Partial Differential Equation - HW 2

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Problem 1

Step 1. Let a linear transformation from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ sending $s \mapsto r^2 s$ and $y_i \mapsto r y_i$, then T sends $E(0, \mathbf{0}, r)$ to $E(0, \mathbf{0}, 1)$. Then

$$\begin{aligned}\frac{1}{4}\iint_{E(0,\mathbf{0},1)} \frac{|y|^2}{s^2} dyds &= \frac{1}{4}\iint_{E(0,\mathbf{0},r)} \frac{|y|^2/r^2}{s^2/r^4}\frac{1}{r^{n+2}} dyds \\ &= \frac{1}{4r^n}\iint_{E(0,\mathbf{0},r)} \frac{|y|^2}{s^2} dyds\end{aligned}$$

Step 2. Since $e^x \leq 1$ if $x \leq 0$,

$$\begin{aligned}&\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{|y|^2}{4s}} \geq 1 \text{ for } s \leq 0 \\ &\Leftrightarrow 4\pi(-s) \leq 1 \text{ and } e^{\frac{|y|^2}{4s}} \geq (4\pi(-s))^{n/2} \text{ and } s \leq 0 \\ &\Leftrightarrow -\frac{1}{4\pi} \leq s \leq 0 \text{ and } |y|^2 \leq 2sn\ln(-4\pi s)\end{aligned}$$

Therefore, $E(0,0,1) = \left\{(s,y) \in R^{n+1} : -\frac{1}{4\pi} \leq s \leq 0, |y|^2 \leq -2sn\ln\left(\frac{1}{-4\pi s}\right)\right\}$

Step 3. Since $|y|^2 \leq -2sn\ln\left(\frac{1}{-4\pi s}\right)$ is the ball with radius $\left(-2sn\ln\left(\frac{1}{-4\pi s}\right)\right)^{1/2}$,

$$\begin{aligned}\iint_{E(0,\mathbf{0},1)} \frac{|y|^2}{s^2} dyds &= \int_{-1/4\pi}^{0}\int_{B_s} \frac{|y|^2}{s^2} dyds \text{ (By Fubini theorem)} \\ &= \int_{-1/4\pi}^{0} \frac{1}{s^2}\int_{S^{n-1}}\int_0^{\left(-2sn\ln\left(\frac{1}{-4\pi s}\right)\right)^{1/2}} r^{n+1} dr d\sigma ds \text{ (By Spherical Coord.)} \\ &= \int_{-1/4\pi}^{0} \frac{1}{s^2}\frac{\frac{n}{2}\pi^{n/2}}{\Gamma(\frac{n}{2}+2)}\left(-2sn\ln\left(\frac{1}{-4\pi s}\right)\right)^{n/2+1} ds\end{aligned}$$

Let $t = \ln\left(\frac{1}{-4\pi s}\right)$, then

$$\begin{aligned}&\frac{1}{4}\left(\frac{n}{2\pi}\right)^{n/2+2}\frac{\pi^{n/2}}{\Gamma(n/2+2)}\int_0^\infty \frac{(4\pi)^2}{e^{-2t}}\left(e^{-t}t\right)^{n/2+1} e^{-t} dt \\ &= 4\left(\frac{n}{2}\right)^{n/2+2}\frac{1}{\Gamma(n/2+2)}\int_0^\infty t^{n/2+1}e^{-(n/2)t} dt \\ &= 4\left(\frac{n}{2}\right)^{n/2+2}\frac{1}{\Gamma(n/2+2)}\left(\frac{2}{n}\right)^{n/2+2}\Gamma(n/2+2) = 4.\end{aligned}$$

Therefore, $\frac{1}{4}\iint_{E(0,\mathbf{0},1)} \frac{|y|^2}{s^2} dyds = 1$.

Summarising the results, $\frac{1}{4r^n}\iint_{E(0,\mathbf{0},r)} \frac{|y|^2}{s^2} dydx = 1$ for all $r > 0$.

