HW8

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1(**S** 9.1) Let's write

$$\varphi = \sum_{\chi_i:\text{irr. char}} c_i\chi_i \tag{1}$$

for $c_i \in \mathbb{C}$; this is possible since the set of irreducible representation of G forms a basis of the class function space. The first condition is translated as follows:

$$\sum_{s\in G} \varphi(s) = 0. \tag{2}$$

In the problem 6.7, we showed that $|\chi(s)| \leq \chi(1)$ for irreducible character χ. For each irreducible representation χ,

$$\begin{aligned}\operatorname{Re}(\langle\varphi,\chi\rangle) &= \sum_{s\in G}\varphi(s^{-1})\operatorname{Re}(\chi(s)) = \chi(1)\varphi(1) + \sum_{s\neq 1}\varphi(s^{-1})\chi(s)\\ &\geq \chi(1)\varphi(1) + \chi(1)\sum_{s\neq 1}\varphi(s^{-1}) = \chi(1)\sum_{s\in G}\varphi(s^{-1}) = 0\end{aligned} \tag{3}$$

as $\varphi(s) \leq 0$ for $s \neq 1$.

If $\varphi \in R(G)$, the above conditions say that $\varphi \in R^+(G)$, which is a character by the previous homework.

2(**S** 9.2) If χ is an irreducible representation, it satisfies the conditions in the problem, so I'll prove the reverse direction.

Let's write χ by

$$\sum_{\chi_i:\text{irr. char}} n_i\chi_i, \tag{4}$$

where $n_i \in \mathbb{Z}$ for each i. Note that

$$\langle\chi,\chi\rangle = \sum_i n_i^2. \tag{5}$$

Therefore, $\langle\chi,\chi\rangle = 1$ means that only one i satisfies $n_i = \pm 1$ and 0 otherwise. Let the i be i_0. Since $\chi_{i_0}(1) \geq 1$, $\chi(1) \geq 0$ means that $n_{i_0} = 1$. Therefore, χ is an irreducible representation.

3(**S** 9.5) For the subgroup $H = 1$, there are only one irreducible character 1_H. The induced character is r_G since $\operatorname{Ind}_H^G 1_H(1) = 12$ and 0 elsewhere. It is $3\psi + \chi_0 + \chi_1 + \chi_2 = (\psi + \chi_0 + \chi_1 + \chi_2) + 2\psi$.

For the subgroup generated by $H = \langle(1\ 2)(3\ 4)\rangle$, there are two irreducible characters: the trivial one 1_H and non-trivial one φ mapping $(1\ 2)(3\ 4)$ to -1. The centralizer of $(1\ 2)(3\ 4)$ in $\mathfrak{A}_4$ is $\{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Also, $\{(3\ 2\ 1), (1\ 2\ 4), (4\ 3\ 1), (2\ 3\ 4)\}$ maps it to $(1\ 3)(2\ 4)$. Therefore, the induced character of each one is

$$\begin{aligned}\operatorname{Ind}_H^G 1_H(s) &= \frac{1}{2}\sum_{\substack{t\in\mathfrak{A}_4\\ t^{-1}st\in H}} 1_H(t^{-1}st)\begin{cases}6 & s = 1\\ 2 & s = (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\\ 0 & o.w.\end{cases}\\ \operatorname{Ind}_H^G \varphi(s) &= \frac{1}{2}\sum_{\substack{t\in\mathfrak{A}_4\\ t^{-1}st\in H}} 1_H(t^{-1}st)\begin{cases}6 & s = 1\\ -2 & s = (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\\ 0 & o.w.\end{cases}\end{aligned} \tag{6}$$

The first one is $\chi_0+\chi_1+\chi_2+\psi$ and latter one is 2ψ. For the subgroups generated by (1 3)(2 4) and (1 4)(2 3), the same computation yields the same characteristic function. (Or it can be deduced from the fact that the groups is the conjugation of H by $t = (1\ 2\ 3)$ or t^2.) For precise computation, see exercise 9.6 (b).

