90% Confidence Level Upper Bound

Overview of Feldman & Cousins

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Outline

1. Confidence Interval (CI)
2. Bayesian

Confidence Interval (CI)

CI Definition

- Goal: estimate parameter μ whose true value is μ_t
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- More experiments; get a bunch of intervals. *i.e.* we get a set

$$C \equiv \{[\mu_l(x_0), \mu_u(x_0)], [\mu_l(x_1), \mu_u(x_1)], [\mu_l(x_2), \mu_u(x_2)] \ldots\}$$

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- The members of C are called *confidence intervals*.

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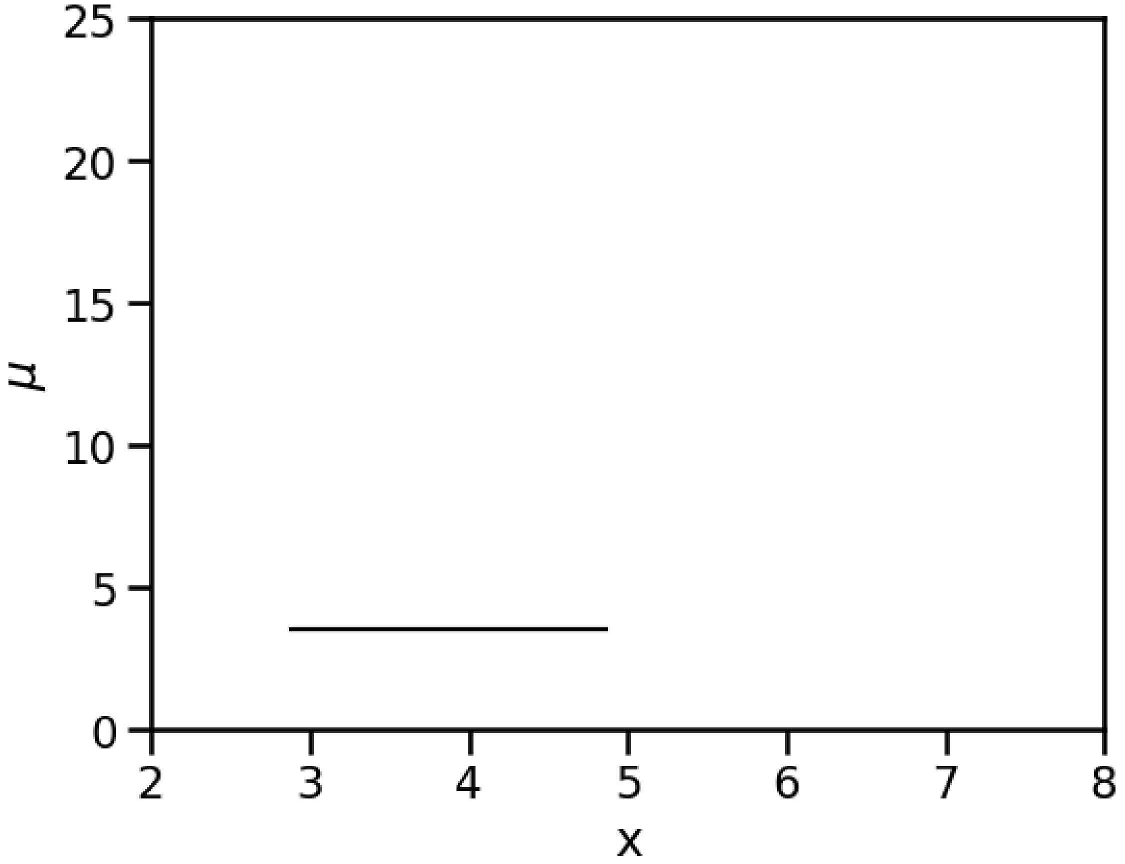

