# Partial Differential Equation - HW 4

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## Problem 1

I'll use some theorem from topology and functional analysis, and I'll follow the proof in Real Analysis, Gerland B. Folland.

**Theorem 1.** *(The Baire Category Theorem) Let $X$ be a complete metric space. Then,*

*(a) If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of $X$, i,e, $\overline{\cup_1^\infty U_n} = X$, then $\cap_1^\infty U_n$ is dense in $X$.*

*(b) $X$ is not a countable union of nowhere dense sets.*

*Proof.* For (a), I'll show that nonempty open set $V$ in $X$ have intersection with $\cap_1^\infty U_n$ using induction. Since $U_1$ is open dense subset of $X$, $U_1 \cap W$ is open and nonempty, so there exists $B(x_0, r_0) \subset U_1 \cap W$ such that $0 < r_0 < 1$. For $n > 0$, choose $x_n$ and $r_n$ by follows: assume that for $j < n$, $x_j$ and $r_j$ are chosen. Then, $B(x_{n-1}, r_{n-1}) \cap U_n$ is nonempty and open. Choose $x_n \in B(x_{n-1}, r_{n-1}) \cap U_n$ and choose $0 < r_n < 2^{-n}$ such that $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$. For $n, m \geq N$, $x_n, x_m \in B(x_N, r_N)$, and $r_N \to 0$ as $N \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in $X$ and have limit point $x \in X$. Since $x_n \in B(r_N, x_N)$ for all $n \geq N$, $x \in \overline{B(x_N, r_N)} \subset U_N \cap B(x_0, r_0) \subset U_N \cap W$ for all $N$, implying $(\cap_{n=1}^\infty U_n) \cap W \neq \phi$.

For (b), If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense subsets in $X$, then $\{\overline{E_n}^c\}$ are a open dense sets since for fixed $n$ and for any $x \in \overline{E}$, $B(x, r) \cap \overline{E}^c \neq \phi$ for all $r > 0$. Since $\cap \left(\overline{E_n}\right)^c \neq \phi$, $\cup E_n \subset \cup \overline{E_n} \neq X$. $\square$

**Theorem 2.** *(Open Mapping Theorem) Let $X$ and $Y$ be Banach spaces. If $T$ is a surjective bounded linear functional from $X$ to $Y$, $T$ is a open map.*

*Proof.* If $T(B(0,1))$ contains a ball of radius $r > 0$, $T$ is open map since for any open set $x \in U$ in $X$, it contains open ball $B(x, r)$, $r > 0$ and $T(B(x, r)) = T(x) + T(B(0, r)) = T(x) + rT(B(0, 1))$, which is open set in $Y$ centered at $T(x)$. Therefore, I'll show that $T(B(0, 1))$ contains a ball of radius $r > 0$. I'll denote $B(0, r) = B_r$.

Since $X = \cup_{n=1}^\infty B_n$ and $T$ is surjective, $Y = T(\cup_{n=1}^\infty B_n)$. If $T(B_1)$ is nowhere dense, $T(B_n)$ are nowhere dense since dialation in $Y$ induces homeomorphism. However, this is impossible by the Baire Category Theorem since $Y$ is complete. Therefore, $\overline{T(B_1)}$ contains an open ball in $Y$ and let it $B(y_0, 4r)$ for some $y_0 \in Y$, $r > 0$. Choose $y_1 = Tx_1 \in T(B_1)$ such that $\|y_1 - y_0\| \leq 2r$; then $B(y_1, 2r) \subset B(y_0, 4r) \subset \overline{T(B_1)}$, so if $\|y\| < 2r$,

$$y = -Tx_1 + (y_1 + y) \in \overline{T(-x_1 + B_1)} \subset \overline{T(B_2)}.$$

Dividing both by 2, we get if $\|y\| < r$, $y \in \overline{T(B_1)}$. I need to change $\overline{T(B_1)}$ to $T(B_1)$.

As noted above, dialation induces homeomorphism, so if $\|y\| < r/2^n$, $y \in \overline{T(B_{1/2^n})}$. Suppose that $y \in r/2$, then there exists $x_1 \in B_{1/2}$ such that $\|y - Tx_1\| \leq r/4$. Proceeding this argument, we can find $x_n$ satisfying $\left\| y - T \sum_{i=1}^n x_i \right\| \leq r/2^{n+1}$. Since $X$ is complete and $\left\| \sum_{i=n}^m x_i \right\| \leq 2^{-n+1}$, it has convergent point $x$ and $\|x\| \leq 1$ and $Tx = y$ since $\|Tx - y\| = 0$ in $Y$. Therefore, $T(B_1)$ contains all $y$ in $\|y\| < r/2$. $\square$

**Theorem 3.** *(Closed Graph Theorem) If $X$ and $Y$ are Banach spaces and $T : X \to Y$ is a closed linear map, then $T$ is bounded.*

*Proof.* Let $\pi_1$ and $\pi_2$ be the projection of $\Gamma(T) := \{(x, Tx) \in X \times Y\}$ to $X$ and $Y$, then these maps are onto. It is clear that $\pi_1$ and $\pi_2$ are bounded linear functional. ALso, $X \times Y$ and $\Gamma(T)$ is complete since $X$ and $Y$ are complete and $\Gamma(T)$ is closed as $T$ is closed. Since $\pi_1$ is a bijection between $\Gamma(T)$ and $X$, $\pi_1^{-1}$ is continuous since $\pi_1$ bounded and continuous. Thus, $T = \pi_2 \circ \pi_1^{-1}$ is bounded. $\square$

If $\eta \in \rho(A)$, $A - \eta\mathrm{Id}$ is one-to-one and onto by definition. Therefore, we can consider the inverse $(A - \eta\mathrm{Id})^{-1}$. I'll first show that this is closed linear map.

Linearity: Fix $y_1, y_2 \in X$ and $r \in \mathbb{R}$ or $\mathbb{C}$. Then, there uniquely exists $x_1$ and $x_2$ in $X$ such that $(A - \eta\mathrm{Id})(x_i) = y_i$ for each $i$ and $(A - \eta\mathrm{Id})(rx_i) = ry_i$. Therefore, $(A - \eta\mathrm{Id})^{-1}(y_i) = x_i$ and $(A - \eta\mathrm{Id})^{-1}(y_1 + ry_2) = x_1 + rx_2$. Thus, $(A - \eta\mathrm{Id})$ is bijective and linear.

Closedness: Suppose $y_n \to y$ in $X$ and $(A - \eta\mathrm{Id})^{-1}(y_n) \to x$. We need to show that $(A - \eta\mathrm{Id})^{-1}(y) = x$. The second assumption implies that $y_n \to (A - \eta\mathrm{Id})(x)$ since $(A - \eta\mathrm{Id})$ is continuous. (Note that boundedness implies continuity if $X$ is normed vector space and the map is linear.) The normed topology in Banach space gives Hausdorff property: if $x_1 \neq x_2$, $r = \|x_1 - x_2\| > 0$ and $B(x_1, r/3)$, $B(x_2, r/3)$ gives disjoint neighborhoods of each $x_i$. Therefore, $y_n$ converges to same point and $(A - \eta\mathrm{Id})(x) = y$.

Since $X$ is Banach space and $(A - \eta\mathrm{Id})^{-1} : X \to X$ is closed linear map, it is bounded by the Closed Graph Theorem.

# Problem 2

(a) I'll check the condition for inner product.

