Implicitization of tensor product surfaces via virtual projective resolutions

Alexandra Seceleanu [1] and Eliana Duarte [2]

[1]Department of Mathematics
University of Nebraska–Lincoln

[2]Max-Planck-Institute for Mathematics in the Sciences and
Otto-von-Guericke-Universität Magdeburg

July 11, 2019

Implicitization

A **tensor product surface** is the closed image of a rational map

$$\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$

The **implicitization problem** consists in finding the implicit equation of the image of λ given its parametrization.

Example

$$\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$

$$[s:t] \times [u:v] \mapsto [\underbrace{s^2v}_{X} : \underbrace{stv}_{Y} : \underbrace{stu}_{Z} : \underbrace{t^2u}_{W}]$$

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Algebraic statement of implicitization

- $\mathbb{K}$ a field
- $R = \mathbb{K}[s,t;u,v]$ is the $\mathbb{Z}^2$-graded coordinate ring for $\mathbb{P}^1 \times \mathbb{P}^1$.
- $S = \mathbb{K}[X,Y,Z,W]$ is the coordinate ring for $\mathbb{P}^3$.
- $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ defined by
  $[s:t] \times [u:v] \mapsto [p_0 : p_1 : p_2 : p_3]$

$$(X:Y:Z:W) \in \overline{\text{image}(\lambda)} = \Lambda \iff \begin{cases} Xp_3 - Wp_0 = 0 \\ Yp_3 - Wp_1 = 0 \\ Zp_3 - Wp_2 = 0 \end{cases}$$

Implicitization amounts to computing

$$I_\Lambda = \langle Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2 \rangle \cap S.$$

Resultants and residual resultants

Methods for implicitization

- **Gröbner bases** to find $\langle Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2 \rangle \cap S$ – are computationally expensive.

- **Resultants** to find image(λ) thought of as

  $$\{(X : Y : Z : W) \mid \mathbf{V}(Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2) \neq \emptyset\}$$

  – fail in the presence of base points, where the set of base points of λ is $\mathbf{V}(p_0, p_1, p_2, p_3) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

- **Residual resultants** – i.e. resultants that "remove" the base points are the focus of this talk

Resultant

The image of a parametric surface can be thought of as

$$\Lambda = \{(X : Y : Z : W) \mid \mathbf{V}(\underbrace{Xp_3 - Wp_0}_{f_0}, \underbrace{Yp_3 - Wp_1}_{f_1}, \underbrace{Zp_3 - Wp_2}_{f_2}) \neq \emptyset\}$$

Definition

The **resultant** $\mathbf{Res(f_0, f_1, f_2)}$ is a homogeneous polynomial in the coefficients of f_0, f_1, f_2 that vanishes whenever the system $f_i = 0$ has a solution.

Hence $\Lambda = \overline{\text{image}(\lambda)} \subseteq \mathbf{V}\left(\text{Res}(f_0, f_1, f_2)\right)$.

Residual resultant

Removing the base point locus $\mathbf{V}(G)$ corresponds to looking for common vanishing of the elements of the **residual ideal** $F : G$.

$$\Lambda = \overline{\{(X : Y : Z : W) \mid \mathbf{V}\left((f_0, f_1, f_2) : G\right) \neq \emptyset\}}$$

Definition (Busé–Elkladi-Mourrain)

The **residual resultant** $\mathbf{Res}_{\mathbf{G:deg(f_i)}}$ is a homogeneous polynomial in the coefficients of f_0, f_1, f_2 that vanishes whenever the system $f_i = 0$ has a solution **outside** $\mathbf{V}(G)$.

Theorem (Busé–Elkadi–Mourrain[2001], Busé[2001])

The residual resultant exists if G is (locally) a complete intersection ideal in a standard graded ring (in $\mathbb{P}^2$).

