AUG. 26, 2013

⋗ (Handwritten)

AUG. 28, 2013

⋗ **Ideals and Varieties:** Let R be a commutative ring.

⋗ DEFN: A subset $I \subseteq R$ is called an *ideal* if the following hold

1. $0 \in I$[1]
2. $f, g \in I$ implies that $f + g \in I$
3. $f \in I$, $r \in R$ implies $rf \in I$.

⋗ DEFN: Let $f_1, ..., f_s \in R$. The set $(f_1, ..., f_s) = \{\sum_{i=1}^{s} r_i f_i : r_i \in R\}$ is the *ideal generated by* $\{f_1, ..., f_s\}$.

⋗ DEFN: Let I be an ideal of R. We say that I is *finitely generated* if there are $f_1, ..., f_s \in R$ such that $I = (f_1, ..., f_s)$.

⋗ NOTATION:[2] $R = k[x_1, ..., x_n]$ is the polynomial ring with coefficients in k (where k is a field that is sometimes algebraically closed). Also, let $\mathbb{A}^n = k^n = \{(a_1, ..., a_n) : a_i \in k\}$ represent affine space.

⋗ DEFN: Let $I = (f_1, ..., f_s)$ be an ideal in R. Define the *affine variety* corresponding to I as $\mathbb{V}(I) = \{a = (a_1, ..., a_n) : f_1(a_1, ..., a_n) = \cdots = f_s(a_1, ..., a_n)\}$.

⋗ EXAMPLES:

1. (Twisted Cubic) $I = (y - x^2, z - x^3) \subseteq R = k[x, y, z]$. In this case,[3]
$$TC := \mathbb{V}(I) = \{a = (a_1, a_2, a_3) : a_2 - a_1^2 = a_3 - a_1^3 = 0\} = \{(a_1, a_1^2, a_1^3) : a_1 \in k\}.$$
2. Hypersurfaces: $\mathbb{V}((f))$ is called a *hypersurface*.

⋗ PROPERTY OF $\mathbb{V}(-)$: Inclusion reversing: $I \subseteq J \implies \mathbb{V}(J) \subseteq \mathbb{V}(I)$. For example, $(y - x^2) \subseteq (y - x^2, z - x^3)$ will imply that $V(TC) \subseteq \mathbb{V}(y - x^2) =: V_1$ (the latter is a parabolic cylinder). Also, since $(z - x^3) \subseteq (y - x^2, z - x^3)$ and so $TC \subseteq \mathbb{V}(z - x^3) =: V_2$ (also some kind of cylinder). In fact, $TC = V_1 \cap V_2$.

⋗ DEFN: Given an affine variety $V \subseteq \mathbb{A}^n$, we define the ideal corresponding to it
$$\mathbb{I}(V) = \{f \in R = k[x_1, ..., x_n] : f(a_1, ..., a_n) = 0 \forall (a_1, ..., a_n) \in V\}.$$

⋗ RMK: $\mathbb{I}(V)$ is also inclusion reversing: $V \subseteq W \iff \mathbb{I}(V) \supseteq \mathbb{I}(W)$. Furthermore, $V = W \iff \mathbb{I}(V) = \mathbb{I}(W)$.

⋗ RMK: $\mathbb{I}(\mathbb{V}(f_1, ..., f_s)) \supseteq (f_1, ..., f_s)$. (Prove this!)

⋗ To see that this inclusion can be strict, consider the following example: (in $R = \mathbb{C}[x, y]$, $\mathbb{A}^2$), consider $\mathbb{V}(x^2, y^2) = \{(0, 0)\}$ and $\mathbb{I}(\mathbb{V}(x^2, y^2)) = \mathbb{I}(\{(0, 0)\}) = (x, y)$.

⋗ **Problems:**

1. Ideal description:
   - Is every ideal $I \subseteq R$ finitely generated? (Hilbert Basis Theorem)
   - How about a "nice" set of generators?
2. Ideal membership: Given some ideal $I = (f_1, ..., f_s)$ and polynomial $f \in R$, is $f \in I$?
3. Ideals/Varieties: Given "nice" sets of generators for two ideal I and J, can we find sets of generators for $I \cap J$ or $I : J$ or I^{sat}?[4]

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[1] This ensures that $I \neq \varnothing$.
[2] For the first half of this class
[3] Reference "numerical semigroup rings."
[4] I^{sat} is the saturation of the ideal I.

4. Implicitization / Elimination: Given a variety $V \subseteq \mathbb{A}^n$ defined parametrically, i.e., $\{x_i = g_i(y_1, ..., y_m\}_{i=1}^n$, can we find $\mathbb{I}(V)$ (equivalently, find relations between the x_i's that don't involve the y_i's).

- **Monomial Orders:** A monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R = k[x_1, ..., x_n]$, $\mathbb{Z}_{\geq 0}^n$.
- DEFN: A *monomial order* on R is a binary relation "$>$" on the set of monomials of R satisfying:
  1. $>$ is a *total ordering* (any two monomials are comparable),
  2. If $x^\alpha > x^\beta \implies x^\alpha x^\gamma > x^\beta x^\gamma$ for any $\gamma \in \mathbb{Z}_{\geq 0}^n$,
  3. $>$ is a *well-ordering* (every nonempty subset of monomials has a smallest element under $>$).

SEP. 4, 2013

- DEFN: (*From last time...*) A *monomial order* on R is a binary relation "$>$" on the set of monomials of R satisfying:
  1. $>$ is a *total ordering* (any two monomials are comparable),
  2. If $x^\alpha > x^\beta \implies x^\alpha x^\gamma > x^\beta x^\gamma$ for any $\gamma \in \mathbb{Z}_{\geq 0}^n$,
  3. $>$ is a *well-ordering* (every nonempty subset of monomials has a smallest element under $>$).
- RMKS:
  - (3) is equivalent to (3'): Every strictly descending sequence of monomials must terminate.
  - (3) is equivalent to (3"): $>$ is a *global ordering*, i.e., $x^\alpha > 1$ for all $\alpha \neq (0, ..., 0)$.
  - (3) is equivalent to (3"'): $>$ refines the partial order given by divisibility, i.e., $x^\beta | x^\alpha \implies x^\beta < x^\alpha$.
- RMK: There is another natura partial order on R given by degree:

$$\deg(x^\alpha) = \sum_{i=1}^n \alpha_i = |\alpha|.$$

  Some monomial orderings refine the degree order, some don't.
- All examples below depend on an ordering of the the variables: $x_1 > x_2 > \cdots > x_n$.
- Examples:
  1. <u>Lex</u>: $x^\alpha >_{\text{Lex}} x^\beta \iff$ (defn) in first coordinate where α and β differ, we have $\alpha_i > \beta_i$. Equivalenetly, the leftmost non-zero entry of $\alpha - \beta$ must be positive. For example, $xy^2 >_{\text{Lex}} y^3z^4$, since the left has $(1, 2, 0)$ and the right has $(0, 3, 4)$. Note that this does not refine the degree order (since the one on the left has degree 3, the one on the right has degree 7). For another example, $xy^2 >_{\text{Lex}} xy$.

     Note that this is similar to the "phonebook ordering," but is only identical when restricted to monomials of a fixed degree.
  2. <u>GrLex</u>: We say $x^\alpha >_{\text{GrLex}} x^\beta \iff \deg(x^\alpha) > \deg(x^\beta)$ OR $\deg(x^\alpha) = \deg(x^\beta)$ and $x^\alpha >_{\text{Lex}} x^\beta$.

     Back to the example, we have $y^3z^4 >_{\text{GrLex}} xy^2$ here.

     The other example, $x^2y >_{\text{GrLex}} xy$.

  (*) <u>RevLex</u>: $x^\alpha >_{\text{RevLex}} x^\beta \iff$ rightmost non-zero entry of $\alpha - \beta$ is negative. *HOWEVER,* this is NOT a monomial order! Note that (3') doesn't hold, since we have:

$$x >_{\text{RevLex}} > x^2 >_{\text{RevLex}} x^3 >_{\text{RevLex}} \cdots,$$

  since this is an infinite descending chain of monomials. Also (3") doesn't hold, since this is NOT a global order: $1 >_{\text{RevLex}} x$ (we have $(0, 0, ..., 0)$ and $(1, 0, ..., 0)$).

(3) GrRevLex: $x^\alpha >_{\text{GrRevLex}} x^\beta \iff \deg(x^\alpha) > \deg(x^\beta)$ OR $\deg(x^\alpha) = \deg(x^\beta)$ and $x^\alpha >_{\text{RevLex}} x^\beta$.

Exercise: Check this is a monomial order.

(4) Weighted orders: Take $w = (w_1, ..., w_n) \in \mathbb{R}^n_{\geq 0}$. Then define $x^\alpha >_w x^\beta \iff \alpha \cdot w > \beta \cdot w \iff \sum_{i=1}^n \alpha_i w_i > \sum_{i=1}^n \beta_i w_i$.

If the entries of w are rationally-independent, then $>_w$ is a monomial order. Also note that rationally independent is equivalent to $>_w$ is a total order.

We could have problems.. pick $w = (1, 1)$. Then x^2 and xy can't be compared.

What if we don't want to work with w having $\mathbb{Q}$-independent entries? Then start with any w (which may give a partial order), then use a w' to refine, continue ... use $w^{(n)}$ to refine. Then this will give a total order.

For example, you can recover Lex by using $w = (1, 0, ..., 0)$, $w' = (0, 1, 0, ..., 0)$, ..., $w^{'\cdots'} = (0, ..., 0, 1)$.

Note!: Any monomial order is equivalent to a refined weighted order.

3. Block order: Two blocks of variables $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_m\}$. Then $>_1$=monomial order on $k[x_1, ..., x_n]$ and $>_2$=monomial order on $k[y_1, ..., y_m]$. On $R = k[x_1, ..., x_n, y_1, ..., y_m]$, we have

$$x^\alpha y^{\alpha'} >_{1,2} x^\beta y^{\beta'} \iff x^\alpha >_1 x^\beta \text{ OR } x^\alpha = x^\beta \text{ and } y^{\alpha'} >_2 y^{\beta'}.$$

⋗ DEFN: Let $f = \sum_\alpha \underbrace{a_\alpha}_{\text{constants}} \underbrace{x^\alpha}_{\text{monomials}} = a_{\text{multideg}(f)} x^{\text{multideg}(f)} +$ lower terms, where $>$ is a monomial order, and

$$\text{multideg}(f) = \max\{\alpha : a_\alpha \neq 0\}.$$

The *leading coefficient* is $LC(f) = a_{\text{multideg}(f)}$.
The *leading monomial* is $LM(f) = x^{\text{multideg}(f)}$.
The *leading term* is $LT(f) = LC(f)LM(f)$.

⋗ THM: (DIVISION ALGORITHM) Fix a monomial ordering "$>$" on $R = k[x_1, ..., x_n]$. Let $f_1, ..., f_x$ be an ordered s tuple of non-zero polynomials in R. Then every polynomial $f \in R$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in R$ such that

1. $\text{multideg}(f) \geq \text{multideg}(a_i f_i)$ for all i such that $a_i \neq 0$. (In fact, $LT(f) = \max\{LT(a_i)LT(f_i) : a_i \neq 0\}$.)
2. no monomial appearing in r is divisible by any of $LT(f_1)$,..., $LT(f_s)$.
3. for $i > j$, no monomial of $a_i LT(f_i)$ is divisible by $LT(f_j)$.

Proof. (Constructive) An algorithm that constructs a_i, r.
$a_i = 0$; $r = 0$.
while $f \neq 0$, do
look at $LT(f)$: if $M = \{i : LT(f_i) : LT(f)\} \neq \varnothing$, then (letting $i = \min M$) $f := f - \underbrace{\frac{LT(f)}{LT(f_i)} \cdot f_i}_{LT \text{ of this } = LT(f)}$ and $a_i := a_i + \frac{LT(f)}{LT(f_i)}$. Else, $f := f - LT(f)$ and $r := r + LT(f)$. □

SEP. 6, 2013

⋗ NOTATION: $r = f\%(f_1, ..., f_s)$ denotes remainder.

- Proof restarted.

Proof. Algorithm:
$a_i := 0$; $r := 0$;
while $f \neq 0$, do look at $LT(f)$: if $(U = \{j : LT(f_j)|LT(f)\} \not\varnothing)$ then { $i := \min U$.[5]
$a_i := a_i + LT(f)/LT(f_i)$
$f := f - (LT(f)/LT(f_i))f_i$ }
else[6] {
$r := r + LT(f)$
$f := f - LT(f)$. }

Claim: This algorithm terminates (in a finite number of steps). Denote by $f^{(0)} = f$, $f^{(t)} = f$ obtained after the tth iteration of the algorithm. The sequence of monomials

$$LT(f^{(0)}), LT(f^{(1)}), ...$$

because either (*) $f^{(t+1)} = f^{(t)} - (LT(f)/LT(f_i))f_i$ when $LT(f_i)|LT(f^{(t)})$ OR (**) $f^{(t+1)} = f^{(t)} - LT(f^{(t)}) \implies LT(f^{(t+1)}) < LT(f^{(t)})$.
Clarifying:
(*) $f^{(t+1)} = f^{(t)} - (LT(f)/LT(f_i))f_i$

$$\begin{aligned} LT(f^{(t+1)}) &= LT(f^{(t)} - (LT(f)/LT(f_i))f_i) \\ &= LT(LT(f^{(t)} + \text{ lower order terms } - (LT(f^{(t)})/LT(f_i))(LT(f_i) + \text{ terms that are smaller than } LT(f_i)\ ) \\ &= LT(\text{ terms that are smaller than } LT(f^{(t)}) - \text{ term that is smaller than } LT(f^{(t)})) \\ &< LT(f^{(t)}). \end{aligned}$$

By well-ordering ((3) of monomial orders) the sequence $\{LT(f^{(t)})\}_{t\geq 0}$ must terminate (i.e., after a number of steps N, $f^{(N)} = 0$).

This algorithm returns polynomials a_i, r. ☐

- Exercise: Check that (1), (3) in Theorem also hold.
- Rmk: Having fixed $>$ and order of the f_i's, the division algorithm in the deterministic form produces a unique remainder r. However, if we change the order of the f_i's or if we change the term order, then the algorithm will produce a different remainder r.
- Example 1: $f_1 = x^3$, $f_2 = x^2y - y^3$, $R = k[x, y]$, $>$=Lex with $x > y$. Let $f = x^3y$.
Division Algorithm:
  - *Initialize:* $f^{(0)} = x^3y$, $a_1 = a_2 = 0 = r$
  - *Iteration 1:* $U = \{1, 2\}$, so $i = 1$. Then $a_1 = 0 + \frac{x^3y}{x^3} = y$. Then $f^{(1)} = x^3y - \frac{x^3y}{x^3}x^3 = 0$ (STOP).

  Return: $a_1 = y$; $a_2 = 0$; $r = 0$, so $f - y \cdot f_1 + 0 \cdot f_2 + 0 = y(x^3) + 0(x^2y - y^3) + 0$
- Example 2: $f_1 = x^2y - y^3$, $f_2 = x^3$ with same other hypotheses as in Ex. 1. Also let $f = x^3y$. Division Algorithm:
  - *Initialize:* $f^{(0)} = x^3y$; $a_1 = a_2 = 0 = r$.
  - *Iteration 1:* $U = \{1, 2\}$, so $i = 1$. Then $a_1 = 0 + \frac{x^3y}{x^2y} = x$. Then $f^{(1)} = x^3y - \frac{x^3y}{x^2y} \cdot (x^2y - y^3) = x^3y - x^3y + xy^3 = xy^3$.
  - *Iteration 2:* $U = \varnothing$ (so we move to else branch). Now $r = 0 + xy^3$ and $f^{(2)} = xy^3 - xy^3 = 0$ (STOP).

[5] this makes the algorithm "determinate" (indeterminate version is pick some $i \in U$).
[6] This ensures that (2) holds.

Return $a_1 = x$, $a_2 = 0$, $r = xy^3$. Hence $f = x(x^2y - y^3) + 0(x^3) + xy^3$.

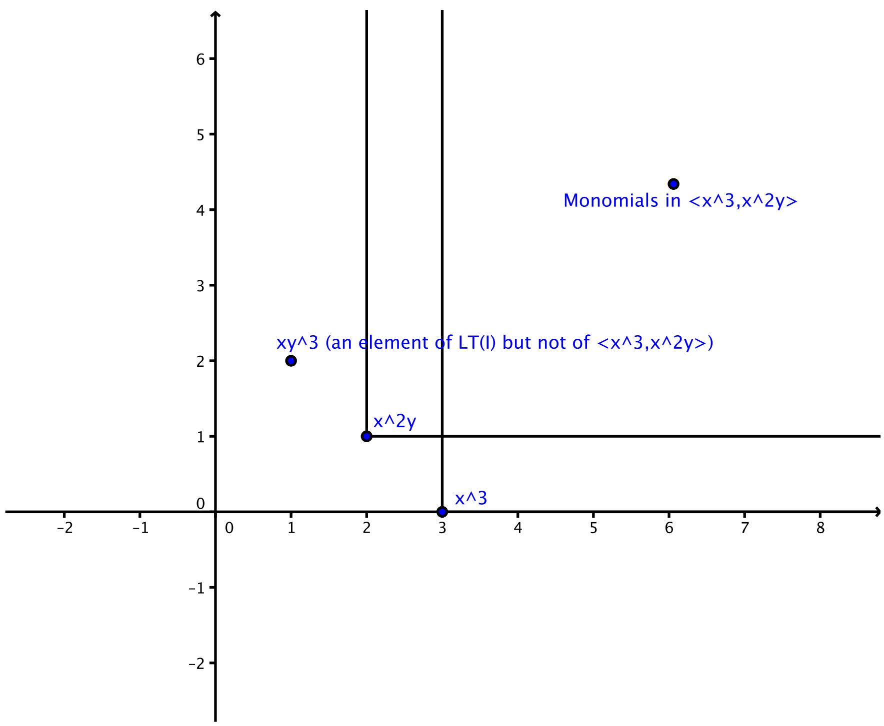
> **Initial ideals.**

> Defn: Given an ideal $I \subseteq R$ and a monomial order $>$, then the ideal of leading terms (the initial ideal) of I is $LT(I) =< LT(f) : f \in I >$.

> Rmks:

1. The set of monomial elements in the $LT(I)$ is $\{LT(f) : f \in I\}$ (Easy exercise).
2. If $I = (f_1, ..., f_s)$, then $< LT(f_1), ..., LT(f_s) > \subseteq LT(I)$.
3. Equality need not hold.

> Example: $I = (\underbrace{x^3}_{f_1}, \underbrace{x^2y - y^3}_{f_2})$ with Lex. Then $< LT(f_1), LT(f_2) > = < x^3, x^2y >$. But $f = yf_1 - xf_2 = yx^3 - x^3y + xy^3 = xy^3 \in I$. Thus $LT(f) = xy^3 \in LT(I)$



> Defn: Fix a monomial order. A finite subset $G = \{g_1, ..., g_s\}$ of an ideal I is called a Gröbner basis or standard basis if $LT(I) =< LT(g_1), ..., LT(g_s) >$

> Question:

1. Does every ideal have a GB?
2. how can we find a GB?

Sep. 9, 2013

> **Monomials ideals, Dickson's Lemma, Hilbert Basis Theorem**

> Defn: A *monomial ideal* is an ideal generated by a (not necessarily finite) set of monomials.

$$I =< x^\alpha : \alpha \in A >,$$

where A is a set of elements of $\mathbb{Z}^n_{\geq 0}$.

