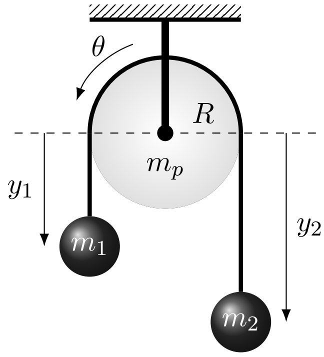
Constraints

Exercises marked with (*) have extra difficulty, don't hesitate to ask for help.



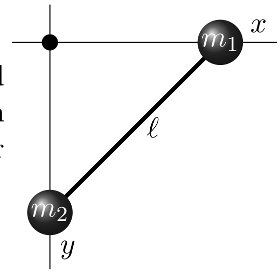
1. **Atwood machine**
   The figure shows a string of length ℓ and a pulley of radius R_p and mass m_p. Find the acceleration of the masses attached at each end of the string.

   (a) The string is inextensible, so it establishes a relation between y_1 and y_2. Write the equation for this constraint.

   (b) If the string slides over the pulley without friction, the pulley will not move. Write the Euler-Lagrange equation for y_1 using the constraint from the previous item and write the masses' acceleration.

   (c) Usually, the string won't slide and the pulley will rotate. This constraint adds a relation between θ and the displacement of the string. Using that constraint, write the pulley's rotational kinetic energy in terms of $\dot{y}_1$, modeling the pulley as an homogeneous cylinder with a moment of inertia of $(m/2)R^2$.

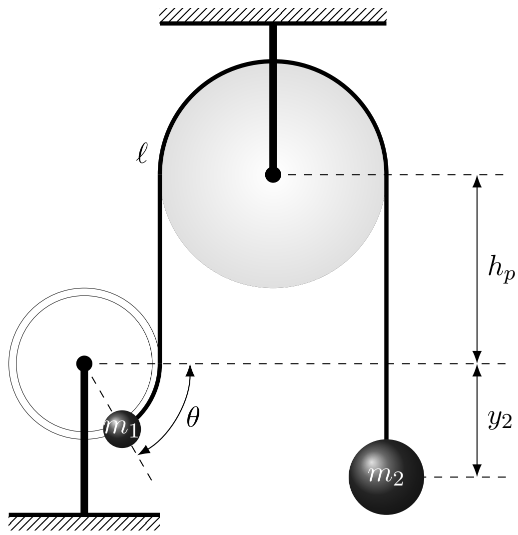
   (d) Use the Euler-Lagrange equation for y_1 to write the masses' accelerations.



2. **Pendulum with sliding and coupled masses**
   Two weights of masses m_1 and m_2 are linked together by a rigid rod of length ℓ and negligible mass. m_1 can slide over an horizontal axis and m_2 over a vertical axis, both without friction. The rod sets a constraint between the coordinates that define their positions, x and y.

   (a) Use the constraint equation to express both positions only in terms of y.

   (b) Calculate the acceleration of m_2.
   Result: $\ddot{y} = \frac{-\ell^2 m_1 y \dot{y}^2 + g m_2 \left(\ell^2 - y^2\right)^2}{\ell^4 m_2 + \ell^2 m_1 y^2 - 2\ell^2 m_2 y^2 - m_1 y^4 + m_2 y^4}$



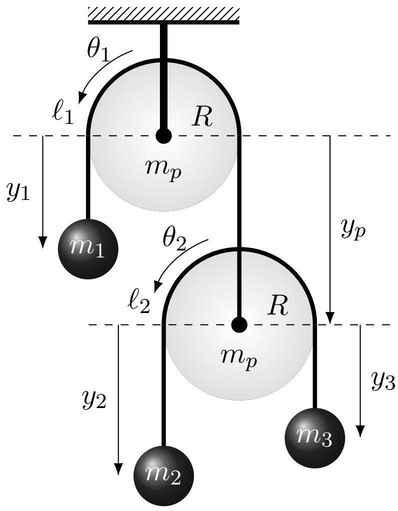
3. **Ring and pulley**
   A weight of mass m_2 hangs from the free end of the string that passes over the pulley of radius R_p and mass m_p. The string moves without sliping, its length is ℓ and its mass is negligible. The other end of the string is attached to mass m_1, fixed to a ring of mass m_r, coiling over the ring by an angle θ. The center of the pulley is at a height h_p over the center of the ring. The radius of the ring is R_r and rotates freely with a moment of inertia $m_r R_r^2$. The generalized coordinates are y_2 and θ, constrained because of the length ℓ.

   (a) Write the position for each object in terms of the generalized coordinates, in a frame of reference with origin at the center of the ring.

   (b) Find an expression for the constraint and use it to rewrite the positions in terms of θ. Verify your solution by checking that a variation in θ approaching its zero value implies that the other object moves down.

(c) Write the Euler-Lagrange equation.
Result: $R_r^2 m_1 \ddot{\theta} + R_r^2 m_r \ddot{\theta} + R_r g m_1 \cos(\theta) + R_p^2 m_2 \ddot{\theta} + \frac{R_p^2 m_p \ddot{\theta}}{2} - R_p g m_2 = 0$

4. **Double Atwood machine** [Marion ex. 7.8]



(a) Write the positions for the three hanging objects and that of the lower pulley in terms of the generalized coordinates shown in the figure: y_i with $i = 1, 2, 3, p$.

(b) Write the equations for the constraints provided by both strings.

(c) Use the constraint equations to express all positions in terms of y_1 and y_2.

(d) The strings don't slide over the pulleys. This is an extra constraint that must be used to relate y_i and θ_i.

(e) Calculate the kinetic and potential energies.

(f) Find both Euler-Lagrange equations.
Results:
$-g m_1 + g m_2 + g m_3 + g m_p + m_1 \ddot{y}_1 + m_2 \ddot{y}_1 - m_2 \ddot{y}_2 + m_3 \ddot{y}_1 + m_3 \ddot{y}_2 + \frac{3 m_p \ddot{y}_1}{2} = 0$
$-g m_2 + g m_3 - m_2 \ddot{y}_1 + m_2 \ddot{y}_2 + m_3 \ddot{y}_1 + m_3 \ddot{y}_2 + \frac{m_p \ddot{y}_2}{2} = 0$

(g) Solve for the generalized accelerations.
Results:
$\ddot{y}_1 = \frac{2g\left(2m_1m_2+2m_1m_3+m_1m_p-8m_2m_3-3m_2m_p-3m_3m_p-m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}$
$\ddot{y}_2 = \frac{2g(4m_1+m_p)(m_2-m_3)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}$

(h) Write the accelerations of the three masses.
Results:
$\ddot{\vec{r}}_1 = \ddot{y}_1(-\hat{e}_y)$
$\ddot{\vec{r}}_2 = -\frac{2g\left(2m_1m_2-6m_1m_3-m_1m_p+8m_2m_3+4m_2m_p+2m_3m_p+m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}\mathbf{\hat{e}_y}$
$\ddot{\vec{r}}_3 = -\frac{2g\left(-6m_1m_2+2m_1m_3-m_1m_p+8m_2m_3+2m_2m_p+4m_3m_p+m_p^2\right)}{4m_1m_2+4m_1m_3+2m_1m_p+16m_2m_3+8m_2m_p+8m_3m_p+3m_p^2}\mathbf{\hat{e}_y}$