(1) For all $u, u_* \in \mathcal{H}$,

$$\begin{aligned}
\overline{(u_*, u)_{\mathcal{H}}} &= \overline{\int_\Omega (v_* + iw_*)(v - iw) + (Dv_* + iDw_*)(Dv - iDw)dx} \\
&= \int_\Omega (v_* - iw_*)(v + iw) + (Dv_* - iDw_*)(Dv + iDw)dx \quad \text{since } v, w, v_*, w_* : X \to \mathbb{R} \\
&= (u, u_*)_{\mathcal{H}}.
\end{aligned}$$

(2) For $a, b \in \mathbb{C}$ and $u, u_*, u_{**} \in \mathcal{H}$,

$$\begin{aligned}
(au + bu_*, u_{**})_{\mathcal{H}} &= \int_\Omega ((av + bv_*) + i(aw + bw_*))(v_{**} - iw_{**}) \\
&\quad + (D(av + bv_*) + iD(aw + bw_*))(Dv_{**} - iDw_{**})dx \\
&= a\int_\Omega (v + iw)(v_{**} - iw_{**}) + (Dv + iDw)(Dv_{**} - iDw_{**})dx \\
&\quad + b\int_\Omega (v_* + iw_*)(v_{**} - iw_{**}) + (Dv_* + iDw_*)(Dv_{**} - iDw_{**})dx \\
&= a(u, u_{**})_{\mathcal{H}} + b(u_*, u_{**})_{\mathcal{H}}.
\end{aligned}$$

(3) For nonzero $u \in \mathcal{H}$,

$$\begin{aligned}
(u, u)_{\mathcal{H}} &= \int_\Omega (v + iw)(v - iw) + (Dv + iDw)(Dv - iDw)dx \\
&= \int_\Omega v^2 + w^2 + |Dv|^2 + |Dw|^2 dx \\
&= \|v\|^2_{H_0^1(\Omega)} + \|w\|^2_{H_0^1(\Omega)} \in (0, \infty).
\end{aligned}$$

If both $\|v\|^2_{H^1_0(\Omega)}$ and $\|w\|^2_{H^1_0(\Omega)}$ are zero, $v = w = 0$ in $H^1_0(\Omega)$ and it means $u = 0$, which is contradiction.

Therefore, $(\cdot,\cdot)_{\mathcal{H}}$ yields an inner product in $\mathcal{H}$.

(b) First, note that $\mathcal{H}$ is a Hilbert space.(First, $H^1_0(\Omega)$ is a Hilbert space as a closed subspace of Hilbert space $H^1(\Omega)$. Assume that $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$ about the norm defined above. By (3) in (a), $(u_i - u_j, u_i - u_j)_{\mathcal{H}} = \|\mathrm{Re}(u_i - u_j)\|^2_{H^1_0(\Omega)} + \|\mathrm{Im}(u_i - u_j)\|^2_{H^1_0(\Omega)}$ and both $\mathrm{Re}(u_i - u_j)$ and $\mathrm{Im}(u_i - u_j)$ are Cauchy sequence. Therefore, it has a convergent subsequence and $\mathcal{H}$ is a Hilbert space.)

I'll first show that the bilinear form $B$ is bounded.

Bilinearity: For $u_1, u_2, v_1, v_2 \in \mathcal{H}$ and $c \in \mathbb{C}$,

$$\begin{aligned}
B[cu_1 + u_2, v_1] &= \int_\Omega a_{ij}\partial_i(cu_1 + u_2)\partial\overline{v}_1 \, dx \\
&= c\int_\Omega a_{ij}\partial_i u_1 \partial\overline{v}_1 \, dx + \int_\Omega a_{ij}\partial_i u_2 \partial\overline{v}_1 \, dx \\
&= cB[u_1, v_1] + B[u_2, v_1]
\end{aligned}$$

and

$$\begin{aligned}
B[u_1, cv_1 + v_2] &= \int_\Omega a_{ij}\partial_i u_1 \partial(c\overline{v}_1 + \overline{v}_2) \, dx \\
&= \overline{c}\int_\Omega a_{ij}\partial_i u_1 \partial\overline{v}_1 \, dx + \int_\Omega a_{ij}\partial_i u_2 \partial\overline{v}_1 \, dx \\
&= \overline{c}B[u_1, v_1] + B[u_2, v_1]
\end{aligned}$$

Boundedness: For $u, v \in \mathcal{H}$,

$$\begin{aligned}
|B[u,v]| &= \left|\int_\Omega a_{ij}\partial_i u \partial_j \overline{v} \, dx\right| \\
&= \left|\int_\Omega a_{ij}\partial_i(\mathrm{Re}(u) + i\mathrm{Im}(u))\partial_j(\mathrm{Re}(v) - i\mathrm{Im}(v)) \, dx\right| \\
&= \left|\int_\Omega a_{ij}(\partial_i\mathrm{Re}(u)\partial_j\mathrm{Re}(v) - \partial_i\mathrm{Im}(u)\partial_j\mathrm{Im}(v))) \, dx\right| \\
&+ \left|\int_\Omega a_{ij}(\partial_i\mathrm{Re}(u)\partial_j\mathrm{Im}(v) + \partial_i\mathrm{Im}(u)\partial_j\mathrm{Re}(v))) \, dx\right| \quad (*) \\
&\le \int_\Omega \frac{1}{\mu}(|D(\mathrm{Re}(u))||D(\mathrm{Re}(v))| + |D(\mathrm{Im}(u))||D(\mathrm{Im}(v))|) \, dx \\
&+ \int_\Omega \frac{1}{\mu}(|D(\mathrm{Re}(u))||D(\mathrm{Im}(v))| + |D(\mathrm{Im}(u))||D(\mathrm{Re}(v))|) \, dx \\
&= \frac{1}{\mu}\left(\|D(\mathrm{Re}(u))\|_{L^2(\Omega)} + \|D(\mathrm{Im}(u))\|_{L^2(\Omega)}\right)\left(\|D(\mathrm{Re}(v))\|_{L^2(\Omega)} + \|D(\mathrm{Im}(v))\|_{L^2(\Omega)}\right) \\
&\le \frac{1}{\mu}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}.
\end{aligned}$$

In the middle step $(*)$, I used the diagonalizable of positive definite matrix of $A$: For unit vector $\eta_1 \in \mathbb{R}^n$, it can be decomposed by the eigenvectors of $A$, so if

$$\eta_1 = \sum_{i=1}^n b_i \xi_i$$

for unit eigenvectors of $A$ $\{\xi_i\}$, then

$$A\eta_1 = \sum_{i=1}^{n} \lambda_i b_i \xi_i$$

and, for unit vector $\eta_2 \in \mathbb{R}^n$ with decomposition $\eta_2 = \sum_{i=1}^{n} c_i \xi_i$,

$$\eta_2^T A\eta_1 = \sum_{i=1}^{n} \lambda_i c_i b_i \le \max\{|\lambda_i|\} = \frac{1}{\mu}$$

since all the eigenvalues are not bigger than $\frac{1}{\mu}$ and are not smaller than $\mu > 0$.
Therefore, it is bounded in $\mathcal{H}$. Fix $u \in \mathcal{H}$ and consider a bounded linear functional $T$ on $\mathcal{H}$ such that $T(v) = B[u, v]$ for all $v \in \mathcal{H}$. I can use the Riesz Representation Thereom and find a unique element $w \in \mathcal{H}$ satisfying

$$B[u, v] = (w, v)_{\mathcal{H}}$$

for all $v \in \mathcal{H}$. It means we can find $w \in \mathcal{H}$ satisfying above for each $u \in \mathcal{H}$, so we can construct $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{L}(u) = w$. I need to show that this is bounded linear functional.

