Connected Sums of Graded Artinian Gorenstein Algebras and the Lefschetz Properties

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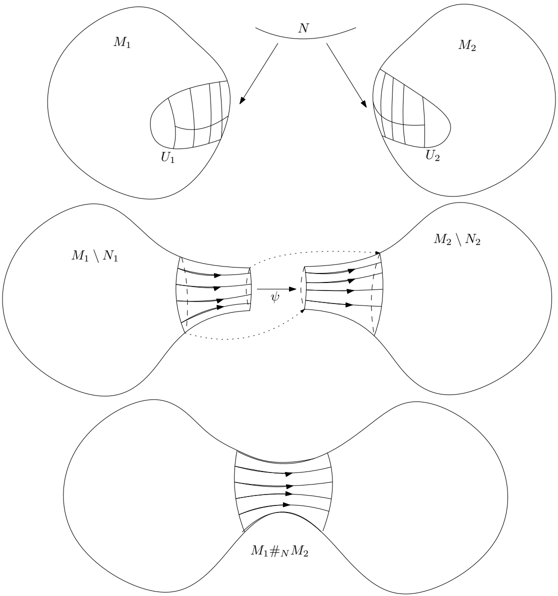
Motivation

- Cohomology rings of smooth complex projective varieties have two important properties:
  - Poincare duality $\Longleftrightarrow$ artinian Gorenstein (AG)
  - Lefschetz hyperplane theorem $\Longleftrightarrow$ strong Lefschetz property

Motivation

- Cohomology rings of smooth complex projective varieties have two important properties:
  - Poincare duality $\iff$ artinian Gorenstein (AG)
  - Lefschetz hyperplane theorem $\iff$ strong Lefschetz property
- The connected sum is a topological construction
  - Algebraic construction for connected sums due to Ananthnarayan-Avramov-Moore (2012)
  - Preserves the Gorenstein property.
  - Does it preserve the strong Lefschetz property ?

Connected Sum of Manifolds



Outline

1. Algebraic constructions

Fibered product

Setup – we focus on the *graded* setting:

- AG $\mathbb{F}$-algebras A, B (socle degree d), T (socle degree k)
- $\pi_A : A \to T$, $\pi_B : B \to T$ graded algebra homomorphisms

Fibered product

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- AG $\mathbb{F}$-algebras A, B (socle degree d), T (socle degree k)
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- **Fibered Product:** the pullback in the diagram

$$\begin{tikzcd} A \times_T B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & T \end{tikzcd}$$

given by $A \times_T B := \{(a, b) \in A \oplus B \mid \pi_A(a) = \pi_B(b)\}$.

More maps

- The given maps
$$\begin{tikzcd} A \arrow[dr, "\pi_A"] & \\ & T \\ B \arrow[ur, "\pi_B"] & \end{tikzcd}$$

- Dualize π_A, π_B via
$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}}(T,\mathbb{F}) \arrow[r, "\pi_A^*"] & \mathrm{Hom}_{\mathbb{F}}(A,\mathbb{F}) \\ T(k) \arrow[u, "\cong"] \arrow[r, "\iota_A(d)"'] & A(d) \arrow[u, "\cong"'] \end{tikzcd}$$
to give homomorphisms $\iota_A : T(-d+k) \to A$, $\iota_B : T(-d+k) \to B$

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to give homomorphisms $\iota_A : T(-d+k) \to A$, $\iota_B : T(-d+k) \to B$
- Overall
$$\begin{tikzcd} & A \arrow[dr, "\pi_A"] & \\ T(-d+k) \arrow[ur, "\iota_A"] \arrow[dr, "\iota_B"'] & & T \\ & B \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

Connected sum

- The complete diagram

$$\begin{tikzcd} & A \arrow[dr, "\pi_A"] & \\ T(-d+k) \arrow[ur, "\iota_A"] \arrow[dr, "\iota_B"'] & & T \\ & B \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

- **Connected Sum:** $A\#_T B := \dfrac{A \times_T B}{\{(\iota_A(t), \iota_B(t)) \mid t \in T\}}$

- Fact: $H^*(M_1 \#_N M_2) = H^*(M_1)\#_{\mathbb{F}} H^*(M_2)$.