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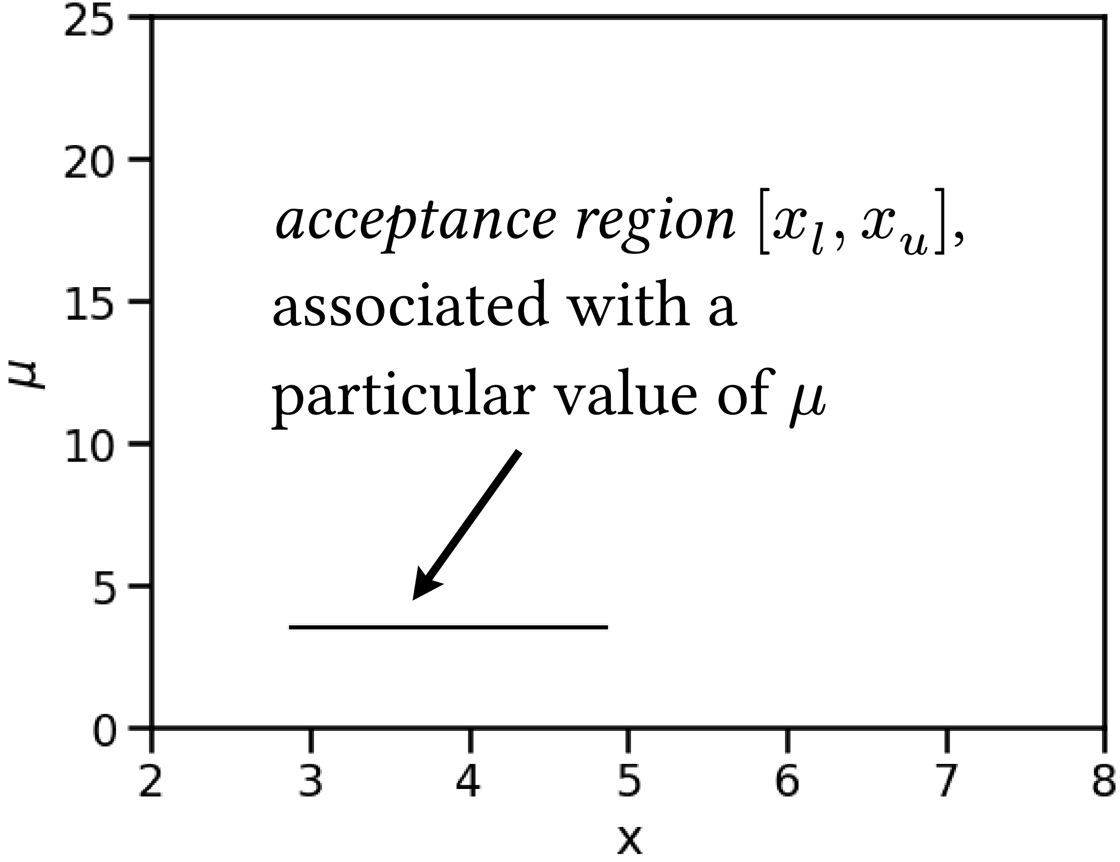

