Partial Differential Equation - HW 4

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Problem 1

I'll use some theorem from topology and functional analysis, and I'll follow the proof in Real Analysis, Gerland B. Folland.

Theorem 1. *(The Baire Category Theorem) Let X be a complete metric space. Then,*

(a) *If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, i,e, $\overline{\cup_1^\infty U_n} = X$, then $\cap_1^\infty U_n$ is dense in X.*

(b) *X is not a countable union of nowhere dense sets.*

Proof. For (a), I'll show that nonempty open set V in X have intersection with $\cap_1^\infty U_n$ using induction. Since U_1 is open dense subset of X, $U_1 \cap W$ is open and nonempty, so there exists $B(x_0, r_0) \subset U_1 \cap W$ such that $0 < r_0 < 1$. For $n > 0$, choose x_n and r_n by follows: assume that for $j < n$, x_j and r_j are chosen. Then, $B(x_{n-1}, r_{n-1}) \cap U_n$ is nonempty and open. Choose $x_n \in B(x_{n-1}, r_{n-1}) \cap U_n$ and choose $0 < r_n < 2^{-n}$ such that $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$. For $n, m \geq N$, $x_n, x_m \in B(x_N, r_N)$, and $r_N \to 0$ as $N \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X and have limit point $x \in X$. Since $x_n \in B(r_N, x_N)$ for all $n \geq N$, $x \in \overline{B(x_N, r_N)} \subset U_N \cap B(x_0, r_0) \subset U_N \cap W$ for all N, implying $(\cap_{n=1}^\infty U_n) \cap W \neq \phi$.

For (b), If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense subsets in X, then $\{\overline{E_n}\}$ are a open dense sets since for fixed n and for any $x \in \overline{E}$, $B(x, r) \cap \overline{E}^c \neq \phi$ for all $r > 0$. Since $\cap \left(\overline{E_n}\right)^c \neq \phi$, $\cup E_n \subset \cup \overline{E_n} \neq X$. $\square$

Theorem 2. *(Open Mapping Theorem) Let X and Y be Banach spaces. If T is a surjective bounded linear functional from X to Y, T is a open map.*

Proof. If $T(B(0,1))$ contains a ball of radius $r > 0$, T is open map since for any open set $x \in U$ in X, it contains open ball $B(x, r)$, $r > 0$ and $T(B(x, r)) = T(x) + T(B(0, r)) = T(x) + rT(B(0, 1))$, which is open set in Y centered at $T(x)$. Therefore, I'll show that $T(B(0,1))$ contains a ball of radius $r > 0$. I'll denote $B(0, r) = B_r$.

Since $X = \cup_{n=1}^\infty B_n$ and T is surjective, $Y = T(\cup_{n=1}^\infty B_n)$. If $T(B_1)$ is nowhere dense, $T(B_n)$ are nowhere dense since dialation in Y induces homeomorphism. However, this is impossible by the Baire Category Theorem since Y is complete. Therefore, $\overline{T(B_1)}$ contains an open ball in Y and let it $B(y_0, 4r)$ for some $y_0 \in Y$, $r > 0$. Choose $y_1 = Tx_1 \in T(B_1)$ such that $\|y_1 - y_0\| \leq 2r$; then $B(y_1, 2r) \subset B(y_0, 4r) \subset \overline{T(B_1)}$, so if $\|y\| < 2r$,

$$y = -Tx_1 + (y_1 + y) \in \overline{T(-x_1 + B_1)} \subset \overline{T(B_2)}.$$

Dividing both by 2, we get if $\|y\| < r$, $y \in \overline{T(B_1)}$. I need to change $\overline{T(B_1)}$ to $T(B_1)$.

As noted above, dialation induces homeomorphism, so if $\|y\| < r/2^n$, $y \in \overline{T(B_{1/2^n})}$. Suppose that $y \in r/2$, then there exists $x_1 \in B_{1/2}$ such that $\|y - Tx_1\| \leq r/4$. Proceeding this argument, we can find x_n satisfying $\left\|y - T\sum_{i=1}^n x_i\right\| \leq r/2^{n+1}$. Since X is complete and $\left\|\sum_{i=n}^m x_i\right\| \leq 2^{-n+1}$, it has convergent point x and $\|x\| \leq 1$ and $Tx = y$ since $\|Tx - y\| = 0$ in Y. Therefore, $T(B_1)$ contains all y in $\|y\| < r/2$. $\square$

Theorem 3. *(Closed Graph Theorem) If X and Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.*

Proof. Let π_1 and π_2 be the projection of $\Gamma(T) := \{(x, Tx) \in X \times Y\}$ to X and Y, then these maps are onto. It is clear that π_1 and π_2 are bounded linear functional. ALso, $X \times Y$ and $\Gamma(T)$ is complete since X and Y are complete and $\Gamma(T)$ is closed as T is closed. Since π_1 is a bijection between $\Gamma(T)$ and X, π_1^{-1} is continuous since π_1 bounded and continuous. Thus, $T = \pi_2 \circ \pi_1^{-1}$ is bounded. $\qquad\square$

If $\eta \in \rho(A)$, $A - \eta\mathrm{Id}$ is one-to-one and onto by definition. Therefore, we can consider the inverse $(A - \eta\mathrm{Id})^{-1}$. I'll first show that this is closed linear map.

Linearity: Fix $y_1, y_2 \in X$ and $r \in \mathbb{R}$ or $\mathbb{C}$. Then, there uniquely exists x_1 and x_2 in X such that $(A - \eta\mathrm{Id})(x_i) = y_i$ for each i and $(A - \eta\mathrm{Id})(rx_i) = ry_i$. Therefore, $(A - \eta\mathrm{Id})^{-1}(y_i) = x_i$ and $(A - \eta\mathrm{Id})^{-1}(y_1 + ry_2) = x_1 + rx_2$. Thus, $(A - \eta\mathrm{Id})$ is bijective and linear.

Closedness: Suppose $y_n \to y$ in X and $(A - \eta\mathrm{Id})^{-1}(y_n) \to x$. We need to show that $(A - \eta\mathrm{Id})^{-1}(y) = x$. The second assumption implies that $y_n \to (A - \eta\mathrm{Id})(x)$ since $(A - \eta\mathrm{Id})$ is continuous. (Note that boundedness implies continuity if X is normed vector space and the map is linear.) The normed topology in Banach space gives Hausdorff property: if $x_1 \neq x_2$, $r = \|x_1 - x_2\| > 0$ and $B(x_1, r/3)$, $B(x_2, r/3)$ gives disjoint neighborhoods of each x_i. Therefore, y_n converges to same point and $(A - \eta\mathrm{Id})(x) = y$.

Since X is Banach space and $(A - \eta\mathrm{Id})^{-1} : X \to X$ is closed linear map, it is bounded by the Closed Graph Theorem.

Problem 2

(a) I'll check the condition for inner product.

