HW7

박성빈, 수학과, 20202120

I'll first show a (trivial) proposition.

Proposition 1. *Let $(\rho_1, V_1), (\rho_2, V_2)$ are representations on a group G. If ρ_2 has degree 1, $\rho_1 \otimes \rho_2 \simeq \rho_2\rho_1$, which act on V_1 by $\rho_2(s)\,(\rho_1(s))\,(v)$ identifying $\rho_2(s) \in \mathbb{C}^\times$.*

Proof. Since $\dim V_2 = 1$, $V_2 \simeq \mathbb{C}$. Identifying V_2 with $\mathbb{C}$, for each $s \in G$, $\rho_2(s)$ acts on $\mathbb{C}$ by scalar multiplication, so in $\mathbb{C}^\times$. Let $\varphi : V_1 \otimes_{\mathbb{C}} V_2 \to V_1$ by $(v_1, c_2) \mapsto c_2 v_1$. This is definitely vector space isomorphism. Furthermore, it is representation isomorphism since

$$\begin{aligned}
\varphi\left((\rho_1 \otimes \rho_2(s))\,((v_1, c_2))\right) &= \varphi\left((\rho_1(s))\,(v_1), (\rho_2(s))\,(c_2)\right) \\
&= (\rho_2(s))\,(c_2)\,(\rho_1(s))\,(v_1) \\
&= (\rho_2(s))\,(1)\,(\rho_1(s))\,(c_2 v_1) \\
&= (\rho_2\rho_1(s)) \circ \varphi(v_1, c_2).
\end{aligned} \tag{1}$$

for any $s \in G$, $v_1 \in V_1$, and $c_2 \in \mathbb{C}$. □

Remark 2. If ρ_2 is trivial, then $\rho_1 \otimes \rho_2 \simeq \rho_1$.

Remark 3. Since $\rho_1 \otimes \rho_2 \simeq \rho_2 \otimes \rho_1$, we get $\rho_1 \otimes \rho_2 \simeq \rho_2$ if ρ_1 is trivial.

1(**S** 8.2)

D_n The textbook denotes the dihedral group with order $2n$ by $D_n = \{r^k s^\sigma; \sigma \in \{0,1\}, 0 \le k \le n-1\}$. For the generator $s, r \in D_n$, let's write $A = \{r^k : 0 \le k < n\}$, $H = \{1, s\}$. Note that both subgroups are abelian and normal to D_n. Since $A \cap H = \{1\}$ and $|A||H| = 2n = |D_n|$, we get $D_n = AH$ as AH forms a subgroup in D_n and it has same cardinal as D_n. Finally, it has inner semi-direct product structure: for any $a_1, a_2 \in A$ and $h_1, h_2 \in H$,

$$a_1 h_1 a_2 h_2 = (a_1 h_1 a_2 h_1^{-1})(h_1 h_2). \tag{2}$$

Note that for $0 \le m \le n-1$, $\chi_m(r^k) = \exp\left(\frac{2\pi i m k}{n}\right)$ are well-defined irreducible group representations from A to $\mathbb{C}^\times$. Each χ_i are distinct since

$$(\chi_i, \chi_j) = \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\frac{2\pi(i-j)mk}{n}\right) = \delta_{ij}. \tag{3}$$

Also, note that χ_i forms a group X under multiplication since $\chi_i\chi_j = \chi_{i+j \mod n}$. The conjugation action of s to r^k is $s^{-1} r^k s = r^{-k}$, so $s\chi_i = \chi_{n-i}$.

If n is even, $\chi_{0 \le i \le n/2}$ is a system of representatives for the orbits of H in X with $\chi_0 \equiv 1$ and $s\chi_{n/2} = \chi_{n/2}$. For $1 \le i < n/2$, $1 \in H$ only fixes χ_i. Let' set $H_i = 1$, then any irreducible representatinos of H_i is trivial and we just need to consider $\mathrm{Ind}_A^G \chi_i \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[A]} \mathbb{C}$. Under basis $1 \otimes 1$ and $s \otimes 1$,

$$\begin{aligned}
\mathrm{Ind}_A^G \chi_m(s) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\mathrm{Ind}_A^G \chi_m(r) &= \begin{pmatrix} \exp\left(\frac{2\pi i m}{n}\right) & 0 \\ 1 & \exp\left(-\frac{2\pi i m}{n}\right) \end{pmatrix}
\end{aligned} \tag{4}$$

for $1 \leq m < n/2$.

