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1 Let $\xi = e^{\frac{2\pi i}{g}}$ and c be the generator of the cyclic group G, i.e. $|c| = g$. Define

$$\rho^k : G \to \mathbb{C}^\times := \{c \in \mathbb{C} : |c| = 1\} \subset GL_1\mathbb{C} \tag{1}$$

by $\rho^k(c^h) = e^{\frac{2\pi i hk}{g}}$ for $k, h \in \{0, 1, \ldots, g-1\}$. For each k, it is a group homomorphism from G to $\mathbb{C}^\times$ since

$$\rho^k(c^{h_1}c^{h_2}) = e^{\frac{2\pi i h_1 k}{g}} e^{\frac{2\pi i h_2 k}{g}} = e^{\frac{2\pi i(h_1+h_2)k}{g}} = \rho^k(c^{h_1+h_2 \mod g}), \tag{2}$$

therefore it is a representation for each k. Since it is degree 1, ρ^k are itself a characterstic function for each k, and irreducible. Finally, each characterstic are pairwisely non-isomorphic: for $k_1, k_2 \in \{0, \ldots, g-1\}$,

$$\begin{aligned} \langle \rho^{k_1}, \rho^{k_2} \rangle &= \frac{1}{g} \sum_{h=0}^{g-1} \exp\left(\frac{-2\pi i h k_1}{g}\right) \exp\left(\frac{2\pi i h k_2}{g}\right) \\ &= \frac{1}{g} \sum_{h=0}^{g-1} \exp\left(\frac{2\pi i (k_2 - k_1) h}{g}\right) \\ &= g^{-1} \frac{\exp(2\pi i (k_2 - k_1)) - 1}{\exp(2\pi i (k_2 - k_1)/g) - 1} = 0 \end{aligned} \tag{3}$$

if $k_1 \neq k_2$. The ρ^k are all the irreducible representations of G since the number of ρ^k is same as the degree of g, i.e. from the corollary 2 of proposition 5, $\sum_{i=0}^{g-1} 1 = g$.

2 Let $F = \{f : G \to \mathbb{C}\}$. This is a $\mathbb{C}$ vector space as it satisfies all the vector space axioms. Set $f_s \in F$

$$f_s(t) = \begin{cases} 1 & s = t \\ 0 & s \neq t. \end{cases} \tag{4}$$

Now, let's give a G action to F as following. For $f \in F$ and $s, t \in G$,

$$(s \cdot f)(t) = f(s^{-1}t), \tag{5}$$

then it is a well-defined group action on F since for $s_1, s_2 \in G$,

$$(s_1 s_2 \cdot f)(t) = f((s_1 s_2)^{-1} t) = f(s_2^{-1} s_1^{-1} t) = (s_1 \cdot (s_2 \cdot f))(t). \tag{6}$$

Also, using the basis f_t for $t \in G$, s acts on f_t transitively. It shows that ρ_s in the basis F_t is in $GL(F)$. Finally, consider a group ring $\mathbb{C}[G]$ by extending the action linearly on $\mathbb{C}$: for $u = \sum_{s \in G} u_s s$ for $c_s \in \mathbb{C}$,

$$u \cdot f = \sum_{s \in G} c_s (s \cdot f), \tag{7}$$

then it makes F have $\mathbb{C}[G]$ module structure, which generates the G-representation on F.

I'll show that this is equivalent to regular representation in representation sense. Let's construct a vector space homomorphism $\varphi : F \to \mathbb{C}[G]$. Set $\varphi(f_s) = s$. Since F forms a basis of the vector space, extending the domain $\mathbb{C}$ linearly defines the well-defined vector space homomorphism φ. Also, for any $u = \sum_{s\in G} u_s s \in \mathbb{C}[G]$, $\varphi : u_s f_s \mapsto u$, so it is surjection. Since both spaces have same dimension, it is an vector space isomorphism.

