HW10

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1-2(**S** 9.3)

(a) In each proof, I'll concentrate on the symmetric power part since alternating power part has the same proof struecuture. To prove the results, it is enough to show that $\sigma_T(\chi)$ is well-defined for $|T| < 1/n$ where $n = \deg\chi$. Since

$$\left|\chi_\sigma^k(s)\right| \le \prod_{i=1}^{k}(|\lambda_1| + \ldots + |\lambda_n|) = n^k \tag{1}$$

for $s \in G$ where λ_i are the eigenvalues of $\rho(s)$, for $T = a/n$ for $|a| < 1$

$$\left|\chi_\sigma^k T^k\right| \le |\chi_\sigma T|^k < |a|^k, \tag{2}$$

and the power series converges absolutely in the domain. Now, I can use the uniqueness and calculus properties of power series. By the similar argument, I can repeat the statement for $\lambda_T(\chi)$.

For eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $\rho(s)$, we get

$$\chi_\sigma^k(s) = \sum_{1\le n_1\le n_2\le\cdots\le n_k\le n} \lambda_{n_1}\cdots\lambda_{n_k} \tag{3}$$

Also,

$$\frac{1}{\det(I-\rho(s)T)} = \prod_{i=1}^{n}\frac{1}{1-\lambda_i T} = \prod_{i=1}^{n}\left(\sum_{j=0}^{\infty}(\lambda_i T)^j\right). \tag{4}$$

Now, let's show a lemma.

Lemma 1. *For $s \in G$, we get*

$$\sum_{k=0}^{\infty}\chi_\sigma^k(s)T^k = \frac{1}{\det(I-\rho(s)T)} \tag{5}$$

for $|T| < 1/(\deg\chi)$.

Proof. Note that (4) converges absolutely for $|T| < 1/n$, so I just need to check whether the coefficients of T^k coincide. To show this, I'll state a proposition.

Proposition 2. *For $\{c_i\}_{i=1}^N \subset \mathbb{C}^\times$ and an undeterminate T in the domain $|T| < 1$, it satisfies*

$$\prod_{i=1}^{N}\sum_{j=0}^{\infty}(c_i T)^j = \sum_{k=0}^{\infty}\sum_{1\le n_1\le n_2\le\cdots\le n_k\le N} c_{n_1}\cdots c_{n_k}T^k. \tag{6}$$

Proof. Since $c_i \in \mathbb{C}^\times$ and $|T| < 1$, the series in LHS converges absolutely, so it is well-defined and change of the order of summation does not change the result. Therefore, we again need to check whether the coefficients of T^k for both side coincide.

Now, let's change the view point of RHS. The RHS can be rewritten by

$$\sum_{k=0}^{\infty} \sum_{1\le n_1\le n_2\le\cdots\le n_k\le N} c_{n_1}\cdots c_{n_k}T^k = \sum_{k=0}^{\infty} T^k \sum_{\substack{\sum_{i=1}^N d_i=k \\ d_i\ge 0}} c_1^{d_1}\cdots c_N^{d_N}. \tag{7}$$

To check this, it is enough to show that any element $(d_1, \ldots, d_N)$ such that $\sum_{i=1}^N d_i = k$ bijectively correspond to $c_1^{d_1}\cdots c_N^{d_N}$ and the set of $c_{n_1}\cdots c_{n_k}$ with

$$\underbrace{c_1\cdots c_1}_{d_1}\cdots\underbrace{c_N\cdots c_N}_{d_N}. \tag{8}$$

Finally, the coefficient of T^k in the LHS (6) is same as the RHS in (7), so it proves the result. □

