

# Partial Differential Equation - HW 3

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## Problem 1

I'll imitate the proof in Evans.

*Proof.* Since each  $\Gamma_j$  is compact,  $\partial\Omega$  is compact and we can choose finite points  $x_i \in \partial\Omega$  with radius  $r_i > 0$  and  $\partial\Omega \subset \cup_{i=1}^n B(x_i, \frac{r_i}{2})$ . If  $x_i$  is not in end point of some  $\Gamma_j$  for all  $j$ , then we can use the argument in the Evans, so we only need to consider the case that  $x_i$  is in end point of  $\Gamma_j$  for some  $j$ .

Fix  $x^0$  be in end point of  $\Gamma_j$  and assume that  $x^0$  is also a end point of  $\Gamma_{j+1}$ . Let  $v_j$  be the tangential vector of  $\Gamma_j$  such that  $v_j$  is toward the  $\Gamma_j$ . In other words, if  $\Gamma_j : [a, b] \rightarrow \mathbb{R}^2$  and  $x_0 = \Gamma_j(b)$ , then  $v_j = \lim_{h \rightarrow 0+} \frac{\Gamma_j(b-h) - \Gamma_j(b)}{h}$ . For  $v_{j+1}$ , set it be tangential vector of  $\Gamma_{j+1}$  toward  $\Gamma_{j+1}$  curve. If  $v_j$  and  $v_{j+1}$  are parallel,  $\Gamma_j$  and  $\Gamma_{j+1}$  can be connected with  $C^1$  property, I'll ignore the case. Let the angle between  $v_j$  and  $v_{j+1}$  be  $\theta > 0$ . Let  $e_0$  be a unit vector such that parallel with  $\frac{v_j + v_{j+1}}{2}$  and inward direction, i.e.,  $x_0 + \lambda e_0 \in \text{int } \Omega$  for small enough  $\lambda$ . (This requires Jordan curve theorem.)

As  $\Gamma_j, \Gamma_{j+1}$  are  $C^1$ , there exists  $r > 0$  such that with  $B(x_0, r)$ , the tangential direction of  $\Gamma_j$  and  $\Gamma_{j+1}$  does not change much. More precisely, if we let  $w_j$  be a tangential vector of  $\Gamma_j$  at  $y \in B(x_0, r)$ , then the angle between  $w_j$  and  $v_j$  is less than  $\theta/10$ , and this is true for  $\Gamma_{j+1}$ . Let  $w_j^1$  (resp.  $w_j^2$ ) be the vector made by rotating  $v_j$  by  $\theta/10$  clockwise (resp. counter-clockwise). Do the same for  $w_{j+1}^1, w_{j+1}^2$ .

Now, let's repeat proof in Evans. Let's consider  $U \cap B(x_0, r)$  and  $V := U \cap B(x_0, r/2)$ . Define  $x_\epsilon := x + \epsilon e_0$  for  $x \in V$ , small enough  $\epsilon > 0$  satisfying  $x + \epsilon e_0 \in U \cap B(x_0, r)$ . WLOG, I'll assume that  $x$  is inside the interior enclosed by  $\Gamma_j$  and the line through  $x_0$  with tangential vector  $\frac{v_j + v_{j+1}}{2}$ . Now, we draw lines through  $x$  such that the tangential vectors  $w_j^1$  and  $w_j^2$ . Then, we know that the angle between  $e_0$  and  $w_j^1$  or  $w_j^2$  is  $(1/2 - 1/10)\theta$  and there is a room to set small enough  $\lambda < 1$  such that  $B(x + \epsilon e_0, \lambda\epsilon) \subset U \cap B(x_0, r)$ . In this room, we can mollify  $u_\epsilon(x) = u(x_\epsilon)$  and denote it  $v_\epsilon$ , and make  $\epsilon \rightarrow 0$ . The remaining part is same as the proof in Evans: Since  $\partial U$  is compact, we can choose finitely many points  $x_0^i \in \partial U$  including end point of  $\Gamma_j$  and make Global approximation.  $\square$

