General Topology - HW 3

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Problem 1

A. I'll first check the metric axioms.

   (a) $\bar{d}$ is a function
   $$\bar{d} : X \times X \longrightarrow \mathbb{R}$$
   and $\bar{d}(x, y) = \min\{d(x, y), 1\} \geq 0$

   (b) $\bar{d}(x, x) = d(x, x) = 0$ for all x. Conversely, if $\bar{d}(x, y) = 0$, then $\min\{d(x, y), 1\} = 0 \Rightarrow d(x, y) = 0$ and $x = y$.

   (c) Symmetric: $\bar{d}(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = \bar{d}(y, x)$.

   (d) $\bar{d}(x, y) + \bar{d}(y, z) = \min\{d(x, y), 1\} + \min\{d(y, z), 1\} = \min\{d(x, y) + d(y, z), d(x, y) + 1, d(y, z) + 1, 2\} \geq \min\{d(x, y) + d(y, z), 1\} \geq \min\{d(x, z), 1\} = \bar{d}(x, z)$

   Therefore, $\bar{d}$ is a metric.

B. I'll first check the metric axioms.

   (a) ρ is a function
   $$\rho : X \times X \longrightarrow \mathbb{R}$$
   and $\rho(x, y) = \frac{d(x,y)}{d(x,y)+1} \geq 0$

   (b) $\rho(x, x) = \frac{d(x,x)}{d(x,x)+1} = 0$ for all x. Conversely, if $\rho(x, y) = 0$, then $d(x, y) = 0$ and $x = y$.

   (c) Symmetric: $\rho(x, y) = \frac{d(x,y)}{d(x,y)+1} = \frac{d(y,x)}{d(y,x)+1} = \rho(y, x)$.

   (d) $\rho(x, y) + \rho(y, z) = \frac{d(x,y)}{d(x,y)+1} + \frac{d(y,z)}{d(y,z)+1} = 2 - \left(\frac{2+d(x,y)+d(y,z)}{d(x,y)d(y,z)+d(x,y)+d(y,z)+1}\right) \geq 2 - \left(\frac{2+d(x,y)+d(y,z)}{d(x,y)+d(y,z)+1}\right) = 2 - \left(1 + \frac{1}{d(x,y)+d(y,z)+1}\right) \geq 2 - \left(1 + \frac{1}{d(x,z)+1}\right) = \rho(x, z)$

   Therefore, ρ is a metric. Since f is bounded by 1 for $t \geq 0$, ρ is bounded metric.

Problem 2

Claim: $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in $\mathbb{R}^\omega$.

Proof. Let $x \in \mathbb{R}^\omega$ and U be an open neighborhood of x in $\mathbb{R}^\omega$. Since $p_i(U) = \mathbb{R}$ for all i but finitely many, let $i_{\max}$ be a natural number such that $p_i(U) = \mathbb{R}$ for $i \geq i_{\max}$. Let $y \in R^\infty$ that $y_i = p_i(x)$ for $i < i_{\max}$ and $y = 0$ for elsewhere. Then, $y \in U$. This is true for all open neighborhood of x, so $x \in \overline{\mathbb{R}^\infty}$. $\square$

Problem 3

Claim: $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty \cup \{(x_i) \in \mathbb{R}^\omega \mid \lim_{i\to\infty} x_i \to 0\}$ in $\mathbb{R}^\omega$.

Proof. Let $x \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$ and U be an open neighborhood of x in $\mathbb{R}^\omega$. Since $x \notin \mathbb{R}^\infty$, $x_i \neq 0$ for infinitely many i. Let the nonzero sequence $\{x_j\}$. Let $\limsup|x_j| \neq 0$, then there exists subsequence of $\{|x_j|\}$ such that converges to $c \in (0, \infty]$ and $B(x, \min\{c/2, 1\})$ is a open neighborhood of x disjoint with $\mathbb{R}^\infty$.

