Partial Differential Equation - HW 3

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Problem 1

I'll imitate the proof in Evans.

Proof. Since each Γ_j is compact, $\partial\Omega$ is compact and we can choose finite points $x_i \in \partial\Omega$ with radius $r_i > 0$ and $\partial\Omega \subset \cup_{i=1}^n B\left(x_i, \frac{r_i}{2}\right)$. If x_i is not in end point of some Γ_j for all j, then we can use the argument in the Evans, so we only need to consider the case that x_i is in end point of Γ_j for some j.

Fix x^0 is in end point of Γ_j and assume that x^0 is also a end point of Γ_{j+1}. As Γ_j, Γ_{j+1} are C^1, there exists $r_1, r_2 > 0$ and a C^1 function γ_1, $\gamma_2 : \mathbb{R} \to \mathbb{R}$ implicit function theorem. $\square$

Problem 2

1. $W_0^{1,p}(\Omega)$ is a vector space: For $f = 0$, $f \in W_0^{1,p}(\Omega)$, so $W_0^{1,p}(\Omega) \neq \phi$. For $f_1, f_2 \in W_0^{1,p}(\Omega)$, there exists f_1^j, f_2^j such that $(f_1^j), (f_2^j) \in C_c^\infty(\Omega)$ and $f_1^j \to f_1$, $f_2^j \to f_2$ in $W^{1,p}(U)$. Since union of two compact set in Ω is compact in Ω, $f_1^j + f_2^j \in C_c^\infty(\Omega)$ and for large enough N satisfying $\left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)}, \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon/2$ for $j > N$, $\left\|f_1^j + f_2^j - f_1 - f_2\right\|_{W^{1,p}(\Omega)} \leq \left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)} + \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon$. Therefore, $f_1^j + f_2^j \to f_1 + f_2$ and $f_1 + f_2 \in W^{1,p}(\Omega)$. Also, $\lambda f^j \to \lambda f$ in $W^{1,p}(\Omega)$ for scalar λ. Therefore, $W^{1,p}$ is vector space.(Other ...)

2. With the norm $\|\cdot\|_{W^{1,p}(\Omega)}$, $W_0^{1,p}(\Omega)$ is Banach space: Let f_j be a cauchy sequence in $W_0^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is Banach space, $f_j \to f$ in $W^{1,p}(\Omega)$. Since Ω is bounded and $\partial\Omega$ is C^1, there exists bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and $Tf_j \equiv 0$ on ∂U as $f_j \in W_0^{1,p}(\Omega)$. Then,

$$\lim_{j\to\infty} \|Tf_j - Tf\|_{W^{1,p}(\Omega)} = \lim_{j\to\infty} \|T(f_j - f)\|_{W^{1,p}(\Omega)} \leq \lim_{j\to\infty} \|T\|_{W^{1,p}(\Omega)} \|f_j - f\|_{W^{1,p}(\Omega)} = 0$$

as $\|T\|_{W^{1,p}(\Omega)}$ is bounded. Therefore, $Tf_j \to Tf$ and $\lim_{j\to\infty} \|Tf_j\|_{W^{1,p}(\Omega)} = \|Tf\|_{W^{1,p}(\Omega)} = 0$. As a result, $f \in W_0^{1,p}(\Omega)$ implying Cauchy sequence in $W_0^{1,p}(\Omega)$ converges.

Therefore, $W_0^{1,p}(\Omega)$ is Banach space.

Problem 3

For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$,

$$C^{k,\alpha}(\bar{\Omega}) \coloneqq \{u \in C^k(\bar{\Omega}) : \|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty\}$$

Before starting, I need to show that $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ is a norm on $C^{k,\alpha}(\bar{\Omega})$.

Proof. 1. By the definition of $C^{k,\alpha}(\bar{\Omega})$, we know that $\|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$ for any $u \in C^{k,\alpha}(\bar{\Omega})$. Let $u, v \in C^{k,\alpha}(\bar{\Omega})$. Then

$$\begin{aligned}
\|u+v\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha|\le k} \|D^\alpha(u+v)\|_{C(\bar{\Omega})} + \sum_{|\alpha|=k} [D^\alpha(u+v)]_{C^{0,\alpha}(\bar{\Omega})} \\
&= \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha(u+v)| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha(u+v)(x) - D^\alpha(u+v)(y)|}{|x-y|^\alpha} \right\} \\
&\le \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + \sup_{x\in\Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)| + |D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\alpha} \right\} \\
&\le \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + \sup_{x\in\Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \right\} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\alpha} \right\} \\
&= \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})}
\end{aligned}$$

Therefore, $\|u+v\|_{C^{k,\alpha}(\bar{\Omega})} \le \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})}$.