## Problem 2

1.  $W_0^{1,p}(\Omega)$  is a vector space: For  $f = 0$ ,  $f \in W_0^{1,p}(\Omega)$ , so  $W_0^{1,p}(\Omega) \neq \emptyset$ . For  $f_1, f_2 \in W_0^{1,p}(\Omega)$ , there exists  $f_1^j, f_2^j$  such that  $(f_1^j), (f_2^j) \in C_c^\infty(\Omega)$  and  $f_1^j \rightarrow f_1, f_2^j \rightarrow f_2$  in  $W^{1,p}(U)$ . Since union of two compact set in  $\Omega$  is compact in  $\Omega$ ,  $f_1^j + f_2^j \in C_c^\infty(\Omega)$  and for large enough  $N$  satisfying  $\|f_1^j - f_1\|_{W^{1,p}(\Omega)}, \|f_2^j - f_2\|_{W^{1,p}(\Omega)} \leq \epsilon/2$  for  $j > N$ ,  $\|f_1^j + f_2^j - f_1 - f_2\|_{W^{1,p}(\Omega)} \leq \|f_1^j - f_1\|_{W^{1,p}(\Omega)} + \|f_2^j - f_2\|_{W^{1,p}(\Omega)} \leq \epsilon$ . Therefore,  $f_1^j + f_2^j \rightarrow f_1 + f_2$  and  $f_1 + f_2 \in W^{1,p}(\Omega)$ . Also,  $\lambda f^j \rightarrow \lambda f$  in  $W^{1,p}(\Omega)$  for scalar  $\lambda$ . Therefore,  $W^{1,p}$  is vector space. (Other ...)
2. With the norm  $\|\cdot\|_{W^{1,p}(\Omega)}$ ,  $W_0^{1,p}(\Omega)$  is Banach space: Let  $f_j$  be a cauchy sequence in  $W_0^{1,p}(\Omega)$ . Since  $W^{1,p}(\Omega)$  is Banach space,  $f_j \rightarrow f$  in  $W^{1,p}(\Omega)$ . Since  $\Omega$  is bounded and  $\partial\Omega$  is  $C^1$ , there exists bounded

linear operator  $T : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$  and  $Tf_j \equiv 0$  on  $\partial U$  as  $f_j \in W_0^{1,p}(\Omega)$ . Then,

$$\lim_{j \rightarrow \infty} \|Tf_j - Tf\|_{W^{1,p}(\Omega)} = \lim_{j \rightarrow \infty} \|T(f_j - f)\|_{W^{1,p}(\Omega)} \leq \lim_{j \rightarrow \infty} \|T\|_{W^{1,p}(\Omega)} \|f_j - f\|_{W^{1,p}(\Omega)} = 0$$

as  $\|T\|_{W^{1,p}(\Omega)}$  is bounded. Therefore,  $Tf_j \rightarrow Tf$  and  $\lim_{j \rightarrow \infty} \|Tf_j\|_{W^{1,p}(\Omega)} = \|Tf\|_{W^{1,p}(\Omega)} = 0$ . As a result,  $f \in W_0^{1,p}(\Omega)$  implying Cauchy sequence in  $W_0^{1,p}(\Omega)$  converges.

Therefore,  $W_0^{1,p}(\Omega)$  is Banach space.

### Problem 3

For  $k \in \mathbb{N}$  and  $\alpha \in (0, 1]$ ,

$$C^{k,\alpha}(\bar{\Omega}) := \{u \in C^k(\bar{\Omega}) : \|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty\}$$

Before starting, I need to show that  $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$  is a norm on  $C^{k,\alpha}(\bar{\Omega})$ .

