# Dummit&Foote Abstract Algebra section 15.4 Solutions for Selected Problems by 3 mod(8)

SungBin Park, 20150462

3.  (a)  $(\varphi^{-1}\left((R/I)[x_1, \ldots, x_i]\right) \supset R[x_1, \ldots, x_i] + I[x_1, \ldots, x_n])$:

$\varphi(R[x_1, \ldots, x_i] + I[x_1, \ldots, x_n]) = \varphi(R[x_1, \ldots, x_i]) = \varphi|_{R[x_1,\ldots,x_i]}(R[x_1, \ldots, x_i]) = (R/I)[x_1, \ldots, x_i]$ since $\varphi : rx_1^{d_1} \cdots x_i^{d_i} \mapsto \varphi(r)x_1^{d_1} \cdots x_i^{d_i}$ for $r \in R$ and $d_i$ nonnegative integers. Therefore, $\varphi^{-1}((R/I)[x_1, \ldots, x_i]) \supset R[x_1, \ldots, x_i] + I[x_1, \ldots, x_n]$.

$(\varphi^{-1}\left((R/I)[x_1, \ldots, x_i]\right) \subset R[x_1, \ldots, x_i] + I[x_1, \ldots, x_n])$:

Let $\varphi(p) \in (R/I)[x_1, \ldots, x_i]$ for some $p \in R[x_1, \ldots, x_n]$. Then, all the terms of $p$ having one of $x_{i+1}, \ldots, x_n$ should have coefficient in $I$, unless $\varphi(p)$ have nonzero terms having one of $x_{i+1}, \ldots, x_n$, so $\varphi(p) \notin (R/I)[x_1, \ldots, x_i]$, which is contradiction. Therefore, there exists $q \in I[x_1, \ldots, x_n]$ such that $p - q \in R[x_1, \ldots, x_n]$ and $p \in R[x_1, \ldots, x_i] + I[x_1, \ldots, x_n]$.

(b)  $(\varphi(A \cap R[x_1, \ldots, x_i]) \subset \overline{A} \cap (R/I)[x_1, \ldots, x_i])$:

$\varphi(A) = \overline{A}$ and $\varphi(R[x_1, \ldots, x_i]) = (R/I)[x_1, \ldots, x_i]$, so $\varphi(A \cap R[x_1, \ldots, x_i]) \subset \overline{A} \cap (R/I)[x_1, \ldots, x_i]$.

$(\varphi(A \cap R[x_1, \ldots, x_i]) \supset \overline{A} \cap (R/I)[x_1, \ldots, x_i])$:

Let $\bar{p} \in \overline{A} \cap (R/I)[x_1, \ldots, x_i]$ since $\overline{A} \cap (R/I)[x_1, \ldots, x_i] \neq \phi$ because it contains 0. Let's write $\bar{p} = \sum_{\alpha} \bar{r}_\alpha x^\alpha$, $\alpha$ is multiindex, such that $\bar{r}_\alpha \in R/I$ and there is no terms containing $x_{i+1}, \ldots, x_n$ since $\bar{p} \in (R/I)[x_1, \ldots, x_i]$. Take $p = \sum_{\alpha} r_\alpha x^\alpha$. Then, $\varphi(p) = \bar{p}$, $p \in \varphi^{-1}(\overline{A}) = A$, $p \in R[x_1, \ldots, x_i]$. Therefore, $\varphi(A \cap R[x_1, \ldots, x_i]) \supset \overline{A} \cap (R/I)[x_1, \ldots, x_i]$.

11. I'll use proposition 42 in section 15.4. Since $D = R \setminus P$ in $R_P$, $P \cap D = \phi$. If $Q$ is a $P$-primary ideal of $R$, by proposition 42 (3), $^c(^e(Q)) = Q$ in $R_P$. For the same reason, $P' \cap D = \phi$ if $P' \subset P$, and $D^{-1}P'$ is a prime in $R_P$. Also, $^c(^e(Q)) = Q$ in $R_P$ for $P'$-primary $Q$.

