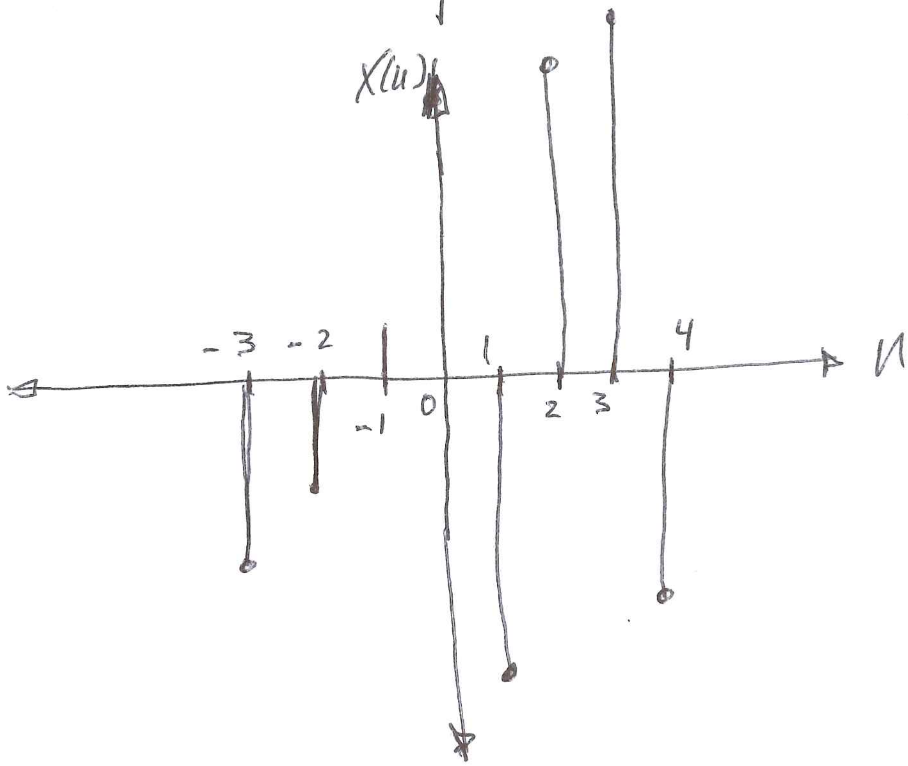
ece 439 Exam 1

① $x(n) = \{-3, -2, 1, \underset{\uparrow}{0}, -5, 6, 7, -4\}$

a)



b) DTFT

$$X(e^{j\omega}) = -3e^{j3\omega} - 2e^{j2\omega} + e^{j\omega} - 5e^{-j\omega} + 6e^{-j2\omega} + 7e^{-j3\omega} - 4e^{-j4\omega}$$

c) ZT and ROC

$$X(z) = -3z^{3} - 2z^{2} + z - 5z^{-1} + 6z^{-2} + 7z^{-3} - 4z^{-4}$$

ROC = all except 0 and ∞

(2) linear and shift-invariant

$$y(n) = x(n+2) - x(2-n)$$

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 x_1(n+2) + a_2 x_2(n+2) - a_1 x_1(2-n) - a_2 x_2(2-n)$$

$$= a_1 [x_1(n+2) - x_1(2-n)] + a_2 [x_2(n+2) - x_2(2-n)]$$

$$= a_1 y_1(n) + a_2 y_2(n)$$

System is Linear

$$L[x(n)] = y(n) = x(n+2) - x(2-n)$$

$$y_k(n) = L[x(n-k)] = x(n-k+2) - x(2-n-k)$$

Response to shifted input

Shifted output is $y(n-k)$

$$y(n-k) = x(n+2-k) - \cancel{x(n-2} - x(2-n+k)$$

System is shift variant

(2) linear and shift-invariant

$$y(n) = -2x(n-2)+7$$

$$T[a_1x_1(n)+a_2x_2(n)] = -2[a_1x_1(n-2)+a_2x_2(n-2)]+7$$

$$= -2a_1x_1(n-2) - 2a_2x_2(n-2) + 7 + 7 - 7$$

$$= -2a_1x_1(n-2)+7 - 2a_2x_2(n-2) + 7 - 7$$

$$= a_1y_1(n) + a_2y_2(n) - 7$$

Non Linear

$$y_k(n) = L[x(n-k)] = -2x(n-k-2)+7$$

$$y(n-k) = -2x(n-k-2)+7$$

shift invariant

(4) solve difference equation

$$y(n) + a^2 y(n-2) = 0 \quad n \geq 0$$

$$|a| < 1 \quad y(-1) = 0 \quad y(-2) = -1$$

$$Y(z) + a^2 \left[z^{-2} Y(z) + \underbrace{y(-1)}_{0} z^{-1} + \underbrace{y(\cancel{-}2)}_{-1} \right] = 0$$

$$Y(z) + a^2 z^{-2} Y(z) - a^2 = 0$$

$$\frac{Y(z) \left[z^2 + a^2 \right]}{z^2} = a^2$$

$$Y(z) = \frac{z^2 a^2}{z^2 + a^2}$$

$$G(z) = z^{n-1} Y(z) = \frac{z^{n+1} a^2}{z^2 + a^2}$$

$$G(z) = \frac{z^{n+1} a^2}{(z + ja)(z - ja)}$$

(4) Residue at $z = ja$

$$R_1 = \left. \frac{z^{n+1} a^2}{(z+ja)} \right|_{z=ja} = \frac{(ja)^{n+1} a^2}{2ja}$$

