

# Zero-inflated models: Poisson, Binomial, negative Binomial and BetaBinomial

## Parametrisation

There is support two types of zero-inflated models, which we name type 0 and type 1. These are defined for both the Binomial, the Poisson, the negative Binomial and BetaBinomial likelihood. For simplicity we will describe only the Poisson as the other cases are similar.

### Type 0

The (type 0) likelihood is defined as

$$\text{Prob}(y \mid \dots) = p \times 1_{[y=0]} + (1 - p) \times \text{Poisson}(y \mid y > 0)$$

where  $p$  is a hyperparameter where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and  $\theta$  is the internal representation of  $p$ ; meaning that the initial value and prior is given for  $\theta$ . This is model is called `zeroinflatedpoisson0` (and `zeroinflatedbinomial0`).

### Type 1

The (type 1) likelihood is defined as

$$\text{Prob}(y \mid \dots) = p \times 1_{[y=0]} + (1 - p) \times \text{Poisson}(y)$$

where  $p$  is a hyperparameter where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and  $\theta$  is the internal representation of  $p$ ; meaning that the initial value and prior is given for  $\theta$ . This is model is called `zeroinflatedpoisson1` (and `zeroinflatedbinomial1`).

## Link-function

As for the Poisson, the Binomial the negative Binomial and the BetaBinomial.

## Hyperparameters

For Poisson and the Binomial, there is one hyperparameter; where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and the prior and initial value is is given for  $\theta$ .

For the negative Binomial and BetaBinomial, there are two hyperparameters. The overdispersion parameter  $n$  for the negative Binomial is represented as

$$\theta_1 = \log(n)$$

and the prior is defined on  $\theta_1$ . The zero-inflation parameter  $p$ , is represented as

$$p = \frac{\exp(\theta_2)}{1 + \exp(\theta_2)}$$

and the prior and initial value is is given for  $\theta_2$ . For the BetaBinomial it is similar.

## Specification

- family = zeroinflatedbinomial0
- family = zeroinflatedbinomial1
- family = zeroinflatednbinomial0
- family = zeroinflatednbinomial1
- family = zeroinflatedpoisson0
- family = zeroinflatedpoisson1
- family = zeroinflatedbetabinomial0
- family = zeroinflatedbetabinomial1
- Required arguments: As for the Binomial, the negative Binomial, BetaBinomial and Poisson likelihood.

## Hyperparameter specification and default values

### Zeroinflated Binomial Type 0

**hyper**

**theta**

**hyperid** 90001  
**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** FALSE

**link** default logit cauchit probit cloglog loglog

**pdf** zeroinflated

### Zeroinflated Binomial Type 1

**hyper**

**theta**

**hyperid** 91001  
**name** logit probability  
**short.name** prob  
**initial** -1

```

    fixed FALSE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default logit cauchit probit cloglog loglog

pdf zeroinflated

```

### Zeroinflated NegBinomial Type 0

```

hyper
  theta1
    hyperid 95001
    name log size
    short.name size
    initial 2.30258509299405
    fixed FALSE
    prior loggamma
    param 1 1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)

  theta2
    hyperid 95002
    name logit probability
    short.name prob
    initial -1
    fixed FALSE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

```

## Zeroinflated NegBinomial Type 1

**hyper**

**theta1**

**hyperid** 96001  
**name** log size  
**short.name** size  
**initial** 2.30258509299405  
**fixed** FALSE  
**prior** loggamma  
**param** 1 1  
**to.theta** function(x) log(x)  
**from.theta** function(x) exp(x)

**theta2**

**hyperid** 96002  
**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** FALSE

**link** default log

**pdf** zeroinflated

## Zeroinflated BetaBinomial Type 0

**hyper**

**theta1**

**hyperid** 88001  
**name** overdispersion  
**short.name** rho  
**initial** 0  
**fixed** FALSE  
**prior** gaussian  
**param** 0 0.4  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**theta2**

