

## The ME model: details

This note gives the missing details in the ME model.

The model is

$$y = \beta x + \epsilon$$

where  $y$  is the response,  $\beta$  the effect of the true covariate  $x$  with zero mean Gaussian noise  $\epsilon$ . The issue is that  $x$  is not observed directly, but only through  $x_{\text{obs}}$ , where

$$x_{\text{obs}} = x + \nu$$

where  $\nu$  is zero mean Gaussian noise. The parameters are:  $\beta$  has prior  $\pi(\beta)$ ,  $x$  is apriori  $\mathcal{N}(\mu_x I, \tau_x I)$ , and  $\tau_{\text{obs}}$  is the observation precision for  $x$  (ie  $\text{Prec}(x_{\text{obs}}|x)$ )<sup>1</sup>.

Assume that the precision of the observations  $y$ ,  $\tau_y$ , is known as it does not influence the calculations. Let  $\theta = (\beta, \tau_x, \tau_{\text{obs}}, \mu_x)$ . The full posterior is

$$\pi(x, \theta | y, x_{\text{obs}}) \propto \pi(\theta) \pi(x | \theta) \pi(x_{\text{obs}} | x, \theta) \pi(y | x, \theta)$$

Using that

$$\pi(x | \theta) \pi(x_{\text{obs}} | x, \theta) = \pi(x | x_{\text{obs}}, \theta) \pi(x_{\text{obs}} | \theta)$$

we get

$$\pi(x, \theta | y, x_{\text{obs}}) \propto \pi(\theta) \pi(x | x_{\text{obs}}, \theta) \pi(x_{\text{obs}} | \theta) \pi(y | x, \theta).$$

This means that  $x$  only enters in *one term* (apart from the likelihood) hence can be used as an ordinary latent model  $\mathbf{f}()$ . Its easy to derive that

$$x | x_{\text{obs}}, \theta \sim \mathcal{N}\left(\frac{\tau_x \mu_x I + \tau_{\text{obs}} x_{\text{obs}}}{\tau_x + \tau_{\text{obs}}}, (\tau_x + \tau_{\text{obs}}) I\right).$$

and

$$x_{\text{obs}} | \theta \sim \mathcal{N}\left(\mu_x I, \frac{1}{1/\tau_x + 1/\tau_{\text{obs}}} I\right).$$

Note that  $x_{\text{obs}} | \theta$  does not depend on  $x$ , hence conditionally on  $\theta$ , its a constant. But it do need to be included in the model, as its log-density is

$$-\frac{n}{2} \log(2\pi) + \frac{n}{2} \log\left(\frac{1}{1/\tau_x + 1/\tau_{\text{obs}}}\right) - \frac{1}{2} \frac{1}{1/\tau_x + 1/\tau_{\text{obs}}} (x_{\text{obs}} - \mu_x I)^T (x_{\text{obs}} - \mu_x I)$$

and do depend on  $\theta$ .

The last tweak, is that we do the change of variable from  $(x, \beta)$  to  $(u, \beta)$ , where  $u = \beta x$ , so that

$$y = u + \epsilon.$$

This makes the implementation more convenient. Then we get

$$\pi(u, \theta | y, x_{\text{obs}}) \propto \pi(\theta) \tag{1}$$

$$\pi(u | x_{\text{obs}}, \theta) \pi(x_{\text{obs}} | \theta) \tag{2}$$

$$\pi(y | u, \theta). \tag{3}$$

where

$$u | \theta, x_{\text{obs}} \sim \mathcal{N}\left(\beta \frac{\tau_x \mu_x I + \tau_{\text{obs}} x_{\text{obs}}}{\tau_x + \tau_{\text{obs}}}, \frac{\tau_x + \tau_{\text{obs}}}{\beta^2} I\right).$$

---

<sup>1</sup>Note: The second argument in  $\mathcal{N}()$  is the precision not the variance.