Assignment 3

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_3/tree/master/CODES

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_3

1 Question No. 2.59

Draw a line segment **AB** of length 8 units. Taking **A** as centre draw a circle of radius 4 units and taking **B** as centre draw a circle of radius 3 units. Construct tangents to each circle from the centre of other circle.

2 Solution

The given data is tabularised in table 2.1:

	Symbols	Circle1	Circle2
Centre	**A**,**B**	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}8\\0\end{pmatrix}$
Radius	r_1,r_2	4	3

TABLE 2.1: Input values

- Also, it is given that line segment AB=8 units.So,Let:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix} \tag{2.0.1}$$

1) To find coordinates of points where tangent touches the circle.
   - Let **M** be any point on x-axis whose coordinates are:

$$\mathbf{M} = \begin{pmatrix}M\\0\end{pmatrix} \tag{2.0.2}$$

   - Tangents are drawn from **M** to any circle with centre **O** at origin and radius $\mathbf{r_n}$.

Lemma 2.1. *The coordinates of points* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *where tangent touches the circle are given by:*

$$\mathbf{N} = \mathbf{n} + \lambda\mathbf{m} \tag{2.0.3}$$

$$where,\ \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.4}$$

$$\mathbf{n} = \begin{pmatrix}\frac{r_n{}^2}{\mathbf{M}}\\0\end{pmatrix} \tag{2.0.5}$$

$$\lambda = \pm\sqrt{\frac{r_n{}^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.6}$$

Proof. We know a tangent is always perpendicular to the radius .

$$\therefore N_1O \perp N_1M, \tag{2.0.7}$$

$$N_2O \perp N_2M \tag{2.0.8}$$

- Now,

$$\implies (\mathbf{O} - \mathbf{N})^T(\mathbf{N} - \mathbf{M}) = 0 \tag{2.0.9}$$

$$\mathbf{N}^T(\mathbf{N} - \mathbf{M}) = 0 \quad (\because \mathbf{O} = 0) \tag{2.0.10}$$

$$\mathbf{N}^T\mathbf{N} - \mathbf{N}^T\mathbf{M} = 0 \tag{2.0.11}$$

$$\mathbf{N}^T\mathbf{M} = \|\mathbf{N}\|^2 \tag{2.0.12}$$

$$\implies \mathbf{M}^T\mathbf{N} = \|\mathbf{N}\|^2(\because \mathbf{N}^T\mathbf{M} = \mathbf{M}^T\mathbf{N}) \tag{2.0.13}$$

- Now,using (2.0.2) in above equation we get:

$$\implies \begin{pmatrix}M & 0\end{pmatrix}\mathbf{N} = r_{\mathrm{n}}{}^2 \tag{2.0.14}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{N} = \begin{pmatrix}\frac{r_{\mathrm{n}}{}^2}{M}\\0\end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{N} = \begin{pmatrix}\frac{r_{\mathrm{n}}{}^2}{M}\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.16}$$

$$\implies \mathbf{N} = \mathbf{n} + \lambda\mathbf{m} \tag{2.0.17}$$

$$\text{where, } \mathbf{n} = \begin{pmatrix}\frac{r_{\mathrm{n}}{}^2}{M}\\0\end{pmatrix} \text{and } \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.18}$$

- Also we know,

$$||\mathbf{n}+\lambda\mathbf{m}||^2 = {r_n}^2 \quad (2.0.19)$$
$$(\mathbf{n}+\lambda\mathbf{m})^T(\mathbf{n}+\lambda\mathbf{m}) = {r_n}^2 \quad (2.0.20)$$

$$\lambda^2 = \frac{{r_n}^2 - ||\mathbf{n}||^2}{||\mathbf{m}||^2} \quad (2.0.21)$$
$$\lambda = \pm\sqrt{\frac{{r_n}^2 - ||\mathbf{n}||^2}{||\mathbf{m}||^2}} \quad (2.0.22)$$

□

2) For tangents at **Circle1** from point **B**:-
   - Here, we have r_1=4
   - Now,referencing (2.0.3),we have

$$\mathbf{P} = \mathbf{p} + \lambda_1\mathbf{m} \quad (2.0.23)$$

where,

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{p} = \begin{pmatrix}\frac{{r_1}^2}{M}\\0\end{pmatrix} \quad (2.0.24)$$

