An Exploration of Angular Momentum in 2 Dimensions

Keshav Anand, Blake Hodges, Songming Liu, Inna Lobanova

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Abstract

This experiment investigated the relationship between the angular momentum of a gyroscope tilted at various angles and the angular momentum of a platform in water to which the gyroscope was attached - the overall gyroscope-apparatus system. The rotation of the gyroscope-apparatus system was then tracked for multiple angles of tilt and compared to theoretical modeling. From this, a conclusion concerning the conservation of angular momentum was reached.

1 Introduction

In our project, we chose to examine a system of a mounted gyroscope floating in a tub of water. The aim of our experiment was to quantify and understand the behavior of the interactions of the different variables in the system, such as the angular momentum, angular velocities, and the tilt of the gyroscope. The inspiration for this experiment was the use of gyroscopes in satellite attitude control. Satellite attitude control systems are primarily used to store angular momentum to help turn the satellite and help it resist rotation. These gyroscopes are in low friction environments, which inspired our setup seen in Section 2, where ideally the only significant frictional force is the drag force of the water.

2 Experimental Design

In order to simulate how a gyroscope may behave in space, we designed an apparatus where the axis of rotation of the gyroscope was fixed, but the gyroscope-apparatus system would float on water and be able to almost freely, with only the drag force of the water inhibiting its movement. A render of the experimental setup can be seen in Figure 1. The axis of the gyroscope varied between tests in order to observe the effect the angle had on the system.

2.1 Materials

For the construction of the system, this is a list of materials used:

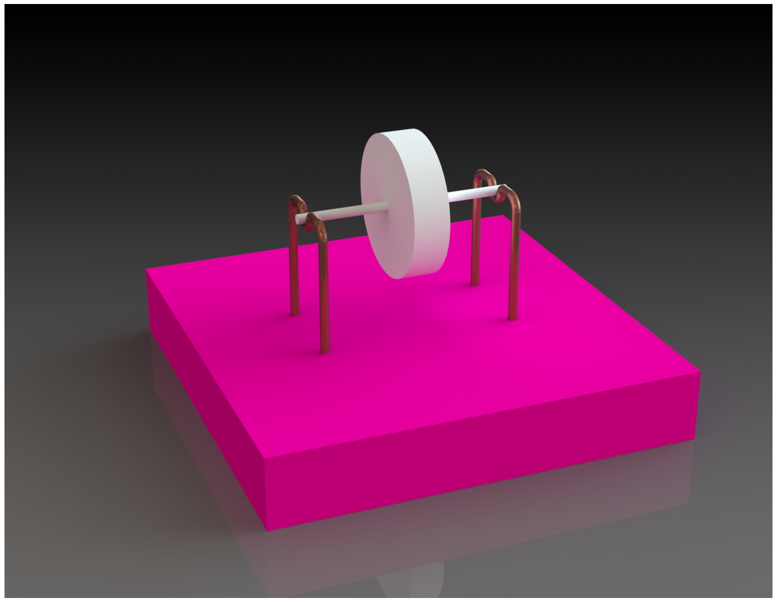

Figure 1: Render of the float's full assembly

- 1 gyroscope
- 15cm x 15cm of insulating foam board
- 1 large plastic bin, filled with water
- Approximately 30cm of 10 gauge copper wire, cut into two pieces and shaped to support the gyroscope
- Duct tape, to secure the gyroscope onto the copper wire
- Hot glue, to anchor the copper wire in the insulating foam board

Tools used in the construction of the apparatus include a wire cutter, hot glue gun, and a mallet. The total mass of the apparatus was found to be 115 ± 5g.

2.2 Measurements

In order to get measurements of the variables we need in our analysis, we employed the use of two cameras. With this setup, one camera would record the spinning flywheel, which was marked with a black dot for tracking. Meanwhile, the other camera recorded the motion of the platform, with the duct tape on the copper wires for references. The resulting videos could then be analyzed using Tracker software, with the data then exported to MATLAB for further processing, analysis, and plotting.

