90% Confidence Level Upper Bound

Brief discussion of Feldman & Cousins

10-31-2024

Outline

Parameter Estimataion

Poisson Distribution

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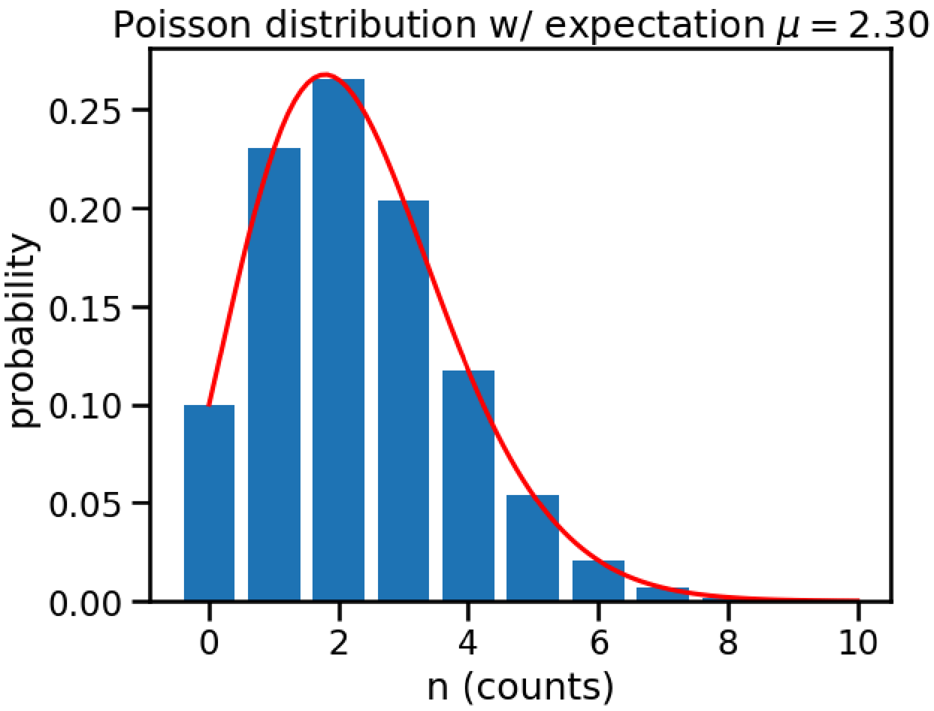
- Example:



Confidence Interval (CI)

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- Repeat experiment; get outcome $x_1 \longrightarrow$ construct $[\mu_l(x_1), \mu_u(x_1)]$
- More experiments; get a bunch of intervals. *i.e.* we get a set

$$C \equiv \{[\mu_l(x_0), \mu_u(x_0)], [\mu_l(x_1), \mu_u(x_1)], [\mu_l(x_2), \mu_u(x_2)]...\}$$

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- The members of C are called *confidence intervals*.

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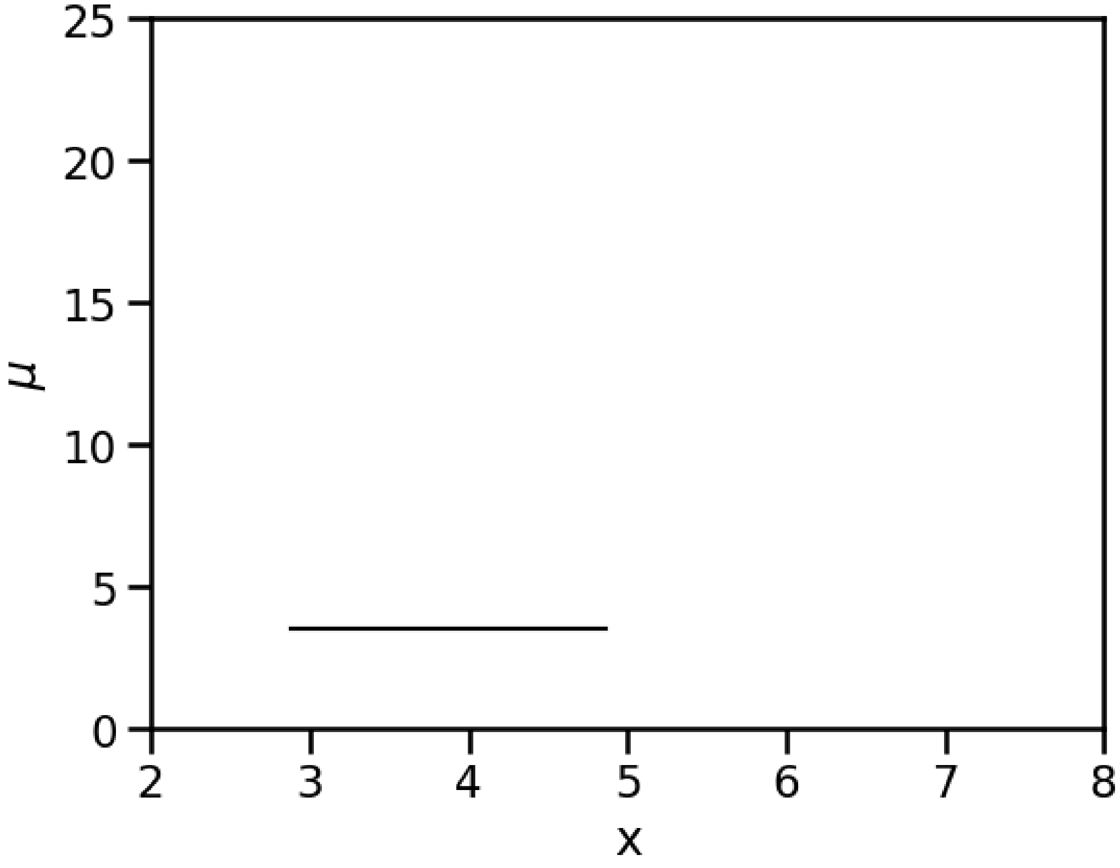

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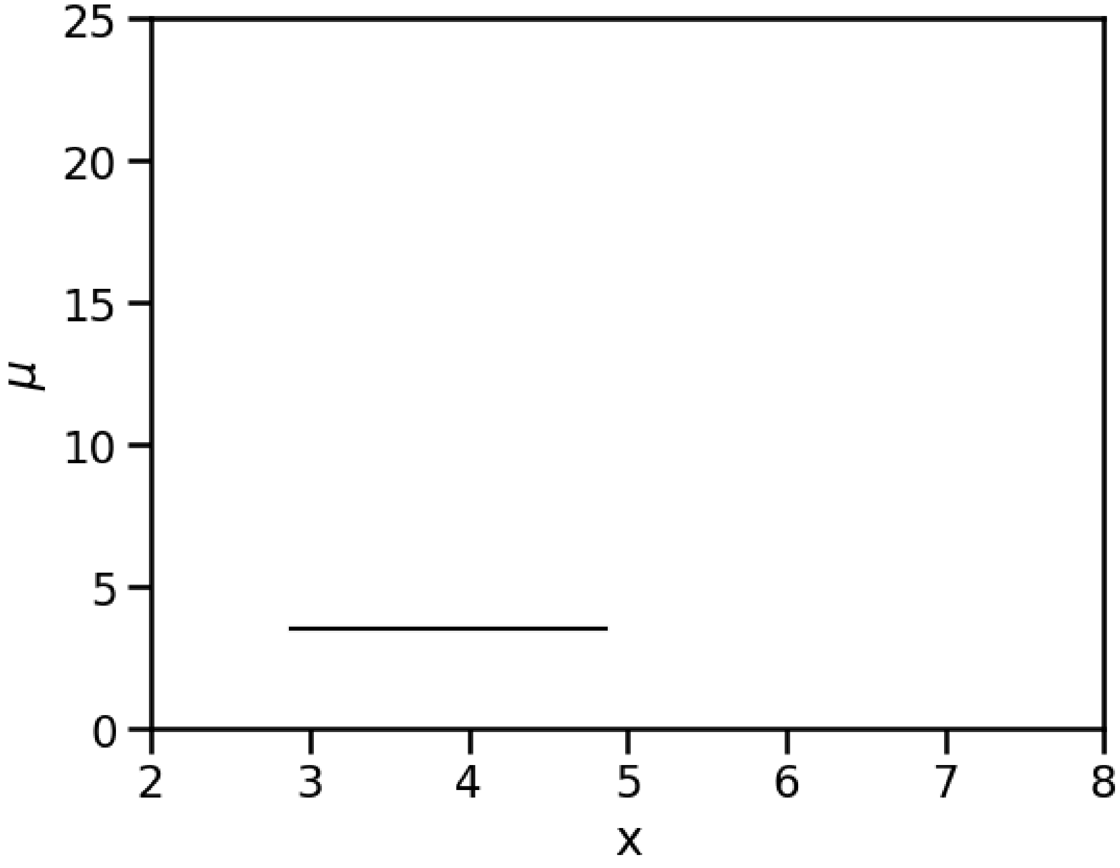
- Select a region $[x_l, x_u]$ such that the probability of measuring $x \in [x_l, x_u]$ is, say, 80%.