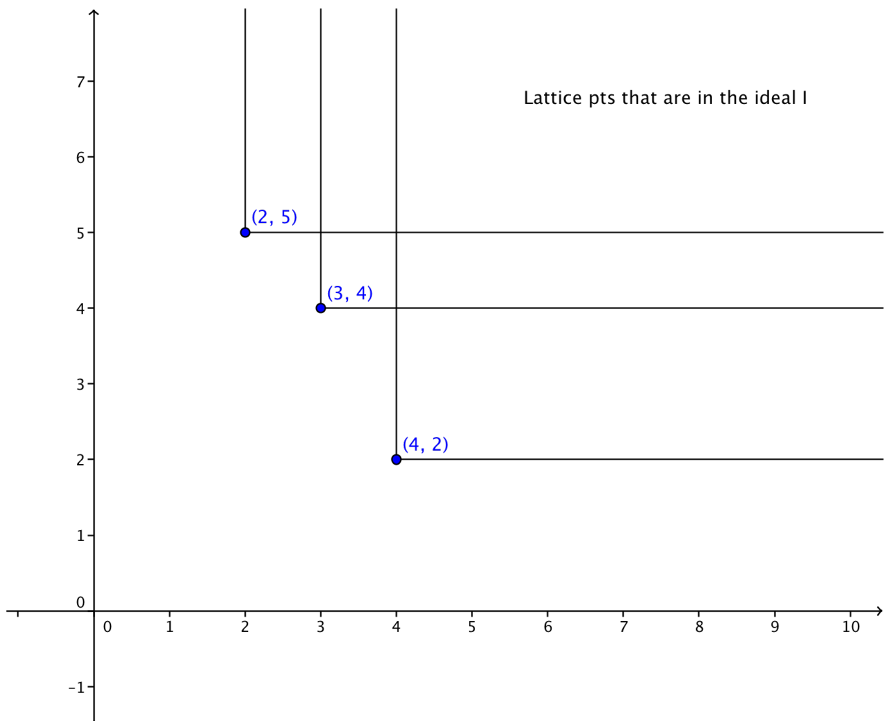
- Rmk: For I to be an ideal, the set of all exponents of monomials in I must be closed under translation by vectors with integer coordinates in the positive orthant.
- Lemma: (The membership problem for monomial ideals.)
  1. $x^\beta \in I =< x^\alpha : \alpha \in A > \iff \exists \alpha \in A$ s.t. $x^\alpha | x^\beta$.
  2. $f \in I =< x^\alpha : \alpha \in A > \iff$ every term of f is divisible by some monomial in I.

  Proof. Left as an exercise. □
- Thm (Dickson's Lemma): Every monomial ideal is finitely generated.

First, a proof by example/picture:

$$I =< x^4y^2, x^3y^4, x^2y^5 >$$

Consider the following diagram that represents the ideal I:



Project points down on the x axis, getting the ideal $< x^2 > \subseteq k[x]$. The point lying over $(2,0)$ is $(2,5)$. Next, go to looking at $< x^3 > \subseteq k[x]$. Using this we rewrite I as:

$$I =< x^2y^5 > + < x^3y^4 > + < x^4y^3 > + < x^4y^2 >,$$

i.e., $x^2y^5, x^3y^4, x^4y^3, x^4y^2$ is a finite set of generators for I (not minimal).

Proof of Dickson's Lemma. By induction on $n=$ the number of variables of R.

$n = 1$ Every ideal in $k[x]$ is principal (see Homework #1), hence finitely generated.

$n > 1$ Now assume that every monomial ideal in $k[x_1, ..., x_{n-1}]$ is finitely generated. We will prove that every monomial ideal in $k[x_1, ..., x_{n-1}, y]$ is finitely generated.

Let $f : k[x_1, ..., x_{n-1}, y] \to k[x_1, ..., x_{n-1}]$ be defined by $f(x_i) = x_i$ for $i = 1, ..., n-1$ and $f(y) = 1$. Let I be an ideal of $k[x_1, ..., x_{n-1}, y]$. Then $f(I) = J \subseteq k[x_1, ..., x_{n-1}]$ is an ideal. By inductive hypothesis, $J =< x^{\alpha_1}, ..., x^{\alpha_s} >$ (which is finitely generated). Then there exist numbers $m_1, ..., m_s$ such that

$x^{\alpha_i} y^{m_i} \in I$. Let $m = \max\{m_i : 1 \leq i \leq s\}$.

Let $J_k := < x^\alpha y^k : x^\alpha y^k \in I >$, where $0 \leq k \leq n-1$. Then $f(J_k)$ is an ideal in $k[x_1, ..., x_{n-1}]$ and so $f(J_k) = < x^{\alpha_1^{(k)}}, ..., x^{\alpha_{s_k}^{(k)}} >$.

Claim: $I = < y^m x^{\alpha_1}, ..., y^m x^{\alpha_s} > + \sum_{k=0}^{m-1} < y^k x^{\alpha_1^{(k)}}, ..., y^k x^{\alpha_{s_k}^{(k)}} >$.
Indeed, we only need to show the "$\subseteq$" containment (the opposite containment is obvious by construction). Moreover, it's enough to show that if $x^\alpha y^\beta \in I$, then $x^\alpha y^\beta$ is in the RHS.

- ⋄ If $\beta \geq m$, then since $x^\alpha \in J$, $x^\alpha \in < x^{\alpha_1}, ..., x^{\alpha_s} >$ and so $x^\alpha y^\beta \in < x^\alpha y^m > \subseteq < x^{\alpha_1} y^m, ..., x^{\alpha_s} y^m >$.
- ⋄ If $\beta \in \{0, ..., m-1\}$, then $x^\alpha y^\beta \in J_\beta$, and so $x^\alpha \in < x^{\alpha_1^{(\beta)}}, ..., x^{\alpha_{s_\beta}^{(\beta)}} >$, hence $x^\alpha y^\beta \in < x^{\alpha_1^{(\beta)}} y^\beta, ..., x^{\alpha_{s_\beta}^{(\beta)}} y^\beta >$.

□

⋗ Aside: We can also cover it with a disjoint union of copies of various dimensions of k:

$$x^2 y^5 k[x,y] \oplus x^3 y^4 k[x] \oplus x^4 y^3 k[x] \oplus x^4 y^2 k[x] := \mathcal{D},$$

we we'll call the Stanley decomposition. We define the Stanley depth, sdepth, as the minimal dimension of a component:

$$\text{sdepth}(\mathcal{D}) = \min\{2, 1, 1, 1\} = 1$$

Also,

$$\text{sdepth}(I) = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} = \text{Stanley decomposition of } I\}.$$

⋗ Conjecture: (Stanley): Let M be an R-module. Then $\text{depth}(M) \leq \text{sdepth}(M)$. (Here, let's just view $M = I$ as an R-module.) For our ideal I above, we have $\text{depth}(I) = 1 = \text{sdepth}(I)$.

Sep. 11, 2013

⋗ Defn: A Groebner Basis $G = \{g_1, ..., g_t\}$ is minimal (respectively reduced) if

(1) $LC(g_i) = 1$ for all $g_i \in G$.
(2 - minimal) For all $g_i \in G$, $LT(g_i) \notin < LT(g_1), ..., LT(g_{i-1}), LT(g_{i+1}), ..., LT(g_t) >$.
(2 - reduced) For all $g_i \in G$, no term of g_i is in $< LT(G \backslash \{g_i\}) >$.

⋗ **Hilbert basis, existence of GB**

⋗ Hilbert Basis Theorem: Every ideal I in $k[x_1, ..., x_n]$ is finitely generated. (Equivalently: $k[x_1, ..., x_n]$ is Noetherian i.e., ACC satisfied).

Proof. If $I = \{0\}$, we're done
Otherwise, $LT(I) \underbrace{=}_{\text{Dickson's Lemma}} = < m_1, ..., m_s >$ for some monomials m_i. By Remark after defn of $LT(I)$, every monomial in $LT(I)$ is of the form $LT(g)$, $g \in I$. This implies that there exists $g_1, ..., g_s \in I$ such that $m_i = LT(g_i)$ for $1 \leq i \leq s$.

Claim: $I = < g_1, ..., g_s >$. To see this, let $f \in I$. By the Division Algorithm applied to f w.r.t. any order of the set $\{g_1, ..., g_s\}$, we have:

$$f = \sum_{i=1}^{s} a_i g_i + r,$$

such that either $r = 0$ or no term in r is divisible by any of the $LT(g_i)$. Note that

$$r = f - \sum_{i=1}^{s} a_i g_i \in I \implies LT(r) \in LT(I) = < LT(g_1), ..., LT(g_s) >,$$

so $LT(r)$ is divisible by at least one of $LT(g_i)$, which contradicts the second possibility. Hence $r = 0$, which means $f = \sum_{i=1}^{s} a_i g_i \in < g_1, ..., g_s >$. □

⋗ Cor: If I is an ideal in $k[x_1,...,x_]$, then a Groebner basis for I exists.

Proof. The set $\{g_1,...,g_s\}$ from the proof of HBT is a GB for I. □

⋗ Prop (Normal form): Let $G=\{g_1,...,g_t\}$ be a GB of an ideal I and $f\in R=k[x_1,...,x_n]$. Then there is a unique r (independent of the order of elements of G) such that:

(1) Either $r=0$ or no term of r is divisible by any of $LT(g_i)$.
(2) $f=g+r$ with $g\in I$.

⋗ Defn: The normal form of f w.r.t. G (or I) is r from Proposition.

⋗ Notation: $r=f\%G=f\%I=\overline{f}^G$ all mean normal form.

Proof of Prop: Existence is given by the Division Algorithm (use any ordering of the g_i's to apply the division algorithm).

Uniqueness: Assume $f=g+r$ and $f=g'+r'$, where g,r,g',r' satisfy (1) and (2). Since $g+r=g'+r'$, we have $\underbrace{r-r'}_{\text{No terms are divisible by any of } LT(g_i) \text{ by (1)}} = \underbrace{g'-g}_{LT(g'-g)\in LT(I)=<LT(g_1),...,LT(g_s)>} \in I$. Therefore there are no nonzero monomials in $r-r'$, hence $r-r'=0$ and so $r=r'$. □

⋗ Cor (Ideal membership): Given $I\subseteq R=k[x_1,...,x_n]$, $f\in R$, then TFAE

(1) $f\in I$
(2) $f\%G=0$ for some GB G of I.
(3) $f\%G=0$ for any GB G of I.

Proof. Fix "$<$" a monomial order.

(1) $\Longrightarrow$ (2): Let $f\in I$, G be a GB of I, say $\{g_1,...,g_s\}$. Then $f\in I=<g_1,...,g_s>$ and so $f=\underbrace{\sum_{i=1}^s a_ig_i}_{g}+0$.

The uniqueness of normal form implies $f\%G=0$.

(2) $\Longrightarrow$ (1): $f\%G=0$ implies $f=g+0$, $g\in I$, i.e., $f\in I$. □

⋗ **How to find GB?**

⋗ Buchberger's Criterion & Algorithm

⋗ Defn: Let $f,g\in R$. Let $x^\gamma=LCM(LM(f),LM(g))$. The S-polynomial[7] of f,g is $S(f,g)=\frac{x^\gamma}{LT(f)}\cdot f-\frac{x^\gamma}{LT(g)}\cdot g$. The leading term of the first part is $\frac{x^\gamma}{LT(f)}\cdot LT(f)=x^\gamma$. The leading term of the second part is $\frac{x^\gamma}{LT(g)}\cdot LT(g)=x^\gamma$. Then multideg$(S(f,g))<\gamma$.

⋗ Theorem (Buchberger's Criterion): Let $I\subseteq R$ be an ideal. A generating set $G=\{g_1,...,g_s\}$ for I is a GB of I if and only if $S(g_i,g_j)\%G=0$, for every $i\neq j$.

⋗ Theorem (Buchberger's Algorithm): Let $I=<f_1,...,f_s>\neq\{0\}$. Then a GB for I is constructed in a finite number of steps following the algorithm below:
$G:=\{f_1,...,f_s\}$.
Repeat: $G':=G=\{g_1,...,g_t\}$. For every pair $1\le i\neq j\le t$, if $S(g_i,g_j)\%G'\neq 0$, then $G=G\cup\{S(g_i,g_j)\}$.
Until $G'=G$.

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[7] some people think S stands for Syzygy.

- Worked through M2: GBs.m2
- Consider $v_3 : \mathbb{P}^2 \to \mathbb{P}^9$ (the 3rd Veronese). Instead, consider $pv_3 : \mathbb{P}^2 \to \mathbb{P}^8$, called the Pinched Veronese. Consider:
$$0 \to I \to k[a_0, ..., a_8] \to k[x, y, z]$$
where the maps are $a_0 \mapsto x^3$, ..., $a_8 \mapsto yz^2$. (We've thrown out the degree 3 term xyz.) The pinched veronese is Koszul, which means when you resolve k over this ring, you get a linear resolution.

  If I has a quadratic GB, then this is Koszul.

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- **Theorem (Buchberger's Criterion):** If I is an ideal in a polynomial ring and $G = \{g_1, ..., g_s\}$ is a generating set for I, then TFAE:
  (i) G is a Gröbner basis for I.
  (ii) For every $f, g \in G$, $S(f, g)\%G = 0$ (some order on G).

  Proof. $(i) \implies (ii)$: Recall that $S(f,g) := \frac{x^\gamma}{LT(f)} \cdot f - \frac{x^\gamma}{LT(g)} \cdot g \in I$. By the Ideal Membership Criterion (using the fact that G is a GB for I), we get that $S(f, g)\%G = 0$.

  $(ii) \implies (i)$: We want to show that $LT(I) =< LT(g_1), ..., LT(g_s) >$. Let $f \in I$ and write $f = \sum_{i=1}^s a_i g_i$ (which we can do since G is a generating set for I). Here, $\text{multideg}(f) \le \max\{\text{multideg}(a_i g_i) : 1 \le i \le s\}$.

Case 1: If $\text{multideg}(f) = \max\{\text{multideg}(a_i g_i) : 1 \le i \le s\}$, then $LM(f) = LM(a_i g_i)$ for some i, so $LM(g_i)|LM(f)$ for some i, hence $LT(f) \in< LT(g_1), ..., LT(g_s) >$.

Case 2: If $\text{multideg}(f) < \max\{\text{multideg}(a_i g_i) : 1 \le i \le s\} := \delta$, our aim is to show that this cannot occur. Start with an expression (*) that achieves the minimum possible δ. Among all expressions (*) with minimum possible δ start with one that has the property that $\#\{i : \text{multideg}(a_i g_i) = \delta\}$ is minimum possible (for this fixed δ). We now have, possibly relabeling the g_is,
$$(*)\ f = \underbrace{a_1 g_1 + \cdots a_m g_m}_{\text{multideg}=\delta} + \underbrace{a_{m+1} g_{m+1} + \cdots + a_s g_s}_{\text{multideg}<\delta}. \tag{1}$$

Note that we must have $m \ge 2$ because cancellation must occur in the first piece.

$S(g_1, g_2) = \frac{x^\gamma}{LT(g_1)} \cdot g_1 - \frac{x^\gamma}{LT(g_2)} \cdot g_2$. By (2), we have $S(g_1, g_2)\%G = 0$, so $S(g_1, g_2) = \sum_{i=1}^s b_i g_i + 0$, where $\text{multideg}(b_i g_i) \le \underbrace{\text{multideg}\, S(g_1, g_2)}_{<\gamma}$ (using condition (2) in the Division Algorithm). Now,
$$\frac{x^\gamma}{LT(g_1)} \cdot g_1 - \frac{x^\gamma}{LT(g_2)} \cdot g_2 - \sum_{i=1}^s b_i g_i = 0. \tag{2}$$

Recall, $x^\gamma = LCM(LM(g_1), LM(g_2))$ and $x^\delta = LM(a_1 g_1) = LM(a_2 g_2)$, hence x^δ is a common multiple of $LM(g_1)$ and $LM(g_2)$. Therefore $x^\gamma | x^\delta$, i.e., $x^\gamma \cdot x^\mu = x^\delta$ for some μ. Multiplying through (2) by $LC(a_1 g_1) x^\mu$. This gives
$$\underbrace{LC(a_1 g_1) \frac{x^\gamma x^\mu}{LT(g_1)} \cdot g_1}_{\text{multideg}=\delta,\text{lead.coeff}=LC(a_1 g_1)} - \underbrace{LC(a_1 g_1) \frac{x^\gamma x^\mu}{LT(g_2)} \cdot g_2}_{\text{multideg}=\delta} - \underbrace{\sum_{i=1}^s LC(a_1 g_1) \underbrace{b_i g_i}_{\text{multideg}<\gamma\text{, from div alg.}} x^\mu}_{\text{multideg}<\delta} = 0. \tag{3}$$

Now subtract (3) from (1) to get:

$$\underbrace{\left(a_1 - LC(a_1g_1)\frac{x^\delta}{LT(g_1)}\right)g_1}_{\text{multideg}<\delta} + \underbrace{\sum_{i=2}^{m}(BLAH)\cdot g_i}_{\text{multideg}\leq\delta} + \underbrace{\sum_{i=m+1}^{s}(BLEH)\cdot g_i}_{\text{multideg}<\delta} = f\ (*)$$

However, this now has $\leq m-1$ terms of multidegree δ, contradicting minimiality, so this case does not actually occur.

So, by Case 1, $LT(f) \in < LT(g_1), ..., LT(g_s) >$, hence $LT(I) \subseteq < LT(g_1), ..., LT(g_s) >$, implying that $LT(I) =< LT(g_1), .., LT(g_s) >$. Therefore G is a GB for I. □

⋗ **Buchberger's Algorithm:**
$G := \{f_1, ..., f_s\}$ is your set of generators for I.
Repeat $G' := G = \{g_1, ..., g_s\}$
for $i \neq j$ do: compute $S(g_i, g_j)$ if $S(g_i, g_j)\%G \neq 0$, $G = G \cup \{S(g_i, g_j)\%G\}$[8]
Until $G' = G$ (i.e., at some iteration all S-polys give remainder 0.

⋗ *Proof of Correctness of Bucherberger Algorithm.* First of all, the fact that it computes a GB is a consequence of Buchberger's Criterion.

Let's prove that this algorithm terminates in finitely many steps.

Claim: If $G' \neq G$, then $< LT(G') > \subsetneq < LT(G) >$.

To see this, note that $G' \neq G$ gives that there exists $g_1, g_2 \in G'$ such that $S(g_1, g_2)\%G' \neq 0$. Let $r = S(g_1, g_2)\%G'$. Then (3) in the Division Algorithm implies that no term in r is in $< LT(G') >$. In particular, $LT(r) \notin < LT(G') >$. However, $r \in G$ so $LT(r) \in < LT(G) >$. Therefore we have $< LT(G') > \subsetneq < LT(G) >$. We now have an ascending chain of monomial ideals

$$< LT(G^{(0)}) > \subset < LT(G^{(1)}) > \subset \cdots$$

where $G^{(i)}$ is G after the ith iteration. This ascending chain must terminate. Therefore there exists i such that $< LT(G^{(i)}) > = < LT(G^{(i+1)}) >$, hence $G^{(i)} = G^{(i+1)}$, implying that the algorithm stops after iteration $i+1$. □

Sep. 18, 2013

⋗ Prop: Fix a monomial order and an ideal I. Then there is a unique reduced GB for I.

Proof. Existence: homework problem.

Uniqueness: Assume G and G' are reduced GB's for I, hence G and G' are minimal. By a homework #1 problem, this implies $LT(G)$ and $LT(G')$ are minimal sets of monomial generators for $LT(I)$.

Notation:
$LT(I)$ is the ideal generated by the set of leading terms of all elements of I (I here is an ideal).
$LT(G)$ is the set of leading terms of all elements of G (where G is just a set).

Fact (left unproven): A monomial ideal has a unique minimal set of monomial generators.

This Fact then implies that $LT(G) = LT(G')$. Let $g \in G'$. Therefore there exists $g' \in G'$ such that $LT(g) = LT(g')$. Consider $g - g'$. Note that no terms in $g - g'$ are divisible by elements of $LT(G)$, since G and G' are reduced. Note that:

[8]This was INCORRECT previously; it is the remainder, not the S-poly itself.

(1) $(g-g')\%G = g-g'$
(2) $g-g' \in I$ implies $(g-g')\%G = 0$.

Therefore, by (1) and (2), $g-g'=0$, so $g=g'$. By a symmetric argument, we then obtain that $G=G'$. □

⩥ **Ideal - Variety Correspondence (Improved):**

⩥ Ideals$\subseteq k[x_1,...,x_n]$ correspond to affine varieties $\subseteq \mathbb{A}^n$ by $\mathbb{V}$ and $\mathbb{I}$.
$\mathbb{V}(I)$ = set of common solutions of $f_1,...,f_s$, where $I =< f_1,...,f_s >$.
$\mathbb{I}(V)$ = set of polynomials vanishing at every point of V.
Facts: we have $\mathbb{I}(\mathbb{V}(I)) \supseteq I$ (where strict inequality can occur).

