Complex Analysis - HW6

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1. Suppose v is a continuous real-valued function. Prove that v is subharmonic if and only if $\triangle v \geq 0$.

Proof. ($\Leftarrow$) Assume v is subharmonic, but $\triangle v < 0$ for some point $x_0 + y_0 i$. Since $v \in C^2$, $\triangle v$ is continuous function, so there exists $R > 0$ such that $\triangle v < 0$ in $D(x_0 + y_0 i, r)$, which is the disc centered at $x_0 + y_0 i$ with radius $r \leq R$. I'll denote the disc $D(r)$ for $r \leq R$. Let

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + y_0 i + re^{i\theta}) d\theta,$$

then the integral is well defined since v is continuous. Since $v \in C^1$, we can interchange ∂_r and $\int$, in other words,

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{1}{2\pi} \frac{\partial}{\partial r} \int_0^{2\pi} v(x_0 + y_0 i + re^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial r}(x_0 + y_0 i + re^{i\theta}) d\theta
\end{aligned}$$

Since v is real-valued function, we can regard the domain of v to be a subset of $\mathbb{R}^2$ and $v(x_0 + y_0 i + re^{i\theta})$ as $v(x_0 + r\cos\theta, y_0 + r\sin\theta)$. Then, $\frac{\partial v}{\partial r} = \nabla v \cdot \hat{r}$ and by stokes' theorem,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial r}(x_0 + y_0 i + re^{i\theta}) d\theta &= \frac{1}{2\pi r} \int_{\partial D(r)} \nabla v \cdot \hat{r} dS \\
&= \frac{1}{2\pi r} \int_{D(r)} \triangle v dx < 0
\end{aligned}$$

and it means $f(r)$ is strictly decreasing function for $r \leq R$. Thus,

$$f(x_0 + y_0 i) > \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial r}(x_0 + y_0 i + re^{i\theta}) d\theta$$

for $0 < r < R$ since

$$f(x_0 + y_0 i) = \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial r}(x_0 + y_0 i + re^{i\theta}) d\theta$$

by the Lebesgue differentiation theorem. It is contradiction. Therefore, $\triangle v \geq 0$.

($\Rightarrow$) Let Ω be a region such that v is defined on. Assume $\triangle v \geq 0$, then for any $\epsilon > 0$, $\triangle(v + \epsilon x^2) = \triangle v + 2\epsilon > 0$. It implies that $v + \epsilon x^2$ can not have local maximum since $\frac{\partial^2}{\partial x^2}(v + \epsilon x^2)$ or $\frac{\partial^2}{\partial y^2}(v + \epsilon x^2)$ is positive. For any harmonic function u which is defined on $\Omega' \subset \Omega$, then $\triangle(v + \epsilon x^2 - u) = \triangle(v + \epsilon x^2) - \triangle u > 0$, so it satisfies maximum principle in Ω' and it means $v + \epsilon x^2$ is subharmonic function on Ω. Since v is limit of subharmonic functions by letting $\epsilon \to 0$, v is subharmonic.(1)

(1): Suppose v is not subharmonic, so there exists a harmonic function u such that $v-u$ has a local maximum inside the region Ω', which is the domain of u. Let the point p. Take an small enough open disc D centered at the local maximum with radius $r>0$. Since p is local maximum, $0<\delta=(v-u)(p)-\max_{z\in\partial D}(v-u)(z)$, and take $\epsilon=\frac{\delta}{8(\|p\|^2+r^2)}$, where $\|p\|$ is the distance between p and 0 in $\mathbb{C}$. Then, $0<(v+\epsilon x^2-u)(p)-\max_{z\in\partial D}(v+\epsilon x^2-u)(z)$ since ϵ is small enough, and there still exists local maximum of $(v+\epsilon x^2-u)$ in D, which is contradiction since $(v+\epsilon x^2-u)$ does not have local maximum in the Ω'. Therefore, v is subharmonic.

