Roots: From Rick to Recent Research

Brian Harbourne

Cather Professor of Mathematics
Fellow of the AMS
Department of Mathematics
University of Nebraska-Lincoln (UNL)

Joy in Algebraic Geometry:
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Levico Terme

Introduction

Root systems have a history of appearing in algebraic geometry in interesting ways.

They have come up (at least tangentially) in Rick's work and they have come up in some recent research, in a way related to a problem Rick has worked on.

I want to tell you about this recent research, but first let me highlight some of Rick's related work.

So let's go way back...

Rick, Jeanne and I shared a grad school office in the 70s

Me Rick Jeanne MIT, 2-229

The office is no longer there. James Simons gave MIT a ton of money to get rid of it.

We aren't the same either! Can you spot Rick?

From 1976:

Thanks to Barbara Peskin for the photo!

In those days Rick worked on GIT for cubic pencils

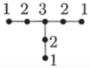
Thesis: "On the stability of rational elliptic surfaces with section". '79

1st paper: "On the stability of pencils of cubic curves". Amer. J. '80

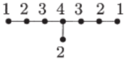
Theorem: A pencil is stable (GIT sense) if and only if it contains a smooth member and every fiber of the elliptic surface obtained by blowing up base points is reduced.

The non-stable fibers are reducible, classified by Dynkin diagrams of affine root systems:

IV^* $(\tilde{E}_6)$

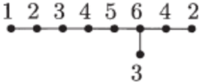


$III^*(\tilde{E}_7)$

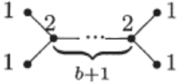


I_0^* $(\tilde{D}_4)$

II^* $(\tilde{E}_8)$



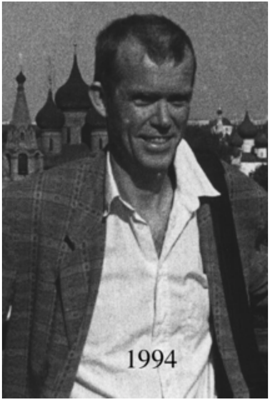
I_b^* $(\tilde{D}_{4+b})$



Rick and Ulf Persson

"On extremal rational elliptic surfaces," Math. Z. '86

Results: Rick and Ulf studied cubic pencils with finitely many sections.



extremal = finitely many (-1)-curves

σ = number of (-1)-curves

Jacobian = has sections so no multiple fibers

This paper gives a nice formula for σ in terms of the fibers F.

r_F^2 = number of reduced components of F

Theorem: $\sigma = \prod_F r_F$

The non-Jacobian case...Rick and Me

"Exceptional curves on rational numerically elliptic surfaces," J. Alg. '90

A non-Jac. extremal rational elliptic surf. has unique multiple fiber.

m = multiplicity of multiple fiber

$r_F(t)$ = power series defined in terms of the Dynkin diagram of F

$\pi(t) = \prod_F r_F(t)$ = the Hadamard product: $\sum_i a_i t^i \sum_i b_i t^i = \sum_i a_i b_i t^i$

$(\pi(t))_m$ = coefficient of t^m in $\pi(t)$

Theorem: Except in special cases

$$\sigma = \left(\prod_F r_F(t) \right)_m .$$

(Results are given in all cases, but the full result is more complicated.)

Interpolation problems

Ciro, Quim, Rick and Olivia have all done work on this.

Interpolation problems (continued)

R_d: all forms on $\mathbb{P}^2$ of degree d, so $R = \mathbb{C}[\mathbb{P}^2] = \oplus_d R_d$.

$\overline{m}\overline{P} = m_1 P_1 + \cdots + m_s P_s \subset \mathbb{P}^2$: scheme defined by all forms vanishing to order $\geq m_i$ at general points P_i.

$[I(\overline{m}\overline{P})]_d = I(\overline{m}\overline{P}) \cap R_d$, so $I(\overline{m}\overline{P}) \subset R$ is the ideal of $\overline{m}\overline{P}$.

