Dummit&Foote Abstract Algebra section 15.4 Solutions for Selected Problems by 3 mod(8)

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3. (a) $(\varphi^{-1}((R/I)[x_1,\dots,x_i]) \supset R[x_1,\dots,x_i] + I[x_1,\dots,x_n])$:
$\varphi(R[x_1,\dots,x_i] + I[x_1,\dots,x_n]) = \varphi(R[x_1,\dots,x_i]) = \varphi|_{R[x_1,\dots,x_i]}(R[x_1,\dots,x_i]) = (R/I)[x_1,\dots,x_i]$ since $\varphi : rx_1^{d_1}\cdots x_i^{d_i} \mapsto \varphi(r)x_1^{d_1}\cdots x_i^{d_i}$ for $r \in R$ and d_i nonnegative integers. Therefore, $\varphi^{-1}((R/I)[x_1,\dots,x_i]) \supset R[x_1,\dots,x_i] + I[x_1,\dots,x_n]$.
$(\varphi^{-1}((R/I)[x_1,\dots,x_i]) \subset R[x_1,\dots,x_i] + I[x_1,\dots,x_n])$:
Let $\varphi(p) \in (R/I)[x_1,\dots,x_i]$ for some $p \in R[x_1,\dots,x_n]$. Then, all the terms of p having one of $x_{i+1},\dots,x_n$ should have coefficient in I, unless $\varphi(p)$ have nonzero terms having one of $x_{i+1},\dots,x_n$, so $\varphi(p) \notin (R/I)[x_1,\dots,x_i]$, which is contradiction. Therefore, there exists $q \in I[x_1,\dots,x_n]$ such that $p - q \in R[x_1,\dots,x_i]$ and $p \in R[x_1,\dots,x_i] + I[x_1,\dots,x_n]$.

(b) $(\varphi(A \cap R[x_1,\dots,x_i]) \subset \overline{A} \cap (R/I)[x_1,\dots,x_i])$:
$\varphi(A) = \overline{A}$ and $\varphi(R[x_1,\dots,x_i]) = (R/I)[x_1,\dots,x_i]$, so $\varphi(A \cap R[x_1,\dots,x_i]) \subset \overline{A} \cap (R/I)[x_1,\dots,x_i]$.
$(\varphi(A \cap R[x_1,\dots,x_i]) \supset \overline{A} \cap (R/I)[x_1,\dots,x_i])$:
Let $\bar{p} \in \overline{A} \cap (R/I)[x_1,\dots,x_i]$ since $\overline{A} \cap (R/I)[x_1,\dots,x_i] \neq \phi$ because it contains 0. Let's write $\bar{p} = \sum\limits_{\alpha} \bar{r}_\alpha x^\alpha$, α is multiindex, such that $\bar{r}_\alpha \in R/I$ and there is no terms containing $x_{i+1},\dots,x_n$ since $\bar{p} \in (R/I)[x_1,\dots,x_i]$. Take $p = \sum\limits_{\alpha} r_\alpha x^\alpha$. Then, $\varphi(p) = \bar{p}$, $p \in \varphi^{-1}(\overline{A}) = A$, $p \in R[x_1,\dots,x_i]$. Therefore, $p \in \varphi(A \cap R[x_1,\dots,x_i])$ and $\varphi(A \cap R[x_1,\dots,x_i]) \subset \overline{A} \cap (R/I)[x_1,\dots,x_i]$.

11. I'll use proposition 42 (3) in section 15.4. Since $D = R \setminus P$ in R_P, $P \cap D = \phi$. If Q is a P-primary ideal of R, by proposition 42 (3), ${}^c({}^e(Q)) = Q$ in R_P. For the same reason, $P' \cap D = \phi$ if $P' \subset P$, and $D^{-1}P'$ is a prime in R_P. Also, ${}^c({}^e(Q)) = Q$ in R_P for P'-primary Q.

19. As R is an integral domain with 1, $R \setminus \{0\}$ is a multiplicative set containing 1. Let $F = \mathrm{Frac}(R) = R_{R\setminus\{0\}}$ be a fraction field of R. Then, $\mathrm{Frac}(D^{-1}R) \simeq F$. ($D^{-1}R$ is integral domain: if $\frac{r_1}{d_1}\frac{r_2}{d_2} = 0$, $dr_1r_2 = 0$ for some $d \in D$, but it means $r_1 = 0$ or $r_2 = 0$ and $\frac{r_1}{d_1} = 0$ or $\frac{r_2}{d_2} = 0$, which is contradiction. Therefore, we can define the fractional field fo $D^{-1}R$. Let $\Phi : F \to \mathrm{Frac}(D^{-1}R)$ by $\Phi : \frac{r_1}{r_2} \mapsto \frac{\frac{r_1}{1}}{\frac{r_2}{1}}$. It is well-defined since if $\frac{r_1}{r_2} = \frac{r_3}{r_4}$, $r_1r_4 - r_2r_4 = 0$ and it means $\frac{\frac{r_1}{1}}{\frac{r_2}{1}} = \frac{\frac{r_3}{1}}{\frac{r_4}{1}}$. Also, it is ring homomorphism since $\Phi\left(\frac{r_1}{r_2} + \frac{r_3}{r_4}\right) = \Phi\left(\frac{r_1r_4+r_2r_3}{r_2r_4}\right) = \frac{\frac{r_1r_4+r_2r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1r_4}{1}+\frac{r_2r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1}{1}}{\frac{r_2}{1}} + \frac{\frac{r_3}{1}}{\frac{r_4}{1}} = \Phi\left(\frac{r_1}{r_2}\right) + \Phi\left(\frac{r_3}{r_4}\right)$ and $\Phi\left(\frac{r_1}{r_2}\frac{r_3}{r_4}\right) = \frac{\frac{r_1r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1}{1}}{\frac{r_2}{1}}\frac{\frac{r_3}{1}}{\frac{r_4}{1}} = \Phi\left(\frac{r_1}{r_2}\right)\Phi\left(\frac{r_3}{r_4}\right)$. Also, it is surjective since for any $\frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} \in \mathrm{Frac}(D^{-1}R)$, $\Phi\left(\frac{r_1d_2}{r_2d_1}\right) = \frac{\frac{r_1d_2}{1}}{\frac{r_2d_1}{1}} = \frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}}$ since $\frac{r_1d_2r_2}{d_2} = \frac{r_1r_2}{1} = \frac{r_1r_2d_1}{d_1}$. As a surjective field homomorphism, it is a field isomorphism, and F and $\mathrm{Frac}(D^{-1}R)$ is isomorphic.)

