

Assignment 3

Unnati Gupta

Download all python codes from

https://github.com/unnatigupta2320/Assignment_3/tree/master/CODES

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_3

1 QUESTION No. 2.59

Draw a line segment **AB** of length 8 units. Taking **A** as centre draw a circle of radius 4 units and taking **B** as centre draw a circle of radius 3 units. Construct tangents to each circle from the centre of other circle.

2 SOLUTION

The given data is tabularised in table 2.1:

	Symbols	Circle1	Circle2
Centre	A,B	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 0 \end{pmatrix}$
Radius	r_1, r_2	4	3

TABLE 2.1: Input values

- Also, it is given that line segment $AB=8$ units. So, Let:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad (2.0.1)$$

- To find coordinates of points where tangent touches the circle.

- Let **M** be any point on x-axis whose coordinates are:

$$\mathbf{M} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad (2.0.2)$$

- Tangents are drawn from **M** to any circle with centre **O** at origin and radius r_n .

Lemma 2.1. The coordinates of points \mathbf{N}_1 and \mathbf{N}_2 where tangent touches the circle are given by:

$$\mathbf{N} = \mathbf{n} + \lambda \mathbf{m} \quad (2.0.3)$$

$$\text{where, } \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{n} = \begin{pmatrix} \frac{r_n^2}{x_1} \\ 0 \end{pmatrix} \quad (2.0.5)$$

$$\lambda = \pm \sqrt{\frac{r_n^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \quad (2.0.6)$$

Proof. We know a tangent is always perpendicular to the radius .

$$\therefore N_1O \perp N_1M, \quad (2.0.7)$$

$$N_2O \perp N_2M \quad (2.0.8)$$

- Now,

$$\Rightarrow (\mathbf{O} - \mathbf{N})^T (\mathbf{N} - \mathbf{M}) = 0 \quad (2.0.9)$$

$$\mathbf{N}^T (\mathbf{N} - \mathbf{M}) = 0 \quad (\because \mathbf{O} = 0) \quad (2.0.10)$$

$$\mathbf{N}^T \mathbf{N} - \mathbf{N}^T \mathbf{M} = 0 \quad (2.0.11)$$

$$\mathbf{N}^T \mathbf{M} = \|\mathbf{N}\|^2 \quad (2.0.12)$$

$$\Rightarrow \mathbf{M}^T \mathbf{N} = \|\mathbf{N}\|^2 (\because \mathbf{N}^T \mathbf{M} = \mathbf{M}^T \mathbf{N}) \quad (2.0.13)$$

- Now, using (2.0.2) in above equation we get:

$$\Rightarrow \begin{pmatrix} x_1 & 0 \end{pmatrix} \mathbf{N} = r_n^2 \quad (2.0.14)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{N} = \begin{pmatrix} \frac{r_n^2}{x_1} \\ 0 \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow \mathbf{N} = \begin{pmatrix} \frac{r_n^2}{x_1} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.16)$$

$$\Rightarrow \mathbf{N} = \mathbf{n} + \lambda \mathbf{m} \quad (2.0.17)$$

$$\text{where, } \mathbf{n} = \begin{pmatrix} \frac{r_n^2}{x_1} \\ 0 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.18)$$

- Also we know,

$$\|\mathbf{n} + \lambda \mathbf{m}\|^2 = r_n^2 \quad (2.0.19)$$

$$(\mathbf{n} + \lambda \mathbf{m})^T (\mathbf{n} + \lambda \mathbf{m}) = r_n^2 \quad (2.0.20)$$

$$\lambda^2 = \frac{r_n^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2} \quad (2.0.21)$$

$$\lambda = \pm \sqrt{\frac{r_n^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \quad (2.0.22)$$

□

2) For tangents at **Circle1** from point **B**:-

- Here, we have $r_1=4$
- Now,referencing (2.0.3),we have

$$\mathbf{P} = \mathbf{p} + \lambda_1 \mathbf{m} \quad (2.0.23)$$

where,

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} r_1 \\ x_1 \\ 0 \end{pmatrix} \quad (2.0.24)$$

- Using $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$, we get:

$$\mathbf{p} = \begin{pmatrix} 16 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (2.0.25)$$

And

$$\lambda_1 = \pm \sqrt{\frac{r_1^2 - \|\mathbf{p}\|^2}{\|\mathbf{m}\|^2}} \quad (2.0.26)$$

$$\lambda_1 = \pm \sqrt{\frac{16 - 4}{1}} \quad (2.0.27)$$

$$\lambda_1 = \pm \sqrt{12} = \pm 3.46 \quad (2.0.28)$$

3) Substituting λ_1 , \mathbf{p} and \mathbf{m} in (2.0.23) we get the coordinates of \mathbf{P} and \mathbf{Q} as :

$$\mathbf{P} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \quad (2.0.29)$$

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3.46 \end{pmatrix} \quad (2.0.30)$$

4) For tangents at **Circle2** from **A**:-

- Here, we have $r_2=3$, So \mathbf{d}_2 is given by,

$$\mathbf{d}_2 = |d_{CO}^2 - r_2^2| \quad (2.0.31)$$

- As centre is at $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ So $d_{CO}=8$,we get:

$$\Rightarrow d_2 = |8^2 - r_2^2| \quad (2.0.32)$$

$$\therefore d_2 = |64 - 9| = 55 \quad (2.0.33)$$

- Referencing (2.0.3),we have

$$\mathbf{S} = \mathbf{s} + \lambda_2 \mathbf{m} \quad (2.0.34)$$

where,

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} d_2 \\ x_1 \\ 0 \end{pmatrix} \quad (2.0.35)$$

- Using (2.0.33) and $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ we get :

$$\mathbf{s} = \begin{pmatrix} 55 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} \quad (2.0.36)$$

And

$$\lambda_2 = \pm \sqrt{\frac{d_2 - \|\mathbf{s}\|^2}{\|\mathbf{m}\|^2}} \quad (2.0.37)$$

$$\lambda_2 = \pm \sqrt{55 - 47.265} \quad (2.0.38)$$

$$\lambda_2 = \pm \sqrt{7.735} = \pm 2.78 \quad (2.0.39)$$

5) Substituting λ_2 , \mathbf{r} and \mathbf{m} in (2.0.34) we get the coordinates of \mathbf{R} and \mathbf{S} as :

$$\mathbf{R} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} + 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 2.78 \end{pmatrix} \quad (2.0.40)$$

$$\mathbf{S} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} - 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ -2.78 \end{pmatrix} \quad (2.0.41)$$

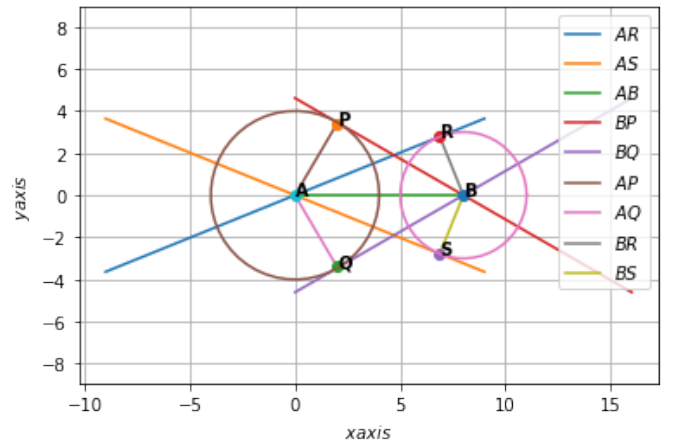


Fig. 2.1: Tangents from centres of Circle