

$$\begin{aligned}
\text{In[2]}:= \mathbf{M}[\mathbf{k}_-, \mathbf{\bar{k}}_-] &= \frac{e^{-i\pi(k+\bar{k})}}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k} \star \\
&\left\{ \left\{ -\left(2e^{2ik\pi}k+2e^{2i\pi(k+2\bar{k})}k-e^{2i\pi(k+\bar{k})}i\alpha+e^{2i\pi(2k+\bar{k})}(-4k+i\alpha)\right), \right. \right. \\
&\quad \left. \left( -2(-1+e^{2i\pi\bar{k}})(-1+e^{2i\pi(k+\bar{k})})k-e^{2i\pi\bar{k}}(-1+e^{2ik\pi})i\alpha \right) \right\}, \\
&\quad \left\{ \left( 2e^{4ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(2k+\bar{k})}(-2k+i\alpha)-e^{2i\pi(k+\bar{k})}(2k+i\alpha) \right), \right. \\
&\quad \left. \left( 2e^{2ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi\bar{k}}(4k+i\alpha) \right) \right\} \Big\} \\
\text{Out[2]}= &\left\{ \left\{ \frac{e^{-i\pi(k+\bar{k})}(-2e^{2ik\pi}k-2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi(2k+\bar{k})}(-4k+i\alpha))}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k}, \right. \right. \\
&\frac{e^{-i\pi(k+\bar{k})}(-2(-1+e^{2i\pi\bar{k}})(-1+e^{2i\pi(k+\bar{k})})k-e^{2i\pi\bar{k}}(-1+e^{2ik\pi})i\alpha)}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k} \Big\}, \\
&\left\{ \frac{e^{-i\pi(k+\bar{k})}(2e^{4ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(2k+\bar{k})}(-2k+i\alpha)-e^{2i\pi(k+\bar{k})}(2k+i\alpha))}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k}, \right. \\
&\left. \frac{e^{-i\pi(k+\bar{k})}(2e^{2ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi\bar{k}}(4k+i\alpha))}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k} \right\} \Big\}
\end{aligned}$$

$$\begin{aligned}
\text{In[1]}:= \mathbf{Eigenvalues} &\left[ \left\{ \left\{ -\left(2e^{2ik\pi}k+2e^{2i\pi(k+2\bar{k})}k-e^{2i\pi(k+\bar{k})}i\alpha+e^{2i\pi(2k+\bar{k})}(-4k+i\alpha)\right), \right. \right. \right. \\
&\quad \left. \left( -2(-1+e^{2i\pi\bar{k}})(-1+e^{2i\pi(k+\bar{k})})k-e^{2i\pi\bar{k}}(-1+e^{2ik\pi})i\alpha \right) \right\}, \\
&\quad \left\{ \left( 2e^{4ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(2k+\bar{k})}(-2k+i\alpha)-e^{2i\pi(k+\bar{k})}(2k+i\alpha) \right), \right. \\
&\quad \left. \left( 2e^{2ik\pi}k+2e^{2i\pi(k+2\bar{k})}k+e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi\bar{k}}(4k+i\alpha) \right) \right\} \Big\} \Big]
\end{aligned}$$

