A few notes extracted from *The Variational Principles of Mechanics* by Cornelius Lanczos

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Resumen

The book *The variational principles of mechanics*[I] uses a historical approach for addressing the principles of analytical mechanics. This helped me understand the foundations of the tools taught in each course on classical mechanics. In this document, I present some *notes* that I wrote while I was trying to understand this work by Cornelius Lanczos[II]. Following the title of each section, I reference the corresponding section about the topic in the best work about this subject, *Mechanics*[III] The numbering of each mathematical expression in this document found in these references matches the original work.

Índice

[I] Cornelius Lanczos. The Variational Principles of Mechanics. University of Toronto press, 1952.
[II] Lánczos Kornél. Hungarian mathematician and physicist (Székesfehérvár, 1893 - Budapest, 1974).
[III] L. D. Landau and E. M. Lifshitz. Volume 1 of Course of Theoretical Physics, 3rd ed., 1976.

1. Mechanics: vector vs. analytical approaches

Reference: Lanczos I§1

For Isaac Newton[IV], the fundamental law for mechanics is the one that sets $\vec{\mathbf{F}}_i = m_i \vec{\mathbf{a}}_i$ for particle i.

Thus, the linear momentum of this particle moving in free space is conserved until a force is exerted upon it. This approach is simple and adequate for a single particle. But when dealing with systems formed by many particles, we have to isolate each one of them and find the forces they exert upon each other. We call these forces *interactions*.

The lack of knowledge about the nature of many of these interaction forces makes the use of additional postulates necessary. Newton thought that his third law, "action equals reaction", would take care of all these dynamical problems. This did not turn out to be the case, as demonstrated by the example of rigid body dynamics, for whose analysis one must *constrain oneself* to assumptions such as that the forces between components of the system are all central[V].

In the analytical approach, the *mechanical system* is analyzed as a whole, without isolating forces on individual particles. If in the vector approach the force on each point must be analyzed, in the analytical approach it suffices to know a single function that depends on the position (and sometimes also the velocity) of the particles; this *work function* contains in implicit form all the forces acting on the particles of the system. To obtain such forces, a simple differentiation of this function suffices.

Another advantage of this latter approach is the handling of auxiliary conditions. Often there exist *dynamical conditions* that are known *a priori* from the analysis of a system. For example, the distance between any two points of a rigid body remains unchanged. Such a condition is maintained thanks to strong constraint forces between particles of the system. The analytical approach does not require determining such forces; it suffices to state the condition between the positions of the particles. The same applies when analyzing a fluid; one disregards the internal forces.

But perhaps the most crucial difference becomes evident in a complex system. In the vector approach, it is required to separately formulate a high number of differential equations and then concatenate them. The *variational principles* of analytical mechanics allow discovering the basis that all these equations must respect. There exists a principle, which holds that a quantity called *action* must be *stationary* (see section **??**) and fulfilling only this condition allows obtaining the differential equations that correctly describe the dynamics of the system.

In summary:

1. Vector mechanics isolates particles; analytical mechanics considers the system as a whole.
2. Vector mechanics constructs a force exerted on each particle; analytical mechanics considers a single function that contains all the information about the forces of interest.
3. If there are forces that maintain a condition between coordinates, the vector approach requires obtaining these. In the analytical approach, it suffices to mathematically state such a relationship.
4. With the analytical method, the complete set of equations describing the dynamics is obtained from a single principle. It suffices to minimize the quantity called *action*.

2. The Analytical Approach to Mechanics

Reference: Lanczos 0§1

[IV] English mathematician and physicist (Woolsthorpe, Lincolnshire, 1642 - Kensington, Middlesex 1726).

[V] In central forces these point from one particle to the other with which it interacts, therefore, the definition excludes forces observed inside solids, caused by torsion or shear stresses that are not collinear with a line passing between the interacting elements.

For Newton, that is in vectorial mechanics, forces act producing a change in the momentum of a particle

$$\sum \vec{\mathbf{F}}(t) = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t},^{\text{VI}} \tag{1}$$

while for his contemporary Gottfried Leibniz[VII] the action of these was to produce a change in a quantity he called *vis viva* (living force) which he quantified as mv^2 which is nothing other than twice what we call *kinetic energy*. And just as Leibniz replaced Newtonian momentum with kinetic energy, following such a line of reasoning, he replaced force with the *work of the force*. The latter was subsequently replaced with the concept of *work function*. Thus Leibniz is the initiator of a new branch of mechanics study called *analytical mechanics* which bases its study of equilibrium and motion on two quantities that are not vectorial, but scalar, the *kinetic energy* and the *work function*, the latter sometimes replaceable by a *potential energy*.

It may be difficult to accept that two scalar quantities are sufficient to determine a motion being this a phenomenon so relatable to a certain direction. The energy theorem that establishes that the sum of kinetic and potential energies does not vary during motion gives only *one* equation, when it is evident that describing the motion of just one particle requires *three* (the spatial dimensions), increasing such number if the system is composed of more particles. But in fact these two scalar quantities are sufficient to describe the dynamics of the most complex system if they are used in conjunction with a *principle*.

Before presenting such principle, we will discuss three fundamental concepts within the analytical approach: the work of a force, generalized coordinates, and the work function.

2.1. Work of a Force

In the Newtonian equation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ each side of the equality responds to two different aspects of a mechanical problem. The right side responds to the inertial quality of mass, which is captured in the kinetic energy in the analytical treatment. The left side is the effect of an external field acting on the particle. Although we are accustomed to thinking of force as something primitive and irreducible, in the analytical approach the *work* done by the force is our main concern while the force itself is a secondary quantity that we derive from the former.

Work is a scalar quantity that is obtained by integrating the force exerted over the trajectory of a particle

$$W = \int \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}. \tag{2}$$

Any force can be decomposed according to the coordinate system used. For the case of the Cartesian system

$$\vec{\mathbf{F}} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}, \tag{3}$$

But if we analyze only one force, nothing prevents orienting an axis with it and the work can be written as $W = \int F_x \mathrm{d}x$. Thus in differential form an infinitesimal work can be expressed as $\mathrm{d}w = F_x \mathrm{d}x$, which will only be *integrable* if F_x is the same when traversing the same x. A classic counterexample is that of the force that brakes a locked wheel of an automobile. The sign of the friction force changes to oppose the displacement.