2.2.1 Problems

Unfortunately, the footage of the spinning flywheel was not at a high enough frame rate to track the motion of the flywheel in Tracker software. As a result, our analysis related to the angular velocity of the flywheel is mostly qualitative, as we are unable to extract any quantitative data or trends from the videos. Luckily, the platform was spinning slowly enough that we did not run into the same issue for the other footage.

3 Theoretical Behavior

In order to have a better understanding of our experiment, this section will consider some theoretical modeling of our experiment, and the analysis from this section will be used to compare against some of features from the data collected by the experiment. This part will consists of two models, with the second model built on top of the first one. The first section serves as the simplest theoretical model, and then more complexity and more variables will be taken into the consideration. Finally, we will arrive at a model which roughly behaves like the system we built for the experiment.

3.1 Simple model

Our first naive model will describe a straightforward, oversimplified system. It will serve as the corner stone for more complicated elements. Some of the assumptions for this system are:

- There is no friction between the system and the environment or internally.
- The shape of gyroscope is simplified to regular nice geometric components for sake of convenience of moment of inertia computation.
- The mass is uniformly distributed among the gyroscope.

To begin, let's state some notations. We will use ω_g to denote the angular velocity of the gyroscope about its axis, and Ω_s to denote the angular velocity of the whole system about a vertical axis through the center of the board. θ will represent the angle tilt between the gyroscope's axis of rotation and the horizontal. Now, the magnitude of total angular momentum normal to the plane spanned by gyroscope is the following equation:

$$L_g = I_r \omega_g \tag{1}$$

where I_r is the moment of inertia of the flywheel part of the gyroscope. We use our second assumption to idealize the round circle part of the gyroscope to be a thin disk. Therefore, the moment of inertia can be represented by:

$$I_r = \frac{1}{2} m_g R_g^2 \tag{2}$$

where R_g is the radius of the disk and m_g is the mass of the gyroscope. Then, the horizontal component of the angular momentum can be represented by:

$$L_h = \frac{1}{2} m_g R_g^2 \omega_g sin\theta \tag{3}$$

By the law of conservation of angular momentum, the whole system should have no net change in angular momentum. To obey this law, the system has to redistribute its angular momentum until it reaches some equilibrium. Thus, we will establish, $L_s = -L_h$ with extra assumption that the center of the mass of the system is just the center of the rotation of the gyroscope. Now, suppose the system is rotating against the angular momentum. Then:

$$I_s \Omega_s = \frac{1}{2} m_g R_g^2 \omega_g sin\theta \tag{4}$$

Here, again, we take the platform's shape as thin disk stacked on rectangular, so

$$I_s = \frac{1}{12} m_s (w^2 + d^2) + \frac{1}{2} m_g R_g^2 \tag{5}$$

where m_s is the mass of the system, w is the width of the platform while d is the depth of the platform. The first part is due to the large cube, and the second part is due to the rotating gyro. Thus the angular velocity of the platform should be:

$$\Omega_s = \frac{6 m_g R_g^2 \omega_g sin\theta}{m_s (w^2 + d^2) + 6 m_g R_g^2} \tag{6}$$

3.2 Approaching reality

The first model relies heavily on the assumption that the mass of the gyroscope is uniformly distributed, i.e. have constant density everywhere. Now, in this part, our theoretical model tries to relax this assumption, we treat the gyro-part with one uniform density ρ_1, and the supporting platform with another different density ρ_2. Also, another change is that we treat the rotating flywheel not as a 2-D thin disk; rather, we view it as a short cylinder with its width d_c. Fortunately, the expression for I_r stays the same, so we simply plug in density in and we get:

$$I_r = \frac{1}{2} R_g^2 \rho_1 \pi R_g^2 d_c \tag{7}$$

Now, we refine our mass for the whole system as, where h is the height of the floating cube.

$$m_s = wdh\rho_2 \tag{8}$$

Thus, the updated moment of inertia for the whole system I_s is

$$I_s = \frac{1}{12} wdh\rho_2 (w^2 + d^2) + \frac{1}{2} R_g^2 \rho_1 \pi R_g^2 d_c \tag{9}$$