$$\begin{aligned}
\text{Out[1]}= &\left\{ \frac{1}{2} \left( -4e^{2i\pi\bar{k}}k+4e^{2i\pi(2k+\bar{k})}k-e^{2i\pi\bar{k}}i\alpha+2e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi(2k+\bar{k})}i\alpha - \right. \right. \\
&\sqrt{\left( -4(4e^{2i\pi(k+\bar{k})}-16e^{4i\pi(k+\bar{k})}+4e^{6i\pi(k+\bar{k})}-8e^{2i\pi(2k+\bar{k})}+4e^{2i\pi(3k+\bar{k})}+ \right. \\
&\quad \left. \left. 8e^{2i\pi(k+2\bar{k})}+8e^{2i\pi(3k+2\bar{k})}+4e^{2i\pi(k+3\bar{k})}-8e^{2i\pi(2k+3\bar{k})} \right) k^2 + \right. \\
&\quad \left. \left. \left( 4e^{2i\pi\bar{k}}k-4e^{2i\pi(2k+\bar{k})}k+e^{2i\pi\bar{k}}i\alpha-2e^{2i\pi(k+\bar{k})}i\alpha+e^{2i\pi(2k+\bar{k})}i\alpha \right)^2 \right) \right), \\
\Box &\frac{1}{2} \left( -4e^{2i\pi\bar{k}}k+4e^{2i\pi(2k+\bar{k})}k-e^{2i\pi\bar{k}}i\alpha+2e^{2i\pi(k+\bar{k})}i\alpha-e^{2i\pi(2k+\bar{k})}i\alpha + \right. \\
&\sqrt{\left( -4(4e^{2i\pi(k+\bar{k})}-16e^{4i\pi(k+\bar{k})}+4e^{6i\pi(k+\bar{k})}-8e^{2i\pi(2k+\bar{k})}+4e^{2i\pi(3k+\bar{k})}+ \right. \\
&\quad \left. \left. 8e^{2i\pi(k+2\bar{k})}+8e^{2i\pi(3k+2\bar{k})}+4e^{2i\pi(k+3\bar{k})}-8e^{2i\pi(2k+3\bar{k})} \right) k^2 + \right. \\
&\quad \left. \left. \left( 4e^{2i\pi\bar{k}}k-4e^{2i\pi(2k+\bar{k})}k+e^{2i\pi\bar{k}}i\alpha-2e^{2i\pi(k+\bar{k})}i\alpha+e^{2i\pi(2k+\bar{k})}i\alpha \right)^2 \right) \right) \Big\}
\end{aligned}$$

$$\text{In[5]:= Together} \left[ \text{Det} \left[ \left\{ \left\{ \frac{e^{-i\pi(k+\bar{k})} \left( -2 e^{2ik\pi} k - 2 e^{2i\pi(k+2\bar{k})} k + e^{2i\pi(k+\bar{k})} i\alpha - e^{2i\pi(2k+\bar{k})} (-4k+i\alpha) \right)}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k}, \right. \right. \right. \right. \\ \left. \frac{e^{-i\pi(k+\bar{k})} \left( -2(-1+e^{2i\pi\bar{k}})(-1+e^{2i\pi(k+\bar{k})})k - e^{2i\pi\bar{k}}(-1+e^{2ik\pi})i\alpha \right)}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k} \right\}, \right. \\ \left. \left\{ \frac{e^{-i\pi(k+\bar{k})} \left( 2e^{4ik\pi}k + 2e^{2i\pi(k+2\bar{k})}k + e^{2i\pi(2k+\bar{k})}(-2k+i\alpha) - e^{2i\pi(k+\bar{k})}(2k+i\alpha) \right)}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k}, \right. \right. \\ \left. \left. \frac{e^{-i\pi(k+\bar{k})} \left( 2e^{2ik\pi}k + 2e^{2i\pi(k+2\bar{k})}k + e^{2i\pi(k+\bar{k})}i\alpha - e^{2i\pi\bar{k}}(4k+i\alpha) \right)}{2(-1+e^{2ik\pi})(1+e^{2i\pi\bar{k}})k} \right\} \right\} \right]$$

$$\text{Out[5]=} \left( 1 / \left( 4(-1+e^{2ik\pi})^2(1+e^{2i\pi\bar{k}})^2k^2 \right) \right) e^{-2i\pi(k+\bar{k})} \\ (-4e^{2i\pi(k+\bar{k})}k^2 + 4e^{4i\pi(k+\bar{k})}k^2 - 4e^{2i\pi(2k+\bar{k})}k^2 + 4e^{2i\pi(k+2\bar{k})}k^2 - \\ 4e^{4i\pi(k+2\bar{k})}k^2 + 8e^{2ik\pi+2i\pi\bar{k}}k^2 - 4e^{4ik\pi+2i\pi\bar{k}}k^2 - 4e^{4ik\pi+2i\pi(k+\bar{k})}k^2 + \\ 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})}k^2 + 4e^{4ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}k^2 - 4e^{2i\pi\bar{k}+4i\pi(k+\bar{k})}k^2 + \\ 8e^{2ik\pi+2i\pi(2k+\bar{k})}k^2 - 12e^{2i\pi\bar{k}+2i\pi(2k+\bar{k})}k^2 + 4e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}k^2 - \\ 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}k^2 - 8e^{2ik\pi+2i\pi(k+2\bar{k})}k^2 + 4e^{2i\pi\bar{k}+2i\pi(k+2\bar{k})}k^2 - \\ 4e^{2i\pi(k+\bar{k})+2i\pi(k+2\bar{k})}k^2 + 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(k+2\bar{k})}k^2 + 8e^{2i\pi(2k+\bar{k})+2i\pi(k+2\bar{k})}k^2 - \\ 2e^{2i\pi(k+\bar{k})}ik\alpha + 2e^{4i\pi(k+\bar{k})}ik\alpha + 2e^{2i\pi(2k+\bar{k})}ik\alpha + 2e^{2ik\pi+2i\pi\bar{k}}ik\alpha - \\ 2e^{4ik\pi+2i\pi\bar{k}}ik\alpha + 2e^{6ik\pi+2i\pi\bar{k}}ik\alpha - 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}ik\alpha - \\ 2e^{2i\pi\bar{k}+4i\pi(k+\bar{k})}ik\alpha - 2e^{2ik\pi+2i\pi(2k+\bar{k})}ik\alpha - 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(2k+\bar{k})}ik\alpha + \\ 2e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}ik\alpha + 2e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}ik\alpha + \\ 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+2\bar{k})}ik\alpha - 2e^{2i\pi(2k+\bar{k})+2i\pi(k+2\bar{k})}ik\alpha + e^{4i\pi(k+\bar{k})}i^2\alpha^2 - \\ e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}i^2\alpha^2 + e^{2ik\pi+2i\pi\bar{k}+2i\pi(2k+\bar{k})}i^2\alpha^2 - e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}i^2\alpha^2)$$

