

# Dummit&Foote Abstract Algebra section 15.4 Solutions for Selected Problems by 3 mod(8)

SungBin Park, 20150462

3. (a)  $(\varphi^{-1}((R/I)[x_1, \dots, x_i]) \supset R[x_1, \dots, x_i] + I[x_1, \dots, x_n]):$   
 $\varphi(R[x_1, \dots, x_i] + I[x_1, \dots, x_n]) = \varphi(R[x_1, \dots, x_i]) = \varphi|_{R[x_1, \dots, x_i]}(R[x_1, \dots, x_i]) = (R/I)[x_1, \dots, x_i]$  since  $\varphi : rx_1^{d_1} \cdots x_i^{d_i} \mapsto \varphi(r)x_1^{d_1} \cdots x_i^{d_i}$  for  $r \in R$  and  $d_i$  nonnegative integers. Therefore,  $\varphi^{-1}((R/I)[x_1, \dots, x_i]) \supset R[x_1, \dots, x_i] + I[x_1, \dots, x_n]$ .  
 $(\varphi^{-1}((R/I)[x_1, \dots, x_i]) \subset R[x_1, \dots, x_i] + I[x_1, \dots, x_n]):$   
 Let  $\varphi(p) \in (R/I)[x_1, \dots, x_i]$  for some  $p \in R[x_1, \dots, x_n]$ . Then, all the terms of  $p$  having one of  $x_{i+1}, \dots, x_n$  should have coefficient in  $I$ , unless  $\varphi(p)$  have nonzero terms having one of  $x_{i+1}, \dots, x_n$ , so  $\varphi(p) \notin (R/I)[x_1, \dots, x_i]$ , which is contradiction. Therefore, there exists  $q \in I[x_1, \dots, x_n]$  such that  $p - q \in R[x_1, \dots, x_i]$  and  $p \in R[x_1, \dots, x_i] + I[x_1, \dots, x_n]$ .
- (b)  $(\varphi(A \cap R[x_1, \dots, x_i]) \subset \overline{A} \cap (R/I)[x_1, \dots, x_i]):$   
 $\varphi(A) = \overline{A}$  and  $\varphi(R[x_1, \dots, x_i]) = (R/I)[x_1, \dots, x_i]$ , so  $\varphi(A \cap R[x_1, \dots, x_i]) \subset \overline{A} \cap (R/I)[x_1, \dots, x_i]$ .  
 $(\varphi(A \cap R[x_1, \dots, x_i]) \supset \overline{A} \cap (R/I)[x_1, \dots, x_i]):$   
 Let  $\bar{p} \in \overline{A} \cap (R/I)[x_1, \dots, x_i]$  since  $\overline{A} \cap (R/I)[x_1, \dots, x_i] \neq \phi$  because it contains 0. Let's write  $\bar{p} = \sum_{\alpha} \bar{r}_{\alpha} x^{\alpha}$ ,  $\alpha$  is multiindex, such that  $\bar{r}_{\alpha} \in R/I$  and there is no terms containing  $x_{i+1}, \dots, x_n$  since  $\bar{p} \in (R/I)[x_1, \dots, x_i]$ . Take  $p = \sum_{\alpha} r_{\alpha} x^{\alpha}$ . Then,  $\varphi(p) = \bar{p}$ ,  $p \in \varphi^{-1}(\overline{A}) = A$ ,  $p \in R[x_1, \dots, x_i]$ . Therefore,  $p \in \varphi(A \cap R[x_1, \dots, x_i])$  and  $\varphi(A \cap R[x_1, \dots, x_i]) \supset \overline{A} \cap (R/I)[x_1, \dots, x_i]$ .

11. I'll use proposition 42 (3) in section 15.4. Since  $D = R \setminus P$  in  $R_P$ ,  $P \cap D = \emptyset$ . If  $Q$  is a  $P$ -primary ideal of  $R$ , by proposition 42 (3),  ${}^e({}^e(Q)) = Q$  in  $R_P$ . For the same reason,  $P' \cap D = \emptyset$  if  $P' \subset P$ , and  $D^{-1}P'$  is a prime in  $R_P$ . Also,  ${}^e({}^e(Q)) = Q$  in  $R_P$  for  $P'$ -primary  $Q$ .

