

# MID

박성빈, 수학과, 20202120

**1** Let  $\xi = e^{\frac{2\pi i}{g}}$  and  $c$  be the generator of the cyclic group  $G$ , i.e.  $|c| = g$ . Define

$$\rho^k : G \rightarrow \mathbb{C}^\times := \{c \in \mathbb{C} : |c| = 1\} \subset GL_1\mathbb{C} \quad (1)$$

by  $\rho^k(c^h) = e^{\frac{2\pi i h k}{g}}$  for  $k, h \in \{0, 1, \dots, g-1\}$ . For each  $k$ , it is a group homomorphism from  $G$  to  $\mathbb{C}^\times$  since

$$\rho^k(c^{h_1} c^{h_2}) = e^{\frac{2\pi i h_1 k}{g}} e^{\frac{2\pi i h_2 k}{g}} = e^{\frac{2\pi i (h_1 + h_2) k}{g}} = \rho^k(c^{h_1 + h_2 \pmod{g}}), \quad (2)$$

therefore it is a representation for each  $k$ . Since it is degree 1,  $\rho^k$  are itself a characterstic function for each  $k$ , and irreducible. Finally, each characterstic are pairwise non-isomorphic: for  $k_1, k_2 \in \{0, \dots, g-1\}$ ,

$$\begin{aligned} \langle \rho^{k_1}, \rho^{k_2} \rangle &= \frac{1}{g} \sum_{h=0}^{g-1} \exp\left(\frac{-2\pi i h k_1}{g}\right) \exp\left(\frac{2\pi i h k_2}{g}\right) \\ &= \frac{1}{g} \sum_{h=0}^{g-1} \exp\left(\frac{2\pi i (k_2 - k_1) h}{g}\right) \\ &= g^{-1} \frac{\exp(2\pi i (k_2 - k_1)) - 1}{\exp(2\pi i (k_2 - k_1)/g) - 1} = 0 \end{aligned} \quad (3)$$

if  $k_1 \neq k_2$ . The  $\rho^k$  are all the irreducible representations of  $G$  since the number of  $\rho^k$  is same as the degree of  $g$ , i.e. from the corollary 2 of proposition 5,  $\sum_{i=0}^{g-1} 1 = g$ .

**2** Let  $F = \{f : G \rightarrow \mathbb{C}\}$ . This is a  $\mathbb{C}$  vector space as it satisfies all the vector space axioms. Set  $f_s \in F$

$$f_s(t) = \begin{cases} 1 & s = t \\ 0 & s \neq t. \end{cases} \quad (4)$$

Now, let's give a  $G$  action to  $F$  as following. For  $f \in F$  and  $s, t \in G$ ,

$$(s \cdot f)(t) = f(s^{-1}t), \quad (5)$$

then it is a well-defined group action on  $F$  since for  $s_1, s_2 \in G$ ,

$$(s_1 s_2 \cdot f)(t) = f((s_1 s_2)^{-1}t) = f(s_2^{-1} s_1^{-1}t) = (s_1 \cdot (s_2 \cdot f))(t). \quad (6)$$

Also, using the basis  $f_t$  for  $t \in G$ ,  $s$  acts on  $f_t$  transitively. It shows that  $\rho_s$  in the basis  $F_t$  is in  $GL(F)$ . Finally, consider a group ring  $\mathbb{C}[G]$  by extending the action linearly on  $\mathbb{C}$ : for  $u = \sum_{s \in G} u_s s$  for  $c_s \in \mathbb{C}$ ,

$$u \cdot f = \sum_{s \in G} c_s (s \cdot f), \quad (7)$$

then it makes  $F$  have  $\mathbb{C}[G]$  module structure, which generates the  $G$ -representation on  $F$ .

I'll show that this is equivalent to regular representation in representation sense. Let's construct a vector space homomorphism  $\varphi : F \rightarrow \mathbb{C}[G]$ . Set  $\varphi(f_s) = s$ . Since  $F$  forms a basis of the

vector space, extending the domain  $\mathbb{C}$  linearly defines the well-defined vector space homomorphism  $\varphi$ . Also, for any  $u = \sum_{s \in G} u_s f_s \in \mathbb{C}[G]$ ,  $\varphi : u_s f_s \mapsto u$ , so it is surjection. Since both spaces have same dimension, it is an vector space isomorphism.