In this one-dimensional case a closed trajectory forces us to traverse the same points in one direction and another, but in the general case it is clear that the work depends on the trajectory and not on its initial and final points. And since usually the forces involved $\vec{F} = \vec{F}(x, y, z)$ present values that make $W \neq 0$ in a closed trajectory (same initial and final point) it becomes clear that the infinitesimal work is not a true differential whose integral depends only on the limits of integration, and is essentially a *non-integrable* quantity.

[VI] Mass can only be taken as constant if we discard relativistic effects.

[VII] German mathematician and physicist (Leipzig 1646 - Hanover 1716)

Thus work is expressed as an infinitesimal quantity that is a first-order differential form, but not integrable[VIII],

$$\overline{\mathrm{d}w} = \sum_{i=1}^{N} F_{xi}\mathrm{d}x_i + F_{yi}\mathrm{d}y_i + F_{zi}\mathrm{d}z_i, \qquad \text{(Lanczos 17.2)}$$

summing over the N forces impressed upon the particle.

2.2. Generalized Coordinates

In the analytical approach, the definition of coordinate is a purely mathematical construction without relation to the position vectors of the Newtonian approach. The coordinates need only maintain a *one-to-one* relationship with the points of the physical space being analyzed, then we can operate algebraically on these while forgetting their physical meaning.

A system of P particles that has no restrictions on its motion presents in the Cartesian system x_i, y_i, z_i $(i = 1, 2, \ldots, P)$ totaling $3P$ coordinates. Its dynamics will be determined if we know x_i, y_i, z_i as functions of time t. The same problem will be solved if these latter are expressed as functions of some quantities $q_1, q_2, \ldots, q_{3P}$ and we know how these depend on t. An example of this is if we take advantage of the relationship with polar coordinates

$$\begin{aligned} x &= r\cos\theta\sin\varphi \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\varphi \end{aligned} \qquad \text{(Lanczos 12.3)}$$

and obtain $r(t), \theta(t), \varphi(t)$. Ultimately, it is not important which coordinate system is used, as long as for P particles we define some $3P$ independent coordinates. These receive the name of *generalized coordinates* q_i, and their derivatives $\dot{q}_i = \frac{\partial q_i}{\partial t}$, the name of *generalized velocities*. With these we must be able to write relations f_{3P} that allow us to recover the Cartesian variables

$$\begin{aligned} x_1 &= f_1(q_i, \ldots, q_n) \\ y_1 &= f_2(q_i, \ldots, q_n) \\ &\ldots \\ y_P &= f_{3P-1}(q_i, \ldots, q_n) \\ z_P &= f_{3P}(q_i, \ldots, q_n) \end{aligned} \qquad \text{(Lanczos 12.8)}$$

In practice, the number of q_i that are used is usually less than $3P$ when considering the conditions that limit the dynamics. For example, if a system of two particles, $P = 2$, is limited to move in a plane, we have two fewer *degrees of freedom*, so we need only a number of generalized coordinates $n = 3 \times 2 - 2$. Thus the number of degrees of freedom of a system is

$$n = 3P - K, \qquad (4)$$

where K is the number of kinematic conditions, *constraints* or *linkages*, imposed on the system.

2.3. Work Function — Monogenic and Polygenic Forces

Just as in equation (Lanczos 17.2) the infinitesimal work was expressed as a function of the coordinates of a Cartesian system, the same can be done with generalized coordinates

$$\overline{\mathrm{d}w} = F_1\mathrm{d}q_1 + F_2\mathrm{d}q_2 + \ldots + F_n\mathrm{d}q_n = \sum_{i=1}^{n} F_i\mathrm{d}q_i, \qquad \text{(Lanczos 17.3)}$$

where the F_i are the components of the so-called *generalized forces*.

[VIII] The notation of writing a bar over a non-integrable differential quantity is used.

While $\overline{\mathrm{d}w}$ is in the general case a first-order differential form, there are cases where it turns out to be a true differential of a function, such that we can remove the bar over it, and rename it as $\mathrm{d}w = \mathrm{d}U$, where

$$U = U(q_1, q_2, \ldots, q_n), \tag{Lanczos 17.6}$$

is the so-called *work function*. In this way

$$\sum_{i=1}^{n} F_i \mathrm{d}q_i = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} \mathrm{d}q_i \Rightarrow F_i = \frac{\partial U}{\partial q_i}, \tag{Lanczos 17.8}$$

where usually the negative of the work function $V = -U$ is used to interpret it as a *potential energy*, so that

$$F_i = -\frac{\partial V}{\partial q_i}, \tag{Lanczos 17.10}$$

where $V = V(q_1, q_2, \ldots, q_n)$.

Forces of this class present two notable characteristics:

1. they satisfy *energy conservation* and therefore receive the name of *conservative forces*;
2. although they have n components, they are calculated from a *single scalar function* U.

The definition of the work function as only dependent on the generalized coordinates is too restrictive, since we well know cases of forces derivable from work functions in which time t appears explicitly $U = U(q_1, q_2, \ldots, q_n, t)$. In such cases, energy conservation is lost, but equations (Lanczos 17.8) remain valid. An example of this case is that of a particle with electric charge in a *cyclotron* that after each cycle returns to the same starting point but with greater kinetic energy. Clearly, the Lorentz forces[IX] involved can be obtained from work functions[X], but in the particular case of the *cyclotron* the electric field varies in time, and therefore does not result in a conservative force.

The opposite case, where energy is conserved and yet no U can be defined, is possible. This is the case of static friction forces that intervene in the rolling phenomenon, which, since there is no displacement involved, do no work at all. The case of this type of forces for which no scalar function can be devised from which to derive them are called *polygenic*, due to their multiple possible origins. This term is in contrast to that which corresponds to those forces that we can derive from a scalar quantity, the work function U, which we call *monogenic* to have a term that encompasses them regardless of whether or not they are conservative.

The most general case of a monogenic force can depend on both the coordinates and the generalized velocities, $U = U(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$. In fact, this is the case of the aforementioned Lorentz forces. In such a case, the force components can be obtained as

$$F_i = \frac{\partial U}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}_i}, \tag{Lanczos 17.13}$$

using a procedure similar to that presented in section 6.