Linearity: For $\lambda_1, \lambda_2 \in \mathbb{C}$, $u_1, u_2, v \in \mathcal{H}$,

$$\begin{aligned}
(\mathcal{L}(\lambda_1 u_1 + \lambda_2 u_2), v)_{\mathcal{H}} &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\
&= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\
&= \lambda_1 (\mathcal{L}(u_1), v)_{\mathcal{H}} + \lambda_2 (\mathcal{L}(u_2), v)_{\mathcal{H}} \\
&= (\lambda_1 \mathcal{L}(u_1) + \lambda_2 \mathcal{L}(u_2), v)_{\mathcal{H}}
\end{aligned}$$

for all $v \in \mathcal{H}$. By the uniqueness part of the Riesz Representation Theorem, $\mathcal{L}(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \mathcal{L}(u_1) + \lambda_2 \mathcal{L}(u_2)$.

Boundedness: For $u \in \mathcal{H}$,

$$\|\mathcal{L}u\|_{\mathcal{H}}^2 = (\mathcal{L}u, \mathcal{L}u)_{\mathcal{H}} = B[u, \mathcal{L}u] \le \frac{1}{\mu} \|u\|_{\mathcal{H}} \|\mathcal{L}u\|_{\mathcal{H}}$$

Therefore, $\|\mathcal{L}u\|_{\mathcal{H}} \le \frac{1}{\mu} \|u\|_{\mathcal{H}}$ and $\mathcal{L}$ is bounded.

(c) First, $\mathcal{L}^*$ is well-defined since if $v_1 = v_2$ in $\mathcal{H}$, $(u, \mathcal{L}^* v_1) = (\mathcal{L}u, v_1) = (\mathcal{L}u, v_2) = (u, \mathcal{L}^* v_2)$ for all $u \in \mathcal{H}$, so $\mathcal{L}^* v_1 = \mathcal{L}^* v_2$.
Since $(a_{ij})$ is real valued and $a_{ij} = a_{ji}$, $B[u, v] = B[\overline{v}, \overline{u}] = \overline{B[v, u]}$ as

$$\begin{aligned}
B[\overline{u}, \overline{v}] &= \int_\Omega a_{ij} \partial_i \overline{u} \partial_j v dx \\
&= \int_\Omega a_{ji} \partial_j v \partial_i \overline{u} dx \\
&= B[v, u]
\end{aligned}$$

Using this relation, for fixed $u \in \mathcal{H}$ and all $v \in \mathcal{H}$,

$$(\mathcal{L}^* u, v)_{\mathcal{H}} = \overline{(v, \mathcal{L}^* u)_{\mathcal{H}}} = \overline{(\mathcal{L}v, u)_{\mathcal{H}}} = \overline{B[v, u]} = B[\overline{v}, \overline{u}] = B[u, v] = (\mathcal{L}u, v)_{\mathcal{H}}.$$

Therefore, $(\mathcal{L}u, v)_{\mathcal{H}} = (\mathcal{L}^* u, v)_{\mathcal{H}}$ for all $v$ and by uniqueness part of Riesz Representation Theorem again, $\mathcal{L}^* u = \mathcal{L}u$ for all $u \in \mathcal{H}$.(More precisely, we can define $T'v = (u, \mathcal{L}^* v)$ and since $Tv = T'v$, $T'$ is bounded linear operator, so by Riesz representation theorem, $\exists w' \in \mathcal{H}$ such that $(w', v)_{\mathcal{H}} = T'v$. Since $T'v = Tv$ for all $v$, $w' = w$ in $\mathcal{H}$.)

(d) Suppose $u$ is a nontrivial weak solution for the eigenvalue problem. By computing inner product;

$$\int_\Omega (Lu)\overline{v} = \int_\Omega -\partial_j(a_{ij}\partial_i u)\overline{v} = \int_\Omega a_{ij}\partial_i u \partial_j \overline{v} = B[u,v]$$

for all $v \in \mathcal{H}$ since the real part and imaginary part of $u, v$ are in $H_0^1(\Omega)$. Therefore, we can define the condition of weak solution $u$ to satisfy

$$B[u,v] = (\lambda u, v)_{L^2(\Omega)}$$

for all $v \in \mathcal{H}$. However,

$$B[u,u] = (\mathcal{L}u, u)_{\mathcal{H}} = (\mathcal{L}^* u, u)_{\mathcal{H}} = (u, \mathcal{L}u)_{\mathcal{H}} = \overline{(\mathcal{L}u, u)_{\mathcal{H}}} = \overline{B[u,u]}$$

and it makes $\lambda(u,u)_{L^2(\Omega)} = (\lambda u, u)_{L^2(\Omega)} = \overline{(\lambda u, u)_{L^2(\Omega)}} = \overline{\lambda (u,u)_{L^2(\Omega)}}$. The only case to consider is that $(u,u)_{L^2(\Omega)} \neq 0$. Assume that $(u,u)_{L^2(\Omega)} = 0$, then $\|u\|_{\mathcal{H}} = \|Du\|_{H_0^1(\Omega)} > 0$. However, it means

$$|B[u,u]| = \left| \int_\Omega a_{ij}\partial_i u \partial_j \overline{u} \, dx \right| \geq \mu \|Du\|^2 > 0$$

which is contradiction since $B[u,u] = (\lambda u, u)_{L^2(\Omega)} = \lambda(u,u)_{L^2(\Omega)} = 0$. Therefore, $\lambda \in \mathbb{R}$.

# Problem 3

Since $u \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega, \end{cases}$$

for all $v \in H_0^1(\Omega)$,

$$B[u,v] = \int_\Omega fv \, dx$$

where the bilinear form $B[\cdot,\cdot]$ is defined by

$$\int_\Omega \sum_{i,j} a_{ij}\partial_i u \partial_j v + \sum_i b_i v \partial_i u + cuv \, dx.$$

I'll show the boundedness by contradiction. Assume that there exists $\{u_k'\}_{k=1}^\infty$, $\{f_k'\}$, $\{g_k'\}$ satisfying

$$\|u_k'\|_{L^2(\Omega)} > k\left(\|f_k'\|_{L^2(\Omega)} + \|g_k'\|_{H^1(\Omega)}\right)$$

and the boundary value problem. Let $u_k = \frac{u_k'}{\|u_k'\|_{L^2(\Omega)}}$, $f_k = f_k'/\|u_k'\|_{L^2(\Omega)}$, $g_k = g_k'/\|u_k'\|_{L^2(\Omega)}$, then $u_k$ also satisfies the boundary value problem for $f_k$ and $g_k$ and the inequality

$$\|u_k\|_{L^2(\Omega)} > k\left(\|f_k\|_{L^2(\Omega)} + \|g_k\|_{H^1(\Omega)}\right).$$