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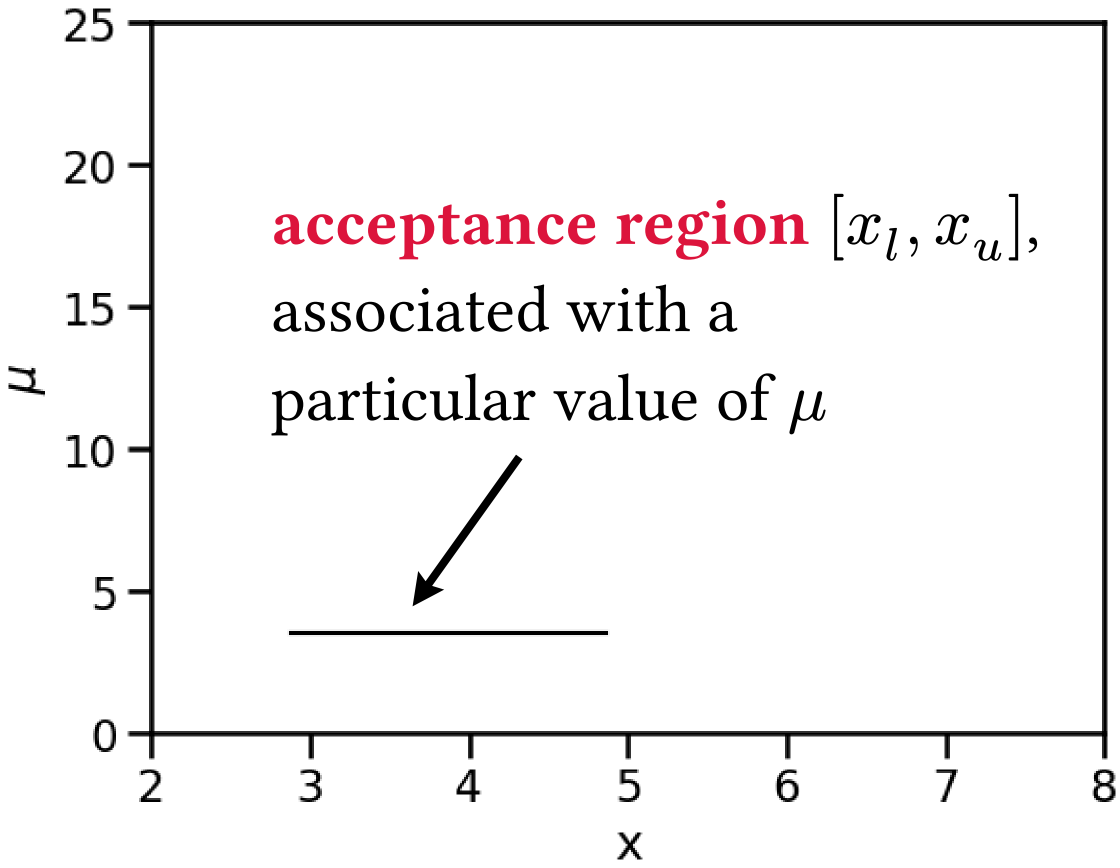

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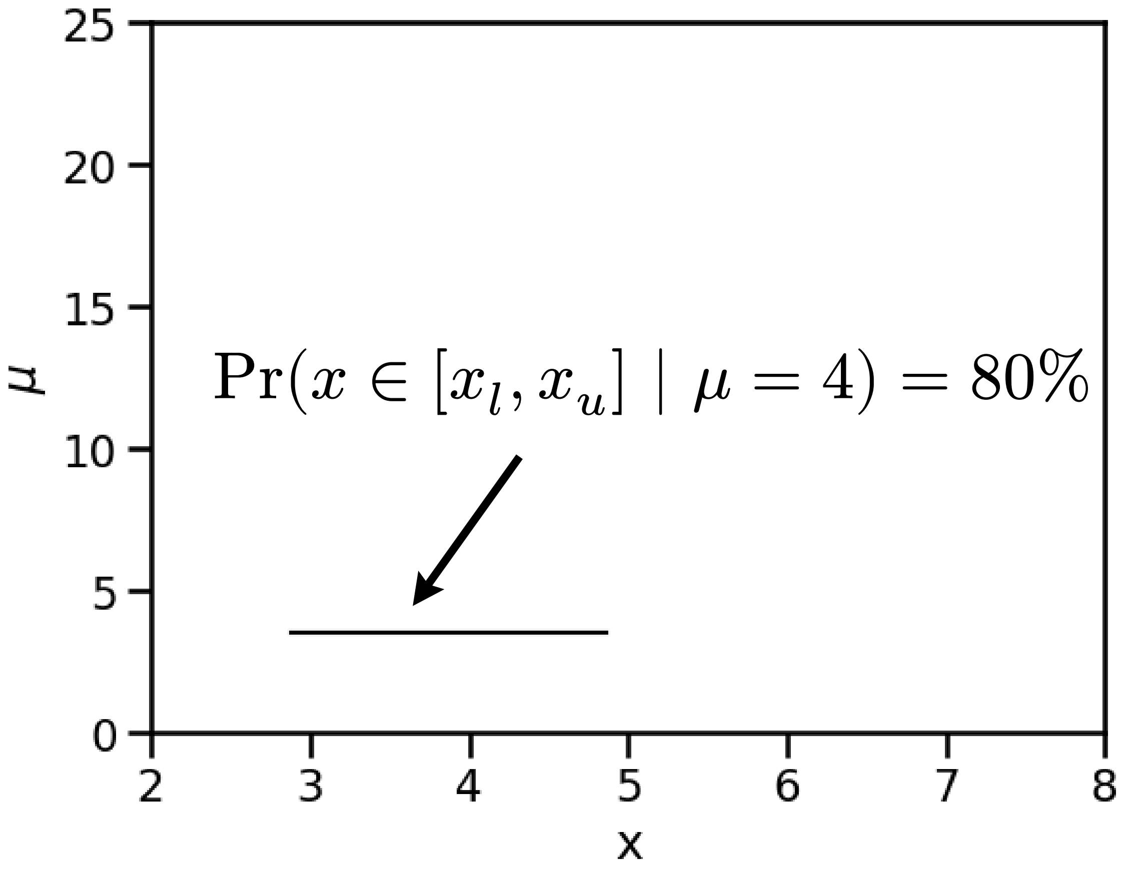