$$= \frac{(j)^{n+1} a^{n+3}}{2ja} = \frac{(j)^{\cancel{n+1}\, n} a^{n+2}}{2}$$

$$j = e^{j\pi/2}$$

$$R_1 = \frac{a^{n+2} e^{jn\pi/2}}{2}$$

Residue at $z = -ja$

$$R_2 = \left. \frac{z^{n+1} a^2}{(z-ja)} \right|_{z=-ja} = \frac{(-j)^{n+1} a^{n+3}}{-2j\cancel{a}}$$

$$= \frac{a^{n+2} (-j)^n}{2} = \frac{a^{n+2} e^{-jn\pi/2}}{2}$$

$$y(n) = a^{n+2} \left[\frac{e^{jn\pi/2} + e^{-jn\pi/2}}{2} \right]$$

$$= a^{n+2} \cos\left(\frac{n\pi}{2}\right) ; \quad n \geq 0$$

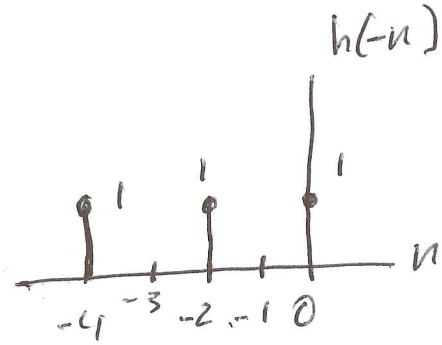
(5) find $y(n) = h(n) * x(n)$

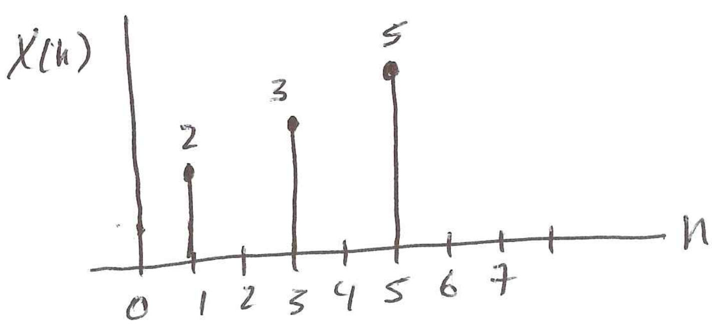
$h(n) = \{\underset{\uparrow}{1}, 0, 1, 0, 1\}$ $N = 5$

$x(n) = \{\underset{\uparrow}{0}, 2, 0, 3, 0, 5, 0\}$ $M = 7$

$L = M + N - 1 = 11$

output $y(n) = \{\underset{\uparrow}{0}, 2, 0, 5, 0, 10, 0, 8, 0, 5, 0\}$





slide and multiply add

(6) $x(n) = \left(0.85\, e^{j\pi/3}\right)^n \qquad -10 \leq n \leq 10$

find $X(e^{j\omega})$ periodicity and conjugate symmetry

DTFT $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

IDTFT $\quad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$

$$X(e^{j\omega}) = \sum_{n=-10}^{10} \left(0.85 e^{j\pi/3}\right)^n e^{-j\omega n}$$

$$= \sum_{n=-10}^{1} \left(0.85 e^{j\pi/3}\right)^n e^{-j\omega n} + \sum_{n=0}^{10} \left(0.85 e^{j\pi/3}\right)^n e^{-j\omega n}$$

$$= \underbrace{\sum_{n=1}^{10} \left(0.85 e^{j\pi/3}\right)^{-n} e^{j\omega n}}_{A} + \underbrace{\sum_{n=0}^{10} \left(0.85 e^{j\pi/3}\right)^n e^{-j\omega n}}_{B}$$

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$$(A) = \sum_{n=1}^{10} \left(0.85^{-n} e^{j(\omega-\pi/3)n}\right)$$

$$(B) = \sum_{n=0}^{10} \left(0.85^{n} e^{j(\pi/3-\omega)n}\right)$$

look at $e^{j(\omega-\pi/3)n}$ and $e^{j(\omega-\pi/3\pm 2\pi)n}$

$$\cos\left(\underline{(\omega-\pi/3)}n\right) + j\sin\left(\underline{(\omega-\pi/3)}n\right)$$

$$\cos\left((\omega-\pi/3 \pm 2\pi)n\right) + j\sin\left((\omega-\tfrac{\pi}{3}\pm 2\pi)n\right)$$

$$\downarrow$$

$$\cos\left((\omega-\tfrac{\pi}{3})n\right)\cos(2\pi n) \mp \sin\left((\omega-\tfrac{\pi}{3})n\right)\cancelto{0}{\sin(2\pi n)} = \underline{\underline{\cos\left(\omega-\tfrac{\pi}{3}\right)n}}$$

$$j\sin\left((\omega-\tfrac{\pi}{3}\pm 2\pi)n\right) = j\sin\left((\omega-\tfrac{\pi}{3})n\right)\cos(2\pi n) \pm \cos\left((\omega-\tfrac{\pi}{3})n\right)\cancelto{0}{\sin(2\pi n)}$$

$$= \underline{\underline{j\sin\left((\omega-\tfrac{\pi}{3})n\right)}}$$

It is periodic.