3. Principle of Virtual Work

In Newtonian mechanics, a particle is in equilibrium if the sum of the N forces acting on it vanishes, $\sum_{i=1}^{N} \vec{\mathbf{F}} = 0$, that is, the resultant force on it is zero. In this approach to mechanics, the

[IX] $\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$

[X] $\vec{\mathbf{B}} = \boldsymbol{\nabla} \times \vec{\mathbf{A}}$ and $\vec{\mathbf{E}} = -\boldsymbol{\nabla}\varphi - \frac{\partial \vec{\mathbf{A}}}{\partial t}$, where $\vec{\mathbf{A}}$ is the so-called vector potential and φ the electric potential.

particle is isolated and all constraints on its motion are replaced with *reaction forces* that enforce these *constraints*. In the analytical treatment, these forces are discarded and only the external ones are considered. For this purpose, *virtual displacements* are made that are in harmony with the *constraints*.

If resultant external forces $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2, \ldots, \vec{\mathbf{F}}_P$ act on each of the P particles that compose a system, we will denote their corresponding *virtual displacements* as $\delta\vec{\mathbf{r}}_1, \delta\vec{\mathbf{r}}_2, \ldots, \delta\vec{\mathbf{r}}_P$[XI]. With this notation, the *principle of virtual work* can be summarized as:

> "A system will be in equilibrium if, and only if, the sum of the virtual works of all the impressed forces acting on it vanishes",

$$\overline{\delta\omega} = \vec{\mathbf{F}}_1 \cdot \delta\vec{\mathbf{r}}_1 + \vec{\mathbf{F}}_2 \cdot \delta\vec{\mathbf{r}}_2 + \ldots + \vec{\mathbf{F}}_P \cdot \delta\vec{\mathbf{r}}_P = 0. \qquad \text{(Lanczos 31.2)}$$

Since in analytical language we use *generalized coordinates* (see section 2.2) we rewrite this last expression as

$$\overline{\delta\omega} = F_1\delta q_1 + F_2\delta q_1 + \ldots + F_n\delta q_n = \sum_{i=1}^{n} F_i\delta q_i = 0. \qquad \text{(Lanczos 31.4)}$$

If from the Newtonian point of view the equilibrium principle of a system required that *the sum of the impressed forces with the reaction forces be zero*, from the analytical point of view the virtual work of the impressed forces can be replaced with the negative value of the work of the reaction forces. Expressed in this form, we arrive at postulating the principle as:

> "The virtual work of the reaction forces is always zero for any virtual displacement that is in harmony with the given kinematic constraints".

3.1. What is a virtual displacement? — Stationary value

Reference: Lanczos II§2

In section 3 the use of the symbol δ associated with a *virtual displacement* was introduced. The term *virtual* responds to the fact that it is an *infinitesimal variation* of position that would be performed with the intention of *mathematically exploring* the values that a continuous and differentiable function in the coordinates q_i assumes in the neighborhood of a point $q_1, q_2, \ldots, q_n$.

To exemplify the difference between such a displacement and a *real* one associated with differentiation, dq, let us imagine a sphere at the deepest point of a *bowl*. We can explore how the *potential energy* V changes by evaluating it after taking the sphere to a position adjacent to the current one, but in fact we do not seek to produce any displacement of the sphere. We will have made a *virtual displacement*, which was done with the intention of exploring the consequence of all possible displacements *in any kinematically admissible form*. The term that condenses the concept of a displacement that is simultaneously *virtual* and *infinitesimal* is called *variation*, and is denoted by δ at the suggestion of Joseph-Louis Lagrange[XII].

Thus a variation of the potential energy would represent

$$\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \ldots + \frac{\partial V}{\partial q_n}\delta q_n = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i}\delta q_i. \qquad \text{(Lanczos 22.3,6)}$$

When the minimum (or maximum) of a function, such as V, is found, the rate of change of this function with respect to an infinitesimal displacement in any of its coordinates must be zero, that is

$$\frac{\partial V}{\partial q_i} = 0 \ (i = 1, 2, \ldots, n). \qquad \text{(Lanczos 22.7)}$$

[XI] The symbol δ denotes virtual variations. See section 3.1.

[XII] Giuseppe Lodovico Lagrangia. Italian mathematician and astronomer (Turin 1736 - Paris 1813)

And while a *saddle* point of V would also satisfy such a condition, each of the points that satisfy it are considered exceptional; it is said that the function presents a *stationary* value there. In short, we agree that for V to present a stationary value at a point, it must be satisfied that $\delta V = 0$ there.

4. d'Alembert's Principle

Reference: Lanczos IV§1

Newton's 2nd law can be written as $\vec{\mathbf{F}} - m\vec{\mathbf{a}} = 0$, where $\vec{\mathbf{F}}$ is the resultant force. Then one can assume that the force created by the motion is an *inertial force* $\vec{\mathbf{I}} = -m\vec{\mathbf{a}}$. Thus expressed $\vec{\mathbf{F}} + \vec{\mathbf{I}} = 0$ is more than a reformulation of Newton's 2nd law, it is the expression of a *principle*. Just as in Newtonian mechanics the resultant force being zero, $\vec{\mathbf{F}} = 0$, means equilibrium, adding the *inertial force* to a system in motion always allows achieving this desired "equilibrium". In this way any criterion we have for a system in equilibrium can now be applied to a system in motion. Jean le Rond d'Alembert[XIII] stated it as:

> "Any system of forces is in equilibrium if the inertial forces are added to the *applied* forces".

If we add to the resultant applied forces $\vec{\mathbf{F}}_i$ on a particle the inertial force we obtain the so-called *effective force*

$$\vec{\mathbf{F}}_i^e = \vec{\mathbf{F}}_i + \vec{\mathbf{I}}_i. \qquad \text{(Lanczos 41.5)}$$

With this definition, and summing over the P particles of a system, *d'Alembert's principle* can be reformulated as

$$\overline{\delta\omega^e} = \sum_{i=1}^{P} \vec{\mathbf{F}}_i^e \cdot \delta\vec{\mathbf{r}}_i \equiv \sum_{i=1}^{P} (\vec{\mathbf{F}}_i - m_i\vec{\mathbf{a}}_i) \cdot \delta\vec{\mathbf{r}}_i = 0, \qquad \text{(Lanczos 41.6)}$$

which put in words is,

> "The total virtual work of the effective forces is zero for any reversible variation that satisfies the given kinematic conditions."