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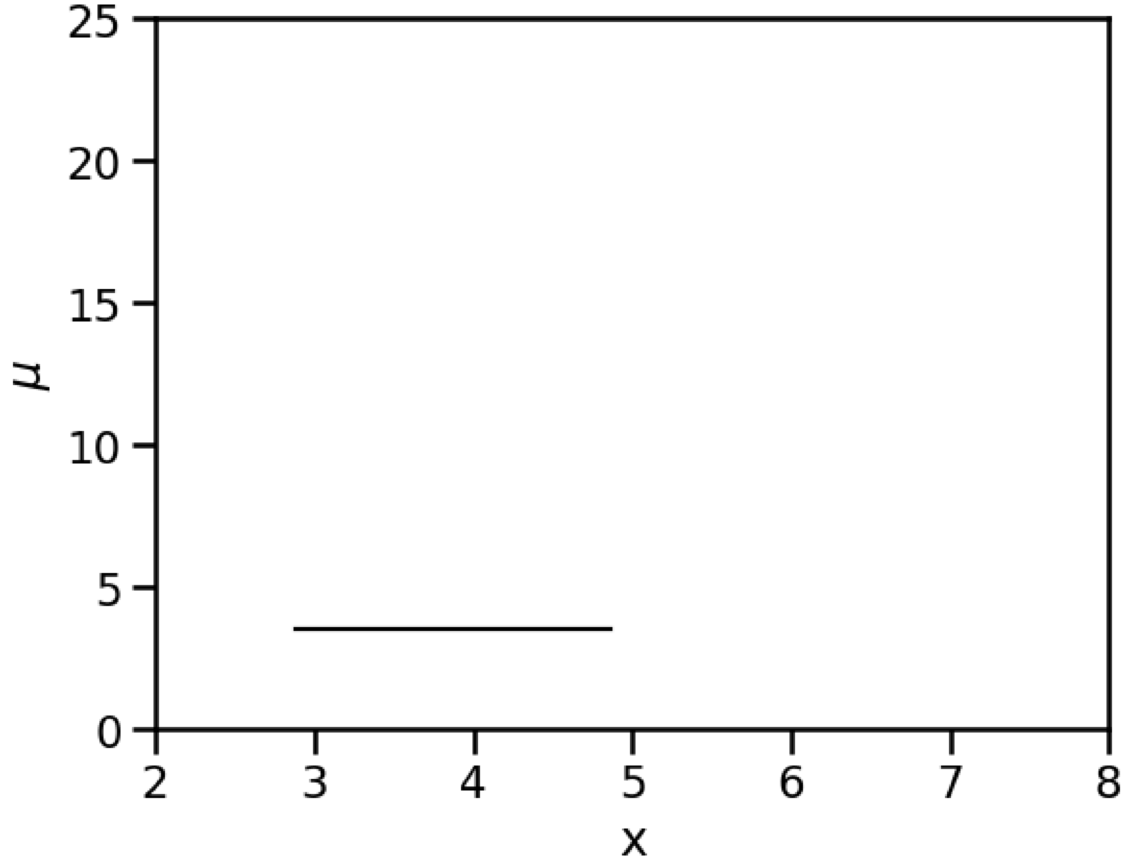

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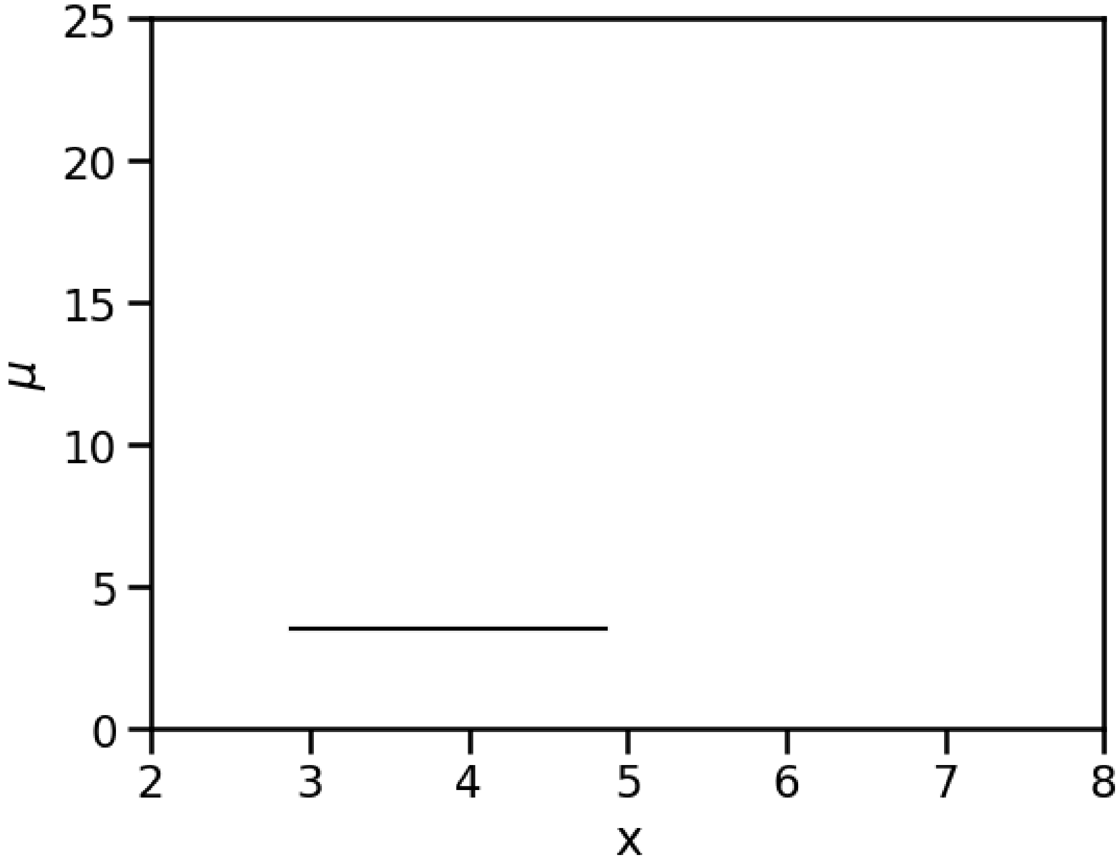

