General Topology - HW 4

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Problem 8 in HW 3

Problem 1

Take arbitrary two point $x, y \in f(X)$. f is surjection for codomain $f(X)$, so there exists $\alpha, \beta \in X$ such that $f(\alpha) = x$ and $f(\beta) = y$. Since X is path connected, there exists continuous $\gamma : [a, b] \to X$, $a, b \in \mathbb{R}$ such that $\gamma(a) = \alpha$, $\gamma(b) = \beta$. Consider $f \circ \gamma$, then $f \circ \gamma : [a, b] \to f(X)$ such that $f \circ \gamma(a) = f(\alpha) = x$ and $f \circ \gamma(b) = f(\beta) = y$. Then, it is path from x to y, and $f(X)$ is path connected.

Problem 2

Let C, D be separation of $\bigcup A_n$, then for each i, $A_i \subset C$ or D since A_i's are connected. WLOG, assume $A_1 \subset C$, and j be the smallest integer such that $A_j \subset D$ since $\mathbb{N}$ is well-ordered set. Then, $A_{j-1} \subset C$ and it means connected $A_{j-1} \cup A_j$ have a separation C, D. Since $A_{j-1} \cap A_j \neq \phi$, it is connected, so it generates contradiction. Therefore, $A_i \subset C$ for all i, and it is contradiction to existence of separation. Therefore, $\bigcup A_n$ is connected.

Problem 3

Fix $x \in A_{\alpha_0}$ and let $\mathcal{C}_x = \{A_\alpha | x \in A_\alpha\}$, then $A_{\alpha_0} \bigcup_{A \in \mathcal{C}_x} A$ is connected. Let it B_x. Then, $\bigcup_{y \in A_{\alpha_0}} B_y$ is connected since $x \in A_{\alpha_0} \subset B_y$ for each $y \in A_{\alpha_0}$. Therefore, $\bigcup A_\alpha$ is connected.

Problem 4

Let $\overline{A} \cap \overline{X - A} \cap C = \phi$. Then, $\left(\overline{A} \cap C\right) \cap \left(\overline{X - A} \cap C\right) = \phi$. However, $\left(\overline{A} \cap C\right)^c$ and $\left(\overline{X - A} \cap C\right)^c$ is open set in C and forms separation of C since $\left(\overline{A} \cap C\right) \cup \left(\overline{A} \cap C\right) = \left(\overline{A} \cup \overline{X - A}\right) \cap C = C$ and they are disjoint. Therefore, C is not connected, which is contradiction. Therefore, $\overline{A} \cap \overline{X - A} \cap C \neq \phi$.

Problem 5

I'll use modify the proof of the theorem in Munkers: *A finite cartesian product of connected space is connected.*

Let's fix $a \in A^c$, $b \in B^c$ since they are proper subset. We know that $X \times b$ (resp. $a \times Y$) is connected since it is homeomorphic with X (resp. Y). As a result, $(X \times b) \cup (a \times Y)$ is connected. Define

$$T_x = (X \times b) \cup (x \times Y)$$
$$S_y = (X \times y) \cup (a \times Y),$$

then $\bigcup_{x\in A^c} T_x$ and $\bigcup_{y\in B^c} S_y$ are connected since they shares common point (a,b). By the same reason, $\left(\bigcup_{x\in A^c} T_x\right) \cup \left(\bigcup_{y\in B^c}\right) S_y$ is connected. For any point $(c,d) \in (X\times Y) - (A\times B)$, $c \notin A$ or $d \notin B$, and it means $(c,d) \in \left(\bigcup_{x\in A^c} T_x\right) \cup \left(\bigcup_{y\in B^c}\right) S_y$. Therefore, $(X\times Y)-(A\times B) = \left(\bigcup_{x\in A^c} T_x\right) \cup \left(\bigcup_{y\in B^c}\right) S_y$ and it is connected.

Problem 6

A. Let $K = \{\alpha_1, \dots, \alpha_n\}$ and $\psi_K : X_K \to X^n$ such that $\psi_K = (\pi_{\alpha_1}, \dots, \pi_{\alpha_n})$. By the definition of X_K, ψ_K is bijective and bicontinuous. Thus X_K is connected.