Applying above proposition with (3) and (5), we get

$$\begin{aligned}
\frac{1}{\det(I-\rho(s)T)} &= \prod_{i=1}^{N}\sum_{j=0}^{\infty}(\lambda_i T)^j \\
&= \sum_{k=0}^{\infty}\sum_{1\le n_1\le n_2\le\cdots\le n_k\le N}\lambda_{n_1}\cdots\lambda_{n_k}T^k \\
&= \sum_{k=0}^{\infty}\chi_\sigma^k(s)T^k.
\end{aligned} \tag{9}$$

for $|T| < 1/n$. □

Let's reproduce the same argument for $\lambda_T(\chi)$. For $k > n$, we know that $\chi_\lambda^k = 0$ by the property of the symmetric power, so we can assume $k \le n$. For eigenvalues of $\rho(s)$, we get

$$\begin{aligned}
\lambda_T(\chi)(s) &= \sum_{k=0}^{n}\chi_\lambda^k(s)T^k = \sum_{1=n_1<n_2<\ldots<n_k=n}\lambda_{n_1}\lambda_{n_2}\cdots\lambda_{n_k}T^k \\
\det(1+\rho(s)T) &= \prod_{i=1}^{n}(1+\lambda_i T).
\end{aligned} \tag{10}$$

We can easily check that the two have same coefficient for T^k by noticing that it is equivalent to choosing k distinct element from $\{\lambda_1, \ldots, \lambda_n\}$.

To proceed next step, I need some fact from linear algebra. Fortunately, we are dealing with diagonalizable matrices, so we can easily check the fact from the linear algebra.

Let's define

$$\frac{1}{1-A} := \sum_{k=0}^{\infty} A^k \tag{11}$$

for a diagonalizable matrix A with eigenvalues λ_i with $|\lambda_i| < 1$. This is well-defined since writing $A = V\Lambda V^{-1}$ which is diagonalization,

$$\sum_{k=0}^{N} A^k = V\left(\sum_{k=0}^{N} \Lambda^k\right)V^{-1} \tag{12}$$

and the diagonal part have $\sum_{k=0}^{N} \lambda_i^k$, which converges absolutely as $N \to \infty$. By the similar mean, we define

$$-\ln(1-A) := \sum_{k=1}^{\infty} \frac{A^k}{k}. \tag{13}$$

for the same restriction on A. Finally, we define

$$\exp A := \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{14}$$

with arbitrary restriction on A: to check the convergence, see "Matrix exponential" article in the Wikipedia. Note that if A are diagonalizable, then the three operations preserves the diagonalizability. For diagonalizable matrix A, we know that

$$\det\left(\exp(A)\right) = \exp\left(\operatorname{tr} A\right), \tag{15}$$

so replacing A by $-\ln(1-\rho(s)T)$ for $|T| < 1/n$, we get

$$= \det\left(\exp(-\ln(1-\rho(s)T))\right) = \exp\left(\operatorname{tr}(-\ln(1-\rho(s)T))\right) \tag{16}$$

Let's calculate both sides. For LHS with diagnalization $\rho(s) = V\Lambda V^{-1}$,

$$\begin{aligned}\det\left(\exp(-\ln(1-\rho(s)T))\right) &= \det\left(\exp\left(V\left(\sum_{k=0}^{\infty}\left(\frac{\Lambda T}{k}\right)^k\right)V^{-1}\right)\right)\\ &= \det\left(V\exp\left(\sum_{k=0}^{\infty}\left(\frac{\Lambda T}{k}\right)^k\right)V^{-1}\right),\end{aligned} \tag{17}$$

and the center term is

$$\begin{aligned}\exp\left(\sum_{k=0}^{\infty}\left(\frac{\Lambda T}{k}\right)^k\right) &= \begin{pmatrix}\exp\left(\sum_k \frac{\lambda_1 T}{k}\right)^k & 0 & \dots & \vdots\\ 0 & \exp\left(\sum_k \frac{\lambda_2 T}{k}\right)^k & \dots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \dots & \dots & 0 & \exp\left(\sum_k \frac{\lambda_n T}{k}\right)^k\end{pmatrix}\\ &= \begin{pmatrix}\exp\left(-\ln(1-\lambda_1 T)\right) & 0 & \dots & \vdots\\ 0 & \exp\left(-\ln(1-\lambda_2 T)\right) & \dots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \dots & \dots & 0 & \exp\left(-\ln(1-\lambda_n T)\right)\end{pmatrix}\\ &= \frac{1}{1-\Lambda T}.\end{aligned} \tag{18}$$

It shows that

$$\det\left(\exp(-\ln(1-\rho(s)T))\right) = \det\left(\frac{1}{1-\Lambda T}\right). \tag{19}$$

