Practice Midterm 1 - Solutions

1. (a) Find the equation of the plane in $\mathbb{R}^3$ that passes through the origin and has normal vector

$$\vec{\mathbf{n}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The normal equation $\vec{\mathbf{n}} \cdot \vec{\mathbf{x}} = \vec{\mathbf{n}} \cdot \vec{\mathbf{p}}$, where $\vec{\mathbf{p}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the position vector of the origin yields

$$x + 2y + 3z = 0$$

(b) Find a and b such that the plane in $\mathbb{R}^3$ of equation $ax + by + z = 2$ is parallel to the plane in part a).

Solution: $\vec{\mathbf{n}}' = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} \Longrightarrow \begin{cases} a = c \\ b = 2c \\ 1 = 3c \end{cases} \Longrightarrow$
$\begin{cases} c = 1/3 \\ a = 1/3 \\ b = 2/3 \end{cases}$

(c) Find the vector equation of the line in $\mathbb{R}^3$ which passes through the point $P = (1, 1, 1)$ and is perpendicular to the plane in a).

Solution: The vector equation $\vec{\mathbf{x}} = t\vec{\mathbf{d}} + \vec{\mathbf{p}}$, where $\vec{\mathbf{p}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the position vector of the point P and $\vec{\mathbf{d}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the direction perpendicular to the plane in part a) yields

$$\vec{\mathbf{x}} = t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t+1 \\ 2t+1 \\ 3t+1 \end{bmatrix}, t \in \mathbb{R}.$$

2. Find all the values of k such that the linear system

$$\begin{cases} y + 2kz & = & 0 \\ x + 2y + 6z & = & 2 \\ kx + 2z & = & 1 \end{cases}$$

has

(a) no solutions.

(b) a unique solution.

(c) infinitely many solutions.

Solution: The augmented matrix of the system row reduces as follows:

$$\begin{bmatrix} 0 & 1 & 2k & | & 0 \\ 1 & 2 & 6 & | & 2 \\ k & 0 & 2 & | & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ k & 0 & 2 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 - kR_1} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ 0 & -2k & 2-6k & | & 1-2k \end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 + 2kR_2} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ 0 & 0 & 2-6k+4k^2 & | & 1-2k \end{bmatrix}$$

Note that in order to row-reduce further we need to know whether the entry $2 - 6k + 4k^2$ is zero. Also note $2 - 6k + 4k^2 = 2(2k-1)(k-1)$.

Case 1: if $k \neq 1/2, k \neq 1$ then the coefficient matrix is invertible because it has rank 3. It follows by the theorem of invertible matrices that the system has unique solution.

Case 2: if $k = 1/2$ then the last variable is free and therefore there are infinitely many solutions.

Case 3: if $k = 1$ then the last row gives the equation $0 = -1$ which says the system is inconsistent (no solutions).

3. (a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -2 \\ -2 & -7 & 0 \end{bmatrix}$.

Solution: We row reduce the multi-augmented matrix $[A|I_3]$.

$$\left[\begin{array}{ccc|ccc} 1 & 5 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -2 & -7 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3'=R_3+2R_1} \left[\begin{array}{ccc|ccc} 1 & 5 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_3'=R_3-3R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$
$$\xrightarrow{R_1'=R_1-5R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & -5 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$

Since A does not row-reduce to I_3, A does not have an inverse.

(b) Check that the matrix you found is really A^{-1}.

Solution: Nothing to do. Trick question.

(c) Use part a) to write $\vec{\mathbf{b}} = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$ as a linear combination of the vectors

$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$.

Solution: By part a) we know that a basis for $Span\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is given by the first two vectors, so it's enough to solve the system $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{2}} = \vec{\mathbf{b}}$ whose augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 5 & -4 \\ 0 & 1 & -1 \\ -2 & -7 & 5 \end{array}\right] \xrightarrow{R_3'=R_3+2R_1} \left[\begin{array}{cc|c} 1 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{cc|c} 1 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-5R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $c_1 = 1, c_2 = -1$ and the required linear combination is

$$\vec{\mathbf{b}} = 1\vec{\mathbf{v}}_1 + (-1)\vec{\mathbf{v}}_2 + 0\vec{\mathbf{v}}_3.$$

4. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$. You may assume that $RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for the row space of A.