For $m = 1$, χ_0 is trivial, so any element in H fixes χ_0. Therefore, extend χ_0 to $G = AH$ by setting $\chi_0(r^k s^\sigma) = \chi_0(r^k)$. For the trivial representation ρ of H, let $\tilde{\rho}(r^k s^\sigma) = \rho(s^\sigma)$, which becomes a representation on $G = AH$. Now, $\chi_0 \otimes \tilde{\rho}$ is a trivial representation on G. For another representation ρ' on H which sends s to -1, we can repeat the above argument and get a representation $\chi_0 \otimes \tilde{\rho'} \simeq \tilde{\rho'}$ on G since χ_0 is trivial.

For $m = n/2$, repeat the above argument and get

$$\begin{aligned} (\chi_{n/2} \otimes \tilde{\rho})(r^k s^\sigma) &= (-1)^k \\ (\chi_{n/2} \otimes \tilde{\rho'})(r^k s^\sigma) &= (-1)^k(-1)^\sigma. \end{aligned} \tag{5}$$

Finally, apply proposition 25 to say that we found all the irreducible representations of G.

For odd n, we can repeat the above argument except considering $n/2$ case: $\mathrm{Ind}_A^G \chi_m$ for $1 \leq m \leq \frac{n-1}{2}$, $\chi_0 \otimes \tilde{\rho}$, and $\chi_0 \otimes \tilde{\rho'}$.

$\mathfrak{A}_4$ Let's consider $\mathfrak{A}_4$ as the group of even permutation of a set $\{1, 2, 3, 4\}$. Set $t = (1\ 2\ 3)$. For $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ and $H = \{1, t, t^2\}$, I'll show that $\mathfrak{A}_4 \simeq A \rtimes H$ which is again given by inner semidirect product structure. Note that A is a normal subgroup of $\mathfrak{A}_4$ since conjugation action just permutes the numbers in the orbits, for example, for $\sigma \in \mathfrak{A}_4$,

$$\sigma(1\ 2)(3\ 4)\sigma^{-1} = (\sigma(1)\ \sigma(2))(\sigma(3)\ \sigma(4)). \tag{6}$$

It shows that AH forms a subgroup of $\mathfrak{A}_4$. Since $A \cap H = \{1\}$ and $|A||H| = 12 = |\mathfrak{A}_4|$, $\mathfrak{A}_4 = AH$. Finally, it satisfies the inner semidirect product structure following the argument in D_n.

Let's construct group representation of A by following:

	1	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Note that $(\chi_i, \chi_j) = \delta_{ij}$, so each are irreducible and distinct to each other. It forms a group X about multiplication: χ_1 is identity, $\chi_2\chi_3 = \chi_3\chi_2 = \chi_4$, $\chi_3\chi_4 = \chi_4\chi_3 = \chi_2$, $\chi_4\chi_2 = \chi_2\chi_4 = \chi_3$, and $\chi_{2\leq i\leq 4}^2 = \chi_1$. For conjugation about t by $t^{-1}st$ for $s \in A$,

$$\begin{aligned} 1 &\mapsto 1 \\ (1\ 2)(3\ 4) &\mapsto (1\ 3)(2\ 4) \\ (1\ 3)(2\ 4) &\mapsto (1\ 4)(2\ 3) \\ (1\ 4)(2\ 3) &\mapsto (1\ 2)(3\ 4). \end{aligned} \tag{7}$$

It shows that $t\chi_4 = \chi_3$, $t\chi_3 = \chi_2$, and $t\chi_2 = \chi_4$ where the action is given by $(t\chi)(s) = \chi(t^{-1}st)$. It shows that χ_1, χ_2 are a system of representatives for the orbits of H in X. For χ_1, all $h \in H$ satisfies $h\chi_1 = \chi_1$ and for χ_2, only 1 makes $1\chi_2 = \chi_2$.

For χ_1, consider irreducible representations $\rho_m : H \to \mathbb{C}^\times$ such that $\rho_m(t) = \exp\left(\frac{2\pi i m}{3}\right)$ for $0 \le m \le 2$; since H is cyclic group, ρ_m are all the irreducible representations of H. For $G = AH$, extend χ_i to G by $\chi_i(ah) = 1$, and let $\tilde{\rho}_m(ah) = \rho_m(h)$. Since $\chi_1 \otimes \tilde{\rho}_m$ is already a representation on G having degree 1, the induced representation is again $\chi_1 \otimes \tilde{\rho}_m$. Also, it is isomorphic to $\tilde{\rho}_m$ since χ_1 is trivial.

