Assignment 3

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_3/tree/master/CODES

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_3

1 Question No. 2.59

Draw a line segment **AB** of length 8 units. Taking **A** as centre draw a circle of radius 4 units and taking **B** as centre draw a circle of radius 3 units. Construct tangents to each circle from the centre of other circle.

2 Solution

The given data is tabularised in table 2.1:

	Symbols	Circle1	Circle2
Centre	**A**,**B**	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}8\\0\end{pmatrix}$
Radius	r_1,r_2	4	3

TABLE 2.1: Input values

- Also, it is given that line segment AB=8 units.So,Let:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix} \tag{2.0.1}$$

1) To find coordinates of points where tangent touches the circle.
   - Let **M** be any point on x-axis whose coordinates are:

$$\mathbf{M} = \begin{pmatrix}x_1\\0\end{pmatrix} \tag{2.0.2}$$

   - Tangents are drawn from **M** to any circle with centre **O**.

Lemma 2.1. *The coordinates of points* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *where tangent touches the circle are given by:*

$$\mathbf{N} = \begin{pmatrix}x_n\\y_n\end{pmatrix} = \mathbf{n} + \lambda\mathbf{m} \tag{2.0.3}$$

$$\tag{2.0.4}$$

$$where,\ \lambda = \pm\sqrt{\frac{\|d_n\|^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.5}$$

And d_n *will be distance of point* **N** *where tangent touches the circle from origin* **O**.

$$\therefore d_n = \|\mathbf{N}\|^2 = \|{x_n}^2 + {y_n}^2\| \tag{2.0.6}$$

$$\mathbf{n} = \begin{pmatrix}\frac{d_n}{x_1}\\0\end{pmatrix} \tag{2.0.7}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.8}$$

Proof. We know a tangent is always perpendicular to the radius .

$$\therefore N_1O \perp N_1M, \tag{2.0.9}$$

$$N_2O \perp N_2M \tag{2.0.10}$$

- Now,

$$\implies (\mathbf{O} - \mathbf{N})^T(\mathbf{N} - \mathbf{M}) = 0 \tag{2.0.11}$$

$$\mathbf{N}^T(\mathbf{N} - \mathbf{M}) = 0 \quad (\because \mathbf{O} = 0) \tag{2.0.12}$$

$$\mathbf{N}^T\mathbf{N} - \mathbf{N}^T\mathbf{M} = 0 \tag{2.0.13}$$

$$\mathbf{N}^T\mathbf{M} = \|\mathbf{N}\|^2 \tag{2.0.14}$$

$$\implies \mathbf{M}^T\mathbf{N} = \|\mathbf{N}\|^2(\because \mathbf{N}^T\mathbf{M} = \mathbf{M}^T\mathbf{N}) \tag{2.0.15}$$

- Now,let distance of point **N** from origin be $\mathrm{d_n}$.

$$\implies \|\mathbf{N} - \mathbf{O}\|^2 = \|\mathbf{N}\|^2 = d_\mathrm{n} \tag{2.0.16}$$

- Now, putting value of $\mathbf{M} = \begin{pmatrix}x_1\\0\end{pmatrix}$ from (2.0.2) in (2.0.15) we get:

$$\implies \begin{pmatrix} x_1 & 0 \end{pmatrix} \mathbf{N} = \begin{pmatrix} {d_\text{n}}^2 \\ 0 \end{pmatrix} \tag{2.0.17}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{N} = \begin{pmatrix} \frac{{d_\text{n}}^2}{x_1} \\ 0 \end{pmatrix} \tag{2.0.18}$$

$$\implies \mathbf{N} = \begin{pmatrix} \frac{{d_\text{n}}^2}{x_1} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.19}$$

$$\implies \mathbf{N} = \mathbf{n} + \lambda \mathbf{m} \tag{2.0.20}$$

$$\text{where, } \mathbf{n} = \begin{pmatrix} \frac{{d_\text{n}}^2}{x_1} \\ 0 \end{pmatrix} \text{and } \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.21}$$

- Also we know,

$$\|\mathbf{n} + \lambda \mathbf{m}\|^2 = {d_\text{n}}^2 \tag{2.0.22}$$

$$(\mathbf{n} + \lambda \mathbf{m})^T (\mathbf{n} + \lambda \mathbf{m}) = {d_\text{n}}^2 \tag{2.0.23}$$

$$\lambda^2 = \frac{{d_\text{n}}^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2} \tag{2.0.24}$$

$$\lambda = \pm \sqrt{\frac{{d_\text{n}}^2 - \|\mathbf{l}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.25}$$

