Algebraic K-theory of Rings

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1 Introduction

It is not wrong to say that algebraic K-theory arose from attempts and solution of Serre's conjecture (See 8.0.10). Our goal is simply to understand the monoid: isomorphism classes of projective modules over a given ring R. So, make it into an abelian group called $K_0(R)$ and compute as much as possible. Next, study automorphisms of all projective modules over R upto certain convenient 'isomorphisms' (See 4.1.2 and 4.0.2). This information is stored in the group called $K_1(R)$. Now, because of certain quotienting operations while defining $K_1(R)$, many automorphisms of projective modules are marked as trivial. So, the next target is to study more about these automorphisms and their inter-relations. This is captured by the group $K_2(R)$ (See 5.0.2), defined years later by Milnor. Now, similar to the homology long exact sequence of pairs or the homotopy fibre long exact equence, we had the first chunk of a possible long exact sequence involving K_0, K_1 and K_2 for I and ideal of R.

$$K_2(R) \longrightarrow K_2(R/I) \longrightarrow \cdots \longrightarrow K_0(R,I) \longrightarrow K_0(R) \longrightarrow K_0(R/I)$$

The task was now to to define $K_n(R)$ for all n such that we can extend this long exact sequence. This was done by Quillen in 1973. He defined the higher K groups to be the homotopy groups of a certain space (See 6.3.3) and the homotopy fibre long exact sequence now translated to the long exact sequence of K groups fitting perfectly with the initial chunk of the long exact sequence.Now, the problem is that computing K-groups is extremely difficult even for the cases when the ring is $\mathbb{Z}$ or a field. For finite fields, we have 6.3.4. Kurihara proved that $K_{4n}(\mathbb{Z}) = 0, \forall n$ is equivalent to the following conjecture by Kummer.
Vandiver's Conjecture: Any prime p doesn't divide the class number h_K of the maximal real subfield of the p^{th} cyclotomic field.

This report encapsulates all of the above along while exploring more computations of the lower K-groups. Towards the end, there is a detailed description of the + construction by Quillen, thus defining the higher K-groups. Most of the contents were delivered as part of a lecture series at Indian Statistical Institute, Kolkata. Many remarks in the report arose from discussions with the attendees to whom I am grateful.

2 $\mathbf{K}_0$

Definition 2.0.1 (K_0 of a ring)**.** Let $P(R)$:= Isomorphism classes of finitely generated projective modules over a commutative ring with unity R. It is an abelian monoid with direct sum as the addition. It is also a commutative semi-ring with tensor product as the product. Its Grothendieck completion gives us a commutative ring $K_0(R)$.

Remark 2.0.2.

- Just defining K_0 to be the set of all finitely generated projective modules won't work as that doesn't form a set.
- Any element in $K_0(R)$ is of the form $[P]-[Q]$, and adding a suitable direct summand it is equal to $[P']-[R^n]$ for some n and some projective module P'. $[M]$ denotes the class of M, a finitely generated projective module, in K_0.
- As in case of any Grothendieck completion, $[M]=[N]\in K_0(R)$ iff $\exists P\in P(R)$: $M+N=N+P\in P(R)$. By adding a suitable direct summand, we have $[M]=[N]$ in $K_0(R)$ iff $\exists R^n$ such that $M+R^n=N+R^n$ in $P(R)$, i.e., $M\bigoplus R^n\cong N\bigoplus R^n$.(stably isomorphic)

$$\begin{tikzcd}
P(R) \arrow[r, "f"] \arrow[d, "i"'] & G \\
K_0 \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

- In the above diagram coming from the universal property of Grothendieck completion, where G is a commutative ring in this case, the map i need not be injective. For example, by 8.0.6, there are $M\in P(R)$ which are not free but stably free.
- However, if $i(M)=0$ in $K_0(R)$, $M=0$ in $P(R)$, This is because, if $M\bigoplus R^n\cong R^n$, localising at every $p\in \mathrm{Spec}(R)$, we get by 8.0.3 and 8.0.4, $M_p=0, \forall p\in \mathrm{Spec}(R)$, hence $M=0$. This doesn't prove injectivity of i as $P(R)$ is not a group.
- If $f:R\to S$ is a ring map, it induces a map $P\mapsto P\bigotimes_R S$, a semiring map from $P(R)$ to $P(S)$, which by universal property of group completion induces a ring map f_* from $K_0(R)$ to $K_0(S)$.

Example 2.0.3.

- If R is a commutative local ring (for example $\mathbb{Z}/p^k\mathbb{Z}$), by 8.0.4,

$K_0(R) \cong \mathbb{Z}$. Same for fields.

- For a PID R, by 8.0.5, $K_0(R) \cong \mathbb{Z}$.
- By Theorem 8.0.10 and induction, for a PID R, $K_0(R[t_1, ..., t_n]) = \mathbb{Z}$.

2.1 K_0 and Idempotents

Theorem 2.1.1. *There is a monoidal isomorphism between the $P(R)$ and $I(R)$, where $I(R)$= Orbits in $Idem(R)$ under the conjugation action of $Gl(R)$. $I(R)$ is a monoid where the addition of two matrices A and B is simply* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

Let $\theta : P(R) \to I(R)$, be the isomorphism. Then, given $f : R \to S$, a ring map, we have the following commutative diagram.

$$\begin{array}{ccc} P(R) & \xrightarrow{f_*} & P(S) \\ \theta\downarrow & & \downarrow\theta \\ I(R) & \xrightarrow[f_*]{} & I(S) \end{array}$$

Here, $Idem(R) = \varinjlim Idem(n, R)$ where $Idem(n, R) \subset M_n(R)$ is the set of idempotent matrices over R. And $Idem(n, R) \to Idem(m, R)$ is $M \mapsto M \bigoplus 0_{m-n}, m \geq n$. Here, $M \in M_k(R), N \in M_l(R), M \bigoplus N \in M_{k+l}(R)$ denotes the matrix with the diagonal blocks being M and N (in this order). $0_k \in M_k(R)$ denotes the zero matrix.

The map θ is defined as follows:
Given a finitely generated projective module P representing some class $[P]$ in $P(R)$, choose Q, a finitely generated projetive module such that $P \bigoplus Q \cong R^n$ for some n. Choose an isomorphism $\phi : P \bigoplus Q \to R^n$. So, $R^n = \phi(P) \bigoplus \phi(Q)$. Let $\pi : R^n \to R^n$ be the projection onto $\phi(P)$.
Define: $\theta([P]) = [\pi] \in I(R)$. A cumbersome checking shows it is well-defined and indeed defines an isomorphism between the monoids. (See 9)

Using the above theorem, we can easily prove the following:

Proposition 2.1.2. *If $\{R_\alpha\}$ is a direct system of rings with $R = \varinjlim R_\alpha$, then $K_0(R) = \varinjlim K_0(R_\alpha)$.*

Proposition 2.1.3. *Define $D(R, I) = \{(x, y) \in R^2 : x - y \in I\}$ a subring of R^2. We have a map $p_1 : D(R, I) \to R$, the projection onto the first co-ordinate. Define $K(R, I) := Ker((p_1)_* : K_0(R) \to K_0(R/I))$. Then, we have the following exact sequence:*

$$K_0(R, I) \xrightarrow[(p_2)_*]{} K_0(R) \xrightarrow[q_*]{} K_0(R/I)$$

Proof. Let $[M] \in K_0(R, I) \subset K_0(D)$.

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "p_i"'] & R^2 \arrow[dl, "p_i"] \\ R & \end{tikzcd} \qquad \begin{tikzcd} K_0(D) \arrow[r, "i_*"] \arrow[d, "(p_i)_*"'] & K_0(R^2) \arrow[dl, "(p_i)_*"] \\ K_0(R) & \end{tikzcd}$$

If $i_*([M])$ corresponds to (e_1, e_2) under θ, then by Theorem 2.1.1, $(p_1)_*([M])$ corresponds to e_1. $[M] \in K_0(R, I) \implies e_1 = 0$. $(p_2)_*([M]) = e_2$ and $e_1 - e_2$ has all entries in I, by definition of D. So, $q_* \circ (p_2)_*([M])$ corresponds to $\bar{e_2} = \bar{e_1} = 0$. So, $q_* \circ (p_2)_* = 0$.

Let $[M] - [N] \in Ker(q_*)$ corresponding to $e - f$. So, $\bar{e} = \bar{f}$. Then, by possibly adding zeroes, we have $\bar{e} \bigoplus 1_k = \bar{A}(\bar{f} \bigoplus 1_k)\bar{A}^{-1} \in Gl(n, R/I)$.

Lemma 2.1.4. $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Since, each of the matrices in the right are invertible, $\exists B \in Gl(2n, R) : \bar{B} = \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{A}^{-1} \end{pmatrix}$. So, an easy matrix multiplication gives, $\bar{e} \bigoplus 1_k \bigoplus 0_n = \bar{B}(\bar{f} \bigoplus 1_k \bigoplus 0_n)\bar{B}^{-1}$. Denote $g = B(f \bigoplus 1_k \bigoplus 0_n)B^{-1}$ and $h := e \bigoplus 1_k \bigoplus 0_n$. So, when seen as elements of $I(R)$, $f \bigoplus 1_k \bigoplus 0_n = g$ and $\bar{h} = \bar{g} \in I(R/I)$. So, (g, h) and (g, g) belong to $I(D)$. Also, $(p_1)_*((g, h) - (g, g)) = g - g = 0$ and $(p_2)_*((g, h) - (g, g)) = h - g = e + 1_k + 0 - f - 1_k - 0 = e - f$. So, $[M] - [N] \in Im((p_2)_*)$. So, the exactness is proven.

□

Remark 2.1.5. In the above sequence, exactness at the right is not guaranteed. For, example it is easy to see from Theorem 2.1.1, $K_0(R\times S)\cong K_0(R)\times K_0(S)$. So, $K_0(\mathbb{Z}/n\mathbb{Z})\cong \mathbb{Z}^k$ where k is the number of distinct prime factors of n. But the map $q_*:\mathbb{Z}=K_0(\mathbb{Z})\to\mathbb{Z}^k$ is just $1\mapsto(1,...,1)$ which is not surjective.

2.2 Idempotent Lifting

Example 2.2.1. $K_0(R[X]/<x^n>)\cong K_0(R)$ follows from the next proposition.

Proposition 2.2.2. *$K_0(R)\cong K_0(R/I)$, if I is a nilpotent ideal.*

Proof. Let I be a nilpotent ideal, i.e., $I^N=0$ for some N. So, $I\subset Nil(R)\subset Jac(R)$. We will first prove the injectivity. Let $\phi:M/IM\to N/IN$ be an isomorphism for some finitely generated projective R-modules M and N. N being projective, we have

$$\begin{tikzcd} & & N \arrow[d] \arrow[dll, dashed, "f"'] \\ M \arrow[r] & \frac{M}{IM} \arrow[r, "\phi"'] & \frac{N}{IN} \end{tikzcd}$$

Now, the induced map $\phi\circ f:N/IN\to N/IN$ is clearly the identity map. So, $f=\phi^{-1}\circ\phi\circ f=\phi^{-1}$. As $I\subset Jac(R)$, by Nakayama's lemma we get f is surjective. Similarly, we get a surjective $g:M\to N$ using the fact that M is projective. Hence, $M\bigoplus A\cong N$ and $N\bigoplus B\cong M$, for some finitely generated projective R-modules. So, in $K_0(R)$, we have $[N]+[B]+[A]=[N]\implies[A\bigoplus B]=0\implies A=B=0$. Note that injectivity only required $I\subset Jac(R)$.

Now, to prove surjectivity we use the bijection in Theorem 1.1.1. Inductively, we can assume, $I^2=0$. So, it is enough to prove that given an element $e\in Idem(R/I)$, we can get an $f\in Idem(R)$ such that $f=e$ mod I, where $I^2=0$. Now, say $g\in M(R)=\varinjlim M_n(R)$ such that $g=e$ mod I. As $e^2=e$, we have $g^2-g\in I$. Now, let $s:=3g^2-2g^3$. Then, $s^2-s=(2g+1)(2g-3)(g^2-g)^2\in I^2=0$ and clearly $s=r=e$ mod I.

Another method to prove surjectivity is by using Milnor patching (See 8.0.9). Consider the following Milnor square:

$$\begin{array}{ccc} R/IJ & \longrightarrow & R/J \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & R/(I+J) \end{array}$$

Take $I = J$. The maps are the usual quotient maps. Now, given an isomorphism between two R/I projective modules, we get a projective module M over R (as $IJ = I^2 = 0$) such that M/IM is isomorphic to the two R/I modules. Take the two modules to be the same and the isomorphism to be identity and we get the result.

□

2.3 K_0 of polynomial rings

Theorem 2.3.1. *If R is a regular noetherian ring, then $K_0(R[t]) \cong K_0(R)$.*

Proof.

- By the following theorem, it's clear that the map $i_* : K_0(R) \to K_0(R[t])$ induced by the inclusion map i is surjective.
- $j : R[t] \to R$ given by $f(t) \mapsto f(0)$ is such that $j_* \circ i_* = Id$, so i_* is in fact a split injection.

□

Theorem 2.3.2 (Swan)**.** *If R is a regular noetherian ring, then for any finitely generated $R[t]$-module M, there exists a resolution*

$$0 \longrightarrow Z_n \longrightarrow ... \longrightarrow Z_0 \longrightarrow M \longrightarrow 0$$

where each $Z_i \in P^R(R[t]) := \{M$, finitely generated $R[t]$-module such that $M \cong N \bigotimes_R R[t]$ where N is a finitely generated R-module $\}$.

Proof. Let M as in the statement. Then, we have

$$0 \longrightarrow N \longrightarrow R[t]^n \longrightarrow M \longrightarrow 0$$

We have by Swan's lemma (See 8.0.8),

$$0 \longrightarrow X \longrightarrow Y \longrightarrow N \longrightarrow 0$$

where $X = A \bigotimes_R R[t], Y = B \bigotimes_R R[t]$. As R is regular, we have finite projective resolutions

$$0 \longrightarrow A_n \longrightarrow ... \longrightarrow A_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow B_m \longrightarrow ... \longrightarrow B_0 \longrightarrow B \longrightarrow 0$$

Now, tensoring with $R[t]$ gives finite projective resolution for X and Y.

Now, take the mapping cone:

$$\cdots \longrightarrow X_1 \bigoplus Y_2 \longrightarrow X_0 \bigoplus Y_1 \xrightarrow{(x_0,y_1)\mapsto g(x_0)+\partial_Y(y_1)} Y_0 \xrightarrow{y_0 \mapsto f\circ\partial_Y(y_0)} N \longrightarrow 0$$

Using exactness of the resolutions for X and Y, it is easy to see that the above sequence is also exact, which gives the required resolution for N and hence for M.

$\square$

3 $\mathbf{K}_1$

Definition 3.0.1 (K_1 of a ring)**.** Let $Gl(R) = \varinjlim Gl(n, R)$ where $Gl(n, R) \to Gl(m, R)$ is the map sending $M \in Gl_n(R)$ to $M \bigoplus 1_{m-n} \in Gl_(m, R), m \geq n$. $K_1(R) := \frac{Gl(R)}{E(R)}$, where $E(R) = \varinjlim E(n, R) =< e_{ij}(a), a \in R >$. $e_{ij}(a)$ is the matrix with 1's on the diagonal and a in the (i, j)-th place, $i \neq j$. The definition makes sense because of the following lemma.

