

Recent results on geproci sets

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Organizing Committee: Łucja Farnik, Giovanna Ilardi, Grzegorz Malara, Piotr Pokora, Hal Schenck, Paweł Solarz, Tomasz Szemberg, Justyna Szpond

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Slides available eventually at my website (green text is clickable):

<https://unlblh.github.io/BrianHarbourne/>

Main references (arXiv), reverse chronologically

University of Nebraska 2024 PhD thesis: Allison Ganger

2312.04644: Pietro De Poi, Giovanna Ilardi and POLITUS

2308.00761: POLITUS

2307.04857: Jake Kettinger

2303.16263: POLITUS

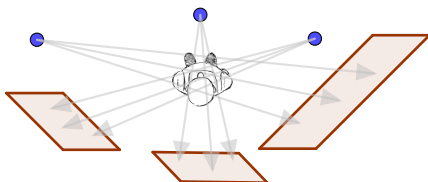
2209.04820: POLITUS

2107.08107: Paulina Fraś and Maciej Zięba

1904.02047: Luca Chiantini and Juan Migliore

POLITUS: Luca Chiantini, Lucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Tomasz Szemberg, Justyna Szpond

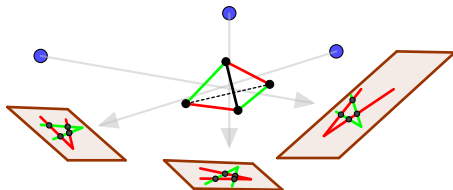
Tomography: an inverse scattering example



Apply Inverse Scattering perspective in Algebraic Geometry:

GePro- \mathcal{P} : Pick a property \mathcal{P} and classify finite point sets $Z \subset \mathbb{P}^n$ whose General Projection \bar{Z} to a hyperplane H satisfy \mathcal{P} .

Example: Geproci (i.e., \mathcal{P} means: \bar{Z} is a complete intersection).



Trivial examples of geproci

If Z is contained in a hyperplane and already a complete intersection, then it is geproci.

If $Z \subset \mathbb{P}^2$, then Z is geproci.

Question: What nontrivial examples of geproci $Z \subset \mathbb{P}^n$ are there (i.e., nondegenerate with $n > 2$)?

We know examples only for $n = 3$, in which case we say Z is (a, b) -geproci if \overline{Z} is an (a, b) complete intersection with $a \leq b$.

Relevance to WLP

Examples (see, e.g., arXiv:1904.02047) suggest if $Z = \{p_1, \dots, p_s\} \subset \mathbb{P}^3$ is nontrivial (a, b) -geproci with $a \geq 2$ and $b > 2$, then Z gives an example of failure of WLP.

In particular, these examples suggest $\frac{R}{(L_{p_1}^a, \dots, L_{p_s}^a)}$ fails the Weak Lefschetz Property (WLP) in degree $a - 1$.

This means $\times L_P$ (where P is a general point) does not have maximal rank:

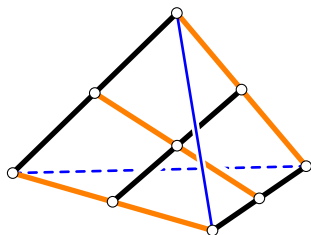
$$\left[\frac{R}{(L_{p_1}^a, \dots, L_{p_s}^a)} \right]_{a-1} \xrightarrow{\times L_P} \left[\frac{R}{(L_{p_1}^a, \dots, L_{p_s}^a)} \right]_a$$

Open Problem: Prove this rigorously.

There are 3 kinds of nontrivial geproci in \mathbb{P}^3

Grids: An (a, b) -grid Z has $2 \leq a \leq b$.

It is $Z = A \cap B$ where A is a space curve consisting of a skew black lines and B is a space curve consisting of b skew orange lines and each black line intersects each orange line in exactly 1 point. Note that $\overline{Z} = \overline{A} \cap \overline{B}$. (In the figure $a = b = 3$.)

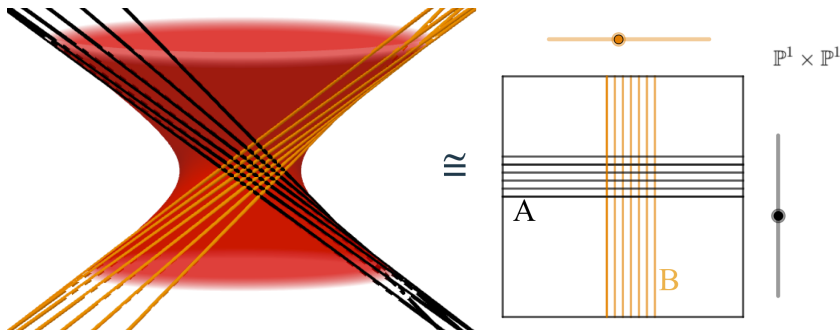


Half grids: Here Z is (a, b) -geproci, not a grid and consists of a points on each of b skew lines (i.e., we have B) or it consists of b points on each of a skew lines (i.e., we have A), but we don't have both A and B . I.e., $\overline{Z} = C \cap D$ is a complete intersection of curves $C, D \subset H$ but only one of the curves is the image of a space curve containing Z and consisting of lines.

