HW#5

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I'll first write the definition of terminologies related to semisimplicty and simplicity.(Ref. *Algebra*, Lang.)

For a division ring R with 1, (as we only consider $R = K[G]$, where K is a field in this homework.)

Definition 1. R-module E is simple if it is non-zero and has no submodule other than 0 or E.

Note that this is same as irreducibile module.

Definition 2. R-module E is semisimple if for any submodule F of E, there exists a submodule F' such that $E = F \oplus F'$. For a ring R with $1 \neq 0$, it is semisimple if it is semisimple as a left module over itself.

1(**S** 6.1) I'll first prove (i)→(ii). Let $F = \{\sum_{s\in G} a_s s \in K[G] : \sum_{s\in G} a_s = 0\}$. It is $k[G]$ submodule of $K[G]$ since for any $s' \in G$,

$$s' \cdot \left(\sum_{s\in G} a_s s\right) = \sum_{s\in G} a_s s' s = \sum_{s\in G} a_{(s')^{-1}s} s \in F. \tag{1}$$

By the definition of semisimple module (or ring), there should exists a submodule F' such that $K[G] = F \oplus F'$. Since F, F' are K-vector space, it can be viewed as a decomposition of the vector space. For K-linear $\phi : K[G] \to K$ by $\phi(\sum_{s\in G} a_s s) = \sum_{s\in G} a_s$, it is surjective and F is the kernel, so $\dim F = |G| - 1$. It means $\dim F' = 1$. Now, assume it is spanned by $u = \sum_{s\in G} a_s s \in F'$, then $\sum_{s'\in G} s' u = \sum_{s\in G} \sum_{s'\in G} a_{s'} s \in F' \cap F$ as

$$\sum_{s\in G} \sum_{s'\in G} a_{s'} = g \sum_{s'\in G} a_{s'} = 0. \tag{2}$$

It means $\sum_{s\in G} \sum_{s'\in G} a_{s'} s = 0$ and $\sum_{s\in G} a_s = 0$, implying that $u \in F' \cap F$, so zero. This is impossible. Therefore, F is not a direct summand of $K[G]$ and $K[G]$ is not semisimple.

Conversely, assume char $K \nmid g$, then $\frac{1}{g}$ is non-zero in K, so $p^0 = g^{-1} \sum_{s\in G} sps^{-1}$ for K-linear projection from $K[G]$ to F is well-defined and the same argument in theorem 1 in 1.3 is well-applied. It shows that $K[G]$ is a semisimple.

2(**S** 6.2) By the definition of $\langle \cdot, \cdot \rangle$, it is bilinear. Also, by the construction of $\tilde{\rho}_i$ and linearlity of Tr_{W_i}, the formula for $\langle u, v \rangle$ is also bilinear. Therefore, I can reduce to the case $u, v \in G$. For $a, b \in G$,

$$\langle a, b \rangle = g \sum_{s\in G} \delta_{s^{-1}a} \delta_{sb} = g\delta_{ab}. \tag{3}$$

Also, by the corollary 5.2 in the chapter 2,

$$\langle a, b \rangle = \sum_{i=1}^{h} n_i \mathrm{Tr}_{W_i}(\rho_i(ab)) = \sum_{i=1}^{h} n_i \chi_i(ab) = g\delta_{ab}. \tag{4}$$

Therefore, we get

$$\langle u, v\rangle = \sum_{i=1}^{h} n_i \mathrm{Tr}_{W_i}(\tilde{\rho}_i(uv)) \tag{5}$$

3-6(**S** 6.3) Note: Since $\mathbb{C}[G]$ is $\mathbb{C}$-algebra isomorphic to product of matrix algebras over $\mathbb{C}$, it is not multiplicative group as some elements does not have inverse. Therefore $\mathbb{C}[G]$ is not itself a multiplicative group.

(a) Since U contains G, $s^{-1}u \in U$ for $s \in G$. As U is finite, $(s^{-1}u)^{|U|} = 1$, which implies that $(\tilde{\rho}_i(s^{-1}u))^{|U|} = \tilde{\rho}_i((s^{-1}u)^{|U|}) = I$ and the minimal polynomial of $\tilde{\rho}_i(s^{-1}u)$ should divide $x^{|U|} - 1 = 0$. Since $\mathbb{C}$ is algebraically closed field of characteristic 0 and $x^{|U|} - 1$ is separable, $\tilde{\rho}_i(s^{-1}u)$ is diagonalizable and eigenvalues are roots of unity.

Since $u'u = 1$, $\tilde{\rho}_i(u's)\tilde{\rho}_i(s^{-1}u) = I$. As an inverse matrix of a diagonalizable matrix, each eigenvalue of $\tilde{\rho}_i(u's)$ is inverse of an eigenvalue of $\tilde{\rho}_i(s^{-1}u)$, and we have shown that each eigenvalues have absolute value 1 above. Therefore,

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \mathrm{Tr}_{W_i}(\tilde{\rho}_i(s^{-1}u))^* = \mathrm{Tr}_{W_i}(\tilde{\rho}_i(u's)) = \mathrm{Tr}_{W_i}(u_i'\rho_i(s)) \tag{6}$$

As $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for matrices A and B, we get

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \mathrm{Tr}_{W_i}(\rho_i(s)u_i') \tag{7}$$

Using Fourier inversion formula, we get $u(s)^* = u'(s^{-1})$.

(b) Note that

$$uu' = \sum_{s\in G}\left(\sum_{s'\in G} u(ss')u'((s')^{-1})\right)s. \tag{8}$$

Since $uu' = 1$, $\sum_{s'\in G} u(s')u'((s')^{-1}) = \sum_{s'\in G} u(s')u(s')^* = \sum_{s\in G} |u(s)|^2 = 1$.

