Assignment 3

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_3/tree/master/CODES

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_3

1 Question No. 2.59

Draw a line segment **AB** of length 8 units. Taking **A** as centre draw a circle of radius 4 units and taking **B** as centre draw a circle of radius 3 units. Construct tangents to each circle from the centre of other circle.

2 Solution

The given data is tabularised in table 2.1:

	Symbols	Circle1	Circle2
Centre	**A**,**B**	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}8\\0\end{pmatrix}$
Radius	r_1,r_2	4	3

TABLE 2.1: Input values

- Also, it is given that line segment AB=8 units.So,Let:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix} \tag{2.0.1}$$

1) To find coordinates of points where tangent touches the circle.
   - Let **M** be any point on x-axis whose coordinates are:

$$\mathbf{M} = \begin{pmatrix}x_1\\0\end{pmatrix} \tag{2.0.2}$$

   - Tangents are drawn from **M** to any circle with centre **C**.

Lemma 2.1. *The coordinates of points* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *where tangent touches the circle are given by:*

$$\mathbf{N} = \mathbf{n} + \lambda\mathbf{m} \tag{2.0.3}$$

$$where,\ \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.4}$$

$$\mathbf{n} = \begin{pmatrix}\frac{d}{x_1}\\0\end{pmatrix} \tag{2.0.5}$$

$$\lambda = \pm\sqrt{\frac{d - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.6}$$

$$d = |{d_{CO}}^2 - {r_n}^2| \tag{2.0.7}$$

- *In* $\mathbf{d} = |{d_{CO}}^2 - {r_n}^2|$ *we have:*
  d_{CO} = *distance of* ***centre*** **C** *of circle from origin* **O**.
- *And*
  r_n = ***radius*** *of circle*

Proof. We know a tangent is always perpendicular to the radius .

$$\therefore N_1O \perp N_1M, \tag{2.0.8}$$

$$N_2O \perp N_2M \tag{2.0.9}$$

- Now,

$$\implies (\mathbf{O} - \mathbf{N})^T(\mathbf{N} - \mathbf{M}) = 0 \tag{2.0.10}$$

$$\mathbf{N}^T(\mathbf{N} - \mathbf{M}) = 0 \quad (\because \mathbf{O} = 0) \tag{2.0.11}$$

$$\mathbf{N}^T\mathbf{N} - \mathbf{N}^T\mathbf{M} = 0 \tag{2.0.12}$$

$$\mathbf{N}^T\mathbf{M} = \|\mathbf{N}\|^2 \tag{2.0.13}$$

$$\implies \mathbf{M}^T\mathbf{N} = \|\mathbf{N}\|^2(\because \mathbf{N}^T\mathbf{M} = \mathbf{M}^T\mathbf{N}) \tag{2.0.14}$$

- Let,
  d_{CO} = distance of **centre** of circle **C** from origin **O**.
  and, r_n = **radius** of circle

- We get $\mathbf{d}$ as
$$d = |{d_{\mathrm{CO}}}^2 - {r_{\mathrm{n}}}^2| \tag{2.0.15}$$
- Now, putting value of $\mathbf{M} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ from (2.0.2) in (2.0.14) we get:
$$\implies \begin{pmatrix} x_1 & 0 \end{pmatrix} \mathbf{N} = \begin{pmatrix} d \\ 0 \end{pmatrix} \tag{2.0.16}$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{N} = \begin{pmatrix} \frac{d}{x_1} \\ 0 \end{pmatrix} \tag{2.0.17}$$
$$\implies \mathbf{N} = \begin{pmatrix} \frac{d}{x_1} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.18}$$
$$\implies \mathbf{N} = \mathbf{n} + \lambda \mathbf{m} \tag{2.0.19}$$
$$\text{where, } \mathbf{n} = \begin{pmatrix} \frac{d}{x_1} \\ 0 \end{pmatrix} \text{and } \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.20}$$
- Also we know,
$$\|\mathbf{n} + \lambda \mathbf{m}\|^2 = d^2 \tag{2.0.21}$$
$$(\mathbf{n} + \lambda \mathbf{m})^T (\mathbf{n} + \lambda \mathbf{m}) = d^2 \tag{2.0.22}$$
$$\lambda^2 = \frac{d^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2} \tag{2.0.23}$$
$$\lambda = \pm \sqrt{\frac{d^2 - \|\mathbf{n}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.24}$$

