

# Autoregressive model of order $p$ (AR( $p$ ))

## Parametrization

The autoregressive model of order  $p$  (AR1( $p$ )) for the Gaussian vector  $\mathbf{x} = (x_1, \dots, x_n)$  is defined as (in obvious notation)

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + \epsilon_t$$

for  $t = p, \dots, n$ , and where the innovation process  $\{\epsilon_t\}$  has fixed precision.

The AR( $p$ ) process has an awkward parameterisation, as there are severe non-linear constraints on the  $\phi$ -parameters for it to define a stationary model. Therefore we re-parameterized using the partial autocorrelation autocorrelation function,  $\{\psi_k, k = 1, \dots, p\}$ , where  $|\psi_k| < 1$  for all  $k$ <sup>1</sup> and its *marginal (NOT conditional) precision*  $\tau$ . Furthermore, the joint distribution for  $\{x_t, t = 1, \dots, p\}$ , is set to the stationary distribution for the process, hence there are no boundary issues.

## Hyperparameters

The marginal precision parameter  $\tau$  is represented as

$$\theta_1 = \log(\tau)$$

and the prior for the marginal precision is defined on  $\theta_1$ . The partial autocorrelation function  $\{\psi_k\}$  is represented

$$\psi_k = 2 \frac{\exp(\theta_{k+1})}{1 + \exp(\theta_{k+1})} - 1$$

for  $k = 1, \dots, p$ . The prior for  $\{\theta_{k+1}, k = 1, \dots, p\}$  is *defined* to be multivariate normal with mean  $\mu$  and precision matrix  $Q$ .

## Specification

The AR( $p$ ) model is specified inside the `f()` function as

```
f(<whatever>, model="ar", order=<p>, hyper = <hyper>)
```

The option `order` ( $> 0$ ) is required. The multivariate normal prior for  $\{\theta_{k+1}, k = 1, \dots, p\}$ , is specified as the parameters to the prior for  $\theta_2$  (the first pacf-parameter), and the parameters to the multivariate normal prior (`mvnorm`), is `c( $\mu, Q$ )`; see the example below.

## Hyperparameter spesification and default values

**hyper**

**theta1**

**name** log precision

**short.name** prec

**initial** 4

**fixed** FALSE

**prior** loggamma

**param** 1 5e-05

**to.theta** function(x) log(x)

**from.theta** function(x) exp(x)

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<sup>1</sup>See for example [https://en.wikipedia.org/wiki/Partial\\_autocorrelation\\_function](https://en.wikipedia.org/wiki/Partial_autocorrelation_function). For  $p = 1$ , then  $\psi_1 = \phi_1$ , and for  $p = 2$ , then  $\psi_1 = \phi_1/(1 - \phi_2)$  and  $\psi_2 = \phi_2$ .

#### theta2

```
name pacf1
short.name pacf1
initial 2
fixed FALSE
prior mvnorm
param 0 0.15
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
```

#### theta3

```
name pacf2
short.name pacf2
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
```

#### theta4

```
name pacf3
short.name pacf3
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
```

#### theta5

```
name pacf4
short.name pacf4
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
```

#### theta6

```
name pacf5
short.name pacf5
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
```

```

    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
  name pacf6
  short.name pacf6
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
  name pacf7
  short.name pacf7
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
  name pacf8
  short.name pacf8
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10
  name pacf9
  short.name pacf9
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta11
  name pacf10
  short.name pacf10
  initial 0
  fixed FALSE
  prior none
  param

```

```

to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required FALSE

set.default.values FALSE

status experimental

pdf ar

```

## Example

```

n = 100L
p = 2L
pacf = runif(p)
phi = inla.ar.pacf2phi(pacf)
y = arima.sim(n, model = list(ar = phi)) +
  rnorm(n, sd=sd(y)/100.0)
idx = 1L:n

param.prec = c(1, 0.01)
param.psi.mean = rep(0, p)
param.psi.prec = 0.15 * diag(p)
param.psi = c(param.psi.mean, param.psi.prec)

r = inla(y ~ -1 + f(
  idx, model='ar',
  order = p,
  hyper = list(
    ## marginal precision
    prec = list(param = param.prec),
    ## the parameters for the joint normal prior for the
    ## transformed pacf's, goes here.
    pacf1 = list(param = param.psi))),
  family = "gaussian",
  data = data.frame(y, idx))

## we will now estimate the posterior marginals of the phi-parameters
## using the (experimental) function 'inla.hyperpar.sampler', which
## creates samples from the approximated joint distribution for the
## hyperparameters.
nsamples = 100000

```

```

pacfs = inla.hyperpar.sampler(nsamples, r)[, 3L:(3L+(p-1L))]
phis = apply(pacfs, 1L, inla.ar.pacf2phi)
for(i in 1:p) {
  inla.dev.new()
  plot(density(phis[i, ]), main = paste("phi", i, sep=""))
  abline(v = phi[i])
}

```

## Notes

- The functions `inla.ar.pacf2phi` and `inla.ar.phi2pacf` converts from the  $\phi$ -parameters to the  $\psi$ -parameters, using the Durbin-Levinson recursions. These can also be used to compute the marginal posteriors of the  $\phi$ -parameters from an approximation of the joint of the  $\phi$ -parameters; see the example for a simulation based approach.
- Currently, the order  $p$  is limited to 10. If this creates a problem, let us know.
- If some of the  $\psi_k$ -parameters are fixed, and  $k < p$ , then the marginal (log-)likelihood is wrong; The joint normal prior for all the  $p$   $\psi$ -parameters is used and not the conditional normal prior condition on the fixed  $\psi_k$ -parameters. If this creates a problem, let us know.
- The prior specification for the multivariate normal is a bit awkward. Hopefully, we will come up with a better way to do this in the future.
- This model is currently marked as experimental.