Math 918 – Computational Algebra Homework 1

Due Friday, September 6, before 5 pm.

Work in groups (of 1 or more). Each group should turn in ONE solution set, preferably by email in LaTeX format. Each group should solve at least max(*# of group members, 4*) *problems. Write down what group members contributed to which problems.*

*AG=alg. geom., CO=combinatorial, AP=applied, M2= computational, B=beginner, *=advanced.*

1 (B -Polynomials in one variable) (a) Prove that every ideal of $\mathbb{C}[x]$ is principal.

(b) Let $f \in \mathbb{C}[x]$. Describe $V((f))$ and $I(V(f))$.

2 (B -Properties of multideg) Let $f, g \in R$ be nonzero polynomials. Prove that:

(a) $\mathrm{multideg(fg)} = \mathrm{multideg(f)} + \mathrm{multideg(g)}$

(b) If $f + g \neq 0$,then $\mathrm{multideg(f + g)} \leq \max(\mathrm{multideg(f)}, \mathrm{multideg(g)})$; in addition, if $\mathrm{multideg(f)} \neq \mathrm{multideg(g)}$, then equality occurs.

(c) Suppose that $\mathrm{multideg(f)} = \mathrm{multideg(g)}$ and$f + g \neq 0$. Give examples to show that $\mathrm{multideg(f + g)}$ may or may not equal $\max(\mathrm{multideg(f)}, \mathrm{multideg(g)})$.

3 (AG) Let $I = (y - x^2, z - x^3) \subset R = k[x, y, z]$ be the ideal of the affine twisted cubic.

(a) Compute a Gröbner basis of I. Is it minimal? Reduced?

(b) For every $f \in R$, show that $f\%I$ is a polynomial in $\mathbb{R}[x]$

(c) Adapt the idea above to find $\mathbf{I}(V)$, where V is parametrized by (t, t^m, t^n), $m, n \geq 2$.

4 (B, CO - 2-dimensional monomial ideals) Let $I = (x^6, x^2y^3, xy^7) \subset k[x, y]$.

(a) In the (x, y)-plane, plot the set of exponent vectors (m, n) of monomials $x^m y^n \in I$

(b) If $f \in k[x, y]$, use the picture in part (a) to explain what terms can appear in the remainder $f\%I$.

5 (B -True or false?) If true, give a proof, otherwise give a counterexample.

(a) If G and G' are two Gröbner bases of the same ideal I with respect to the *same monomial order* then $f\%G = f\%G'$ for any polynomial f.

(b) $S(f, g)$ does not depend on the monomial order for any polynomials f, g.

(c) $(fg)\%G = ((f\%G)(g\%G))\%G$ for any polynomials f, g and any Gröbner basis G.

6 (Minimal Gröbner basis) (a) Show that a Gröbner basis G of I is minimal if and only if $LC(g) = 1$ for all $g \in G$ and $LT(G)$ is a minimal generating set of the monomial ideal $LT(I)$.

(b) Conclude that two minimal Gröbner bases of the same ideal have the same number of elements.

7 (*) Suppose you travel to a country whose currency has four coins valued 20, 24, 25, and 31. What is the largest amount of money which cannot be expressed by these coins?

Hint: knowing about Hilbert functions may help. It is ok to use Macaulay2 *for this problem. I am specifically requiring a computational commutative algebra solution for this problem.*

8 (*, M2) Find (experimentally) all the possible initial ideals with respect to all the possible monomial orders on $\mathbb{C}[x, y, z]$ for I, where I is the ideal of a set of 7 general points in $\mathbf{P}^2$ (use the code below to build this ideal).

Hint: every monomial order is given by a weight vector $\mathbf{w}$*. Can you reduce the possibilities to be considered for* $\mathbf{w}$*? Experiment with a large number of vectors to find several distinct* $LT_{\mathbf{w}}(I)$*. Explore the continuity of* $LT_{\mathbf{w}}(I)$ *as a function of* $\mathbf{w}$*. What seem to be the cut-off weights? I am not looking for a proof, just a well documented guess.*

The code computes the defining ideal of a set of 7 general points in $\mathbf{P}^2$ *out of their coordinates (stored in the matrix M) and its initial ideal w.r.t. the weights 3,2,1. The ideal of the 7 points is stored in the variable G and the initial ideal is stored in the variable inG.*

```
loadPackage "Points";
M = random(ZZ^3, ZZ^7);
R = QQ[x,y,z, Weights =>{3,2,1}];
(Q,inG,G) = points(M,R);
```