□

2) For tangents at **Circle1** from point **B**:-
   - Here, we have r_1=4, So $\mathbf{d_1}$ is given by,
$$\mathbf{d_1} = |{d_{\mathrm{CO}}}^2 - {r_1}^2| \tag{2.0.25}$$
   - As centre is at origin so d_{CO}=0,we get:
$$\implies d_1 = |-{r_1}^2| \tag{2.0.26}$$
$$\therefore d_1 = 16 \tag{2.0.27}$$
   - Referencing (2.0.3),we have
$$\mathbf{P} = \mathbf{p} + \lambda_1 \mathbf{m} \tag{2.0.28}$$
     where,
$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} \frac{d_1}{x_1} \\ 0 \end{pmatrix} \tag{2.0.29}$$
   - Using (2.0.27) and $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$, we get
$$\mathbf{p} = \begin{pmatrix} \frac{16}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.30}$$
     And
$$\lambda_1 = \pm \sqrt{\frac{d_1 - \|\mathbf{p}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.31}$$
$$\lambda_1 = \pm \sqrt{16 - 4} \tag{2.0.32}$$
$$\lambda_1 = \pm \sqrt{12} = \pm 3.46 \tag{2.0.33}$$
3) Substituting λ_1, $\mathbf{p}$ and $\mathbf{m}$ in (2.0.28) we get the coordinates of **P** and **Q** as :
$$\mathbf{P} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \tag{2.0.34}$$
$$\mathbf{Q} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 3.46 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3.46 \end{pmatrix} \tag{2.0.35}$$
4) For tangents at **Circle2** from **A**:-
   - Here, we have r_2=3, So $\mathbf{d_2}$ is given by,
$$\mathbf{d_2} = |{d_{\mathrm{CO}}}^2 - {r_2}^2| \tag{2.0.36}$$
   - As centre is at $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ So d_{CO}=8,we get:
$$\implies d_2 = |8^2 - {r_2}^2| \tag{2.0.37}$$
$$\therefore d_2 = |64 - 9| = 55 \tag{2.0.38}$$
   - Referencing (2.0.3),we have
$$\mathbf{S} = \mathbf{s} + \lambda_2 \mathbf{m} \tag{2.0.39}$$
     where,
$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} \frac{d_2}{x_1} \\ 0 \end{pmatrix} \tag{2.0.40}$$
   - Using (2.0.38) and $\mathbf{B} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ we get :
$$\mathbf{s} = \begin{pmatrix} \frac{55}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} \tag{2.0.41}$$
     And
$$\lambda_2 = \pm \sqrt{\frac{d_2 - \|\mathbf{s}\|^2}{\|\mathbf{m}\|^2}} \tag{2.0.42}$$
$$\lambda_2 = \pm \sqrt{55 - 47.265} \tag{2.0.43}$$
$$\lambda_2 = \pm \sqrt{7.735} = \pm 2.78 \tag{2.0.44}$$

5) Substituting λ_2, $\mathbf{r}$ and $\mathbf{m}$ in (2.0.39) we get the coordinates of $\mathbf{R}$ and $\mathbf{S}$ as :

$$\mathbf{R} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} + 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ 2.78 \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{S} = \begin{pmatrix} 6.875 \\ 0 \end{pmatrix} - 2.78 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.875 \\ -2.78 \end{pmatrix} \tag{2.0.46}$$

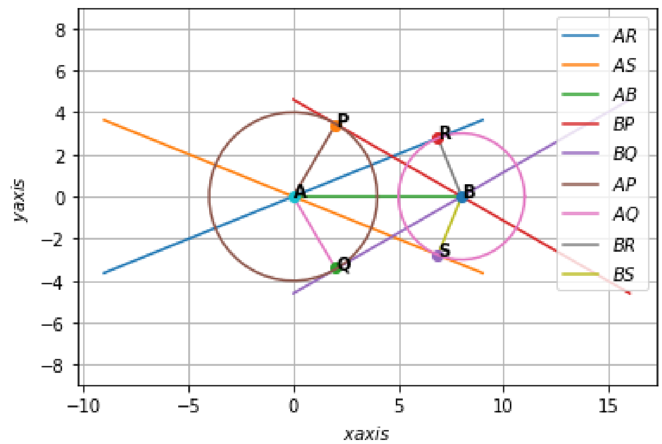

Fig. 2.1: Tangents from centres of Circle