# Partial Differential Equation - HW 3

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## Problem 1

I'll use the proof in Evans.

*Proof.* Before starting, let's arrange the index of $\Gamma_j$ so that the adjacent curve of $\Gamma_j$ be $\Gamma_{j-1}$ and $\Gamma_{j+1}$.

Fix $x^0$ be in end point of $\Gamma_j$ and assume that $x^0$ is also a end point of $\Gamma_{j+1}$. Let $v_j$ be the tangential vector of $\Gamma_j$ such that $v_j$ is toward the $\Gamma_j$. In other words, if $\Gamma_j : [a,b] \to \mathbb{R}^2$ and $x_0 = \Gamma_j(b)$, then $v_j = \lim_{h\to 0+} \frac{\Gamma_j(b-h)-\Gamma_j(b)}{h}$. For $v_{j+1}$, set it be tangential vector of $\Gamma_{j+1}$ toward $\Gamma_{j+1}$ curve. If $vj$ and $v_{j+1}$ are parallel, $\Gamma_j$ and $\Gamma_{j+1}$ can be connected with $C^1$ property, I'll ignore the case. Let the angle between $v_j$ and $v_{j+1}$ be $\theta > 0$. Let $e_0$ be a unit vector such that parallel with $\frac{v_j+v_{j+1}}{2}$ and inward direction, i.e., $x_0 + \lambda e_0 \in \operatorname{int}\Omega$ for small enough $\lambda$.(This requires Jordan curve theorem.)

As $\Gamma_j$, $\Gamma_{j+1}$ are $C^1$, there exists $r > 0$ such that with $B(x_0, r)$, the tangential direction of $\Gamma_j$ and $\Gamma_{j+1}$ does not change much. More precisely, if we let $w_j$ be a tangential vector of $\Gamma_j$ at $y \in B(x_0, r)$, then the angle between $w_j$ and $v_j$ is less than $\theta/10$, and this is true for $\Gamma_{j+1}$. Let $w_j^1$ (resp. $w_j^2$) be the vector made by rotating $v_j$ by $\theta/10$ clockwise (resp. counter-clockwise). Do the same for $w_{j+1}^1$, $w_{j+1}^2$.

Now, let's repeat proof in Evans. Let's consider $U \cap B(x_0, r)$ and $V := U \cap B(x_0, r/2)$. Define $x_\epsilon := x + \epsilon e_0$ for $x \in V$, small enough $\epsilon > 0$ satisfying $x + \epsilon e_0 \in U \cap B(x_0, r)$. WLOG, I'll assume that $x$ is inside the interior enclosed by $\Gamma_j$ and the line through $x_0$ with tangential vector $\frac{v_j+v_{j+1}}{2}$. Now, we draw lines through $x$ such that the tangential vectors $w_j^1$ and $w_j^2$. Then, we know that the angle between $e_0$ and $w_j^1$ or $w_j^2$ is $(1/2 - 1/10)\theta$ and there is a room to set small enough $\lambda < 1$ such that $B(x + \epsilon e_0, \lambda\epsilon) \subset U \cap B(x_0, r)$. In this room, we can mollify $u_\epsilon(x) = u(x_\epsilon)$ and denote it $v_\epsilon$, and make $\epsilon \to 0$. The remaining part is same as the proof in Evans: Since $\partial U$ is compact, we can choose finitely many points $x_0^i \in \partial U$ including end point of $\Gamma_j$ and make Global approximation. $\square$

## Problem 2

1. $W_0^{1,p}(\Omega)$ is a vector space: For $f = 0$, $f \in W_0^{1,p}(\Omega)$, so $W_0^{1,p}(\Omega) \neq \phi$. For $f_1, f_2 \in W_0^{1,p}(\Omega)$, there exists $f_1^j, f_2^j$ such that $(f_1^j), (f_2^j) \in C_c^\infty(\Omega)$ and $f_1^j \to f_1$, $f_2^j \to f_2$ in $W^{1,p}(\Omega)$. Since union of two compact set in $\Omega$ is compact in $\Omega$, $f_1^j + f_2^j \in C_c^\infty(\Omega)$ and for large enough $N$ satisfying $\left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)}, \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon/2$ for $j > N$, $\left\|f_1^j + f_2^j - f_1 - f_2\right\|_{W^{1,p}(\Omega)} \leq \left\|f_1^j - f_1\right\|_{W^{1,p}(\Omega)} + \left\|f_2^j - f_2\right\|_{W^{1,p}(\Omega)} \leq \epsilon$. Therefore, $f_1^j + f_2^j \to f_1 + f_2$ in $W^{1,p}(\Omega)$, so $f_1 + f_2 \in W_0^{1,p}(\Omega)$. Also, $\lambda f^j \to \lambda f$ in $W^{1,p}(\Omega)$ for scalar $\lambda$. Therefore, $W_0^{1,p}$ is vector space.

