Some characterisations of flatness using Tor functor

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VSRP-2022

11th July, 2022

Pre-requisites and Notations

- R is a commutative ring with identity. All the modules are $R-$ modules and tensoring is done over R.
- $A : \ldots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \ldots \xrightarrow{d_1} X_0 \longrightarrow 0$ is called a chain if $Im(d_{n+1}) \subset Ker(d_n)\ \forall n$. If A is a chain, $M \otimes A : \ldots \xrightarrow{1 \otimes d_{n+1}} M \otimes X_n \xrightarrow{1 \otimes d_n} \ldots \xrightarrow{1 \otimes d_1} M \otimes X_0 \longrightarrow 0$ is also a chain. $[(1 \otimes f)(a \otimes b) = a \otimes f(b)]$
- If $Im(d_{n+1}) = Ker(d_n)$, then the above sequence is **exact** at X_n.
- $M \xrightarrow{f} N \xleftarrow{g} P$, if f is surjective, P is called a projective module if $\exists h : P \longrightarrow M$ such that $f \circ h = g$.
- Direct sum of projective modules is projective.
- Every free module is projective.

Tor(A,B)

- Take a projective resolution of B
  $\ldots \longrightarrow P_n \xrightarrow{d_n} \ldots \longrightarrow P_0 \xrightarrow{d_0} B \longrightarrow 0$
  P_i's are projective modules and the sequence is exact.
- $X : \ldots \longrightarrow A \otimes P_n \xrightarrow{1 \otimes d_n} \ldots \longrightarrow A \otimes P_0 \longrightarrow 0$ is also a chain. So, take its homology groups.
- $\mathrm{Tor}_i(A, B) := H_i(X) = \frac{Ker(1 \otimes d_i)}{Im(1 \otimes d_{i+1})}$
- Is it well-defined ?
  Only thing to check is that Tor is independent of the projective resolution.

Map between projective resolutions

- $\ldots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \longrightarrow 0$

$$\ldots \xrightarrow{e_{n+1}} Q_n \xrightarrow{e_n} \ldots \xrightarrow{e_3} Q_2 \xrightarrow{e_2} Q_1 \xrightarrow{e_1} Q_0 \xrightarrow{e_0} C \longrightarrow 0$$

Hail, Projective Modules!

- $$\ldots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \longrightarrow 0$$

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Hail, Projective Modules!

- Inductively we can define functions $s_n : P_n \longrightarrow Q_{n+1}$ such that $e_{n+1} \circ s_n + s_{n-1} \circ d_n = f_n - g_n$.
- The same relation holds for the maps after tensoring with M, i.e., $(1 \otimes e_{n+1}) \circ (1 \otimes s_n) + (1 \otimes s_{n-1}) \circ (1 \otimes d_n) = (1 \otimes f_n) - (1 \otimes g_n)$.
- Map between two chains, induces map between their homologies.
- Easy to check that if such relation is satisfied by f_n and g_n, then thei induced maps on the homologies are equal.
- Suppose X and X' are two chains for two different projective resolutions of B and $f : X \longrightarrow X'$ and $g : X' \longrightarrow X$ are maps between them. Then, both $g \circ f$ and Id are maps from X to X. The induced map of $g \circ f$ on the homology is $Id \implies H_n(X) \cong H_n(X')$. Same results are obtained even after tensoring. So, $Tor_i(A, B)$ is well-defined.

Flat Modules

- $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact.
  $\implies A \otimes L \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow 0$ is exact.
- A is flat iff tensoring it to an exact sequence is exact sequence.
- **Target: TFAE**
  1. $Tor_i(A, B) = 0$ for all modules B, $i \geq 1$.
  2. $Tor_1(A, B) = 0$ for all modules B.
  3. A is flat module.
- $3 \implies 1$ is clear from definitions. Also, $1 \implies 2$. Need to show $2 \implies 3$.

