Real Analysis II - MID

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Problem 1

Let b_i be a sequence in c_0 such that 1 in i th position and 0 elsewhere. Let $T \in (c_0)^*$ and $c_i = T(b_i)$. Since c_0 have uniform norm, $1 = \left|\sum_{i=1}^{N} b_i\right|$ for fixed N. Let $\sum_{i=1}^{\infty} |c_i| = \infty$, and $\|T\| \leq C$ for some $C > 0$. Then there exists N such that $\sum_{i=1}^{N} |c_i| > C$ and it means $T(\sum_{i=1}^{N} \frac{|c_i|}{c_i} b_i) > C$ with $\left\|\sum_{i=1}^{N} \frac{|c_i|}{c_i} b_i\right\| = 1$. This is contradiction to boundedness of T. Therefore, $(c_i) \in l^1$ converges. Also, this argument shows that $\|(c_i)\|_{l^1} \leq \|T\|$.

Conversely, if $(c_i) \in l^1$, we can make linear functional T by setting $T(b_i) = c_i$ and $T(\sum_{i=1}^{\infty} \lambda_i b_i) = \sum_{i=1}^{\infty} \lambda_i c_i$. Then, this is definitely linear, so we only need to check the boundedness. Let $(a_i) \in c_0$ and $\|(a_i)\|_{c_0} = 1$, then $|a_i| \leq 1$ for all i, and $T((a_i)) = \sum_{i=1}^{\infty} a_i c_i \leq \sum_{i=1}^{\infty} |a_i c_i| \leq \sum_{i=1}^{\infty} |c_i| < \infty$. Therefore, T is bounded and $\|T\| \leq \|(c_i)\|_{l^1}$.

Finally, if we let $\phi : (c_0)^* \to l^1$ by $T \mapsto (c_i)$, $T(b_i) = c_i$, then it is bijective, $\lambda T \mapsto (\lambda c_i)$, $T_1 + T_2 \mapsto (T_1(b_i) + T_2(b_i))$. Therefore, it is vector space isomorphism. Also, $\|T\| = \|\phi(T)\|_{l^1}$ and ϕ is isometry isomorphism.

Let e_i be a sequence in l^1 such that 1 in i th position and 0 elsewhere. Let $T \in (l^1)^*$ and $f_i = T(e_i)$ Let $\sup f_i = \infty$, then find a subsequence (f_{i_j}) that $f_{i_1} = f_1$ and $f_{i_j} > \max\{|f_{i_{j-1}}|, 2^j\}$. Then, $\sum_{j=1}^{\infty} \left|\frac{|f_{i_j}|}{2^j f_{i_j}} e_{i_j}\right| = 1$, but $T\left(\sum_{j=1}^{\infty} \frac{|f_{i_j}|}{2^j f_{i_j}} e_{i_j}\right) = \sum_{j=1}^{\infty} \frac{|f_{i_j}|}{2^j} = \infty$. Therefore, $\sup f_i < \infty$ and $(f_i) \in l^\infty$.

Conversely, let $(f_i) \in l^\infty$ such that $\|f_i\|_{l^\infty} = M$ and make linear functional T of l^1 by setting $T(e_i) = f_i$. Linearity is given as before, so I'll prove the boundedness. Let $(c_i) \in l^1$ such that $\|(c_i)\|_{l^1} = 1$. Then, $T((c_i)) = \sum_{i=1}^{\infty} c_i f_i$ and $\|T((c_i))\| \leq \sum_{i=1}^{\infty} |c_i f_i| \leq M \sum_{i=1}^{\infty} |c_i| = M$. Therefore, T is bounded linear functional and $\|T\| \leq \|(f_i)\|_{l^\infty}$. Furthermore, for any $\epsilon > 0$, there exists j such that $|f_j| \geq M - \epsilon$ and $T\left(\frac{|f_j|}{f_j} e_j\right) \geq M - \epsilon$. This is true for any ϵ, so $\|T\| \geq M$. Therefore, $\|T\| = \|(f_i)\|_{l^\infty}$.

Finally, if we let $\phi : (l^1)^* \to l^\infty$ by $T \mapsto (f_i)$, $T(e_i) = f_i$, then it is bijective, $\lambda T \mapsto (\lambda f_i)$, $T_1 + T_2 \mapsto (T_1(e_i) + T_2(e_i))$. Therefore, it is vector space isomorphism. Also, $\|T\| = \|\phi(T)\|_{l^\infty}$ and ϕ is isometry isomorphism.