B. For all K, $a \in X_K$, so $\bigcup_{K\in F} X_K$ is connected.

C. Since $X \supset \overline{\bigcup_{K\in F} X_K}$, we need to show that $X \subset \overline{\bigcup_{K\in F} X_K}$. Let $|K| < \infty$, then we can set $K = \{\alpha_1, \dots, \alpha_{|K|}\}$, so $X = \overline{\bigcup_{K\in F} X_K}$. Let K is infinite set. Fix $x \in X$. For any open neighborhood of x U, $\pi_\alpha(U) = X$ for all but finitely many. For the finite set L, let

$$y = \begin{cases} a_\alpha & \text{For } \alpha \notin L \\ x_\alpha & \text{For } \alpha \in L \end{cases}$$

Then, $y \in U$. Therefore, $x \in \overline{\bigcup_{K\in F} X_K}$ and $X = \overline{\bigcup_{K\in F} X_K}$.

Problem 7

The nontrivial term is $\{0\}\times[-1,1]$, so I'll show that $\overline{S}$ contains the term. Let $a \in (-1,1)$. For any $r > 0$, there exists n such that $\frac{1}{2\pi n + \frac{3\pi}{2}} < r$. For the n, $\left(\frac{1}{2\pi n+\frac{\pi}{2}}, 1\right)$, $\left(\frac{1}{2\pi n+\frac{3\pi}{2}}, -1\right) \in S$. We know that $\sin\frac{1}{x}$ is continuous(using real analysis and $\epsilon - \delta$ argument) on connected space $\left[\frac{1}{2\pi n+\frac{\pi}{2}}, \frac{1}{2\pi n+\frac{3\pi}{2}}\right]$, so there exists (b,a) such that $b \in \left(\frac{1}{2\pi n+\frac{\pi}{2}}, \frac{1}{2\pi n+\frac{3\pi}{2}}\right)$. It means $(b,a) \in B_r((0,a))$ and it means $(0,a) \in \overline{S}$ for $a \in (-1,1)$. For $a = 1$ or -1, we can use $x = \frac{1}{2\pi n+\frac{\pi}{2}}$ or $\frac{1}{2\pi n+\frac{3\pi}{2}}$ and make $B_r((0,a))$ contains it. Therefore, $\{0\}\times[-1,1] \subset \overline{S}$

Problem 8

Consider $f : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}^{n+1}$ such that $f(x) = \frac{x}{|x|}$. Then, the codomain of f is S^n and this is surjective on S^n since $f(x) = x$ for $x \in S^n$. For $x \neq 0$, $f(x)$ is continuous since $|x| = \sqrt{\sum_{i=1}^{n+1} x_i^2}$ is nonzero and continuous. If we show that $\mathbb{R}^{n+1}\setminus\{0\}$ is path connected, we can show that S^n is path connected.

Let $x, y \in \mathbb{R}^{n+1}\setminus\{0\}$. If $(1-t)x+ty$ for $0 \le t \le 1$ does not contains 0, take $\gamma : [0,1] \to \mathbb{R}^{n+1}$ such that $\gamma(t) = (1-t)x+ty$. This is a path from x to y.

Assume $(1-t)x+ty$ contains 0 at $t \in (0,1)$. If $(1-t)x+ty$ contains $(1,0,\dots,0)$ for some $t \in (0,1)$, then take $\gamma_1(t) = (1-t)x + (0,1,0,\dots,0)t$ for $0 \le t \le 1$ and $\gamma_2(t) = (0,1,0,\dots,0)(2-t) + (t-1)y$ for $1 \le t \le 2$. Then, γ_1, γ_2 does not contains 0 since x, y is contained in a straight line through 0 and $(1,0,\dots,0)$. Taking

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{For } 0 \le t \le 1 \\ \gamma_2(t) & \text{For } 1 \le t \le 2 \end{cases}$$

we can take path from x to y.

If $(1-t)x+ty$ does not contain $(1,0,\dots,0)$, take $\gamma_1(t)=(1-t)x+(1,0,0,\dots,0)t$ for $0\le t\le 1$ and $\gamma_2(t)=(1,0,0,\dots,0)(2-t)+(t-1)y$ for $1\le t\le 2$. Then, γ_1,γ_2 does not contains 0 since x,y is does not be contained in a straight line through 0 and $(1,0,\dots,0)$. Taking

$$\gamma(t)=\begin{cases}\gamma_1(t) & \text{For } 0\le t\le 1\\ \gamma_2(t) & \text{For } 1\le t\le 2\end{cases}$$

we can take path from x to y.