(6) $\mathrm{Re}\{X(e^{-j\omega})\} = \mathrm{Re}\{X(e^{j\omega})\}$

$\mathrm{Im}\{X(e^{-j\omega})\} = -\mathrm{Im}\{X(e^{j\omega})\}$

$$X(e^{j\omega}) = \sum_{n=-10}^{10} \underbrace{(0.85)^n e^{jn(\pi/3-\omega)}}_{\text{complex-valued signal}}$$

$$X(e^{-j\omega}) = \sum_{n=-10}^{10} (0.85)^n e^{jn(\pi/3+\omega)}$$

$$\mathrm{Re}\left\{e^{jn(\pi/3-\omega)}\right\} = \cos\left[n\left(\frac{\pi}{3}-\omega\right)\right]$$

$$\mathrm{Re}\left\{e^{jn(\frac{\pi}{3}+\omega)}\right\} = \cos\left[n\left(\frac{\pi}{3}+\omega\right)\right]$$

not conjugate symmetric

only valid for real-valued signals

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⑦ $x(n) = (0.8)^n \quad 0 \le n \le 10$

find $X(e^{j\omega})$ periodicity and conjugate-symmetry

DTFT $X(e^{j\omega}) = \sum_{n=0}^{10} (0.8)^n e^{-j\omega n}$

Kernel is periodic

look at $e^{-j\omega n}$ and $e^{-j(\omega \pm 2\pi)n}$

$\underline{\cos}(\omega n) - j\, \underline{\sin}(\omega n)$

$\underbrace{\cos((\omega \pm 2\pi)n)} - j \sin((\omega \pm 2\pi)n)$

$\cos(\omega n)\cos(2\pi n) \mp \sin(\omega n)\sin(2\pi n) = \underline{\cos(\omega n)}$

$\sin((\omega \pm 2\pi)n) = \sin(\omega n)\cos(2\pi n) \pm \cos(\omega n)\sin(2\pi n)$

$= \underline{\sin(\omega n)}$

It is periodic

$\text{Real}\{X(e^{-j\omega})\} = \text{Real}\{X(e^{j\omega})\}$

(7) 11

$$X(e^{j\omega}) = \sum_{n=0}^{10} (0.8)^n e^{-j\omega n}$$

$$X(e^{-j\omega}) = \sum_{n=0}^{10} (0.8)^n e^{j\omega n}$$

look at Real $\{ \cancel{x} e^{-j\omega n} \} = \cos(-n\omega) = \cos(n\omega)$

look at Real $\{ e^{j\omega n} \} = \cos(n\omega)$

It is conjugate symmetric

look at Im $\{ \cancel{x} e^{-j\omega n} \} = -j \sin(\omega n)$

look at Im $\{ e^{j\omega n} \} = +j \sin(\omega n)$

$\text{Im}\{ X(e^{-j\omega}) \} = -\text{Im}\{ X(e^{j\omega}) \}$

$-j \sin(\omega n) = -j \sin(\omega n)$

It is conjugate symmetric

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(9) $X_a(t) = 5 + 2\cos(200\pi t + \pi/3) - 3\sin(400\pi t)$

100 Hz   200 Hz

$F_s = 300$ s/sec $= 300$ sps

find $x(n)$ and $X(e^{j\omega})$ similar to ex 3.17 textbook

highest frequency in $X_a(t) = 200$ Hz

$F_s \geq 2 \cdot 200 = 400$ Hz or smples/sec = sps

problem ⟹ aliasing

$x(n) = X_a(nT_s) = X_a(n/300)$

$x(n) = 5 + 2\cos(0.66\pi n + \pi/3) - 3\sin(1.33 n\pi)$

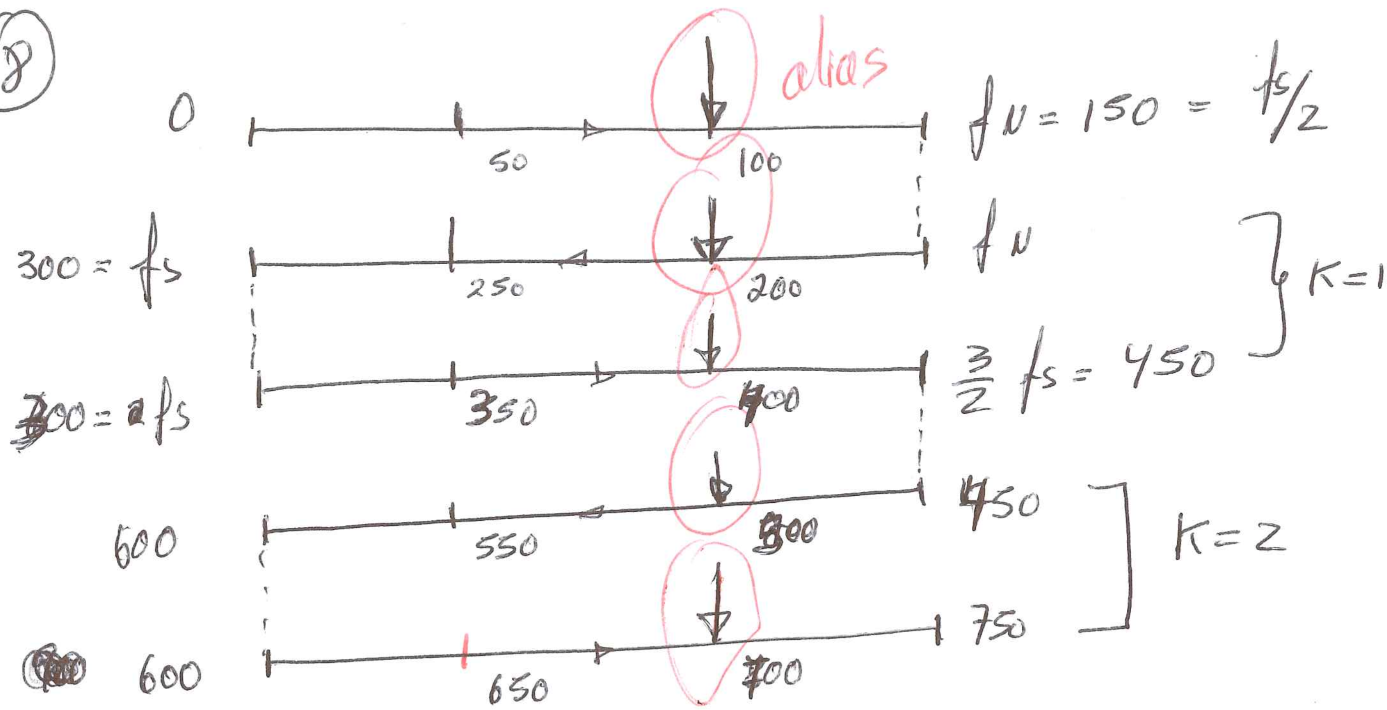
greater $> |\pi|$

Recall: aliasing

$f_2 = 2k f_N \pm f_1$ f_N = Nyquist freq.
$k \geq 1$ $f_N = f_s/2$

$f_2 = k\,300 \pm f_1$
$k \geq 1$

(8)



