

Complex Analysis - HW1

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1. Consider $\prod_{i=2}^{\infty} \left(1 + \frac{(-1)^i}{i}\right)$, then it converges since all the terms are nonzero, and

$$\prod_{i=2}^N \left(1 + \frac{(-1)^i}{i}\right) = \begin{cases} \frac{3}{2} \frac{2}{3} \cdots \frac{N+1}{N} \frac{N}{N+1} = 1 & \text{For odd } N \\ \frac{3}{2} \frac{2}{3} \cdots \frac{N+1}{N} = \frac{N+1}{N} & \text{For even } N \end{cases}$$

and both goes to 1 as $N \rightarrow \infty$.

However, it does not converges absolutely since

$$\prod_{i=2}^N \left(1 + \frac{1}{i}\right) = \frac{3}{2} \frac{4}{3} \cdots \frac{N+1}{N} = \frac{N+1}{2}$$

and it goes to infinity as $N \rightarrow \infty$.

2. I'll start from $n = 2$. For even $N \geq 4$,

$$\begin{aligned} 0 < \prod_{i=3}^N \left(1 + \frac{(-1)^i}{\sqrt{i}}\right) &= \left(1 - \frac{1}{\sqrt{3}}\right) \left(1 + \frac{1}{\sqrt{4}}\right) \cdots \left(1 - \frac{1}{\sqrt{N-1}}\right) \left(1 + \frac{1}{\sqrt{N}}\right) \\ &= \left(1 + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{12}}\right) \cdots \left(1 + \frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N-1}} - \frac{1}{\sqrt{N(N-1)}}\right) \\ &\leq \left(1 - \frac{1}{\sqrt{12}}\right) \cdots \left(1 - \frac{1}{\sqrt{N(N-1)}}\right) \\ &\leq \prod_{i=2}^{N/2} \left(1 - \frac{1}{2i}\right) \leq \exp\left(\sum_{i=2}^{N/2} -\frac{1}{2i}\right) \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$. For odd N , the partial product also goes to 0 since

$$\prod_{i=3}^N \left(1 + \frac{(-1)^i}{\sqrt{i}}\right) = \left(\prod_{i=3}^{N-1} \left(1 + \frac{(-1)^i}{\sqrt{i}}\right)\right) \left(1 - \frac{1}{\sqrt{N}}\right) \rightarrow 1 \cdot 0 = 0$$

as $N \rightarrow \infty$. Therefore, the product diverges to 0. However, $\sum_n \frac{(-1)^n}{\sqrt{n}}$ converges since it is alternating sequence such that the absolute value decreases and $\frac{(-1)^n}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$.

3. Let

$$a_n = \begin{cases} \frac{-1}{\sqrt{n}} & \text{for even } n \\ \frac{1}{\sqrt{n}} + \frac{1}{n} & \text{for odd } n. \end{cases}$$

For odd N ,

$$\begin{aligned}
1 + \sum_{i=1}^N a_i &= \left(\frac{1}{\sqrt{2}} + \frac{1}{2} \right) - \frac{1}{\sqrt{3}} + \cdots + \left(\frac{1}{\sqrt{N-1}} + \frac{1}{N-1} \right) - \frac{1}{\sqrt{N}} \\
&\geq \left(\frac{1}{\sqrt{3}} + \frac{1}{2} \right) - \frac{1}{\sqrt{3}} + \cdots + \left(\frac{1}{\sqrt{N}} + \frac{1}{N-1} \right) - \frac{1}{\sqrt{N}} \\
&= \sum_{i=1}^{(N-1)/2} \frac{1}{2i}
\end{aligned}$$

and RHS goes to infinity as $N \rightarrow \infty$. Therefore, $\sum_n a_n$ diverges.