   (1) For all $u, u_* \in \mathcal{H}$,

$$\overline{(u_*, u)}_{\mathcal{H}} = \overline{\int_\Omega (v_* + iw_*)(v - iw) + (Dv_* + iDw_*)(Dv - iDw)dx}$$

$$= \int_\Omega (v_* - iw_*)(v + iw) + (Dv_* - iDw_*)(Dv + iDw)dx \quad \text{since } v, w, v_*, w_* : X \to \mathbb{R}$$

$$= (u, u_*)_{\mathcal{H}}.$$

   (2) For $a, b \in \mathbb{C}$ and $u, u_*, u_{**} \in \mathcal{H}$,

$$(au + bu_*, u_{**})_{\mathcal{H}} = \int_\Omega ((av + bv_*) + i(aw + bw_*))(v_{**} - iw_{**})$$

$$+ (D(av + bv_*) + iD(aw + bw_*))(Dv_{**} - iDw_{**})dx$$

$$= a \int_\Omega (v + iw)(v_{**} - iw_{**}) + (Dv + iDw)(Dv_{**} - iDw_{**})dx$$

$$+ b \int_\Omega (v_* + iw_*)(v_{**} - iw_{**}) + (Dv_* + iDw_*)(Dv_{**} - iDw_{**})dx$$

$$= a(u, u_{**})_{\mathcal{H}} + b(u_*, u_{**})_{\mathcal{H}}.$$

   (3) For nonzero $u \in \mathcal{H}$,

$$(u, u)_{\mathcal{H}} = \int_\Omega (v + iw)(v - iw) + (Dv + iDw)(Dv - iDw)dx$$

$$= \int_\Omega v^2 + w^2 + |Dv|^2 + |Dw|^2 dx$$

$$= \|v\|_{H_0^1(\Omega)}^2 + \|w\|_{H_0^1(\Omega)}^2 \in (0, \infty).$$

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If both $\|v\|^2_{H^1_0(\Omega)}$ and $\|w\|^2_{H^1_0(\Omega)}$ are zero, $v = w = 0$ in $H^1_0(\Omega)$ and it means $u = 0$, which is contradiction.

Therefore, $(\cdot, \cdot)_{\mathcal{H}}$ yields an inner product in $\mathcal{H}$.

(b) First, note that $\mathcal{H}$ is a Hilbert space.(First, $H^1_0(\Omega)$ is a Hilbert space as a closed subspace of Hilbert space $H^1(\Omega)$. Assume that $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$ about the norm defined above. By (3) in (a), $(u_i - u_j, u_i - u_j)_{\mathcal{H}} = \|\text{Re}(u_i - u_j)\|^2_{H^1_0(\Omega)} + \|\text{Im}(u_i - u_j)\|^2_{H^1_0(\Omega)}$ and both $\text{Re}(u_i - u_j)$ and $\text{Im}(u_i - u_j)$ are Cauchy sequence. Therefore, it has a convergent subsequence and $\mathcal{H}$ is a Hilbert space.)

I'll first show that the bilinear form B is bounded.

Bilinearity: For $u_1, u_2, v_1, v_2 \in \mathcal{H}$ and $c \in \mathbb{C}$,

$$B[cu_1 + u_2, v_1] = \int_\Omega a_{ij} \partial_i (cu_1 + u_2) \partial \overline{v}_1 \ dx$$

$$= c \int_\Omega a_{ij} \partial_i u_1 \partial \overline{v}_1 \ dx + \int_\Omega a_{ij} \partial_i u_2 \partial \overline{v}_1 \ dx$$

$$= cB[u_1, v_1] + B[u_2, v_1]$$

and

$$B[u_1, cv_1 + v_2] = \int_\Omega a_{ij} \partial_i u_1 \partial (c\overline{v}_1 + \overline{v}_2) \ dx$$

$$= \overline{c} \int_\Omega a_{ij} \partial_i u_1 \partial \overline{v}_1 \ dx + \int_\Omega a_{ij} \partial_i u_2 \partial \overline{v}_1 \ dx$$

$$= \overline{c}B[u_1, v_1] + B[u_2, v_1]$$

Boundedness: For $u, v \in \mathcal{H}$,

$$|B[u, v]| = \left| \int_\Omega a_{ij} \partial_i u \partial_j \overline{v} \ dx \right|$$

$$= \left| \int_\Omega a_{ij} \partial_i (\text{Re}(u) + i\text{Im}(u)) \partial_j (\text{Re}(v) - i\text{Im}(v)) \ dx \right|$$

$$= \left| \int_\Omega a_{ij} (\partial_i \text{Re}(u) \partial_j \text{Re}(v) - \partial_i \text{Im}(u) \partial_j \text{Im}(v)) \ dx \right|$$

$$+ \left| \int_\Omega a_{ij} (\partial_i \text{Re}(u) \partial_j \text{Im}(v) + \partial_i \text{Im}(u) \partial_j \text{Re}(v)) \ dx \right| \quad (*)$$

$$\leq \int_\Omega \frac{1}{\mu} (|D(\text{Re}(u))||D(\text{Re}(v))| + |D(\text{Im}(u))||D(\text{Im}(v))|) \ dx$$

$$+ \int_\Omega \frac{1}{\mu} (|D(\text{Re}(u))||D(\text{Im}(v))| + |D(\text{Im}(u))||D(\text{Re}(v))|) \ dx$$

$$= \frac{1}{\mu} \left(\|D(\text{Re}(u))\|_{L^2(\Omega)} + \|D(\text{Im}(u))\|_{L^2(\Omega)} \right) \left(\|D(\text{Re}(v))\|_{L^2(\Omega)} + \|D(\text{Im}(v))\|_{L^2(\Omega)} \right)$$

$$\leq \frac{1}{\mu} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

In the middle step $(*)$, I used the diagonalizable of positive definite matrix of A: For unit vector $\eta_1 \in \mathbb{R}^n$, it can be decomposed by the eigenvectors of A, so if

$$\eta_1 = \sum_{i=1}^n b_i \xi_i$$

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for unit eigenvectors of A $\{\xi_i\}$, then

$$A\eta_1 = \sum_{i=1}^{n} \lambda_i b_i \xi_i$$

and, for unit vector $\eta_2 \in \mathbb{R}^n$ with decomposition $\eta_2 = \sum_{i=1}^{n} c_i \xi_i$,

$$\eta_2^T A \eta_1 = \sum_{i=1}^{n} \lambda_i c_i b_i \leq \max\{|\lambda_i|\} = \frac{1}{\mu}$$

since all the eigenvalues are not bigger than $\frac{1}{\mu}$ and are not smaller than $\mu > 0$.

Therefore, it is bounded in $\mathcal{H}$. Fix $u \in \mathcal{H}$ and consider a bounded linear functional T on $\mathcal{H}$ such that $T(v) = B[u, v]$ for all $v \in \mathcal{H}$. I can use the Riesz Representation Thereom and find a unique element $w \in \mathcal{H}$ satisfying

$$B[u, v] = (w, v)_{\mathcal{H}}$$

for all $v \in \mathcal{H}$. It means we can find $w \in \mathcal{H}$ satisfying above for each $u \in \mathcal{H}$, so we can construct $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{L}(u) = w$. I need to show that this is bounded linear functional.