⩥ **Thm (Weak Nullstellensatz):** If $k=\overline{k}$, then $V(I)=\varnothing$ if and only if $I=(1)=k[x_1,...,x_n]$.

⩥ **Cor (Consistency Theorem):** - *A practical way to check when* $V(I)=\varnothing$.
If $k=\overline{k}$, then TFAE:

(1) $\mathbb{V}(I)=\varnothing$
(2) $I=(1)$
(3) Any GB G of I contains a constant among its elements.
(4) The reduced GB of I is just $\{1\}$.

This isn't a silly theorem; consider the ideal $I=(x-1,x+1)=(1)$ (char not 2).

⩥ **Thm (Finiteness Theorem):** Suppose $k=\overline{k}$ and $R=k[x_1,...,x_n]$. Then TFAE:

(1) $\mathbb{V}(I)$ is finite.
(2) R/I is a finite dimensional k-vector space.
(3) There exists finitely many monomials outside $LT(I)$.
(4) If G is a GB for I, then for every i, there exists $n_i \geq 1$ such that $x_i^{n_i}$ is the leading term of some element of $G \iff LT(I)$ contains pure powers of every variable.

⩥ **Thm (Strong Nullstellensatz):** Let $k=\overline{k}$ and $I\subseteq k[x_1,...,x_n]$. TFAE:

(1) $f\in \mathbb{I}(\mathbb{V}(I))$
(2) There exists $m\geq 1$ such that $f^m\in I$ (if and only if (by defn) $f\in\sqrt{I}$).

In other words, $\mathbb{I}(\mathbb{V}(I))=\sqrt{I}$.

Proof. (1) $\implies$ (2) : Suppose $I=< f_1,...,f_s >$. Consider $\widetilde{I} =< f_1,...,f_s,1-gf >\subseteq k[x_1,...,x_n,y]$.[9]

CLAIM: $\mathbb{V}(\widetilde{I})\neq\varnothing$. Suppose not. Let $(a_1,...,a_n,b)\in\mathbb{V}(\widetilde{I})$. By definition of $\mathbb{V}$, we have $f_i(a_1,...,a_n)=0$ for all $1\leq i\leq s$, hence $(a_1,...,a_n\in\mathbb{V}(I)$. Then $1-b\cdot f(a_1,...,a_n)=0$. Since $f\in\mathbb{I}(\mathbb{V}(I))$, we get $f(a_1,...,a_n)=0$. Now $1-b\cdot f(a_1,...,a_n)=0$ implies $1=0$, a contradiction.

By Weak Nullstellensatz, $\mathbb{V}(\widetilde{I})=\varnothing \iff \widetilde{I}=(1)=k[x_1,...,x_n,y]$. Hence $1=\sum_{i=1}^s p_i(x_1,...,x_n,y)\cdot f_i(x_1,...,x_n)+g(x_1,...,x_n,y)\cdot(1-y\cdot f(x_1,...,x_n)$. Substitute $y=\frac{1}{f(x_1,...,x_n)}$. Then

$$1=\sum_{i=1}^s p_i\left(x_1,...,x_n,\frac{1}{f(x_1,...,x_n)}\right)\cdot f_i(x_1,...,x_n)+0.$$

Let $m=\max\{n_i\}$ (where $n_i=\deg(p_i)$).

$$f^m=\sum g_i(x_1,...,x_n)\cdot f^{m-n_i}\cdot f_i,$$

hence $f^m\in I$. □

[9]Coming up with $\widetilde{I}$ is usually referred to as Rabinowitz's Trick.

- **Thm (Radical Membership):**[10] Let $k = \overline{k}$, I and ideal, and $f \in R$. TFAE:
  (1) $f \in \sqrt{I}$
  (2) $\widetilde{I} = I + (1 - y \cdot f) = k[x_1, ..., x_n, y]$.
  (3) The reduced GB of $\widetilde{I}$ is just $\{1\}$.
- **Cor (Improved Ideal-Variety correspondence):** The following maps are inclusion-reversing bijections that are inverse to each other.

$$\{\underbrace{\text{Radical Ideals}}_{(\text{i.e., } I=\sqrt{I})} \subseteq k[x_1, ..., x_n]\} \leftrightarrow \{\text{Varieties in } \mathbb{A}^n\}.$$

- Next: Elimination. Looked at graph of $I =< xy = 1 >$. Then $I(\pi(V)) = I \cap k[x]$.

SEP. 20, 2013

- **Elimination Theory** (Elimination of variables = Projecting onto coordinate hyperplanes)
- **Example:** $V =< y - z, zy - 1 >$. Consider the projection $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ defined by $\pi(x, y, z) = (y, z)$. Then $\pi(V) =$ the line $y = z$ in the yz-plane, except at $(0, 0)$. This line is the ideal $< y - z > \subseteq k[y, z]$. Notice that $\pi(V)$ is not a variety: $\mathbb{V}(y - z)$ is the whole line.
- GOALS:
  1. Give an algorithmic way for finding generators for the ideal I_ℓ "describing" $\pi(V)$.
  2. Relate $\pi(V)$ to $\mathbb{V}(I_\ell)$.
  3. Extending partial solutions (lifting from $\mathbb{V}(I_\ell)$ to $\mathbb{V}(I)$).
- DEFN: Let $t \in \mathbb{N}$, $1 \leq t \leq n$. An *elimination order* of $R = k[x_1, ..., x_n]$ w.r.t. $x_1, ..., x_t$ is a monomial order which satisfies the following:

$$(E) : LT(f) \in k[x_{t+1}, ..., x_n] \implies f \in k[x_{t+1}, ..., x_n]$$

- EXAMPLES:
  1. Lex with $x_1 > x_2 > \cdots x_n$ is an elimination order for any t.
  2. Block order (product order) given by 2 arbitrary monomial orders $>_1$ and $>_2$ on $k[x_1, ..., x_t]$ and $k[x_{t+1}, ..., x_n]$. The block order = first compare using $>_1$ then break ties using $>_2$ ((E) holds because any monomial ≥ 1.)
  3. Weighted order with $w = (1, ..., 1, 0, ..., 0)$. First compare using $>_w$ then break ties using some other arbitrary monomial order on R.
- DEFN: The ideal $I_t = I \cap k[x_{t+1}, ..., x_n]$ is called the *tth elimination ideal* of I. ($I_t \subseteq k[x_{t+1}, ..., x_n]$.)
- **Thm (Elimination Theorem):** Let I be an ideal in $R = k[x_1, ..., x_n]$, let G be a GB of I w.r.t. an elimination order for $x_1, ..., x_t$. Then $G_t = G \cap k[x_{t+1}, ..., x_n]$ is a GB for I_t for the induced monomial order on $k[x_{t+1}, ..., x_n]$.
- EXAMPLE: $I =< y - z, xy - 1 >$. Compute $I_1 := I \cap k[y, z]$.
  - Use Lex with $x > z > y$. Then $G = \{y - z, xy - 1\}$ is a GB, the Elimination Theorem gives $G_1 = \{y - z\}$ is a GB for I_1, hence $I_1 =< y - z >$. Then

$$S(y - z, xy - 1)\%\{y - z, xy - 1\} = 0$$

  (by a homework problem).

[10] Using part (3), this allows us to test if an element is in the radical of an ideal.

- Use Lex with $x > y > z$. Then

$$S(y-z, xy-1) = x(y-z) - (xy-1) = -xz+1$$

$$-xz+1\%\{y-z, xy-1\} = -xz+1$$

$$G = \{y-z, xy-z, -xz+1\}$$

(check Buchberger stops here). By Elimination Theorem, $G_1 = \{y-z\}$, hence $I_1 =< y-z >$.

Proof of Elimination Theorem. We need to show:

- $< G_t >= I_t$
- $< LT(G_t) >= LT(I_t)$. (Clearly, $G_t \subseteq I_t$ gives $< LT(G_t) > \subseteq LT(I_t)$.) For the converse, let $f \in I_t$, and so $f \in k[x_{t+1}, ..., x_n]$, hence $LT(f) \in k[x_{t+1}, ..., x_n]$. $f \in I$ and G is a GB for I, and so $LT(f) \in < LT(G) >$, which implies there is $g \in G$ such that $LT(g)|LT(f)$. This means we can write $LT(g) \in k[x_{t+1}, ..., x_n]$, hence by (E), $g \in k[x_{t+1}, ..., x_n]$ and so $g \in G \cap k[x_{t+1}, ..., x_n] = G_t$. Therefore $LT(f) \in < LT(G_t) >$. Hence $LT(I_t) \subseteq < LT(G_t) >$.

□

- Defn: A point $(a_{t+1}, ..., a_n) \in \mathbb{V}(I_t) \subseteq \mathbb{A}^{n-t}$ is a *partial solution* to the equations given by (a finite set of generators) of I.
- Defn: A set of points is Zariski closed if it is an affine variety. We say a set of points is Zariski open if its complement is Zariski closed.
- Rmk: Zariski open sets form a topology on $\mathbb{A}^n$.
- Defn: Given a set of points S, the *Zariski closure* of S is the smallest Zariski closed set containing S, denoted by $\overline{S}$.
- Example: The Zariski closure of a line missing a point in the yz-plane is the entire line in the plane.
- Next time: $\overline{\pi(V)} = \mathbb{V}(I_t)$

Sep. 23, 2013

- Given an ideal $I \subseteq k[x_1, ..., x_n]$; $V = \mathbb{V}(I)$. We defined $I_t = I \cap k[x_{t+1}, ..., x_n]$ (the t^{th} elimination ideal). We defined $V(I_t) =$ the variety of partial solutions.

  We have $\pi_t : \mathbb{A}^n \to \mathbb{A}^{n-t}$ is the projection onto last $(n-t)$-coordinates.
- **Closure Theorem:** If k is algebraically closed, then $\overline{\pi_t(V)} = \mathbb{V}(I_t)$.
- Defn: If S is a set of points in $\mathbb{A}^n$, we can define $\mathbb{I}(S) = \{f \in k[x_1, ..., x_n] : f \text{ vanishes at every point in } S\}$.
- Lemma: If S is a set of points in $\mathbb{A}^n$, then $\overline{S} = \mathbb{V}(\mathbb{I}(S))$.

Proof. We need to show:

(1) $\mathbb{V}(\mathbb{I}(S))$ is an affine variety containing S. (By defn).

(2) $\mathbb{V}(\mathbb{I}(S))$ is the smallest (w.r.t. containment) affine variety containing S. Let W be an affine variety containing S. We'll show $\mathbb{V}(\mathbb{I}(S)) \subseteq W$. We have

$$W \supseteq S \implies \mathbb{I}(W) \subseteq \mathbb{I}(S) \implies W = \mathbb{V}(\mathbb{I}(W)) \supseteq \mathbb{V}(\mathbb{I}(S)),$$

hence $W \supseteq \mathbb{V}(\mathbb{I}(S))$.

□

Proof of Closure Theorem: Claim 1: $\pi_t(V) \subseteq \mathbb{V}(I_t)$.

If $(a_{t+1}, ..., a_n) \in \pi_t(V)$, then there exists $(a_1, ..., a_n) \in V = \mathbb{V}(I)$ such that for all $f \in I$, $f(a_1, ..., a_n) = 0$. Therefore for all $f \in I \cap k[x_{t+1}, ..., x_n]$, $f(a_1, ..., a_n) = 0$. Then for any $f \in I_t$, $f(a_{t+1}, ..., a_n) = 0$. Thus $(a_{t+1}, ..., a_n) \in \mathbb{V}(I_t)$.

$\pi_t(V) \subseteq \mathbb{V}(I_t)$, and so $\overline{\pi_t(V)} \subseteq \mathbb{V}(I_t)$.

Claim 2: $\mathbb{I}(\pi_t(V)) \subseteq \mathbb{I}(\mathbb{V}(I_t))$. Let $f \in \mathbb{I}(\pi_t(V))$, then f vanishes at every point of $\pi_t(V)$. View f as a polynomial in $k[x_1, ..., x_n]$. Then f vanishes at every point of V.

$$f(a_1, ..., a_t, a_{t+1}, ..., a_n) = f(a_{t+1}, ..., a_n) = 0$$

Thus $f \in \mathbb{I}(V) = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$, and so there exists $m \geq 1$ such that $f^m \in I$. Then $f \in k[x_{t+1}, ..., x_n] \implies f^m \in k[x_{t+1}, ..., x_n]$, and so $f^m \in I \cap k[x_{t+1}, ..., x_n] = I_t$. Therefore $f^m \in I_t$, hence $f \in \sqrt{I_t} = \mathbb{I}(\mathbb{V}(I_t))$. Here we're using the Strong Nullstellensatz.

So $\mathbb{I}(\pi_t(V)) \subseteq \mathbb{I}(\mathbb{V}(I_t)) \implies \mathbb{V}(\mathbb{I}(\pi_t(V))) \supseteq \mathbb{V}(\mathbb{I}(\mathbb{V}(I_t)))$. Therefore $\overline{\pi_t(V)} \supseteq \mathbb{V}(I_t)$. □

- The Prop says: "most" partial solutions come from actual solutions. $\pi_t(V)$ fills up "most" of $\mathbb{V}(I_t) = \overline{\pi_t(V)}$.
- Prop: There exists an affine variety $W \subseteq V(I_t)$ such that
  - $\overline{\mathbb{V}(I_t) \backslash W} = \mathbb{V}(I_t)$ (i.e., W is "small")
  - $V(I_t) \backslash W \subseteq \pi_t(V)$ (i.e., $V(I_t)$ differs from $\pi_t(V)$ by some set that is even smaller than W.
- **Thm (Extension Theorem):** Let k be algebraically closed; let I_1 be the first elimination ideal. Say $I =< f_1, ..., f_s >\subseteq k[x_1, ..., x_n]$. Write $f_i = x_1^{N_i} g_i(x_2, ..., x_n) +$ terms where degree in x_1 is $< N_i$. Let $(a_2, ..., a_n) \in \mathbb{V}(I_1)$ be a partial solution. Then $(a_2, ..., a_n)$ extends to a solution $(a_1, ..., a_n) \in \mathbb{V}(I)$ if and only if $(a_2, ..., a_n) \notin \mathbb{V}(< g_1, ..., g_s >)$.
- Example: (from last time) $I =< y - z, xy - 1 >$. We saw $I_1 =< y - z >$. A point $(a, a) \in \mathbb{V}(I_1)$ extends to a point $(a_1, a, a) \in \mathbb{V}(I)$ if and only if $(a, a) \notin \mathbb{V}(y - z, y) = \mathbb{V}(y, z) = \{(0, 0)\}$ (where in the theorem, $g_1 = y - z$ and $g_2 = y$). Hence a partial solution extends if and only if it is not the origin.

Sep. 25, 2013

- absent / see Kat's notes

Sep. 27, 2013

- Computer day.

Sep. 30, 2013

- Dickson's Lemma $\implies$ Proof of Hilbert Basis Theorem $\implies$ Existence of GBs.
- Also, Dickson's Lemma $\implies$ the well ordering property for total orderings on monomials that refine divisibility.
- Monomial Ordering (with well-ordering to insure termination of the division algorithm in finitely many steps) $\implies$ Division Algorithm (determinate form - involves an order on the set $\{g_1, ..., g_s\}$) $\implies$ Division Algorithm (indeterminate form) (to see if $S(f, g) = \sum a_i g_i$, multideg$(a_i g_i) \leq$ multideg $S(f, g)$) $\implies$ Buchberger's Algorithm / Criterion $\implies$ Construction of GBs
- **Gröbner bases for modules**

⪢ Let $R = k[x_1, ..., x_n]$. Use u, v to denote monomials in R (formerly x^α).
Fix a free R-module F with basis $\{e_1, ..., e_r\}$

⪢ Defn: We say $m \in F$ is a *monomial* in F if $m = u \cdot e_i$, where u is a monomial in R.

⪢ Defn: We say that U is a *monomial* submodule of F if it is generated by monomials of F.

⪢ Prop: (Characterization of monomial submodules):

$U \subseteq F$ is a monomial submodule if and only if for every $1 \leq i \leq r$ there is a monomial ideal $I_i \subseteq R$ such that $U = I_1 e_1 \oplus \cdots \oplus I_r e_r$.

⪢ Cor 1: Any monomial submodule of a free R-module is finitely generated. (use the Prop and Dickson's lemma, taking finite generators for each of the I_i).

⪢ Cor 2: Any submodule of a finitely generated free R-module is finitely generated (from Cor. 1 and the argument from Dickson's Lemma to the proof of the Hilbert Basis Thm.)

⪢ Cor 3: Gröbner bases for modules exist.

⪢ Defn: A *monomial ordering* on the monomials of the free module F is a total order satisfying:

(1) $m < um$, for any monomial $m \in F$, for any $u \in R$, u a monomial, $u \neq 1$.
(2) $m_1 < m_2$ implies $um_1 < um_2$ for all $m_1, m_2 \in F$ monomials and for all $u \in R$, a monomial.

⪢ Examples of monomial orderings on F: We fix a monomial order "$>$" on R.

1. Position over Coefficient: $ue_i > ue_j$ if $i < j$ OR $i = j$ and $u > v$.
2. Coefficient over Position: $ue_i > ue_j$ if $u > v$ OR $u = v$ and $i < j$.

For example, take $R = k[[x_1, x_2]$ and $F = Re_1 \oplus Re_2$. Then $x_2e_1 >_{\text{PoC}} x_1e_2$ but $x_2e_1 <_{\text{CoP with Lex on } R} x_1e_2$.

⪢ The notions of LT,LC,LM have the same definition.

⪢ Defn: Given a submodule U of a finitely generated free module F, a set $G = \{g_1, ..., g_s\}$ is a Gröbner basis of U if

(1) G generates U (as an R-module).
(2) $LT(U) =< LT(g_1), ..., LT(g_2) >$. Where $LT(U)$ is the "initial module of U."

⪢ Defn: S-elements can bej defined for $f, g \in F$ such that $LT(f) = ue_i$ and $LT(g) = ve_j$ where $i = j$. For such f, g we define:

$$S(f,g) = \frac{LCM(u,v)}{u} f - \frac{LCM(u,v)}{v} g.$$

(This is defined so that cancellation of LTs occurs.)

⪢ **Syzygies:**

⪢ Prop: Given an R-module M there exists a free R-module F and a submodule U of F such that $M \cong F/U$. Moreover, if M is finitely generated, then F can be chosen to be finitely generated.

Proof. Use first iso theorem. □

Oct. 2, 2013

⪢ **Syzygies:**

- Last time: Given an R-module M, we can iterate the procedure in the proof of the presentation (Prop.) to come up with a sequence of the free R-modules and R-module maps:

$$\begin{tikzcd}
\cdots \arrow[r, "\phi_3"] & F_2 \arrow[rr, "\phi_2"] \arrow[dr] & & F_1 \arrow[rr, "\phi_1"] \arrow[dr] & & F_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& & U_2 \arrow[ur] & & U_1 \arrow[ur] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & & &
\end{tikzcd}$$

  such that

  - $M = \operatorname{coker}(\phi_1) = F_0/\operatorname{im}(\phi_1)$
  - $\ker(\phi_i) = \operatorname{im}(\phi_{i+1})$ (This implies $\phi_i \circ \phi_{i+1} = 0$ for all i)

- Defn: A sequence of free R-modules and R-modules maps as above is called a *free resolution* of M over R. The module $U_i = \ker(\phi_i) = \operatorname{im}(\phi_{i+1})$ is called the ith *syzygy module of* M with respect to the resolution $F_\bullet$.

- Question: How to compute (find generators or presentations) for U_i?

- Defn: Say $U =< f_1, ..., f_s >$ is an R-module. We denote by $\operatorname{syz}(f_1, ..., f_s) = \ker(R^s \to U)$.

- Lemma: (Buchberger's criterion gives us syzygies for free.) If $\{f_1, ..., f_s\}$ is a GB for an R-modules U, then we can use $S(f_i, f_j)$ to come up with elements $r_{ij} \in \operatorname{syz}(f_1, ..., f_s)$.

  Proof. (Also defining r_{ij}.)

  Whenever the leading terms of f_i, f_j are supported on the same basis element of F we defined $S(f_i, f_j) = u_{ij}f_i - u_{ji}f_j$ (where $u_{ij} = LCM(-)/LT(-)$ and $u_{ji} = LCM(-)/LT(-)$).