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After incorporating those two changes to our system, we will finally arrive at:

$$\Omega_s = \frac{6R_g^2\rho_1\pi R_g^2 d_c\omega_g \sin\theta}{wdh\rho_2(w^2+d^2)+6R_g^2\rho_1\pi R_g^2 d_c} \tag{10}$$

3.3 Some predictions

Through our observation, the most obvious and important conclusion we want to draw is that as the gyroscope rotates, the underlying foam platform will also rotate as consequence of conservation of angular momentum. This is explained by the previous two sections of theoretical models. Another important corollary follows that as the angle tilts more, the rotation will become greater in magnitude. Mathematically, this is controlled by the $\sin\theta$ term of our expression for angular velocity of the whole system. As θ ranges from zero to $\frac{\pi}{2}$, we have Ω range from 0 to the constant in front of $sin\theta$ in equation 10. It turns out that both of the major predictions from our theoretical model coincides with our experimental results, which indicates that our theoretical model agrees with reality.

4 Observed Behavior and Analysis

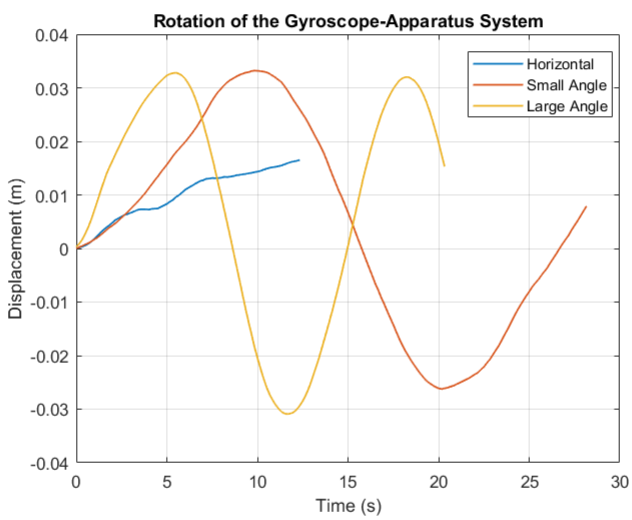

Figure 2: Angular displacement of the float for different gyroscope tilts

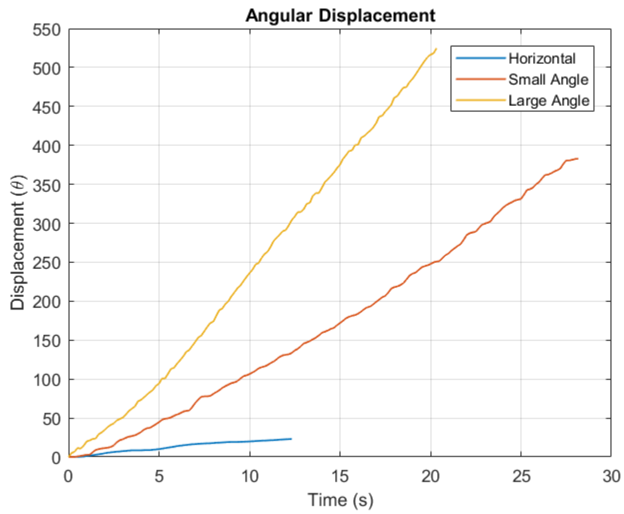

Figure 3: Displacement of the float in a horizontal gyroscope test

4.1 Horizontal

As can be seen in the Figure 2, there is a small rotation of the platform when the gyroscope is horizontal. The small amount of rotation that is measured is most likely the result of applying an initial force on the system while pulling the string of the gyroscope. According to our theoretical model, this lack of rotation is to be expected; more specifically, since the angular momentum of the gyroscope in this case does not have a component parallel to the plane of the water, the change in angular momentum along the vertical axis is zero.

4.2 Small Angle

For a nonzero angle of tilt, it can be seen that the platform has an angular velocity that allows it to complete a full rotation about the vertical axis. The presence of this angular velocity can be explained by the conservation of angular momentum; the angular momentum of the gyroscope has a component perpendicular to the plane of rotation, which allows the platform to rotate in such a way that would, if not for friction, conserve the angular momentum of the system.

