HW9

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Notation: In this paper, I used superscript for character χ^i just for indexing, not multiplication.

1-2(**S** 10.5)

(a) From the assumption, there exists $r_{i,j} \in \mathbb{R}_+$ and degree 1 representations $(A_j, \chi_{i,j})$ where $A_j \leq G$ such that

$$\chi = \sum_j \sum_i r_{i,j} \operatorname{Ind}_{A_j}^G \chi_{i,j}. \tag{1}$$

(I used i to express distinct irreducible characters in same A_j.) As χ is an irreducible character, we get

$$1_G = \langle \chi, \chi \rangle = \sum_j \sum_i r_{i,j} \langle \operatorname{Ind}_{A_j}^G \chi_{i,j}, \chi \rangle, \tag{2}$$

and for the other irreducible representations χ',

$$0 = \langle \chi, \chi' \rangle = \sum_j \sum_i r_{i,j} \langle \operatorname{Ind}_{A_j}^G \chi_{i,j}, \chi' \rangle. \tag{3}$$

Since $r_{i,j} > 0$ by deleting the terms with $r_{i,j} = 0$ and $\langle \operatorname{Ind}_{A_j}^G \chi_{i,j}, \chi' \rangle \in \mathbb{Z}_{\geq 0}$ as both are characters, we get $\langle \operatorname{Ind}_{A_j}^G \chi_{i,j}, \chi' \rangle = 0$ for all irreducible representations of G except χ. Choose one i, j and denote it i_0, j_0. Since the irreducible representations forms an orthonomal basis of class function, and $\operatorname{Ind}_{A_{j_0}}^G \chi_{i_0,j_0} \in R^+(G)$, we get $\operatorname{Ind}_{A_{j_0}}^G \chi_{i_0,j_0} = n\chi$ for some $n \in \mathbb{N}$. As a result, we conclude that $n\chi$ is a monomial.

(b) Before start, note that $\mathfrak{A}_5$ has elements

(a) identity element;

(b) 15 elements of type like (1 2)(3 4);

(c) 20 elements of type like (1 2 3);

(d) 24 elements of type like (1 2 3 4 5).

Let $G = \mathfrak{A}_5$. The permutation representation of $\mathfrak{A}_5$ on $\{e_i\}_{i=1}^5$ G action defined by by $\sigma \cdot e_i = e_{\sigma(i)}$ have character η. Note that the permutation representation have degree 1 subrepresentation for the basis $\sum_{i=1}^5 e_i$ since any element in $\mathfrak{A}_5$ fix it, so trivial representation. Therefore, $\eta - 1_G$ is again an character of $\mathfrak{A}_5$, and

$$\langle \eta - 1_G, \eta - 1_G \rangle = \frac{1}{60}(4 * 4 + 0 * 15 + 1 * 20 + 1 * 24) = 1. \tag{4}$$

It shows that $\eta - 1_G$ is irreducible having degree 4. Set $\chi = \eta - 1_G$.

Now, assume there exists $m \geq 1$ such that $m\chi = \operatorname{Ind}_H^G \chi_H$ for some subgroup H such that the degree of χ_H is 1. Since it has degree $4m$, $[G : H] = 4m$, and $|H| = 15/m$; so the possible m is $1, 3, 5, 15$. Also,

$$\langle \operatorname{Res}_H \chi, \chi_H \rangle = \langle \chi, \operatorname{Ind}_H^G \chi_H \rangle = m. \tag{5}$$

As the degree of $\mathrm{Res}_H \chi$ is 4 and χ_H is irreducible, $m \leq 4$, so the possible m is 1 or 3. If m were 3, then $|H| = 5$ and it should be a cyclic group having element like $(1\ 2\ 3\ 4\ 5)$ as the other element (except identity) have order coprime to 5. However, this is impossible: by the definition,

$$\mathrm{Ind}_H^G \chi_H((1\ 2\ 3)) = \frac{1}{5} \sum_{\substack{t \in G \\ t(1\ 2\ 3)t^{-1} \in H}} \chi_H(t(1\ 2\ 3)t^{-1}), \tag{6}$$

but the conjugacy class of $(1\ 2\ 3)$ does not have intersection with H, so it is zero. However, $(\eta - 1_G)(1\ 2\ 3) = 2 - 1 = 1$.