Since $\|u_k\|_{L^2(\Omega)} = 1$ for all $k$, $\|f_k\|_{L^2(\Omega)}, \|g_k\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. By a variation of energy estimation,

$$\|u_k\|_{H^1(\Omega)}^2 \leq C\left(\|f_k\|_{L^2(\Omega)}\|u_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)}^2\right)$$

for the constant $C$ not depending on $f_k$ and $u_k$.(By the ellipticity condition, there exists $\theta > 0$ such that

$$\begin{aligned}
\theta \int_\Omega |Du|^2 dx &\le \int_\Omega a_{ij}\partial_i u \partial_j u dx \\
&= B[u,u] - \int_\Omega b_i u \partial_i u + cu^2 dx \\
&= B[u,u] + \sum_i \|b_i\|_{L^\infty} \int_\Omega \int_\Omega |Du||u| dx + \|c\|_{L^\infty} \int_\Omega u^2 dx
\end{aligned}$$

and by Cauchy's inequality,

$$\int_\Omega \int_\Omega |Du||u| dx \le \epsilon \int_\Omega |Du|^2 dx + \frac{1}{4\epsilon} \int_\Omega u^2 dx$$

for $\epsilon > 0$. By choosing small enough $\epsilon$ to satisfy

$$\epsilon \sum_i \|b_i\|_{L^\infty} < \theta/2,$$

we get

$$\frac{\theta}{2} \int_\Omega |Du|^2 dx \le B[u,u] + C \int_\Omega u^2 dx$$

for constant $C$. By adding $\frac{\theta}{2}\int_\Omega u^2 dx$ on both side, we get

$$\|u\|^2_{H^1(\Omega)} \le C(B[u,u] + \|u\|^2_{L^2(\Omega)}) \le C' \left( \|f_k\|_{L^2(\Omega)} \|u_k\|_{L^2(\Omega)} + \|u_k\|^2_{L^2(\Omega)} \right)$$

which is the above inequality.)

Since $\|f_k\|_{L^2(\Omega)} \to 0$ and $\|u_k\|^2_{L^2(\Omega)} = 1$, $\|u_k\|_{H^1(\Omega)}$ is uniformly bounded. Therefore, by compact embedding theorem,(For $n > 2$, we can use Rellich-Kondrachov Theorem and for $n = 2$, we choose $p$ close enough to $n = 2$ such that $2 < p^*$ and as $H^1(\Omega) \subset W^{1,p}(\Omega)$, we again use Rellich-Kondrachov Theorem. For $n = 1$, since $n < p = 2$, it can be done by Morrey's inequality and Arzela-Ascoli compactness theorem, which was the last HW.) there exists subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that

$$\begin{cases} u_{k_j} \rightharpoonup u & \text{weakly in } H^1(\Omega) \\ u_{k_j} \to u & \text{in } L^2(\Omega) \end{cases}$$

where $u \in H^1(\Omega)$. (Note that $H^1(\Omega)$ is a Hilbert space and bounded sequence in it have weakly convergent subsequence.)

Since $\left\|u_{k_j}\right\|_{L^2(\Omega)} = 1$ for all $j$, $\|u\|_{L^2(\Omega)} = 1$. Also,

$$\begin{cases} \lim\limits_{k_j \to \infty} B[u_{k_j}, \eta] = \lim\limits_{k_j \to \infty} (f_{k_j}, \eta)_{L^2(\Omega)} = 0 & \text{by } L^2 \text{ convergence} \\ \lim\limits_{k_j \to \infty} B[u_{k_j}, \eta] = B[u, \eta] & \text{by } H^1 \text{ weak convergence.} \end{cases}$$

for all $\eta \in H^1_0(\Omega)$. Therefore, $u$ is a nontrivial weak solution of the BVP:

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

However, this is impossible by the given condition. Therefore, there exists $C$ not depending on $u, f, g$ such that

$$\|u\|_{L^2(\Omega)} \le C \left( \|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} \right)$$

Also, by the above equation,

$$\begin{aligned}\|u\|_{H^1(\Omega)}^2 &\le C\left(\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2\right) \\ &\le C\left(\|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\right) \\ &\le C'\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}\right)\|u\|_{H^1(\Omega)}\end{aligned}$$

and

$$\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}\right)$$

where the constant $C$ not depending on $u, f, g$.

# Problem 4

(a) In the computation, I'll assume that $u, v$ are smooth function.

$$\begin{aligned}\int_\Omega -v\,\triangle u + cuv\ dx &= \int_\Omega -\sum_{i=1}^n \partial_i(v\partial_i u) + \partial_i v\partial_i u + cuv\ dx \text{ Integration by parts} \\ &= \int_{\partial\Omega} -\sum_{i=1}^n (v\partial_i u)\,\nu^i dS + \int_\Omega -\partial_i v\partial_i u + cuv\ dx \text{ Stokes' theorem} \\ &= \int_{\partial\Omega} v\nabla u\cdot\boldsymbol{\nu}_{\text{in}} dS + \int_\Omega \partial_i v\partial_i u + cuv\ dx \\ &= \int_{\partial\Omega} gv\ dS + \int_\Omega \partial_i v\partial_i u + cuv\ dx \\ &= \int_\Omega fv\ dx\end{aligned}$$

for $u, v \in H^1(\Omega)$ and $\boldsymbol{\nu}_{\text{out}} = (\nu^1, \nu^2, \ldots, \nu^n)$ outward unit normal vector on $\partial\Omega$. Therefore, the definition of weak solution of this problem:

$$B[u, v] = \int_\Omega \nabla u\cdot\nabla v + cuv\ dx = \int_\Omega fv\ dx - \int_{\partial\Omega} gv\ dS$$

makes sense.

(b) First, note that $H^1(\Omega)$ is a Hilbert space. Define a bilinear form $B[\cdot,\cdot]$ by

$$\int_\Omega \nabla u\cdot\nabla v + cuv\ dx.$$

I'll show that this is well-defined elliptic form on $H^1(\Omega)$.

Bilinear: For $u_1, u_2, v_1, v_2 \in H^1(\Omega)$ and $a \in \mathbb{R}$ or $\mathbb{C}$,

$$\begin{aligned}B[au_1 + u_2, v_1] &= \int_\Omega \nabla(au_1 + u_2)\cdot\nabla v_1 + c(au_1 + u_2)v_1\ dx \\ &= a\int_\Omega \nabla u_1\cdot\nabla v_1 + cu_1 v_1\ dx + \int_\Omega \nabla u_2\cdot\nabla v_1 + cu_2 v_1\ dx \\ &= aB[u_1, v_1] + B[u_2, v_1]\end{aligned}$$

and

$$\begin{aligned}
B[u_1, av_1+v_2] &= \int_\Omega \nabla u_1 \cdot \nabla(av_1+v_2) + cu_1(av_1+v_2)\ dx \\
&= a\int_\Omega \nabla u_1 \cdot \nabla v_1 + cu_1 v_1\ dx + \int_\Omega \nabla u_1 \cdot \nabla v_2 + cu_1 v_2\ dx \\
&= aB[u_1, v_1] + B[u_1, v_2]
\end{aligned}$$

Boundedness: For $u, v \in H^1(\Omega)$,

$$\begin{aligned}
|B[u,v]| &= \left| \int_\Omega \nabla u \cdot \nabla v + cuv\ dx \right| \\
&\leq \|Du\|_{L^2(\Omega)}^{1/2} \|Dv\|_{L^2(\Omega)}^{1/2} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq C(\|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)})(\|v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)}) \\
&= C\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

for constant $C$ depending only on $\Omega$ and $c$.

Coercivity: For $u \in H^1\Omega$,

$$\begin{aligned}
B[u,u] &= \int_\Omega \nabla u \cdot \nabla u + cu^2\ dx \\
&= \|Du\|_{L^2(\Omega)}^2 + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 \\
&\geq \min\{1, \mu_0\} \|u\|_{H^1(\Omega)}^2
\end{aligned}$$

Furthermore, define $I(f,g) : H^1(\Omega) \to \mathbb{R}$ by

$$I : v \mapsto \int_\Omega fv\ dx - \int_\Omega gv\ dS.$$

Then, this is a bounded linear functional on $H^1(\Omega)$ since it is definitely linear and $I(v) = \left|\int_\Omega fv\ dx\right| + \left|\int_\Omega gv\ dS\right| \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + C\|g\|_{L^2(\partial\Omega)} \|v\|_{H^1(\Omega)}$. (In the final step, I used trace inequality.)

