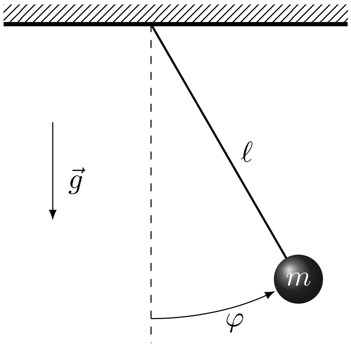
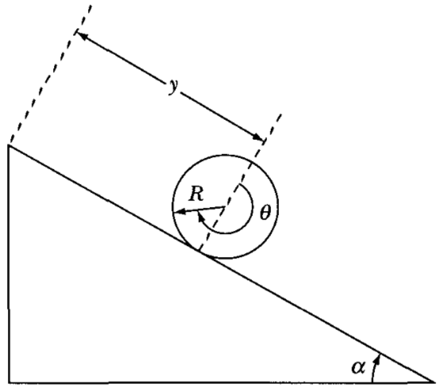
Constraint forces | Lagrange multipliers

1. **Ideal pendulum**
   Calculate the tension in the string using the method of Lagrange multipliers. The constraint is that the bead is always at $\vec{r} = \ell\hat{\rho}$, ergo, the function expressing this is $f(\rho) = \rho - \ell = 0$.



2. **Cylinder rolling over an inclined plane** [Marion ex. 7.5]

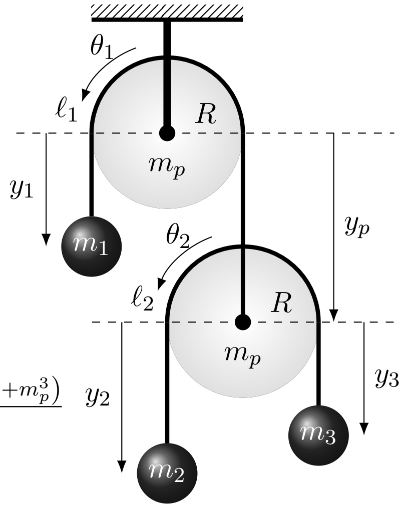


   (a) Write the equations of motion,

   (b) find the angular acceleration,

   (c) and the constraint forces.

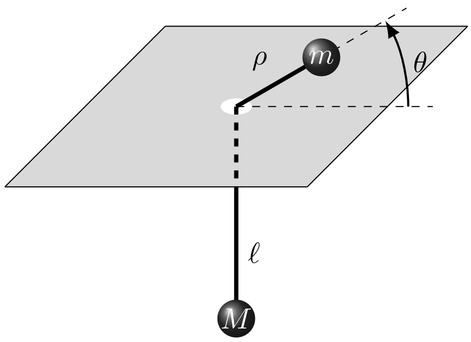
3. **Double Atwood machine** [Marion ex. 7.8 and 7-37]
   Use the method of Lagrange multipliers to find the equations of motion and the tensions in the strings.



   (a) Verify that you obtain the same generalized accelerations as when this is solved without using Lagrange multipliers. Results:
   $\ddot{y}_1 = \frac{2g\left(2m_1m_2+2m_1m_3+m_1m_p-8m_2m_3-3m_2m_p-3m_3m_p-m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}$
   $\ddot{y}_2 = \frac{2g(4m_1+m_p)(m_2-m_3)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}$

   (b) Find the tension in both strings. Results:
   $Q_1 = \frac{g\left(32m_1m_2m_3+8m_1m_2m_p+20m_1m_3m_p+4m_1m_p^2+8m_2m_3m_p+2m_2m_p^2+4m_3m_p^2+m_p^3\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+6m_2m_p+14m_3m_p+3m_p^2}$
   $Q_2 = \frac{gm_3\cdot\left(16m_1m_2+6m_1m_p+4m_2m_p-m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+6m_2m_p+14m_3m_p+3m_p^2}$

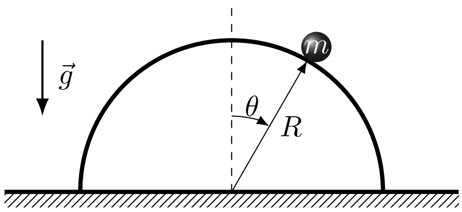
4. **Weights linked by a rope** [Taylor 7.50]
   A particle of mass m that lies over a table is linked to another particle of mass M by a rope of length l that passes through a hole in the table, without friction. The second one is hanging vertically at a distance $y = \ell - \rho$ from the table, where ρ is the distance between the first particle and the hole.



   (a) Assuming that θ is not necessarily constant, find the Lagrange equations for ρ and y. Result:
   $-Mg + M\ddot{y} + \lambda_1 = 0 \qquad \lambda_1 - m\rho\dot{\theta}^2 + m\ddot{\rho} = 0$

   (b) Solve the system for ρ, y and the Lagrange multiplier λ_1 finding the tensions exerted upon both particles.
   Result: $Q_\rho = \frac{Mm\left(g+\rho\dot{\theta}^2\right)}{M+m}$

5. **Particle sliding over a semi-sphere** [Marion ex. 7.10]
   The particle of mass m, considered as a point particle, slides over a semi-sphere of radius R without friction.



   (a) Find the constraint force.
   Result: $F_\rho^{\text{constraint}} = m\left(-R\dot{\theta}^2 + g\cos(\theta)\right)$

   (b) Calculate the angle at which the particle leaves the semi-sphere.
   Result: $\approx 48.19°$

To find the angle at which the particle leaves the semi-sphere, you must solve the differential equation you'll get after working out the rather tough constraint force, which will be $\ddot{\theta} = \frac{g\sin(\theta)}{R}$. This expression can be integrated for the perticle's trajectory. This is easier after applying the chain rule to insert derivatives with respect to θ.

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\theta}{dt}\frac{d\dot{\theta}}{d\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

Since the particle starts at $\theta(t=0)=0$ with $\dot{\theta}(t=0)=0$:

$$\begin{aligned}
\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} &= \frac{g}{R}\sin(\theta) \\
\dot{\theta}d\dot{\theta} &= \frac{g}{R}\sin(\theta)d\theta \\
\int_0^{\dot{\theta}_{\text{leaving}}} \dot{\theta}d\dot{\theta} &= \frac{g}{R}\int_0^{\theta_{\text{leaving}}} \sin\theta d\theta \\
\left.\frac{\dot{\theta}^2}{2}\right|_0^{\dot{\theta}_{\text{leaving}}} &= \frac{g}{R}(-\cos\theta)\Big|_0^{\theta_{\text{leaving}}} \\
\frac{\dot{\theta}^2_{\text{leaving}}}{2} &= \frac{g}{R}(-\cos(\theta_{\text{leaving}})+1)
\end{aligned}$$

After this, you have to substitute $\dot{\theta}^2$ in an expression for $F_\rho^{\text{constraint}}$, that must vanish at the moment of leaving the semi-sphere.

 Updated July 24, 2025