Recent results on geproci sets

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Organizing Committee: Łucja Farnik, Giovanna Ilardi, Grzegorz Malara, Piotr Pokora, Hal Schenck, Paweł Solarz, Tomasz Szemberg, Justyna Szpond

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Slides available eventually at my website (green text is clickable):
https://unlblh.github.io/BrianHarbourne/

Main references (arXiv), reverse chronologically

University of Nebraka 2024 PhD thesis: Allison Ganger

2312.04644: Pietro De Poi, Giovanna Ilardi and POLITUS

2308.00761: POLITUS

2307.04857: Jake Kettinger

2303.16263: POLITUS

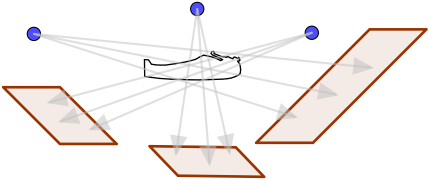
2209.04820: POLITUS

2107.08107: Paulina Fraś and Maciej Zięba

1904.02047: Luca Chiantini and Juan Migliore

POLITUS: Luca Chiantini, Lucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Tomasz Szemberg, Justyna Szpond

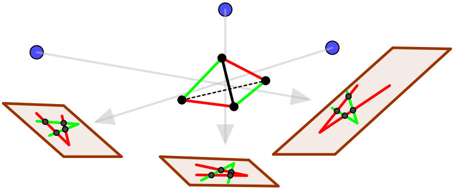
Tomography: an inverse scattering example



Apply Inverse Scattering perspective in Algebraic Geometry:

GePro-$\mathcal{P}$: Pick a property $\mathcal{P}$ and classify finite point sets $Z \subset \mathbb{P}^n$ whose $\boxed{\text{Ge}}$neral $\boxed{\text{Pro}}$jections $\overline{Z}$ to a hyperplane H satisfy $\boxed{\mathcal{P}}$.

Example: Geproci (i.e., $\mathcal{P}$ means: $\overline{Z}$ is a complete intersection).



Trivial examples of geproci

If Z is contained in a hyperplane and already a complete intersection, then it is geproci.

If $Z \subset \mathbb{P}^2$, then Z is geproci.

Question: What nontrivial examples of geproci $Z \subset \mathbb{P}^n$ are there (i.e., nondegenerate with $n > 2$)?

We know examples only for $n = 3$, in which case we say Z is (a, b)-geproci if $\overline{Z}$ is an (a, b) complete intersection with $a \leq b$.

Relevance to WLP

Examples (see, e.g., arXiv:1904.02047) suggest if $Z = \{p_1, \ldots, p_s\} \subset \mathbb{P}^3$ is nontrivial (a, b)-geproci with $a \geq 2$ and $b > 2$, then Z gives an example of failure of WLP.

In particular, these examples suggest $\frac{R}{(L_{p_1}^a, \ldots, L_{p_s}^a)}$ fails the Weak Lefschetz Property (WLP) in degree $a - 1$.

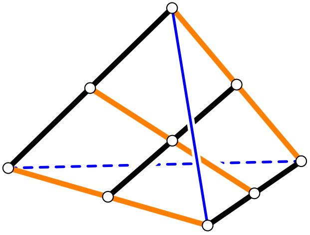
This means $\times L_P$ (where P is a general point) does not have maximal rank:

$$\left[\frac{R}{(L_{p_1}^a, \ldots, L_{p_s}^a)}\right]_{a-1} \xrightarrow{\times L_P} \left[\frac{R}{(L_{p_1}^a, \ldots, L_{p_s}^a)}\right]_{a}$$

Open Problem: Prove this rigorously.

There are 3 kinds of nontrivial geproci in $\mathbb{P}^3$

Grids: An (a, b)-grid Z has $2 \leq a \leq b$. It is $Z = A \cap B$ where A is a space curve consisting of a skew black lines and B is a space curve consisting of b skew orange lines and each black line intersects each orange line in exactly 1 point. Note that $\overline{Z} = \overline{A} \cap \overline{B}$. (In the figure $a = b = 3$.)

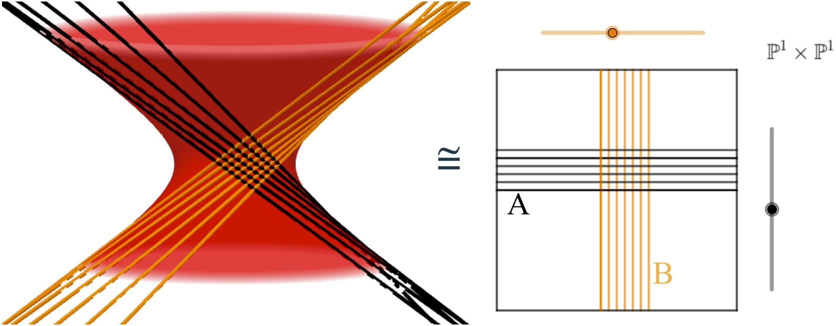


Half grids: Here Z is (a, b)-geproci, not a grid and consists of a points on each of b skew lines (i.e., we have B) or it consists of b points on each of a skew lines (i.e., we have A), but we don't have both A and B. I.e., $\overline{Z} = C \cap D$ is a complete intersection of curves $C, D \subset H$ but only one of the curves is the image of a space curve containing Z and consisting of lines.