Lemma 3.0.2 (Whitehead)**.** $E(R) = [Gl(R), Gl(R)] = [E(R), E(R)]$.

Proof. From $\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$, we have by Lemma 2.1.4, $[Gl(R), Gl(R)] \subset E(R)$. Now, for i, j, k distinct, $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$. So, $E(R) = [E(R), E(R)] \subset [Gl(R), Gl(R)]$. □

Remark 3.0.3.
- Since R is commutative, we have a determinant map $det : Gl(R) \to R^*$ which factors through $det : K_1(R) \to R^*$ which is clearly surjective. Denote its kernel by $SK_1(R)$. Then, $K_1(R) \cong R^* \bigoplus SK_1(R)$ as abelian groups.

3.1 Stability of K_1

Observe the following: We have a matrix $A \in Gl_n(R)$ and by multiplication with elements of $E(R)$, we get to the following matrix $\begin{pmatrix} a_1 & * \\ 0 & A' \end{pmatrix}$ where $a_1 \in R^*$. Now, this is possible in the obvious case of when there is an $r \in R^*$ in the first coloumn of A (or can be brought there by multiplication with elements of $E(R)$). This is clearly true when R is a field. Now, if $[a_{11}, ..., a_{nn}]$ is the first coloumn of A and R is a semi-local ring. As A is invertible $< a_{11}, ..., a_{nn} >= R$ and by a form of prime avoidance lemma, $\exists b_j \in R, 2 \leq j \leq n$ such that $a_1 + b_2a_2 + ... + b_na_n \in R^*$. So, in case we have R local, semi-local, Euclidean domain or a field, the earlier mentioned matrix transformation can be done. Now, by coloumn transformations, we get to the matrix $\begin{pmatrix} a_1 & 0 \\ 0 & A' \end{pmatrix}$ where $A' \in Gl_{n-1}(R)$. Now, as long as the above process can be repeated for A', we can proceed and after a finite number of steps, we reach the diagonal matrix with elements $a_i, 1 \leq i \leq n$. Now, by the proof of Lemma 2.1.4, the matrix $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a_i & & & \\ & & & a_i^{-1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ is in $E(R)$. So, consecutively multiplying these matrices, we get to the following matrix: $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & det(A) \end{pmatrix}$ which is obtained from A by multiplying matrices from $E(R)$. So, if A has determinant 1, $[A] = 1 \in K_1(R)$. So, in this case, $K_1(R) = R^*$.

For dedekind domains, a similar thing can be done provided the matrix has size at least 3. So, in this case $K_1(R)$ is generated by the image in $Gl(R)$ of R^* and $Sl(2, R)$. (See 9)

3.2 K$_1$ of polynomial rings

Theorem 3.2.1. *If R is a noetherian regular ring, then $K_1(R[t]) \cong K_1(R)$.*

Proof. $S := R[t]$. Let $M \in Gl_n(S)$. So, $M = M_0 + M_1 + ...M_d$, where $(M_i)_{kl} = a_{kl}t^i, a_{kl} \in R$. So, M_d contains entries which are homogeneous elements of order d. $M_d = t\tilde{M_{d-1}}$ where $\tilde{M_{d-1}}$ contains entries homogeneous of degree $(d-1)$. We carry out elementary row operations on the following matrix. $\begin{pmatrix} M & 0_n \\ 0_n & I_n \end{pmatrix}$. The resultant matrix is $\begin{pmatrix} M & \tilde{M_{d-1}} \\ 0_n & I_n \end{pmatrix}$.
Futher applying elementary coloumn operations on it we get,

$\begin{pmatrix} M - t\tilde{M_{d-1}} & \tilde{M_{d-1}} \\ -tI_n & I_n \end{pmatrix}$. Note that every entry of this matrix has degree less or equal to d. Also, as elements of $Gl(S)$, $M = \begin{pmatrix} M & 0_n \\ 0_n & I_n \end{pmatrix}$. So, continuing in a similar manner, we get to a matrix $\beta_0 + \beta_1$ which is equal (as elements of $Gl(S)$) to M times some elements of $E(S)$

(multiplied both in the left and right); where $\beta_i \in M_N(S)$ is of degree i, $i = 0, 1$. We denote this by $M \sim \beta_0 + \beta_1 = \beta_0(I_N + \beta_0^{-1}\beta_1)$. This can be written since $\beta_0 = p_*(\beta_0 + \beta_1)$ where $p_* : Gl_N(S) \to Gl_N(R)$ is the group homomorphism where $p_*(f_{ij}) = f_{ij}(0)$.

Claim:$\gamma := \beta_0^{-1}\beta_1$ Then, $\gamma^k = 0$ for some k as elements of $M_N(S)$.

To prove this write $(I_N + \gamma)^{-1} = \gamma_0 + ... + \gamma_m$. Then, $I = \gamma_0 + ... + \gamma_m + \gamma\gamma_0 + ... + \gamma\gamma_m$ and γ has degree 1. For the sake of illustartion, we prove the claim for $m = 2$ and the general case folllows similarly by induction. In this case, we then have $I = \gamma_0 + \gamma_1 + \gamma_2 + \gamma\gamma_0 + \gamma\gamma_1 + \gamma\gamma_2$. $\gamma\gamma_2$ has degree 3, so comparing degrees on both sides (entry-wise), we get $\gamma\gamma_2 = 0$. Now, $\gamma_2 + \gamma\gamma_1$ has degree 2. So, again $\gamma_2 + \gamma\gamma_1 = 0 \implies \gamma\gamma_2 + \gamma^2\gamma_1 = 0 \implies \gamma^2\gamma_1 = 0$. Proceeding similarly, we get $\gamma^3\gamma_0 = 0$ but again by comparing degrees we get $\gamma_0 = I_N$. So, $\gamma^3 = 0$ in this case.

Now, we will be defining in the next section a group $G_1(R)$ and an isomorphism $\psi : K_1(R) \to G_1(R)$ mapping $\alpha \in Gl_n(R)$ to the class of (R^n, α) where is α is treated as an element of $Aut(R^n)$.

Claim: $\psi(I_N + \gamma) = 0 \in G_1(R)$

Note that as ψ is an isomorphism, proving the claim proves that $I_N + \gamma = 1 \in K_1(S)$. So, as elements of $K_1(S), M = \beta_0^{-1} \in i_*(K_1(R))$, where i_* is induced by the inclusion $i : R \to R[t]$. So, i_* is surjective. It is clearly injective as before in the case of K_0 since it has a left inverse q_* induced by the quotient map $R[t] \to R; t \mapsto 0$. Hence, the above claim implies our theorem.

Now, we prove this claim. We illustrate this for the case $k = 2, 3$ where $\gamma^k = 0$ and the general case follows from induction.

k=2

We have the following diagram where each row is clearly exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \gamma S^N & \hookrightarrow & S^N & \longrightarrow & S^N/\gamma S^N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle I_N+\gamma} & & \Big\downarrow{\scriptstyle I_N+\gamma} & & \Big\downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \gamma S^N & \hookrightarrow & S^N & \longrightarrow & S^N/\gamma S^N & \longrightarrow & 0 \end{array}$$

Clearly, the left and the right squares commute. Now, if $x \in \gamma S^N \implies x = \gamma(y), y \in S^n \implies (I_N + \gamma)(x) = \gamma(y) + \gamma^2(y) = \gamma(y) = x$ as $\gamma^2 = 0$. So, $I_N + \gamma : \gamma S^N \to \gamma S^N$ is simply the identity map. So, by Remark 4.0.3, in $G_1(S)$, $(S^N, I_N + \gamma) = (S^N/\gamma S^N, Id) + (\gamma S^N, Id) = 0$.

k=3

Now, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \gamma^2 S^N & \hookrightarrow & \gamma S^N & \longrightarrow & \gamma S^N/\gamma^2 S^N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle I_N+\gamma} & & \Big\downarrow{\scriptstyle I_N+\gamma} & & \Big\downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \gamma^2 S^N & \hookrightarrow & \gamma S^N & \longrightarrow & \gamma S^N/\gamma^2 S^N & \longrightarrow & 0 \end{array}$$

As in the previous case, we have $I_N + \gamma : \gamma^2 S^N \to \gamma^2 S^N$ is the identity map. Clearly the rows are exact and squares commute. Then, again by Remark 4.0.3 in $G_1(S)$, we have $(\gamma S^N, I_N + \gamma) = 0$. Now, going back to the previous diagram, we have $(S^N, I_N + \gamma) = 0 \in G_1(S)$.

$\square$

4 G-theory

The contents of this section give a brief introduction to G-theory of a ring R. It will complete our proof of Theorem 3.2.1 of the previous section.

Definition 4.0.1 (G$_0$ of a ring)**.** First note that the following is a set: $S_0 = \{M : M = R^n/K$ for some n and some R-submodule K of $R^n\}$. Now let F_0 denote the free abelian group on this set. Define $G_0(R)$ to be the quotient of this free abelian group by the following relations:

- $[M] = [N]$ if $M \cong N$ as R-modules.
- For any short exact sequence:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

  $[M_3] + [M_1] = [M_2]$.

Definition 4.0.2 (G$_1$ of a ring)**.** Again note that the following is a set: $S_1 = \{(M, \alpha) : M = R^n/K$ for some n and some R-submodule K of R^n and $\alpha \in Aut(M)\}$. Now let F_1 denote the free abelian group on this set. Define $G_1(R)$ to be the quotient of this free abelian group by the following relations:

- $[(M, \alpha)] + [(M, \beta)] = [(M, \alpha \circ \beta)]$
-

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "\alpha_1"'] & M \arrow[r, "g"] \arrow[d, "\alpha"'] & M_2 \arrow[r] \arrow[d, "\alpha_2"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"'] & M \arrow[r, "g"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

  If the rows are exact and the squares are commuative, then $[(M_1, \alpha_1)] + [(M_2, \alpha_2)] = [(M, \alpha)]$.

Remark 4.0.3.

1. Note that $[(M, Id)] = [(M, Id)] + [(M, Id)] \implies [(M, Id)] = 0 \in G_1(R)$.
2. In most cases, we will have commutative squares of exact sequences where the finitely generated R-modules are not necessarily of the form R^n/K, but only isomorphic. Still we can gather information by the following:

$$\begin{tikzcd}
0 \arrow[r, dashed] & N_1 \arrow[r, dashed, "g_1"] \arrow[d, "\phi_1"'] & N \arrow[r, dashed, "g_2"] \arrow[d, "\phi"'] & N_2 \arrow[r, dashed] \arrow[d, "\phi_2"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "\alpha_1"'] & M \arrow[r, "g"] \arrow[d, "\alpha"'] & M_2 \arrow[r] \arrow[d, "\alpha_2"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"'] \arrow[d, "\phi_1^{-1}"'] & M \arrow[r, "g"'] \arrow[d, "\phi^{-1}"'] & M_2 \arrow[r] \arrow[d, "\phi_2^{-1}"'] & 0 \\
0 \arrow[r, dashed] & N_1 \arrow[r, dashed, "h_1"'] & N \arrow[r, dashed, "h_2"'] & N_2 \arrow[r, dashed] & 0
\end{tikzcd}$$

Here, the R-modules N and N_i's are of the form R^n/K for some n's and K's. The dotted maps are induced by the pre-existing maps. For example, $g_1 := \phi^{-1} \circ f \circ \phi_1$ and similarly for g_2. $h_1 := \phi^{-1} \circ f \circ \phi_1$ and similarly for h_2. The maps ϕ and the ϕ_i's are some chosen isomorphism between the finitely generated modules and modules of the form R^n/K. Since the maps in the top and bottom row are induced from those in between, it is easy to check that the top and bottom rows are exact and all squares are commutative. So, in $G_1(R)$, we have $[(N_2, \phi_2^{-1} \circ \alpha_2 \circ \phi_2)] + [(N_1, \phi_1^{-1} \circ \alpha_1 \circ \phi_1)] = [(N, \phi^{-1} \circ \alpha \circ \phi)]$.

3. Now, suppose as in the previous remark, we have a finitely generated $R-$module, M and pick two isomorphisms ϕ and ψ. $\phi : R^n/K \to M$, $\psi : R^m/L \to M$. We have $\psi^{-1} \circ \phi : R^n/K \to R^m/L$ as isomorphism. If in the set up of the previous remark, we pick 2 choices $[(R^n/K, \phi^{-1} \circ \alpha \circ \phi)]$ and $[(R^m/L, \psi^{-1} \circ \alpha \circ \psi)]$ 'representing' the pair (M, α).

   Then, $g := \psi^{-1} \circ \alpha \circ \psi = (\psi^{-1} \circ \phi) \circ f \circ (\psi^{-1} \circ \phi)^{-1}$, where $f = \phi^{-1} \circ \alpha \circ \phi$.

$$\begin{tikzcd}
0 \arrow[r] & R^n/K \arrow[r, "\psi^{-1}\circ\phi"'] \arrow[d, "f"'] & R^m/L \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & R^n/K \arrow[r, "\psi^{-1}\circ\phi"'] & R^m/L \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

   So, in $G_1(R)$, $[(R^n/K, f)] = [(R^m/L, g)]$.

4.1 Another view of K_0 and K_1

Definition 4.1.1. Consider the following set $S_0 := \{M : M$ is a direct summand of some $R^n\}$. And let $K_0(PR)$ be the free abelian group on S_0 with relations as in the case of G_0. Similarly, let $S_1 := \{(M, \alpha) : M$ is a direct summand of some R^n and $\alpha \in Aut(M)\}$. Consider the free abelian group on this set and put relations as in the definition of G_1; call it $K_1(PR)$.

Theorem 4.1.2. *1. If $[P] \in K_0(R)$ is a generator, $\exists M \cong P$ and M is a direct summand of some R^n. So, there is a map $\phi : K_0(R) \to K_0(PR)$ sending generators to generators whose well-definedness follows from the first relation the definition.*

2. Consider a map ψ sending a matrix $A \in Gl(R)$ to $[(R^n, A)]$ where A is treated as an element of $Aut(R^n)$ if $A \in Gl_n(R)$ for some n. So, there is a map $\psi : K_1(R) \to K_1(PR)$.

Then, ψ is well-defined and both ϕ and ψ are isomorphisms.

Proof. **Proof of 1**

As well-definedness is clear from the definition, we only need to check ϕ is an isomorphism. ϕ is induced from the monoid map from $P(R)$ to $K_0(PR)$. It is a monoid map because: in $P(R), [P] + [Q] = [P \bigoplus Q]$, and we have an exact sequence

$$0 \longrightarrow P \longrightarrow P \bigoplus Q \longrightarrow Q \longrightarrow 0$$

So, by the defining relations, we have $[P] + [Q] = [P \bigoplus Q]$ even in $K_0(PR)$. Now, this shows the induced map ϕ is actually a group homomorphism. Clearly by the above observation, relations in the generators of $P(R)$ map to 0 in $K_0(PR)$. Now, conversely, if we have a relation representing the following exact sequence,

$$0 \longrightarrow P \longrightarrow M \longrightarrow Q \longrightarrow 0$$

As Q is projective, this splits and we have $P \bigoplus Q \cong M$; so in $K_0(R), [P]+[Q]=[M]$. Hence, ϕ is an isomorphism.

Proof of 2

We have to first show the well definedness of ψ.

