Complex Analysis - HW3

SungBin Park, 20150462

November 28, 2018

1. If f is holomorphic on the disk $D(0,R)$, $|f| \le M$ on the disc, and $f(z_0) = w_0$, then f satisfies

$$\begin{cases} \left|\frac{M(f(z)-w_0)}{M^2-\bar{w}_0 f(z)}\right| \le \left|\frac{z-z_0}{1-\bar{z}_0 z}\right| \\ |f'(z)| \le \frac{1-|w_0|^2}{1-z_0^2} \end{cases}$$

Proof. In the class, we showed that

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}$$

is automorphism between unit open disc. For open disc $D(0,R)$, we can set the automorphism:

$$\phi_a(z) = R\frac{z/R - a/R}{1-\bar{a}z/R^2} = \frac{z-a}{1-\bar{a}z/R^2}$$

... Let's construct g as

$$g(z) = \frac{1}{R}\left(\phi_{\frac{R}{M}w_0} \circ \left(\frac{R}{M}f\right) \circ \phi_{-z_0}\right)$$

For $z \in D(0,R)$, $\left|\frac{R}{M}f\right| \le R$ and $|g(z)| \le 1$. Also, $g(0) = 0$. Therefore, using Schwarz lemma, we can know that

$$\begin{cases} |g(z)| \le |z| \\ |g'(0)| \le 1. \end{cases}$$

Therefore,

$$\begin{aligned} &\left|\phi_{\frac{R}{M}w_0} \circ \left(\frac{R}{M}f\right) \circ \phi_{-z_0}\right| \le R|z| \\ \Rightarrow &\left|\phi_{\frac{R}{M}w_0} \circ \left(\frac{R}{M}f\right)\right| \le R|\phi_{z_0}(z)| \\ \Rightarrow &\left|\frac{\frac{R}{M}f - \frac{R}{M}w_0}{1-\frac{R}{M}\bar{w}_0\frac{R}{M}f/R^2}\right| \le \left|R\frac{z-z_0}{1-\bar{z}_0 z/R^2}\right| \\ \Rightarrow &\left|\frac{M(f-w_0)}{M^2-\bar{w}_0 f}\right| \le \left|\frac{z-z_0}{R^2-\bar{z}_0 z}\right| \end{aligned}$$

Therefore,

$$\left|\frac{M(f(z)-w_0)}{M^2-\bar{w}_0 f(z)}\right| \le \left|\frac{z-z_0}{R^2-\bar{z}_0 z}\right|$$

□

2. Describe all the automorphisms on upper half plane.

Proof. I'll figure out the mappings between upper half plane and open unit disc. □