HW#4

박성빈, 수학과, 20202120

Notation: For each problem, I'll follow the notations in the problem if there is no additional mention.

1(**D&F** 18.3.11) Let ϕ be the irreducible representation of G on V generating χ. Let z is in the center of G, then we get

$$\phi(z)\phi(g) = \phi(g)\phi(z) \tag{1}$$

for all $g \in G$, which means that $\phi(z)$ is an $\mathbb{C}G$ module automorphism on V. By schur's lemma, it shows that $\phi(z)$ is a homothety, i.e., for $\lambda \in \mathbb{C}$, $\phi(z) = \lambda \cdot \mathrm{id}$, so $\chi(z) = \lambda\chi(1)$. Since $|z| < \infty$, $\lambda^{|z|} = 1$ and λ is some root of unity in $\mathbb{C}$.

2(**D&F** 18.3.15) This is basis-dependent argument(private communication with TA): For a cyclic group $G =< \sigma : \sigma^3 = 1 >$, consider a group representation ϕ on F^2 given by

$$\rho_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{3}\right) \end{pmatrix}. \tag{2}$$

I'll denote $u = \exp\left(\frac{2\pi i}{3}\right)$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2F$, the similar transformation is given by

$$A^{-1}\rho_\sigma A = \frac{1}{ad-bc}\begin{pmatrix} ad - ubc & bd(1-u) \\ ac(-1+u) & uad - bc \end{pmatrix}. \tag{3}$$

For arbitrary nonzero $f \in F$, if I set $a = 1 + f$, $b = c = d = 1$, then

$$\frac{ad - ubc}{ad - bc} = \frac{f - u + 1}{f} = \frac{-u+1}{f} + 1. \tag{4}$$

If $\mathbb{Q}(\varphi)$ implies collecting all entries of φ upto similar transformation, it means that F is finite extension of $\mathbb{Q}[u]$, which is contradiction. Therefore, I'll fix a basis for $GL_n(F)$.

Define

$$A = \cup_{s\in G} \cup_{a_{ij}^s \text{ is an entry of } \varphi(s)} \{a_{ij}^s\}, \tag{5}$$

then it is finite since $|G| < \infty$ and $\varphi(s) \in GL_n(F)$. Furthermore, $a_{ij}^s \in F$, so each $[\mathbb{Q}(a_{ij}^s) : \mathbb{Q}] < \infty$. Therefore, $[\mathbb{Q}(A) : \mathbb{Q}] < \infty$, which shows that $\mathbb{Q}(\varphi)$ is finite extension of $\mathbb{Q}$.

3(**D&F** 18.3.16) Fix $s \in G$. Since σ is a automorphism on F, for $a_i, b_i \in F$,

$$\sigma\left(\sum_i a_i b_i\right) = \sum_i \sigma(a_i)\sigma(b_i). \tag{6}$$

Therefore, $\sigma(AB) = \sigma(A)\sigma(B)$ for $A, B \in GL_n(F)$, which shows that φ^σ is a group homomorphism from G to $GL_n(F)$, so φ^σ is a representation if φ is a representation. Furthermore, $\mathrm{tr}(\sigma(A)) = \sigma(\mathrm{tr}(A))$ by the same reason, so we get the character of $\varphi^\sigma = \sigma \circ \psi$.

4(**D&F** 18.3.17) Since $(\varphi^\sigma)^{\sigma^{-1}} = \varphi$, it is enough to show that φ is irreducible implies φ^σ is irreducible.

Assume φ^σ is not irreducible, so there exists a proper subspace W of V such that $\varphi^\sigma|_W$ is an automorphism on W. It means that there exists a complement W^0 of W which also satisfies $\varphi^\sigma|_{W^0} \in Aut(W^0)$. Therefore, the matrix form of φ can be decomposed into smaller block matrices, which have determinant non-zero. Taking σ^{-1} to each entries of the decomposed matrix, we get the same block decomposed matrices with non-zero determinant since σ is field isomorphism. It shows that φ is not irreducible, which is contradiction. It ends the proof.

5(**D&F** 19.3.1) For basis $1\otimes e_1, 1\otimes e_2, 1\otimes e_3, (1\ 2)\otimes e_1, (1\ 2)\otimes e_2, (1\ 2)\otimes e_3\}$, the matrix representation is given by the following: for

$$\begin{aligned} P_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ P_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{7}$$

we get

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \\ (1\ 2) &\mapsto \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \\ (1\ 3) &\mapsto \begin{pmatrix} 0 & \varphi((1\ 2\ 3)) \\ \varphi((1\ 3\ 2)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & P_1 \\ P_2 & 0 \end{pmatrix}, \\ (2\ 3) &\mapsto \begin{pmatrix} 0 & \varphi((1\ 3\ 2)) \\ \varphi((1\ 2\ 3)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & P_2 \\ P_1 & 0 \end{pmatrix}, \\ (1\ 2\ 3) &\mapsto \begin{pmatrix} \varphi((1\ 2\ 3)) & 0 \\ 0 & \varphi((1\ 3\ 2)) \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \text{ and} \\ (1\ 3\ 2) &\mapsto \begin{pmatrix} \varphi((1\ 3\ 2)) & 0 \\ 0 & \varphi((1\ 2\ 3)) \end{pmatrix} = \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix}. \end{aligned} \tag{8}$$