  Since $\{f_1, ..., f_s\}$ is a GB, Buchberger's criterion for modules tells us $S(f_i, f_j) = \sum a_{ijk}f_k$, with $LT(S(f_i, f_j)) \geq LT(a_{ijk}f_k)$ for all k. So,

$$u_{ij}f_i - u_{ji}f_j = \sum a_{ijk}f_k \implies \sum_{k=1}^{s} a_{ijk}f_k - u_{ij}f_i + u_{ji}f_j = 0.$$

  Define: $r_ij = \sum_{k=1}^{s} a_{ijk}e_k - u_{ij}e_i + u_{ji}e_j$.

  Clearly, $\phi(r_{ij}) = 0$, so $r_{ij} \in \ker(\phi) = \operatorname{syz}(f_1, ..., f_s)$. □

- **Thm:** The elements r_{ij} generate $\operatorname{syz}(f_1, ..., f_s)$ (if $\{f_1, ..., f_s\}$ is a GB).

  Sketch of Proof. Assign to $r \in \operatorname{syz}(f_1, ..., f_s)$ (write $r = \sum_{i=1}^{s} h_i e_i$) the monomial $u_r = \max\{LT(h_i f_i)\}_{1 \leq i \leq s}$.

  Then the proof goes by contradiction: Suppose there exists $r \in \operatorname{syz}(f_1, ..., f_s) \setminus < r_{ij} >$. Consider among such r one that has minimum possible u_r.

  Since $\phi(r) = 0$, there exists at least 2 terms $h_1 f_1$ and $h_2 f_2$ such that $u_r = LT(h_1 f_1) = LT(h_2 f_2)$.

  Use r_{12} and r to fabricate $r' \in \operatorname{syz}(f_1, ..., f_s)$ such that $u_{r'} < u_r$, contradicting minimality. □

- Cor: If U is a monomial submodule of F and $\{f_1, ..., f_n\}$ is a set of monomial generators for U (in particular $\{f_1, ..., f_n\}$ is a GB for U), then $\operatorname{syz}(f_1, ..., f_n)$ is generated by $r_{ij} = u_{ij}e_i - u_{ji}e_j$, where u_{ij}, u_{ji} are the coefficients from $S(f_i, f_j)$ if this exists. (i.e., r_{ij} is gotten from $S(f_i, f_j)$ by replacing f_i by e_i and f_j by e_j).[11]

- Lemma: U is a free R-module if and only if $LT(U) = \oplus_{j=1}^{m} I_j e_j$ with all I_j being principal ideals.

  Proof. Homework. □

[11] Key point: If f_i, f_j are monomials, then $S(f_i, f_j) = 0$, hence $a_{ijk} = 0$ for all k.

- **Algorithm for computing a free resolution for M:**
  - Start with a presentation $M = F/U$.
  - Set $i = 1$, $U_1 = U$.
  - Repeat until U_i is free:
    - Compute a GB of U_i; compute $LT(U_i)$ and decide if U_i is free.
    - If U_i is not free, then $U_{i+1} :=< r_{jk} >$.
    - $i = i + 1$

- Example: Set $M = R/ < x^2, xy, y^2 >$, where $R = k[x, y]$. Then (use a monomial order such that $xy > y^2$): $f_1 = x^2$ and $f_2 = xy + y^2$. Then $U_1 =< f_1, f_2 >$ but we want a GB for U_1. $F_0 = R$.

  Compute $S(f_1, f_2) = yf_1 - xf_2 = -xy^2 = -yf_2 + y^3$. Buchberger's algorithm says: throw in $f_3 = y^3$.

  $r_{12} = ye_1 - xe_2 + ye_2 - e_3 = ye_1 + (y - x)e_2 - e_3$

  $r_{13} = y^3e_1 - x^2e_3$ (easy because f_1, f_3 are monomials – see Cor.)

  r_{23} comes from $S(f_2, f_3) = y^2f_2 - xf_3 = y^4 = yf_3$, so $r_{23} = y^2e_2 - xe_3 - ye_3 = y^2e_2 - (x + y)e_3$.

  So $\{f_1, f_2, f_3\}$ is a GB and $\text{syz}(f_1, f_2, f_3) =< r_{12}, r_{13}, r_{23} >$. We want $\text{syz}(f_1, f_2)$....

Oct. 4, 2013

- See Haydee's notes.

Oct. 7, 2013

- Used to be following Cochslittle / oshea (?)
- Now following: V. Ene & J. Herzog called "Gröbner bases in Commutative Algebra."
- Also following: D. Eisenbud called "Commutative Algebra with a view towards Algebraic Geometry."
- Example: Compute a free resolution of $M = R/ < x^2, xy + y^2 >$ where $R = k[x, y]$ with Lex such that $x > y$.
  - Start with the presentation of $M = R^1/U$, where $U =< x^2, xy + y^2 >\subseteq R^1$.
  - We then have $R^2 \to U$ a surjection, mapped to x^2 and $xy + y^2$. This gives a map

    $$R^2 \to R \to M \to 0,$$

    where the first map is given by $\begin{bmatrix} x^2 & xy + y^2 \end{bmatrix}$. We now need $\text{syz}(x^2, xy + y^2) \subseteq R^2$.
  - Need to compute GB of U. Put $f_1 = x^2$ and $f_2 = xy + y^2$.
    - First iteration: Compute:

      $$S(f_1, f_2) = yf_1 - xf_2 = -xy^2 = -yf_2 + y^3.$$

      The remainder in the last term is $y^3 \neq 0$. Buchberger's Algorithm tells us we need to include $f-3 = y^3$ in the GB. Now $G = \{f_1, f_2, f_3\}$.
    - Second iteration:
      $S(f_1, f_2) = -yf_2 + f_3 + 0$ (nothing new).
      $S(f_1, f_3) = y^3f_1-x^2f_2+0$. (Nothing new; This is always the case if you start with coprime monomials.)
      $S(f_2, f_3) = y^2f_2 - xf_3 = y^4 = yf_3 + 0$. (Again, nothing new.) B. Criterion STOP.

    So, we have that $G = \{f_1, f_2, f_3\}$ is a GB for U.

- Turning S-polys into generators for the syzygy module $\text{syz}(f_!, f_2, f_3)$. (Then at the end we'll prune down.)

$$yf_1 - xf_2 = -yf_2 + f_3 \implies ye_1 - xe_2 + ye_2 - e_3 \in \text{syz}(f_1, f_2, f_3)$$

So here we're mapping three copies of R onto U via $e_i \mapsto f_i$. The kernel of this map is $\text{syz}(f_!, f_2, f_3)$. Here
$r_{12} = ye_1 + (y-x)e_2 - e_3$.
$r_{13} = y^3e_1 - x^2e_2$
$r_{23} = y^2e_2 - (x+y)e_3$
$\text{syz}(f_1, f_2, f_3) =< r_{12}, r_{13}, r_{23} >\subseteq R^3$.

- Pruning step: Plug in $e_3 = ye_1 + (y-x)e_2$ into r_{12}, r_{13}, r_{23}. Then r_{12} becomes trivial.. call the latter two r_{13}' and r_{23}'.
Then:

$$r_{13}' = y^3e_1 - x^2(ye_1 + (y-x)e_2) = y^3e_1 - x^2ye_1 - x^2(y-x)e_2$$

$$r_{13}' = (y^3 - x^2y)e_1 - x^2(y-x)e_2 = y(y+x)(y-x)e_1 - x^2(y-x)e_2$$

$$r_{23}' = y^2e_2 - (x+y)(ye_1 + (y-x)e_2) = -(xy+y^2)e_1 + x^2e_2,$$

$$r_{23}' = -y(x+y)e_1 + x^2e_2.$$

Hence

$$\text{syz}(f_1, f_2) =< r_{13}', r_{23}' >\subseteq R^2$$

But: $r_{13}' = (y-x)r_{23}'$. So, $\text{syz}(f_1, f_2) =< r_{23}' >\subseteq R^2$. Hence $\text{syz}(f_1, f_2) \cong R$, so it's a free R-module!
Now,

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix}-y(x+y)\\ x^2\end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix}x^2 & xy+y^2\end{bmatrix}} R \longrightarrow M \longrightarrow 0$$

is a free resolution of M (over R). Check the composition of the two maps in the middle are indeed 0.

- **Schreyer's Theorem:** The idea: change monomial order at each step of computing a free resolution so that $\{r_{ij}\}$ form a GB for the syzygy module.
- Defn: Let $U =< f_1, ..., f_s >$ be a submodule of a free R-module F (we already have a given monomial order on F). Let $F' = R^s \to U$ by sending $e_i \to f_i$. We define a monomial order on F' as follows:

$$ue_i <_{\{f_1,...,f_s\}} ve_j$$

if $LM(uf_i) < LM(vf_j)$ in F or $LM(uf_i) = LM(vf_j)$ and $j < i$.

- (Check: this is a monomial order on F'.)
- **Theorem (Schreyer):** If $\{f_1, ..., f_s\}$ is a GB of U, then $\{r_{ij}\}$ (as defined last time) form a GB for $\text{syz}(f_1, ..., f_s)$ with respect to $>_{\{f_1,...,f_s\}}$. Moreover if $i < j$, $LT(r_{ij}) = u_{ij}e_i$, where $LT(f_i) = ue_k$, $LT(f_j) = ve_k$ implies $u_{ij} = LCM(u, v)/u$. (u_{ij} come from $S(f_i, f_j) = u_{ij}f_i - u_{ji}f_j$.)

Oct. 9, 2013

- **Theorem (Schreyer):** If $\{f_1, ..., f_s\}$ is a GB of U, then $\{r_{ij}\}$ (as defined last time) form a GB for $\text{syz}(f_1, ..., f_s)$ with respect to $>_{\{f_1,...,f_s\}}$. Moreover if $i < j$, $LT(r_{ij}) = u_{ij}e_i$, where $LT(f_i) = ue_k$, $LT(f_j) = ve_k$ implies $u_{ij} = LCM(u, v)/u$. (u_{ij} come from $S(f_i, f_j) = u_{ij}f_i - u_{ji}f_j$.)

Proof. Recall $\{f_1, ..., f_s\}$ is a GB for U so by Buchberger's Criterion, $S(f_i, f_j) = \sum_{k=1}^{s} a_{ijk}f_k$, where $LT(a_{ijk}f_k) \leq LT(S(f_i, f_j))$ for every k such that $a_{ijk} \neq 0$.

By definition, whenever f_i and f_j are of the form $LT(f_i) = ue_k$ and $LT(f_j) = ve_k$,

$$S(f_i, f_j) = (LCM(u, v)/u)f_i - (LCM(u, v)/v)f_i$$

Hence $u_{ij}f_i - u_{ji}f_j = \sum_{k=1}^s a_{ijk}f_k$, so $\sum_{k=1}^s a_{ijk}f_k - u_{ij}f_i - u_{ji}f_j = 0$. Then

$$r_{ij} = \sum_{k=1}^s a_{ijk}e_k - u_{ij}e_i - u_{ji}e_j.$$

Claim 1: Every monomial in $\sum_{k=1}^s a_{ijk}e_k$ is $<_{\{f_1,...,f_s\}}$ than $u_{ij}e_i$. Indeed, $LT(a_{ijk}f_k) \leq LT(S(f_i,f_j))$. Therefore $LT(a_{ijk}e_k) <_{\{f_1,...,f_s\}} LT(u_{ij}e_i) = u_{ij}e_i$.

Claim 2: $u_{ji}e_j <_{\{f_1,...,f_s\}} ju_{ij}e_i$.

$$LM(u_jif_j) = LM(u_{ij}f_i)$$

and

$$i < j \implies u_{ji}e_j <_{\{f_1,...,f_s\}} u_{ij}e_i$$

Claims 1 & 2 then imply $LT(r_{ij}) = u_{ij}e_i$. Next, we show $\{r_{ij}$ form a GB. We need to show that $LT(\text{syz}\{f_1,...,f_s\}) =< LT(r_{ij}) >$. Let $r \in \text{syz}\{f_1,...,f_s\} \subseteq F'$.

Hence $r = \sum_{j=1}^s r_je_j$. Suppose $LT(r) = v_ie_i$ for some fixed i, v_i is a monomial in $R = k[x_1,...,x_n$.

Denote $LT(r_je_j) = v_je_j$ for every $1 \leq j \leq s$.

$r \in \text{syz}(f_1,...,f_s)$, so $\phi(r) = 0$ and so $\sum_{j=1}^s r_jf_j = 0$, in particular, $LT(v_if_i)$ appears in the sum and is cancelled by other summands.

Let $S = \{j | LM(v_jf_j) = LM(v_if_i)\}$.
Claim 3: $i = \min s$.

For every $j \in S$, $r_je_j <_{\{f_1,...,f_s\}} r_ie_i$ because $LT(r) = v_ie_i$. Also for every $j \in S$, $LM(r_je_j) = LM(r_ie_i)$. Hence $j > i$ (breaking ties using position).

Let $r' = \sum_{j\in S} v_je_j$. But $\sum_{j\in S} v_jLT(f_j) = 0$ (because $\sum r_jf_j = 0$). Therefore $r' \in \text{syz}(LT(f_{j_1}),...,LT(f_{j_t}))$. By the Cor, $r' = \sum_{k,l\in S} b_{kl}(u_{kl}e_k - u_{lk}e_l)$. Then $LT(r')$ is divisible by $u_{ki}e_i = LT(r_{ij})$ for some $k \in S$, and so $LT(r_{ij}) | LT(r') = LT(r)$. This means $LT(r) \in < LT(r_{ij}) >$. □

- Cor: Re-index $\{f_1,...,f_s\}$ such that whenever $LT(f_i)$ and $LT(f_j)$ involve the same basis element, say $LT(f_i) = ue_k$ and $LT(f_j) = ve_k$, then $u >_{\text{Lex}} v$. Then if $x_1,...,x_t$ do not appear in $LT(f_j)$, then $x_1,...,x_{t+1}$ do not appear in $LT(r_{ij})$.

  Proof. By the Theorem, $LT(r_{ij}) = u_{ij}e_i = (LCM(u,v)/u)e_i$. Then $u >_{\text{Lex}} v \implies$ exponent of x_{t+1} in u is bigger than the exponent of x_{t+1} in v, we get exponent of x_{t+1} in $LCM(u,v)$ is exponent of x_{t+1} in u, and so this power cancels in $LCM(u,v)/u$. □

- **Thm (Hilbert's Syzygy Theorem):** Let M be a finitely generated R-module ($R = k[x_1,...,x_n]$). Then M admits a free resolution over R of length at most n.

  For the resolution
  $$0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$
  we define p to be the length of the resolution.

  Proof. Let t= largest index such that $x_1,...,x_t$ do not appear in $LT(U_1)$. By the Corollary, $x_1,...,x_{t+1}$ do not appear in $LT(U_2)$. Inductively, $x_1,...,x_{t+i-1}$ do not appear in $LT(U_i)$. Set $i = n-t+1$. Then $x_1,...,x_n$ do not appear in $LT(U_{n-t+1})$. Hence $LT(U_{n-t+1}) = 0$, and so $U_{n-t+1} = 0$. Then note $n - t = p \leq n$. □

OCT. 11, 2013

⩾ Computer day.

OCT. 14, 2013

⩾ **Plan of what's to come:**

1. Graded rings, modules, resolutions
2. (Multi) Graded free resolutions for monomial ideals
3. The relationship between the free resolution of I and that of $LT(I)$.

⩾ Graded rings and modules / Graded resolutions.

⩾ DEFN: Let k be a field. A ring R is a *graded k-algebra (graded ring)* if

1. $R = \oplus_{i\geq 0} R_i$, each R_i is a k-vector space.
2. $R_0 = k$
3. $R_i R_j \subseteq R_{i+j}$.

We say R is *standard graded* if $R = k[R_1]$ and $\dim_k R_1 < \infty$.

⩾ EXAMPLE: $R = [x_1, ..., x_n] = k \oplus \text{span}_k < x_1, ..., x_n > \oplus \text{span}_k < x_i^2, x_i x_j > \oplus \cdots \oplus \text{span}_k <$ deg i monomials $>$.

⩾ PROP: Let R be a graded k-algebra. Then TFAE:

1. R is standard graded
2. $R = \frac{k[x_1,...,x_n]}{I}$, where I is a homogeneous ideal contained in $k[x_1, ..., x_n]$ and $n = \dim_k R_1$.

⩾ DEFN: Let R be a graded ring. An R-module M is called a *graded R-module* if $M = \oplus_{i\in\mathbb{Z}} M_i$ and $R_i M_j \subseteq M_{i+j}$.

⩾ RMK: Any finitely generated graded R-module can be generated by a finite system of homogeneous elements. (Homogeneous elements are elements in M_i for some i.)

⩾ From now on, $R = k[x_1, ..., x_n]$, $\mathfrak{m} = (x_1, ..., x_n)$.

⩾ PROP: (NAK): Let M be a finitely generated R-module and let $m_1, ..., m_r$ be homogeneous elements whose residue classes modulo $\mathfrak{m}M$ form a k-basis for $M/\mathfrak{m}M$. Then $m_1, ..., m_r$ generate M.

⩾ COR: Let M be a finitely generated R-module. Then ALL homogeneous minimal systems of generators of M have the same cardinality, namely, $\dim_k M/\mathfrak{m}M$.

⩾ DEFN: (DEGREE SHIFTING): Let M be a graded R-module. Define $M(j)$ to be the graded module whose graded components are given by $M(j)_i = M_{i+j}$.

⩾ EXAMPLE: Let $d \in \mathbb{N}$. $R = R_0 \oplus R_1 \oplus \cdots$ Then $R(-d)_i = R_{-d+i}$, i.e., $R(-d)_0 = R_{-d} = 0 \cdots R(-d)_{d+1} = R_1 \cdots$:

degree	0	1	2	$\cdots$	d	$\cdots$	$d+i$	$\cdots$
R	R_0	R_1	R_2	$\cdots$	R_d	$\cdots$	R_{d+i}	$\cdots$
$R(-d)$	0	0	0	$\cdots$	R_0	$\cdots$	R_i	$\cdots$

⩾ DEFN: An R-module homomorphism $\phi : M \to N$ is called *homogeneous* if $\phi(M_i) \subseteq N_i$. (This is also sometimes called *degree preserving*.)

- Example: The R-module homomorphism $\phi : R(-d) \to R$ given by $\phi(x) = f \cdot x$ (where f is a homogeneous poly of degree $d > 0$) is homogeneous.
- Defn: A free resolution

$$F_\bullet \cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of a graded R-module M is a *graded free resolution* if each ϕ as well as ε are homogeneous R-module homomorphisms.

- Prop: Let $U \subseteq F$ be a graded submodule of a free R-module F. Then the reduced GB of U consists of homogeneous elements.

Proof. (sketch)

- S-elements between a pair of homogeneous elements are homogeneous.
- Remainders under division algorithm of a homogeneous element w.r.t. a set of homogeneous elements are homogeneous.
- This implies there exists a GB of U that consists of homogeneous elements.
- Furthermore, to get a reduced GB:
  - we discard some of the elements in the GB
  - we take further remainders
  - we multiply by constants.

These all yield homogeneous elements.

□

- Cor: Let M be a graded R-module. Then M admits a graded free resolution of length $\leq n = \#$ variablles.

Proof. (sketch) Previous Prop + Prop that r_{ij} generate syz(U)+HST. □

- Defn: A *minimal* graded free resolution of a graded R-module M is a graded free resolution

$$F_\bullet : \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

such that $\phi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all $i \geq 1$.

- Rmk: The ranks of the free R-modules in a minimal graded free resolution of M are minimal among the ranks of free modules in any given graded resolution of M.
- Example: Fix $f \in R$, $f \in R_d$, $d \geq 1$, (i.e., f is a homogeneous polynomial of degree d).

$$0 \to R(-d) \xrightarrow{f} R \xrightarrow{\varepsilon} R/(f) \to 0$$

- is a homogeneous free resolution of $R/(f)$.
- is also minimal because $\phi_1(R(-d)) \subseteq f \cdot R \subseteq \mathfrak{m} \cdot R$. (Or, look at the matrix $R(-d) \xrightarrow{[f]} R$ and check all of its entries are in $\mathfrak{m}$).

