MATH 517 PARTIAL DIFFERENTIAL EQUATIONS
HOMEWORK 4

Due date: NOON on Tuesday, Dec. 24, 2018.

TeX-typed Homework is accepted only.(No hand-written Homework accepted)

1. Let X be a real Banach space, and let $A : X \to X$ be a bounded linear map. For $\eta \in \rho(A)$, prove that $(A - \eta \mathrm{Id})^{-1}$ is linear and bounded.

2. For a Hilbert space H, let $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ be a bounded linear map. A linear map $\mathcal{L}^* : \mathcal{H} \to \mathcal{H}$ is called the *adjoint of* $\mathcal{L}$ if

$$(\mathcal{L}x, y)_{\mathcal{H}} = (x, \mathcal{L}^* y)_{\mathcal{H}} \quad \text{for } \forall x, y \in \mathcal{H}.$$

Here, $(\cdot, \cdot)_{\mathcal{H}}$ is the inner product in $\mathcal{H}$. If $\mathcal{L} = \mathcal{L}^*$, then $\mathcal{L}$ is called *self-adjoint.*

Now, let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. For a complex valued function $u : \Omega \to \mathbb{C}$, we define

$$\mathcal{H} := \{u : \Omega \to \mathbb{C} \,|\, \mathrm{Re}\, u, \mathrm{Im}\, u \in H_0^1(\Omega)\}.$$

(a) For each $u \in \mathcal{H}$, set $v := \mathrm{Re}\, u$ and $w := \mathrm{Im}$. Show that the bilinear operator $(\cdot, \cdot)_{\mathcal{H}}$ given by

$$(u, u_*)_{\mathcal{H}} := \int_\Omega (v + iw)(v_* - iw_*) + (Dv + iDw)(Dv_* - iDw_*)\, dx$$

yields an inner product in $\mathcal{H}$.

(b) For each $i, j = 1, \cdots, n$, assume that $a_{ij} \in L^\infty(\Omega)$, and $a_{ij} = a_{ji}$ in Ω. Also, assume that there exists a constant $\mu \in (0, 1)$ satisfying

$$\mu|\boldsymbol{\xi}|^2 \le a_{ij}\xi_i\xi_j \le \frac{1}{\mu}|\boldsymbol{\xi}|^2 \quad \text{a.e. in } \Omega,\ \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

Here, each a_{ij} is a real-valued function. Define

$$Lu := -\partial_j(a_{ij}\partial_i u),$$

and define a bilinear operator $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$B[u, v] = \int_\Omega a_{ij}\partial_i u \partial_j \bar{v}\, \mathrm{dx}.$$

Prove that there exists a bounded linear map $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ satisfying

$$(\mathcal{L}u, v)_{\mathcal{H}} = B[u, v] \quad \text{for all } u, v \in \mathcal{H}.$$

(c) Prove that the $\mathcal{L}$ found in (b) is self-adjoint.

(d) By using the result obtained from (c), prove that the eigenvalue problem

$$\begin{cases} Lu = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has only real eigenvalues.

3. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with $\partial\Omega \in C^1$. For each $i, j = 1, \cdots, n$, assume that $a_{ij}, b_i, c \in L^\infty(\Omega)$, and assume that there exists a constant $\mu \in (0, 1)$ satisfying

$$\mu|\boldsymbol{\xi}|^2 \leq a_{ij}\xi_i\xi_j \leq \frac{1}{\mu}|\boldsymbol{\xi}|^2 \quad \text{a.e. in } \Omega, \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

Here, each of a_{ij}, b_i, c is a real-valued function. Define

$$Lu := -\partial_j(a_{ij}\partial_i u) + b_i\partial_i u + cu$$

For given functions $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$, suppose that $u \in H^1(\Omega)$ is a weak solution to the following boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}.$$

If the homogeneous boundary value problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

has ONLY trivial weak solution, then prove that there exists a constant $C > 0$ (independent of u, f and g) so that

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}).$$

4. Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with $\partial\Omega \in C^1$. Given functions $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, consider a boundary value problem

$$\begin{cases} -\Delta u + cu = f & \text{in } \ \Omega, \\ \nabla u \cdot \mathbf{n}_{in} = g & \text{on } \ \partial\Omega \end{cases}$$

where $\mathbf{n}_{in}$ is the inward unit normal on $\partial\Omega$. $u \in H^1(\Omega)$ is called *a weak solution* to this boundary value problem if u satisfies

$$\int_\Omega \nabla u \cdot \nabla v + cuv \, d\mathrm{x} = \int_\Omega fv \, d\mathrm{x} - \int_{\partial\Omega} gv \, dS_{\mathrm{x}} \quad \text{for all} \quad v \in H^1(\Omega).$$

Suppose that $c \in C^0(\overline{\Omega})$ satisfies

$$c \geq \mu_0 \quad \text{in} \quad \Omega$$

for some constant $\mu_0 > 0$.