6(**D&F** 19.3.2(a)) Since induced representation is unique upto isomorphism, it is enough to calculate the character with some fixed representatives of G/H. Since $< (1\ 2) >$ has order 2, it has two irreducible representations: trivial and non-trivial one. The non-trivial irreducible character ψ is $\psi(1) = 1$ and $\psi((1\ 2)) = -1$. For representatives $\{1, (1\ 3), (2\ 3)\}$, the induced character Ψ is

$$\begin{aligned} \Psi(1) &= 3\psi(1) = 3 \\ \Psi((1\ 2)) = \Psi((1\ 3)) = \Psi((2\ 3)) &= \psi((1\ 2)) = -1 \\ \Psi((1\ 2\ 3)) = \Psi((1\ 3\ 2)) &= 0. \end{aligned} \tag{9}$$

Using the character table of S_3 in section 19.1, we get $\Psi = \chi_2 + \chi_3$.

7(**D&F** 19.3.4) Let $R = \{1, g_1, \ldots, g_m\}$ be the representation set of G/H and Φ be the induced representation of φ which is a representation on V. Fix $g_i \in R$. For $v \in V$ and $n \in N \leq H$, there exists $n' \in N$ such that $ng_i = g_i n'$ since N is a normal subgroup of G. Therefore,

$$n \cdot (g_i \otimes v) = ng_i \otimes v = g_i n' \otimes v = g_i \otimes (n' \cdot v) = g_i \otimes v. \tag{10}$$

It shows that N is contained in the kernel of induced presentation.

8(**S** 3.3.4) From example 1, we know that the regular representation ρ on G is induced by the regular representation θ on H. Let's decompose $\rho = \oplus_{i=1}^{n}\rho_i$ and $\theta = \oplus_{i=1}^{m}\theta_i$ into irreducible representations. By Corollary 5.1 in chapter 2, every irreducible representations on G and H is contained in ρ and θ.

Let ρ_i' be the induced representation of θ_i, then by example 3, we know that $\oplus_{i=1}^{m}\rho_i'$ is induced by $\oplus_{i=1}^{m}\theta_i$. By the uniqueness of induced representation, $\oplus_{i=1}^{m}\rho_i'$ and ρ are isomorphic, so each irreducible components of ρ is contained in some ρ_i', which shows the statement in the problem.

Since $A \leq G$ is abelian, each θ_i has degree 1. It means that the induced representation ρ_i' of θ_i has degree g/a for all i, so each ρ_i should have degree not greater than g/a.

9(**S** 3.3.5) Before start, I'll show that ρ is a well-defined representation. For $s_1, s_2 \in G$, $(\rho_{s_1}(\rho_{s_1^{-1}})f)(u) = f(us_1s_1^{-1}) = f(u)$ and by the same reason, $(\rho_{s_1^{-1}}(\rho_{s_1})f)(u) = f(u)$, so $\rho_s \in GL(V)$ for all $s \in G$. Also,

$$(\rho_{s_1s_2^{-1}}f)(u) = f(us_1s_2^{-1}) = (\rho_{s_1}(\rho_{s_2^{-1}}f))(u) = (\rho_{s_1}(\rho_{s_2}^{-1}f))(u). \tag{11}$$

Therefore, it is a well-defined group action, i.e. group homomorphism from G to $GL(V)$.

To show $w \mapsto f_w$ is an isomorphism, I'll first show that $f_w \in V$. For $t \in H$ and $u \in G$, if $tu \in H$, which means that $u \in H$,

$$f_w(tu) = \theta_{tu}w = \theta_t\theta_u w = \theta_t f_w(u). \tag{12}$$

If $tu \notin H$, $f_w(u) = 0$ since $u \notin H$, so it also satisfies $f_w(tu) = \theta_t f_w(u)$. Therefore, $f_w \in V$.

To show the isomorphism from W to W_0, it is enough to show that $\varphi : w \mapsto f_w$ is injection from W. If $w_1 \neq w_2$, then $f_{w_1}(1) = w_1 \neq w_2 = f_{w_2}(1)$, so $f_{w_1} \neq f_{w_2}$.

