Complex Analysis - HW3

SungBin Park, 20150462

November 21, 2018

1. Prove that $\int_{H_\epsilon} \frac{t^{z-1}}{e^t-1}dt$ is entire for $\epsilon < 2\pi$, where H_ϵ is Hankel contour as Fig. 1.

   Proof. I'll use Morera's theorem. Let γ be a closed piecewise C^1 curve in $\mathbb{C}$. Then,

$$\int_\gamma \int_{H_\epsilon} \frac{t^{z-1}}{e^t-1}\, dtdz = \int_{H_\epsilon} \frac{1}{e^t-1} \int_\gamma t^{z-1}\, dzdt = \int_{H_\epsilon} 0\, dt = 0$$

   since t^{z-1} is entire in $\mathbb{C}$.($\because \frac{\partial t^{z-1}}{\partial \bar{z}} = 0$.) By Morera's theorem, $\int_{H_\epsilon} \frac{t^{z-1}}{e^t-1}\, dt$ is entire in $\mathbb{C}$.
   Let's see why we can exchange integral. First, let's divide H_ϵ by C_ϵ, $x : \infty \sim \delta(\epsilon)$, $x : \delta(\epsilon) \sim \infty$ where $\delta(\epsilon)$ is defined for connecting the straight lines and C_ϵ. Assume the straight line have imaginary part $\sigma > 0$. Then, $\frac{t^{z-1}}{e^t-1}$ is continuous on the contour.

$$\int_\gamma \int_{H_\epsilon} \frac{t^{z-1}}{e^t-1}\, dtdz = \int_\gamma \left(\int_{C_\epsilon} \frac{t^{z-1}}{e^t-1}\, dt + \int_\delta^\infty \frac{(x-i\sigma)^{z-1}}{e^{x-i\sigma}-1}\, dx + \int_\infty^\delta \frac{(x+i\sigma)^{z-1}}{e^{x+i\sigma}-1}\, dt \right) dz.$$

   For $z = a + bi$, $\left|(x \pm i\sigma)^{z-1}\right| = \left|e^{(a-1+bi)\log(x\pm i\sigma)}\right| \le \left|e^{(a-1)\log\left(\sqrt{x^2+\sigma^2}\right)+|b\theta|}\right| \le 2\sqrt{x^2+\sigma^2}^{a-1}$ for sufficiently large x, where $\theta = \arg(x \pm i\sigma)$, so

$$\left| \int_\delta^\infty \frac{(x \pm i\sigma)^{z-1}}{e^{x\pm i\sigma}-1}\, dx \right| \le \left| \int_\delta^R \frac{(x \pm i\sigma)^{z-1}}{e^{x\pm i\sigma}-1}\, dx \right| + \int_R^\infty \frac{2\sqrt{x^2+\sigma^2}^{a-1}}{e^x-1}\, dx < \infty$$

   for sufficiently large R. and it means $\frac{t^{z-1}}{e^t-1}$ is integrable on each domain. Therefore, we can use Fubini theorem and yield

$$\int_\gamma \int_{H_\epsilon} \frac{t^{z-1}}{e^t-1}\, dtdz = \int_{H_\epsilon} \frac{1}{e^t-1} \int_\gamma t^{z-1}\, dzdt.$$

□

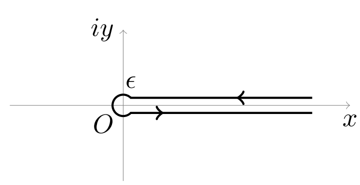

Figure 1: Hankel Contour for $\epsilon < 2\pi$. The branch cut is Re ≥ 0, Im $= 0$.

2. Prove that $\zeta(s) \neq 0$ for $\operatorname{Re} s = 0$.

Proof. I'll use the functional equation:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\cdot\Gamma(s)\zeta(s). \tag{1}$$

Put $s = 1+it$ for $t \in \mathbb{R}$, then

$$\zeta(-it) = 2^{-it}\pi^{-1-it}\cos\frac{\pi(1+it)}{2}\cdot\Gamma(1+it)\zeta(1+it).$$

If $t = 0$, problem 3 proves that $\zeta(0) \neq 0$, so I'll assume that $t \neq 0$. For $t \neq 0$, $\cos\frac{\pi(1+it)}{2} = \frac{i}{2}\left(e^{-\pi t/2} - e^{\pi t/2}\right) \neq 0$. Using identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s},$$

$\Gamma(-it)\Gamma(1+it) = \frac{\pi}{\sin\pi it}$ and $\sin\pi it < \infty$. Also, $|\Gamma(it)| < \infty$ since $1/\Gamma(s)$ only have zeros in $0, -1, \cdots$. Therefore, $\Gamma(1+it) \neq 0$ for $t \in \mathbb{R}\setminus\{0\}$. In the class, we proved that $\zeta(s) \neq 0$ at $\operatorname{Re} s = 1$. Hence,

$$\zeta(it) \neq 0 \quad \text{for } t \in \mathbb{R}\,.$$

□

3. Prove that $\zeta(0) = -1/2$.

Proof. I'll use (1). We know that ζ has a simple pole at $s = 1$ and the residue is 1. Therefore,

$$\lim_{s\to 1}\zeta(1-s) = \lim_{s\to 1}2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\cdot\Gamma(s)\zeta(s) = \lim_{s\to 1}2^{1-s}\pi^{-s}\left(\frac{\pi}{2} - \frac{\pi}{2}s\right)\Gamma(s)\zeta(s) = -\frac{1}{2}\Gamma(1),$$

and $\Gamma(1) = 1$. Therefore, $\lim\limits_{s\to 1}\zeta(1-s)$ exists and $\zeta(0) = -1/2$ since $\zeta(s)$ is entire except $s = 1$. □