Complex Analysis - Mid Term

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Before starting, I'll prove Jordan's inequality, which will be used throughout this paper.

Lemma 1. *For $R > 0$,*

$$\int_0^\pi e^{-R\sin\theta} d\theta < \pi/R$$

Proof. Since $\sin\theta$ is a convex function in $0 < \theta < \pi/2$, $\frac{2}{\pi}\theta \leq \sin\theta$ in $0 \leq< \theta \leq \pi/2$. Therefore,

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq \int_0^{\pi/2} e^{-R\frac{2}{\pi}\theta} d\theta \leq \frac{\pi}{2R}\left(1 - e^{-R}\right) < \frac{\pi}{2R}$$

Since $\int_0^\pi e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta$,

$$\int_0^\pi e^{-R\sin\theta} d\theta < \frac{\pi}{R}$$

□

Problem 1

Let's separate each term in the integral and calculate. I'll use the Fig. (1) as a contour in the integral. Since there is no singularity in the contour for all $R > 1$, the integral is well defined for $R > 1$.

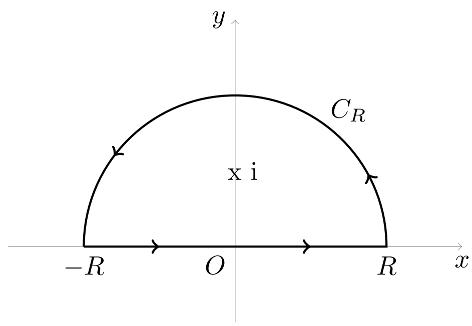

Figure 1: The contour used in problem 1. x represent a pole of $\frac{1}{z^2+1}$ in the domain enclosed by contour.

By Contour integral for fixed $R > 1$,

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+1} dx + \int_{C_R} \frac{e^{iz}}{z^2+1} dz = 2\pi i \operatorname{Res}_{z=i}\left(\frac{e^{iz}}{z^2+1}\right) = \pi e^{-1} \tag{1}$$

where x is real valued variable. By the same way,

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+1}dx + \int_{C_R} \frac{ze^{iz}}{z^2+1}dz = 2\pi i \operatorname{Res}_{z=i}\left(\frac{ze^{iz}}{z^2+1}\right) = i\pi e^{-1}. \tag{2}$$

We know that

$$\begin{aligned}
\operatorname{Re}\int_{-R}^{R} \frac{e^{ix}}{x^2+1}dx &= \int_{-R}^{R} \operatorname{Re}\frac{e^{ix}}{x^2+1}dx = \int_{-R}^{R} \frac{\cos(x)}{x^2+1}dx \quad \text{and} \\
\operatorname{Im}\int_{-R}^{R} \frac{xe^{ix}}{x^2+1}dx &= \int_{-R}^{R} \operatorname{Im}\frac{xe^{ix}}{x^2+1}dx = \int_{-R}^{R} \frac{x\sin(x)}{x^2+1}dx.
\end{aligned}$$

Therefore, if each $\int_{C_R}$ term in (1) and (2) converges to 0 as $R \to \infty$, we can get the desired result.

By the for remained term in (1),

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{z^2+1}dz\right| &= \left|\int_0^{\pi} \frac{e^{iRe^{i\theta}}}{(R^2e^{i2\theta}+1)}Rie^{i\theta}d\theta\right| \le \lim_{R\to\infty}\int_0^{\pi} R\left|\frac{e^{iRe^{i\theta}}}{(R^2e^{i2\theta}+1)}\right|d\theta \\
&\le \lim_{R\to\infty}\int_0^{\pi} R\left|\frac{e^{-R\sin\theta}}{(R^2-1)}\right|d\theta \le \lim_{R\to\infty}\int_0^{\pi} R\left|\frac{\pi/R}{(R^2-1)}\right|d\theta \text{ (Using Jordan's inequality)} \\
&= 0.
\end{aligned}$$

By the same reason, $\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{z^2+1}dz\right| = 0$ since z produce a R term, making $\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{z^2+1}dz\right| \le \lim_{R\to\infty}\left|\int_0^{\pi} \frac{\pi R}{R^2-1}\right|d\theta = 0.$

Therefore,

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\cos(x)+x\sin(x)}{x^2+1}dx = \frac{2\pi}{e}$$

Problem 2

I'll use the Fig. (2) as a contour in the integral.

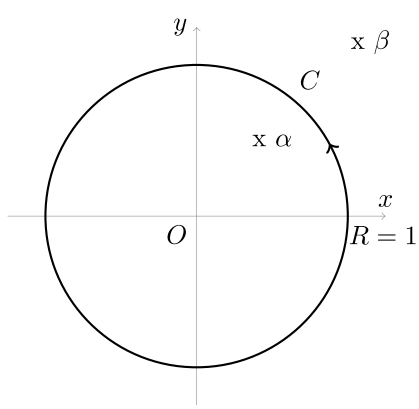

Figure 2: The contour used in Problem 2. x represent poles and α and β is some root of polynomial in the denominator.