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- Take another value μ, say $\mu = 5$
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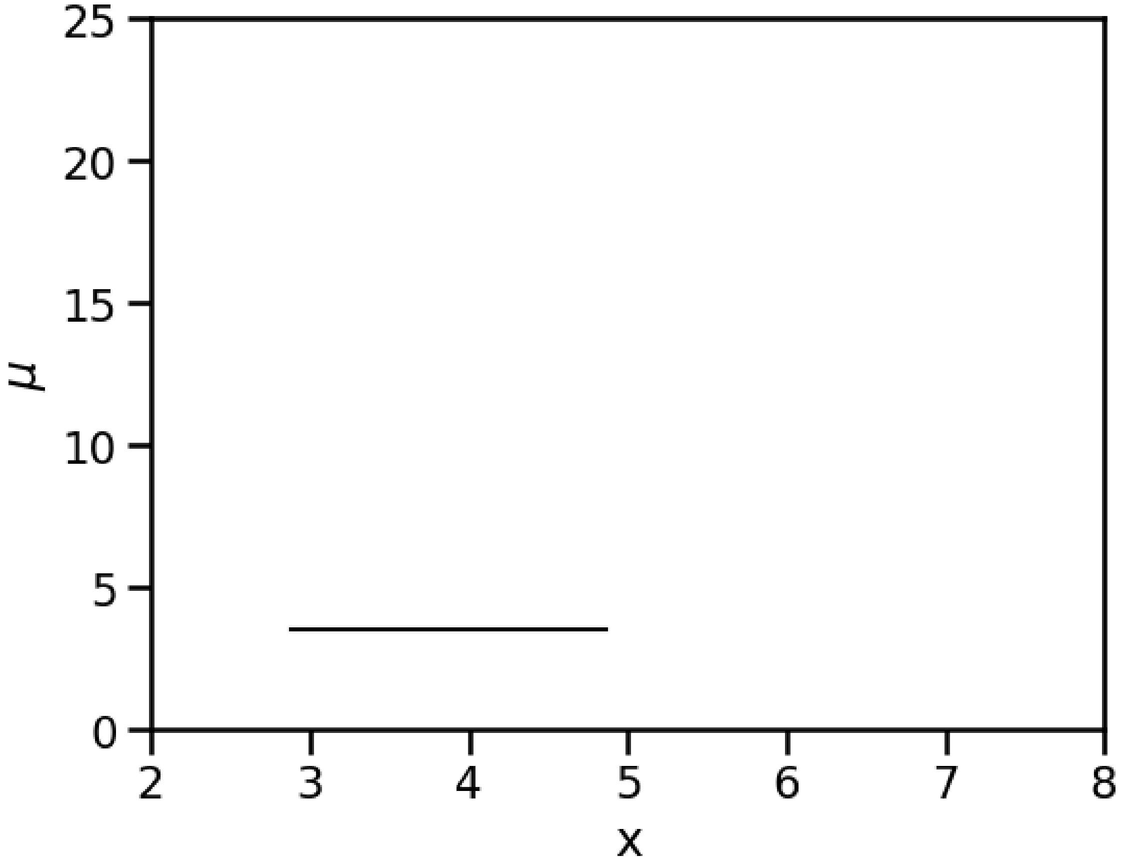
- Do this many times.
- Get confidence belt.



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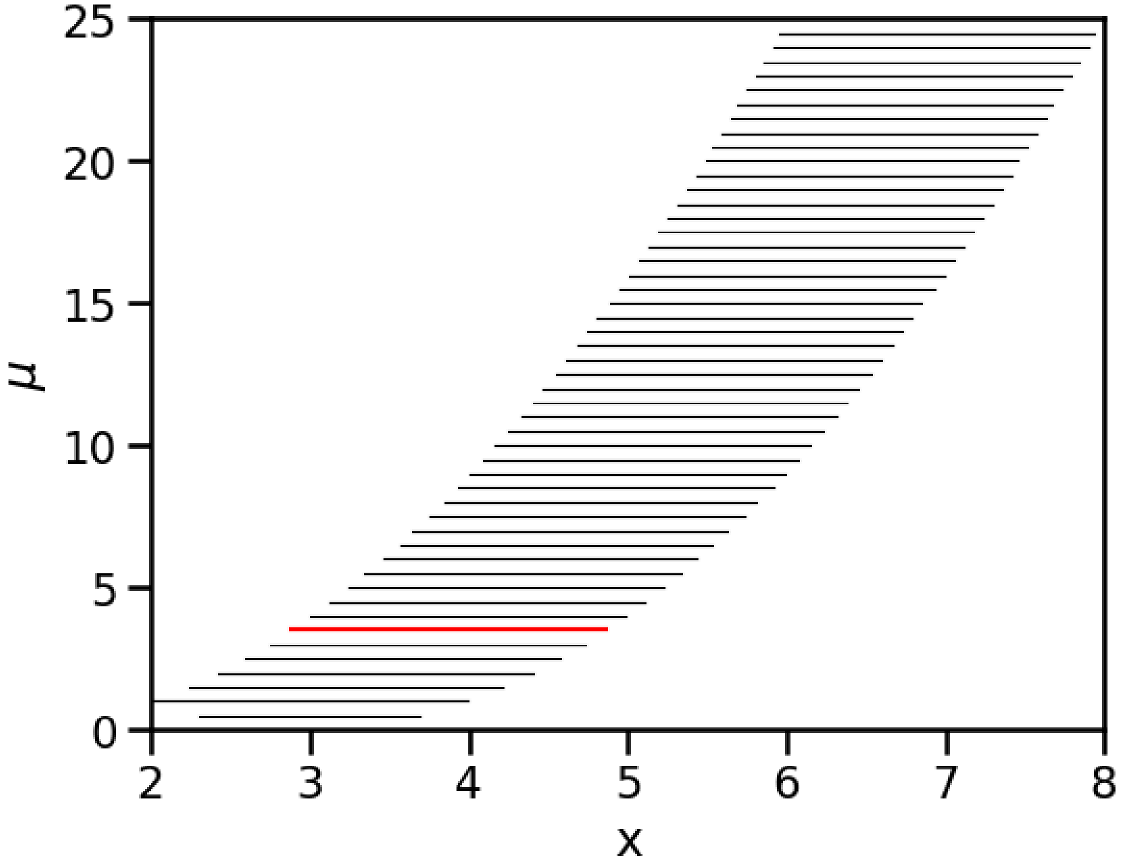