By Lax-Millgram theorem, there exists a unique element $u \in H^1(\Omega)$ such that

$$B[u,v] = I(v)$$

for all $v \in H^1(\Omega)$. Therefore, there exists unique weak solution to the boundary value problem.

(c) If $u \in H^1(\Omega)$ is a weak solution to the boundary value problem,

$$\min\{1, \mu_0\} \|u\|_{H^1(\Omega)}^2 \leq |B[u,u]| = |I(u)| \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} + C\|g\|_{L^2(\partial\Omega)} \|u\|_{H^1(\Omega)}$$

so,

$$\|u\|_{H^1(\Omega)} \leq C'(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)})$$

for some $C < C'$ constants depending only on $c, \Omega$.

## Problem 5

$n = 1$일 때 Boundary condition이 잘 안 맞고, $n = 1$일 때 Boundary가 $C^\infty$라는 것이 non-sense라고 생각하여 $n \geq 2$를 가정하고 하였습니다.

(a) For $u, v \in H^1(\Omega)$,

(i)

$$
\begin{aligned}
(v,u)_{H^1} &= \int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} uv \; dS \\
&= \int_\Omega \nabla v \cdot \nabla u \; dx + \int_{\partial\Omega} vu \; dS \\
&= (u,v)_{H^1}
\end{aligned}
$$

Therefore, $(u,v)_{H^1} = (v,u)_{H^1}$.

(ii) Since $\Omega$ is bounded and $\partial U$ is $C^\infty$, consider a trace operator, which is bounded linear operator,

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ for $u \in H^2(\Omega) \cap C(\bar{\Omega})$ and $\|Tu\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$ with the constant $C$ only depending on $p$ and $\Omega$.

$$
\begin{aligned}
|(u,v)_{H^1}| &= \left| \int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} uv \; dS \right| \\
&\leq \int_\Omega |Du||Dv| \; dx + \int_{\partial\Omega} |u||v| \; dS \\
&\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|Tu\|_{L^2(\Omega)}\|Tv\|_{L^2(\Omega)} \\
&\leq C(\|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}) \\
&\leq C'(\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)})
\end{aligned}
$$

for some $C, C'$ constant independent on $u, v$.

(iii) For nonzero $u \in H^1(\Omega)$,

$$
\begin{aligned}
(u,u)_{H^1} &= \int_\Omega \nabla u \cdot \nabla u \; dx + \int_{\partial\Omega} u^2 \; dS \\
&= \|Du\|^2_{L^2(\Omega)} + \|Tu\|^2_{L^2(\partial\Omega)} < \infty
\end{aligned}
$$

If $normDu^2_{L^2(\Omega)} = \|Tu\|^2_{L^2(\partial\Omega)} = 0$, then $u \in H^1_0(\Omega)$ and if $n > 2$, it means $\|u\|_{H^1} = 0$ since $\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$ for constant $C$. For $n = 2$, choose $p$ close enough to 2, then $\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \leq C'\|Du\|_{L^2(\Omega)}$. Therefore, $(\cdot,\cdot)_{H^1}$ is an inner product on $H^1(\Omega)$.(Linearity: For $a, b \in \mathbb{R}$ and $u_1, u_2, v \in H^1(\Omega)$, $(au_1 + bu_2, v)_{H^1} = \int_\Omega \nabla(au_1 + bu_2) \cdot \nabla v \; dx + \int_{\partial\Omega}(au_1 + bu_2)v \; dS = a\int_\Omega \nabla u_1 \cdot \nabla v \; dx + a\int_{\partial\Omega} u_1 v \; dS + b\int_\Omega \nabla u_2 \cdot \nabla v \; dx + b\int_{\partial\Omega} u_2 v \; dS = a(u_1,v)_{H^1} + b(u_2,v)_{H^1}$.) I need to show the coercivity.

I'll prove the coercivity by contradiction. Suppose there exists a sequence $\{u_k\}_{k=1}^\infty \subset H^1(\Omega)$ such that

$$\|u_k\|^2_{H^1(\Omega)} > k(u_k,u_k)_{H^1}.$$

For such $u_k$, take $v_k = u_k/\|u_k\|_{H^1(\Omega)}$. Then, $\|v_k\|_{H^1(\Omega)} = 1$ and satisfies the above relation. As $k \to \infty$, $(v_k, v_k)_{H^1} \to 0$.

Since $\{v_k\}$ is a bounded sequence in $H^1(\Omega)$, which is compactly embedded in $L^2(\Omega)$, (I already explained how to deal for various dimension $n$ in Problem 3.) so there exists a subsequence $\{v_{k_j}\}_{j=1}^\infty$ such that

$$
\begin{cases}
v_{k_j} \rightharpoonup v & \text{weakly in } H^1(\Omega) \\
v_{k_j} \to v & \text{in } L^2(\Omega)
\end{cases}
$$

since $H^1(\Omega)$ is Hilbert space.(Since strong convergence implies weak convergence and weak limit in Hilbert space is unique. Therefore, the limit coincide in $L^2(\Omega)$.) As $v \in H^1(\Omega)$ and $\|v_{k_j}\|_{H^1(\Omega)} = 1$ for all $k_j$, $\|v\|_{H^1(\Omega)} \leq 1$. Since $\|Dv_{k_j}\|^2_{L^2(\Omega)} \leq (v_{k_j}, v_{k_j})_{H^1} \leq \frac{1}{k_j}$, $\|Dv_{k_j}\|^2_{L^2(\Omega)} \to 0$ as $k_j \to \infty$ and it means $\|v\|_{L^2(\Omega)} = 1$, $\|v\|_{H^1(\Omega)} = 0$ and $\|Dv\|_{L^2(\Omega)} = 0$. (If not, for some $N$, $\|v_N\|^2_{H^1(\Omega)} = \|v_N\|^2_{L^2(\Omega)} + \|Dv_N\|^2_{L^2(\Omega)} < 1$, which is contradiction.) Also,

$$\|v_k - v\|^2_{H^1(\Omega)} = \|v_{k_j} - v\|^2_{L^2(\Omega)} + \|D(v_{k_j} - v)\|^2_{L^2(\Omega)} \leq \|v_{k_j} - v\|^2_{L^2(\Omega)} + \|Dv_{k_j}\|^2_{L^2(\Omega)} + \|D(v_{k_j} - v)\|^2_{L^2(\Omega)} \to 0$$

as $k_j \to \infty$, $v_{k_j} \to v$ in $H^1(\Omega)$.

Since $\|Tv_{k_j}\|^2_{L^2(\partial\Omega)} \leq (v_{k_j}, v_{k_j})_{H^1} < \frac{1}{k_j}$,

$$\|Tv\|_{L^2(\partial\Omega)} \leq \|T(v - v_{k_j})\|_{L^2(\partial\Omega)} + \|Tv_{k_j}\|_{L^2(\partial\Omega)} \leq \frac{1}{\sqrt{k_j}} + C\|v - v_{k_j}\|_{H^1(\Omega)}$$

where the $C$ is given by the Trace Theorem. Therefore, $\|Tv\|_{L^2(\partial\Omega)} = 0$.

