General Topology - HW 7

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Problem 1

Let X be a simply ordered set. Fix $p \in X$ and closed set $C \subset X$ such that $p \notin C$. Since $x \in C^c$, there exists $(a,b) \cap C = \phi$, $a \le p \le b$. If $a, b \notin C$, take $\{U_c\}$ by

$$U_c = \begin{cases} (-\infty, a) & \text{if } c \le a \\ (b, \infty) & \text{if } b \le c \end{cases}$$

and let $U = \cup_c U_c$. Then, $C \subset U$ and $U \cap (a,b) = \phi$ since $a, b \notin U$

WLOG, I'll deal with the case $a \in C$.(The other case uses similar argument.) If there exists $d \in (a,b)$ such that $a < d < p$, then take (d,b) be a open neighborhood of p and take U_c by

$$U_c = \begin{cases} (-\infty, d) & \text{if } c \le a \\ (b, \infty) & \text{if } b \le c. \end{cases}$$

Then, $(d,b) \cap \cup_c U_c = \phi$ since $d \notin U$.

Assume there is no d such that $a < d < p$, then take $U_a = (-\infty, p)$ and the neighborhood of p be (a,b). Then, they are disjoint, and by taking $\{U_c\}$ by

$$U_c = \begin{cases} (-\infty, a) & \text{if } c \le a \\ (b, \infty) & \text{if } b \le c \end{cases}$$

for $c \neq a$, we again get open neighborhood of C disjoint with (a,b).

Hence, X is regular.

Problem 2

Let $C_1, C_2 \subset Y$ two disjoint closed sets. I need to choose two disjoint open sets that each contains C_1 and C_2. Take inverse image of C_1, C_2 by f: $f^{-1}(C_1)$ and $f^{-1}(C_2)$. Since f is continuous, they are closed sets in X and there exists open sets $f^{-1}(C_1) \subset U_1$, $f^{-1}(C_2) \subset U_2$. As U_1^c and U_2^c are closed subset such that each disjoints with C_1, there exists $C_1 \subset V_1 \subset U_1$ and $C_2 \subset V_2 \subset U_2$ such that $\overline{V_1} \subset U_1$ and $\overline{V_2} \subset U_2$. Take $(f(V_1^c))^c$ and $(f(V_2^c))^c$ be open neighborhood of C_1 and C_2. (This is possible since $f^{-1}(C_i)$ are saturated set.) Then, there is am obstacle that there can exists intersection of two open neighborhoods. To remove the intersection, take $f(\overline{V_1})$ and $f(\overline{V_2})$ and let $V_1' = V_1 \setminus f(\overline{V_2})$ and $V_2' = V_2 \setminus f(\overline{V_1})$. (Since $V_1 \subset \overline{V_1}$, $(f(V_1^c))^c \subset f(\overline{V_1})$ as $y \notin (f(V_1^c))$ means there exists $x \in V_1$ such that $f(x) = y$ (as f is surjective) and $y = f(x) \in f(\overline{V_1})$. It implies that $C_1 \subset (f(V_1^c))^c \subset f(\overline{V_1})$ and $f(\overline{V_1})$ is disjoint with C_2.) Then, $V_1' \cap V_2' = \phi$ and $C_1 \subset V_1'$, $C_2 \subset V_2$. Therefore, Y is normal space.

Problem 3

Let X be a locally compact Hausdorff space. Fix $x \in X$ and closed set $C \subset X$. Take one point compactification of X and denote it Y. As a closed set in compact Hausdorff Y, C is compact, and there exist disjoint two open sets $x \in U_1$, $C \subset U_2$. To remove ∞ from U_1 and U_2, take two disjoint open neighborhoods $\infty \in V_1$, $x \in V_2$ and let $U_1' = U_1 \cap V_2$. Also, make $C \subset V_3$ disjoint from ∞ and let $U_2' = U_2 \cap V_3$. Then $\infty \notin U_1', U_2'$ and $U_1' \cap U_2' = \phi$. Therefore, locally compact Hausdorff space is regular.

Problem 4

Let X be a connected normal space and take two different point $a, b \in X$ since X contains at least two points. By the Urysohn lemma, there exists continuous function $f : X \to [0,1]$ such that $f(a) = 0$, $f(b) = 1$. (Note that T_4 implies T_1.) If there exists $r \in [0,1]$ such that $f^{-1}(r) = \phi$, $f^{-1}([0,r))$ and $f^{-1}((r,1])$ forms a separation of X and they are nonempty since they contains a and b. This is contradiction to connectedness of X. Therefore, f is surjective map and X is uncountable.