I'll show that it is an representation isomorphism: it is enough to show that for any  $t \in G$ ,  $\varphi(t \cdot f_s) = t \cdot \varphi(f_s)$  as we'll see at last, but

$$\varphi(t \cdot f_s) = \varphi(f_{ts}) = t \cdot s = t \cdot \varphi(f_s) \quad (8)$$

Finally, for  $u = \sum_{t \in G} u_t t$  and  $f = \sum_{s \in G} c_s f_s$ ,

$$\varphi(u \cdot f) = \sum_{t,s \in G} u_t c_s \varphi(t \cdot f_s) = \sum_{t,s \in G} u_t c_s \varphi(f_{ts}) = \left( \sum_{t \in G} u_t t \right) \cdot \left( \sum_{s \in G} \varphi(c_s f_s) \right) = u \cdot \varphi(f) \quad (9)$$

It shows that the two representations are isomorphic.

**3** One is using conjugacy classes: let  $\{c_i\}_{i=1}^h$  be the set of conjugacy classes of  $G$ . By the definition of the class function, for any class function  $f$ , it has same values on the same conjugacy classes. Choose a representatives for each conjugacy classes and write the set  $R = \{r_1, \dots, r_h\}$ , and let's denote  $e_i : G \rightarrow \mathbb{C}$  a function such that  $e_i(s) = 1$  if  $s \in c_i$  and 0 else. Now, we can write

$$f(s) = \sum_{i=1}^h f(r_i) e_i. \quad (10)$$

Now, the  $e_i$  is the basis of the class function.

Second is using irreducible characteristic functions. In the class, we already checked that the irreducible characteristic functions on  $G$  forms a orthonormal basis in the space of class functions under the inner product  $(\phi_1, \phi_2) = g^{-1} \sum_{s \in G} \phi_1(s) \phi_2(s)^*$  for  $\phi_1, \phi_2$  class functions.

**4** Let's extend the space  $X$  and  $X \times X$  to  $\mathbb{C}[X]$  and  $\mathbb{C}[X \times X]$  to apply linear algebra; the group action is well-defined in the extended space. Fix  $s \in G$  and consider  $\rho_s$ . Since  $|s| < \infty$ ,  $(\rho_s)^{|s|} = I$ , and the minimal polynomial of  $\rho_s$  should divide  $x^{|s|} - 1$ . Since our scalar is  $\mathbb{C}$ ,  $x^{|s|} - 1$  is a separable polynomial and completely split into degree one monic polynomials in  $\mathbb{C}[x]$ . Using standard theory of linear algebra,  $\rho_s$  is diagonalizable with eigenvalues root of unity. Since our vector space is  $\mathbb{C}[X] = \{\sum_{x \in X} c_x x : c_x \in \mathbb{C}\}$ , let  $\{\xi_i\}_{i=1}^{|X|}$  be eigenvectors of  $\rho_s$  with corresponding eigenvalues  $\lambda_i$  accepting multiplicity, i.e.  $\lambda_i$  are not necessarily distinct. Note that  $\{\xi_i\}_{i=1}^{|X|}$  spans  $\mathbb{C}[X]$  and

$$\chi(s) = \sum_{i=1}^{|X|} \lambda_i. \quad (11)$$

For further analysis, I'll write  $\xi_i = \sum_{x \in X} c_{ix} x$ . Note that  $s \xi_i = \sum_{x \in X} c_{ix} s x = \lambda_i \sum_{x \in X} c_{ix} x$

Now, consider  $\sum_{x, x' \in X} c_{ix} c_{jx'}(x, x') \in \mathbb{C}[X \times X]$  for  $1 \leq i, j \leq n$ . For readability, I'll write the element  $(\xi_i, \xi_j)$ . Now, we get

$$\begin{aligned} s \cdot (\xi_i, \xi_j) &= \sum_{x, x' \in X} c_{ix} c_{jx'}(sx, sx') = \sum_{x' \in X} c_{jx'} \sum_{x \in X} c_{ix}(sx, sx') \\ &= \sum_{x' \in X} c_{jx'} \sum_{x \in X} \lambda_i c_{ix}(x, sx') = \lambda_i \sum_{x \in X} c_{ix} \sum_{x' \in X} c_{jx'}(x, sx') \\ &= \lambda_i \sum_{x \in X} c_{ix} \sum_{x' \in X} \lambda_j c_{jx'}(x, x') = \lambda_i \lambda_j \sum_{x, x' \in X} c_{ix} c_{jx'}(x, x') \\ &= \lambda_i \lambda_j (\xi_i, \xi_j). \end{aligned} \quad (12)$$