Problem 2

a. Since $f \in L^2(\mathbb{T})$, $\hat{f}(k) = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi i k x} dx = \frac{i}{2\pi n}$ for $k \neq 0$ If $k = 0$, $\hat{f}(0) = \int_0^1 \left(\frac{1}{2} - x\right) dx = 0$.

b. Using Parseval's identity, $\|f\|_{L^2(\mathbb{T})} = \sum_{k=-\infty}^{k=\infty} \left|\hat{f}(k)\right|^2$

$\frac{1}{12} = 2\sum\limits_{n=1}^{n=\infty} \frac{1}{4\pi^2 n^2} = \sum\limits_{n=1}^{n=\infty} \frac{1}{2\pi^2 n^2}$. Therefore,

$$\frac{\pi^2}{6} = \sum_{n=1}^{n=\infty} \frac{1}{n^2}$$

Problem 3

Since f is continuous on compact $\mathbb{T}$, $f \in L^1(\mathbb{T})$. Therefore, $f * K_n$ exists for almost x and is in L^1. For any small enough $\epsilon > 0$, there exists δ such that $|f(x) - f(y)| \leq \epsilon$ when $|x - y| \leq \delta$. Also, there exists N such that for all $n \geq N$, $\int_{\delta}^{1-\delta} |K_n| dx \leq \epsilon/3$. Write

$$f * K_n(x) = \int_0^1 f(x-y)K_n(y) = \left(\int_0^{\delta} + \int_{\delta}^{1-\delta} + \int_{1-\delta}^{1}\right) f(x-y)K_n(y)dy$$

Then,

$$\left|\int_{\delta}^{1-\delta} f(x-y)K_n(y)dy\right| \leq \|f\|_{C^0}\epsilon/3$$

The remaining term can be rewritten by

$$\left(\int_0^{\delta} + \int_{1-\delta}^{1}\right) f(x-y)K_n(y)dy = \int_{x-\delta}^{x} f(y)K_n(x-y)dy + \int_{x-1}^{x-1+\delta} f(y)K_n(x-y)dy = \int_{x-\delta}^{x+\delta} f(y)K_n(x-y)dy$$

Since $\int_0^1 K_n(y)dy = 1$, $\int_{-\delta}^{\delta} K_n(y) \geq 1 - \epsilon/3$. Then,

$$\left|\int_{x-\delta}^{x+\delta} (f(y) - f(x))K_n(x-y)dy + \int_{x-\delta}^{x+\delta} f(x)K_n(x-y)dy - f(x)\right| \leq \epsilon \int_{x-\delta}^{x+\delta} |K_n(x-y)|dy + \epsilon|f(x)| \leq \epsilon\left(M + |f(x)|\right).$$

Therefore, for fixed $\epsilon > 0$, there exists N such that for all $n \geq N$,

$$|f * K_n(x) - f(x)| \leq \epsilon\left(\frac{1}{3} + M - f(x)\right) \leq \epsilon\left(\frac{1}{3} + M + \|f(x)\|_{C^0}\right)$$

for all $x \in \mathbb{T}$. It means $f * K_n(x) \to f(x)$ uniformly.

Problem 4

(1) For convenience, assume $e^{2\pi i x} \neq 1$.

$$\begin{aligned}
NF_N(x) + N = 2\sum_{k=0}^{N-1}(N-k)e^{2\pi ikx} &= 2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) - \frac{2}{2\pi i}\frac{\partial}{\partial x}\sum_{k=0}^{N-1} e^{2\pi ikx} \\
&= 2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) - \frac{2}{2\pi i}\frac{\partial}{\partial x}\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) = \\
2N\left(\frac{1-e^{2\pi iNx}}{1-e^{2\pi ix}}\right) &- 2\frac{(1-e^{2\pi iNx})(e^{2\pi ix}) - Ne^{2\pi iNx}(1-e^{2\pi ix})}{(1-e^{2\pi ix})^2} \\
&= 2Ne^{\pi i(N-1)x}\frac{\sin \pi Nx}{\sin \pi x} + 2\frac{e^{\pi i(N-1)x}}{2i}\frac{\sin(\pi Nx)e^{\pi ix} - Ne^{\pi iNx}\sin(\pi x)}{\sin^2 \pi x}
\end{aligned}$$

Since $\overline{F_N(x)} = F_N(x)$, it is real valued function, so

$$NF_N(x) + N = 2N\cos{(N-1)\pi x}\frac{\sin\pi Nx}{\sin\pi x} + \frac{(\sin(\pi Nx))^2 - N\sin(\pi(2N-1)x)\sin(\pi x)}{\sin^2(\pi x)}$$
$$= \frac{\sin^2(\pi Nx)}{\sin^2(\pi x)} + N\frac{\cos((N-1)\pi x)\sin(N\pi x) - \sin((N-1)\pi x)\cos(N\pi x)}{\sin(\pi x)} = \frac{\sin^2(\pi Nx)}{\sin^2(\pi x)} + N$$

Therefore, $F_N(x) = \frac{1}{N}\frac{\sin^2(\pi Nx)}{\sin^2(\pi x)}$ for $e^{2\pi ix} \neq 1$. If $e^{2\pi ix} = 1$, then $NF_N(x) + N = N(N+1)$, so $F_N(x) = N$. Since $\lim\limits_{x\to n} F_N(x) = \frac{1}{N}N^2 = N$, we can get

$$F_N(x) = \frac{1}{N}\frac{\sin^2(N\pi x/2)}{\sin^2{(\pi x/2)}}$$

(2) First, $F_N(x+1) = F_N(x)$ for all x since $e^{2\pi ikx}$ is periodic function with period 1. Therefore, $F_N(x)$ is on $\mathbb{T}$.