Suppose $[A]=[B] \in K_1(R), \implies A \bigoplus 1_k = (B \bigoplus 1_l)E; E \in E(N,R)$, as elements of $Gl(N,R)$, where $A \in Gl(N-k,R), B \in Gl(N-l,R)$. By the second relation in Definition 4.0.2 and Remark 1, $[(R^{N-k},A)] = [(R^N, A \bigoplus 1_k)]$ and similarly $[(R^{N-l},A)] = [(R^N, B \bigoplus 1_l)]$. Now, clearly by the first relation of Definition 4.0.2, $[(R^N, A \bigoplus 1_k)] = [(R^N,(B \bigoplus 1_l)E)] = [(R^N, B \bigoplus 1_l)] + [(R^N,E)]$. So, showing $[(R^N,E)]=0$ suffices and this is enough if $E = e_{ij}(\lambda)$ again by the first relation of Definition 4.0.2.

$$\begin{tikzcd}
0 \arrow[r] & R^{N-1} \arrow[r, "f"] \arrow[d, "Id"] & R^N \arrow[r, "\pi"] \arrow[d, "e_{ij}(\lambda)"] & R \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & R^{N-1} \arrow[r, "f"'] & R^N \arrow[r, "\pi"'] & R \arrow[r] & 0
\end{tikzcd}$$

Here, π is the projection to the i^{th} coordinate and f is the map sending $(x_1,..,x_{N-1})$ to $(x_1,..,x_{j-1},0,x_j,..,x_{N-1})$. So, the above diagram has rows exact and squares commutative. So, in $K_1(PR)$, by Remark 1, $[(R^N, e_{ij}(\lambda))]=0$. The above arguments also show that ψ is a homomorphism.

Now, we show surjectivity of ψ. Note that if $[(P,\alpha)] \in K_1(PR)$, $\exists Q : P \bigoplus Q = R^n$.

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "\alpha"] & P \bigoplus Q \arrow[r] \arrow[d, "\alpha \bigoplus 1_Q"] & Q \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & P \arrow[r] & P \bigoplus Q \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

So, by Remark 1, $[(P,\alpha)] = [(R^n, \alpha \bigoplus 1_Q)] \in \text{Image}(\psi)$.

To show injectivity, we find a left inverse to ψ. Define $\eta : K_1(PR) \to K_1(R)$ as follows: if $[(P,\alpha)] \in K_1(PR) \implies \exists Q : P \bigoplus Q = R^n, \eta([(P,\alpha)]) := [\alpha \bigoplus 1_Q] \in K_1(R)$. Note that $P \bigoplus Q \bigoplus R^k = R^{n+k}$ with the obvious equality. And $[\alpha \bigoplus 1_Q] = [\alpha \bigoplus 1_Q \bigoplus 1_{R^k}]$ as elements of $K_1(R)$ by the definition of $Gl(R)$. So, if $P \bigoplus S = P \bigoplus Q = R^n$ and $[(P,\alpha)] \in$

$K_1(PR), Q = R^n/P \cong R^n/P = S$, let the isomorphism be $f : Q \to S$. Now, $(1, f) : R^n \to R^n$ be the isomorphism defined by $(1, f)(p + q) := p + f(q), p \in P; q \in Q$. Then, as elements of $Aut(R^n)$, $(1, f)^{-1} \circ (\alpha \bigoplus 1_S) \circ (1, f) = (\alpha \bigoplus 1_Q)$. So, since $K_1(R)$ is abelian, $[\alpha \bigoplus 1_Q] = [\alpha \bigoplus 1_S]$. This proves the well defined-ness of the map η as a map from the free abelian group on the set S_1 as in Definition 4.1.1. Now, to show well-definedness of η as a map from $K_1(PR)$, we will simply show that the relations map to 0 under η. If $\alpha, \beta \in Aut(P)$, clearly as elements of $Aut(R^n)$, $(\alpha \circ \beta) \bigoplus 1_Q = (\alpha \bigoplus 1_Q) \circ (\beta \bigoplus 1_Q)$. Now, if the second relation holds, i.e.,

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"] & P \arrow[r] \arrow[d, "\alpha"] & P_2 \arrow[r] \arrow[d, "\alpha_2"] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & P \arrow[r] & P_2 \arrow[r] & 0
\end{tikzcd}$$

as per the earlier notation. Say, $P_1 \bigoplus Q_1 = R^n; P_2 \bigoplus Q_2 = R^m$. So, we have,

$$\begin{tikzcd}
 & R^n \arrow[d] & R^{n+m} \arrow[d] & R^m \arrow[d] & \\
0 \arrow[r] & P_1 \bigoplus Q_1 \arrow[r] \arrow[d, "\alpha_1"] & P \bigoplus Q_1 \bigoplus Q_2 \arrow[r] \arrow[d, "\alpha"] & P_2 \bigoplus Q_2 \arrow[r] \arrow[d, "\alpha_2"] & 0 \\
0 \arrow[r] & P_1 \bigoplus Q_1 \arrow[r] & P \bigoplus Q_1 \bigoplus Q_2 \arrow[r] & P_2 \bigoplus Q_2 \arrow[r] & 0 \\
 & R^n \arrow[u] & R^{n+m} \arrow[u] & R^m \arrow[u] &
\end{tikzcd}$$

which becomes the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R^n \arrow[r, "f"] \arrow[d, "\alpha_1 = A_1"'] & R^{n+m} \arrow[r, "g"] \arrow[d, "A"] & R^m \arrow[r] \arrow[d, "\alpha_2 = A_2"] & 0 \\
0 \arrow[r] & R^n \arrow[r, "f"'] & R^{n+m} \arrow[r, "g"'] & R^m \arrow[r] & 0
\end{tikzcd}$$

Here, $R^{n+m} = \{(x, y) : x \in R^n, y \in R^m\}$. Let $j : R^m \to R^{n+m}$ be the right inverse of g. So, $R^{n+m} = f(R^n) \bigoplus j(R^m)$. Note that $A \circ j(y) - j \circ A_2(y) \in Im(f)$ so there exists a unique $a \in R^n$ such that $f(a) = A \circ j(y) - j \circ A_2(y)$ and this map sending y to this a is R-linear, denote it by the matrix $M \in M_{n \times m}$. Now, define $H : R^{n+m} \to R^{n+m}$ by $H((x, y)) := f(x) + j(y)$.

Then, $H^{-1} \circ A \circ H((x,y)) = H^{-1}(A(f(x)) + A(j(y))) = H^{-1}(f \circ A_1(x)) + (My, A_2(y)) = (A_1(x), 0) + (My, 0) + (0, A_2(y))$. So, if B is the following matrix $\begin{pmatrix} A_1 & M \\ 0 & A_2 \end{pmatrix}$, as elements of $K_1(R), B = A$. Now, since A_2 is invertible, clearly by elementary row operations we get from B to the matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, and this matrix represents the product of elements A_1 and A_2 in $K_1(R)$ since

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \text{ and Lemma 2.1.4.}$$

Thus, the relations all map to 0. Hence, we have a group homomorphism from $K_1(PR)$ to $K_1(R)$. Now, $\eta \circ \psi([A]) = \eta([(R^n, A)]) = [A]$. So, ψ is injective, hence an isomorphism.

□

4.2 $G_i(R)$ and $K_i(R)$

Proposition 4.2.1. *The map sending generators to generators from $K_i(PR) \to G_i(R)$ is well-defined for $i = 0, 1$.*

Proof. This is clear as the relation satisfied by the generators is exactly the same for both groups. □

If R is regular, given any M finitely generated R-module, we have a finite projective resolution of M, as follows:

$$0 \longrightarrow Q_m \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow M \longrightarrow 0$$

So, we may define a map from $G_0(R)$ to $K_0(PR)$ by $[M] \mapsto \sum_{i=1}^{m} (-1)^{j+1} [Q_j]$.

As detailed in 9, given any $\alpha \in Aut(M)$, we have a finite projective resolution and an automorphism of the resolution:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_n & \xrightarrow{f_n} & \cdots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & P_n & \xrightarrow[f_n]{} & \cdots & \longrightarrow & P_2 & \xrightarrow[f_2]{} & P_1 & \xrightarrow[f_1]{} & M & \longrightarrow & 0
\end{array}$$

where $\alpha_i \in Aut(P_i)$ such that all squares commute. So, we may define a map from $G_1(R)$ to $K_1(PR)$ by $[(M, \alpha)] \mapsto \sum_{i=1}^{n}(-1)^{i+1}[(P_i, \alpha_i)]$.

Theorem 4.2.2 (Grothendieck's Resolution Theorem)**.** *For R regular noetherian, the above two maps are well-defined.*

Theorem 4.2.3. *For R regular noetherian, the above maps are isomorphisms.*

Proof. Note that the above two maps are well-defined by the previous theorem and when composed with the inclusion of $K_i(PR)$ in $G_i(R)$ is clearly identity. $\square$

5 K$_2$

Definition 5.0.1 (K_2 of a ring)**.** Define the nth Steinberg group $St(n, R)$ as follows: Consider the free group on generator $x_{ij}(r); i \neq j, 1 \geq i, j \leq n, n \geq 3, r \in R$ and relations:

- $x_{ij}(a)x_{ij}(b) = x_{ij}(a + b)$
- $[x_{ij}(a), x_{kl}(b)] = 1, i \neq l$ and $j \neq k$.
- $[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab)$, i, j, k are distinct.
- $[x_{ij}(a), x_{ki}(b)] = x_{kj}(-ab)$, i, j, k are distinct.

Note that $e_{ij}(\lambda)$'s also satisfy the above four relations. Hence, there is a surjective homomorphism from $\phi_n : St(n, R) \to E(n, R)$, $x_{ij}(a) \mapsto e_{ij}(a)$.

$K_2(n, R) := Ker(\phi_n)$. there are natural maps ψ_{nm} from the n^{th} Steinberg group to the m^{th} , $m \geq n$. We have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & K_2(n,R) \arrow[r, hook] \arrow[d] & St(n,R) \arrow[r, "\phi_n"'] \arrow[d, "\psi_{nm}"] & E(n,R) \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & K_2(m,R) \arrow[r, hook] & St(m,R) \arrow[r, "\phi_m"'] & E(m,R) \arrow[r] & 1
\end{tikzcd}$$

where $K_2(R) = \varinjlim K_2(n, R) = Ker(\phi)$. where $\phi : St(R) \to E(R)$, $St(R) := \varinjlim St(n, R)$.

Remark 5.0.2.

- For example, if $R = \mathbb{F}_2$, $St(2, R) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$
- Note that $K_2(R)$ precisely captures the information about all the non-trivial relations satisfied by the $e_{ij}(a)$'s.

Theorem 5.0.3. *$K_2 = Z(St(R))$, the centre of the Steinberg group.*

Proof. Clearly, $Z(E(R)) = 1$ and as ϕ is onto, $Z(St(R)) \subset \phi^{-1}(Z(E(R))) = K_2(R)$.

Now, for the converse, let $x \in K_2(R) \implies \phi(x) = 1$. Consider the following subgroup of $St(R)$. $P_n :=< x_{in}(r) : i < n, r \in R >$. From the Steinberg relations, it is clear that these generators commute among themselves. Further, using the first Steinberg relation, any element of P_n has a unique representation of the form $x_{1n}(r_1)...x_{(n-1)n}(r_{n-1})$. Now, ϕ restricted to P_n maps a general element to the following matrix:

$$\begin{pmatrix} 1 & & & r_1 \\ & 1 & & r_2 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}.$$

Clearly, multiplication in P_n translates to addition in the last coloumn (except the $(n,n)^{th}$ entry) of the matrices in $E_n(R)$.

So, if $\phi(y) = 1, y \in P_n$, then $r_i = 0, \forall i \implies y = 1$. So, ϕ restricted to P_n is injective. Now, for the $x \in K_2(R)$, we have x as a word in the $x_{ij}(r)$'s. Pick n to be larger than all the i,j's appearing in that word. Again from the Steinberg relations, we have that x normalises P_n. So, if $y \in P_n$, then $xyx^{-1} \in P_n$ and $\phi(y) = \phi(xyx^{-1})$. As ϕ restricted to P_n is injective, we have that $xy = yx, \forall y \in P_n$. Similarly, we can show that $[x, x_{nj}(r)] = 1$. So, as $x_{ij}(r) = [x_{in}(r), x_{nj}(1)]$ for i,j distinct, we have x commutes with all $x_{ij}(r)$. So, $x \in Z(St(R))$.

$\square$

Remark 5.0.4. We have:

$$\begin{tikzcd} St(n,R) \arrow[r, "\psi_{nm}"] \arrow[d, "\phi_n"'] & St(m,R) \arrow[d, "\phi_m"] \\ E(n,R) \arrow[r, hook] & E(m,R) \end{tikzcd}$$

A careful observation of the proof that $K_2(R) \subset Z(St(R))$ yields that actually we have for $m > n \geq 2, \psi_{nm}(K_2(n,R)) \subset Z(St(m,R))$.

Theorem 5.0.5 (Dennis and Vsserstein)**.** *For $n \geq sr(R) + 2$ (See 8.0.1), the map $\phi_n : K_2(n,R) \to K_2(R)$ is an isomorphism. (See 11)*

Theorem 5.0.6 (K_2 of $\mathbb{Z}$)**.** *Consider the element $w = x_{12}(1)x_{21}(-1)x_{12}(1) \in St(R)$. Then, clearly $\phi(w^4) = 1 \implies w^4 \in K_2(R)$. In fact, $< w^4 >= K_2(\mathbb{Z}) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$. (See 8)*

Theorem 5.0.7 (K_2 of fields)**.** *It follows from a result of Matsumoto that $K_2(F) = 0$ for all finite fields. (See 9)*

5.1 Universal central extensions

The main goal of this section is to show that $St(R)$ is the universal central extension of $E(R)$.

Definition 5.1.1 (Category of central extensions)**.** Given a group G. This category has objects: the central extensions of G, (E, ϕ) i.e., a group E and a surjective homomorphism $\phi : E \to G$ such that $Ker(\phi) \subset Z(E)$.
A morphism f in this category between (E, ϕ) and (F, ψ) are commutative diagrams like

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ f\downarrow & & \downarrow Id \\ F & \xrightarrow[\psi]{} & G \end{array}$$

Definition 5.1.2 (Universal Central extension)**.** A universal central extension (E, ϕ) of G is a central extension of G such that for any other central extension (F, ψ) of G, $\exists! f : (E, \phi) \to (F, \psi)$.

Example 5.1.3.
- Given any abelian group A, the projection $p : G \times A \to G$ is a cenral extension. This is called a trivial extension. In particular, every group G has a central extension.
- Every group may not have a universal central extension. They need to be perfect.
- The kernel of the universal central extension of G is isomorphic to $H_2(G)$.

Proposition 5.1.4. *G has a universal central extension iff $G = [G, G]$.*

Proof. Note that if G has a universal central extension, then by the next lemma, it is the image of a perfect group, hence is perfect.
Conversely, choose a free group F surjecting onto G with kernel R. Note that if (X, ψ) is a central extension of a perfect group G, then, then the commutator $[X, X]$ is perfect and maps onto G via ψ. So, $[R, F]$ is a normal subgroup of F and $F/[R, F]$ maps onto G, a central extension. So, by our previous observation, $[F, F]/[R, F]$ is a central extension of G.