The remainder is to show that $\mathfrak{A}_5$ does not have order 15 group. (To do this, I refered "https://math.stackexchange.com/questions/135654/why-a-5-has-no-subgroup-of-order-15") By the application of Sylow's theorem, cf. *Abstract Algebra*, Dummit and Foote, Chapter 4.5 p. 143, we know that order 15 group is cyclic group, but we know that $\mathfrak{A}_5$ does not have order 15 element. Since $m\chi$ can not be a monomial, it shows that χ cannot be a linear combination with positive real coefficients of monomial characters.

3-6(**S** 10.6)

(a) In the previous homework, I showed that for any $E \leq H \leq G$, and $\chi \in R(E)$, we get $\mathrm{Ind}_E^G \chi = \mathrm{Ind}_H^G \mathrm{Ind}_E^H \chi$. Therefore, for $\mathrm{Ind}_E^H(\alpha - 1_E)$ having same notation as problem with $E \leq H$, we get

$$\mathrm{Ind}_H^G \mathrm{Ind}_E^H(\alpha - 1_E) = \mathrm{Ind}_E^G(\alpha - 1_E). \tag{7}$$

Also, note that E is again an elementary subgroup of G since writing $E \simeq \langle x \rangle \times P$ with p-group $P \leq Z_H(x)$, which is center in H, $P \leq Z_G(x)$ and again p-group in G.

(b) Let $\rho = \mathrm{Ind}_H^G(1_H)$. Set $\{\sigma_i\}$ be the representatives of G/H with $\sigma_1 = 1$. Since H is normal in G and $1_H(h) = 1$ for all $h \in H$, considering $\mathbb{C}[G]$ action on $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}$, for $g = \sigma_i h$,

$$g \cdot (\sigma \otimes c) = \sigma_j h \sigma \otimes c = \sigma_j \sigma \otimes (h' \cdot c) = \sigma_j \sigma \otimes c, \tag{8}$$

where $h\sigma = \sigma h'$. It shows that $\rho(h) = \mathrm{id}$ for $h \in H$, and the group homomorphism $\rho : G \to \mathrm{Ind}_H^G(\mathbb{C})$ induces a homomorphism $\tilde{\rho} : G/H \to \mathrm{Ind}_H^G(\mathbb{C})$ which factor through the canonical projection $\pi : G \to G/H$. Computing the character of $\tilde{\rho}$, we get $\tilde{\rho}(1) = (G : H)$ and zero elsewhere. It means that $\tilde{\rho}$ is isomorphic to the regular representation of G/H. Since G/H is abelian, we know that the character $\tilde{\chi}$ of $\tilde{\rho}$ is the sum of degree 1 characters of G/H; let's denote the degree 1 characters $\tilde{\chi}_i$, then we can write

$$\tilde{\rho} = \sum_{i=1}^{(G:H)} \tilde{\chi}_i \tag{9}$$

Finally, set functions χ_i by $\chi_i = \tilde{\chi}_i \circ \pi$. These are character functions: for $g_1, g_2 \in G$,

$$\chi_i(g_1 g_2) = \tilde{\chi}_i(\pi(g_1 g_2)) = \tilde{\chi}_i(\pi(g_1)\pi(g_2)) = \tilde{\chi}_i(\pi(g_1))\tilde{\chi}_i(\pi(g_2)) = \chi_i(g_1)\chi_i(g_2), \tag{10}$$

and we can identify characters and representations since χ_i have codomain $\mathbb{C}^\times$. Note that χ_i all have degree 1, and $\chi_1 = 1_G$. Finally, we get

$$\mathrm{Ind}_H^G 1_H = \sum_{i=1}^{(G:H)} \chi_i, \tag{11}$$

Now, let's constructively show that $\mathrm{Ind}_H^G 1_H \in R'(G)$. First, using Brauer's theorem, we choose irreducible characters η_E^i and n_E^i for elementary subgroups E satisfying

$$1_G = \sum_E \sum_i n_E^i \, \mathrm{Ind}_E^G \, \eta_E^i. \tag{12}$$

Note that I used sup-script i to distinguish distinct irreducible characters in the same E. To go to next step, I need a lemma.