Problem 5

Let $f : X \to [0,1]$ by $f(x,y) = \frac{1}{2}\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{2}$. Then, it is continuous on X, the image is in $[0,1]$, $f(A) = \frac{x}{|x|} = 0$, $f(B) = \frac{x}{|x|} = 1$.

Problem 6

In the class, we showed that if X is a normal space and A is a closed subspace of X, any continuous function of A into the closed interval $[a,b]$ of $\mathbb{R}$ may be extended to a continuous function of all of X into $[a,b]$. I need change $[a,b]$ to $\mathbb{R}$.

Let $f : A \to \mathbb{R}$, then using $\tan^{-1}(x)$, we can make continuous function $\tilde{f} = \tan^{-1}(x) \circ f : A \to (-\pi/2, \pi/2)$. Abusing notation, I'll denote $\tilde{f}$ by f, then f is a continuous function from A to $(-\pi/2, \pi/2)$. We can enlarge the codomain of f by $[-\pi/2, \pi/2]$ and apply Tietze extension theorem to get $g : X \to [-\pi/2, \pi/2]$. What I want to do is modify the codomain $[-\pi/2, \pi/2]$ to $(-\pi/2, \pi/2)$ and sends it by $\tan(x)$ to make codomain $\mathbb{R}$. To do this, first, note that $B = g^{-1}(-\pi/2) \cup g^{-1}(\pi/2)$ is closed subspace in X, which is normal space. Also, it is disjoint from A. Thus, we can get continuous function $\phi(x) : X \to [0,1]$, whose value on B is 0 and on A is 1. Our desired function is $g\phi$ since $g\phi = 0$ on B and $g\phi = g = f$ on A. Therefore, $g\phi : X \to (-\pi/2, \pi/2)$ and $\tan \circ g\phi : X \to \mathbb{R}$. Finally, we can easily check $(\tan \circ g\phi)(a) = f(a)$ on $a \in A$.

Problem 7

If X is normal, the Tietze extension theorem is satisfied, so I'll show that if X is not normal, the conclusion of the Tietze extension theorem does not satisfied.

Consider $\mathbb{R}$ with co-finite topology. Fix closed subspace $A = \{0,1\} \subset \mathbb{R}$ and $f : A \to [-1,2]$, $f(0) = 0$, $f(1) = 1$. Since there exists open set $U = \mathbb{R} \setminus \{0\}$, $V = \mathbb{R} \setminus \{1\}$ in $\mathbb{R}$, A has discrete topology, so f is continuous. Assume there exists continuous function $\tilde{f} : \mathbb{R} \to [-1,2]$ such that $\tilde{f}|_A = f$. Let $\tilde{U} = f^{-1}((1/2, 3/2))$, then $1 \in \tilde{U}$ and $\tilde{U}^c$ is finite set. By the same reason, $\tilde{V} = f^{-1}((-1/2, 1/2))$ is open set containing 0 and $\tilde{V}^c$ is finite. However, it means $\tilde{U} \cap \tilde{V} \neq \phi$ and $f^{-1}((-1/2, 1/2)) \cap f^{-1}((1/2, 3/2)) \neq \phi$, which is contradiction. Therefore, the conclusion of Tietze extension theorem does not hold.

Problem 8

(a) At nth step, the number of segment is $2(4^n - 2) + 2$. Fix $N > 3$, $0 \leq n_1 \leq 2(4^N - 2) + 1$ and $t \in \left[\frac{n_1}{2(4^N-2)+2}, \frac{n_1+1}{2(4^N-2)+2}\right]$. Then, for $n > N$, $\frac{4^{n-N}n_1}{2(4^n-2)+2} = \frac{n_1}{2(4^N-2)+4-2\cdot 4^{N-n}} < \frac{n_1}{2(4^N-2)+2}$ and $\frac{4^{n-N}(n_1+1)+4^{n-N}}{2(4^n-2)+2} = \frac{n_1+2}{2(4^N-2)+4-2\cdot 4^{N-n}} > \frac{n_1+1}{2(4^N-2)+2}$. Geometrically, $\frac{4^{n-N}n_1}{2(4^n-2)+2} < \frac{n_1}{2(4^N-2)+2} < t < \frac{n_1+1}{2(4^N-2)+2} < \frac{4^{n-N}(n_1+1)+4^{n-N}}{2(4^n-2)+2}$ means that $f_n(t)$ is contained in the index of rectangle of $n_1 - 2$ from n_1+2 at Nth step for all $n > N$. (If $n_1 = 2(4^N-2)+1$, then take upper bound as n_1+1.) Since the size of rectangle goes to 0 by $\frac{1}{4^n}$ speed, it means $|f_{n_1}(t) - f_{n_2}(t)| \leq \frac{2}{2^N}$ for all $n_1, n_2 > N$. Therefore, this is Cauchy sequence in $\mathbb{R}^2$ for fixed t and converges to some function f. As $n_1 \to \infty$, $|f - f_n(t)| \leq \frac{2}{2^N}$ for $n > N$ for all $t \in [0,1]$ and it means $\{f_n\} \to f$ uniformly. Therefore, f is continuous as f_n are continuous.