I'll show that it is an representation isomorphism: it is enough to show that for any $t \in G$, $\varphi(t \cdot f_s) = t \cdot \varphi(f_s)$ as we'll see at last, but

$$\varphi(t \cdot f_s) = \varphi(f_{ts}) = t \cdot s = t \cdot \varphi(f_s) \tag{8}$$

Finally, for $u = \sum_{t\in G} u_t t$ and $f = \sum_{s\in G} c_s f_s$,

$$\varphi(u \cdot f) = \sum_{t,s\in G} u_t c_s \varphi(t \cdot f_s) = \sum_{t,s\in G} u_t c_s \varphi(f_{ts}) = \left(\sum_{t\in G} u_t t\right) \cdot \left(\sum_{s\in G} \varphi(c_s f_s)\right) = u \cdot \varphi(f) \tag{9}$$

It shows that the two representations are isomorphic.

3 One is using conjugacy classes: let $\{c_i\}_{i=1}^h$ be the set of conjugacy classes of G. By the definition of the class function, for any class function f, it has same values on the same conjugacy classes. Choose a representatives for each conjugacy classes and write the set $R = \{r_1, \ldots, r_h\}$, and let's denote $e_i : G \to \mathbb{C}$ a function such that $e_i(s) = 1$ if $s \in c_i$ and 0 else. Now, we can write

$$f(s) = \sum_{i=1}^{h} f(r_i) e_i. \tag{10}$$

Now, the e_i is the basis of the class function.

Second is using irreducible characteristic functions. In the class, we already checked that the irreducible characteristic functions on G forms a orthonormal basis in the space of class functions under the inner product $(\phi_1, \phi_2) = g^{-1} \sum_{s\in G} \phi_1(s)\phi_2(s)^*$ for ϕ_1, ϕ_2 class functions.

4 Let's extend the space X and $X \times X$ to $\mathbb{C}[X]$ and $\mathbb{C}[X \times X]$ to apply linear algebra; the group action is well-defined in the extended space. Fix $s \in G$ and consider ρ_s. Since $|s| < \infty$, $(\rho_s)^{|s|} = I$, and the minimal polynomial of ρ_s should divide $x^{|s|} - 1$. Since our scalar is $\mathbb{C}$, $x^{|s|} - 1$ is a separable polynomial and completely split into degree one monic polynomials in $\mathbb{C}[x]$. Using standard theory of linear algebra, ρ_s is diagonalizable with eigenvalues root of unity. Since our vector space is $\mathbb{C}[X] = \{\sum_{x\in X} c_x x : c_x \in \mathbb{C}\}$, let $\{\xi_i\}_{i=1}^{|X|}$ be eigenvectors of ρ_s with corresponding eigenvalues λ_i accepting multiplicity, i.e. λ_i are not necessarily distinct. Note that $\{\xi_i\}_{i=1}^{|X|}$ spans $\mathbb{C}[X]$ and

$$\chi(s) = \sum_{i=1}^{|X|} \lambda_i. \tag{11}$$

For further analysis, I'll write $\xi_i = \sum_{x\in X} c_{ix} x$. Note that $s\xi_i = \sum_{x\in X} c_{ix} sx = \lambda_i \sum_{x\in X} c_{ix} x$

Now, consider $\sum_{x,x'\in X} c_{ix} c_{jx'} (x, x') \in \mathbb{C}[X \times X]$ for $1 \le i, j \le n$. For readability, I'll write the element (ξ_i, ξ_j). Now, we get

$$\begin{aligned}
s \cdot (\xi_i, \xi_j) &= \sum_{x,x'\in X} c_{ix} c_{jx'} (sx, sx') = \sum_{x'\in X} c_{jx'} \sum_{x\in X} c_{ix} (sx, sx') \\
&= \sum_{x'\in X} c_{jx'} \sum_{x\in X} \lambda_i c_{ix} (x, sx') = \lambda_i \sum_{x\in X} c_{ix} \sum_{x'\in X} c_{jx'} (x, sx') \\
&= \lambda_i \sum_{x\in X} c_{ix} \sum_{x'\in X} \lambda_j c_{jx'} (x, x') = \lambda_i \lambda_j \sum_{x,x'\in X} c_{ix} c_{jx'} (x, x') \\
&= \lambda_i \lambda_j (\xi_i, \xi_j).
\end{aligned} \tag{12}$$