2. With the norm $\|\cdot\|_{W^{1,p}(\Omega)}$, $W_0^{1,p}(\Omega)$ is Banach space: Let $f_j$ be a cauchy sequence in $W_0^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is Banach space, $f_j \to f$ in $W^{1,p}(\Omega)$. Since $\Omega$ is bounded and $\partial\Omega$ is $C^1$, there exists bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and $Tf_j \equiv 0$ on $\partial U$ as $f_j \in W_0^{1,p}(\Omega)$. Then,

$$\lim_{j\to\infty} \|Tf_j - Tf\|_{W^{1,p}(\Omega)} = \lim_{j\to\infty} \|T(f_j - f)\|_{W^{1,p}(\Omega)} \leq \lim_{j\to\infty} \|T\|_{W^{1,p}(\Omega)} \|f_j - f\|_{W^{1,p}(\Omega)} = 0$$

as $\|T\|_{W^{1,p}(\Omega)}$ is bounded. Therefore, $Tf_j \to Tf$ in $W^{1,p}(\Omega)$ and $\lim_{j\to\infty} \|Tf_j\|_{W^{1,p}(\Omega)} = \|Tf\|_{W^{1,p}(\Omega)} = 0$. As a result, $f \in W_0^{1,p}(\Omega)$ and it implies Cauchy sequence in $W_0^{1,p}(\Omega)$ converges.

Therefore, $W_0^{1,p}(\Omega)$ is Banach space.

# Problem 3

For $k \in \mathbb{N}$ and $\alpha \in (0,1]$,

$$C^{k,\alpha}(\bar{\Omega}) \coloneqq \{u \in C^k(\bar{\Omega}) : \|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty\}$$

Before starting, I need to show that $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ is a norm on $C^{k,\alpha}(\bar{\Omega})$.

*Proof.* 1. By the definition of $C^{k,\alpha}(\bar{\Omega})$, we know that $\|u\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$ for any $u \in C^{k,\alpha}(\bar{\Omega})$. Let $u, v \in C^{k,\alpha}(\bar{\Omega})$. Then

$$\begin{aligned}
\|u+v\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha|\le k} \|D^\alpha(u+v)\|_{C(\bar{\Omega})} + \sum_{|\alpha|=k} [D^\alpha(u+v)]_{C^{0,\alpha}(\bar{\Omega})} \\
&= \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha(u+v)| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha(u+v)(x) - D^\alpha(u+v)(y)|}{|x-y|^\alpha} \right\} \\
&\le \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + \sup_{x\in\Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)| + |D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\alpha} \right\} \\
&\le \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + \sup_{x\in\Omega} |D^\alpha v| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \right\} + \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\alpha} \right\} \\
&= \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})}
\end{aligned}$$

Therefore, $\|u+v\|_{C^{k,\alpha}(\bar{\Omega})} \le \|u\|_{C^{k,\alpha}(\bar{\Omega})} + \|v\|_{C^{k,\alpha}(\bar{\Omega})}$.

2. For $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\|\lambda\|_{C^{k,\alpha}(\bar{\Omega})} &= \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha \lambda u| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha \lambda u(x) - D^\alpha \lambda u(y)|}{|x-y|^\alpha} \right\} \\
&= |\lambda| \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u| + |\lambda| \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \right\} \\
&= \lambda \|u\|_{C^{k,\alpha}(\bar{\Omega})}.
\end{aligned}$$

3. For $u = 0$, $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$. Conversely, if $\|u\|_{C^{k,\alpha}(\bar{\Omega})} = 0$, then $\|u\|_{C(\Omega)} = 0$ with continuity of $u$, so $u = 0$ on $\bar{\Omega}$.

Therefore, $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ is a norm.