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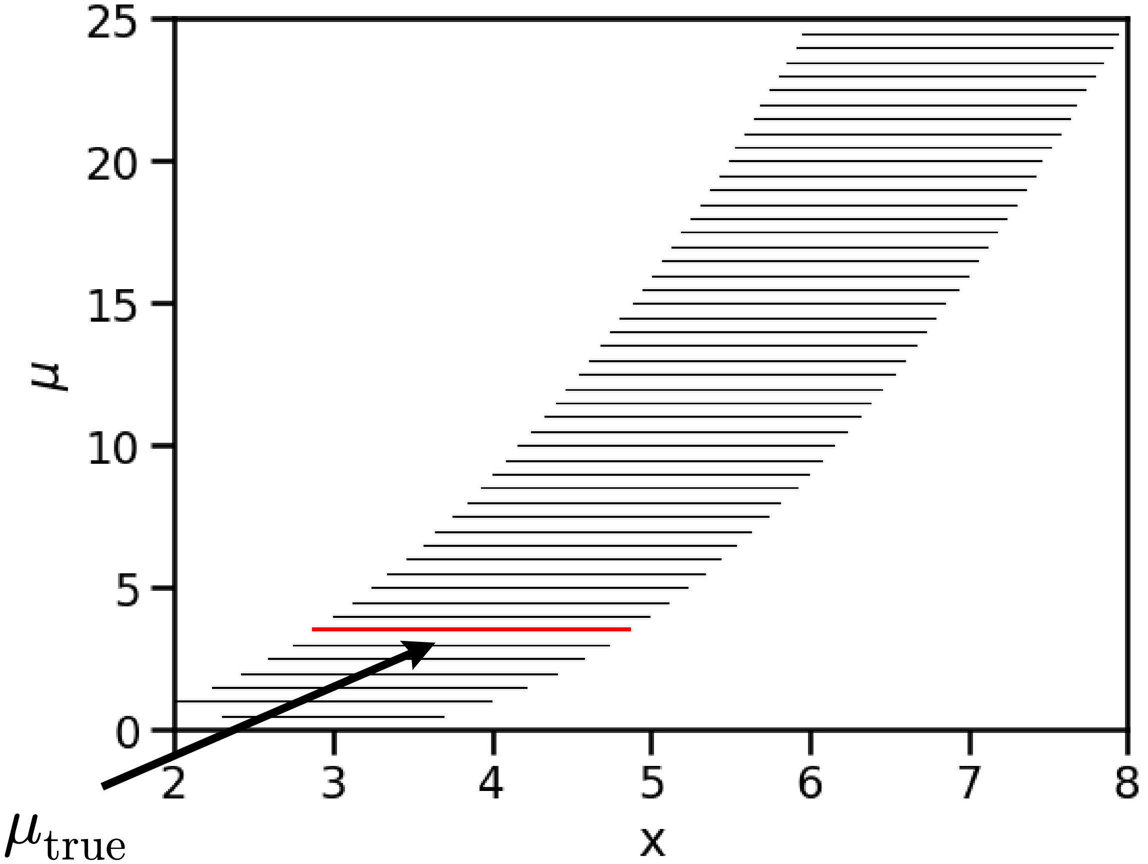

