ASSIGNMENT 7

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Download all python codes from

https://github.com/unnatigupta2320/Assignment_7/blob/master/codes.py

and latex-tikz codes from

https://github.com/unnatigupta2320/Assignment_7

1 Question No 2.74(d)

In each of the following find the equation for the ellipse that satisfies the given conditions:

a. Conjugate axis length= 8, Foci = $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$

2 Solution

Given that,

$$\text{Conjugate axis length} = 2b = 8 \tag{2.0.1}$$

$$\therefore \text{Length of semi major axis, } b = 4 \tag{2.0.2}$$

$$\text{Foci} = \mathbf{F} = \begin{pmatrix} \pm 5 \\ 0 \end{pmatrix} \tag{2.0.3}$$

Lemma 2.1. *The standard equation of an ellipse is given by:*

$$\frac{\mathbf{y}^\top D \mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{2.0.4}$$

$$\textit{where, } D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.5}$$

Lemma 2.2. *The coordinates of foci* $\mathbf{F}$ *of ellipse with x-axis as major axis are:*

$$\mathbf{F} = \begin{pmatrix} \pm\left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}}\right) \\ 0 \end{pmatrix} \tag{2.0.6}$$

Also,the length of semi major axis, a is

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.7}$$

and the length of semi minor axis, b is

$$b = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} \tag{2.0.8}$$

1) From (2.0.8) length of semi-minor axis is:

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = b \tag{2.0.9}$$

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2} = b^2 \tag{2.0.10}$$

$$\implies \lambda_2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{b^2} \tag{2.0.11}$$

2) From (2.0.6), the focus of ellipse is given as:

$$\mathbf{F} = \begin{pmatrix} \pm\left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}}\right) \\ 0 \end{pmatrix} \tag{2.0.12}$$

or

$$\|\mathbf{F}\|^2 = \frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2} \tag{2.0.13}$$

$$\|\mathbf{F}\|^2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} - \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2} \tag{2.0.14}$$

3) Putting value of λ_2 from (2.0.11) in above equation, we get:

$$\|\mathbf{F}\|^2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} - b^2 \tag{2.0.15}$$

$$\|\mathbf{F}\|^2 + b^2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} \tag{2.0.16}$$

$$\implies \lambda_1 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\|\mathbf{F}\|^2 + b^2} \tag{2.0.17}$$

4) *For finding* λ_2:
   - From (2.0.11) we have:

$$\lambda_2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{b^2} \tag{2.0.18}$$

$$\implies \lambda_2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{16} \ (\because b = 4) \tag{2.0.19}$$

5) *For finding* λ_1*:*
   - Putting value of b from (2.0.2) and $\mathbf{F}$ from

(2.0.3) in equation (2.0.17),we get:

$$\lambda_1 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\left(\sqrt{5^2 + 0^2}\right)^2 + 4^2} \tag{2.0.20}$$

$$\lambda_1 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{25 + 16} \tag{2.0.21}$$

$$\lambda_1 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{41} \tag{2.0.22}$$

6) Using lemma (2.1),the standard equation of ellipse is given by :

$$\frac{\mathbf{y}^\top D\mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{2.0.23}$$

$$\implies \frac{\mathbf{y}^\top \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{2.0.24}$$

7) Putting (2.0.19) and (2.0.22) in above equation we get:

$$\implies \mathbf{y}^\top \begin{pmatrix} \frac{1}{41} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{y} = 1 \tag{2.0.25}$$

which is the required equation of ellipse.

8) The Plot of ellipse is:

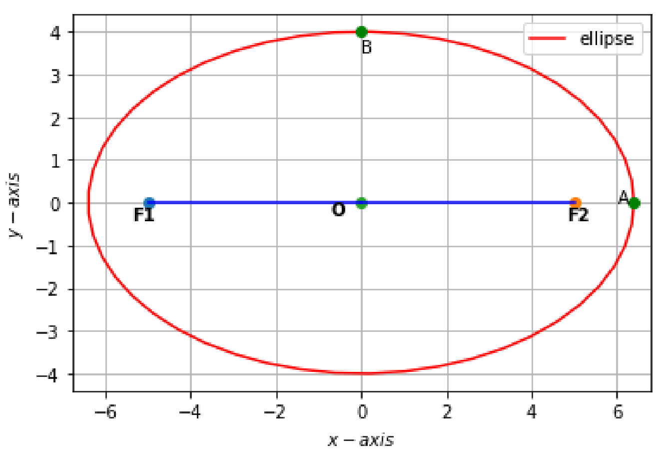

Fig. 2.1: Ellipse $\frac{x^2}{41} + \frac{y^2}{16} = 1$