19/5/2022

Thursday

Ut-3 → CONES and CYLINDERS

Keypoints

1) Intersection of a line with a cone $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

2) The 2 pts of intersection of a cone are

$(\alpha + lr_1, \beta + mr_1, \gamma + nr_1)$ $(\alpha + lr_2, \beta + mr_2, \gamma + nr_2)$

$r_1 r_2$ = roots.

3) Eqn of tangent plane is $x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) +$
$z(g\alpha + f\beta + c\gamma) = 0$

4) Reciprocal Cone.

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

given cone is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{h} & \overset{+}{g} \\ \overset{-}{h} & \overset{+}{b} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{f} & \overset{+}{c} \end{vmatrix}$$

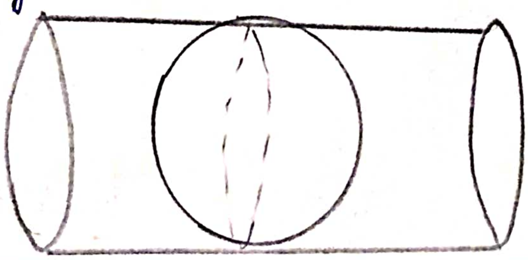
$A = (bc - f^2)$ $\quad G = (hf - bg)$

$B = (ac - g^2)$ $\quad H = -(hc - fg)$

$C = (ab - h^2)$

$F = -(af - hg)$

5) Enveloping Cylinder: The locus of the tangent lines of sphere which are parallel to a given line is a cylinder known as the eveloping cylinder of sphere.



⑥ Equation of an Enveloping Cylinder

The locus of (α, β, γ) is $(lx+my+nz)^2 = (l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$ is the eqn of E. cylinder

⑦ The Right Circular Cylinder

Let (l, m, n) be the direction numbers of the normal to the plane π containing a circle C. Let L be a normal line to the plane π and passing through h point P.

⑧ The eqn of the Right circular cylinder with radius r and axis line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is

$$[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2][(l^2+m^2+n^2)] = [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$

PRACTICAL NO:- 05

Solve the following questions

Q.2.1) Prove that the cones $fyz+gzx+hxy=0$ and $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ are reciprocal.

Sol Given eqn $fyz+gzx+hxy=0$

$$\Rightarrow 2fyz+2gzx+2hxy=0 \longrightarrow ①$$

Comparing with general eqn $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

$a=b=c=0$

$f=f, g=g, h=h$

Let the reciprocal eqn $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

$A = bc-f^2 = -f^2$ $\quad$ $F = hg-af = hg$

$B = ac-g^2 = -g^2$ $\quad$ $G = hf-bg = hf$

$C = ab-h^2 = -h$ $\quad$ $H = gf-ch = gf$

$-f^2x^2-g^2y^2-h^2z^2+2hgyz+2hfzx+2gfxy=0$

$f^2x^2+g^2y^2+h^2z^2+2fgxy-2ghyz-2hfzx = 4fgxy$

$(fx+gy-hz)^2 = 4fgxy$

$fx+gy-hz = \pm\sqrt{4fgxy} = \pm 2\sqrt{fgxy}$

$fx \pm 2\sqrt{fgxy} + gy = h \Rightarrow fx \pm 2\sqrt{fgxy} + gy = hz$

$(\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$$

Q 22) Find the equation of the circular cone whose vertex is (origin and passes through $(1,1,2)$, axis the line $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$

Sol Given vertex of circular (right) cone $= (0,0,0)$

Eq^n of axis $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$. $\longrightarrow$ (1)

$\therefore$ Dr's of the axes are $(2,-4,3)$

Let the semivertical angle be θ

Then the eqn of the cone from angle

$$\cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}} \longrightarrow (2)$$

$(l_1, m_1, n_1) = (2,-4,3)$

$(l_2, m_2, n_2) = (x-0, y-0, z-0) = (x,y,z)$

$$\cos^2\theta = \frac{(l_1l_2 + m_1m_2 + n_1n_2)^2}{\left[\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}\right]^2}$$

$$\cos^2\theta = \frac{(2x-4y+3z)^2}{(x^2+y^2+z^2)(4+16+9)} \Rightarrow \frac{(2x-4y+3z)^2}{(x^2+y^2+z^2)(29)}$$

$$(x^2+y^2+z^2)(29)\cos^2\theta = (2x-4y+3z)^2 \longrightarrow (3)$$

Eq (3) passes through $(1,1,2)$

$(1+1+4)\,29\cos^2\theta = (2-4+6)^2$

$(6)(29)\cos^2\theta = 16$

$$\cos^2\theta = \frac{8}{87} \longrightarrow (4)$$

put (4) in eq (3)

$(x^2+y^2+z^2)(29)\left(\frac{8}{87}\right) = (2x-4y+3z)^2$

$(29x^2+29y^2+29z^2)\left(\frac{8}{87}\right) = (4x^2+16y^2+9z^2-16xy+12xz-24yz)$

$(232x^2+232y^2+232z^2) = 348x^2+1392y^2+783z^2-1392xy+1044xz$
$-2088yz$

$-116x^2-1160y^2-551z^2+1392xy-1044xz+2088yz = 0$

$\div -29$

$-29\left[4x^2+40y^2+19z^2-72yz+36zx-48xy\right) = 0$ //.