suppose this is μ_{true}

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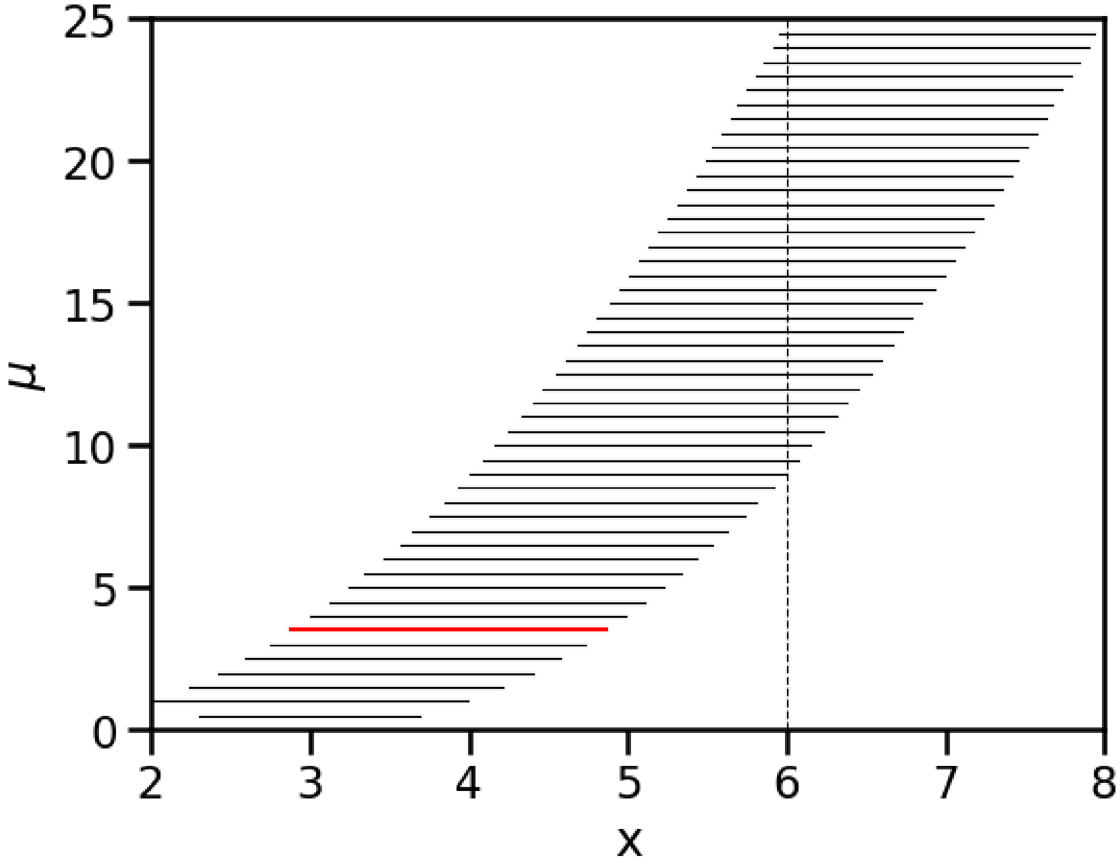