Conversely, let $\limsup|x_j| = 0$, then $\lim_{j\to\infty} x_j = 0$ and for any $\epsilon > 0$, $B(x, \epsilon)$ contains 0 in ith coordinate for all i but finitely many. Let $i_{\max} \in \mathbb{N}$ such that $0 \in p_i(B(x, \epsilon))$ for $i \geq i_{\max}$. Then, $(y_i) \in \mathbb{R}^\infty$ such that $y_i = x_i$ for $i < i_{\max}$ and $y = 0$ for $i \geq i_{\max}$, so $(y_i) \in B(x, \epsilon)$. Therefore, $x \in \overline{R^\infty}$.

Consequently, $\overline{R^\infty} = \mathbb{R}^\infty \cup \{(x_i) \in \mathbb{R}^\omega \mid \lim_{i\to\infty} x_i \to 0\}$. □

Problem 4

First, I'll show that D is a metric.

1. $\bar{d}$ is a function
$$D : X \times X \longrightarrow \mathbb{R}$$
and $D(x, y) \geq 0$

2. $D(x, x) = \sum_{i=1}^{n} d(x_i, x_i) = 0$ for all x. Conversely, if $D(x, y) = 0$, then $0 = D(x, y) \geq d(x_i, y_i) \geq 0$ for each i, so $x_i = y_i$ and $x = y$.

3. Symmetric: $D(x, y) = \sum_{i=1}^{n} d(x_i, y_i) = \sum_{i=1}^{n} d(y_i, x_i) = D(y, x)$.

4. $D(x, y) + D(y, z) = \sum_{i=1}^{n} d(x_i, y_i) + d(y_i, z_i) \geq \sum_{i=1}^{n} d(x_i, z_i) = D(x, z)$

Therefore, $\bar{D}$ is a metric. Let $\mathcal{B} = \{B_1 \times B_2 \times \cdots \times B_n \subset \mathbb{R}^n \,|B_i$ is an open interval of $\mathbb{R}\}$ be a basis of $\mathbb{R}^n$. Let $x \in B \in \mathcal{B}$, then there exists $\epsilon > 0$ such that $B_d(x_i, \epsilon) \subset p_i(B)$ for all i for each $p_i(B)$ is an open neighborhood and there exists small $\epsilon_i > 0$ such that $B_d(x_i, \epsilon_i) \subset p_i(B)$ and we can set $\epsilon = \min\{\epsilon_i\}$. Therefore, $B_D(x, \epsilon) \subset B$ since for any $y \in B_D(x, \epsilon)$, $d(x_i, y_i) < \epsilon$.

Problem 5

Let d be a euclidean metric on $\mathbb{R}$ and let $\bar{d}$ be a bounded metric on $\mathbb{R}$ as in problem 1 (A). Let $\bar{d}_1$ be a function on $\mathbb{R}^2$ such that
$$\bar{d}_1(x, y) = \begin{cases} \bar{d}(x_2, y_2) & \text{if } x_1 = y_1 \\ 1 & \text{if } x_1 \neq y_1. \end{cases}$$

I'll show that $\bar{d}_1$ is a metric.

1. $\bar{d}_1(x, y) \geq 0$

2. $\bar{d}_1(x, x) = \bar{d}(x_2, x_2) = 0$ for all x. Conversely, if $\bar{d}_1(x, y) = 0$, then $x_1 = y_1$ and $\bar{d}(x_2, y_2) = 0$ implies $x_2 = y_2$. Therefore, $x = y$.

3. Symmetric: If $x_1 \neq y_1$, $\bar{d}_1(x, y) = 1 = \bar{d}_1(y, x)$. If $x_1 = y_1$, $\bar{d}_1(x, y) = \bar{d}(x_2, y_2) = \bar{d}(y_2, x_2) = \bar{d}_1(y, x)$.