*Proof.* 1. By the definition of  $C^{k,\alpha}(\bar{\Omega})$ , we know that  $\|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$  for any  $u \in C^{k,\alpha}(\bar{\Omega})$ . Let  $u, v \in C^{k,\alpha}(\bar{\Omega})$ . Then

$$\begin{aligned} \|u + v\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha| \leq k} \|D^\alpha(u + v)\|_{C(\bar{\Omega})} + \sum_{|\alpha|=k} [D^\alpha(u + v)]_{C^{0,\alpha}(\bar{\Omega})} \\ &= \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha(u + v)| + \sum_{|\alpha|=k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha(u + v)(x) - D^\alpha(u + v)(y)|}{|x - y|^\alpha} \right\} \\ &\leq \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u| + \sup_{x \in \Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)| + |D^\alpha v(x) - D^\alpha v(y)|}{|x - y|^\alpha} \right\} \\ &\leq \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u| + \sup_{x \in \Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\alpha} \right\} + \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x - y|^\alpha} \right\} \\ &= \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})} \end{aligned}$$

Therefore,  $\|u + v\|_{C^{k,\alpha}(\bar{\Omega})} \leq \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})}$ .

2. For  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} \|\lambda u\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha \lambda u| + \sum_{|\alpha|=k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha \lambda u(x) - D^\alpha \lambda u(y)|}{|x - y|^\alpha} \right\} \\ &= |\lambda| \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u| + |\lambda| \sum_{|\alpha|=k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\alpha} \right\} \\ &= \lambda \|u\|_{C^{k,\alpha}(\bar{\Omega})}. \end{aligned}$$

3. For  $u = 0$ ,  $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$ . Conversely, if  $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$ , then  $\|u\|_{C(\Omega)} = 0$  with continuity of  $u$ , so  $u = 0$  on  $\bar{\Omega}$ .

Therefore,  $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$  is a norm.

(a) Clearly,  $0 \in C^{k,p}(\bar{\Omega})$ . For  $f_1, f_2 \in C^{k,p}(\bar{\Omega})$ ,  $f_1 + f_2 \in C^k(\Omega)$  and  $\|f_1 + f_2\|_{C^{k,\alpha}(\bar{\Omega})} \leq \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} + \|f_2\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$ . Therefore,  $f_1 + f_2 \in C^{k,\alpha}(\bar{\Omega})$ .  $f_1 + f_2 = f_2 + f_1$  and for scalar  $\lambda$ ,  $\lambda f_1 \in C^{k,\alpha}(\bar{\Omega})$  for  $\|\lambda f_1\|_{C^{k,\alpha}(\bar{\Omega})} = |\lambda| \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$ . Therefore,  $C^{k,p}(\bar{\Omega})$  is a vector space.

- (b) Fix  $x \in \Omega$  and take an open neighborhood  $B(x, r) \subset \Omega$  for some  $r > 0$ . Then there exists  $N \in \mathbb{N}$  such that for  $\frac{1}{N} < \epsilon$ , then  $B(x, \frac{1}{n}) \subset \Omega$  for  $n > N$ . I'll use  $C^\infty$  Urysohn lemma to show that there exists infinitely many linearly independent elements in  $C^{k, \alpha}(\bar{\Omega})$ . For  $n > N$ , take  $K_n = \overline{B(x, \frac{1}{n+1})}$  and  $U_n = B(x, \frac{1}{n+1} + (\frac{1}{n} - \frac{1}{n+1})/2)$ . Using  $C^\infty$  Urysohn lemma, take  $\phi^n \in C^\infty$  such that 1 on  $K_n$  and has support in  $U$ . Take finite elements in the set:  $\{\phi^j\}_{j=N_1}^{N_n}$ ,
- (c) Let  $\{u_i\}$  be a Cauchy sequence in  $C^{k, p}(\bar{\Omega})$ . For fixed  $\epsilon > 0$ , there exists  $N$  such that  $i, j > N \Rightarrow \|u_i - u_j\|_{C^{k, p}(\bar{\Omega})} \leq \epsilon$ . It implies

$$\begin{cases} \|D^\alpha u_i - D^\alpha u_j\|_{C(\bar{\Omega})} \leq \epsilon & \text{For } |\alpha| \leq k \\ [D^\alpha u_i - D^\alpha u_j]_{C^{0, \gamma}(\bar{\Omega})} \leq \epsilon & \text{For } |\alpha| = k. \end{cases}$$

Since  $D^\alpha u_i$  is uniformly Cauchy for  $|\alpha| \leq k$ ,  $D^\alpha u_i$  converges to  $u_\alpha$  for  $|\alpha| \leq k$  pointwisely. Also, these convergences are uniform. Therefore,  $D^\alpha u = u_\alpha$  for all  $|\alpha| \leq k$ .