It shows that  $(\xi_i, \xi_j)$  are the eigenvectors of  $\rho_s$  acting on  $\mathbb{C}[X \times X]$ . Furthermore,  $(\xi_i, \xi_j)$  spans  $\mathbb{C}[X \times X]$ : for any  $(a, b) \in \mathbb{C}[X \times X]$ , there exists  $c_i$  and  $d_j$  such that  $\sum_i c_i \xi_i = a$  and  $\sum_j d_j \xi_j = b$ , and

$$\sum_{i,j} c_i d_j (\xi_i, \xi_j) = \sum_j d_j \sum_i c_i (\xi_i, \xi_j) = \sum_j d_j (a, \xi_j) = (a, b). \quad (13)$$

For linearly independence, the linearly independency of  $\xi_i$  and the same technique can be applied to show it. It shows that  $(\xi_i, \xi_j)$  is the complete eigenvector set of the representation of  $s$  on  $X \times X$ .

Computing the trace, we get

$$\sum_{i,j} \lambda_i \lambda_j = \sum_j \lambda_j \sum_i \lambda_i = \chi^2(s). \quad (14)$$

It ends the proof.

**5** Let  $\chi$  be a character of  $G$  such that it is zero except  $s = 1$ . Let  $1_G(s) = 1$  for all  $s$ , then it is an irreducible character of unit representation. Therefore,

$$\langle \chi, 1_G \rangle = g^{-1} \sum_{s \in G} \chi(s^{-1}) 1_G(s) = g^{-1} \chi(1) \in \mathbb{Z}_{\geq 0}. \quad (15)$$

(By decomposing  $\chi$  into the sum of irreducible characters of  $G$ , the above equation is just counting the same characters as  $1_G$ .) Therefore  $r_G(1) = g \mid \chi(1)$ , and as  $r_G(s) = 0$  except  $s = 1$ ,  $\chi$  is an integer multiple of  $r_G$ .

**6** Let  $(\rho, V)$  be a irreducible representation of an abelian group  $G$ . Fix  $s \in G$ , then for any  $t \in G$ ,  $st = ts$ , so

$$\rho_s \rho_t = \rho_{st} = \rho_{ts} = \rho_t \rho_s. \quad (16)$$

This is true for all  $t \in G$ , and  $\rho_s$  can be viewed as a linear map from  $V$  to  $V$ . Therefore, by Schur's lemma,  $\rho_s$  is a homothety. This is true for all  $s \in G$  implying all the  $\rho^s$  are homothety. Let's write  $\rho^s = \lambda_s I$ .

Choose a non-zero element  $v \in V$ , then  $\rho^s(v) = \lambda_s v$  for all  $s$ . It shows that  $\text{span}\{v\}$  forms a  $\mathbb{C}[G]$  stable subspace of  $V$ . Since  $V$  is irreducible, it means that  $V = \text{span}\{v\}$  and  $\dim_{\mathbb{C}} V = 1$ . Therefore, the degree of  $\rho$  is 1.

**7** I'll use two fact: 1. Let  $H$  be a subgroup of  $G$ . The regular representation  $r_G$  of  $G$  is induced by the regular representation  $r_H$  of  $H$ . 2. For representation  $\theta_1, \theta_2$  of  $H$ ,  $\text{Ind}_H^G(\theta_1 \oplus \theta_2) = \text{Ind}_H^G \theta_1 \oplus \text{Ind}_H^G \theta_2$ .

Let's write  $\{(\theta_i, W_i)\}_{i=1}^m$  be all the irreducible representations of  $H$  and  $\{(\rho_j, V_j)\}_{j=1}^l$  be all the irreducible representations of  $G$ , then we know that

$$r_H \simeq \bigoplus_{i=1}^m \bigoplus_{j=1}^{n_i} \theta_i \quad (17)$$

where  $n_i$  is the degree of  $\theta_i$ . Also, using fact 2, we get

$$\bigoplus_{i=1}^l \bigoplus_{j=1}^{\deg \rho_i} \rho_i \simeq r_G \simeq \text{Ind}_H^G r_H \simeq \bigoplus_{i=1}^m \bigoplus_{j=1}^{n_i} \text{Ind}_H^G \theta_i. \quad (18)$$

Now, decompose  $\text{Ind}_H^G \theta_i$  into irreducible representations in  $G$ . Since both side are isomorphic in representation sense, each irreducible representations in LHS should be corresponds to some irreducible components of some  $\text{Ind}_H^G \theta_i$ . It proves the statement of the problem.

