

## Correlated random effects: iid1d, iid2d, iid3d, iid4d and iid5d

This model is available for dimensions  $p = 1, 2, 3, 4$  and  $5$ . We describe in detail the case for  $p = 2$ , and then the changes required for  $p = 1, p = 3, p = 4$  and  $p = 5$

### Parametrization

The 2-dimensional Normal-Wishard model is used if one want to define two vectors of “random effects”,  $u$  and  $v$ , say, for which  $(u_i, v_i)$  are iid bivariate Normals

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{W}^{-1})$$

where the covariance matrix  $\mathbf{W}^{-1}$  is

$$\mathbf{W}^{-1} = \begin{pmatrix} 1/\tau_a & \rho/\sqrt{\tau_a\tau_b} \\ \rho/\sqrt{\tau_a\tau_b} & 1/\tau_b \end{pmatrix} \quad (1)$$

and  $\tau_a, \tau_b$  and  $\rho$  are the hyperparameters.

Note that  $\rho$  is the correlation coefficient, and that  $\tau_a$  and  $\tau_b$  are the *marginal precisions*, *not* the elements in the precision matrix.

For these models the precision matrix  $\mathbf{W}$  is Wishart distributed

$$\mathbf{W} \sim \text{Wishart}_p(r, \mathbf{R}^{-1}), \quad p = 2$$

with density

$$\pi(\mathbf{W}) = c^{-1} |\mathbf{W}|^{(r-(p+1))/2} \exp \left\{ -\frac{1}{2} \text{Trace}(\mathbf{W}\mathbf{R}) \right\}, \quad r > p + 1$$

and

$$c = 2^{(rp)/2} |\mathbf{R}|^{-r/2} \pi^{(p(p-1))/4} \prod_{j=1}^p \Gamma((r+1-j)/2).$$

Then,

$$\text{E}(\mathbf{W}) = r\mathbf{R}^{-1}, \quad \text{and} \quad \text{E}(\mathbf{W}^{-1}) = \mathbf{R}/(r - (p + 1)).$$

### Hyperparameters

The hyperparameters are

$$\theta = (\log \tau_a, \log \tau_b, \tilde{\rho})$$

where

$$\rho = 2 \frac{\exp(\tilde{\rho})}{\exp(\tilde{\rho}) + 1} - 1$$

The prior-parameters are

$$(r, R_{11}, R_{22}, R_{12})$$

where

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

and  $r_{12} = R_{21}$  due to symmetry.

The `inla` function reports posterior distribution for the hyperparameters  $\tau_a, \tau_b, \rho$  in equation (1).

The prior for  $\theta$  is **fixed** to be `wishart2d`

## Specification

The model `iid2d` is specified as

```
y ~ f(i, model="iid2d", n = <length>) + ...
```

and the `iid2d` model is represented internally as one vector of length  $n$ ,

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m)$$

where  $n = 2m$ , and  $n$  is the (required) argument in `f()`.

For this model the argument `constr=TRUE` is interpreted as

$$\sum u_i = 0, \quad \text{and} \quad \sum v_i = 0.$$

## Hyperparameter specification and default values

**hyper**

**theta1**

```
name  log precision1
short.name  prec1
initial  4
fixed  FALSE
prior  wishart2d
param  4 1 1 0
to.theta  function(x) log(x)
from.theta  function(x) exp(x)
```

**theta2**

```
name  log precision2
short.name  prec2
initial  4
fixed  FALSE
prior  none
param
to.theta  function(x) log(x)
from.theta  function(x) exp(x)
```

**theta3**

```
name  logit correlation
short.name  cor
initial  4
fixed  FALSE
prior  none
param
to.theta  function(x) log((1+x)/(1-x))
from.theta  function(x) 2*exp(x)/(1+exp(x))-1
```

```

constr FALSE
nrow.ncol FALSE
augmented TRUE
aug.factor 1
aug.constr 1 2
n.div.by 2
n.required TRUE
set.default.values TRUE
pdf iid123d

```

## Example

In this example we implement the model

$$y|\eta \sim \text{Pois}(\exp(\eta))$$

where

$$\eta = a + b + 1$$

and  $a$  and  $b$  are correlated as described above.

```

n = 1000
N = 2*n
rho = 0.5
## set variances
Sigma = matrix(c(1/1, NA, NA, 1/2), 2, 2)
## and the correlation
Sigma[1,2] = Sigma[2,1] = rho*sqrt(Sigma[1,1]*Sigma[2,2])

## need it to simulate data
library(mvtnorm)

if (TRUE)
{
  ## first example

  y = yy = rmvnorm(n, sigma=Sigma)
  y = c(y[,1], y[,2])

  i = 1:N
  formula = y ~ f(i, model="iid2d", n=N)

  r = inla(formula, data = data.frame(i,y),
           control.family=list(initial=10,fixed=TRUE))
  print(summary(r))
}

```

```

    print(1/diag(cov(yy)))
    print(cor(yy)[1,2])
}

if (TRUE)
{
  ## second example

  y = yy = rmvnorm(n, sigma=Sigma)
  z = rnorm(n)
  zz = rnorm(n)
  y = y[,1] + z*y[,2] + zz
  i = 1:n
  j = n + 1:n
  formula = y ~ f(i, model="iid2d", n=N) + f(j,z,copy="i") + zz

  r = inla(formula, data = data.frame(i,j,y,z,zz),
           control.family=list(initial=10,fixed=TRUE),keep=T)
  print(summary(r))
  print(1/diag(cov(yy)))
  print(cor(yy)[1,2])
}