4. If $x_1 \neq y_1$ or $y_1 \neq z_1$, $\bar{d}_1(x, y) + \bar{d}_1(y, z) \leq 1 \leq \bar{d}_1(x, z)$. Conversely, if $x_1 = y_1 = z_1$, $\bar{d}_1(x, y) + \bar{d}_1(y, z) = \bar{d}(x_2, y_2) + \bar{d}(y_2, z_2) =\leq \bar{d}(x_2, z_2) = \bar{d}_1(x, z)$

Therefore, $\bar{d}_1$ is a metric.

I need to show that $\bar{d}_1$ generate dictionary topology. In the last homework, we showed that the product topology on $\mathbb{R}^d \times \mathbb{R}$ is equal to the dictionary topology on $\mathbb{R}^2$. Therefore, we can set the basis of dictionary topology on $\mathbb{R}^2$ be $\mathcal{B} = \{\{a\} \times (b,c) | a,b,c \in \mathbb{R}, -\infty < b < c < \infty\}$. For $x \in B = \{a\} \times (b,c) \in \mathcal{B}$, we can set $\epsilon = \min\{\frac{x_2-b}{2}, \frac{c-x_2}{2}\}$ so that $B_{\bar{d}_1}(x,\epsilon) \subset B$. Conversely, for any $y \in B_{\bar{d}_1}(x,\epsilon)$ for fixed ϵ, we can set $\delta = \min\{\frac{\epsilon-(y_2-x_2)}{4}, \frac{\epsilon-(x_2-y_2)}{4}\}$ so that $\{y_1\} \times (y-\delta, y+\delta) \subset B_{\bar{d}_1}(x,\epsilon)$. (For $\epsilon \leq 1$, this can be viewed as setting small interval in the interval, and for $\epsilon > 1$, we can arbitrary set $\delta > 0$ since $B_{\bar{d}_1}(x,\epsilon)$ is the whole set.)

Problem 6

Let V be an open set in Y. I'll show that $f^{-1}(V)$ is open in X.

Let $x_0 \in f^{-1}(V)$ and $y_0 = f(x_0) \in V$. Since V is an open set, there exists $\epsilon > 0$ such that $B(y_0,\epsilon) \subset V$. Let N be a natural number such that $|f_n(x) - f(x)| \leq \epsilon/3$ for all $n \geq N$. Let $U = f_N^{-1}(B(y_0,\epsilon))$. Then U is an open set containing x_0 since $|f_N(x_0) - f(x_0)| = |f_N(x_0) - y_0| \leq \epsilon$. If I show that U is contained in $f^{-1}(V)$, then it implies f is continuous function.

Let $x \in U$, I need to show that $f(x) \in V$ to show that $x \in f^{-1}(V)$, but $|f(x) - y_0| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Therefore, $f(x) \in B(y_0,\epsilon) \subset V$ and $x \in f^{-1}(V)$.

Consequently, f is a continuous function.

Problem 7

($\Rightarrow$) Let a sequence $\{f_n\}$ of functions $f_n : Y \to \mathbb{R}$ converges uniformly to a function $f : Y \to \mathbb{R}$. Then, for any $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ as a function of ϵ such that $|f_n(y) - f(y)| < \epsilon$ for all $y \in Y$ for all $n \geq N$, so $\sup\{|f_n(y) - f(y)|\} \leq \epsilon$ for all $n \geq N$ and it means $\rho(f_n - f) \leq \epsilon$ for $n \geq N$ in the metric space $(\mathbb{R}^Y, \rho)$. Therefore, $\{f_n\}$ converges to f in the metric space $(\mathbb{R}^Y, \rho)$.

($\Leftarrow$) Let $\{f_n\}$ converges to f in the metric space $(\mathbb{R}^Y, \rho)$. Then for fixed $\epsilon > 0$, there exists N such that $\rho(f_n, f) = \sup\{|f_n(y) - f(y)|\} \leq \epsilon/2$ for all $n \geq N$. It means $|f_n(y) - f(y)| \leq \epsilon$ for all $y \in Y$ for all $n \geq N$. Therefore, the sequence $\{f_n\}$ converges uniformly to a function f.