I'll show the fact I used: first, the regular representation  $(r_H, \mathbb{C}[H])$  of  $H$  induces the regular representation  $(r_G, \mathbb{C}[G])$  of  $G$ . To show this, it is enough to show that  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H]$  is  $\mathbb{C}[G]$  module isomorphic to  $\mathbb{C}[G]$ . Let  $\varphi : \mathbb{C}[G] \times \mathbb{C}[H] \rightarrow \mathbb{C}[G]$  by defining  $(s, t) \mapsto st$  for  $s \in G$  and  $t \in H$ , and extending to satisfy  $\mathbb{C}$ -linearity. Since  $H \leq G$ , it is well-defined  $\mathbb{C}[H]$ -balanced map, so by the universal property, it can be extended to a group homomorphism  $\Phi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \rightarrow \mathbb{C}[G]$ . Furthermore,  $\mathbb{C}[G]$  has  $(\mathbb{C}[G], \mathbb{C}[H])$  bimodule structure, so  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H]$  has well-defined left  $\mathbb{C}[G]$  action given by  $s_1(s_2 \otimes t) = s_1 s_2 \otimes t$  for  $s_1, s_2 \in \mathbb{C}[G]$  and  $t \in \mathbb{C}[H]$ . Finally,  $\Phi(s \otimes 1) = s$ , so it is surjective and if  $\Phi(\sum_i s_i \otimes t_i) = \Phi(\sum_i s_i t_i \otimes 1) = \sum_i s_i t_i = 0$ , then  $\sum_i s_i t_i \otimes 1 = 0$ , so it is injective. Also, it conserves the  $\mathbb{C}[G]$  action. Therefore, it is  $\mathbb{C}[G]$  module isomorphism, implying it is a group representation isomorphism of  $G$ .

Second one is just the elementary property of tensor product:

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} (W_1 \oplus W_2) \simeq (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_1) \oplus (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_2) \quad (19)$$

as left  $\mathbb{C}[G]$  module since  $\mathbb{C}[G]$  is  $(\mathbb{C}[G], \mathbb{C}[H])$  bimodule and  $W_i$  are left  $\mathbb{C}[H]$  module.

8 Since the eigenvalues of  $\rho(s)$  have absolute value 1,

$$|\chi(s)| = \left| \sum_{i=1}^n \lambda_i \right| \leq \sum_{i=1}^n |\lambda_i| = n. \quad (20)$$

( $\lambda_i$  are the eigenvalues of  $\rho(s)$ .) The equality holds if and only if  $\lambda_i = \sigma_i \lambda$  for some  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$  and  $\sigma_i = \pm 1$  with  $\sigma_i \sigma_j > 0$  for all  $i, j$ ; considering  $\mathbb{C}$  by  $\mathbb{R}^2$ ,  $\lambda = \sum_i \lambda_i$  and  $\lambda \cdot \lambda = \sum_i |\lambda_i|^2 + \sum_{i < j} \lambda_i \cdot \lambda_j$ . To make  $\lambda \cdot \lambda = n^2$ , we need to impose  $\lambda_i \cdot \lambda_j = 1$ , which implies all the  $\lambda_i$  points same direction since all  $\lambda_i$  have absolute value 1. It shows that  $\rho(s)$  is homothety if and only if  $|\chi(s)| = n$  as the multiple of identity matrix have the same form under any similar transformation. If  $\chi(s) = n$ , then  $\lambda_i = 1$  for all  $i$  and  $\rho(s)$  is a homothety, which means that  $\rho(s) = 1$ .

9

- (a) Let  $V'$  be a  $\mathbb{C}[G]$ -module and  $f : W \rightarrow V$  be a  $\mathbb{C}[H]$  module homomorphism. Let  $i : W \rightarrow \text{Ind}_H^G W$  by inclusion, i.e. if I use a representatives  $R = \{\sigma_1, \dots, \sigma_n\}$  of  $G/H$  and  $\text{Ind}_H^G W = \bigoplus_{i=1}^n \sigma_i W_i$  as in the textbook,  $i(w) = (w, 0, 0, \dots, 0)$  for  $w \in W$ . Note that  $i$  is  $\mathbb{C}[H]$  homomorphism. The universal property of  $\text{Ind}_H^G W$  means that there exists a well-defined unique  $\mathbb{C}[G]$ -homomorphism satisfying the commutative diagram.

$$\begin{array}{ccc} W & \xrightarrow{i} & \text{Ind}_H^G W \\ & \searrow f & \downarrow F \\ & & V \end{array}$$

- (b) The Frobenius reciprocity for two representations  $(\rho_1, W)$  of  $H$  and  $(\rho_2, V)$  of  $G$  can be given as following:

$$\text{Hom}_H(W, \text{Res}_H V) \simeq \text{Hom}_G(\text{Ind}_H^G W, V). \quad (21)$$

