Dummit&Foote Abstract Algebra section 15.2 Solutions for Selected Problems by 3 mod(8)

SungBin Park, 20150462

3. This is exactly same as exercise 2 (c) since $\operatorname{rad}(I \cap J) = \operatorname{rad} I \cap \operatorname{rad} J = I \cap J$ if I, J are radical ideal.

11. Assume that $V \cap U_1 \neq \phi$ and $V \cap U_2 \neq \phi$, but $V \cap U_1 \cap U_2 = \phi$. Then,

$$V \cap (U_1 \cap U_2) = \phi \Leftrightarrow V = V \cap (U_1 \cap U_2)^c = V \cap (U_1^c \cup U_2^c) = (V \cap U_1^c) \cup (V \cap U_2^c).$$

If $(V \cap U_1^c) \neq \phi$ and $(V \cap U_2^c) \neq \phi$, then it is contradiction to irreducibility of V since $(U_1^c \cap V)$ and $(U_2^c \cap V)$ is proper algebraic sets in V. If $(V \cap U_1^c) = V$, then $U_2 = \phi$ and $V \cap U_2 = \phi$ which is contradiction. This is same for U_2. Therefore, $V \cap U_1 \cap U_2 \neq \phi$.

Let U be an nonempty open set in a variety V and assume $\overline{U}$ is proper in V. Then, $V \setminus \overline{U}$ is open in V which is nonempty and $V \cap U \cap \left(V \setminus \overline{U}\right) = \phi$, which is contradiction. Therefore, $\overline{U} = V$.

19. Let $V = \{(x, y) \in \mathbb{R}^2 \,|x^2 + y^2 - 1 = 0\}$ in $\mathbb{R}^2$ and $W = \mathbb{R}$. Let $\varphi : V \to W$ such that $\varphi(x, y) = x$. Then, this is not surjective since projection of V to x-axis is $[-1, 1]$. The k-algebra homomorphism $\tilde{\varphi} : k[x] \to k[V]$, $\tilde{\varphi} : x \mapsto x$. is definitely injective.

27. I'll start with lemma about topology.

Lemma 1. *X is Hausdorff iff diagonal set $\Delta = \{(x,x) | x \in X\}$ is closed in product topology.*

Proof. If X is Hausdorff, then for any (x,y), $x \neq y$, there exist two disjoint open sets $x \in U$, $y \in V$ and $(x,y) \in U \times V$ is disjoint with Δ. Therefore, Δ is closed.

Conversely, suppose that Δ is closed, then for any (x,y), $x \neq y$, there exists open set containing (x,y) disjoint with Δ. By the definition of product topology, there exist two disjoint open set $x \in U$, $y \in V$. Therefore, X is Hausdorff space. $\square$

Consider $\mathcal{Z}((y-x))$ in k^2. This is diagonal set in k^2 and closed by the definition of Zarisky topology. However, this is not closed in product topology of k^2 since Zarisky topology on k is not Hausdorff: all the polynomials on $k[x]$ has finitely many roots, which means this is co-finite topology.

35. (a) I'll first prove that $\varphi(Q_1 \cap \cdots \cap Q_m) = \varphi(Q_1) \cap \cdots \varphi(Q_m)$.

($\subset$): Let $a \in \varphi(Q_1 \cap \cdots \cap Q_m)$, then there exists $b \in Q_1 \cap \cdots \cap Q_m$ such that $\varphi(b) = a$. Therefore, $a = \varphi(b) \in \varphi(Q_1) \cap \cdots \varphi(Q_n)$.

($\supset$): Let $\varphi(Q_1 \cap \cdots \cap Q_m) \not\supset \varphi(Q_1) \cap \cdots \varphi(Q_m)$, then there exists $a_1, a_2 \in Q_1 \cup \cdots \cup Q_m$ such that $a_1 \notin Q_k$ and $a_2 \in Q_k$ for some k and $\varphi(a_1) = \varphi(a_2)$. Then, $\varphi(a_1 - a_2) = 0$ and $a_1 - a_2 \in \ker \varphi \subset Q_k$, which is contradiction. Therefore $\varphi(Q_1 \cap \cdots \cap Q_m) \supset \varphi(Q_1) \cap \cdots \varphi(Q_m)$.

Therefore, $\varphi(Q_1 \cap \cdots \cap Q_m) = \varphi(Q_1) \cap \cdots \varphi(Q_m)$. Using previous exercise, $\operatorname{rad} \varphi(Q_i) = \varphi(P_i)$ which is prime ideal in S.

Consider isomorphism $\overline{\varphi} : R/\ker \varphi \to S$ and $\overline{Q_i} = Q_i/\ker \varphi$. Then, $\varphi(Q_i) \simeq \overline{\varphi}(\overline{Q_i})$. By lattice isomorphism theorem, $Q_i \not\supseteq \cap_{j \neq i} Q_j$ for all j implies $\overline{Q_i} \not\supseteq \cap_{j \neq i} \overline{Q_j}$ and $\varphi(Q_i) \not\supseteq \cap_{j \neq i} \varphi(Q_j)$. Also, $P_i \neq P_j$ for $i \neq j$ implies $\varphi(P_i) \neq \varphi(P_j)$ for $i \neq j$ by the same reason. Therefore, this primary decomposition is minimal.