Solution:

A basis for the row space of A is given by the rows in R containing leading 1's:

$$\{\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}\}.$$

(b) Find a basis for the null space of A.

Solution:

We compute the nullspace of A by solving the system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$A = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ and $x_5 = t$ are free variables. The nullspace is

$$\begin{aligned} null(A) &= \left\{ \begin{bmatrix} s+t \\ -s-t \\ s \\ -t \\ t \end{bmatrix}, s, t \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, s, t \in \mathbb{R} \right\} \\ &= Span \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \& \quad \text{a basis is given by} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

(c) Check that the rank theorem holds for A.

Solution: From a) $rank(A) = 3$, from b) $nullity(A) = 2$ and $2 + 3 = 5 = \#$ of columns (this verifies the rank theorem: $rank(A) + nullity(A) = \#$ of columns of A).

5. (a) Define what a *subspace* is.

Solution:

A set S is a subspace of $\mathbb{R}^n$ if it satisfies:

(a) $\vec{\mathbf{0}} \in S$

(b) for any two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S \Longrightarrow \vec{\mathbf{u}} + \vec{\mathbf{v}} \in S$

(c) for any vector $\vec{\mathbf{u}} \in S$ and any scalar $c \in \mathbb{R} \Longrightarrow c\vec{\mathbf{u}} \in S$

(b) Is $S = \{ \begin{bmatrix} x \\ y \end{bmatrix}$ with $x \geq 0, y \leq 0\}$ a subspace of $\mathbb{R}^2$? Why or why not?

Solution:

S is not a subspace or $\mathbb{R}^2$ because it is not closed under scalar multiplication. For example if $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in S$ and $c = -1$ then $c\vec{\mathbf{u}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin S$.

6. Are the following statements *true* or *false*. If you believe a statement is false give a counterexample. If you believe the statement is true state why it is true.

(a) Let A be a 3×4 matrix. Then the columns of A must be linearly dependent vectors.

Solution: True. The rows are 4 vectors in $\mathbb{R}^3$ and since 4¿3 they will be linearly dependent by a theorem we learned (it is denoted by Theorem 2.8 in the textbook and class notes). Another way to see this is that the rank of A is at most 3, hence the rank of A is less than the number of vectors (4).

(b) Let A be a 3×4 matrix. Then the nullity of A must be 1.

Solution: False. For $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $rank(A) = 2$ and since $rank(A) + nullity(A) = 4$ we have $nullity(A) = 2$.

(c) If A, B, C are all $n \times n$ matrices and $AB = AC$ then $B = C$

Solution: False. Let A be the zero matrix and $B \neq C$ any two distinct matrices. Then $AB = AC = \mathbf{O}_{n\times n}$ but $B \neq C$.

(d) If A, B are $n \times n$ matrix and $AB = \mathbf{O}_{n\times n}$, then $A = \mathbf{O}_{n\times n}$ or $B = \mathbf{O}_{n\times n}$. (Here $\mathbf{O}_{n\times n}$ is the $n \times n$ zero matrix.)

Solution: False. Consider $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = \mathbf{O}_{n\times n}$ but neither A nor B are the zero matrix.

7. Prove that if A is an invertible $n \times n$ matrix then

$$(A^T)^{-1} = (A^{-1})^T.$$

(Recall A^T means the matrix transpose and A^{-1} means the matrix inverse.)

Solution: All we need to show is that

$$(A^{-1})^T \cdot A^T = I_n \text{ and } A^T \cdot (A^{-1})^T = I_n$$

$(A^{-1})^T \cdot A^T = (A \cdot A^{-1})^T = (I_n)^T = I_n$
$A^T \cdot (A^{-1})^T = (A^{-1}A)^T = (I_n)^T = I_n$