$$x(n) = 5 + 2\cos\left(0.66\pi n + \frac{\pi}{3}\right) - 3\sin(1.33\pi n)$$

$$= 5 + 2\cos\left(0.66\pi n + \frac{\pi}{3}\right) - 3\sin(1.33\pi n - 2\pi n)$$ correction

$$= 5 + 2\cos(0.66\pi n + \pi/3) - 3\sin(-0.66\pi n)$$

$$= 5 + 2\left[\frac{e^{j(0.66\pi n + \pi/3)} + e^{-j(0.66\pi n + \pi/3)}}{2}\right] +$$

$$+ 3\left[\frac{e^{j0.66\pi n} - e^{-j0.66\pi n}}{2j}\right]$$

$$= 5 + e^{j0.66\pi n} e^{j\pi/3} + e^{-j0.66\pi n} e^{-j\pi/3}$$

$$- 1.5j\, e^{j0.66\pi n} + 1.5j\, e^{-j0.66\pi n}$$

look up table

(8)

$$X(e^{j\omega}) = 10\pi\delta(\omega) + 2\pi\,\delta(\omega - 0.66\pi)e^{j\pi/3}$$
$$+ 2\pi\,\delta(\omega + 0.66\pi)e^{-j\pi/3}$$
$$- j3\pi\,\delta(\omega - 0.66\pi) + j3\pi\,\delta(\omega + 0.66\pi)$$
$$-\pi < \omega < \pi$$

(9) find IZT of $X(z)$ and ROC

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

Example 4.7 in textbook

$$X(z) = \frac{z}{3(z^2 - \frac{4}{3}z + \frac{1}{3})} = \frac{\frac{1}{3}z^{-1}}{\underbrace{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}_{\text{factor, roots}}}$$

$$X(z) = \frac{\frac{1}{3}z^{-1}}{(1 - z^{-1})(1 - \frac{1}{3}z^{-1})}$$

poles at $z = 1$ and $z = 1/3$

look at Residue Theorem

(9) or $\dfrac{A}{(1-\frac{1}{3}z^{-1})} + \dfrac{B}{(1-z^{-1})}$ PFE

$$A(1-z^{-1}) + B(1-\tfrac{1}{3}z^{-1}) = \tfrac{1}{3}z^{-1}$$

$$A - Az^{-1} + B - \tfrac{B}{3}z^{-1} = \tfrac{1}{3}z^{-1}$$

$$A + B = 0 \Rightarrow A = -B$$

$$Bz^{-1} - \tfrac{B}{3}z^{-1} = \tfrac{1}{3}z^{-1}$$

$$B(1-1/3) = \tfrac{1}{3}$$

$$B\tfrac{2}{3} = \tfrac{1}{3} \Rightarrow B = 1/2$$

$$A = \tfrac{-1}{2} \quad B = 1/2$$

$$X(z) = \frac{1/2}{(1-z^{-1})} - \frac{1/2}{(1-\frac{1}{3}z^{-1})}$$ poles $1/3$ and 1

$$x(n) = \tfrac{1}{2}u(n) - \tfrac{1}{2}\left(\tfrac{1}{3}\right)^n u(n)$$

Three possible ROCs

(9) a) ROC_1: $1 < |z| < \infty$ both poles on the interior side of the ROC_1

$$|z_1| \le R_{x-} = 1 \quad |z_2| \le 1$$

$$x_1(n) = \frac{1}{2} u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

causal sequence

b) ROC_2: $0 < |z| < 1/3$ both poles on the exterior side of ROC_2 $\quad |z_1| \ge R_{x+} = \frac{1}{3} \quad |z_2| \ge 1/3$

$$x_2(n) = -\frac{1}{2} u(-n-1) + \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2} u(-n-1)$$

anticausal sequence

c) ROC_3: $\frac{1}{3} < |z| < 1$ z_1 on exterior side of ROC_3 and z_2 on interior side

$$|z_1| \ge R_{x+} = 1 \quad |z_2| \le \frac{1}{3}$$

$$x_3(n) = -\frac{1}{2} u(+n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

causal and anticausal sequence

#17

(10) Given the causal system $H(z)$

$$H(z) = \frac{z+1}{z^2 - 0.9z + 0.81}$$

a) find transfer function representation

b) difference equation

c) impulse response

example 4.12 in textbook

a) find poles to see if unit circle is in the ROC

root of denominator $(0.45+0.78j)$ $(0.45-0.78j)$

magnitude $= 0.9$ angle $\pm \pi/3$

causal system thus $|z| > 0.9$

unit circle is in the ROC thus

the DTFT exists $H(e^{jw})$

$$H(e^{jw}) = \frac{e^{jw}+1}{e^{j2w} - 0.9e^{jw} + 0.81}$$

$$= \frac{e^{jw}+1}{(e^{jw} - 0.9e^{j\pi/3})(e^{jw} - 0.9e^{-j\pi/3})}$$

transfer function

10 b) $H(z) = \frac{Y(z)}{X(z)} = \frac{z+1}{z^2 - 0.9z + 0.81} \left(\frac{z^{-2}}{z^{-2}}\right)$ 18

