The + Construction

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1 Classifying space of a group

Definition 1.0.1 (Classifing space of a group G)**.** Any path-connected CW complex X such that $\pi_n(X) = 1, n > 1$ and $\pi_1(X) = G$.

Construction: Given a group G with presentation $< a_\gamma | b_\delta >$, let $X = \bigvee_\gamma S^1$. b_δ are words in a_γ. Attach 2-cells to X via maps specified by these words to get the space X_1. By Proposition 5.0.7 $\pi_1(X_1) = G$. Also, by construction, X_1 is a path-connected CW complex with cells of dimension at most 2. Now, for each element f in $\pi_2(X_1)$, i.e., f is the class of a map $S^2 \to X_1$ in $\pi_2(X_1)$, attach a 3-cell via this map to X_1. Call the new space X_2. By 5.0.8, $\pi_2(X_2) = 1$ and by Proposition 5.0.7, $\pi_1(X_2) = G$. Continue this process, but now with 4-cells attached to X_2 via maps representing elements of $\pi_3(X_2)$ and so on. Taking the direct limit, we have formed a path-connected CW complex BG such that $\pi_n(BG) = 1, \forall n > 1$ and $\pi_1(BG) = G$.

Uniqueness: If $\tilde{X}$ is the universal cover of X, where X is some classifying space of G, then by Proposition 5.0.12, $\tilde{X}$ is contractible and X is obtained by the usual action of $\pi_1(X)$ on $\tilde{X}$. Then, by Proposition 5.0.13 any two classifying spaces are homotopy equivalent and $G \mapsto BG$ is functorial.

2 Some examples of classifying spaces

- If $G = \{1\}$, then $BG = *$.
- If $G = \mathbb{Z}$, then $BG = S^1$.
- If $G = F_n$, the free group on n generators, $BG = \bigvee_{i=1}^n S^1$.
- If $G = \mathbb{Z}_2$, $BG = \mathbb{RP}^\infty$.

3 Acyclic maps

Definition 3.0.1 (Acyclic spaces)**.** A topological space X such that $H_\bullet(X) = H_\bullet(*)$.

Definition 3.0.2 (Acyclic map)**.** Assume X and Y are path-connected. Then, $f : X \to Y$ is called acyclic if the homotopy fibre H_f (which is unique upto homotopy equivalence here; see Definition 5.0.3) is an acyclic space.

Proposition 3.0.3 (Equivalent conditions for acyclic maps)**.** *TFAE:*

1. *$f : X \to Y$ is acyclic.*
2. *For $\tilde{Y}$, the universal cover of Y, in the following pullback diagram, f' induces isomorphism between $H_\bullet(X \times_Y \tilde{Y})$ and $H_\bullet(\tilde{Y})$.*

$$\begin{tikzcd}
X \times_Y \tilde{Y} \arrow[r, "f'"'] \arrow[d] & \tilde{Y} \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proposition 3.0.4. *If $f : X \to Y$ is acyclic, then it induces isomorphism at the homology level.*

Proposition 3.0.5. *If $f : X \to Y$ is acyclic, then $\pi_1(Y) \cong \frac{\pi_1(X)}{P}$, where P is some perfect (i.e., $[P, P] = P$) normal subgroup of $\pi_1(X)$.*

Proof. As $\tilde{H}_\bullet(H_f) = 0$, $\pi_1(H_f)$ is perfect. From the long exact sequence in Proposition 5.0.4, $\pi_1(Y) \cong \frac{\pi_1(X)}{Ker(f_*)}$ where $Ker(f_*)$ is the image of the perfect group $\pi_1(H_f)$, hence is perfect. □

Proposition 3.0.6. *If $f : X \to Y$ is acyclic and X and Y are CW complexes. If f also induces isomorphism of fundamental groups, then f is a homotopy equivalence.*

