Math 918 – Computational Algebra **Homework 3**

Due (tentatively) Friday, October 11.

Work in groups (of 1 or more). Each group should turn in ONE solution set, preferably by email in LaTeX format. Each group should solve at least max(*# of group members, 2*) *problems.*

*AG=alg. geom., CO=comb., AP=applied, M2= computational, B=beginner, *=advanced.*

1. (M2 - Kernel of a polynomial map) Let $\phi : R = k[x_1, \dots, x_n] \to k[y_1, \dots, y_m]$ be a ring map given by $x_i \mapsto f_i(y_1, \dots, y_m)$. Describe how to compute the kernel of ϕ using some of the algorithms we have discussed.

2. (M2 - Algebraic dependence) Let $f_1, \dots, f_n$ be elements of a polynomial ring R. Describe how to check whether there is an algebraic dependence relation between these polynomials, i.e. whether there is a polynomial $h \in k[y_1, \dots, y_n]$ such that $h(f_1, \dots, f_n) = 0$ using some of the algorithms we have discussed.

3. (B - A syzygy computation) Let $f_1 = x^2, f_2 = y^2, f_3 = xy + yz$ be elements of $R = k[x, y, z]$. Find $Syz(f_1, f_2, f_3)$ using a Gröbner basis with respect to the graded reverse lexicographic order with $x > y > z$. Note that you will likely have additional elements in your Gröbner basis so you will need a "pruning" step like in the example that we did in class.

4. (M2 - Another syzygy computation) Let $R = k[x, y, z]$, $F = Re_1 \oplus Re_2$. Let $f_1 = (y - z)e_1 + (x + 1)e_2, f_2 = (y - 1)e_2, f_3 = (y - 1)e_1 + (z - 1)e_2$ be elements of F.

(a) find (using M2) a Gröbner basis for the submodule of F generated by f_1, f_2, f_3. Use these commands `M=matrix{{y-z, 0, y-1},{x+1,y-1,z-1}}; N=gens gb image M`.

(b) Say the elements of th Gröbner basis in part (a) are f_1, f_2, f_3, f_4, f_5. Find (using M2) $Syz(f_1, f_2, f_3, f_4, f_5)$. The generators of this module are the columns of the matrix obtained by using the command `(res image N).dd_1`.

(c) Use part (b) to deduce in what way the S-elements of f_1, f_2, f_3, f_4, f_5 can be written as combinations f_1, f_2, f_3, f_4, f_5.

5. (B - A criterion for freeness) Let R be a polynomial ring, let $F = \oplus_{i=1}^n Re_i$ be a free R-module and let U be a submodule of F.

(a) show that $LT(U)$ can be written as $LT(U) = \oplus I_i e_i$, with I_i monomial ideals in R

(b) show that U is a free R-module if the I_i in the expression above are principal ideals.

6. (* - Syzygies of monomial submodules) Let R be a polynomial ring, let F be a free R-module and let M be a submodule of F generated by monomials $m_1, \dots m_t$ (these are "generalized monomials" i.e. elements of F not of R). Let $\phi : \oplus_{i=1}^t Re_i \to M$ be defined by $\phi(e_i) = m_i$. For each pair i, j such that m_i and m_j involve the same basis element of F, set

$$u_{ij} = \frac{LCM(m_i, m_j)}{m_i}, \quad u_{ji} = \frac{LCM(m_i, m_j)}{m_j}, \quad r_{ij} = u_{ij}e_i - u_{ji}e_j.$$

Prove that the elements r_{ij} generate the kernel of ϕ from first principles (i.e. use only the definitions above, do not use the more general theorem that we used in class).

7. (* - Syzygies of monomial submodules - continued) Show that the generators r_{ij} of $\ker(\phi)$ as described above form a Gröbner basis with respect to any monomial order that gives priority to the position (e.g. Position over Coefficient).