MATH 314 Fall 2011
Section 001

Practice Midterm 2 - Solutions

1. Compute the following:

(a) This is a triangular matrix, so the determinant is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot (-3) = -6$$

(b) Method 1: We use elementary row operations:

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \overset{R_2' = R_2 - 2R_1}{\longrightarrow} B = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \text{so } det(A) = det(B).$$

Since B has a row of zero entries, $det(B) = 0$ and so $det(A) = 0$.

Method 2: We use the cofactor expansion along the third column of the 4×4 determinant followed by a cofactor expansion along the third row of the 3×3 determinant:

$$\begin{vmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = (-1)^{3+3} \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 0 & 0 & -3 \end{vmatrix} = (-1)^{3+3}(-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

Solution notes: there are many other valid approaches.

(c) We use cofactor expansions along the first row of the matrix then we notice the smaller matrices are diagonal:

$$\begin{vmatrix} a & b & 0 & \dots & 0 \\ c & d & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = (-1)^{1+1} \cdot a \begin{vmatrix} d & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{vmatrix} + (-1)^{1+2} \cdot b \begin{vmatrix} c & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{vmatrix} =$$

$= ad - bc.$

Solution notes: there are other possible solutions; for example by keeping expanding along the last row.

(d) Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, compute:

(a)

$$\begin{array}{rcc} det(A) & = & 0+2+0-1-1-0=0 \\ det(5A) & = & 5^3 det(A) = 125 \cdot 0 = 0 \\ det(A^{-1}) & = & \text{does not exist} \\ det(A^{10}) & = & det(A)^{10} = 0^{10} = 0. \end{array}$$

(b) The minors of A (arranged in a matrix); m_{ij} is the minor obtained by eliminating row i and column j of A:

$$M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

(c) The cofactors of A ($c_{ij} = (-1)^{i+j} m_{ij}$):

$$C = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

(d) The adjoint matrix of A ($adj(A) = C^T$):

$$adj(A) = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

(e) The inverse matrix A^{-1} does not exits (since $det(A) = 0$, A is not invertible).

2. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the characteristic equation of A.

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 2-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$= (3-\lambda)[(2-\lambda)^2 - 1] = (3-\lambda)(\lambda^2 - 4\lambda + 3) = (3-\lambda)(\lambda-3)(\lambda-1) = -(\lambda-3)^2(\lambda-1)$

The characteristic equation is: $-(\lambda-3)^2(\lambda-1) = 0$.

(b) Find the eigenvalues of A.
The eigenvalues are: $\lambda_1 = 3, \lambda_2 = 1$.

(c) Find the eigenspaces of A.
$E_{\lambda_1} = Null(A - 3I)$:

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{cases} x_1 &= s + 2t \\ x_2 &= s \\ x_3 &= t \end{cases}$$

$$E_{\lambda_1} = \left\{ \begin{bmatrix} s+2t \\ s \\ t \end{bmatrix} ; s, t \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$E_{\lambda_2} = Null(A - I)$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{cases} x_1 &= -t \\ x_2 &= t \\ x_3 &= 0 \end{cases}$$

$$E_{\lambda_1} = \left\{ \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} ; t \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(d) Find for each eigenvalue the algebraic and geometric multiplicity. Is A diagonalizable?
For $\lambda_1 = 3$: algebraic multiplicity 2, geometric multiplicity 2.
For $\lambda_1 = 1$: algebraic multiplicity 1, geometric multiplicity 1.
Since for each of the two eigenvalues the algebraic multiplicity is equal to the geometric multiplicity, A is diagonalizable.

3. Let A be an unknown 3x3 matrix that has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding eigenspaces $E_{\lambda_1} = Span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}, E_{\lambda_2} = Span\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$.

(a) What are the algebraic and geometric multiplicities of λ_1, λ_2? Explain. A quick check reveals that $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$ are linearly independent, hence $dim(E_{\lambda_1}) = 2$. So the geometric multiplicity of λ_1 is 2 and the geometric multiplicity of λ_2 is $dim(E_{\lambda_2}) = 1$.

(b) Find an invertible P and a diagonal D such that $P^{-1}AP = D$.

$$P = \begin{bmatrix}1 & 2 & 3\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}, D = \begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 1\end{bmatrix}$$

(c) Compute A^{100}.

$$P^{-1}AP = D \Rightarrow (P^{-1}AP)^{100} = D^{100} \Rightarrow P^{-1}A^{100}P = D^{100} \Rightarrow A^{100} = PD^{100}P^{-1}.$$

$$D^{100} = \begin{bmatrix}3^{100} & 0 & 0\\0 & 3^{100} & 0\\0 & 0 & 1^{100}\end{bmatrix}, P^{-1} = \frac{1}{2}\begin{bmatrix}-1 & 3 & 2\\0 & 0 & 2\\1 & -1 & -2\end{bmatrix}$$

$$\begin{aligned}A^{100} &= \begin{bmatrix}1 & 2 & 3\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}3^{100} & 0 & 0\\0 & 3^{100} & 0\\0 & 0 & 1^{100}\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}-1 & 3 & 2\\0 & 0 & 2\\1 & -1 & -2\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}3 - 3^{100} & 3^{101} - 3 & 2(3^{101} - 3)\\1 - 3^{100} & 3^{101} - 1 & 2(3^{100} - 1)\\0 & 0 & 3^{100}\end{bmatrix}.\end{aligned}$$

(d) Find a *probability* vector $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{x}}$.