Proof:

- Suppose for all short exact sequences $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, we have a long exact sequence $\ldots \longrightarrow Tor_n(A, N) \longrightarrow Tor_{n-1}(A, L) \longrightarrow Tor_{n-1}(A, M) \longrightarrow Tor_{n-1}(A, N) \longrightarrow \ldots \longrightarrow Tor_1(A, N) \longrightarrow A \otimes L \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow 0$.
- If (2) is true, then $Tor_1(A, N) = 0 \implies 0 \longrightarrow A \otimes L \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow 0$ is exact. $\implies A$ is flat (3).

Proof:

- We know that if A, B and C are chains,
  $A : ... \longrightarrow X_n \longrightarrow ... \longrightarrow X_0 \longrightarrow 0$
  $B : ... \longrightarrow Y_n \longrightarrow ... \longrightarrow Y_0 \longrightarrow 0$
  $C : ... \longrightarrow Z_n \longrightarrow ... \longrightarrow Z_0 \longrightarrow 0$, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$
  is exact, then we have the long exact sequence:
  $\longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C) \longrightarrow ... \longrightarrow H_0(C) \longrightarrow 0.$
- Follows from the definition that $Tor_0(A, B) \cong A \otimes B$.

Proof:

Proof:

- $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ **[Exact]**
- Tensoring projective resolutions of L, M and N, still remain chains.
- Problem is that we may not have **exact maps between the chains**. So, we have to cleverly pick the projective resolutions.
- $A : \ldots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \ldots \xrightarrow{d_1} X_0 \xrightarrow{d_0} L \longrightarrow 0$
  $B : \ldots \longrightarrow (X_n \oplus Z_n) \longrightarrow \ldots \longrightarrow (X_0 \oplus Z_0) \xrightarrow{\mathbf{?}} M \longrightarrow 0$
  $C : \ldots \xrightarrow{e_{n+1}} Z_n \xrightarrow{e_n} \ldots \xrightarrow{e_1} Z_0 \xrightarrow{e_0} N \longrightarrow 0.$
- $f \circ d_0 : X_0 \longrightarrow M$.
  g is surjective and Z_0 is projective module $\implies \exists t : Z_0 \longrightarrow M$ such that $g \circ t = e_0$.
- $h : X_0 \oplus Z_0 \longrightarrow M$ with $h(x + z) = f \circ d_0(x) + t(z)$

Proof:

Proof:

- Except the first row, all other rows are split exact.
- Tensoring preserves split exact sequences.
- $\mathbf{S} : \ldots \xrightarrow{1 \otimes d_{n+1}} A \otimes X_n \xrightarrow{1 \otimes d_n} \ldots \xrightarrow{1 \otimes d_1} A \otimes X_0 \longrightarrow 0$

  $\mathbf{T} : \ldots \longrightarrow A \otimes (X_n \oplus Z_n) \longrightarrow \ldots \longrightarrow A \otimes (X_0 \oplus Z_0) \longrightarrow 0$

  $\mathbf{U} : \ldots \xrightarrow{1 \otimes e_{n+1}} A \otimes Z_n \xrightarrow{1 \otimes e_n} \ldots \xrightarrow{1 \otimes e_1} A \otimes Z_0 \longrightarrow 0$
- $0 \longrightarrow S \longrightarrow T \longrightarrow U \longrightarrow 0$ is exact.