Furthermore, using (18), we get

$$\det\left(\exp(-\ln(1-\rho(s)T))\right) = \frac{1}{\det(1-\Lambda T)} = \frac{1}{\det(1-\rho(s)T)}. \tag{20}$$

For the RHS of (16), we again get

$$\begin{aligned}\exp\left(\operatorname{tr}(-\ln(1-\rho(s)T))\right) &= \exp\left(\operatorname{tr}\left(\sum_{k=1}^{\infty}\frac{(\Lambda T)^k}{k}\right)\right)\\ &= \exp\left(\sum_{k=1}^{\infty}\operatorname{tr}\Lambda^k\frac{T^k}{k}\right) = \exp\left(\sum_{k=1}^{\infty}\operatorname{tr}\rho^k(s)\frac{T^k}{k}\right)\\ &= \exp\left(\sum_{k=1}^{\infty}\operatorname{tr}\rho(s^k)\frac{T^k}{k}\right) = \exp\left(\sum_{k=1}^{\infty}\Psi^k(\chi)(s)\frac{T^k}{k}\right).\end{aligned} \tag{21}$$

For $\lambda_T(\chi)(s)$, we repeat the similar computation. Pluggin $A = \ln(1+\rho(s)T)$ for (15), we get

$$\det\left(\exp(\ln(1+\rho(s)T))\right) = \exp\left(\operatorname{tr}\ln(1+\rho(s)T)\right). \tag{22}$$

The RHS is

$$\exp\left(\operatorname{tr}\ln(1+\rho(s)T)\right) = \exp\left(\sum_{k=1}^{\infty}(-1)^{k-1}\Psi^k(\chi)T^k/k\right), \tag{23}$$

and the LHS is

$$\det\left(\exp(\ln(1+\rho(s)T))\right) = \det(1+\rho(s)T). \tag{24}$$

Finally, for $|T| < 1/n$, $\sigma_T(\chi)(s)$ is smooth and has well-defined series form derivative, so we get

$$(\ln\sigma_T(\chi)(s))' = \frac{(\sigma_T(\chi)(s))'}{\sigma_T(\chi)(s)}, \tag{25}$$

and we know that $\sigma_T(\chi)(s) \neq 0$ for all $|T| < 1$ since it has exponential form, and $\sum_{k=1}^{\infty}\Psi^k(\chi)T^k/k$ converges absolutely for $|T| < 1/n$ as $\left|\Psi^k(\chi)(s)\right| \leq n$. Therefore,

$$\left(\sum_{k=1}^{\infty}\Psi^k(\chi)T^k/k\right)'\left(\sum_{k=0}^{\infty}\chi_\sigma^k T^k\right) = \sum_{n=0}^{\infty}T^n\sum_{k=1}^{n+1}\Psi^k(\chi)\chi_\sigma^{n+1-k} = \sum_{n=1}^{\infty}n\chi_\sigma^n T^{n-1}. \tag{26}$$

It shows that

$$n\chi_\sigma^n = \sum_{k=1}^{n}\Psi^k(\chi)\chi_\sigma^{n-k}. \tag{27}$$

Repeating same calculation, we again get

$$n\chi_\lambda^n = \sum_{k=1}^{n}(-1)^{k-1}\Psi^k(\chi)\chi_\lambda^{n-k}. \tag{28}$$

(b) Since Ψ^k is $\mathbb{Z}$ linear map, it is enough to show that $\Psi^k(\chi) \in R(G)$ for an irreducible character χ on G. I'll show the result for $k \geq 0$, and extend it to $\mathbb{Z}$. Let's use induction on k. For $k = 1$, it is trivial, so assume it is true for $k < K$. For $k = K$, note that

$$\Psi^K(\chi) = K\chi_\sigma^K - \sum_{k=1}^{K-1} \Psi^k(\chi)\chi_\sigma^{K-k}. \tag{29}$$

We know that $R(G)$ is closed under addition and multiplication, and $\chi_\sigma^n \in R(G)$ for all $n \geq 1$. Therefore, we get $\Psi^K(\chi) \in R(G)$ as $\Psi^k(\chi) \in R(G)$ for $k < K$ by the induction hypothesis.

