2**, taking $m_i = j - t + 1$ for all i and defining D_j to be $D_j = jE_0 - (j-t+1)(E_1 + \cdots + E_n)$, we thus have:

$$\dim_{\mathbf{K}} I_j = \begin{cases} n\binom{r+j-t-1}{r-1} - h^1(\mathcal{I}_Z(j)) = n\binom{r+j-t-1}{r-1} - h^1(D_j) & \text{for } j \geq t \\ 0 & \text{for } 0 \leq j < t \end{cases} \tag{3.2}$$

Alternatively, this can be stated for the quotient $S/I = A$ as:

$$\dim_{\mathbf{K}} A_j = \begin{cases} h^0(D_j) & \text{for } j \geq t \\ \binom{r-1+j}{r-1} & \text{for } 0 \leq j < t \end{cases} \tag{3.3}$$

Definition 3. *We will say that I has* expected dimension *in degree j if either $I_j = 0$ or $h^1(D_j) = 0$. We say D_j is* irregular *if $h^1(D_j) > 0$ and* regular *otherwise. We say D_j is* special *if $h^0(D_j)$ and $h^1(D_j)$ are both positive.*

A landmark result on the dimension of linear systems is:

Theorem 4 (Alexander–Hirschowitz [AH92])**.** *Fix $m, n-1 \geq 2$, and consider the linear system of hypersurfaces of degree m in $\mathbf{P}^{n-1}$ passing through n general points with multiplicity two. Then*

1. *For $m = 2$, the system is special iff $2 \leq r \leq n-1$.*
2. *For m greater than two, the only special systems are*

$$(n-1, m, r) \in \{(2,4,5), (3,4,9), (4,4,14), (4,3,7)\}.$$

In each of these four cases, the linear system is expected to be empty but in fact has projective dimension 0.

Example 5. Let A be the quotient of $\mathbf{K}[x_1, x_2, x_3]$ by the cubes of five general linear forms. The corresponding five points in $\mathbf{P}^2$ are general, and the first interesting computation involves $D_4 = 4E_0 - \sum\limits_{i=1}^{5} 2E_i$, for which we have

$$\dim_{\mathbf{K}} A_4 = h^0(D_4) = \binom{6}{2} - 15 + h^1(D_4).$$

Since $H^0(D_4)$ contains the double of a conic through the five points, D_4 is special, and in fact we have $h^0(D_4) = 1 = h^1(D_4)$.

A famous open conjecture on the Hilbert function of fat points in $\mathbf{P}^2$ is expressed in terms of (-1)-curves (i.e., smooth rational curves E with $E^2 = -1$):

Conjecture 6 (Segre-Harbourne-Gimigliano-Hirschowitz [**?**])**.** *Suppose that $\{p_1, \ldots, p_n\} \subseteq \mathbf{P}^2$ is a collection of points in general position, X is the blowup of $\mathbf{P}^2$ at the points, and E_i the exceptional divisor over p_i. If $F_j = jE_0 - \sum_{i=1}^{n} a_i E_i$ is special, then there exists a (-1)-curve E with $E \cdot F_j \leq -2$.*

Example 7. Let $C = 2(2E_0 - \sum\limits_{i=1}^{5} E_i) + (E_0 - E_1 - E_2)$. Then $h^0(C) = 1$ and $h^1(C) = 1$, so C is special, but $E = 2E_0 - \sum\limits_{i=1}^{5} E_i$ is rational by adjunction with $E^2 = -1$ and $E \cdot C = -2$.