Now, let (X, ψ) be a central extension of G. Since F is free and this is a central extension, we have a map from $[F, F]/[R, F]$ whose uniqueness is by the proof of the next lemma. $\square$

Proposition 5.1.5. *A central extension (E, ϕ) of G is universal iff $E = [E, E]$ and all central extensions of E are trivial.*

Proof. Suppose every central extension of (E, ϕ) splits. So, take a central extension (X, ψ) of G. Consider the following pull-back diagram.

$$\begin{tikzcd} E \times_G X \arrow[r, "(e,x)\mapsto e"'] \arrow[d, "(e,x)\mapsto x"'] & E \arrow[d, "\phi"] \\ X \arrow[r, "\psi"'] & G \end{tikzcd}$$

Then, $(E \times_G X)$ is a central extension of E, hence it splits via a map $h : E \to E \times_G X$. Composing with the projection, we get the required map. Now, we prove its uniqueness.

Now, we use the perfectness of E. Let f_1, f_2 be two maps from (E, ϕ) to (X, ψ). So, for any $y, z \in E$, $f_1(y) = f_2(y)c, f_1(z) = f_2(z)d, c, d \in Ker(\phi) \subset Z(E)$. So, $f_1([y, z]) = f_2([y, z])$ and as E is perfect $f_1 = f_2$.

Conversely, let (E, ϕ) be a universal central extension of G. Suppose E is not perfect. Then, $\exists f : E \to A$ a non-zero homomorphism to some abelian group A. Then, $f_1(y) := (\phi(y), f(y)); f_2(y) := (\phi(y), 1)$ are two maps from (E, ϕ) to ($G \times A$,projection) such that $f_1 \neq f_2$.

Now, let (X, ψ) be a central extension of E. Then, $p : \phi \circ \psi : X \to G$. If $y \in Ker(p)$, then $\psi(y) \in Z(E)$. So, the map $x \mapsto yxy^{-1}$ is a homomorphism from X to itself over E. Restrict it to the commutator subgroup of X which surjects onto E and is perfect, as E is perfect. But by one of our earlier observations, since $[X, X]$ is perfect, there can be only 1 homomorphism to itself over E. Hence, that has to be identity. So, y commutes with all elements of $[X, X]$. But as $[X, X]$ surjects onto E, X is generated by $[X, X]$ and $Ker(\psi)$. So, $y \in Z(X)$. Hence, (X, p) is a central extension of G. So, there is a unique map s from E to X. Then, $\psi \circ s : E \to E$ over G, hence is equal to identity. Hence, s is the required splitting. $\square$

Theorem 5.1.6 (($St(R), \phi$) is a universal central extension)**.**

Proof. Let $[Y, \phi]$ be a central extension of $St(R)$.

$$1 \longrightarrow C \longrightarrow Y \xrightarrow{\phi} St(R) \longrightarrow 1$$

We have to show that this is a trivial extension. So, since C is abelian, it is enough to find a splitting map for ϕ. Let $x_{ij}(\lambda) \in St(R)$. Let $h \notin \{i, j\}$. Pick $y_1 \in \phi^{-1}(x_{ih}(1)), y_2 \in \phi^{-1}(x_{hj}(\lambda))$. Define $s(x_{ij}(\lambda)) := [y_1, y_2]$. CLearly, as $Ker(\phi) \subset Z(St(R))$, this is well-defined.

Lemma 5.1.7. *If $j \neq k, l \neq i$, then $[\phi^{-1}(x_{ij}(\lambda)), \phi^{-1}(x_{kl}(\mu))] = 1$.*

Proof. Let $h \notin \{i, j, k, l\}$ and $y \in \phi^{-1}(x_{ih}(1)), y' \in \phi^{-1}(x_{hj}(\lambda)), y'' \in \phi^{-1}(x_{kl}(\mu)) \implies [y', y''] \in C, [[y, y'], y''] = 1$. □

It is an easy check that $[u, v][u, w] = [u, vw][v, [w, u]]$

Lemma 5.1.8. $[\phi^{-1}(x_{ih}(1)), \phi^{-1}(x_{hj}(\mu))] = [\phi^{-1}(x_{ik}(1)), \phi^{-1}x_{kj}(\mu)]$.

Proof. Let $u \in \phi^{-1}(x_{ik}(1)), v \in \phi^{-1}(x_{kh}(1)), w \in \phi^{-1}(x_{hj}(\mu))$. By the previous lemma, $[u, w] = 1$ $G :=< u, v, w >$. $[u, v] \in \phi^{-1}(x_{ih}(1)), [v, w] \in \phi^{-1}(x_{kj}(\mu)) \implies [[u, v], [v, w]] = 1, [[u, v], v] = 1 = [[u, v], u] = [[v, w], u] = [[v, w], v]$. So, $[G, G]$ is abelian. By an identity of Jacobi, if $[G, G]$ is abelian, then $[u, [v, w]][v, [w, u]][w, [u, v]] = 1, \forall u, v, w \in G$. This instantly proves our lemma by putting the expressions for u, v and w. □

This proves the well-definedness of s.

Lemma 5.1.9. *The $s(x_{ij}(\lambda))$'s satisfy Steinberg relations.*

Proof. Lemma 5.1.7 shows the 2nd Steinberg relation. Now, $s(x_{ij}(a))s(x_{ij}(v)) = [u, v][u, w] = [u, vw][v, [w, u]]$ where $u \in \phi^{-1}(x_{ij}(1)), v \in \phi^{-1}(x_{ij}(a)), w \in \phi^{-1}(x_{ij}(b))$. So, as $[w, u] \in Z(St(R))$, we have shown the 1st Steinberg relation. The third relation follows from the above Lemma. □

□

- If we had defined $St(2, R)$ similarly, then, for $R = \mathbb{F}_2$, it is isomorphic to $\mathbb{F}_2 * \mathbb{F}_2$ which is clearly not perfect, so can't be a universal central extension.
- Note that while showing that $St(n, R)$ is the universal central extension of $E(n, R)$, we need it first to be a central extension and then our proof works for $n \geq 5$. This is indeed the case for fields as we show in the next section.
- The above is true for $n = 3, 4$ for fields except the examples $St(3, \mathbb{F}_2), St(4, \mathbb{F}_2), St(3, \mathbb{F}_4)$. (See 8)

5.2 Matsumoto's theorem

The purpose of this section is to prove a part of the Matsumoto's theorem. We will simply find a set of simple-looking generators of $K_2(F)$ where F is a field instead of finding its presentation which is the actual theorem.

Definition 5.2.1 (Steinberg symbols)**.** Let $d_{12}(u) := \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $d_{13}(v) := \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}$.
Note that $[d_{12}(u), d_{13}(v)] = 1$. Define $\{u, v\} := [\phi^{-1}(d_{12}(u)), \phi^{-1}(d_{13}(v))]$
Define $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u); h_{ij}(u) := w_{ij}(u)w_{ij}(-1)$. Then, it is an easy check that $[h_{12}(u), h_{13}(v)] = \{u, v\} \in K_2(R)$.

Proposition 5.2.2. *The map $\{u, v\} : R^* \times R^* \to K_2(R)$ is skew-symmetric and bilinear.*

Proof. To prove skew-symmetry observe that $\phi(w_{23}(1))d_{13}(v)\phi(w_{23}(1))^{-1} = d_{12}(v)$. So,
$\{v, u\} = [w_{23}(1)\phi^{-1}(d_{13}(v))w_{23}(1)^{-1}, w_{23}(1)\phi^{-1}(d_{12}(u))w_{23}(1)] =$
$w_{23}(1)[\phi^{-1}(d_{13}(v)), \phi^{-1}(d_{12}(u))]w_{23}(1)^{-1} = \{u, v\}^{-1}$.
For bilinearity, we have, $\{u, v_1v_2\} = [\phi^{-1}(d_{12}(u)), \phi^{-1}(d_{13}(v_1))\phi^{-1}(d_{13}(v_2))] =$
$[\phi^{-1}(d_{12}(u)), \phi^{-1}(d_{13}(v_1))][\phi^{-1}(d_{12}(u)), \phi^{-1}(d_{13}(v_2))][\phi^{-1}(d_{13}(v_1)), [\phi^{-1}(d_{13}(v_2)), \phi^{-1}(d_{12}(u))]]^{-1} =$
$\{u, v_1\}\{u, v_2\}[\phi^{-1}(d_{13}(v_1)), \{u, v_2\}^{-1}]^{-1} = \{u, v_1\}\{u, v_2\}$. □

Proposition 5.2.3. $\{u, -u\} = 1 = \{u, 1 - u\}$

Proof. If $v = (1 - u)$ or $(-u)$, we have to prove $h_{12}(u)h_{12}(v) = h_{12}(uv)$. This follows from an observation in a later proof. Using the definition of $h_{12}(u)$, it suffices to prove that $w_{12}(u)w_{12}(-1)w_{12}(v) = w_{12}(uv)$. Now, it's a matter of brute force calculation using the conjugation formulas for $w_{ij}(u)$'s listed later. □

Theorem 5.2.4. *(K_2 of finite fields) For a finite field F, $K_2(F) = 0$.*

Proof. F^* is cyclic, say generated by u. By bilinearity, it is enough to show $\{u, u\} = 1$. By skew-symmetry, $\{u, u\}^2 = 1$. Now, by the above lemma, if char(F)=2, $\{u, u\} = 1$. If char(F) $= p$, an odd prime. Then, $\{u, u\} = \{u, -u\}\{u, -1\} = \{u, -1\} = \{u, u^{(q-1)/2}\} =$

$\{u,u\}^{(q-1)/2}$ where $|F^*| = q-1$. If $(q-1)/2$ is even, we are done. If not, (-1) is not a square. If $\forall a \in F$ such that a is a square, then $(a+1)$ is a perfect square. Then, as 1 is a square, so is $2, 3, .., (p-1) = (-1)$. Hence, a contradiction. So, $\exists a$ such that a is a square and $(a+1)$ is not a square. Note that $(-a)$ is not a square as (-1) is not. Then, $\{a+1, -a\} = 1 = \{u^{2k+1}, u^{2l+1}\}$ a contradiction to $\{u,u\} = 1$ unless it is trivial. So, in all cases, $\{u,u\} = 1$. Now, by Theorem 5.2.6 below, $K_2(F) = 0$. □

Remark 5.2.5.

- Using Hensel lifting, the same proof shows that the Steinberg symbols vanish in $K_2(\mathbb{Z}/p^k\mathbb{Z})$ for odd primes p.
- From —, we have this long exact sequence for $R = \mathbb{Z}, I = p^k\mathbb{Z}$.

$$\cdots \longrightarrow K_2(I) \longrightarrow K_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{Z}/I) \longrightarrow K_1(I) \longrightarrow \cdots$$

  It is known that $K_1(I) = 0$. So, for odd primes p, $K_2(\mathbb{Z}/p^k\mathbb{Z})$ is generated by the Steinberg symbol $\{-1,-1\}$ (See 5.0.6) which then is trivial.
- It is known that $\{-1,-1\} \neq 0$ in $K_2(\mathbb{R})$. But in $K_2(\mathbb{C}), \{-1,-1\} = \{i^2, i^2\} = \{i,i\}^4 = 1$ as $\{u,u\}^2 = 1$ for any u. For example, if $R = \mathbb{Z}[\zeta]$, $R^* =<\zeta>$ of order 6. Then, $1 = \{\zeta, 1-\zeta\} = \{\zeta, \zeta^{-1}\}$ but $\{\zeta,\zeta^{-1}\}\{\zeta,\zeta\} = \{\zeta, 1\} = 1 \implies \{\zeta,\zeta\} = 1$.

Theorem 5.2.6. $K_2(F) =< \{u,v\} >$

Proof. Let $W :=< w_{ij}(u) >$. The following lemma is proven by brute force calculations.

Lemma 5.2.7. *If $w \in W$, then $wx_{ij}(\lambda)w^{-1} = x_{kl}(\mu)$ for some k,l and $\mu \in R$.*

Define $C_n := Ker(\phi_{|W})$. Then, if $w \in C_n$, by the previous lemma, $wx_{ij}(\lambda)w^{-1} = x_{kl}(\mu)$. Operating ϕ on both sides, we get $e_{ij}(\lambda) = e_{kl}(\mu) \implies \{i,j\} = \{k,l\}, \lambda = \mu$. So, $C_n \subset Z(St(n,R))$. We have the following easy computational observations:

- If $w \in W, \phi(w) = PD$ where P is a permutation matrix and D is a diagonal matrix (diag$(v_1, v_2, ..., v_n)$) and this decomposition is unique. This follows from the fact that $\phi(w_{ij}(u))$ is a matrix with u at the $(i,j)th$ place and $(-u^{-1})$ at the $(j,i)th$ place and the fact that $PD = D_1P$ for some other diagonal matrix D_1.
- $wx_{ij}(\lambda)w^{-1} = x_{\pi(i),\pi(j)}(v_i\lambda v_j^{-1})$

- $ww_{ij}(u)w^{-1} = w_{\pi(i)\pi(j)}(v_i u v_j^{-1})$
- $wh_{ij}(u)w^{-1} = h_{\pi(i),\pi(j)}(v_i u v_j^{-1})h_{\pi(i),\pi(j)}(v_i v_j^{-1})^{-1}$.

Lemma 5.2.8. $C_n =< \{u, v\} >$.

Proof. Let $H :=< h_{ij}(u) >$, a normal subgroup of W, by the previous observations. Observe that $w_{ij}(u) = w_{ji}(-u^{-1})$ and $w_{ij}(u)^{-1} = w_{ij}(-u)$. So, in W/H, $\overline{w_{ij}(u)} = \overline{w_{ij}(1)}, \forall u$. So, define $w_{ij} := \overline{w_{ij}(1)}$. Then, as $w_{ij}(-1) = w_{ji}(1)$, $w_{ij} = w_{ji}$. Let $c = w_{i_1 j_1}(u_1)...w_{i_k, j_k}(u_k) \in C_n$.

Claim: $c \equiv 1$ **mod** H**.**

Proof of claim: Going mod H, $c = w_{i_1 j_1}...w_{i_k, j_k}$. As $w_{ij} = w_{ji}$, assume $i_l < j_l, \forall l$. If $i > 1, w_{ij}w_{1k}w_{ij}^{-1} = w_{1k'}$. So, shift all the w_{1k}'s to the left. Now, $w_{1j}w_{1l} = w_{jl}w_{1j}$. Now, as $j > 1$, shift the w_{jl} to the right again as before. Also, $w_{1l}w_{1l} = 1$. So, in the end we get rid of all the $w_{1l}'s$ for all varying l till there is at most only 1 in the left most end. For an illustration: $w_{14}w_{12}w_{13} = w_{14}w_{13}w_{23} = w_{13}w_{34}w_{23}$.
So, $c = w_{1l}w'h$ where $w' \in W$ and in the expression of w' there is no appearance of any w_{1l}'s. $1 = \phi(c) = \phi(w_{1l})\phi(w')\phi(h)$. Taking $\phi(h)$ which is a diagonal matrix to the left, we get a contradiction as $\phi(w_{1l})$ permutes columns 1 and l, but that can't be reversed by $\phi(w')$ by the expression of w'. In this way, we get rid of all the w_{ij}'s appearing in the expression of c. So, $c \in H$.

