Assignment 3

Unnati Gupta

Download all python codes from

```
https://github.com/unnatigupta2320/Assignment_3/tree/master/CODES
```

and latex-tikz codes from

```
https://github.com/unnatigupta2320/Assignment_3
```

1 Question No. 2.59

Draw a line segment **AB** of length 8 units. Taking **A** as centre draw a circle of radius 4 units and taking **B** as centre draw a circle of radius 3 units. Construct tangents to each circle from the centre of other circle.

2 Solution

The given data is tabularised in table 2.1:

	Symbols	Circle1	Circle2
Centre	**A**,**B**	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}8\\0\end{pmatrix}$
Radius	r_1,r_2	4	3

TABLE 2.1: Input values

- Also, it is given that line segment AB=8 units.So,Let:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix} \tag{2.0.1}$$

1) To find coordinates of points where tangent touches the circle.
   - Let **M** be any point on x-axis whose coordinates are:

$$\mathbf{M} = \begin{pmatrix}x_1\\0\end{pmatrix} \tag{2.0.2}$$

   - Tangents are drawn from **M** to any circle with centre **O**.

Lemma 2.1. *The coordinates of points $\mathbf{L}_1$ and $\mathbf{L}_2$ where tangent touches the circle are given by:*

$$\mathbf{L} = \mathbf{l} + \lambda\mathbf{m} \tag{2.0.3}$$

$$where,\ \lambda = \pm\sqrt{\frac{||\mathbf{L}||^2 - ||\mathbf{l}||}{||\mathbf{m}||^2}} \tag{2.0.4}$$

$$\mathbf{l} = \begin{pmatrix}\frac{||\mathbf{L}||^2}{x_1}\\0\end{pmatrix} \tag{2.0.5}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.6}$$

Proof. We know a tangent is always perpendicular to the radius .

$$\therefore L_1O \perp L_1M, \tag{2.0.7}$$

$$L_2O \perp L_2M \tag{2.0.8}$$

- Now,

$$\implies (\mathbf{O} - \mathbf{L})^T(\mathbf{L} - \mathbf{M}) = 0 \tag{2.0.9}$$

$$\implies \mathbf{L}^T(\mathbf{L} - \mathbf{M}) = 0 \quad (\because \mathbf{O} = 0) \tag{2.0.10}$$

$$\implies \mathbf{L}^T\mathbf{L} - \mathbf{L}^T\mathbf{M} = 0 \tag{2.0.11}$$

$$\implies \mathbf{L}^T\mathbf{M} = ||\mathbf{L}||^2 \tag{2.0.12}$$

$$\implies \mathbf{M}^T\mathbf{L} = ||\mathbf{L}||^2 \tag{2.0.13}$$

- Now, using $\mathbf{M} = \begin{pmatrix}x_1\\0\end{pmatrix}$ from (2.0.2) we get:

$$\implies \begin{pmatrix}x_1 & 0\end{pmatrix}\mathbf{L} = ||\mathbf{L}||^2 \tag{2.0.14}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{L} = \begin{pmatrix}\frac{||\mathbf{L}||^2}{x_1}\\0\end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{L} = \begin{pmatrix}\frac{||\mathbf{L}||^2}{x_1}\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.16}$$

$$\implies \mathbf{L} = \mathbf{l} + \lambda\mathbf{m} \tag{2.0.17}$$

$$\text{where, } \mathbf{l} = \begin{pmatrix}\frac{||\mathbf{L}||^2}{x_1}\\0\end{pmatrix}, \text{and } \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.18}$$

- Also we know,

$$\|\mathbf{l}+\lambda\mathbf{m}\|^2 = \|\mathbf{L}\|^2 \tag{2.0.19}$$
$$(\mathbf{l}+\lambda\mathbf{m})^T(\mathbf{l}+\lambda\mathbf{m}) = \|\mathbf{L}\|^2 \tag{2.0.20}$$

$$\lambda^2 = \frac{\|\mathbf{L}\|^2 - \|\mathbf{l}\|^2}{\|\mathbf{m}\|^2} \tag{2.0.21}$$
$$\lambda = \pm\sqrt{\frac{\|\mathbf{L}\|^2 - \|\mathbf{l}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.22}$$

□

2) For tangents at **Circle1** from point **B**:-
   - Referencing (2.0.3),we have

$$\mathbf{P} = \mathbf{p} + \lambda\mathbf{m} \tag{2.0.23}$$

   where,

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{p} = \begin{pmatrix}\frac{\|\mathbf{P}\|^2}{x_1}\\0\end{pmatrix} \tag{2.0.24}$$

