Real Analysis II - FINAL

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Problem 1

Let $\mu \in M(\mathbb{R}^n)$. Construct $\phi_t \in C_c^\infty(\mathbb{R}^n)$ for $t > 0$ such that $\int \phi_t dx = 1$. More explicitly, we can set $\phi_t(x) = t^{-n}\phi(t^{-1}x)$ where $\phi(x)$ defined as

$$\phi(x) = \begin{cases} \exp\left[(|x|^2 - 1)^{-1}\right] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Since $\phi_t \in L^1(\mathbb{R}^n)$ for all $t > 0$, $\phi_t * \mu(x) = \int f(x-y)d\mu(y)$ exists for a.e. x, $\phi_t * \mu \in L^1$, and $\|\phi_t * \mu\|_1 \leq \|\mu\| = |\mu|(\mathbb{R}^n)$. Since $\phi_t * \mu \in L^1$, we can identify $\phi_t * \mu \in L^1$ as a Radon measure on $\mathbb{R}^n$ and denote $d\mu_t = (\phi_t * \mu)dm$. I'll show that $\mu_t \to \mu$ vaguely by showing that $\int f d\mu_t \to \int f d\mu$ for all $f \in C_0(\mathbb{R}^n)$.

Fix $f \in C_0(\mathbb{R}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} f d\mu_t = \int_{\mathbb{R}^n} f(\phi_t * \mu)dm &= \int_{\mathbb{R}^n} f(x)\left(\int_{\mathbb{R}^n} \phi_t(x-y)d\mu(y)\right)dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\phi_t(x-y)dxd\mu(y) \text{ (By Fubini theorem)} \\ &= \int_{\mathbb{R}^n} (f * \tilde{\phi_t})(y)d\mu(y) \end{aligned}$$

The Fubini theorem can be applied since $f(x)\phi_t(x-y)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and $\left|\int_{\mathbb{R}^n} f(\phi_t * \mu)dm\right| \leq \|f\|_\infty \int_{\mathbb{R}^n} |(\phi_t * \mu)|dm < \infty$, implying $f(x)\phi_t(x-y) \in L^1(\mu \times m)$. (Since the function is continuous, we can consider m be a Borel measure restricting to $\mathcal{B}_{\mathbb{R}^n}$.)

I claim that $f * \tilde{\phi_t} \in C_0(\mathbb{R}^n)$. Since $f \in C_0(\mathbb{R}^n)$, there exists a sequence $\{f_j\} \subset C_c(\mathbb{R}^n)$ such that $\|f_j - f\|_u \to 0$ as $j \to \infty$. Note that $f \in L^\infty$ and $\tilde{\phi_t} \in L^1$, so $f * \tilde{\phi_t}$ is bounded and uniform continuous. As $f_j, \tilde{\phi_t} \in C_c$, $f_j * \tilde{\phi_t} \in C_c$ and

$$\left\|f_j * \tilde{\phi_t} - f * \tilde{\phi_t}\right\|_u = \left\|(f_j - f) * \tilde{\phi_t}\right\|_u \leq \|(f_j - f)\|_\infty \|\phi_t\|_1 = \|(f_j - f)\|_u \|\phi_t\|_1 \to 0$$

Therefore, $f * \tilde{\phi_t} \in C_0(\mathbb{R}^n)$.

Since $f \in C_0(\mathbb{R}^n)$, it is uniformly continuous, (Let $U \subset \mathbb{R}^n$ such that $f(x) \leq \epsilon$, then U^c is bounded as $f \in C_0$, so U^c is compact. Therefore, $|f(x) - f(y)|$ can be controlled by controlling $|x-y|$ uniformly.) and so, $f * \tilde{\phi_t} \to f$ uniformly as $t \to 0$. As $\mu \in M(\mathbb{R}^n)$, $f \mapsto \int f d\mu$ is bounded linear functional, so $\int f * \tilde{\phi_t} d\mu \to \int f d\mu$ as $t \to \infty$. It means

$$\int_{\mathbb{R}^n} f d\mu_t = \int_{\mathbb{R}^n} (f * \tilde{\phi_t})(y)d\mu(y) \to \int_{\mathbb{R}^n} f d\mu$$

as $t \to \infty$ and $\mu_t \to \mu$. Since μ_t is generated by L^1 functions, this result shows that $L^1(\mathbb{R}^n)$ is dense in $M(\mathbb{R}^n)$.

