Real Analysis II - MID

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Problem 1

Problem 2

a. Since $f \in L^2(\mathbb{T})$, $\hat{f}(k) = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi ikx} dx = \frac{1i}{2\pi n}$ for $k \neq 0$ If $k = 0$, $\hat{f}(0) = \int_0^1 \left(\frac{1}{2} - x\right) dx = 0$.

b. Using Parseval's identity, $\|f\|_{L^2(\mathbb{T})} = \sum_{k=-\infty}^{k=\infty} \left|\hat{f}(k)\right|^2$. $\frac{1}{12} = 2\sum_{n=1}^{n=\infty} \frac{1}{4\pi^2 n^2} = \sum_{n=1}^{n=\infty} \frac{1}{2\pi^2 n^2}$. Therefore,

$$\frac{\pi^2}{6} = \sum_{n=1}^{n=\infty} \frac{1}{n^2}$$

Problem 3

Since f is continuous on compact $\mathbb{T}$, $f \in L^1(\mathbb{T})$. Therefore, $f * K_n$ exists for almost x and is in L^1. Therefore, small enough $\epsilon > 0$, there exists δ such that $|f(x) - f(y)| \leq \epsilon$ when $|x - y| \leq \delta$. Also, there exists N such that for all $n \geq N$,

$$f*K_n(x) = \int_0^1 f(x-y)K_n(y) = \left(\int_0^\delta + \int_\delta^{1-\delta} + \int_{1-\delta}^1\right) f(x-y)K_n(y)dy = \left(\int_0^\delta + \int_{1-\delta}^1\right) f(x-y)K_n(y)dy + \epsilon/3$$

The last term can be rewritten by

$$\left(\int_0^\delta + \int_{1-\delta}^1\right) f(x-y)K_n(y)dy = \int_{x-\delta}^x f(y)K_n(x-y)dy + \int_{x-1}^{x-1+\delta} f(y)K_n(x-y)dy = \int_{x-\delta}^{x+\delta} f(y)K_n(x-y)dy$$

Since $\int_0^1 K_n(y)dy = 1$, $\int_{-\delta}^\delta K_n(y) \geq 1 - \epsilon$.

$$\int_{x-\delta}^{x+\delta} (f(y)-f(x))K_n(x-y)dy + \int_{x-\delta}^{x+\delta} f(x)K_n(x-y)dy \leq \epsilon \int_{x-\delta}^{x+\delta} K_n(x-y)dy + f(x)(1-\epsilon) \leq \epsilon M + f(x)(1-\epsilon).$$

Therefore, for fixed $\epsilon > 0$, there exists N such that for all $n \geq N$,

$$|f * K_n(x) - f(x)| \leq \epsilon\left(\frac{1}{3} + M - f(x)\right) \leq \epsilon\left(\frac{1}{3} + M + \|f(x)\|_{C^0}\right)$$

for all $x \in \mathbb{T}$. It means $f * K_n(x) \to f(x)$ uniformly.

Problem 4

(1) For convenience, assume $e^{2\pi ix} \neq 1$.

$$NF_N(x) + N = 2\sum_{k=0}^{N-1}(N-k)e^{2\pi ikx} = 2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) - \frac{2}{2\pi i}\frac{\partial}{\partial x}\sum_{k=0}^{N-1}e^{2\pi ikx}$$
$$= 2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) - \frac{2}{2\pi i}\frac{\partial}{\partial x}\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) =$$
$$2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) - 2\frac{(1-e^{2\pi iNx})(e^{2\pi ix}) - Ne^{2\pi iNx}(1-e^{2\pi ix})}{(1-e^{2\pi ix})^2}$$
$$= 2Ne^{\pi i(N-1)x}\frac{\sin \pi Nx}{\sin \pi x} + 2\frac{e^{\pi i(N-1)x}}{2i}\frac{\sin(\pi Nx)e^{\pi ix} - Ne^{\pi iNx}\sin(\pi x)}{\sin^2 \pi x}$$

Since $\overline{F_N}(x) = F_N(x)$, it is real valued function, so

$$NF_N(x) + N = 2N\cos(N-1)\pi x\frac{\sin \pi Nx}{\sin \pi x} + \frac{(\sin(\pi Nx))^2 - N\sin(\pi(2N-1)x)\sin(\pi x)}{\sin^2(\pi x)}$$
$$= \frac{\sin^2(\pi Nx)}{\sin^2(\pi x)} + N\frac{\cos((N-1)\pi x)\sin(N\pi x) - \sin((N-1)\pi x)\cos(N\pi x)}{\sin(\pi x)} = \frac{\sin^2(\pi Nx)}{\sin^2(\pi x)} + N$$

Therefore, $F_N(x) = \frac{1}{N}\frac{\sin^2(\pi Nx)}{\sin^2(\pi x)}$ for $e^{2\pi ix} \neq 1$. If $e^{2\pi ix} = 1$, then $NF_N(x) + N = N(N+1)$, so $F_N(x) = N$. Since $\lim_{x\to n} F_N(x) = \frac{1}{N}N^2 = N$,...

$$F_N(x) = \frac{1}{N}\frac{\sin^2(N\pi x/2)}{\sin^2(\pi x/2)}$$

(2) First, $F_N(x+1) = F_N(x)$ for all x since $e^{2\pi ikx}$ is periodic function with period 1. Therefore, $F_N(x)$ is on $\mathbb{T}$.

(i) For $k \neq 0$,

$$\int_0^1 e^{2\pi ikx}dx = 0,$$

so $\int_0^1 F_N(x)dx = \frac{1}{N}\int_0^1 Ndx = N$.

(ii) Since $F_N > 0$, $\int_0^1 F_N(x)dx = 1 < 2$.

(iii) For $\delta > 0$, there exists $N > 0$ such that $\frac{1}{N} < \delta$. For the N,

$$\int_\delta^{1-\delta}|F_{N^2}(x)|dx \leq \sum_{k=N}^{N^2-N}\frac{1}{N^2}\frac{1}{\sin^2(\frac{2k+1}{4N^2}\pi)}\frac{1}{N} \leq \sum_{k=N}^{N^2-N}\frac{1}{N^2}\frac{2N^2}{2k+1}\frac{1}{N} \leq \frac{1}{N}\ln\left(\frac{2(N^2-N+1)+1}{2N+1}\right)$$

Therefore, $\lim_{n\to\infty}\int_\delta^{1-\delta}|F_n(x)|dx = 0$

Problem 5

Problem 6

Problem 7

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