General Topology - HW 6

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Problem 1

($\Rightarrow$) Let $\prod_{\alpha\in J} X_\alpha$ is nonempty locally compact set and fix $x \in \prod_{\alpha\in J} X_\alpha$. Then, there exists compact subspace C of $\prod_{\alpha\in J} X_\alpha$ containing a open neighborhood U of x, which is X_α for all $\alpha \in J$ but finitely many. Let's index $I = \{i_1, \dots, i_n\}$ for the set $\pi_\alpha(U) \neq X_\alpha$. Since $C \supset U$, $\pi_\alpha(C) = X_\alpha$ and X_α is compact for $\alpha \notin I$ as π_α is continuous.

WLOG, I'll show that X_{i_1} is locally compact. Fix $x_{i_1} \in X_{i_1}$ and choose arbitrary point x_α, $\alpha \neq i_1$ in X_α.(This requires AC.) For $x = (x_\alpha)$, there exists an open neighborhood U for x and compact set C containing U. By the definition of product topology, there exists a basis B containing x and $\pi_{i_1}(B)$ is open set in x_{i_1}. Also, $\pi_{i_1}(C)$ is compact set containing $\pi_{i_1}(B)$.(π_α is continuous function.) Therefore, X_{i_1} is locally compact.

If $\prod_{\alpha\in J} X_\alpha$ is empty set, it is trivially locally compact since ϕ is open, compact set.

($\Leftarrow$) Let $x \in \prod_{\alpha\in J} X_\alpha$. Let's index $I = \{i_1, \dots, i_n\}$ if X_α is not compact. For $\alpha \notin I$, let $U_\alpha = X_\alpha$ and for $x \in I$, find open neighorhood U_α with compact set C_α containing the neighhood. Finally, set $U = \prod_\alpha U_\alpha$ and $C = \prod_\alpha C_\alpha$, the $x \in U$ is open and C is compact set containing U by Tychnoff theorem. Therefore, $\prod_{\alpha\in J} X_\alpha$ is locally compact.

Problem 2

A. Let's define $f : \mathbb{R}^n \to S^n \setminus \{N\}$ by

$$f(x) = \frac{2}{\|x\|^2 + 1}\left(x_1, x_2, \dots, x_n, \frac{\|x\|^2 - 1}{2}\right)$$

for usual Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. Then, $\|f\| = 1$ for all $x \in \mathbb{R}^n$ and f is continuous. If $f(x^1) = f(x^2)$, $\frac{\|x^1\|^2-1}{\|x^1\|^2+1} = \frac{\|x^2\|^2-1}{\|x^2\|^2+1}$, and $\|x_1\| = \|x_2\|$. Also, it means $x_i^1 = x_i^2$ for $i = 1, \dots, n$. Therefore, f is injective. For any point in $S^n \setminus \{N\}$ $p = (p_1, \dots, p_{n+1})$, $f(x) = p$ for $x = \frac{1}{1-p_{n+1}}(p_1, \dots, p_n, 0)$. Therefore, f is bijective. Since the inverse of f is

$$f^{-1}(x) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n, 0)$$

which is continuous without $x_{n+1} = 1$, i.e., $\{N\}$, f is homeomorphism and $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$.

B. Let $r : \mathbb{R}^{n+1} \to [0, \infty)$ by sending $r(x) = \|x\|$, then r is continuous, so $r^{-1}(1) = S^n$ is closed in $\mathbb{R}^{n+1}$. By Heine-Borel theorem, S^n is compact in $\mathbb{R}^{n+1}$. As a subspace of $\mathbb{R}^{n+1}$, S^n is Hausdorff. Also, $f(\mathbb{R}^n)$ is proper subspace of S^n whose closure is equal to S^n: For any neighborhood of $\{N\}$, it intersect with S^n. Therefore, we can regard S^n is the 1-point compatification of $\mathbb{R}^n$ since it is 1-point compactification of $S^n \setminus \{N\}$ which is homeomorphic to $\mathbb{R}^n$.

Problem 3

Each point in $\mathbb{Z}_{>0}$ is closed and open, and compact. Therefore, it is locally compact Hausdorff. Let $f : \mathbb{Z}_{>0} \to \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$, $f(n) = \frac{1}{n}$, then it is bijective and bicontinuous since each point is open and closed. Therefore, $\mathbb{Z}_{>0}$ and $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$ are homeomorphic. Let $Y = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$, then it is compact since any open neighborhood of $\{0\}$ contains infinite points of $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. As a subspace of $\mathbb{R}$, it is Hausdorff. Therefore, Y is one point compactification of $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$ and unique upto homeomorphism.