Proof. From the long exact sequence in Proposition 5.0.4, if we show, $\pi_n(H_f) = 1, \forall n$, then f induces isomorphism of all homotopy groups of X and Y. Now, by Proposition 5.0.5, f is a homotopy equivalence.
To show, $\pi_n(H_f) = 1, \forall n$, observe that again by Proposition 5.0.4, at $n = 1$ level, we have $\pi_1(H_f)$ is the image of the abelian group $\pi_2(Y)$, hence it is abelian. But it is also a perfect group, hence it is 1. $H_0(H_f) = 0$ implies $\pi_0(H_f) = 1$. By Proposition 5.0.6, $\pi_2(H_f) = 1$ and inducively $\pi_n(H_f) = 1.\forall n$. So, we are done. □

Proposition 3.0.7 (Pullbacks and Acyclic maps)**.** *If f or g is a fibration (See 5.0.1), then f is acyclic iff f' is.*

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Proposition 3.0.8 (Pushouts and acyclic maps)**.** *If f_1 is a cofibration (See 5.0.9), then f_i is acyclic implies f_i' is acyclic.*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_0'} \\ Y_0 & \xrightarrow{f_1'} & Y_0 \cup_X Y_1 \end{array}$$

4 + Construction

Definition 4.0.1 (+ Construction)**.** Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based path connected CW complex. An acyclic map $f : X \to Y$ is called a + Construction on X (relative to P) if P is the kernel of $f_* : \pi_1(X) \to \pi_1(Y)$.

Existence of + Construction. First we prove it for a path connected CW complex X whose fundamental group is perfect. By Proposition 5.0.7, attaching 2-cells along all the elements of $\pi_1(X)$, we get a path connected CW complex W with $\pi_1(W) = 1$. We have the following long exact sequence of the pair (W, X) (also observing that $\pi_1(X)$ is perfect implies $H_1(X) = 0$).

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_2(W) & \longrightarrow & H_2(W, X) & \longrightarrow & H_1(X) = 0 \\ & & \uparrow & & & & \\ & & \pi_2(W) & & & & \end{array}$$

Now, by excision, $H_2(W,X) \cong H_2(W/X) \cong H_2(\bigvee_\lambda S^2) \cong \bigoplus_\lambda H_2(S^2)$. Hence, let $\{1_\lambda\}$ be the generators of $H_2(\bigvee_\lambda S^2)$, which correspond to elements $\{b_\lambda\}$ in $H_2(W)$. By Proposition 5.0.6, pick elements $\beta_\lambda \in \pi_2(W)$ mapping to $b_\lambda \in H_2(W)$. Pick some representative of β_λ; call it the same. So, $\beta_\lambda : S^2 \to W$. Attach 3-cells via these maps to get another CW complex Y.
Claim: $H_\bullet(Y,X) = 0$
We have the following exact sequence of the triplet (Y,W,X). We will show that $H_3(Y,W) \to H_2(W,X)$ is an isomorphism. This clearly implies $H_i(Y,X) = 0$, for $i = 2,3$.

$$0 = H_3(W,X) \longrightarrow H_3(Y,X) \longrightarrow H_3(Y,W) \longrightarrow H_2(W,X) \longrightarrow H_2(Y,X) \longrightarrow H_2(Y,W) = 0$$

Denote by $\phi_\lambda : D^3 \to Y$, the corresponding characteristic map associated with the attaching map $\beta_\lambda : S^2 \to W$. The following commutative diagram induces a commutative diagram for the corresponding homology groups of the triplets.

$$\begin{tikzcd}
* \arrow[r, hook] \arrow[d] & \bigvee_\lambda S^2 \arrow[r, hook] \arrow[d, "H(\bigvee_\lambda \beta_\lambda)"] & \bigvee_\lambda D^3 \arrow[d, "H(\bigvee_\lambda \phi_\lambda)"] \\
X \arrow[r, hook] & W \arrow[r, hook] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
H_3(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "Diagram\ 1"] & H_2(\bigvee_\lambda S^2, *) \arrow[d] \\
H_3(Y,W) \arrow[r] & H_2(W,X)
\end{tikzcd}$$

$$\begin{tikzcd}
(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r] \arrow[d] & \bigvee_\lambda S^3 \arrow[d, "\cong"] \\
(Y,W) \arrow[r] & Y/W
\end{tikzcd}$$

The above diagram of obvious maps induces the following by excision as (Y,W) is CW pair.