19. As  $R$  is an integral domain with 1,  $R \setminus \{0\}$  is a multiplicative set containing 1. Let  $F = \text{Frac}(R) = R_{R \setminus \{0\}}$  be a fraction field of  $R$ . Then,  $\text{Frac}(D^{-1}R) \simeq F$ . ( $D^{-1}R$  is integral domain: if  $\frac{r_1}{d_1} \frac{r_2}{d_2} = 0$ ,  $dr_1r_2 = 0$  for some  $d \in D$ , but it means  $r_1 = 0$  or  $r_2 = 0$  and  $\frac{r_1}{d_1} = 0$  or  $\frac{r_2}{d_2} = 0$ , which is contradiction. Therefore, we can define the fractional field for  $D^{-1}R$ . Let  $\Phi : F \rightarrow \text{Frac}(D^{-1}R)$  by  $\Phi : \frac{r_1}{r_2} \mapsto \frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}}$ . It is well-defined since if  $\frac{r_1}{r_2} = \frac{r_3}{r_4}$ ,  $r_1r_4 - r_2r_3 = 0$  and it means  $\frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} = \frac{\frac{r_3}{d_1}}{\frac{r_4}{d_2}}$ . Also, it is ring homomorphism since  $\Phi\left(\frac{r_1}{r_2} + \frac{r_3}{r_4}\right) = \Phi\left(\frac{r_1r_4 + r_2r_3}{r_2r_4}\right) = \frac{\frac{r_1r_4 + r_2r_3}{d_1}}{\frac{r_2r_4}{d_2}} = \frac{\frac{r_1r_4}{d_1} + \frac{r_2r_3}{d_1}}{\frac{r_2r_4}{d_2}} = \frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} + \frac{\frac{r_3}{d_1}}{\frac{r_4}{d_2}} = \Phi\left(\frac{r_1}{r_2}\right) + \Phi\left(\frac{r_3}{r_4}\right)$  and  $\Phi\left(\frac{r_1}{r_2} \frac{r_3}{r_4}\right) = \frac{\frac{r_1r_3}{d_1}}{\frac{r_2r_4}{d_2}} = \frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} \frac{\frac{r_3}{d_1}}{\frac{r_4}{d_2}} = \Phi\left(\frac{r_1}{r_2}\right) \Phi\left(\frac{r_3}{r_4}\right)$ . Also, it is surjective since for any  $\frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} \in \text{Frac}(D^{-1}R)$ ,  $\Phi\left(\frac{r_1d_2}{r_2d_1}\right) = \frac{\frac{r_1d_2}{d_1}}{\frac{r_2d_1}{d_2}} = \frac{r_1}{r_2}$  since  $\frac{r_1d_2r_2}{d_2} = \frac{r_1r_2}{1} = \frac{r_1r_2d_1}{d_1}$ . As a surjective field homomorphism, it is a field isomorphism, and  $F$  and  $\text{Frac}(D^{-1}R)$  is isomorphic.) Choose  $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right) \in \text{Frac}(D^{-1}R)$ . Since  $R$  is integrally closed, there exists monic polynomial  $p(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$ ,  $a_i \in R$ , such that  $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$ . My claim is that  $\pi(p(x)) = \tilde{p}(x) = x^n + \sum_{i=0}^{n-1} \pi(a_i) x^i \in D^{-1}R[x]$  has a root  $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$ . (Define  $\pi : R \rightarrow D^{-1}R$ ,  $\pi : r \mapsto \frac{r}{1}$ .) With the field isomorphism  $\Phi^{-1}$ ,

$$\begin{aligned} \Phi^{-1}\left(\tilde{p}\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)\right) &= \Phi^{-1}\left(\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^n + \sum_{i=0}^{n-1} \pi(a_i) \left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^i\right) \\ &= \left(\frac{r_1d_2}{r_2d_1}\right)^n + \sum_{i=0}^{n-1} \frac{a_i}{1} \left(\frac{r_1d_2}{r_2d_1}\right)^i = 0 \end{aligned}$$

since  $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$ . Since  $\Phi$  is isomorphism, it means that  $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$  is integral over  $D^{-1}R$  and  $D^{-1}R$  is integrally closed.

