Ordinary and symbolic Rees algebras for Fermat configurations of points

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and Uwe Nagel

Outline

The goal for this talk is:

For a homogeneous ideal I, study

- the Rees algebra of I, $\mathcal{R} = \bigoplus_{i \geq 0} I^i t^i$
- the symbolic Rees algebra of I, $\mathcal{R}_s = \bigoplus_{i \geq 0} I^{(i)} t^i$

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Why? To understand the ordinary and symbolic powers all at once.

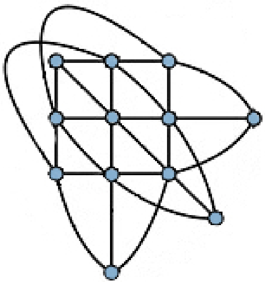
Fermat configuration

Definition

The Fermat configuration of n^2+3 points in $\mathbb{P}^2$ is the zero locus of

$$I=\left(x(y^n-z^n),y(z^n-x^n),z(x^n-y^n)\right).$$

Example: n=3



Resolutions for ordinary powers

The Hilbert-Burch resolution for I is:

$$0 \longrightarrow S(-6)^2 \xrightarrow{\begin{bmatrix} x^2 & yz \\ y^2 & xz \\ z^2 & xy \end{bmatrix}} S(-4)^3 \longrightarrow I \longrightarrow 0$$

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The resolutions for I^2, I^3, I^4 are:

m	2				3				4			
betti(I^m)		0	1	2		0	1	2		0	1	2
	8:	6	.	.	12:	10	.	.	16:	15	.	.
	9:	.	6	.	13:	.	12	.	17:	.	20	.
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<tr><td>m</td><td colspan="4">2</td><td colspan="4">3</td><td colspan="4">4</td></tr>
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What is the pattern?

Rees algebra - equations

For $S = k[x, y, z]$ and I an ideal with resolution

$$0 \longrightarrow S^v \xrightarrow{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}} S^u \longrightarrow I \longrightarrow 0$$

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We have the diagram

$$\begin{tikzcd}
T = S[T_1, T_2, T_3] \arrow[d, two heads] \arrow[rr, two heads] & & \mathcal{R}(I) \\
\mathrm{Sym}(I) = T/(a_1T_1 + a_2T_2 + a_3T_3, b_1T_1 + b_2T_2 + b_3T_3) \arrow[urr, two heads] & &
\end{tikzcd}$$

Rees algebra - aci's

Theorem (Herzog-Simis-Vasconcelos)

If I is a homogeneous almost complete intersection that is locally a complete intersection at each of its minimal associated primes, then

$$Sym(I) = \mathcal{R}(I).$$

Corollary

If I is a 3-generated ideal defining a reduced set of points in $\mathbb{P}^2$, then

$$Sym(I) = \mathcal{R}(I).$$

Moreover, $\mathcal{R}(I)$ is a complete intersection, whose equations are determined by the Hilbert-Burch matrix of I.

Resolutions for ordinary powers

Theorem (Nagel-S.)

If I is a 3-generated ideal defining a reduced set of points in $\mathbb{P}^2$, then

- *the resolution of $\mathcal{R}(I)$ is*

$$0 \to T(-3d,-2) \to T(-d-d_1,-1) \oplus T(-d-d_2,-1) \to T \to \mathcal{R}(I) \to 0$$

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- *the regularity of I^m when $m \geq 2$ is*

$$reg(I^m) = (m+1)d - 2 = md + (d-2).$$

Symbolic powers

Definition

The n-th **symbolic power** $I^{(n)}$ of an ideal $I \subset R$ of the form

$$I = I(p_1) \cap I(p_2) \cap \ldots \cap I(p_s)$$

is defined as

$$I^{(m)} = I(p_1)^m \cap I(p_2)^m \cap \ldots \cap I(p_s)^m.$$

Recall that the symbolic Rees algebra of I is

$$\mathcal{R}_s(I) = \bigoplus_{i \geq 1} I^{(i)} t^i.$$

Main problem: $\mathcal{R}_s(I)$ is often not finitely generated.

Finite generation of the symbolic Rees algebra

Theorem (Nagel-S.)

If I is the ideal defining the Fermat configuration of $n^2 + 3$ points then

$$I^{(nk)} = \left(I^{(n)}\right)^k \text{ for all } k \geq 1.$$

Consequences:

- $\mathcal{R}_s(I^{(n)}) = \mathcal{R}(I^{(n)})$, thus $\mathcal{R}_s(I^{(n)})$ is finitely generated
- $\mathcal{R}_s(I)$ is a finitely-generated $\mathcal{R}_s(I^{(n)})$-module, thus $\mathcal{R}_s(I)$ is Noetherian.

Regularity for symbolic powers

Theorem (Nagel-S.)

If I is the ideal defining the Fermat configuration *of $n^2 + 3$ points then*

$$reg(I^{(m)}) = m(n+1) \text{ for } m \gg 0.$$

In general, we only know $reg(I^{(m)})$ is periodic-linear whenever $\mathcal{R}_s(I)$ is finitely generated by Cutkoski-Herzog-Trung.

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Thank you!