**Midterm exam: Real analysis II**

There are 6 problems and the due date is 10/29.

1. Let $c_0(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} : \lim_n a_n = 0\}$. Show that $c_0^* \cong l^1$ and $(l^1)^* \cong l^\infty$.

2. Let $f(x) = \frac{1}{2} - x$ on $[0, 1)$ and extend $f$ to be periodic on $\mathbb{R}$.

a. Compute $\hat{f}(k)$.

b. Use Parseval identity to derive

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

3. Suppose that $K_n(x)$ on $\mathbb{T}$ satisifies that

(i) For all $n$,

$$\int_0^1 K_n(x)\,dx = 1$$

(ii) There is $M > 0$ such that for all $n$,

$$\int_0^1 |K_n(x)|\,dx < M$$

(iii) For every $\delta > 0$,

$$\lim_{n\to\infty} \int_\delta^{1-\delta} |K_n(x)|\,dx = 0.$$

Show that if $f$ is continuous on $\mathbb{T}$, then $f * K_n \to f$ uniformly.

4. Let

$$F_N(x) = \frac{1}{N}\left(D_0(x) + D_1(x) + \cdots + D_{N-1}(x)\right),$$

where $D_m(x) = \sum_{k=-m}^{m} e^{2\pi ikx}$. Show that

(1) $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$

(2) $F_N$ satisfies conditions (i) - (iii) in problem 2.

5. Use Problems 3 and 4 to show that:

(1) If $f$ is continuous on $\mathbb{T}$ and $\hat{f} = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

(2) Show that continuous functions on $\mathbb{T}$ can be uniformly approximated by trigonometric polynomials.

6. Let $f \in C(\mathbb{R})$ with $f \in L^1$ and $\hat{f} \in L^1$. Suppose that for all $x$

$$\int_{-\infty}^{\infty} f(y)e^{-\pi y^2 + 2\pi xy}\,dy = 0.$$

Show that $f = 0$. (Hint: Consider $f * g$, where $g(x) = e^{-\pi x^2}$.)