Problem 2

By change of variable shifting x and t to 0, we can rearrange the formula by

$$\begin{aligned}\frac{1}{r^n}\iint_{E(t,x;r)} u(s,y)\frac{|s-y|^2}{|t-s|^2} dyds &= \frac{1}{r^n}\iint_{E(0,\mathbf{0};r)} u'(s,y)\frac{|y|^2}{s^2} dyds \\ &= \iint_{E(0,\mathbf{0};1)} u'(r^2s', ry')\frac{|y'|^2}{s'^2} dy'ds'\end{aligned}$$

where $u'(s,y) = u(s+t, y+x)$ and $s = r^2s'$, $y = ry'$. For simplicity, I'll write u by u' . Let $\phi(r)$ be the RHS of above equation. Our strategy is showing that $\phi(r)$ is constant function, and it means $\phi(r) = \lim_{r\to 0}\phi(r) = r^n u(0,0)$ using continuity of u and the result from problem 1. I'll write $E(r) = E(0,\mathbf{0};r)$. Computing $\phi'(r)$,

$$\begin{aligned}\phi'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\iint_{E(1)} u(r^2s', ry')\frac{|y'|^2}{s'^2} dy'ds' \\ &= \iint_{E(1)} (2rs'u_s(r^2s', ry') + y'\cdot\nabla_y u(r^2s', ry'))\frac{|y'|^2}{s'^2} dy'ds' \text{ (Since } \frac{|y|^2}{s^2} \in L^1(E(1)) \text{ and } u \in C(\overline{E(1)})\text{)} \\ &= \frac{1}{r^{n+1}}\iint_{E(r)} (2su_s(s,y) + y\cdot\nabla_y u)\frac{|y|^2}{s^2} dyds\end{aligned}$$

Let the first term A and last term B.

For simple calculation, I'll introduce a function

$$\varphi(s,y) = -\frac{n}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + n\ln r$$

Then, we can get a relation $|y|^2 = \sum_{i=1}^{n} 2sy_i\varphi_{y_i}$, and

$$\begin{aligned}r^{n+1}A &= \iint_{E(r)} 4u_s\sum_{i=1}^{n} y_i\varphi_{y_i} dyds \\ &= \iint_{E(r)} 4\sum_{i=1}^{n}\left(\frac{\partial}{\partial y_i}(y_iu_s\varphi) - u_s\varphi - u_{sy_i}y_i\varphi\right) dyds \\ &= -4\iint_{E(r)} nu_s\varphi + \varphi\sum_{i=1}^{n} u_{sy_i}y_i dyds\end{aligned}$$

Since

$$\begin{aligned}\iint_{E(r)}\sum_{i=1}^{n}\frac{\partial}{\partial y_i}(y_iu_s\varphi)\, dyds &= \sum_{i=1}^{n}\int_{-\frac{r^2}{4\pi}}^{0}\int_{B\left(\mathbf{0},-2ns\ln\left(-\frac{4\pi s}{r^2}\right)\right)}\frac{\partial}{\partial y_i}(y_iu_s\varphi)\, dyds \\ &= \sum_{i=1}^{n}\int_{-\frac{r^2}{4\pi}}^{0}\int_{\partial B\left(\mathbf{0},-2ns\ln\left(-\frac{4\pi s}{r^2}\right)\right)}(y_iu_s\varphi)\,\nu^i dyds \\ &= 0\end{aligned}$$

for $\varphi \equiv 0$ in $\partial E - \{0, \mathbf{0}\}$.($\nu^i$ is ith component of outward pointing unit normal vector.)