Therefore, S^n is path connected.

Problem 9

Define $g:S^1\to\mathbb{R}$ such that $g(x)=f(x)-f(-x)$. Then, $g(x)$ is continuous since $f(-x)$ is continuous.($h: x\to -x$ is continuous on domain $\mathbb{R}^2$ so on domain S^1.) Fix $(1,0)\in S^1$ and assume $g((1,0))>0$, then $g((-1,0))=-g((1,0))<0$. Since g is continuous and S^1 is connected by previous problem, there exists c in S^1 such that $g(c)=0$ and it means $f(c)=f(-c)$.

Problem 10

Before staring, I'll prove an easy lemma.

Lemma 1. *Let A and B are homeomorphic topological space and φ be the homeomorphism. For $A'=A\setminus\{a_1,a_2,\dots,a_n\}$, A' is hemeomorphic with $\varphi(A')=B\setminus\varphi(\{a_1,a_2,\dots,a_n\})=B'$.(Assume A',B' have subspace topology.)*

Proof. For simplicity, let $S=\{a_1,a_2,\dots,a_n\}$. Modify φ: $\varphi'=\varphi|_{A'}$. This is bijective, so I'll show that this is bicontinuous. For any open set V' in B', there exists V in B such that $V\setminus V'\subset\varphi(S)$. $\varphi^{-1}(V)$ is open in A and $(\varphi')^{-1}(V')=\varphi^{-1}(V)\setminus S$. Therefore, φ' is continuous and same argument for $(\varphi')^{-1}$ prove that the inverse is continuous. Therefore, φ' is hemeomorphism. □

Let's subtract $(-1,0)$, $(1,0)$, $(0,-1)$ from T and let it T', then T' is connected since $T'=((-1,1)\times\{0\})\cup(\{0\}\times(-1,0])$. However, subtracting three points from $[0,1]$ is not connected: if we subtract a point in $(0,1)$, it generate separation, so all we can do is subtracting $0,1$ from $[0,1]$, but finally, we should subtract a point from $(0,1)$, making the interval disconnected.

Problem 11

Let $(a,b),(c,d)\in X\times Y$. Since X,Y is path connected, there exists path $\gamma_X:[\alpha,\beta]\to X$, $\gamma_Y:[\gamma,\delta]\to Y$ such that $\gamma_X(\alpha)=a$, $\gamma_X(\beta)=c$, $\gamma_Y(\gamma)=b$, $\gamma_Y(\delta)=d$. Normalise the domain by $[0,1]$ using $h_X,h_Y:\mathbb{R}\to\mathbb{R}$, $h_X=(\beta-\alpha)x+\alpha$, $h_Y=(\delta-\gamma)x+\gamma$, $\gamma'_X=\gamma_X\circ h_X$, $\gamma'_Y=\gamma_Y\circ h_Y$. Make $\gamma:[0,1]\to X\times Y$ by $\gamma(t)=(\gamma_X(t),\gamma_Y(t))$, then it is continuous and $\gamma(0)=(a,b)$, $\gamma(1)=(c,d)$. Therefore, $X\times Y$ is path connected.

Problem 12

Let $p,q\in\mathbb{R}^2\setminus A$ and consider $f_a(x)=(1,a)x+p$, $x\ge 0$. Let $f_a(\mathbb{R})\cap A\ne\phi$ for all $a\in\mathbb{R}^+$, then we can set $h:\mathbb{R}^+\to A$ such that $h(a)$ is an element in $f_a(\mathbb{R})\cap A$. Since $f_a(\mathbb{R})\setminus\{p\}$ are disjoint for different a, h is injective and A is uncountable which is contradiction. Therefore, there exists a (even uncountable) that $f_a(\mathbb{R})\cap A=\phi$. By the same argument there exists straight line g_b such that $g_b(0)=q$ and $g_b(\mathbb{R})\cap A=\phi$. If $a=b$, we can choose another b different from a, so $g_b(\mathbb{R})\cap f_a(\mathbb{R})\ne\phi$. Let the intersection of point r and glue two path from p to r by f_a and r to q by g_b. This is a path from p to q. Since it is true for arbitrary p and q, $\mathbb{R}^2 \setminus \{A\}$ is path connected.

