Partial Differential Equation - HW 2

SungBin Park, 20150462

October 25, 2018

Problem 1

Step 1. Let a linear transformation from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ sending $s \mapsto r^2 s$ and $y_i \mapsto ry_i$, then T sends $E(0, \mathbf{0}, 1)$ to $E(0, \mathbf{0}, r)$ bijectively. To check it, let $(s, y) \in E(0, \mathbf{0}, 1)$, then $r^2 s \leq 0$ and $\frac{1}{(4\pi(-r^2 s))^{n/2}} e^{\frac{|ry|^2}{4r^2 s}} = \frac{1}{r^n} \frac{1}{(4\pi(-s))^{n/2}} e^{\frac{|y|^2}{4s}} \geq \frac{1}{r^n}$. Since T is injective, T is bijective between $E(0, \mathbf{0}, r)$ and $E(0, \mathbf{0}, 1)$. Therefore,

$$\begin{aligned}
\frac{1}{4} \iint_{E(0,\mathbf{0},1)} \frac{|y|^2}{s^2} dyds &= \frac{1}{4} \iint_{E(0,\mathbf{0},r)} \frac{|y|^2/r^2}{s^2/r^4} \frac{1}{r^{n+2}} dyds \\
&= \frac{1}{4r^n} \iint_{E(0,\mathbf{0},r)} \frac{|y|^2}{s^2} dyds
\end{aligned}$$

Step 2. Since $e^x \leq 1$ for $x \leq 0$,

$$\begin{aligned}
&\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{|y|^2}{4s}} \geq 1 \text{ for } s \leq 0 \\
&\Leftrightarrow 4\pi(-s) \leq 1 \text{ and } e^{\frac{|y|^2}{4s}} \geq (4\pi(-s))^{n/2} \text{ and } s \leq 0 \\
&\Leftrightarrow -\frac{1}{4\pi} \leq s \leq 0 \text{ and } |y|^2 \leq 2sn \ln(-4\pi s)
\end{aligned}$$

Therefore, $E(0, 0, 1) = \left\{ (s, y) \in R^{n+1} : -\frac{1}{4\pi} \leq s \leq 0, |y|^2 \leq -2sn \ln\left(\frac{1}{-4\pi s}\right) \right\}$

Step 3. Since $|y|^2 \leq -2sn \ln\left(\frac{1}{-4\pi s}\right)$ is the ball with radius $\left(-2sn \ln\left(\frac{1}{-4\pi s}\right)\right)^{1/2}$,

$$\begin{aligned}
\iint_{E(0,\mathbf{0},1)} \frac{|y|^2}{s^2} dyds &= \int_{-1/4\pi}^{0} \int_{B_s} \frac{|y|^2}{s^2} dyds \text{ (By Fubini theorem)} \\
&= \int_{-1/4\pi}^{0} \frac{1}{s^2} \int_{S^{n-1}} \int_{0}^{\left(-2sn \ln\left(\frac{1}{-4\pi s}\right)\right)^{1/2}} r^{n+1} dr d\sigma ds \text{ (By Spherical Coord.)} \\
&= \int_{-1/4\pi}^{0} \frac{1}{s^2} \frac{\frac{n}{2}\pi^{n/2}}{\Gamma(\frac{n}{2}+2)} \left(-2sn \ln\left(\frac{1}{-4\pi s}\right)\right)^{n/2+1} ds
\end{aligned}$$

Let $t = \ln\left(\frac{1}{-4\pi s}\right)$, then

$$\begin{aligned}
&\frac{1}{4}\left(\frac{n}{2\pi}\right)^{n/2+2}\frac{\pi^{n/2}}{\Gamma(n/2+2)}\int_0^\infty \frac{(4\pi)^2}{e^{-2t}}\left(e^{-t}t\right)^{n/2+1}e^{-t}dt\\
&=4\left(\frac{n}{2}\right)^{n/2+2}\frac{1}{\Gamma(n/2+2)}\int_0^\infty t^{n/2+1}e^{-(n/2)t}dt\\
&=4\left(\frac{n}{2}\right)^{n/2+2}\frac{1}{\Gamma(n/2+2)}\left(\frac{2}{n}\right)^{n/2+2}\Gamma(n/2+2)=4.
\end{aligned}$$

Therefore, $\frac{1}{4}\iint_{E(0,\mathbf{0},1)}\frac{|y|^2}{s^2}dyds = 1$.