$$H(z) = \frac{z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$Y(z)\left[1 - 0.9z^{-1} + 0.81z^{-2}\right] = X(z)\left[z^{-1} + z^{-2}\right]$$

$$Y(z) - Y(z)0.9z^{-1} + Y(z)0.81z^{-2} = X(z)z^{-1} + X(z)z^{-2}$$

take IZT

$$y(n) - 0.9y(n-1) + 0.81y(n-2) = x(n-1) + x(n-2)$$

$$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n-1) + x(n-2)$$

c) Using MATLAB find the residues $|z| > 0.9$

$$H(z) = 1.2346 + \frac{-0.6173 + j0.9979}{1 - 0.9e^{-j\pi/3}z^{-1}} + \frac{-0.6173 - j0.9979}{1 - 0.9e^{j\pi/3}z^{-1}}$$

look up table

$$h(n) = 1.2346\,\delta(n) + \left([-0.6173 + j0.9979]\,0.9^n e^{-j\pi n/3} + [-0.6173 - j0.9979]\,0.9\,e^{j\pi n/3}\right) u(n)$$

$$h(n) = 1.12346\,\delta(n) + 0.9^n\left[-1.12346\cos(\pi n/3) + 1.9958\sin(\pi n/3)\right] u(n)$$

$$= 0.9^n\left[-1.12346\cos(\pi n/3) + 1.9958\sin(\pi n/3)\right] u(n-1)$$

$h(0) = 0$

10 c) $H(z) = \dfrac{z+1}{z^2 - 0.9z^{-1} + 0.81}$

roots: cartesian $(0.45 + j0.78), (0.45 - j0.78)$

magnitude $= 0.9$ phase $= \pm \pi/3$

PFE $\dfrac{H(z)}{z} = \dfrac{A}{z} + \dfrac{B}{(z-p)} + \dfrac{C}{(z-p^*)}$

$$A = \left.\frac{H(z)}{z} z \right|_{z=0} = \frac{z+1}{(z-p)(z-p^*)}$$

$$= \frac{1}{-p p^*} = \frac{1}{0.81} = 1.2346$$

$$B = \left.\frac{H(z)}{z} (z-p) \right|_{z=p} = \frac{z+1}{z(z-p^*)}$$

$p = 0.45 + j0.78$

$$= \frac{1 + 0.45 + j0.78}{(0.45+j0.78)(0.45+j0.78 - 0.45 + j0.78)}$$

$$= \frac{1.45 + j0.78}{(0.45+j0.78)(1.56j)}$$

$$= \frac{1.45 + j0.78}{-1.22 + 0.702j}$$

10 c) $$B = \frac{1.45 + j0.78}{-1.22 + 0.702j}$$

$$B = \frac{(1.45 + j0.78)(-1.22 - 0.702j)}{(-1.22 + 0.702j)(-1.22 - 0.702j)}$$

$$B = \frac{-1.22 - 1.97j}{1.98} = -0.617 - 0.998j$$

$$C = B^* \Rightarrow -0.617 + 0.998j$$

$$H(z) = 1.12346 + \frac{(-0.617 - 0.998j)}{1 - 0.9e^{j\pi/3}z^{-1}} + \frac{(-0.617 + 0.998j)}{1 - 0.9e^{-j\pi/3}z^{-1}}$$

IZT look up table

$$h(n) = 1.12346\,\delta(n) + (-0.617 - 0.998j) \times 0.9^n e^{j\frac{\pi n}{3}} u(n)$$
$$+ (-0.617 + 0.998j) \times 0.9^{-j\frac{\pi n}{3}} u(n)$$

$$= 1.12346\,\delta(n) - 0.9^n \times 0.617 \times 2\cos(n\pi/3)\, u(n)$$
$$- 0.9^n \times 0.998j \times 2j \sin(n\pi/3)\, u(n)$$

$$= 1.12346\,\delta(n) + 0.9^n \left[-1.12346 \cos\left(\frac{n\pi}{3}\right) + 1.996 \sin\left(\frac{n\pi}{3}\right)\right] u(n)$$

(10) at $n=0$

$$h(n) = 1.2346 - 1.2346 = 0$$

at $n>0$

$$h(n) = 0.9^n\left[-1.2346\cos\left(\frac{\pi n}{3}\right) + 1.996\sin\left(\frac{n\pi}{3}\right)\right]\mu(n-1)$$

(11) given $x(n) = \{\underset{\uparrow}{-1}, 0, 2, 3\}$ example textbook 5.3

a) find DTFT $X(e^{j\omega})$

b) sample $X(e^{j\omega})$ at $k\omega_1 = \frac{2\pi k_1}{4}$; $k=0,1,2,3$
show it is equal to DFS $\tilde{X}(k)$ of $\tilde{x}(n)$

a) DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{3} x(n)e^{-j\omega n}$$

$$= x(0)e^{-j\omega 0} + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega}$$

$$= -1 + 2e^{-j2\omega} + 3e^{-j3\omega}$$

(11) b) sampling $X(e^{j\omega})$ at $k\omega_1 = \frac{2\pi k}{4}$; $k=0,1,2,3$

$$X(e^{j\frac{2k\pi}{4}k}) = -1 + 2e^{-j2(\frac{2k}{4}\pi)} + 3e^{-j3(\frac{2k\pi}{4})}$$

$$X(e^{j\frac{k\pi}{2}}) = -1 + 2e^{-j\pi k} + 3^{-j\frac{\pi 3}{2}k} \quad ; N=4$$