Let $H = \langle(1\ 2\ 3)\rangle$. There are three irreducible characters: 1_H, $\varphi(t) = w = \exp\left(\frac{2\pi i}{3}\right)$, $\varphi'(t) = \exp\left(\frac{4\pi i}{3}\right)$. The centralizer of (1 2 3) is only $\{1, (1\ 2\ 3), (1\ 3\ 2)\}$ in $\mathfrak{A}_4$. The induced characters are

$$
\begin{aligned}
\operatorname{Ind}_H^G 1_H(s) &= \frac{1}{3}\sum_{\substack{t\in\mathfrak{A}_4\\ t^{-1}st\in H}} 1_H(t^{-1}st)\begin{cases}4 & s=1\\ 0 & s=(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\\ 1 & o.w.\end{cases}\\
\operatorname{Ind}_H^G \varphi(s) &= \frac{1}{3}\sum_{\substack{t\in\mathfrak{A}_4\\ t^{-1}st\in H}} 1_H(t^{-1}st)\begin{cases}4 & s=1\\ 0 & s=(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\\ w & s=(1\ 2\ 3),(1\ 2\ 4),(1\ 4\ 3),(4\ 2\ 3)\\ w^2 & o.w.\end{cases}\\
\operatorname{Ind}_H^G \varphi'(s) &= \frac{1}{3}\sum_{\substack{t\in\mathfrak{A}_4\\ t^{-1}st\in H}} 1_H(t^{-1}st)\begin{cases}4 & s=1\\ 0 & s=(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\\ w^2 & s=(1\ 2\ 3),(1\ 2\ 4),(1\ 4\ 3),(4\ 2\ 3)\\ w & o.w.\end{cases}
\end{aligned}
\tag{7}
$$

The first one is $\chi_0+\psi$, the second one is $\chi_1+\psi$, and the third one is $\chi_2+\psi$. Any element in $\mathfrak{A}_4 \setminus \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ can be made by conjugation of (1 2 3) or (1 3 2), so any cyclic subgroup generated by one element in the set have above induced character. It shows that image of $\oplus_{H\in X}R^+(H)$ under Ind is generated by the five characters.

Note that above characters all have even number at $s = 1$. Conversely, assume χ is a character of $\mathfrak{A}_4$ having $\chi(1) \equiv 0 \mod 2$. Since ψ, χ_i all have odd degree, χ is generated by

$$2\psi, 2\chi_1, 2\chi_2, 2\chi_3, \psi+\chi_0, \psi+\chi_1, \psi+\chi_2, \chi_0+\chi_1, \chi_0+\chi_2, \chi_1+\chi_2. \tag{8}$$

Since all the characters are generated by the five characters, we know that χ is generated by the five character and is in the image.

According to the above computation, we know that any non-zero characters induced from $R^+(H)$ where H is a cyclic subgroup have non-zero ψ part, so χ_0, χ_1, and χ_2 can not be generated by linear combination with positive rational coefficients of characters induced from cyclic subgroups.

4-6(**S** 9.6)

(a) For irreducible $\mathbb{C}[H']$ module V, by the universal property of the induced representation, there exists a unique $\mathbb{C}[G]$ module homomorphism Ψ such that the diagram commutes; i, i', i'', and $i^{(3)}$ are the inclusion map.

$$
\begin{tikzcd}
V \arrow[rr, "i"] \arrow[d, "i'"'] \arrow[drr, "i^{(3)}"] & & \mathbb{C}[G]\otimes_{\mathbb{C}[H']} V \arrow[d, dashed, "\Psi"] \\
\operatorname{Res}_{H'} \mathbb{C}[G]\otimes_{\mathbb{C}[H]} (\mathbb{C}[H]\otimes_{\mathbb{C}[H']} V) \arrow[rr, "i''"] & & \mathbb{C}[G]\otimes_{\mathbb{C}[H]} (\mathbb{C}[H]\otimes_{\mathbb{C}[H']} V)
\end{tikzcd}
$$

Ψ is surjective map since for any $g \otimes (h \otimes v) \in \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (\mathbb{C}[H] \otimes_{\mathbb{C}[H']} V)$, $\Psi(gh \otimes v) = gh \cdot \Psi(1 \otimes v) = gh \cdot (1 \otimes (1 \otimes v)) = g \otimes (h \otimes v)$. By the dimensional analysis, the Ψ is an bijective map, so it is $\mathbb{C}[G]$ module isomorphism. It shows that

$$\mathrm{Ind}_H^G \mathrm{Ind}_{H'}^H \chi' = \mathrm{Ind}_H^G \chi, \tag{9}$$

and $\mathrm{Ind}_{H'}^H \chi' - \chi \in N$.