By $\frac{\partial}{\partial s}(u_{y_i} y_i \varphi) = u_{sy_i} y_i \varphi + u_{y_i} \varphi y_i \varphi_s$

$$\iint_{E(r)} \varphi \sum_{i=1}^{n} u_{sy_i} y_i dyds = \iint_{E(r)} \sum_{i=1}^{n} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi) - u_{y_i} y_i \varphi_s dyds$$

Let $s_0(y)$ and $s_1(y)$ is the the boundary points of s for fixed $y \neq \mathbf{0}$, and $\{y\} \times (s_0(y), s_1(y))$ is in the heat ball. In the integration on $E(r)$, we can neglect $y = 0$ case since it $\{\mathbf{0}\} \times (-1/4\pi, 0)$ is measure zero set in Ω. Then,

$$\begin{aligned}\iint_{E(r)} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi)\, dyds &= \int_{|y|^2 \leq \frac{nr^2}{2\pi e}} \int_{s_0(y)}^{s_1(y)} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi)\, dsdy \\ &= \int_{|y|^2 \leq \frac{nr^2}{2\pi e}} u_{y_i}(s_1(y), y), y) y_i \varphi(s_1(y)) - u_{y_i}(s_0(y), y), y) y_i \varphi(s_0(y)) dy = 0.\end{aligned}$$

because $\varphi(s_0(y)) = \varphi(s_1(y)) = 0$ for $y \neq \mathbf{0}$.

Therefore,

$$\begin{aligned}r^{n+1} A = -4 \iint_{E(r)} n u_s \varphi + \varphi \sum_{i=1}^{n} u_{sy_i} y_i dyds &= -4 \iint_{E(r)} n u_s \varphi - \varphi \sum_{i=1}^{n} u_{y_i} y_i \varphi_s dyds \\ &= -4 \iint_{E(r)} n u_s \varphi - \sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) dyds \\ &= \iint_{E(r)} -4 n u_s \varphi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds - B.\end{aligned}$$

Since u is a solution for heat equation,

$$\begin{aligned}\phi'(r) = A + B &= \iint_{E(r)} -4 n u_s \varphi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds \\ &= \iint_{E(r)} -4 n \varphi \triangle_y u - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds \\ &= \iint_{E(r)} \sum_{i=1}^{n} -4 n u_{y_i} \varphi_{y_i} - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds \\ &= \iint_{E(r)} \sum_{i=1}^{n} \frac{2n}{s} u_{y_i} y_i - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds = 0.\end{aligned}$$

The intermediate step $\varphi \triangle_y u \to u_{y_i} \varphi_{y_i}$ uses the fact that $\varphi = 0$ on $\partial E(r) - \{(0, \mathbf{0}\}$ as in s case.

Consequently, ϕ is a constant function. Since u is continuous and by problem 1,

$$\lim_{r \to 0+} \frac{1}{r^n} \iint_{E(0,\mathbf{0};r)} u(s,y) \frac{|y|^2}{s^2} dyds = 4u(0,0)$$

Hence,

$$\frac{1}{4r^n} \iint_{E(t,x;r)} u(s,y) \frac{|s-y|^2}{|t-s|^2} dyds = u(t,x)$$

for all $r > 0$.

Problem 3

Step 1. First, we need to assume that the convergence radius of u is ∞, so that we can safely differentiate the series term by term. To satisfy $u_t - u_{xx} = 0$ for $t > 0, x \in \mathbb{R}$,

$$u_t - u_{xx} = \sum_{j=0}^{\infty} \left(\frac{\partial g_j(t)}{\partial t} - (j+1)(j+2)g_{j+2}(t) \right) x^j = 0.$$

By uniqueness of power series in the convergence region, $\frac{\partial g_j(t)}{\partial t} - (j+1)(j+2)g_{j+2} = 0$ for all $j \geq 0$ iff $u_t - u_{xx} = 0$.

To satisfy initial condition $u(0,x) = 0$ for $x \in \mathbb{R}$, $\sum_{j=0}^{\infty} g_j(0)x^j = 0$, $g_j(0)$ should be 0 for all j followed by uniqueness of power series.

Step 2. I'll use induction for proof. Let g_j be

$$g_j(t) = \begin{cases} 0 & j \text{ is odd.} \\ \frac{1}{(2(j/2))!} \frac{d^{(j/2)}}{dt^{(j/2)}} g(t) & j \text{ is even.} \end{cases} \tag{1}$$

for $j < 2n$. The starting case is given in the problem. For $j = 2n$,

$$g_{2n} = \frac{1}{(2n-1)(2n)} \frac{\partial g_{2n-2}(t)}{\partial t} = \frac{1}{2n!} \frac{d^n}{dt^n} g(t).$$

For $j = 2n+1$,

$$g_{2n+1} = \frac{1}{(2n)(2n+1)} \frac{\partial g_{2n-1}(t)}{\partial t} = 0.$$

Therefore, (1) is valid.

