MATH 314 Fall 2011
Section 001

Practice Final - Solutions

1. Check your notes or book for answer to problem 1.

2. Give examples of:

(a) Four different vector spaces each having dimension 10.

Solution: $\mathbb{R}^{10}, \mathcal{P}_9, \mathcal{M}_{2\times 5}, \mathcal{M}_{5\times 2}, \mathcal{M}_{1\times 10}, \mathcal{M}_{10\times 1},$ the set of symmetric 4×4 matrices, etc.

(b) A set of linearly independent polynomials in $\mathcal{P}_2$ which is not a basis.

Solution: $\{1, x\}$ is a set of two linearly independent polynomials in $\mathcal{P}_2$ which is not a basis of $\mathcal{P}_2$ because any basis of $\mathcal{P}_2$ has 3 elements whereas the set $\{1, x\}$ only has 2 elements. *Your example may be different.*

(c) A basis of $\mathcal{P}_2$ obtained by extending the set in part (b).

Solution: $\{1, x, x^2\}$ is the standard basis that extends the independent set $\{1, x\}$. *Your example may be different.*

(d) A set of matrices that span $\mathcal{M}_{2\times 2}$ but do not form a basis.

Solution: $A = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\}$ is a set of matrices that span $\mathcal{M}_{2\times 2}$ because in fact the subset $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ is enough to span $\mathcal{M}_{2\times 2}$. A is not a basis of $\mathcal{M}_{2\times 2}$ beacause any basis of $\mathcal{M}_{2\times 2}$ has 4 elements, whereas A has 5 elements.

Your example may be different.

(e) A subspace W of $\mathcal{M}_{2\times 2}$ of such that the dimension of W is 2.

Solution: $W = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix}, a, b \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ or $W = \left\{ \begin{bmatrix} a & 5a+b \\ 7b & a \end{bmatrix}, a, b \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 7 & 0 \end{bmatrix} \right\}$. Because the given sets are bases (you need to check this!) these are 2-dimensional subspaces of $\mathcal{M}_{2\times 2}$. *Your example may be different.*

3. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 5 & 2 & 4 \\ 0 & -2 & -1 \\ 5 & 3 & 5 \end{bmatrix}$.

(a) Find bases for $Row(A)$ and $Null(A)$.

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 5 & 2 & 4 \\ 0 & -2 & -1 \\ 5 & 3 & 5 \end{bmatrix} \xrightarrow{RREF} R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so a basis for $Row(A)$ is $\left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$ (we can see from here that $Row(A) = \mathbb{R}^3$). $Null(A) = (Row(A))^{\perp} = (\mathbb{R}^3)^{\perp} = \{\vec{\mathbf{0}}\}$ because the only vector in $\mathbb{R}^3$ that is orthogonal to the whole $\mathbb{R}^3$ is $\vec{\mathbf{0}}$.

(b) Find bases for $Col(A)$ and $Null(A^T)$. *Solution:*

$$[A^T|0] = \left[\begin{array}{ccccc|c} 1 & -3 & 5 & 0 & 5 & 0 \\ -1 & 1 & 2 & -2 & 3 & 0 \\ 0 & -1 & 4 & -1 & 5 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right]$$

hence $Null(A^T) = \left\{ \begin{bmatrix} -3s-4t \\ -s-3t \\ -2t \\ s \\ t \end{bmatrix}, s,t \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and this is in fact a basis. A basis for $Col(A)$ is a basis for $Row(A^T)$ so a possible choice of basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. (Another choice would be the three original columns of A.)

(c) Find the rank of A and the nullity of A. Check the rank theorem.

Solution: $rank(A) = 3$ and $nullity(A) = dim(Null(A)) = 0$. The rank theorem for A asserts: $rank(A) + nullity(A) = 3$ and indeed $3 + 0 = 3$.

(d) Find the rank and the nullity of A^T. Check the rank theorem for A^T.

Solution: $rank(A^T) = rank(A) = 3$ and $nullity(A^T) = dim(Null(A^T)) = 2$. The rank theorem for A^T asserts: $rank(A^T) + nullity(A^T) = 5$ and indeed $3 + 2 = 5$.

4. Consider $W = Span\left\{ \vec{\mathbf{x}}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{x}}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$ as a subspace of $\mathbb{R}^3$.

(a) Find $dim(W)$ and say what W is as a geometric object.

