In the literature you will find two entirely different approaches to this problem. The first is a straightforward "brute force" solution to the differential equation, using the **power series method**; it has the virtue that the same strategy can be applied to many other potentials. The second is a diabolically clever algebraic technique, using so-called **ladder operators**. I'll show you the algebraic method first, because it is quicker and simpler (and a lot more fun);[17] if you want to skip the power series method for now, that's fine, but you should certainly plan to study it at some stage.

$\hat{H}\psi = E\psi$

3.1 Algebraic Method

To begin with, let's rewrite Equation 44 in a more suggestive form:

$$\frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi, \tag{45}$$

where $p \equiv (\hbar/i)d/dx$ is, of course, the momentum operator. The basic idea is to *factor* the Hamiltonian,

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2]. \tag{46}$$

$\frac{p^2}{2m} + \frac{(m\omega x)^2}{2m}$

$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = T + V$

SHM Potential Energy function

If these were *numbers*, it would be easy:

$$u^2 + v^2 = (iu + v)(-iu + v).$$

Here, however, it's not quite so simple, because p and x are *operators*, and operators do not, in general, **commute** (xp is not the same as px). Still, this does motivate us to examine the quantities

$$\boxed{a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)} \tag{47}$$

(the factor in front is just there to make the final results look nicer).

Well, what *is* the product $a_- a_+$?

$$\begin{aligned} a_- a_+ &= \frac{1}{2\hbar m\omega}(ip + m\omega x)(-ip + m\omega x) \\ &= \frac{1}{2\hbar m\omega}[p^2 + (m\omega x)^2 - im\omega(xp - px)]. \end{aligned}$$

[17] Some of the same strategies are encountered in the theory of angular momentum, and the technique generalizes to a broad class of potentials in **super-symmetric quantum mechanics** (see, for example, Richard W. Robinett, *Quantum Mechanics*, (Oxford U.P., New York, 1997), Section 14.4).