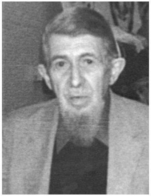
Problem: Given $\overline{m} = (m_1, \ldots, m_s)$ and d, $\boxed{\text{find } \dim[I(\overline{m}\overline{P})]_d.}$

There's a Cremona group G which acts to reduce the data $(d, m_1, \ldots, m_s)$ to the case (*) $d \geq m_1 + m_2 + m_3$ with $m_1 \geq m_2 \geq \cdots \geq m_s \geq 0$.

SHGH Conjecture: Given (*), then

$$\dim[I(\overline{m}\overline{P})]_d = \max\left\{0, \dim R_d - \sum_{m_i > 0} \binom{m_i + 1}{2}\right\}.$$

This group G is the Weyl group of a Dynkin diagram. See "On the Kantor group of a set of points in a plane", PLMS '37,
Patrick Du Val, right, for the case $s \leq 8$.



SHGH: Current status

B. Segre 1961

BH 1986

A. Gimigliano 1987

A. Hirschowitz 1989

Theorem All four versions are equivalent. (Ciro and Rick, 2001)

Situation understood for $s \leq 9$. (Castelnuovo, 1891)

The conjecture is true when $12 \geq m_1 = \cdots = m_s$ (Ciro and Rick, 2000) and for $m_1 = \cdots = m_{10}$ when $\frac{d}{m_1} < \frac{117}{37}$ (Ciro, Rick, Quim, Olivia, 2011).

But it's still open!

Idea: Could it help to study versions of a more general problem?

Unexpectedness: Cook, H., Migliore, Nagel, Compositio '18

Original SHGH problem: $n = 2$, $R_d = [I(Z)]_d$ for $Z = \varnothing$, $\overline{m}\overline{P} \subset \mathbb{P}^2$, P_i general. We say Z has *unexpected curves* in degree d and multiplicity $\overline{m}$ if

$$\dim[I(Z \cup \overline{m}\overline{P})]_d > \max\{0, \dim[I(Z)]_d - \sum_i \binom{m_i + 1}{2}\}.$$

The SHGH Conjecture accounts for all known unexpectedness.

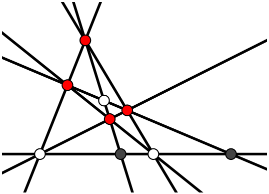
New problem: any n, $Z = q_1 + \cdots + q_r \subset \mathbb{P}^n$ any given points q_i, $mP \subset \mathbb{P}^n$, P general. We say Z has *unexpected hypersurfaces* in degree d and multiplicity m if

$$\dim[I(Z \cup mP)]_d > \max\{0, \dim[I(Z)]_d - \binom{m + n - 1}{n}\}.$$

Problem (CHMN): To understand when Z, m, d is unexpected.

CHMN

First example (Di Gennaro, Ilardi, Vallès, '14)



Z is a set of 9 points in $\mathbb{P}^2$ consisting of:

4 general points (red) which give a pencil pf conics;

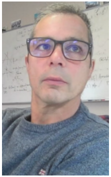
3 points (white), singular points of the singular conics;

2 points (black) where a singular conic meets the line through the singular points of the other 2 singular conics.

Fact: Z has an unexpected quartic with a general triple point; we expect no such quartic. What does this have to do with roots?

R. Di Gennaro

G. Ilardi



J. Vallès

Root systems and unexpectedness

See H., Migliore, Nagel, Teitler, Mich. J. '21

Projectivizing the B_3 root system gives the 9 points; $Z = Z(B_3)$!

Projectivizing D_4 gives 12 points $Z(D_4) \subset \mathbb{P}^3$ with two unexpected cones: a cubic and a quartic.

Projectivizing F_4 gives 24 points $Z(F_4) \subset \mathbb{P}^3$ with two unexpected cones: a quartic and a sextic.

Projectivizing H_4 gives 60 points $Z(H_4) \subset \mathbb{P}^3$ with two unexpected cones: a sextic and decic (see Wiśniewska-Zięba, arXiv:2107.08107).

Trying to understand unexpectedness is an expanding area of research. For example ...