$$\begin{tikzcd}
H_3(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r, "\cong"] \arrow[d] & H_3(\bigvee_\lambda S^3) \arrow[d, "\cong"] \\
H_3(Y,W) \arrow[r, "\cong"'] & H_3(Y/W)
\end{tikzcd}$$

By the above diagram, we have an isomorphism in the left arrow in Diagram 1. We also have $H_2(\bigvee_\lambda S^2, *) \to H_2(W,X)$, sending 1_λ to $(\beta_\lambda)_*(1_\lambda) = b_\lambda$ by Proposition 5.0.6. Hence, it is an isomorphism in Diagram 1. Also, from the homology long exact sequence of the triplet $(*, \bigvee_\lambda S^2, \bigvee_\lambda D^3)$ and the fact that $\bigvee_\lambda D^3$ is contractible we have the isomorphism in the top arrow in Diagram 1. So, in Diagram 1, we have isomorphism in the bottom arrow as well. Hence, as observed before, $H_i(Y,X) = 0; i = 2,3$. It is also easy to see that $H_n(Y,X) = 0$ for all other n. So, we are done.

Now, we show $X \hookrightarrow Y$ is an acyclic map using Proposition 3.0.3. As Y is simply connected, it is its own universal cover. So, as in Proposition 3.0.3, we have the following diagram of pullback and the map induced at homology level.

$$\begin{tikzcd} X \times_Y Y \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "Id"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd} \qquad \begin{tikzcd} H_n(X \times_Y Y) \arrow[r] \arrow[d, "\cong"'] & H_n(Y) \\ H_n(X) \arrow[ur, "\cong"'] & \end{tikzcd}$$

Note that $X \times_Y Y = \{(x, x); x \in X\} \subset X \times Y$, clearly homeomorphic to X via the projection in the pullback diagram. Also, by the previous claim, $H_\bullet(Y, X) = 0 \implies H_n(X) \cong H_n(Y), \forall n$. So, by Proposition 3.0.3, the inclusion $X \hookrightarrow Y$ is an acyclic map. As by construction (Y, X) is CW pair, the inclusion is also a cofibration.

Now, for the general case, let $P \subset \pi_1(X)$, a perfect normal subgroup. Let X' be the cover of X with $\pi_1(X') = P$. Clearly, X' is also a path connected CW complex with $P = [P, P]$. Hence, by the previous construction, $\exists X' \hookrightarrow Y'$ where Y' is a simply connected CW complex and the inclusion is an acyclic cofibration. Now, by Proposition 3.0.8, in the following pushout diagram, we have $X \dashrightarrow X \cup_{X'} Y'$ an acyclic co-fibration.

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[d, dashed] \\ X \arrow[r, dashed, "f^+"'] & X \cup_{X'} Y' \end{tikzcd}$$

Now, by Van Kampen's theorem, if $X^+ := X \cup_{X'} Y'$, then $\mathrm{Ker}(f^+) = P$ and by Proposition 3.0.5, $\pi_1(X^+) = \frac{\pi_1(X)}{P}$.

$\square$

Uniqueness of + Construction.

Lemma 4.0.2. *Suppose $f : X \to Y$ is an acyclic cofibration and $g : X \to Z$ is any continuous map with Ker(f_*) $\subset$Ker(g_*), where X, Y, Z are path-connected CW complexes. Then, there exists a map $h : Y \to Z$ such that $h \circ f = g$. Moreover, any two such maps are homotopic.*

Proof of lemma

We first prove it for the case when g is the inclusion $i : X \to M_g$, the mapping cylinder, which we know is a cofibration. We have the following pushout diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i'"] \\ M_g \arrow[r, "f'"'] & M_g \cup_X Y \end{tikzcd}$$