Consequently, $\frac{1}{4r^n}\iint_{E(0,\mathbf{0},r)}\frac{|y|^2}{s^2}dydx = 1$ for all $r > 0$.

Problem 2

By change of variable shifting x and t to 0, we can rearrange the formula by

$$\begin{aligned}
\frac{1}{r^n}\iint_{E(t,x;r)}u(s,y)\frac{|s-y|^2}{|t-s|^2}dyds &= \frac{1}{r^n}\iint_{E(0,\mathbf{0};r)}u'(s,y)\frac{|y|^2}{s^2}dyds\\
&=\iint_{E(0,\mathbf{0};1)}u'(r^2s',ry')\frac{|y'|^2}{s'^2}dy'ds'
\end{aligned}$$

where $u'(s,y) = u(s+t, y+x)$ and $s = r^2s'$, $y = ry'$. For simplicity, I'll write u by u' . Let $\phi(r)$ be the RHS of above equation. Our strategy is showing that $\phi(r)$ is constant function, and it means $\phi(r) = \lim_{r\to 0}\phi(r) = r^n u(0,0)$ using continuity of u and the result from problem 1. I'll write $E(r) = E(0,\mathbf{0};r)$. Computing $\phi'(r)$,

$$\begin{aligned}
\phi'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\iint_{E(1)}u(r^2s',ry')\frac{|y'|^2}{s'^2}dy'ds'\\
&=\iint_{E(1)}(2rs'u_s(r^2s',ry') + y'\cdot\nabla_y u(r^2s',ry'))\frac{|y'|^2}{s'^2}dy'ds' \text{ (Since } \frac{|y|^2}{s^2}\in L^1(E(1)) \text{ and } u\in C(\overline{E(1)})\text{)}\\
&=\frac{1}{r^{n+1}}\iint_{E(r)}(2su_s(s,y) + y\cdot\nabla_y u)\frac{|y|^2}{s^2}dyds
\end{aligned}$$

Let the first term A and last term B.

For simple calculation, I'll introduce a function

$$\varphi(s,y) = -\frac{n}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + n\ln r$$

Then, we can get a relation $|y|^2 = \sum_{i=1}^n 2sy_i\varphi_{y_i}$, and

$$\begin{aligned}
r^{n+1}A &= \iint_{E(r)}4u_s\sum_{i=1}^n y_i\varphi_{y_i}dyds\\
&=\iint_{E(r)}4\sum_{i=1}^n\left(\frac{\partial}{\partial y_i}(y_iu_s\varphi) - u_s\varphi - u_{sy_i}y_i\varphi\right)dyds\\
&=-4\iint_{E(r)}nu_s\varphi + \varphi\sum_{i=1}^n u_{sy_i}y_idyds
\end{aligned}$$

Since

$$\iint_{E(r)} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} (y_i u_s \varphi)\, dyds = \sum_{i=1}^{n} \int_{-\frac{r^2}{4\pi}}^{0} \int_{B\left(\mathbf{0},-2ns\ln\left(-\frac{4\pi s}{r^2}\right)\right)} \frac{\partial}{\partial y_i} (y_i u_s \varphi)\, dyds$$
$$= \sum_{i=1}^{n} \int_{-\frac{r^2}{4\pi}}^{0} \int_{\partial B\left(\mathbf{0},-2ns\ln\left(-\frac{4\pi s}{r^2}\right)\right)} (y_i u_s \varphi)\, \nu^i dyds$$
$$= 0$$

for $\varphi \equiv 0$ in $\partial E - \{0, \mathbf{0}\}$.($\nu^i$ is ith component of outward pointing unit normal vector.)