Let $u_n = 1 + a_n$. Computing $\sum_{i=2}^N u_i$ for odd N ,

$$\begin{aligned}
\prod_{i=2}^N u_i &= \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} \right) \left(1 - \frac{1}{\sqrt{3}} \right) \cdots \left(1 + \frac{1}{\sqrt{N-1}} + \frac{1}{N-1} \right) \left(1 - \frac{1}{\sqrt{N}} \right) \\
&= \left(1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{2} - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{2}} + \frac{1}{2} \right) \right) \cdots \left(1 + \frac{1}{\sqrt{N-1}} - \frac{1}{\sqrt{N}} + \frac{1}{N-1} - \frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N-1}} + \frac{1}{N-1} \right) \right) \\
&= \prod_{i=1}^{(N-1)/2} \left(1 + \frac{1}{2i} - \frac{1}{2i+1} + \frac{1}{2i} - \frac{1}{\sqrt{2i+1}} \left(\frac{1}{\sqrt{2i}} + \frac{1}{2i} \right) \right).
\end{aligned}$$

Also,

$$\begin{aligned}
&\prod_{n=2}^N \left| \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} + \frac{1}{n-1} - \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n-1}} + \frac{1}{n-1} \right) \right| \\
&\leq \sum_{n=2}^N \left| \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right| + \left| \frac{1}{n-1} - \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n-1}} \right| + \left| \frac{1}{\sqrt{n}(n-1)} \right| \\
&\leq \sum_{n=2}^N \left| \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right| + \left| \frac{1}{n-1} - \frac{1}{n} \right| + \left| \frac{1}{\sqrt{n-1}^3} \right|
\end{aligned}$$

and it converges as $N \rightarrow \infty$ since $\sum_{i=1}^{\infty} \frac{1}{i^{3/2}}$ converges. For even N , it also converges to the same value since

$$\prod_{i=1}^N u_i = \left(\prod_{i=1}^{N-1} u_i \right) \left(1 + \frac{1}{\sqrt{N}} + \frac{1}{N} \right) \rightarrow \left(\prod_{i=1}^{\infty} u_i \right) \cdot 1$$

as $N \rightarrow \infty$. Therefore, $\prod_{i=2}^N u_n$ converges.

4. I'll prove a proposition.

Proposition 1. *Let U be an open set in \mathbb{C} . Suppose that f_j defined on U are holomorphic and $\sum_{j=1}^{\infty} |f_j|$ converges uniformly on any compact sets in U . Then,*

$$\Phi_n(z) = \prod_{j=1}^n (1 + f_j(z))$$

converges uniformly to $\Phi(z) = \prod_{i=1}^{\infty} (1 + f_i(z))$ on compact sets and the limit is holomorphic on U .

Proof. Fix a compact set $K \subset U$. Since $\sum_{j=1}^{\infty} |f_j|$ converges uniformly on K and $|f_j|$ are continuous function on the compact set, there exists $C > 0$ such that $\sum_{j=1}^{\infty} |f_j|$ is uniformly bounded about C and it means $\prod_{j=1}^{\infty} (1 + |f_j|)$ is uniformly bounded about e^C .

For any $\epsilon > 0$, there exists N_0 such that for any $N_2 > N_1 > N_0$,

$$\sum_{j=N_1}^{N_2} |f_j| \leq \epsilon$$

since it is uniformly converging sequence, and it implies

$$|\Phi_{N_2} - \Phi_{N_1}| \leq |\Phi_{N_1}| \left| \prod_{j=N_1+1}^{N_2} (1 + f_j(z)) - 1 \right| \leq \prod_{j=1}^{N_1} (1 + |f_j|) \left(\exp \left(\sum_{j=N_1}^{N_2} |f_j| \right) - 1 \right) \leq e^C (e^\epsilon - 1)$$

Therefore, Φ_n is uniformly cauchy and uniformly converges to Φ on K .

Finally, by the uniform boundedness of $\sum_{j=1}^{\infty} |f_j|$, the infinite product in Φ is well-defined. Also, Φ is defined on open set U and is holomorphic since it is uniform limit of holomorphic functions on compact sets. \square

Using the proposition, we can easily prove the problem. Since $\Phi_n \rightarrow \Phi(z)$ uniformly on a compact set C , we know that $\Phi'_n \rightarrow \Phi'$ uniformly on the set, and

$$\Phi'_n(z) = \sum_{k=1}^n f'_k \prod_{j \neq k}^n (1 + f_j)$$

Therefore,

$$\Phi'(z) = \lim_{n \rightarrow \infty} \Phi'_n(z) = \sum_{k=1}^{\infty} f'_k \prod_{j \neq k}^{\infty} (1 + f_j)$$