(a) Clearly, $0 \in C^{k,p}(\bar{\Omega})$. For $f_1, f_2 \in C^{k,p}(\bar{\Omega})$, $f_1 + f_2 \in C^k(\Omega)$ and $\|f_1 + f_2\|_{C^{k,\alpha}(\bar{\Omega})} \le \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} + \|f_2\|_{C^{k,\alpha}(\bar{\Omega})} < \infty$. Therefore, $f_1 + f_2 \in C^{k,\alpha}(\bar{\Omega})$. $f_1 + f_2 = f_2 + f_1$ and for scalar $\lambda$, $\lambda f_1 \in C^{k,\alpha}(\bar{\Omega})$ for $\|\lambda f_1\|_{C^{k,\alpha}(\bar{\Omega})} = |\lambda| \|f_1\|_{C^{k,\alpha}(\bar{\Omega})} \le \infty$. Therefore, $C^{k,p}(\bar{\Omega})$ is a vector space.

(b) Fix $x \in \Omega$ and take an open neighborhood $B(x,r) \subset \Omega$ for some $r > 0$. Then there exists $N \in \mathbb{N}$ such that for $\frac{1}{N} < \epsilon$, then $B(x, \frac{1}{n}) \subset \Omega$ for $n > N$. I'll use $C^\infty$ Urysohn lemma to show that there exists infinitely many linearly independent elements in $C^{k,\alpha}(\bar{\Omega})$. For $n > N$, take $K_n = \overline{B(x, \frac{1}{n+1})}$ and $U_n = B\left(x, \frac{1}{n+1} + \left(\frac{1}{n} - \frac{1}{n+1}\right)/2\right)$. Using $C^\infty$ Urysohn lemma, take $\phi^n \in C^\infty$ such that 1 on $K_n$ and has support in $U$. Take finite elements in the set: $\{\phi^j\}_{j=N_1}^{N_n}$ with $N_i < N_j$ for $i < j$ and let $\sum_{i=1}^{n} \lambda_i \phi^i = 0$. For $x \in U_{N_1} \setminus B_{N_1+1}$, $\phi^{N_1}(x) = 1$ but $\phi^{N_i}(x) = 0$ for $i > 1$. Therefore, $\lambda_1 = 0$. Repeating this argument, we can show that $\lambda_i = 0$ for all $i$ and it means $\phi^n$ is linearly independent for all $n > N$ and consequently, $C^{k,\alpha}(\bar{\Omega})$ has infinite dimension.

(c) Let $\{u_i\}$ be a Cauchy sequence in $C^{k,p}(\bar{\Omega})$. For fixed $\epsilon > 0$, there exists $N$ such that $i, j > N \Rightarrow \|u_i - u_j\|_{C^{k,p}(\bar{\Omega})} \leq \epsilon$. It implies

$$\begin{cases} \|D^\alpha u_i - D^\alpha u_j\|_{C(\bar{\Omega})} \leq \epsilon & \text{For } |\alpha| \leq k \\ [D^\alpha u_i - D^\alpha u_j]_{C^{0,\gamma}(\bar{\Omega})} \leq \epsilon & \text{For } |\alpha| = k. \end{cases}$$

Since $D^\alpha u_i$ is uniformly Cauchy for $|\alpha| \leq k$, $D^\alpha u_i$ converges to $u_\alpha$ for $|\alpha| \leq k$ pointwisely. Also, these convergences are uniform. Therefore, $D^\alpha u = u_\alpha$. for all $|\alpha| \leq k$.

Letting $i \to \infty$, $[D^\alpha u - D^\alpha u_j]_{C^{0,\gamma}(\bar{\Omega})} \leq \epsilon$ for $j > N$. Also,

$$\begin{aligned} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} - \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|D^\alpha u_j(x) - D^\alpha u_j(y)|}{|x-y|^\gamma} \right\} &\leq \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} - \frac{|D^\alpha u_j(x) - D^\alpha u_j(y)|}{|x-y|^\gamma} \\ &\leq \frac{|D^\alpha (u-u_j)(x) - D^\alpha (u-u_j)(y)|}{|x-y|^\gamma} \\ &\leq [D^\alpha u - D^\alpha u_j]_{C^{0,\gamma}(\bar{\Omega})} \leq \epsilon \end{aligned}$$

for all $x, y \in \Omega$, $x \neq y$. Therefore, $\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \leq [D^\alpha u_j]_{C^{0,\gamma}(\bar{\Omega})} + \epsilon$ and $[D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})} < \infty$. Therefore, $u \in C^{k,\alpha}(\bar{\Omega})$. It means $C^{k,\alpha}(\bar{\Omega})$ is Banach space.