Problem 8

A. Since $p(\text{Im}q) = Y$, p is surjective. Let U be a open set in X. By the continuity of q, $q^{-1}(U)$ is open set in Y. Since $p \circ q = 1_Y$, $p(U) = p \circ q(q^{-1}(U)) = q^{-1}(U)$. Therefore p is open map with continuity, and p is quotient map.

B. Let the retraction p. It is definitely surjective and continuous. For a set $V \subset A$, let $p^{-1}(V)$ is open in X. Then $p^{-1}(V) \cap A = p_A^{-1}(V) = V$, and by subspace topology, V is open in A. Therefore, p is quotient topology.

Problem 9

A. I'll write the equivalence class of $\sim$ by $[(x,y)] = \{(z,w) \in \mathbb{R}^2 \,|z^2 + w^2 = x^2 + y^2\}$. Let $p(z,w) = [(z,w)]$ and $\mathbb{R}^2 / \sim$ have quotient topology induced by p. Then, $p^{-1}[(z,1)] = g^{-1}(z)$ by the definition of p and g.

Let's define $f([x,y]) = g(x,y)$ for the domain $\mathbb{R}^2 / \sim$. I need to show that this is a well-defined function. The f definitely have function value for each element in $\mathbb{R}^2 / \sim$. Let $[(x,y)] = [(x',y')]$, then $x^2 + y^2 = x'^2 + y'^2$ and $g(x,y) = g(x',y')$. Also, the codomain of f is $[0,\infty)$. Therefore f is well defined function. For $x^2 + y^2 \neq x'^2 + y'^2$, $g(x, y) \neq g(x', y')$ so $f([(x, y)]) \neq f([(x', y')])$. Also, for any $r \in [0, \infty)$, there exists $[(\sqrt{r}, 0)]$ such that $f([(\sqrt{r}, 0)]) = r$. Therefore f is bijective. For a basis (a, b) or $[0, b)$, $a < b$, in $[0, \infty)$, considering the subspace topology as a subset of $\mathbb{R}$, $g^{-1}(U)$ is open set in $\mathbb{R}^2$.(Recall that we already showed such function is continuous in previous HW.) If we compute $p \circ g^{-1}(U)$, $p \circ g^{-1}(U) = \bigcup_{z \in U} p \circ g^{-1}(z) = \bigcup_{z \in U} [(z, 1)] = \bigcup_{z \in U} f^{-1}(z) = f^{-1}(U)$. Also, $g^{-1}(z) = p^{-1}([(z, 1)])$, $g^{-1}(U) = p^{-1}\left(\bigcup_{z \in U} [(z, 1)]\right)$ and $p \circ g^{-1}(U)$ is open in $\mathbb{R}^2$. Therefore, f is continuous.

B. Define $h(r) = (r, 0)$ for $r \geq 0$. Then, $f(p(h(r))) = f(p(r, 0)) = f([(r, 0)]) = r$. Therefore, $f \circ p \circ h = 1_{[0,\infty)}$. Also, h is continuous since for any basis $(a, b) \times (c, d)$, $a < b$, $c < d$ of $\mathbb{R}^2$,

$$h^{-1}((a, b) \times (c, d)) = \begin{cases} \phi & \text{if } c \leq 0 \leq d \\ (a, b) & \text{if } c < d < 0 \text{ or } 0 < c < d. \end{cases}$$

Therefore, $p \circ h$ is continuous left inverse of f, so f is quotient map. Since f is bijective, $\left(f^{-1}\right)^{-1} = f$. For any open set U in $\mathbb{R}^2 / \sim$, there exists a set V in $[0, \infty)$ such that $f^{-1}(V) = U$ and V is open by the quotient map property of f. It implies $f(U) = V$, and f is bicontinuous. Therefore, f is homeomorphism.

Problem 10

Before starting, I'll prove a short lemma.