It shows that (ξ_i, ξ_j) are the eigenvectors of ρ_s acting on $\mathbb{C}[X \times X]$. Furthermore, (ξ_i, ξ_j) spans $\mathbb{C}[X \times X]$: for any $(a, b) \in \mathbb{C}[X \times X]$, there exists c_i and d_j such that $\sum_i c_i\xi_i = a$ and $\sum_j d_j\xi_j = b$, and

$$\sum_{i,j} c_i d_j(\xi_i, \xi_j) = \sum_j d_j \sum_i c_i(\xi_i, \xi_j) = \sum_j d_j(a, \xi_j) = (a, b). \tag{13}$$

For linearly independence, the linearly independency of ξ_i and the same technique can be applied to show it. It shows that (ξ_i, ξ_j) is the complete eigenvector set of the representation of s on $X \times X$.

Computing the trace, we get

$$\sum_{i,j} \lambda_i\lambda_j = \sum_j \lambda_j \sum_i \lambda_i = \chi^2(s). \tag{14}$$

It ends the proof.

5 Let χ be a character of G such that it is zero except $s = 1$. Let $1_G(s) = 1$ for all s, then it is an irreducible character of unit representation. Therefore,

$$\langle \chi, 1_G \rangle = g^{-1} \sum_{s \in S} \chi(s^{-1})1_G(s) = g^{-1}\chi(1) \in \mathbb{Z}_{\geq 0}. \tag{15}$$

(By decomposing χ into the sum of irreducible characters of G, the above equation is just counting the same characters as 1_G.) Therefore $r_G(1) = g \mid \chi(1)$, and as $r_G(s) = 0$ except $s = 1$, χ is an integer multiple of r_G.

6 Let (ρ, V) be a irreducible representation of an abelian group G. Fix $s \in G$, then for any $t \in G$, $st = ts$, so

$$\rho_s\rho_t = \rho_{st} = \rho_{ts} = \rho_t\rho_s. \tag{16}$$

This is true for all $t \in G$, and ρ_s can be viewed as a linear map from V to V. Therefore, by Schur's lemma, ρ_s is a homothety. This is true for all $s \in G$ implying all the ρ^s are homothety. Let's write $\rho^s = \lambda_s I$.

Choose a non-zero element $v \in V$, then $\rho^s(v) = \lambda_s v$ for all s. It shows that $\mathrm{span}\{v\}$ forms a $\mathbb{C}[G]$ stable subspace of V. Since V is irreducible, it means that $V = \mathrm{span}\{v\}$ and $\dim_{\mathbb{C}} V = 1$. Therefore, the degree of ρ is 1.

7 I'll use two fact: 1. Let H be a subgroup of G. The regular representation r_G of G is induced by the regular representation r_H of H. 2. For representation θ_1, θ_2 of H, $\mathrm{Ind}_H^G(\theta_1 \oplus \theta_2) = \mathrm{Ind}_H^G \theta_1 \oplus \mathrm{Ind}_H^G \theta_2$.

Let's write $\{(\theta_i, W_i)\}_{i=1}^m$ be all the irreducible representations of H and $\{(\rho_j, V_j)\}_{j=1}^l$ be all the irreducible representations of G, then we know that

$$r_H \simeq \oplus_{i=1}^m \oplus_{j=1}^{n_i} \theta_i \tag{17}$$

where n_i is the degree of θ_i. Also, using fact 2, we get

$$\oplus_{i=1}^l \oplus_{j=1}^{\deg \rho_i} \rho_i \simeq r_G \simeq \mathrm{Ind}_H^G r_H \simeq \oplus_{i=1}^m \oplus_{j=1}^{n_i} \mathrm{Ind}_H^G \theta_i. \tag{18}$$

Now, decompose $\mathrm{Ind}_H^G \theta_i$ into irreducible representations in G. Since both side are isomorphic in representation sense, each irreducible representations in LHS should be corresponds to some irreducible components of some $\mathrm{Ind}_H^G \theta_i$. It proves the statement of the problem.