(i) For $k \neq 0$,
$$\int_0^1 e^{2\pi ikx}dx = 0,$$
so $\int_0^1 F_N(x)dx = \frac{1}{N}\int_0^1 Ndx = N$.

(ii) Since $F_N > 0$, $\int_0^1 F_N(x)dx = 1 < 2$.

(iii) For $\delta > 0$, there exists $N > 0$ such that $\frac{1}{N} < \delta$. For the N,
$$\int_\delta^{1-\delta} |F_{N^2}(x)|dx \leq \sum_{k=N}^{N^2-N+1} \frac{1}{N^2}\frac{1}{\sin^2(\frac{2k+1}{4N^2}\pi)}\frac{1}{N} \leq \sum_{k=N}^{N^2-N+1}\frac{1}{N^2}\frac{2N^2}{2k+1}\frac{1}{N} \leq \frac{1}{N}\ln\left(\frac{2(N^2-N+3)+1}{2N+1}\right)$$
$$\to 0$$
as $N \to 0$. Therefore, $\lim\limits_{n\to\infty}\int_\delta^{1-\delta}|F_n(x)|dx = 0$

Problem 5

(1) $f * K_n(x) = \int_0^1 f(x-y)K_n(y)dy = \sum\limits_{k=-n+1}^{n-1}\int_0^1 kf(x-y)e^{2\pi iky}dy = \sum\limits_{k=-n+1}^{n-1} ke^{2\pi ikx}\int_{x-1}^x f(y)e^{-2\pi iky} = \sum\limits_{k=-n+1}^{n-1} ke^{2\pi ikx}\hat{f}(k) = 0$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{T}$, and by problem 3, $f * K_n \to 0$ uniformly, therefore, $f = 0$.

(2) Let f be a continuous function on $\mathbb{T}$ and fix $\epsilon > 0$. Then, there exists N such that $\|f * F_n - f\|_u \leq \epsilon$ for all $n \geq N$. However, $f*F_n(x) = \int_0^1 f(x-y)F_n(y)dy = \int_{x-1}^x f(y)KF_n(x-y)dy = \left(\int_{x-1}^0 + \int_0^x\right) f(y)F_n(x-y)dy = \int_0^1 f(y)F_n(x-y)dy = \sum\limits_{k=-N+1}^{N-1} ke^{2\pi ikx}\int_0^1 f(y)e^{-2\pi iky}dy$ and this is trigonometric polynomial since $f(y)e^{-2\pi iky}$ is continuous on compact $\mathbb{T}$ for all k. Therefore, continuous functions on $\mathbb{T}$ can be uniformly approximated by trigonometric polynomials.

Problem 6

$\int_{\mathbb{R}} f(y)e^{\pi(-y^2+2xy)}dy = 0 \Leftrightarrow e^{-\pi x^2}\int_{\mathbb{R}} f(y)e^{\pi(-y^2+2xy)}dy = \int_{\mathbb{R}} f(y)e^{-\pi(x-y)^2}dy = (f*g)(x) = 0$ for $g(x) = e^{-\pi x^2}$. Let $\phi(\xi) = \exp\left(2\pi i\xi x - \pi\xi^2\right)$, then

$$\hat{\phi}(y) = e^{-\pi x^2}\left(e^{-\pi(\xi - ix)^2)}\right)^{\wedge} = e^{-\pi x^2}\left(\tau_{ix}e^{-\pi\xi^2}\right)^{\wedge} = e^{-\pi(x^2-2yx+y^2)} = e^{-\pi(x-y)^2}$$

Therefore, $\hat{\phi}(y) = g(x-y)$ and $f*g(x) = \int_{\mathbb{R}} f\hat{\phi} = \int_{\mathbb{R}} \hat{f}\phi = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x - \pi\xi^2}d\xi$. Let $h(\xi) = \hat{f}(\xi)e^{-\pi\xi^2}$, then $\int_{\mathbb{R}} h(\xi)e^{2\pi i\xi x} = 0$ for all x. It means $\hat{h}(x) = 0$ for all x. Since $h \in L^1$ and $\hat{h} = 0$, $h = 0$ a.e. and it means $\hat{f}(\xi) = 0$ a.e. Since $f \in L^1$ and $\hat{f} = 0$, $f = 0$ a.e. and $f = 0$ since $f \in C(\mathbb{R})$.(Since $f \neq 0$ is measure zero set, for any x in the set, any open neighborhood of x should contains a point p such that $f(p) = 0$ and there exists $(p_n) \to x$, $f(p_n) = 0$ for all n. Since f is continuous, $\lim\limits_{n\to\infty} f(p_n) = 0 = f(x)$.)