Math 918 – Computational Algebra Homework 4

Due (tentatively) Friday, November 8.

Work in groups (of 1 or more). Each group should turn in ONE solution set, preferably by email in LaTeX format. Each group should solve at least max(*# of group members, 2) problems.*

*AG=alg. geom., CO=comb., AP=applied, M2= computational, B=beginner, *=advanced.*

1. (B, CO - Operations with monomial ideals) Let I and J be monomial ideals of given by monomial generators $m_1, \ldots, m_r$ and $n_1 \ldots n_s$, respectively, and let m be a monomial.

(a) Show that
$$I \cap J = \langle LCM(m_i, n_j) | 1 \leq i \leq r, 1 \leq j \leq s \rangle.$$

(b) Show that $I : m$ is generated by the monomials
$$\frac{LCM(m_i; m)}{m} = \frac{m_i}{GCD(m_i; m)}; 1 \leq i \leq r.$$

2. (B, CO - Associated primes of monomial ideals) Given a subset $S = \{i_1, \ldots i_s\} \subseteq \{1, 2, \ldots n\}$, define P_S to be the following (prime) ideal of $k[x_1, \ldots, x_n]$: $P_S = \langle x_{i_1}, \ldots, x_{i_s} \rangle$.

(a) Show that, for any simplicial complex Δ, $I_\Delta = \cap_{\sigma \in \Delta} P_{\overline{\sigma}}$, where $\overline{\sigma}$ denotes the complement of σ with respect to the set $\{1, 2, \ldots n\}$.

(b) Let G be a graph on n vertices and define the *edge ideal* of G to be $I(G) = \langle x_i x_j | (i, j) \in E(G) \rangle$. Show that $I(G) = \cap_C P_C$ where the set C varies over the *minimal vertex covers* of G (a vertex cover is a set of vertices such that every edge in G has at least one end point in this set and the word minimal is meant with respect to containment).

3. (B, CO - Alexander duality) Recall that for a simplicial complex Δ we defined the Alexander dual simplicial complex to be $\Delta^* = \{\overline{\tau} | \tau \notin \Delta\}$ and for a squarefree monomial ideal $I = \langle x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_t} \rangle$ we defined the Alexander dual ideal of I to mean $I^* = \cap_{i=1}^t P_{\sigma_i}$. Show that

(a) $(\Delta^*)^* = \Delta$

(b) $I_\Delta^* = I_{\Delta^*}$

Hint: You may want to use part (a) of the previous problem here.

4. (Hochster's Theorem) Use Hochster's Theorem to compute all the $\mathbb{Z}^n$-graded Betti numbers of S/I_Δ, where $I_\Delta = \langle x_1x_4, x_2x_3x_4, x_1x_5, x_2x_5, x_3x_5, x_4x_5 \rangle$.

5. (Stanley's Triangle & Dehn-Sommerville relations)

(a) Prove that Stanley's triangle (as defined in class) indeed computes the h-vector of a Stanley-Reisner ring.

(b) Prove that if the f-vector of a 2-dimensional simplicial complex satisfies Euler's relations $f_0 - f_1 + f_2 = 2$ and $3f_2 = 2f_1$ (this is the case, for example for the boundary complex of a 3-dimensional simplicial polytope), then the h-vector of the Stanley-Reisner ring associated to this simplicial complex is symmetric (i.e $h_i = h_{3-i}$ for $i = 0, 1$).

6. (* - Borel-fixed ideals in positive characteristic) Suppose p is a prime number and $a, b \in \mathbb{N}$. Define $a <_p b$ if each digit in the base p expansion of a is $\leq$ the corresponding digit in the base p expansion of b. Let I be a monomial ideal in $k[x_1 \dots x_n]$ with $char(k) = p$. Show that I is Borel-fixed iff the folowing condition is satisfied for all $i < j$ and all monomial minimal generators m of I: if m is divisible by x_j^t but no higher power of x_j, then $(x_i/x_j)^s m \in I$ for all $i < j$ and $s <_p t$.

7. (* - Distractions) Let I be an arbitrary monomial ideal in $k[x_1, \dots, x_n]$ (k algebraically closed) and let $B \subset \mathbb{N}^n$ be the set of all vectors b such that x^b is not in I. The distraction of I is the radical ideal $D(I)$ of all polynomials in $k[x_1, \dots, x_n]$ which vanish on the set B.

(a) Determine a finite generating set of $D(I)$.

(b) Show that I is the initial monomial ideal of $D(I)$ with respect to any term order.

(c) Determine the prime decomposition of $D(I)$.