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Residual resultant for $\mathbb{P}^1 \times \mathbb{P}^1$

Let $G = \langle g_1, \ldots, g_n \rangle$, $F = \langle f_0, f_1, f_2 \rangle \subset R = \mathbb{K}[s,t;u,v]$ with
$\deg g_i = (k_j, l_j), 1 \leq j \leq n$, $\deg f_i = (a_i, b_i), 0 \leq i \leq 2$

Theorem (Duarte – S. following Busé–Elkadi–Mourrain ['01])

Suppose

- *$G \subseteq R$ is locally a complete intersection (e.g. the reduced ideal of a finite set of points) and*
- *$(a_i, b_i) \geq (k_{j_1} + 1, l_{j_1})$ and $(a_i, b_i) \geq (k_{j_2}, l_{j_2} + 1)$ for some j_1, j_2.*

Then there exists a polynomial $\mathrm{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}$ which satisfies

$$\mathrm{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}(f_0, f_1, f_2) = 0 \iff \mathbb{V}(F : G) \neq \emptyset$$

and it has multihomogeneous degree in the coefficients of f_k

$$\deg\left(\mathrm{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}(f_0, f_1, f_2)\right) = \deg(f_i, f_j) - \deg(G).$$

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Example

For the system below the base point locus in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathbf{V}(s, v)$.

$$\begin{cases} f_0 = (ua_{00} + va_{01})s + (sa_{02} + ta_{03})v \\ f_1 = (ua_{10} + va_{11})s + (sa_{12} + ta_{13})v \\ f_2 = (ua_{20} + va_{21})s + (sa_{22} + ta_{23})v \end{cases}$$

The system has a solution outside $\mathbf{V}(G)$, $G = \langle s, v \rangle$ whenever

$$\mathrm{Res}_{G,(1,1)}(f_0, f_1, f_2) = \begin{vmatrix} a_{00} & a_{01} + a_{02} & a_{03} \\ a_{10} & a_{11} + a_{12} & a_{13} \\ a_{20} & a_{21} + a_{22} & a_{23} \end{vmatrix} = 0$$

$$\deg\left(\mathrm{Res}_{G,(1,1)}(f_0, f_1, f_2)\right) = 1 = \underbrace{1 \cdot 1 + 1 \cdot 1}_{\deg(f_i, f_j)} - \underbrace{1}_{\deg(G)} \text{ in } a_{k*}.$$

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Virtual resolutions

Virtual resolutions in $\mathbb{K}[\mathbb{P}^1 \times \mathbb{P}^1]$

$R = \mathbb{K}[s,t;u,v]$ is $\mathbb{Z}^2$-graded with $\begin{cases} \deg(s) = \deg(t) = (1,0) \\ \deg(u) = \deg(v) = (0,1) \end{cases}$,

$B = \langle s,t \rangle \cap \langle u,v \rangle$ is geometrically irrelevant ideal of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition

A complex of free $\mathbb{Z}^2$-graded modules

$$\mathbf{F} : \quad F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_m \leftarrow 0,$$

is called a **virtual resolution** if its homology groups $H_i(\mathbf{F})$ are B-torsion modules for $i > 0$.

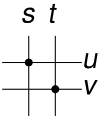
- M is B-torsion if $B^i M = 0$ for some i.
- Every free resolution is a virtual resolution.

Projective vs. virtual resolutions

Projective resolution / $\mathbb{K}[\mathbb{P}^n]$	Proj. res. / $\mathbb{K}[\mathbb{P}^1 \times \mathbb{P}^1]$	Virtual resolution / $\mathbb{K}[\mathbb{P}^1 \times \mathbb{P}^1]$
length $\leq \dim(\mathbb{P}^n) + 1$	length ≤ 4 ✗	length $\leq \dim(\mathbb{P}^1 \times \mathbb{P}^1) + 1$ ✓
if $\dim(Z) = 0$ $I_Z \leftarrow A^m \leftarrow A^{m-1} \leftarrow 0$ is a Hilbert-Burch resolution $(A = k[x, y, z])$	✗	if $\dim(Z) = 0$ I_Z has a Hilbert-Burch **virtual** resolution ✓ [Berkesch–Erman–Smith, 2017]