Lemma 1. *$g : S^1 \times S^1 \to S^1$ by $g(z, w) = zw$ is continuous function.*

Proof. First, I'll give a metric topology on $\mathbb{C}$ with the usual topology $d(z, w) = |z - w|$. Seeing $z = x + yi$, $w = s + ti$ and $|z - w| = (x - s)^2 + (t - y)^2$, the topological structure of $\mathbb{C}$ is same as $\mathbb{R}^2$ with euclidean metric. Therefore, I'll see $\mathbb{C}$ as $\mathbb{R}^2$ and $S^1 = \{(x, y) \in \mathbb{R}^2 \,|x^2 + y^2 = 1\}$. Let $h(x, y) = x^2 + y^2$, then $S^1 = h^{-1}(1)$, so S^1 is closed set in $\mathbb{R}^2$.

Let $G : \mathbb{R}^4 \to \mathbb{R}^2$ by $G(a, b, c, d) = (ac - bd, ad + bc)$, then G is continuous map by the previous homework. Since $\mathbb{R}^4$ with product topology has same topological structure with $\mathbb{R}^2 \times \mathbb{R}^2$ with product topology, replacing the domain of G by $\mathbb{R}^2 \times \mathbb{R}^2$ does not change the continuity. By the restricting the domain of $\mathbb{R}^2 \times \mathbb{R}^2$ by $S^1 \times S^1$ with subspace topology, we can get a continuous function $g' : S^1 \times S^1 \to \mathbb{R}^2$ and since the codomain of g' can be restrict to S^1, $g : S^1 \times S^1 \to S^1$ is continuous map. Since $g((a, b), (c, d)) = (ac - bd, ad + bc)$ is the same as $g(z, w) = zw$ with $z = a + bi$, $w = c + di$, it does not depends on the defining space of S^1 whether it is $\mathbb{R}^2$ or $\mathbb{C}$. Therefore, g is continuous. $\square$

Let $p : S^1 \times S^1 \to X^*$ quotient map generating quotient topology on X^* and define $g : S^1 \times S^1 \to S^1$ by $g(z, w) = zw$. Let f be $f : X^* \to S^1$ by $f([(z, w)]) = g(z, w)$. Then, f have function value for each element in X^* and $[(z, w)] = [(z', w')] \Rightarrow f([(z, w)]) = f([(z', w')])$. Therefore, it is well-defined. Also it is bijective since $f(z, 1) = z$ and $[(z, w)] \neq [(z', w')] \Rightarrow f([(z, w)]) \neq f([(z', w')])$.

Second, I'll prove that $g^{-1}(z) = p^{-1}([(z, 1)])$ for $z \in S^1$. Let $\alpha \in g^{-1}(z)$ s,t, $\alpha \in S^1 \times S^1$, then $g(\alpha) = z$ and $p(z) = [(z, 1)]$. Conversely, if $\alpha \in p^{-1}([(z, 1)])$, then $p(\alpha) = [(z, 1)]$ and for $\alpha = (a, b)$, $ab = z$, so $\alpha \in g^{-1}(z)$. Therefore $g^{-1}(z) = p^{-1}([(z, 1)])$.

Since g is continuous by the lemma above, for any open set U in S^1, then $g^{-1}(U)$ is open set in $S^1 \times S^1$. Since $g^{-1}(U) = \bigcup_{z \in U} g^{-1}(z)$, $p \circ g^{-1}(U) = \bigcup_{z \in U} p \circ g^{-1}(z) = \bigcup_{z \in U} [(z, 1)] = \bigcup_{z \in U} f^{-1}(z) = f^{-1}(U)$ and $g^{-1}(U) = \bigcup_{z \in U} g^{-1}(z) = \bigcup_{z \in U} p^{-1}([(z, 1)]) \Rightarrow p \circ g^{-1}(U)$ is open in X^* and therefore, f is continuous.

Let's define $h : S^1 \to S^1 \times S^1$ by $h(z) = (z, 1)$. Then, $f \circ p \circ h(z) = z$, so it is identity on S^1. Therefore, it is the same case as problem 9 B., and f is hemeomorphism. Consequently, X^* is hemeomorphic to S^1.