Lemma 1. *Any subgroup of p-elementary group is p-elementary group. More precisely, for $G \simeq C \times P$ where $C = \langle x \rangle$ is a cyclic subgroup having order prime to p and P a p-group of $Z(x)$, any subgroup of G is written as $H \simeq A \times B$ where $A \leq C$ and $B \leq P$.*

Proof. Let $G = \langle x \rangle \cdot P$ where $x \in G$ having order prime to p and P being a p-group in $Z(x)$. For a subgroup $H \leq G$, choose $y = (\alpha, \beta) \in H$ identifying $\langle x \rangle \cdot P \simeq \langle x \rangle \times P$. Since $(|\alpha|, |\beta|) = 1$, there exists n_1, n_2 in $\mathbb{Z}$ such that $n_1|\alpha| + n_2|\beta| = 1$. It shows that

$$\begin{aligned} y^{n_1|\alpha|} &= (1, \beta^{1-n_2|\beta|}) = (1, \beta) \\ y^{n_2|\beta|} &= (\alpha^{1-n_1|\alpha|}, 1) = (\alpha, 1), \end{aligned} \tag{13}$$

so $(\alpha, 1), (1, \beta) \in H$. Denote each α, β of y by α_y, β_y. Set A (resp. B) be the subgroup of $\langle x \rangle$ (resp. P) generated by α_y. (resp. β_y) Note that A is cyclic group since any subgroup of cyclic group is again cyclic group, and B is a p-subgroup of $Z(a)$ by the similar reason. I claim that $H \simeq A \times B$ with the same identification as G, then H is p-elementary subgroup of G.

By the construction of A, B, it is enough to show that $H \supset A \times B$, but any generator of $A \times 1$ and $1 \times B$ is in H, so $H \supset A \times B$. □

Since E are super-solvable groups, η_E^i are induced by a representation of degree 1 of a subgroup of E, which is again elementary subgroup by the lemma. Abusing notation by setting η_E^i degree 1 character, we again write (12). From the basic property of induced representation, we know that

$$\mathrm{Ind}_E^G(\eta_E^i \, \mathrm{Res}_E(\chi_j - 1_G)) = (\mathrm{Ind}_E^G \, \eta_E^i)(\chi_j - 1_G), \tag{14}$$

so

$$\begin{aligned} \sum_{j=2}^{(G:H)} \sum_E \sum_i n_E^i \, \mathrm{Ind}_E^G(\eta_E^i \, \mathrm{Res}_E(\chi_j - 1_G)) &= \sum_{j=2}^{(G:H)} \sum_E \sum_i n_E^i \, \mathrm{Ind}_E^G(\eta_E^i)(\chi_j - 1_G) \\ &= \sum_{j=2}^{(G:H)} ((\chi_j - 1_G)1_G) \\ &= \sum_{j=2}^{(G:H)} (\chi_j - 1_G). \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}\operatorname{Ind}_H^G 1_H &= (G:H)1_G + \sum_{j=2}^{(G:H)} (\chi_j - 1_G) \\ &= (G:H)1_G + \sum_{j=2}^{(G:H)} \sum_E \sum_i n_E^i \operatorname{Ind}_E^G(\eta_E^i \operatorname{Res}_E(\chi_j - 1_G)) \\ &= (G:H)1_G + \sum_{j=2}^{(G:H)} \sum_E \sum_i n_E^i \left(\operatorname{Ind}_E^G(\eta_E^i \operatorname{Res}_E \chi_j) - \operatorname{Ind}_E^G(\eta_E^i)\right) \\ &= (G:H)1_G + \sum_{j=2}^{(G:H)} \sum_E \sum_i n_E^i \left(\operatorname{Ind}_E^G(\eta_E^i \operatorname{Res}_E \chi_j - 1_E) - \operatorname{Ind}_E^G(\eta_E^i - 1_E)\right).\end{aligned} \tag{16}$$

Since multiplication of two degree 1 characters is again a degree 1 character, we get the result.