(I'll specify the sort of isomorphism.) (If we consider  $W$  as a  $\mathbb{C}[H]$  module and  $V$  as a  $\mathbb{C}[G]$  module, the information of the representation on each vector space is already contained in

the space.) The proof is simple if we use the universal property. Any  $\mathbb{C}[H]$  module homomorphism from  $W$  to  $\text{Res}_H V$  considering  $V$  as a  $\mathbb{C}[H]$  module is uniquely extends to  $\mathbb{C}[G]$  module homomorphism  $F : \text{Ind}_H^G W \rightarrow V$ . Conversely, any  $\mathbb{C}[G]$  module homomorphism  $F : \text{Ind}_H^G W \rightarrow V$  can makes  $f : W \rightarrow V$  by setting  $f = F \circ i$ . Finally, the uniqueness of the universal property guarantees that the extension of  $F \circ i$  is again  $F$ . It shows that if I define  $\Psi : \text{Hom}_H(W, \text{Res}_H V) \rightarrow \text{Hom}_G(\text{Ind}_H^G W, V)$  by  $\Psi(f) = F$ , then it is a bijective function. Furthermore, for any  $f_1, f_2 \in \text{Hom}_H(W, \text{Res}_H V)$  and  $c \in \mathbb{C}$ ,  $\Psi(cf_1 + f_2) = c\Psi(f_1) + \Psi(f_2)$ , which shows that  $\Psi$  can be considered as  $\mathbb{C}$  vector space isomorphism.

**10** Let's first configure what is  $H_s$ . For  $s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  and  $t = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  in  $SL_2(k)$ ,  $sts^{-1}$  is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} a\alpha\delta - b\alpha\gamma - d\gamma\beta & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ a\gamma\delta - b\gamma^2 - d\gamma\delta & -a\beta\gamma + b\alpha\gamma + d\alpha\delta \end{pmatrix} \quad (22)$$

To make it in  $H$ , we need to impose  $a\gamma\delta - b\gamma^2 - d\gamma\delta = 0$ . Assume  $s \notin H$ , then  $\gamma \neq 0$ , so  $\delta(a - d) - b\gamma = 0$ . Therefore,

$$\begin{pmatrix} a\alpha\delta - b\alpha\gamma - d\gamma\beta & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ a\gamma\delta - b\gamma^2 - d\gamma\delta & -a\beta\gamma + b\alpha\gamma + d\alpha\delta \end{pmatrix} = \begin{pmatrix} d & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ 0 & a \end{pmatrix} \quad (23)$$

using  $\alpha\delta - \gamma\beta = 1$ . Also,

$$\gamma(-a\alpha\beta + b\alpha^2 + d\alpha\beta) = \alpha(-a\beta\gamma + \delta(a - d)\alpha + d\gamma\beta) = \alpha(a - d), \quad (24)$$

so

$$\begin{pmatrix} d & -a\alpha\beta + b\alpha^2 + d\alpha\beta \\ 0 & a \end{pmatrix} = \begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} \quad (25)$$

Therefore,

$$H_s = \left\{ \begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} : ad = 1 \right\}. \quad (26)$$

Therefore, for  $s \notin H$ ,

$$\begin{aligned} \langle \rho^s, \text{Res}_{H_s}(\rho) \rangle &= \frac{1}{|H_s|} \sum_{t \in H_s} \rho^s(t^{-1}) (\text{Res}_{H_s} \rho)(t) \\ &= \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \chi_\omega \left( \begin{pmatrix} a & -\alpha\gamma^{-1}(a - d) \\ 0 & d \end{pmatrix} \right) \chi_\omega \left( \begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} \right) \\ &= \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \chi_\omega \left( \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \right) \chi_\omega \left( \begin{pmatrix} d & \alpha\gamma^{-1}(a - d) \\ 0 & a \end{pmatrix} \right) = \frac{1}{|H_s|} \sum_{d \in k \setminus \{0\}} \omega^2(d) \end{aligned} \quad (27)$$

Assume  $\omega^2 \neq 1$ . Since  $k$  is finite field,  $k^*$  is a cyclic group about an element  $x \in k$ . Therefore,  $\omega^2(x) \neq 1$  is a root of unity such that the order of  $\omega^2(x)$  divides  $|k| - 1$ . It shows that

$$\sum_{d \in k \setminus \{0\}} \omega^2(d) = \sum_{i=1}^{|k|-1} (\omega^2(x))^i = 0. \quad (28)$$

Now, the induced representation satisfies all the conditions in proposition 23, so it is irreducible. (Since  $\chi_\omega$  is the character of degree 1, the condition (a) is automatically satisfied.)