Linearity: For $\lambda_1, \lambda_2 \in \mathbb{C}$, $u_1, u_2, v \in \mathcal{H}$,

$$\begin{aligned}
(\mathcal{L}(\lambda_1 u_1 + \lambda_2 u_2), v)_{\mathcal{H}} &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\
&= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\
&= \lambda_1 (\mathcal{L}(u_1), v)_{\mathcal{H}} + \lambda_2 (\mathcal{L}(u_2), v)_{\mathcal{H}} \\
&= (\lambda_1 \mathcal{L}(u_1) + \lambda_2 \mathcal{L}(u_2), v)_{\mathcal{H}}
\end{aligned}$$

for all $v \in \mathcal{H}$. By the uniqueness part of the Riesz Representation Theorem, $\mathcal{L}(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \mathcal{L}(u_1) + \lambda_2 \mathcal{L}(u_2)$.

Boundedness: For $u \in \mathcal{H}$,

$$\|\mathcal{L}u\|_{\mathcal{H}}^2 = (\mathcal{L}u, \mathcal{L}u)_{\mathcal{H}} = B[u, \mathcal{L}u] \leq \frac{1}{\mu} \|u\|_{\mathcal{H}} \|\mathcal{L}u\|_{\mathcal{H}}$$

Therefore, $\|\mathcal{L}u\|_{\mathcal{H}} \leq \frac{1}{\mu} \|u\|_{\mathcal{H}}$ and $\mathcal{L}$ is bounded.

(c) First, $\mathcal{L}^*$ is well-defined since if $v_1 = v_2$ in $\mathcal{H}$, $(u, \mathcal{L}^* v_1) = (\mathcal{L}u, v_1) = (\mathcal{L}u, v_2) = (u, \mathcal{L}^* v_2)$ for all $u \in \mathcal{H}$, so $\mathcal{L}^* v_1 = \mathcal{L}^* v_2$.

Since (a_{ij}) is real valued and $a_{ij} = a_{ji}$, $B[u, v] = B[\bar{v}, \bar{u}] = \overline{B[v, u]}$. Using this relation, for fixed $u \in \mathcal{H}$ and all $v \in \mathcal{H}$,

$$(\mathcal{L}^* u, v)_{\mathcal{H}} = \overline{(v, \mathcal{L}^* u)_{\mathcal{H}}} = \overline{(\mathcal{L}v, u)_{\mathcal{H}}} = \overline{B[v, u]} = B[\bar{v}, \bar{u}] = B[u, v] = (\mathcal{L}u, v)_{\mathcal{H}}.$$

Therefore, $(\mathcal{L}u, v)_{\mathcal{H}} = (\mathcal{L}^* u, v)_{\mathcal{H}}$ for all v and by Riesz Representation Theorem again, $\mathcal{L}^* u = \mathcal{L}u$ for all $u \in \mathcal{H}$.

(d) Suppose u is a nontrivial weak solution for the eigenvalue problem. By computing inner product;

$$\int_{\Omega} (Lu)\bar{v} = \int_{\Omega} -\partial_j(a_{ij}\partial_i u)\bar{v} = \int_{\Omega} a_{ij}\partial_i u \partial_j \bar{v} = B[u, v]$$

for all $v \in \mathcal{H}$ since the real part and imaginary part of u, v are in $H_0^1(\Omega)$. Therefore, we can define the condition of weak solution u to satisfy

$$B[u, v] = (\lambda u, v)_{L^2(\Omega)}$$

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for all $v \in \mathcal{H}$. However, as $u = 0$ on $\partial\Omega$, $u \in \mathcal{H}$ and

$$B[u, u] = (\mathcal{L}u, u)_{\mathcal{H}} = (\mathcal{L}^*u, u)_{\mathcal{H}} = (u, \mathcal{L}u)_{\mathcal{H}} = \overline{(\mathcal{L}u, u)_{\mathcal{H}}} = \overline{B[u, u]}$$

and it makes $\lambda(u, u)_{L^2(\Omega)} = (\lambda u, u)_{L^2(\Omega)} = \overline{(\lambda u, u)_{L^2(\Omega)}} = \bar{\lambda}(u, u)_{L^2(\Omega)}$. The only case to consider is that $(u, u)_{L^2(\Omega)} \neq 0$. Assume that $(u, u)_{L^2(\Omega)} = 0$, then $\|u\|_{\mathcal{H}} = \|Du\|_{H_0^1(\Omega)} > 0$. However, it means

$$|B[u, u]| = \left| \int_\Omega a_{ij}\partial_i u \partial_j \bar{u} \ dx \right| \geq \mu\|Du\|^2 > 0$$

which is contradiction since $B[u, u] = (\lambda u, u)_{L^2(\Omega)} = \lambda(u, u)_{L^2(\Omega)} = 0$. Therefore, $\lambda \in \mathbb{R}$.

Problem 3

Since $u \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega, \end{cases}$$

for all $v \in H_0^1(\Omega)$,

$$B[u, v] = \int_\Omega fv \ dx$$

where the bilinear form $B[\cdot, \cdot]$ is defined by

$$\int_\Omega \sum_{i,j} a_{ij}\partial_i u \partial_j v + \sum_i b_i v \partial_i u + cuv \ dx.$$

Define $w = u - g \in H^1(\Omega)$ (since $g \in H^1(\Omega)$), then $B[u, v] = B[w + g, v] = B[w, v] + B[g, v]$, and

$$B[w, v] = \int_\Omega fv - \sum_{i,j} a_{ij}\partial_i g \partial_j v + \sum_i b_i v \partial_i g + cgv \ dx$$

and w is a weak solution for

$$\begin{cases} Lw = Lu - Lg = f - Lg & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

.

I'll show the boundedness by contradiction. Assume that there exists $\{u'_k\}_{k=1}^\infty$, $\{f'_k\}$, $\{g'_k\}$ satisfying

$$\|u'_k\|_{L^2(\Omega)} > k \left(\|f'_k\|_{L^2(\Omega)} + \|g'_k\|_{H^1(\Omega)} \right)$$

and the boundary value problem. Let $u_k = \frac{u'_k}{\|u'_k\|_{L^2(\Omega)}}$, $f_k = f'_k/\|u'_k\|_{L^2(\Omega)}$, $g_k = g'_k/\|u'_k\|_{L^2(\Omega)}$, then u_k also satisfies the boundary value problem for f_k and g_k and the inequality

$$\|u_k\|_{L^2(\Omega)} > k \left(\|f_k\|_{L^2(\Omega)} + \|g_k\|_{H^1(\Omega)} \right).$$

Since $\|u_k\|_{L^2(\Omega)} = 1$ for all k, $\|f_k\|_{L^2(\Omega)}, \|g_k\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. By a variation of energy estimation,

$$\|u_k\|_{H^1(\Omega)}^2 \leq C \left(\|f_k\|_{L^2(\Omega)}\|u_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)}^2 \right)$$

for the constant C not depending on f_k and u_k.(By the ellipticity condition, there exists $\theta > 0$ such that

$$\theta \int_\Omega |Du|^2 dx \leq \int_\Omega a_{ij} \partial_i u \partial_j u dx$$

$$= B[u, u] - \int_\Omega b_i u \partial_i u + cu^2 dx$$

$$= B[u, u] + \sum_i \|b_i\|_{L^\infty} \int_\Omega \int_\Omega |Du||u| dx + \|c\|_{L^\infty} \int_\Omega u^2 dx$$

and by Cauchy's inequality,

$$\int_\Omega \int_\Omega |Du||u| dx \leq \epsilon \int_\Omega |Du|^2 dx + \frac{1}{4\epsilon} \int_\Omega u^2 dx$$

for $\epsilon > 0$. By choosing small enough ϵ to satisfy

$$\epsilon \sum_i \|b_i\|_{L^\infty} < \theta/2,$$

we get

$$\frac{\theta}{2} \int_\Omega |Du|^2 dx \leq B[u, u] + C \int_\Omega u^2 dx$$

for constant C. By adding $\frac{\theta}{2} \int_\Omega u^2 dx$ on both side, we get

$$\|u\|_{H^1(\Omega)}^2 \leq C(B[u, u] + \|u\|_{L^2(\Omega)}^2) \leq C' \left(\|f_k\|_{L^2(\Omega)} \|u_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)}^2 \right)$$

which is the above inequality.)