$k=0 \Rightarrow -1 + 2 + 3 = 4$

$k=1 \Rightarrow -1 + 2e^{-j\pi} + 3e^{-j\pi\frac{3}{2}}$

$-1 - 2 + 3j = -3 + j3$

$k=2 \Rightarrow -1 + 2e^{-j2\pi} + 3e^{-j3\pi}$

$-1 + 2 - 3 = -2$

complex conjugate

$k=3 \Rightarrow -1 + 2e^{-j\pi 3} + 3e^{-j\frac{9\pi}{2}}$

$-1 - 2 - 3j = -3 - 3j$

$W_4 = e^{-j\frac{2\pi}{4}}$ $\quad \tilde{X}(k) = \sum_{n=0}^{3} \tilde{x}(n) W_4^{nk}$

$\tilde{X}(0) = -1 + 0 + 2 + 3 = 4$

$\tilde{X}(1) = -1 + 0 - 2 + 3j = -3 + 3j$ (X)

(11) $\tilde{X}(2) = -1 + 2 - 3 = -2$

$\tilde{X}(3) = -1 - 2 - 3j = -3 - 3j$ (*) complex conjugate

$$X(e^{j\omega}) = \tilde{X}(k) \; ; \; N = 4$$

(12) example 5.8 textbook

Matlab code

(13) circular vs linear convolution

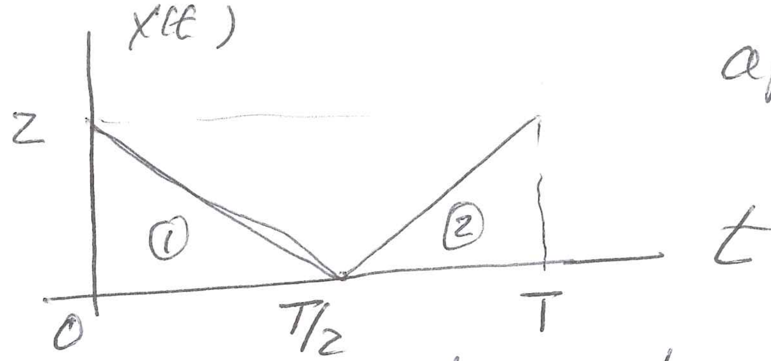
linear $L = N + M - 1$

$x(n)$ is length N

$h(n)$ is length M

$y(n)$ is L in linear convolution

(14) aperiodic CT signal



Find: a) mag. spectra b) phase spectra c) Parseval's theorem

$$x_1(t) = -\frac{4}{T}t + 2 \qquad x_2(t) = \frac{4}{T}t - 2$$

$$0 \le t \le T/2 \qquad T/2 \le t \le T$$

$$X_1(\omega) = \int_0^{T/2} \left(\frac{-4t}{T} + 2\right) e^{-j\omega t}\, dt$$

$$= \int_0^{T/2} \frac{-4t}{T} e^{-j\omega t}\, dt + \int_0^{T/2} 2e^{-j\omega t}\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$$t e^{-j\omega t} dt$$

$$u = t \qquad dv = e^{-j\omega t}$$

$$du = dt \qquad v = \frac{e^{-j\omega t}}{-j\omega}$$

$$\frac{te^{-j\omega t}}{-j\omega} + \int \frac{e^{-j\omega t}}{j\omega}\, dt = \left.\frac{te^{-j\omega t}}{-j\omega} \cancel{+} \frac{1}{(j\omega)(-j\omega)} e^{-j\omega t}\right|_0^{T/2}$$

(14) $$X_1(\omega) = \frac{-4}{T}\left(\frac{t}{-j\omega} - \frac{1}{(-j\omega)^2}\right)e^{-j\omega t}\Bigg|_0^{T/2} + 2\underbrace{\int_0^{T/2} e^{-j\omega t}\,dt}_{\frac{e^{-j\omega t}}{-j\omega}\Big|_0^{T/2}}$$

$$X_1(\omega) = -\frac{4}{T}\left\{\left(\frac{1}{\omega^2} - \frac{T}{j2\omega}\right)e^{-j\omega T/2} - \frac{1}{\omega^2}\right\} + \frac{2}{j\omega} - \frac{2e^{-j\omega T/2}}{j\omega}$$

$$X_2(\omega) = \int_{T/2}^{T} \frac{4t}{T} e^{-j\omega t}\,dt - 2\int_{T/2}^{T} e^{-j\omega t}\,dt$$

$$X_2(\omega) = \frac{4}{T}\left\{\left(\frac{T}{-j\omega} + \frac{1}{\omega^2}\right)e^{-j\omega T} - \left(\frac{T}{-2j\omega} + \frac{1}{\omega^2}\right)e^{-j\omega T/2}\right\} - \frac{2e^{j\omega T/2}}{j\omega} + \frac{2e^{-j\omega T}}{j\omega}$$

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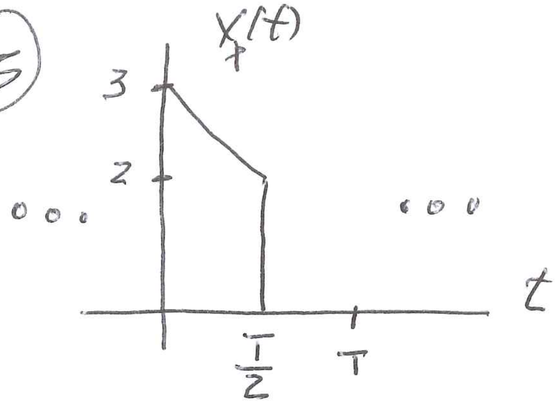