For $k = 0$, it is just $\Psi^0(\chi)(s) = \chi(s^0) = \chi(1)$, so it is $\chi(1)1_G$. For $k < 0$, choose sufficiently large $m > 0$ such that $k + mg > 0$, then

$$\Psi^{k+mg}(\chi)(s) = \chi(s^{k+mg}) = \chi(s^k) = \Psi^k(\chi)(s) \tag{30}$$

for all $s \in G$. Therefore, $\Psi^k(\chi) = \Psi^{k+mg}(\chi) \in R(G)$. It ends the proof.

(If the stability means $\Psi^k : R(G) \to R(G)$ is bijective for all k, it is false: $\Psi^g(\chi) = \chi(1)1_G$ for all irreducible representation χ on G.)

3-4(**S** 9.4)

(a) I'll first show a proposition.

Proposition 3. *Let's define $\varphi : G \to G$ by $\varphi(s) = s^n$. If $(n, g) = 1$, then φ is a bijective map.*

Proof. Since $(n, g) = 1$, there exists $k \in \mathbb{N}$ such that $kn \equiv 1 \mod g$. If $g_1^n = g_2^n$, then $g_1 = g_1^{kn} = g_2^{kn} = g_2$, so φ is injective. Since the domain and codomain have same finite cardinality, φ is bijective. □

Corollary 4. *Let c_1 be a conjugacy class in G. Then the φ maps c_1 to another conjugacy class bijectively, i.e. if I write $c_1' = \operatorname{Im}\varphi(c_1)$, then $\varphi|_{c_1} : c_1 \to c_1'$ is bijective.*

Proof. Let's consider $\varphi|_{c_i}$, then it is contained in some conjugacy class, in fact, it is surjective on the conjugacy class: if $s_1 \in c_i$, then for any $s \in G$, $\varphi(ss_1s^{-1}) = ss_1^n s^{-1}$, so it is contained in some conjugacy class c_1' containing s_1^n and generated any element in the class. Since φ is injective, $\varphi|_{c_1}$ is bijective. □

Corollary 5. *Let $\{c_1, \ldots, c_h\}$ be the set of conjugacy classes in G. Let's define*

$$\Phi : \{c_1, \ldots, c_h\} \to \{c_1, \ldots, c_h\} \tag{31}$$

by $\Phi(c_i) = \operatorname{Im}\varphi(c_i)$. Then, Φ is a bijective map.

Proof. From the above consideration, the map $\Phi : \{c_1, \ldots, c_h\} \to \{c_1, \ldots, c_h\}$ is well-defeind. Since φ is bijective, Φ is again surjective, so bijective. □

Using the proposition, we get

$$\langle \Psi^n(\chi), \Psi^n(\chi) \rangle = \frac{1}{g} \sum_{s \in G} \chi(s^n)\chi(s^{-n}) = \frac{1}{g} \sum_{s \in G} \chi(s)\chi(s^{-1}) = \langle \chi, \chi \rangle = 1. \tag{32}$$

Also, $\Psi^n(\chi)(1) = \chi(1) > 0$. By the problem 9.2, we know that $\Psi^n\chi$ is an irreducible character of G.

(b) The center of the algebra $\mathbb{C}[G]$ is spanned by $e_c = \sum_{s \in c} s$ where c is a conjugacy class of G; in fact, it is a basis. Now, I'll prove a lemma.

Lemma 6. *For two conjugacy classes c_1, c_2 in G, we get*

$$\sum_{s \in c_1} \sum_{s' \in c_2} s^n (s')^n = \sum_{s \in c_1} \sum_{s' \in c_2} (ss')^n. \tag{33}$$

Proof. If G were abelian, then it is easy to see, so assume G is non-abelian. Let's use proposition 13 and algebra homomorphisms ω_i which sends $\sum_{s \in G} u(s)s \in \text{Cent.}\ \mathbb{C}[G]$ to $\mathbb{C}$ by

$$\omega_i\left(\sum_{s \in G} u(s)s\right) = \frac{1}{n_i} \sum_{s \in G} u(s)\chi_i(s), \tag{34}$$

where χ_i is the irreducible character corresponding to ω_i and $n_i = \deg \chi_i$. (For detailed explanation, see Chapter 6.3, **S**.) Since $(\omega_i)_{i=1}^h$, where h is the number of conjugacy classes in G, defines an isomorphism of the center of $\mathbb{C}[G]$ onto the algebra $\mathbb{C}^h$, it is enough to show that

$$\omega_i\left(\sum_{s \in c_1} \sum_{s' \in c_2} s^n (s')^n\right) = \omega_i\left(\sum_{s \in c_1} \sum_{s' \in c_2} (ss')^n.\right) \tag{35}$$

for all i.