2. For $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\|\lambda\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha \lambda u| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha \lambda u(x) - D^\alpha \lambda u(y)|}{|x-y|^\alpha} \right\} \\
&= |\lambda| \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + |\lambda| \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \right\} \\
&= \lambda \|u\|_{C^{k,\alpha}(\bar{\Omega})}.
\end{aligned}$$

3. For $u = 0$, $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$. Conversely, if $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$, then $\|u\|_{C(\Omega)} = 0$ with continuity of u, so $u = 0$ on $\bar{\Omega}$.

Therefore, $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ is a norm.

(a) Clearly, $0 \in C^{k,p}(\bar{\Omega})$. For $f_1, f_2 \in C^{k,p}(\bar{\Omega})$, $f_1 + f_2 \in C^k(\Omega)$ and $\|f_1 + f_2\|_{C^{k,\alpha}(\bar{\Omega})} \le \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} + \|f_2\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$. Therefore, $f_1 + f_2 \in C^{k,\alpha}(\bar{\Omega})$. $f_1 + f_2 = f_2 + f_1$ and for scalar λ, $\lambda f_1 \in C^{k,\alpha}(\bar{\Omega})$ for $\|\lambda f_1\|_{C^{k,\alpha}(\bar{\Omega})} = |\lambda| \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} \le \infty$. Therefore, $C^{k,p}(\bar{\Omega})$ is a vector space.

(b) C^∞ Uryshon lemma

(c) Let $\{u_i\}$ be a Cauchy sequence in $C^{k,p}(\bar{\Omega})$. For fixed $\epsilon > 0$, there exists N such that $i, j > N \Rightarrow \|u_i - u_j\|_C^{k,p}(\bar{\Omega}) \le \epsilon$. It implies

$$\begin{cases} \|D^\alpha u_i - D^\alpha u_j\|_{C(\bar{\Omega})} \le \epsilon & \text{For } |\alpha| \le k \\ [D^\alpha u_i - D^\alpha u_j]_{C^{0,\gamma}(\bar{\Omega})} \le \epsilon & \text{For } |\alpha| = k. \end{cases}$$

Since $D^\alpha u_i$ is uniformly Cauchy for $|\alpha| \le k$, $D^\alpha u_i$ converges to u_α for $|\alpha| \le k$ pointwisely. Also, this convergence is uniform...

$\square$

Problem 4

I'll follow the proof in Evans.

Proof. Since U is bounded, open subset of $\mathbb{R}^n$, and $\partial\Omega$ is C^1,

$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\bar{\Omega}), \ \|u\|_{C^{0,\alpha}(\bar{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for $\alpha = 1 - n/p$ and C depends only on p, n and Ω. Now, we need to show that each bounded sequence in $W^{1,p}(\Omega)$ is precompact in $C^{0,\alpha}(\bar{\Omega})$. Let a bounded sequence in $W^{1,p}(\Omega)$: $\{u_m\}_{m=1}^{\infty}$.

Using Extension Theorem, we can assume that $\Omega = \mathbb{R}^n$, all $\{u_m\}$ have compact support in some bounded open set $V \subset \mathbb{R}^n$, and

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty$$

... (make support B_R $\square$

Problem 5

Fix $\epsilon > 0$. Define $\Omega_\epsilon \coloneqq \{x \in \Omega | d(x, \partial\Omega) > \epsilon\}$. Let's mollify the u with standard mollifier η_ϵ and denote it u^ϵ. Then,

$$Du^\epsilon = \eta_\epsilon * Du = 0$$

in Ω_ϵ. It implies that if $B(x, r) \in \Omega_\epsilon$ for small enough $r > 0$, u_ϵ is constant on $B(x, r)$ since the derivative of u^ϵ is zero on the set. In other words, it is locally constant in Ω_ϵ.

Let $x \in U$ and $B(x, r)$ be an open neighborhood of x in Ω, then there exists ϵ such that $B(x, r) \subset \Omega_\epsilon$ and by previous, we know that u^ϵ is constant on $B(x, r)$. Let the constant value c^ϵ. We know that $u^\epsilon \to u$ as $\epsilon \to 0$ and it means on u is constant a.e. on $B(x, r)$.(If not, there always exists non measure zero set such that u^ϵ is different with u on $B(x, r)$.) Therefore, u is locally constant function in a.e. sense).

Let take a partition such that $x \sim y$ if $u(x) = u(y)$. Since Ω is locally constant, any element in partition is open set. Assume that there exists at least two element in the partition. This is impossible since Ω is connected set. Therefore, u is a.e. constant function.