□

2) For tangents at **Circle1** from point **B**:-
   - Here, we have r_1=4, So $\|\mathbf{P}\|^2$ is given by,

$$\|\mathbf{P} - \mathbf{O}\|^2 = \|\mathbf{P}\|^2 = r_1 \tag{2.0.26}$$

$$\|\mathbf{P}\|^2 = r_1 = 4 \tag{2.0.27}$$

$$\implies d_{\text{n}1} = 4 \tag{2.0.28}$$

$$\therefore {d_{\text{n}1}}^2 = 16 \tag{2.0.29}$$

   - Referencing (2.0.3),we have

$$\mathbf{P} = \mathbf{p} + \lambda \mathbf{m} \tag{2.0.30}$$

   where,

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} \frac{{d_{\text{n}1}}^2}{x_1} \\ 0 \end{pmatrix} \tag{2.0.31}$$

   - Using (2.0.29) and $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$, we get

$$\mathbf{p} = \begin{pmatrix} \frac{16}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.32}$$

   And

$$\lambda = \pm \sqrt{\frac{{d_{\text{n}1}}^2 - \|\mathbf{p}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.33}$$

$$\lambda = \pm \sqrt{16 - 4} \tag{2.0.34}$$

$$\lambda = \pm \sqrt{12} = \pm 3.46 \tag{2.0.35}$$

3) Substituting λ, **p** and **m** in (2.0.30) we get the coordinates of **P** and **Q** as :

$$\mathbf{P} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \tag{2.0.36}$$

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad = \begin{pmatrix} 2 \\ -3.46 \end{pmatrix} \tag{2.0.37}$$

4) For tangents at **Circle2** from **A**:-
   - In △ ABR, using Pythagoras Theorem,we get:

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{R}\|^2 + \|\mathbf{B} - \mathbf{R}\|^2 \tag{2.0.38}$$

$$\|\mathbf{A} - \mathbf{R}\|^2 = \|\mathbf{A} - \mathbf{B}\|^2 - \|\mathbf{B} - \mathbf{R}\|^2 \tag{2.0.39}$$

   - As, $\mathbf{A} = 0$, above equation get reduced to:

$$\|\mathbf{R}\|^2 = \|\mathbf{A} - \mathbf{B}\|^2 - \|\mathbf{B} - \mathbf{R}\|^2 \tag{2.0.40}$$

$$\|\mathbf{R}\|^2 = \sqrt{8^2 - 3^2} = 7.42 \tag{2.0.41}$$

$$\implies d_{\text{n}2} = 7.42 \tag{2.0.42}$$

$$\therefore {d_{\text{n}2}}^2 = 55 \tag{2.0.43}$$

   - Referencing (2.0.3),we have

$$\mathbf{R} = \mathbf{r} + \lambda \mathbf{m} \tag{2.0.44}$$

   where,

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} \frac{{d_{\text{n}2}}^2}{x_1} \\ 0 \end{pmatrix} \tag{2.0.45}$$

   - Using (2.0.43) and $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ we get :

$$\mathbf{r} = \begin{pmatrix} \frac{55}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} \tag{2.0.46}$$

   And

$$\lambda = \pm \sqrt{\frac{\|d_{\text{n}2}\|^2 - \|\mathbf{r}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.47}$$

$$\lambda = \pm \sqrt{55 - 47.265} \tag{2.0.48}$$

$$\lambda = \pm \sqrt{7.735} = \pm 2.78 \tag{2.0.49}$$

5) Substituting λ, **r** and **m** in (2.0.44) we get the

coordinates of **R** and **S** as :

$$\mathbf{R} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} + 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 2.78 \end{pmatrix} \quad (2.0.50)$$

$$\mathbf{S} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} - 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ -2.78 \end{pmatrix} \quad (2.0.51)$$

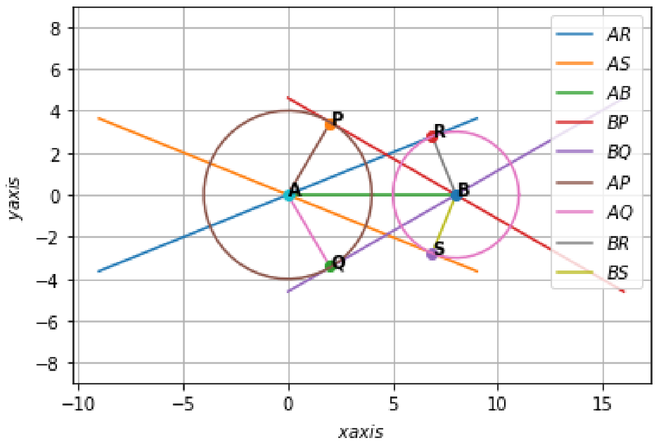

Fig. 2.1: Tangents from centres of Circle