Q 23) Find the equation of the right circular cone whose vertex is $(3,2,1)$ axis the line $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$ and semi-vertical angle 30°.

Sol Given

eqn of the axis $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$

Semi vertical angle is 30°

$\therefore$ Eqn to the cone from the angle

$$\cos\theta = \frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$\cos^2\theta = \frac{(l_1l_2+m_1m_2+n_1n_2)^2}{(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)}$$

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)\cos^2\theta = (l_1l_2+m_1m_2+n_1n_2)^2$$

$(l_1, m_1, n_1) = (4, 1, 3)$

$(l_2, m_2, n_2) = (x-3, y-2, z-1)$

$\therefore$ eqn of cone is $[(x-3)^2+(y-2)^2+(z-1)^2](4^2+1^2+3^2)\cos^2 30 =$

$$[4(x-3)+1(y-2)+3(z-1)]^2$$

$= \cos 30 = \frac{\sqrt{3}}{2}$ $\quad \cos^2 30 = \frac{3}{4}$

$$[x^2+y^2+z^2+9+4+1-6x-4y-2z](16+1+9)\left(\frac{3}{4}\right) = [4x+y+3z-12-2-3]^2$$

$$[x^2+y^2+z^2+14-6x-4y-2z](26)\left(\frac{3}{4}\right) = (4x+y+3z-17)^2$$

The req eqn $(x^2+y^2+z^2-6x-4y-2z+14)(26)(3/4) = (4x+y+3z-17)^2$

Q24) Find the equation of the right circular cone which passes through point $(1,1,1)$, whose vertex is $(1,0,1)$ and axis of the cone makes equal angles with the co-ordinate axes.

Sol Given Vertex $= (1,0,1)$

Dr's of the axis $(1,1,1)$ passes through $P(1,1,1)$,

Eqn of axis $\frac{x-1}{1} = \frac{y-0}{1} = \frac{z-1}{1}$

$\therefore$ dr's of $VP = (1-1, 1-0, 1-1) = (0,1,0)$

Semi vertical angle θ is angle b/w VP and axis

$$\cos\theta = \frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}} = \frac{1(0)+(1)(1)+1(0)}{\sqrt{1+1+1}\sqrt{0+1+0}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Eqn of cone is $(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)\cos^2\theta = (l_1l_2+m_1m_2+n_1n_2)^2 \longrightarrow (1)$

$(l_1, m_1, n_1) = (1,1,1)$

$(l_2, m_2, n_2) = (x-1, y, z-1)$

put in (1)

$$[(x-1)^2+(y)^2+(z-1)^2](1+1+1)\left(\frac{1}{\sqrt{3}}\right)^2 = [1(x-1)+y+(z-1)]^2$$

$$[x^2+y^2+z^2-2x-2z+2] = (x+y+z-2)^2$$

$$xy+yz+zx-x-2y-z+1=0 \;//.$$

Q25) Find the eqn of the ~~[cylinder whose]~~ right circular cone with its vertex at the origin, axis along z-axis and semi-vertical angle θ.

Sol Let $P(x,y,z)$ be any point on the surface of the cone

Hence dr's of OP are (x,y,z)

dr's of z axes $(0,0,1)$

$$\cos\theta = \frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$\cos\theta = \frac{(0)x + y(0) + z(1)}{\sqrt{x^2+y^2+z^2}\ \sqrt{0+0+01}} = (x^2+y^2+z^2)\cos^2\theta = z^2$$

$$(x^2+y^2)\cos^2\theta = z^2 - z^2\cos^2\theta$$
$$= z^2[1-\cos^2\theta]$$
$$= z^2\sin^2\theta$$

$$x^2+y^2 = z^2\tan^2\theta \,//.$$

Note;- Vertex at origin, axis along x-axis, angle θ

$\Rightarrow$ eq^n of R. Circular cone $\Rightarrow y^2+z^2 = x^2\tan^2\theta$

Vertex at origin, axis along y-axis, angle θ

$\Rightarrow$ eqn of R. Circular Cone $\Rightarrow x^2+z^2 = y^2\tan^2\theta$

Vertex at origin, axis along z-axis, angle θ

$\Rightarrow$ eqn of R-Circular Cone $= x^2+y^2 = z^2\tan^2\theta \,//.$

19/5/2022 Unit 3- CONES and CYLINDERS

Thursday

PRACTICAL-NO: 06

Solve the following

Q.26 Find the eqn of the cylinder whose generators are $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which passes through the curve $x^2+y^2=16, z=0$