```

## The case $p = 1$

For  $p = 1$  the hyperparameter is the marginal precision

$$\theta = \log \tau_1$$

The prior is fixed to be `wishart1d` with parameters

$$param = r R_{11}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} \end{bmatrix}$$

## Hyperparameter spesification and default values

**hyper**

**theta**

```

name precision
short.name prec
initial 4
fixed FALSE
prior wishart1d
param 2 1e-04
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

```

constr FALSE
nrow.ncol FALSE
augmented FALSE
aug.factor 1
aug.constr
n.div.by
n.required FALSE
set.default.values TRUE
pdf iid123d

```

**The case  $p = 3$**

For  $p = 3$  the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23})$$

The prior is fixed to be **wishart3d** with parameters

$$param = r \begin{matrix} R_{11} & R_{22} & R_{33} & R_{12} & R_{13} & R_{23} \end{matrix}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & R_{33} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions  $\tau_1, \tau_2$  and  $\tau_3$  and the correlations  $\rho_{12}, \rho_{13}$  and  $\rho_{23}$ .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m)$$

where  $n = 3m$  is a required argument, and where  $(u_i, v_i, w_i)$  are trivariate iid Normal.

## Hyperparameter spesification and default values

**hyper**

```

theta1
  name log precision1
  short.name prec1
  initial 4
  fixed FALSE
  prior wishart3d
  param 7 1 1 1 0 0 0
  to.theta function(x) log(x)

```

```

    from.theta function(x) exp(x)
theta2
  name log precision2
  short.name prec2
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta3
  name log precision3
  short.name prec3
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta4
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta5
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta6
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param

```

```

    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1

constr FALSE

nrow.ncol FALSE

augmented TRUE

aug.factor 1

aug.constr 1 2 3

n.div.by 3

n.required TRUE

set.default.values TRUE

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```

**The case  $p = 4$**

For  $p = 4$  the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \log \tau_4, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{14}, \tilde{\rho}_{23}, \tilde{\rho}_{24}, \tilde{\rho}_{34})$$

The prior is fixed to be **wishart4d** with parameters

$$param = r \ R_{11} \ R_{22} \ R_{33} \ R_{44} \ R_{12} \ R_{13} \ R_{14} \ R_{23} \ R_{24} \ R_{34}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12} & R_{22} & R_{23} & R_{24} \\ R_{13} & R_{23} & R_{33} & R_{34} \\ R_{14} & R_{24} & R_{34} & R_{44} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions  $\tau_1, \tau_2, \tau_3$  and  $\tau_4$ , and the correlations  $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}$  and  $\rho_{34}$ .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m, x_1, x_2, \dots, x_m)$$

where  $n = 4m$  is a required argument, and where  $(u_i, v_i, w_i, x_i)$  are fourvariate iid Normal.

## Hyperparameter spesification and default values

**hyper**

```

theta1
  name log precision1
  short.name prec1
  initial 4
  fixed FALSE

```

```

    prior wishart4d
    param 11 1 1 1 1 0 0 0 0 0 0
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta2
    name log precision2
    short.name prec2
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta3
    name log precision3
    short.name prec3
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta4
    name log precision4
    short.name prec4
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta5
    name logit correlation12
    short.name cor12
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta6
    name logit correlation13
    short.name cor13
    initial 0

```



```

fixed FALSE
prior none
param
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
  name logit correlation14
  short.name cor14
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
  name logit correlation24
  short.name cor24
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10
  name logit correlation34
  short.name cor34
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
constr FALSE
nrow.ncol FALSE

```

```

augmented TRUE
aug.factor 1
aug.constr 1 2 3 4
n.div.by 4
n.required TRUE
set.default.values TRUE
pdf iid123d

```

The case  $p = 5$

The case  $p = 5$  follows by a direct extension of  $p = 3$  and  $p = 4$ , and is therefore not included.

## Hyperparameter spesification and default values

hyper

```

theta1
  name log precision1
  short.name prec1
  initial 4
  fixed FALSE
  prior wishart5d
  param 16 1 1 1 1 1 0 0 0 0 0 0 0 0 0
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta2
  name log precision2
  short.name prec2
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta3
  name log precision3
  short.name prec3
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)

```

```

    from.theta function(x) exp(x)
theta4
  name log precision4
  short.name prec4
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta5
  name log precision5
  short.name prec5
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta6
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
  name logit correlation14
  short.name cor14
  initial 0
  fixed FALSE
  prior none
  param

```

```

    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
  name logit correlation15
  short.name cor15
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta11
  name logit correlation24
  short.name cor24
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta12
  name logit correlation25
  short.name cor25
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1+x)/(1-x))
  from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta13
  name logit correlation34
  short.name cor34
  initial 0
  fixed FALSE
  prior none

```

```

    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
  theta14
    name logit correlation35
    short.name cor35
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
  theta15
    name logit correlation45
    short.name cor45
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1

  constr FALSE

  nrow.ncol FALSE

  augmented TRUE

  aug.factor 1

  aug.constr 1 2 3 4 5

  n.div.by 5

  n.required TRUE

  set.default.values TRUE

  pdf iid123d

```

## Notes

The model `iid1d` is similar to the model `iid` (and included for completeness only). The prior for `iid1d` is fixed to be Wishart-distributed, which reduces to a Gamma-distribution for the precision with parameters

$$a = r/2 \quad \text{and} \quad b = R_{11}/2$$

hence

```
y ~ f(i, model="iid1d", hyper = list(theta=list(param=c(3, 4))))
```

is equivalent to

```
y ~ f(i, model="iid", hyper = list(theta=list(param=c(1.5, 2), prior="loggamma")))
```