

# Autoregressive model of order $p$ (AR( $p$ ))

## Parametrization

The autoregressive model of order  $p$  (AR1( $p$ )) for the Gaussian vector  $\mathbf{x} = (x_1, \dots, x_n)$  is defined as (in obvious notation)

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + \epsilon_t$$

for  $t = p, \dots, n$ , and where the innovation process  $\{\epsilon_t\}$  has fixed precision.

The AR( $p$ ) process has an awkward parameterisation, as there are severe non-linear constraints on the  $\phi$ -parameters for it to define a stationary model. Therefore we re-parameterized using the partial autocorrelation autocorrelation function,  $\{\psi_k, k = 1, \dots, p\}$ , where  $|\psi_k| < 1$  for all  $k$ <sup>1</sup> and its *marginal (NOT conditional) precision*  $\tau$ . Furthermore, the joint distribution for  $\{x_t, t = 1, \dots, p\}$ , is set to the stationary distribution for the process, hence there are no boundary issues.

## Hyperparameters

The marginal precision parameter  $\tau$  is represented as

$$\theta_1 = \log(\tau)$$

and the prior for the marginal precision is defined on  $\theta_1$ . The partial autocorrelation function  $\{\psi_k\}$  is represented

$$\psi_k = 2 \frac{\exp(\theta_{k+1})}{1 + \exp(\theta_{k+1})} - 1$$

for  $k = 1, \dots, p$ . The prior for  $\{\theta_{k+1}, k = 1, \dots, p\}$  is *defined* to be multivariate normal with mean  $\mu$  and precision matrix  $Q$ .

## Specification

The AR( $p$ ) model is specified inside the `f()` function as

```
f(<whatever>, model="ar", order=<p>, hyper = <hyper>)
```

The option `order` ( $> 0$ ) is required. The multivariate normal prior for  $\{\theta_{k+1}, k = 1, \dots, p\}$ , is specified as the parameters to the prior for  $\theta_2$  (the first pacf-parameter), and the parameters to the multivariate normal prior (`mvnorm`), is `c( $\mu, Q$ )`; see the example below.

## Hyperparameter spesification and default values

**doc** Auto-regressive model of order  $p$  (AR( $p$ ))

**hyper**

**theta1**

**hyperid** 15001

**name** log precision

**short.name** prec

**initial** 4

**fixed** FALSE

**prior** pc.prec

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<sup>1</sup>See for example [https://en.wikipedia.org/wiki/Partial\\_autocorrelation\\_function](https://en.wikipedia.org/wiki/Partial_autocorrelation_function). For  $p = 1$ , then  $\psi_1 = \phi_1$ , and for  $p = 2$ , then  $\psi_1 = \phi_1/(1 - \phi_2)$  and  $\psi_2 = \phi_2$ .

```

    param 3 0.01
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta2
    hyperid 15002
    name pacf1
    short.name pacf1
    initial 1
    fixed FALSE
    prior pc.cor0
    param 0.5 0.5
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta3
    hyperid 15003
    name pacf2
    short.name pacf2
    initial 0
    fixed FALSE
    prior pc.cor0
    param 0.5 0.4
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta4
    hyperid 15004
    name pacf3
    short.name pacf3
    initial 0
    fixed FALSE
    prior pc.cor0
    param 0.5 0.3
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta5
    hyperid 15005
    name pacf4
    short.name pacf4
    initial 0
    fixed FALSE
    prior pc.cor0
    param 0.5 0.2
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta6

```

```

hyperid 15006
name pacf5
short.name pacf5
initial 0
fixed FALSE
prior pc.cor0
param 0.5 0.1
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta7
hyperid 15007
name pacf6
short.name pacf6
initial 0
fixed FALSE
prior pc.cor0
param 0.5 0.1
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta8
hyperid 15008
name pacf7
short.name pacf7
initial 0
fixed FALSE
prior pc.cor0
param 0.5 0.1
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta9
hyperid 15009
name pacf8
short.name pacf8
initial 0
fixed FALSE
prior pc.cor0
param 0.5 0.1
to.theta function(x) log((1+x)/(1-x))
from.theta function(x) 2*exp(x)/(1+exp(x))-1
theta10
hyperid 15010
name pacf9
short.name pacf9
initial 0

```

```

    fixed FALSE
    prior pc.cor0
    param 0.5 0.1
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1
  theta11
    hyperid 15011
    name pacf10
    short.name pacf10
    initial 0
    fixed FALSE
    prior pc.cor0
    param 0.5 0.1
    to.theta function(x) log((1+x)/(1-x))
    from.theta function(x) 2*exp(x)/(1+exp(x))-1

  constr FALSE

  nrow.ncol FALSE

  augmented FALSE

  aug.factor 1

  aug.constr

  n.div.by

  n.required FALSE

  set.default.values FALSE

  pdf ar

```

## Example I

```
n = 100L
p = 2L
pacf = runif(p)
phi = inla.ar.pacf2phi(pacf)
y = c(scale(arima.sim(n, model = list(ar = phi)))) +
      rnorm(n, sd=1/100.0)
idx = 1L:n

param.prec = c(1, 0.01)
param.psi.mean = rep(0, p)
param.psi.prec = 0.15 * diag(p)
param.psi = c(param.psi.mean, param.psi.prec)

r = inla(y ~ -1 + f(
  idx, model='ar',
  order = p,
  hyper = list(
    ## marginal precision
    prec = list(param = param.prec),
    ## the parameters for the joint normal prior for the
    ## transformed pacf's, goes here.
    pacf1 = list(param = param.psi))),
  family = "gaussian",
  data = data.frame(y, idx))

## we will now estimate the posterior marginals of the phi-parameters using
## 'inla.hyperpar.sampler', which creates samples from the approximated joint distribution for
## the hyperparameters.
nsamples = 100000
pacfs = inla.hyperpar.sampler(nsamples, r)[, 3L:(3L+(p-1L))]
phis = apply(pacfs, 1L, inla.ar.pacf2phi)
for(i in 1:p) {
  inla.dev.new()
  plot(density(phis[i, ]), main = paste("phi", i, sep=""))
  abline(v = phi[i])
}
```