I'll show the fact I used: first, the regular representation $(r_H, \mathbb{C}[H])$ of H induces the regular representation $(r_G, \mathbb{C}[G])$ of G. To show this, it is enough to show that $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H]$ is $\mathbb{C}[G]$ module isomorphic to $\mathbb{C}[G]$. Let $\varphi : \mathbb{C}[G] \times \mathbb{C}[H] \to \mathbb{C}[G]$ by defining $(s, t) \mapsto st$ for $s \in G$ and $t \in H$, and extending to satisfy $\mathbb{C}$-linearlity. Since $H \leq G$, it is well-defined $\mathbb{C}[H]$-balanced map, so by the universal property, it can be extended to a group homomorphism $\Phi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \to \mathbb{C}[G]$. Furthermore, $\mathbb{C}[G]$ has $(\mathbb{C}[G], \mathbb{C}[H])$ bimodule structure, so $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H]$ has well-defined left $\mathbb{C}[G]$ action given by $s_1(s_2 \otimes t) = s_1 s_2 \otimes t$ for $s_1, s_2 \in \mathbb{C}[G]$ and $t \in \mathbb{C}[H]$. Finally, $\Phi(s \otimes 1) = s$, so it is surjective and if $\Phi(\sum_i s_i \otimes t_i) = \Phi(\sum_i s_i t_i \otimes 1) = \sum_i s_i t_i = 0$, then $\sum_i s_i t_i \otimes 1 = 0$, so it is injective. Also, it conserves the $\mathbb{C}[G]$ action. Therefore, it is $\mathbb{C}[G]$ module isomorphism, implying it is a group representation isomorphism of G.

Second one is just the elementary property of tensor product:

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} (W_1 \oplus W_2) \simeq (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_1) \oplus (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_2) \tag{19}$$

as left $\mathbb{C}[G]$ module since $\mathbb{C}[G]$ is $(\mathbb{C}[G], \mathbb{C}[H])$ bimodule and W_i are left $\mathbb{C}[H]$ module.

8 Since the eigenvalues of $\rho(s)$ have absolute value 1,

$$|\chi(s)| = \left| \sum_{i=1}^{n} \lambda_i \right| \leq \sum_{i=1}^{n} |\lambda_i| = n. \tag{20}$$

(λ_i are the eigenvalues of $\rho(s)$.) The equality holds if any only if $\lambda_i = \sigma_i \lambda$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\sigma_i = \pm 1$ with $\sigma_i \sigma_j > 0$ for all i, j; considering $\mathbb{C}$ by $\mathbb{R}^2$, $\lambda = \sum_i \lambda_i$ and $\lambda \cdot \lambda = \sum_i |\lambda_i|^2 + \sum_{i<j} \lambda_i \cdot \lambda_j$. To make $\lambda \cdot \lambda = n^2$, we need to impose $\lambda_i \cdot \lambda_j = 1$, which implies all the λ_i points same direction since all λ_i have absolute value 1. It shows that $\rho(s)$ is homothety if and only if $|\chi(s)| \leq n$ as the multiple of identity matrix have the same form under any similar transformation. If $\chi(s) = n$, then $\lambda_i = 1$ for all i and $\rho(s)$ is a homothety, which means that $\rho(s) = 1$.

9

(a) Let V' be a $\mathbb{C}[G]$-module and $f : W \to V$ be a $\mathbb{C}[H]$ module homomorphism. Let $i : W \to \operatorname{Ind}_H^G W$ by inclusion, i.e. if I use a representatives $R = \{\sigma_1, \ldots, \sigma_n\}$ of G/H and $\operatorname{Ind}_H^G W = \oplus_{i=1}^n \sigma_i W_i$ as in the textbook, $i(w) = (w, 0, 0, \ldots, 0)$ for $w \in W$. Note that i is $\mathbb{C}[H]$ homomorphism. The universal property of $\operatorname{Ind}_H^G W$ means that there exists a well-defined unique $\mathbb{C}[G]$-homomorphism satisfying the commutative diagram.