Since $\|f_k\|_{L^2(\Omega)} \to 0$ and $\|u_k\|_{L^2(\Omega)}^2 = 1$, $\|u_k\|_{H^1(\Omega)}$ is uniformly bounded. Therefore, by compact embedding theorem,(For $n > 2$, we can use Rellich-Kondrachov Theorem and for $n = 2$, we choose p close enough to $n = 2$ such that $2 < p^*$ and as $H^1(\Omega) \subset W^{1,p}(\Omega)$, we again use Rellich-Kondrachov Theorem. For $n = 1$, since $n < p = 2$, it can be done by Morrey's inequality and Arzela-Ascoli compactness theorem, which was the last HW.) there exists subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that

$$\begin{cases} u_{k_j} \rightharpoonup u & \text{weakly in } H^1(\Omega) \\ u_{k_j} \to u & \text{in } L^2(\Omega) \end{cases}$$

where $u \in H^1(\Omega)$. (Note that $H^1(\Omega)$ is a Hilbert space.)

Since $\|u_{k_j}\|_{L^2(\Omega)} = 1$ for all j, $\|u\|_{L^2(\Omega)} = 1$. Also,

$$\begin{cases} \lim_{k_j \to \infty} B[u_{k_j}, \eta] = \lim_{k_j \to \infty} (f_{k_j}, \eta)_{L^2(\Omega)} = 0 & \text{By } L^2 \text{ convergence} \\ \lim_{k_j \to \infty} B[u_{k_j}, \eta] = B[u, \eta] & \text{By } H^1 \text{ weak convergence.} \end{cases}$$

for all $\eta \in H_0^1(\Omega)$. Therefore, u is a nontrivial weak solution of the BVP:

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

However, this is impossible by the given condition. Therefore, there exists C not depending on u, f, g such that

$$\|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} \right)$$

Also, by the above equation,

$$\|u\|_{H^1(\Omega)}^2 \le C \left(\|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2 \right)$$

$$\le C \left(\|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \right)$$

$$\le C' \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} \right) \|u\|_{H^1(\Omega)}$$

and

$$\|u\|_{H^1(\Omega)} \le C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} \right) \|u\|_{H^1(\Omega)}$$

where the constant C not depending on u, f, g.

Problem 4

(a) By computation,

$$\int_\Omega -v \triangle u + cuv \ dx = \int_\Omega -\sum_{i=1}^n \partial_i(v\partial_i u) + \partial_i v \partial_i u + cuv \ dx$$

$$= \int_{\partial\Omega} -\sum_{i=1}^n (v\partial_i u) \, \nu^i dS + \int_\Omega -\partial_i v \partial_i u + cuv \ dx$$

$$= \int_{\partial\Omega} v\nabla u \cdot \nu_{\text{in}} dS + \int_\Omega \partial_i v \partial_i u + cuv \ dx$$

$$= \int_{\partial\Omega} gv \ dS + \int_\Omega \partial_i v \partial_i u + cuv \ dx$$

$$= \int_\Omega fv \ dx$$

for $u, v \in H^1(\Omega)$ and $\nu_{\text{out}} = (\nu^1, \nu^2, \dots, \nu^n)$ outward unit normal vector on $\partial\Omega$. Therefore, the definition of weak solution of this problem:

$$B[u, v] = \int_\Omega \nabla u \nabla v + cuv \ dx = \int_\Omega fv \ dx - \int_{\partial\Omega} gv \ dS$$

makes sense.

(b) First, note that $H^1(\Omega)$ is a Hilbert space. Define a bilinear form $B[\cdot, \cdot]$ by

$$\int_\Omega \nabla u \nabla v + cuv \ dx.$$

I'll show that this is well-defined elliptic form on $H^1(\Omega)$.

Bilinear: For $u_1, u_2, v_1, v_2 \in H^1(\Omega)$ and $a \in \mathbb{R}$ or $\mathbb{C}$,

$$B[au_1 + u_2, v_1] = \int_\Omega \nabla(au_1 + u_2)\nabla v_1 + c(au_1 + u_2)v_1 \ dx$$

$$= a \int_\Omega \nabla u_1 \nabla v_1 + cu_1 v_1 \ dx + \int_\Omega \nabla u_2 \nabla v_1 + cu_2 v_1 \ dx$$

$$= aB[u_1, v_1] + B[u_2, v_1]$$

and

$$B[u_1, av_1 + v_2] = \int_\Omega \nabla u_1 \nabla (av_1 + v_2) + cu_1(av_1 + v_2) \ dx$$

$$= a \int_\Omega \nabla u_1 \nabla v_1 + cu_1 v_1 \ dx + \int_\Omega \nabla u_1 \nabla v_2 + cu_1 v_2 \ dx$$

$$= aB[u_1, v_1] + B[u_1, v_2]$$

Boundedness: For $u, v \in H^1(\Omega)$,

$$|B[u,v]| = \left| \int_\Omega \nabla u \nabla v + cuv \ dx \right|$$

$$\leq \|Du\|_{L^2(\Omega)}^{1/2} \|Dv\|_{L^2(\Omega)}^{1/2} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

$$\leq C(\|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)})(\|v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)})$$

$$= C\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

for constant C depending only on Ω and c.

Coercivity: For $u \in H^1\Omega$,

$$B[u,u] = \int_\Omega \nabla u \nabla u + cu^2 \ dx$$

$$= \|Du\|_{L^2(\Omega)}^2 + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2$$

$$\geq \min\{1, \mu_0\} \|u\|_{H^1(\Omega)}^2$$

Furthermore, define $I(f,g) : H^1(\Omega) \to \mathbb{R}$ by

$$I : v \mapsto \int_\Omega fv \ dx - \int_\Omega gv \ dS.$$

Then, this is bounded linear function on $H^1(\Omega)$ since it is definitely linear and $I(v) = \left| \int_\Omega fv \ dx \right| + \left| \int_\Omega gv \ dS \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + C\|g\|_{L^2(\partial\Omega)} \|v\|_{H^1(\Omega)}$. (In the final step, I used trace inequality.)

By Lax-Millgram theorem, there exists a unique element $u \in H^1(\Omega)$ such that

$$B[u,v] = I(v)$$

for all $v \in H^1(\Omega)$. Therefore, there exists unique weak solution to the boundary value problem.