Essentially, the principle postulates that since the virtual work of the impressed forces on a system is usually different from zero, the system will react by moving in such a way that the inertial forces make $\overline{\delta\omega^e}$ zero. The radical importance of d'Alembert's principle is that it allows applying the *principle of virtual work* to any dynamic system.

5. Hamilton's Principle: from d'Alembert to the Lagrangian

Reference: Lanczos V§1

D'Alembert's principle establishes that $\overline{\delta\omega^e} = 0$, i.e., that the virtual work of effective forces is zero. The virtual work of impressed forces can be obtained from a monogenic differential form, but inertial forces cannot, meaning they cannot be deduced from a work function U. William Rowan Hamilton[XIV] showed that a simple integration with respect to time allows relating the virtual work of inertial forces to a monogenic form. His proposal basically consists in that, if d'Alembert's principle is respected, the integration of $\overline{\delta\omega^e}$ over any time interval should also vanish

$$\int_{t_1}^{t_2} \overline{\delta\omega^e}\, \mathrm{d}t = \int_{t_1}^{t_2} \sum_{i=1}^{P} \left(\vec{\mathbf{F}}_i - m_i \frac{\mathrm{d}\vec{\mathbf{v}}_i}{\mathrm{d}t} \right) \cdot \delta\vec{\mathbf{r}}_i \, \mathrm{d}t = 0 \qquad \text{(Lanczos 51.1)}$$

[XIII] French mathematician and physicist (Paris 1717 - Paris 1783).
[XIV] Irish physicist and mathematician (Dublin 1805 - Dublin 1865).

The integral of the resulting difference could be solved as the difference of separate integrals. To analyze the first, associated with the P resultant forces exerted on each particle, we must remember that these can be decomposed into n monogenic generalized forces corresponding to the n generalized coordinates[XV], and that these, in turn, are related to the potential energy according to what is expressed in equations (Lanczos 17.8) and (Lanczos 17.10)

$$\sum_{i=1}^{n} F_i \,\mathrm{d}q_i = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\,\mathrm{d}q_i \Rightarrow F_i = \frac{\partial U}{\partial q_i} = -\frac{\partial V}{\partial q_i} \Rightarrow \sum_{i=1}^{P} \vec{\mathbf{F}}_i \cdot \delta\vec{\mathbf{r}}_i = -\delta V. \qquad \text{(Lanczos 17.7)}$$

By integrating this relation and making use of the commutativity between differentiation (integration) and differentiation[XVI] the first integral turns out to be

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \vec{\mathbf{F}}_i \cdot \delta\vec{\mathbf{r}}_i \,\mathrm{d}t = -\int_{t_1}^{t_2} \delta V \,\mathrm{d}t = -\delta \int_{t_1}^{t_2} V \,\mathrm{d}t\,, \qquad \text{(Lanczos51.2)}$$

which represents a variation of the system's potential energy between t_1 and t_2 for forces that are commonly called conservative.

As for the inertial forces term, if we assume non-relativistic conditions, we write it as

$$\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(m_i\vec{\mathbf{v}}_i) \cdot \delta\vec{\mathbf{r}}_i \,\mathrm{d}t\,. \qquad \text{(Lanczos51.3A)}$$

If we make use of integration by parts[XVII] to rewrite it as

$$-\left[\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(m_i\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{r}}_i)\,\mathrm{d}t - \int_{t_1}^{t_2} m_i\vec{\mathbf{v}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\mathbf{r}}_i \,\mathrm{d}t\right], \qquad \text{(Lanczos51.3B)}$$

whose first term integrates to $-\left[m\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{r}}_i\right|_{t_1}^{t_2}$, which is what is called a boundary term, and its value will depend on its evaluation at t_1 and t_2.

To solve the second term of equation (Lanczos51.3B) we make use of the fact that the operations of differentiation and variation are interchangeable, which is called the commutative property[XVIII]

$$\int_{t_1}^{t_2} m_i\vec{\mathbf{v}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\mathbf{r}}_i \,\mathrm{d}t = \int_{t_1}^{t_2} m_i\vec{\mathbf{v}}_i \cdot \delta\frac{\mathrm{d}\vec{\mathbf{r}}_i}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_1}^{t_2} m_i\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{v}}_i \,\mathrm{d}t\,. \qquad \text{(Lanczos 51.5A)}$$

The result of the scalar product can be obtained taking into account that the variation of a product is performed in the same way as a differentiation, i.e., $\delta(\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i) = \vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{v}}_i + \delta\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i = 2\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{v}}_i$,

[XV] An analogous demonstration can be followed for forces that depend explicitly on time t and/or generalized velocities $\dot{q}_i$, but these are omitted here for clarity.

[XVI] Such commutativity is demonstrated later in this section.

[XVII] Integration by parts is nothing other than a rewriting of the differentiation of a product,

$$\int (u\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t}v) = \int \frac{\mathrm{d}(uv)}{\mathrm{d}t} \Rightarrow \int \frac{\mathrm{d}u}{\mathrm{d}t}v = \int \frac{\mathrm{d}(uv)}{\mathrm{d}t} - \int u\frac{\mathrm{d}v}{\mathrm{d}t} = uv - \int u\frac{\mathrm{d}v}{\mathrm{d}t}. \qquad (5)$$

In this case $u = m_i\vec{\mathbf{v}}_i$ and $v = \delta\vec{\mathbf{r}}_i$.