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[dr, "f"'] & \operatorname{Ind}_H^G W \arrow[d, dashed, "F"] \\ & V \end{tikzcd}$$

(b) The Frobenius reciprocity for two representations (ρ_1, W) of H and (ρ_2, V) of G can be given as following:

$$\operatorname{Hom}_H (W, \operatorname{Res}_H V) \simeq \operatorname{Hom}_G \left(\operatorname{Ind}_H^G W, V\right). \tag{21}$$

(I'll specify the sort of isomorphism.) (If we consider W as a $\mathbb{C}[H]$ module and V as a $\mathbb{C}[G]$ module, the information of the represenation on each vector space is already contained in the space.) The proof is simple if we use the universal property. Any $\mathbb{C}[H]$ module homomorphism from W to $\operatorname{Res}_H V$ considering V as a $\mathbb{C}[H]$ module is uniquely extends to $\mathbb{C}[G]$ module homomorphism $F : \operatorname{Ind}_H^G W \to V$. Conversely, any $\mathbb{C}[G]$ module homomorphism $F : \operatorname{Ind}_H^G W \to V$ can makes $f : W \to V$ by setting $f = F \circ i$. Finally, the uniqueness of the universal property guarantees that the extension of $F \circ i$ is again F. It shows that if I define $\Psi : \operatorname{Hom}_H (W, \operatorname{Res}_H V) \to \operatorname{Hom}_G \left(\operatorname{Ind}_H^G W, V\right)$ by $\Psi(f) = F$, then it is a bijective function. Furthermore, for any $f_1, f_2 \in \operatorname{Hom}_H (W, \operatorname{Res}_H V)$ and $c \in \mathbb{C}$, $\Psi(cf_1 + f_2) = c\Psi(f_1) + \Psi(f_2)$, which shows that Ψ can be considered as $\mathbb{C}$ vector space isomorphism.

10 Let's first configure what is H_s. For $s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $t = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ in $SL_2(k)$, sts^{-1} is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} a\alpha\delta - b\alpha\gamma - d\gamma\beta & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ a\gamma\delta - b\gamma^2 - d\gamma\delta & -a\beta\gamma + b\alpha\gamma + d\alpha\delta \end{pmatrix} \tag{22}$$

To make it in H, we need to impose $a\gamma\delta - b\gamma^2 - d\gamma\delta = 0$. Assume $s \notin H$, then $\gamma \neq 0$, so $\delta(a - d) - b\gamma = 0$. Therefore,

$$\begin{pmatrix} a\alpha\delta - b\alpha\gamma - d\gamma\beta & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ a\gamma\delta - b\gamma^2 - d\gamma\delta & -a\beta\gamma + b\alpha\gamma + d\alpha\delta \end{pmatrix} = \begin{pmatrix} d & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ 0 & a \end{pmatrix} \tag{23}$$

using $\alpha\delta - \gamma\beta = 1$. Also,

$$\gamma(-a\alpha\beta + b\alpha^2 + d\alpha\beta) = \alpha(-a\beta\gamma + \delta(a - d)\alpha + d\gamma\beta) = \alpha(a - d), \tag{24}$$

so

$$\begin{pmatrix} d & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ 0 & a \end{pmatrix} = \begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} \tag{25}$$

Therefore,

$$H_s = \left\{\begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} : ad = 1\right\}. \tag{26}$$

Therefore, for $s \notin H$,

$$\begin{aligned} \langle \rho^s, \operatorname{Res}_{H_s}(\rho) \rangle &= \frac{1}{|H_s|} \sum_{t \in H_s} \rho^s(t^{-1}) \left(\operatorname{Res}_{H_s} \rho\right)(t) \\ &= \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \chi_\omega^s \left(\begin{pmatrix} a & -\alpha\gamma^{-1}(a - d) \\ 0 & d \end{pmatrix}\right) \chi_\omega \left(\begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix}\right) \\ &= \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \chi_\omega \left(\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}\right) \chi_\omega \left(\begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix}\right) = \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \omega^2(d) \end{aligned} \tag{27}$$

Assume $\omega^2 \neq 1$. Since k is finite field, k^* is a cyclic group about an element $x \in k$. Therefore, $\omega^2(x) \neq 1$ is a root of unity such that the order of $\omega^2(x)$ divides $|k| - 1$. It shows that

$$\sum_{d \in k \setminus \{0\}} \omega^2(d) = \sum_{i=1}^{|k|-1} \left(\omega^2(x)\right)^i = 0. \tag{28}$$

Now, the induced representation satisfies all the conditions in proposition 23, so it is irreducible. (Since χ_ω is the character of degree 1, the condition (a) is automatically satisfied.)