Sol Given the dr's of generators $(1, 2, 3)$ from line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

Let $P(x_1, y_1, z_1)$ be a point on the cylinder

$\therefore$ Eqn to the generator be $\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{z-z_1}{3} = (r)$ say)

Any point on the line is $(r+x_1, 2r+y_1, 3r+z_1)$

This pt lies on the ~~line~~ curve

$$x^2+y^2=16,\ z=0$$

$\frac{x-x_1}{1} = r$	$\frac{z-z_1}{3} = r$
$x = r+x_1$	$z = 3r+z_1$
$\frac{y-y_1}{2} = r$	
$y = 2r+y_1$	

$$(x_1+r)^2 + (y_1+2r)^2 = 16 \longrightarrow ① \quad ; \quad 3r+z_1 = 0$$

$r = \frac{-z_1}{3}$ put in eq①

$$\left(x_1 - \frac{z_1}{3}\right)^2 + \left(y_1 - \frac{2z_1}{3}\right)^2 = 16$$

$$(3x-z_1)^2 + (3y-2z)^2 = 9(16)$$

$9x^2+9y^2+5z^2-6zx-12yz-144=0$ is the locus of P is the cylinder.

Q27) Find the eqn of the enveloping cylinder of sphere $x^2+y^2+z^2=25$ whose generators are parallel to line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

Sol Let $P(x_1, y_1, z_1)$ be any point on cylinder

$\therefore$ Eqn of generator is $\frac{x-x_1}{1}=\frac{y-y_1}{2}=\frac{z-z_1}{3}=k$ (say)

The point $(k+x_1, 2k+y_1, 3k+z_1)$ lies on the sphere

(k- given eqn of sphere $x^2+y^2+z^2=25 \longrightarrow (1)$

Substitae $(k+x_1, 2k+y_1, 3k+z_1)$ in eq (1)

$(k+x_1)^2+(2k+y_1)^2+(3k+z_1)^2=25$

$k^2+x_1^2+2x_1k+4k^2+y_1^2+4ky_1+9k^2+z_1^2+6kz_1-25=0$

$14k^2+2k(x_1+2y_1+3z_1)+(x_1^2+y_1^2+z_1^2-25)=0$ //.

$(x_1+2y_1+3z_1)^2-14(x_1^2+y_1^2+z_1^2-25)=0$

$x_1^2+4y_1^2+9z_1^2+4xy_1+12yz_1+6x_1z_1-14x_1^2-14y_1^2-14z_1^2+350=0$

$(x_1, y_1, z_1)=(x, y, z)$

$13x^2+10y^2+5z^2-4xy-12yz-6zx-350=0$ //.

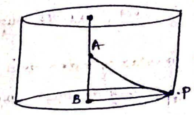
Q28) Find the right circular cylinder radius of 2 and axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

Sol Given Radius $=2$

axis $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

Let $P(x,y,z)$ be any pt on cylinder

$A(1,2,3)$ on axis



Phythogras theorem $AP^2=AB^2+BP^2$

$AP^2=$ distance b/w 2 pts

$(x-1)^2+(y-2)^2+(z-3)^2$

$BP=2$

AB is projection of AP

$AB=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}}$

l_1, m_1, n_1 are the dc's of axis i-e $(l_1, m_1, n_1)=(2,1,2)$

l_2, m_2, n_2 are dc's of AP i-e $(x-1, y-2, z-3)$

$(x-1)^2+(y-2)^2+(z-3)^2=4+\frac{2(x-1)+1(y-2)+2(z-3)}{\sqrt{1+4+4}}$

$x^2+y^2+z^2+1+4+9-2x-4y-6z=4+\left[\frac{2x-2+y-2+2z-6}{3}\right]^2$

$x^2+y^2+z^2+14-2x-4y-6z=4+\left[\frac{4x^2+y^2+4z^2+100+4xy+8zx-40x}{9}+\frac{+4yz-20y-40z}{9}\right]$

$9x^2+9y^2+9z^2+126-18x-36y-54z-36-4x^2-y^2-4z^2-100-4xy-8xz$
$+40x-4yz+20y+40z=0$

$\Rightarrow 5x^2+8y^2+5z^2+22x-16y-14z-4xy-8xz-4yz-10=0$ is req eqn //

Q. Q29) Find the eqn of Right circular cylinder whose axis is $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$ and fass through $(0,0,3)$