Letting  $i \rightarrow \infty$ ,  $[D^\alpha u - D^\alpha u_j]_{C^{0, \gamma}(\bar{\Omega})} \leq \epsilon$  for  $j > N$ . Also,

$$\begin{aligned} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} - \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha u_j(x) - D^\alpha u_j(y)|}{|x - y|^\gamma} \right\} &\leq \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} - \frac{|D^\alpha u_j(x) - D^\alpha u_j(y)|}{|x - y|^\gamma} \\ &\leq \frac{|D^\alpha(u - u_j)(x) - D^\alpha(u - u_j)(y)|}{|x - y|^\gamma} \\ &\leq [D^\alpha u - D^\alpha u_j]_{C^{0, \gamma}(\bar{\Omega})} \leq \epsilon \end{aligned}$$

for all  $x, y \in \Omega, x \neq y$ . Therefore,  $\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} \leq [D^\alpha u_j]_{C^{0, \gamma}(\bar{\Omega})} + \epsilon$  and  $[D^\alpha u]_{C^{0, \gamma}(\bar{\Omega})} < \infty$ . Therefore,  $u \in C^{k, \alpha}(\bar{\Omega})$ . It means  $C^{k, \alpha}(\bar{\Omega})$  is Banach space.

□

## Problem 4

I'll follow the proof in Evans.

*Proof.* Since  $U$  is bounded, open subset of  $\mathbb{R}^n$ , and  $\partial\Omega$  is  $C^1$ ,

$$W^{1, p}(\Omega) \subset C^{0, \alpha}(\bar{\Omega}), \quad \|u\|_{C^{0, \alpha}(\bar{\Omega})} \leq C \|u\|_{W^{1, p}(\Omega)}$$

for  $\alpha = 1 - n/p$  and  $C$  depends only on  $p, n$  and  $\Omega$ . Also,  $C^{0, \alpha}(\bar{\Omega}) \subset C^{0, \tilde{\alpha}}(\bar{\Omega})$  since  $\|u\|_{C(\bar{\Omega})}$  is same for both norm and if  $[u]_{C^{0, \alpha}(\bar{\Omega})} < \infty$ , then as  $|x - y| \rightarrow 0$ ,  $\frac{|u(x) - u(y)|}{|x - y|^\alpha} \rightarrow 0$  because  $\frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$  for all  $x, y \in \Omega, x \neq y$ ,  $[u]_{C^{0, \tilde{\alpha}}(\bar{\Omega})} < \infty$  and  $u \in C^{0, \tilde{\alpha}}(\bar{\Omega})$ .

Now, we need to show that each bounded sequence in  $W^{1, p}(\Omega)$  is precompact in  $C^{0, \alpha}(\bar{\Omega})$ . Let a bounded sequence in  $W^{1, p}(\Omega)$ :  $\{u_m\}_{m=1}^\infty$  and  $\sup_m \left\{ \|u_m\|_{W^{1, p}(\Omega)} \right\} = K$ . By Morney's inequality, we can assume that  $\{u_m\} \subset C^{0, \alpha}(\bar{\Omega}) \subset C^{0, \tilde{\alpha}}(\bar{\Omega})$  and there exists constant  $K'$  such that  $\|u\|_{C^{0, \alpha}(\bar{\Omega})} \leq K'$  for all  $m$ . Let's check the condition for Arzela-Ascoli theorem

1. For each  $m$ ,  $u_m$  is continuous on compact set  $\bar{\Omega}$ .
2. Since  $\|u_m\|_{C(\bar{\Omega})} \leq K'$ ,  $\{u_m\}$  is pointwisely bounded.

3.  $\frac{|u_m(x) - u_m(y)|}{|x - y|^\alpha} \leq K'$  for all  $x, y \in \Omega$ ,  $x \neq y$ . Therefore,  $|u_m(x) - u_m(y)| \leq K'|x - y|^\alpha$  for  $x, y \in \bar{\Omega}$  for all  $m$  and it means  $\{u_m\}$  is equicontinuous on  $\bar{\Omega}$ .

