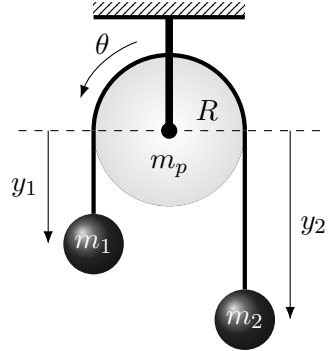


## CONSTRAINTS

Exercises marked with (\*) have extra difficulty, don't hesitate to ask for help.

## 1. Atwood machine

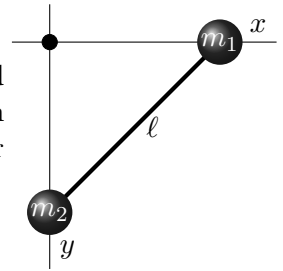
The figure shows a string of length  $\ell$  and a pulley of radius  $R_p$  and mass  $m_p$ . Find the acceleration of the masses attached at each end of the string.



- The string is inextensible, so it establishes a relation between  $y_1$  and  $y_2$ . Write the equation for this constraint.
- If the string slides over the pulley without friction, the pulley will not move. Write the Euler-Lagrange equation for  $y_1$  using the constraint from the previous item and write the masses' acceleration.
- Usually, the string won't slide and the pulley will rotate. This constraint adds a relation between  $\theta$  and the displacement of the string. Using that constraint, write the pulley's rotational kinetic energy in terms of  $\dot{y}_1$ , modeling the pulley as an homogeneous cylinder with a moment of inertia of  $(m/2)R^2$ .
- Use the Euler-Lagrange equation for  $y_1$  to write the masses' accelerations.

## 2. Pendulum with sliding and coupled masses

Two weights of masses  $m_1$  and  $m_2$  are linked together by a rigid bar of length  $\ell$  and negligible mass.  $m_1$  can slide over an horizontal axis and  $m_2$  over a vertical axis, both without friction. The bar sets a constraint between the coordinates that define their positions,  $x$  and  $y$ .

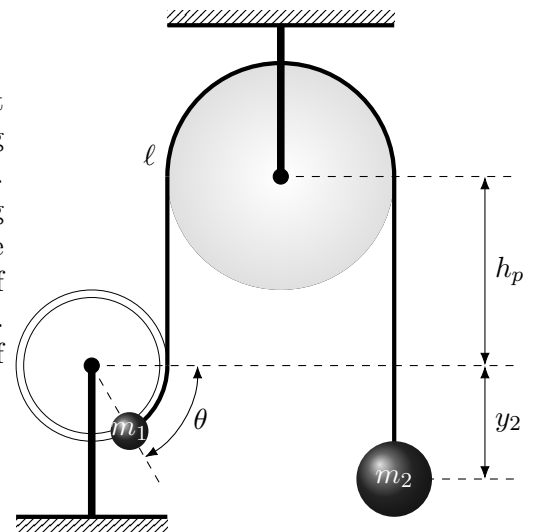


- Use the constraint equation to express both positions only in terms of  $y$ .
- Calculate the acceleration of  $m_2$ .

$$\text{Result: } \ddot{y} = \frac{-\ell^2 m_1 y \dot{y}^2 + g m_2 (\ell^2 - y^2)^2}{\ell^4 m_2 + \ell^2 m_1 y^2 - 2\ell^2 m_2 y^2 - m_1 y^4 + m_2 y^4}$$

## 3. Ring and pulley

A weight of mass  $m_2$  hangs from the free end of the string that passes over the pulley of radius  $R_p$  and mass  $m_p$ . The string moves without slipping, its length is  $\ell$  and its mass is negligible. The other end of the string is attached to mass  $m_1$ , fixed to a ring of mass  $m_r$ , coiling over the ring by an angle  $\theta$ . The center of the pulley is at a height  $h_p$  over the center of the ring. The radius of the ring is  $R_r$  and rotates freely with a moment of inertia  $m_r R_r^2$ . The generalized coordinates are  $y_2$  and  $\theta$ , constrained because of the length  $\ell$ .



