

# The Ornstein-Uhlenbeck process

## Parametrization

The Ornstein-Uhlenbeck process is defined with (mean zero), as the SDE

$$dx_t = -\phi x_t + \sigma dW_t$$

where  $\phi > 0$  and  $W_t$  is the Wiener process. This is the continuous time analogue to the discrete time AR(1) model.

The process has a Markov property. Let  $x = (x_1, x_2, \dots, x_n)$  be value of the process at increasing time-points  $t = (t_1, t_2, \dots, t_n)$ , then the conditional distribution

$$x_i \mid x_1, \dots, x_{i-1}, \quad i = 2, \dots, n,$$

is Gaussian with mean

$$x_{i-1} \exp(-\phi \delta_i)$$

and precision

$$\tau (1 - \exp(-2\phi \delta_i))^{-1}$$

where

$$\delta_i = t_i - t_{i-1}, \quad i = 2, \dots, n$$

and

$$\tau = 2\phi/\sigma^2.$$

The marginal distribution for  $x_1$  is taken to be the stationary distribution, which is a zero mean Gaussian with precision  $\tau$ .

## Hyperparameters

The precision parameter  $\tau$  is represented as

$$\theta_1 = \log(\tau)$$

where  $\tau$  is the *marginal* precision for the Ornstein-Uhlenbeck process given above.

The parameter  $\phi$  is represented as

$$\theta_2 = \log(\phi)$$

and the prior is defined on  $\theta = (\theta_1, \theta_2)$ .

## Specification

The Ornstein-Uhlenbeck model is specified inside the `f()` function as

```
f(<whatever>, model="ou", values=<values>, hyper = <hyper>)
```

The optional argument `values` gives the time-points where the process is defined/observed on (default is `unique(sort(<whatever>))`).

## Hyperparameter specification and default values

hyper

theta1

```
name  log precision
short.name  prec
prior  loggamma
param  1 5e-05
initial  4
fixed  FALSE
to.theta  function(x) log(x)
from.theta  function(x) exp(x)
```

theta2

```
name  log phi
short.name  phi
prior  normal
param  -2 1
initial  -1
fixed  FALSE
to.theta  function(x) log(x)
from.theta  function(x) exp(x)
```

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required FALSE

set.default.values FALSE

pdf ou

## Example

```
## simulate an OU-process and estimate its parameters back.
phi = -log(0.95)
sigma = 1
marg.prec = 2*phi/sigma^2
n = 1000
locations = cumsum(sample(c(1, 2, 5, 20),n, replace=TRUE))

## do it sequentially and slow (for clarity)
x = numeric(n)
```

```

x[1] = rnorm(1, mean=0, sd = sqrt(1/marg.prec))
for(i in 2:n) {
  delta = locations[i] - locations[i-1]
  x[i] = x[i-1] * exp(-phi * delta) +
    rnorm(1, mean=0, sd = sqrt(1/marg.prec * (1-exp(-2*phi*delta))))
}

## observe it with a little noise
y = 1 + x + rnorm(n, sd= 0.01)
plot(locations, x, type="l")

formula = y ~ 1 + f(locations, model="ou", values=locations)
r = inla(formula, data = data.frame(y, locations))
summary(r)

```

## Notes

The Ornstein-Uhlenbeck process is the continuous-time analogue to the discrete AR(1) model (for positive lag-one correlation only), but they are parameterised slightly different.