ASSIGNMENT-6

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_6/blob/master/codes.py

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_6

1 Question No-2.105 (quadforms)

Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.

2 Solution

1) Given equation of parabola is:

$$4y = 3x^2 \tag{2.0.1}$$
$$3x^2 - 4y = 0 \tag{2.0.2}$$
$$x^2 - \frac{4}{3}y = 0 \tag{2.0.3}$$

2) Comparing with the standard equation :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.4}$$
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix}, f = 0 \tag{2.0.5}$$

3) So, the equation of parabola is:

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -\frac{2}{3} \end{pmatrix} \mathbf{x} = 0 \tag{2.0.6}$$

4) We can find the eigen values corresponding to the $\mathbf{V}$,

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \tag{2.0.7}$$
$$\implies (1-\lambda)(-\lambda) = 0 \tag{2.0.8}$$

$\therefore$ Eigen values are

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.9}$$

5) Calculating the eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 1$ respectively.

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.10}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.11}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.12}$$

6) By eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.13}$$

Where,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.14}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.15}$$

7) To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$
$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \frac{-2}{3} \tag{2.0.17}$$
$$\implies \begin{pmatrix} 0 & \frac{-4}{3} \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

8) The above equation implies that,

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

9) To find the point of intersection:
   - Let $\mathbf{K}$ and $\mathbf{L}$ are point of intersection.
   - The given line is:

$$\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12 \tag{2.0.20}$$
$$\implies \mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.21}$$

- In parametric form Line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.22}$$

$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.23}$$

- The points of intersection of this line with the parabola in (2.0.6) are given by:

$$\mathbf{x}_i = \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \mu_i \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.24}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big) \tag{2.0.25}$$

- Finding μ_i :
  a. Value of $\mathbf{m}^T\mathbf{V}\mathbf{m}$:

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.0.27}$$

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = 4 \tag{2.0.28}$$

  b. Value of $\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})$:

$$\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) = \begin{pmatrix} 2 & 3 \end{pmatrix} \left(\begin{pmatrix} -4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-2}{3} \end{pmatrix} \right) \tag{2.0.29}$$

$$\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ \frac{-2}{3} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) = -10 \tag{2.0.31}$$

  c. Value of $\mathbf{q}^T\mathbf{V}\mathbf{q}$:

$$\mathbf{q}^T\mathbf{V}\mathbf{q} = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{q}^T\mathbf{V}\mathbf{q} = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{q}^T\mathbf{V}\mathbf{q} = 16 \tag{2.0.34}$$

  d. Value of:$2\mathbf{u}^T\mathbf{q}$:

$$2\mathbf{u}^T\mathbf{q} = 2 \begin{pmatrix} 0 & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.0.35}$$

$$2\mathbf{u}^T\mathbf{q} = 0 \tag{2.0.36}$$

- Putting values from (2.0.28), (2.0.31), (2.0.34), (2.0.36) in equation (2.0.25), we get:

$$\mu_i = \frac{1}{4} \Big(-(-10) \pm \sqrt{[(-10)]^2 - (16 + 0 + 0)(4)}\Big) \tag{2.0.37}$$

$$\mu_i = \frac{1}{4}(10) \pm 6 \tag{2.0.38}$$

- So, we get μ_1 and μ_2 as:

$$\mu_1 = \frac{1}{4}(10 - 6) \tag{2.0.39}$$

$$\implies \mu_1 = 1 \tag{2.0.40}$$

$$\text{Similarly, } \mu_2 = \frac{1}{4}(10 + 6) \tag{2.0.41}$$

$$\implies \mu_2 = 4 \tag{2.0.42}$$

- Putting μ_1 and μ_2 in (2.0.23) we get point of intersection as:

$$\implies \mathbf{K} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.43}$$

$$\implies \mathbf{L} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{2.0.44}$$

10) For finding area enclosed by parabola and line: Area required,

$$A = \text{Area under line} - \text{Area under parabola} \tag{2.0.45}$$

$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{2.0.46}$$

$$A = A_1 - A_2 \tag{2.0.47}$$

11) Area under the line 2y=3x+12 i.e, A_1-

$$A_1 = \int_{-2}^{4} y dx \tag{2.0.48}$$

$$A_1 = \frac{1}{2}\int_{-2}^{4} (3x + 12) dx \tag{2.0.49}$$

$$A_1 = \frac{3}{2}\int_{-2}^{4} x + \frac{1}{2}\int_{-2}^{4} 12dx \tag{2.0.50}$$

$$A_1 = \frac{3}{4}\left(4^2 - 2^2\right) + \frac{12}{2}(4 + 2) \tag{2.0.51}$$

$$A_1 = \frac{3}{4}(12) + \frac{12}{2}(6) \tag{2.0.52}$$

$$A_1 = 9 + 36 \tag{2.0.53}$$

$$A_1 = 45 \text{ units} \tag{2.0.54}$$

12) Area under the parabola that is A_2-

$$A_2 = \int_{-2}^{4} y dx \tag{2.0.55}$$
$$A_2 = \int_{-2}^{4} \frac{3}{4} x^2 dx \tag{2.0.56}$$
$$A_2 = \frac{3}{4} \int_{-2}^{4} x^2 dx \tag{2.0.57}$$
$$A_2 = \frac{3}{4 \times 3} \left(4^3 - (-2)^3\right) \tag{2.0.58}$$
$$A_2 = \frac{1}{4} (64 + 8) \tag{2.0.59}$$
$$A_2 = \frac{72}{4} \tag{2.0.60}$$
$$A_2 = 18 \text{ units} \tag{2.0.61}$$

13) Putting (2.0.54) and (2.0.61) in (2.0.47) we get required area A as:

$$A = A_1 - A_2 \tag{2.0.62}$$
$$A = 45 - 18 \tag{2.0.63}$$
$$A = 27 \text{ units} \tag{2.0.64}$$

So, the required area A is **27 units**

14) Plot of line and parabola is:

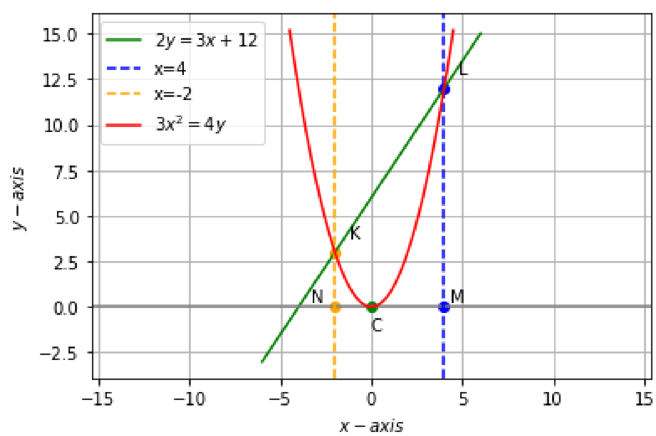

Fig. 2.1: Plot of the parabola and line