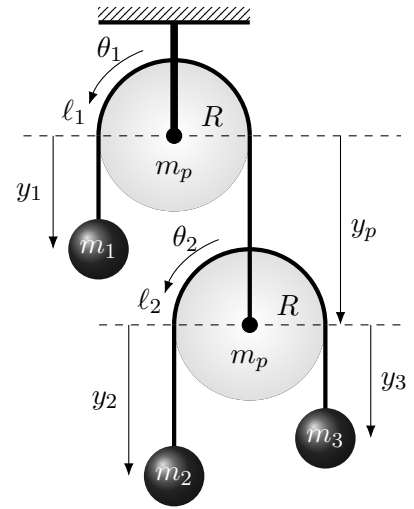
- Write the position for each object in terms of the generalized coordinates, in a frame of reference with origin at the center of the ring.
- Find an expression for the constraint and use it to rewrite the positions in terms of  $\theta$ . Verify your solution by checking that a variation in  $\theta$  approaching its zero value implies that the other object moves down.

- (c) Write the Euler-Lagrange equation.

$$\text{Result: } R_r^2 m_1 \ddot{\theta} + R_r^2 m_r \ddot{\theta} + R_r g m_1 \cos(\theta) + R_p^2 m_2 \ddot{\theta} + \frac{R_p^2 m_p \ddot{\theta}}{2} - R_p g m_2 = 0$$

## 4. Double Atwood machine [Marion ex. 7.8]

- (a) Write the positions for the three hanging objects and that of the lower pulley in terms of the generalized coordinates shown in the figure:  $y_i$  with  $i = 1, 2, 3, p$ .
- (b) Write the equations for the constraints provided by both strings.
- (c) Use the constraint equations to express all positions in terms of  $y_1$  and  $y_2$ .
- (d) The strings don't slide over the pulleys. This is an extra constraint that must be used to relate  $y_i$  and  $\theta_i$ .



- (e) Calculate the kinetic and potential energies.

- (f) Find both Euler-Lagrange equations.

Results:

$$-gm_1 + gm_2 + gm_3 + gm_p + m_1 \ddot{y}_1 + m_2 \ddot{y}_1 - m_2 \ddot{y}_2 + m_3 \ddot{y}_1 + m_3 \ddot{y}_2 + \frac{3m_p \ddot{y}_1}{2} = 0$$

$$-gm_2 + gm_3 - m_2 \ddot{y}_1 + m_2 \ddot{y}_2 + m_3 \ddot{y}_1 + m_3 \ddot{y}_2 + \frac{m_p \ddot{y}_2}{2} = 0$$

- (g) Solve for the generalized accelerations.

Results:

$$\ddot{y}_1 = \frac{2g(2m_1 m_2 + 2m_1 m_3 + m_1 m_p - 8m_2 m_3 - 3m_2 m_p - 3m_3 m_p - m_p^2)}{4m_1 m_2 + 4m_1 m_3 + 2m_1 m_p + 16m_2 m_3 + 8m_2 m_p + 8m_3 m_p + 3m_p^2}$$

$$\ddot{y}_2 = \frac{2g(4m_1 + m_p)(m_2 - m_3)}{4m_1 m_2 + 4m_1 m_3 + 2m_1 m_p + 16m_2 m_3 + 8m_2 m_p + 8m_3 m_p + 3m_p^2}$$

- (h) Write the accelerations of the three masses.

Results:

$$\ddot{\vec{r}}_1 = \ddot{y}_1 (-\hat{e}_y)$$

$$\ddot{\vec{r}}_2 = -\frac{2g(2m_1 m_2 - 6m_1 m_3 - m_1 m_p + 8m_2 m_3 + 4m_2 m_p + 2m_3 m_p + m_p^2)}{4m_1 m_2 + 4m_1 m_3 + 2m_1 m_p + 16m_2 m_3 + 8m_2 m_p + 8m_3 m_p + 3m_p^2} \hat{e}_y$$

$$\ddot{\vec{r}}_3 = -\frac{2g(-6m_1 m_2 + 2m_1 m_3 - m_1 m_p + 8m_2 m_3 + 2m_2 m_p + 4m_3 m_p + m_p^2)}{4m_1 m_2 + 4m_1 m_3 + 2m_1 m_p + 16m_2 m_3 + 8m_2 m_p + 8m_3 m_p + 3m_p^2} \hat{e}_y$$