19. As $R$ is an integral domain with $1$, $R \setminus \{0\}$ is a multiplicative set containing $1$. Let $F = \text{Frac}(R) = R_{R \setminus \{0\}}$ be a fraction field of $R$. Then, $\text{Frac}(D^{-1}R) \simeq F$. ($D^{-1}R$ is integral domain: if $\frac{r_1}{d_1} \frac{r_2}{d_2} = 0$, $dr_1r_2 = 0$ for some $d \in D$, but it means $r_1 = 0$ or $r_2 = 0$ and $\frac{r_1}{d_1} = 0$ or $\frac{r_2}{d_2} = 0$, which is contradiction. Therefore, we can define the fractional field fo $D^{-1}R$. Let $\Phi : F \to \text{Frac}(D^{-1}R)$ by $\Phi : \frac{r_1}{r_2} \mapsto \frac{\frac{r_1}{1}}{\frac{r_2}{1}}$. It is well-defined since if $\frac{r_1}{r_2} = \frac{r_3}{r_4}$, $r_1r_4 - r_2r_4 = 0$ and it means $\frac{\frac{r_1}{1}}{\frac{r_2}{1}} = \frac{\frac{r_3}{1}}{\frac{r_4}{1}}$. Also, it is ring homomorphism since

$$\Phi\left(\frac{r_1}{r_2} + \frac{r_3}{r_4}\right) = \Phi\left(\frac{r_1r_4 + r_2r_3}{r_2r_4}\right) = \frac{\frac{r_1r_4 + r_2r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1r_4}{1} + \frac{r_2r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1}{1}}{\frac{r_2}{1}} + \frac{\frac{r_3}{1}}{\frac{r_4}{1}} = \Phi\left(\frac{r_1}{r_2}\right) + \Phi\left(\frac{r_3}{r_4}\right)$$

and $\Phi\left(\frac{r_1}{r_2}\frac{r_3}{r_4}\right) = \frac{\frac{r_1r_3}{1}}{\frac{r_2r_4}{1}} = \frac{\frac{r_1}{1}}{\frac{r_2}{1}}\frac{\frac{r_3}{1}}{\frac{r_4}{1}} = \Phi\left(\frac{r_1}{r_2}\right)\Phi\left(\frac{r_3}{r_4}\right)$. Also, it is surjective since for any $\frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}} \in \text{Frac}(D^{-1}R)$, $\Phi\left(\frac{r_1d_2}{r_2d_1}\right) = \frac{\frac{r_1d_2}{1}}{\frac{r_2d_1}{1}} = \frac{\frac{r_1}{d_1}}{\frac{r_2}{d_2}}$ since $\frac{r_1d_2r_2}{d_2} = \frac{r_1r_2}{1} = \frac{r_1r_2d_1}{d_1}$. As a surjective field homomorphism, it is a field isomorphism, and $F$ and $\text{Frac}(D^{-1}R)$ is isomorphic.)

Choose $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right) \in \text{Frac}(D^{-1}R)$. Since $R$ is integrally closed, there exists monic polynomial $p(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$, $a_i \in R$, such that $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$. My claim is that $\pi(p(x)) = \tilde{p}(x) = x^n + \sum_{i=0}^{n-1} \pi(a_i)x^i \in D^{-1}R[x]$ has a root $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$. With the field isomorphism $\Phi$,

$$\Phi\left(\tilde{p}\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)\right) = \Phi\left(\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^n + \sum_{i=0}^{n-1} \pi(a_i)\left(\frac{\left(\frac{r_1}{d_1}\right)}{\left(\frac{r_2}{d_2}\right)}\right)^i\right)$$

$$= \left(\frac{r_1d_2}{r_2d_1}\right)^n + \sum_{i=0}^{n-1}\frac{a_i}{1}\left(\frac{r_1d_2}{r_2d_1}\right)^i = 0$$

sine $p\left(\frac{r_1d_2}{r_2d_1}\right) = 0$. Since $\Phi$ is isomorphism, it means that $\left(\frac{r_1}{d_1}\right) / \left(\frac{r_2}{d_2}\right)$ is integral over $D^{-1}R$ and $D^{-1}R$ is integrally closed.