Now, let's use corollary 4. Let's set $c_1' = \operatorname{Im}\varphi(c_1)$ and $c_2' = \operatorname{Im}\varphi(c_2)$, then we get

$$\begin{aligned} \sum_{s \in c_1} s^n &= \sum_{s \in c_1'} s \\ \sum_{s' \in c_2} (s')^n &= \sum_{s' \in c_2'} s', \end{aligned} \tag{36}$$

and it shows that both are in the center of $\mathbb{C}[G]$. Now, we get

$$\begin{aligned} \omega_i\left(\sum_{s \in c_1} \sum_{s' \in c_2} s^n (s')^n\right) &= \omega_i\left(\left(\sum_{s \in c_1} s^n\right)\left(\sum_{s' \in c_2} (s')^n\right)\right) \\ &= \omega_i\left(\sum_{s \in c_1} s^n\right)\omega_i\left(\sum_{s' \in c_2} (s')^n\right) \\ &= \frac{1}{n_i^2} \sum_{s \in c_1} \Psi^n\chi_i(s) \sum_{s' \in c_2} \Psi^n\chi_i(s'). \end{aligned} \tag{37}$$

From (a), we know that $\Psi^n\chi_i = \chi_j$ for some j since it is irreducible, and $n_i = n_j$ since $\Psi^n\chi_i(1) = \chi_i(1)$. It shows that

$$\begin{aligned}\omega_i\left(\sum_{s\in c_1}\sum_{s'\in c_2} s^n(s')^n\right) &= \frac{1}{n_i^2}\sum_{s\in c_1}\Psi^n\chi_i(s)\sum_{s'\in c_2}\Psi^n\chi_i(s') \\ &= \frac{1}{n_j^2}\sum_{s\in c_1}\chi_j(s)\sum_{s'\in c_2}\chi_j(s') \\ &= \omega_j\left(\sum_{s\in c_1} s\right)\omega_j\left(\sum_{s'\in c_2} s'\right) \\ &= \omega_j\left(\sum_{s\in c_1}\sum_{s'\in c_2} ss'\right).\end{aligned} \tag{38}$$

Also,

$$\begin{aligned}\omega_i\left(\sum_{s\in c_1}\sum_{s'\in c_2}(ss')^n.\right) &= \frac{1}{n_i}\sum_{s\in c_1, s'\in c_2}\Psi^n\chi_i(ss') \\ &= \frac{1}{n_j}\sum_{s\in c_1, s'\in c_2}\chi_j(ss') \\ &= \omega_j\left(\sum_{s\in c_1, s'\in c_2} ss'\right).\end{aligned} \tag{39}$$

Therefore, (35) holds and the lemma is true for $(n, g) = 1$. □

The lemma shows that ψ_n is algebra endomorphism on the center of $\mathbb{C}[G]$ as it shows

$$\psi_n(e_{c_1})\psi_n(e_{c_2}) = \psi_n(e_{c_1}e_{c_2}) \tag{40}$$

for any conjugacy classes c_1 and c_2 in G. By the corollary 5, we know that $\operatorname{Im}\psi_n$ maps the basis $\{e_c\}$ to the basis $\{e_c\}$ surjectively. Since the domain and codomain have same dimension, it shows that φ is an algebra automorphism on the center of $\mathbb{C}[G]$.

5(**S** 11.1) Assume I showed the following proposition: P2: "Let f be a class function on cyclic group G with values in $\mathbb{Q}$ such that $f(x^m) = f(x)$ for all m primes to g, then $f \in \mathbb{Q}\otimes R(G)$". Let the original statement P1. It is trivial that P1 implies P2. Also, P2 implies P1: using th. 21' in the textbook, it is enough to show that for any cyclic subgroup $H \le G$, $\mathrm{Res}_H f \in \mathbb{Q}\otimes R(H)$.