Solution: the two vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ are linearly independent (not a multiple of each other) and span W, therefore the set $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ is a basis of W and so $dim(W) = 2$. Any 2-dimensional subspace of $\mathbb{R}^3$ is a plane.

(b) Find and **orthonormal** basis for W.

Solution: Apply Gramm-Schmidt to find an **orthogonal** basis:

$$\vec{\mathbf{v}}_1 = \vec{\mathbf{x}}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \ \vec{\mathbf{v}}_2 = \vec{\mathbf{x}}_2 - \tfrac{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{x}}_2}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1} \vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \tfrac{-3}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ -3/2 \end{bmatrix}.$$

Then **normalize**:

$$\vec{\mathbf{q}}_1 = \frac{\vec{\mathbf{v}}_1}{||\vec{\mathbf{v}}_1||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \vec{\mathbf{q}}_2 = \frac{\vec{\mathbf{v}}_2}{||\vec{\mathbf{v}}_2||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(c) Find the projection of the vector $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto W.

Solution:

$$proj_W(\vec{\mathbf{v}}) = (\vec{\mathbf{q}}_1 \cdot \vec{\mathbf{v}})\vec{\mathbf{q}}_1 + (\vec{\mathbf{q}}_2 \cdot \vec{\mathbf{v}})\vec{\mathbf{q}}_2 = \frac{4}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/\sqrt{6} \\ 4/\sqrt{6} + 2/\sqrt{2} \\ 4/\sqrt{6} - 2/\sqrt{2} \end{bmatrix}$$

(d) Find the distance from the point $(1, 2, 0)$ to the subspace W.

Solution: The required distance is $d = ||perp_W(\vec{\mathbf{v}})||$.

From the Pithagorean Theorem $||perp_W(\vec{\mathbf{v}})||^2 + ||proj_W(\vec{\mathbf{v}})||^2 = ||\vec{\mathbf{v}}||^2$.

$$||proj_W(\vec{\mathbf{v}})||^2 = ||\frac{4}{\sqrt{6}}\vec{\mathbf{q}}_1+\frac{2}{\sqrt{2}}\vec{\mathbf{q}}_2||^2 = ||\frac{4}{\sqrt{6}}\vec{\mathbf{q}}_1||^2+||\frac{2}{\sqrt{2}}\vec{\mathbf{q}}_2||^2+2 < \frac{4}{\sqrt{6}}\vec{\mathbf{q}}_1, \frac{2}{\sqrt{2}}\vec{\mathbf{q}}_2 >$$

$$= \frac{16}{6}||\vec{\mathbf{q}}_1||^2 + \frac{4}{2}||\vec{\mathbf{q}}_2||^2 + 0 = \frac{8}{3} + 2 = \frac{14}{3}.$$

$d^2 = ||perp_W(\vec{\mathbf{v}})||^2 = ||\vec{\mathbf{v}}||^2 - ||proj_W(\vec{\mathbf{v}})||^2 = 5 - \frac{14}{3} = \frac{1}{3}$, so $d = \frac{1}{\sqrt{3}}$.

(e) Describe $W^\perp$ (the orthogonal complement of W) by giving a basis for $W^\perp$ and describing what it is as a geometric object.

Solution: We know $dim(W) + dim(W^\perp) = 3$, so $dim(W^\perp) = 1$ and we get that $W^\perp$ is a line. A basis for $W^\perp$ is given by $\{perp_W(\vec{\mathbf{v}})\} = \left\{ \begin{bmatrix} 8/\sqrt{6} \\ 4/\sqrt{6} + 2/\sqrt{2} \\ 4/\sqrt{6} - 2/\sqrt{2} \end{bmatrix} \right\}$ or any multiple of it like $\frac{\sqrt{6}}{2} perp_W(\vec{\mathbf{v}}) = \begin{bmatrix} 4 \\ 2+\sqrt{3} \\ 2-\sqrt{3} \end{bmatrix}$.

(f) Write the linear equation for which the solution set is W.

Solution: Use the normal vector $perp_W(\vec{\mathbf{v}})$ to write the normal form of the equation for the plane W.

$$\frac{8}{\sqrt{6}}x + (4/\sqrt{6} + 2/\sqrt{2})y + (4/\sqrt{6} - 2/\sqrt{2})z = 0$$

or

$$4x + (2+\sqrt{3})y + (2-\sqrt{3})z = 0$$

5. For this problem we work in the vector space $\mathcal{P}_2$ with the inner product $\langle P(x), Q(x)\rangle = \int_0^1 P(x)Q(x)dx$.