**hyperid** 88002

**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** TRUE

**link** default logit cauchit probit cloglog loglog

**pdf** zeroinflated

### Zeroinflated BetaBinomial Type 1

**hyper**

**theta1**

**hyperid** 89001  
**name** overdispersion  
**short.name** rho  
**initial** 0  
**fixed** FALSE  
**prior** gaussian  
**param** 0 0.4  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**theta2**

**hyperid** 89002  
**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** TRUE

**link** default logit cauchit probit cloglog loglog

**pdf** zeroinflated

## Zeroinflated Poisson Type 0

**hyper**

**theta**

**hyperid** 85001  
**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** FALSE

**link** default log

**pdf** zeroinflated

## Zeroinflated Poisson Type 1

**hyper**

**theta**

**hyperid** 86001  
**name** logit probability  
**short.name** prob  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x/(1-x))  
**from.theta** function(x) exp(x)/(1+exp(x))

**survival** FALSE

**discrete** FALSE

**link** default log

**pdf** zeroinflated

## Example

In the following example we estimate the parameters in a simulated example for both type 0 and type 1.

## Poisson

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
E = sample(c(1,5,10,15), size=n, replace=TRUE)
lambda = E*exp(eta)

## first sample y|y>0
y = rpois(n, lambda = lambda)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
  y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
  is.zero = (y == 0)
}
## then set some of these to zero
y[ rbinom(n, size=1, prob=p) == 1 ] = 0

data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedpoisson0", data = data, E=E)
summary(result0)

## type 1
y = rpois(n, lambda = lambda)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedpoisson1", data = data, E=E)
summary(result1)
```

## Binomial

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
Ntrials = sample(c(1,5,10,15), size=n, replace=TRUE)
prob = exp(eta)/(1 + exp(eta))

y = rbinom(n, size = Ntrials, prob = prob)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
```

```

    y[is.zero] = rbinom(sum(is.zero), size = Ntrials[is.zero], prob = prob[is.zero])
    is.zero = (y == 0)
}
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedbinomial0", data = data, Ntrials = Ntrials)
summary(result0)

## type 1
y = rbinom(n, size = Ntrials, prob = prob)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedbinomial1", data = data, Ntrials=Ntrials)
summary(result1)

```



## Advanced example

In the following example we estimate the parameters in a simulated example for a type0 likelihood, where one linear predictor enters the zero-probability and one other linear predictor enters the non-zero Poisson for example. The same trick can be used for other models of type0. The trick is that the likelihood

$$p^* 1_{[y=0]} + (1 - p^*) P(y|y > 0)$$

can be reformulated as a Bernoulli likelihood for the “class”-variable

$$z = \begin{cases} 1, & \text{if } y = 0 \\ 0, & \text{if } y > 0. \end{cases}$$

where  $p^*$  is the probability for success, and zero-inflated type0 likelihood (with fixed  $p = 0$ ) for those  $y > 0$ . Since  $p^*$  and the linear predictor in  $P$  is separated into two likelihoods, we can apply one linear predictor to each one, hence extend the basic model to cases where  $p^*$  also depends on a linear predictor. Here is a small simulated example doing this.

```
require(INLA)

n = 100
a = 0.5
b = 1.5
x1 = rnorm(n, sd = 0.5)

eta.z = -a - b*x1
z = rbinom(n, 1, inla.link.logit(eta.z, inverse=TRUE))
n.y = sum(z)

x2 = rnorm(n.y, sd = 0.5)
eta.y = a + b*x2
lambda = exp(eta.y)
y = rpois(n.y, lambda)

is.zero = (y == 0)
while(sum(is.zero) > 0)
{
  y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
  is.zero = (y == 0)
}

Y = matrix(NA, n + n.y, 2)
Y[1:n, 1] = z
Y[n + 1:n.y, 2] = y

form = Y ~ 0 + mu.z + mu.y + cov.z + cov.y
ldat = list(
  Y=Y,
  mu.z=rep(1:0, c(n, n.y)),
  mu.y=rep(0:1, c(n, n.y)),
  cov.z=c(x1, rep(NA,n.y)),
  cov.y=c(rep(NA, n), x2))
```

```

res <- inla(form, data=ldat,
            family=c('binomial', 'zeroinflatedpoisson0'),
            control.family=list(
              list(),
              list(hyper = list(
                prob = list(
                  initial = -20,
                  fixed = TRUE))))))
round(res$summary.fix, 4)

```

## Notes

None.

