MATH 517 PARTIAL DIFFERENTIAL EQUATIONS
HOMEWORK 2

Updated on October 17, 2018 Due 3:30pm, Thursday, Ocrober 25, 2018.
TeX-typed Homework is accepted only.(No hand-written Homework accepted)

1. Let $\Phi(t, \mathrm{x})$ be the fundamental solution of the n-dimensional Heat equation
$$u_t - \Delta_{\mathrm{x}} u = 0 \quad \text{for } t > 0,\, \mathrm{x} \in \mathbb{R}^{\mathrm{n}}.$$
For $r > 0$, define a *Heat ball* $E(t, \mathrm{x}; r)$ with the center (t, x) by
$$E(t, \mathrm{x}; r) := \left\{ (s, \mathrm{y}) \in \mathbb{R}^{n+1} : s \leq t,\ \Phi(t - s, \mathrm{x} - \mathrm{y}) \geq \frac{1}{r^n} \right\}.$$
Show that
$$\frac{1}{4r^n} \int\int_{E(0,\mathbf{0};r)} \frac{|\mathrm{y}|^2}{s^2}\, d\mathrm{y}ds = 1 \quad \forall\, r > 0$$
by following the steps described below:
   - Step 1: Show that
   $$\frac{1}{4r^n} \int\int_{E(0,\mathbf{0};r)} \frac{|\mathrm{y}|^2}{s^2}\, d\mathrm{y}ds = \frac{1}{4} \int\int_{E(0,\mathbf{0};1)} \frac{|\mathrm{y}|^2}{s^2}\, d\mathrm{y}ds \quad \forall\, r > 0.$$
   - Step 2: Show that
   $$E(0, 0; 1) = \left\{ (s, y) \in \mathbb{R}^{n+1} : -\frac{1}{4\pi} \leq s \leq 0,\ |\mathrm{y}|^2 \leq -2sn \ln\left(\frac{1}{-4\pi s} \right) \right\}$$
   - Step 3*: Using the definition of *the Gamma function* $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$, prove that
   $$\frac{1}{4} \int\int_{E(0,\mathbf{0};1)} \frac{|\mathrm{y}|^2}{s^2}\, d\mathrm{y}ds = 1.$$
   Here, you may use the following properties:
   $$\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad \Gamma(\frac{n}{2} + 2) = (\frac{n}{2} + 1)\Gamma(\frac{n}{2} + 1),$$
   where ω_n denotes the volume of $B_1(\mathbf{0}) \subset \mathbb{R}^n$.

2. Suppose that u is C^1 w.r.t t, and C^2 w.r.t. $\mathrm{x} \in \mathbb{R}^{\mathrm{n}}$, and that u solves the Heat equation

$$u_t - \Delta_{\mathrm{x}} u = 0 \quad \text{for } t > 0,\, \mathrm{x} \in \mathbb{R}^{\mathrm{n}}.$$

Prove that u satisfies the following mean value property:

$$u(t,\mathrm{x}) = \frac{1}{4r^n} \int\int_{E(t,\mathrm{x};r)} u(s,\mathrm{y}) \frac{|\mathrm{x}-\mathrm{y}|^2}{|t-s|^2}\, d\mathrm{y}ds$$

for any $t > 0$, $\mathrm{x} \in \mathbb{R}^n$, and $r > 0$. (Note. This mean value formula yields the *Strong maximum principle of the Heat equation.*)

3. Find a *nontrivial solution* to the initial value problem

$$u_t - u_{xx} = 0 \text{ for } t > 0,\, x \in \mathbb{R}, \quad u(0,x) = 0 \text{ for } x \in \mathbb{R}$$

by following the steps described below:

- Step 1. Set $u(t,x) := \sum_{j=0}^{\infty} g_j(t)x^j$, and find a recursive condition for g_j so that u solves the initial value problem given above.
- Step 2. By using the recursive condition for g_j found in the previous step, show that if $g_0(t) = g(t)$ and $g_1(t) = 0$, then

$$g_{2k+1}(t) = 0, \quad g_{2k}(t) = \frac{1}{(2k)!} \frac{d^k}{dt^k} g(t) \quad \text{for all } k = 0, 1, 2, \cdots.$$

- Step 3. Choose $g(t)$ as

$$g(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}.$$

By using *Cauchy integral formula*, show that there exists a sufficiently small constant $\theta > 0$ such that

$$\left|\frac{d^k}{dt^k} g(t)\right| \le k! e^{-\frac{1}{t}} (\theta t)^{-k} \quad \text{for all } k = 0, 1, 2, \cdots.$$

- Step 4. By using the results obtained from steps 1–3, show that $u(t,x)$ given in Step 1 converges for all $t > 0$ and $x \in \mathbb{R}$, and also show that

$$\lim_{t \to 0+} u(t,x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

4. Given smooth functions f and g with compact supports, write down an explicit formula for a solution of

$$u_t - \Delta_{\mathrm{x}} u + cu = f(t,\mathrm{x}) \quad \text{for } t > 0,\, \mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \quad u(0,\mathrm{x}) = g(\mathrm{x})$$

where $c \in \mathbb{R}$.

5. Given smooth functions f, ϕ and ψ, write down an explicit formula for a solution of

$$\begin{cases} u_{tt} - u_{xx} = f(t,x) & \text{for } t > 0,\ x \in \mathbb{R} \\ u(0,x) = \phi(x),\ u_t(0,x) = \psi(x) & \text{for } x \in \mathbb{R} \end{cases}$$

where $c \in \mathbb{R}$.

6. Assume that $u \in W^{1,p}(\Omega)$ for $\Omega = (0,1) \subset \mathbb{R}$. Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \|u'\|_{L^p(\Omega)}$$

for a.e. $x, y \in \Omega$.

7. * Prove that

$$\|Du\|_{L^p(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1/2} \|D^2 u\|_{L^p(\Omega)}^{1/2}$$

for $2 \leq p < \infty$, and all $u \in C_0^\infty(\Omega)$.