Let $Z = \mathbb{Z}_{>0} \cup \{\infty\}$ be a one point compacitifcation of X, then take a function $h : Y \to Z$ such that $h(\frac{1}{n}) = n$ and $h(0) = \infty$. Then $h = f$ on $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. For open set $U \subset Y$ does not containing $\{0\}$, $h(U) = f(U)$, which is open in Z, and for open set U containing $\{0\}$, $h(Y \setminus U) = f(Y \setminus U)$ is compact in $\mathbb{Z}_{>0}$ and Z. Since Z is Hausdorff, $h(Y \setminus U)$ is closed, and $h(U)$ is open in Z. Therefore h is open map and by the same argument on h^{-1}, we can show that h^{-1} is also open map. Therefore, h is homeomorphism, and Y and Z are homeomorphic.

Problem 4

A. If X is compact, X is compact in Y and closed since Y is Hausdorff. Therefore, $\{\infty\}$ is open in Y. Since Hausdorff implies T_1, $\{\infty\}$ is closed in Y.

B. Let $\{\infty\}$ is open in Y, then $Y \setminus \{\infty\}$ is closed in Y, so compact. Since X is not compact, it is contradiction. Therefore, any open neighborhood of $\{\infty\}$ should intersect with X and $\overline{X} = Y$.

Problem 5

($\Rightarrow$) Fix $x \in X$. Let C be a compact set containing open neighborhood U' of x. For any open neighborhood U of x, $V = U' \cap U$ is a open neighborhood of x contained in C. Since X is Hausdorff, C is closed in X, and $C \setminus V$ is closed in X and C. Therefore, $C \setminus V$ is compact, and there exists an open neighborhood V' of x such that $\overline{V'}$ is disjoint from $C \setminus V$. Take $V' \cap V$ as a open neighborhood of x, then $\overline{V' \cap V} \subset V \subset U$ since $\overline{V'}$ is disjoint from $C \setminus V$.(note that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.)

($\Leftarrow$) For any $x \in V$, choose V as an open neighborhood such that contained in compact set $\overline{V}$.

Problem 6

If $X = \phi$ or one point set, it is trivially locally compact Hausdorff, so I'll assume that X has at least two points. Since X is a subspace of Hausdorff space, X is Hausdorff. Choose $x \in X$, and open neighborhood $x \in U$ in X. Choose an open neighborhood V of ∞ in Y which is disjoint from U, then V^c is closed in Y, which implies compactness. Since $V^c \subset X$, V^c is compact in X. Therefore, $x \in U \subset V^c$ and X is locally compact.

Problem 7

For $n \in \mathbb{N}$, let $\mathcal{A}_n = \left\{B\left(x, \frac{1}{n}\right) \mid x \in X\right\}$, then it forms an open cover S_n of X for each n. Since X is compact, choose finite subcover for each n and construct $\mathcal{B} = \bigcup_{i=1}^{\infty} S_n$. I'll show that this is a basis for the topology on X.

Choose an open set in X, and choose a point x in U. There exists $N \in \mathbb{N}$ such that $B\left(x, \frac{1}{N}\right) \subset U$ since X has metric topology. Choose $B \in S_{3N}$ such that $x \in B$, then B is contained in $B\left(x, \frac{1}{N}\right)$ since the center of the B and x is smaller than $\frac{1}{3N}$ and the radius of ball is $\frac{1}{3N}$.

For $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \phi$, then the distance between center is smaller than the summation of radius of each ball. Let the center of each ball x_1, x_2 and radius r_1, r_2. There exists $x = \frac{x_1+x_2}{2}$ and $r = \frac{r_1+r_2-d(x_1,x_2)}{8} > 0$ such that $B(x,r) \in B_1 \cap B_2$. Therefore, $\mathcal{B}$ forms a countable basis of X and X is 2nd countable.

Problem 8

Let A be the dense subset and $\mathcal{B} = \left\{B\left(x, \frac{1}{n}\right) \mid x \in A, n \in \mathbb{N}\right\}$. I'll show that $\mathcal{B}$ is the basis.

Fix x in X and take an open neighborhood $x \in U$ in X. Then, there exists $N \in \mathbb{N}$ such that $B\left(x, \frac{1}{N}\right) \subset U$. Since A is dense in X, choose a point $p \in A$ such that $d(x,p) < \frac{1}{3N}$.(If not, $B\left(x, \frac{1}{3N}\right)$ is not in $\overline{A}$, which is contradiction.) For $B\left(p, \frac{1}{2N}\right) \in \mathcal{B}$, it is contained in $B\left(x, \frac{1}{N}\right)$ by the same reason in problem 7.

