# Lefschetz properties, unexpectedness and geprociness

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Organizers: Emilia Mezzetti, Karim Adiprasito, Roberta Di Gennaro, Sara Faridi, Satoshi Murai & Uwe Nagel

## September 13, 2022

Slides will be available at my website (green text is clickable):
https://www.math.unl.edu/~bharbourne1/

# Abstract: AG Problems Motivated by Inverse Scattering

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry (AG).

Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.



Idea: classify structures algebro-geometrically based on properties of projected images.
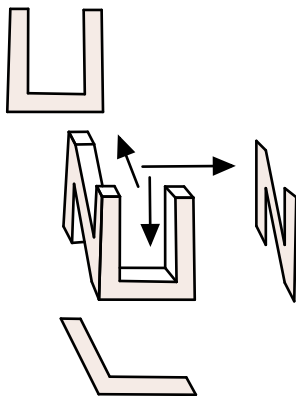
# This talk is on joint work of the Geproci Team (GT):



Luca
Chiantini

Łucja
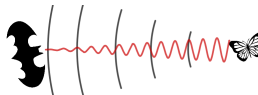Farnik

Giuseppe
Favacchio

Brian
Harbourne

Juan
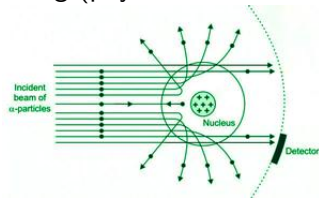Migliore

Tomasz
Szemberg

Justyna
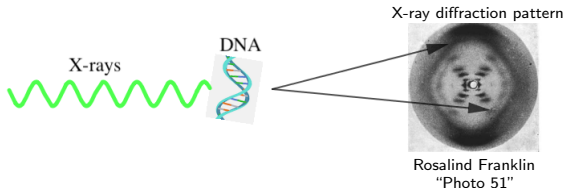Szpond

# Some examples of ISP

Echolocation (biology):



Rutherford scattering (physics; led to Bohr model of atom):
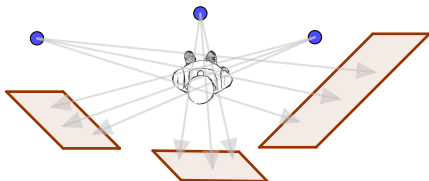


X-ray crystallography (chem/bio; led to DNA double helix model):

# More examples

Tomography (medicine):



GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets in $\mathbb{P}^n$ whose $\boxed{\text{Ge}}$ neral $\boxed{\text{Pro}}$ jections to a hyperplane satisfy $\boxed{\mathcal{P}}$.

Here are 12 points (10 visible) in space whose projections from general points to a plane are complete intersections (so $\mathcal{P}$ is "being a CI"). These 12 points (known as $D_4$) are "geproci."
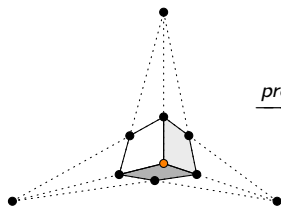


$\xrightarrow{\textit{project to plane}}$
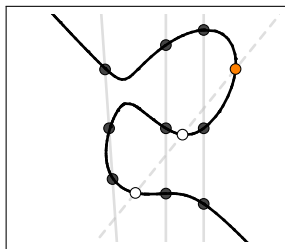
# Why $D_4$ is geproci

The quartic comes from lines through collinear points:



*project to plane*

The cubic is one in a pencil of cubics:



*project to plane*
(3D visualization)

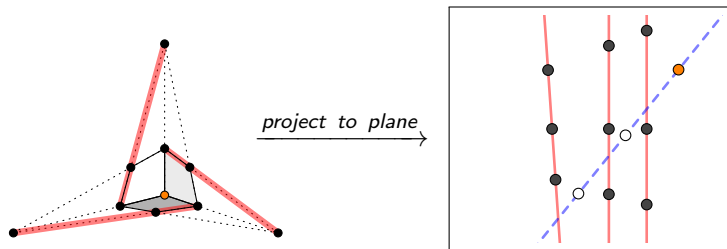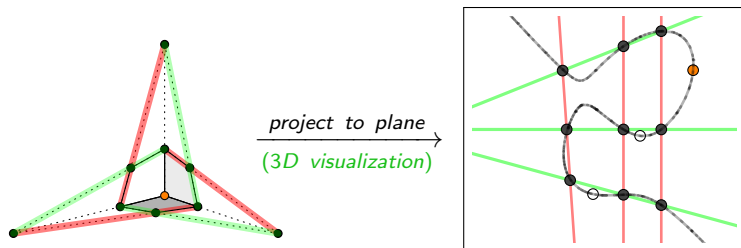# Gepro-$\mathcal{P}$