Choose $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right) \in \mathrm{Frac}(D^{-1}R)$. Since R is integrally closed, there exists monic polynomial $p(x) = x^n + \sum_{i=0}^{n-1} a_ix^i$, $a_i \in R$, such that $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$. My claim is that $\pi(p(x)) = \tilde{p}(x) = x^n + \sum_{i=0}^{n-1} \pi(a_i)x^i \in D^{-1}R[x]$ has a root $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$.(Define $\pi : R \to D^-1R$, $\pi : r \mapsto \frac{r}{1}$.) With the field isomorphism Φ,

$$\Phi\left(\tilde{p}\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)\right) = \Phi\left(\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^n + \sum_{i=0}^{n-1}\pi(a_i)\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^i\right)$$

$$= \left(\frac{r_1d_2}{r_2d_1}\right)^n + \sum_{i=0}^{n-1}\frac{a_i}{1}\left(\frac{r_1d_2}{r_2d_1}\right)^i = 0$$

sine $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$. Since Φ is isomorphism, it means that $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$ is integral over $D^{-1}R$ and $D^{-1}R$ is integrally closed.

27. First, I'll compute $\mathbb{T}_{t,V}$, $\mathbb{T}_{(w_1,w_2,w_3),W}$. Since $V = I(0)$, $D_v(0) = 0$ and $\mathbb{T}_{t,V} = \mathbb{A}^1$. If $w = (w_1, w_2, w_3) \neq 0$,

$$D_w(f)(x,y,z) = (w_3 x - 2w_2 y + w_1 z, -3w_1^2 x + w_3 y + w_2 z, -2w_1 w_2 x - w_1^2 y + 2w_3 z)$$

For $\varphi(t)$,

$$D_{\varphi(t)}(f)(x,y,z) = (t^5 s - 2t^4 y + t^3 z, -3t^6 x + t^5 y + t^4 z, -2t^7 x - t^6 y + 2t^5 z)$$

Computing $\mathcal{Z}\left(D_{\varphi(t)}(f)(x,y,z)\right)$, $\mathbb{T}_{(\varphi(t),W)} = \left(1, \frac{4}{3}t, \frac{5}{3}t^2\right) x$. If $w = 0$, then $D_0(f)(x,y,z) = 0$ and $\mathbb{T}_{(0,W)} = \mathbb{A}^3$. It describes all $\mathbb{T}_{w,W}$, $w \in W$ since φ is set-theoretically bijective between $\mathbb{A}^1$ and W.

Using previous exercise, for each $t \in \mathbb{A}^1$, $d\varphi : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{\varphi(t),V}$ is given explicitly by

$$d\varphi(a) = (D_t(t\varphi_1)(a), D_t(\varphi_2)(a), D_t(\varphi_3)(a)) = (3t^2 a, 4t^3 a, 5t^4 a) = 3t^2 \left(1, \frac{4}{3}t, \frac{5}{3}t^2\right) a$$

It shows that $d\phi$ is vector space isomorphism for $t \neq 0$. However, for $t = 0$, $\mathbb{T}_{0,\mathbb{A}^1} = \mathbb{A}^1$, but $\mathbb{T}_{0,W} = \mathbb{A}^3$ and $d\varphi(a) = 0$.

Finally, assume that V and W are isomorphic and there exists isomorphism Φ. Then, $d\Phi : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{\varphi(t),V}$ should be isomorphism on $t = 0$, but there can not exists such isomorphism between $\mathbb{A}^1$ and $\mathbb{A}^3$ at $w = 0$. Therefore, V and W are not isomorphic.

35. Let P be a prime ideal in R such that $P \cap D = \phi$ and $P \in \mathrm{Ass}_R(M)$. First, $D^{-1}P$ is a prime ideal in $D^{-1}R$ by proposition 38 (3). Let an annihilated element in R be m. Then, $R/P \simeq Rm$ since $P \in \mathrm{Ass}_R(M)$ with annihilated element m.

Consider an exact sequence such that i is inclusion of P into R and $\varphi : R \to Rm$ by $r \mapsto rm$. Since $D^{-1}R$ is a flat R-module, the following sequence is also exact.

$$0 \longrightarrow P \xrightarrow{i} R \xrightarrow{\varphi} Rm \longrightarrow 0$$

$$0 \longrightarrow D^{-1}P \xrightarrow{1\otimes i} D^{-1}R \xrightarrow{1\otimes\varphi} D^{-1}(Rm) \longrightarrow 0$$

Since $1 \otimes \varphi : \frac{r}{d} \mapsto \frac{rm}{d}$, $\ker(1 \otimes \varphi) = D^{-1}P$ means that $D^{-1}P$ is the annihilator of an element $\frac{m}{1}$, and $D^{-1}P \in \mathrm{Ass}_{D^{-1}R}(D^{-1}M)$.