

Correlated random effects: iid1d, iid2d, iid3d, iid4d and iid5d

This model is available for dimensions $p = 1, 2, 3, 4$ and 5 . We describe in detail the case for $p = 2$, and then the changes required for $p = 1, p = 3, p = 4$ and $p = 5$

Parametrization

The 2-dimensional Normal-Wishard model is used if one want to define two vectors of “random effects”, u and v , say, for which (u_i, v_i) are iid bivariate Normals

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{W}^{-1})$$

where the covariance matrix \mathbf{W}^{-1} is

$$\mathbf{W}^{-1} = \begin{pmatrix} 1/\tau_a & \rho/\sqrt{\tau_a\tau_b} \\ \rho/\sqrt{\tau_a\tau_b} & 1/\tau_b \end{pmatrix} \quad (1)$$

and τ_a, τ_b and ρ are the hyperparameters.

Note that ρ is the correlation coefficient, and that τ_a and τ_b are the *marginal precisions*, *not* the elements in the precision matrix.

For these models the precision matrix \mathbf{W} is Wishart distributed

$$\mathbf{W} \sim \text{Wishart}_p(r, \mathbf{R}^{-1}), \quad p = 2$$

with density

$$\pi(\mathbf{W}) = c^{-1} |\mathbf{W}|^{(r-(p+1))/2} \exp \left\{ -\frac{1}{2} \text{Trace}(\mathbf{W}\mathbf{R}) \right\}, \quad r > p + 1$$

and

$$c = 2^{(rp)/2} |\mathbf{R}|^{-r/2} \pi^{(p(p-1))/4} \prod_{j=1}^p \Gamma((r+1-j)/2).$$

Then,

$$\text{E}(\mathbf{W}) = r\mathbf{R}^{-1}, \quad \text{and} \quad \text{E}(\mathbf{W}^{-1}) = \mathbf{R}/(r - (p + 1)).$$

Hyperparameters

The hyperparameters are

$$\theta = (\log \tau_a, \log \tau_b, \tilde{\rho})$$

where

$$\rho = 2 \frac{\exp(\tilde{\rho})}{\exp(\tilde{\rho}) + 1} - 1$$

The prior-parameters are

$$(r, R_{11}, R_{22}, R_{12})$$

where

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

and $r_{12} = R_{21}$ due to symmetry.

The `inla` function reports posterior distribution for the hyperparameters τ_a, τ_b, ρ in equation (??).

The prior for θ is **fixed** to be `wishart2d`

Specification

The model `iid2d` is specified as

```
y ~ f(i, model="iid2d", n = <length>) + ...
```

and the `iid2d` model is represented internally as one vector of length n ,

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m)$$

where $n = 2m$, and n is the (required) argument in `f()`.

For this model the argument `constr=TRUE` is interpreted as

$$\sum u_i = 0, \quad \text{and} \quad \sum v_i = 0.$$

Hyperparameter specification and default values

hyper

theta1

```
name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart2d
param 4 1 1 0
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

theta2

```
name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

theta3

```
name logit correlation
short.name cor
initial 4
fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
```

constr FALSE

nrow.ncol FALSE

```

augmented TRUE
aug.factor 1
aug.constr 1 2
n.div.by 2
n.required TRUE
set.default.values TRUE
pdf iid123d

```

Example

In this example we implement the model

$$y|\eta \sim \text{Pois}(\exp(\eta))$$

where

$$\eta = a + b + 1$$

and a and b are correlated as described above.

```

n = 1000
N = 2*n
rho = 0.5
## set variances
Sigma = matrix(c(1/1, NA, NA, 1/2), 2, 2)
## and the correlation
Sigma[1,2] = Sigma[2,1] = rho*sqrt(Sigma[1,1]*Sigma[2,2])

## need it to simulate data
library(mvtnorm)

if (TRUE)
{
  ## first example

  y = yy = rmvnorm(n, sigma=Sigma)
  y = c(y[,1], y[,2])

  i = 1:N
  formula = y ~ f(i, model="iid2d", n=N)

  r = inla(formula, data = data.frame(i,y),
           control.family=list(initial=10,fixed=TRUE))
  print(summary(r))

  print(1/diag(cov(yy)))
  print(cor(yy)[1,2])
}

```

```

if (TRUE)
{
  ## second example

  y = yy = rmvnorm(n, sigma=Sigma)
  z = rnorm(n)
  zz = rnorm(n)
  y = y[,1] + z*y[,2] + zz
  i = 1:n
  j = n + 1:n
  formula = y ~ f(i, model="iid2d", n=N) + f(j,z,copy="i") + zz

  r = inla(formula, data = data.frame(i,j,y,z,zz),
           control.family=list(initial=10,fixed=TRUE),keep=T)
  print(summary(r))
  print(1/diag(cov(yy)))
  print(cor(yy)[1,2])
}

```

The case $p = 1$

For $p = 1$ the hyperparameter is the marginal precision

$$\theta = \log \tau_1$$

The prior is fixed to be `wishart1d` with parameters

$$param = r R_{11}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} \end{bmatrix}$$