Step 3. Since $\frac{1}{z^2}$ is holomorphic without $z = 0$, $e^{-\frac{1}{z^2}}$ is holomorphic without $z = 0$. For $t > 0$, let $r = t/8$ be a radius of open disk centred at $t \in \mathbb{R}$ in $\mathbb{C}$. Let the boundary of the disk C. Then by Cauchy integral formula,

$$\left| \frac{d^k}{dt^k} g(t) \right| \leq \frac{k!}{2\pi} \oint_C \left| \frac{e^{-\frac{1}{z^2}}}{(z-t)^{k+1}} \right| dz \leq \frac{k!}{2\pi} r^{-k} \max_C \left| e^{-\frac{1}{z^2}} \right|$$

and if we let $z = x + iy$, $x, y \in \mathbb{R}$,

$$\left| e^{-\frac{1}{z^2}} \right| = \left| e^{-\frac{x^2-y^2}{(x^2+y^2)^2}} \right| \leq \left| e^{-\frac{1280}{1681} t^{-2}} \right| \leq \left| e^{-\frac{1}{2t^2}} \right|.$$

Therefore, if we set $\theta = \frac{1}{8}$,

$$\left| \frac{d^k}{dt^k} g(t) \right| \leq k! \left(\frac{1}{8} t \right)^{-k} e^{-\frac{1}{2t^2}}$$

If $t \leq 0$, it is obvious inequality since $g^(k)(t) = 0$ for $t < 0$ and $\lim_{t \to 0} g(k)(t) = 0$....

Step 4. Let g_i as in Step 2. Then,

$$\left| g_{2k}(t) x^{2k} \right| \leq \frac{k!}{(2k)!} e^{-\frac{1}{2t^2}} \left(\frac{8x^2}{t} \right)^k.$$

The radius of convergence of RHS is

$$R^2 \leq \lim_{k \to \infty} (2k+1) \frac{t}{4},$$

and it means $\sum g_{2k}(t)x^{2k}$ converges for all $t > 0$ and $x \in \mathbb{R}$.

Since $\frac{k!}{(2k)!}e^{-\frac{1}{2t^2}}\left(\frac{8x^2}{t}\right)^k \leq \frac{1}{k!}e^{-\frac{1}{2t^2}}\left(\frac{8x^2}{t}\right)^k$, by replacing $\frac{8x^2}{t}$ by z, we can get

$$\left|\sum g_{2k}(t)x^{2k}\right| \leq e^{-\frac{1}{2t^2}} \sum \frac{1}{k!}z^k = e^{-\frac{1}{2t^2}+\frac{8x^2}{t}}$$

for fixed $t > 0$ with convergence of radius ∞. As $t \to 0$, $e^{-\frac{1}{2t^2}+\frac{8x^2}{t}} \to 0$. Therefore,

$$\sum g_{2k}(t)x^{2k} \to 0 \quad \text{as } t \to 0.$$

Problem 4

Consider the following PDE:

$$u_t - \triangle_x u = f(t,x)e^{ct} \quad \text{for } t > 0, x \in \mathbb{R}^n, \quad u(0,x) = g(x) \tag{2}$$

Then, $f(t,x)e^{ct}$ and $g(x)$ are smooth and have compact supports. Therefore, there exists a solution u satisfying (2):

$$u(x,y) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)e^{ct}dyds$$

Let's consider $u' = ue^{-ct}$, then

$$\begin{aligned}
&u'_t - \triangle_x u' + cu' = u_t e^{-ct} - cue^{ct} - \triangle_x ue^{ct} + cue^{ct} = (u_t - \triangle_x u)e^{ct} = f(t,x) \\
&u'(0,x) = ue^{c\cdot 0} = u = g(x)
\end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}^n$. Therefore,

$$e^{-ct}\left(\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)e^{ct}dyds\right)$$

is a solution for the problem.