Nondegenerate nongrid non-half grids: more on these later

Grids are well understood

Fact: For an (a, b) -grid with $3 \leq a \leq b$, the grid lines **come from the rulings** on a smooth quadric.



Fact: A $(2, b)$ -grid consists of $b \geq 2$ points on each of two skew lines (but the grid lines need not all lie on a smooth quadric).

Half Grids are partly understood

Theorem (POLITUS): For every $n \geq 3$, there is an $(n, n+1)$ -geproci half grid of n points on each of $n+1$ skew lines (which POLITUS calls the “standard construction”). For $n = 3$, this is the only half grid and comes from the D_4 root system.

Theorem (De Poi, Ilardi, POLITUS): All complex $(4, r)$ -geproci half grids on r skew lines with transversals have $r \leq 6$ and arise in only two explicitly described ways.

Theorem (Kettinger): For any finite field F , let $|F| = q$. Then $Z = \mathbb{P}_F^3 \subset \mathbb{P}_{\overline{F}}^3$ is a $(q+1, q^2+1)$ -geproci half grid on q^2+1 skew lines; these skew lines come from a kind of Hopf fibration. In particular, if $q = 3$, Z is a $(4, 10)$ -geproci half grid on 10 skew lines.

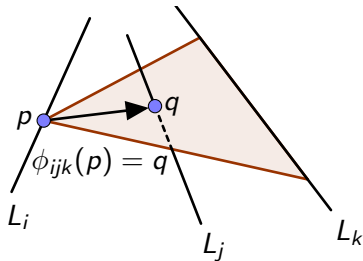
Theorem (Ganger): The half grid skew lines of the standard construction also come (up to projective equivalence) from the Hopf fibration.

Combinatorics of skew lines: groupoids

Open Question: When are finitely many skew lines the half grid lines of a half grid?

Groupoid: A category \mathcal{G} whose arrows all are invertible.

Example: Skew lines $\mathcal{L} = \{L_1, \dots, L_r\}$ give a groupoid $\mathcal{G}_{\mathcal{L}}$. The lines L_i are the Objects. Define arrows $\phi_{ijk} : L_i \xrightarrow{L_k} L_j$:



Then $\text{Hom}(L_i, L_j) =$ all possible compositions
 $\phi_{j_s j_{s+1}} \cdots \phi_{j_1 j_2} \phi_{ij_1 k_1}.$

Note: $\text{Hom}(L_i, L_i)$ is a group, the group of the groupoid.

Open Problem: When is the group finite?

Groupoid orbits, geproci half grids and the Hopf fibration

The groupoid $\mathcal{G}_{\mathcal{L}}$ acts on points of the skew lines

$\mathcal{L} = \{L_1, \dots, L_r\}$, so we can talk about groupoid orbits.

Theorem (POLITUS): A geproci half grid is a union of groupoid orbits on the half grid lines.

Examples (Ganger's thesis):

- (1) If F is a finite field, then the points $Z = \mathbb{P}_F^3$ form a single groupoid orbit on the skew lines coming from the Hopf fibration.
- (2) Up to projective equivalence, the half grid lines of the standard construction can be chosen to be fibers of the Hopf fibration and then the half grid points form a single groupoid orbit on these lines.

The Hopf fibration

The original Hopf fibration comes from the field extension $\mathbb{R} \subset \mathbb{C}$:

$$S^3 \rightarrow \mathbb{P}_{\mathbb{R}}^3 = \mathbb{P}(\mathbb{C} \oplus \mathbb{C})_{\mathbb{R}} \rightarrow \mathbb{P}(\mathbb{C} \oplus \mathbb{C})_{\mathbb{C}} = \mathbb{P}_{\mathbb{C}}^1 = S^2.$$

More generally: let $F \subset K$ be any degree 2 field extension. Then:

- K is a 1 dimensional K and a 2 dimensional F vector space;
- $K \oplus K$ is a 2 dimensional K vector space;
- $K \oplus K$ is a 4 dimensional F vector space;

and we get a canonical “Hopf fibration” map

$$\mathbb{P}_F^3 = \mathbb{P}(K \oplus K)_F \rightarrow \mathbb{P}(K \oplus K)_K = \mathbb{P}_K^1$$

where the fibers are collinear sets of points.

Theorem (Gangner): When $F \subset K$ is a degree 2 extension of finite fields, the group of the groupoid on the fibers of the Hopf fibration is K^*/F^* , hence cyclic of order $\frac{|F|^2-1}{|F|-1} = |F| + 1$.

More combinatorics

Consider \mathbb{P}_F^3 over a finite field F . In combinatorics, skew lines L_1, \dots, L_r in \mathbb{P}_F^3 with each L_i defined over F is called a *spread*.