4.3 Large Angle

We can see that for an even larger angle of tilt, the platform rotates slightly beyond one cycle before losing its energy to friction. Additionally, the platform reaches a larger angular velocity than in the previous case, as can be seen from the periods of the two graphs. This, again, is to be expected according to our theoretical modeling; according to equation 6 as the angle of tilt increases, the angular velocity of the platform should also increase. A platform with a larger angular velocity is then able to complete more rotations before losing energy to friction, as seen in the plot of displacement.

5 Individual Commentary

The use of an assembly of flywheels to store angular momentum is not a novel idea. There have been various documented uses of such a system throughout humanity's push into the space age- notable uses are on the International Space Station, Hubble Space Telescope, and virtually every satellite launched into space that needs to stay stable.

There are two main reasons to use such a system: the need to orient a spacecraft in the three dimensions of space and keep it stable once it is there. Our experiment shows that this first case is taken care of through our system: we saw that with the rotation of our gyroscope, the system rotated in the opposite direction to conserve angular momentum. To examine the second case, we will look at various uses of these systems in practice. For example, there are six gyroscopes mounted on the Hubble Space Telescope that spin at 19,200 revolutions per minute. This extremely high rate of rotation stores a substantial amount of angular momentum- in order to push the spacecraft, one would have to not only apply enough torque to shift the mass itself but also counter the total angular momentum of the gyroscope. In practice, this system allows for highly stable crafts.

This system is not infallible. One main issue with this system is the worry of "saturating" the flywheels - spinning them up to their maximum speed, which leaves no room to add any additional force. If a craft were nudged with some torque, the wheel could compensate by spinning faster. If the spacecraft kept being nudged in this direction, eventually, the wheel would no longer have the ability to spin faster, thus resulting in its "saturation." We can demonstrate this problem with our setup by just opposing the rotation of the float with a finger. Eventually, we could apply more force than the gyroscope could generate, resulting in this "saturation." This problem is solved in practice with chemical propellants, which could hold the craft stable while the wheel spun down. In practice, this has been used on many satellites.

There has been extensive research on the use of flywheels in control moment gyroscopes (as seen on the International Space Station) and reaction wheels (as seen on virtually every satellite). In our experiment, we saw that they could reasonably be used to push a spacecraft through the conservation of angular momentum. Various other research has arrived at this conclusion as well. In a paper studying the control moment gyroscopes on the International Space Station, the authors mention that "Prior to CMG activation, the ISS attitude control was accomplished using thrusters. Thrusters are now used as a backup for CMG control, for large attitude maneuvers, or for translational control" (Gurrisi et al., 164). The usage of the CMGs as primary attitude control shows that this strategy is viable and used in practice.

Though our system was not entirely accurate to reality, our model drew strong parallels to in-situ uses of the theory we had. From a simple postulate that angular momentum must be conserved, we were able to show that finely controlling the orientation and speed of rotation was possible with just a wheel and an angle. In the future, improving the design with a more isolated environment, finer control of the gyroscope angle, and more precise data will help refine these results and further show the efficacy of our model.

6 Conclusion

Through our testing in this experiment, we have shown that angular momentum must be conserved in an isolated system. We have also seen that spinning up a gyroscope at different angles is a viable method of controlling the amount of angular momentum affecting the system as a whole, allowing for distinctly different changes in velocity. One important thing to note, however, is that in our experiment, angular momentum was lost to the water. We expected to see this, though, since water is not frictionless, and it forms waves. Furthermore, this shows that the in-situ usage of these systems is justified- they hold angular momentum and convert it into different directions, as expected.

7 References

- Gurrisi, C., Seidel, R., Dickerson, S., Didziulis, S., Frantz, P., and Ferguson, K. (2013, August 29). Space Station Control Moment Gyroscope Lessons Learned.

- Kleppner, D., and Kolenkow, R. J. (2014). An introduction to mechanics (2nd ed.). Cambridge: Cambridge Univ. Press.