Oct. 16, 2013

- Example: f is a homogeneous polynomial with degree d. We came up with 2 resolutions for $R/(f)$:

$$0 \to R(-d) \xrightarrow{[f]} R \to R/(f) \to 0$$

We could also resolve (non-minimally) like:

$$F_\bullet : 0 \to R^2 \xrightarrow{\begin{bmatrix} 1 & f \\ -1 & 0 \end{bmatrix}} R^2 \to R/(f) \to 0$$

Note that this second one has elements (in the first column of the matrix) that are not in the maximal ideal $\mathfrak{m}$. So, we actually have that the following exact sequence injects into the last one:

$$G_\bullet : 0 \to R \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} R(e_1 - e_2) \to 0$$

Let's view this injection as a map of complexes:

$$0 \to G_\bullet \to F_\bullet \to F_\bullet/G_\bullet \to 0$$

Coincidentally, the cokernels give the minimal resolution!

- Prop: Every finitely generated graded R-module M has a minimal graded free resolution.

  Sketch of Proof: Start with any graded resolution $F_\bullet$ of M. (We know such $F_\bullet$ exists.) If there exists $x \in F_i$ such that $\phi_i(x) \notin \mathfrak{m}F_{i-1}$ (i.e., $F_\bullet$ is non-minimal). Then let

  $$G_\bullet : 0 \to 0 \to \cdots \to 0 \to G_i \to G_{i-1} \to 0$$

  is an exact complex. There is a sequence of complexes

  $$0 \to G_\bullet \to F_\bullet \to F_\bullet/G_\bullet \to 0$$

  The l.e.s. in homology corresponding to this sequence of complexes implies $F_\bullet/G_\bullet$ is exact, except for the 0 th spot, where the cokernel is M, i.e., $F_\bullet/G_\bullet$ is a resolution of M.

  Continue this process with $F_\bullet/G_\bullet$ instead of $F_\bullet$ until a minimal resolution is obtained. □

- Prop: Let M be a finitely generated R-module. Then any two graded minimal free resolutions of M are isomorphic, i.e., if $F_\bullet$ and $G_\bullet$ are minimal free resolutions of M, there exist degree-preserving isomorphisms $\mu_i : F_i \to G_i$ that make the following commute:

$$\begin{tikzcd}
F_\bullet : & \cdots \arrow[r] & F_i \arrow[r, "\phi_i"] \arrow[d, "\cong"] & F_{i-1} \arrow[r] \arrow[d, "\cong"] & \cdots \\
G_\bullet : & \cdots \arrow[r] & G_i \arrow[r, "\phi_i"] & G_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

- Cor: The ranks of the modules F_i in a minimal free resolution of M only depend on M (not on the choice of minimal resolution).
- Refinement: Each $F_i = \oplus_{j=1}^{\infty} R^{\beta_{ij}}(-j)$ and the β_{ij} only depend on M (not on the choice of minimal free resolution).
- Defn: The numbers β_{ij} as above are called the *graded Betti numbers* of M.
- Numerical data attached to a finitely generated graded R-module M. Graded Betti numbers (often summarized in a Betti diagram (or Betti table) is a matrix in which $\beta_{i,i+j}$ appears in position (i, j).
- Example: $I = x_1^2 - x_2x_3, x_3^2x_4, x_1x_2x_3, x_4^3)$. A graded minimal free resolution of R/I is:

$$0 \to R(-8) \to R^2(-6) \oplus R^3(-7) \to R^6(-5) \oplus R(-6) \to R(-2) \oplus R^3(-3) \to R \to R/I \to 0$$

This gives Betti numbers:

$$\beta_{4,8} = 1 \qquad \beta_{3,6} = 2 \qquad \beta_{2,5} = 6 \qquad \beta_{1,2} = 1 \qquad \beta_{00} = 1$$

$$\beta_{3,7} = 3 \qquad \beta_{2,6} = 1 \qquad \beta_{1,3} = 3$$

Putting them in a table, we have something like:

	0	1	2	3	4
0	1				
1		1			
2		3			
3			6	2	
4			1	3	
5					1

- The *total Betti numbers*: $\beta_i = \sum_{j \geq 0} \beta_{ij}$
- The *projective dimension* is the index of last column in the Betti table.

$$\operatorname{pd}(M) = \max\{i : \exists j, \beta_{ij} \neq 0\}.$$

- The *regularity* is the index of the last row in the Betti table:

$$\operatorname{reg}(M) = \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}.$$

- Defn: The numerical function $H_M : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with $H_M(i) = \dim_k M_i$ is called the *Hilbert function* of the graded module M.

  The formal (Laurent) series $HS_M(t) = \sum_{i \in \mathbb{Z}} H_M(i) t^i$.
- Facts:
  1. $HS_R(t) = \frac{1}{(1-t)^n}$, where $n =$ number of variables of $R = k[x_1, ..., x_n]$.
  2. $HS_{R(-d)}(t) = \frac{t^d}{(1-t)^n}$, where $R = k[x_1, ..., x_n]$.
  3. A s.e.s. of graded R-modules and homogeneous R-module maps

     $$0 \to A \to B \to C \to 0$$

     gives

     $$HS_B(t) = HS_A(t) + HS_C(t).$$

     Proof. We can restrict to s.e.s. of k-vector spaces $0 \to A_i \to B_i \to C_i \to 0$, and then done by dimensions of vector spaces: $\dim_k B_i = \dim_k A_i + \dim_k C_i$. □
- Prop: If $0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ is a minimal graded free resolution of M (over R), then $HS_M(t) = HS_{F_0}(t) - HS_{F_1}(t) + HS_{F_2}(t) - \cdots + (-1)^p HS_{F_p}(t)$. Each of these is a sum $HS_{R^{\beta_{ij}}(-j)}(t) = \frac{\beta_{ij} t^j}{(1-t)^n}$.

  Thus the $HS_M(t) = \sum_{i,j} (-1)^i \frac{\beta_{ij} t^j}{(1-t)^n}$.
- Example: $HS_{S/I}(t) = \frac{1 - t^2 - 3t^3 + 6t^5 - t^6 - 3t^7 + t^8}{(1-t)^4}$.

⋗

Oct. 23, 2013

⋗ **Computing graded Betti numbers using Tor:**

⋗ Let $R = k[x_1, ..., x_n]$.

⋗ Defn: Let M and N be finitely generated graded R-modules. Take a free resolution of N:

$$F_\bullet : 0 \to F_p \xrightarrow{d_p} F_{p-1} \to \cdots \to F_1 \to F_0 \to N \to 0$$

Tensor $F_\bullet$ with M and get a (no longer exact) complex:

$$M \otimes F_\bullet : 0 \to M \otimes_R F_p \xrightarrow{\widetilde{d_p}} \cdots \to M \otimes_R F_1 \to M \otimes_R F_0$$

Then

$$\operatorname{Tor}_i^R(M, N) = \frac{\ker(\widetilde{d_i})}{\operatorname{im}(\widetilde{d_{i+1}})}$$

is the ith homology of the complex $M \otimes F_\bullet$.

Note: If M and N are graded and $F_\bullet$ is a homogeneous (graded) resolution, then $\operatorname{Tor}_i^R(M, N)$ are graded R-modules, i.e., $\operatorname{Tor}_i^R(M, N) = \oplus_{j \in \mathbb{Z}} \operatorname{Tor}_i^R(M, N)_j$.

⋗ Prop: $\beta_{ij}(M) = \dim_k(\operatorname{Tor}_i^R(k, M))_j$.

Proof. To compute $\operatorname{Tor}_i^R(k, M)$ one considers a minimal graded free resolution of M

$$F_\bullet : 0 \to F_p \to \mathbb{F}_{p-1} \to \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \to F_0 \to M \to 0$$

Then

$$k \otimes F_\bullet : 0 \to k \otimes F_p \to k \otimes F_{p-1} \to \cdots \to k \otimes F_{i+1} \xrightarrow{\widetilde{d_{i+1}}} k \otimes F_i \to \cdots \to k \otimes F_0 \to k \otimes M \to 0$$

is a complex of k-vector spaces, where $\widetilde{d_i}(\lambda \otimes f) = \lambda \otimes d_i(f)$.

Claim: $\tilde{d_i} \equiv 0$. To see this: If $F_\bullet$ is a minimal graded free resolution, then $\operatorname{im} d_i \subseteq \mathfrak{m} F_{i-1}$, and so $d_i(f) = \sum_{i=1}^r m_i g_i$, $\{g_1, ..., g_r\}$ is a basis for F_{i-1} as a free R-module. Hence

$$\tilde{d_i}(\lambda \otimes f) = \lambda \otimes d_i(f) = \lambda \otimes (\sum m_i g_i) = \sum_{i=1}^r (\lambda m_i \otimes g_i) = \sum (0 \otimes g_i) = 0.$$

Therefore,

$$\operatorname{Tor}_i^R(k, M) = k \otimes F_i = k \otimes_R (\bigoplus_{j \in \mathbb{Z}} R^{\beta_{ij}}(-j)) = \bigoplus_{j \in \mathbb{Z}} k^{\beta_{ij}}(-j),$$

where this last is the decomposition of $\operatorname{Tor}_i^R(k, M)$ into graded pieces. Hence $\operatorname{Tor}_i^R(k, M)_j = k^{\beta_{ij}}(-j)$. Thus $\dim_k(\operatorname{Tor}_i^R(k, M))_j = \beta_{ij}$. □

⋗ Remarks:

1. Tensor product is symmetric, i.e., $\operatorname{Tor}_0^R(M, N) = M \otimes_R N \cong N \otimes_R M = \operatorname{Tor}_0^R(N, M)$.
2. Tor is also symmetric: $\operatorname{Tor}_i^R(M, N) \cong T_i^R(N, M)$.

⋗ Cor 1: $\beta_{ij}(M) = \dim_k(\operatorname{Tor}_i^R(k, M))_j = \dim_k(\operatorname{Tor}_i^R(M, k))_j$.

- Cor 2: One can compute β_{ij}'s by taking a minimal free resolution of k (i.e., the Koszul complex $K_\bullet$) and tensoring with M.
$$\mathrm{Tor}_i(M,k) = H_i(M \otimes_R K_\bullet).$$
- **Multigraded ($\mathbb{Z}^n$-graded, fine graded) modules:**
-
  - $$R = \bigoplus_{\alpha\in\mathbb{N}^n} R_\alpha = \bigoplus_{\alpha\in\mathbb{N}^n} \mathrm{span}_k < x^\alpha >$$
    gives R a $\mathbb{N}^n$-graded structure.
  - $$M = \bigoplus_{\beta\in\mathbb{Z}^n} M_\beta$$
    such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ is a $\mathbb{Z}^n$-graded R-module.
- Examples:
  1. If I is a monomial ideal, then I is a $\mathbb{Z}^n$-graded R-module.
  2. If I is a monomial ideal, then R/I is a $\mathbb{Z}^n$-graded R-module.
- $\mathbb{Z}^n$-graded Hilbert Function:
$$HF_M(\beta) = \dim_k(M_\beta)$$
- $\mathbb{Z}^n$-graded Hilbert Series
$$HS_M(t_1,...,t_n) = \sum_{\beta\in\mathbb{Z}^n} HF_M(\beta)\cdot t^\beta$$
- $\mathbb{Z}^n$-graded modules admit resolutions that are multi-graded (i.e., the differentials preserve multi-degrees).
- Multigraded Betti numbers: $\beta_{i\alpha}$, $i \in \mathbb{N}\cup\{0\}$, $\alpha\in\mathbb{Z}^n$, are given by $\beta_{i\alpha}(M) = \dim_k(F_i)_\alpha$, where F_i= the gree R-module in position i of a minimal free multigraded resolution of M.
- Goal: Describe $\beta_{i\alpha}(R/I)$, where I is a monomial ideal.
- **(Abstract) Simplicial complexes and their homology:**
- Defn: An (abstract) simplicial complex Δ on $\{1,2,...,n\}$ is a collection of subsets of $\{1,2,...,n\}$ closed under the operation of taking subsets, i.e., if $T\in\Delta$ and $\tau\subseteq T$, then $\tau\in\Delta$.
- Example: The abstract 2-simplex (a.k.a. the triangle) is a simplicial complex on $\{1,2,3\}$ given by
$$\Delta = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\varnothing\}$$
- The abstract n-simplex $\Delta_n = \mathcal{P}(\{1,...,n\})$.
- The abstract n-simplex has a geometric realization given by all complex combinations of $n+1$ affinely independent points.
- Any abstract simplicial complex has a geometric realization (see Topology).
- Defn: Given a simplicial complex Δ, an element $\sigma\in\Delta$ such that σ has cardinality $i+1$ is called an *i-face* (or an i-dimensional face). (Note: $\varnothing$ is the unique -1-face.)
- The dimension of Δ is $\dim(\Delta) = \max\{i : i-\text{faces exist in }\Delta\}$.
- The f-vector of Δ is the vector $(f_{-1}, f_0, f_1, ..., f_{\dim(\Delta)})$, where f_i is the number of i-faces of Δ.
- Maximal faces (with respect to containment) are called *facets*.
- Example: Consider the shape Δ with 5 vertices, with edges $\{13\},\{3,4\},\{1,2\},\{2,3\}$, and $\{1,2,3\}$. Then this has f-vector $(1,5,5,1)$. (i.e., 1 empty set, 5 vertices, 5 edges, and 1 triangle.) The facets in this example are $\{1,2,3\}$ and $\{3,4\}$ and $\{2,4\}$ and $\{5\}$. Also $\dim(\Delta) = 2$.

Oct. 25, 2013

⋗

Oct. 28, 2013

⋗ Last time: Stanley-Reisher correspondence

⋗

$$\{\text{simplicial complexes}\} \leftrightarrow_{\text{bij}} \{\text{squarefree monomial ideals}\},$$

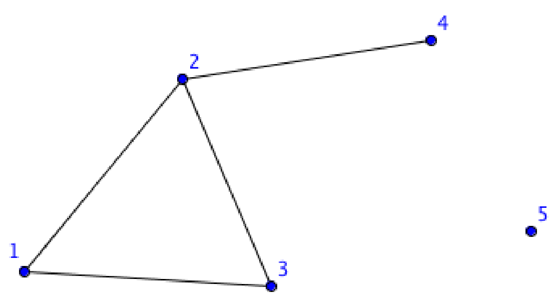
where the map is given by:

$$\Delta \mapsto I_\Delta := < x_\tau : \tau \notin \Delta >$$

⋗ Notation:

- If $\alpha = (\alpha_1, ..., \alpha_n)$, then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
- If $\sigma \subset \{1, ..., n\}$, $\sigma = \{i_1, i_2, ..., i_t\}$, then $x_\sigma = x_{i_1} x_{i_2} \cdots x_{i_t}$ is a squarefree monomial.
- If $\alpha = (\alpha_1, ..., \alpha_n)$, then $\mathrm{supp}(\alpha) = \{i : \alpha_i \neq 0\}$ (supp : multi-exponents $\to$ subsets of $\{1, ..., n\}$).
- If $\sigma \subset \{1, ..., n\}$, then $\mathrm{char}(\sigma) = (\alpha_1, ..., \alpha_n)$, where $\alpha_i = \begin{cases} 0 & i \notin \sigma \\ 1 & i \in \sigma \end{cases}$.

⋗ Example: Δ = simplicial complex from before.



(including 123 also).

We computed the f-vector is $(1, 5, 5, 1)$. Also,

$$HS_{R/I_\Delta} = 1 + 5\frac{1}{(1-t)} + 5\frac{t^2}{(1-t)^2} + 1\frac{t^3}{(1-t)^3} = \frac{1 + 2t - 2t^2}{(1-t)^3}$$

The h-vector is $(1, 2, -2)$. (The coefficients of the numerator of HS.)

Stanley's Magic Triangle: Rule: Entry to the NE-entry the NW. Build a triangle with rows from $f_{-1}, f_0, f_1, f_2, ...$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 5 & & & \\ & & 1 & & 4 & & 5 & & \\ & 1 & & 3 & & 1 & & 1 & \\ 1 & & 2 & & -2 & & 0 & & \end{array}$$

⩾ **Theorem:**

(1)

$$HS_{R/I_\Delta}(x_1, ..., x_n) = \frac{\sum_{\sigma\in\Delta}(\prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j))}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

(2)

$$HS_{R/I_\Delta}(t) = \frac{\sum_{i=0}^{d+1} f_{i-1}t^i(1-t)^{n-i}}{(1-t)^n} = \sum_{i=0}^{d+1} \frac{f_{i-1}t^i}{(1-t)^i}.$$

Proof. (1) By Lemma,

$$HS_{R/I_\Delta}(x_1, ..., x_n) = \sum_{x^\alpha \notin I_\Delta} x^\alpha = \sum_{\sigma\in\Delta}\left(\sum_{\mathrm{supp}(\alpha)=\sigma} x^\alpha\right) = \sum_{\sigma\in\Delta} \frac{\prod_{i\in\sigma} x_i}{\prod_{i\in\sigma}(1-x_i)} = \frac{\sum_{\sigma\in\Delta}\prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j)}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

Some people call the numerator of the last expression the k-polynomial of R/I_Δ.

(2)

$$HS_{R/I_\Delta}(t) = HS_{R/I_\Delta}(t, t, ..., t) = \frac{\sum_{\sigma\in\Delta} t^{|\sigma|}(1-t)^{n-|\sigma|}}{(1-t)^n} = \frac{\sum_{i=0}^{d+1} f_{i-1}t^i(1-t)^{n-i}}{(1-t)^n} = \sum_{i=0}^{d+1} f_{i-1}\frac{t^i}{(1-t)^i} = \frac{h(t)}{(1-t)^{d+1}}$$

(d=dimension of Δ – largest face has cardinality $d+1$.)

□

⩾ Defn: The last line above shows that we can always write HS_{R/I_Δ} as

$$HS_{R/I_\Delta}(t) = \frac{h_0 + h_1 t + \cdots + h_{d+1}t^{d+1}}{(1-t)^{d+1}},$$

where $d = \dim(\Delta)$. The vector $(h_0, h_1, ..., h_{d+1})$ is called the h-vector of R/I_Δ.

⩾ Rmk: The Krull dimension of R/I_Δ is $d+1$. (In general, we can write for a standard graded R-module M: $HS_M(t) = \frac{h_0+h_1t+\cdots+h_{\dim(M)}t^{\dim(M)}}{(1-t)^{\dim(M)}}$.)

⩾ Cor: (Going between h-vector and f-vector.)

$$h_j = \sum_{i=0}^{j}(-1)^{j-i}\binom{d-i}{j-i}f_{i-1}$$

(This first is equivalent to doing Stanley's triangle.)

$$f_j = \sum_{i=0}^{j+1} \binom{d-i}{j+1-i} h_i$$

⋗ **Alexander duality:**

⋗ Defn: If Δ is a simplicial complex, then the *Alexander dual simplicial complex* is $\Delta^* = \Delta^\vee = \{\overline{\tau} : \tau \notin \Delta\}$

⋗ Notation: Given $\sigma \subset \{1, 2, ..., n\}$, $\sigma = \{i_1, ..., i_s\}$, denote by

$$P_\sigma = < x_{i_1}, ..., x_{i_s} >$$

(this is a prime ideal).

⋗ Defn: Suppose I_Δ is a squarefree monomial ideal. Say $I_\Delta = < x_{\sigma_1}, ..., x_{\sigma_n} > = < x^{\text{char}(\sigma_1)}, ..., x^{\text{char}(\sigma_s)} >$. Define the *Alexander dual ideal* of I_Δ to be

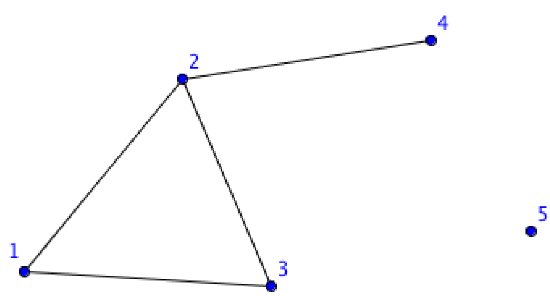
$$I_\Delta^* = \bigcap_{i=1}^{s} P_{\sigma_i}.$$

⋗ **Theorem:**

(1) $(\Delta^*)^* = \Delta$ (duality)

(2) $I_{\Delta^*} = I_\Delta^* (= (I_\Delta)^*)$ (homework)
Consequently, $I_\Delta^* = < x^{\overline{\sigma}} : \sigma \in \Delta >$ (the SR ideal I_{Δ^*}).