   - Using $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$, we get:

$$\mathbf{p} = \begin{pmatrix}\frac{16}{8}\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \quad (2.0.25)$$

And

$$\lambda_1 = \pm\sqrt{\frac{{r_1}^2 - ||\mathbf{p}||^2}{||\mathbf{m}||^2}} \quad (2.0.26)$$
$$\lambda_1 = \pm\sqrt{\frac{16-4}{1}} \quad (2.0.27)$$
$$\lambda_1 = \pm\sqrt{12} = \pm 3.46 \quad (2.0.28)$$

3) Substituting λ_1, **p** and **m** in (2.0.23) we get the coordinates of **P** and **Q** as :

$$\mathbf{P} = \begin{pmatrix}2\\0\end{pmatrix} + 3.46\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\3.46\end{pmatrix} \quad (2.0.29)$$
$$\mathbf{Q} = \begin{pmatrix}2\\0\end{pmatrix} - 3.46\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-3.46\end{pmatrix} \quad (2.0.30)$$

4) For tangents at **Circle2** from **A**:-
   - Here,we have r_2=3. If d_{CO} = distance of centre of circle from origin.
   - Then $\mathbf{d_2}$ is given by,

$$\mathbf{d_2} = |{d_{CO}}^2 - {r_2}^2| \quad (2.0.31)$$

   - As centre is at $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$ So d_{CO}=8,we get:

$$\implies d_2 = |8^2 - {r_2}^2| \quad (2.0.32)$$
$$\therefore d_2 = |64-9| = 55 \quad (2.0.33)$$

   - Referencing (2.0.3),we have

$$\mathbf{S} = \mathbf{s} + \lambda_2\mathbf{m} \quad (2.0.34)$$

where,

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{s} = \begin{pmatrix}\frac{d_2}{M}\\0\end{pmatrix} \quad (2.0.35)$$

   - Using (2.0.33) and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$ we get :

$$\mathbf{s} = \begin{pmatrix}\frac{55}{8}\\0\end{pmatrix} = \begin{pmatrix}6.875\\0\end{pmatrix} \quad (2.0.36)$$

And

$$\lambda_2 = \pm\sqrt{\frac{d_2 - ||\mathbf{s}||^2}{||\mathbf{m}||^2}} \quad (2.0.37)$$
$$\lambda_2 = \pm\sqrt{55 - 47.265} \quad (2.0.38)$$
$$\lambda_2 = \pm\sqrt{7.735} = \pm 2.78 \quad (2.0.39)$$

5) Substituting λ_2, **r** and **m** in (2.0.34) we get the coordinates of **R** and **S** as :

$$\mathbf{R} = \begin{pmatrix}6.875\\0\end{pmatrix} + 2.78\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}6.875\\2.78\end{pmatrix} \quad (2.0.40)$$
$$\mathbf{S} = \begin{pmatrix}6.875\\0\end{pmatrix} - 2.78\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}6.875\\-2.78\end{pmatrix} \quad (2.0.41)$$

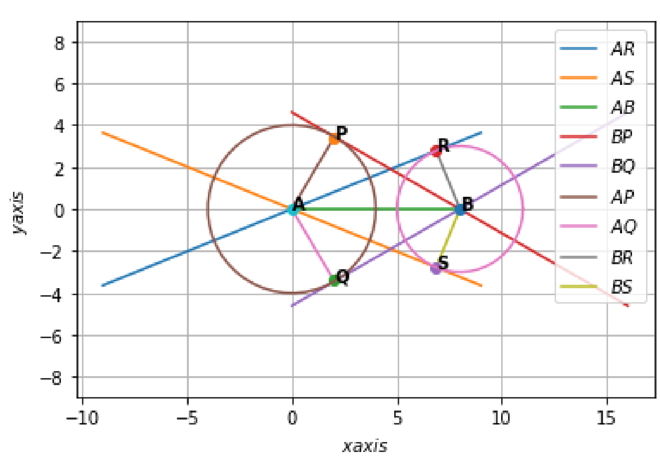

Fig. 2.1: Tangents from centres of Circle