27. First, I'll compute $\mathbb{T}_{t,V}$, $\mathbb{T}_{(w_1,w_2,w_3),W}$. Since $V = I(0)$, $D_v(0) = 0$ and $\mathbb{T}_{t,V} = \mathbb{A}^1$. If $w = (w_1, w_2, w_3) \neq 0$,

$$D_w(f)(x, y, z) = (w_3 x - 2w_2 y + w_1 z, -3w_1^2 x + w_3 y + w_2 z, -2w_1 w_2 x - w_1^2 y + 2w_3 z)$$

For $\varphi(t)$,

$$D_{\varphi(t)}(f)(x, y, z) = (t^5 s - 2t^4 y + t^3 z, -3t^6 x + t^5 y + t^4 z, -2t^7 x - t^6 y + 2t^5 z)$$

Computing $\mathcal{Z}\left(D_{\varphi(t)}(f)(x, y, z)\right)$, $\mathbb{T}_{(\varphi(t),W)} = \left(1, \frac{4}{3}t, \frac{5}{3}t^2\right) x$. If $w = 0$, then $D_0(f)(x, y, z) = 0$ and $\mathbb{T}_{(0,W)} = \mathbb{A}^3$. It describes all $\mathbb{T}_{(w,W)}$, $w \in W$ since $\varphi$ is set-theoretically bijective between $\mathbb{A}^1$ and $W$.

Using previous exercise, for each $t \in \mathbb{A}^1$, $d\varphi : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{\varphi(t),V}$ is given explicitly by

$$d\varphi(a) = (D_t(t\varphi_1)(a), D_t(\varphi_2)(a), D_t(\varphi_3)(a)) = (3t^2 a, 4t^3 a, 5t^4 a) = 3t^2 \left(1, \frac{4}{3}t, \frac{5}{3}t^2\right) a$$

It shows that $d\phi$ is vector space isomorphism for $t \neq 0$. However, for $t = 0$, $\mathbb{T}_{0,\mathbb{A}^1} = \mathbb{A}^1$, but $\mathbb{T}_{0,W)} = \mathbb{A}^3$ and $d\varphi(a) = 0$.

Finally, assume that $V$ and $W$ are isomorphic and there exists isomorphism $\Phi$. Then, $d\Phi : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{\varphi(t),V}$ should be isomorphism on $t = 0$, but there can not exists such isomorphism between $\mathbb{A}^1$ and $\mathbb{A}^3$. Therefore, $V$ and $W$ are not isomorphic.

35. Let $P$ be a prime ideal in $R$ such that $P \cap D = \phi$ and $P \in \mathrm{Ass}_R(M)$. First, $D^{-1}P$ is a prime ideal in $D^{-1}R$ by proposition 38 (3). Let an annihilated element in $R$ be $m$. Then, $R/P \simeq Rm$ since $P \in \mathrm{Ass}_R(M)$ with annihilated element $m$.

Consider an exact sequence such that $i$ is inclusion of $P$ into $R$ and $\varphi : R \to Rm$ by $r \mapsto rm$. Since $D^{-1}R$ is a flat $R$-module, the following sequence is also exact. Since

$$0 \longrightarrow P \xrightarrow{\ i\ } R \xrightarrow{\ \varphi\ } Rm \longrightarrow 0$$

$$0 \longrightarrow D^{-1}P \xrightarrow{\ 1\otimes i\ } D^{-1}R \xrightarrow{\ 1\otimes\varphi\ } D^{-1}(Rm) \longrightarrow 0$$

$1 \otimes \varphi : \frac{r}{d} \mapsto \frac{rm}{d}$, $\ker(1 \otimes \varphi) = D^{-1}P$ means that $D^{-1}P$ is the annihilator of an element $\frac{m}{1}$, and $D^{-1}P \in \mathrm{Ass}_{D^{-1}R}(D^{-1}M)$.