(b) For $s, s' \in G$,

$$\begin{aligned} \mathrm{Ind}_{{}^sH}^G {}^s\chi(s') &= \frac{1}{\left|{}^sH\right|} \sum_{\substack{t \in G \\ t^{-1}s't \in {}^sH}} {}^s\chi(t^{-1}s't) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}s't \in sHs^{-1}}} \chi(s^{-1}t^{-1}s'ts) \\ &= \frac{1}{|H|} \sum_{\substack{t \in G \\ (ts)^{-1}s'(ts) \in H}} \chi((ts)^{-1}s'(ts)) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}s't \in H}} \chi(t^{-1}s't) \\ &= \mathrm{Ind}_H^G \chi(s'), \end{aligned} \tag{10}$$

we get $\chi - {}^s\chi \in N$.

(c) Let S is the collection of functions of type (a) and (b) and consider the submodule of $\oplus_{H \in X} \mathbb{Q} \otimes R(H)$ spanned by S. Let's rewrite the submodule by S. What I want to do is to show that $S = N$. To use theory of class function, let's extend the scalar to $\mathbb{C}$; if $\mathbb{C} \otimes S = \mathbb{C} \otimes N$ in $\oplus_{H \in X} \mathbb{C} \otimes R(H)$, then for a basis $\{s_\alpha\}$ of S, $1 \otimes s_\alpha$ forms a basis of $\mathbb{C} \otimes S$ and so $\mathbb{C} \otimes N$. It shows that s_α is a basis of N, and $S = N$ in $\oplus_{H \in X} \mathbb{Q} \otimes R(H)$. By (a) and (b), we know that $S \subset N$.

I'll show what the hint says: let A be the collection of $(f_H) \in \oplus_{H \in X} \mathbb{C} \otimes R(H)$ such that if $H' \subset H$, then $f_{H'} = \mathrm{Res}_{H'} f_H$ and $f_{{}^sH}(sts^{-1}) = f_H(t)$ for any $s \in G$. It is well-defined subspace in $\oplus_{H \in X} \mathbb{C} \otimes R(H)$. Also, it is not empty set since 0 is in the set.

To use Hilbert space's property, I'll first check that $\mathbb{C} \otimes R(H)$ is a Hilbert space, but we know that $(f, g) = \sum_{s \in H} f(s)\overline{g(s)}$ is a inner product with $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$ using the fact that the irreducible characters (χ_i) forms a basis of the class function and $(\chi_i, \chi_j) = \delta_{ij}$. Since product of Hilbert space is again Hilbert space with the sum of inner product,-which will be clear writing the proof- I can take orthogonal decomposition of $\oplus_{H \in X} \mathbb{C} \otimes R(H)$ about A by $A^\perp$; note that the orthogonal spaces are unique. By the same reason, we can consider $N^\perp$.

From now on, I'll use another bilinear form $\langle \cdot, \cdot \rangle$. The only difference from $(\cdot, \cdot)$ is that the second one take complex conjugate of the coefficient of characters in right side in the sum. To avoid it, I'll take the basis of each $\mathbb{Q} \otimes R(H)$ by the irreducible characters of H and only use them in the equation since any operation such as Ind_H^G, Res_H, and the operations ${}^sH$ are $\mathbb{C}$-linear and maps a $\mathbb{Z}$ linear combination of characters to a $\mathbb{Z}$ linear combination of characters. In other words, we can safely treat only basis element in the computation identifying $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$.