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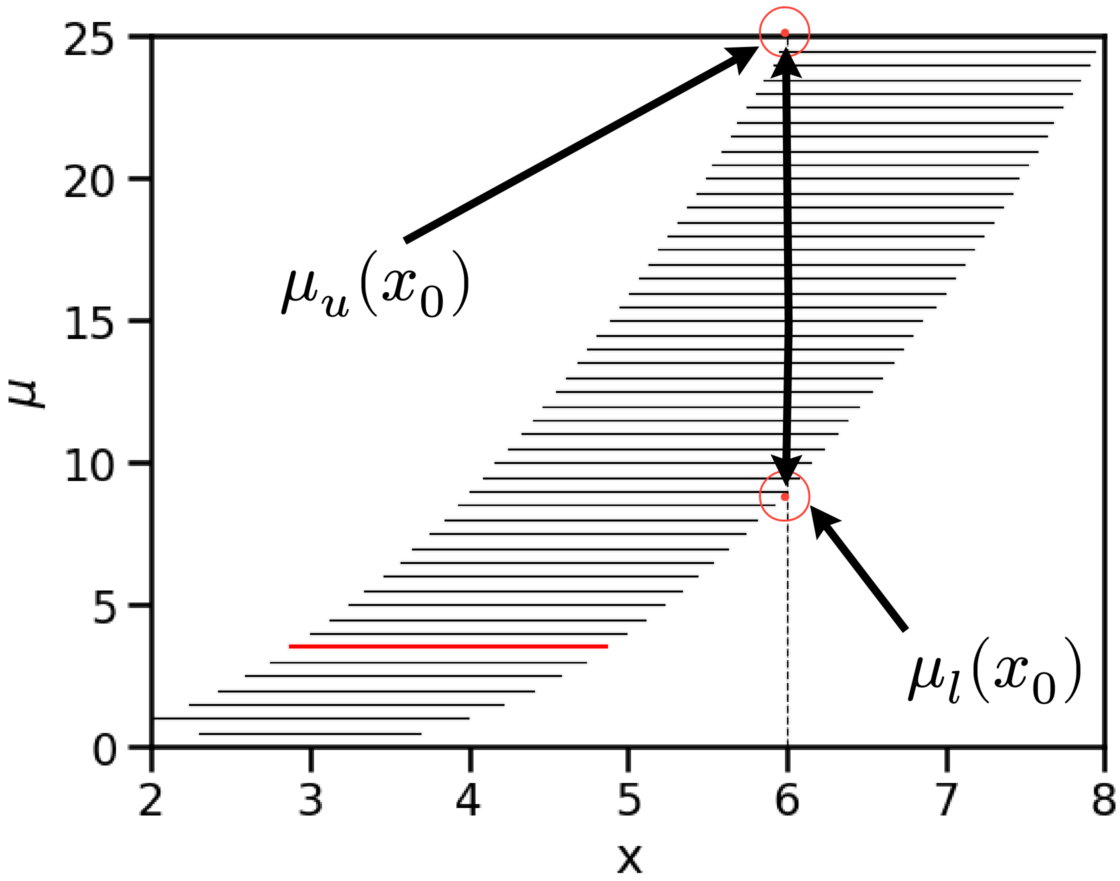
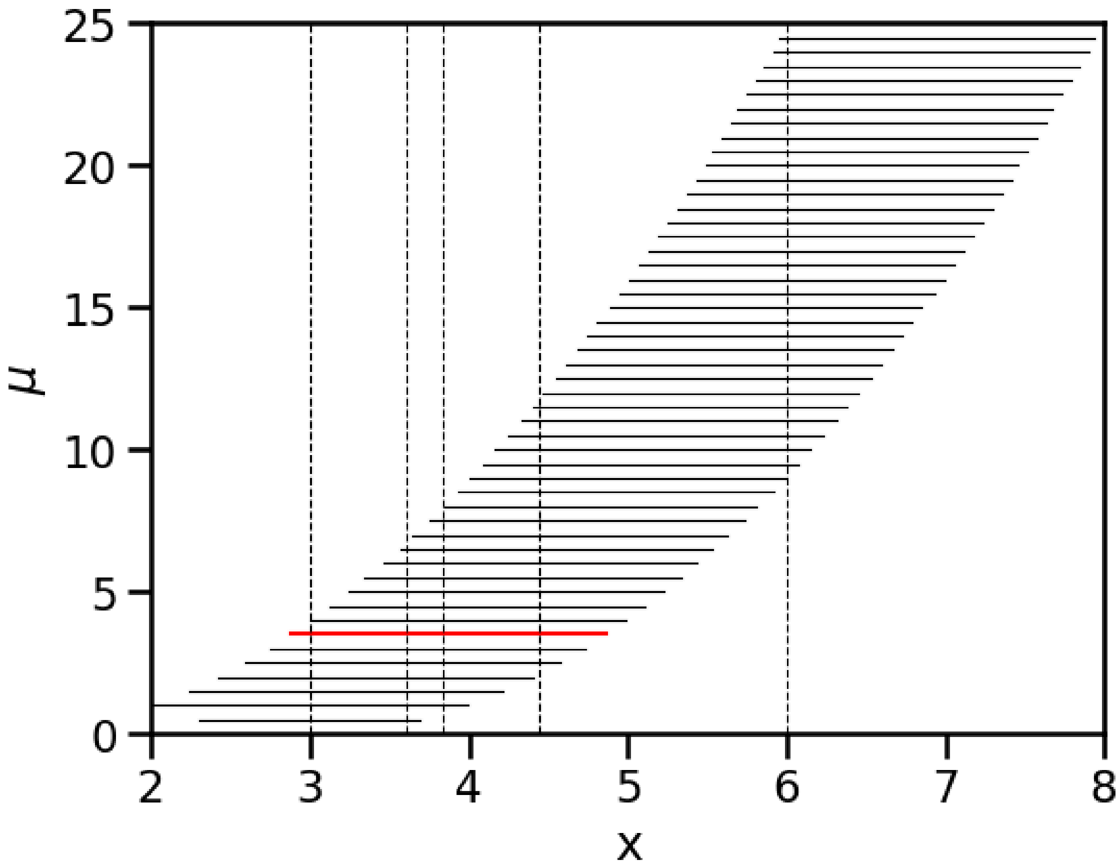