Teitler and a new recruit

P. Wiśniewska

M. Zięba

General projections to a complete intersection: Geproci

Definition: We say a finite set $Z \subset \mathbb{P}^3$ is (a, b)-*geproci* if its image $\overline{Z}$ under projection to a plane from a general point $P \in \mathbb{P}^3$ is an (a, b)-complete intersection.

Theorem (Levico Terme Working Group, 2018): A finite noncoplanar $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $|Z| = ab$ and Z has unexpected cones of degrees $a \leq b$ with no common components.

Corollary: $Z(D_4)$, $Z(F_4)$ and $Z(H_4)$ are geproci!

Open Problems: (1) What other kinds of sets are geproci?

(2) If $Z \subset \mathbb{P}^3$ is noncoplanar and (a, b)-geproci with $3 \leq a \leq b$, must Z have unexpected cones of degrees a and b?

(3) Does every noncoplanar geproci set Z have a nontrivial matroid (e.g., does it have subsets of 3 collinear points)?

Levico Terme Working Group, 2018

Alessandra
Bernardi

Luca
Chiantini

Graham
Denham

Giuseppe
Favacchio

Brian
Harbourne

Juan
Migliore

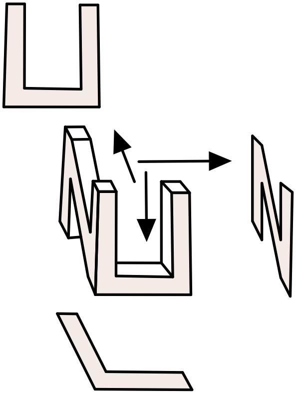
Tomasz
Szemberg

Justyna
Szpond

Classifying geproci sets is an inverse scattering problem

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry.

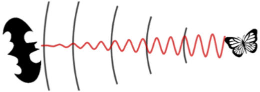
Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.



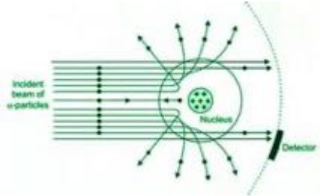
Idea: classify structures algebro-geometrically based on properties of projected images.

Some examples of ISP

Echolocation (biology):

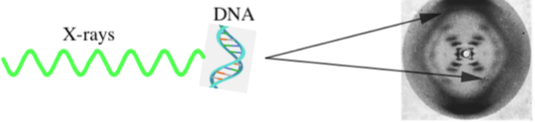


Rutherford scattering (physics; led to Bohr model of atom):



X-ray crystallography (chem/bio; led to DNA double helix model):

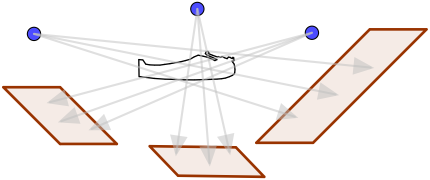
X-ray diffraction pattern



Rosalind Franklin
"Photo 51"

More examples

Tomography (medicine):



GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets $Z \subset \mathbb{P}^n$ whose $\boxed{\text{Ge}}$neral $\boxed{\text{Pro}}$jections $\overline{Z}$ to a hyperplane satisfy $\boxed{\mathcal{P}}$.

Example 1: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in H, which is Gorenstein.

Open Problem 1: Classify gepro-Gorenstein sets Z.

Geproci is when $\mathcal{P}$ means "$\overline{Z}$ is a complete intersection". Every geproci set is also gepro-Gorenstein but not conversely.

Open Problem 2: Classify geproci sets in $\mathbb{P}^n$ for $n \geq 3$.

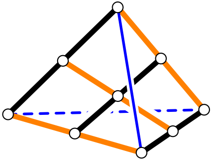
Some history: Polizzi and Panov

We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

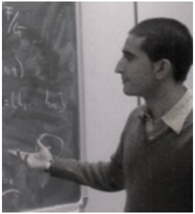
A geproci set Z in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in H.