(c) By (b), we get $u(s)$ are all zero except one which is equal to ± 1. As it contains G and U is contained in $G \cup (-G)$.

(d) Let $u \in Z[G]$ has finite order about multiplication, then $U = \langle u, G\rangle$ is a finite subgroup of multiplicative group of $Z[G]$ as the generators are commutative and have finite order. By (c), U is contained in $G \cup (-G)$, so $u \in G \cup (-G)$. It proves the proposition.

7(**S** 6.4) Note that χ_i is a class function on G for each i, so for any conjugacy class $c \subset G$, $\chi_i(s_1^{-1}) = \chi_i(s_2^{-1})$ for $s_1, s_2 \in c$. Therefore, p_i is in the center of $\mathbb{C}[G]$. Also,

$$\omega_i(p_j) = g^{-1}\sum_{s\in G}\chi_j(s^{-1})\chi_i(s) = \delta_{ij} \tag{9}$$

from the theorem 3 in chapter 2. Since $(\omega_i)_{1\le i\le h}$ defines an algebra isomorphism from center of $\mathbb{C}[G]$ to $\mathbb{C}^h$, which is $\mathbb{C}$-vector space isomorphism, and each $(p_i)_{1\le i\le h}$ maps onto the basis of $\mathbb{C}^h$, p_i forms a basis of center of $\mathbb{C}[G]$.

The rest properties are the consequence of calculations. Since

$$\omega_i(p_jp_k) = \omega_i(p_j)\omega_i(p_k) = \delta_{ij}\delta_{ik}, \tag{10}$$

and $(\omega_i)_{1\le i\le h}$ is an isomorphism, $p_i^2 = p_i$ and $p_i p_j = 0$. Also, $\omega_i(1) = 1$ for all i, and

$$(\omega_i)_{1\le i\le h} : \sum_{j=1}^{h} p_j \mapsto (1, 1, \dots, 1), \tag{11}$$

so $\sum_{j=1}^{h} p_j = 1$.

Now, I'll prove the theroem 8 of 2.6. For a representation $\rho : G \to GL(V)$, $\rho(p_i)^2 = \rho(p_i)$, so $\rho(p_i)$ is a projection matrix. Also, $\operatorname{Im}\rho(p_i) \cap \operatorname{Im}\rho(p_j) = 0$ for $i \neq j$ for $p_i p_j = 0$. Finally, as $\sum_{i=1}^{h} p_i = 1$, $\oplus_{i=1}^{h} \operatorname{Im}\rho(p_i) = V$. Now, I need to show $\operatorname{Im}\rho(p_i) = V_i$, which is constructed by collecting irreducible submodules isomorphic to W_i.

For $j \neq i$, assume that there exists a irreducible submodule $v \in L \subset V$ which is isomorphic to W_j and $\rho(p_i)(v) \neq 0$. Since p_i is in center of $\mathbb{C}[G]$ and L is irreducible, we can restrict the domain of $\rho(p_i)$ by L and get $\operatorname{End}_{\mathbb{C}[G]}(L)$. By Schur's lemma, $\rho(p_i)$ is a homothety and $\operatorname{Tr}(\rho(p_i)) = \frac{n_i}{g} \sum_{s\in G} \chi_i(s^{-1})\chi_j(s) = 0$. Therefore, it is contradiction, and it implies $\sum_{j\neq i} V_j \subset \ker\rho(p_i)$. It shows that $\ker\rho(p_i) = V_i$, which ends the proof.

8(**S** 6.5) Let ϕ be the algebra homomorphism from center of $\mathbb{C}[G]$ to $\mathbb{C}$. Since $\sum_{i=1}^{h} p_i = 1 \in \mathbb{C}[G]$ is maps to 1 in $\mathbb{C}$, there should exists p_{i_0} such that p_{i_0} is maps to non-zero a. Assume it is not 1, then $\frac{1}{a} p_{i_0}$ maps to 1, so $\sum_{i=1}^{h} p_i - \frac{1}{a} p_{i_0}$ maps to 0. However,

$$\phi : \left(\sum_{i=1}^{h} p_i - \frac{1}{a} p_{i_0}\right) p_{i_0} = \left(1 - \frac{1}{a}\right) p_{i_0}^2 = \left(1 - \frac{1}{a}\right) p_{i_0} \mapsto a - 1 \neq 0, \tag{12}$$

which is contradiction. Therefore, $a = 1$. Since $\phi(p_{i_0} p_j) = \phi(p_{i_0})\phi(p_j) = 0$ for $j \neq i_0$, $\phi(p_j) = 0$ except i_0, and it should be same as ω_{i_0}. It shows that each homomorphism of center of $\mathbb{C}[G]$ is equal to one of the ω_i.

9(**S** 6.6) Let $\{c_i\}_{i=1}^{h}$ be the conjugacy classes of G. The center of $\mathbb{C}[G]$ is $\oplus_{i=1}^{h} \mathbb{C} e_i$ where $e_i = \sum_{s\in c_i} s$. Therefore, $\oplus_{i=1}^{h} \mathbb{Z} e_i$ is contained in the center of $\mathbb{Z}[G]$. Conversely, if u is in the center of $\mathbb{Z}[G]$, then $us = su$ for all $s \in G$ and $\mathbb{C}$ is in the center of $\mathbb{C}[G]$, so $uu' = u'u$ for all $u' \in \mathbb{C}[G]$. Therefore, $u \in \left(\oplus_{i=1}^{h} \mathbb{C} e_i\right) \cap \mathbb{Z}[G] = \oplus_{i=1}^{h} \mathbb{Z} e_i$.