□

# Problem 4

*Proof.* Since $U$ is bounded, open subset of $\mathbb{R}^n$, and $\partial\Omega$ is $C^1$,

$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\bar{\Omega}), \ \|u\|_{C^{0,\alpha}(\bar{\Omega})} \leq C \|u\|_{W^{1,p}(\Omega)}$$

for $\alpha = 1 - n/p$ and $C$ depends only on $p, n$ and $\Omega$. Also, $C^{0,\alpha}(\bar{\Omega}) \subset C^{0,\tilde{\alpha}}(\bar{\Omega})$ since $\|u\|_{C(\bar{U})}$ is same for both norm and if $[u]_{C^{0,\alpha}(\bar{\Omega})} < \infty$, then as $|x-y| \to 0$, $\frac{|u(x)-u(y)|}{|x-y|^{\tilde{\alpha}}} \to 0$ because $\frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty$ for all $x, y \in \Omega$, $x \neq y$, $[u]_{C^{0,\tilde{\alpha}}(\bar{\Omega})} < \infty$ and $u \in C^{0,\tilde{\alpha}}(\bar{\Omega})$.

Now, we need to show that each bounded sequence in $W^{1,p}(\Omega)$ is precompact in $C^{0,\alpha}(\bar{\Omega})$. Let a bounded sequence in $W^{1,p}(\Omega)$: $\{u_m\}_{m=1}^\infty$ and $\sup_m \left\{ \|u_m\|_{W^{1,p}(\Omega)} \right\} = K$. By Morney's inequality, we can assume that $\{u_m\} \subset C^{0,\alpha}(\bar{\Omega}) \subset C^{0,\tilde{\alpha}}(\bar{\Omega})$ and there exists constant $K'$ such that $\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leq K'$ for all $m$: For $|x-y| \leq 1$, $\frac{|u(x)-u(y)|}{|x-y|^{\tilde{\alpha}}} \leq \frac{|u(x)-u(y)|}{|x-y|^\alpha} |x-y|^{-\tilde{\alpha}+\alpha} \leq \frac{|u(x)-u(y)|}{|x-y|^\alpha} R^{-\tilde{\alpha}+\alpha}$ where $R$ is a constant such that $\Omega \subset B(0,R)$.

To use Arzela-Ascoli theorem, we need functions having compact domain. I'll denote $\bar{u_m}$ be a a function such that $\bar{u}_m = u_m$ in $\Omega$ and for $x \in \partial\Omega$, $u_m(x) = \lim_{r \to 0} u_m(y)$ where $y \in B(x,r) \cap \Omega$. I'll show that $\bar{u}_m$ is continuous function on $\bar{\Omega}$. Fix $x \in \partial\Omega$. Since $u$ is bounded, $u_m(x)$ is uniformly bounded in $\partial\Omega$ if they exist. Fix $r > 0$ and let $a = \lim_{r\to 0}\left\{\inf_{y\in B(x,r)\cap\Omega} u_m(y)\right\}$ and $b = \lim_{r\to 0}\left\{\sup_{y\in B(x,r)\cap\Omega} u_m(y)\right\}$. If $a \neq b$, then it means there exists $x, y \in \Omega$ such that $|x-y| < r$ but $|f(x) - f(y)| > (b-a)/2$ for all $r > 0$ which is contradiction to continuity of $u$. Therefore, the limit $u_m(x)$ for $x \in \partial\Omega$ is well defined and $\bar{u}_m$ is continuous on $\bar{\Omega}$.

Let's check the condition for Arzela-Ascoli theorem for $\bar{u}_m$.

1. For each $m$, $\bar{u}_m$ is continuous on compact set $\bar{\Omega}$.
2. Since $\|\bar{u}_m\|_{C(\bar{\Omega})} \leq K'$, $\{\bar{u}_m\}$ is pointwisely bounded.
3. Assume $\tilde{\alpha} > 0$ $\frac{|\bar{u}_m(x)-\bar{u}_m(y)|}{|x-y|^{\alpha}} \leq K'$ for all $x, y \in \Omega$, $x \neq y$. Therefore, $|\bar{u}_m(x) - \bar{u}_m(y)| \leq K'|x-y|^{\alpha}$ for $x, y \in \bar{\Omega}$ for all $m$ and it means $\{u_m\}$ is equicontinuous on $\bar{\Omega}$.

