# *Math 918 – Computational Algebra* Homework 2

**Due (tentatively) Friday, September 27.**

*Work in groups (of 1 or more). Each group should turn in ONE solution set, preferably by email in LaTeX format. Each group should solve at least* max(**# of group members, 2)** *problems.*

AG=alg. geom., CO=comb., AP=applied, M2= computational, B=beginner, *=advanced.

**1 (B - Hilbert Basis Theorem)** Give a proof by contradiction of the Hilbert Basis Theorem (this is a different proof than the one given in class). Proceed by induction on the number of variables. Let $I$ be an ideal and assume that $I$ is not finitely generated. Inductively construct a sequence $f_1, f_2, \dots$ of elements of $I$ such that $f_{i+1}$ has minimal degree in $I \setminus J_i$, where $J_i$ is the ideal generated by $f_1, \dots, f_i$. Use the inductive hypothesis to derive a contradiction.

**2 (B - Discard criterion for S-polynomials)** Let $G = \{g_1, \dots, g_s\}$ be a set of polynomials (not assumed to be a Gröbner basis ). Let $f, g \in G$ be such that the leading monomials of $f$ and $g$ are coprime, i.e.

$$LCM(LM(f), LM(g)) = LM(f) \cdot LM(g).$$

Show that there are polynomials $a_i$ such that $S(f,g) = \sum_{i=1}^{s} a_i g_i$ and $\mathrm{multideg(a_ig_i)} \leq \mathrm{multideg(S(f,g))}$ whenever $a_i \neq 0$.

**3 (B - Construction of reduced Gröbner bases)** Prove that the result of the following procedure is a reduced Gröbner basis : start with any Gröbner basis $G = \{g_1, \dots, g_s\}$ of $I$.

- **Step 1:** for $i$ from 1 to $s$ do: if $LT(g_i) \in \langle LT(G \setminus \{g_i\})\rangle$, then set $G = G \setminus \{g_i\}$;
- **Step 2:** Suppose that at the end of Step 1 you have a new $G = \{g_1', \dots, g_t'\}$.
  For $i$ from 1 to $t$ do : $G = (G \setminus \{g_i\}) \cup \{g_i \% (G \setminus \{g_i\})\}$.

**4 (B - LCM criterion for Gröbner bases)** Fix a monomial order $<$. Show that a finite set $G = \{g_i, \dots, g_s\}$ is a Gröbner basis of $I = \langle g_i, \dots, g_s\rangle$ if and only if for every $f, h \in G$, $S(g,h) = \sum_{i=1}^{s} a_i \cdot g_i$, where $a_i \neq 0$ implies $LT(a_i \cdot g_i) < LCM(LM(f), LM(h))$.

**5 (B - Standard representation criterion for Gröbner bases)** Fix a monomial order $>$. Show that a finite set $G = \{g_i, \dots, g_s\}$ is a Gröbner basis of $I = \langle g_i, \dots, g_s\rangle$ if and only if every $f \in I$, can be written as $f = \sum_{i=1}^{s} a_i \cdot g_i$, where $LT(f) = \max_{>}\{LT(a_i)LT(g_i) | a_i \neq 0\}$.

**6 (CO, AP - Binomial ideals)** A polynomial is called a binomial if it has at most two terms. An ideal is called a binomial ideal if it is generated by binomials. Given any monomial order $>$ and an ideal $I$, show that the following are equivalent:

(a) $I$ is a binomial ideal.

(b) $I$ has a binomial Gröbner basis , that is, a Gröbner basis consisting of binomials

(c) the normal form with respect to $I$ of any monomial is a term (a term is a constant times a monomial).

**7 (*, AG - Ideals of minors)** Consider the ideal generated by the two by two minors of the matrix

$$A = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$$

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(a) Show that the minors form a Gröbner basis with respect to the lexicographic order with $x_1 > x_2 > \ldots > x_n > y_1 > y_2 > \ldots > y_n$?

(b) Can you think of a different term order for which your proof holds?

**8 (* - Artinian test)** Let $>$ be a fixed monomial order on a polynomail ring $R = k[x_1, \ldots, x_n]$. Let $I \subset R$ be an ideal and let $G$ be a Gröbner basis of $I$ with respect to $>$. Show that the following are equivalent:

(a) $\dim_k(R/I) < \infty$,

(b) $\forall\ 1 \le i \le n, \exists\ n_i \in \mathbb{Z}_{\ge 0}$ such that $x_i^{n_i}$ is a leading monomial of an element of $G$.