By $\frac{\partial}{\partial s}(u_{y_i} y_i \varphi) = u_{sy_i} y_i \varphi + u_{y_i} \varphi y_i \varphi_s$

$$\iint_{E(r)} \varphi \sum_{i=1}^{n} u_{sy_i} y_i dyds = \iint_{E(r)} \sum_{i=1}^{n} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi) - u_{y_i} y_i \varphi_s dyds$$

Let $s_0(y)$ and $s_1(y)$ is the the boundary points of s for fixed $y \neq \mathbf{0}$, and $\{y\} \times (s_0(y), s_1(y))$ is in the heat ball. In the integration on $E(r)$, we can neglect $y = 0$ case since it $\{\mathbf{0}\} \times (-1/4\pi, 0)$ is measure zero set in Ω. Then,

$$\iint_{E(r)} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi)\, dyds = \int_{|y|^2 \le \frac{nr^2}{2\pi e}} \int_{s_0(y)}^{s_1(y)} \frac{\partial}{\partial s}(u_{y_i} y_i \varphi)\, dsdy$$
$$= \int_{|y|^2 \le \frac{nr^2}{2\pi e}} u_{y_i}(s_1(y), y), y) y_i \varphi(s_1(y)) - u_{y_i}(s_0(y), y), y) y_i \varphi(s_0(y)) dy = 0.$$

because $\varphi(s_0(y)) = \varphi(s_1(y)) = 0$ for $y \neq \mathbf{0}$.

Therefore,

$$r^{n+1} A = -4 \iint_{E(r)} nu_s \varphi + \varphi \sum_{i=1}^{n} u_{sy_i} y_i dyds = -4 \iint_{E(r)} nu_s \varphi - \varphi \sum_{i=1}^{n} u_{y_i} y_i \varphi_s dyds$$
$$= -4 \iint_{E(r)} nu_s \varphi - \sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) dyds$$
$$= \iint_{E(r)} -4nu_s \varphi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds - B.$$

Since u is a solution for heat equation,

$$\phi'(r) = A + B = \iint_{E(r)} -4nu_s \varphi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds$$
$$= \iint_{E(r)} -4n\varphi \triangle_y u - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds$$
$$= \iint_{E(r)} \sum_{i=1}^{n} -4nu_{y_i} \varphi_{y_i} - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds$$
$$= \iint_{E(r)} \sum_{i=1}^{n} \frac{2n}{s} u_{y_i} y_i - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dyds = 0.$$

The intermediate step $\varphi \triangle_y u \to u_{y_i} \varphi_{y_i}$ uses the fact that $\varphi = 0$ on $\partial E(r) - \{(0, \mathbf{0}\}$ as in s case.

Consequently, ϕ is a constant function. Since u is continuous and by problem 1,

$$\lim_{r\to 0+}\frac{1}{r^n}\iint_{E(0,\mathbf{0};r)} u(s,y)\frac{|y|^2}{s^2}dyds = 4u(0,0)$$

Hence,

$$\frac{1}{4r^n}\iint_{E(t,x;r)} u(s,y)\frac{|s-y|^2}{|t-s|^2}dyds = u(t,x)$$

for all $r > 0$.

Problem 3

Step 1. First, we need to assume that the convergence radius of u is ∞, so that we can safely differentiate the series term by term. To satisfy $u_t - u_{xx} = 0$ for $t > 0, x \in \mathbb{R}$,

$$u_t - u_{xx} = \sum_{j=0}^{\infty}\left(\frac{\partial g_j(t)}{\partial t} - (j+1)(j+2)g_{j+2}(t)\right)x^j = 0.$$

By uniqueness of power series in the convergence region, $\frac{\partial g_j(t)}{\partial t} - (j+1)(j+2)g_{j+2} = 0$ for all $j \geq 0$ iff $u_t - u_{xx} = 0$.

To satisfy initial condition $u(0,x) = 0$ for $x \in \mathbb{R}$, $\sum_{j=0}^{\infty} g_j(0)x^j = 0$, $g_j(0)$ should be 0 for all j followed by uniqueness of power series.

Step 2. I'll use induction for proof. Let g_j be

$$g_j(t) = \begin{cases} 0 & j \text{ is odd.} \\ \frac{1}{(2(j/2))!}\frac{d^{(j/2)}}{dt^{(j/2)}}g(t) & j \text{ is even.} \end{cases} \tag{1}$$

for $j < 2n$. The starting case is given in the problem. For $j = 2n$,

$$g_{2n} = \frac{1}{(2n-1)(2n)}\frac{\partial g_{2n-2}(t)}{\partial t} = \frac{1}{2n!}\frac{d^n}{dt^n}g(t).$$

For $j = 2n+1$,

$$g_{2n+1} = \frac{1}{(2n)(2n+1)}\frac{\partial g_{2n-1}(t)}{\partial t} = 0.$$

Therefore, (1) is valid.

