Complex Analysis - HW1

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1. Consider $\prod_{i=2}^{\infty}\left(1+\frac{(-1)^i}{i}\right)$, then it converges since all the terms are nonzero, and

$$\prod_{i=2}^{N}\left(1+\frac{(-1)^i}{i}\right)=\begin{cases}\frac{3}{2}\frac{2}{3}\cdots\frac{N+1}{N}\frac{N}{N+1}=1 & \text{For odd } N\\ \frac{3}{2}\frac{2}{3}\cdots\frac{N+1}{N}=\frac{N+1}{N} & \text{For even } N\end{cases}$$

and both goes to 1 as $N\to 0$.

However, it does not converges absolutely since

$$\prod_{i=2}^{N}\left(1+\frac{1}{i}\right)=\frac{3}{2}\frac{4}{3}\cdots\frac{N+1}{N}=\frac{N+1}{2}$$

and it goes to infinity as $N\to\infty$.

2. I'll start from $n=2$. For even $N\geq 4$,

$$\begin{aligned}0<\prod_{i=3}^{N}\left(1+\frac{(-1)^i}{\sqrt{i}}\right)&=\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\cdots\left(1-\frac{1}{\sqrt{N-1}}\right)\left(1+\frac{1}{\sqrt{N}}\right)\\&=\left(1+\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{12}}\right)\cdots\left(1+\frac{1}{\sqrt{N}}-\frac{1}{\sqrt{N-1}}-\frac{1}{\sqrt{N(N-1)}}\right)\\&\leq\left(1-\frac{1}{\sqrt{12}}\right)\cdots\left(1-\frac{1}{\sqrt{N(N-1)}}\right)\\&\leq\prod_{i=2}^{N/2}\left(1-\frac{1}{2i}\right)\leq\exp\left(\sum_{i=2}^{N/2}-\frac{1}{2i}\right)\to 0\end{aligned}$$

as $N\to\infty$. For odd N, the partial product also goes to 0 since

$$\prod_{i=3}^{N}\left(1+\frac{(-1)^i}{\sqrt{i}}\right)=\left(\prod_{i=3}^{N-1}\left(1+\frac{(-1)^i}{\sqrt{i}}\right)\right)\left(1-\frac{1}{\sqrt{N}}\right)\to 1\cdot 0=0$$

as $N\to\infty$. Therefore, the product diverges to 0. However, $\sum_n\frac{(-1)^n}{\sqrt{n}}$ converges since it is alternating sequence such that the absolute value decreases and $\frac{(-1)^n}{\sqrt{n}}\to 0$ as $n\to\infty$.

3. Let

$$a_n=\begin{cases}\frac{-1}{\sqrt{n}} & \text{for even } n\\ \frac{1}{\sqrt{n}}+\frac{1}{n} & \text{for odd} n.\end{cases}$$

For odd N,

$$\begin{aligned}
1+\sum_{i=1}^{N} a_i &= \left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)-\frac{1}{\sqrt{3}}+\cdots+\left(\frac{1}{\sqrt{N-1}}+\frac{1}{N-1}\right)-\frac{1}{\sqrt{N}} \\
&\geq \left(\frac{1}{\sqrt{3}}+\frac{1}{2}\right)-\frac{1}{\sqrt{3}}+\cdots+\left(\frac{1}{\sqrt{N}}+\frac{1}{N-1}\right)-\frac{1}{\sqrt{N}} \\
&= \sum_{i=1}^{(N-1)/2} \frac{1}{2N}
\end{aligned}$$

and RHS goes to infinity as $N \to \infty$. Therefore, $\sum_n a_n$ diverges.