(a) Derive the definition of weak solution stated above by integration by parts.

(b) Prove that there exists a unique weak solution $u \in H^1(\Omega)$ of this boundary value problem.

(c) Show that if $u \in H^1(\Omega)$ is a weak solution to the boundary value problem, then it satisfies the estimate

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)})$$

for some constant $C > 0$ depending only on c, Ω.

5. Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$. Suppose that $\partial\Omega$ is C^∞. Consider an eigenvalue problem

$$(EVP) \begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ \nabla u \cdot \mathbf{n}_{in} - u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\mathbf{n}_{in}$ is the inward unit normal on $\partial\Omega$. Define a set

$$\Sigma := \{\lambda : \lambda \text{ is an eigenvalue of (EVP)}\}.$$

(a) Define a bilinear operator $(\cdot,\cdot)_{H^1} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$(u, v)_{H^1} := \int_\Omega \nabla u \cdot \nabla v \, d\mathrm{x} + \int_{\partial\Omega} uv \, dS_{\mathrm{x}}.$$

Show that $(\cdot,\cdot)_{H^1}$ is an inner product in $H^1(\Omega)$. Equivalently, prove that the operator $(\cdot,\cdot)_{H^1}$ satisfies the following properties:

(i) $(u, v)_{H^1} = (v, u)_{H^1}$ for all $u, v \in H^1(\Omega)$.

(ii) There exists a constant $C > 0$ independent of u, v such that

$$|(u, v)_{H^1}| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

for all $u, v \in H^1(\Omega)$.

(iii) There exists a constant $\theta > 0$ independent of u, v such that

$$(u, u)_{H^1} \geq \theta \|u\|^2_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$.

(b) Show that every eigenvalue of (EVP) is a positive real number.

(c) Show that $\Sigma = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty\}$ is a monotonically increasing sequence which diverges to ∞.

(d) For each eigenvalue $\lambda_k \in \Sigma$(repeating each eigenvalue according to its multiplicity), let $w_k \in H^1(\Omega) \cap C^\infty(\overline{\Omega})$ be an eigenfunction corresponding to λ_k. Prove that one can choose $\{w_k\}_{k=1}^\infty$ such that

(i) $\{w_k\}_{k=1}^\infty$ forms an orthornormal basis of $L^2(\Omega)$, and

(ii) $\{w_k\}_{k=1}^\infty$ forms an orthogonal basis of $H^1(\Omega)$ with respect to the inner product $(\cdot, \cdot)_{H^1}$.

6. Let $H^1_*(\Omega)$ denote the dual space of $H^1(\Omega)$.(Of course, $H^1_*(\Omega)$ is isomorphic to $H^1(\Omega)$ by Riesz representation theorem. But we do not need to care about this fact for this problem.) For each $\mathcal{L} \in H^1_*(\Omega)$, the norm $\|\mathcal{L}\|_{H^1_*(\Omega)}$ in $H^1_*(\Omega)$ is defined by

$$\|\mathcal{L}\|_{H^1_*(\Omega)} := \sup_{v \in H^1(\Omega), \|v\|_{H^1(\Omega)}=1} |\mathcal{L}(v)|$$

Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected with $\partial\Omega \in C^\infty$. Fix a positive constant $T > 0$, and set

$$\Omega_T := \Omega \times (0, T]$$

Consider the following initial boundary value problem of linear parabolic equation: Given smooth functions $f \in C^\infty(\overline{\Omega_T})$, $g \in C^\infty(\overline{\Omega})$,

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega_T \\ \nabla u \cdot \mathbf{n}_{in} - u = 0 & \text{on} \quad \partial\Omega \times [0, T] \\ u = g & \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

Prove that there exists a weak solution u to this initial boundary value problem. In other words, show that there exists a function $u : \overline{\Omega_T} \to \mathbb{R}$ such that

(i) $u \in L^2(0, T; H^1(\Omega))$, $u' \in L^2(0, T; H^1_*(\Omega))$,

(ii) $u = g$ on $\Omega \times \{t = 0\}$,

(iii) For each $v \in H^1(\Omega)$, and a.e. $0 \leq t \leq T$, u satisfies

$$(u'(t), v)_{L^2(\Omega)} + B[(u(t)), v] = (f(t), v)_{L^2(\Omega)}$$

for $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}(v, w)_{L^2(\Omega)} &:= \int_\Omega vw \; d\mathrm{x}, \\ B[v, w] &:= \int_\Omega \nabla v \cdot \nabla w \; d\mathrm{x} + \int_{\partial\Omega} vw \; dS_{\mathrm{x}}.\end{aligned}$$