ASSIGNMENT-6

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_6/blob/master/codes.py

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_6

1 Question No-2.105 (quadforms)

Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12$.

2 Solution

1) Given equation of parabola is:

$$4y = 3x^2 \tag{2.0.1}$$

$$3x^2 - 4y = 0 \tag{2.0.2}$$

$$x^2 - \frac{4}{3}y = 0 \tag{2.0.3}$$

Lemma 2.1. *If equation of a parabola is:*

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.4}$$

Then its vertex can be calculated as:

$$\begin{pmatrix}\mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.5}$$

$$\textit{where, } \eta = \mathbf{u^T}\mathbf{p_1} \tag{2.0.6}$$

Lemma 2.2. *The points of intersection of* ***Line*** $L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m}$ *with* ***parabola****,are given by:*

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.7}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big) \tag{2.0.8}$$

Lemma 2.3. *The area between line* $\mathbf{l}$ *:*$y_l = mx + c$ *and the parabola* $\mathbf{P}$ *:*$y_p = ax^2$ *can be given as:*

$$Area = \int y_l dx - \int y_p dx \tag{2.0.9}$$

2) Comparing (2.0.3) with (2.0.4) we get :

$$\mathbf{V} = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}, \mathbf{u} = \begin{pmatrix}0 \\ -\frac{2}{3}\end{pmatrix}, f = 0 \tag{2.0.10}$$

3) So, the equation of parabola is:

$$\implies \mathbf{x}^T\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}0 & -\frac{2}{3}\end{pmatrix}\mathbf{x} = 0 \tag{2.0.11}$$

4) We can find the eigen values corresponding to the $\mathbf{V}$,

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 0 \\ 0 & -\lambda\end{vmatrix} = 0 \tag{2.0.12}$$

$$\implies (1-\lambda)(-\lambda) = 0 \tag{2.0.13}$$

$\therefore$ Eigen values are

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.14}$$

5) Calculating the eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 1$ respectively.

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.15}$$

$$\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix}0 \\ 1\end{pmatrix} \tag{2.0.16}$$

$$\begin{pmatrix}0 & 0 \\ 0 & -1\end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix}1 \\ 0\end{pmatrix} \tag{2.0.17}$$

6) Using equation (2.0.5) the vertex of the parabola can be given as,

$$\begin{pmatrix}\mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.18}$$

$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \frac{-2}{3} \tag{2.0.19}$$

$$\implies \begin{pmatrix}0 & \frac{-4}{3} \\ 1 & 0 \\ 0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix} \tag{2.0.20}$$

7) So, the vertex of parabola is:
$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.21}$$
8) To find the point of intersection:
- Let **K** and **L** are point of intersection.
- The given line is:
$$\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12 \tag{2.0.22}$$
- In parametric form line can be written as:
$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{2.0.23}$$
$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.24}$$
- Putting values of **V**, **q** from (2.0.10) and **m** and **q** from (2.0.24) in equation (2.0.8)we get:
$$\mu_i = \frac{1}{4}(10) \pm 6 \tag{2.0.25}$$
- Now we can get μ_1 and μ_2 as:
$$\mu_1 = \frac{1}{4}(10 - 6) \tag{2.0.26}$$
$$\implies \mu_1 = 1 \tag{2.0.27}$$
$$\text{Similarly, } \mu_2 = \frac{1}{4}(10 + 6) \tag{2.0.28}$$
$$\implies \mu_2 = 4 \tag{2.0.29}$$
- Putting μ_1 and μ_2 in (2.0.24) we get point of intersection as:
$$\implies \mathbf{K} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.30}$$
$$\implies \mathbf{L} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{2.0.31}$$
9) Using Lemma (2.3) area enclosed by parabola and line can be given as:
$$A = \text{Area under line} - \text{Area under parabola} \tag{2.0.32}$$
$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{2.0.33}$$
$$A = A_1 - A_2 \tag{2.0.34}$$
10) Area under the line 2y=3x+12 i.e, A_1-
$$A_1 = \int_{-2}^{4} y dx \tag{2.0.35}$$
$$A_1 = \frac{1}{2} \int_{-2}^{4} (3x + 12)\, dx \tag{2.0.36}$$
$$A_1 = \frac{3}{2} \int_{-2}^{4} x + \frac{1}{2} \int_{-2}^{4} 12dx \tag{2.0.37}$$
$$A_1 = \frac{3}{4}\left(4^2 - 2^2\right) + \frac{12}{2}(4 + 2) \tag{2.0.38}$$
$$A_1 = \frac{3}{4}(12) + \frac{12}{2}(6) \tag{2.0.39}$$
$$A_1 = 9 + 36 \tag{2.0.40}$$
$$A_1 = 45 \text{ units} \tag{2.0.41}$$
11) Area under the parabola that is A_2-
$$A_2 = \int_{-2}^{4} y dx \tag{2.0.42}$$
$$A_2 = \int_{-2}^{4} \frac{3}{4} x^2 dx \tag{2.0.43}$$
$$A_2 = \frac{3}{4} \int_{-2}^{4} x^2 dx \tag{2.0.44}$$
$$A_2 = \frac{3}{4 \times 3}\left(4^3 - (-2)^3\right) \tag{2.0.45}$$
$$A_2 = \frac{1}{4}(64 + 8) \tag{2.0.46}$$
$$A_2 = \frac{72}{4} \tag{2.0.47}$$
$$A_2 = 18 \text{ units} \tag{2.0.48}$$
12) Putting (2.0.41) and (2.0.48) in (2.0.34) we get required area A as:
$$A = A_1 - A_2 \tag{2.0.49}$$
$$A = 45 - 18 \tag{2.0.50}$$
$$A = 27 \text{ units} \tag{2.0.51}$$
So, the required area A is **27 units**

13) Plot of line and parabola is:

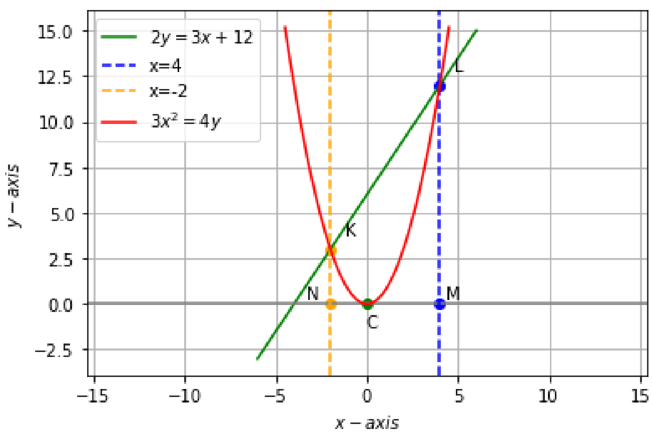

Fig. 2.1: Plot of the parabola and line