ASSIGNMENT-6

Unnati Gupta

Download all python codes from

https://github.com/unnatigupta2320/Assignment_6/blob/master/codes.py

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_6

1 Question No-2.105 (quadforms)

Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12$.

2 Solution

1) Given equation of parabola is:

$$4y = 3x^2 \tag{2.0.1}$$
$$3x^2 - 4y = 0 \tag{2.0.2}$$
$$x^2 - \frac{4}{3}y = 0 \tag{2.0.3}$$

2) Comparing with the standard equation :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.4}$$
$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = \begin{pmatrix}0\\-\frac{2}{3}\end{pmatrix}, f = 0 \tag{2.0.5}$$

3) We can find the eigen values corresponding to the $\mathbf{V}$,

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 0\\0 & -\lambda\end{vmatrix} = 0 \tag{2.0.6}$$
$$\implies (1-\lambda)(-\lambda) = 0 \tag{2.0.7}$$

$\therefore$ Eigen values are

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.8}$$

4) Calculating the eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 1$ respectively.

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.9}$$
$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.10}$$
$$\begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.0.11}$$

5) By eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.12}$$

Where,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{2.0.13}$$
$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.0.14}$$

6) To find the vertex of the parabola ,

$$\begin{pmatrix}\mathbf{u^T} + \eta\mathbf{p_1^T}\\\mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f}\\\eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.15}$$
$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \frac{-2}{3} \tag{2.0.16}$$
$$\implies \begin{pmatrix}0 & \frac{-4}{3}\\1 & 0\\0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.17}$$

7) The above equation implies that,

$$\mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.18}$$

8) To find the point of intersection:
   - Let $\mathbf{K}$ and $\mathbf{L}$ are point of intersection.
   - The given line is:

$$\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12 \tag{2.0.19}$$
$$\text{or, } 2y = 3x + 12 \tag{2.0.20}$$
$$\therefore \mathbf{x} = \begin{pmatrix}x\\\frac{12+3x}{2}\end{pmatrix} \text{satisfies it.} \tag{2.0.21}$$

   - From (2.0.4) we have:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.22}$$
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}0 & \frac{-2}{3}\end{pmatrix}\mathbf{x} + 0 = 0 \tag{2.0.23}$$

   - Putting (2.0.21) in above equation we get:

$$\begin{pmatrix}x & \frac{12+3x}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}x\\\frac{12+3x}{2}\end{pmatrix} + \begin{pmatrix}0 & \frac{-4}{3}\end{pmatrix}\begin{pmatrix}x\\\frac{12+3x}{2}\end{pmatrix} = 0 \tag{2.0.24}$$

$$\implies \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ \frac{12+3x}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{-4}{3} \end{pmatrix} \begin{pmatrix} x \\ \frac{12+3x}{2} \end{pmatrix} = 0 \quad (2.0.25)$$

$$\implies x^2 + \frac{(-4)}{3}\left(\frac{12+3x}{2}\right) = 0 \quad (2.0.26)$$

$$\implies x^2 - 2x - 8 = 0 \quad (2.0.27)$$

$$\implies x = -2, 4 \quad (2.0.28)$$

- Putting values of x in (2.0.20), we get:

$$\text{When x = -2} \implies y = 3 \quad (2.0.29)$$

$$\implies \mathbf{K} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.0.30)$$

$$\text{When x = 4,} \implies y = 12 \quad (2.0.31)$$

$$\implies \mathbf{L} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad (2.0.32)$$

9) For finding area enclosed by parabola and line: Area required,

$$A = \text{Area under line} - \text{Area under parabola} \quad (2.0.33)$$

$$A = Ar(KLMNK) - Ar(KCLMCNK) \quad (2.0.34)$$

$$A = A_1 - A_2 \quad (2.0.35)$$

10) For finding area under the line that is A_1- **KLMN** is a trapezium. So its area can be given as:

$$A_1 = MN \times \frac{(KN + ML)}{2} \quad (2.0.36)$$

$$A_1 = 6 \times \frac{(3+12)}{2} \quad (2.0.37)$$

$$A_1 = 45 \text{ units} \quad (2.0.38)$$

11) For finding area under the parabola that is A_2-

$$A_2 = \int_{-2}^{4} y dx \quad (2.0.39)$$

$$A_2 = \int_{-2}^{4} \frac{3}{4} x^2 dx \quad (2.0.40)$$

$$A_2 = \frac{3}{4} \int_{-2}^{4} x^2 dx \quad (2.0.41)$$

$$A_2 = \frac{3}{4 \times 3}\left(4^3 - (-2)^3\right) \quad (2.0.42)$$

$$A_2 = \frac{1}{4}(64 + 8) \quad (2.0.43)$$

$$A_2 = \frac{72}{4} \quad (2.0.44)$$

$$A_2 = 18 \text{ units} \quad (2.0.45)$$

12) Putting (2.0.38) and (2.0.45) in (2.0.35) we get required area A as:

$$A = A_1 - A_2 \quad (2.0.46)$$

$$A = 45 - 18 \quad (2.0.47)$$

$$A = 27 \text{ units} \quad (2.0.48)$$

So, the required area A is **27 units**

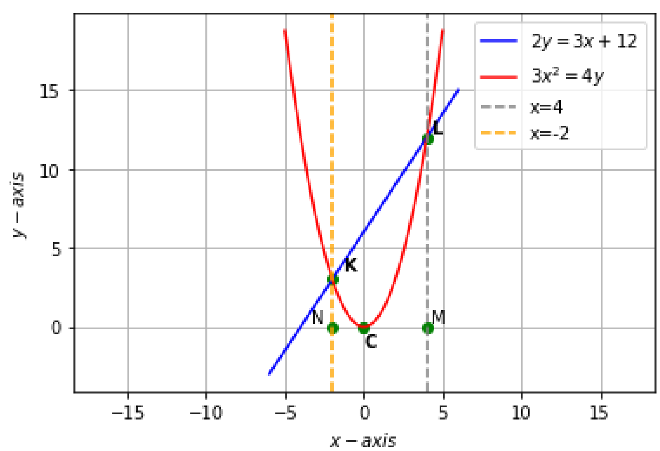

Fig. 2.1: Plot of the parabola and line