(c) Let's write $G \simeq C \times P$ where $C = \langle x \rangle$ is a cyclic subgroup having order prime to p and P a p-group of $Z(x)$. I'll first show a general lemma.

Lemma 2. *If G is nilpotent and $H < G$, then $H < N_G(H)$.*

Proof. Let's use the definition of nilpotent group given in the textbook. Given a sequence

$$\{1\} = G_0 \subsetneq G_1 \subsetneq \cdots G_{n-1} \subsetneq G_n = G \tag{17}$$

of subgroups of G satisfying $G_{i-1} \trianglelefteq G_i$ and $G_i/G_{i-1} \subset Z(G/G_{i-1})$, take quotient of G_1 in the sequence and get

$$\{1\} = G_1/G_1 \subsetneq G_2/G_1 \subsetneq \cdots G_{n-1}/G_1 \subsetneq G_n/G_1 = G/G_1. \tag{18}$$

I need to show that this is well-defined. First, G_1 is normal to $G_{i\geq 2}$ since $G_1/G_0 = G_1 \leq Z(G)$. Also, by the fourth isomorphism theorem, the normal property does not change. Finally, for $\varphi : G/G_1 \to G/G_{i-1}$ mapping $gG_1 \mapsto gG_{i-1}$, we get $\ker\varphi = G_{i-1}/G_1$ by the third isomorphism theorem for $i > 2$, and we know that $\varphi(G_i/G_1) = G_i/G_{i-1}$ with the same kernel by the same reason. Let $\bar{\varphi}$ be the induced isomorphism by taking quotient of G_{i-1}/G_1. It shows that $G_i/G_{i-1} \leq Z(G/G_{i-1})$ implies $(G_i/G_1)/(G_{i-1}/G_1) \leq Z((G/G_1)/(G_{i-1}/G_1))$: for any $\bar{a} \in (G_i/G_1)/(G_{i-1}/G_1)$ and $\bar{b} \in (G/G_1)/(G_{i-1}/G_1)$ with $\bar{\varphi}(\bar{a}) = a$ and $\bar{\varphi}(\bar{b}) = b$,

$$\bar{\varphi}(\bar{a}\bar{b}) = \bar{\varphi}(\bar{a})\bar{\varphi}(\bar{b}) = ab = ba = \bar{\varphi}(\bar{b}\bar{a}), \tag{19}$$

so $\bar{a}\bar{b} = \bar{b}\bar{a}$ for all $\bar{b}$. It shows that G/G_1 is nilpotent.

Now, let's take induction on $|G|$ for the main result. If $G = 1$, it is trivial, so assume the statement is true for $|G| < n$ for some n. For $|G| = n$, take an proper subgroup H of G. Note that $Z(G)$ is non-trivial; unless, it is not nilpotent. If $G_1 \nleq H$, then $N_G(H) \supset \langle H, G_1 \rangle$, which ends the proof, so we can assume $G_1 \leq H$. Consider $\bar{G} = G/G_1$, which is again nilpotent having smaller order, so $\bar{H} < N_{\bar{G}}(\bar{H})$. Again, by taking lattice isomorphism theorem, we know that the preimage of $N_{\bar{G}}(\bar{H})$ is $N_G(H)$: more precisely, for the canonical projection $\pi : G \to G/G_1 = \bar{G}$ and $g \in \pi^{-1}\left(N_{\bar{G}}(\bar{H})\right)$ with $h \in H$,

$$\pi(ghg^{-1}) = \bar{g}\bar{h}\bar{g}^{-1} \in \bar{H}, \tag{20}$$

so $ghg^{-1} \in \pi^{-1}(\bar{H}) = H$. Conversely, if $g \in N_G(H)$, then by the same reason, $\pi(ghg^{-1}) \in \bar{H}$, so $g \in \pi^{-1}\left(N_{\bar{G}}(\bar{H})\right)$. It shows that $H < N_G(H)$ and ends the proof. □