Example

- $A = \mathbb{F}[x]/(x^5), B = \mathbb{F}[y]/(y^5), T = \mathbb{F}$
- Surjective Maps $A = \mathbb{F}[x]/(x^5) \xrightarrow{\pi_A} \mathbb{F} \xleftarrow{\pi_B} \mathbb{F}[y]/(y^5) = B$
- **Fibered Product:** $A \times_T B = \mathbb{F}[\underbrace{(x,0)}_{x}, \underbrace{(0,y)}_{y}]$

$$A \times_T B \cong \mathbb{F}[x,y]/(x^5, y^5, xy).$$

- Dual maps $\iota_A(1) = x^4, \iota_B(1) = y^4$
- **Connected Sum:** $(\iota_A(1), \iota_B(1)) = (x^4, y^4) \cong x^4 + y^4$

$$A\#_T B \cong \mathbb{F}[x,y]/(xy, x^4 + y^4)$$

- The connected sum is standard graded.

More Complicated Example

- $A = \mathbb{F}[x]/(x^5), B = \mathbb{F}[y]/(y^5), T = \mathbb{F}[z]/(z^2)$
- Surjective Maps $A \xrightarrow{\pi_A} T \xleftarrow{\pi_B} B, \ \ \pi_A(x) = \pi_B(y) = z$
- **Fibered Product:** $A \times_T B = \mathbb{F}[\underbrace{(x, y)}_{t}, \underbrace{(x^2, 0)}_{u}, \underbrace{(0, y^2)}_{v}]$

$$A \times_T B \cong \mathbb{F}[t, u, v]/(t^5, u^3, v^3, t^3u, tu^2, t^3v, tv^2, uv, t^2 - (u + v)).$$

- Dual maps $\iota_A(1) = x^3, \iota_B(1) = y^3$
- **Connected Sum:**$(\iota_A(1), \iota_B(1)) = (x^3, y^3) \cong t^3$

$$\begin{aligned} A\#_T B \cong &\mathbb{F}[t, u, v]/(t^5, u^3, v^3, t^3u, tu^2, t^3v, tv^2, uv, t^2 - (u + v), t^3) \\ \cong &\mathbb{F}[t, u]/(t^3, t^2u - u^2) \end{aligned}$$

- The connected sum is not standard graded.

Basic Properties

Theorem (Ananthnarayan-Avramov-Moore '12)

If A, B are AG algebras of socle degree d and T is also AG then

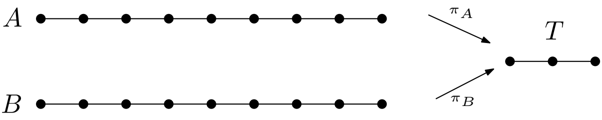
- *$A \times_T B$ is graded Artinian level algebra of socle degree d and socle dimension 2.*

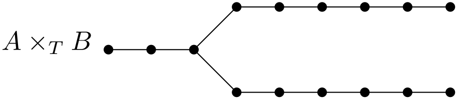
$$H(A \times_T B) = H(A) + H(B) - H(T)$$

- *$A \#_T B$ is a graded Artinian Gorenstein algebra of socle degree d.*

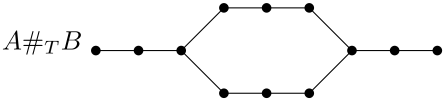
$$H(A \#_T B) = H(A) + H(B) - H(T) - H(T)(d-k)$$

Pictures of Hilbert Functions





$$H(A \times_T B) = H(A) + H(B) - H(T)$$



$$H(A\#_T B) = H(A) + H(B) - H(T) - H(T(d-k))$$

Outline

1. Algebraic constructions

2. The Lefschetz properties

Strong Lefschetz Properties

Definition

$A = \bigoplus_{i=0}^{d} A_i$ has the strong Lefschetz property if there exists $\ell \in A_1$ such that $\times \ell^{d-2i} \colon A_i \xrightarrow{\cong} A_{d-i}, \;\; 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.