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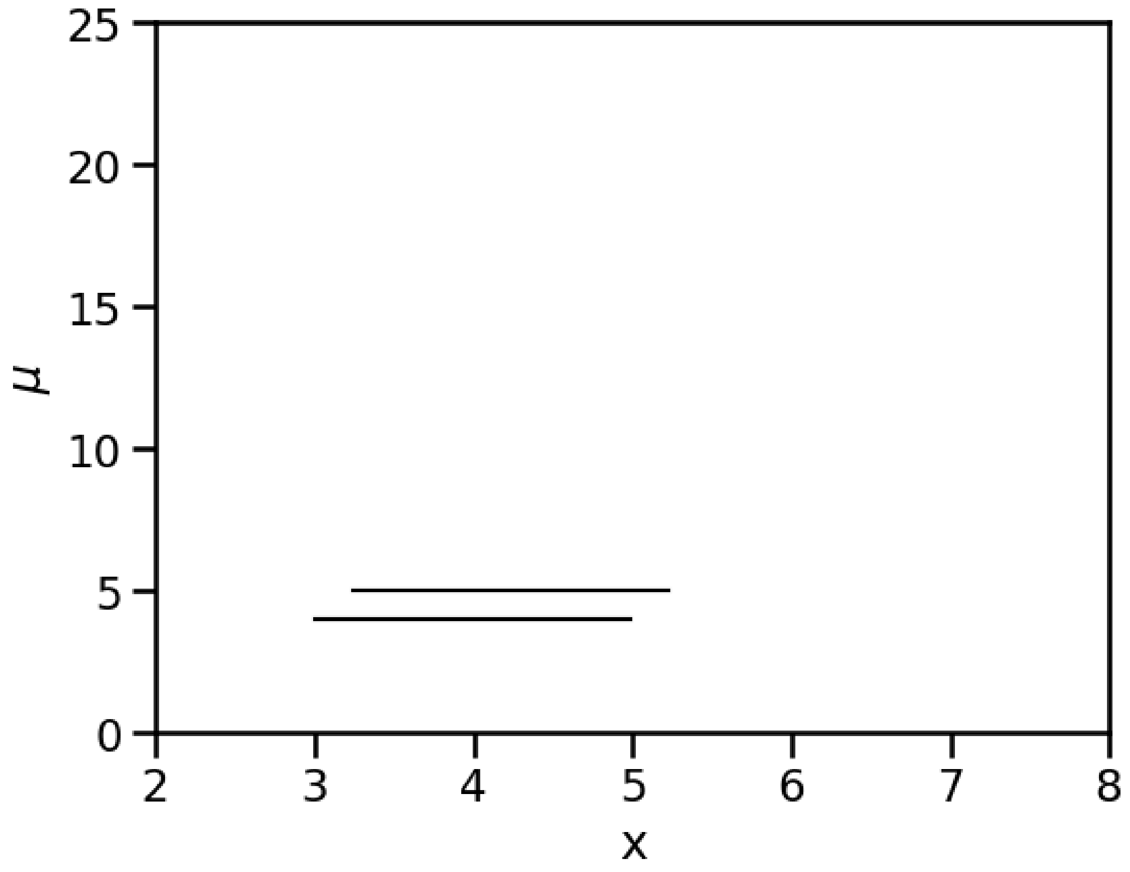

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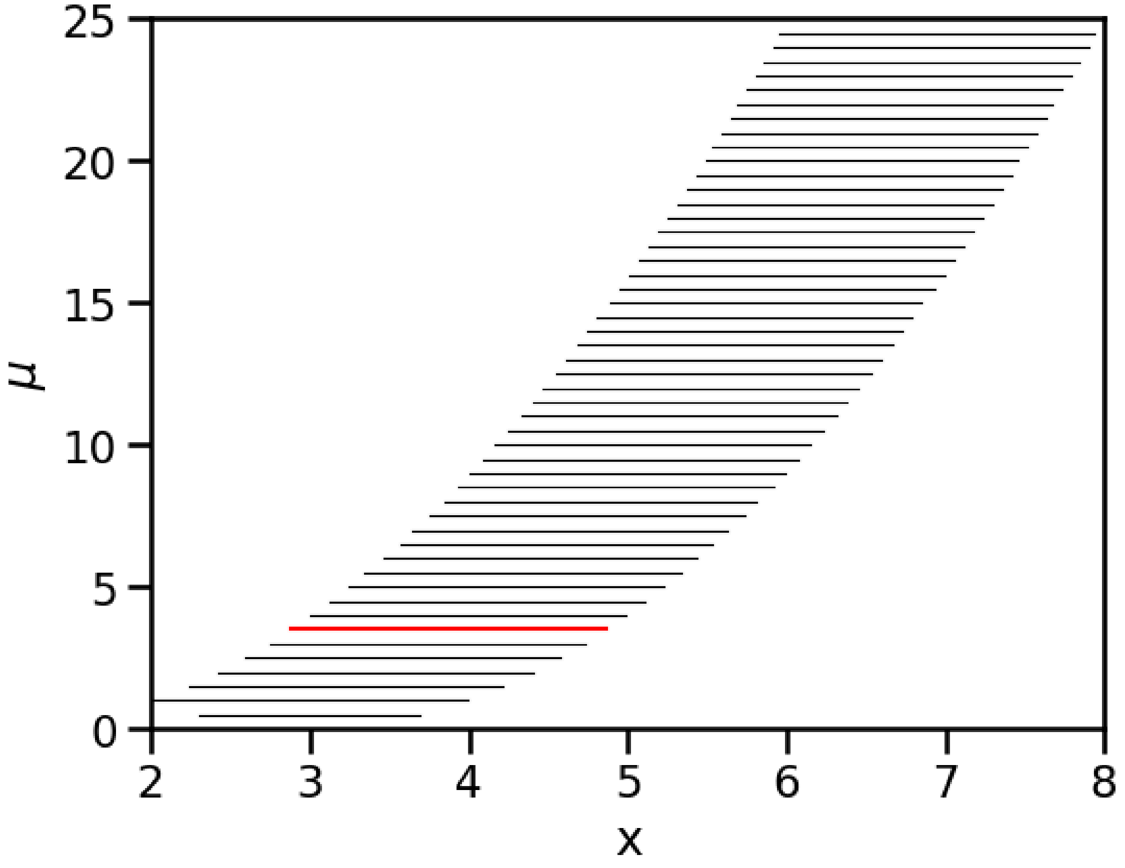

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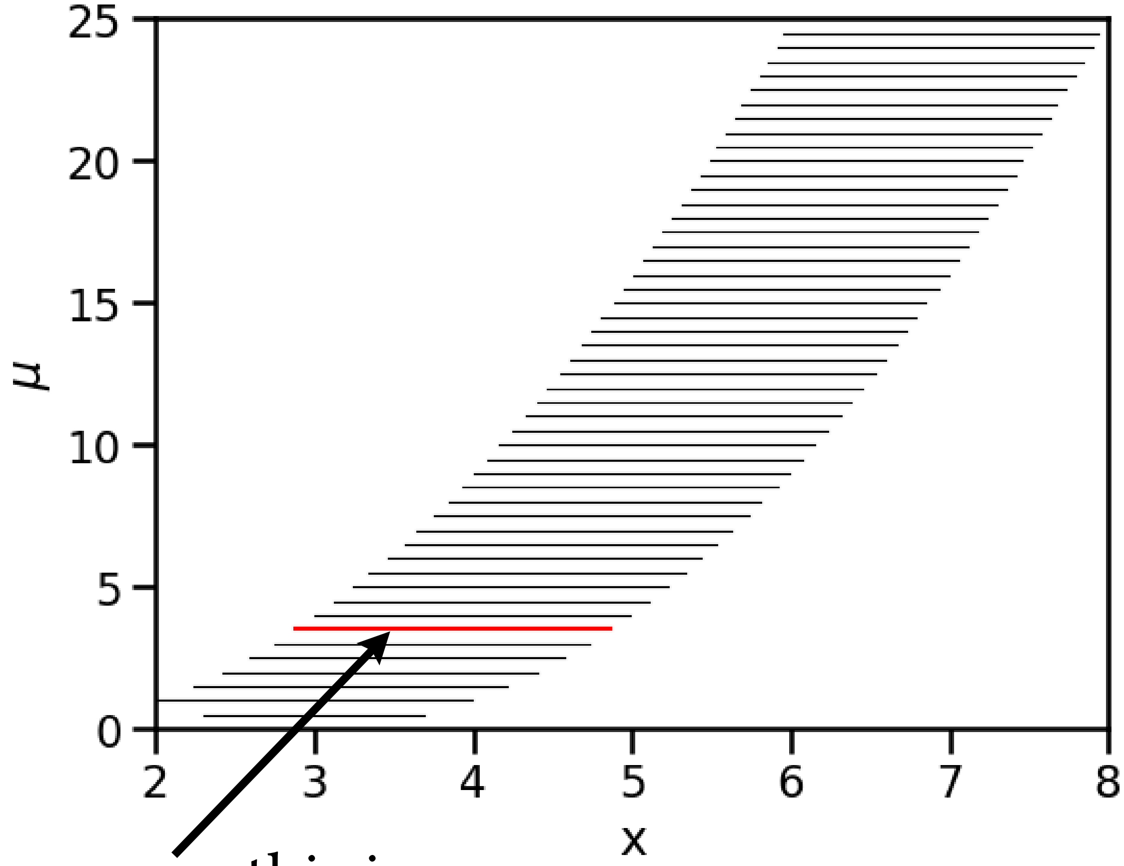