A probability vector is an eigenvector for $\lambda_2 = 1$ therefore $\vec{\mathbf{v}} = c\begin{bmatrix}3\\1\\0\end{bmatrix}$. So $3c + c = 1$, hence $c = 1/4$ and $\vec{\mathbf{v}} = \begin{bmatrix}3/4\\1/4\\0\end{bmatrix}$.

(e) Can A be a stochastic matrix?
A cannot be a stochastic matrix because the eigenvalues of a stochastic matrix must obey $|\lambda| \leq 1$ (1 is the dominant eigenvalue) , whereas $|\lambda_1| = 3 > 1$.

4. (a) Give an example of three linearly independent vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^3$ such that $\vec{\mathbf{u}}$ is orthogonal to $\vec{\mathbf{v}}$, $\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{w}}$, but $\vec{\mathbf{w}}$ is not orthogonal to $\vec{\mathbf{u}}$. You will use these vectors for the rest of the problem.

Solution notes: *your choice of* $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ *will probably be different.*

Let $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and check that $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are not orthogonal since $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 1 \neq 0$. Notice that $\vec{\mathbf{v}}$ must be orthogonal to *both* $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$. A vector with this property is $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$ so set

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} \times \vec{\mathbf{w}} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = -3j + 2k = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}.$$

(b) Let $S = Span\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$. Compute $proj_S(\vec{\mathbf{w}})$ and $perp_S(\vec{\mathbf{w}})$.
Note that $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ is an orthogonal basis of S. Therefore

$$proj_S(\vec{\mathbf{w}}) = (\frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}}{\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}})\vec{\mathbf{u}} + (\frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}})\vec{\mathbf{v}} = \vec{\mathbf{u}} + 0 \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$perp_S(\vec{\mathbf{w}}) = \vec{\mathbf{w}} - proj_S(\vec{\mathbf{w}}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

(c) Find an orthogonal basis $\mathcal{B}$ of $\mathbb{R}^3$ that contains the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.
Since $\vec{\mathbf{p}} = perp_S(\vec{\mathbf{w}})$ is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, it is convenient to pick

$$\mathcal{B} = \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{p}}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(d) Find the coordinates of $\vec{\mathbf{w}}$ with respect to the basis $\mathcal{B}$ described above.

$$\vec{\mathbf{w}} = (\frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}}{\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}})\vec{\mathbf{u}} + (\frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}})\vec{\mathbf{v}} + (\frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{p}}}{\vec{\mathbf{p}} \cdot \vec{\mathbf{p}}})\vec{\mathbf{p}} = 1 \cdot \vec{\mathbf{u}} + 0 \cdot \vec{\mathbf{v}} + 1 \cdot \vec{\mathbf{p}}$$

Therefore $[\vec{\mathbf{w}}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

5. The purpose of this problem is to prove that eigenspaces are subspaces.

(a) Let A be an $n \times n$ matrix. Define what an eigenvalue λ of A is and what an eigenvector $\vec{\mathbf{v}}$ corresponding to the eigenvalue λ is.
A scalar λ and a non-zero vector $\vec{\mathbf{v}}$ are called an eigenvalue of A and an eigenvector corresponding to the eigenvalue λ respectively if they satisfy $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$.

(b) Let $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ be eigenvectors corresponding to the *same* eigenvalue λ. Prove that $\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$ is an eigenvector corresponding to the eigenvalue λ as well.

$$\begin{cases} A\vec{\mathbf{v_1}} = \lambda\vec{\mathbf{v_1}} \\ A\vec{\mathbf{v_2}} = \lambda\vec{\mathbf{v_2}} \end{cases} \Longrightarrow A\vec{\mathbf{v_1}} + A\vec{\mathbf{v_2}} = \lambda\vec{\mathbf{v_1}} + \lambda\vec{\mathbf{v_2}} \Longrightarrow A(\vec{\mathbf{v_1}} + \vec{\mathbf{v_2}}) = \lambda(\vec{\mathbf{v_1}} + \vec{\mathbf{v_2}})$$

$\Longrightarrow \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$ is an eigenvector corresponding to the eigenvalue λ.

(c) Let $\vec{\mathbf{v}}$ be an eigenvectors corresponding to the eigenvalue λ and let c be a scalar. Prove that $c\vec{\mathbf{v}}$ is an eigenvector corresponding to the eigenvalue λ.

$$A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \Longrightarrow cA\vec{\mathbf{v}} = c\lambda\vec{\mathbf{v}} \Longrightarrow A(c\vec{\mathbf{v}}) = \lambda(c\vec{\mathbf{v}})$$

$\Longrightarrow c\vec{\mathbf{v}}$ is an eigenvector corresponding to the eigenvalue λ.

(d) Define the eigenspace E_λ and explain how the facts above prove it is a subspace of $\mathbb{R}^n$.
The eigenspace E_λ is the set of all eigenvectors corresponding to the eigenvalue λ:

$$E_\lambda = \{\vec{\mathbf{v}}; A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}\}.$$

Part (b) shows that E_λ is closed under vector addition and part (c) shows that E_λ is closed under scalar multiplication, therefore E_λ is a subspace of $\mathbb{R}^n$.