Summarising the fact, $v \in H^1(\Omega)$ and $Tv = 0$ on $\partial\Omega$, so $v \in H^1_0(\Omega)$ and $\|Dv\|_{L^2(\Omega)} = 0$. For $n > 2$, $\|v\|_{L^2(\Omega)} \leq C\|Dv\|_{L^2(\Omega)}$ for the constant $C$ depending only on $n$, and $\Omega$, so $\|v\|_{L^2(\Omega)} = 0$. For $n = 2$, we can find $p$ near to 2 and get $\|v\|_{L^2(\Omega)} \leq C\|Dv\|_{L^p(\Omega)}$. Since $\|Dv\|_{L^2(\Omega)} = 0 \Rightarrow \|Dv\|_{L^p(\Omega)} = 0$, $\|v\|_{L^2(\Omega)} = 0$. Therefore, $\|v\|_{L^2(\Omega)} = 0$ also in $n = 1$ case. Finally, this result shows that $v = 0$, but $\|v\|_{H^1(\Omega)} \neq 0$, which is contradiction.

Thus, there exists $C$ not depending on $u \in H^1(\Omega)$ such that

$$\|u\|^2_{H^1(\Omega)} \leq C(u, u)_{H^1}.$$

(b) I'll rewrite $(u, v)_{H^1}$ by $B[u, v]$. Consider a BVP:

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n}_{\text{in}} - u = 0 & \text{on } \partial\Omega \end{cases}$$

for $f \in L^2(\Omega)$. Then, the weak solution of the BVP should satisfy

$$B[u, v] = \int_\Omega fv \, dx$$

for all $v \in H^1(\Omega)$. (This result is from problem 4 (a).)

Assume that there exists eigenvalue $\lambda$ and the corresponding nontrivial weak solution $u \in H^1(\Omega)$. Then, for all $v \in H^1(\Omega)$,

$$B[u, v] = (\lambda u, v)_{L^2(\Omega)}$$

If we put $u$ as $v$,

$$B[u, u] = (\lambda u, u)_{L^2(\Omega)} = \lambda\|u\|_{L^2(\Omega)} > 0$$

since $\|u\|_{H^1(\Omega)} > 0$. If $\lambda$ is not real, it is nonsense. Also, if $\lambda \leq 0$, it is contradiction to coercivity of $B$. Therefore, every eigenvalue of (EVP) is a positive real.

(c) Since $B$ is symmetric bounded linear function with coercivity, By the Lax-Millgram Theorem, there exists unique weak solution $u \in H^1(\Omega)$ to the BVP for each $f$. Let's define $L^{-1} : L^2(\Omega) \to L^2(\Omega)$ by sending $f$ to $u$. Now, I claim that $L^{-1}$ is bounded, linear, compact operator. It is linear since if $L^{-1}(f_1) = u_1$ and $L^{-1}(f_2) = u_2$, then the unique weak solution to $rf_1 + f_2$ is $ru_1 + u_2$ for $r \in \mathbb{R}$ and

$L^{-1}(rf_1+f_2)=ru_1+u_2$. Also, $L^{-1}$ is bounded. It requires same argument used in problem 3. If there is no constant $C$ such that $\|u\|_{L^2(\Omega)}\le C\|f\|_{L^2(\Omega)}$, there is a nontrivial weak solution to

$$\begin{cases} Lu=0 & \text{in } \Omega \\ \nabla u\cdot \boldsymbol{n}_{\text{in}}-u=0 & \text{on } \partial\Omega \end{cases}$$

and since we just showed that $\lambda=0$ can not be an eigenvalue, this is impossible. Finally, $L^{-1}$ is compact: if $\{f_k\}_{k=1}^{\infty}$ is uniformly bounded in $L^2(\Omega)$ and let $u_k=L^{-1}f_k$, then by coercivity, there exists $\beta>0$ such that $\beta\|u_k\|^2_{H^1(\Omega)}\le B[u_k,u_k]=(f_k,u_k)_{L^2(\Omega)}\le\|f_k\|_{L^2(\Omega)}\|u_k\|_{H^1(\Omega)}$ and $\|u_k\|_{H^1(\Omega)}\le C$ for all $k$. By Rellich-Kondrachov Compactness Theorem (and for $n=2,1$ cases as described in previous) there exists $u_k\to u$ in $L^2(\Omega)$ and it means $L^{-1}$ is compact.

$u$ is a weak solution of the EVP in the problem if and only if $L^{-1}(\lambda u)=u$ if and only if $u-\lambda L^{-1}u=0$. Since $\lambda>0$, we can denote $\left(L^{-1}-\frac{1}{\lambda}\mathrm{Id}\right)u=0$. Therefore, $\lambda\in\Sigma$ if and only if $\frac{1}{\lambda}\in\sigma_p(L^{-1})\setminus\{0\}$ and $\Sigma=\left\{\lambda\in\mathbb{R}\mid\frac{1}{\lambda}\in\sigma_p\left(L^{-1}\right)\setminus\{0\}\right\}$.

Since $L^{-1}:L^2(\Omega)\to L^2(\Omega)$ and $L^2(\Omega)$ separable as the domain of the function is open set $\Omega\subset\mathbb{R}^n$. Also, $L^{-1}$: symmetric: Fix $f,g\in L^2(\Omega)$ and let $\omega=L^{-1}f$, $\xi=L^{-1}g$, then $\omega,\xi\in H^1(\Omega)$ and

$$\begin{aligned}(L^{-1}f,g)_{L^2(\Omega)}&=\int_\Omega g\omega dx\\&=\int_\Omega\nabla\xi\cdot\nabla\omega dx+\int_{\partial\Omega}\xi\omega dS\\&=\int_\Omega\nabla\omega\cdot\nabla\xi dx+\int_{\partial\Omega}\omega\xi dS\\&=\int_\Omega f\xi dx\\&=(f,L^{-1}g).\end{aligned}$$

Therefore, there exists countable orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of $L^{-1}$. By Fredholm alternative, the dimension of null space of $\left(\mathrm{Id}-\lambda L^{-1}\right)$ for $\lambda\in\Sigma$ is finite. Therefore, $|\Sigma|=\infty$. Since $L^{-1}$ is compact and the dimension of $L^2(\Omega)$ is infinite, $\sigma(L^{-1})\setminus\{0\}=\sigma_p(L^{-1})\setminus\{0\}$ and $\sigma(L^{-1})\setminus\{0\}$ is a sequence tending to 0. Therefore, $\Sigma$ consists of the eigenvalues monotonic increasing to $\infty$.

(d) In the (c), I showed that there exists countable orthonormal basis of separable space $L^2(\Omega)$ consisting of eigenvectors of compact and symmetric bounded linear operator $L^{-1}$. I need to show that this is orthogonal basis of $H^1(\Omega)$.

$$B[w_i,w_j]=(Lw_i,w_j)_{L^2(\Omega)}=(Lw_j,w_i)_{L^2(\Omega)}=B[w_j,w_i]$$

If $i\neq j$, $(Lw_i,w_j)_{L^2(\Omega)}=\lambda_i(w_i,w_j)_{L^2(\Omega)}=0$. If $i=j$, $(Lw_i,w_i)_{L^2(\Omega)}=\lambda_i$.

Finally, assume that $\{w_i\}$ can not span $H^1(\Omega)$. Then, there exists nontrivial $\xi\in H^1(\Omega)$ such that $B[w_i,\xi]=0$ for all $w_i$. Then, $B[w_i,\xi]=(\lambda_i w_i,\xi)_{L^2(\Omega)}=0$ and $\lambda_i>0$ for all $i$. Therefore, $\xi=0$, which is contradiction. Therefore, $\{w_i\}$ spans $H^1(\Omega)$. Thus, $\{w_i\}$ forms a orthogonal basis of $H^1(\Omega)$.

