Partial Differential Equation - HW 3

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November 17, 2018

Problem 1

I'll imitate the proof in Evans.

Proof. Since each Γ_j is compact, $\partial\Omega$ is compact and we can choose finite points $x_i \in \partial\Omega$ with radius $r_i > 0$ and $\partial\Omega \subset \cup_{i=1}^{n} B\left(x_i, \frac{r_i}{2}\right)$. If x_i is not in end point of some Γ_j for all j, then we can use the argument in the Evans, so we only need to consider the case that x_i is in end point of Γ_j for some j.

Fix x^0 is in end point of Γ_j and assume that x^0 is also a end point of Γ_{j+1}. As Γ_j, Γ_{j+1} are C^1, there exists $r_1, r_2 > 0$ and a C^1 function γ_1, $\gamma_2 : \mathbb{R} \to \mathbb{R}$ implicit function theorem.

Problem 2

1. $W_0^{1,p}(\Omega)$ is a vector space: For $f = 0$, $f \in W_0^{1,p}(\Omega)$, so $W_0^{1,p}(\Omega) \neq \phi$. For $f_1, f_2 \in W_0^{1,p}(\Omega)$, there exists f_1^j, f_2^j such that $(f_1^j), (f_2^j) \in C_c^\infty(\Omega)$ and $f_1^j \to f_1$, $f_2^j \to f_2$ in $W^{1,p}(U)$. Since union of two compact set in Ω is compact in Ω, $f_1^j + f_2^j \in C_c^\infty(\Omega)$ and for large enough N satisfying $\left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)}, \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon/2$ for $j > N$, $\left\|f_1^j + f_2^j - f_1 - f_2\right\|_{W^{1,p}(\Omega)} \leq \left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)} + \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon$. Therefore, $f_1^j + f_2^j \to f_1 + f_2$ and $f_1 + f_2 \in W^{1,p}(\Omega)$. Also, $\lambda f^j \to \lambda f$ in $W^{1,p}(\Omega)$ for scalar λ. Therefore, $W^{1,p}$ is vector space.(Other ...)

2. With the norm $\|\cdot\|_{W^{1,p}(\Omega)}$, $W_0^{1,p}(\Omega)$ is Banach space: Let f_j be a cauchy sequence in $W_0^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is Banach space, $f_j \to f$ in $W^{1,p}(\Omega)$. Since Ω is bounded and $\partial\Omega$ is C^1, there exists bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and $Tf_j \equiv 0$ on ∂U as $f_j \in W_0^{1,p}(\Omega)$. Then,
$$\lim_{j\to\infty} \|Tf_j - Tf\|_{W^{1,p}(\Omega)} = \lim_{j\to\infty} \|T(f_j - f)\|_{W^{1,p}(\Omega)} \leq \lim_{j\to\infty} \|T\|_{W^{1,p}(\Omega)} \|f_j - f\|_{W^{1,p}(\Omega)} = 0$$
as $\|T\|_{W^{1,p}(\Omega)}$ is bounded. Therefore, $Tf_j \to Tf$ and $\lim_{j\to\infty} \|Tf_j\|_{W^{1,p}(\Omega)} = \|Tf\|_{W^{1,p}(\Omega)} = 0$. As a result, $f \in W_0^{1,p}(\Omega)$ implying Cauchy sequence in $W_0^{1,p}(\Omega)$ converges.

Therefore, $W_0^{1,p}(\Omega)$ is Banach space.

Problem 3

For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$,
$$C^{k,p}(\bar{\Omega}) \coloneqq \{u \in C^k(\bar{\Omega}) : \|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty\}$$

(a) Clearly, $0 \in C^{k,p}(\bar{\Omega})$. For $f_1, f_2 \in C^{k,p}(\bar{\Omega})$,

Problem 4

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