suppose this is μ_t

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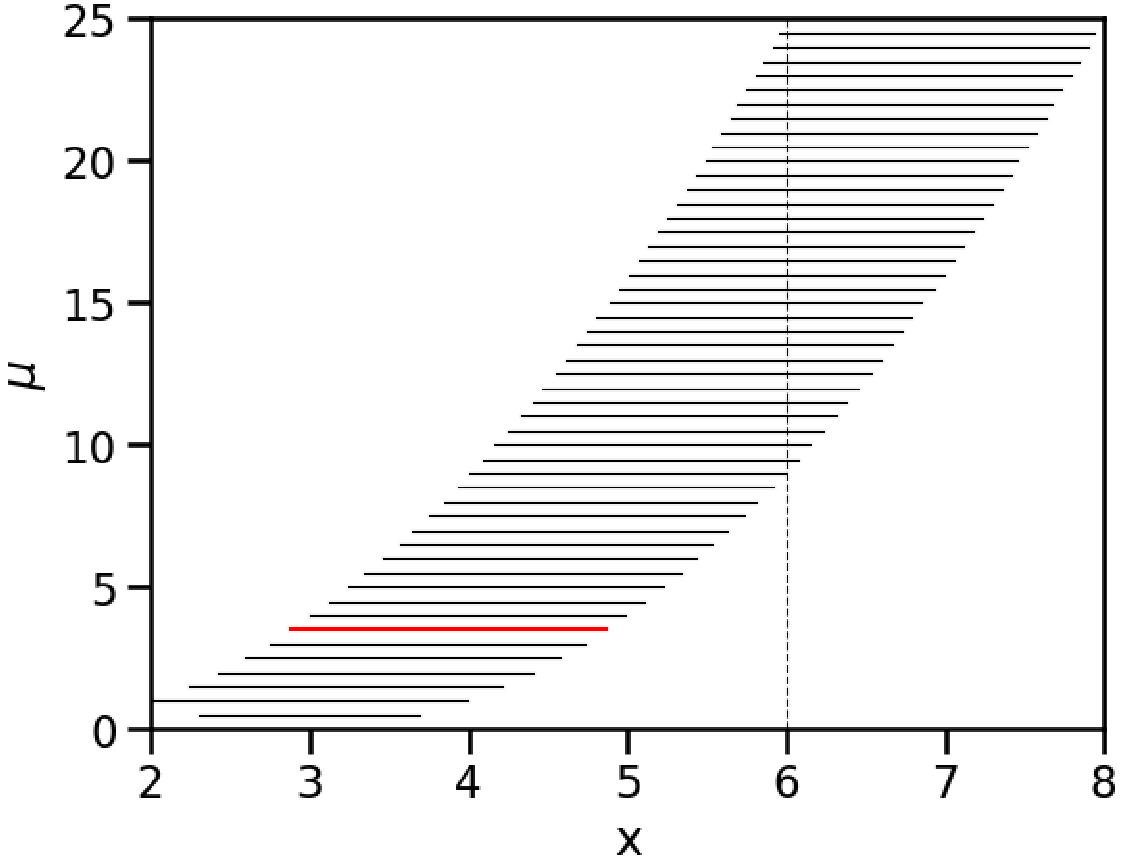
- Make a measurement, get result $x_0 = 6$



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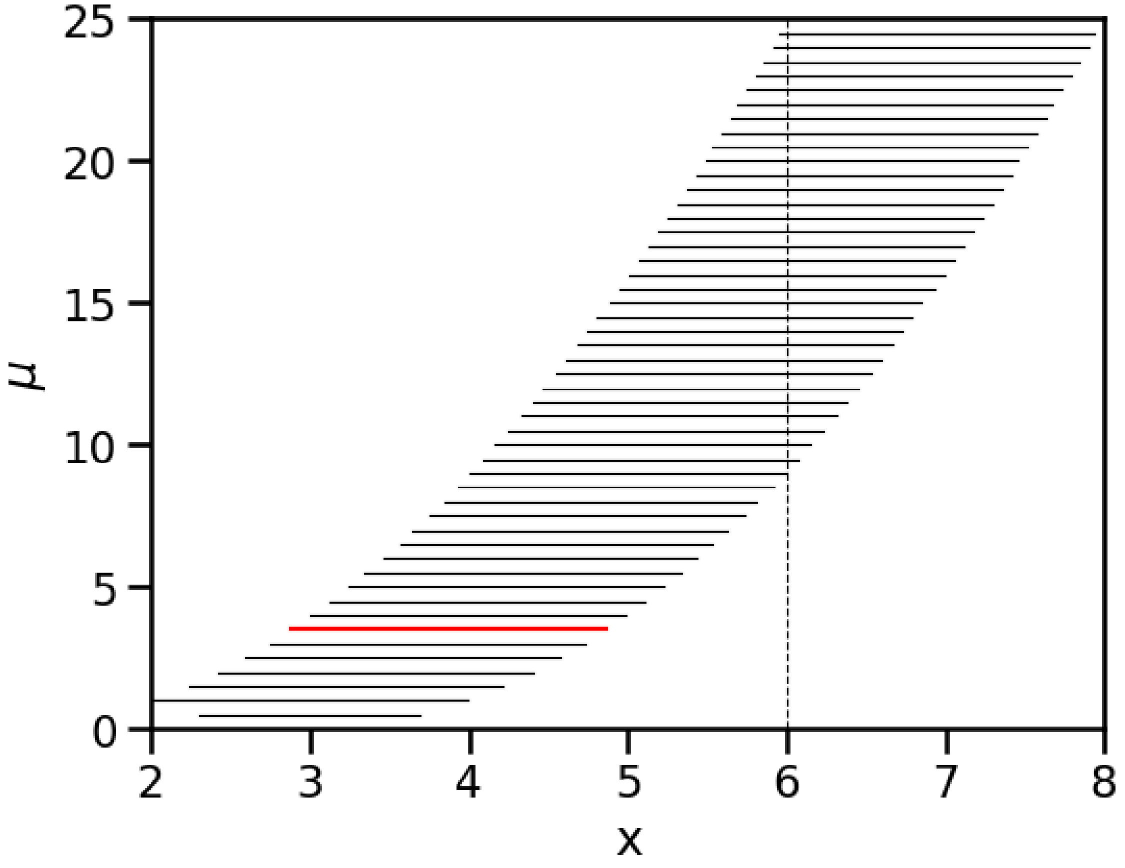