Now, note that the $h_{jk}(u)$'s can be expressed as a product of $h_{1k}(u)$'s. This is because: $(h_{ik}(u)w_{jk}(1)h_{ik}(u)^{-1})w_{jk}(-1) = w_{jk}(u)w_{jk}(-1) = h_{jk}(u)$ and also $h_{ik}(u)(w_{jk}(1)h_{ik}(u)^{-1}w_{jk}(-1)) = h_{ik}(u)h_{ik}(u)^{-1}$. So, $h_{jk}(u) = h_{ik}(u)h_{ij}(u)^{-1}$. Putting $i = 1$ proves our observation. Also, this implies $h_{jk}(u)h_{kj}(u) = 1$ and $h_{ij}(u)^{-1}h_{jk}(u)^{-1}h_{ki}(u)^{-1} = 1$.
Clearly, $< \{u, v\} >\subset C_n \subset H$. So, by our previous observations, $c = h_{1j}^{\pm 1}...h_{1k}^{\pm 1}$. Now, observe $\{u, v\} = [h_{12}(u), h_{13}(v)] = h_{12}(u)h_{13}(v)h_{12}(u)^{-1}h_{13}(v)^{-1} = (h_{12}(u)h_{13}(v)h_{12}(u)^{-1})h_{13}(v)^{-1} = h_{13}(uv)h_{13}(u)^{-1}h_{13}(v)^{-1} = h_{12}(u)(h_{13}(v)h_{12}(u)^{-1}h_{13}(v)^{-1}) = h_{12}(u)h_{12}(uv)h_{12}(u)^{-1}$. So, this expression is independent of the index "2" in $h_{12}(u)$ and the index "3" in $h_{13}(v)$. So, $\{u, v\} = [h_{1j}(u), h_{1k}(v)], j \neq k$, i.e., $h_{1l}(uv)h_{1l}(u)^{-1}h_{1l}(v)^{-1} \in< \{u, v\} >$. Observe that $h_{ij}(u)^{-1} = h_{ji}(u) \neq h_{ij}(u^{-1})$ in general. But $h_{ij}(u)h_{ij}(u^{-1}) \equiv h_{ij}(1) \equiv 1 \bmod < \{u, v\} >$.

So, $h_{ij}(u)^{-1} = h_{ij}(u^{-1})$ mod $< \{u,v\} >$. So, mod $< \{u,v\} >$, $c = h_{12}(u_1)...h_{1n}(u_n)$. Operating ϕ, we get $c \in < \{u,v\} >$.

Let $T := < x_{ij}(\lambda) : i < j > \subset St(n,R)$

Claim: Any element of St(n,R) belongs to $TWT = \{twt'; t,t' \in T, w \in W\}$

Proof of claim: First observe that $St(n,R)$ is generated by elements $x_{ij}(\lambda)$ where $|i-j| = 1$. So, it is enough to show that TWT is closed under right multiplication by $w_{i,(i+1)}(\pm 1)$ as, $w_{ij}(1)x_{ij}(\lambda)w_{ij}(-1) = x_{ji}(-\lambda)$. Also, it is easy to see that any element of T has a unique expression of the form $x_{(n-1),n}(*)x_{(n-2),n}(*)...x_{1,n}(*)x_{(n-2),(n-1)}(*)...x_{12}(*)$. This can be obtained by observing that ϕ is an isomorphism between T and upper triangular matrices with 1's on the diagonal. So, fix a $w_{ij}(-1); j = i+1$ (WLOG as the same proof works for $w_{ij}(1)$). Then, we claim that if $t_1wt_2 \in TWT$, then $t_2 = x_{ij}(\lambda)t'$ for some $t' \in T$ where t' is a product of x_{kl}'s with $(k,l) \neq (i,j), k < l$. For an illustration, let $(i,j) = (n-2, n-1)$. Then, $t_2 = x_{(n-1),n}(*)x_{(n-2),n}(*)...x_{1,n}(*)x_{(n-2),(n-1)}(*)...x_{12}(*) = x_{(n-1),n}(*)x_{(n-2),(n-1)}(*)x_{(n-2),n}(*)...$ as $[x_{in}(*), x_{(n-2),(n-1)}] = 1, 1 \leq i \leq (n-2)$. Now, $[x_{(n-2),(n-1)}(*), x_{(n-1),n}(*)] = x_{(n-2),n}(*)$ and $[x_{(n-2),n}, x_{(n-2),(n-1)}] = 1$. Hence, we get $x_{(n-2),(n-1)}(*)$ in the front. The same idea applies to any other $x_{ij}(*)$.

So, $t_1wt_2w_{ij}(-1) = t_1wx_{ij}(\lambda)w_{ij}(-1)w_{ij}(1)t'w_{ij}(-1)$. But $w_{ij}(1)t'w_{ij}(-1)$ is in T by the conjugation properties of $w_{ij}(1)$. So, it is enough to show that $wx_{ij}(\lambda)w_{ij}(-1) \in TWT$. Now, if w represents the permutation π. Then, we have $wx_{ij}(\lambda) = x_{\pi(i)\pi(j)}(\lambda')w$. If $\pi(i) < \pi(j)$, then $wx_{ij}(\lambda)w_{ij}(-1) = x_{\pi(i)\pi(j)}(\lambda')ww_{ij}(-1) \in TW$.

If $\pi(i) > \pi(j)$ and $\lambda \in R^*$, then $x_{ij}(\lambda) = w_{ij}(\lambda)x_{ij}(-\lambda)x_{ji}(\lambda^{-1}) \implies wx_{ij}(\lambda)w_{ij}(-1) = x_{\pi(i)\pi(j)}(\lambda')ww_{ij}(\lambda)w_{ij}(-1)x_{ij}(\lambda'') \in TWT$. If λ is not a unit, as R is a field, $\lambda = 0$. Here, our claim is trivially true. Hence, the claim is proven.

Now, let $t_1wt_2 \in Ker(\phi)$. Operating ϕ, we get $\phi(w)$ is an upper triangular matrix. This implies that the permutation part of $\phi(w)$ is trivial. Hence, $\phi(w) = \phi((t_2t_1)^{-1})$, a diagonal matrix. But then by definition of T, $\phi(w) = 1 = \phi((t_2t_1)^{-1}) \implies t_2 = t_1^{-1}$. So, the element is $t_1wt_1^{-1}$ where $w \in C_n \subset Z(St(n,R))$. So, $Ker(\phi) \subset C_n$, thus proving the theorem. $\square$

$\square$

5.3 New approach to Matsumoto's theorem

Let F be an infinite field and $G := Gl(2,F)$, B :=upper triangular matrices in G, T :=diagonal matrices in G and F^* denote the scalar matrices in G. The following are the major steps of this approach:

- Let $P(F) := \mathbb{Z}\{\{z\}; z \in F^* - \{1\}\}$ subject to the following relation:
  $\{x_1\} - \{x_2\} + \{\frac{x_2}{x_1}\} - \{\frac{1-x_2}{1-x_1}\} + \{\frac{(1-x_2)x_1}{(1-x_1)x_2}\} = 0, x_1 \neq x_2$.
  Then, there is an exact sequence:
  $$P(F) \longrightarrow \bigwedge^2 T/(w^{-1}-1) \longrightarrow H_2(Gl(2,F)) \longrightarrow 0$$
  where $w^{-1} : \bigwedge^2 T \to \bigwedge^2 T$ is the map $(a,b) \wedge (c,d) \mapsto (b,a) \wedge (d,c)$
- Let $(F^* \bigotimes F^*)_{sym} := (F^* \bigotimes F^*)/(x \otimes y + y \otimes x)$. Then, we have the following exact sequence:
  $$P(F) \xrightarrow{\phi} (F^* \bigotimes F^*)_{sym} \longrightarrow H_0(F^*, H_2(Sl(2,F))) \longrightarrow 0$$
  where $\phi(\{z\}) = (1-z^{-1}) \otimes z^{-1}$.
- The last step is to show that $H_0(F^*, H_2(Sl(2,F))) \cong H_0(F^*, H_2(Sl(n,F))), \forall n \geq 3$ and that F^* acts trivially on $H_2(Sl(n,F)), \forall n \geq 3$. This is not done here, but can be referred from 23.

Note that the three steps will prove that $H_2(Sl(n,F)) \cong H_2(Sl(F)), \forall n \geq 3$ and hence isomorphic to $K_2(F) = H_2(Sl(F))$. So, $K_2(F)$ has a presentation as a free abelian group on generators $\{x,y\}, x,y \in F^*$ subject to the following relations:

- $\{1-x, x\} = 0, x \neq 1$
- $\{xy, z\} = \{x,z\} + \{y,z\}$
- $\{x, yz\} = \{x,y\} + \{x,z\}$

Note that these relations together imply $\{x,-x\} = 0$ and $\{x,y\} = -\{y,x\}$. $\{x, 1-x^{-1}\} = -\{x^{-1}, 1-x^{-1}\} = 0 \implies \{x,-x\} = \{x,1-x\} - \{x,1-x^{-1}\} = 0$. So, $\{x,y\} = \{x,y\} + \{x,-x\} - \{xy,-xy\} + \{y^{-1},-y^{-1}\} = \{y^{-1},x\} = -\{y,x\}$.

5.3.1 Preparatory steps

Now, we prove and collect few of the results related to group homology in our context which we will be frequently using.

Let X be an infinite set.

Define $C_q(X)$ =Free abelian group on the (q+1)-tuples $(x_0, ..., x_q); x_i \in X$ **where the x_i's are distinct**. Then, we have the following chain:

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $d_n((x_0, ..., x_n)) := \sum_{i=0}^{n}(-1)^i(x_0, ..., \hat{x}_i, ..., x_n)$ and $\epsilon((x)) = 1, \forall x \in X$. Note that, as in case of simplicial homology, this is a chain complex.

Lemma 5.3.1. *The above chain is exact.*

Proof. Let $z \in Ker(d_n), n \geq 0$. Then, z is a finite integer sum of some $(n+1)$-tuples in C_n. Since X is infinite, pick x such that x doesn't appear in any of the tuples in the expression for z. Define $S_x : C_n \to C_{n+1}$ by $(x_0, ..., x_n) \mapsto (x, x_0, ..., x_n)$ if $x \neq x_i, \forall i$ and 0 otherwise. Then, it is an easy check that $z = d_{n+1}(S_x(z))$.

Also, ϵ is clearly surjective. So, the above chain is exact everywhere. $\square$

In particular, we have, the following chain C has $H_n(C) = 0$, if $n \geq 1$ and $H_0(C) = \mathbb{Z}$.

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

Now, let G be a group acting on X, hence also on $C_q(X)$ both on left and right, component-wise. Consider the left action. Hence, $C_q(X)$ is a $\mathbb{Z}[G]$-module. It is a quick check that the $d_n's$ are $\mathbb{Z}[G]$-linear. Now, let E_q be the set of orbit representatives of the action of G on $X_q := \{(x_0, ..., x_q); x_i \in X \text{ distinct}\}$. Let $G_y := Stab(y)$ for $y \in E_q$. Then, by Shapiro's lemma, there is an isomorphism $\bigoplus_{y \in E_q} H_p(G_y) \to H_p(G, C_q)$ where the map restricted to individual components is $z \otimes 1 \mapsto z \otimes y$ where $z \in P_*$, a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Lemma 5.3.2. *Let G act transitively on sets X_1 and X_2, and x_i's be the representatives with stabilizers H_i respectively. Let $\phi : \mathbb{Z}[X_1] \to \mathbb{Z}[X_2]$ induced by $x_1 \mapsto \sum_{g\in G/H_2} n_g g.x_2$. Then, if $H_1 \leq gH_2g^{-1}$ for $g : n_g \neq 0$, then $\phi_* : H_*(H_1) \to H_*(H_2)$ is given by $\phi_*(z) = \sum_{g\in G/H_2} n_g i_g \circ c_{g^{-1}}(z)$ where i_g is the map induced on homology by the inclusion $g^{-1}H_1 g \leq H_2$ and $c_{g^{-1}}$ is the map induced on homology by the map $H_1 \to g^{-1}H_1 g; x \mapsto g^{-1}xg$.*

Proof. First of all, note that if P_* is a right $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then by Shapiro's lemma, if $z \otimes 1 \in P_* \bigotimes_{\mathbb{Z}H_1} \mathbb{Z}$ represents an element of $H_*(H_1, \mathbb{Z})$, it corresponds to $z \otimes x_1 \in P_* \bigotimes_{\mathbb{Z}G} \mathbb{Z}[X_1]$. So, $\phi_*(z \otimes x_1) = z \otimes \sum_{g\in G/H_2} n_g g.x_2 = \sum_{g\in G/H_2} n_g(z.g \otimes x_2)$. This is further represented as a element of $H_*(H_2)$ by the element $\sum_{g\in G/H_2} n_g(z.g \otimes 1)$ in $P_* \bigotimes_{\mathbb{Z}H_2} \mathbb{Z}$ by Shapiro's lemma. Now, by 9.0.1, this is equal to $\sum_{g\in G/H_2} n_g i_g \circ c_{g^{-1}}(z \otimes 1)$.

$$\begin{tikzcd}
H_*(H_1) \arrow[rr, "\phi_*"'] \arrow[dr, "c_{g^{-1}}"'] & & H_*(H_2) \\
& H_*(g^{-1}H_1 g) \arrow[ur, "i_g"'] &
\end{tikzcd}$$

□

Now, let $X := P^1(F) = F \cup \infty$. Then, we have the following:

1. G acts transitively on X_0 and $Stab(\infty) = B$.
2. G acts transitively on X_1 and $Stab((\infty, 0)) = T$.
3. G acts transitively on X_2 and $Stab((\infty, 0, 1)) = F^*$.
4. Orbits of G on X_3 are $\{(\infty, 0, 1, x); x \in F^* - \{1\}\}$.
5. Orbits of G on X_4 are $\{(\infty, 0, 1, x, y); x, y \in F^* - \{1\}\}$.

The transitivity part of 1 and 2 will be proven in 3.

Proof of 1: This is clear as the vector $(a, c) \in F^2$ represents $\infty \in X$ iff $c = 0$.

Proof of 2: Similar to 1.

Proof of 3: F^* stabilises $(\infty, 0, 1)$ is clear. Now, if $g \in Gl(2, F)$ stabilises $(\infty, 0, 1)$, by 1 and 2 above, $g = diag(a, d)$. But then $g.1 = 1 \implies a = d$. Now, to prove transitivity, take $(x, y, z) \in X_2$. Working backwards, if there is a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : g.(\infty, 0, 1) =$

$(x,y,z) \implies (a,c) = x, (b,d) = y$ and $(a+b, c+d) = z$ as elements of X. So, pick $(A,B),(C,D) \in F^2$ such that they represent x and y in X. Now, as $x \neq y$, the two vectors are linearly indpendent. So, $\exists \lambda, \mu \in F^*$ such that $\lambda(A,B) + \mu(C,D)$ represents z in X. The coefficients are both non-zero because z is distinct from x, y. Now let the coloumns of g be $\lambda(A,B)$ and $\mu(C,D)$.

Proof of 4: First note that, if g as before and $g.(\infty,0,1,z) = (\infty,0,1,w)$ then by above, $g \in F^* \implies z = w \in X$. So, the orbits of $(\infty,0,1,z)$ are distinct for distinct $z \in F^* - \{1\}$. Now, using an element of G as in 3 above, we can take any $(x,y,z,w) \in X_3$ to $(\infty,0,1,a)$ for some a.

Proof of 5: Same as part 4 above.

