Complex Analysis - HW2

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1. Define
$$F(s) = \int_0^\infty t^{s-1}e^{-t}dt$$
defined for Re $s > 0$. Prove that
$$\frac{F(s+n+1)}{n!n^s} \to 1$$
for $0 < s \leq 1$ as $n \to \infty$.

Proof. Since $0 < s \leq 1$, $t^s \leq n^s$ and $t^{s-1} \geq n^{s-1}$ iff $0 \leq t \leq n$. Therefore,
$$n^{s-1}\int_0^n t^{n+1}e^{-t}dt + n^s\int_n^\infty t^n e^{-t}dt \leq F(s+n+1) \leq n^s\int_0^n t^n e^{-t}dt + n^{s-1}\int_n^\infty t^{n+1}e^{-t}dt.$$
By integration by parts,
$$n^{s-1}\int_0^n t^{n+1}e^{-t}dt + n^s\int_n^\infty t^n e^{-t}dt = n^{s-1}(n+1)\int_0^n t^n e^{-t}dt - e^{-n}n^{n+s} + n^s\int_n^\infty t^n e^{-t}dt$$
$$n^s\int_0^n t^n e^{-t}dt + n^{s-1}\int_n^\infty t^{n+1}e^{-t}dt = n^s\int_0^n t^n e^{-t}dt + n^{s-1}(n+1)\int_n^\infty t^n e^{-t}dt + n^{n+s}e^{-n}.$$
Therefore,
$$n^{s-1}\int_0^n t^n e^{-t}dt + n^s\int_0^\infty t^n e^{-t}dt - n^{n+s}e^{-n} \leq F(s+n+1) \leq n^s\int_0^\infty t^n e^{-t}dt + n^{s-1}\int_n^\infty t^n e^{-t}dt + n^{n+s}e^{-n}.$$
Since $\int_0^n t^n e^{-t} \leq n!$ and $\int_n^\infty t^n e^{-t}dt \leq n!$, as $n \to \infty$,
$$1 - \frac{n^n e^{-n}}{n!} \leq \frac{F(s+n+1)}{n^s n!} \leq 1 + \frac{n^n e^{-n}}{n!}.$$
Using Taylor series of e^n,
$$\frac{e^n n!}{n^n} \geq \frac{n!}{n^n}\left(\sum_{i=n}^{2n}\frac{1}{i!}i^i\right) \geq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$
for large enough n, and it means that $\frac{e^{-n}n^n}{n!} \to 0$ as $n \to \infty$. Therefore,
$$\frac{F(s+n+1)}{n^s n!} \to 1$$
as $n \to \infty$. The above inequality is established by
$$\frac{(n+k)!}{n!} \leq (\sqrt[k]{k}n)^k = kn^k$$
for large enough n and $n \leq k \leq 2n$. □

2. Show that

$$\left|\Gamma\left(\frac{1}{2}+it\right)\right| = \sqrt{\frac{2\pi}{e^t+e^{-t}}}$$

Proof. Since

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

for all s,

$$\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right) = \frac{\pi}{\sin\left(\pi(\frac{1}{2}+it)\right)} = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

for all $t \in \mathbb{R}$. As the real part is positive,

$$\overline{\Gamma\left(\frac{1}{2}+it\right)} = \overline{\int_0^\infty e^{-u}u^{-\frac{1}{2}+it}du} = \int_0^\infty e^{-u}u^{-\frac{1}{2}-it}du = \Gamma\left(\frac{1}{2}-it\right).$$

Therefore,

$$\left|\Gamma\left(\frac{1}{2}+it\right)\right| = \sqrt{\frac{2\pi}{e^t+e^{-t}}}$$

□

3. Calculate $\int_0^\infty t^{s-1}\cos t dt$ and $\int_0^\infty t^{s-1}\sin t dt$ for $0 < \text{Re } s < 1$.

Proof. Take a contour as Fig. 1, then the contour integral for $z^{s-1}e^{-z}$ is

$$\int_\epsilon^R t^{s-1}e^{-t}dt + \int_{C_R} z^{s-1}e^{-z}dz + \int_R^\epsilon (it)^{s-1}e^{-it}i\ dt + \int_{C_\epsilon} z^{s-1}e^{-z}dz = 0$$

since there is no pole in the interior of the contour. As $\epsilon \to 0$,

$$\left|\int_{C_\epsilon} z^{s-1}e^{-z}dz\right| \le \epsilon^{s-1}\epsilon\frac{\pi}{2} = \epsilon^s\frac{\pi}{2} \to 0$$

As $R \to \infty$,

$$\left|\int_{C_R} z^{s-1}e^{-z}dz\right| \le R^s\int_0^{\frac{\pi}{2}} e^{-R\cos\theta}d\theta \le R^s\int_0^{\frac{\pi}{2}} e^{R(\theta-\pi/2)}d\theta \le R^{s-1} \to 0.$$

The inequality is established by $\cos\theta \ge \pi/2-\theta$ for $0 \le \theta \le \pi/2$. Therefore,

$$\int_0^\infty t^{s-1}e^{-t}dt = i^s\int_0^\infty t^{s-1}e^{-it}dt$$

and $i^{-s}\Gamma(s) = \int_0^\infty t^{s-1}e^{-it}dt$.
Hence,

$$\text{Re}\int_0^\infty t^{s-1}e^{-it}\ dt = \int_0^\infty t^{s-1}\cos t\ dt = \text{Re } e^{-\pi i s/2}\Gamma(s) = \cos(\pi s/2)\Gamma(s)$$
$$\text{Im}\int_0^\infty t^{s-1}e^{-it}\ dt = -\int_0^\infty t^{s-1}\sin t\ dt = -\text{Im } e^{-\pi i s/2}\Gamma(s) = \sin(\pi s/2)\Gamma(s)$$

□

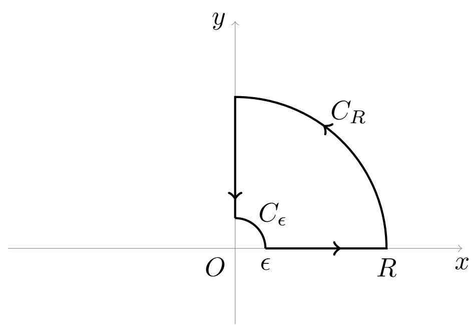

Figure 1: The contour used in problem 3.