⋗ Example: Let Δ be as above:



Then facets of Δ^* are complements of minimal non-faces of Δ. So,

$$\Delta^* =$$

(Image here: we end up with triangles 134, 142, 235, and 15, 12)

Then

$$I_\Delta = < x_2x_3x_4, x_1x_4, x_1x_5, x_2x_5, x_3x_5, x_4x_5 >$$

and

$$I_\Delta^* = < x_2, x_3, x_4 > \cap < x_1, x_4 > \cap < x_1, x_5 > \cap < x_2, x_5 > \cap < x_3, x_5 > \cap < x_4, x_5 >$$

But also, $I_\Delta^* = I_{\Delta^*} = < x_1x_2x_3x_4, x_1x_2x_5, x_1x_3x_5, x_4x_5 >$. Note also,

$$I_\Delta = I_\Delta^{**} = < x_1, x_2, x_3, x_4 > \cap < x_1, x_2, x_5 > \cap < x_1, x_3, x_5 > \cap < x_4, x_5 > .$$

Oct. 30, 2013

- **Hochster's Theorem:** Given a simplicial complex Δ, all non-zero Betti numbers of I_Δ and of R/I_Δ occur in squarefree (multi) degrees and are given by:

$$\beta_{i,\alpha}(I_\Delta) = \beta_{i+1,\alpha}(R/I_\Delta) = \dim_k(\widetilde{H}_{i-1}(\mathrm{link}_{\Delta^*}\overline{\mathrm{supp}(\alpha)}),$$

for α a squarefree multi-degree.

- Defn: Given a simplicial complex Δ and a set σ,

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}.$$

- Rmk: links are simplicial complexes.
- Example: Same as before: Δ and Δ^*. Here,

$$\mathrm{link}_{\Delta^*}(\{1\}) = \{(24), (23), (43), (5)\}.$$

Also,

$$\mathrm{link}_{\Delta^*}(\{3\}) = \{(14), (24), (12), (25)\}.$$

The reduced homologies:

$$\widetilde{H}_i(\mathrm{link}_{\Delta^*}(\{1\})) = \begin{cases} 0 & i=-1 \\ k & i=0 \\ k & i=1 \\ 0 & i \geq 2 \end{cases}$$

and

$$\widetilde{H}_i(\mathrm{link}_{\Delta^*}(\{3\})) = \begin{cases} 0 & i=-1 \\ 0 & i=0 \\ k & i=1 \\ 0 & i \geq 2 \end{cases}$$

This tells us about the betti numbers :

$$\beta_{i,(0,1,1,1,1)}(I_\Delta) = \begin{cases} 0 & i=0 \\ 1 & i=1 \\ 1 & i=2 \\ 0 & i \geq 3 \end{cases}$$

and

$$\beta_{i,(1,1,0,1,1)}(I_\Delta) = \begin{cases} 0 & i=0,1 \\ 1 & i=2 \\ 0 & i \geq 3 \end{cases}$$

- **The Koszul complex** $R = k[x_1, ..., x_n]$
  - For each variable x_i, define a new variable e_i
  - For each monomial $g = x_{i_1}x_{i_2}\cdots x_{i_j}$, set $Dg = e_{i_1} \wedge \cdots \wedge e_{i_j}$ where we require $e_r \wedge e_s = -e_s \wedge e_r$ (in char 2 also $e_r \wedge e_r = 0$).
    So, $Dg = 0$ whenever g is not square free.
    Assign multideg(Dg) = multideg(g). (Or for standard graded: $\deg(Dg) = \deg(g)$.)

⩾ Defn: The Koszul complex is a complex of R-modules:

$$K_\bullet : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to F_{-1} = k \to 0$$

(subscripts correspond to homological degree) where

- F_i is the free R-module on basis $\{Dg : g \text{ is square free and } \deg(g) = i\}$, so $F_i = R^{\binom{n}{i}}(-i) = \bigoplus_{\deg(g)=i} R(-\operatorname{multideg}(g))$.
- $d_i(Dg) = \sum_{j \in \operatorname{supp}(g)} \operatorname{sign}(j, \operatorname{supp}(g)) \times_j Dg/\times_j$ (This is similar to the topological differential of the chain complex of Δ_{n-1}.)

⩾ Fact: $K_\bullet$ is a minimal free resolution of k.

⩾ Recall: $\beta_{i,\alpha}(M) = \dim_k(\operatorname{Tor}_i(k, M))_\alpha = \dim_k(H_i(K_\bullet \otimes M))_\alpha$

⩾ Defn: Let I be a monomial ideal. Then $K_\bullet(I) := I \otimes_R K_\bullet$ is the complex (not necessarily exact)

$$0 \to I \otimes_R F_n \to I \otimes_R F_{n-1} \to \cdots \to I \otimes_R F_0 \to I/\mathfrak{m}I \to 0.$$

⩾ Note: The module $I \otimes_R F_i$ has basis $\{f \otimes Df : f \text{ is a monomial in } I\}$ (i.e., $\deg(g) = i$, g is squarefree).

⩾ Note: $K_\bullet(I)$ is a multi-graded complex (d_i's preserve multidegree).

⩾ key point: $K_\bullet(I)$ will break into a direct sum of complexes of k-vector spaces

$$K_\bullet(I) = \bigoplus_{\alpha \in \mathbb{Z}^n} (K_\bullet(I))_\alpha \implies H_i(K_\bullet(I)) = \bigoplus_{\alpha \in \mathbb{Z}^n} H_i(K_\bullet(I))_\alpha,$$

i.e., $H_i(K_\bullet(I))_\alpha) = H_i(K_\bullet(I)_\alpha)$.

⩾ Outcome: $\beta_{i,\alpha}(I) = \dim_k H_i(K_\bullet(I)_\alpha)$.

Proof of Hochster's Theorem: We proceed as follows:

Step 1. Claim: There is a bijection between the k-basis of $(K_\bullet(I))_\alpha$ and faces of $\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)})$.

A basis for $(I_\Delta \otimes F_i)_\alpha$ is $B = \{f \otimes Dg : f \in I_\Delta, g \text{ sqfree}, \deg(g) = i, \operatorname{multideg}(f \otimes Dg) = \alpha \iff \operatorname{multideg}(fg) = \alpha \iff fg = x^\alpha\}$.
So there is a bijection:

$B \leftrightarrow \{g : g \text{ is square-free }, g | x^\alpha, \deg(g) = i, x^\alpha/g \in I_\Delta\} = \{g : g \text{ is square-free }, \deg(g) = i, g | x^\alpha, \operatorname{supp}(x^\alpha/g) \notin \Delta\}$

DIAGRAM: $[n] \supseteq \operatorname{supp}(\alpha) \supseteq \operatorname{supp}(g)$.

Note $\overline{\overline{\operatorname{supp}(\alpha)} \cup \operatorname{supp}(g)} \notin \Delta$. Hence $\overline{\operatorname{supp}(\alpha)} \cup \operatorname{supp}(g)$ is a complement of a non-face of Δ. By definition, this is true if and only if $\overline{\operatorname{supp}(\alpha)} \cup \operatorname{supp}(g) \in \Delta^*$. Again by definition of the Alexander dual, if and only if $\operatorname{supp}(g) \in \operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}$. (The fact that $\overline{\operatorname{supp}(\alpha)} \cap \operatorname{supp}(g) = \varnothing$ is given by $g | x^\alpha$.
We therefore have the bijection

$$B \leftrightarrow \{g : g \text{ sqfree }, \deg(g) = i, \operatorname{supp}(g) \in \operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)})\}.$$

Therefore $(I \otimes F_i)_\alpha = k^{f_{i-1}(\operatorname{link}_{\Delta^*} \overline{\operatorname{supp}(\alpha)})}$, where the one on the left is the entry of $K_\bullet(I)_\alpha$ in homological degree i and on the right is the entry of something else.

□

Nov. 1, 2013

⩾ **Theorem (Hochster):** $\beta_{i\alpha}(I_\Delta) = \beta_{i+1,\alpha}(R/I_\Delta) = \dim_k \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^*} \overline{\operatorname{supp}(\alpha)})$

Proof. Starting with a review from last time...

Step 1: We found that a basis for the free module in homological degree i of $(K_\bullet(I_\Delta))_\alpha$ is given by $\{fDg = Bg : \deg(g) = i, \operatorname{supp}(g) \in \operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}), f = x^\alpha/g\}$.
Let $Bg = fDg$.

Step 2: Consider the chain complex of $\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)})$.

$$\widetilde{C}_\bullet : 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

where C_i is a k-vector space with basis corresponding to i-dimensional faces of $\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)})$.
Let C_g be the basis element in C_i corresponding to $\operatorname{supp}(g)$ (here g is a squarefree monomial such that $g|x^\alpha$).
This means that $\dim(\operatorname{supp}(g)) = i$, hence $|\operatorname{supp}(g)| = -1 = i$, and so $|\operatorname{supp}(g)| = i+1 \implies \deg(g) = i+1$.
Define a map: $\phi : (K_\bullet(I_\Delta))_\alpha \to \widetilde{C}$ by setting $\phi(Bg) = C_g$, for any squarefree g with $g|x^\alpha$ and $\deg(g) = i+1$.

$$\begin{tikzcd}
0 \arrow[r] & I_\Delta \otimes_R F_n \arrow[r] \arrow[dr] & I_\Delta \otimes F_{n-1} \arrow[r] \arrow[dr] & I_\Delta \otimes_R F_{n-2} \arrow[r] \arrow[dr] & \cdots & I/mI \arrow[r] & 0 \\
0 \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \arrow[r] & \cdots \arrow[r] & C_{-1} \arrow[r] & C_{-2}
\end{tikzcd}$$

Differential of $K_\bullet(I)$ was $d(fDg) = \sum_{j \in \operatorname{supp}(g)} \operatorname{sign}(j) f x_j Dg/x_i$ (Koszul differential).
Differential of $\widetilde{C}_\bullet$ was $\partial(Cg) = \sum_{j \in \operatorname{supp}(g)} \operatorname{sign}(j) Cg/x_j$ (topological differential).
Therefore:

$$(\operatorname{Tor}_i(I_\Delta)_\alpha) = H_i(K \bullet (I)_\alpha) = H_{i-1}(\widetilde{C}(\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}))) = \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}))$$

This implies

$$\beta_{i\alpha}(I_\Delta) = \dim \operatorname{Tor}_i(I_\Delta)_\alpha = \dim \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}))$$

□

- Note: Same proof for non-squarefree I shows that

$$\beta_{i\alpha}(I) = \widetilde{H}_{i-1}(K^\alpha(I)),$$

where $K^\alpha(I) =$ simplicial complex consisting of $\{\operatorname{supp}(g), g \text{ is squarefree}, \alpha/g \in I\}$.

- **Theorem (Alexander duality - topological):** If Δ is a simplicial complex on n vertices, then $\widetilde{H}_{n-i-2}(\Delta; k) = \widetilde{H}^{i-1}(\Delta^*; k) = (\widetilde{H}_{i-1}(\Delta^*; k))^*$. Consequently, $\dim_k \widetilde{H}_{n-i-2}(\Delta) = \dim_k \widetilde{H}_{i-1}(\Delta^*)$.
- Defn: If Δ is a simplicial complex, then $\Delta[\alpha] = \{\tau : \tau \in \Delta, \tau \subseteq \operatorname{supp}(\alpha)\}$ is a simplicial complex.
- Lemma: $\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)}) = (\Delta[\alpha])^*$
- **Theorem (Hochster's Theorem - dual version):**

$$\beta_{i\alpha}(I_\Delta) = \dim \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^*}(\overline{\operatorname{supp}(\alpha)})) = \dim \widetilde{H}_{i-1}((\Delta[\alpha])^*) = \dim_k \widetilde{H}_{n-i-2}(\Delta[\alpha]).$$

- **Theorem (Terai):**

$$\operatorname{reg}(R/I_\Delta) - 1 = \operatorname{reg}(I_\Delta) = \operatorname{pd}(R/I_{\Delta^*}) = \operatorname{pd}(I_{\Delta^*}) + 1$$

Proof.

$$\begin{aligned}
\operatorname{reg}(I_\Delta) &= \max\{j : \beta_{i,i+j}(I_\Delta) \neq 0 \text{ for some value of } i\} \\
&= \max\{j : \text{there exists a squarefree multidegree } \alpha \text{ and } i \geq 0 \text{ s/t } \deg(x^\alpha) = i+j \text{ and } \beta_{i\alpha}(I_\Delta) \neq 0\} \\
&= \max\{j : \widetilde{H}_{n-i-2}(\Delta[\alpha]) \neq 0 \text{ for some } i \geq 0,\ n = |\operatorname{supp}(\alpha)| = \deg(x^\alpha) = i+j\} \\
&= \max\{j : \widetilde{H}_{j-2}(\Delta[\alpha]) \neq 0 \text{ for some } i \geq 0,\ \alpha \text{ squarefree},\ |\operatorname{supp}(\alpha)| = i+j\} \\
&= \max\{j : \beta_{j-1,\alpha}(I_{\Delta^*}) \neq 0 \text{ for some squarefree } \alpha\} \\
&= \operatorname{pd}(I_{\Delta^*}) + 1 \\
&= \operatorname{pd}(R/I_{\Delta^*}).
\end{aligned}$$

□

⩾ **Theorem (Eagon-Reiner):** I_Δ has a linear resolution if and only if R/I_{Δ^*} is CM.

⩾ DEFN: An R-module has a linear resolution if the Betti table looks like: (If $\operatorname{reg}(M) = d$.)

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \cdots \\ \hline \vdots & & & & & \\ d & * & * & * & * & \\ \vdots & & & & & \end{array}$$

equivalently,

- all the differentials in the minimal free resolution of M over R are linear (matrices representing these maps have linear entries).
- $\operatorname{reg}(M)$ is the degree of the generators of M.

Proof of Eagen-Reiner: I_Δ has linear resolution $\iff$ $\operatorname{reg}(I_\Delta)$ = degree of the minimal generators of I_Δ (All mingens of I_Δ must have same degree.)
$\iff$ $\operatorname{pd}(R/I_{\Delta^*})$ = cardinality of the minimal non-faces of Δ.
$\iff$ $\operatorname{pd}(R/I_{\Delta^*}) = n-$cardinality of the facets of Δ^*
$\iff$ cardinality of the facets of Δ^* is $n - \operatorname{pd}(R/I_{\Delta^*})$
$\iff$ $\dim(\Delta^*) + 1 = n - \operatorname{pd}(R/I_{\Delta^*})$
$\iff$ $\dim(R/I_{\Delta^*}) = n - \operatorname{pd}(R/I_{\Delta^*})$ (dimension is Krull here).
$\iff$ (Auslander Buchsbaum) R/I_{Δ^*} is CM. □

NOV. 4, 2013

⩾ This week: **Borel fixed monomial ideals; generic initial ideals.**

⩾ In the following, the characteristic of k is 0 and all the ideals are standard graded ($\mathbb{Z}$-graded).

⩾ The matrix group $GL_n(k)$ acts on $R = k[x_1, ..., x_n]$ as follows:
if $g = (g_{ij}) \in GL_n(k)$, then $gf = f(gx_1, ..., gx_n)$, where $gx_j = \sum_{i=1}^n g_{ij}x_i$.

⩾ Examples:

Matrix group	Name	Invariant ideals
$GL_n(k)$	General linear	$0, m^d$, $\forall d$
$B_n(k)$	Borel group	Borel-fixed ideals?
$T_n(k)$	Torus group	all monomial ideals

Where $B_n(k)$ is the group of upper triangular invertible matrices; $T_n(k)$ is the group of invertible diagonal matrices.

⩾ DEFN: An ideal $I \subseteq R$ is Borel fixed if $gI = I$ for every $g \in B_n(k)$.

⩾ PROP: (CHARACTERIZATION OF BOREL-FIXED IDEALS)
I is aBorel fixed ideal if and only if

(1) I is a monomial ideal, and
(2) - for all monomials $m \in I$, for every $i < j$, and
  - if m is divisible by x_j^t but not by x_j^{t+1}, then $x_i^s \frac{m}{x_j^s} \in I$, for all $s \leq t$.[12]

⩾ NOTE: In the proposition, actually (2) $\iff$ (2′): If $m \in I$ is a monomial divisible by x_j and $i < j$, then $x_i \frac{m}{x_j} \in I$.

[12] In characteristic p, we would need to change this to $s <_p t$.

⋗ Rmk: $B_n(k)$ is generated by $T_n(k)$ together with the upper-triangular elementary matrices: Γ^c_{ij} is the matrix with 1s on the diagonal and c in the i,j spot, so that $\Gamma^c_{ij}(x_j) = x_j + cx_i$, $\Gamma^c_{ij}x_l = x_l$ for all $l \neq j$.

Proof of Prop: $\Rightarrow$: Le I be a Borel-fixed ideal. Since $T_n(k) \subseteq B_n(k)$, we also have I is $T_n(k)$-fixed. Therefore I is a monomial ideal.

Let $m \in I$. Suppose $m = x_j^t \cdot m'$, where m' is not divisible by x_j^t. Then $\Gamma^c_{ij}m = \Gamma^c_{ij}(x_j^t) \cdot \Gamma_{ij}(m') = (x_j + cx_i)^t \cdot m' = m' \sum_{s=0}^t \binom{t}{s} x_j^{t-s} \cdot (cx_i)^s = m' \sum_{s=0}^t c_s \binom{t}{s} \left(\frac{x_i}{x_j}\right)^s \cdot x_j^t = m' x_j^t \sum_{s=0}^t c^s \binom{t}{s} \left(\frac{x_i}{x_j}\right)^s \in I$. Since I is Borel-fixed, we have $\Gamma^c_{ij}m \in I$. As I is a monomial ideal, $m \cdot \left(\frac{x_i}{x_j}\right)^s \in I$, for every $0 \leq s \leq t$. This implies (2).

$\Leftarrow$: The above equation and (2) imply $\Gamma^c_{ij}m \in I$ for any monomial $m \in I$. Since I is a monomial ideal, $T_n(k)I = I$. Therefore $B_n(k)$ fixes I. □

⋗ Examples of Borel-fixed ideals:

(1) When $\mathrm{char}(k) = 0$ in $R = k[x_1, x_2]$, the Borel-fixed ideals are "initial lex-segments," e.g., $(x_1^3, x_1^2 x_2, x_1 x_2^2)$.

(2) In 3 variables, above not true any more. E.g., $(x_1^3, x_1^2 x_2, x_1^2 x_3)$ is Borel-fixed but not lex-segment.

(3) In characteristic p, $(x_1^{p^e}, ..., x_n^{p^e})$ is Borel-fixed.

(4) Products, sums, and intersections of Borel-fixed ideals are Borel-fixed.

⋗ **Generic initial ideals:** (Fix a monomial order $<$ on R.)

⋗ **Theorem:** Let I be a homogeneous ideal. There is a Zariski open set $\varnothing \neq U \subseteq GL_n(k) = \mathbb{A}^{n^2}$ and a monomial ideal J, such that

$$LT(gI) = J, \text{ for any } g \in I$$

⋗ Defn: J as in the theorem is called the generic initial ideal of I. Usually, we write $J = gin(I)$. (Depends on monomial ordering.)

⋗ Notations: Say $V \subseteq R_d$ is a vector space of homogeneous polynomials of degree d, $\dim(V) = t$. Then V can be represented as a 1-dimensional vector space

$$L = \wedge^t V \subseteq \wedge^t R_d$$

with basis of L given by $f_1 \wedge f_2 \wedge \cdots \wedge f_t$, where $\{f_1, ..., f_t\}$ is a basis of V.

If $m_1, ..., m_t$ are monomials in R_d we say $m_1 \wedge \cdots \wedge m_t$ is a monomial in $\bigwedge^t R_d$.

We say $m_1 \wedge \cdots \wedge m_t$ is a *normal expression* if $m_1 > \cdots > m_t$.

We order monomials of $\bigwedge^t R_d$ be ordering their normal expressions lexicographically (i.e., if $m = m_1 \wedge \cdots \wedge m_t$ and $m' = m'_1 \wedge \cdots \wedge m'_t$ are normal expressions, then $m > m'$ in $\bigwedge^t R_d$ if for the smallest i such taht $m_i \neq m'_i$ we have that $m_i > m'_i$ w.r.t. the monomial order on R.