Problem 2

(1) By the fundamental Theorem of Calculus for Lebesgue Integrals, F is absolutely continuous on $[0,1]$. Also, F is bounded variation $[0,1]$. I need to show that $F(0) = F(1)$ to state that $F \in C(\mathbb{T})$, but $\hat{f}(0) = 0$ implies

$$\hat{f}(0) = \int_0^1 f(x)dx = 0.$$

Therefore, $F(1) = 0 = F(0)$ and $F \in C(\mathbb{T})$.

By the fundamental Theorem of Calculus for Lebesgue Integrals again, F is differentiable a.e. on $[0,1]$ and $F' = f$. Therefore,

$$\begin{aligned}\hat{F}(n) = \int_0^1 F(x)e^{-2\pi inx}dx &= \left[\frac{1}{-2\pi in}F(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 f(x)e^{-2\pi inx}dx \\ &= \frac{1}{2\pi in}f(n)\end{aligned}$$

for $n \neq 0$ and if $n = 0$, $\hat{F}(0) < \infty$ since F is continuous on $[0,1]$, which is compact set.

(2) Since F is bounded variation on $\mathbb{T}$, we can apply Fejér's theorem and get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(S_nF)(0) = F(0).$$

By the definition of $(S_nF)(0)$,

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{F}(k) &= \lim_{N\to\infty}\frac{1}{N}\left(N\hat{F}(0) + \sum_{n=1}^{N-1}(N-n)(\hat{F}(n)+\hat{F}(-n))\right) \\ &= \hat{F}(0) + \lim_{N\to\infty} 2\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)\hat{F}(n) \\ &= \hat{F}(0) + \lim_{N\to\infty} 2\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)\frac{\hat{f}(n)}{2i\pi n} \\ &= F(0).\end{aligned}$$

(In the calculation, I used $\hat{F}(-n) = \frac{1}{-2\pi in}\hat{f}(-n) = \frac{1}{2\pi in}\hat{f}(n) = \hat{F}(n)$.) Therefore,

$$\lim_{N\to\infty} 2\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)\frac{\hat{f}(n)}{n} = 2\pi i(F(0)-\hat{F}(0)) = -2\pi i\hat{F}(0)$$

since $F(0) = 0$.

(3) By Hausdorff-Young Inequality, if $f \in L^1(\mathbb{T})$, $\hat{f} \in l^\infty(\mathbb{Z})$ and $\left\|\hat{f}\right\|_\infty \le \|f\|_1$. Therefore,

$$\lim_{N\to\infty} 2\sum_{n=1}^{N-1}\left(\frac{\hat{f}(n)}{N}\right) \le \|f\|_1$$

and

$$\left\|\lim_{n\to\infty}\frac{1}{n}\hat{f}(n)\right\| \le \|f\|_1 + \left\|\pi i\hat{F}(0)\right\| < \infty$$

for the norm on $\mathbb{C}$, and $\lim_{n\to\infty}\frac{1}{n}\hat{f}(n) < \infty$ since $\hat{f}(n) \in \mathbb{R}$ for all $n \in \mathbb{Z}$.

(4) By integral test of $\sum_{n=2}^{\infty} \frac{1}{n \log n}$,

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \leq \frac{1}{2 \log 2} + \int_{2}^{\infty} \frac{1}{x \log x} dx = \frac{1}{2 \log 2} + [\ln t]_{\ln 2}^{\infty} = \infty,$$

so $\sum_{n=2}^{\infty} \frac{1}{n \log n} \to \infty$ and it is not a Fourier series of L^1 function. However,

$$\sum_{n=2}^{\infty} \frac{\sin 2\pi n x}{\log n}$$

converges for all x: computing $\sum_{i=2}^{N} e^{2\pi i n x}$, it is $\frac{e^{2\pi i (N+1)x} - e^{4\pi i x}}{-1+e^{2\pi i x}}$, whose absolute value is upper bounded by $\frac{2}{-1+e^{2\pi i x}}$ for fixed x without $x = 0, 1$ for all N. Therefore, $\sum_{i=2}^{N} \sin(2\pi i n x)$ is bounded by $|M(x)| < \infty$ for fixed x in $[0, 1]$ for all N. Since $\frac{1}{\log n}$ is decreasing sequence, by Dirichlet test, $\sum_{n=2}^{\infty} \frac{\sin 2\pi n x}{\log n}$ converges for all x.