Therefore, we can use Arzela-Ascoli theorem and find a uniformly convergent subsequence  $\{u_{m_j}\}$  in  $C^{0,\tilde{\alpha}}(\bar{\Omega})$ . Since  $C^{0,\tilde{\alpha}}(\bar{\Omega})$  is Banach space, the converging point is in  $C^{0,\tilde{\alpha}}(\bar{\Omega})$ . Hence,

$$W^{1,p}(\Omega) \subset\subset C^{0,\tilde{\alpha}}(\bar{\Omega})$$

for all  $\tilde{\alpha} \in [0, \alpha)$ . □

## Problem 5

Fix  $\epsilon > 0$ . Define  $\Omega_\epsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \epsilon\}$ . Let's mollify the  $u$  with standard mollifier  $\eta_\epsilon$  and denote it  $u^\epsilon$ . Then,

$$Du^\epsilon = \eta_\epsilon * Du = 0$$

in  $\Omega_\epsilon$ . It implies that if  $B(x, r) \subset \Omega_\epsilon$  for small enough  $r > 0$ ,  $u^\epsilon$  is constant on  $B(x, r)$  since the derivative of  $u^\epsilon$  is zero on the set. In other words, it is locally constant in  $\Omega_\epsilon$ .

Let  $x \in U$  and  $B(x, r)$  be an open neighborhood of  $x$  in  $\Omega$ , then there exists  $\epsilon$  such that  $B(x, r) \subset \Omega_\epsilon$  and by previous, we know that  $u^\epsilon$  is constant on  $B(x, r)$ . Let the constant value  $c^\epsilon$ . We know that  $u^\epsilon \rightarrow u$  as  $\epsilon \rightarrow 0$  and it means on  $u$  is constant a.e. on  $B(x, r)$ . (If not, there always exists non measure zero set such that  $u^\epsilon$  is different with  $u$  on  $B(x, r)$ .) Therefore,  $u$  is locally constant function in a.e. sense).

Let take a partition such that  $x \sim y$  if  $u(x) = u(y)$ . Since  $\Omega$  is locally constant, any element in partition is open set. Assume that there exists at least two element in the partition. This is impossible since  $\Omega$  is connected set. Therefore,  $u$  is a.e. constant function.

## Problem 6

First, I'll show that  $u \in L^n(B_1(\mathbf{0}))$ . Note that  $u$  is symmetric function about rotation, so we can show that integral on  $B_1(\mathbf{0})$  is finite by showing that integral is finite for  $r$ . Also, we can restrict the range of  $r$  to  $(0, \frac{1}{e-1})$  since  $u$  is bounded in outside of the range. In other words,

$$\int_{B_1(\mathbf{0})} u dx \leq C \int_0^{\frac{1}{e-1}} \left( \left| \log \log \left( 1 + \frac{1}{r} \right) \right| \right)^n r^{n-1} dr$$

for some constant  $C < \infty$ . Let  $y = \log \left( 1 + \frac{1}{r} \right)$ , then

$$\begin{aligned} \left| \int_0^{\frac{1}{e-1}} \left( \log \log \left( 1 + \frac{1}{r} \right) \right)^n r^{n-1} dr \right| &\leq \int_1^\infty (\log y)^n \frac{e^y}{(e^y - 1)^{n+1}} dy \\ &\leq \int_1^\infty (\log y)^n \frac{2^{n+1} e^y}{e^{(n+1)y}} dy \\ &\leq \int_1^\infty y^n 2^{n+1} e^{-ny} dy < \infty \end{aligned}$$

Therefore,  $u \in L^n(B_1(\mathbf{0}))$ , and  $u \in L^1(B_1(\mathbf{0}))$ .