[XVIII] To prove that differentiation and variation commute, we must interpret what the derivative of a variation of $\vec{\mathbf{r}}_i(t)$ means. If we depart in an arbitrary direction $\vec{\rho}(t)$ by a magnitude ϵ we get $\overline{\vec{\mathbf{r}}_i(t)} = \vec{\mathbf{r}}_i(t) + \epsilon\vec{\rho}(t)$. Thus the derivative of such variation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\mathbf{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\vec{\mathbf{r}}_i(t)} - \vec{\mathbf{r}}_i(t)\right) = \epsilon\dot{\vec{\rho}}_i(t) = \epsilon\vec{\mathbf{v}}_\rho(t), \qquad \text{(Lanczos 29.1)}$$

where $\vec{\mathbf{v}}_\rho(t)$ would be a velocity of the difference with respect to a $\vec{\mathbf{v}}_r(t)$. Now we see what the variation of a derivative would be,

$$\delta\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}_i = \overline{\dot{\vec{\mathbf{r}}}_i(t)} - \dot{\vec{\mathbf{r}}}_i(t) = (\vec{\mathbf{v}}_r(t) + \epsilon\vec{\mathbf{v}}_\rho(t)) - \vec{\mathbf{v}}_r(t) = \epsilon\vec{\mathbf{v}}_\rho(t), \qquad \text{(Lanczos 29.3)}$$

which by coinciding with equation (Lanczos 29.1) demonstrates the commutativity between variation and differentiation.

therefore

$$\int_{t_1}^{t_2} m_i \vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{v}}_i \, \mathrm{d}t = \int_{t_1}^{t_2} m_i \frac{1}{2} \delta \left(\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i \right) \mathrm{d}t = \delta \int_{t_1}^{t_2} \frac{1}{2} m_i v^2 \mathrm{d}t. \qquad \text{(Lanczos 51.5B)}$$

We add the expression we arrived at in equation (Lanczos51.2) with the integration of the first term of (Lanczos51.3B) and with the reformulation of its second term that we arrived at in equation (Lanczos 51.5B) and obtain

$$\int_{t_1}^{t_2} \overline{\delta\omega^e} \mathrm{d}t = -\delta \int_{t_1}^{t_2} V \mathrm{d}t - \left[m\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{r}}_i \right] \Big|_{t_1}^{t_2} + \delta \int_{t_1}^{t_2} \frac{1}{2} \sum m_i v_i^2 \mathrm{d}t. \qquad \text{(Lanczos51.6)}$$

The last term is nothing other than the variation of the system's kinetic energy T between t_1 and t_2, so equation (Lanczos51.6) can be summarized as

$$\int_{t_1}^{t_2} \overline{\delta\omega^e} \mathrm{d}t = \delta \int_{t_1}^{t_2} (T - V) \, \mathrm{d}t - \left[m\vec{\mathbf{v}}_i \cdot \delta\vec{\mathbf{r}}_i \right] \Big|_{t_1}^{t_2}. \qquad \text{(Lanczos 51.8)}$$

Since the objective of performing variations in the traveled positions $\delta\vec{\mathbf{r}}_i$ is to minimize the virtual work of forces to make it *stationary*, i.e., $\overline{\delta\omega^e} = 0$, such variations are performed in the path along the integration, not at the initial and final positions; that is, *we vary between fixed limits*, therefore

$$\delta\vec{\mathbf{r}}_i \ (t_1) = 0 \text{ and } \delta\vec{\mathbf{r}}_i \ (t_2) = 0, \qquad \text{(Lanczos 51.9)}$$

with which the second term of equation (Lanczos 51.8), called the boundary term, is indeed zero.

We finally arrive at

$$\int_{t_1}^{t_2} \overline{\delta\omega^e} \mathrm{d}t = \delta \int_{t_1}^{t_2} (T - V) \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \mathcal{L} \mathrm{d}t = \delta S, \qquad \text{(Lanczos 51.10)}$$

where S is the so-called action obtained from the integration of

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t) = T - V, \qquad \text{(Lanczos 51.11)}$$

which is the so-called *Lagrangian* of the system.

This reformulation of d'Alembert's principle called *Hamilton's principle* simply establishes that $\delta S = 0$, and can be expressed in words as:

> "Given the initial and final configurations resulting from the dynamics, the action must be stationary for all possible variations of the system's configuration".

This procedure, of finding the equations that describe the system's dynamics through a minimization of the action, or *principle of least action*, is common to classical, relativistic, and wave mechanics. The only thing that changes in each approach is how the Lagrangian is written.

6. Ecuación de Euler-Lagrange

Referencia: Lanczos II§10, Landau §I.2

¿Cómo obtener el valor estacionario de la acción S? La respuesta a como asegurar la condición $\delta S = 0$ está ligada al concepto de desplazamientos virtuales (ver sección **??**). Se trata justamente de explorar el conjunto de variaciones sobre $(q_i, \dot{q}_i, t)$ que efectivamente aseguren minimizar $S = \int \mathcal{L}(q_i, \dot{q}_i, t) \mathrm{d}t$.

Empezamos analizando la variación del integrando de S, es decir el Lagrangiano ante una variación de coordenadas generalizadas, $\overline{q_i(t)} = q_i(t) + \delta q_i = q_i(t) + \epsilon\tilde{q}_i(t)$, y velocidades generalizadas

$$\begin{aligned}\delta\mathcal{L}(q_i,\dot{q}_i,t) &= \mathcal{L}(q_i+\delta q_i,\dot{q}_i+\delta\dot{q}_i,t) - \mathcal{L}(q_i,\dot{q}_i,t) = \left(\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i\right)\\ &= \mathcal{L}(q_i+\epsilon\tilde{q}_i,\dot{q}_i+\epsilon\dot{\tilde{q}}_i,t) - \mathcal{L}(q_i,\dot{q}_i,t) = \epsilon\left(\frac{\partial\mathcal{L}}{\partial q_i}\tilde{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{\tilde{q}}_i\right).\end{aligned} \tag{Lanczos 210.1}$$

Con esto puede escribirse la variación de S como

$$\delta S = \delta\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t = \int_{t_1}^{t_2}\delta\mathcal{L}\,\mathrm{d}t = \epsilon\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q_i}\tilde{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{\tilde{q}}_i\right)\mathrm{d}t\,. \tag{Lanczos 210.2}$$

Puesto que son producto de una variación intencional arbitraria, $\tilde{q}(t)$ y $\dot{\tilde{q}}_i(t)$ no son independientes, pero la dependencia entre ambas no puede formularse de forma algebraica. Lo que si se puede es eliminar $\dot{\tilde{q}}_i(t)$ como variable independiente aplicando el método de integración por partes[XIX] al segundo término de (Lanczos 210.2)

$$\int_{t_1}^{t_2}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{\tilde{q}}_i\,\mathrm{d}t = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\tilde{q}_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\tilde{q}_i\,\mathrm{d}t\,. \tag{Lanczos 210.4}$$