Problem 3

I'll follow the steps in exercise 11 in sec 9.1.

(1) There exist $N \in \mathbb{N}$, $C > 0$ such that for all $\phi \in C_c^{\infty}$,

$$|\langle F, \phi \rangle| \leq \sum_{|\alpha| \leq N} \sup_{|x| \leq 1} |\partial^{\alpha} \phi(x)|. \tag{1}$$

Proof. $F : C_c^{\infty}(\mathbb{R}^n) \to \mathbb{R}$, which is continuous linear functional, and $C_c^{\infty}(\mathbb{R}^n)$ is Fréchet space with the topology defined by the norms

$$\phi \to \|\partial^{\alpha} \phi\|_u \quad (\alpha \in \{0, 1, 2, \dots\}^n)$$

for $K \subset U$, $\phi \in C_c^{\infty}(K)$. Therefore, by proposition 5.15 in Folland, there exists $\alpha_1, \dots, \alpha_k \in \{0, 1, 2, \dots\}^n$ and $C > 0$ such that

$$|\langle F, \phi \rangle| \leq C \sum_{j=1}^{k} \|\partial^{\alpha_j} \phi\|_u$$

To concentrate only on $|x| \leq 1$ region, make C_c^{∞} function φ using C^{∞} Urysohn lemma such that $\varphi = 1$ on $|x| \leq 1/2$ and $\varphi = 0$ at $|x| \geq 1$. Denote $\phi_1 = \varphi \phi$ and $\phi_2 = (1 - \varphi)\phi$. Then, $\phi_1 + \phi_2 = \phi$ and

$$|\langle F, \phi \rangle| = |\langle F, \phi_1 \rangle + \langle F, \phi_2 \rangle| = |\langle F, \phi_1 \rangle|$$

since $\text{supp}(F) = \{0\}$. Hence,

$$|\langle F, \phi \rangle| = |\langle F, \phi_1 \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{|x| \leq 1} |\partial^{\alpha} \phi(x)|$$

where $N = |\alpha_k|$. □

(2) Take $\psi \in C_c^{\infty}$ with $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$. Assume $\phi \in C_c^{\infty}$ and $\partial^{\alpha} \phi(0) = 0$ for $|\alpha| \leq N$. Let $\phi_k(x) = \phi(x)(1 - \psi(kx))$, then $\partial_k^{\alpha} \phi \to \partial^{\alpha} \phi$ uniformly as $k \to \infty$ for $|\alpha| \leq N$.

Proof. Since $\partial^\alpha \phi_k(x)$ is compactly supported on the support of ϕ for $|\alpha| \leq N$, we just need to show that $\partial^\alpha \phi_k \to \partial^\alpha \phi$. Fix α such that $|\alpha| \leq N$, then

$$|\partial^\alpha \phi - \partial^\alpha \phi_k| = |\partial^\alpha (\phi - \phi_k)| = |\partial^\alpha (\phi(x)\psi(kx))|$$

and $\partial^\alpha(\psi(kx)) = 0$ on $|x| > 2/k$. Using Leibniz rule and hint, $|\partial^\alpha \phi(x)| \leq C|x|^{N+1-|\alpha|}$ for $|\alpha| \leq N$ on $|x| \leq 1$, which will be proven later,

$$\begin{aligned}
|\partial^\alpha (\phi(x)\psi(kx))| &= \left| \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial_x^\beta(\phi(x)))(\partial_x^\gamma(\psi(kx))) \right| \\
&\leq \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left|(\partial_x^\beta(\phi(x)))(\partial_x^\gamma(\psi(kx)))\right| \\
&\leq C \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} |x|^{N+1-|\beta|} |k|^{|\gamma|} |(\partial^\gamma \psi)(kx)| \\
&= C \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} |kx|^{|\gamma|} |x|^{|N+1-|\beta|-|\gamma||} |(\partial^\gamma \psi)(kx)|
\end{aligned}$$

in the support of $\psi(kx) \subset \{|x| \leq 1\}$. The support of ψ is a subset of $|kx| \leq 2$ and $|N+1-|\beta|-|\gamma|)| = |N+1-|\alpha|| \geq 1$ for $|\alpha| \leq N$. Also, $\partial^\gamma \psi(x)$ is bounded for $|\gamma| \leq N$ since it is C_c^∞ function. Summarising the fact,

$$|\partial^\alpha (\phi(x)\psi(kx))| \leq C'|x| \leq \frac{2C'}{k}$$

for some $C' > 0$ for all $|\alpha| \leq N$ and goes to 0 as $k \to \infty$. Therefore,

$$\partial^\alpha \phi_k \to \partial^\alpha \phi$$

uniformly as $k \to \infty$ for $|\alpha| \leq N$.