CI Construction: Confidence Belt

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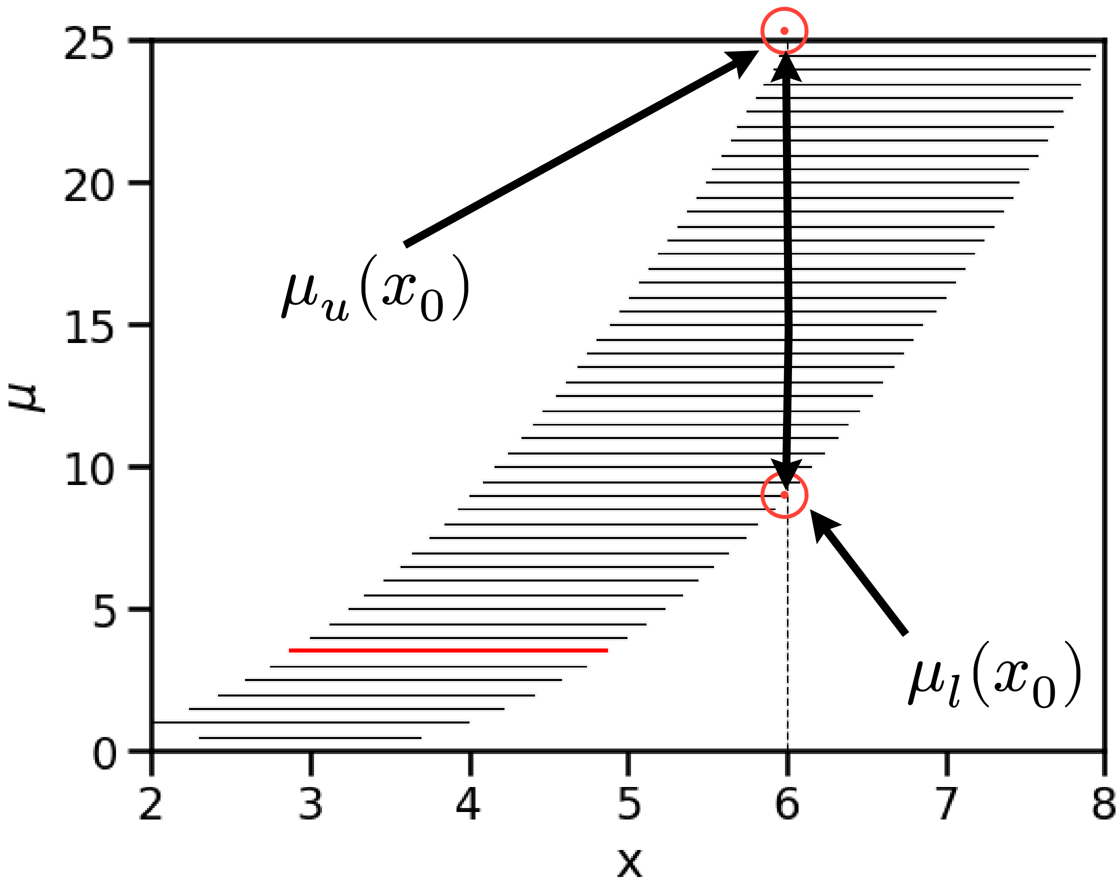
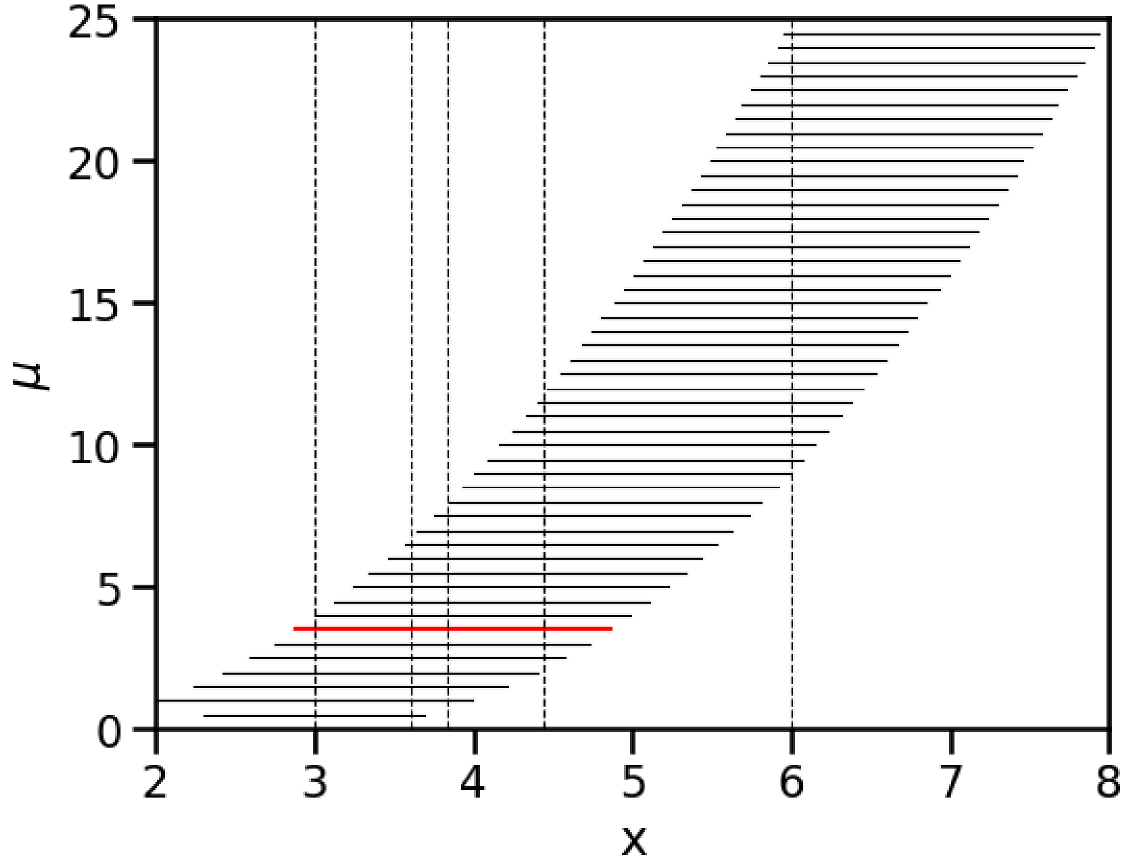
- The **confidence interval** $[\mu_l, \mu_u]$ from this experiment is the vertical intercept.