$x_p(t) = -\frac{2}{T}t + 3$

$0 \le t \le \frac{T}{2}$

$x_p(t) = 0 \quad \cancel{\text{else}} \quad \frac{T}{2} \le t \le T$

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega_0}$$

$$C_n = \frac{1}{T}\int_0^T x_p(t)\, e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T}\int_0^{T/2}\left(-\frac{2}{T}t + 3\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{-2}{T^2}\int_0^{T/2} t\, e^{-jn\omega_0 t}\, dt + \frac{3}{T}\int_0^{T/2} e^{-jn\omega_0 t}\, dt$$

$$= \frac{-2}{T^2}\left[\frac{t}{-jn\omega_0} - \frac{1}{(jn\omega_0)^2}\right] e^{-jn\omega_0 t}\Bigg|_0^{T/2}$$

$$= -\frac{2}{T^2}\left[\frac{t e^{-jn\omega_0 t}}{-jn\omega_0} + \frac{e^{-jn\omega_0 t}}{(n\omega_0)^2}\right]_0^{T/2} + \frac{3}{T}\,\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_0^{T/2}$$

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$$C_n = \frac{-2}{T^2}\left[\frac{T e^{-jn\omega_0 T/2}}{-2jn\omega_0} + \frac{e^{-jn\omega_0 T/2}}{(n\omega_0)^2} + 0 + \frac{1}{(n\omega_0)^2}\right]$$

$$+ \frac{3}{T}\left[\frac{e^{-jn\omega_0 T/2}}{-jn\omega_0} - \frac{1}{-jn\omega_0}\right]$$

$$= +\frac{2T e^{-jn\omega_0 T/2}}{T^2 jn\omega_0 2} - \frac{2}{T^2}\frac{e^{-jn\omega_0 T/2}}{(n\omega_0)^2} - \frac{2}{T^2(n\omega_0)^2}$$

$$+ \frac{3 e^{-jn\omega_0 T/2}}{T jn\omega_0} + \frac{3}{jTn\omega_0}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{\cancel{2}\cancel{T} e^{-jn\frac{\cancel{2}\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}}}{\cancel{T}^2 2jn\frac{2\pi}{\cancel{T}}} - \frac{2 e^{-j\frac{\cancel{2}\pi}{\cancel{T}}\frac{\cancel{T}n}{\cancel{2}}}}{\cancel{T^2}\left(n\frac{2\pi}{\cancel{T}}\right)^2} - \frac{2}{\cancel{T^2}\left(n\frac{2\pi}{\cancel{T}}\right)^2}$$

$$- \frac{3 e^{-jn\frac{\cancel{2}\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}}}{\cancel{T} jn\frac{2\pi}{\cancel{T}}} + \frac{3}{j\cancel{T}n\frac{2\pi}{\cancel{T}}}$$

$$C_n = \frac{e^{-jn\pi}}{jn\pi 2} - \frac{e^{-jn\pi}}{2(n\pi)^2} - \frac{1}{2(n\pi)^2}$$

$$- \frac{3e^{-jn\pi}}{jn2\pi} + \frac{3}{jn2\pi}$$

(15)

$$C_n = \frac{1}{j2\pi n}\left(e^{-jn\pi} - 3e^{-jn\pi} + 3\right)$$

$$+ \frac{1}{2(n\pi)^2}\left(1 + e^{-jn\pi}\right)$$

$$C_n = \frac{1}{j2\pi n}\left(-2e^{-jn\pi} + 3\right) - \frac{1}{2(n\pi)^2}\left(1 + e^{-jn\pi}\right)$$

$$e^{-jn\pi} = \cos(-n\pi) + j\cancel{\sin(-n\pi)}^{\,0}$$

$n =$ even

$\cos(-n\pi) = 1 \quad \sin(-n\pi) = 0$

$n =$ odd $\quad \cos(-n\pi) = -1 \quad \sin(-n\pi) = 0$

For $n =$ even

$$C_n = \frac{1}{j2\pi n}(3-2) - \frac{1}{2(\pi n)^2}(2)$$

$$C_n = \frac{1}{j2\pi n} - \frac{1}{(\pi n)^2}$$

15 for $n=odd$

$$C_n = \frac{1}{j2\pi n}\left[3+2\right] - \frac{1}{2(n\pi)^2}(1-1)$$

$$C_n = \frac{5}{j2\pi n}$$

$$C_n = \begin{cases} \dfrac{1}{j2\pi n} - \dfrac{1}{(\pi n)^2} & \text{for } n=\text{even} \\ \dfrac{5}{j2\pi n} & \text{for } n=\text{odd} \end{cases}$$

<u>n=even</u>

$$C_n = \frac{-1}{(\pi n)^2} - \frac{j}{2\pi n} \qquad |C_n| = \sqrt{\left(\frac{1}{(\pi n)^2}\right)^2 + \left(\frac{1}{(2\pi n)}\right)^2}$$

$$\angle C_n = \left(\frac{-1/2\pi n}{-1/(\pi n)^2}\right)^{\tan^{-1}}$$

<u>n=odd</u>

$$|C_n| = \left(\frac{5}{2\pi n}\right)$$

$$\angle C_n = \left(\frac{\frac{5}{2\pi n}}{0}\right)^{\tan^{-1}}$$

(15) Parseval's Theorem

$$P = \frac{1}{T}\int_0^T |x(t)|^2 dt$$

$$= \frac{1}{T}\int_0^{T/2} \left(-\frac{2t}{T}+3\right)^2 dt$$

$$= \frac{1}{T}\left[\int_0^{T/2} \frac{4t^2}{T^2}\,dt - \int_0^{T/2} \frac{12t}{T}\,dt + \int_0^{T/2} 9\,dt\right]$$

$$= \frac{1}{T^3}\int_0^{T/2} t^2 dt - \frac{12}{T^2}\int_0^{T/2} t\,dt + \frac{9}{T}\int_0^{T/2} dt$$