## Problem 6

(Step 1) In problem 5 (d), we found a set of functions $\{w_k\}_{k=1}^{\infty}\subset H^1(\Omega)\cap C^\infty(\Omega)$ such that

(i) $\{w_k\}$ forms an orthornormal basis of $L^2(\Omega)$, and

(ii) $\{w_k\}$ forms an orthogonal basis of $H^1(\Omega)$ with respect to the inner product $(\cdot,\cdot)_{H^1} = B[\cdot,\cdot]$.
The weak solution of (IBVP-P) should satisfies

$$(u'(t), v)_{L^2(\Omega)} + B[u(t), v] = (f(t), v)_{L^2(\Omega)}$$

for $v \in H^1(\Omega)$ and a.e. time $0 \le t \le T$ and $u(0) = g$.

(Step 2) I'll use Galerkin approximation. Fix $m \in \mathbb{N}$ and set $u_m(t,x) = \sum_{k=1}^{m} d_k(t) w_k(x)$. I need to find $\{d_k\}_{k=1}^m$ satisfying

(i) $(u_m'(t,\cdot), w_k)_{L^2(\Omega)} + B[u_m(t), w_k] = (f(t,\cdot), w_k)_{L^2(\Omega)}$ for $1 \le j \le m$.

(ii) $u_m(0,x) = \sum_{k=1}^{m} d_k(0) w_k(x) = g^{(m)}(x)$ where $g^{(m)}(x) = \sum_{k=1}^{m} G_k w_k(x)$, $G_k = (g, w_k)_{L^2(\Omega)}$.

Note that $u_m$ satisfies boundary condition $\nabla u_m \cdot \boldsymbol{n}_{\text{in}} - u_m = 0$ on $(0,T] \times \partial\Omega$ since $w_k = 0$ on $\partial\Omega$ for all $k$.

Define $f^{(m)}(t,x) \coloneqq \sum_{k=1}^{m} F_k(t) w_k(x)$ where $F_k(t) = (f(t,\cdot), w_k)_{L^2(\Omega)}$.

Note that (i) holds if and only if $d_j'(t) + \lambda_j d_j(t) = F_j(t)$ for $0 < t \le T$ and $1 \le j \le m$, and (ii) holds if and only if $d_j(0) = G_j = (g, w_j)_{L^2(\Omega)}$:

$$(\sum_{k=1}^{m} d_k'(t) w_k(x), w_j)_{L^2(\Omega)} + B[\sum_{k=1}^{m} d_k(t) w_k(x), w_j] = (f(t,\cdot), w_j)_{L^2(\Omega)} \Leftrightarrow d_j'(t) + \lambda_j d_j(t) = F_j(t)$$

and

$$\begin{aligned}
\sum_{k=1}^{m} d_k(0) w_k(x) &= \sum_{k=1}^{m} G_k w_k(x) \\
&\Leftrightarrow \left(\sum_{k=1}^{m} d_k(0) w_k(x), w_j(x)\right)_{L^2(\Omega)} = \left(\sum_{k=1}^{m} G_k w_k(x), w_j(x)\right)_{L^2(\Omega)} \quad \text{For all } 1 \le j \le m \\
&\Leftrightarrow d_j(0) = G_j \text{ For all } 1 \le j \le m
\end{aligned}$$

Since each $F_j(t), G_j(t)$ are smooth, by the uniqueness existence of ODEs, the initial value problem for each $j$:

$$\begin{cases} d_j'(t) + \lambda_j d_j(t) = F_j(t) & 0 < t \le T \\ d_j(0) = G_j = (g, w_j)_{L^2(\Omega)} \end{cases}$$

has a unique smooth solution $d_j$ for $1 \le j \le m$. With these $d_j$, $u_m(t,x) = \sum_{k=1}^{m} d_k(t) w_k(t)$ satisfies (i),(ii) and 0 on $(0,T] \times \partial\Omega$.

What I want is that as $m \to \infty$, $u_m \to u \in L^2((0,T); H^1(\Omega))$, $u' \in L^2((0,T); H^1_*(\Omega))$.

For now, I'll accept the proposition:

**Proposition 1.** *For each $u_m$ satisfies*

$$\max_{t\in[0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u_m'\|_{L^2(0,T;H^1_*(\Omega))} \le C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

*for some constant $C$ depending only on $\Omega, T$.*

(Step 3) By the proposition, we get the fact that $\{u_m\}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ and $\{u'_m\}$ is uniformly bounded in $L^2(0,T;H^1_*(\Omega))$. Since $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;H^1_*(\Omega))$ are Hilbert space since $L^2(0,T)$ is, there is a subsequence such that

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0,T;H^1(\Omega)) \\ u'_{m_k} \rightharpoonup u' & \text{in } L^2(0,T;H^1_*(\Omega)) \end{cases}$$

(We know that $u'_{m_k} \rightharpoonup w \in L^2(0,T;H^1_*(\Omega))$. I'll show that $u' = w$. For $\phi \in C_c^\infty(0,T)$ and $v \in H^1(\Omega)$,

$$\begin{aligned}
\langle \int_0^T \phi'(t)u(t)dt, v\rangle &= \int_0^T \langle \phi'(t)u(t), v\rangle dt = \int_0^T \langle u(t), \phi'(t)v\rangle dt \\
&= \int_0^T \lim_{m_k\to\infty} \langle u_{m_k}(t), \phi'(t)v\rangle dt = \lim_{m_k\to\infty} \int_0^T \langle u_{m_k}(t), \phi'(t)v\rangle dt \\
&= \lim_{m_k\to\infty} \langle \int_0^T u_{m_k}(t)\phi'(t)dt, v\rangle = \lim_{m_k\to\infty} \langle \int_0^T u'_{m_k}(t)\phi(t)dt, v\rangle \\
&= \int_0^T \lim_{m_k\to\infty} \langle u'_{m_k}(t), \phi(t)v\rangle dt = \int_0^T \langle w(t), \phi(t)v\rangle dt \\
&= \langle \int_0^T w(t)\phi(t)dt, v\rangle.
\end{aligned}$$

The $\langle\cdot,\cdot,\rangle$ represent the action of $H^1_*(\Omega)$ to $H^1(\Omega)$ and the commutativity of $\langle\cdot,\cdot,\rangle$ and integral is by smoothness of $\phi$ and riemann sum argument. The commutativity of limit and integral is by the boundedness of $U$ and $(0,T)$ and integrability of $u, v, \phi$ with dominance convergence theorem. Therefore, $u' = w$.)

Now, I'll show that $u$ is the weak solution to (IBVP-P).

Since $\nabla u_{m_k} \cdot \boldsymbol{n}_{\text{in}} - u_{m_k} = 0$ on $(0,T] \times \partial\Omega$ for all $m_k$, $\nabla u \cdot \boldsymbol{n}_{\text{in}} - u = 0$ on $(0,T] \times \partial\Omega$.