By Proposition 3.0.8, f' is an acyclic cofibration. And by Van Kampen's theorem,
$\mathrm{Ker}(f'_*) =< i_*(\mathrm{Ker}(f_*)) >= 1$, as $\mathrm{Ker}(f_*) \subset \mathrm{Ker}(g_*)$ by hypothesis. So, by Proposition 3.0.4, f'_* is an isomorphism of fundamental groups. Also, it can be easily checked that $M_g \cup_X Y$ is a path-connected CW complex. So, by Proposition 3.0.6, f' is a homotopy equivalence. By Proposition 5.0.11, f' is

also a homotopy equivalence under X (See 5.0.10). Let h' be the its homotopy inverse under X, unique upto homotopy. $h := h' \circ i' : Y \to M_g$. Then, $h \circ f(x)= h'(f'((x,1))) = (x,1) = i(x)$, by Proposition 5.0.11.
Now, if $p : Y \to M_g$ such that $p \circ f = i$, then consider the following by universal property of pushouts.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i'"'] \arrow[ddr, bend left, "p"] & \\
M_g \arrow[r, "f'"] \arrow[drr, bend right, "Id"'] & M_g \cup_X Y \arrow[dr, dashed, "j"] & \\
& & M_g
\end{tikzcd}$$

So, $p = j \circ i'$ and $j \circ f' = Id$. Now, $f' \circ h' \approx Id \implies j \circ (f' \circ h') \approx j \implies h' \approx j$. So, $p \approx h$.
Now, for the general case, we have the following commutative diagrams which can be easily checked:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "Id"] & \\
X \times I \arrow[r] \arrow[drr, bend right, "(x,t)\mapsto g(x)"'] & M_g \arrow[dr, "\phi"] & \\
& & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "Id"] & \\
X \times I \arrow[r] \arrow[dr, bend right, "(x,t)\mapsto f(x)"'] & M_g \arrow[dr, "\phi"] & \\
& Y \arrow[u, "h"] \arrow[r, "l"'] & Z
\end{tikzcd}$$

Here, ϕ is the homotopy equivalence between M_g and Z as defined uniquely by the left diagram above by universal property of pushouts. The existence of h is from the previous discussion as Ker(f_*) $\subset$Ker(g_*)=Ker($\phi_* \circ i_*$)=Ker(i_*) as ϕ_* is an isomorphism. So, for any map l with $l \circ f = g$, $l = \phi \circ h$, where h is unique upto homotopy as seen before. $\phi \circ h$ is our required map which is unique upto homotopy.

Coming back to the uniqueness of + construction, it is clear from the lemma, that any two + constructions are homotopy equivalent [**where in the definition we want the map $f : X \to X^+$ to also be a cofibration, which is also guaranteed by our explicit construction**]. □

5 Results from topology

Definition 5.0.1 (Fibration)**.** $f : X \to Y$, a continuous map between topological spaces is called a fibration if

- for any homotopy $F : Z \times I \to Y$ and
- for any map g' lifting $F(\_, 0)$

$\exists F' : Z \times I \to X$ lifting F such that $F'(\_, 0) = g'$.

Proposition 5.0.2 (Fibration long exact sequence)**.** *If $f : X \to Y$ is a fibration, we have this long exact sequence induced by $(f^{-1}(y_0), x_0) \hookrightarrow (X, x_0) \xrightarrow{f} (Y, y_0)$, where $f(x_0) = y_0$.*

$$\cdots \longrightarrow \pi_n(f^{-1}(y_0), x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0) \longrightarrow \cdots$$

Definition 5.0.3 (Homotopy fibre)**.** Let $f : X \to Y$ be any continuous map with $f(x_0) = y_0$ $E_f := \{(a, \gamma) : a \in X, \gamma : I \to Y, \gamma(0) = f(a)\} \subset X \times Y^I$ where Y^I is given the compact-open topology. Let $\phi : E_f \to Y$ be the map $\phi(a, \gamma) = \gamma(1)$. It is a fibration.

Homotopy fibre of f with respect to $y_0 \in Y$ is defined to be $H_f := \phi^{-1}(y_0)$. It is unique upto homotopy equivalence if Y is path connected.