Projective vs. virtual resolution example

Example



$I_Z = \langle s, u \rangle \cap \langle t, v \rangle = \langle st, sv, tu, uv \rangle$

$$0 \longleftarrow I_Z \xleftarrow{\begin{pmatrix} st & tu & sv & uv \end{pmatrix}} R^4 \xleftarrow{\begin{pmatrix} -u & -v & 0 & 0 \\ s & 0 & 0 & -v \\ 0 & t & -u & 0 \\ 0 & 0 & s & t \end{pmatrix}} R^4 \xleftarrow{\begin{pmatrix} v \\ -u \\ -t \\ s \end{pmatrix}} R \longleftarrow 0$$

$$0 \longleftarrow I_Z \cap B = G \xleftarrow{\begin{pmatrix} tu & sv \end{pmatrix}} R^2 \xleftarrow{\begin{pmatrix} -sv \\ tu \end{pmatrix}} R^2 \longleftarrow 0$$

Towards a resolution of the residual ideal

Recall

- $f_0 = Xp_3 - Wp_0, f_1 = Yp_3 - Wp_1, f_2 = Zp_3 - Wp_2$
- $F = \langle f_0, f_1, f_2 \rangle \subseteq G = \langle g_1, \ldots, g_n \rangle$, $\mathbf{V}(G)$=base point locus
- $(f_0 \;\; f_1 \;\; f_2) = (g_1, \ldots, g_n) \underbrace{\begin{bmatrix} h_{10} & h_{11} & h_{12} \\ \vdots & \vdots & \vdots \\ h_{n0} & h_{n1} & h_{n2} \end{bmatrix}}_{\psi}$, $\quad h_{ij}$ polynomials
- may assume $0 \leftarrow G \leftarrow R^n \xleftarrow{\varphi} R^{n-1} \leftarrow 0$ is a Hilbert-Burch resolution.

Theorem (Buchsbaum-Eisenbud)

$$\sqrt{F : G} = \sqrt{\mathrm{Ann}(G/F)} = \sqrt{I_n \left[\; \varphi \;\vdots\; \psi \; \right]}.$$

A virtual projective resolution

Let $F = \langle f_0, f_1, f_2 \rangle \subseteq G = \langle g_1, \ldots, g_n \rangle$ as before.

Theorem (Duarte – S.)

If for every point $\mathfrak{p} \in \mathbb{P}^1 \times \mathbb{P}^1$ there is an equality $F_\mathfrak{p} = G_\mathfrak{p}$ then the **Eagon-Northcott complex** *of the map $\varphi \oplus \psi$ is a virtual projective resolution for $I_n([\varphi\ \psi])$.*

The hypothesis $F_\mathfrak{p} = G_\mathfrak{p}$ for all $p \in \mathbb{P}^1 \times \mathbb{P}^1$ holds when

- the generators of F are generic linear combinations of the generators of G (with coefficients = new variables)
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Thank you!

Eagon-Northcott complex

Example

Let

$$\alpha : \bigoplus_{i=1}^{n-1} R(-c_i, -d_i) \oplus R(-a, -b)^3 \to \bigoplus_{j=1}^{n} R(-e_j, -f_j)$$

and set $(c, d) = \sum_{i=1}^{n-1}(c_i, d_i)$ and $(e, f) = \sum_{j=1}^{n}(e_j, f_j)$. Then the graded shifts in the Eagon-Northcott complex of α are:

degree	shifts
1	$(a + c - e, b + d - f), (2a + c - e - c_i, 2b + d - f - d_i)$
	$(3a + c - e - c_i - c_j, 3b + d - f - d_i - d_j), i \neq j$
2	$(2a + c - e - e_j, 2b + d - f - f_j), (3a + c - e - c_i - e_j, 2b + d - f - d_i - f_j)$
3	$(3a + c - e - e_i - e_j, 3b + d - f - f_i - f_j)$