Now, assume $S \not\supset N$, then there exists non-zero $n = (n_H) \in N$ such that $n \perp S$. It means that for any $\mathrm{Ind}_{H'}^H \chi_{H'} = \chi_H$ where $H' \subset H$ and $\chi_{H'}$ and χ_H are class functions, for $\chi \in \oplus_{H \in X} \mathbb{C} \otimes R(H)$ with 0 except H' and H having $-\chi_{H'}$ and χ_H,

$$\langle n, \chi \rangle = \langle n_H, \chi_H \rangle - \langle n_{H'}, \chi_{H'} \rangle = \langle \mathrm{Res}_{H'} n_H - n_{H'}, \chi_{H'} \rangle = 0. \tag{11}$$

Also, for any $s \in G$ and $\chi = 0$ except χ_H at H, and $-\chi_{^sH}$ at ^sH,

$$\begin{aligned}\langle n, \chi\rangle &= \langle n_H, \chi_H\rangle - \langle n_{^sH}, \chi_{^sH}\rangle \\ &= \langle n_H, \chi_H\rangle - \frac{1}{|H|}\sum_{t\in H} n_{^sH}(sts^{-1})\chi_{^sH}(st^{-1}s^{-1}) \\ &= \langle n_H, \chi_H\rangle - \frac{1}{|H|}\sum_{t\in H} n_{^sH}(sts^{-1})\chi_H(t^{-1}).\end{aligned} \tag{12}$$

If we define $g(t) = n_{^sH}(sts^{-1})$ for $t \in H$, which is again class function in H, we get

$$\langle n, \chi\rangle = \langle n_H - g, \chi_H\rangle = 0 \tag{13}$$

If I choose irreducible representations at each RHS, then it means $\mathrm{Res}_{H'} n_H - n_{H'} = 0$ and $n_H - g = 0$. Checking the definition of A, the second one implies that $n_{^sH}(sts^{-1}) = n_H(t)$, which implies $n \in A$. Finally, if I show that $N^\perp = A$, then it means $n \in N \cap N^\perp = 0$, which ends the proof. Therefore, it is enough to show that $A = N^\perp$.

Let $A' = \{(\mathrm{Res}_H \varphi) \in \oplus_{H\in X} C(H) : \varphi \in C(G)\}$. I'll first show that $A' = A$. $A' \subset A$ is easy to see since $\mathrm{Res}_{H'} \varphi = \mathrm{Res}_{H'} \mathrm{Res}_H \varphi$ for $H' \subset H$, and $\mathrm{Res}_{^sH} \varphi = \mathrm{Res}_H \varphi$ by the definition of class function. Conversely, assume $(f_H) \in A$. Construct $\varphi \in C(G)$ as following: for any $t \in G$, there exists $t \in H \in X$ since $\cup_{H\in X} H = G$. Set $\varphi(t) = f_H(t)$. This is well-defined: assume there exists another $H' \in X$ with $t \in H'$, then $t \in H' \cap H$. Since $\mathrm{Res}_{H'\cap H} f_H(t) = f_{H'\cap H}(t) = \mathrm{Res}_{H'\cap H} f_{H'}(t)$, $f_H(t) = f_{H'}(t)$. (I interpreted that "passage to subgroups" means that $H, H' \in X$ implies $H \cap H' \in X$.)

Since A is a subspace, so we can decompose $\oplus_{H\in X} C(H) = A \oplus A^\perp$. For fixed $\varphi \in C(G)$ and $n \in N$, we get

$$\sum_{H\in X} \langle n_H, \mathrm{Res}_H \varphi\rangle = \sum_{H\in X} \langle \mathrm{Ind}_H^G n_H, \varphi\rangle = 0. \tag{14}$$

It shows that $A \leq N^\perp$.

Conversely, fix $(f_H) \in N^\perp$, then $(f_H) \in S^\perp$, and the above calculations (11), (12), and (13) show that $(f_H) \in A$. Therefore, $A = N^\perp$. It ends the proof.

7(**S** 9.7) Let $S = \{(H, \chi) : H \in X, \chi \in \mathbb{Q} \otimes R(H)\}$. Consier the free $\mathbb{Q}$-module $F(S)$, and i be the inclusion map from S to $F(S)$. A function $\varphi : S \to \oplus_{H\in X} \mathbb{Q} \otimes R(H)$ be a map of set defined as follows: $\varphi((H, \chi)) = \chi_H$. By the universal property of free module, there exists a well-defined $\mathbb{Q}$-module homomorphism Φ satisfying

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\varphi"'] & F(S) \arrow[d, "\Phi"] \\ & \oplus_{H\in X} \mathbb{Q} \otimes R(H) \end{tikzcd}$$