Nondegenerate nongrid non-half grids: more on these later

Grids are well understood

Fact: For an (a, b)-grid with $3 \leq a \leq b$, the grid lines come from the rulings on a smooth quadric.



Fact: A $(2, b)$-grid consists of $b \geq 2$ points on each of two skew lines (but the grid lines need not all lie on a smooth quadric).

Half Grids are partly understood

Theorem (POLITUS): For every $n \geq 3$, there is an $(n, n+1)$-geproci half grid of n points on each of $n+1$ skew lines (which POLITUS calls the "standard construction"). For $n = 3$, this is the only half grid and comes from the D_4 root system.

Theorem (De Poi, Ilardi, POLITUS): All complex $(4, r)$-geproci half grids on r skew lines with transversals have $r \leq 6$ and arise in only two explicitly described ways, related to the D_4 and F_4 root systems.

Theorem (Kettinger): For any finite field F, let $|F| = q$. Then $Z = \mathbb{P}^3_F \subset \mathbb{P}^3_{\overline{F}}$ is a $(q+1, q^2+1)$-geproci half grid on q^2+1 skew lines (which can be taken to come from a kind of Hopf fibration). E.g., if $q = 3$, Z is a $(4, 10)$-geproci half grid on 10 skew lines.

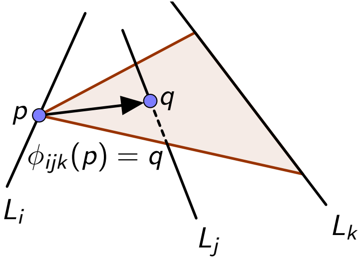
Theorem (Ganger): The half grid skew lines of the standard construction also can (up to projective equivalence) be taken to come from the Hopf fibration.

Combinatorics of skew lines: groupoids

Open Question: When are finitely many skew lines the half grid lines of a half grid?

Groupoid: A category $\mathcal{G}$ whose arrows all are invertible.

Example: Skew lines $\mathcal{L} = \{L_1, \ldots, L_r\}$, $r \geq 3$, give a groupoid $\mathcal{G}_{\mathcal{L}}$.
The lines L_i are the Objects. Define arrows $\phi_{ijk} : L_i \xrightarrow{L_k} L_j$:



Then $\mathrm{Hom}(L_i, L_j) =$ all possible compositions $\phi_{j_s j k_{s+1}} \cdots \phi_{j_1 j_2 k_2} \phi_{i j_1 k_1}$.

Note: $\mathrm{Hom}(L_i, L_i)$ is a group, the group of the groupoid.

Open Problem: When is the group finite?

Groupoid orbits, geproci half grids and the Hopf fibration

The groupoid $\mathcal{G}_{\mathcal{L}}$ acts on points of the skew lines $\mathcal{L} = \{L_1, \dots, L_r\}$, so we can talk about groupoid orbits.

Theorem (POLITUS): A geproci half grid is a union of groupoid orbits on the half grid lines.

Examples (Ganger's thesis):

(1) If F is a finite field, then the points $Z = \mathbb{P}^3_F$ form a single groupoid orbit on the skew lines coming from the Hopf fibration.

(2) Up to projective equivalence, the half grid lines of the standard construction can be chosen to be fibers of the Hopf fibration and then the half grid points form a single groupoid orbit on these lines.

The Hopf fibration

The original Hopf fibration comes from the field extension $\mathbb{R} \subset \mathbb{C}$:

$$S^3 \to \mathbb{P}^3_{\mathbb{R}} = \mathbb{P}(\mathbb{C} \oplus \mathbb{C})_{\mathbb{R}} \to \mathbb{P}(\mathbb{C} \oplus \mathbb{C})_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} = S^2.$$

More generally: let $F \subset K$ be any degree 2 field extension. Then:

- K is a 1 dimensional K and a 2 dimensional F vector space;
- $K \oplus K$ is a 2 dimensional K vector space;
- $K \oplus K$ is a 4 dimensional F vector space;

and we get a canonical "Hopf fibration" map

$$\mathbb{P}^3_F = \mathbb{P}(K \oplus K)_F \to \mathbb{P}(K \oplus K)_K = \mathbb{P}^1_K$$

where the fibers are collinear sets of points.