If H is maximal subgroup of G, then $N_G(H) = G$ by the lemma, so H is normal in G. Also, using the lemma 1, let's write $H \simeq A \times B$ where $A \leq C$ and $B \leq P$. If $B \neq P$, then $A < C$ and $[C : A]$ should be prime applying the classification of finitely generated abelian group with the lattice theorem to C/A since $G/H \simeq C/A \times B/P$. If $C = A$, then $[P : B]$ should be prime since P/B is again a p group and any p group P have subgroup having order dividing $|P|$. If $C < A$ and $B < P$, then $C \times P$ is the bigger subgroup of G, so it is contradiction. As a result, $[G : H]$ is prime.

Using theorem 16 since G is super-solvable group, we can write each irreducible representation χ of G by

$$\chi = \sum_E \sum_i n_E^i \operatorname{Ind}_E^G \eta_E^i, \tag{21}$$

where E are subgroups of G, $n_E^i \in \mathbb{Z}$, and η_E^i are characters of degree 1 of E. Since any proper subgroup of G is contained in some maximal subgroup and by (a), we get

$$\begin{aligned}\chi &= \sum_i n_G^i \eta_G^i + \sum_{E<G} \sum_i n_E^i \operatorname{Ind}_E^G \eta_E^i \\ &= \sum_i n_G^i \eta_G^i + \sum_{E<G} \sum_i n_E^i \operatorname{Ind}_{H_E}^G \left(\operatorname{Ind}_E^{H_E} \eta_E^i\right),\end{aligned} \tag{22}$$

where H_E is the maximal subgroup containing E. It shows that $R(G)$ is generated by the characters of degree 1 of G together with the $\operatorname{Ind}_H^G(R(H))$, where H runs over Y.

Now, I'll prove a proposition

Proposition 3. *If G is an elementary group, then $R(G) = R'(G)$.*

Proof. Let's use induction on $|G|$. If $|G| = 1$, then it is trivial, so let's assume that the statement is true for $|G| < n$ for some $n \in \mathbb{N}_{\geq 2}$. It is enough to show that $R(G) \subset R'(G)$. Since $[G : H]$ is prime order, it is abelian, cf. *Abstract Algebra*, Dummit and Foote, Section 4.3 theorem 8, so we can apply (b). By induction, we know that $R(H) = R'(H)$ since H are all proper elementary subgroup of G, so $\operatorname{Ind}_H^G(R(H)) \subset R'(G)$. Since G is an elementary subgroup, by setting $E = G$ in the generating element $\operatorname{Ind}_E^G(\alpha - 1_E)$ of $R_0'(G)$ with 1_G, we know that any degree 1 character of G is contained in $R'(G)$. Therefore, it shows that $R(G) \subset R'(G)$ and we get $R(G) = R'(G)$. □

(d) In (b), I already justified the argument to write

$$\varphi = \sum_{E \in X} \operatorname{Ind}_E^G(\varphi_E) \tag{23}$$

where $\varphi_E = f_E \cdot \mathrm{Res}_E(\varphi)$ following the notation in the problem. If $\varphi(1) = 0$, then $\varphi_E(1) = 0$ for all E since $\mathrm{Res}_E(\varphi)(1) = \varphi(1) = 0$. Since we already showed that $\varphi_E \in R(E) = R'(E)$, it means that $\varphi_E \in R'_0(E)$ as $\varphi_E(1) = 0$. By (a), it shows that $\varphi \in R'_0(G)$. Finally, for general $\varphi \in R(G)$, consider $\varphi - \varphi(1)1_G$, then it is again in $R'_0(G)$ and $\varphi \in R'(G)$. Therefore, $R(G) \subset R'(G)$. Conversely, $R'(G) \subset R(G)$ since it is generated by the element of $R(G)$. It shows that $R'(G) = R(G)$.