(c) If $u \in H^1(\Omega)$ is a weak solution to the boundary value problem,

$$\min\{1, \mu_0\} \|u\|_{H^1(\Omega)}^2 \leq |B[u,u]| = |I(u)| \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} + C\|g\|_{L^2(\partial\Omega)} \|u\|_{H^1(\Omega)}$$

so,

$$\|u\|_{H^1(\Omega)} \leq C'(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)})$$

for some $C < C'$ constants depending only on c, Ω.

Problem 5

(a) For $u, v \in H^1(\Omega)$,

(i)

$$(v, u)_{H^1} = \int_\Omega \nabla u \cdot \nabla v \ dx + \int_{\partial\Omega} uv \ dS$$
$$= \int_\Omega \nabla v \cdot \nabla u \ dx + \int_{\partial\Omega} vu \ dS$$
$$= (u, v)_{H^1}$$

Therefore, $(u, v)_{H^1} = (v, u)_{H^1}$.

(ii) Since Ω is bounded and ∂U is C^∞, consider a trace operator, which is bounded linear operator,

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ for $u \in H^2(\Omega) \cap C(\bar\Omega)$ and $\|Tu\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$ with the constant C only depending on p and Ω.

$$|(u, v)_{H^1}| = \left| \int_\Omega \nabla u \cdot \nabla v \ dx + \int_{\partial\Omega} uv \ dS \right|$$
$$\le \int_\Omega |Du||Dv| \ dx + \int_{\partial\Omega} |u||v| \ dS$$
$$\le \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|Tu\|_{L^2(\Omega)}\|Tv\|_{L^2(\Omega)}$$
$$\le C(\|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)})$$
$$\le C'(\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)})$$

for some C, C' constant independent on u, v.

(iii) For nonzero $u \in H^1(\Omega)$,

$$(u, u)_{H^1} = \int_\Omega \nabla u \cdot \nabla u \ dx + \int_{\partial\Omega} u^2 \ dS$$
$$= \|Du\|_{L^2(\Omega)}^2 + \|Tu\|_{L^2(\partial\Omega)}^2 \in (0, \infty)$$

Therefore, $(\cdot, \cdot)_{H^1}$ is an inner product on $H^1(\Omega)$.(Linearity: For $a, b \in \mathbb{R}$ and $u_1, u_2, v \in H^1(\Omega)$, $(au_1+bu_2, v)_{H^1} = \int_\Omega \nabla(au_1+bu_2) \cdot \nabla v \ dx + \int_{\partial\Omega}(au_1+bu_2)v \ dS = a\int_\Omega \nabla u_1 \cdot \nabla v \ dx + a\int_{\partial\Omega} u_1 v \ dS + b\int_\Omega \nabla u_2 \cdot \nabla v \ dx + b\int_{\partial\Omega} u_2 v \ dS = a(u_1, v)_{H^1} + b(u_2, v)_{H^1}$.) I need to show the coercivity.

I'll prove the coercivity by contradiction. Suppose there exists a sequence $\{u_k\}_{k=1}^\infty \subset H^1(\Omega)$ such that

$$\|u_k\|_{H^1(\Omega)}^2 > k(u_k, u_k)_{H^1}.$$

For such u_k, take $v_k = u_k/\|u_k\|_{H^1(\Omega)}$. Then, $\|v_k\|_{H^1(\Omega)} = 1$ and satisfies the above relation. As $k \to \infty$, $(v_k, v_k)_{H^1} \to 0$.

Since $\{v_k\}$ is a bounded sequence in $H^1(\Omega)$, which is compactly embedded in $L^2(\Omega)$, (I already explained how to deal for various dimension n in Problem 3.) so there exists a subsequence $\{v_{k_j}\}_{j=1}^\infty$ such that

$$\begin{cases} v_{k_j} \rightharpoonup v & \text{weakly in } H^1(\Omega) \\ v_{k_j} \to v & \text{in } L^2(\Omega) \end{cases}$$

since $H^1(\Omega)$ is Hilbert space...(Should I write the unique limit point?) As $v \in H^1(\Omega)$ and $\|v_{k_j}\|_{H^1(\Omega)} = 1$ for all k_j, $\|v\|_{H^1(\Omega)} \le 1$. Since $\|Dv_{k_j}\|_{L^2(\Omega)}^2 \le (v_{k_j}, v_{k_j})_{H^1} \le \frac{1}{k_j}$, $\|Dv_{k_j}\|_{L^2(\Omega)}^2 \to 0$

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as $k_j \to \infty$ and it means $\|v\|_{L^2(\Omega)} = 1$, $\|v\|_{H^1(\Omega)} = 0$ and $\|Dv\|_{L^2(\Omega)} = 0$. (If not, for some N, $\|v_N\|_{H^1(\Omega)}^2 = \|v_N\|_{L^2(\Omega)}^2 + \|Dv_N\|_{L^2(\Omega)}^2 < 1$, which is contradiction.) Also,

$$\|v_k - v\|_{H^1(\Omega)}^2 = \|v_{k_j} - v\|_{L^2(\Omega)}^2 + \|D(v_{k_j} - v)\|_{L^2(\Omega)}^2 \leq \|v_{k_j} - v\|_{L^2(\Omega)}^2 + \|Dv_{k_j}\|_{L^2(\Omega)}^2 + \|D(v_{k_j} - v)\|_{L^2(\Omega)}^2 \to 0$$

as $k_j \to \infty$, $v_{k_j} \to v$ in $H^1(\Omega)$.

Since $\|Tv_{k_j}\|_{L^2(\partial\Omega)}^2 \leq (v_{k_j}, v_{k_j})_{H^1} < \frac{1}{k_j}$,

$$\|Tv\|_{L^2(\partial\Omega)} \leq \|T(v - v_{k_j})\|_{L^2(\partial\Omega)} + \|Tv_{k_j}\|_{L^2(\partial\Omega)} \leq \frac{1}{\sqrt{k_j}} + C\|v - v_{k_j}\|_{H^1(\Omega)}$$

where the C is given by the Trace Theorem. Therefore, $\|Tv\|_{L^2(\partial\Omega)} = 0$.

Summarising the fact, $v \in H^1(\Omega)$ and $Tv = 0$ on $\partial\Omega$, so $v \in H_0^1(\Omega)$ and $\|Dv\|_{L^2(\Omega)} = 0$. For $n > 2$, $\|v\|_{L^2(\Omega)} \leq C\|Dv\|_{L^2(\Omega)}$ for the constant C depending only on n, and Ω, so $\|v\|_{L^2(\Omega)} = 0$. For $n = 2$, we can find p near to 2 and get $\|v\|_{L^2(\Omega)} \leq C\|Dv\|_{L^p(\Omega)}$. Since $\|Dv\|_{L^2(\Omega)} = 0 \Rightarrow \|Dv\|_{L^p(\Omega)} = 0$, $\|v\|_{L^2(\Omega)} = 0$. For $n = 1$, since $Dv = 0$ a.e. in Ω, v is constant a.e. in Ω and since $p = 2 > 1$, we can identify this as a continuous function.(Modify the measure zero part.) Since $v \in C(\overline{U})$, $Tv = v|_{\partial U}$ and if $\|v\|_{L^2(\Omega)} = \neq 0$, $Tv \neq 0$, which is contradiction. Therefore, $\|v\|_{L^2(\Omega)} = 0$ also in $n = 1$ case. Finally, this result shows that $v = 0$, but $\|v\|_{H^1(\Omega)} \neq 0$, which is contradiction.