Since we assumed P2 is true, it is again enough to show that $f(x^m) = f(x)$ for all m prime to $|H|$ for $x \in H$. To show it, let's fix $x \in H$ and m such that $(m, |H|) = 1$. Note that $(m, |G|)$ need not to be 1. However, there always exists $k \in \mathbb{N}$ such that $(m + k|H|, |G|) = 1$ by the Dirichlet's theorem on arithmetic progressions: the original statment of the Dirichelt's theorem is that if $(m, |H|) = 1$, then $m + k|H|$ contains infinitely many primes, which means that there exists k_0 such that $(m + k_0|H|, |G|) = 1$. It means that

$$f(x^m) = f(x^{m+k_0|H|}) = f(x). \tag{41}$$

(The second equality is by the assumption of P1.) Therefore, the condition for P2 is satisfied and I showed that P2 implies P1. Now, I can safely reduce G to a cyclic group.

Let the generator of G by x and $g = |G|$. Choose any irreducible character χ_k from $0 \leq k \leq g-1$ such that $\chi_k(x) = \exp(2\pi ik/g)$ of G, then

$$\langle f, \chi_k \rangle = \sum_{m=1}^{g} f(x^m) \exp\left(\frac{-2\pi ikm}{g}\right). \tag{42}$$

Now, take partition of $\{1, \ldots, g\}$ such that $(a, g) = q$ for $1 \leq q \leq g$, for example,

$$A_q = \{a \in \{1, \ldots, g\} : (a, g) = q\}. \tag{43}$$

Since $(a/q, g/q) = 1$, we get

$$\sum_{m \in A_q} \exp\left(\frac{-2\pi ikm}{g}\right) = \sum_{m \in A_q} \exp\left(\frac{-2\pi ik(m/q)}{g/q}\right) = \sum_{m \in (\mathbb{Z}/(g/q)\mathbb{Z})^\times} \exp\left(\frac{-2\pi ikm}{g/q}\right) \in \mathbb{Z} \tag{44}$$

for each q since the nth cyclotomic polynomial is in $\mathbb{Z}[x]$ for any $n \geq 1$. Now, I'll show a proposition.

Proposition 7. *For any $a_1, a_2 \in \{1, \ldots, g\}$ such that $(a_1, g) = (a_2, g) = q$ for some $q \in \mathbb{Z}$, there exists $m \in \mathbb{N}$ such that $a_1 m \equiv a_2 \mod g$.*

Proof. Consider $a_1/q, a_2/q \in (\mathbb{Z}/(g/q)\mathbb{Z})^\times$, so take $m \in \mathbb{Z}$ such that $(m, g/q) = 1$ and $a_1 m/q - a_2/q \equiv 0 \mod g/q$. It shows that $a_1 m - a_2 \equiv 0 \mod g$. Also, $(m, g) = 1$ since $(a_1 m, g) = (a_1, g)(m, g) = (a_2, g)$. □

The above proposition shows that $f(x^{a_1}) = f(x^{a_2})$. Therefore,

$$\begin{aligned}
\langle f, \chi \rangle &= \frac{1}{g} \sum_{m=1}^{g} f(x^m) \exp\left(\frac{-2\pi ikm}{g}\right) \\
&= \frac{1}{g} \sum_{q=1}^{g} \sum_{m \in A_q} f(x^m) \exp\left(\frac{-2\pi ikm}{g}\right) \\
&= \frac{1}{g} \sum_{q=1}^{g} \sum_{m \in A_q} f(x^q) \exp\left(\frac{-2\pi ikm}{g}\right) \\
&= \frac{1}{g} \sum_{q=1}^{g} f(x^q) \sum_{m \in A_q} \exp\left(\frac{-2\pi ikm}{g}\right) \in \mathbb{Q},
\end{aligned} \tag{45}$$

for any irreducible character χ and it shows that $f \in \mathbb{Q} \otimes R(G)$.