(b) Since f is continuous and $[0,1] \times [0,1]$ is Hausdorff, $f([0,1])$ is compact in $[0,1] \times [0,1]$. For any point $p \in [0,1] \times [0,1]$, it is contained in some rectangle, and the image of f_n pass through the rectangle for all n. Also, I showed that if there exists $f_N(t)$ in some rectangle for some N, the image of $f_n(t)$ does not move too much for all $n > N$, so there exists $t \in [0,1]$ such that for any rectangle at Nth step, the $f(t)$ is in the rectangle which is determined at Nth step. It means for any $p \in [0,1] \times [0,1]$, there exists t_n such that $|f_n - p| \leq 1/n$. Therefore, $\overline{f([0,1])} = [0,1] \times [0,1]$. However, $f([0,1])$ is closed, so $f([0,1]) = [0,1] \times [0,1]$ and f is surjective.

Problem 9

Since path component containing x_0 is X, X is path connected. Since $\pi_1(x, X) \simeq \pi_1(x_0, X)$ as group isomorphic sense for all $x \in X$, I'll show that $\pi_1(x_0, X)$ is trivial. Let $\alpha : [0,1] \to X$ be a loop based at x_0, i.e. $\alpha(0) = \alpha(1) = x_0$, then we can make path homotopy by

$$H(s,t) = (1-t)\alpha(s) + tx_0.$$

Then, $H(s,0) = \alpha(s)$, $H(s,1) = x_0$, $H(0,t) = H(1,t) = x_0$. Also, it is continuous on $I \times I$ since $\alpha(s)$ is continuous. Therefore, $[\alpha] = 1$ and $\pi_1(x_0, X)$ is trivial.

Problem 10

Since $\alpha(1) = \beta(0)$, $\gamma = \alpha * \beta$ is well-defined. $\hat{\alpha} : \pi_1(\alpha(0), X) \to \pi_1(\alpha(1), X)$, $\hat{\beta} : \pi_1(\alpha(0), X) \to \pi_1(\beta(1), X)$, so $\hat{\gamma} : \pi_1(\alpha(0), X) \to \pi_1(\beta(1), X)$ and $\hat{\gamma} = [\bar{\gamma}] * [f] * [\gamma]$ for $f \in \pi_1(\alpha(0), X)$. If we expand it,

$$\hat{\gamma}[f] = [\bar{\gamma}] * [f] * [\gamma] = [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha * \beta] = [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta] = [\bar{\beta}] * \hat{\alpha}[f] * [\beta] = \hat{\beta} \circ \hat{\alpha}[f].$$

Therefore, $\hat{\beta} \circ \hat{\alpha} = \hat{\gamma}$

Problem 11

Assume $\pi_1(X, x_0)$ is trivial and $f : S^1 \to X$ is a continuous function. Construct $h : [0,1] \to S^1$ by $h(r) = e^{2\pi i r}$. Then, $f' = f \circ h = [0,1] \to X$ is a path based at $f(0)$ and path homotopic to $f(0)$. Construct homotopy $F(s,t)$ such that $F(s,0) = f'(s)$, $F(s,1) = f(0)$. Consider the following universal property: ,where the p is given by $p(s,t) = (1-t)e^{2\pi i s}$. This is quotient map from $I \times I$ to D^2 since p is surjective, continuous, and closed map since $I \times I$ is compact and D^2 is Hausdorff space. Also, $\{s = 0\} \cup \{t = 1\} \cup \{s = 1\} \subset F^{-1}(f(0))$, so we can define g. Since F is continuous, by universal property, g is continuous and it extends f.