Thus, there exists C not depending on $u \in H^1(\Omega)$ such that

$$\|u\|_{H^1(\Omega)}^2 \leq C(u, u)_{H^1}.$$

(b) I'll rewrite $(u, v)_{H^1}$ by $B[u, v]$. Consider a BVP:

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n}_{\text{in}} - u = 0 & \text{on } \partial\Omega \end{cases}$$

for $f \in L^2(\Omega)$. Then, the weak solution of the BVP should satisfy

$$B[u, v] = \int_\Omega fv \; dx$$

for all $v \in H^1(\Omega)$. (This result is from problem 4 (a).)

Assume that there exists eigenvalue λ and the corresponding nontrivial weak solution $u \in H^1(\Omega)$. Then, for all $v \in H^1(\Omega)$,

$$B[u, v] = (\lambda u, v)_{L^2(\Omega)}$$

If we put u as v,

$$B[u, u] = (\lambda u, u)_{L^2(\Omega)} = \lambda\|u\|_{L^2(\Omega)} > 0$$

since $\|u\|_{H^1(\Omega)} > 0$. If λ is not real, it is nonsense. Also, if $\lambda \leq 0$, it is contradiction to coercivity of B. Therefore, every eigenvalue of (EVP) is a positive real.

(c) Since B is symmetric bounded linear function with coercivity, By the Lax-Millgram Theorem, there exists unique weak solution $u \in H^1(\Omega)$ to the BVP for each f. Let's define $L^{-1} : L^2(\Omega) \to L^2(\Omega)$ by sending f to u. Now, I claim that L^{-1} is bounded, linear, compact operator. It is linear since if $L^{-1}(f_1) = u_1$ and $L^{-1}(f_2) = u_2$, then the unique weak solution to $rf_1 + f_2$ is $ru_1 + u_2$ for $r \in \mathbb{R}$ and

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$L^{-1}(rf_1 + f_2) = ru_1 + u_2$. Also, L^{-1} is bounded. It requires same argument used in problem 3. If there is no constant C such that $\|u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$, there is a nontrivial weak solution to

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n}_{\text{in}} - u = 0 & \text{on } \partial\Omega \end{cases}$$

and since we just showed that $\lambda = 0$ can not be an eigenvalue, this is impossible. Finally, L^{-1} is compact: if $\{f_k\}_{k=1}^\infty$ is uniformly bounded in $L^2(\Omega)$ and let $u_k = L^{-1}f_k$, then by coercivity, there exists $\beta > 0$ such that $\beta\|u_k\|_{H^1(\Omega)}^2 \le B[u_k, u_k] = (f_k, u_k)_{L^2(\Omega)} \le \|f_k\|_{L^2(\Omega)}\|u_k\|_{H^1(\Omega)}$ and $\|u_k\|_{H^1(\Omega)} \le C$ for all k. By Rellich-Kondrachov Compactness Theorem (and for $n = 2, 1$ cases as described in previous) there exists $u_k \to u$ in $L^2(\Omega)$ and it means L^{-1} is compact.

u is a weak solution of the EVP in the problem if and only if $L^{-1}(\lambda u) = u$ if and only if $u - \lambda L^{-1}u = 0$. Since $\lambda > 0$, we can denote $\left(L^{-1} - \frac{1}{\lambda}\text{Id}\right)u = 0$... Therefore, $\lambda \in \Sigma$ if and only if $\frac{1}{\lambda} \in \sigma_p(L^{-1}) \setminus \{0\}$ and $\Sigma = \left\{\lambda \in \mathbb{R} \mid \frac{1}{\lambda} \in \sigma_p\left(L^{-1}\right) \setminus \{0\}\right\}$.

Since $L^{-1} : L^2(\Omega) \to L^2(\Omega)$ and $L^2(\Omega)$ separable as the domain of the function is open set $\Omega \subset \mathbb{R}^n$. Also, L^{-1}: symmetric: Fix $f, g \in L^2(\Omega)$ and let $\omega = L^{-1}f$, $\xi = L^{-1}g$, then $\omega, \xi \in H^1(\Omega)$ and

$$\begin{aligned} (L^{-1}f, g)_{L^2(\Omega)} &= \int_\Omega g\omega dx \\ &= \int_\Omega \nabla\xi \cdot \nabla\omega dx + \int_{\partial\Omega} \xi\omega dS \\ &= \int_\Omega \nabla\omega \cdot \nabla\xi dx + \int_{\partial\Omega} \omega\xi dS \\ &= \int_\Omega f\xi dx \\ &= (f, L^{-1}g). \end{aligned}$$

Therefore, there exists countable orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of L^{-1}. By Fredholm alternative, the dimension of null space of $\left(\text{Id} - \lambda L^{-1}\right)$ for $\lambda \in \Sigma$ is finite. Therefore, $|\Sigma| = \infty$. Since L^{-1} is compact and the dimension of $L^2(\Omega)$ is infinite, $\sigma(L^{-1}) \setminus \{0\} = \sigma_p(L^{-1}) \setminus \{0\}$ and $\sigma(L^{-1}) \setminus \{0\}$ is a sequence tending to 0. Therefore, Σ consists of the eigenvalues monotonic increasing to ∞.

(d) In the (c), I showed that there exists countable orthonormal basis of separable space $L^2(\Omega)$ consisting of eigenvectors of compact and symmetric bounded linear operator L^{-1}. I need to show that this is orthogonal basis of $H^1(\Omega)$.

$$B[w_i, w_j] = (Lw_i, w_j)_{L^2(\Omega)} = (Lw_j, w_i)_{L^2(\Omega)} = B[w_j, w_i]$$

If $i \ne j$, $(Lw_i, w_j)_{L^2(\Omega)} = \lambda_i(w_i, w_j)_{L^2(\Omega)} = 0$. If $i = j$, $(Lw_i, w_i)_{L^2(\Omega)} = \lambda_i$.

Finally, assume that $\{w_i\}$ can not span $H^1(\Omega)$. Then, there exists nontrivial $\xi \in H^1(\Omega)$ such that $B[w_i, \xi] = 0$ for all w_i. Then, $B[w_i, \xi] = (\lambda_i w_i, \xi)_{L^2(\Omega)} = 0$ and $\lambda_i > 0$ for all i. Therefore, $\xi = 0$, which is contradiction. Therefore, $\{w_i\}$ spans $H^1(\Omega)$. Thus, $\{w_i\}$ forms a orthogonal basis of $H^1(\Omega)$.

Problem 6

(Step 1) In problem 5 (d), we found a set of functions $\{w_k\}_{k=1}^\infty \subset H^1(\Omega) \cap C^\infty(\Omega)$ such that

    (i) $\{w_k\}$ forms an orthornormal basis of $L^2(\Omega)$, and

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(ii) $\{w_k\}$ forms an orthogonal basis of $H^1(\Omega)$ with respect to the inner product $(\cdot,\cdot)_{H^1} = B[\cdot,\cdot]$. The weak solution of (IBVP-P) should satisfies

$$(u'(t), v)_{L^2(\Omega)} + B[u(t), v] = (f(t), v)_{L^2(\Omega)}$$

for $v \in H^1(\Omega)$ and a.e. time $0 \le t \le T$ and $u(0) = g$.