$$= \frac{4}{T^3}\,\frac{t^3}{3}\Big|_0^{T/2} - \frac{12}{T^2}\,\frac{t^2}{2}\Big|_0^{T/2} + \frac{9}{T}\,t\Big|_0^{T/2}$$

$$= \frac{4}{3\cancel{T^3}}\left(\frac{\cancel{T^3}}{8}\right) - \frac{6}{\cancel{T^2}}\left(\frac{\cancel{T^2}}{4}\right) + \frac{9}{\cancel{T}}\,\frac{\cancel{T}}{2}$$

$$= \frac{4}{24} - \frac{6}{4} + \frac{9}{2}$$

$$P = 19/6$$

(15) $x(t) = -\frac{2}{T} + 3 \qquad 0 \le t \le T/2$

$x(t) = 0 \qquad \frac{T}{2} \le t \le T$

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x_p(t) e^{-jn\omega_0 t} dt$$

$$x_p'(t) = 3\delta(t) - \frac{2}{T}\left[u(t) - u(t-\tfrac{T}{2})\right] - 2\delta(t - T/2)$$

$$x_p''(t) = 3\delta'(t) - \frac{2}{T}\left[\delta(t) - \delta(t-\tfrac{T}{2})\right] - 2\delta'(t - T/2)$$

$$\alpha_n = C_n (jn\omega_0)^2 ; \quad \omega_0 = \frac{2\pi}{T}$$

$$\alpha_n = \frac{1}{T}\int_0^T x_p''(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_0^T \left[3\delta'(t) - \frac{2}{T}\left[\delta(t) - \delta(t - T/2)\right] - 2\delta'(t-\tfrac{T}{2})\right] e^{-jn\omega_0 t} dt$$

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$$\alpha_n = \frac{1}{T}\left[3(jn\omega_0) - \frac{2}{T}\left(1-e^{-jn\omega_0 T/2}\right) - 2jn\omega_0 e^{-jn\omega_0 T/2}\right]$$

$$\alpha_n = \frac{-2}{T^2}\left(1-e^{-jn\omega_0 T/2}\right) + \frac{jn\omega_0}{T}\left(3-2e^{-jn\omega_0 T/2}\right)$$

$$C_n = \frac{\alpha_n}{(jn\omega_0)^2}$$

$$C_n = \frac{-2}{(jn\omega_0 T)^2}\left(1-e^{-jn\omega_0 T/2}\right) + \frac{1}{jn\omega_0 T}\left(3-2e^{-jn\omega_0 T/2}\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{-2}{\left(jn\frac{2\pi}{\cancel{T}}\cancel{T}\right)^2}\left(1-e^{-jn\frac{2\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}}\right) + \frac{1}{jn\frac{2\pi}{\cancel{T}}\cancel{T}}\left(3-2e^{-jn\frac{\cancel{2}\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}}\right)$$

$$C_n = \frac{-2}{(jn2\pi)^2}\left(1-e^{-jn\pi}\right) + \frac{1}{jn2\pi}\left(3-2e^{-jn\pi}\right)$$

$$C_n = \frac{2}{(n2\pi)^2}\left(1-e^{-jn\pi}\right) - \frac{j}{(n2\pi)}\left(3-2e^{-jn\pi}\right)$$

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$$e^{-jn\pi} = \cos(-n\pi) - j\,\overset{0}{\cancel{\sin(-n\pi)}}$$

$$e^{-jn\pi} = \begin{cases} 1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases}$$

<u>n = even</u>

$$C_n = \frac{1}{2(n\pi)^2}(1-1) - \frac{j}{(2n\pi)}(3-2)$$

$$C_n = -\frac{j}{2n\pi} = \frac{1}{j2n\pi}$$

<u>n = odd</u>

$$C_n = \frac{1}{2(n\pi)^2}(2) - \frac{j}{(2n\pi)}(5)$$

$$C_n = \frac{1}{(n\pi)^2} - \frac{5j}{(2n\pi)}$$

(16) Connection between FS and FT

look at FS pair; L = period

$$x_p(t) = \frac{1}{L} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \int_L x_p(t) e^{-jn\omega_0 t} \, dt$$

as $L \to \infty$

$$\omega_0 = \frac{2\pi}{L} \longrightarrow d\omega$$

$$n\omega_0 \longrightarrow \omega$$

$$x_p(t) \longrightarrow x(t)$$

spacing between lines ~~bet~~ (spectral lines) get closer and closer and the signal becomes periodic

(16)

$$\lim_{L \to \infty} C_n = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$$

Forward FT

and

$$\lim_{L \to \infty} x_p(t) = \lim_{L \to \infty} \frac{1}{L} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \lim_{L \to \infty} \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \lim_{L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} C_n e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Inverse FT

(16) Consider a function defined $[-L/2, L/2]$

FT is $$X(\omega) = \int_{-L/2}^{L/2} x(t) e^{-j\omega t} dt$$

define $\omega_0 = \frac{2\pi}{L}$, then, at integer multiples of ω_0 we get

$$X(n\omega_0) = \int_{-L/2}^{L/2} x(t) e^{-jn\omega_0 t} dt$$

Consider periodic extension of $x(t)$, that is, $x_p(t)$

$$C_n = \int_{-L/2}^{L/2} x_p(t) e^{-jn\omega_0 t} dt$$

$$= \int_{-L/2}^{L/2} x(t) e^{-jn\omega_0 t} dt$$

The FT of $x(t)$ at equally spaced points on the ω axis is exactly specified by the FS.

So $X(n\omega_0) = C_n$

Distance between points is $\frac{2\pi}{L}$ radians

(17) look at problem 8
example 3.17 textbook