Step 3. Since $\frac{1}{z^2}$ is holomorphic without $z = 0$, $e^{-\frac{1}{z^2}}$ is holomorphic without $z = 0$. For $t > 0$, let $r = t/8$ be a radius of open disk centred at $t \in \mathbb{R}$ in $\mathbb{C}$. Let the boundary of the disk C. Then by Cauchy integral formula,

$$\left|\frac{d^k}{dt^k}g(t)\right| \leq \frac{k!}{2\pi}\oint_C\left|\frac{e^{-\frac{1}{z^2}}}{(z-t)^{k+1}}\right|dz \leq \frac{k!}{2\pi}r^{-k}\max_C\left|e^{-\frac{1}{z^2}}\right|$$

and if we let $z = x + iy$, $x, y \in \mathbb{R}$,

$$\left|e^{-\frac{1}{z^2}}\right| = \left|e^{-\frac{x^2-y^2}{(x^2+y^2)^2}}\right| \leq \left|e^{-\frac{1280}{1681}t^{-2}}\right| \leq \left|e^{-\frac{1}{2t^2}}\right|.$$

Therefore, if we set $\theta = \frac{1}{8}$,

$$\left|\frac{d^k}{dt^k}g(t)\right| \leq k!\left(\frac{1}{8}t\right)^{-k}e^{-\frac{1}{2t^2}}$$

If $t \leq 0$, it is obvious inequality since $g^{(k)}(t) = 0$ for $t < 0$ and $\lim_{t\to 0} g^{(k)}(t) = 0$.(For all $k \in \mathbb{N}$, there exists polynomial about $\frac{1}{t}$ $P_k\left(\frac{1}{t}\right)$ such that $g^{(k)}(t) = P_k\left(\frac{1}{t}\right)e^{-\frac{1}{t^2}}$ and we know that for any $j \in \mathbb{N}$, $\frac{1}{t^j}e^{-\frac{1}{t^2}} \to 0$ as $t \to 0+$. Therefore, $g^{(k)}(t) \to 0$ as $t \to 0$.)

Step 4. Let g_i as in Step 2. Then,

$$\left|g_{2k}(t)x^{2k}\right| \leq \frac{k!}{(2k)!}e^{-\frac{1}{2t^2}}\left(\frac{8x^2}{t}\right)^k.$$

The radius of convergence of RHS is

$$R^2 \leq \lim_{k\to\infty}(2k+1)\frac{t}{4},$$

and it means $\sum g_{2k}(t)x^{2k}$ converges for all $t > 0$ and $x \in \mathbb{R}$.

Since $\frac{k!}{(2k)!}e^{-\frac{1}{2t^2}}\left(\frac{8x^2}{t}\right)^k \leq \frac{1}{k!}e^{-\frac{1}{2t^2}}\left(\frac{8x^2}{t}\right)^k$, by replacing $\frac{8x^2}{t}$ by z, we can get

$$\left|\sum g_{2k}(t)x^{2k}\right| \leq e^{-\frac{1}{2t^2}}\sum \frac{1}{k!}z^k = e^{-\frac{1}{2t^2}+\frac{8x^2}{t}}$$

for fixed $t > 0$ with convergence of radius ∞. As $t \to 0$, $e^{-\frac{1}{2t^2}+\frac{8x^2}{t}} \to 0$. Therefore,

$$\sum g_{2k}(t)x^{2k} \to 0 \quad \text{as } t \to 0.$$

Problem 4

Consider the following PDE:

$$u_t - \triangle_x u = f(t,x)e^{ct} \quad \text{for } t > 0, x \in \mathbb{R}^n, \quad u(0,x) = g(x) \tag{2}$$

Then, $f(t,x)e^{ct}$ and $g(x)$ are smooth and have compact supports. Therefore, there exists a solution u satisfying (2):

$$u(x,y) = \int_{\mathbb{R}^n}\Phi(x-y,t)g(y)dy + \int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)e^{ct}dyds$$

Let's consider $u' = ue^{-ct}$, then

$$\begin{aligned} &u'_t - \triangle_x u' + cu' = u_t e^{-ct} - cue^{ct} - \triangle_x ue^{ct} + cue^{ct} = (u_t - \triangle_x u)e^{ct} = f(t,x) \\ &u'(0,x) = ue^{c\cdot 0} = u = g(x) \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}^n$. Therefore,

$$e^{-ct}\left(\int_{\mathbb{R}^n}\Phi(x-y,t)g(y)dy + \int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)e^{ct}dyds\right)$$

is a solution for the problem.