(Step 2) I'll use Galerkin approximation. Fix $m \in \mathbb{N}$ and set $u_m(t, x) = \sum\limits_{k=1}^{m} d_k(t) w_k(x)$. I need to find $\{d_k\}_{k=1}^{m}$ satisfying

(i) $(u_m'(t, \cdot), w_k)_{L^2(\Omega)} + B[u_m(t), w_k] = (f(t, \cdot), w_k)_{L^2(\Omega)}$ for $1 \le j \le m$.

(ii) $u_m(0, x) = \sum\limits_{k=1}^{m} d_k(0) w_k(x) = g^{(m)}(x)$ where $g^{(m)}(x) = \sum\limits_{k=1}^{m} G_k w_k(x)$, $G_k = (g, w_k)_{L^2(\Omega)}$.

Note that u_m satisfies boundary condition $\nabla u_m \cdot \boldsymbol{n}_{\text{in}} - u_m = 0$ on $(0, T] \times \partial\Omega$ since $w_k = 0$ on $\partial\Omega$ for all k.

Define $f^{(m)}(t, x) := \sum\limits_{k=1}^{m} F_k(t) w_k(x)$ where $F_k(t) = (f(t, \cdot), w_k)_{L^2(\Omega)}$.

Note that (i) holds if and only if $d_j'(t) + \lambda_j d_j(t) = F_j(t)$ for $0 < t \le T$ and $1 \le j \le m$, and (ii) holds if and only if $d_j(0) = G_j = (g, w_j)_{L^2(\Omega)}$:

$$(\sum_{k=1}^{m} d_k'(t) w_k(x), w_j)_{L^2(\Omega)} + B[\sum_{k=1}^{m} d_k(t) w_k(x), w_j] = (f(t, \cdot), w_j)_{L^2(\Omega)} \Leftrightarrow d_j'(t) + \lambda_j d_j(t) = F_j(t)$$

and

$$\sum_{k=1}^{m} d_k(0) w_k(x) = \sum_{k=1}^{m} G_k w_k(x)$$
$$\Leftrightarrow \left(\sum_{k=1}^{m} d_k(0) w_k(x), w_j(x) \right)_{L^2(\Omega)} = \left(\sum_{k=1}^{m} G_k w_k(x), w_j(x) \right)_{L^2(\Omega)} \quad \text{For all } 1 \le j \le m$$
$$\Leftrightarrow d_j(0) = G_j \quad \text{For all } 1 \le j \le m$$

Since each $F_j(t), G_j(t)$ are smooth, by the uniqueness existence of ODEs, the initial value problem for each j:

$$\begin{cases} d_j'(t) + \lambda_j d_j(t) = F_j(t) & 0 < t \le T \\ d_j(0) = G_j = (g, w_j)_{L^2(\Omega)} \end{cases}$$

has a unique smooth solution d_j for $1 \le j \le m$. With these d_j, $u_m(t, x) = \sum\limits_{k=1}^{m} d_k(t) w_k(t)$ satisfies (i),(ii) and 0 on $(0, T] \times \partial\Omega$.

What I want is that as $m \to \infty$, $u_m \to u \in L^2((0, T); H^1(\Omega))$, $u' \in L^2((0, T); H^1_*(\Omega))$.

For now, I'll accept the proposition:

Proposition 1. *For each u_m satisfies*

$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u_m'\|_{L^2(0,T;H^1_*(\Omega))} \le C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)} \right)$$

for some constant C depending only on Ω, T.

(Step 3) By the proposition, we get the fact that $\{u_m\}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ and $\{u'_m\}$ is uniformly bounded in $L^2(0,T;H^1_*(\Omega))$. Since $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;H^1_*(\Omega))$ are Hilbert space since $L^2(0,T)$ is..., there is a subsequence such that

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0,T;H^1(\Omega)) \\ u'_{m_k} \rightharpoonup u' & \text{in } L^2(0,T;H^1_*(\Omega)) \end{cases}$$

(We know that $u'_{m_k} \rightharpoonup w \in L^2(0,T;H^1_*(\Omega))$. I'll show that $u' = w$. For $\phi \in C_c^\infty(0,T)$ and $v \in H^1(\Omega)$,

$$\begin{aligned}
\langle \int_0^T \phi'(t)u(t)dt, v \rangle &= \int_0^T \langle \phi'(t)u(t), v \rangle dt = \int_0^T \langle u(t), \phi'(t)v \rangle dt \\
&= \int_0^T \lim_{m_k \to \infty} \langle u_{m_k}(t), \phi'(t)v \rangle dt = \lim_{m_k \to \infty} \int_0^T \langle u_{m_k}(t), \phi'(t)v \rangle dt \\
&= \lim_{m_k \to \infty} \langle \int_0^T u_{m_k}(t)\phi'(t)dt, v \rangle = \lim_{m_k \to \infty} \langle \int_0^T u'_{m_k}(t)\phi(t)dt, v \rangle \\
&= \int_0^T \lim_{m_k \to \infty} \langle u'_{m_k}(t), \phi(t)v \rangle dt = \int_0^T \langle w(t), \phi(t)v \rangle dt \\
&= \langle \int_0^T w(t)\phi(t)dt, v \rangle.
\end{aligned}$$

The $\langle \cdot, \cdot \rangle$ represent the action of $H^1_*(\Omega)$ to $H^1(\Omega)$ and the commutativity of $\langle \cdot, \cdot \rangle$ and integral is by smoothness of ϕ and riemann sum argument. The commutativity of limit and integral is by the boundedness of U and $(0,T)$ and integrability of u, v, ϕ with dominance convergence theorem. Therefore, $u' = w$.)

Now, I'll show that u is the weak solution to (IBVP-P).

Since $\nabla u_{m_k} \cdot \boldsymbol{n}_{\text{in}} - u_{m_k} = 0$ on $(0,T] \times \partial\Omega$ for all m_k, $\nabla u \cdot \boldsymbol{n}_{\text{in}} - u = 0$ on $(0,T] \times \partial\Omega$.

Fix $N \in \mathbb{N}$ and fix $m_k \geq N$. Consider $u_{m_k} = \sum_{i=1}^{m_k} d_i(t)w_i(x)$. Fix $j \in \{1, \cdots, N\}$ and let $\alpha(t)$ be a smooth function. Then,

$$\int_0^T \alpha(s) \left((u'_{m_k}(s), w_j)_{L^2(\Omega)} + B[u_{m_k}(s), w_j] \right) ds = \int_0^T (f^{(m_l)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$
$$\Leftrightarrow \int_0^T (u'_{m_k}(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$

By the weak convergence, as $m_k \to \infty$,

$$\int_0^T (u'(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$

Since $\alpha(s)$ is arbitrary, we can let $\alpha(s) \to \delta(s-t)$ (such as standard mollifier) and get

$$(u'(t), w_j)_{L^2(\Omega)} + B[u(t), w_j] = (f(t), w_j)_{L^2(\Omega)}$$

Since N is arbitrarily fixed, this is true for all j. Since the function $\sum_{i=1}^n d_i(x)w_i(x)$ with coefficient $d_i(x) \in H^1(\Omega)$ is dense in $H^1(\Omega)$, we get

$$(u'(t), v)_{L^2(\Omega)} + B[u(t), v] = (f(t), v)_{L^2(\Omega)}$$

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for $v \in H^1(\Omega)$ and a.e. time $0 \le t \le T$.