## Extentions

There are some extentions available which currently is only implemented for the cases where its needed/requested.

**Type 2** Is like Type 1 but where (for the Poisson)

$$p = 1 - \left( \frac{E \exp(x)}{1 + E \exp(x)} \right)^\alpha$$

where  $\alpha > 0$  is the hyperparameter instead of  $p$  (and  $E \exp(x)$  is the mean). Available for Poisson as **zeroinflatedpoisson2**, for binomial as **zeroinflatedbinomial2** and for the negative binomial as **zeroinflatednbinomial2**.

The internal representation is  $\theta = \log(\alpha)$  and prior is defined on  $\log(\alpha)$ .

## Zeroinflated Poisson Type 2

### hyper

#### theta

```

hyperid 87001
name log alpha
short.name a
initial 0.693147180559945
fixed FALSE
prior gaussian
param 0.693147180559945 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

## **Zeroinflated Binomial Type 2**

**hyper**

**theta**

**hyperid** 92001  
**name** alpha  
**short.name** alpha  
**initial** -1  
**fixed** FALSE  
**prior** gaussian  
**param** -1 0.2  
**to.theta** function(x) log(x)  
**from.theta** function(x) exp(x)

**survival** FALSE

**discrete** FALSE

**link** default logit cauchit probit cloglog loglog

**pdf** zeroinflated

## **Zeroinflated Negative Binomial Type 2**

**hyper**

**theta1**

**hyperid** 99001  
**name** log size  
**short.name** size  
**initial** 2.30258509299405  
**fixed** FALSE  
**prior** loggamma  
**param** 1 0.1  
**to.theta** function(x) log(x)  
**from.theta** function(x) exp(x)

**theta2**

**hyperid** 99002  
**name** log alpha  
**short.name** a  
**initial** 0.693147180559945  
**fixed** FALSE  
**prior** gaussian  
**param** 2 1  
**to.theta** function(x) log(x)  
**from.theta** function(x) exp(x)

**survival** FALSE

**discrete** FALSE

**link** default log

**pdf** zeroinflated

## Zeroinflated Negative Binomial Type 1 Strata 2

hyper

theta1

hyperid 97001  
name log size  
short.name size  
initial 2.30258509299405  
fixed FALSE  
prior loggamma  
param 1 0.1  
to.theta function(x) log(x)  
from.theta function(x) exp(x)

theta2

hyperid 97002  
name logit probability 1  
short.name prob1  
initial -1  
fixed FALSE  
prior gaussian  
param -1 0.2  
to.theta function(x) log(x/(1-x))  
from.theta function(x) exp(x)/(1+exp(x))

theta3

hyperid 97003  
name logit probability 2  
short.name prob2  
initial -1  
fixed FALSE  
prior gaussian  
param -1 0.2  
to.theta function(x) log(x/(1-x))  
from.theta function(x) exp(x)/(1+exp(x))

theta4

hyperid 97004  
name logit probability 3  
short.name prob3  
initial -1  
fixed TRUE  
prior gaussian  
param -1 0.2  
to.theta function(x) log(x/(1-x))  
from.theta function(x) exp(x)/(1+exp(x))

theta5

```

hyperid 97005
name logit probability 4
short.name prob4
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta6
hyperid 97006
name logit probability 5
short.name prob5
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta7
hyperid 97007
name logit probability 6
short.name prob6
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta8
hyperid 97008
name logit probability 7
short.name prob7
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta9
hyperid 97009
name logit probability 8
short.name prob8
initial -1

```