In[6]:=

$$\text{FullSimplify} \left[ \left( 1 / \left( 4(-1+e^{2ik\pi})^2(1+e^{2i\pi\bar{k}})^2k^2 \right) \right) e^{-2i\pi(k+\bar{k})} \right. \\ (-4e^{2i\pi(k+\bar{k})}k^2 + 4e^{4i\pi(k+\bar{k})}k^2 - 4e^{2i\pi(2k+\bar{k})}k^2 + 4e^{2i\pi(k+2\bar{k})}k^2 - \\ 4e^{4i\pi(k+2\bar{k})}k^2 + 8e^{2ik\pi+2i\pi\bar{k}}k^2 - 4e^{4ik\pi+2i\pi\bar{k}}k^2 - 4e^{4ik\pi+2i\pi(k+\bar{k})}k^2 + \\ 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})}k^2 + 4e^{4ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}k^2 - 4e^{2i\pi\bar{k}+4i\pi(k+\bar{k})}k^2 + \\ 8e^{2ik\pi+2i\pi(2k+\bar{k})}k^2 - 12e^{2i\pi\bar{k}+2i\pi(2k+\bar{k})}k^2 + 4e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}k^2 - \\ 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}k^2 - 8e^{2ik\pi+2i\pi(k+2\bar{k})}k^2 + 4e^{2i\pi\bar{k}+2i\pi(k+2\bar{k})}k^2 - \\ 4e^{2i\pi(k+\bar{k})+2i\pi(k+2\bar{k})}k^2 + 4e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(k+2\bar{k})}k^2 + 8e^{2i\pi(2k+\bar{k})+2i\pi(k+2\bar{k})}k^2 - \\ 2e^{2i\pi(k+\bar{k})}ik\alpha + 2e^{4i\pi(k+\bar{k})}ik\alpha + 2e^{2i\pi(2k+\bar{k})}ik\alpha + 2e^{2ik\pi+2i\pi\bar{k}}ik\alpha - \\ 2e^{4ik\pi+2i\pi\bar{k}}ik\alpha + 2e^{6ik\pi+2i\pi\bar{k}}ik\alpha - 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}ik\alpha - \\ 2e^{2i\pi\bar{k}+4i\pi(k+\bar{k})}ik\alpha - 2e^{2ik\pi+2i\pi(2k+\bar{k})}ik\alpha - 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(2k+\bar{k})}ik\alpha + \\ 2e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}ik\alpha + 2e^{2i\pi\bar{k}+2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}ik\alpha + \\ 2e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+2\bar{k})}ik\alpha - 2e^{2i\pi(2k+\bar{k})+2i\pi(k+2\bar{k})}ik\alpha + e^{4i\pi(k+\bar{k})}i^2\alpha^2 - \\ e^{2ik\pi+2i\pi\bar{k}+2i\pi(k+\bar{k})}i^2\alpha^2 + e^{2ik\pi+2i\pi\bar{k}+2i\pi(2k+\bar{k})}i^2\alpha^2 - e^{2i\pi(k+\bar{k})+2i\pi(2k+\bar{k})}i^2\alpha^2) \left. \right]$$

Out[6]= 1