   - As $\|\mathbf{P}\|^2$=16 and $\mathbf{B} = \begin{pmatrix}x_1\\0\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}$, we get

$$\mathbf{p} = \begin{pmatrix}\frac{16}{8}\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.0.25}$$

   And

$$\lambda = \pm\sqrt{\frac{\|\mathbf{P}\|^2 - \|\mathbf{p}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.26}$$
$$\lambda = \pm\sqrt{16-4} \tag{2.0.27}$$
$$\lambda = \pm\sqrt{12} = \pm 3.46 \tag{2.0.28}$$

3) Substituting λ, **p** and **m** in (2.0.23) we get the coordinates of **P** and **Q** as :

$$\mathbf{P} = \begin{pmatrix}2\\0\end{pmatrix} + 3.46\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\3.46\end{pmatrix} \tag{2.0.29}$$
$$\mathbf{Q} = \begin{pmatrix}2\\0\end{pmatrix} - 3.46\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-3.46\end{pmatrix} \tag{2.0.30}$$

4) For tangents at **Circle2** from **A**:-
   - In $\triangle$ ABR, using Pythagoras Theorem,we get:

$$\|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{R}\|^2 + \|\mathbf{B}-\mathbf{R}\|^2 \tag{2.0.31}$$
$$\|\mathbf{A}-\mathbf{R}\|^2 = \|\mathbf{A}-\mathbf{B}\|^2 - \|\mathbf{B}-\mathbf{R}\|^2 \tag{2.0.32}$$

   - As, **A** = 0, above equation get reduced to:

$$\|\mathbf{R}\|^2 = \|\mathbf{A}-\mathbf{B}\|^2 - \|\mathbf{B}-\mathbf{R}\|^2 \tag{2.0.33}$$
$$\|\mathbf{R}\|^2 = 8^2 - 3^2 = 55 \tag{2.0.34}$$

   - Referencing (2.0.3),we have

$$\mathbf{R} = \mathbf{r} + \lambda\mathbf{m} \tag{2.0.35}$$

   where,

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{r} = \begin{pmatrix}\frac{\|\mathbf{R}\|^2}{x_1}\\0\end{pmatrix} \tag{2.0.36}$$

   - As $\mathbf{B} = \begin{pmatrix}x_1\\0\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}$ and from (2.0.34) $\|\mathbf{R}\|^2$=55, we get

$$\mathbf{r} = \begin{pmatrix}\frac{55}{8}\\0\end{pmatrix} = \begin{pmatrix}6.875\\0\end{pmatrix} \tag{2.0.37}$$

   And

$$\lambda = \pm\sqrt{\frac{\|\mathbf{R}\|^2 - \|\mathbf{r}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.38}$$
$$\lambda = \pm\sqrt{55 - 47.265} \tag{2.0.39}$$
$$\lambda = \pm\sqrt{7.735} = \pm 2.78 \tag{2.0.40}$$

5) Substituting λ, **r** and **m** in (2.0.35) we get the coordinates of **R** and **S** as :

$$\mathbf{R} = \begin{pmatrix}6.875\\0\end{pmatrix} + 2.78\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}6.875\\2.78\end{pmatrix} \tag{2.0.41}$$
$$\mathbf{S} = \begin{pmatrix}6.875\\0\end{pmatrix} - 2.78\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}6.875\\-2.78\end{pmatrix} \tag{2.0.42}$$

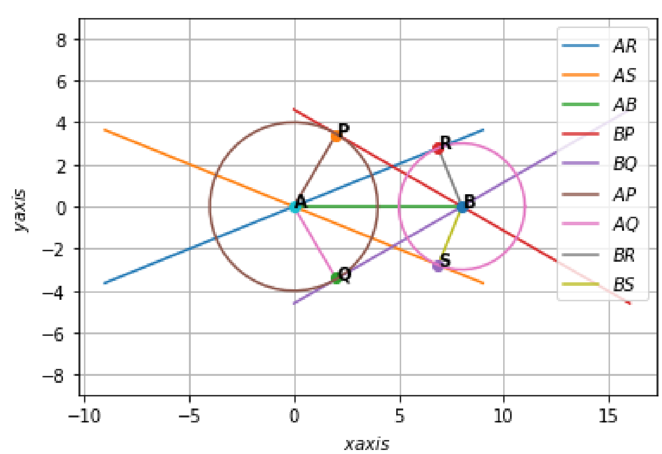

Fig. 2.1: Tangents from centres of Circle