Hyperparameter spesification and default values

hyper

theta

name precision

short.name prec

initial 4

fixed FALSE

prior wishart1d

param 2 1e-04

to.theta function(x) log(x)

from.theta function(x) exp(x)

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required FALSE

set.default.values TRUE

pdf iid123d

The case $p = 3$

For $p = 3$ the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23})$$

The prior is fixed to be **wishart3d** with parameters

$$param = r \ R_{11} \ R_{22} \ R_{33} \ R_{12} \ R_{13} \ R_{23}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & R_{33} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions τ_1 , τ_2 and τ_3 and the correlations ρ_{12} , ρ_{13} and ρ_{23} .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m)$$

where $n = 3m$ is a required argument, and where (u_i, v_i, w_i) are trivariate iid Normal.

Hyperparameter spesification and default values

hyper

theta1

name log precision1

short.name prec1

initial 4

fixed FALSE

prior wishart3d

param 7 1 1 1 0 0 0

to.theta function(x) log(x)

from.theta function(x) exp(x)

theta2

name log precision2

short.name prec2

initial 4

fixed FALSE

prior none

param

```

    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta3
  name log precision3
  short.name prec3
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta4
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta5
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta6
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
constr FALSE
nrow.ncol FALSE
augmented TRUE
aug.factor 1

```

```

aug.constr 1 2 3
n.div.by 3
n.required TRUE
set.default.values TRUE
pdf iid123d

```

The case $p = 4$

For $p = 4$ the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \log \tau_4, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{14}, \tilde{\rho}_{23}, \tilde{\rho}_{24}, \tilde{\rho}_{34})$$

The prior is fixed to be **wishart4d** with parameters

$$param = r \ R_{11} \ R_{22} \ R_{33} \ R_{44} \ R_{12} \ R_{13} \ R_{14} \ R_{23} \ R_{24} \ R_{34}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12} & R_{22} & R_{23} & R_{24} \\ R_{13} & R_{23} & R_{33} & R_{34} \\ R_{14} & R_{24} & R_{34} & R_{44} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions τ_1, τ_2, τ_3 and τ_4 , and the correlations $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}$ and ρ_{34} .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m, x_1, x_2, \dots, x_m)$$

where $n = 4m$ is a required argument, and where (u_i, v_i, w_i, x_i) are fourvariate iid Normal.

Hyperparameter spesification and default values

hyper

theta1

```

name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart4d
param 11 1 1 1 1 0 0 0 0 0 0
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

theta2

```

name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param

```

```

    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta3
  name log precision3
  short.name prec3
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta4
  name log precision4
  short.name prec4
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta5
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta6
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
  name logit correlation14
  short.name cor14
  initial 0
  fixed FALSE
  prior none

```



```

    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
    name logit correlation23
    short.name cor23
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
    name logit correlation24
    short.name cor24
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10
    name logit correlation34
    short.name cor34
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1

constr FALSE

nrow.ncol FALSE

augmented TRUE

aug.factor 1

aug.constr 1 2 3 4

n.div.by 4

n.required TRUE

set.default.values TRUE

pdf iid123d

```

The case $p = 5$

The case $p = 5$ follows by a direct extension of $p = 3$ and $p = 4$, and is therefore not included.

Hyperparameter specification and default values

hyper

theta1

name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart5d
param 16 1 1 1 1 1 0 0 0 0 0 0 0 0 0
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta2

name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta3

name log precision3
short.name prec3
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta4

name log precision4
short.name prec4
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta5

name log precision5

```

    short.name prec5
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta6
    name logit correlation12
    short.name cor12
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
    name logit correlation13
    short.name cor13
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
    name logit correlation14
    short.name cor14
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
    name logit correlation15
    short.name cor15
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10

```

```

    name logit correlation23
    short.name cor23
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta11
    name logit correlation24
    short.name cor24
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta12
    name logit correlation25
    short.name cor25
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta13
    name logit correlation34
    short.name cor34
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta14
    name logit correlation35
    short.name cor35
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1

```

theta15

name logit correlation45

short.name cor45

initial 0

fixed FALSE

prior none

param

to.theta function(x) log((1+x)/(1-x))

from.theta function(x) 2*exp(x)/(1+exp(x))-1

constr FALSE

nrow.ncol FALSE

augmented TRUE

aug.factor 1

aug.constr 1 2 3 4 5

n.div.by 5

n.required TRUE

set.default.values TRUE

pdf iid123d

Notes

The model `iid1d` is similar to the model `iid` (and included for completeness only). The prior for `iid1d` is fixed to be Wishart-distributed, which reduces to a Gamma-distribution for the precision with parameters

$$a = r/2 \quad \text{and} \quad b = R_{11}/2$$

hence

```
y ~ f(i, model="iid1d", hyper = list(theta=list(param=c(3, 4))))
```

is equivalent to

```
y ~ f(i, model="iid", hyper = list(theta=list(param=c(1.5, 2), prior="loggamma")))
```