I still need to check the initial condition. For the same circumstance with one more condition $\alpha(T) = 0$,

$$\int_0^T (u'(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$

$$\Rightarrow \int_0^T -(u(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (u(0), \alpha(0)w_j))$$

and

$$\int_0^T (u'_{m_k}(s), \alpha(s)w_j)_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds$$

$$\Rightarrow \int_0^T -(u_{m_k}(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u_{m_k}(s), \alpha(s)w_j]ds = \int_0^T (f^{(m_k)}(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (u_{m_k}(0), (\alpha(0)w_j)).$$

Letting $m_k \to \infty$ and by weak convergence,

$$\int_0^T -(u(s), (\alpha(s)w_j)')_{L^2(\Omega)} + B[u(s), \alpha(s)w_j]ds = \int_0^T (f(t), w_j)_{L^2(\Omega)}\alpha(s)ds + (g(0), (\alpha(0)w_j))$$

as $u_{m_k}(0) = g^{(m_k)}$ and $\lim_{m_k \to \infty} u_{m_k}(0) = \lim_{m_k \to \infty} g^{(m_k)} = g$. This is true for all $\alpha \in C^\infty((0,T))$ and for all j, $u(0) = g(0)$.

The left one is to show the proposition. For readability, I again write the proposition:

For each u_m satisfies

$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u'_m\|_{L^2(0,T;H^1_*(\Omega))} \le C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)} \right)$$

for some constant C depending only on Ω, T.

I'll start from this equation:

$$(\sum_{k=1}^m d'_k(t)w_k(x), w_j)_{L^2(\Omega)} + B[\sum_{k=1}^m d_k(t)w_k(x), w_j] = (f(t, \cdot), w_j)_{L^2(\Omega)}$$

Note that $d_k(t)$ are calculated for fixed m. On each side for fixed j, multiply $d_j(t)$ and sum about j, then we get

$$(u'_m(t), u_m(t))_{L^2(\Omega)} + B[u_m(t), u'_m(t)] = (f(t, \cdot), u_m(t))_{L^2(\Omega)}$$

for a.e. $0 \le t \le T$. For fixed $t \in [0,T]$, $(u'_m, u_m)_{L^2(\Omega)} = \frac{d}{dt}\left(\frac{1}{2}\|u_m\|^2_{L^2(\Omega)} \right)$ and $|(f, u_m)| \le \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_m\|^2_{L^2(\Omega)}$. Furthermore, I'll use the proof of variation of energy estimation used in problem 3: there exists $\beta, \gamma > 0$ such that

$$\beta\|u_m\|^2_{H^1(\Omega)} \le B[u_m, u_m, t] + \gamma\|u_m\|^2_{L^2(\Omega)}.$$

Mixing these inequalities, we get

$$\frac{d}{dt}\left(\|u_m\|^2_{L^2(\Omega)} \right) + 2\beta\|u_m\|^2_{H^1(\Omega)} \le C_1\|u_m\|^2_{L^2(\Omega)} + C_2\|f\|^2_{L^2(\Omega)}. \tag{1}$$

for a.e. $0 \le t \le T$ and constant C_1, C_2 not depending on u_m, f.

Now, let

$$\eta(t) := \|u_m(t)\|^2_{L^2(\Omega)}$$

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and
$$\xi(t) := \|f(t)\|_{L^2(\Omega)}^2.$$

Then,
$$\eta'(t) \le C_1 \eta(t) + C_2 \xi(t)$$

for a.e. $0 \le t \le T$ and by differential form of Gronwall's inequality,
$$\eta(t) \le e^{C_1 t}\left(\eta(0) + C_2 \int_0^t \xi(s)ds\right)$$

for $0 \le t \le T$. Since $\eta(0) = \|u_m(0)\|_{L^2(\Omega)}^2 \le \|g\|_{L^2(\Omega)}^2$, (Note that $u_m(0)$ is the projection of g.)
$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(\Omega)}^2 \le C\left(\|g\|_{L^2(\Omega)}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2\right).$$

Also, integrating (1) from 0 to T generates the relation:
$$\|u_m\|_{L^2(0,T;H^1(\Omega))}^2 = \int_0^T \|u_m\|_{H^1(\Omega)}^2 dt \le C\left(\|g\|_{L^2(\Omega)}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2\right)$$

Finally, I'll estimate $\|u_m'\|_{L^2(0,T;H_*^1(\Omega))}$. Fix $v \in H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \le 1$ and denote $v = v^1 + v^2$ by $v_1 \in \text{span}\left(\{w_i\}_{i=1}^m\right)$ and $(v_2, w_i) = 0$ for $i \in \{1, \ldots, m\}$. Since v_1 is the projection to $\text{span}\left(\{w_i\}_{i=1}^m\right)$, $\|v^1\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)} \le 1$. Using the previous equation: $(u_m'(t, \cdot), w_k)_{L^2(\Omega)} + B[u_m(t), w_k] = (f(t, \cdot), w_k)_{L^2(\Omega)}$, we can write
$$(u_m'(t, \cdot), v^1)_{L^2(\Omega)} + B[u_m(t), v^1] = (f(t, \cdot), v^1)_{L^2(\Omega)}$$

for a.e. $0 \le t \le T$. Since $u_m' \in \text{span}\left(\{w_i\}_{i=1}^m\right)$,
$$(u_m', v)_{L^2(\Omega)} = (u_m', v^1)_{L^2(\Omega)} = (f, v^1) - B[u_m(t), v^1].$$

Therefore,
$$\left|(u_m', v)_{L^2(\Omega)}\right| \le C\left(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1(\Omega)}\right)$$

since $\|v^1\|_{H^1(\Omega)} \le 1$.(Also, by boundedness of $B[\cdot, \cdot]$: $\left|B[u_m, v^1; t]\right| \le \theta \|u_m(t)\|_{H^1(\Omega)}$ for some constant θ.) Since $\|v\|_{H^1(\Omega)} = 1$ and RHS is independent to v,
$$\|u_m'\|_{H_*^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1(\Omega)}\right)$$

and
$$\|u_m'\|_{L^2(0,T;H_*^1(\Omega))} = \int_0^T \|u_m'\|_{H_*^1(\Omega)} dt \le C'\left(\|g\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}\right)$$

for some constant C'.(Note that we already showed that $\|u_m\|_{L^2(0,T;H^1(\Omega))}^2$ is bounded by $\|g\|_{L^2(\Omega)}$ and $\|f\|_{L^2(0,T;L^2(\Omega))}$.)

Summarizing all the results, we get
$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{L^2(0,t;H^1(\Omega))} + \|u_m'\|_{L^2(0,T;H_*^1(\Omega))} \le C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

for some C. This is true for all m since the estimation constant does not depending on u_m, f, g.