Therefore, we can use Arzela-Ascoli theorem and find a uniformly convergent subsequence $\{\bar{u}_{m_j}\}$ in $C^{0,\tilde{\alpha}}(\bar{\Omega})$ and it means $\{u_{m_j}\}$ is uniformly converges in $C^{0,\tilde{\alpha}}(\bar{\Omega})$. Since $C^{0,\tilde{\alpha}}(\bar{\Omega})$ is Banach space, the converging point is in $C^{0,\tilde{\alpha}}(\bar{\Omega})$. Hence,

$$W^{1,p}(\Omega) \subset\subset C^{0,\tilde{\alpha}}(\bar{\Omega})$$

for all $\tilde{\alpha} \in (0, \alpha)$. If $\tilde{\alpha} = 0$, then find $0 < \tilde{\alpha}' < \alpha$ and do the same procedure above. Since $C^{0,\tilde{\alpha}'}(\bar{\Omega}) \subset C^{0,0}(\bar{\Omega})$, the above compact inclusion is true for $\tilde{\alpha} = 0$. ☐

# Problem 5

Fix $\epsilon > 0$. Define $\Omega_\epsilon := \{x \in \Omega | d(x, \partial\Omega) > \epsilon\}$. Let's mollify the $u$ with standard mollifier $\eta_\epsilon$ and denote it $u^\epsilon$. Then,

$$Du^\epsilon = \eta_\epsilon * Du = 0$$

in $\Omega_\epsilon$. It implies that if $B(x,r) \subset \Omega_\epsilon$ for small enough $r > 0$, $u^\epsilon$ is constant on $B(x,r)$ since the derivative of $u^\epsilon$ is zero on the set. In other words, it is locally constant in $\Omega_\epsilon$.

Let $x \in U$ and $B(x,r)$ be an open neighborhood of $x$ in $\Omega$ and it is compactly embedded, then there exists $\epsilon$ such that $B(x,r) \subset \Omega_\epsilon$ and by previous, we know that $u^\epsilon$ is constant on $B(x,r)$. Let the constant value $c^\epsilon$. We know that $u^\epsilon \to u$ as $\epsilon \to 0$ and it means on $u$ is constant a.e. on $B(x,r)$.(If not, there always exists non measure zero set such that $u^\epsilon$ is different with $u$ on $B(x,r)$.) Also, it behave well since all any compactly embedded open neighborhood $B(x,r')$, the constant value should be same as $B(x,r)$ since $u$ is constant in $B(x,r') \cup B(x,r)$. Therefore, $u$ is locally constant function in a.e. sense).

Let take a partition such that $x \sim y$ if for $B(x,r_x) \subset\subset \Omega$ and $B(y,r_y) \subset\subset \Omega$, the constant values of the functions on the balls are same. Since $\Omega$ is locally constant, any element in partition is open set. Assume that there exists at least two element in the partition. This is impossible since $\Omega$ is connected set. Therefore, $u$ is a.e. constant function.

# Problem 6

First, I'll show that $u \in L^n(B_1(\mathbf{0}))$. Note that $u$ is symmetric function about rotation, so we can show that integral on $B_1(\mathbf{0})$ is finite by showing that integral is finite for $r$. Also, we can restrict the range of $r$ to $(0, \frac{1}{e-1}$ since $u$ is bounded in outside of the range. In other words,

$$\int_{B_1(\mathbf{0})} udx \leq C \int_0^{\frac{1}{e-1}} \left(\left|\log\log\left(1 + \frac{1}{r}\right)\right|\right)^n r^{n-1}dr$$

for some constant $C < \infty$. Let $y = \log\left(1 + \frac{1}{r}\right)$, then

$$\begin{aligned}
\left| \int_0^{\frac{1}{e-1}} \left( \log\log\left(1 + \frac{1}{r}\right)\right)^n r^{n-1} dr \right| &\leq \int_1^\infty (\log y)^n \frac{e^y}{(e^y - 1)^{n+1}} dy \\
&\leq \int_1^\infty (\log y)^n \frac{2^{n+1} e^y}{e^{(n+1)y}} dy \\
&\leq \int_1^\infty y^n 2^{n+1} e^{-ny} dy < \infty
\end{aligned}$$

Therefore, $u \in L^n(B_1(\mathbf{0}))$, and $u \in L^1(B_1(\mathbf{0}))$.