Problem 6

First, I'll show that $u \in L^n(B_1(\mathbf{0})$. Note that u is symmetric function about rotation, so we can show that integral on $B_1(\mathbf{0})$ is finite by showing that integral is finite for r. Also, we can restrict the range of r to $(0, \frac{1}{e-1}$ since u is bounded in outside of the range. In other words,

$$\int_{B_1(\mathbf{0})} u dx \leq C \int_0^{\frac{1}{e-1}} \left(\left| \log\log\left(1 + \frac{1}{r}\right) \right| \right)^n r^{n-1} dr$$

for some constant $C < \infty$. Let $y = \log\left(1 + \frac{1}{r}\right)$, then

$$\begin{aligned}
\left| \int_0^{\frac{1}{e-1}} \left(\log\log\left(1 + \frac{1}{r}\right) \right)^n r^{n-1} dr \right| &\leq \int_1^\infty (\log y)^n \frac{e^y}{(e^y - 1)^{n+1}} dy \\
&\leq \int_1^\infty (\log y)^n \frac{2^{n+1} e^y}{e^{(n+1)y}} dy \\
&\leq \int_1^\infty y^n 2^{n+1} e^{-ny} dy < \infty
\end{aligned}$$

Therefore, $u \in L^n(B_1(\mathbf{0}))$, and $u \in L^1(B_1(\mathbf{0}))$.

Next, I'll show that u has weak derivative in $B_1(\mathbf{0})$ and belongs to $L^n(B_1(\mathbf{0}))$. Since u goes to ∞ as $x \to 0$, we need to care when we compute weak derivative. However, we can ignore at $\mathbf{0}$ by the following argument. Let V be a compactly embedded set in U and ϕ be a C^∞ function having support V. Assume $\mathbf{0} \in V$. Without $\mathbf{0}$, Du should be $\partial_{x_i} u$ for some i. Since u, $D^\alpha \phi$ for all α are L^1 function on V, we can use Fubini theorem, and rewrite the integral by

$$\int_U uD\phi dx = \int_{-1}^{1} (\cdots) dx_1.$$

Since $n > 1$, we know that the $n-1$ dim plane through 0 is measure zero set and it does not effect integral to delete 0 from integral range of x_1. Therefore, the weak derivative is just derivative of u except $\mathbf{0}$...(Fundamental of Calculus? d/dx1-¿ int int dx1 dx2-¿Explicitly show)

I'll show that Du is in L^n. Computing partial derivative:

$$|\partial_{x_i} u| = \left| \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \frac{1}{1 + \frac{1}{|x|}} \frac{x_i}{|x|^3} \right| \leq \frac{1}{\left|\log\left(1 + \frac{1}{r}\right)\right|} \frac{1}{r+1} \frac{1}{r}.$$

Then, by the same reason before, we just need to check whether the integral in finite for r in $\left(0, \frac{1}{e-1}\right)$.

$$\int_0^{\frac{1}{e-1}} \left(\frac{1}{\log\left(1 + \frac{1}{r}\right)} \frac{1}{r+1} \frac{1}{r} \right)^n r^{n-1} dx \leq \int_0^{\frac{1}{e-1}} \left(\frac{1}{\log\left(1 + \frac{1}{r}\right)} \right)^n \frac{1}{r} dr$$

Let $x = \log\left(1 + \frac{1}{r}\right)$, then the integral becomes

$$\int_1^\infty \frac{1}{x^n} \frac{e^x}{e^x - 1} dx$$

For sufficiently large R, $\frac{e^x}{e^x-1} < 2$ for $x > R$ and we know that $\int_1^\infty \frac{1}{x^n}$ converges for $n > 2$. Therefore, $Du \in L^n(B_1(\mathbf{0}))$ and $u \in W^{1,n}(B_1(\mathbf{0}))$.

Problem 7

Since $u \in L^2(\mathbb{R}^n)$, $u = (\hat{u})^\vee$ by Theorem 2 in chapter 4.3 Evans. Then,

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} \left|e^{ikx} \hat{u}(k)\right| dk \leq \int_{\mathbb{R}^n} |\hat{u}(k)| dk \\
&= \int_{\mathbb{R}^n} (1 + |k|^2)^{s/2} (1 + |k|^2)^{-s/2} |\hat{u}(k)| dk \\
&\left(\leq \int_{\mathbb{R}^n} (1 + |k|^2)^s |\hat{u}|^2 dk \right)^{1/2} \left(\int_{\mathbb{R}^n} (1 + |k|^2)^{-s} dk \right)^{1/2}
\end{aligned}$$

For $|k| > 1$, $(1 + |k|^2)^s > (2|k|)^{2s}$

$$\int_{|k|>1} k^{-2s} dk = \sigma(S^{n-1}) \int_1^\infty r^{-2s} r^{n-1} dr < \infty$$

since $-2s + n - 1 < -1$ and $\int_1^\infty r^\alpha dr < \infty$ for $\alpha < -1$. Therefore,

$$|u(x)| \leq C \left(\int_{\mathbb{R}^n} (1 + |y|^2)^s |\hat{u}|^2 dy \right)^{1/2} = C \|u\|_{H^s(\mathbb{R}^n)}$$

This is true for a.e. x, so

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \|H^s(\mathbb{R}^n)\|$$