MATH 517 PARTIAL DIFFERENTIAL EQUATIONS
HOMEWORK 3

Updated on November 7, 2018 Due date: 17:00 on Tuesday, Nov. 20, 2018.

TeX-typed Homework is accepted only.(No hand-written Homework accepted)

1. Let Ω be an open and bounded set in $\mathbb{R}^2$. Assume that the boundary $\partial\Omega$ of Ω satisfies
$$\partial\Omega = \bigcup_{j=1}^{n} \Gamma_j,$$
where each Γ_j for $j = 1, \cdots, n$ is a C^1 curve. Also, assume that if $j \neq j'$, then Γ_j and $\Gamma_{j'}$ do not intersect except at their endpoints. Under these assumptions, prove the following statement:

   If $u \in W^{k,p}(\Omega)$ for some $k \in \mathbb{N}$ and $1 \leq p < \infty$, then there exists a sequence $\{u^{(m)}\} \subset C^\infty(\overline{\Omega})$ so that
$$u^{(m)} \to u \quad \textit{in } W^{k,p}(\Omega) \textit{ as } m \to \infty.$$

2. Let Ω be an open and bounded set in $\mathbb{R}^n$, and sssume that the boundary $\partial\Omega$ of Ω is C^1. Is $W_0^{1,p}(\Omega)$ a *Banach space*?

3. Let Ω be an open set in $\mathbb{R}^n$. (The set Ω is not necessarily bounded.) For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, define a space
$$C^{k,\alpha}(\overline{\Omega}) := \{u \in C^k(\overline{\Omega}) : \|u\|_{C^{k,\alpha}(\overline{\Omega})} < \infty\}.$$
   (a) Prove that $C^{k,\alpha}(\overline{\Omega})$ is a vector space.
   (b) Is $C^{k,\alpha}(\overline{\Omega})$ finite dimensional?
   (c) Prove that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.

4. Let Ω be an open and bounded set in $\mathbb{R}^n$, and sssume that the boundary $\partial\Omega$ of Ω is C^1. Assume that $n < p < \infty$. Prove that $W^{1,p}(\Omega)$ is compactly embedded into $C^{0,\tilde{\alpha}}(\overline{\Omega})$ for any $\tilde{\alpha} \in [0, \alpha)$ for $\alpha = 1 - \frac{n}{p}$.

5. Suppose that Ω is connected and $u \in W^{1,p}(\Omega)$ satisfies
$$Du = 0 \quad \text{a.e. in } \Omega.$$
Prove that u is constant a.e. in Ω

6. Verify that if $n > 1$, the unbounded function

$$u = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(B_1(\mathbf{0}))$.

7. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$, then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant $C > 0$ depending only on s and n.