Fix $N \in \mathbb{N}$ and fix $m_k \geq N$. Consider $u_{m_k} = \sum_{i=1}^{m_k} d_i(t)w_i(x)$. Fix $j \in \{1, \cdots, N\}$ and let $\alpha(t)$ be a smooth function. Then,

$$\begin{aligned}
&\int_0^T \alpha(s)\left((u'_{m_k}(s), w_j)_{L^2(\Omega)} + B[u_{m_k}(s), w_j]\right) ds = \int_0^T (f^{(m_l)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds \\
\Leftrightarrow &\int_0^T (u'_{m_k}(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds
\end{aligned}$$

By the weak convergence, as $m_k \to \infty$,

$$\int_0^T (u'(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$

Since $\alpha(s)$ is arbitrary, we can let $\alpha(s) \to \delta(s-t)$ (such as standard mollifier) and get

$$(u'(t), w_j)_{L^2(\Omega)} + B[u(t), w_j] = (f(t), w_j)_{L^2(\Omega)}$$

Since $N$ is arbitrarily fixed, this is true for all $j$. Since the function $\sum_{i=1}^{n} d_i(x)w_i(x)$ with coefficient $d_i(x) \in H^1(\Omega)$ is dense in $H^1(\Omega)$, we get

$$(u'(t), v)_{L^2(\Omega)} + B[u(t), v] = (f(t), v)_{L^2(\Omega)}$$

for $v \in H^1(\Omega)$ and a.e. time $0 \leq t \leq T$.

I still need to check the initial condition. For the same circumstance with one more condition $\alpha(T) = 0$,

$$\int_0^T (u'(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$
$$\Rightarrow \int_0^T -(u(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (u(0), \alpha(0)w_j))$$

and

$$\int_0^T (u'_{m_k}(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$
$$\Rightarrow \int_0^T -(u_{m_k}(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (u_{m_k}(0), (\alpha(0)w_j)).$$

Letting $m_k \to \infty$ and by weak convergence,

$$\int_0^T -(u(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (g(0), (\alpha(0)w_j))$$

as $u_{m_k}(0) = g^{(m_k)}$ and $\lim_{m_k\to\infty} u_{m_k}(0) = \lim_{m_k\to\infty} g^{(m_k)} = g$. This is true for all $\alpha \in C^\infty((0,T))$ and for all $j$, $u(0) = g(0)$.

The left one is to show the proposition. For readability, I again write the proposition:

For each $u_m$ satisfies

$$\max_{t\in[0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u'_m\|_{L^2(0,T;H^1_*(\Omega))} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

for some constant $C$ depending only on $\Omega, T$.

I'll start from this equation:

$$(\sum_{k=1}^m d'_k(t)w_k(x), w_j)_{L^2(\Omega)} + B[\sum_{k=1}^m d_k(t)w_k(x), w_j] = (f(t,\cdot), w_j)_{L^2(\Omega)}$$

Note that $d_k(t)$ are calculated for fixed $m$. On each side for fixed $j$, multiply $d_j(t)$ and sum about $j$, then we get

$$(u'_m(t), u_m(t))_{L^2(\Omega)} + B[u_m(t), u'_m(t)] = (f(t,\cdot), u_m(t))_{L^2(\Omega)}$$

for a.e. $0 \leq t \leq T$. For fixed $t \in [0,T]$, $(u'_m, u_m)_{L^2(\Omega)} = \frac{d}{dt}\left(\frac{1}{2}\|u_m\|^2_{L^2(\Omega)}\right)$ and $|(f, u_m)| \leq \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_m\|^2_{L^2(\Omega)}$. Furthermore, I'll use the proof of variation of energy estimation used in problem 3: there exists $\beta, \gamma > 0$ such that

$$\beta\|u_m\|^2_{H^1(\Omega)} \leq B[u_m, u_m, t] + \gamma\|u_m\|^2_{L^2(\Omega)}.$$

Mixing these inequalities, we get

$$\frac{d}{dt}\left(\|u_m\|^2_{L^2(\Omega)}\right) + 2\beta\|u_m\|^2_{H^1(\Omega)} \leq C_1\|u_m\|^2_{L^2(\Omega)} + C_2\|f\|^2_{L^2(\Omega)}. \tag{1}$$

for a.e. $0 \leq t \leq T$ and constant $C_1, C_2$ not depending on $u_m, f$.

Now, let

$$\eta(t) := \|u_m(t)\|^2_{L^2(\Omega)}$$

and

$$\xi(t) := \|f(t)\|^2_{L^2(\Omega)}.$$

Then,

$$\eta'(t) \le C_1\eta(t) + C_2\xi(t)$$

for a.e. $0 \le t \le T$ and by differential form of Gronwall's inequality,

$$\eta(t) \le e^{C_1 t}\left(\eta(0) + C_2\int_0^t \xi(s)ds\right)$$

for $0 \le t \le T$. Since $\eta(0) = \|u_m(0)\|^2_{L^2(\Omega)} \le \|g\|^2_{L^2(\Omega)}$, (Note that $u_m(0)$ is the projection of $g$.)

$$\max_{t\in[0,T]} \|u_m(t)\|^2_{L^2(\Omega)} \le C\left(\|g\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right).$$

Also, integrating (1) from 0 to $T$ generates the relation:

$$\|u_m\|^2_{L^2(0,T;H^1(\Omega))} = \int_0^T \|u_m\|^2_{H^1(\Omega)}dt \le C\left(\|g\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right)$$

Finally, I'll estimate $\|u_m'\|_{L^2(0,T;H^1_*(\Omega)}$. Fix $v \in H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \le 1$ and denote $v = v^1 + v^2$ by $v_1 \in \operatorname{span}(\{w_i\}_{i=1}^m)$ and $(v_2, w_i) = 0$ for $i \in \{1,\ldots,m\}$. Since $v_1$ is the projection to $\operatorname{span}(\{w_i\}_{i=1}^m)$, $\left\|v^1\right\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)} \le 1$. Using the previous equation: $(u_m'(t,\cdot), w_k)_{L^2(\Omega)} + B[u_m(t), w_k] = (f(t,\cdot), w_k)_{L^2(\Omega)}$, we can write

$$(u_m'(t,\cdot), v^1)_{L^2(\Omega)} + B[u_m(t), v^1] = (f(t,\cdot), v^1)_{L^2(\Omega)}$$

for a.e. $0 \le t \le T$. Since $u_m' \in \operatorname{span}(\{w_i\}_{i=1}^m)$,

$$(u_m', v)_{L^2(\Omega)} = (u_m', v^1)_{L^2(\Omega)} = (f, v^1) - B[u_m(t), v^1].$$

Therefore,

$$\left|(u_m', v)_{L^2(\Omega)}\right| \le C\left(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1(\Omega)}\right)$$

since $\left\|v^1\right\|_{H^1(\Omega)} \le 1$.(Also, by boundedness of $B[\cdot,\cdot]$: $\left|B[u_m, v^1; t]\right| \le \theta\|u_m(t)\|_{H^1(\Omega)}$ for some constant $\theta$.) Since $\|v\|_{H^1(\Omega)} = 1$ and RHS is independent to $v$,

$$\|u_m'\|_{H^1_*(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1(\Omega)}\right)$$

and

$$\|u_m'\|_{L^2(0,T;H^1_*(\Omega))} = \int_0^T \|u_m'\|_{H^1_*(\Omega)}\, dt \le C'\left(\|g\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}\right)$$

for some constant $C'$.(Note that we already showed that $\|u_m\|^2_{L^2(0,T;H^1(\Omega)}$ is bounded by $\|g\|_{L^2(\Omega)}$ and $\|f\|_{L^2(0,T;L^2(\Omega))}$.

Summarizing all the results, we get

$$\max_{t\in[0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u_m'\|_{L^2(0,T;H^1_*(\Omega))} \le C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

for some $C$. This is true for all $m$ since the estimation constant does not depending on $u_m, f, g$.