## Example II

In this example we demonstrate how the parameters in the PC-prior for  $AR(p)$  can be set using the idea in S. H. Sørbye and H. Rue (2016)<sup>2</sup>.

```
library(gsl)

inla.pc.ar.lambda = function(a = 0.5, b = 0.8, p = 4)
{
  inla.pc.ar.solve.lambda = function(pred.err.factors, nseq = 1000L) {
    ## pred.err.factor = E(1-rho^2)
    pred.err = function(lambda) {
      return(0.5 * lambda * sqrt(pi) * exp(lambda^2/4 + log_erfc(lambda/2)))
    }

    ## find lower and upper limit of lambda for the pred.err.factors given
    lambda.min = lambda.max = 1
    val.max = max(pred.err.factors)
    val.min = min(pred.err.factors)
    while(pred.err(lambda.min) > val.min) {
      lambda.min = lambda.min / 2.0
    }
    while(pred.err(lambda.max) < val.max) {
      lambda.max = lambda.max * 2.0
    }

    lambda = lambda.min * exp(seq(0, log(lambda.max/lambda.min), length = nseq))
    fun = splinefun(pred.err(lambda), lambda, method = "monoH.FC")
    lambdas = fun(pred.err.factors)

    return (lambdas)
  }

  pred.err.factors = 1.0 - (1.0-a)*b^(0:(p-1))
  lambda = inla.pc.ar.solve.lambda(pred.err.factors)

  return (lambda)
}

inla.pc.ar.test1 = function(p=1, p.est = p+1, n = 100)
{
  pacf = c(runif(p), rep(0, p.est-p))
  phi = inla.ar.pacf2phi(pacf)
  sd.x = sqrt(1/prod(1-pacf^2))
  x = arima.sim(n = n, model = list(ar = phi))/sd.x
  lambda = c(inla.pc.ar.lambda(p = p.est, b = 0.5), rep(1, 10))
  initial = c(inla.models()$latent$ar$hyper$theta2$to.theta(pacf), rep(0, 10))
  r = (inla(
    x ~ -1 +
```

---

<sup>2</sup>Penalised complexity priors for stationary autoregressive processes. arXiv preprint arXiv:1608.08941, UiT The Arctic University of Norway, 2016

```

f(time, model = "ar", order = p.est,
  hyper = list(
    prec = list(param = c(3, 0.01), initial = 0),
    pacf1 = list(param = c(lambda[1], 0), initial = initial[1]),
    pacf2 = list(param = c(lambda[2], 0), initial = initial[2]),
    pacf3 = list(param = c(lambda[3], 0), initial = initial[3]),
    pacf4 = list(param = c(lambda[4], 0), initial = initial[4]),
    pacf5 = list(param = c(lambda[5], 0), initial = initial[5]),
    pacf6 = list(param = c(lambda[6], 0), initial = initial[6]),
    pacf7 = list(param = c(lambda[7], 0), initial = initial[7]),
    pacf8 = list(param = c(lambda[8], 0), initial = initial[8]),
    pacf9 = list(param = c(lambda[9], 0), initial = initial[9]),
    pacf10 = list(param = c(lambda[10], 0), initial = initial[10]))),
data = data.frame(x=x, time = 1:length(x)),
control.family = list(hyper = list(prec = list(initial = 12,
                                              fixed=TRUE))))))
result = cbind(est = r$summary.hyperpar$mean[-1], true = pacf)
inside = c()
for(i in 1:p.est) {
  int = inla.hpdmarginal(0.95, r$marginals.hyperpar[[i+1]])
  inside[i] = (int[1] < pacf[i] && pacf[i] < int[2])
}
result = cbind(result, coverage = inside)
print(round(result, digits=3))

result = (cbind(acf.est = inla.ar.pacf2acf(r$summary.hyperpar$mean[-1], lag.max = 10),
  acf.emp = c(acf(x, lag.max=10, plot=FALSE)$acf),
  acf.true = inla.ar.pacf2acf(pacf, lag.max=10)))
print(round(result, digits=3))

return (invisible())
}

```

## Notes

- The functions `inla.ar.pacf2phi` and `inla.ar.phi2pacf` converts from the  $\phi$ -parameters to the  $\psi$ -parameters, using the Durbin-Levinson recursions. These can also be used to compute, the marginal posteriors of the  $\phi$ -parameters from an approximation of the joint of the  $\phi$ -parameters; see the example for a simulation based approach.
- Currently, the order  $p$  is limited to 10. If this creates a problem, let us know.
- If some of the  $\psi_k$ -parameters are fixed, and  $k < p$ , then the marginal (log-)likelihood is wrong; The joint normal prior for all the  $p$   $\psi$ -parameters is used and not the conditional normal prior condition on the fixed  $\psi_k$ -parameters. If this creates a problem, let us know.
- The prior specification for the multivariate normal is a bit awkward. Hopefully, we will come up with a better way to do this in the future.