El primer término de la ecuación (Lanczos 210.4) se anula, puesto que se evalúa en límites temporales definidos donde precisamente no hay variación posible ($\tilde{q}(t_1) = \tilde{q}(t_2) = 0$). Por tal razón la ecuación (Lanczos 210.2) se reduce a

$$\begin{aligned}\delta S &= \epsilon\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q_i}\tilde{q}_i - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\tilde{q}_i\right)\mathrm{d}t = \epsilon\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\tilde{q}_i\,\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\delta q_i\,\mathrm{d}t\,.\end{aligned} \tag{Lanczos 210.5}$$

Si S debe ser estacionaria, es decir $\delta S = 0$, es claro que la integral debe ser nula ante δq_i arbitrarios. Esto se cumple si y solo si la cantidad entre paréntesis se anula[XX]

$$\delta S = 0 \iff \frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0 \quad \forall i. \tag{6}$$

[XIX] En la expresión de integración por partes, cuya derivación figura en la nota al pie [XVII], $u = \dot{\tilde{q}}_i$ y $v = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$.

[XX] Si buscamos que S sea estacionaria, es decir $\delta S = 0$, tenemos que asegurar que la integral a la derecha de la ecuación (Lanczos 210.5) sea nula para valores arbitrarios de $\tilde{q}_i(t)$, y esto se cumplirá solo si la cantidad entre paréntesis $E(t) = \frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$, se anula entre t_1 y t_2.

La demostración matemática aprovecha el hecho de que podemos elegir un caso particular de todas las arbitrarias, $\tilde{q}_i(t)$, para que esta sea nula con la posible excepción de un corto intervalo ξ en torno a un determinado t del intervalo ($t_1 < t < t_2$). Esto haría que solo en este intervalo $q_i(t)$ pueda variar. En un intervalo corto de t cualquier función continua es casi constante, y por tanto $E(t)$ podría sacarse de la integral para aproximar la igualdad de la ecuación (Lanczos 210.5) como

$$\delta S \simeq \epsilon E(\xi)\int_{\xi+\rho}^{\xi-\rho}\tilde{q}_i(t)\,\mathrm{d}t\,. \tag{Lanczos 210.8}$$

El error respecto de la igualdad tiende a desaparecer a medida que ρ tiene a cero. Así que de respetarse la igualdad no quedaría más remedio que establecer que $E(\xi) = 0$, para cumplir con el supuesto de que $\delta S = 0$. Como $t = \xi$ puede ser cualquiera en el intervalo $t_1 < t < t_2$, concluimos que dentro de este intervalo siempre debiera cumplirse que $E(t) = 0$.

En definitiva, basta con variar los $q_i(t)$ hasta lograr dar con aquellos para que $E(t) = 0$ en todo el intervalo para asegurar $\delta S = 0$, es decir, esa es condición *necesaria* para asegurar que S sea estacionario. Además, esta condición es *suficiente* pues, como es evidente de revisar la ecuación (Lanczos 210.5), no importa que valor asuma $\tilde{q}_i(t)$ dentro del intervalo temporal, si allí $E(t) = 0$ la integración dará un valor nulo. Al probar que $E(t) = 0$ es condición es necesaria y suficiente para que $\delta S = 0$ significa que si la primera es cierta *si y solo si* la segunda también lo es; en resumen, una asegura la otra.

La demostración de esta relación fue realizada por Lagrange, y es fruto de aplicar a la dinámica el *principio de acción estacionaria* bajo la formulación que estableció Leonhard Euler[XXI]. En honor a ambos la ecuación a la derecha de (6) se denomina *ecuación de Euler-Lagrange*.

Llegamos así a construir una poderosa herramienta para determinar la dinámica de *cualquier sistema físico*: basta con idear su Lagrangiano $\mathcal{L}(q_i, \dot{q}_i, t)$, ecuación (Lanczos 51.11), y luego resolver la correspondiente ecuación de Euler-Lagrange.

7. Cantidades conservadas

Un concepto básico del análisis matemático es que si la derivada total de una función f respecto a alguna variable x es nula, $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ su integral da cuenta de que tal función es constante respecto a esa variable, $\int f \mathrm{d}x = \text{const.}$. En la mecánica es útil determinar tales constantes cuando la variable es el tiempo, pues hablamos entonces de cantidades que se mantienen inalteradas durante la dinámica del sistema. En particular si tal función es la derivada respecto de t de otra cantidad, $f = \frac{\partial g}{\partial t} = \dot{g}$, y se cumple $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$, esta última cantidad *se conserva* durante la dinámica, $\int f \mathrm{d}t = \int \dot{g} \mathrm{d}t = g = \text{const.}$

7.1. Coordenadas ignorables— Momentos generalizados

Referencia: Lanczos V§4

Puede que en el Lagrangiano no figure explícitamente alguna de las coordenadas generalizadas q_n, aunque si lo haga la correspondiente velocidad $\dot{q}_n$, así $\mathcal{L} = \mathcal{L}(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t)$. En ese caso la ecuación de Euler-Lagrange (6) se reduce, pues,

$$\frac{\partial \mathcal{L}}{\cancel{\partial} q_n} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_n} = \text{const.} \qquad \text{(Lanczos 54.2)}$$

Por ejemplo una partícula libre en el espacio tendrá una $T = \frac{m}{2}\dot{x}^2$ con $V = 0$, por lo que la coordenada x no figurará en el Lagrangiano, y $\frac{\partial \mathcal{L}}{\partial x} = mx$, cantidad escalar que corresponde a la magnitud del momento lineal $\vec{p}_x$. De forma similar la cantidad conservada de la ecuación (Lanczos 54.2) recibe el nombre de *momento generalizado*,

$$p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}, \qquad \text{(Lanczos 53.4)}$$

y permite escribir la correspondiente velocidad generalizada en función de este

$$\dot{q}_n = f(q_1, \ldots, q_{n_1}; \dot{q}_1 \ldots, \dot{q}_{n-1}; p_n; t). \qquad \text{(Lanczos 54.5)}$$

El obtener los momentos generalizados p_i antes de resolver las ecuaciones diferenciales de Euler-Lagrange permite simplificar el Lagrangiano y que estas ecuaciones resulten más sencillas de integrar. Básicamente cada vez que encontramos $\dot{q}_n$ en el Lagrangiano lo reemplazamos por el lado derecho de la igualdad (Lanczos 54.5).