Conversely, assume that there exists g extending f. Let i be a inclusion from S^1 to B^2, then $g \circ i = f$. Therefore, $g_* \circ i_* = f_*$. Since $i_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$ for $b_0 \in S^1$, i_* should be trivial and f_* is trivial.

$$\begin{tikzcd} I \times I \arrow[r, "p"] \arrow[dr, "F"'] & D^2 \arrow[d, dashed, "g"] \\ & X \end{tikzcd}$$

Problem 12

Let α and β are path-homotopic, then by definition of equivalence class by path-homotopy relation, $[\alpha] = [\beta]$. Let $x_0 = \alpha(0) = \beta(0)$ and $x_1 = \alpha(1) = \beta(1)$, then $\hat{\alpha}, \hat{\beta} : \pi_1(X, x_0) \to \pi_1(X, x_1)$. For any $[f] \in \pi_1(X, x_0)$, $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] = [\bar{\beta}] * [f] * [\beta] = \hat{\beta}([f])$ since $[\alpha] = [\beta] \Rightarrow [\bar{\alpha}] * [\alpha] * [\bar{\beta}] = [\bar{\alpha}] * [\bar{\beta}] * [\bar{\beta}] \Rightarrow [\bar{\beta}] = [\bar{\alpha}]$. Therefore, $\hat{\alpha} = \hat{\beta}$.

Problem 13

Lemma 1. *If $\alpha, \beta : I \to X$ be two paths such that $\alpha(1) = \beta(0)$. Then, $\overline{\alpha * \beta} = \bar{\beta} * \bar{\alpha}$.*

Proof. By the definition of $\overline{\alpha * \beta}$,

$$\overline{\alpha * \beta}(t)(\alpha * \beta)(1-t) = \begin{cases} \alpha(2(1-t)) & \text{if } 0 < 1-t < 1/2 \\ \beta(1-2(1-t)) & \text{if } 1/2 < 1-t < 1. \end{cases}$$

Then,

$$\overline{\alpha * \beta}(t)(\alpha * \beta)(1-t) = \begin{cases} al\bar{p}ha(2t) & \text{if } 1/2 < t < 1 \\ be\bar{t}a(1-2t) & \text{if } 0 < t < 1/2. \end{cases}$$

which is $\bar{\beta} * \bar{\alpha}$. $\square$

($\Rightarrow$) Suppose $\pi_1(X, x_0)$ is Abelian group for all $x_0 \in X$. Then, for any path $\alpha, \beta : I \to X$ with $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, $[\alpha * \bar{\beta}] \in \pi_1(X, \alpha(0))$. Let $[f] \in \pi_1(X, \alpha(0))$, $[\overline{\alpha * \bar{\beta}}] * [f] * [\alpha * \bar{\beta}] = [\bar{\bar{\beta}} * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] = \hat{\bar{\beta}} \circ \hat{\alpha}([f])$ by the lemma above. On the other hands, $[\bar{\bar{\beta}} * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] = [\bar{\bar{\beta}} * \bar{\alpha}] * [\alpha * \bar{\beta}] * [f] = [f]$ since $\pi_1(X, \alpha(0))$ is Abelian group. Thus, $\hat{\bar{\beta}} \circ \hat{\alpha}([f]) = [f]$ and $\hat{\alpha}([f]) = \hat{\beta}([f])$ since $\hat{\beta}^{-1} = \hat{\bar{\beta}}$. Therefore, $\hat{\alpha} = \hat{\beta}$.

($\Leftarrow$) Suppose $\hat{\alpha} = \hat{\beta}$ for all paths α, β in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then, $\hat{\bar{\beta}} \circ \hat{\alpha}$ is identity on $\pi_1(X, \alpha(0))$ and $[\bar{\bar{\beta}} * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] = [f]$ for all $[f] \in \pi_1(X, \alpha(0))$.

Fix $x_0 \in X$ and $[f_1], [f_2] \in \pi_1(X, x_0)$. For f_2, which is selected by representative. Make two paths $\alpha, \beta : I \to X$ by $\alpha(t) = f_2(t/2)$ and $\beta(t) = f_2((t+1)/2)$. Then, $\alpha * \beta = f_2$ since

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) = f_2(t) & \text{if } 0 \le t \le 1/2 \\ \beta(2(t-1/2)) = f_2(t) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Therefore, $[\bar{f_2}] * [f_1] * [f_2] = [\bar{\beta} * \bar{\alpha}] * [f_1] * [\alpha * \beta] = \hat{\bar{\beta}} \hat{\alpha}([f_1]) = [f_1]$ and $[f_2] * [\bar{f_2}] * [f_1] * [f_2] = [f_1] * [f_2] = [f_2] * [f_1]$. This is true for all $[f_1], [f_2]$ and for all x_0. Therefore, $\pi_1(X, x_0)$ is abelian for all $x_0 \in X$.