Problem 5

Let u be:

$$u(t,x) = \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t}h(y)dy + \frac{1}{2}\int_0^t\int_{x+t-s}^{x-t+s}f(y,s)dyds. \tag{3}$$

First, u is a smooth function since g, h, f are. Also,

$$
\begin{aligned}
u_t &= \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \left(h(x+t) + h(x-t)\right) + \frac{1}{2}\int_0^t f(x+t-s,s) + f(x-t+s,s)ds \\
u_{tt} &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + f(x) + \int_0^t f_t(x+t-s) + f_t(x-t+s)ds \\
u_x &= \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \left(h(x+t) - h(x-t)\right) + \frac{1}{2}\int_0^t f(x+t-s,s) - f(x-t+s,s)ds \\
u_{xx} &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + \frac{1}{2}\int_0^t f_x(x+t-s,s) - f_x(x-t+s,s)ds \\
&= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \left(h'(x+t) - h'(x-t)\right) + \frac{1}{2}\int_0^t f_t(x+t-s,s) + f_t(x-t+s,s)ds \\
&= u_{tt} - f(x,t)
\end{aligned}
$$

Therefore, $u_{tt} - u_{xx} = f(x,t)$.

Also,

$$
\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(t,x) = g(x_0,0), \lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0,0)
$$

since u_t and u is continuous on $\mathbb{R}^2$.

Hence, (3) is a solution for the problem.

Problem 6

WLOG, let $x \le y$. I'll show that

$$
u(y) - u(x) = \int_y^x u' dt
$$

Since $u' \in L^p(0,1)$, $\int_x^y |u'|dt \le \|u'\|_{L^p(\Omega)}$, and $u' \in L^1(\Omega)$.

If $x = y$, it is trivially equal, so I'll set $x < y$. Fix small enough $0 < \epsilon < \frac{x+y}{2}$ and consider $\phi_\epsilon(t) \in C_c^\infty((x,y))$ such that $\phi_\epsilon \equiv 1$ in $[x+\epsilon, y-\epsilon]$. Define

$$
\eta_\delta(x) = \begin{cases} \frac{1}{\delta} e^{\frac{1}{\frac{x^2}{\delta^2}-1}} & \text{(For } |x| < \delta) \\ 0 & \text{(For } |x| \ge \delta) \end{cases}
$$

I'll set the $\phi_\epsilon(t)$ by

$$
\phi_\epsilon(t) = \begin{cases} \int_x^t \eta_\epsilon(\sigma - x)d\sigma / \int_x^{x+\epsilon} \eta_\epsilon(\sigma - x)d\sigma & \text{(For } x \le t < \frac{x+y}{2}) \\ 1 - \left(\int_{y-\epsilon}^t \eta_\epsilon(\sigma - (y-\epsilon))d\sigma / \int_{y-\epsilon}^y \eta_\epsilon(\sigma - (y-\epsilon))d\sigma\right) & \text{(For } \frac{x+y}{2} \le t \le y) \end{cases}
$$

Then,

$$
\begin{aligned}
\int_\Omega u\phi_\epsilon' dt &= \int_x^{x+\epsilon} u\phi_\epsilon' dt + \int_{y-\epsilon}^y u\phi_\epsilon' dt \\
&= -\int_\Omega u'\phi_\epsilon dt = -\int_x^y u'\phi_\epsilon dt.
\end{aligned}
$$

Since $u' \in L^1(\Omega)$, $|u'\phi_\epsilon| \le |u'|$, and $\phi_\epsilon \to 1$ in $[x,y]$ a.e., so by Lebesgue dominance convergence theorem, $\lim_{\epsilon\to 0} \int_x^y u'\phi_\epsilon dt = \int_x^y u' dt$.