If every point of \mathbb{P}_F^3 is in some line it is a *full spread*, otherwise a *partial spread*.

A spread L_1, \dots, L_r is *maximal* if every F -line L meets some line L_i .

Problems partially addressed by combinatorists:

Count the number of full spreads up to projective equivalence.

(The Hopf fibration always gives 1; usually there are others. Hence $Z = \mathbb{P}_F^3$ is usually a half grid in more than one way.)

More generally, count the number of maximal spreads up to projective equivalence.

Problems not yet addressed by combinatorists:

Study the groupoid for maximal spreads. For example, when is the group nonabelian?

Nondegenerate nongrid non-half grid geproci sets

Very few examples are known in characteristic 0:

- (1) The H_4 root system gives a $(5, 12)$ -geproci (Fraś and Zięba).
- (2) A $(5, 8)$ -geproci set (arxiv:2209.04820).
- (3) A $(10, 12)$ -geproci set (arxiv:2209.04820).

Kettinger gives more examples in characteristic $p > 0$ using maximal partial spreads.

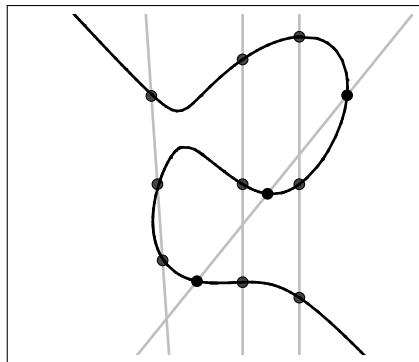
Open Problem: Are there more examples in char 0?

The $Z = Z_{D_4}$ half grid

This Z is in the intersection of combinatorics, representation theory and algebraic geometry:

It's the smallest complex half grid, given by the standard construction for $n = 3$ (and hence by groupoid action on fibers of the Hopf fibration).

\overline{Z} is the complete intersection of 4 lines with an irreducible cubic:

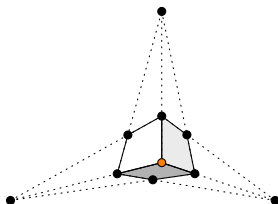


Visualizing Z_{D_4}

The D_4 root system consists of the 24 vectors obtained by permuting $(\pm 1, \pm 1, 0, 0) \in \mathbb{R}^4$.

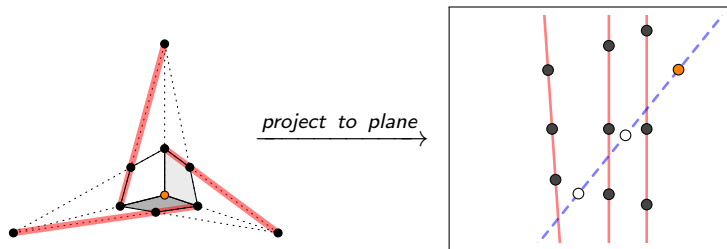
These give the 12 points of $Z_{D_4} \subset \mathbb{P}_{\mathbb{R}}^3$ (i.e., the permutations of $[\pm 1 : \pm 1 : 0 : 0]$, but note that $[1 : 1 : 0 : 0] = [-1 : -1 : 0 : 0]$).

Up to change of coordinates these 12 points can be visualized as a cube in 3 point perspective:

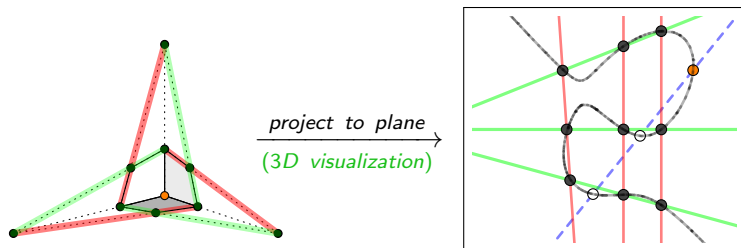


Why Z_{D_4} is geproci

The quartic comes from lines through collinear points:

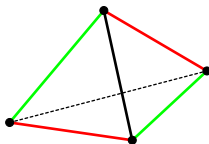


The cubic is one in a pencil of cubics:



Some open problems

A $(2, 2)$ -grid is a nontrivial geproci set of 4 linearly general points:



No other nontrivial geproci set that we know of is linearly general.

Open problem: Find a nontrivial linearly general geproci set or prove none exist.

Example: Say \mathcal{P} means “ \overline{Z} is Gorenstein”. Then a set Z of $n + 1$ general points in \mathbb{P}^n is gepro- \mathcal{P} since the image \overline{Z} is a set of $n + 1$ general points in a hyperplane, which is Gorenstein.

Open Problem: Classify gepro-Gorenstein sets Z .

Every geproci set is also gepro-Gorenstein but not conversely.

Thanks for your attention!

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