Next, I'll show that  $u$  has weak derivative in  $B_1(\mathbf{0})$  and belongs to  $L^n(B_1(\mathbf{0}))$ . Since  $u$  goes to  $\infty$  as  $x \rightarrow 0$ , we need to care when we compute weak derivative. However, we can ignore at  $\mathbf{0}$  by the following argument. Let  $V$  be a compactly embedded set in  $U$  and  $\phi$  be a  $C^\infty$  function having support  $V$ . Assume

$\mathbf{0} \in V$ . Without  $\mathbf{0}$ ,  $Du$  should be  $\partial_{x_i} u$  for some  $i$ . Since  $u, D^\alpha \phi$  for all  $\alpha$  are  $L^1$  function on  $V$ , we can use Fubini theorem, and rewrite the integral by

$$\int_U u D\phi dx = \int_{-1}^1 (\cdots) dx_1.$$

Since  $n > 1$ , we know that the  $n - 1$  dim plane through  $0$  is measure zero set and it does not effect integral to delete  $0$  from integral range of  $x_1$ . Therefore, the weak derivative is just derivative of  $u$  except  $\mathbf{0}$ ... (Fundamental of Calculus?  $d/dx_1 \int \int dx_1 dx_2 \dots$  Explicitly show)

I'll show that  $Du$  is in  $L^n$ . Computing partial derivative:

$$|\partial_{x_i} u| = \left| \frac{1}{\log \left(1 + \frac{1}{|x|}\right)} \frac{1}{1 + \frac{1}{|x|}} \frac{x_i}{|x|^3} \right| \leq \frac{1}{\left| \log \left(1 + \frac{1}{r}\right) \right|} \frac{1}{r+1} \frac{1}{r}.$$

Then, by the same reason before, we just need to check whether the integral is finite for  $r$  in  $\left(0, \frac{1}{e-1}\right)$ .

$$\int_0^{\frac{1}{e-1}} \left( \frac{1}{\log \left(1 + \frac{1}{r}\right)} \frac{1}{r+1} \frac{1}{r} \right)^n r^{n-1} dr \leq \int_0^{\frac{1}{e-1}} \left( \frac{1}{\log \left(1 + \frac{1}{r}\right)} \right)^n \frac{1}{r} dr$$

Let  $x = \log \left(1 + \frac{1}{r}\right)$ , then the integral becomes

$$\int_1^\infty \frac{1}{x^n} \frac{e^x}{e^x - 1} dx$$

For sufficiently large  $R$ ,  $\frac{e^x}{e^x - 1} < 2$  for  $x > R$  and we know that  $\int_1^\infty \frac{1}{x^n}$  converges for  $n > 2$ . Therefore,  $Du \in L^n(B_1(\mathbf{0}))$  and  $u \in W^{1,n}(B_1(\mathbf{0}))$ .

## Problem 7

Since  $u \in L^2(\mathbb{R}^n)$ ,  $u = (\hat{u})^\vee$  by Theorem 2 in chapter 4.3 Evans. Then,

$$\begin{aligned} |u(x)| &\leq \int_{\mathbb{R}^n} |e^{ikx} \hat{u}(k)| dk \leq \int_{\mathbb{R}^n} |\hat{u}(k)| dk \\ &= \int_{\mathbb{R}^n} (1 + |k|^2)^{s/2} (1 + |k|^2)^{-s/2} |\hat{u}(k)| dk \\ &\left( \leq \int_{\mathbb{R}^n} (1 + |k|^2)^s |\hat{u}|^2 dk \right)^{1/2} \left( \int_{\mathbb{R}^n} (1 + |k|^2)^{-s} dk \right)^{1/2} \end{aligned}$$

For  $|k| > 1$ ,  $(1 + |k|^2)^s > (2|k|)^{2s}$

$$\int_{|k|>1} k^{-2s} dk = \sigma(S^{n-1}) \int_1^\infty r^{-2s} r^{n-1} dr < \infty$$

since  $-2s + n - 1 < -1$  and  $\int_1^\infty r^\alpha dr < \infty$  for  $\alpha < -1$ . Therefore,

$$|u(x)| \leq C \left( \int_{\mathbb{R}^n} (1 + |y|^2)^s |\hat{u}|^2 dy \right)^{1/2} = C \|u\|_{H^s(\mathbb{R}^n)}$$

This is true for a.e.  $x$ , so

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \|H^s(\mathbb{R}^n)\|$$