Repeating the same argument in problem 7, we can verify that $\mathcal{B}$ is a basis for X.

Since $\mathcal{B}$ is a countable union of countable sets, it is countable, and X is 2nd countable.

Problem 9

Let A_i is a dense subset of X_i. Let $\mathcal{A} \coloneqq \{x \in \prod_{i=1}^{\infty} X_i \mid x_j \in A_j \text{ finitely many, and } x_j = 0 \text{ elsewhere}\}$. Then, this is countable.(1)

I'll show that this is dense subset of $\prod_{i=1}^{\infty} X_i$. Fix $x \in \prod_{i=1}^{\infty} X_i$ and choose an element of basis of product topology $x \in B$. Then, $\pi_i(B) = X_i$ for all but finitely many. For each index j such that $\pi_j(B) \neq X_j$, we can choose a_j in A_j such that $\pi_j(B)$ contains since A_j is dense subset of X_j. Let $a = a_j$ for the indexes and $a_j = 0$ elsewhere, then $a \in \mathcal{A}$ and $a \in B$. Therefore, $\mathcal{A}$ is dense subset of $\prod_{i=1}^{\infty} X_i$.

(1): Let $\mathcal{B}_n$ be a collection such that $x_i = 0$ for all $i > n$. Then, it is finite set. Therefore, $\bigcup_{i=1}^{n} \mathcal{B}_n = \mathcal{A}$ is a countable set.

Problem 10

I'll show that X has a countable dense subset, then X is 2nd countable by problem 8. Let $I = [0,1]$. I'll denote functions as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$, $a_i, b_i \in \mathbb{Q}$, $0 = a_1 < a_2 < \ldots < a_n = 1$ to say a function $f : I \to \mathbb{R}$ such that

$$f(x) = \frac{b_j - b_{j-1}}{a_j - a_{j-1}}(x - a_{j-1}) + b_{j-1} \text{ for } a_{j-1} \leq x \leq a_j$$

By pasting lemma, $f(x)$ is continuous function. I'll show that the collection of such f forms a countable dense subset of f.

By Heine-Borel theorem, I is compact subspace, so f is uniformly continuous.(1) For each $\epsilon > 0$, choose small enough $\delta > 0$ such that $\delta < \frac{1}{3}$, $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Let's divide I by $\{a_1 = 0, a_2 = 1/N, \ldots, a_N = (N-1)/N, a_{N+1} = 1\}$. Then, the distance between a_j and a_{j+1} is smaller than δ. Choose $b_j \in \mathbb{Q}$ such that $|b_j - f(a_j)| \leq \epsilon/2$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. For a continuous function $h = \{(a_1, b_1), \ldots, (a_{N+1}, b_{N+1})\}$, the uniform norm of $f - h$ is smaller or equal to 3ϵ since in each interval $[a_{j-1}, a_j]$, the difference between maximum and minimum of $f - h$ is smaller or equal to 3ϵ.(the max/min value of $f - h$ in the interval $[a_{j-1}, a_j]$ is not bigger than $\max f - \min h$ for max and $\min f - \max h$ for min in the interval, but $f(x_1) - h(x_2) - (f(x_3) - h(x_4)) = (f(x_1) - f(x_2)) - (h(x_2) - h(x_4)) \leq 3\epsilon$ for all $x_1, x_2, x_3, x_4 \in [a_{j-1}, a_j]$.) It means for any open neighborhood of f, there exists intersection with $\mathcal{A}$, and $\mathcal{A}$ is dense subset.

Let $I = [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$. For any function f defined on the I, we can modify it by $f((\beta - \alpha)x + \alpha)$ which is defined on $[0, 1]$ and approximate it by the countable dense subset. After it, we can sends the functions by $\frac{1}{\beta - \alpha}(x - \alpha)$ which does not change uniform metric value. Therefore, $\mathcal{A}' = \{f \in \mathcal{A} \mid f(\frac{1}{\beta - \alpha}(x - \alpha))\}$ forms the countable dense subset.

(1): For given $\epsilon > 0$, cover Y by $B(y, \epsilon/2)$ for all $y \in Y$. Let $\mathcal{A}$ be the inverse images of each ball in Y, then it forms an open cover of X. By Lebesgue number lemma, there exists $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$, $f(x_1), f(x_2) \in B(y, \epsilon/2)$ for some y. Therefore, $d_Y(f(x_1), f(x_2)) < \epsilon$ and it implies that f is uniformly continuous.