(b) I'll first prove that $\varphi^{-1}(I) = \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$. Let $b \in \varphi^{-1}(I)$, then there exists $a \in I$ such that $\varphi(b) = a$. It means $a \in Q_1 \cap \cdots \cap Q_m$ and $a \in \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$. Conversely, let $b \in \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$, then there exists a_i in Q_i such that $\varphi(a_i) = b$. Since $\varphi(a_i - a_j) = 0$, $a_i - a_j \in \varphi^{-1}(0) = \ker \varphi \subset \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$ for all j, $a_i \in Q_1 \cap \cdots \cap Q_m$ and $a_i \in \varphi^{-1}(I)$.(By exercise 24 in Section 7.3, $\varphi^{-1}(J)$ is an ideal containing $\ker \varphi$ for any ideal J in S.) Hence, $\varphi^{-1}(I) = \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$.

By exercise 13 in Section 7.4, $\varphi^{-1}(P) = R$ or $\varphi^{-1}(P)$ is a prime ideal of R for any prime ideal P in S. Therefore, $\operatorname{rad} \varphi(Q_i) \subset \varphi(P_i)$ since $\operatorname{rad} \varphi(Q_i)$ is the intersection of all prime ideals containing $\operatorname{rad} \varphi(Q_i)$. Conversely, let $a \in \varphi^{-1}(P_i)$, then $\varphi(a) \in P_i$ and $(\varphi(a))^n \in P_i^n \subset Q_i$ for some i. Therefore, $a \in \operatorname{rad} \varphi^{-1}(Q_i)$ and $\operatorname{rad} \varphi^{-1}(Q_i) = \varphi^{-1}(P_i)$. Hence, $\varphi^{-1}(I) = \varphi^{-1}(Q_1) \cap \cdots \varphi^{-1}(Q_m)$ is primary decomposition of $\varphi^{-1}(I)$.

If φ is surjective, $\varphi^{-1}(P_i) \neq R$ and there is no substitution on the decomposition. If $\varphi^{-1}(P_i) = \varphi^{-1}(P_j)$ for $i \neq j$, then $\varphi(\varphi^{-1}(P_i)) = P_i = P_j = \varphi(\varphi^{-1}(P_i))$, so $P_i \neq P_j$ for $i \neq j$. By the same reason, $\varphi^{-1}(Q_i) \not\supseteq \cap_{j \neq i} \varphi^{-1}(Q_j)$ for all i. Therefore, this is minimal primary decomposition.

43. ($\Leftarrow$) Let $I = P_1 \cap \cdots \cap P_m$ be a minimal primary decomposition such that P_i are prime ideals. For $a \in \operatorname{rad} I$, there exists integer k such that $a^k \in P_1 \cap \cdots \cap P_m$ and it means $a \in P_1 \cap \cdots \cap P_m$ since all the components are prime ideal. Therefore, $a \in I$ and I is a radical ideal.

($\Rightarrow$) I'll follow the hint.

Let $I = Q_1 \cap \cdots \cap Q_m$ is radical with Q_i a P_i-primary component of a minimal decomposition. Since Q_i are primary, there exists n_i such that $P_i^{n_i} \subset Q_i \subset P_i$. Fix $a \in P_1 \cap \cdots \cap P_m$, then $a^{\max\{n_1,\ldots,n_m\}} \in Q_1 \cap \cdots \cap Q_m = I$. Hence, $a \in I$ since I is radical. It implies $P_1 \cap \cdots \cap P_m \subset I \subset P_1 \cap \cdots \cap P_m$ and $I = P_1 \cap \cdots \cap P_m$.

I need to show that this is a minimal primary decomposition. Fix arbitrary i, then there exists $b \notin Q_i$ such that $b \in Q_j$ for $j \neq i$. Let's assume that $b \in \operatorname{rad} Q_i = P_i$, then $b \in \bigcap_i P_i = I$, but it means $b \in Q_i$ by the construction of I. Therefore, $b \notin P_i$ and $b \in P_j$ for $j \neq i$. Since all P_i are distinct, $I = P_1 \cap \cdots \cap P_m$ is minimal primary decomposition.

If $a \in P_i$, then $ab \in P_i$ and $ab \in I$. Thus, $ab \in Q_i$ and $a^k \in Q_i$ for some k since $b \notin P_i$. I couldn't find a reason for $a \in Q_i$, but I'll assume it for further step.

Therefore, $Q_i = P_i$. So, the primary components of a minimal primary decomposition are all prime ideals.

Let $I = P_1 \cap \cdots \cap P_m = P_1' \cap \cdots P_{m'}'$, P are prime ideals, and I is a radical ideal. By the minimality of primary decomposition, $\{P_1, \ldots, P_m\} = \{P_1', \ldots, P_{m'}'\}$ and it means $m = m'$ and $P_1' \cap \cdots P_{m'}'$ is just a permutation of $P_1 \cap \cdots \cap P_m$. Therefore, the minimal primary decomposition is unique.