Recognizing $\sin\theta$ as $\frac{e^{i\theta}-e^{-i\theta}}{2i}$, we can rewrite the integral by

$$\int_0^{2\pi} \frac{1}{(1+2a\sin\theta+a^2)^2}d\theta = \int_0^{2\pi} \frac{1}{\left(1+2a\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)+a^2\right)^2}d\theta$$
$$= -\frac{1}{a^2 i}\int_C \frac{z}{\left(z^2+\frac{i}{a}(1+a^2)z-1\right)^2}dz$$

where $z = e^{i\theta}$. Since $z^2+\frac{i}{a}(1+a^2)z-1 = 0$ has roots at $-i/a$ and $-ai$ and the C is S^1 in $\mathbb{C}$, only $-ai$ is inside C. Therefore,

$$\int_C \frac{z}{\left(z^2+\frac{i}{a}(1+a^2)z-1\right)^2}dz = 2\pi i\operatorname{Res}_{z=-ai}\frac{z}{\left(z^2+\frac{i}{a}(1+a^2)z-1\right)^2} = 2\pi i\left(\frac{z}{(z+i/a)^2}\right)'\Bigg|_{z=-ia} = -\frac{2\pi i}{a^2}\frac{a^2+1}{(-a^2+1)^3}$$

Therefore,

$$int_0^{2\pi}\frac{1}{(1+2a\sin\theta+a^2)^2}d\theta = \frac{2\pi(a^2+1)}{(-a^2+1)^3}.$$

Problem 3

Using real analysis, we first reformulate the formula.

$$\int_0^\infty \frac{\sin x}{x^{3/2}}dx = -\lim_{x\to\infty}\frac{2\sin x}{x^{1/2}} + \lim_{x\to 0+}\frac{2\sin x}{x^{1/2}} + 2\int_0^\infty \frac{\cos x}{x^{1/2}}dx = 4\int_0^\infty \cos t^2 dt$$

the last equality is derived by change x by t^2. The limit is calculated by

$$\lim_{x\to\infty}\left|\frac{2\sin x}{x^{1/2}}\right| \le \lim_{x\to\infty}\frac{2}{|x^{1/2}|} = 0$$
$$\lim_{x\to 0+}\frac{2\sin x}{x^{1/2}} = \lim_{x\to 0+}\frac{2\sin x}{x}\frac{x}{x^{1/2}} = 1\cdot 0 = 0.$$

I'll evaluate $\int_0^\infty \cos t^2 dt$. Consider a contour as in Fig. (3). There is no singularity of e^{iz^2} in the region enclosed by the contour. Therefore,

$$\int_0^R e^{ir^2}dr + \int_R^0 e^{i(re^{i\pi/4})^2}e^{i\pi/4}dr + \int_0^{\pi/4} e^{iR^2e^{2\theta i}}Rie^{i\theta}d\theta = 0$$

Since $\int_0^{\pi/2} e^{-R^2\sin(2\theta)} \le \frac{\pi}{2R^2}$ by Jordan's inequality, $\left|\int_0^{\pi/4} e^{iR^2e^{2\theta i}}Rie^{i\theta}d\theta\right| \to 0$ as $R\to\infty$. Also, $\int_R^0 e^{(re^{i\pi/4})^2}dr = \int_R^0 e^{i^2r^2}dr = -\int_0^R e^{-r^2}dr$ and the integral value goes to $-\sqrt{\pi}/2$ as $R\to\infty$. Therefore,

$$\int_0^\infty e^{ir^2}dr = -e^{i\pi/4}\int_\infty^0 e^{i(re^{i\pi/4})^2}dr = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

and $\int_0^\infty cos(x^2)dx = \operatorname{Re}\int_0^\infty e^{ix^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Consequently,

$$\int_0^\infty \frac{\sin x}{x^{3/2}}dx = \sqrt{2\pi}.$$

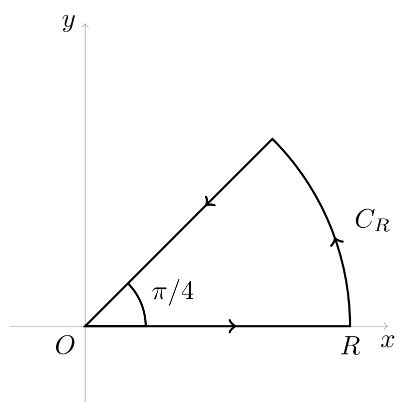

Figure 3: The contour used in problem 3. There is no pole of e^{ir^2} in the domain enclosed by the contour.