Problem 5

Let u be:

$$u(t,x) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + \frac{1}{2}\int_0^t \int_{x+t-s}^{x-t+s} f(y,s)dyds. \tag{3}$$

First, u is a smooth function since g, h, f are. Also,

$$\begin{aligned}
u_t &= \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \left(h(x+t) + h(x-t)\right) + \frac{1}{2}\int_0^t f(x+t-s,s) + f(x-t+s,s)ds \\
u_{tt} &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + f(x) + \int_0^t f_t(x+t-s) + f_t(x-t+s)ds \\
u_x &= \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \left(h(x+t) - h(x-t)\right) + \frac{1}{2}\int_0^t f(x+t-s,s) - f(x-t+s,s)ds \\
u_{xx} &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + \frac{1}{2}\int_0^t f_x(x+t-s,s) - f_x(x-t+s,s)ds \\
&= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + \frac{1}{2}\int_0^t f_t(x+t-s,s) + f_t(x-t+s,s)ds \\
&= u_{tt} - f(x,t)
\end{aligned}$$

Therefore, $u_{tt} - u_{xx} = f(x,t)$.

Also,

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(t,x) = g(x_0,0), \lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0,0)$$

since u_t and u is continuous on $\mathbb{R}^2$.

Hence, (3) is a solution for the problem.

Problem 6

WLOG, let $x \le y$. I'll show that

$$u(y) - u(x) = \int_y^x u' dt$$

Since $u' \in L^p(0,1)$, $\int_x^y |u'|dt \le \|u'\|_{L^p(\Omega)}$, and $u' \in L^1(\Omega)$.

If $x = y$, it is trivially equal, so I'll set $x < y$. Fix small enough $0 < \epsilon < \frac{x+y}{2}$ and consider $\phi_\epsilon(t) \in C_c^\infty((x,y))$ such that $\phi_\epsilon \equiv 1$ in $[x+\epsilon, y-\epsilon]$. Define

$$\eta_\delta(x) = \begin{cases} \frac{1}{\delta} e^{\frac{1}{\frac{x^2}{\delta^2}-1}} & (\text{For } |x| < \delta) \\ 0 & (\text{For } |x| \ge \delta) \end{cases}$$

I'll set the $\phi_\epsilon(t)$ by

$$\phi_\epsilon(t) = \begin{cases} \int_x^t \eta_\epsilon(\sigma - x)d\sigma / \int_x^{x+\epsilon} \eta_\epsilon(\sigma - x)d\sigma & (\text{For } x \le t < \frac{x+y}{2}) \\ 1 - \left(\int_{y-\epsilon}^t \eta_\epsilon(\sigma - (y-\epsilon))d\sigma / \int_{y-\epsilon}^y \eta_\epsilon(\sigma-(y-\epsilon))d\sigma \right) & (\text{For } \frac{x+y}{2} \le t \le y) \end{cases}$$

Then,

$$\begin{aligned} \int_\Omega u\phi'_\epsilon dt &= \int_x^{x+\epsilon} u\phi'_\epsilon dt + \int_{y-\epsilon}^y u\phi'_\epsilon dt \\ &= -\int_\Omega u'\phi_\epsilon dt = -\int_x^y u'\phi_\epsilon dt. \end{aligned}$$

Since $u' \in L^1(\Omega)$, $|u'\phi_\epsilon| \le |u'|$, and $\phi_\epsilon \to 1$ in $[x,y]$ a.e., so by Lebesgue dominance convergence theorem, $\lim_{\epsilon\to 0} \int_x^y u'\phi_\epsilon dt = \int_x^y u'dt$.