Applications

- $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ **[Exact]**
  Suppose N is flat. M is flat iff L is flat.
- **Proof:**
  Take a projective resolution of any module A,
  $\ldots \longrightarrow P_n \ldots \longrightarrow A \longrightarrow 0.$
  - $S : \ldots \longrightarrow L \otimes P_n \longrightarrow \ldots \longrightarrow L \otimes P_0 \longrightarrow 0$
  - $T : \ldots \longrightarrow M \otimes P_n \longrightarrow \ldots \longrightarrow M \otimes P_0 \longrightarrow 0$
  - $U : \ldots \longrightarrow N \otimes P_n \longrightarrow \ldots \longrightarrow N \otimes P_0 \longrightarrow 0$
- We have the following exact sequence $0 \longrightarrow S \longrightarrow T \longrightarrow U \longrightarrow 0$, exact because $P_i's$ are projective, hence flat.
- So, we have the long exact sequence,
  $\ldots \longrightarrow Tor_2(N, A) \longrightarrow Tor_1(L, A) \longrightarrow Tor_1(M, A) \longrightarrow$
  $Tor_1(N, A) \longrightarrow L \otimes A \longrightarrow M \otimes A \longrightarrow N \otimes A \longrightarrow 0.$

Applications

- By assumption, $Tor_2(N, A) = 0$ and $Tor_1(N, A) = 0$. So, we have the exact sequence $0 \longrightarrow Tor_1(L, A) \longrightarrow Tor_1(M, A) \longrightarrow 0$.
  $\implies Tor_1(L, A) \cong Tor_1(M, A)$ for any module A.
- If L and M are flat, N needn't be flat.
- Exact sequence of $\mathbb{Z}$-modules $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.
- $\mathbb{Z}_2$ is **NOT** torsion-free, hence **NOT** flat. **(Why?)**

Torsion

- If A is flat and suppose it has a torsion element a. $\Longrightarrow$ $\exists r \in R$, not a zero-divisor such that $ra = 0$.

  $0 \longrightarrow R \xrightarrow{\times r} R \longrightarrow \frac{R}{(r)} \longrightarrow 0$. **[Exact]**
  By the *Tor* long exact sequence and the fact that R is flat module (free, hence flat), we have
  $\longrightarrow Tor_1(A, R) \longrightarrow Tor_1(A, \frac{R}{(r)}) \longrightarrow A \otimes R \longrightarrow A \otimes R \longrightarrow$
  So, $0 \longrightarrow Tor_1(A, \frac{R}{(r)}) \longrightarrow A \xrightarrow{\times r} A \longrightarrow 0$.
- $Tor_1(A, \frac{R}{(r)}) \cong \{a \in A : ra = 0\}$. Flat $\Longrightarrow$ $Tor_1(A, \frac{R}{(r)}) = 0 \Longrightarrow$ $\{a \in A : ra = 0\} = 0, \Longrightarrow$ Torsion-free.

Torsion

- Converse not true.
- $R = k[x, y], I = (x, y), k = \frac{R}{I}$. Clearly, I is torsion-free.
  $0 \longrightarrow R \xrightarrow{(-yr, xr)} R^2 \xrightarrow{(xr_1+yr_2)} R \longrightarrow k \longrightarrow 0$ is a free, hence, a projective resolution of k.
- Tensoring with k over R, we get
  $0 \longrightarrow R \otimes k \longrightarrow R^2 \otimes k \longrightarrow R \otimes k \longrightarrow k \otimes k \longrightarrow 0$ which is same as
  $0 \longrightarrow k \xrightarrow{f} k^2 \longrightarrow k \longrightarrow k \otimes k \longrightarrow 0$
  $\implies Tor_2(k, k) = Ker(f)$.
  Easy to check that if $\bar{a} \in k, f(\bar{a}) = ((-ya)$ mod I,(xa)mod I)=(0,0)
  $\implies Ker(f) = k \implies Tor_2(k, k) \cong k$.
- $0 \longrightarrow I \longrightarrow R \longrightarrow k \longrightarrow 0$, yields the *Tor* long exact sequence
  $\longrightarrow Tor_2(R, k) \longrightarrow Tor_2(k, k) \longrightarrow Tor_1(I, k) \longrightarrow Tor_1(R, k) \longrightarrow$
  which means $0 \longrightarrow k \longrightarrow Tor_1(I, k) \longrightarrow 0$ as R is flat.
  $k \neq 0 \implies Tor_1(I, k) \neq 0$, so I is not flat.