In problem 3, we showed that Ψ^n maps $R(G)$ to $R(G)$, so extending the scalar to $\mathbb{Q}$, we can treat that Ψ^n maps $\mathbb{Q} \otimes R(G)$ to $\mathbb{Q} \otimes R(G)$. Also, if $\operatorname{Im} f \subset \mathbb{Z}$, so $\operatorname{Im} \Psi^n f \subset \mathbb{Z} \subset A$, then $(g/(g, n))\Psi^n f \in A \otimes R(G)$ by theorem 23. It means that for any irreducible character χ of G,

$$g/(g, n)\langle \Psi^n f, \chi \rangle \in \mathbb{Q} \cap A = \mathbb{Z}. \tag{46}$$

Therefore, $(g/(g, n))\Psi^n f \in R(G)$.

For a class function $f(s) = \delta_{s=1}$, $\Psi^n f$ captures elements $s \in G$ such that $|s| \mid n$, i.e. $\Psi^n f = 1$ if $|s| \mid n$ and 0 elsewhere. The above result shows that $g/(g,n)1_{\{s:|s||n\}} \in R(G)$, which generalize the result $g\delta_{s=1}$ is the character of regular representation.

6(Thm 23) From the class, what I need to show is the following:

Proposition 8. *For each conjugacy class c of p-group G, and each irreducible character χ of G, we have $1/(g,n)\sum_{x^n \in c}\chi(x) \in A$.*

and what we actually proved is the following

Proposition 9. *Let c be a conjugacy class of a p-group G, let χ be a character of degree 1 of G,and let $a_c \sum_{x^n \in c}\chi(x)$. Then $a_c \in (g,n)A$.*

To end the proof, I need to show that proposition 9 implies 8.
Before start, I'll prove some propositions.

Proposition 10. *For a conjugacy class c of G, set $c^{-1} = \{s^{-1} : s \in c\}$, then c^{-1} is again a conjugacy class.*

Proof. If $a_1, a_2 \in c^{-1}$, then $a_1^{-1}, a_2^{-1} \in c$, so there exists $s \in G$ such that $sa_1^{-1}s^{-1} = a_2^{-1}$. It shows that $sa_1s^{-1} = a_2$. Conversely, for fixed $a_1 \in c^{-1}$, $sa_1s^{-1} \in c^{-1}$ by the same reason. □

Proposition 11. *For a conjugacy class c of G and a subgroup $H \le G$, $c \cap H$ is a disjoint union of the conjugacy classes in H if $c \cap H \neq \emptyset$.*

Proof. Assume $a \in c \cap H$, then for any $s \in H$, $sas^{-1} \in c \cap H$. It ends the proof. □

From the fact that any irreducible character of p-group is induced by a character of degree 1, for an irreducible character χ of G, choose degree 1 character η of H such that $H \le G$ satisfying $\chi = \mathrm{Ind}_H^G \eta$. Note that any subgroup of p-group is again p-group, so H is p-group. Now,

$$\langle \chi, \Psi^n f_{c^{-1}} \rangle = \frac{1}{g}\sum_{x^n \in c}\chi(x). \tag{47}$$

On the other hands,

$$\langle \chi, \Psi^n f_{c^{-1}} \rangle = \langle \eta, \mathrm{Res}_H \Psi^n f_{c^{-1}} \rangle = \frac{1}{h}\sum_{x \in H}\eta(x) f_{c^{-1}}(x^{-n}) = \frac{1}{h}\sum_{x \in H, x^n \in c \cap H}\eta(x). \tag{48}$$

By the proposition 11, $c \cap H$ is a disjoint union of c_H^i, where these are conjugacy classes of H, so

$$\frac{1}{h}\sum_{x \in H, x^n \in c \cap H}\eta(x) = \frac{1}{h}\sum_i \sum_{x \in H, x^n \in c_H^i}\eta(x) \in (h,n)A \tag{49}$$

by the 9. Finally, it implies

$$\langle \chi, \Psi^n f_{c^{-1}} \rangle \in (h,n)A \subset (g,n)A, \tag{50}$$

and we get

$$\frac{1}{(g,n)}\sum_{x^n \in c}\chi(x) \in A. \tag{51}$$

It ends the proof.