CI Construction: Confidence Belt

- Make some more measurements



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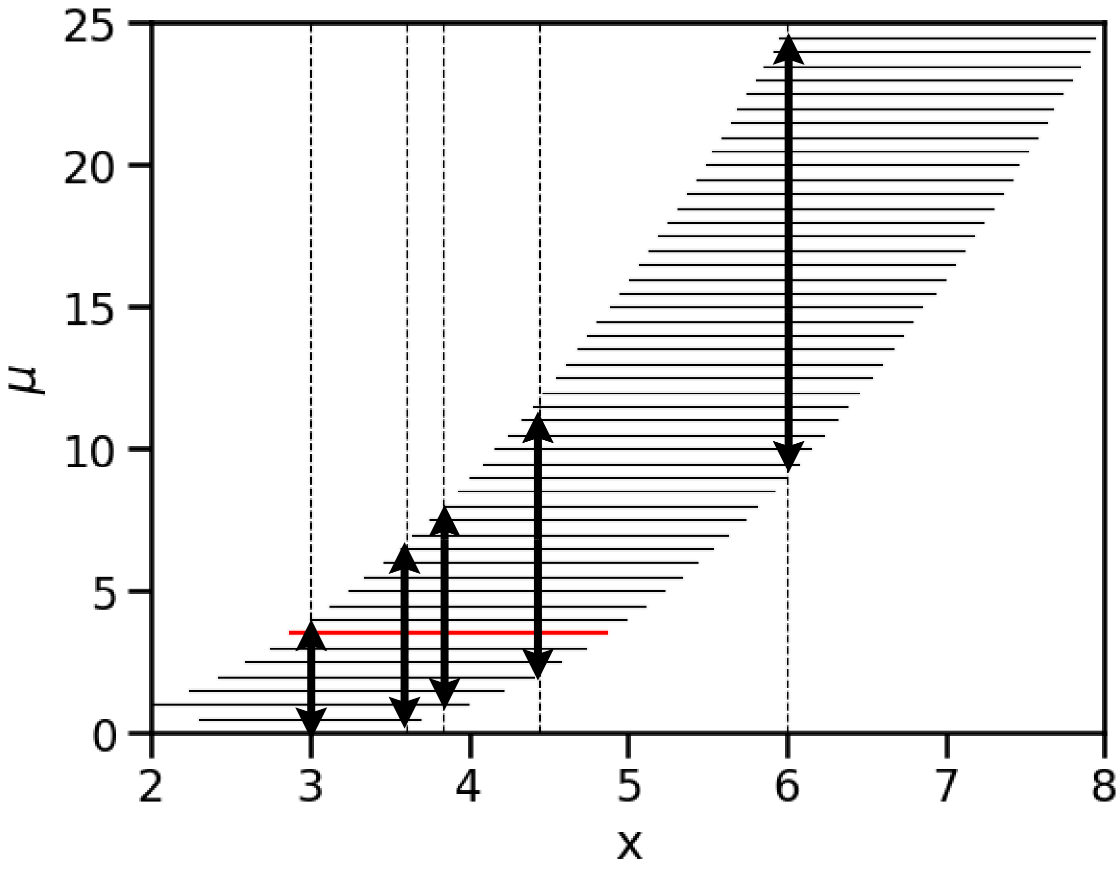
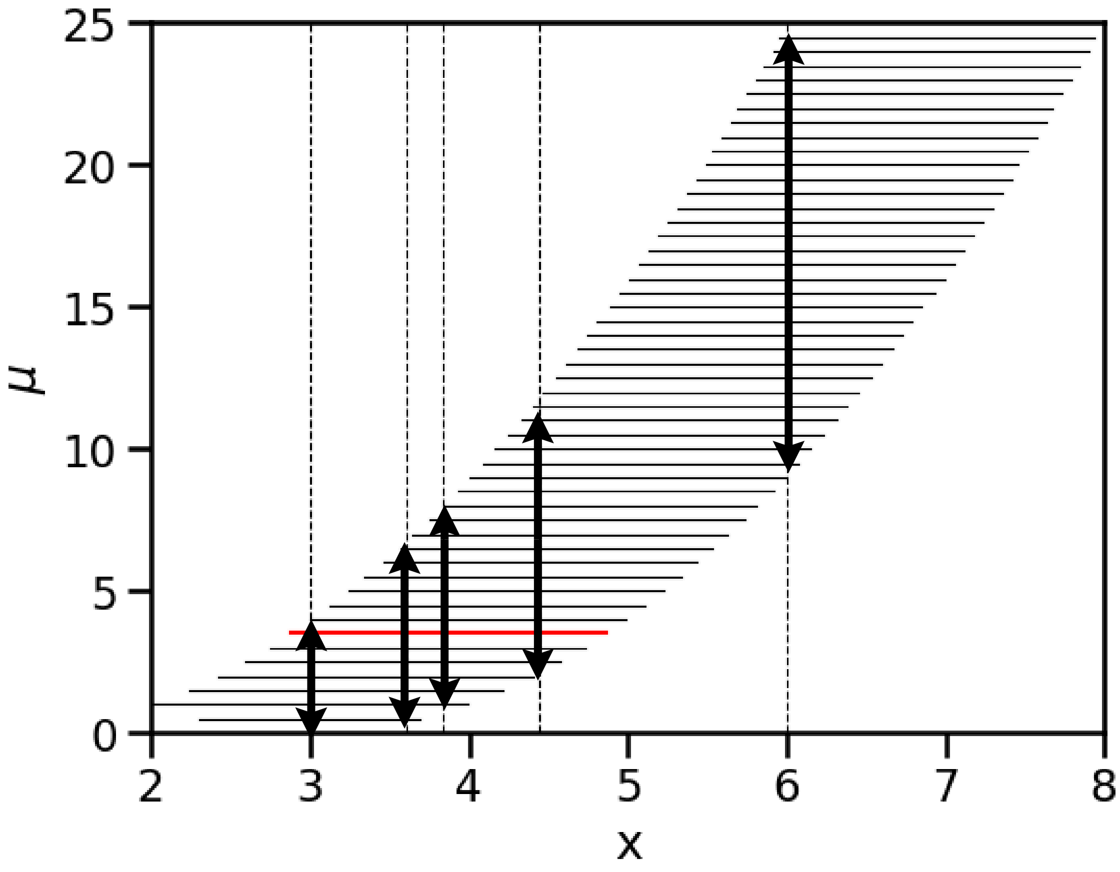


CI Construction: Confidence Belt

- Make some more measurements
- Get some more confidence intervals.
- Have a set

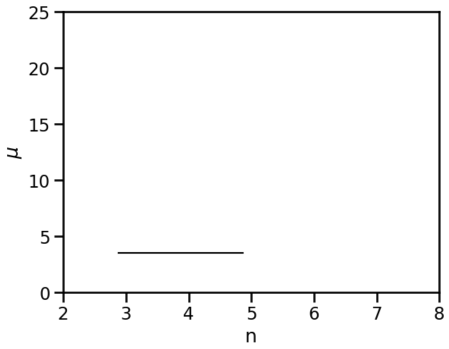
$$C = \{\mathrm{CI}_1, \mathrm{CI}_2, \mathrm{CI}_3, \mathrm{CI}_4, \mathrm{CI}_5\}$$

- 80% of this set would cover the true value, μ_t.



Acceptance Region

Maximum Likelihood



- Recall acceptance region:

$$\Pr(n \in [n_1, n_2] \mid \mu_{\text{fixed}}) = 90\%$$

- Complete freedom in choosing how to construct the acceptance regions.
- Consider likelihood: Poisson with background b:

$$\mathcal{L} \equiv \Pr(n \mid \mu) = \frac{(\mu + b)^n e^{-(\mu+b)}}{n!}$$

- F&C propose to compute a likelihood ratio R
  - This needs a "best fit" $\mu_{\text{best}} \equiv \max(0, n - b)$

Maximum Likelihood

Derivation (skip me!)

- Likelihood is a Poisson in this case.

$$\mathcal{L} \equiv \Pr(n \mid \mu) = \frac{(\mu + b)^n e^{-(\mu+b)}}{n!}$$

- Find maximum (fixing n, vary μ):

$$\left.\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\mu}\right|_{\mu=\mu_{\text{best}}} = 0$$

- Result: "best fit" $\mu = \mu_{\text{best}} = n - b$
- Require physical $\mu \geq 0 \Rightarrow \mu_{\text{best}} = \max(0, n - b)$

Likelihood Ratio

- Do this for representative values of μ; say we start with $\mu = 0.5$
  - As an example background, $b = 3$
  - $\Rightarrow \mu_{\text{best}} \equiv \max(0, n - b) = \max(0, n - 3)$
- Procedure:
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  -
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n	$\Pr(n\|\mu)$	μ_{best}	$\Pr(n\|\mu_{\text{best}})$	R	**rank**
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  - Divide likelihoods to get R.