```

    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta10
    hyperid 97010
    name logit probability 9
    short.name prob9
    initial -1
    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta11
    hyperid 97011
    name logit probability 10
    short.name prob10
    initial -1
    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

```

**status** experimental

**survival** FALSE

**discrete** FALSE

**link** default log

**pdf** zeroinflated

### **Zeroinflated Negative Binomial Type 1 Strata 3**

**hyper**

```

theta1
    hyperid 98001
    name logit probability
    short.name prob
    initial -1
    fixed FALSE
    prior gaussian
    param -1 0.2

```

```

    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta2
    hyperid 98002
    name log size 1
    short.name size1
    initial 2.30258509299405
    fixed FALSE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta3
    hyperid 98003
    name log size 2
    short.name size2
    initial 2.30258509299405
    fixed FALSE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta4
    hyperid 98004
    name log size 3
    short.name size3
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta5
    hyperid 98005
    name log size 4
    short.name size4
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta6
    hyperid 98006

```

```

    name log size 5
    short.name size5
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta7
    hyperid 98007
    name log size 6
    short.name size6
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta8
    hyperid 98008
    name log size 7
    short.name size7
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta9
    hyperid 98009
    name log size 8
    short.name size8
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta10
    hyperid 98010
    name log size 9
    short.name size9
    initial 2.30258509299405
    fixed TRUE

```



```

    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta11
    hyperid 98011
    name log size 10
    short.name size10
    initial 2.30258509299405
    fixed TRUE
    prior loggamma
    param 1 0.1
    to.theta function(x) log(x)
    from.theta function(x) exp(x)

status experimental

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

```

### 0.0.1 Zero and $N$ -inflated Binomial likelihood: type 3

This is the case where

$$\begin{aligned}
 \text{Prob}(y|\dots) = & p_0 \times 1_{[y=0]} + \\
 & p_N \times 1_{[y=N]} + \\
 & (1 - p_0 - p_N) \times \text{binomial}(y, N, p)
 \end{aligned}$$

where:

$$p = \frac{\exp(\eta)}{1 + \exp(\eta)} \quad p_0 = \frac{p^{\alpha_0}}{1 + p^{\alpha_0} + (1 - p)^{\alpha_N}} \quad p_N = \frac{(1 - p)^{\alpha_N}}{1 + p^{\alpha_0} + (1 - p)^{\alpha_N}}$$

There are 2 hyperparameters,  $\alpha_0$  and  $\alpha_N$ , governing zero-inflation where: The zero-inflation parameters  $\alpha_0$  and  $\alpha_N$  are represented as  $\theta_0 = \log(\alpha_0)$ ;  $\theta_N = \log(\alpha_N)$  and the prior and initial value is given for  $\theta_0$  and  $\theta_N$  respectively.

Here is an example

```

nsim<-10000
x<-rnorm(nsim)
alpha0<-1.5
alphaN<-2.0
p = exp(x)/(1+exp(x))
p0 = p^alpha0 / (1 + p^alpha0 + (1-p)^alphaN)
pN = (1-p)^alphaN / (1 + p^alpha0 + (1-p)^alphaN)
P<-cbind(p0, pN, (1-p0 -pN))
N<-rpois(nsim,20)
y<-rep(0,nsim)

```

```

for(i in 1:nsim)
  y[i]<-sum(rmultinom(1,size = 1,P[i,])*c(0,N[i],rbinom(1,N[i],p[i]))))
formula = y ~1 + x
r = inla(formula, family = "zeroninflatedbinomial3", Ntrials = N, verbose = TRUE,
        data = data.frame(y, x))

```

and the default settings

**hyper**

**theta1**

```

hyperid 93101
name alpha0
short.name alpha0
initial 1
fixed FALSE
prior loggamma
param 1 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

**theta2**

```

hyperid 93102
name alphaN
short.name alphaN
initial 1
fixed FALSE
prior loggamma
param 1 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

**status** experimental

**survival** FALSE

**discrete** FALSE

**link** default logit cauchit probit cloglog loglog

**pdf** zeroinflated