Let $u_n = 1 + a_n$. Computing $\sum_{i=2}^{N} u_n$ for odd N,

$$\begin{aligned}
\prod_{i=2}^{N} u_n &= \left(1+\frac{1}{\sqrt{2}}+\frac{1}{2}\right)\left(1-\frac{1}{\sqrt{3}}\right)\cdots\left(1+\frac{1}{\sqrt{N-1}}+\frac{1}{N-1}\right)\left(1-\frac{1}{\sqrt{N}}\right) \\
&= \left(1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{2}-\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)\right)\cdots\left(1+\frac{1}{\sqrt{N-1}}-\frac{1}{\sqrt{N}}+\frac{1}{N-1}-\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{N-1}}+\frac{1}{N-1}\right)\right) \\
&= \prod_{i=1}^{(N-1)/2}\left(1+\frac{1}{2i}-\frac{1}{2i+1}+\frac{1}{2i}-\frac{1}{\sqrt{2i+1}}\left(\frac{1}{\sqrt{2i}}+\frac{1}{2i}\right)\right).
\end{aligned}$$

Also,

$$\begin{aligned}
&\prod_{n=2}^{N}\left|\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}+\frac{1}{n-1}-\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n-1}}+\frac{1}{n-1}\right)\right| \\
&\quad\leq \sum_{n=2}^{N}\left|\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right|+\left|\frac{1}{n-1}-\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n-1}}\right|+\left|\frac{1}{\sqrt{n}(n-1)}\right| \\
&\quad\leq \sum_{n=2}^{N}\left|\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right|+\left|\frac{1}{n-1}-\frac{1}{n}\right|+\left|\frac{1}{\sqrt{n-1}^{3}}\right|
\end{aligned}$$

and it converges as $N \to \infty$ since $\sum_{i=1}^{\infty} \frac{1}{i^{3/2}}$ converges. Therefore, $\prod_{i=2}^{N} u_n$ converges.

4. I'll prove the proposition.

Proposition 1. *Let U be an open set in $\mathbb{C}$. Suppose that f_j defined on U are holomorphic and $\sum_{j=1}^{\infty} |f_j|$ converges uniformly on any compact sets in U. Then,*

$$\Phi_n(z) = \prod_{j=1}^{n}(1 + f_j(z))$$

converges uniformly to $\Phi(z) = \prod_{i=1}^{\infty}(1 + f_j(z))$ on compact sets and the limit is holomorphic.

Proof. Fix a compact set $K \subset U$. Since $\sum_{j=1}^{\infty} |f_j|$ converges uniformly on K and $|f_j|$ are continuous function on the compact set, there exists $C > 0$ such that $\sum_{j=1}^{\infty} |f_j|$ is uniformly bounded about C and it means $\prod_{j=1}^{\infty}(1 + |f_j|)$ is uniformly bounded about e^C.

For any $\epsilon > 0$, there exists N_0 such that for any $N_2 > N_1 > N_0$,

$$\sum_{j=N_1}^{N_2} |f_j| \leq \epsilon$$

since it is uniformly converging sequence, and it implies

$$|\Phi_{N_2} - \Phi_{N_1}| \leq |\Phi_{N_1}| \left| \prod_{j=N_1+1}^{N_2} (1 + f_j(z)) - 1 \right| \leq \prod_{j=1}^{N_1} (1 + |f_j|) \left(\exp\left(\sum_{j=N_1}^{N_2} |f_j| \right) - 1 \right) \leq e^C (e^\epsilon - 1)$$

Therefore, Φ_n is uniformly cauchy in $\mathbb{C}$ and uniformly converges to Φ.

Finally, by the uniform boundedness of $\sum_{j=1}^{\infty} |f_j|$, the infinite product in Φ is well-defined. Also, Φ is defined on open set U and is holomorphic since it is uniform limit of holomorphic functions. $\square$

Using the proposition, we can easily prove the problem. Since $\Phi_n \to \Phi(z)$ uniformly on a compact set C, we know that $\Phi_n' \to \Phi'$ uniformly on the set, and

$$\Phi_n'(z) = \sum_{k=1}^{n} f_k' \prod_{j \neq k}^{n} (1 + f_j)$$

Therefore,

$$\Phi'(z) = \lim_{n \to \infty} \Phi_n'(z) = \sum_{k=1}^{\infty} f_k' \prod_{j \neq k}^{\infty} (1 + f_j)$$