Let's check what is the kernel of Φ. It is clear that the relation (i) should be contained in the $\ker \Phi$. Taking quotient using relation (i), as $F(S)$ is $\mathbb{Q}$ vector space, the quotient is again $\mathbb{Q}$ vector space. Furthermore, it has dimension at most $\sum_{H\in X} \mathbb{Q} \otimes R(H)$: for fixed $H \in X$ and irreducible characters χ_i, (H, χ_i) spans $\{(H, \chi) : \chi \in \mathbb{Q} \otimes R(H)\}$ using the linear relation. However, we know that Φ is surjective map, which means that the dimension is same and Φ induces the $\mathbb{Q}$ module isomorphism between two module. Using ex. 9.6, finally, we get the relation (i), (ii), and (iii) for $F(S)$ makes it isomorphic to $\mathbb{Q} \otimes R(G)$ since the relation (ii) and (iii) for $\otimes_{H \in X} \mathbb{Q} \otimes R(H)$ makes it isomorphic to $\mathbb{Q} \otimes R(G)$.

8(**S** 9.8) I'll check the conditions in exercise 9.1. λ_A is real-valued function on A, and it is in $R(A)$ by proposition 28. Also, $\langle \varphi(a) r_A - \theta_A, 1_A \rangle = \frac{1}{a}(\varphi(a)a - \varphi(a)a) = 0$, so it is orthogonal to 1_A. Finally, $r_A(s) = 0$ for $s \neq 1$, so $\lambda_A(s) \leq 0$ for $s \neq 1$. Therefore, λ_A is a character of A which is orthogonal to unit character 1_A. Using proposition 27,

$$\sum_{A \subset G} \operatorname{Ind}_A^G(\lambda_A) = \sum_{A \subset G} \operatorname{Ind}_A^G(\varphi(a) r_A - \theta_A) = \sum_{A \subset G} \varphi(a) \operatorname{Ind}_A^G(r_A) - g \tag{15}$$

Since 1 is itself a conjugacy class, $\operatorname{Ind}_A^G(r_A)(1) = \frac{1}{a} ga = g$ and 0 if $s \neq 1$. Finally, $\sum_{A \subset G} \varphi(a) = g$ since any element in G uniquely corresponds to a generator of a cyclic group in G. Therefore, we get

$$\sum_{A \subset G} \operatorname{Ind}_A^G(\lambda_A) = g(r_G - 1). \tag{16}$$

9(**S** 10.1) For any $h = x^k p \in C \cdot P$, where $k \in \mathbb{Z}$, $hxh^{-1} = x^k p x p^{-1} x^{-k} = x$ since $xp = px$ in H having inner direct product structure. Therefore, $H \subset Z(x)$ and P should be contained in a Sylow p-subgroup of $Z(x)$ by the Sylow theorem. Choosing Sylow p-subgroup of $Z(x)$ containing H, we prove the statement.

10(**S** 10.2) If $|x| = p^k$ for $k \in \mathbb{Z}_{\geq 0}$, $(1 - x)^{p^k} = 1 - x^{p^k} = 0$, so $(1 - x)$ is nilpontent; cf. Frobenius endomorphism. Conversely, assume $1 - x$ is nilpotent, then there exists large enough $k \geq 1$ such that $(1 - x)^{p^k} = 0$. It means that $x^{p^k} = 1$, and $|x| \mid p^k$, which implies x is p-element. If x is p'-element, then $x^N = 1$ for $(N, p) = 1$. The minimum polynomial should divide $q(x) = x^N - 1$, which is separable as $(q'(x), q(x)) = (Nx^{N-1}, x^N - 1) = (Nx^{N-1}, -1) = 1$. It shows that the minimal polynomial is separable and x is diagonalizable in a finite extension of k. Conversely, assume x is diagonalizable in some finite extension of k having order p^k. By little Fermat's theorem, any non-zero element in the field have order dividing $p^k - 1$, which is coprime to p. Therefore, writing

$$x = V \Lambda V^{-1}, \tag{17}$$

where $V \in GL_n(k)$ and Λ is a diagonal matrix having eigenvalues in the diagonal part, we get the order of Λ coprime to p. Therefore, x have order coprime to p, and it is an p'-element.