7.2. Conservación de la energía

Analizaremos que cantidad se conserva al diferenciar el Lagrangiano en el tiempo. Consideremos el caso particular de un sistema que se denomina *esclerónomo*, en que el tiempo no figura explícitamente ni en la energía cinética ni en la función trabajo. Ergo en tales sistemas el Lagrangiano no tiene dependencia explícita con el tiempo, es decir $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$ y su derivada respecto al tiempo será

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\right) = \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right). \qquad \text{(Landau 6.1a)}$$

[XXI] Matemático suizo (Basilea 1707 - San Petersburgo 1783)

De la ecuación de Euler-Lagrange recordamos que $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}$. Realizamos ese reemplazo para obtener

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right) = \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}\dot{q}_i\right) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}\dot{q}_i\right). \tag{Landau 6.1b}$$

En esta última expresión identificamos $\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i} = p_i$, es decir, que se trata de un momento generalizado. Juntando ambas derivadas con el tiempo a izquierda de la igualdad llegamos a escribir una cantidad que no presenta variación en el tiempo

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}\dot{q}_i - \mathcal{L}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}\right) = 0, \tag{Landau 6.1c}$$

es decir, la cantidad entre paréntesis se conserva en el tiempo.

La cantidad a la derecha de la expresión (Landau 6.1c) resulta ser algo muy familiar en los sistemas mecánicos más usuales, en los que se cumple que:

1. T es cuadrático en las velocidades: $T = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(q_i, q_k)\dot{q}_i\dot{q}_k$ (e.g. $T = \frac{m}{2}\dot{x}^2$), y
2. V no depende de las velocidades: $\frac{\mathrm{d}V}{\mathrm{d}\dot{q}_i} = 0$.

Para estos sistemas la cantidad entre paréntesis de la ecuación (Landau 6.1c) se reduce a

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}_i}\dot{q}_i - \mathcal{L} &= \frac{\mathrm{d}(T-V)}{\mathrm{d}\dot{q}_i}\dot{q}_i - T + V\left(\frac{\mathrm{d}}{\mathrm{d}\dot{q}_i}T - \frac{\mathrm{d}}{\mathrm{d}\dot{q}_i}V\right)\dot{q}_i - T + V \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}\dot{q}_i}\frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(q_i, q_k)\dot{q}_i\dot{q}_k\right)\dot{q}_i - T + V \\
&= \left(\sum_{i,k=1}^{n} a_{ik}(q_i, q_k)\dot{q}_k\right)\dot{q}_i - T + V = 2T - T + V = T + V = E,
\end{aligned} \tag{Landau 6.2}$$

que no es otra cosa que la familiar *energía mecánica* E.

8. De la formulación Lagrangiana a la Hamiltoniana

8.1. Ecuaciones canónicas del movimiento — Hamiltoniano

Referencia: Lanczos VI§3

Vimos que podemos determinar la dinámica de un sistema descrito por un Lagrangiano con n coordenadas generalizadas q_i no ignorables resolviendo n ecuaciones de Euler-Lagrange. Para finalmente obtener $q_i = q_i(t)$ tendremos que resolver un sistema de n ecuaciones son de 2.º grado, lo que puede resultar complicado.

La formulación Hamiltoniana nos ofrece un camino alternativo por el que solo tendremos que resolver ecuaciones de 1.er grado, pero al costo de enfrentarnos a un sistema de $2n$ ecuaciones diferenciales, lo que puede resultar tedioso.

La cantidad entre paréntesis de la ecuación (Landau 6.1c)

$$\sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}, \tag{Lanczos 62.3}$$

se denomina Hamiltoniano cuando se expresan todos las velocidades generalizas $\dot{q}_i$ en función de los correspondientes momentos generalizados p_i

$$H = H(q_1, \ldots, q_n; p_i, \ldots, p_n; t). \qquad \text{(Lanczos 62.4)}$$

Para llegar a la expresión de tales ecuaciones primero analizamos el diferencial del Hamiltoniano[XXII]

$$\mathrm{d}H = \mathrm{d}\sum_{i=1}^{n}(p_i\dot{q}_i) - \mathrm{d}\mathcal{L} = \sum_{i=1}^{n}\mathrm{d}p_i\dot{q}_i + \sum_{i=1}^{n}p_i\mathrm{d}\dot{q}_i - \mathrm{d}\mathcal{L}. \qquad \text{(Landau 40.3)}$$

Recordemos que el Lagrangiano tiene por variables independientes las coordenadas y velocidades generalizadas, $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$, por lo que el diferencial del último término de la ecuación (Landau 40.3) es

$$\mathrm{d}\mathcal{L} = \sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial q_i}\mathrm{d}q_i + \sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\mathrm{d}\dot{q}_i = \sum_{i=1}^{n}\dot{p}_i\mathrm{d}q_i + \sum_{i=1}^{n}p_i\mathrm{d}\dot{q}_i, \qquad \text{(Landau 40.1)}$$

esto recordando la ecuación de Euler-Lagrange y la definición de momento generalizado,

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}p_i = \dot{p}_i. \qquad (7)$$

Reemplazamos ahora el resultado de la ecuación (Landau 40.1) en la (Landau 40.3)

$$\mathrm{d}H = \sum_{i=1}^{n}\mathrm{d}p_i\dot{q}_i + \cancel{\sum_{i=1}^{n}p_i\mathrm{d}\dot{q}_i} - \sum_{i=1}^{n}\dot{p}_i\mathrm{d}q_i - \cancel{\sum_{i=1}^{n}p_i\mathrm{d}\dot{q}_i} = \sum_{i=1}^{n}(\dot{q}_i\mathrm{d}p_i - \dot{p}_i\mathrm{d}q_i), \qquad \text{(Landau 40.3)}$$

El hecho de que en el Hamiltoniano coordenadas y momentos generalizados son variables independientes permite analizar el diferencial respecto a cada uno por separado

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ ,\ \dot{p}_i = -\frac{\partial H}{\partial q_i}. \qquad \text{(Lanczos 63.5)}$$

Estas son las 2 ecuaciones diferenciales de primer orden que corresponden a cada q_i. Carl Gustav Jacob Jacobi[XXIII] las nombró *ecuaciones canónicas* por la simple simetría que demuestran entre las así llamadas *variables conjugadas* q_i y p_i.