Next, I'll show that $u$ has weak derivative in $B_1(\mathbf{0})$ and belongs to $L^n(B_1(\mathbf{0}))$. Since $u$ goes to $\infty$ as $x \to 0$, we need to care when we compute weak derivative. However, we can ignore at $\mathbf{0}$ by the following argument. Let $V$ be a compactly embedded set in $U$ and $\phi$ be a $C^\infty$ function having support $V$. Assume $\mathbf{0} \in V$. Without $\mathbf{0}$, $Du$ should be $\partial_{x_i} u$ for some $i$. Since $u$, $D^\alpha \phi$ for all $\alpha$ are $L^1$ function on $V$, we can use Fubini theorem, and rewrite the integral by

$$\int_U u D\phi \; dx = \int_{-1}^1 (\cdots) dx_i$$

for $1 \leq i \leq n$. Since $n > 1$, we know that the $(n-1)$ dim plane through 0 is measure zero set and it does not effect integral to delete 0 from integral range of $x_1$. Therefore, the weak derivative is just derivative of $u$ except $\mathbf{0}$ More explicitly, for $\partial_{x_i} \phi$, take $j \neq i$. Then,

$$\begin{aligned}
\int_U u \partial_{x_i} \phi dx &= \int_{(-1,1)} \cdots \int_{x_1}^{x_2} u \partial_{x_i} \phi \; dx_i \cdots dx_j \\
&= \int_{(-1,1)\setminus\{0\}} \cdots \int_{x_1}^{x_2} u \partial_{x_i} \phi \; dx_i \cdots dx_j \\
&= \int_{(-1,1)\setminus\{0\}} \cdots \int_{x_1}^{x_2} \phi \partial_{x_i} u \; dx_i \cdots dx_j = \int_U \phi \partial_{x_i} u \; dx.
\end{aligned}$$

Also, for any compact set not containing $\mathbf{0}$, we can just use $\int_U u \partial_{x_i} \phi \; dx = \int_U \phi \partial_{x_i} u \; dx$. Thus, $\partial_{x_i} u$ is weak derivative of $u$ except $\mathbf{0}$.

I'll show that $Du$ is in $L^n$. Computing partial derivative:

$$|\partial_{x_i} u| = \left| \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \frac{1}{1 + \frac{1}{|x|}} \frac{x_i}{|x|^3} \right| \leq \frac{1}{\left|\log\left(1 + \frac{1}{r}\right)\right|} \frac{1}{r+1} \frac{1}{r}.$$

Then, by the same reason before, we just need to check whether the integral in finite for $r$ in $\left(0, \frac{1}{e-1}\right)$.

$$\int_0^{\frac{1}{e-1}} \left( \frac{1}{\log\left(1 + \frac{1}{r}\right)} \frac{1}{r+1} \frac{1}{r} \right)^n r^{n-1} dx \leq \int_0^{\frac{1}{e-1}} \left( \frac{1}{\log\left(1 + \frac{1}{r}\right)} \right)^n \frac{1}{r} dr$$

Let $x = \log\left(1 + \frac{1}{r}\right)$, then the integral becomes

$$\int_1^\infty \frac{1}{x^n} \frac{e^x}{e^x - 1} dx$$

For sufficiently large $R$, $\frac{e^x}{e^x - 1} < 2$ for $x > R$ and we know that $\int_1^\infty \frac{1}{x^n}$ converges for $n > 1$. Therefore, $Du \in L^n(B_1(\mathbf{0}))$ and $u \in W^{1,n}(B_1(\mathbf{0}))$.

## Problem 7

Since $u \in L^2(\mathbb{R}^n)$, $u = (\hat{u})^\vee$ by Theorem 2 in chapter 4.3 Evans. Then,

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} \left|e^{ikx}\hat{u}(k)\right| dk \leq \int_{\mathbb{R}^n} |\hat{u}(k)| dk \\
&= \int_{\mathbb{R}^n} (1+|k|^2)^{s/2}(1+|k|^2)^{-s/2}|\hat{u}(k)| dk \\
&\left(\leq \int_{\mathbb{R}^n} (1+|k|^2)^s |\hat{u}|^2 dk\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|k|^2)^{-s} dk\right)^{1/2}
\end{aligned}$$

For $|k| > 1$, $(1+|k|^2)^s > |k|^{2s}$ and

$$\int_{|k|>1} k^{-2s} dk = \sigma(S^{n-1}) \int_1^\infty r^{-2s} r^{n-1} dr < \infty$$

since $-2s + n - 1 < -1$ and $\int_1^\infty r^\alpha dr < \infty$ for $\alpha < -1$. Therefore,

$$|u(x)| \leq C \left(\int_{\mathbb{R}^n} (1+|y|^2)^s |\hat{u}|^2 dy\right)^{1/2} = C\|u\|_{H^s(\mathbb{R}^n)}$$

for some constant $C > 0$ depends only on $s$ and $n$. This is true for a.e. $x$, so

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$