4 Main results

The case of ideals generated by powers of linear forms in $\mathbf{K}[x_1, \ldots, x_3]$ has been answered in the affirmative in previous work [**?**].
Recall that one of the strategies (described in previous talk) of analyzing the WLP is through the syzygy bundle of the associated ideal sheaf restricted to a line bundle. In this section, we focus on powers of linear forms in $S = \mathbf{K}[x_1, \ldots, x_4]$ for which the associated (restricted) fat point subscheme is a subset of $\mathbf{P}^2$ and we furthermore restrict to fat point schemes whose Hilbert function is known. This means looking at fat point schemes supported at 8 or less points.
Recall that via the syzygy bundle techniques one obtains a long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{S}(I)(m)) \arrow[r] & H^0(\mathcal{S}(I)(m+1)) \arrow[r, "\phi_m"] & H^0(\mathcal{S}(I)|_L(m+1)) \arrow[dll] \\ & H^1(\mathcal{S}(I)(m)) \arrow[r, "\mu_\ell"'] & H^1(\mathcal{S}(I)(m+1)) \arrow[r] & H^1(\mathcal{S}(I)|_L(m+1)) \arrow[dll, "\psi_m"'] \\ & H^2(\mathcal{S}(I)(m)) \arrow[r] & H^2(\mathcal{S}(I)(m+1)) \arrow[r] & \cdots . \end{tikzcd} \tag{4.1}$$

We observed that surjectivity of the map ϕ_m implies injectivity of the WLP map and injectivity of ψ_m implies surjectivity of the WLP map. The present work translates these observations into the language of divisors on the blowup of $\mathbf{P}^2$.

Theorem 8. *If the divisor D_m (as defined in the previous section) on the blowup of $\mathbf{P}^n$ is non-special and the divisor D_{m+1} on $\mathbf{P}^{n-1}$ is special then WLP fails for the corresponding ideal generated by powers of linear forms.*

Example 9. Here we apply the theorem to obtain the Hilbert function for $A = \mathbf{K}[x_1, x_2, x_3, x_4]/\langle x_1^3, x_2^3, x_3^3, x_4^3, (x_1+x_2+x_3+x_4)^3\rangle$.

j	0	1	2	3	4	5	6	...
$\dim_{\mathbf{K}} A_j$	1	4	10	15	15	6	0	...
$HF(\cap_{i=1}^5 \wp_i^{j-2}, j)$	0	0	0	15	15	6	0	...

We consider the restriction of this example to $\mathbf{P}^2$ in Example 5.

Example 10. *But as in the proof of Proposition **??**(c), the kernel of $A_3 \to A_4$ has dimension $h^1(D_4')$, hence the cokernel has dimension $h^1(D_4')$, so μ_ℓ fails to have full rank, since $h^1(D_4') = 1$ by Theorem 4.*

We employ this theorem together with the AlexanderHirschowitz (for 5 points) and Segre-Harbourne-Gimigliano-Hirschowitz Theorem to fully describe the failure of the WLP in the case of 5,6,7,8 fat points.

Theorem 11. *Let* $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbf{K}[x_1, x_2, x_3, x_4]$ *with* $l_i \in S_1$ *generic. If* $n \in \{5, 6, 7, 8\}$*, then WLP fails, respectively, for* $t \geq \{3, 27, 140, 704\}$*.*

5 Connections with Gelfand-Tsetlin patterns

Definition 12. *A two-row Gelfand-Tsetlin pattern is a non-negative integer* $2 \times n$*-matrix* (λ_{ij}) *that satisfies* $\lambda_{2n} = 0$*,* $\lambda_{1,j+1} \geq \lambda_{2,j}$ *and* $\lambda_{i,j} \geq \lambda_{i,j+1}$ *for* $i = 1, 2$ *and* $j = 1, \dots, n-1$*.*

In Proposition 3.6 of [SX10], Sturmfels-Xu show that for generic forms l_i, the Hilbert function of $\mathbf{K}[x_1, \dots, x_r]/\langle l_1^{u_1}, \dots, l_{r+1}^{u_{r+1}} \rangle$ in degree i is the number of two-rowed Gelfand-Tsetlin patterns with $\lambda_{21} = i$ and $\lambda_{1j} + \lambda_{2j} = u_j + \cdots + u_{r+1}$ for $j = 1, \dots, r+1$.

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