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0	0.03	0	0.05	0.607	

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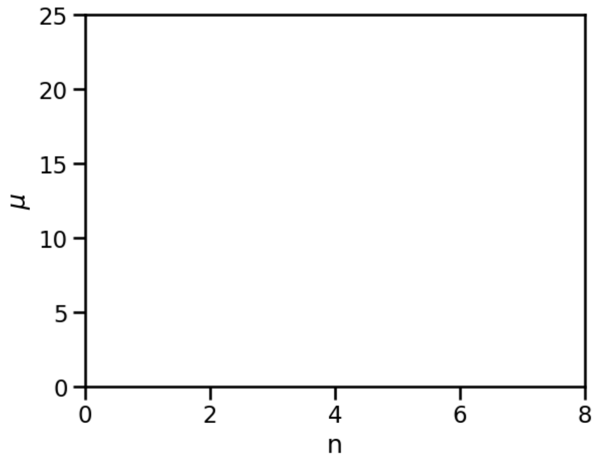

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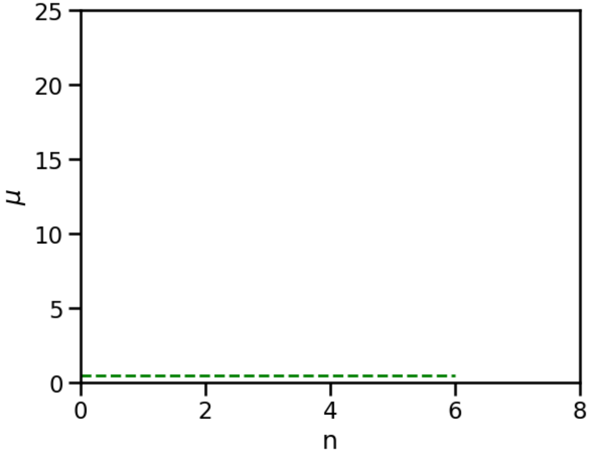

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- To construct the region, make a new row for $n = 1$

n	$\Pr(n\|\mu)$	μ_{best}	$\Pr(n\|\mu_{\text{best}})$	R	**rank**
0	0.030	0	0.050	0.607	
1	0.106	0	0.149	0.708	

Example acceptance region for $\mu = 0.5$

And then for a bunch of other n.

Example acceptance region for $\mu = 0.5$

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0	0.030	0	0.050	0.607	
1	0.106	0	0.149	0.708	
2	0.185	0	0.224	0.826	
3	0.216	0	0.224	0.963	
4	0.189	1	0.195	0.966	
5	0.132	2	0.175	0.753	
6	0.077	3	0.161	0.480	
7	0.039	4	0.149	0.259	

Example acceptance region for $\mu = 0.5$

n	$\Pr(n\|\mu)$	μ_{best}	$\Pr(n\|\mu_{\text{best}})$	R	**rank**
0	0.030	0	0.050	0.607	6
1	0.106	0	0.149	0.708	5
2	0.185	0	0.224	0.826	3
3	0.216	0	0.224	0.963	2
4	0.189	1	0.195	0.966	1
5	0.132	2	0.175	0.753	4
6	0.077	3	0.161	0.480	7
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- Start adding $\Pr(n|\mu)$ in the second column based on the rank.
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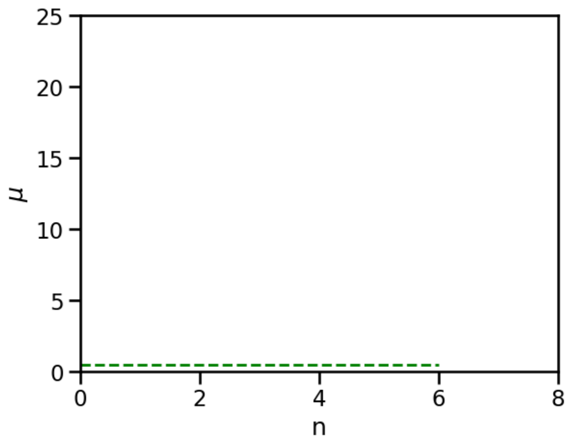

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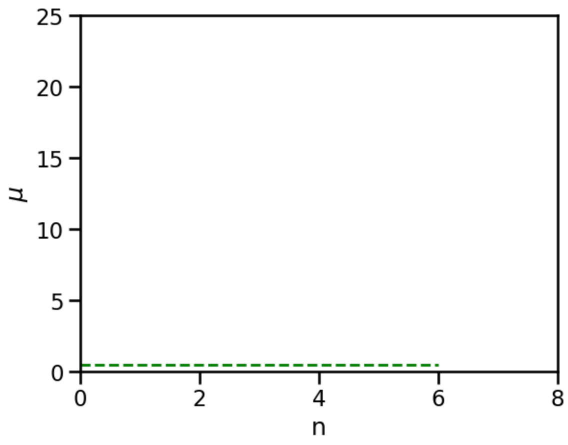
- The n's that contribute to the sum are the ones included in the acceptance region.
- Acceptance region for $\mu = 0.5$ is therefore $n \in [0, 6]$
- Next, construct the acceptance region for other μ as well.



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$$\underbrace{\Pr(\lambda \mid n)}_{\text{Posterior}} = \frac{\Pr(n \mid \lambda) \cdot \Pr(\lambda)}{\underbrace{\Pr(n)}_{\text{evidence}}}$$

- Evidence is typically just a normalization and ignored. Let's call it 1 :)

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- If we specify a prior, we get the posterior.
- "uniform prior" $\Pr(\lambda) = 1 \implies \Pr(\lambda \mid n) = \Pr(n \mid \lambda)$
- Suppose we measure $n = 0$ event, then the posterior is

$$\Pr(\lambda \mid n = 0) = \Pr(n = 0|\lambda) = \frac{e^{-\lambda} \cdot \lambda^n}{n!} = e^{-\lambda}$$

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$$\Rightarrow \int_0^{\lambda_{\max}} e^{-\lambda} = 0.9$$

$$\Rightarrow \lambda_{\max} = \ln(10) \approx 2.3$$

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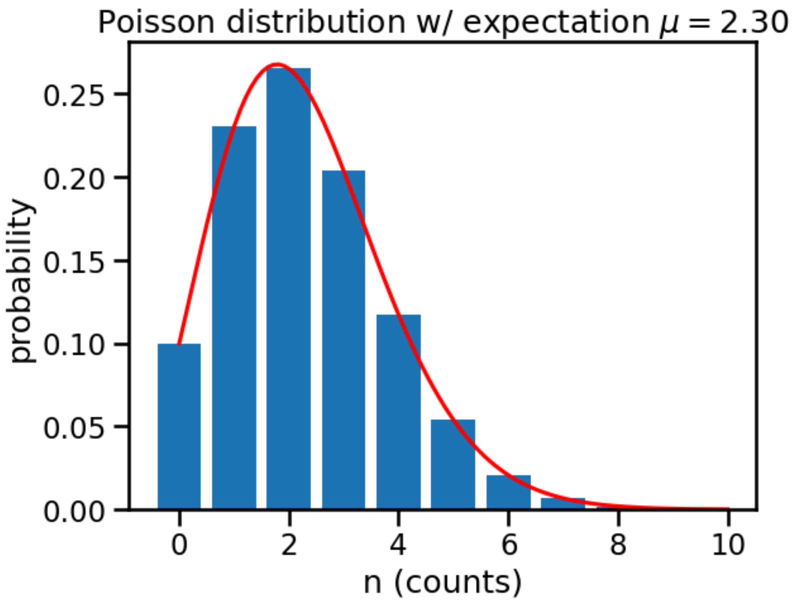
- so we "estemate with 90% confidence that $\lambda \leq 2.3$" base on a non-detection.



Summary: Bayesian Credible Interval

- Let μ denote the unknown parameter we wish to estimate.
- Let x_0 denote the outcome of a single measurement.
- Assume that we know how the measurement outcome depends on the parameter, $x = x(\mu)$.
  - *e.g.* if the neutrino flux is very small, then oftentimes a measurement reports a non-detection.
  - In other words, we know the *likelihood*, $P(x_0|\mu)$.
- From the Bayesian perspective, we can flip things around and say that the parameter is a function of the measurement, $P(\mu|x_0)$, provided that we state our prior beliefs about the parameter, $P(\mu)$.