27. First, I'll compute  $\mathbb{T}_{t,V}$ ,  $\mathbb{T}_{(w_1,w_2,w_3),W}$ . Since  $V = I(0)$ ,  $D_v(0) = 0$  and  $\mathbb{T}_{t,V} = \mathbb{A}^1$ . If  $w = (w_1, w_2, w_3) \neq 0$ ,

$$D_w(f)(x, y, z) = (w_3x - 2w_2y + w_1z, -3w_1^2x + w_3y + w_2z, -2w_1w_2x - w_1^2y + 2w_3z)$$

For  $\varphi(t)$ ,

$$D_{\varphi(t)}(f)(x, y, z) = (t^5s - 2t^4y + t^3z, -3t^6x + t^5y + t^4z, -2t^7x - t^6y + 2t^5z)$$

Computing  $\mathcal{Z}(D_{\varphi(t)}(f)(x, y, z))$ ,  $\mathbb{T}_{(\varphi(t),W)} = (1, \frac{4}{3}t, \frac{5}{3}t^2)x$ . If  $w = 0$ , then  $D_0(f)(x, y, z) = 0$  and  $\mathbb{T}_{(0,W)} = \mathbb{A}^3$ . It describes all  $\mathbb{T}_{w,W}$ ,  $w \in W$  since  $\varphi$  is set-theoretically bijective between  $\mathbb{A}^1$  and  $W$ .

Using previous exercise, for each  $t \in \mathbb{A}^1$ ,  $d\varphi : \mathbb{T}_{t,\mathbb{A}^1} \rightarrow \mathbb{T}_{\varphi(t),V}$  is given explicitly by

$$d\varphi(a) = (D_t(t\varphi_1)(a), D_t(\varphi_2)(a), D_t(\varphi_3)(a)) = (3t^2a, 4t^3a, 5t^4a) = 3t^2 \left(1, \frac{4}{3}t, \frac{5}{3}t^2\right) a$$

It shows that  $d\phi$  is vector space isomorphism for  $t \neq 0$ . However, for  $t = 0$ ,  $\mathbb{T}_{0,\mathbb{A}^1} = \mathbb{A}^1$ , but  $\mathbb{T}_{0,W} = \mathbb{A}^3$  and  $d\varphi(a) = 0$ .

Finally, assume that  $V$  and  $W$  are isomorphic and there exists isomorphism  $\Phi$ . Then,  $d\Phi : \mathbb{T}_{t,\mathbb{A}^1} \rightarrow \mathbb{T}_{\varphi(t),V}$  should be isomorphism on  $t \neq 0$ , but there can not exists such isomorphism between  $\mathbb{A}^1$  and  $\mathbb{A}^3$  at  $w = 0$ . Therefore,  $V$  and  $W$  are not isomorphic.

35. Let  $P$  be a prime ideal in  $R$  such that  $P \cap D = \phi$  and  $P \in \text{Ass}_R(M)$ . First,  $D^{-1}P$  is a prime ideal in  $D^{-1}R$  by proposition 38 (3). Let an annihilated element in  $R$  be  $m$ . Then,  $R/P \simeq Rm$  since  $P \in \text{Ass}_R(M)$  with annihilated element  $m$ .

Consider an exact sequence such that  $i$  is inclusion of  $P$  into  $R$  and  $\varphi : R \rightarrow Rm$  by  $r \mapsto rm$ . Since  $D^{-1}R$  is a flat  $R$ -module, the following sequence is also exact.

$$0 \longrightarrow P \xrightarrow{i} R \xrightarrow{\varphi} Rm \longrightarrow 0$$

$$0 \longrightarrow D^{-1}P \xrightarrow{1 \otimes i} D^{-1}R \xrightarrow{1 \otimes \varphi} D^{-1}(Rm) \longrightarrow 0.$$

$1 \otimes \varphi : \frac{r}{d} \mapsto \frac{rm}{d}$ ,  $\ker(1 \otimes \varphi) = D^{-1}P$  means that  $D^{-1}P$  is the annihilator of the element  $\frac{m}{1}$  in  $D^{-1}R$ , and  $D^{-1}P \in \text{Ass}_{D^{-1}R}(D^{-1}M)$ .