5.3.2 Main results

Now, consider the double complex $\{F_p \bigotimes C_q\}$, where F_p is the standard $\mathbb{Z}G$-resolution of $\mathbb{Z}$ (See 9.0.3). As of now, taking any projective resolution works, but later we will need the standard resolution for computation. Also, we use the transposed notation where the $(p,q)^{th}$ term of this double complex is $F_q \bigotimes_{\mathbb{Z}G} C_p$.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & F_2 \bigotimes C_0 \arrow[l] \arrow[d] & F_2 \bigotimes C_1 \arrow[l] \arrow[d] & F_2 \bigotimes C_2 \arrow[l] \arrow[d] & F_2 \bigotimes C_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & F_1 \bigotimes C_0 \arrow[l] \arrow[d] & F_1 \bigotimes C_1 \arrow[l] \arrow[d] & F_1 \bigotimes C_2 \arrow[l] \arrow[d] & F_1 \bigotimes C_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & F_0 \bigotimes C_0 \arrow[l] \arrow[d] & F_0 \bigotimes C_1 \arrow[l] \arrow[d] & F_0 \bigotimes C_2 \arrow[l] \arrow[d] & F_0 \bigotimes C_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

So, using the second filtering as in 9.0.6, we get the E^1 page which q^{th} row looks like:

$$\begin{tikzcd}
E^1_{4,q} \arrow[r, "d^1_{4,q}"] & E^1_{3,q} \arrow[r, "d^1_{3,q}"] & E^1_{2,q} \arrow[r, "d^1_{2,q}"] & E^1_{1,q} \arrow[r, "d^1_{1,q}"] & E^1_{0,q} \arrow[r] & 0 \\
\bigoplus_{x,y} H_q(F^*).\{x,y\} \arrow[u, "\cong"] & \bigoplus_x H_q(F^*).\{x\} \arrow[u, "\cong"] & H_q(F^*) \arrow[u, "\cong"'] & H_q(T) \arrow[u, "\cong"'] & H_q(B) \arrow[u, "\cong"'] &
\end{tikzcd}$$

Here, the vertical isomorphisms are NOT arbitrary; they all come from Shapiro's lemma. Recall, $E^1_{p,q} = H_q(G, C_p)$.

- $d_2((\infty, 0)) = (0) - (\infty) = w.(\infty) - (\infty)$, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

  Note that $w \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} w^{-1} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$. So, $d^1_{1,q}(z) = cor^B_T(w^{-1}(z) - z)$. By a result of Suslin, cor^B_T is an isomorphism.

  So, keeping track of the maps, $E^2_{0,q} = H_q(T)/(w^{-1} - 1)$.
- Similarly, $d_2((\infty, 0, 1)) = (g_1 - g_2 + 1)(\infty, 0)$ where $g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

  But as $F^* = Z(G)$, $d^1_{2,q}(z) = cor^T_{F^*}(g_1^{-1}.z - g_2^{-1}.z + z) = cor^T_{F^*}(z)$. The map $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ splits the inclusion $F^* \to T$, hence $d^1_{2,q}$ is injective. So, $d^1_{3,q}$ is the zero map. So, $E^2_{2,q} = 0$. Also, $E^2_{1,q} = Ker(w^{-1} - 1)/H_q(F^*)$.
- Note that for $q = 0$, $Ker(w^{-1} - 1) = H_0(T) = \mathbb{Z} = H_0(F^*)$. For $q = 1$, $H_1(T) = T$ and $Ker(w^{-1} - 1) =$ diagonal matrices whose diagonal entries are the same $= F^* = H_1(F^*)$. So, $E^2_{1,q} = 0$ for $q = 0, 1$.
- $d_3((\infty, 0, 1, x, y)) = h_1.(\infty, 0, 1, \frac{(1-y)x}{(1-x)y}) - h_2.(\infty, 0, 1, \frac{1-y}{1-x}) + h_3.(\infty, 0, 1, \frac{y}{x}) - (\infty, 0, 1, y) + (\infty, 0, 1, x)$ for some elements $h_i \in G$, which of course commute with every element of F^*. Note that $E^1_{3,q} = H_q(F^*) \bigotimes \mathbb{Z}\{\{x\}; x \in F^* - \{1\}\}$ and $E^1_{4,q} = H_q(F^*) \bigotimes \mathbb{Z}\{\{x, y\}; x \neq y \in F^* - \{1\}\}$. So, image of $d^1_{4,q}$ is the subgroup generated by $\{x\} - \{y\} + \{\frac{y}{x}\} - \{\frac{1-y}{1-x}\} + \{\frac{(1-y)x}{(1-x)y}\}$. So, $E^2_{3,q} = H_q(F^*) \bigotimes P(F)$.

So, the E^2 page looks like

$$\begin{array}{cccc}
H_2(T)/(w^{-1}-1) & * & 0 & * \\
 & & & \\
H_1(T)/(w^{-1}-1) & 0 & 0 & F^* \bigotimes P(F) \\
 & & & \\
H_0(T)/(w^{-1}-1) & 0 & 0 & P(F)
\end{array}$$

Now, as $E^2_{2,1} = 0$, $E^3_{0,2} = E^2_{0,2} = H_2(T)/(w^{-1}-1)$. As $E^2_{1,1} = 0, E^3_{3,0} = E^2_{3,0} = P(F)$. Next, we compute the map $d^3_{3,0} : E^3_{3,0} \to E^3_{0,2}$.

Lemma 5.3.3. *$d^3_{3,0}(\{z\}) = \lambda((1-z^{-1}) \otimes z^{-1})$ where $\lambda : (F^* \bigotimes F^*)_{sym} \to \bigwedge^2 T/(w^{-1}-1) \cong H_2(T)/(w^{-1}-1)$ given by $(a \otimes b) \mapsto (a,1) \wedge (1,b) - (a,1) \wedge (b,1)$.*

Proof. We follow the method as in 9.0.6. The element $\{z\} \in P(F)$ comes from the element $(1) \otimes (\infty, 0, 1, z) \in F_0 \bigotimes C_3$. This is mapped horizontally to $[(g_1) - (g_2) + (g_3) - (1)] \otimes (\infty, 0, 1) \in F_0 \bigotimes C_2$, where $g_1 = \begin{pmatrix} 0 & z \\ 1-z & z \end{pmatrix}, g_2 = \begin{pmatrix} z(z-1) & z \\ 0 & z \end{pmatrix}, g_3 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. This further lifts to $[(g_2, g_1) - (g_3, 1)] \otimes (\infty, 0, 1) \in F_1 \bigotimes C_2$ which maps to $[(g_2, g_1) - (g_3, 1)](g_1 - g_2 + 1) \otimes (\infty, 0) \in F_1 \bigotimes C_1$ which is again the image of $x \otimes (\infty, 0) \in F_2 \bigotimes C_1$ which maps to $x(w^{-1}-1) \otimes (\infty) \in F_2 \bigotimes C_0$. Here, $x = [(g_3 g_1, g_2, g_1) - (z^{-1} g_3 g_2, z^{-1} g_3 g_1, z^{-1} g_2) - (z^{-2} g_2^2 g_3^2, z^{-1} g_2^2, z^{-1} g_1 g_2) + (z^{-3} g_2^3 g_3^2, z^{-2} g_2^2, 1) - (z^{-3} g_2^2 g_3^2, z^{-1} g_3^2, 1) + (z^{-1} g_2 g_1, z^{-1} g_1^2, 1) - (g_1^2 g_3^{-1}, z g_3^{-1}, 1) + (g_3, z^{-1} g_3^2, 1)]$.

Now, let $s : G/B \to G$ be any set-theoretic section of the projection from G to G/B. Let F, F', F'' be the standard resolution of the groups G, B, T respectively. Define $f : F \to F'$ be the map sending $(g_0, ..., g_n) \mapsto (\bar{g}_0, ..., \bar{g}_n)$ where $\bar{g} := (s(\pi(g)))^{-1} g$. Choose s such that $s(B) = 1, s(\{g \in G : g.(\infty) = 0\}) = w, s(\{g \in G : g.(\infty) = 1\}) = z^{-1} g_2 w, s(\{g \in G : g.(\infty) = z\}) = z^{-2} g_2^2 w$. This can be done as G acts transitively on X with $Stab(\infty) = B$. Then, clearly f is an augmentation preserving chain map from F to F', hence is a homotopy inverse of the map in the reverse direction induced by inclusion of F' in F, by 9.0.1. So, f is an

isomorphism from the homology of the complex $F \bigotimes_{\mathbb{Z}B} \mathbb{Z}$ to that of the complex $F' \bigotimes_{\mathbb{Z}B} \mathbb{Z}$.
It is an easy calculation that $f(x) = [(\begin{pmatrix} z & 1-z \\ 0 & z(1-z)^{-1} \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \begin{pmatrix} z & 1-z \\ 0 & 1 \end{pmatrix}) - ...]$ in $F'_2 \bigotimes_{\mathbb{Z}T} \mathbb{Z}$.
Similarly, the map $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ induces an inverse to the isomorphism cor^B_T. Call this map g. So, $g(f(x)) = [((z, z(1-z)^{-1}), (z, z), (z, 1)) - ...] \otimes 1 \in F''_2 \bigotimes_{\mathbb{Z}T} \mathbb{Z}$. So, using these isomorphism, we get $d^3_{3,0}(\{z\}) = g(f(x))$ in the second homology of the complex $F'' \bigotimes_{\mathbb{Z}T} \mathbb{Z}$, i.e.,$H_2(T)$.
Now, by 9.0.4, $(a, b) \wedge (c, d) \in \bigwedge^2 T$ corresponds to the class of the cycle in $F''_2 \bigotimes \mathbb{Z}$, $[((ac, bd)(c, d), 1) - ((ac, bd), (a, b), 1)] \otimes 1$. Consider the map from F''_2 to $(F^* \bigotimes F^*)_{sym}$ sending $((a_1, a_2), (b_1, b_2), (c_1, c_2))$ to $a_1 b_1^{-1} \otimes b_2 c_2^{-1}$. Call this map h. Then, if $g \in T, x \in F''_2 \bigotimes_{\mathbb{Z}T} \mathbb{Z}$, $h(gx) = h(x)$. So, by 9.0.4, h induces a map from $F''_2 \bigotimes_{\mathbb{Z}T} \mathbb{Z} \to (F^* \bigotimes F^*)_{sym}$. So, we have $h : H_2(T) \to (F^* \bigotimes F^*)_{sym}$. But clearly h vanishes on the image of $(w^{-1} - 1)$, so we have $h : H_2(T)/(w^{-1} - 1) \to (F^* \bigotimes F^*)_{sym}$. Note that $h((a, b) \wedge (c, d)) = a \otimes d - c \otimes b$. So, $h(g(f(x))) = (1 - z^{-1}) \otimes z^{-1} \in (F^* \bigotimes F^*)_{sym}$.
Now, come back to the E^2-page of the spectral sequence. Clearly, $E^2_{1,1} = E^{\infty}_{1,1} = 0 = E^{\infty}_{2,0}$. Note also that $E^4_{0,2} = E^{\infty}_{0,2} = E^3_{0,2}/Image(P(F))$. Also, by 9.0.6, we have the following exact sequence:

$$P(F) \xrightarrow{d^3_{3,0}} \bigwedge^2 T/(w^{-1} - 1) \xrightarrow{i} H_2(G) \longrightarrow 0$$

, where i is induced by the inclusion of T in G.

Now, we also have the following commutative diagram:

$$\begin{tikzcd} P(F) \arrow[r, "d^3_{3,0}"] & \bigwedge^2 T/(w^{-1} - 1) \arrow[r, "i"] \arrow[d, "d"] & H_2(G) \arrow[r] \arrow[d, "d"] & 0 \\ & H_2(F^*) \arrow[r, "Id"'] & H_2(F^*) & \end{tikzcd}$$

Hence, $d^3_{3,0}(\{z\}) \in Ker(d) = Image((F^* \bigotimes F^*)_{sym})$. So, $d^3_{3,0}(\{z\}) = \lambda h(d^3_{3,0}(\{z\})) = \lambda((1 - z^{-1}) \otimes z^{-1})$, by the following Lemma. □

Lemma 5.3.4. *We have an exact sequence:*

$$0 \longrightarrow (F^* \bigotimes F^*)_{sym} \xrightarrow{\lambda} \bigwedge^2 T/(w^{-1}-1) \xrightarrow{d} \bigwedge^2 F^* \longrightarrow 0$$

where d is induced by the determinant map from T to F^*.

Proof. Note clearly that $d \circ \lambda = 0$. Now, we have two splittings one each for λ and d. Let $h : \bigwedge^2 T/(w^{-1}-1) \to (F^* \bigotimes F^*)_{sym}$ be given by $(a,b) \wedge (c,d) \mapsto a \otimes d + b \otimes c$. And $i : \bigwedge^2 F^* \to \bigwedge^2 T/(w^{-1}-1)$ given by $a \wedge b \mapsto (a,1) \wedge (b,1)$. Now, it's an easy check that these are actually the splitting maps. Hence, this is a split exact sequence. □

From the Hochschild-Serre spectral sequence and the fact that $Sl(2,F)$ is perfect, we have $H_0(F^*, H_2(Sl(2,F))) = Ker(d : H_2(Gl(2,F)) \to H_2(F^*))$. So we have the following commutative diagram where the dotted arrows are induced maps.

$$\begin{tikzcd}
0 \arrow[r] & P(F) \arrow[r, "Id"] \arrow[d, dotted, "\phi"'] & P(F) \arrow[r] \arrow[d, "d^3_{3,0}"] & 0 \arrow[d] & \\
0 \arrow[r] & (F^* \bigotimes F^*)_{sym} \arrow[r, "\lambda"] \arrow[d, dotted] & \bigwedge^2 T/(w^{-1}-1) \arrow[r, "d"] \arrow[d, "i"] & H_2(F^*) \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & H_0(F^*, H_2(Sl(2,F))) \arrow[r] \arrow[d] & H_2(Gl(2,F)) \arrow[r, "d"'] \arrow[d] & H_2(F^*) \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Note that $\phi(\{z\}) = (1-z^{-1}) \otimes z^{-1}$. Note that $d \circ d^3_{3,0} = 0 = d \circ i \circ \lambda$. So, the induced maps make sense. Since the rows are exact and the last two coloumns are exact, so is this:

$$P(F) \xrightarrow{\phi} (F^* \bigotimes F^*)_{sym} \longrightarrow H_0(F^*, H_2(Sl(2,F))) \longrightarrow 0$$

6 Higher K-theory

6.1 Classifying space of a group

Definition 6.1.1 (Classifing space of a group G)**.** Any path-connected CW complex X such that $\pi_n(X) = 1, n > 1$ and $\pi_1(X) = G$.

Construction: Given a group G with presentation $< a_\gamma | b_\delta >$, let $X = \bigvee_\gamma S^1$. b_δ are words in a_γ. Attach 2-cells to X via maps specified by these words to get the space X_1. By Proposition 7.0.7 $\pi_1(X_1) = G$. Also, by construction, X_1 is a path-connected CW complex with cells of dimension at most 2. Now, for each element f in $\pi_2(X_1)$, i.e., f is the class of a map $S^2 \to X_1$ in $\pi_2(X_1)$, attach a 3-cell via this map to X_1. Call the new space X_2. By 7.0.8, $\pi_2(X_2) = 1$ and by Proposition 7.0.7, $\pi_1(X_2) = G$. Continue this process, but now with 4-cells attached to X_2 via maps representing elements of $\pi_3(X_2)$ and so on. Taking the direct limit, we have formed a path-connected CW complex BG such that $\pi_n(BG) = 1, \forall n > 1$ and $\pi_1(BG) = G$.

Uniqueness: If $\tilde{X}$ is the universal cover of X, where X is some classifying space of G, then by Proposition 7.0.12, $\tilde{X}$ is contractible and X is obtained by the usual action of $\pi_1(X)$ on $\tilde{X}$. Then, by Proposition 7.0.13 any two classifying spaces are homotopy equivalent and $G \mapsto BG$ is functorial.