Problem 13

I'll state a lemma.

Lemma 2. *For any radius r, ball with radius r in $\mathbb{R}^n$ is path-connected.*

Proof. For any $p, q \in B_r(x)$ for some $x \in \mathbb{R}^n$, $p(1-t) + qt \in B_r(x)$ for $t \in [0,1]$ since $(p(1-t))^2 + (qt)^2 \leq 2(1-t)tp \cdot q \leq |p||q| \leq r^2$. □

Let's fix $p \in U$ which is open connected subset of $\mathbb{R}^n$. Let P be a path connected component in U containing p. This is not empty since open ball in U containing p is path connected. Let $P \neq U$. First, P is open since for any $q \in P$, the ball containing q is path connected, and joining path from p to q and from q to a point in the ball makes path from p to the point in the ball. Since U is connected, P is not closed, so $\overline{P} \setminus P \neq \phi$. Let a point in the set x. Take a ball in U containing x, then it intersect with P. Take a point in the intersection, then there exists path from p to the point and from the point to x. Therefore, $x \in P$, which is contradiction. Therefore, $P = U$ and U is path connected.

Problem 14

A. Since determinant function is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

where S_n is symmetric group and $\operatorname{sgn}(\sigma)$ is the signature of σ. This is finite summation of polynomial, so continuous function and codomain is $\mathbb{R} \setminus \{0\}$. Since codomain is not continuous space, the domain is not continuous.

B. Fix $A \in GL^+(n)$. Since it is nonsignular, it can be diagonalized by $V \Lambda V^{-1}$ such that $\det V = 1$ and Λ is a diagonal matrix with eigenvalues. Consider a path $\gamma : [0,1] \to GL^+(n)$ such that

$$\gamma(t) = V \begin{bmatrix} \frac{\lambda_1}{(1-t)+|\lambda_1|t} & & \cdots & \\ & \frac{\lambda_2}{(1-t)+|\lambda_2|t} & \cdots & \\ \vdots & \vdots & \vdots & \vdots \\ & & \cdots & \frac{\lambda_n}{(1-t)+|\lambda_n|t} \end{bmatrix} V^{-1}.$$

Then, $\gamma(t)$ is continuous since $(1-t) + |\lambda_i|t > 0$ for $t \in [0,1]$. Also, $\gamma(0) = A$, $\gamma(1)$ is a diagonal matrix with ± 1. I'll show that this matrix is path connected with I and conclude that A is path connected to I in $GL^+(n)$.

Let $\gamma(1) = D$. Then, $\det D = 1$. Recognizing each column as a vector in $\mathbb{R}^n$, we can take rotation matrix R making $(-1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$.

I'll let R_n be the rotation matrix in $\mathbb{R}^n$.

We can make D to I using rotation matrix by the algorithm: assume $(1,1)$ element in D is -1, then using R_n, we can make the $(1,1)$ term $+1$ and let the matrix D_1. Assume 1 term appears in (k,k) term, then let $D_k = D_{k-1}$ and if the term is -1, we can take

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & R_{n-k+1} \\ & & \end{bmatrix}$$

to D_{k-1} and make (k,k) term $+1$. Let it D_k. Proceeding this algorithm, we can finally get I since the rotation matrix is determinant 1. Finally the multiplication of all rotation matrix used in this procedure is the path from D to I.(The domain would be $[0,\pi]$.) Therefore, A and I is path connected.

For any $A, B \in GL^+(n)$, we can connect A, B by path through I:

$$\gamma = \begin{cases} \gamma_A(t) & \text{For } 0 \leq t \leq 1 \\ \gamma_B(2-t) & \text{For } 1 \leq t \leq 2 \end{cases}$$

which γ_A, γ_B is a path form A, B to I. Since $\gamma_A(1) = \gamma_B(1)$, by pasting lemma, γ is continuous path from A to B.

C. We know that $GL(n) = GL^-(n) \cup GL^+(n)$. Using the same argument above, we can show that any matrix in $GL^-(n)$ is path connected to I'_n such that the only (n,n) term is -1, and prove that any $A, B \in GL^-(n)$ is path connected. Therefore, $GL(n)$ has two path connected component.