$\square$

2. Show that the solution of the Dirichlet problem for upper half plane, i.e.

$$\begin{cases} \triangle u \equiv 0 & \text{in } \mathbb{H}^+ \\ u = f & \text{on } \partial\mathbb{H}^+ \end{cases}$$

is

$$u(x+iy)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(t-x)^2+y^2}f(t)dt$$

Proof. In the class, we showed that the solution for Dirichlet problem for unit disc D, i.e.

$$\begin{cases} \triangle v \equiv 0 & \text{in } D \\ v = g & \text{on } \partial D \end{cases}$$

is

$$v(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}g(e^{i\theta})d\theta.$$

For after, I'll set $g(e^{i\theta})=f(-\cot(\theta/2))$, which is defined on ∂D a.e. and this will be clear as the proof goes on. Using this solution, I'll find the solution for original problem.
In the previous HW, I showed that

$$h(z)=\frac{z-i}{z+i}$$

is a conformal transform from $\mathbb{H}^+$ to D and $\partial\mathbb{H}^+$ to D without $z=1$. Since v is harmonic on D, $v\circ h$ is harmonic on $\mathbb{H}^+$. I'll denote $u=v\circ h$ and show that $u=f$ on $\partial\mathbb{H}^+$.(1) Evaluating $u(z)$,

$$\begin{aligned} u(z)=v(h(z))&=v\left(\frac{z-i}{z+i}\right)\\ &=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\left|\frac{z-i}{z+i}\right|^2}{\left|e^{i\theta}-\frac{z-i}{z+i}\right|^2}g(e^{i\theta})d\theta\\ &=\frac{1}{2\pi}\int_0^{2\pi}\frac{|z+i|^2-|z-i|^2}{|(z+i)e^{i\theta}-(z-i)|^2}g(e^{i\theta})d\theta. \end{aligned}$$

By h, $e^{i\theta}$ maps to $-\cot(\theta/2)$, and this is why I set $g=f\circ h^{-1}$ on ∂D by setting $g(e^{i\theta})=f(-\cot(\theta/2))$.(If we consider the value at boundary, $\lim_{z\to-\cot(\theta_0/2)}u(z)=\lim_{z\to-\cot(\theta_0/2)}v\circ h(z)=\lim_{t\to e^{i\theta_0}}v(t)=g(e^{i\theta_0})=f(-\cot(\theta/2))$, where $0<\theta_0<2\pi$, since h is conformal mapping.)

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|z+i|^2-|z-i|^2}{|(z+i)e^{i\theta}-(z-i)|^2}g(e^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{|z+i|^2-|z-i|^2}{|(z+i)e^{i\theta}-(z-i)|^2}f\left(-\cot\left(\theta/2\right)\right)d\theta$$

Let $t = -\cot(\theta/2)$, then $e^{i\theta} = \frac{t-i}{t+i}$ and $dt = \frac{1}{2}(\cot^2(\theta/2)+1)d\theta$.

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|z+i|^2 - |z-i|^2}{|(z+i)e^{i\theta} - (z-i)|^2} f\left(\cot\left(\theta/2\right)\right) d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|z+i|^2 - |z-i|^2}{\left|(z+i)\frac{t-i}{t+i} - (z-i)\right|^2} f(t) \left(\frac{2}{t^2+1}\right) dt.$$

Denoting $z = x+iy$,

$$\begin{aligned}
u(x+iy) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |t+i|^2 \frac{|x+(y+1)i|^2 - |x+(y-1)i|^2}{|(x+(y+1)i)(t-i) - (x+(y-1)i)(t+i)|^2} f(t) \left(\frac{2}{t^2+1}\right) dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4y}{4(t-x)^2 + 4y^2} f(t) 2dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(t-x)^2 + y^2} f(t) dt.
\end{aligned}$$

Therefore, it is a solution of the Dirichlet problem for upper half plane.

I need to show that this is the unique solution. Assume there is two solutions u_1 and u_2, then $u_1 - u_2$ is a solution for

$$\begin{cases} \triangle u \equiv 0 & \text{in } \mathbb{H}^+ \\ u = 0 & \text{on } \partial\mathbb{H}^+ \end{cases}$$

By strong maximum principle for harmonic function, $u_1 - u_2 \equiv 0$ in $\mathbb{H}^+$ and it means $u_1 = u_2$. It proves the uniqueness of the solution for Dirichlet problem for upper half plane.

(1): Let $u : U \to \mathbb{R}$ is harmonic and $h : V \to U$ is holomorphic, then there exists harmonic conjugate $u+iv$ which is holomorphic on U, and $(u+iv)\circ h$ is holomorphic on V. Therefore, $\mathrm{Re}(u+iv)\circ h = u \circ h$ is harmonic. $\square$