Let's consider $\int_x^{x+\epsilon} u\phi'_\epsilon dt$. We know that $\int_x^{x+\epsilon} \phi'_\epsilon dt = \phi_\epsilon(x+\epsilon) - \phi_\epsilon(x) = 1$ and $\int_0^\epsilon \eta_\epsilon(\sigma)\sigma = \int_0^1 \eta(\sigma)\sigma = C > 0$. Therefore,

$$\begin{aligned} \int_x^{x+\epsilon} u\phi'_\epsilon dt &= C^{-1} \int_x^{x+\epsilon} (u - u(x))\eta_\epsilon(t-x)dt + \int_0^\epsilon u(x)\phi'_\epsilon dt \\ &= \int_x^{x+\epsilon} (u-u(x))\eta_\epsilon(t-x)dt + u(x). \end{aligned}$$

Since $\|\eta_\epsilon\|_{L^\infty} = \frac{1}{\epsilon}$ and $C^{-1}\left|\int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt\right| \le \frac{\|\eta_\epsilon\|_{L^\infty}}{C}\left|\frac{1}{\epsilon}\int_x^{x+\epsilon}(u-u(x))dt\right|$, by Lebesgue differential theorem, $\int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt \to 0$ as $\epsilon \to 0$. Therefore, $\int_x^{x+\epsilon} u\phi'_\epsilon dt \to u(x)$ as $\epsilon \to 0$. By the same reason, $\int_{y-\epsilon}^y u\phi'_\epsilon dt \to -u(y)$ as $\epsilon \to 0$.

Therefore,

$$\int_x^y u'dt = u(y) - u(x)$$

a.e. and

$$|u(y) - u(x)| \leq \left| \int_x^y u' dt \right| \leq |x-y|^{1-\frac{1}{p}} \|u'\|_{L^p(\Omega)}$$

by Hölder's inequality.

Problem 7

Because u is compactly supported, we can set B be a large ball containing the compact support of u and set $u = 0$ out of compact support for integration; it does not effect the integration. Since $p \geq 2$, we can do integration by parts:

$$\begin{aligned}
\int_U |Du|^p dx &= \sum_{i=1}^n \int_B u_{x_i} u_{x_i} |Du|^{p-2} dx \\
&= -\sum_{i,j=1}^n \int_B u \left(u_{x_i x_i} |Du|^{p-2} + u_{x_i} u_{x_j} u_{x_j x_i} |Du|^{p-4} \right) dx \text{ Since } u \equiv 0 \text{ at boundary} \\
&\leq -\sum_{i,j=1}^n \int_B u \left(u_{x_i x_i} |Du|^{p-2} + B u_{x_j x_i} |Du|^{p-2} \right) dx \text{ Since } \sum_{i,j} u_{x_i} u_{x_j} \leq n \left| Du^2 \right| \text{ by Cauchy-Schwarz inequality} \\
&\leq -C \int_B u \left| D^2 u \right| \left(|Du|^{p-2} \right) dx \text{ By the same reason.} \\
&\leq \left| C \int_U u \left| D^2 u \right| \left(|Du|^{p-2} \right) dx \right|
\end{aligned}$$

for some constant B and C depends on n.

Let $p > 2$, then by the Hölder's inequality,

$$\left| \int_U u \left| D^2 u \right| \left(|Du|^{p-2} \right) dx \right|^p \leq \left(\int_U \left| u \left| D^2 u \right| \right|^{\frac{p}{2}} dx \right)^2 \left(\int_U |Du|^p dx \right)^{p-2}$$

If $\int_U |Du|^p dx = 0$, then the original inequality satisfied, so we can assume $\int_U |Du|^p dx > 0$. Then,

$$\left(\int_U |Du|^p dx \right)^2 \leq C^p \left(\int_U \left| u \left| D^2 u \right| \right|^{\frac{p}{2}} dx \right)^2 \leq C^p \left(\int_U |u|^p dx \right) \left(\int_U \left| D^2 u \right|^p dx \right)$$

Therefore,

$$\|Du\|_{L^p} \leq C \|u\|_{L^p}^{1/2} \|D^u\|_{L^p}^{1/2}$$

for $2 \leq p \leq \infty$ and all $u \in C_c^\infty(U)$.