Hard Lefschetz Theorem

If M is a complex projective variety of (complex) dimension d, and $[\omega] \in H^2(M, \mathbb{C})$ is the class of a hyperplane section $H \cap M$, then

$$\smile [\omega]^{d-2i} \colon H^{2i}(M, \mathbb{C}) \xrightarrow{\cong} H^{2d-2i}(M, \mathbb{C}).$$

In particular, the (even) cohomology ring $H^{2*}(M, \mathbb{C})$ has the strong Lefschetz property.

Strong Lefschetz Property and Connected Sum

Question

If A, B, and T satisfy SLP, must their connected sum $A\#_T B$ also satisfy SLP??

Strong Lefschetz Property and Connected Sum

Question

If A, B, and T satisfy SLP, must their connected sum $A\#_T B$ also satisfy SLP?? *Not always!!*

- $A = \mathbb{F}[x]/(x^5), B = \mathbb{F}[y]/(y^5), T = \mathbb{F}[z]/(z^2)$
- Surjective Maps $A \xrightarrow{\pi_A} T \xleftarrow{\pi_B} B, \;\; \pi_A(x) = \pi_B(y) = z$
- **Connected Sum:**
  $t = (x, y), \deg(t) = 1, u = (x^2, 0), \deg(u) = 2$

$$A\#_T B \cong \mathbb{F}[t, u]/(t^3, t^2u - u^2)$$

$$\times t^4 : \; (A\#_T B)_0 \to (A\#_T B)_4 \quad (\text{zero map})\,.$$

SLP for Connected Sum over $T = \mathbb{F}$

Theorem (Babson-Nevo '10, Watanabe et al. '13, Iarrobino-McDaniel-S. '19)

If A and B have the SLP, then $A\#_{\mathbb{F}}B$ also has the SLP. If A and B have the standard grading, then the converse holds too.

Conjecture (Iarrobino-McDaniel-S. '19)

Suppose that A, B, and $C = A\#_T B$ have the standard grading. Then A and B have SLP $\Rightarrow$ C has SLP.

Weak Lefschetz Properties

Definition

$A = \bigoplus_{i=0}^{d} A_i$ has the weak Lefschetz property if there exists $\ell \in A_1$ such that the following maps are all either injective or surjective

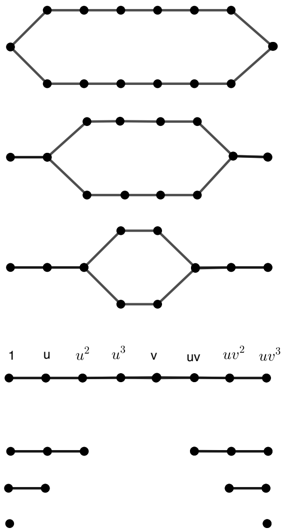
$$\times\ell\colon A_i \longrightarrow A_{i+1}, \quad 0 \leq i \leq d.$$

Theorem (Iarrobino-McDaniel-S. '19)

If A, B have SLP and the standard grading and if $\operatorname{soc.deg}(T) < \frac{1}{2}\operatorname{soc.deg}(A, B)$ then $C = A\#_T B$ has WLP.

WLP example

Hilbert function of $A\#_T B$, $d = 8$



Example

- $A = \mathbb{F}[x]/(x^d), B = \mathbb{F}[y]/(y^d),$
  $T = \mathbb{F}[z]/(z^t)$
- $A = \mathbb{F}[x]/(x^d) \xrightarrow{\pi_A} T = \mathbb{F}[z]/(z^t), x \mapsto z$
  $B = \mathbb{F}[y]/(y^d) \xrightarrow{\pi_B} T = \mathbb{F}[z]/(z^t), y \mapsto z$
- **Connected Sum:**

$$A\#_T B = \mathbb{F}[u,v]/(u^{d-t}, v^2 - u^t v)$$

  where
  $u = (x, y), \deg(u) = 1,$
  $v = (x^t, 0), \deg(v) = t.$
- $A\#_T B$ has WLP $\iff$ $t \neq d/2$.

Thank you!