For χ_2, the fixing group $H_2 = 1$, so the irreducible representation is only trivial representation for H_2. Now, we take $\operatorname{Ind}_A^G \chi_2 \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[A]} \mathbb{C}$, which have matrix form under basis $1 \otimes 1$, $t \otimes 1$, and $t^2 \otimes 1$:

$$\begin{aligned}
\operatorname{Ind}_A^G \chi_2(1) &= 1 \\
\operatorname{Ind}_A^G \chi_2(t) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\operatorname{Ind}_A^G \chi_2((1\ 2)(3\ 4)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\operatorname{Ind}_A^G \chi_2((1\ 3)(2\ 4)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\operatorname{Ind}_A^G \chi_2((1\ 3)(2\ 4)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{8}$$

The above representations are all irreducible, distinct, and shows all the irreducible representations of $\mathfrak{A}_4$ by proposition 25.

$\mathfrak{S}_4$ Again set A as in $\mathfrak{A}_4$ and set $H = \{s \in \mathfrak{S}_4 : s \cdot 4 = 4\}$. By the same reason in $\mathfrak{A}_4$, we get $G = AH$ with inner semi-direct product structure since $|H| = 6$. Unfortunately, H is not an cyclic group now, but we know that H is group isomorphic to D_3, so we can use the first case. χ_1,χ_2 are a system of representatives for the orbits of H in X, and only $(1\ 2)$ fixes χ_2 in H.

Since χ_1 is trivial, any element in H fixes χ_1. For any irreducible representation ρ on H, we can extend it by $\tilde{\rho}$ to $G = AH$ as in $\mathfrak{A}_4$; note that $\chi_1 \otimes \tilde{\rho} \simeq \tilde{\rho}$. For the degree 2 irreducible representation ρ of H, we get

$$\begin{aligned}
\tilde{\rho}(a) &= 1 \\
\tilde{\rho}((1\ 2\ 3)) &= \begin{pmatrix} \exp\left(\frac{2\pi i}{3}\right) & 0 \\ 0 & \exp\left(-\frac{2\pi i}{3}\right) \end{pmatrix} \\
\tilde{\rho}((1\ 2)) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{9}$$

for $a \in A$. For non-trivial degree 1 representation ρ' of H,

$$\begin{aligned}
\tilde{\rho'}(a) &= 1 \\
\tilde{\rho'}((1\ 2\ 3)) &= 1 \\
\tilde{\rho'}((1\ 2)) &= -1
\end{aligned} \tag{10}$$

for $a \in A$. The left one is trivial representation on G.

As χ_2 is only fixed by (1 2) in H, $H_2 = \{1, (1\ 2)\}$ and we can extend χ_2 to AH_2. For trivial representation ρ on H_2, extend it to AH_2, which is again trivial, so the tensor of two representations is isomorphic ot χ_2. Finally, considering $\mathrm{Ind}_{AH_2}^{G} \chi_2 \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[AH_2]} \mathbb{C}$ with basis $1 \otimes 1$, $(1\ 3) \otimes 1$, and $(1\ 4) \otimes 1$,

$$\begin{aligned}
\mathrm{Ind}_{AH_2}^{G} \chi_2(1) &= 1 \\
\mathrm{Ind}_{AH_2}^{G} \chi_2((1\ 2\ 3)) &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} \chi_2((1\ 2)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} \chi_2((1\ 2)(3\ 4)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} \chi_2((1\ 3)(2\ 4)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} \chi_2((1\ 4)(2\ 3)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{11}$$

For the non-trivial irreducible representation ρ' of H_2, repeating above procedure, we get

$$\begin{aligned}
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,(1) &= 1 \\
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,((1\ 2\ 3)) &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,((1\ 2)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,((1\ 2)(3\ 4)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,((1\ 3)(2\ 4)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathrm{Ind}_{AH_2}^{G} (\chi_2 \otimes \tilde{\rho'})\,((1\ 4)(2\ 3)) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{12}$$

Applying proposition 25, we get all the irreducbile distinct representations of $\mathfrak{S}_4$.