Complex Analysis - HW4

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1. If f is holomorphic on the disk $D(0,R)$, $|f| \le M$ on the disc, and $f(z_0) = w_0$, then f satisfies

$$\left|\frac{M(f(z)-w_0)}{M^2 - \bar{w}_0 f(z)}\right| \le \left|\frac{R(z-z_0)}{R^2 - \bar{z}_0 z}\right|$$

Proof. In the class, we showed that

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}$$

is automorphism between unit open disc. Let's construct g as

$$g(z) = \phi_{w_0/M} \circ \left(\frac{1}{M} f\right) \circ Rz \circ \phi_{-z_0/R}$$

For $z \in D(0,1)$, $\left|\frac{1}{M} f\right| \le 1$ and $|g(z)| \le 1$. Also, $g(0) = 0$. Therefore, using Schwarz lemma, we can know that

$$\begin{cases} |g(z)| \le |z| \\ |g'(0)| \le 1. \end{cases}$$

Therefore,

$$\begin{aligned} &\left|\phi_{w_0/M} \circ \left(\frac{1}{M} f\right) \circ Rz \circ \phi_{-z_0/R}\right| \le R|z| \\ \Rightarrow &\left|\phi_{w_0/M} \circ \left(\frac{1}{M} f\right)\right| \le R\left|\phi_{z_0/R}(z/R)\right| \\ \Rightarrow &\left|\frac{\frac{1}{M} f - \frac{w_0}{M}}{1 - \frac{\bar{w}_0}{M}\frac{f}{M}}\right| \le \left|\frac{z/R - z_0/R}{1 - \bar{z}_0 z/R^2}\right| \\ \Rightarrow &\left|\frac{M(f-w_0)}{M^2 - \bar{w}_0 f}\right| \le \left|\frac{R(z-z_0)}{R^2 - \bar{z}_0 z}\right| \end{aligned}$$

Therefore,

$$\left|\frac{M(f(z)-w_0)}{M^2 - \bar{w}_0 f(z)}\right| \le \left|\frac{R(z-z_0)}{R^2 - \bar{z}_0 z}\right|$$

$\square$

2. Describe all the automorphisms on upper half plane.(I'll denote upper half plane $\mathbb{P}$ and unit disc $\mathbb{D}$.)

Proof. I'll show that the automorphism of $\mathbb{P}$ is of form:

$$f(z) = \frac{az+b}{cz+d} \quad a,b,c,d \in \mathbb{R} \text{ and } ad-bc = 1.$$

Let f is in the automorphism of $\mathbb{P}$. I'll assume that

$$\phi(z) = \frac{z-i}{z+i}$$

is transformation from upper half plane to unit disc for now.(1) Then, $\phi \circ f(\mathbb{P}) = \phi(\mathbb{P}) = \mathbb{D}$ as a set-theoretic sense, and $\phi \circ f \circ \phi^{-1}$ is an automorphism of $\mathbb{D}$. We know that all the automorphism of $\mathbb{D}$ is of form:

$$e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. I'll modify this form by

$$\frac{pz+q}{\bar{q}z+\bar{p}} \quad \text{for } p, q \in \mathbb{C} \text{ and } |p|^2 - |q|^2 = 1.$$

I'll also assume this for now.(2) Then,

$$\begin{aligned}
f(z) &= \phi^{-1} \circ \left(\frac{pz+q}{\bar{q}z+\bar{p}}\right) \circ \phi \\
&= \left(\frac{-iz-i}{z-1}\right) \circ \left(\frac{p\frac{z-i}{z+i}+q}{\bar{q}\frac{z-i}{z+i}+\bar{p}}\right) \\
&= \left(\frac{-iz-i}{z-1}\right) \circ \left(\frac{(p+q)z+(-p+q)i}{(\bar{p}+\bar{q})z+(\bar{p}-\bar{q})i}\right) \\
&= \frac{(p+\bar{p}+q+\bar{q})iz/\sqrt{8}-(\bar{p}-\bar{q}-p+q)/\sqrt{8}}{-(p+q-\bar{p}-\bar{q})z/\sqrt{8}-(-p-\bar{p}+q+\bar{q})i/\sqrt{8}}.
\end{aligned}$$

We know that $(z+\bar{z})i, (z-\bar{z}) \in \mathbb{R}$, so each coefficient in numerator and denominator are real. Let's rewrite it as

$$\frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}$$

Also,

$$\begin{aligned}
8(ad-bc) &= (p+\bar{p}+q+\bar{q})(p+\bar{p}-q-\bar{q}) - (\bar{p}-p+\bar{q}-q)(p-\bar{p}-q+\bar{q}) \\
&= (p+\bar{p})^2 - (\bar{p}-p)^2 - (q+\bar{q})^2 + (-q+\bar{q})^2 = 8(|p|^2 - |q|^2) = 8
\end{aligned}$$

Conversely, let if f is such form, then

$$\mathrm{Im}(f) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{(ad-bc)y}{|cz+d|^2} > 0$$

for $z = x+iy$ and $y > 0$. Also, f is linear fractional transformation, so it is conformal mapping. Since it has inverse function on $\mathbb{P}$:

$$f^{-1}(z) = \frac{-dz+b}{cz-a}$$

which is also satisfies $-d, b, c, -a \in \mathbb{R}$, $ad - bc = 1$, the domain and codomain is $\mathbb{P}$ and bijective. Therefore, f is in the automorphism on $\mathbb{P}$, completing the proof.

(1): By geometric analysis, for any $z \in \mathbb{P}$,

$$\frac{|z-i|}{|z+i|} < 1$$

since the distance from i is always smaller than the distance from $-i$. Also, the real axis is mapped to the boundary of unit disc since $|z-i| = |z+i|$. Since any linear fractional is one to one mapping of the extended z plane to extended w plane,(Complex Variables and Applications, James Ward Brown and Ruel V. Churchill) $\frac{z-i}{z+i}$ maps $\mathbb{P}$ to $\mathbb{D}$.

(2): Let's start from

$$f(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}.$$

Let $k = 1 - |\alpha|^2 > 0$ and $a = e^{i\theta/2}/\sqrt{k}$, then

$$\begin{aligned}
f(z) &= e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z} \\
&= \frac{(e^{i\theta/2}/\sqrt{k})z - (e^{i\theta/2}/\sqrt{k})\alpha}{-(e^{-i\theta/2}/\sqrt{k})\bar{\alpha}z + (e^{-i\theta/2}/\sqrt{k})} \\
&= \frac{az+b}{\bar{b}+\bar{a}}
\end{aligned}$$

for $a = (e^{i\theta/2}/\sqrt{k})z$, $b = -(e^{i\theta/2}/\sqrt{k})\alpha$. Also, $|a|^2 - |b|^2 = \frac{1}{k^2} - \frac{|\alpha|^2}{k^2} = 1$. $\square$