Now, I'll show that ρ is induced by θ. Let's first fix a representatives $R = \{1 = \sigma_1, \ldots, \sigma_n\} \in G$ of G/H. I'll first show that $\{\rho_{\sigma_i}f_{w_j}\}$ forms a basis of V. For linearly independence, assume there exists $a_{ij} \in \mathbb{C}$ satisfying

$$\sum_{i,j} a_{ij}\rho_{\sigma_i}f_{w_j} = 0. \tag{13}$$

It means that for $u \in G$,

$$\sum_{i,j} a_{ij}\rho_{\sigma_i}f_{w_j}(u) = \sum_{i,j} a_{ij}f_{w_j}(u\sigma_i) = \sum_{j}\sum_{u\sigma_i \in H} a_{ij}\theta_{u\sigma_i}w_j = 0. \tag{14}$$

Note that u acts on G/H by permutation, so there exists only one i_0 such that $u\sigma_{i_0} \in H$, which means that

$$\sum_{j} a_{i_0 j}\theta_{u\sigma_{i_0}}w_j = 0. \tag{15}$$

Since $\theta_{u\sigma_{i_0}}$ is automorphism on W and w_j is a basis on W, $a_{i_0 j} = 0$ for all j. Since this is true for arbitrary $u \in G$ and G acts on G/H transitively, we get $a_{ij} = 0$ for all i, j. Therefore, $\{\rho_{\sigma_i}f_{w_j}\}$ is linearly independent.

Now, I'll show that the $\{\rho_{\sigma_i}f_{w_j}\}$ spans V. Fix arbitrary $f \in V$. Even though I chose left coset of G/H, it also works as a right coset of $H\backslash G$. Therefore, it is enough to show that I can generate $f(\sigma_i)$ using the basis to generate the f: for any $u \in G$, there exists $\sigma \in R$ and $t \in H$ such that $tu = \sigma$, so $f(u) = f(t^{-1}tu) = \theta_{t^{-1}}f(\sigma)$. Let's write $f(\sigma_i) = \sum_{i,j} a_{ij}w_j$. If I set

$$\phi = \sum_{i,j} a_{ij}\rho_{\sigma_i^{-1}}f_{w_j}, \tag{16}$$

for any $\sigma_k \in R$,

$$\phi(\sigma_k) = \sum_{i,j} a_{ij}\rho_{\sigma_i^{-1}} f_{w_j}(\sigma_k) = \sum_{i,j} a_{ij} f_{w_j}(\sigma_k \sigma_i^{-1}) = \sum_j a_{kj} w_j = f(\sigma_k). \tag{17}$$

Therefore, $\phi = f$.

By writing $W_{\sigma_i} = \text{span}\{\rho_{\sigma_i} f_{w_1}, \dots, \rho_{\sigma_i} f_{w_m}\}$, we get $V = \oplus_{\sigma \in R} W_\sigma$, which means that ρ is induced by θ.

10(S 3.3.6) Since $G = H \times K$, $hk = kh$ for all $h \in H$ and $k \in K$. It shows that $H \trianglelefteq G$ and $G/H \simeq K$. Therefore, We can take the representatives of G/H by the elements of K, and for any $k \in K$ and $h \in H$, we get $h(kH) = kH$. I'll write the representatives $R = \{k_1, \dots, k_n\}$ with $n = |K|$ if enumerating is necessary.

Let W and L be vector spaces such that $\theta : H \to GL(W)$ and $r_K : K \to GL(L)$, i.e. L has basis $\{e_k\}_{k \in K}$. Since $\rho : G \to GL(V)$ is induced by θ, we can write

$$V = \oplus_{k \in K} W_k. \tag{18}$$

Let's construct a vector space isomorphism $\varphi : W \otimes L \to V$ by

$$\varphi(w \otimes l) = \varphi(\sum_{k \in K} a_k (w \otimes e_k)) = \sum_{k \in K} a_k \rho_k w. \tag{19}$$

Indeed, we know that $W \otimes L$ is a vector space having basis as a simple tensor of basis elements in W and L, this is well-defined map, and this is surjective since for any $\sum_{k \in K} w_k \in V$ such that $w_k \in W_k$,

$$\varphi : \sum_{k \in K} \rho_{k^{-1}} w_k \otimes e_k \mapsto \sum_{k \in K} w_k. \tag{20}$$

By dimension analysis, φ is a vector space isomorphism. Finally, this is isomorpshim of representation between $\theta \otimes r_K$ and ρ: with the same notation above and $s = hk_i \in G$ for some $k_i \in K$ and $h \in H$,

$$\begin{aligned} \rho_s(\varphi(w \otimes l)) &= \sum_{k \in K} a_k \rho_s \rho_k w = \sum_{k \in K} a_k \rho_{k_i k} \rho_h w \\ \varphi\left((\theta \otimes r_K)_s (w \otimes l)\right) &= \varphi\left(\theta_h(w) \otimes (r_K)_{k_i}\left(\sum_{k \in K} a_k e_k\right)\right) = \varphi\left(\theta_h(w) \otimes \left(\sum_{k \in K} a_k e_{k_i k}\right)\right) = \sum_{k \in K} a_k \rho_{k_i k} \rho_h w. \end{aligned} \tag{21}$$

(To write it more precise, I need to introduce the inclusion map $i : W \to V$ and use $i(\theta_h(w)) = \rho_h(i(w))$.) Therefore, $\theta \otimes r_K$ and ρ are isomorphic.