Example 6.1.2.

- If $G = \{1\}$, then $BG = *$.
- If $G = \mathbb{Z}$, then $BG = S^1$.
- If $G = F_n$, the free group on n generators, $BG = \bigvee_{i=1}^n S^1$.
- If $G = \mathbb{Z}_2$, $BG = \mathbb{RP}^\infty$.

6.2 Acyclic maps

Definition 6.2.1 (Acyclic spaces)**.** A topological space X such that $H_\bullet(X) = H_\bullet(*)$. This, in particular, implies that X is path-connected.

Definition 6.2.2 (Acyclic map)**.** Assume X and Y are path-connected. Then, $f : X \to Y$ is called acyclic if the homotopy fibre H_f (which is unique upto homotopy equivalence here; see Definition 7.0.3) is an acyclic space. Note that the homotopy fibre may not always be homotopy equivalent to the usual fibre.

Proposition 6.2.3 (Equivalent conditions for acyclic maps)**.** *TFAE:*

1. *$f : X \to Y$ is acyclic.*
2. *For $\tilde{Y}$, the universal cover of Y, in the following pullback diagram, f' induces isomorphism between $H_\bullet(X \times_Y \tilde{Y})$ and $H_\bullet(\tilde{Y})$.*

$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r, "f'"'] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proposition 6.2.4. *If $f : X \to Y$ is acyclic, then it induces isomorphism at the homology level.*

Proposition 6.2.5. *If $f : X \to Y$ is acyclic, then $\pi_1(Y) \cong \frac{\pi_1(X)}{P}$, where P is some perfect (i.e., $[P, P] = P$) normal subgroup of $\pi_1(X)$.*

Proof. As $\tilde{H}_\bullet(H_f) = 0$, $\pi_1(H_f)$ is perfect. From the long exact sequence in Proposition 7.0.4, $\pi_1(Y) \cong \frac{\pi_1(X)}{Ker(f_*)}$ where $Ker(f_*)$ is the image of the perfect group $\pi_1(H_f)$, hence is perfect. $\square$

Proposition 6.2.6. *If $f : X \to Y$ is acyclic and X and Y are CW complexes. If f also induces isomorphism of fundamental groups, then f is a homotopy equivalence.*

Proof. From the long exact sequence in Proposition 7.0.4, if we show, $\pi_n(H_f) = 1, \forall n$, then f induces isomorphism of all homotopy groups of X and Y. Now, by Proposition 7.0.5, f is a homotopy equivalence.

To show, $\pi_n(H_f) = 1, \forall n$, observe that again by Proposition 7.0.4, at $n = 1$ level, we have $\pi_1(H_f)$ is the image of the abelian group $\pi_2(Y)$, hence it is abelian. But it is also a perfect group, hence it is 1. $\tilde{H}_0(H_f) = 0$ implies $\pi_0(H_f) = 1$. By Proposition 7.0.6, $\pi_2(H_f) = 1$ and inducively $\pi_n(H_f) = 1, \forall n$. So, we are done. $\square$

Proposition 6.2.7 (Pullbacks and Acyclic maps)**.** *If f or g is a fibration (See 7.0.1), then f is acyclic iff f' is.*

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Proposition 6.2.8 (Pushouts and acyclic maps)**.** *If f_1 is a cofibration (See 7.0.9), then f_i is acyclic implies f_i' is acyclic.*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_0'} \\ Y_0 & \xrightarrow{f_1'} & Y_0 \cup_X Y_1 \end{array}$$

6.3 + Construction

Definition 6.3.1 (+ Construction)**.** Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based path connected CW complex. An acyclic map $f : X \to Y$ is called a + Construction on X (relative to P) if P is the kernel of $f_* : \pi_1(X) \to \pi_1(Y)$.

Existence of + Construction. First we prove it for a path connected CW complex X whose fundamental group is perfect. By Proposition 7.0.7, attaching 2-cells along all the elements of $\pi_1(X)$, we get a path connected CW complex W with $\pi_1(W) = 1$. We have the following long exact sequence of the pair (W, X) (also observing that $\pi_1(X)$ is perfect implies $H_1(X) = 0$).

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_2(W) & \longrightarrow & H_2(W,X) & \longrightarrow & H_1(X) = 0 \\ & & \uparrow & & & & \\ & & \pi_2(W) & & & & \end{array}$$

Now, by excision, $H_2(W, X) \cong H_2(W/X) \cong H_2(\bigvee_\lambda S^2) \cong \bigoplus_\lambda H_2(S^2)$. Hence, let $\{1_\lambda\}$ be the generators of $H_2(\bigvee_\lambda S^2)$, which correspond to elements $\{b_\lambda\}$ in $H_2(W)$. By Proposition 7.0.6, pick elements $\beta_\lambda \in \pi_2(W)$ mapping to $b_\lambda \in H_2(W)$. Pick some representative of β_λ; call it the same. So, $\beta_\lambda : S^2 \to W$. Attach 3-cells via these maps to get another CW complex Y.

Claim: $H_\bullet(Y, X) = 0$

We have the following exact sequence of the triplet (Y, W, X). We will show that $H_3(Y, W) \to H_2(W, X)$ is an isomorphism. This clearly implies $H_i(Y, X) = 0$, for $i = 2, 3$.

$$0 = H_3(W, X) \longrightarrow H_3(Y, X) \longrightarrow H_3(Y, W) \longrightarrow H_2(W, X) \longrightarrow H_2(Y, X) \longrightarrow H_2(Y, W) = 0$$

Denote by $\phi_\lambda : D^3 \to Y$, the corresponding characteristic map associated with the attaching map $\beta_\lambda : S^2 \to W$. The following commutative diagram induces a commutative diagram for the

corresponding homology groups of the triplets.

$$\begin{tikzcd}
* \arrow[r, hook] \arrow[d] & \bigvee_\lambda S^2 \arrow[r, hook] \arrow[d, "H(\bigvee_\lambda \beta_\lambda)"] & \bigvee_\lambda D^3 \arrow[d, "H(\bigvee_\lambda \phi_\lambda)"] \\
X \arrow[r, hook] & W \arrow[r, hook] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
H_3(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "Diagram\ 1"] & H_2(\bigvee_\lambda S^2, *) \arrow[d] \\
H_3(Y, W) \arrow[r] & H_2(W, X)
\end{tikzcd}$$

$$\begin{tikzcd}
(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r] \arrow[d] & \bigvee_\lambda S^3 \arrow[d, "\cong"] \\
(Y, W) \arrow[r] & Y/W
\end{tikzcd}$$

The above diagram of obvious maps induces the following by excision as (Y, W) is CW pair.

$$\begin{tikzcd}
H_3(\bigvee_\lambda D^3, \bigvee_\lambda S^2) \arrow[r, "\cong"] \arrow[d] & H_3(\bigvee_\lambda S^3) \arrow[d, "\cong"] \\
H_3(Y, W) \arrow[r, "\cong"'] & H_3(Y/W)
\end{tikzcd}$$

By the above diagram, we have an isomorphism in the left arrow in Diagram 1. We also have $H_2(\bigvee_\lambda S^2, *) \to H_2(W, X)$, sending 1_λ to $(\beta_\lambda)_*(1_\lambda) = b_\lambda$ by Proposition 7.0.6. Hence, it is an isomorphism in Diagram 1. Also, from the homology long exact sequence of the triplet

$(*, \bigvee_\lambda S^2, \bigvee_\lambda D^3)$ and the fact that $\bigvee_\lambda D^3$ is contractible we have the isomorphism in the top arrow in Diagram 1. So, in Diagram 1, we have isomorphism in the bottom arrow as well. Hence, as observed before, $H_i(Y,X) = 0; i = 2, 3$. It is also easy to see that $H_n(Y,X) = 0$ for all other n. So, we are done.

Now, we show $X \hookrightarrow Y$ is an acyclic map using Proposition 6.2.3. As Y is simply connected, it is its own universal cover. So, as in Proposition 6.2.3, we have the following diagram of pullback and the map induced at homology level.

$$\begin{tikzcd} X \times_Y Y \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "Id"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd} \qquad \begin{tikzcd} H_n(X \times_Y Y) \arrow[r] \arrow[d, "\cong"'] & H_n(Y) \\ H_n(X) \arrow[ur, "\cong"'] & \end{tikzcd}$$

Note that $X \times_Y Y = \{(x,x); x \in X\} \subset X \times Y$, clearly homeomorphic to X via the projection in the pullback diagram. Also, by the previous claim, $H_\bullet(Y,X) = 0 \implies H_n(X) \cong H_n(Y), \forall n$. So, by Proposition 6.2.3, the inclusion $X \hookrightarrow Y$ is an acyclic map. As by construction (Y,X) is CW pair, the inclusion is also a cofibration.

Now, for the general case, let $P \subset \pi_1(X)$, a perfect normal subgroup. Let X' be the cover of X with $\pi_1(X') = P$. Clearly, X' is also a path connected CW complex with $P = [P,P]$. Hence, by the previous construction, $\exists X' \hookrightarrow Y'$ where Y' is a simply connected CW complex and the inclusion is an acyclic cofibration. Now, by Proposition 6.2.8, in the following pushout diagram, we have $X \dashrightarrow X \cup_{X'} Y'$ an acyclic co-fibration.

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[d, dashed] \\ X \arrow[r, dashed, "f^+"'] & X \cup_{X'} Y' \end{tikzcd}$$

Now, by Proposition 7.0.15, if $X^+ := X \cup_{X'} Y'$, then $\mathrm{Ker}(f^+) = P$ and by Proposition 6.2.5, $\pi_1(X^+) = \frac{\pi_1(X)}{P}$.

□

Uniqueness of + Construction.

Lemma 6.3.2. *Suppose $f : X \to Y$ is an acyclic cofibration and $g : X \to Z$ is any continuous map with Ker(f_*) $\subset$Ker(g_*), where X, Y, Z are path-connected CW complexes. Then, there exists a map $h : Y \to Z$ such that $h \circ f = g$. Moreover, any two such maps are homotopic.*

Proof of lemma

We first prove it for the case when g is the inclusion $i : X \to M_g$, the mapping cylinder, which we know is a cofibration. We have the following pushout diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i'"] \\ M_g \arrow[r, "f'"'] & M_g \cup_X Y \end{tikzcd}$$

By Proposition 6.2.8, f' is an acyclic cofibration. And by Proposition 7.0.15, Ker(f'_*) =< i_*(Ker(f_*)) >= 1, as Ker(f_*) $\subset$Ker(g_*) by hypothesis. So, by Proposition 6.2.4, f'_* is an isomorphism of fundamental groups. Also, it can be easily checked that $M_g \cup_X Y$ is a path-connected CW complex. So, by Proposition 6.2.6, f' is a homotopy equivalence. By Proposition 7.0.11, f' is also a homotopy equivalence under X (See 7.0.10). Let h' be the its homotopy inverse under X, unique upto homotopy. $h := h' \circ i' : Y \to M_g$. Then, $h \circ f(x) = h'(f'((x, 1))) = (x, 1) = i(x)$, by Proposition 7.0.11.

Now, if $p : Y \to M_g$ such that $p \circ f = i$, then consider the following by universal property of pushouts.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i'"'] \arrow[ddr, bend left, "p"] & \\ M_g \arrow[r, "f'"] \arrow[drr, bend right, "Id"'] & M_g \cup_X Y \arrow[dr, dashed, "j"'] & \\ & & M_g \end{tikzcd}$$

So, $p = j \circ i'$ and $j \circ f' = Id$. Now, $f' \circ h' \approx Id \implies j \circ (f' \circ h') \approx j \implies h' \approx j$. So, $p \approx h$.

Now, for the general case, we have the following commutative diagrams which can be easily checked:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "Id"] & \\
X \times I \arrow[r] \arrow[drr, bend right, "{(x,t)\mapsto g(x)}"'] & M_g \arrow[dr, "\phi"] & \\
 & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "Id"] & \\
X \times I \arrow[r] \arrow[dr, bend right, "{(x,t)\mapsto f(x)}"'] & M_g \arrow[dr, "\phi"] & \\
 & Y \arrow[u, "h"] \arrow[r, "l"'] & Z
\end{tikzcd}$$

Here, ϕ is the homotopy equivalence between M_g and Z as defined uniquely by the left diagram above by universal property of pushouts. The existence of h is from the previous discussion as Ker(f_*) $\subset$Ker(g_*)=Ker$(\phi_* \circ i_*)$=Ker(i_*) as ϕ_* is an isomorphism. So, for any map l with $l \circ f = g$, $l = \phi \circ h$, where h is unique upto homotopy as seen before. $\phi \circ h$ is our required map which is unique upto homotopy.

Coming back to the uniqueness of + construction, it is clear from the lemma, that any two + constructions are homotopy equivalent [**Here, in the definition we want the map $f : X \to X^+$ to also be a cofibration, which is also guaranteed by our explicit construction. In general, if we don't take it in our definition, still the uniqueness can be proven. See 16].** □

Definition 6.3.3. Given a ring R, $BGl(R)$ as in Definition 6.1, which is a path connected CW complex unique upto homotopy. Now, carry out the plus construction on this space and the new path-connected space we get is $BGl(R)^+$. Define $K(R) := BGl(R)^+ \times K_0(R)$ where the group $K_0(R)$ is given the discrete topology. Its path components are all homeomorphic to $BGl(R)^+ \times \{0\}$, $0 \in K_0(R)$. Define the higher K-groups by $K_n(R) := \pi_n(K(R), (p, 0)), p \in BGl(R)^+$ arbitrary. The functoriality $R \mapsto K_n(R)$ is clear from Proposition 7.0.14, Lemma 6.3.2 and the fact that if $f : R \to S$ is a ring map, $f(E(R)) \subset E(S)$.

Example 6.3.4 (Quillen)**.** $K_n(F) = 0$, if n is even and non-zero and $K_n(F) = \mathbb{Z}/(q^k - 1)\mathbb{Z}$ for $n = 2k + 1$ odd where F is a finite field with q elements.

