Complex Analysis - HW5

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December 5, 2018

1. Let f be a holomorphic function on $R_1 < |z| < R_2$ and continuous on $R_1 \le |z| \le R_2$. Let $M(r) = \max\limits_{|z|=r} |f(z)|$. Then,

$$M(r) \le M(R_1)^{\alpha} M(R_2)^{1-\alpha}, \quad \alpha = \log(R_2/r)/\log(R_2/R_1)$$

Proof. Consider $g(z) = z^{\lambda} f(z)$, $\lambda \in \mathbb{R}$ and assume that this is not a constant function.(Unless, the theorem is trivial since the second paragraph directly applies to f.) Then, it is holomorphic on $0 < R_1 < |z| < R_2$ except a branch cut. By Maximum modulus principle, $\left|z^{\lambda} f\right| = |z|^{\lambda} |f|$ can not have maximum value inside the region without branch cut since g is not a constant function. Since the $|g|$ is independent from the setting of branch cut and continuous on the closure of the region, which is compact, $|g|$ have maximum on the boundary of the region: $|z| = R_1$ or R_2. Let each maximum values of $|g|$: $R_1^{\lambda} M(R_1)$, $R_2^{\lambda} M(R_2)$.

We can fix λ to satisfy $R_1^{\lambda} M(R_1) = R_2^{\lambda} M(R_2)$. Then, the λ is $\log(M(R_2)/M(R_1))/\log(R_1/R_2)$. By Maximum modulus principle, we know that $r^{\lambda} M(r) \le R_1^{\lambda} M(R_1) = R_2^{\lambda} M(R_2)$ for all $R_1 \le r \le R_2$. Rewriting the inequality,

$$\begin{aligned}
M(r) &\le \left(\frac{R_2}{r}\right)^{\lambda} M(R_2) \\
&= \left(\frac{R_2}{r}\right)^{\log(M(R_2)/M(R_1))/\log(R_1/R_2)} M(R_2) \\
&= \left(\frac{R_2}{r}\right)^{\log_{R_1/R_2}(M(R_2)/M(R_1))} M(R_2) \\
&= \left(\frac{M(R_2)}{M(R_1)}\right)^{\log_{R_1/R_2}(R_2/r)} M(R_2) \\
&= \left(\frac{M(R_2)}{M(R_1)}\right)^{-\log(R_2/r)/\log(R_2/R_1)} M(R_2) \\
&= M(R_1)^{\alpha} M(R_2)^{1-\alpha}
\end{aligned}$$

for $\alpha = \log(R_2/r)/\log(R_2/R_1)$. ∎