(a) Show that the polynomials $P(x) = 1 + x, Q(x) = 1 - x, R(x) = x^2$ are linearly independent.

Solution:

$$c_1P(x) + c_2Q(x) + c_3R(x) = 0$$

$$c_1(1+x)+c_2(1-x)+c_3(x^2)=0$$
$$(c_1+c_2)+(c_1-c_2)x+c_3x^2=0$$
$$\begin{cases} c_1+c_2 &= 0 \\ c_1-c_2 &= 0 \\ \quad c_3 &= 0 \end{cases} \Longrightarrow \begin{cases} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{cases}$$

Since the only solution to $c_1P(x)+c_2Q(x)+c_3R(x)=0$ is the trivial one, the polynomials $P(x), Q(x), R(x)$ are linearly independent.

(b) Show that $Span(P,Q,R)=\mathcal{P}_2$. (Use the same P,Q,R as in part (a)).

Solution:

$$c_1P(x)+c_2Q(x)+c_3R(x)=ax^2+bx+c$$
$$c_1(1+x)+c_2(1-x)+c_3(x^2)=ax^2+bx+c$$
$$(c_1+c_2)+(c_1-c_2)x+c_3x^2=ax^2+bx+c$$
$$\begin{cases} c_1+c_2 &= c \\ c_1-c_2 &= b \\ \quad c_3 &= a \end{cases} \Longrightarrow \begin{cases} c_1 &= \frac{b+c}{2} \\ c_2 &= \frac{b-c}{2} \\ c_3 &= a \end{cases}$$

This shows that any element ax^2+bx+c of $\mathcal{P}_2$ can be written as a linear combination $\frac{b+c}{2}P(x)+\frac{b-c}{2}Q(x)+aR(x)$.

(c) Explain why $\{P,Q,R\}$ is a basis of $\mathcal{P}_2$.

Solution: From part (a) $\{P,Q,R\}$ is a linearly independent set. From part (b) $Span\{P,Q,R\}=\mathcal{P}_2$, hence $\{P,Q,R\}$ is a basis of $\mathcal{P}_2$.

(d) Find an orthogonal basis of $\mathcal{P}_2$ with respect to the inner product described in the beginning of the problem.

Solution: Use Gramm-Schmidt applied to the basis $P(x)=1+x, Q(x)=1-x, R(x)=x^2$.

$$\vec{\mathbf{v}}_1=P(x)=1+x$$

$$\vec{\mathbf{v}}_2=perp_{\vec{\mathbf{v}}_1}Q(x)=Q(x)-\frac{<\vec{\mathbf{v}}_1,Q(x)>}{<\vec{\mathbf{v}}_1,\vec{\mathbf{v}}_1>}\vec{\mathbf{v}}_1=1-x-\frac{\int_0^1(1+x)(1-x)dx}{\int_0^1(1+x)^2dx}(1+x)=$$
$$=1-x-\frac{2/3}{7/3}(1-x)=1-x-\frac{2}{7}-\frac{2}{7}x=\frac{5}{7}-\frac{9}{7}x$$

Rescale to $\vec{\mathbf{v}'}_2 = 5 - 9x$.

$$\vec{\mathbf{v}}_3 = R(x) - \frac{<\vec{\mathbf{v}}_1, R(x)>}{<\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_1>}\vec{\mathbf{v}}_1 - \frac{<\vec{\mathbf{v}'}_2, R(x)>}{<\vec{\mathbf{v}'}_2, \vec{\mathbf{v}'}_2>}\vec{\mathbf{v}'}_2 =$$

$$= x^2 - \frac{\int_0^1 x^2(1+x)dx}{\int_0^1 (1+x)^2 dx}(1+x) - \frac{\int_0^1 x^2(5-9x)dx}{\int_0^1 (5-9x)^2 dx}(5-9x) =$$

$$= x^2 - \frac{7/12}{7/3}(1+x) - \frac{-7/12}{7}(5-9x) = x^2 - \frac{1}{4}(1+x) + \frac{1}{12}(5-9x) = x^2 - x + \frac{1}{6}$$

(e) Find the coordinates of the polynomial $T(x) = x^2 + x + 1$ with respect to the orthogonal basis in (d).

