HW#3

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Notation: For each problem, I'll follow the notations in the problem if there is no additional mention.

1(**D&F** 18.1.10) Assume there is a subgroup of $GL_2(\mathbb{R})$ which is isomorphic to Q_8, and i and j corresponds to matrices A and B. Since $A^4 - I = (A-I)(A+I)(A^2+I) = 0$, the minimal polynomial of A and B can be $x-1$, $x+1$, $(x-1)(x+1) = x^2-1$, or x^2+1 as it should divide x^4-1 in $\mathbb{R}[x]$. Since $A^2 \neq I$, the only possible case is $x^2+1=0$. Fix a basis for A making it rational canonical form. The corresponding rational canonical form is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then,

$$\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

If $b=0$ or $c=0$, then $a^2=-1$, which is impossible. Therefore, $a=-d$. Also,

$$\begin{aligned} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}, \end{aligned} \tag{3}$$

so $b=c$, which means that $a^2+b^2=-1$, which is impossible. Therefore, there is no subgroup of $GL_2(\mathbb{R})$ which is isomorphic to Q_8.

2(**S** 2.7) Let r be the regular representation on W with basis $\{e_s\}_{s\in G}$. Since G acts on $\{e_s\}$ transitively, by the previous homework, we know that W has only one unit representation 1, which is irreducible. Therefore, for any character with $\rho_s = 0$ for all $s \neq 1$, for $c \in \mathbb{C}$ with $cr_G = \rho$,

$$(\rho, 1) = c(r_G, 1) = c \in \mathbb{N}. \tag{4}$$

3-5(**S** 2.8)

(a) Decomposing V into irreducible subspace, let $m_i = \dim V_i / \dim W_i$. Since we are only interested in calculating the dimension of H_i, transform each irreducible subspaces in V_i is copy of W_i by taking isomorphism for each space. Let each copy W_{ij} for $1 \leq j \leq m_i$. Abusing notation, we can treat ρ as a irreducible representation on W_{ij} in each V_i.

Fix an non-zero element $w \in W_i$. Since W_i is irreducible, $\mathbb{C}G \cdot w = W_i$. Now, assume $h(w) \in W_{ij}$ for some j and non-zero; if $h(w) = 0$, then $h \equiv 0$ as $0 = h \circ \rho_s(w)$ for $s \in G$. Since $h \circ \rho_s(w) = \rho_s \circ h(w) \in W_{ij}$ and W_i is irreducible, $\operatorname{Im} h \subset W_{ij}$. Using Schur's lemma, identifying $W_{ij} = W_i$, we get $h = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{C}^\times$. If $h(w) \notin V_i \setminus \{0\}$, then again by Schur's lemma, $h \equiv 0$. This argument illustrates the possible functions in H_i. Finally, if I set $h = \lambda \cdot \mathrm{id} : W_i \to W_{ij}$, it satisfies $h \circ \rho_s = \rho_s \circ h$, so such function exists in H_i.

For any $h \in H_i$ we can decompose h into $P_1 \circ h, \ldots, P_{m_i} \circ h$ such that P_j is the projection onto W_{ij}. Each $P_j \circ h$ is a multiple of id by the above argument, note that $P_j \circ \rho = \rho \circ P_j$ for representation ρ on V_i since W_{ij} is G-stable. It shows that H_i is spanned by $\{h_{ij}\}_{j=1}^{m_i}$ such that $h_{ij} : W_i \to W_{ij}$ and is identity by identifying $W_i = W_{ij}$. (By retrieving using the isomorphism, we can find the actual function $h_{ij} : W_i \to V_{ij}$ where V_{ij} is the jth position of the decomposition of V_i.) It shows $\dim H_i = \dim V_i / \dim W_i$.

(b) Let's define $F' : H_i \times W_i \to V_i$ by

$$F' : (h_\alpha, w_\alpha) \mapsto h_\alpha(w_\alpha) \tag{5}$$

and extend it to satisfy $\mathbb{C}$-linearity. This is $\mathbb{C}$-bilinear, so by the universal property, we get the well-defined vector space homomorphism $F : H_i \otimes W_i \to V_i$ which factors through F'. By (a), we know that that F' is surjective, so F is surjective. By dimension analysis, it means F is vector space isomorphism.

(c) By tensor-hom adjuction, we get natural isomorphism

$$\mathrm{Hom}(H_i \otimes W_i, V_i) \simeq \mathrm{Hom}(H_i, \mathrm{Hom}(W_i, V_i)), \tag{6}$$

which shows that the F maps each $(h_1, \ldots, h_k)$ to a linear map $h : \oplus_{j=1}^{m_i} W_i \mapsto V_i$ by

$$h : (w_{i1}, \ldots, w_{im_i}) \mapsto \sum_{j=1}^{m_i} h_j(w_{ij}). \tag{7}$$

I'll show that h is surjective, then by dimension analysis, it is vector space isomorphism. First, consider the basis $\{e_1, \ldots e_{m_i}\} \in H_i$ I set in (a), which is isomorphic from W_i to jth irreducible component of V_i in representation sense. In this setting, it is easy to see that h is surjective. Now, let $\{h_1, \ldots, h_{m_i}\} \in H_i$ be an aribtrary basis of H_i, and let