**General Problem**: Given a property $\mathcal{P}$ of finite point sets $\overline{Z} \subset \mathbb{P}^{n-1}$,

classify all finite $Z \subset \mathbb{P}^n$ such that $\overline{Z} \subset H \cong \mathbb{P}^{n-1}$ has property $\mathcal{P}$

(where $\overline{Z}$ is the image of $Z$ under projection $\mathbb{P}^n \dashrightarrow H$ from a general point $P$ to a hyperplane $H \subset \mathbb{P}^n$).

**Example 1**: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set $Z$ of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in $H$, which is Gorenstein.

**Open Problem 1**: Classify gepro-Gorenstein sets $Z$.

Every geproci set is also gepro-Gorenstein but not conversely.

**Open Problem 2**: Classify geproci sets in $\mathbb{P}^3$.

# History of geproci

We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

We say $Z \subset \mathbb{P}^3$ is $(a, b)$-geproci if $\overline{Z}$ is the intersection of a curve $A$ of degree $a$ with a curve $B$ of degree $b$, with $a \leq b$.

A geproci set $Z$ in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in $H$.

**Question 1** (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

**Answer** (D. Panov, 2011): No!
$(a, b)$-grids are nondegenerate and
geproci. I.e., $2 \leq a \leq b$ with $A$ being $a$
skew black lines and $B$ being $b$ skew
orange lines, where each black line
intersects each orange line in exactly 1
point. (Here $a = b = 3$.)
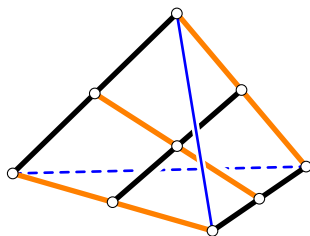
## New Question and a partial answer

For simplicity, call a geproci set in $\mathbb{P}^3$ *trivial* if it is either a grid or contained in a plane.

**Question 1′** (F. Polizzi 2011): Is every geproci $Z \subset \mathbb{P}^3$ trivial? If not, can such $Z$ be classified up to projective equivalence, at least when $|Z|$ is small?

**Answer** (2018, Lefschetz Working Group at Levico Terme): Certain finite sets $Z$ given by root systems (such as $D_4$ and $F_4$) which have unexpected cones (see Harbourne-Migliore-Nagel-Teitler: arXiv:1805.10626, Michigan Math. J. 2020) turn out to be nontrivial geproci sets.

**Theorem** (Levico Terme Working Group, 2018): A finite set $Z \subset \mathbb{P}^3$ is $(a, b)$-geproci if $|Z| = ab$ and it has unexpected cones of degrees $a \leq b$ with no common components.

# The 2018 Levico Terme Working Group (LTWG)



Alessandra
Bernardi

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Graham
Denham

Giuseppe
Favacchio

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Juan
Migliore

Tomasz
Szemberg

Justyna
Szpond

## Unexpected cones are equivalent to WLP failing

**Definition**: A finite set of points $Z \subset \mathbb{P}^n$ satisfies $C(t)$ if it has an "unexpected cone" of degree $t$.

Taking Macaulay duals, $Z = \{p_1, \dots, p_s\}$ satisfies $C(t)$ iff the Weak Lefschetz Property (WLP) fails for $Z$ in degree $t-1$.

(For a point $Q = [q_0 : \cdots : q_n] \in \mathbb{P}^n$, let $L_Q = q_0 x_0 + \cdots + q_n x_n$.)