pf of hint: The Taylor series of ϕ about 0 would be

$$\phi(x) = \sum_{|\beta| \leq N+1} \frac{\partial^\beta \phi(0)}{\beta!} x^\beta + r(x)$$

such that $r(x)$ is differentiable and $\lim_{x \to 0} \frac{r(x)}{|x|^{N+1}} = 0$. Since we only interested in $|x| \leq 1$, $\partial^\beta \phi$ is bounded for all $|\beta| \leq N+1$ and and $\frac{r(x)}{x^{N+1}}$ is bounded in $|x| \leq 1$. As $\partial^\beta \phi(0) = 0$ for $|\beta| \leq N$, we can get

$$|\phi(x)| \leq \left(C + \frac{r(x)}{|x|^{N+1}} \right) |x|^{N+1} \leq C'|x|^{N+1}.$$

for some $0 < C, C' < \infty$. By the same reason, we can get

$$|\partial^\alpha \phi(x)| \leq C|x|^{N+1-|\alpha|}$$

for $|\alpha| \leq N$. □

(3) Considering $\langle F, \phi - \phi_k \rangle$,

$$|\langle F, \phi - \phi_k \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{|x| \leq 1} |\partial^\alpha (\phi - \phi_k)(x)|$$

and we already showed that $|\partial^\alpha(\phi - \phi_k)(x)| \to 0$ uniformly as $k \to \infty$ for $|\alpha| \le N$. Therefore,

$$|\langle F, \phi\rangle| = \lim_{k\to\infty} |\langle F, \phi_k\rangle|.$$

For all k, $\langle F, \phi_k\rangle = 0$ since $\phi_k(x) = 0$ on $|x| \le 1/k$. Therefore, $\langle F, \phi\rangle = 0$ if $\phi \in C_c^\infty$ and $\partial^\alpha\phi(0) = 0$ for $|\alpha| \le N$.

(4) I'll use $\varphi \in C_c^\infty$ in step 1 again. Assume $\phi \in C_c^\infty$ and $\partial^\alpha\phi(0) = a_\alpha$, then let $f(x) = \varphi \sum_{|\alpha|\le N} \frac{a_\alpha}{\alpha!} x^\alpha$. Then, $f \in C_c^\infty$ and $g = \phi - f \in C_c^\infty$ such that $\partial^\alpha g(0) = 0$ for all $|\alpha| \le N$. Therefore $\langle F, g\rangle = 0 = \langle F, \phi\rangle - \langle F, f\rangle$, and $\langle F, \phi\rangle = \langle F, f\rangle = \sum_{|\alpha|\le N} \frac{a_\alpha}{\alpha!}\langle F, x^\alpha\rangle$. Let $c_\alpha = \frac{\langle F, x^\alpha\rangle}{\alpha!}$, then

$$\langle F, \phi\rangle = \sum_{|\alpha|\le N} c_\alpha \partial^\alpha\phi(0) = \langle \sum_{|\alpha|\le N} c_\alpha \partial^\alpha \delta, \phi\rangle.$$

Therefore, $F = \sum_{|\alpha|\le N} c_\alpha \partial^\alpha \delta$. Therefore, $(F * G)\widehat{} = \hat{F}\hat{G}$.

Problem 4

I'll follow the steps in exercise 20 in sec 9.2.

(1) If $\psi \in \mathcal{S}$, which is Schwartz class, then $G * \psi \in \mathcal{S}$ for $G \in \mathcal{E}'$.