Nov. 6, 2013

⋗ Today: any characteristic for k, want k to be infinite.

⋗ **Theorem (Galligo, Bayer-Stillman):** Let I be a homogeneous ideal. Then there is a Zariski open set

$$U \subseteq GL_n(k) \subseteq M_n(k) \cong \mathbb{A}^{n^2}$$

and there is a monomial ideal J such that $LT(gI) = J$, for every $g \in U$. $(J = gin(I))$

⩥ Remark: I being homogeneous means that $I = \bigoplus_{d\geq 0} I_d$ (where I_d is the span of the homogeneous elements of I of degree d). Fix d; say $\{f_1, ..., f_t\}$ is a basis for $I_d \subseteq R_d$ (k-vector subspace). We have a way to identify t-dimensional subspaces of R_d with affine 1-dimensional subspaces $\bigwedge^t R_d$.

$$I_d = \text{span}\{f_1, ..., f_t\} \leftrightarrow \text{span}_k\{f_1 \wedge \cdots \wedge f_t\} \subseteq \bigwedge^t R_d$$

(the last is a $\binom{\binom{n+d-1}{d-1}}{t}$-dimensional k-vector space).

The action of $GL_n(k)$ on R induces the following action on $\bigwedge^t R_d$:

$$g(f_1 \wedge \cdots \wedge f_t) = g(f_1) \wedge \cdots \wedge g(f_t)$$

Proof of theorem: Let $g = (g_{ij})$ be a matrix with g_{ij} as distinct variables.

$$g(f_1) \wedge \cdots \wedge g(f_t) = \sum_{m=\text{ mon in } \wedge^t R_d} P_{m,d}(g_{ij})m,$$

where $P_{m,d}$ is a polynomial, $m = m_{1,d} \wedge \cdots \wedge m_{t,d}$, and $m_{i,d}$ are monomials of degree d.

More concrete way to come up with $P_{m,d}(g_{ij})$: $g(f_1), ..., g(f_t) \in R_d =$ span of the $\binom{n+d-1}{d-1}$ monomials of degree d in R. Therefore there exists a matrix of size $t \times \binom{n+d-1}{d-1}$ in which we label rows by $g(f_i)$ and columns by monomials of R_d. The rows will be the coefficients of $g(f_i)$ written in the monomial basis of R_d. Then $P_{m,d}(g_{ij})$ is the determinant of the $t \times t$ minor of the matrix corresponding to columns indexed by $m_{1,d}, ..., m_{t,d}$.

Let $m_d = \max\{m : P_{m,d}(g_{ij}) \not\equiv 0\}$. Say $m_d = m_{d,1} \wedge \cdots \wedge m_{d,t}$. Let $U_d = \{g \in GL_n(k) : P_{m,d}(g_{i,j}) \neq 0\} \neq \varnothing$ and Zariski open.

We have that for every $g \in U_d$ $(LT(gI))_d = (m_{d,1}, ..., m_{d,t})$.

Let $J_d = (m_{d,1}, m_{d,2}, ..., m_{d,t})$. Also set $J = \bigoplus_{d\geq 0} J_d$.

Claim 1: J is an ideal.

To see this, it suffices to show that $R_1 J \subseteq J$, in fact enough to show $R_1 J_d \subseteq J_{d+1}$ for all d (since we already know it is a k-vector space.) We know $R_1 J_d \subseteq J_d$ (since J_d is an ideal). Note that U_d and U_{d+1} are nonempty Zariski open sets, so $U_d \cap U_{d+1} \neq \varnothing$. Hence there exists $g \in U_d \cap U_{d+1}$. Then

$$R_1 J_d = R_1(LT(gI))_d \subseteq (LT(gI))_{d+1} = J_{d+1}.$$

Set $U = \bigcap_{d\geq 0} U_d$.

Claim 2: U is a in fact a finite intersection. By HBT, J has a (unique) finite set of monomial generators. Let e=maximum of the degrees of these generators.

Claim 2 (refined): $\bigcap_{d\geq 0} U_d = \bigcap_{d=0}^{e} U_d$.

Need to show "$\supseteq$". Let $g \in \bigcap_{d=0}^{e} U_d$. Then $(LT(gI))_d = J_d$, for every $d \leq e$. Hence $LT(gI)$ contains all minimal generators of J. Thus $LT(gI) \supseteq J$.

Trick:

- $LT(gI) \supseteq J$
- $HF_{LT(gI)}(t) = HF_{gI}(t)$, meaning $\dim_k LT(gI)_d = \dim_k(gI)_d = \dim_k(I_d) = \dim_k(J_d)$ for each d.

Thus $LT(gI)_d = J_d$ for each d, and therefore $g \in U_d$ for all d means $g \in \bigcap_{d\geq 0} U_d$.

Claim 2 now tells us that $U = \bigcap_{d\geq 0} U_d = \bigcap_{d=0}^{e}$ is non-empty Zariski open.

□

Nov. 8, 2013

⋗ Gone for panel at augie.

NOV. 11, 2013

⋗ Email with paper.

⋗ The Eliahou-Kervaire Resolution.

⋗ We work in $R = k[x_1, ..., x_n]$ where $\text{char}(k) = 0$, R has the $\mathbb{Z}^n$-grading.

⋗ DEFN: For a monomial $m \in \mathbb{R}$, let $\max(m) = \max \text{supp}(m)$ and $\min(m) = \min \text{supp}(m)$.

⋗ **Theorem (Eliahou-Kervaire):** Let I be a Borel-fixed (monomial) ideal in R. (in char 0) Suppose $I =< m_1, ..., m_r >$ (min gens) and let M be the module of first syzygies on the generators of I. Then

(1) There exists a monomial ordering on R^r such that $LT(M)$ has linear resolution which is a direct sum of Koszul complexes.

(2) $\beta_{i,\alpha}(R^r/M) = \beta_{i,\alpha}(R^r/LT(M))$.

(3) $\beta_{i,j}(I) = \sum_{\deg(m_l)=j-1} \binom{\max(m_l)-1}{i}$ and $\beta_i(I) = \sum_{l=1}^{r} \binom{\max(m_l)-1}{i}$

(4) $\text{pd}(I) = \max\{\max(m_i) - 1 : 1 \le i \le r\}$ (i.e., the max index of a variable appearing in any non minimal gen of I -1.

(5) reg(I)= highest degree of a minimal monomial generator of I.

(6) $HS(R/I) = \dfrac{1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{\max(m_i)-1} (1 - x_j)}{\prod_{i=1}^{r}(1 - x_i)}$

Proof. Method 1: (Iterated mapping cone)
LEMMA: Order the minimal generators $m_1, ..., m_r$ of I in increasing order according to GrRevLex. Then $(m_1, ..., m_i) : m_{i+1} = (x_1, ..., x_{\max(m_{i+1})-1})$ (Proof: follows from exchange property of Borel-fixed ideals.)
Proof of E-K: Proceed by induction on r using

$$0 \to \frac{R}{(m_1, ..., m_{r-1}) : m_r} \to \frac{R}{(m_1, ..., m_{r-1})} \to \frac{R}{(m_1, ..., m_r)} \to 0$$

The first term is $K_.(x_1, ..., x_{\max(m_r)-1})$, the middle one is known by inductive hypothesis. The last one: form the mapping cone and observe that it is a minimal resolution of R/I.

Method 2: (using GBs)
Step 0:
LEMMA: Let $I =< m_1, ..., m_r >$ be Borel fixed. Every monomial $m \in I$ can be written uniquely as a product of the form

$$m = m_i \cdot m'$$

such that $\max(m_i) \le \min(m')$.

Step 1:
Set $u_i = \max(m_i)$. Order the generators $m_1, ..., m_r$ decreasingly by u_i, for generators with the same u_i, decreasingly by the power of x_{u_i} they contain.

Step 2:
Recall $M = \text{syz}(I)$ (i.e., 1st syzygy module of I). We build an element of M for each pair (j, u) so that $1 \le j \le r$ and $u < \max(m_j) = u_j$.

Consider $m = x_u \cdot m_j \in I$. (x_u is NOT as in the lemma.) The Lemma gives a different way to write $m = m_i \cdot m'$ (with $u_i = \max(m_i) \le \min(m')$). Thus $x_u m_j - m' m_i = 0$, so $x_u e_j - m' e_i \in M$, where $M \subseteq R^r$ with basis $\{e_1, ..., e_r\}$, $e_i \mapsto m_i$.

Claim: We must have $i > j$. Note: $\min(m') \leq \max(m) = \max(m_j) = u_j$. Also have $u_i \leq min(m')$. Therefore $u_i \leq u_j$. If $u_i < u_j$, then $i > j$ (by the way we ordered the m_i's) If $u_i = u_j$, then all equalities above, and so, in particular, $\min(m') = u_i = u_j$. This implies $\deg_{x_{u_j}}(m_i) < \deg_{x_{u_j}}(m_j) = \deg_{x_{n_j}}(m)$. Again by the ordering we put on the m_i's, we must have $i > j$.

Step 3:
Consider Position-over-Coefficient ordering on R^r with $e_1 > e_1 > ... > e_r$. This implies $LT(x_u e_j - m' e_i)$ because $i > j \implies e_j > e_i$.

Claim: $\mathcal{B} = \{x_u e_j - m' e_i : 1 \leq j \leq r, u < u_j\}$ is a GB of M with respect to PoC order. (i.e., $LT(M) =< x_u e_j : 1 \leq j \leq r, u < u_j >$ or equiv $LT(M) = \oplus_{j=1}^r < x_1, ..., x_{u_j - 1} > e_j$).

To see this, it's enough to show that we don't have $m'' e_j - m' e_i \in M$ such that neither $m'' e_j$ or $m' e_i$ are in $LT(\mathcal{B})$. □

Nov. 13, 2013

- Finishing up EK resolution proof. I is Borel-fixed, $M = \operatorname{syz}(I)$, $I =< m_1, ..., m_r >$. We proved $LT(M) =< x_u e_j : 1 \leq j \leq r, 1 \leq u \leq u_j = \max(m_j)\}$, i.e., $LT(M) = \bigoplus_{j=1}^r < x_1, ..., x_{u_j - 1} > e_j$. This module $LT(M)$ is resolved by $K_\bullet = \bigoplus_{j=1}^r K_\bullet < x_1, ..., x_{u_j - 1} >$. Every syzygy in K (S_{ij}) produces a syzygy r_{ij} for M. In fact, these r_{ij}s are minimal generators for syzygy modules of M. (They even form GBs for these syzygy modules.) In fact, $\beta_{i,j}(M) = \beta_{i,j}(LT(M)) = \sum_{i=1}^r \binom{u_j - 1}{i}$.
- Relationship between β_{ij}'s for I and $LT_<(I)$. Method for computing β_{ij}: None for I.

  In $LT(I)$, monomial ideal β_{ij} computed by LCM lattice.

  Polarization gives $Pol(LT_<(I))$.

  $$I \to\to LT_<(I) \to\to Pol(LT_<(I))$$

  For the first step, $\beta_{ij}(LT_<(I)) \geq \beta_{ij}(I)$. For the second step, $\beta_{ij}(LT_<(I)) = \beta_{ij}(Pol(LT_<(I)))$.
- We are looking for a "tighter relationship" between $\beta_{ij}(I)$ and $\beta_{ij}(gin(I))$.
- **Facts about gins:**
- **Theorem (Galligo, Bayer-Stillman):** If I is a homogeneous ideal, $<$ any monomial order, then $gin_<(I)$ is Borel-fixed.
- Defn: A sequence of elements $y_1, ..., y_d$ of R is a regular sequence on R/I if
  1. y_n is a nzd on R/I
  2. y_i is a nzd on $R/(I + (y_1, ..., y_{i-1}))$
- Prop: If $I =< m_1, ..., m_r >$ is Borel-fixed. $I \subseteq R = k[x_1, ..., x_n]$. Then there exists a maximal regular sequence on R/I of the form $x_n, x_{n-1}, ..., x_{p+1}$. In characteristic 0, p from above is the maximum index of any variable that appears in the support of the monomial generators $m_1, ..., m_r$, i.e., $p = \operatorname{pd}(R/I)$ as given by E.K.
- Fix $<$=GrRevLex on R.
- Lemma 1: If f is a homogeneous polynomial, then $x_n | f \iff x_n | LT(f)$.

  Proof. $\Rightarrow$ is obvious; $\Leftarrow$: monomials divisible by x_n if $<_{\text{GrRevLex}}$ monomials not divisible by x_n. $x_n | LT(f)$ implies any term in f is divisible by x_n. □
- Lemma 2: Let I be a homogeneous ideal. Then
  1. $LT(I + (x_n)) = LT(I) + (x_n)$. Furthermore, if $\{g_1, ..., g_t\}$ is a GB of I, then $\{g_1, ..., g_t, x_n\}$ is a GB for $I + (x_n)$.

2. $LT(I:(x_n)) = LT(I):(x_n)$. Furthermore, if $\{g_1,...,g_t\}$ is a GB of I, then $\{g_i/GCD(g_i,x_n)\}$ is a GB for $I:(x_n)$.

⋗ Corollary: x_n is a nzd on $R/I \iff x_n$ is a nzd on $R/LT(I)$.

Proof. Uses Lemma 2(2). □

⋗ **Theorem (Bayer-Stillman):** $x_n, x_{n-1}, ..., x_s$ form a regular sequence on $R/I \iff x_n, x_{n-1}, ..., x_s$ form a regular sequence on $R/LT(I)$.

Proof. Iterate the corollary. □

⋗ **Theorem (Bayer-Stillman):** k an infinite field, any characteristic. If I is a homogeneous ideal, then

$$\mathrm{pd}(R/I) = \mathrm{pd}(R/gin_{\text{GrRevLex}}(I))$$

and

$$\mathrm{reg}(R/I) = \mathrm{reg}(R/gin_{\text{GrRevLex}}(I)).$$

Nov. 15, 2013

⋗ Computer day.

Nov. 18, 2013

⋗ Deformations from GB theory[13]

⋗ Example: Let $I =< x^2 - y >\subseteq k[x,y] = R$. Under Lex with $x > y$, we have $LT(I) =< x^2 >$. (pictures of parabolas: $V(LT(I)) - -V(x^2 - \alpha y) - -V(x^2 - y)$, where $\alpha \in (0,1)$.) Then connect this family of parabolas / double line into a surface (a third dimension, t).

Let S be the surface connecting the parabolas $S = V(x^2 - ty)$. The cross-sections of S corresponding to plane $t = \alpha$ are given by the varieties $V(x^2 - \alpha y$.

Goal: Describe the family of varieties $V(x^2 - \alpha y)$ where $\alpha \in k$. It's best to look at the map $S \to B$, where $B = \mathbb{A}^1$ corresponds to the t-axis.

This map gives a ring homomorphism $k[t] \to k[x,y,t]/ < x^2 - ty >$. Therefore we can view $k[x,y,t]/ < x^2 - ty >$ as a $k[t]$-module.

"Good properties" of $k[x,y,t]/ < x^2 - ty >$ as a $k[t]$-module (i.e., flatness) ensures that the cross sections are "not too different" from each other.

⋗ Defn: The fiber of S at a point $B : P_\alpha$=the point $t = \alpha$ is the cross-sections through S by the plane $t = \alpha$. The coordinate ring S_α of the fiber at P_α is

$$S_\alpha = k[x,y,t]/ < x^2 - ty > \otimes_{k[t]} k[t]/ < t - \alpha >\cong k[x,y]/ < x^2 - \alpha y > .$$

⋗ We'll see that (in general):

$$S_\alpha \cong \begin{cases} k[x,y]/ < x^2 > & \alpha = 0 \\ k[x,y]/ < x^2 - y > & if \alpha \neq 0 \end{cases}.$$

(The fiber at $t = 0$ is $k[x,y]/LT(I)$ and the fibers at $t = \alpha \neq 0$ are isomorphic to $k[x,y]/I$.)

⋗ The general setup for the GB deformation.

⋗ **weight orders on monomials / non standard gradings**

[13]Reference for today: Chapter 15 of Eisenbud.

⋗ DEFN: Given a **weight vector** $w = (w_1, ..., w_n) \in \mathbb{Z}^n_{\geq 0}$ we define:

- the weight of a monomial $w(x^\alpha) := \sum_{i=1}^n \alpha_i w_i = \alpha \cdot w$.
- the *partial order on monomials* given by w is defined by $x^\alpha > x^\beta$ if $w(x^\alpha) > w(x^\beta)$.
- the *initial form of a polynomial* $f \in R$ is $in_w(f)$ =sum of terms of f that are maximal w.r.t. the partial order given by w.
- For example, if $f = x^2 - y$, if $w = (1,1)$, then $in_w(f) = x^2$. However, if $w = (1,2)$, then $in_w(f) = x^2 - y$. If $w = (1,5)$, then $in_w(f) = -y$.
- the *ideal of initial forms* of an ideal I is $in_w(I) =< in_w(f) : f \in I >$.

⋗ **Theorem (Bayer):** Let "<" be a monomial order on $R = k[x_1, ..., x_n]$ and let I be an ideal of R. Then there exists $w \in \mathbb{Z}^n_{\geq 0}$ such that $LT_<(I) = in_w(I)$. (i.e., weighted orders generalize total monomial orders). Also, any weighted order can be refined to a total order.

⋗ To construct the deformation: Let $\widetilde{R} = R[t] = k[x_1, ..., x_n, t]$. Fix $w \in \mathbb{Z}^n_{\geq 0}$ For a polynomial $f \in R$, define $\widetilde{f}(x_1, ..., x_n, t) = t^{w(f)} \cdot f(\frac{x_1}{t^{w_1}}, ..., \frac{x_n}{t^{w_n}})$, where $w(f)$ is the max weight of any monomial appearing in f. E.g., $f = x^2 - y$, $w = (1,1)$, $\widetilde{f}(x,y,t) = t^2(\frac{x}{t^2} - \frac{y}{t}x^2 - ty$. Now w.r.t. the new weight vector $w' = (w,1)$, then every monomial m in $\widetilde{f}$ has $w'(m) = w(f)$. Or:

$$\widetilde{f}(\underline{x}, t) = \sum_m c_m \cdot m \cdot t^{w(f) - w(m)}.$$

Given an ideal $I \subseteq R$, set $\widetilde{I} =< \widetilde{f}(x_1, ..., x_n, t) : f \in I >$.

Set $S = V(\widetilde{I})$.

⋗ **Theorem (Flat family):** For any ideal I and any weight vector w,

1. $\widetilde{R}/\widetilde{I}$ is free (hence flat) as a $k[t]$-module.
2. $\widetilde{R}/\widetilde{I} \otimes_{k[t]} k[t]/(t) \cong R/in_w(I)$
3. $\widetilde{R}/\widetilde{I} \otimes_{k[t]} k[t, t^{-1}] \cong R/I$.

NOV. 20, 2013

⋗ Connections between I and $LT(I)$.

⋗ **Theorem (Macaulay):** Let $I \subseteq R$ be an ideal. For any monomial order $>$ on R, the set B of all monomials not in $LT_>(I)$ forms a k-vector space basis for R/I. (Also for $R/LT_<(I)$).

Proof. - LINEAR INDEPENDENCE: Assume that $\sum_{b_i \in B} \lambda_i b_i = 0$ in R/I. This implies $f := \sum_{b_i \in B} \lambda_i b_i \in I$. Hence $LT(f) \in LT_>(I)$. But all monomials in f are from B which are monomials NOT in $LT(I)$. Therefore $f = 0$.

- SPANNING SET: To show $\text{span}_k(B) = R/I \iff \text{span}_k(B \cup I) = R$ (as k-vs.) Suppose $\text{span}_k(B \cup I) \subsetneq R$. Let $f \in R \backslash \text{span}_k(B \cup I)$ of minimal leading term. Consider $LT(f)$.
  1) If $LT(f) \notin LT_<(I)$, hence $LT(f) \in B$. Then $f - LT(f) \notin \text{span}_k(B \cup I)$. (Contradiction
  2) If $LT(f) \in LT_<(I)$. Then there exists $g \in I$ such that $LT(g) = LT(f)$. But then $f - g \notin \text{span}_k(B \cup I)$ and also LT strictly smaller than that of f. (Contradiction)

□

⋗ COROLLARY: If I is a homogeneous ideal, then $HF_{R/I}(i) = HF_{R/LT(I)}(i)$. (To see this: The left hand side is just $\dim_k(R/I)_i$ and the right side is just $\dim_k(R/LT(I))_i$. A basis for $(R/I)_i$ is given by elements of B of degree i. A basis for the right is given by the same monomials, hence the dimensions must be equal, giving the desired equality.)