Theorem (Ganger): When $F \subset K$ is a degree 2 extension of finite fields, the group of the groupoid on the fibers of the Hopf fibration is K^*/F^*, hence cyclic of order $\frac{|F|^2-1}{|F|-1} = |F| + 1$.

More combinatorics

Consider $\mathbb{P}^3_F$ over a finite field F. In combinatorics, skew lines $L_1, \ldots, L_r$ in $\mathbb{P}^3_F$ with each L_i defined over F is called a *spread*.

If every point of $\mathbb{P}^3_F$ is in some line it is a *full* spread, otherwise a partial spread.

A spread $L_1, \ldots, L_r$ is *maximal* if every F-line L meets some line L_i.

Problems partially addressed by combinatorists:
Count the number of full spreads up to projective equivalence.
(The Hopf fibration always gives 1; usually there are others. Hence $Z = \mathbb{P}^3_F$ is usually a half grid in more than one way.)

More generally, count the number of maximal spreads up to projective equivalence.

Problems not yet addressed by combinatorists:
Study the groupoid for maximal spreads. For example, when is the group nonabelian?

Nondegenerate nongrid non-half grid geproci sets

Very few examples are known in characteristic 0:

(1) The H_4 root system gives a $(5, 12)$-geproci (Fraś and Zięba).

(2) A $(5, 8)$-geproci set (arxiv:2209.04820).

(3) A $(10, 12)$-geproci set (arxiv:2209.04820).

Kettinger gives more examples in characteristic $p > 0$ using maximal partial spreads.

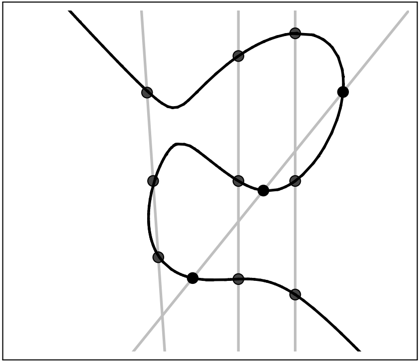
Open Problem: Are there more examples in characteristic 0?

The $Z = Z_{D_4}$ half grid

This Z is in the intersection of combinatorics, representation theory and algebraic geometry:

It's the smallest complex half grid, given by the standard construction for $n = 3$ (and hence by the groupoid action on fibers coming from the Hopf fibration).

$\overline{Z}$ is the complete intersection of 4 lines with an irreducible cubic:

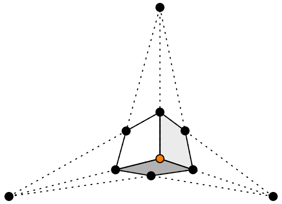


Visualizing Z_{D_4}

The D_4 root system consists of the 24 vectors obtained by permuting $(\pm 1, \pm 1, 0, 0) \in \mathbb{R}^4$.

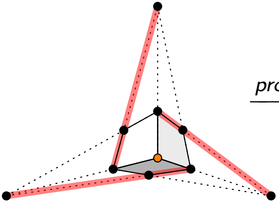
These give the 12 points of $Z_{D_4} \subset \mathbb{P}^3_{\mathbb{R}}$ (i.e., the permutations of $[\pm 1 : \pm 1 : 0 : 0]$, but note that $[1 : 1 : 0 : 0] = [-1 : -1 : 0 : 0]$).

Up to change of coordinates these 12 points can be visualized as a cube in 3 point perspective:

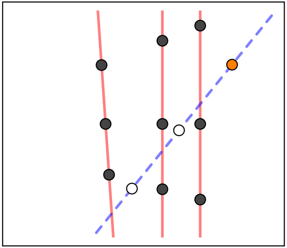


Why Z_{D_4} is geproci

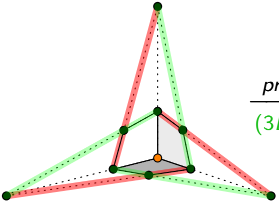
The quartic comes from lines through collinear points:



project to plane $\longrightarrow$

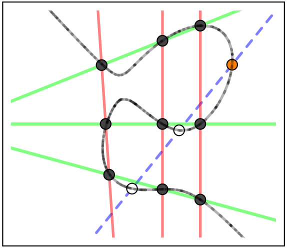


The cubic is one in a pencil of cubics:



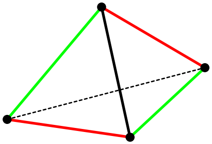
project to plane $\longrightarrow$

(3*D* *visualization*)



Some open problems

A $(2,2)$-grid is a nontrivial geproci set of 4 linearly general points:



No other nontrivial geproci set that we know of is linearly general.

Open problem: Find a nontrivial linearly general geproci set or prove none exist.

Example: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in a hyperplane, which is Gorenstein.

Open Problem: Classify gepro-Gorenstein sets Z.

Every geproci set is also gepro-Gorenstein but not conversely.

Thanks for your attention!

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