Dummit&Foote Abstract Algebra section 15.3 Solutions for Selected Problems by 3 mod(8)

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3. Let $\varphi[x, y] \to k[t]$ be a function sending $x \to t^j$ and $y \to t^i$. Then, it is a ring homomorphism such that the kernel of φ is $(x^i - y^j)$.(1) Clearly, $\operatorname{Im}\varphi \subset k[t]$.

   Choose $a, b \in \mathbb{Z}$ such that $ai - bj = 1$. Since $\frac{t^{ai}}{t^{bj}} = t$, the quotient field of $\operatorname{Im}\varphi$ is $k(t)$. As $\operatorname{Im}\varphi$ is in $k[t]$ having same quotient field, the integral closure of $\operatorname{Im}\varphi$ is smaller than $k[t]$. However, $y^i - t^i \in \operatorname{Im}\varphi[y]$ has a root t, so the integral closure of $\operatorname{Im}\varphi$ is same as $k[t]$.

   Using the ring isomorphism $\varphi : k[x, y]/(x^i - y^j) \to \operatorname{Im}\varphi$, we can extend to field isomorphism $\tilde{\varphi}$ between quotient fields by sending $\overline{\frac{a(x,y)}{b(x,y)}}$ to $\frac{\varphi(\overline{a})}{\varphi(\overline{b})}$.(cf. Theorem 15. (2) in section 7.5.)

   For $p(x, y)$ in the quotient field of $k[x, y]/(x^i - y^j)$, which is a root of monic polynomial $f(z)$ in $(k[x, y]/(x^i - y^j))\,[z]$, $\tilde{\varphi}(p(x, y)) \in k(t)$ is again a root of monic polynomial $\tilde{\varphi}(f(z)) \in (\operatorname{Im}\varphi)\,[z]$ sending $z \mapsto z$ and this is true for vice versa. Therefore, the normalization of $k[x, y]/(x^i - y^j)$ is isomorphic to the normalization of $\operatorname{Im}\varphi$.

   Hence, the normalization of the integral domain R is $k\left[\frac{\bar{x}^a}{\bar{y}^b}\right]$ for $\bar{x} = x + (x^i - y^j)$, $\bar{y} = y + x + (x^i - y^j)$.

   (1): It is clear that the kernel of φ contains $(x^i - y^j)$. For $\overline{f(x, y)} \in k[x, y]/(x^i - y^j)$, we can rewrite it as $\sum_{i=0}^{j-1} y^i f_i(x)$ for some $f_i(x) \in k[x]$ since $y^j = x^i$. $\varphi(x^r y^s) = t^{rj-si}$ for $0 \leq s \leq j-1$ are all distinct since i and j are co-prime. Therefore, $\tilde{\varphi} : k[x, y]/(x^i - y^j) \to k[t]$ is injective and $(x^i - y^j)$ is the kernel of φ.

11. Since $k[x]$ is U.F.D., there exists monic irreducible polynomials $q_i^1(x)$, $q_j^2(x)$ and units u^1, u^2 such that $a(x) = u^1 \prod_{i=1}^n q_i^1(x)$, $b(x) = u^2 \prod_{j=1}^m q_j^2(x)$. Fix an irreducible factor $q(x)$ in the factorization, then there is a root α which is integral over R since α is a root of $p(x)$. By exercise 10 above, $q(x) \in R[x]$. It means $a(x), b(x) \in R[x]$. (As $p(x)$, $a(x)$, $b(x)$ are monic, $u^1 = u^2 = 1$.)

19. Computing reduced Gröbner basis using computer for $(x^3+y^3+z^3, x^2+y^2+z^2, (x+y+z)^3, 1-tx)$ in ordering $x > y > z > t$, it produces (1). In the algorithm computing Gröbner basis, there is no multiplication of constant not appearing in the coefficient of the polynomials, i.e., we only multiply integers having 2 or 3 as prime factors since the coefficient in $(x^3+y^3+z^3, x^2+y^2+z^2, (x+y+z)^3, 1-tx)$ are just $1, -1, 3, 6$. It means if $\mathrm{ch}(k) \neq 2, 3$, then the polynomial does not lose their terms in Gröbner basis algorithm, and we safely get (1) as a reduced Gröbner basis for $(x^3+y^3+z^3, x^2+y^2+z^2, (x+y+z)^3, 1-tx)$. By the symmetry, we also get (1) as the reduced Gröbner basis for $(x^3+y^3+z^3, x^2+y^2+z^2, (x+y+z)^3, 1-ty)$, $(x^3+y^3+z^3, x^2+y^2+z^2, (x+y+z)^3, 1-tz)$ and it means $x, y, z \in \operatorname{rad} I$.

27. (a) By exercise 26 (d) above, we know that $\mathcal{Z}_{\bar{k}}(I)$ is finite. Let $m = |\mathcal{Z}_{\bar{k}}(I)|$. I'll write $a^i = (a_1^i, a_2^i, \ldots, a_n^i)$ for $a^i \in \mathcal{Z}_{\bar{k}}(I)$. For $a^i \in \mathcal{Z}_{\bar{k}}(I)$, $\mathcal{I}_{\bar{k}}(a^i) = (x - a_1^i, x - a_2^i, \ldots, x - a_n^i)$, so $\mathcal{I}_{\bar{k}}\left(\mathcal{Z}_{\bar{k}}(I)\right) = \operatorname{rad} I' = \bigcap_{i=1}^m (x - a_1^i, x - a_2^i, \ldots, x - a_n^i) = \prod_{i=1}^m (x - a_1^i, x - a_2^i, \ldots, x - a_n^i)$ since each ideals is maximal. By Chinese Remainder Theorem,

$$\begin{aligned}
&\bar{k}[x_1, \ldots, x_n]/\operatorname{rad} I' \\
&\qquad \cong \bar{k}[x_1, \ldots, x_n]/(x - a_1^1, \ldots, x - a_n^1) \times \cdots \times \bar{k}[x_1, \ldots, x_n]/(x - a_1^m, \ldots, x - a_n^m) \\
&\qquad \cong \bar{k}^m.
\end{aligned}$$

Therefore, $|\mathcal{Z}_{\bar{k}}(I)| = \dim_{\bar{k}} \bar{k}[x_1, x_2, \ldots, x_n]/\operatorname{rad} I'$.

(b) i. Since $k \subset \bar{k}$, $\mathcal{Z}(I) \subset \mathcal{Z}_{\bar{k}}(I)$.

ii. By exercise 43 in Section 1, $\dim_k k[x_1, \ldots, x_n]/I = \dim_{\bar{k}} \bar{k}[x_1, \ldots, x_n]/I'$.

iii. Since $\operatorname{rad} I' \supset I'$, $\dim_{\bar{k}} \bar{k}[x_1, \ldots, x_n]/\operatorname{rad} I' \leq \dim_{\bar{k}} \bar{k}[x_1, \ldots, x_n]/I'$: If $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)$ is a linearly independent set in $\bar{k}[x_1, \ldots, x_n]/\operatorname{rad} I'$, but $\sum_{i=1}^m k_i b_i \in I'$ for $\{k_i\} \in k$, then $\sum_{i=1}^m k_i b_i \in \operatorname{rad} I'$ and makes contradiction.

Combining these facts, we can get $|\mathcal{Z}(I)| \leq \dim_k k[x_1, \ldots, x_n]/I$.