Sol: Given Right circular cylinder axis $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$

and eqn is passing through $(0,0,3) = (x,y,z)$

Axis $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z-0}{3}$

Vertex $(2,1,0)$

Let $P(x,y,z)$ be any point on cylinder

$AP^2 = (x-2)^2 + (y-1)^2 + z^2$

$BP^2 = ?$

AB is the projection of AP: $AP = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}}$

(l_1, m_1, n_1) are d.r's of axis $(2,1,3)$

(l_2, m_2, n_2) are d.c's of AP $(x-2, y-1, z)$

Phythogras theorem

$AP^2 = AB^2 + BP^2 \longrightarrow ①$

$$(x-2)^2 + (y-1)^2 + z^2 = BP^2 + \left[\frac{2(x-2) + 1(y-1) + 3z}{\sqrt{4+1+9}}\right]^2$$

$$x^2 + y^2 + z^2 + 4 + 1 - 4x - 2y = BP^2 + \left[\frac{2x-4+y-1+3z}{\sqrt{14}}\right]^2$$

$$14(x^2 + y^2 + z^2 + 5 - 4x - 2y) = 14BP^2 + (2x+y+3z-5)^2$$

$$14x^2 + 14y^2 + 14z^2 + 70 - 56x - 28y = 14BP^2 + 4x^2 + y^2 + 9z^2 + 25 + 4xy + 12xz$$
$$-20x + 6yz - 10y - 30z$$

$$x^2(14-4) + y^2(14-1) + z^2(14-9) + x(-56+20) + y(-28+10) + z(30) + (70-25)$$
$$-4xy - 12xz - 6yz = 14BP^2$$

$$10x^2 + 13y^2 + 5z^2 - 36x - 18y + 30z + 45 - 4xy - 12xz - 6yz = 14BP^2 \longrightarrow ②$$

eq is passing through $(0,0,3)$

$0 + 0 + 45 + 90 + 45 = 14BP^2$

$14BP^2 = 180$

$BP^2 = \frac{90}{7}$

Sub in eq ②

$$10x^2 + 13y^2 + 5z^2 - 36x - 18y + 30z + 45 - 4xy - 12xz - 6yz = \cancel{14}^{2}\left(\frac{90}{7}\right)$$

$$10x^2 + 13y^2 + 5z^2 - 36x - 18y + 30z - 4xy - 12xz - 6yz - 135 = 0$$ // is req eqn

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Q30) Find the eqn of Right circular cylinder whose axis is $x-2 = z, y = 0$ and passes through pt $(3,0,0)$.

Sol) Eqn to axis is $\frac{x-2}{1} = \frac{y}{0} = \frac{z}{1}$

Let r be the radius of cylinder

Given point $(3,0,0)$

$A(1,1,1) = (l_1, m_1, n_1)$

$(l_2, m_2, n_2) = (x-2, y, z)$

$AP^2 = (1-3)^2 + (1-0)^2 + (1-0)^2 = (-2)^2 + (1)^2 + (1)^2 = 6$

Distance)

$BP^2 = ?$

$$AB = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}} = \frac{1(x-2) + 1(y) + 1(z)}{\sqrt{3}} = \frac{x+y+z-2}{\sqrt{3}}$$

$$AP^2 = BP^2 + AB^2 \Rightarrow 6 = BP^2 + \frac{(x+y+z-2)^2}{3}$$

$3(6) = 3BP^2 + (x+y+z-2)^2$

$18 = 3BP^2 + x^2 + y^2 + z^2 + 4 + 2xy - 2yz + 2zx - 4x - 4y - 4z$

$18 = 3BP^2 + (x^2 + y^2 + z^2 + 4 + 2xy + 2yz + 2zx - 4x - 4y - 4z)$

$AP^2 = (x-2)^2 + y^2 + z^2$

$[(x^2 + 4 - 4x) + (y^2 + z^2)]3 = 3BP^2 + (x^2 + y^2 + z^2 + 4 + 2xy + 2yz + 2zx - 4x - 4y - 4z]$

$2x^2 + 2y^2 + 2z^2 - 8x + 4y + 4z + 2xy + 2yz + 2zx + 8 = 3BP^2 \longrightarrow (2)$

point $(3,0,0)$ pass by eq (2)

$2(3)^2 + 0 - 8(3) + 8 = 3BP^2$

$26 - 24 = 3BP^2$

$\frac{2}{3} = BP^2 \longrightarrow (3)$ sub in eq (2)

$2x^2 + 2y^2 + 2z^2 - 8x + 4y + 4z + 2xy + 2yz + 2zx + 6 = 0$

$2[x^2 + y^2 + z^2 - 4x + 2y + 2z + xy + yz + zx + 3] = 0$ //.