Proof. Since $G \in \mathcal{E}'$, choose $\phi \in C_c^\infty(U)$ with $\phi = 1$ on $\mathrm{supp}(G)$, and define the linear functional H on $C_c^\infty(U)$ by $\langle H, \psi\rangle = \langle G, \phi\psi\rangle$. Then, it is continuous on $C_c^\infty(U)$ and unique continuous extension of G. Taking restriction of H to $\mathcal{S}$, we can make tempered distribution H and $\langle H, \psi\rangle = \langle G, \phi\psi\rangle$ for $\psi \in \mathcal{S}$. By the definition of the convolution,

$$H * \psi(x) = \langle H, \tau_x\tilde{\psi}\rangle = \langle G, (\tau_x\tilde{\psi})\phi\rangle$$

for $\psi \in \mathcal{S}$. So, we can define $G * \psi(x) = \langle G, (\tau_x\tilde{\psi})\phi\rangle$ and get $H * \psi(x) = G * \psi(x)$

To show that $G * \psi$ is in Schwartz class, I'll compute each semi-norms in Schwartz class. For $N \in \mathbb{N}$, α,

$$\begin{aligned}
\|\partial^\alpha(G * \psi)\|_{N,\alpha} &= \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\partial^\alpha(G * \psi)(x)| \\
&= \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |G * \partial^\alpha\psi| \\
&= \sup_{x\in\mathbb{R}^n} (1 + |x|)^N \left|\langle G, (\tau_x\widetilde{\partial^\alpha\psi})\phi\rangle\right| \\
&= \sup_{x\in\mathbb{R}^n} (1 + |x|)^N \left|\langle G, (\tau_x\widetilde{\partial^\alpha\psi})\phi\rangle\right| \\
&\le C \sup_{x\in\mathbb{R}^n} (1 + |x|)^N \sum_{|\beta|\le M} \sup_{y\in\mathbb{R}^n} (1 + |y|)^m \left|\partial^\beta((\tau_x\widetilde{\partial^\alpha\psi})\phi)\right| \\
&\le C \sup_{x\in\mathbb{R}^n} (1 + |x|)^N \sum_{|\beta|\le M} \sup_{y\in\mathbb{R}^n} (1 + |y|)^m \left|\partial^\beta(\partial^\alpha\psi(x - y))\phi(y))\right|.
\end{aligned}$$

for some m. Since $\phi \in C_c^\infty$, there exists $y_0^\beta(x)$ such that $\sup_{y\in\mathbb{R}^n}(1 + |y|)^m \left|\partial^\beta(\partial^\alpha\psi(x - y))\phi(y))\right|$ has maximum for each β and x. Also, $\left|y_0^\beta(x)\right|$ is uniformly bounded for β and x since it is in the compact set. As $\psi \in \mathcal{S}$, $\sup_{x\in\mathbb{R}^n}(1 + |x|)^N \sum_{|\beta|\le M} (1 + \left|y_0^\beta(x)\right|)^m \left|\partial^\beta(\partial^\alpha\psi(x - y_0^\beta(x)))\phi(y_0^\beta(x)))\right|$ is bounded for all α and N, directly proving $G * \psi \in \mathcal{S}$.

□

(2) Let's define $F * G$ by

$$\langle F * G, \psi\rangle = \langle F, \tilde{G} * \psi\rangle.$$

for $\psi \in \mathcal{S}$. Since $\tilde{G} \in \mathcal{E}'$, $\tilde{G} * \psi \in \mathcal{S}$, and $\langle F, \tilde{G} * \psi\rangle$ is well-defined since F is tempered distribution, i.e., if $\psi_j \to \psi$ in $\mathcal{S}$, then calculating $\|\partial^\alpha (G * (\psi_j - \psi))\|_{N,\alpha}$ for all N, α as above and let $\|\psi - \psi_j\|_{M,\beta}$ for all M, β, we get $G * \psi_j \to G * \psi$ in $\mathcal{S}$ and $\langle F, \tilde{G} * \psi_j\rangle \to \langle F, \tilde{G} * \psi\rangle$ as $F \in \mathcal{S}'$. It means $F * G$ is continuous linear functional on S.