$$h_\alpha = \sum_{\beta=1}^{m_i} a_{\alpha\beta} e_\beta. \tag{8}$$

Let's denote $A = (a_{\alpha\beta}) \in GL_{m_i}(\mathbb{C})$ and $A^{-1} = (b_{\alpha'\beta'})$. For $\sum_j v_j \in V_i$, there exists $(w_{i1}, \ldots, w_{im_i}) \in \oplus_{j=1}^{m_i} W_i$ such that $e_j(w_{ij}) = v_j$ and $e_j(w_{ik}) = 0$ if $j \neq k$. Finally, for

$$w'_{i\alpha'} = \sum_{\beta'=1}^{m_i} b_{\beta'\alpha'} w_{i\beta'} \tag{9}$$

we get

$$\sum_{\alpha=1}^{m_i} h_\alpha(w'_{i\alpha}) = \sum_{\alpha=1}^{m_i} \sum_{\beta=1}^{m_i} \sum_{\beta'=1}^{m_i} a_{\alpha\beta} e_\beta(b_{\beta'\alpha} w_{i\beta'}) = \sum_{\beta=1}^{m_i} \sum_{\alpha=1}^{m_i} a_{\alpha\beta} b_{\beta\alpha} v_\beta = \sum_{\beta=1}^{m_i} v_\beta. \tag{10}$$

Finally, it is isomorphism of representations as we chose h_i to satisfy $\rho \circ h_i = h_i \circ \rho$.

6(**S** 3.1) Let's decompose V into irreducible subspaces V_i with representation function ρ_i. For any $s_1, s_2 \in G$, we get

$$(\rho_i)_{s_1} \circ (\rho_i)_{s_2} = (\rho_i)_{s_2} \circ (\rho_i)_{s_1} \tag{11}$$

as G is abelian group. Using Schur's lemma, $(\rho_i)_s = \lambda_s \circ \mathrm{id}$ for some $\lambda \in \mathbb{C}^\times$ for all $s \in G$, but it means that V_i is not irreducible if $\dim V_i > 1$ since ρ can be decomposed into block matrices. Therefore, all the V_i are degree 1.

7-9(**S** 3.2)

(a) By the same argument in 3.1, ρ_s is a homothety for each $s \in C$. Since eigenvalues of ρ_s are absolute value 1, we get $|\chi(s)| = n$ for $s \in C$.

(b) Since $|G| \geq |C|$,

$$g = \sum_{s\in G} |\chi(s)|^2 \geq \sum_{s\in C} |\chi(s)|^2 = cn^2, \tag{12}$$

so $n^2 \leq g/c$.

(c) For each $s \in C$, we can write the scalar $\lambda = \exp(2\pi i q)$ for some $q \in [0, 2\pi) \cap \mathbb{Q}$. Let $q_0 = \min_{s\in C} q$. If $q_0 = a/b$ with $(a, b) = 1$, then by Fermat's little theorem, we get $1/b \leq a/b$, so q_0 is of form $1/n$ for some $n \in \mathbb{N}$ and corresponding group element s_0. Assume there exists s' which is not in $\{1, s, \ldots, s^{n-1}\}$ and the corresponding phase $q' = 1/b'$. If $b' \nmid n$, we can make phase $1/\mathrm{lcm}(b', n)$ taking combination of s and s', so assume $b' \mid n$. However, it also makes a contradiction since

$$s^{n/b'}(s')^{-1} = 1. \tag{13}$$

Therefore, C is a cyclic group.

10(**S** 3.3) Since G is abelian, any irreducible representation has degree 1. For irreducible representations ρ_1 and ρ_2 on V_1 and V_2, $\rho_1 \otimes \rho_2$ is again irreducible since $V_1 \otimes V_2$ also has degree 1. Therefore, for any irreducible character χ_1 and χ_2, $\chi_1\chi_2$ is also irreducible. As a function from G to $\mathbb{C}$, this operation satisfies associative, has identity element $\chi(s) = 1$ for all s, and inverse $\overline{\chi}$: for a representation $\rho : G \to \mathbb{C}^\times$, $\overline{\rho}$ is also a representation having character $\overline{\chi}$. Since the number of classes of G is g, the number of irreducible representations is g, so $\hat{G}$ is an abelian group of order g.

For fixed $x \in G$, let's define $\varphi_x : \hat{G} \to \mathbb{C}$ by $\varphi_x(\chi) = \chi(x)$. This is well-defined group homomorphism with image in $\mathbb{C}^\times$, so it is an element of the $\hat{\hat{G}}$. Let $h : G \to \hat{\hat{G}}$ by $h(x) = \varphi_x$. h is group homomorphism since $\varphi_{xy^{-1}}(\chi) = \chi(xy^{-1}) = \chi(x)\chi(y^{-1}) = \varphi_x\varphi_{y^{-1}}$. If $\varphi_x \equiv 1$, then it means $\chi(x) = 1$ for all irreducible character $\chi \in \hat{G}$. If $x \neq 1$, then $\sum_{i=1}^{g} \chi_i(1)^*\chi_i(x) = g \neq 0$ according to proposition 7 in chapter 2, which is contradiction. Therefore, $x = 1$, and it shows $\ker h = 0$. Since $\left|\hat{\hat{G}}\right| = g$, h is an isomorphism.