A. Notas

Aquí secciones aún por desarrollar.

A.1. Sistemas holónomos y no holónomos

Referencia: Lanczos II§6

Dado un condicionante a la dinámica del sistema, si la relación entre sus coordenadas puede darse solo en función de los diferenciales de estos, la condición es *no holónoma* en contraposición a aquellas que pueden darse en función de las coordenadas definidas,

$$f(q_1, \ldots, q_n) = 0, \qquad \text{(Lanczos 16.1)}$$

que denominamos *holónomas*. De diferenciar tal relación se obtiene que

$$\frac{\partial f}{\partial q_1}\mathrm{d}q_i + \ldots + \frac{\partial f}{\partial q_n}\mathrm{d}q_n = 0, \qquad \text{(Lanczos 16.2)}$$

[XXII] Para esta demostración obviamos la dependencia explícita con el tiempo del Hamiltoniano, $H = H(q_i, p_i)$

[XXIII] Matemático alemán (Potsdam 1804 - Berlín 1851)

. Pero si la condición se expresa de la forma

$$A_1(q_1, \ldots, q_n)\mathrm{d}q_i + \ldots + A_n(q_1, \ldots, q_n)\mathrm{d}q_n = 0, \qquad \text{(Lanczos 16.3)}$$

tal relación solo puede reducirse a la forma de la ecuación (Lanczos 16.2) correspondiente a una condición *holónoma*, de cumplirse determinadas condiciones de integración (excepto para $n = 2$ que siempre es integrable), de lo contrario estamos ante una condición *no holónoma*.

Un ejemplo de condición *no holónoma* es el de rodadura de una bola de billar sobre el paño bidimensional de la mesa. La posición de la bola puede darse en función de un par (x, y) de tal superficie, y su orientación (cuanto *rotó* respecto a dos ejes perpendiculares) en función de dos ángulos (α, β). La pregunta crucial es si a una determinada coordenada (x, y) corresponde una determinada orientación (α, β), y la respuesta es que no. Hay multitud de caminos partiendo de (x_0, y_0) que resultaran en distintos (α, β) que llevan al mismo (x, y). Entonces si bien los diferenciales de (α, β) son expresables en función de diferenciales de (x, y), tales relaciones diferenciables no son integrables.

A.1.1. Multiplicador de Lagrange

El formalismo analítico puede obtener la magnitud de una fuerza fruto de una condición no holónoma. Volvamos al concepto fundamental del valor estacionario para el Lagrangiano $\delta\mathcal{L} = 0$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \ldots + \frac{\partial\mathcal{L}}{\partial q_n}\delta q_n = 0; \qquad \text{(Lanczos 25.4)}$$

y le adicionamos la expresión de la ecuación (Lanczos 16.3), que expresa la relación a la que es inocuo multiplicar por un factor λ puesto que a fin de cuentas su suma es nula

$$\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \ldots + \frac{\partial\mathcal{L}}{\partial q_n}\delta q_n + \lambda[A_1(q_1, \ldots, q_n)\mathrm{d}q_i + \ldots + A_n(q_1, \ldots, q_n)\mathrm{d}q_n] = 0 \qquad \text{(Lanczos 25.5)}$$

(Aún no terminado.)

A.2. Fuerzas poligénicas vs. monogénicas

Referencia: Lanczos V§1

Para fuerzas poligénicas no puede transformarse el principio de d'Alembert en uno de valor estacionario. Ya que las condiciones holónomas es el equivalente mecánico a fuerzas monogénicas, y las no holónomas a poligénicas:

> El principio de Hamilton solo se sostiene para sistemas con fuerzas monogénicas y condiciones holónomas.

(Aún no terminado.)

A.3. Sistemas reónomos y esclerónomos, conservación de la energía

Referencia: Lanczos I§8

Todo sistema relevante a la dinámica tiene cantidades con dependencia en el tiempo, pero si tal dependencia es *explícita*, sea en T o en U, no podamos cumplir con las condiciones analizadas en la sección 7.2 para asegurar la conservación de la energía.

Ludwig Eduard Boltzmann[XXIV] acuño el término de *reónomo* para las condiciones dinámicas en que el tiempo figura explícitamente. En tal caso al recuperar las coordenadas cartesianas, la ecuación

[XXIV] Físico austríaco (Viena, 1844 - Tybein, 1906).

(Lanczos 12.8) presenta un t ineludible en al menos algunas de las f_{3N} condiciones

$$\begin{aligned} x_1 &= f_1(q_i, \ldots, q_n; t) \\ &\ldots \\ z_N &= f_{3N}(q_i, \ldots, q_n; t) \end{aligned} \qquad \text{(Lanczos 18.3)}$$

Idéntico resultado resulta de usar un marco de referencia en movimiento. Al derivar respecto al tiempo para obtener las velocidades figuran ahora términos $\frac{\partial f_i}{\partial t}$ que hacen que en T aparezcan términos no cuadráticos en $\dot{q}_i$, sean lineales (*términos giroscópicos*) o inclusive constantes.

Asimismo el tiempo puede figurar explícitamente en U. Sea cual fuere el mecanismo, la dependencia explícita en t hace que no podamos cumplir con las condiciones para la conservación de la energía, como analizamos en la sección 7.2, y entonces extendemos el término de reónomo a todo sistema en que esto suceda. El término contrapuesto, sistemas *esclerónomos*, corresponde a aquellos que comúnmente se los denomina *sistemas conservativos*.

Sistema	Condición	Consecuencia
Reonómico	$f(q_i, \ldots, q_n; t) = 0$ o $U = U(\dot{q}_i)$	$E \neq$ const.
Esclerónomo	$f(q_i, \ldots, q_n) = 0$ y $U \neq U(\dot{q}_i)$	$E =$ const.

Aun así la energía en un sistema reónomo sigue siendo la suma de T con una potencial V, pero esta última debe ser definida como

$$V = \sum_{i=1}^{n} \frac{\partial U}{\partial \dot{q}_i} q_i - U, \qquad \text{(Lanczos 18.5)}$$

y por supuesto no es constante.