- Recall the image from last time: surface S; $V(I)$ at $t=1$ and $V(LT(I))$ and $t=0$.
- We proposed a construction $\widetilde{I} = < \tilde{f}(\underline{x}, t) : f \in I >$ w.r.t. weight vector w.
- **Theorem (Flat family):** For any ideal I and any weight vector $w \in \mathbb{Z}^n_{\geq 0}$.

  (1) $\widetilde{R}/\widetilde{I}$ is free (hence flat) as a $k[t]$-module.

  (2) $\widetilde{R}/\widetilde{I} \otimes_{k[t]} k[t]/(t) \cong R/in_w(I)$

  (3) $\widetilde{R}/\widetilde{I} \otimes_{k[t]} k[t, t^{-1}] \cong (R/I)[t, t^{-1}]$.[14]

  Proof. □

- How to compute $\tilde{I}$:

  Method 1: Compute a GB $\{g_1, ..., g_s\}$ of I w.r.t. $<_w$. Then $\tilde{I} = < \tilde{g}_1, ..., \tilde{g}_s >$.

  Method 2: If $I = < f_1, ..., f_t >$, then $\tilde{I} = < \tilde{f}_1, ..., \tilde{f}_t > : (t^\infty)$.

Nov. 22, 2013

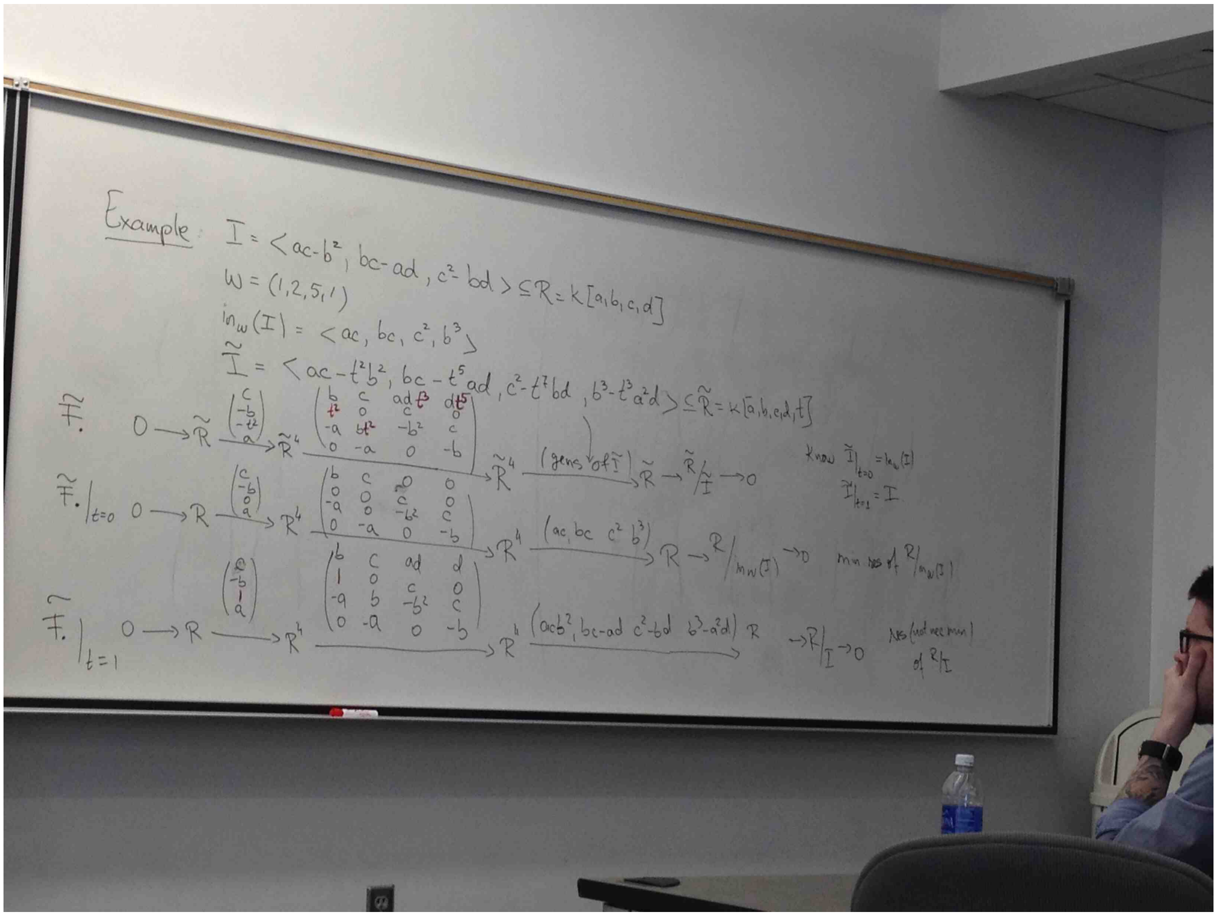
- **Theorem (Peeva, 2005):** Consecutive cancellations.

  Let I be a homogeneous ideal and $w \in \mathbb{Z}^n_{\geq 0}$. Then

  - $\beta_{ij}(R/I) \leq \beta_{ij}(R/in_w(I))$
  - furthermore, the $\beta_{ij}(R/I)$ can be obtained from $\beta_{ij}(R/in_w(I))$ by a sequence of consecutive cancellations (i.e., simultaneously decreasing β_{ij} and $\beta_{i+1,j}$ by 1 unit for some fixed i, j).

- Example:

[14] different from first time



Proof of theorem: Let $\tilde{R}$ and $\tilde{I}$ be like last time.

Fact 1: Thm about flat family implied $\tilde{R}/\tilde{I}$ is $k[t]$-free (in fact $\tilde{R}/I = \bigoplus_{b \in B} b \cdot k[t]$). For $\alpha \in k$, $t - \alpha$ is a nzd on $k[t]$ so $t - \alpha$ is also a nzd on $\tilde{R}/\tilde{I}$. (obviously, $t - \alpha$ is a nzd on $\tilde{R} = k[x_1, ..., x_n, t]$).

Fact 2: If M is an S-module and u is a n.z.d. both on S and on M and if $F.$ is a (minimal) free resolution of M over S, then $F. \otimes_S S/(u)$ is a free resolution of $M \otimes_S S/(u)$ over $S/(u)$.

Fact 3: We have two gradings on $\tilde{R}$: $\deg(x_i) = 1$, $\deg(t) = 0$ or $\deg(x_i) = w_i$, $\deg(t) = 1$. Note that $\tilde{I}$ is homogeneous w.r.t. both of the gradings. It follows that $\tilde{R}/\tilde{I}$ has a graded $\tilde{R}$-free resolution $\tilde{F}.$ that is

- minimal (i.e., entries in the differential maps are in $(x_1, ..., x_n, t)$)
- homogeneous w.r.t. both gradings.

Facts 1 & 2 ($S = \tilde{R}$, $M = \tilde{R}/\tilde{I}$) give $\tilde{F}. \otimes_{\tilde{R}} \tilde{R}/(t - \alpha)$ is a free resolution of $\tilde{R}/\tilde{I} \otimes_{\tilde{R}} \tilde{R}/(t-\alpha) \cong \tilde{R}/(\tilde{I} + (t - \alpha))$ over $\tilde{R}/(t - \alpha)$.

Let $\alpha = 0$. Then $\tilde{F}|_{t=0} = \tilde{F} \otimes_{\tilde{R}} \tilde{R}/(t)$ is a free resolution of $\tilde{R}/\tilde{I} \otimes \tilde{R}/(t) \cong R/in_w(I)$ over $\tilde{R}/(t) = R$ i.e., $\tilde{F}|_{t=0}$ is a minimal (entries of the differentials are now in $(x_1, ..., x_n)$) R-free resolution of $R/in_w(I)$.

Now let $\alpha = 1$. Then $\tilde{F}|_{t=1} = \tilde{F}. \otimes_{\tilde{R}} \tilde{R}/(t-1)$ is a resolution of $\tilde{R}/\tilde{I} \otimes \tilde{R}/(t-1) \cong R/I$ over $\tilde{R}/(t-1) \cong R$. (However, this might not be minimal.)

But then $\tilde{F}|_{t=1} = G. \oplus H.$. (where G is the minimal free resolution of R/I over R and H is a direct sum of trivial complexes.) Therefore $\beta_{ij}(\tilde{R}/\tilde{i}) = \beta_{ij}(R/I) \oplus \beta_{ij}(H.)$. □

⪢ Cor: If there are no possible cancellations, then the $\beta_{ij}(R/I) = \beta_{ij}(R/in_w(I))$. For example if $R/in_w(I)$ has a linear resolution.....

Nov. 25, 2013

⋗ What properties transfer between I and $in_w(I)$ (or $LT(I)$)?

1) $\dim R/I = \dim R/in_w(I) = \dim R/LT(I)$. (Krull dimension) (Pf: Fibers in a flat family have the same dimension; OR use Macaulay's Theorem.)
2) $HF_{R/I}(i) = HF_{R/in_w(I)}(i) = HF_{R/LT(I)}(i)$ (Macaulay)
3) $\operatorname{pd} R/in_w(I) \geq \operatorname{pd} R/I$ and $\operatorname{reg} R/in_w(I) \geq \operatorname{reg} R/I$ (Peeva's Thm).
4) If $R/in_w(I)$ is CM ($\dim R/in_w(I) = n - \operatorname{pd} R/in_w(I)$), then R/I is CM.
5) R/I is CM if and only if $R/gin_{GrRevLex}(I)$ is CM.

⋗ **The Gröbner Fan**

- There exist an uncountable number of monomial orderings. (e.g. every weight vector $w \in \mathbb{R}^n_{\geq 0}$ such that coord. of w are algebraically independent implies $>_w$ is a monomial ordering. $>_= >_{w/} \iff w = \lambda \cdot w'$ for $\lambda \in (0, \infty)$.)
- Fix an ideal I.

⋗ **Thm:** Let I be an ideal. There are only finitely many distinct initial ideals of I.

⋗ Prop: If I is an ideal and $LT_<(I) = LT_{<'}(I)$, then the reduced GBs of I w.r.t. $<$ and $<'$ are identical.

⋗ The theorem + Prop give there exists finitely many reduced GBs for a fixed ideal I (letting monomial order vary).

⋗ Cor: Let I be an ideal. There is a finite set that generates I and is a Gröbner basis for I w.r.t. any monomial ordering.

Proof. This set is the union of the finite set of distinct reduced GBs of I. □

⋗ Defn: The set from the Cor. is called a universal GB for I.

⋗ Fix I and a monomial ordering $>$. Recall, by a theorem of Bayer, there exists $w \in \mathbb{R}^n_{\geq 0}$ s/t $LT_>(I) = in_w(I)$.

⋗ Question: What are all the weight vectors w with the property that $LT_>(I) = in_w(I)$?

⋗ Defn: Let G be the reduced GB of I w.r.t. $>$. $G = \{g_1, ..., g_s\}$. $LT_>(I) =< LT(g_1), ..., LT(g_s)$. Say $g_i = u_i + \sum_j v_{ij}$.

$$C_>(I) := \{w \in \mathbb{R}^n_{\geq 0} : u_i \geq w v_{ij} \text{ for } 1 \leq i \leq s,\ u_i = LT(g_i),\ v_{ij} \text{ any other lower terms in } g\}$$

is the *cone* of weight vectors corresponding to I and the monomial ordering $>$.

⋗ Example: $I =< x^2 - y^3, x^3 - y^2 + x >$, $>= GrLex$ with $x > y$. Then $C_{GrLex}(I) =?$

A reduced GB for I w.r.t. GRLex is $G = \{y^3 - x^2, x^3 - y^2 + x\} = \{g_1, g_2\}$

Then

$$C_{GrLex}(I) = \{w = (w_1, w_2) \in \mathbb{R}^2_{\geq 0} : y^3 \geq_w x^2, x^3 \geq_w y^2, x^3 \geq_w x\}.$$

Recall, $y^3 \geq_w x^2 \iff (0,3) \cdot (w_1, w_2) \geq (2,0) \cdot (w_1, w_2). \iff (-2,3) \cdot (w_1, w_2) \geq 0 \iff -2w_1 + 3w_2 \geq 0$

Similarly, $x^3 \geq_w y^2 \iff (3,0) \cdot (w_1, w_2) \geq (0,2) \cdot (w_1, w_2). \iff (3,-2) \cdot (w_1, w_2) \geq 0 \iff 3w_1 - 2w_2 \geq 0$

Also $x^3 \geq_w x \iff (3,0) \cdot (w_1, w_2) \geq (1,0) \cdot (w_1, w_2). \iff (2,0) \cdot (w_1, w_2) \geq 0 \iff 2w_1 \geq 0$ (superflous since $(w_1, w_2) \in \mathbb{R}^2_{\geq 0}$.)

Picture: two lines with slope of 3/2 and one with slope 2/3, the cone is the region between these two lines.

⋗ Fact: $C_>(I)$ is always a (geometric) cone. i.e., closed under addition of vectors and closed under multiplication by non-negative scalars.

- Rmk: For any w in the interior of $C_>(I)$, we have $LT_>(I) = in_w(I)$. For w on the boundary of $C_>(I)$, $in_w(I)$ is NOT a monomial ideal.
- Fact: The two distinct Gröbner cones of I intersect along a common face of each.
- Example: The Grobner cones of $I =< x^2 - y^3, x^3 - y^2 + x >$: The cones are: regions bounded between the lines with slopes 6, 4, 3/2, 2/3, 1/4, 1/7. In the example, label these cones (1)-(7). (4) corresponds to C_{GrLex} with $x > y$, (1) corresponds to C_{Lex}, w/ $y > x$, and (7) corresponds to C_{Lex} with $x > y$.
- Defn: The Gröbner fan of I is the union of the Gröbner cones of I.

Dec. 2, 2013

- Nathan talk on: **The Gröbner Walk**
- Goal: Convert a RGB (reduced Gröbner basis) of I w.r.t. $<_1$ to a RGB of I w.r.t. $<_2$.
- Example:
  - $R = k[x, y]$, $I =< x^2 - y^3, x^3 - y^2 + x >$.
  - $G_0 =< y^3 - x^2, x^3 - y^2 + x >$ is a RGB of I w.r.t. grevlex, with $x > y$. (call this $<_1$)
  - Connect to RGB of I w.r.t. lex with $x > y$. (call this $<_2$)
  - $\omega_0 = (1, 1) \in C_{<_1}(I)$ and $\tau_0 = (1, 0) \in C_{<_2}(I)$ and $\alpha = (1, 2/3)$.
  - $in_\alpha(G_0) = \{y^3 - x^2, x^3\}$. Does this generated $in_\alpha(I)$? Yes, since for all $f \in I$, $LT_{<_1}(f) = LT_{<_1}(in_\alpha(f))$, by the definition of Grobner cone $C_{<_1}(I)$ and the fact that $\alpha \in C_{<_1}(I)$.
  - Use Buchberger's Algorithm to compute H_1, the RGB of $in_\alpha(I)$ w.r.t. $<_{2_\alpha}$. Note that $\alpha \in C_{<_{2_\alpha}}(I)$. ($f <_{2_\alpha} g \iff$ multideg$(f) \cdot \alpha <$ multideg$(g) \cdot \alpha$ if $= wf <_2 g$???).
  $$H_1 = \{x^2 - y^3, xy^3, y^6\}.$$
  - Examine $S = \{\overline{h}^{G_0} : h \in H_1\} = \{0, y^2 - x, xy^2 - y^3\}$. So $in_\alpha(h) = in_\alpha(h - \overline{h}^{G_0})$ for every $h \in H_1$. $LT_{<_{2_\alpha}}(I) =< LT_{<_{2_\alpha}}(h - \overline{h}^{G_0} : h \in H_1 >$.
  Now, $G' = \{h - \overline{h}^{G_0} : h \in H_1\} = \{x^2 - y^3, xy^3 - (y^2 - x), y^6 - (x_\alpha^2 - y^3)\}$ is a GB of I w.r.t. $<_{2_\alpha}$.
  Now $G_1 = \{x^2 - y^3, xy^3 - y^2 + x, y^6 - xy^2 + x^2\}$ is a RGB of I w.r.t. $<_{2_\alpha}$.
- Lemma 1: If G is a RGB of I w.r.t. $<_1$, then $in_\omega(G)$ is RGB of $in_\omega(I)$ w.r.t. $<_1$, for all $\omega \in \mathbb{R}^n_{\geq 0}$.
- Lemma 3: $C_{<_1}(I) = C_{<_2}(I)$ if and only if $LT_{<_1}(g) = LT_{<_2}(g)$ for all $g \in RGB$ of I w.r.t. $<_1$ (termination condition).
- Louigi: **Maximal Bettei Numbers**
- Setup: k is a field with characteristic 0. $A = k[x_1, ..., x_n]$ (with standard grading). I is monomial ideal. Then define $G(I)$ as the set of minimal monomial generators.
- Defn: I is *strongly stable* if $x_i m \in I$ implies $x_p m \in I$ for every $1 \leq p \leq i$.
- Notice: when char$(k) = 0$, Borel-fixed = strongly stable.
- Defn: L is *Lexicographic* if for every $j \in \mathbb{N}$, L_j is spanned by the first $\dim L_j$ monomials in the lexicographic order.
- Fact: Lexicographic ideal is strongly stable. $x_i m \in I$ for $1 \leq p \leq i$ $x_p < x_i$ implies $x_p m < x_i m$.
- **Theorem (Peeva):** J homogeneous $\beta_{i,i+j}(J) \subseteq \beta_{i,i+j}(inJ)$.
- **Theorem (Galligo, Bayer-Stillman):** If J is homogeneous, C is any monomial order, then $gin_< J$ is Borel-fixed.
- Fact: The Hilbert series of J and $gin_< J$ are the same: $HS_{gin_< J} = HS_{LT(gJ)} = HS_J$.

- We proved that there exists a Borel-fixed ideal I with the same HS of J such that $\beta_{i,i+j}(J) \leq \beta_{i,i+j}(I)$.
- Fact: By Macaulay's Theorem and Kruskal-Katona's Theorem, there exists a Lexicographic ideal L with the same Hilbert series of I.
- **Main Theorem:** Let J be a homogeneous ideal of A. If L is the lexicographic ideal with the same HS, then

$$\beta_{i,i+j}(J) \leq \beta_{i,i+j}(L)$$

for every i, j.

Proof. It suffices to show $\beta_{i,i+j}(I) \leq \beta_{i,i+j}(L$ when I is Borel-fixed.

Notation: m is a monomial, $\max(m) = \max\{i : x_i \text{ divides } m\}$. M a monomial ideal, then $M_J^{\#}$ is the set of all monomials in M_J. If $\mathcal{M}$ is a set of monomials, then $\omega_p(\mathcal{M}) = |\{m \in \mathcal{M} : \max(m) = p\}|$ and $\omega_{\leq p}(\mathcal{M}) = |\{m \in \mathcal{M} : \max(m) \leq p\}|$. In particular, $\omega_{\leq m}(M_J^{\#}) = \dim_k M_J$.

Theorem (Green): If I is strongly stable and L is lexicographic with the same HS as I then $\omega_{\leq p}(L_J^{\#}) \leq \omega_{\leq p}(I_J^{\#})$ for all p, J.

Lemma: I is Borel-fixed, then

$$\beta_{i,i+j} = |I_J^{\#}\binom{n-1}{i} - \sum_{p=1}^{n-1} \omega_{\leq p}(I_J^{\#})\binom{p-1}{i-1} - \sum_{p=1}^{n} \omega_{\leq p}(I_{j-1}^{\#})\binom{p-1}{i}.$$

Proof. By Eliahou-Kervaire,

$$\beta_{i,i+j}(I) = \sum_{m \in G(I)_J} \binom{\max(m)-1}{i} = \sum_{p=1}^{m} \omega_p(G(I)_J)\binom{p-1}{i}.$$

$G(I)_j = I_j^{\#} \backslash I_{j-1}^{\#} \cdot \{x_1, ..., x_n\}$. by the strongly stable property ... □

□

Dec. 4, 2013

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Dec. 6, 2013

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Dec. 2, 2013

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