Proposition 5.0.4 (Homotopy fibre long exact sequence)**.** *It is true that $\psi : X \to E_f$ where $x \mapsto (x, \gamma_{f(x)})$ is an embedding of X in E_f to which E_f deformation retracts, where $\gamma_{f(x)}$ is the constant path at $f(x)$. The same map restricted to $f^{-1}(y_0)$ is a homotopy equivalence between $f^{-1}(y_0)$ and H_f. We have the following:*

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_n(H_f, (x_0, \gamma_{y_0})) \arrow[r, "i_*"] & \pi_n(E_f, (x_0, \gamma_{y_0})) \arrow[r, "\phi_*"] & \pi_n(Y, y_0) \arrow[r] & \cdots \\
& \pi_n(f^{-1}(y_0), x_0) \arrow[u, "\psi_*", "\cong"'] & \pi_n(X, x_0) \arrow[u, "\psi_*", "\cong"'] \arrow[r, "f_*"'] & \pi_n(Y, y_0) \arrow[u, "Id"'] &
\end{tikzcd}
$$

Proposition 5.0.5 (Whitehead's theorem)**.** *If $f : X \to Y$, $f(x_0) = y_0$, where X and Y are path-connected CW complexes, and $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism, $\forall n$, then X is homotopy equivalent to Y, via f.*

Proposition 5.0.6 (Hurewicz's theorem)**.** *Define $\phi : \pi_n(X, x_0) \to H_n(X)$ by $\phi([f]) := f_*(\alpha)$, where $f_* : H_n(S^n) \to H_n(X)$ and α is some fixed generator of $H_n(S^n)$. Then, for $n \geq 2$, if X is $(n-1)$ connected (i.e., $\pi_j(X, x_0) = 1, \forall j < n$), then $\tilde{H}_j(X) = 0, \forall j < n$ and ϕ is an isomorphism.*

Proposition 5.0.7 (Corollary of Van Kampen's theorem)**.** *If X is path connected and Y is obtained from X by attaching 2-cells along the loops based at x_0, say ϕ_α, then $X \hookrightarrow Y$ induces a surjective map from $\pi_1(X, x_0)$ to $\pi_1(Y, x_0)$ and $\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{<\phi_\alpha>}$ And $< \phi_\alpha >$ denotes the normal subgroup in $\pi_1(X, x_0)$ generated by the ϕ_α's. But if Y is obtained by attaching cells of dimension greater than 2, $\pi_1(Y, x_0) \cong \pi_1(X, x_0)$.*

Proposition 5.0.8. *Corollary of cellular approximation theorem If (Y, X) is a CW pair based at $x_0 \in X$ where Y is obtained from X by attaching cells of dimension greater than n, then the inclusion induces isomorphisms of $\pi_j(Y, x_0) \cong \pi_j(X, x_0), j < n$ and a surjection for $j = n$.*

Definition 5.0.9 (Cofibration)**.** $f : X \to Y$, a continuous map between topological spaces is called a cofibration if

- for any homotopy $F : X \times I \to Z$ and
- for any map g' extending $F(\_, 0)$

$\exists F' : Y \times I \to Z$ extending F such that $F'(\_, 0) = g'$.
If (Y, X) is CW pair, the inclusion of X in Y is a cofibration.

Definition 5.0.10 (Homotopy under a space)**.** With respect to the diagram below, $F : X \times I \to Y$ is a homotopy under A if $F(i(a), t) = j(t), \forall a, t$.

$$\begin{tikzcd}
 & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Proposition 5.0.11. *If i and j are cofibrations in the above diagram, and suppose f is a homotopy equivalence, then $\exists g : Y \to X, g \circ j = i$ and both $g \circ f$ and $f \circ g$ are homotopic to Id under A.*

Proposition 5.0.12 (Homotopy groups of a covering space)**.** *If $p : X \to Y$ is a cover, then $\pi_n(X) \cong \pi_n(Y), \forall n > 1$.*

Proposition 5.0.13 (Uniqueness of classifying space)**.** *If G is a group and X_1, X_2 are contractible Hausdorff spaces on which G acts properly discontinuously, where $\frac{X_1}{G}, \frac{X_2}{G}$ are para-compact (a space in which every open cover has an open refinement that is locally finite; every CW complex is para-compact), then the quotient spaces are homotopy equivalent.*

Proposition 5.0.14 (Functoriality of BG)**.** *The classifying space construction gives a functor B from the category of groups and group homomorphisms to the category of CW complexes and homotopy classes of continuous maps.*