Since $F * G$ is tempered distribution, we can define Fourier transform of it, then

$$\langle (F * G)\widehat{\ }, \psi\rangle = \langle (F * G), \hat{\psi}\rangle = \langle F, \tilde{G} * \hat{\psi}\rangle.$$

Also,(the compactly supported function ϕ is omitted.)

$$\langle \tilde{G} * \hat{\psi}, \varphi\rangle = \langle \tilde{G}, \varphi * \tilde{\hat{\psi}}\rangle = \langle \tilde{G}, \varphi * \check{\psi}\rangle = \langle \tilde{G}, (\hat{\varphi}\psi)\check{\ }\rangle = \langle \hat{G}, \hat{\varphi}\psi\rangle = \langle (\hat{G}\psi)\widehat{\ }, \varphi\rangle$$

for $\psi, \varphi \in \mathcal{S}$. Therefore,

$$\langle F, \tilde{G} * \hat{\psi}\rangle = \langle F, (\hat{G}\psi)\widehat{\ }\rangle = \langle \hat{F}, \hat{G}\psi\rangle = \langle \hat{F}\hat{G}, \psi\rangle$$

since we can view $\hat{G}$ a slowly increasing C^∞ function as $G \in \mathcal{E}'$.

Problem 5

(1) Without 0, $(x^2+y^2)^{-1}$ is continuous function, and it is locally integrable function on $\mathbb{C} \setminus \{0\}$, which is second countable. Therefore, $(x^2+y^2)^{-1}dxdy$ is a Radon measure. I'll show the left and right-invariant. For $f \in C_c^+$, $z_0 = x_0 + y_0 i \in \mathbb{C} \setminus \{0\}$

$$\int_{\mathbb{C}\setminus\{0\}} \frac{f((x_0+y_0 i)(x+iy))}{x^2+y^2} dxdy = \int_{\mathbb{C}\setminus\{0\}} \frac{f((x_0 x - y_0 y) + (x_0 y + y_0 x)i)}{x^2+y^2} dxdy.$$

Let $s = x_0 x - y_0 y$, $t = x_0 y + y_0 x$. Then, $dsdt = (x_0^2 + y_0^2)dxdy$ and $\frac{x_0 s + y_0 t}{x_0^2+y_0^2} = x$, $\frac{-y_0 s + x_0 t}{x_0^2+y_0^2} = y$. Since it is linear transformation with full rank, it maps $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$.

$$\begin{aligned}\int_{\mathbb{C}\setminus\{0\}} \frac{f((x_0 x - y_0 y) + (x_0 y + y_0 x)i)}{x^2+y^2} dxdy &= \int_{\mathbb{C}\setminus\{0\}} \frac{f(s+ti)}{\left(\frac{x_0 s + y_0 t}{x_0^2+y_0^2}\right)^2 + \left(\frac{-y_0 s + x_0 t}{x_0^2+y_0^2}\right)^2} (x_0^2+y_0^2)^{-1} dsdt \\ &= \int_{\mathbb{C}\setminus\{0\}} \frac{f(s+ti)}{s^2+t^2} dsdt\end{aligned}$$

It proves left-invariant. Since $\mathbb{C} \setminus \{0\}$ is Abelian group, left-invariant implies right-invariant. Hence, it is Haar measure.

(2) Again, x^{-1}, x^{-2} is continuous function without $x = 0$, and it is locally integrable function on $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. Therefore, $x^{-1}dxdy$ and $x^{-2}dxdy$ are Radon measure on G, which is homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. I'll show left and right-invariant. Let

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$\int_G f(\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) x^{-2}dxdy = \int_G f(\left(\begin{bmatrix} ax & ay+b \\ 0 & 1 \end{bmatrix}\right) x^{-2}dxdy.$$

Let $s = ax$, $t = ay+b$, then $dsdt = a^2 dxdy$ and the integration region does not change since the matrix is invertible.

$$\int_G f(\left(\begin{bmatrix} ax & ay+b \\ 0 & 1 \end{bmatrix}\right) x^{-2} dxdy = \int_G f(\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix}\right) (s/a)^{-2} a^{-2} dsdt = \int_G f(\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix}\right) s^{-2} dsdt.$$

For right-invariant:

$$\int_G f(\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) x^{-1} dxdy = \int_G f(\left(\begin{bmatrix} ax & bx+y \\ 0 & 1 \end{bmatrix}\right) x^{-1} dxdy.$$

Let $s = ax$, $t = bx + y$, then $dsdt = adxdy$. The integration region does not change since the matrix is invertible.

$$\int_G f(\left(\begin{bmatrix} ax & bx+y \\ 0 & 1 \end{bmatrix}\right) x^{-1} dxdy = \int_G f(\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix}\right) (s/a)^{-1} a^{-1} dsdt = \int_G f(\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix}\right) s^{-1} dsdt.$$

Therefore, they are Haar measure.