WLP failing for $Z$ in degree $t-1$ means $\times L_P$ does not have maximal rank:

$$\left[ \frac{R}{(L_{p_1}^t, \dots, L_{p_s}^t)} \right]_{t-1} \xrightarrow{\times L_P} \left[ \frac{R}{(L_{p_1}^t, \dots, L_{p_s}^t)} \right]_t \twoheadrightarrow \left[ \frac{R}{(L_{p_1}^t, \dots, L_{p_s}^t, L_P)} \right]_t$$

# A result and an Open Problem

**Theorem** (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): Every $(a, b)$-grid with $3 \leq a \leq b$ satisfies both $C(a)$ and $C(b)$.

**Open Problem 3**: Does every nontrivial $(a, b)$-geproci $Z \subset \mathbb{P}^3$ satisfy both $C(a)$ and $C(b)$?

# $D_4$ played a special role

**Theorem** (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): The least $|Z|$ for a nontrivial geproci set is $|Z| = 12$. An example is given by the $D_4$ configuration of 12 points; it is $(3, 4)$-geproci.

**Theorem 1** (GT, 2022): The $D_4$ configuration is, up to projective equivalence, the only nontrivial $(a, b)$-geproci set in $\mathbb{P}^3$ with $a \leq 3$.

$D_4$ motivated the following theorem:

**Theorem 2** (GT, 2022): For each $4 \leq a \leq b$, there is a nontrivial $(a, b)$-geproci $Z \subset \mathbb{P}^3$.
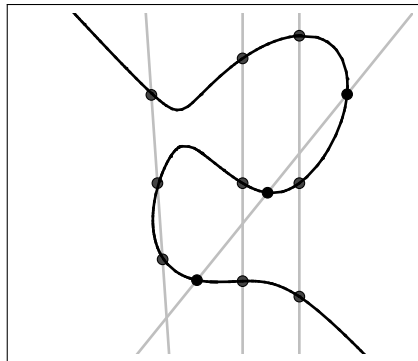
$D_4$ is obtained from a $(3, 3)$-grid by adding a special set of 3 collinear points.

The proof of Theorem 2 comes from adding (and deleting) collinear sets of points from special grids.

# $D_4$ is a half grid

**Definition**: A non-grid $(a, b)$-geproci set $Z$ is a *half grid* if $\overline{Z}$ is the intersection of two curves, exactly one of which can always be taken to be a union of lines.

**Example**: $D_4$ is a half grid.

# Open Problems

(a) We know only a few examples of nontrivial geproci non-half grids:

- A 60 point set coming from the $H_4$ root system (Fraś-Zięba).
- A 40 point $(5, 8)$-geproci set originally constructed by Penrose, who applied it to quantum mechanics (QM).
- A 120 point $(10, 12)$-geproci set also related to QM.

**Open Problem 4**: Are there only finitely many nontrivial geproci non-half grids?

(b) Every nontrivial geproci set in $\mathbb{P}^3$ that we know of has multiple subsets of at least 3 collinear points.

**Open Problem 5**: Can a nontrivial geproci set be linearly general?

(c) The 40 point Penrose set is Gorenstein.

**Open Problem 6**: Are there other finite Gorenstein geproci sets?

# More Open Problems

(a) There are, up to projective equivalence, uncountably many grids.

**Open Problem 7**: Up to projective equivalence, is there any $(a, b)$ with infinitely many nontrivial $(a, b)$-geproci sets?

(b) We know no example of a geproci set in $\mathbb{P}^n$ for $n > 3$.

**Open Problem 8**: Do geproci sets exist in $\mathbb{P}^n$, $n > 3$?

(c) We can define a geproci variety as any variety whose general projection is a complete intersection. A cone with a general vertex over a finite geproci set is a geproci curve, the cone over that is a geproci surface, etc. These geproci varieties all have codimension 3.

**Open Problem 9**: Are there other kinds of geproci varieties? Are there any with codimension greater than 3?

# Terao type problems

Terao's Conjecture concerns whether a certain property of hyperplane arrangements is a combinatorial property. A geproci set also has combinatorics (e.g., its collinear subsets).

**Open Problem 10**: If two geproci sets have the same combinatorics, are they projectively equivalent?

**Open Problem 11**: If a set has the same combinatorics as a geproci set, is it geproci?

Thanks for your attention!

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