CI Construction: Confidence Belt

- Make some more measurements
- Get some more confidence intervals.
- Have a set

$$C = \{\mathrm{CI}_1, \mathrm{CI}_2, \mathrm{CI}_3, \mathrm{CI}_4, \mathrm{CI}_5\}$$

- 80% of this set would cover the true value, μ_t.



Bayesian

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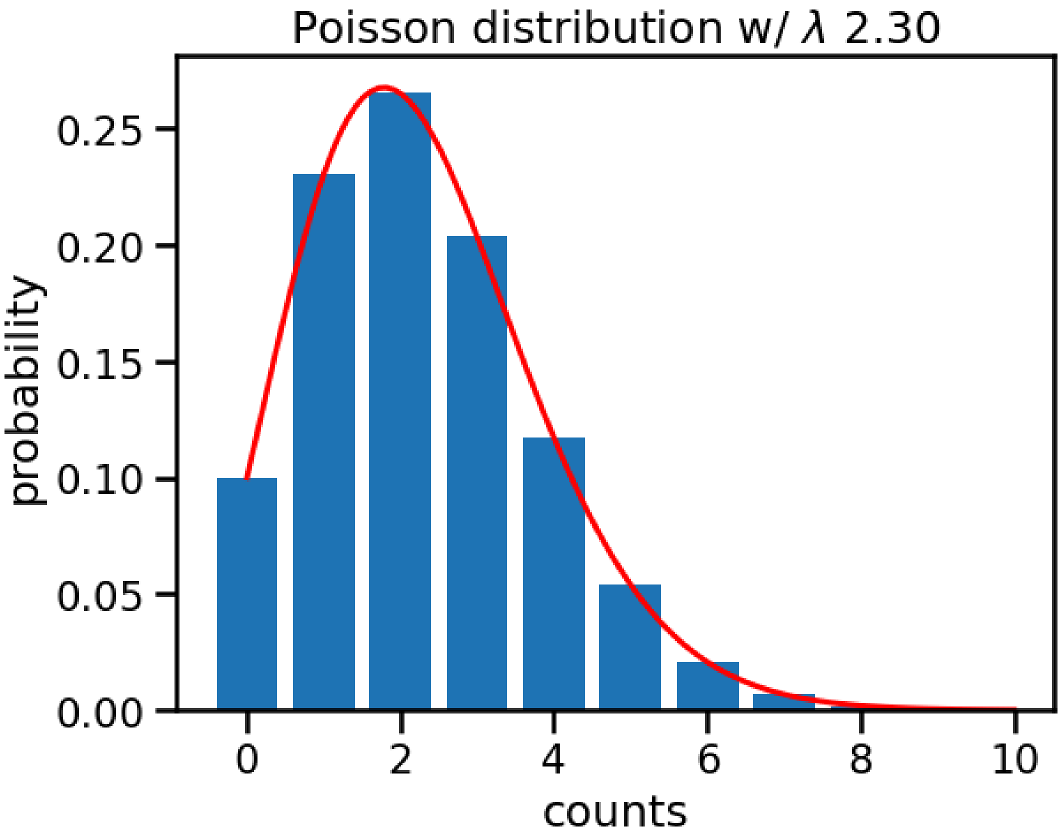
Example:



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$$\underbrace{\Pr(\lambda \mid n)}_{\text{Posterior}} = \frac{\overbrace{\Pr(n \mid \lambda)}^{\text{Likelihood}} \cdot \Pr(\lambda)}{\underbrace{\Pr(n)}_{\text{evidence}}}$$

- Evidence is typically just a normalization and ignored. Let's call it 1 :)

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- "uniform prior" $\Pr(\lambda) = 1 \implies \Pr(\lambda \mid n) = \Pr(n \mid \lambda)$
- Suppose we measure $n = 0$ event, then the posterior is

$$\Pr(\lambda \mid n = 0) = \Pr(n = 0|\lambda) = \frac{e^{-\lambda} \cdot \lambda^n}{n!} = e^{-\lambda}$$

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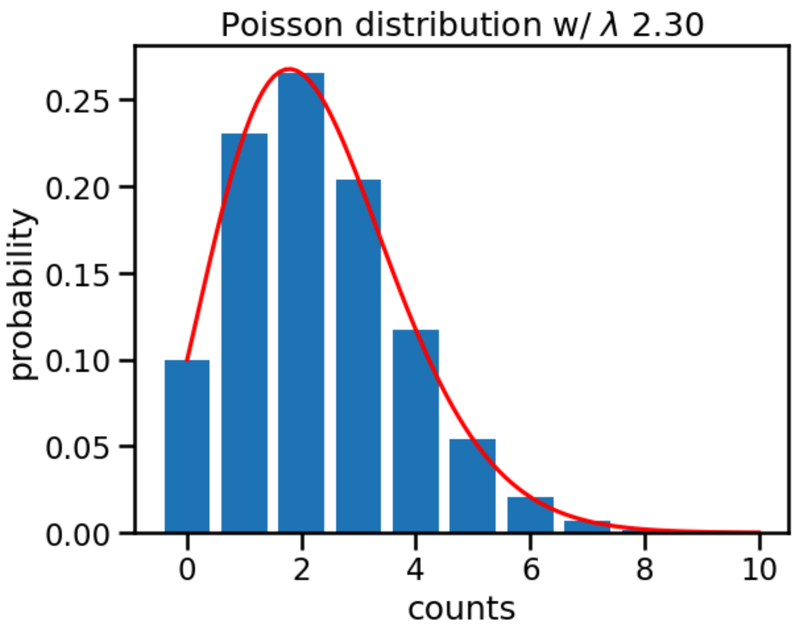
- so we “estemate with 90% confidence that $\lambda \leq 2.3$” base on a non-detection.



Summary: Bayesian Credible Interval

- Let μ denote the unknown parameter we wish to estimate.
- Let x_0 denote the outcome of a single measurement.
- Assume that we know how the measurement outcome depends on the parameter, $x = x(\mu)$.
  - *e.g.* if the neutrino flux is very small, then oftentimes a measurement reports a non-detection.
  - In other words, we know the *likelihood*, $P(x_0|\mu)$.
- From the Bayesian perspective, we can flip things around and say that the parameter is a function of the measurement, $P(\mu|x_0)$, provided that we state our prior beliefs about the parameter, $P(\mu)$.