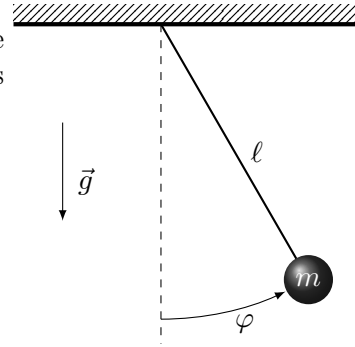


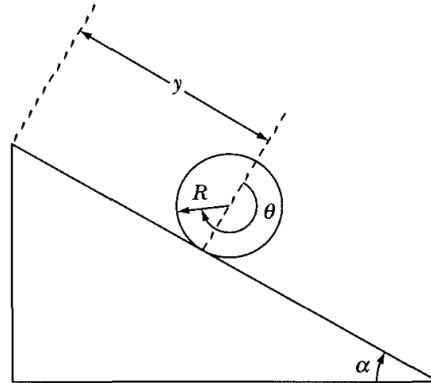
1. **Ideal pendulum**

Calculate the tension in the string using the method of Lagrange multipliers. The constraint is that the bead is always at  $\vec{r} = \ell \hat{\rho}$ , ergo, the function expressing this is  $f(\rho) = \rho - \ell = 0$ .



2. **Cylinder rolling over an inclined plane** [Marion ex. 7.5]

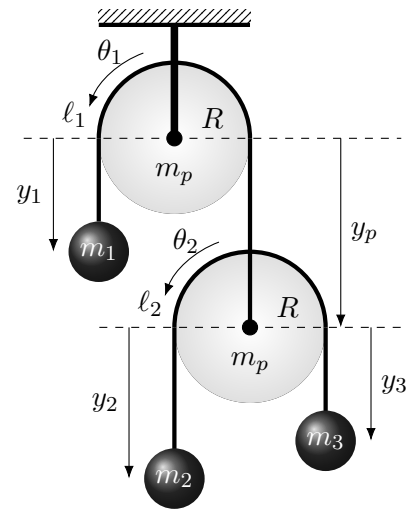
- Write the equations of motion,
- find the angular acceleration,
- and the constraint forces.



3. **Double Atwood machine** [Marion ex. 7.8 and 7-37]

Use the frame of reference shown in the figure. For this system of pulleys, determine:

- the equations of motion,
- and the tensions in both strings using the method of Lagrange multipliers.



Results:

$$Q_1 = \frac{g(32m_1m_2m_3 + 8m_1m_2m_p + 20m_1m_3m_p + 4m_1m_p^2 + 8m_2m_3m_p + 2m_2m_p^2 + 4m_3m_p^2 + m_p^3)}{4m_1m_2 + 4m_1m_3 + 2m_1m_p + 16m_2m_3 + 6m_2m_p + 14m_3m_p + 3m_p^2}$$

$$Q_2 = \frac{gm_3 \cdot (16m_1m_2 + 6m_1m_p + 4m_2m_p - m_p^2)}{4m_1m_2 + 4m_1m_3 + 2m_1m_p + 16m_2m_3 + 6m_2m_p + 14m_3m_p + 3m_p^2}$$

4. **Weights linked by a rope** [Taylor 7.50]

A particle of mass  $m$  that lies over a table is linked to another particle of mass  $M$  by a rope of length  $l$  that passes through a hole in the table, without friction. The second one is hanging vertically at a distance  $y = \ell - \rho$  from the table, where  $\rho$  is the distance between the first particle and the hole.

- Assuming that  $\theta$  is not necessarily constant, find the Lagrange equations for  $\rho$  and  $y$ . Result:  

$$-Mg + M\ddot{y} + \lambda_1 = 0 \quad \lambda_1 - m\rho\dot{\theta}^2 + m\ddot{\rho} = 0$$

- Solve the system for  $\rho, y$  and the lagrange multiplier  $\lambda_1$  finding the tensions exerted upon both particles.

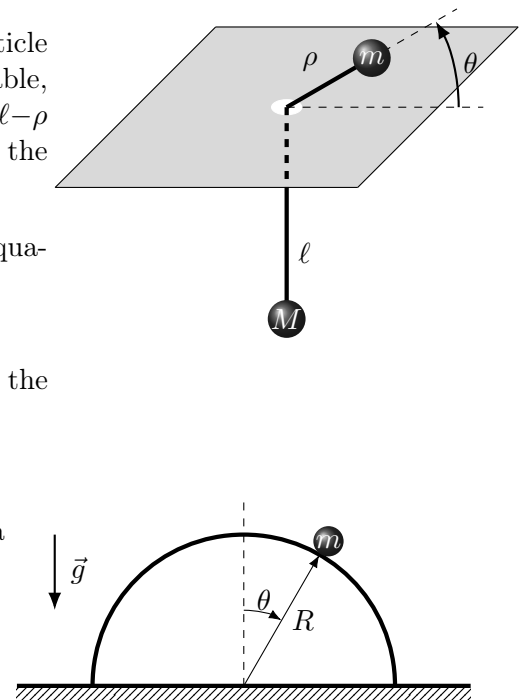
$$\text{Result: } Q_\rho = \frac{Mm(g + \rho\dot{\theta}^2)}{M + m}$$

5. **Particle sliding over a semi-sphere** [Marion ex. 7.10]

The particle of mass  $m$ , considered as a point particle, slides over a semi-sphere of radius  $R$  without friction.

- Find the constraint force.  
 Result:  $F_\rho^{\text{constraint}} = m(-R\dot{\theta}^2 + g \cos(\theta))$

- Calculate the angle at which the particle leaves the semi-sphere.  
 Result:  $\approx 48.19^\circ$



To find the angle at which the particle leaves the semi-sphere, you must solve the differential equation you'll get after working out the rather tough constraint force, which will be  $\ddot{\theta} = \frac{g \sin(\theta)}{R}$ . This expression can be integrated for the particle's trajectory. This is easier after applying the chain rule to insert derivatives with respect to  $\theta$ .

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\theta}{dt} \frac{d\dot{\theta}}{d\theta} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$$

Since the particle starts at  $\theta(t=0) = 0$  with  $\dot{\theta}(t=0) = 0$ :

$$\begin{aligned}\ddot{\theta} &= \dot{\theta} \frac{d\dot{\theta}}{d\theta} = \frac{g}{R} \sin(\theta) \\ \dot{\theta} d\dot{\theta} &= \frac{g}{R} \sin(\theta) d\theta \\ \int_0^{\dot{\theta}_{\text{leaving}}} \dot{\theta} d\dot{\theta} &= \frac{g}{R} \int_0^{\theta_{\text{leaving}}} \sin \theta d\theta \\ \frac{\dot{\theta}^2}{2} \Big|_0^{\dot{\theta}_{\text{leaving}}} &= \frac{g}{R} (-\cos \theta) \Big|_0^{\theta_{\text{leaving}}} \\ \frac{\dot{\theta}_{\text{leaving}}^2}{2} &= \frac{g}{R} (-\cos(\theta_{\text{leaving}}) + 1)\end{aligned}$$

After this, you have to substitute  $\dot{\theta}^2$  in an expression for  $F_{\rho}^{\text{constraint}}$ , that must vanish at the moment of leaving the semi-sphere.