Let's consider $\int_x^{x+\epsilon} u\phi'_\epsilon dt$. We know that $\int_x^{x+\epsilon} \phi'_\epsilon dt = \phi_\epsilon(x+\epsilon) - \phi_\epsilon(x) = 1$ and $\int_0^\epsilon \eta_\epsilon(\sigma)\sigma = \int_0^1 \eta(\sigma)\sigma = C > 0$. Therefore,

$$\begin{aligned}\int_x^{x+\epsilon} u\phi'_\epsilon dt &= C^{-1}\int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt + \int_0^\epsilon u(x)\phi'_\epsilon dt \\ &= \int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt + u(x).\end{aligned}$$

Since $\|\eta_\epsilon\|_{L^\infty} = \frac{1}{\epsilon}$ and $C^{-1}\left|\int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt\right| \leq \frac{\|\eta_\epsilon\|_{L^\infty}}{C}\left|\frac{1}{\epsilon}\int_x^{x+\epsilon}(u-u(x))dt\right|$, by Lebesgue differential theorem, $\int_x^{x+\epsilon}(u-u(x))\eta_\epsilon(t-x)dt \to 0$ as $\epsilon \to 0$. Therefore, $\int_x^{x+\epsilon} u\phi'_\epsilon dt \to u(x)$ as $\epsilon \to 0$. By the same reason, $\int_{y-\epsilon}^y u\phi'_\epsilon dt \to -u(y)$ as $\epsilon \to 0$.

Therefore,

$$\int_x^y u' dt = u(y) - u(x)$$

a.e. and

$$|u(y)-u(x)| \leq \left|\int_x^y u' dt\right| \leq |x-y|^{1-\frac{1}{p}}\|u'\|_{L^p(\Omega)}$$

by Hölder's inequality.

Problem 7

Because u is compactly supported, we can set B be a large ball containing the compact support of u and set $u = 0$ out of compact support for integration; it does not effect the integration. Since $p \geq 2$, we can do integration by parts:

$$\begin{aligned}\int_U |Du|^p dx &= \sum_{i=1}^n \int_B u_{x_i}u_{x_i}|Du|^{p-2}dx \\ &= -\sum_{i,j=1}^n \int_B u\left(u_{x_ix_i}|Du|^{p-2} + u_{x_i}u_{x_j}u_{x_jx_i}|Du|^{p-4}\right)dx \quad \text{Since } u \equiv 0 \text{ at boundary} \\ &\leq -\sum_{i,j=1}^n \int_B u\left(u_{x_ix_i}|Du|^{p-2} + Bu_{x_jx_i}|Du|^{p-2}\right)dx \quad \text{Since } \sum_{i,j} u_{x_i}u_{x_j} \leq n\left|Du^2\right| \text{ by Cauchy-Schwarz inequality} \\ &\leq -C\int_B u\left|D^2u\right|\left(|Du|^{p-2}\right)dx \quad \text{By the same reason.} \\ &\leq \left|C\int_U u\left|D^2u\right|\left(|Du|^{p-2}\right)dx\right|\end{aligned}$$

for some constant B and C depends on n.

Let $p > 2$, then by the Hölder's inequality,

$$\left|\int_U u\left|D^2u\right|\left(|Du|^{p-2}\right)dx\right|^p \leq \left(\int_U \left|u\left|D^2u\right|\right|^{\frac{p}{2}}dx\right)^2\left(\int_U |Du|^p dx\right)^{p-2}$$

If $\int_U |Du|^p dx = 0$, then the original inequality satisfied, so we can assume $\int_U |Du|^p dx > 0$. Then,

$$\left(\int_U |Du|^p dx\right)^2 \leq C^p\left(\int_U \left|u\left|D^2u\right|\right|^{\frac{p}{2}}dx\right)^2 \leq C^p\left(\int_U |u|^p dx\right)\left(\int_U \left|D^2u\right|^p dx\right)$$

Therefore,

$$\|Du\|_{L^p} \leq C\|u\|_{L^p}^{1/2}\|D^u\|_{L^p}^{1/2}$$

for $2 \leq p \leq \infty$ and all $u \in C_c^\infty(U)$.