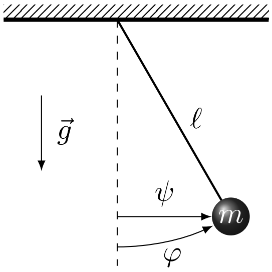
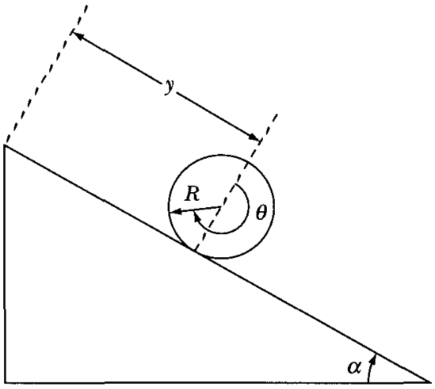
Constraint forces | Lagrange multipliers

1. **Ideal pendulum**
   Calculate the tension in the string using the method of Lagrange multipliers. The constraint is that the bead is always at $\vec{r} = \ell\hat{\rho}$, ergo, the function expressing this is $f(\rho) = \rho - \ell = 0$.



2. **Cylinder rolling over an inclined plane** [Marion ex. 7.5]



   (a) Write the equations of motion,

   (b) find the angular acceleration,

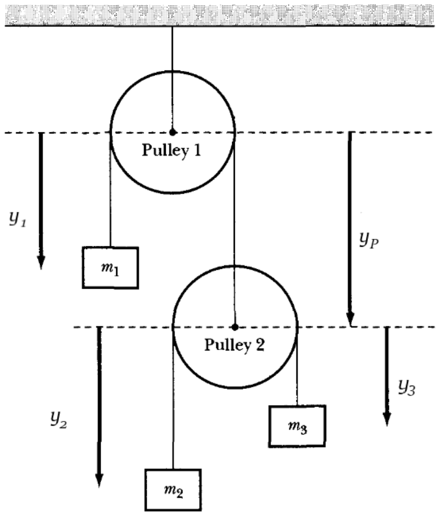
   (c) and the constraint forces.

3. **Double Atwood machine** [Marion ex. 7.8 and 7-37]
   Use the frame of reference shown in the figure. For this system of pulleys, determine:

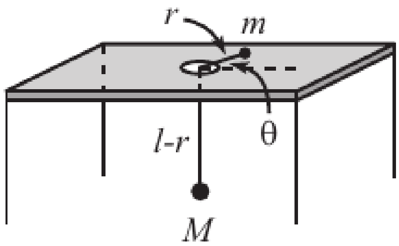
   (a) the equations of motion,

   (b) and the tensions in both strings using the method of Lagrange multipliers.

   Results:
   $Q_1 = \frac{g\left(32m_1m_2m_3+8m_1m_2m_p+20m_1m_3m_p+4m_1m_p^2+8m_2m_3m_p+2m_2m_p^2+4m_3m_p^2+m_p^3\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+6m_2m_p+14m_3m_p+3m_p^2}$
   $Q_2 = \frac{gm_3\cdot\left(16m_1m_2+6m_1m_p+4m_2m_p-m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+6m_2m_p+14m_3m_p+3m_p^2}$



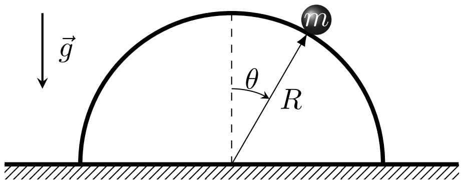
4. **Weights linked by a rope** [Taylor 7.50]
   A particle of mass m that lies over a table is linked to another particle of mass M by a rope of length l that passes through a hole in the table, without friction. The second one is hanging vertically at a distance $y = \ell - \rho$ from the table, where ρ is the distance between the first particle and the hole.



   (a) Assuming that θ is not necessarily constant, find the Lagrange equations for ρ and y. Result:
   $-Mg + M\ddot{y} + \lambda_1 = 0 \qquad \lambda_1 - m\rho\dot{\theta}^2 + m\ddot{\rho} = 0$

   (b) Solve the system for ρ, y and the lagrange multiplier λ_1 finding the tensions exerted upon both particles.
   Result: $Q_\rho = \frac{Mm\left(g+\rho\dot{\theta}^2\right)}{M+m}$

5. **Particle slidding over a semi-sphere** [Marion ex. 7.10]
   The particle of mass m, considered as a point particle, slides over a semi-sphere of radius R without friction.



   (a) Find the constraint force.
   Result: $F_\rho^{\text{constraint}} = m\left(-R\dot{\theta}^2 + g\cos(\theta)\right)$

   (b) Calculate the angle at which the particle leaves the semi-sphere.
   Result: $\approx 48.19°$

To find the angle at which the particle leaves the semi-sphere, you must solve the differential equation you'll get after working out the rather tough constraint force, which will be $\ddot{\theta} = \frac{g\sin(\theta)}{R}$. This expression can be integrated for the perticle's trajectory. This is easier after applying the chain rule to insert derivatives with respect to θ.

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\theta}{dt}\frac{d\dot{\theta}}{d\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

Since the particle starts at $\theta(t=0) = 0$ with $\dot{\theta}(t=0) = 0$:

$$\begin{aligned}
\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} &= \frac{g}{R}\sin(\theta) \\
\dot{\theta}d\dot{\theta} &= \frac{g}{R}\sin(\theta)d\theta \\
\int_0^{\dot{\theta}_{\text{leaving}}} \dot{\theta}d\dot{\theta} &= \frac{g}{R}\int_0^{\theta_{\text{leaving}}} \sin\theta d\theta \\
\left.\frac{\dot{\theta}^2}{2}\right|_0^{\dot{\theta}_{\text{leaving}}} &= \frac{g}{R}(-\cos\theta)\Big|_0^{\theta_{\text{leaving}}} \\
\frac{\dot{\theta}^2_{\text{leaving}}}{2} &= \frac{g}{R}(-\cos(\theta_{\text{leaving}}) + 1)
\end{aligned}$$

After this, you have to substitute $\dot{\theta}^2$ in an expression for $F_\rho^{\text{constraint}}$, that must vanish at the moment of leaving the semi-sphere.

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