Solution:

$$[T]_{\mathcal{B}} = \begin{bmatrix} \frac{<T(x),\vec{\mathbf{v}}_1>}{<\vec{\mathbf{v}}_1,\vec{\mathbf{v}}_1>} \\ \frac{<T(x),\vec{\mathbf{v}'}_2>}{<\vec{\mathbf{v}'}_2,\vec{\mathbf{v}'}_2>} \\ \frac{<T(x),\vec{\mathbf{v}}_3>}{<\vec{\mathbf{v}}_3,\vec{\mathbf{v}}_3>} \end{bmatrix} = \begin{bmatrix} \frac{35/12}{7/3} \\ \frac{-7/12}{7} \\ \frac{1/180}{1/180} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{12} \\ 1 \end{bmatrix}$$

6. True or false? If true give a proof if false say why it fails to be true.

(a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } a + c = b + d \right\}$ is a subspace of $\mathcal{M}_{2\times 2}$ with the usual addition and scalar multiplication.

Solution: True. We check that W is closed under addition and scalar multiplication.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ be elements of W, so that $a + c = b + d, a' + c' = b' + d'$. Then $A + B = \begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}$ has the property $(a + a') + (c + c') = (b + b') + (d + d')$ obtained by adding together the previous two equalities. Therefore $A + B$ is in turn in W.

Let k be a scalar. $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ has the property $ka + kc = kb + kd$ (obtained from $a + c = b + d$) which means kA is in W.

(b) $W = \{A \in \mathcal{M}_{2\times 2}$ with $det(A) = 1\}$ is a subspace of $\mathcal{M}_{2\times 2}$ with the usual addition and scalar multiplication.

Solution: False. W is not closed under addition. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are in W (since $det(A) = 1, det(B) = 1$), but $A + B = \mathbf{0_{2\times 2}}$ is not in W since $det(\mathbf{0_{2\times 2}}) = 0 \neq 1$.

(c) $\mathcal{B} = \{1 + x, 2 - x + x^2, 3x - 2x^2, -1 + 3x + x^2\}$ is a basis for $\mathcal{P}_2$.

Solution: False. Since $dim(\mathcal{P}_2) = 3$ any basis of $\mathcal{P}_2$ must have 3 elements, but $\mathcal{B}$ has 4 elements.

(d) If V and W are subspaces of $\mathbb{R}^3$ then

$$V \cap W = \text{ the set of vectors that are both in } V \text{ and in } W$$

is a subspace of $\mathbb{R}^3$.

Solution: True. We check that $V \cap W$ is closed under addition and scalar multiplication.

Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ be vectors in $V \cap W$. Since $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are in V, $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is in V. Since $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are in W, $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is in W. Therefore $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is in $V \cap W$.

Let $\vec{\mathbf{u}}$ be a vector in $V \cap W$ and c a scalar. Since $\vec{\mathbf{u}}$ is in V, $c\vec{\mathbf{u}}$ is in V. Since $\vec{\mathbf{u}}$ is in W, $c\vec{\mathbf{u}}$ is in W. Therefore $c\vec{\mathbf{u}}$ is in $V \cap W$.

(e) If V and W are subspaces of $\mathbb{R}^3$ then

$$V + W = \{\vec{\mathbf{v}} + \vec{\mathbf{w}} \text{ with } \vec{\mathbf{v}} \in V, \vec{\mathbf{w}} \in W\}$$

is a subspace of $\mathbb{R}^3$.

Solution: True. We check that W is closed under addition and scalar multiplication.

Let $\vec{\mathbf{v}} + \vec{\mathbf{w}}, \vec{\mathbf{v}'} + \vec{\mathbf{w}'}$ be vectors in $V + W$. Then their sum $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) + (\vec{\mathbf{v}'} + \vec{\mathbf{w}'}) = (\vec{\mathbf{v}} + \vec{\mathbf{v}'}) + (\vec{\mathbf{w}} + \vec{\mathbf{w}'})$ is in $V + W$ because $\vec{\mathbf{v}} + \vec{\mathbf{v}'}$ is in V and $\vec{\mathbf{w}} + \vec{\mathbf{w}'}$ is in W.

Let $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ be a vector in $V + W$ and c a scalar. Then $c(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = c\vec{\mathbf{v}} + c\vec{\mathbf{w}}$ is in $V + W$ since $c\vec{\mathbf{v}}$ is in V and $c\vec{\mathbf{w}}$ is in W.