Question 1 (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

Answer (D. Panov, 2011): No!
Nondegenerate geproci sets are given by (a, b)-grids (i.e., sets Z of ab points where $Z = A \cap B$ and A consists of a skew lines and B consists of b skew lines).



Francesco Polizzi



and Dimitri Panov

What we now know: POLITUS and Kettinger

Call a geproci $Z \subset \mathbb{P}^3$ *trivial* if it is a grid or degenerate.

Call a nontrivial geproci Z a *half-grid* when $\overline{Z}$ is the complete intersection of two curves if one curve is a union of lines.

Most known examples over $\mathbb{C}$ are half-grids, including $Z(D_4)$ and $Z(F_4)$. The POLITUS research group has found many more half-grids closely related to the combinatorics of skew lines in $\mathbb{P}^3$.

We know only 3 examples over $\mathbb{C}$ of nontrivial geproci non-half-grids: $Z(H_4)$ and two examples used in quantum mechanics, a $(5, 8)$-geproci and a $(10, 12)$-geproci. Things are different in positive characteristics.

Theorem(Kettinger arXiv:2307.04857): Let $\mathbb{F}$ be a finite field, $q = |\mathbb{F}|$. Then the points of $\mathbb{P}^3_{\mathbb{F}}$ are $(q + 1, q^2 + 1)$-geproci. Moreover, for odd $q \geq 7$ there are nontrivial non-half-grid $(q + 1, d)$-geproci sets $Z \subset \mathbb{P}^3_F$ whenever $q + 1 < d < \frac{q^2+1}{2} - 6$.

Jake Kettinger

Jake's results are closely related to the notion of a *spread* (a topic in combinatorics), namely the $(q+1)b$ points on a set of $b \leq q^2 + 1$ skew lines in $\mathbb{P}^3_{\mathbb{F}}$ over a finite field $\mathbb{F}$ of order q. Thanks to the Hopf fibration, $\mathbb{P}^3_{\mathbb{F}}$ is itself a spread. Jake's nontrivial non-half-grids come from maximal partial spreads.

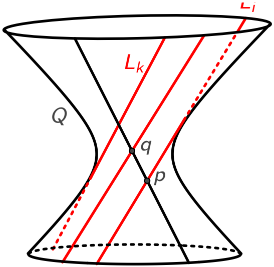
Main POLITUS references

POLITUS 1: arXiv:2209.04820 POLITUS 2: arXiv:2308.00761

Łucja and Karolina Farnik, Tomasz Szemberg, Justyna Szpond, Luca Chiantini, Giuseppe Favacchio, Juan Migliore, Me.

Combinatorics of skew lines? POLITUS 2

Any set $\mathcal{L} = \{L_1, L_2, L_3, \ldots, L_b\}$ of skew lines $L_i \subset \mathbb{P}^3$ has an associated groupoid $G(\mathcal{L})$ generated by the maps $\gamma_{ijk} : L_i \to L_j$ where, given $p \in L_i$, $\gamma_{ijk}(p)$ is the point $q \in L_j$ such that $\overline{pq}$ is contained in the unique quadric Q containing L_i, L_j, L_k.



If $p \in \cup_i L_i$, the images $\phi(p)$ of p under all the maps $\phi \in G(\mathcal{L})$ for which $\phi(p)$ is defined is the *orbit* of p under the groupoid.

Theorem (POLITUS 2) A geproci half-grid of a points on each of the b lines of $\mathcal{L}$ is a finite union of orbits. Conversely, if Z is a finite union of finite orbits consisting of $a \geq b - 1 \geq 3$ points on each of the b lines, then Z is a half-grid (a, b)-geproci set.

Some open problems...

1. Every grid in $\mathbb{P}^3$ is contained in a quadric. If Z is geproci and contained in a quadric, is Z a grid?
2. Are there nontrivial non-half-grids over $\mathbb{C}$ than the 3 we know?
3. The nontrivial non-half-grid $(5,8)$-geproci is Gorenstein. Are there other Gorenstein geproci sets?
4. Is there a finite set of points in $\mathbb{P}^n$ that is a nontrivial geproci set when $n > 3$?

Grazie per l'attenzione!