6.4 New K_2 = Old K_2

From the homotopy long exact sequence associated to the acyclic map $f : BGl(R) \to BGl(R)^+$, we have:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(BGl(R)) \arrow[r] & \pi_2(BGl(R)^+) \arrow[r] & \pi_1(H_f) \arrow[d] & & \\
& & & \pi_1(BGl(R)) \arrow[r] & \pi_1(BGl(R)^+) \arrow[r] & 1
\end{tikzcd}$$

as $\pi_0(H_f) = 0$. Now, $f_* : \pi_1(BGl(R)) \to \pi_1(BGl(R)^+)$ has kernel $E(R)$. Also, by definition $\pi_2(BGl(R)) = 0$. So, we have the exact sequence:

$$1 \longrightarrow \pi_2(BGl(R)^+) \longrightarrow \pi_1(H_f) \longrightarrow E(R) \longrightarrow 1$$

Now, replacing f by the path space fibration, i.e., consider the map between the pairs $(E, H_f) \to (Y, y_0)$ where E is the path space associated to f and $Y := BGl(R)^+$, we have:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(H_f) \arrow[r] \arrow[d, "\phi"] & \pi_n(E) \arrow[r] \arrow[d, "\phi"] & \pi_n(E, H_f) \arrow[r] \arrow[d, "\phi"] & \pi_{n-1}(H_f) \arrow[r] \arrow[d, "\phi"] & \cdots \\
\cdots \arrow[r] & \pi_n(\{y_0\}) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(Y, \{y_0\}) \arrow[r] & \pi_{n-1}(\{y_0\}) \arrow[r] & \cdots
\end{tikzcd}$$

From, $\pi_1(H_f)$ acts on this diagram (See 16). If $g \in \pi_1(H_f), \gamma \in \pi_n(E, H_f), n \geq 1$, then $g.\gamma, \gamma \in \pi_n(E, H_f)$ and $\phi(g.\gamma) = \phi_(g).\phi(\gamma)$. But $\phi(g) = 1, \forall g \in \pi_1(H_f)$ as $\pi_1(\{y_0\}) = 1$. So, $\phi(g.\gamma) = \phi(\gamma)$. As ϕ is an isomorphism, $g.\gamma = \gamma, \gamma \in \pi_n(E, H_f), \forall n \geq 1$. Let j denote the map $\pi_n(E, H_f) \to \pi_{n-1}(H_f)$. Then, $g.j(\gamma) = j(g(\gamma)) = j(\gamma) \implies g(j(\gamma)) = j(\gamma)$. So, the action of g on $Im(\pi_n(E, H_f))$ is trivial. So, we have:

$$1 \longrightarrow \pi_2(BGl(R)^+) \xrightarrow{\partial} \pi_1(H_f) \xrightarrow{i} E(R) \longrightarrow 1$$

and $\forall g \in \pi_1(H_f), \forall \gamma \in \pi_2(BGl(R)^+)$, $g.\partial(\gamma) = \partial(\gamma)$. But we know that g acts on $\pi_1(H_f)$ by conjugation. $\implies \partial(\gamma) \in Z(\pi_1(H_f)), \forall \gamma \in \pi_2(BGl(R)^+)$. Now, as f is acyclic, $\pi_1(H_f)$ is perfect, and from the lower degree terms of Serre's spectral sequence it can be seen that $H_2(\pi_1(H_f)) = 0$. It is also true that X is an universal central extension iff $H_1(X) = 0 = H_2(X)$. So, the above is a universal central extension of $E(R)$, hence by 9.0.4 , $\pi_2(BGl(R)^+) \cong K_2(R)$ (old).

7 Results from topology

Definition 7.0.1 (Fibration). $f : X \to Y$, a continuous map between topological spaces is called a fibration if

- for any homotopy $F : Z \times I \to Y$ and
- for any map g' lifting $F(\_, 0)$

$\exists F' : Z \times I \to X$ lifting F such that $F'(\_, 0) = g'$.

Proposition 7.0.2 (Fibration long exact sequence). *If $f : X \to Y$ is a fibration, we have this long exact sequence induced by $(f^{-1}(y_0), x_0) \hookrightarrow (X, x_0) \xrightarrow{f} (Y, y_0)$, where $f(x_0) = y_0$.*

$$\cdots \longrightarrow \pi_n(f^{-1}(y_0), x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0) \longrightarrow \cdots$$

which ends in

$$\cdots \longrightarrow \pi_0(f^{-1}(y_0)) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y)$$

Definition 7.0.3 (Homotopy fibre). Let $f : X \to Y$ be any continuous map with $f(x_0) = y_0$ $E_f := \{(a, \gamma) : a \in X, \gamma : I \to Y, \gamma(0) = f(a)\} \subset X \times Y^I$ where Y^I is given the compact-open topology. Let $\phi : E_f \to Y$ be the map $\phi(a, \gamma) = \gamma(1)$. It is a fibration.

Homotopy fibre of f with respect to $y_0 \in Y$ is defined to be $H_f := \phi^{-1}(y_0)$. It is unique upto homotopy equivalence if Y is path connected.

Proposition 7.0.4 (Homotopy fibre long exact sequence). *It is true that $\psi : X \to E_f$ where $x \mapsto (x, \gamma_{f(x)})$ is an embedding of X in E_f to which E_f deformation retracts, where $\gamma_{f(x)}$ is the constant path at $f(x)$. The same map restricted to $f^{-1}(y_0)$ is a homotopy equivalence between $f^{-1}(y_0)$ and H_f. We have the following:*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(H_f, (x_0, \gamma_{y_0})) \arrow[r, "i_*"] & \pi_n(E_f, (x_0, \gamma_{y_0})) \arrow[r, "\phi_*"] & \pi_n(Y, y_0) \arrow[r] & \cdots \\
& \pi_n(f^{-1}(y_0), x_0) \arrow[u, "\psi_*", "\cong"'] & \pi_n(X, x_0) \arrow[u, "\psi_*", "\cong"'] \arrow[r, "f_*"'] & \pi_n(Y, y_0) \arrow[u, "Id"'] &
\end{tikzcd}$$

Proposition 7.0.5 (Whitehead's theorem)**.** *If $f : X \to Y$, $f(x_0) = y_0$, where X and Y are path-connected CW complexes, and $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism, $\forall n$, then X is homotopy equivalent to Y, via f.*

Proposition 7.0.6 (Hurewicz's theorem)**.** *Define $\phi : \pi_n(X, x_0) \to H_n(X)$ by $\phi([f]) := f_*(\alpha)$, where $f_* : H_n(S^n) \to H_n(X)$ and α is some fixed generator of $H_n(S^n)$. Then, for $n \geq 2$, if X is $(n-1)$ connected (i.e., $\pi_j(X, x_0) = 1, \forall j < n$), then $\tilde{H}_j(X) = 0, \forall j < n$ and ϕ is an isomorphism.*

Proposition 7.0.7 (Corollary of Van Kampen's theorem)**.** *If X is path connected and Y is obtained from X by attaching 2-cells along the loops based at x_0, say ϕ_α, then $X \hookrightarrow Y$ induces a surjective map from $\pi_1(X, x_0)$ to $\pi_1(Y, x_0)$ and $\pi_1(Y, x_0) \cong \frac{\pi_1(X,x_0)}{<\phi_\alpha>}$ And $< \phi_\alpha >$ denotes the normal subgroup in $\pi_1(X, x_0)$ generated by the ϕ_α's. But if Y is obtained by attaching cells of dimension greater than 2, $\pi_1(Y, x_0) \cong \pi_1(X, x_0)$.*

Proposition 7.0.8. *(Corollary of cellular approximation theorem) If (Y, X) is a CW pair based at $x_0 \in X$ where Y is obtained from X by attaching cells of dimension greater than n, then the inclusion induces isomorphisms of $\pi_j(Y, x_0) \cong \pi_j(X, x_0), j < n$ and a surjection for $j = n$.*

Definition 7.0.9 (Cofibration)**.** $f : X \to Y$, a continuous map between topological spaces is called a cofibration if

- for any homotopy $F : X \times I \to Z$ and
- for any map g' extending $F(\_, 0)$

$\exists F' : Y \times I \to Z$ extending F such that $F'(\_, 0) = g'$.

If (Y, X) is CW pair, the inclusion of X in Y is a cofibration.

Definition 7.0.10 (Homotopy under a space)**.** With respect to the diagram below, $F : X \times I \to Y$ is a homotopy under A if $F(i(a), t) = j(t), \forall a, t$.

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Proposition 7.0.11. *If i and j are cofibrations in the above diagram, and suppose f is a homotopy equivalence, then $\exists g : Y \to X, g \circ j = i$ and both $g \circ f$ and $f \circ g$ are homotopic to Id under A.*

Proposition 7.0.12 (Homotopy groups of a covering space)**.** *If $p : X \to Y$ is a cover, then $\pi_n(X) \cong \pi_n(Y), \forall n > 1$.*

Proposition 7.0.13 (Uniqueness of classifying space)**.** *If G is a group and X_1, X_2 are contractible Hausdorff spaces on which G acts properly discontinuously, where $\frac{X_1}{G}, \frac{X_2}{G}$ are para-compact (a space in which every open cover has an open refinement that is locally finite; every CW complex is para-compact), then the quotient spaces are homotopy equivalent.*

Proposition 7.0.14 (Functoriality of BG)**.** *The classifying space construction gives a functor B from the category of groups and group homomorphisms to the category of CW complexes and homotopy classes of continuous maps. Also, if $f : G \to H$ induces a map $\bar{f} : BG \to BH$ then $\bar{f}_* : \pi_1(BG) \to \pi_1(BH)$ is simply f.*

Proposition 7.0.15.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\tilde{g}"] \\ C \arrow[r, "\tilde{f}"] & D \end{tikzcd}$$

Here, D is the pushout of the maps f and g. Suppose, either g or f is an embedding and a cofibration. Suppose also that $f_* : \pi_1(A) \to \pi_1(B)$ is surjective. [Note that this is true if f is acyclic]. Then, $Ker(\tilde{f}_*) = g_*(Ker(f_*))$.

Proof. We simply replace the maps by their respective mapping cylinders and use Van Kampen's theorem.

$$\begin{tikzcd}
A \arrow[rrr, hook, "f"] \arrow[drr, hook, "i"'] \arrow[ddr, hook, "j"] \arrow[ddd, "g"'] & & & B \arrow[ddd, "\tilde{g}"] \\
& & M_f \arrow[ur, "\sim"] \arrow[d, hook, "\tilde{j}"'] & \\
& M_g \arrow[r, hook, "\tilde{i}"'] \arrow[dl, "\sim"'] & M_f \cup_A M_g \arrow[dr, dashed, "H"] & \\
C \arrow[rrr, "\tilde{f}"'] & & & D
\end{tikzcd}$$

Here, H is induced by the universal property of pushouts. By Theorem 7.5.4 of 12, as either f or g is an embedding and a cofibration, H is a homotopy equivalence. As per the diagram, denote by $\phi : M_g \to C$ and $\psi : M_f \to B$, both being homotopy equivalences. By Van Kampen's theorem, we have $\pi_1(M_f \cup_A M_g) = \pi_1(M_f) *_{\pi_1(A)} \pi_1(M_g)$. As ϕ_* and H_* are isomorphisms, $Ker(\tilde{f}_*) = \phi_*(Ker(\tilde{i}_*))$. So, if we prove that $Ker(\tilde{i}_*) = j_*(Ker(i_*))$, then $Ker(\tilde{f}_*) = \phi_* \circ j_*(Ker(i_*)) = g_*(Ker(f_*))$ as $Ker(f_*) = Ker(i_*)$ since ψ_* is an isomorphism. By the universal property of pushouts, we have $\pi_1(D) \cong \pi_1(B) *_{\pi_1(A)} \pi_1(C)$. Now, as f_* is onto so is i_* and we have $\pi_1(M_f) \cong \pi_1(A)/Ker(i_*)$. Now, from the universal property of pushouts we have the following diagram

$$\begin{tikzcd}
\pi_1(A) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(M_f) \arrow[d, "\tilde{j}_*"] \arrow[dr, "\cong"] & \\
\pi_1(M_g) \arrow[r, "\tilde{i}_*"] \arrow[drr, bend right] & \pi_1(M_f) *_{\pi_1(A)} \pi_1(M_g) \arrow[dr, "r"] & \pi_1(A)/Ker(i_*) \arrow[d, "j_*"] \\
& & \pi_1(M_g)/j_*(Ker(i_*))
\end{tikzcd}$$

Now, define a map $s : \pi_1(M_g)/j_*(Ker(i_*)) \to \pi_1(M_f) *_{\pi_1(A)} \pi_1(M_g)$ by $c + j_*(Ker(i_*)) \mapsto \bar{c}$. s is well-defined because of the definition of amalgamated free product of groups. It is also an easy check that r and s are inverses of each other. So, $Ker(\tilde{i}_*) \subset j_*(Ker(i_*))$. The other inclusion is trivial. Hence, we are done.

$\square$

8 Results from Commutative algebra

Definition 8.0.1 (Stable range)**.** A ring R is said to satisfy S_n is for any row $(r_0, ..., r_n)$ with $< r_0, ..., r_n >= 1$, if there exists $(s_1, ..., s_n)$ such that $< s_1, ..., s_n >= 1$ and $s_i = r_i - r_0 t_i$ for some $t_i \in R$. Stable range $sr(R)$ is defined to be the smallest n such that R satisfies S_n. Any semi-local ring satisfies S_2. Any dedekind domain satisfies S_3.

Proposition 8.0.2 (Invariant basis property)**.** *If $f : R \to F$ is a ring morphism where F is a (skew) field. Then, if M is any free R-module with basis $\{x_\alpha\}$, $\{x_\alpha \otimes 1\}$ is an F-basis of $M \bigotimes_R F$. But this has to be unique as $M \bigotimes_R F$ is an F-vector space.*

Example 8.0.3.

- If R is commutative, then $R \to R/m$, for $m \in Spec_m(R)$.
- If $R = \mathbb{Z}[G]$, $f : \mathbb{Z}[G] \to \mathbb{Q}$ be the map $\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g$.

Proposition 8.0.4. *For finitely presented modules, projective = Locally free*

Proposition 8.0.5. *Any projective module over a PID is free. This follows from the structure theorem of modules over a PID.*

Proposition 8.0.6. *Let $R = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$. If $T := \{(f, g, h) \in R^3 : xf + yg + zh = 1\}$, then $R \bigoplus T \cong R^3$, but $T \ncong R^2$.*

Remark 8.0.7. A stably free line bundle (finitely generated projective module of rank 1) is free. So, for a counterexample to exist, we at least need the rank to be 2.

Proof. Define a dot product on R^3 by $(a, b, c).(e, f, g) := ae + bf + cg$. $r_v := v.(x, y, z) \in R$. Then, $(v - r_v(x, y, z)).(x, y, z) = 0 \implies v - r_v(x, y, z) \in T$. It is easy to check that $R^3 = R(x, y, z) \bigoplus T \cong R \bigoplus T$.

Now, assume $T \cong R^2$. Let $\{(a, b, c), (e, f, g)\}$ be the basis of T. So, determinant of the matrix with rows $(x, y, z), (a, b, c)$ and (e, f, g) is a unit in R. Our target is to construct a nowhere vanishing continuous vector field on S^2. But this is impossible by the hairy ball theorem which says that every continuous vector field on S^2 vanishes at least once. Coming back to the determinant which is a unit in R, we can think of it as a function on S^2 which never vanishes. So, for each $v \in S^2$, $(a(v), b(v), c(v)) \neq (0, 0, 0)$, a non vanishing continuous (as it is a polynomial) vector field as required. $\square$

Lemma 8.0.8 (Swan's Lemma)**.** *Let R be a noetherian ring and M is a submodule of $N \in M^R(R[t])$. then there exists a short exact sequence*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

where $X, Y \in M^R(R[t])$. (See 7)

Theorem 8.0.9 (Milnor patching)**.** *Suppose we have the following pull-back diagram of rings and ring morphisms:*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & S \end{tikzcd}$$

At least one of the j_i's are surjective. Now, given projective modules P_i over R_i , $i = 1, 2$ respectively and an isomorphism $\phi : S \bigotimes_{R_1} P_1 \to S \bigotimes_{R_2} P_2$, consider the set $P = \{(a, b) \in P_1 \times P_2 : \phi(1 \bigotimes_{R_1} a) = 1 \bigotimes_{R_2} b\}$. Then, P can be made into an R-module by $r.(a, b) := (i_1(r)a, i_2(r)b)$. Also, the following is true:

- *P is a projective R-module. If P_i's are finitely generated, so is P.*
- *Every projective module over R is of this form for suitable P_i's and ϕ.*
- *$P_j \cong R_j \bigotimes_R P$ for $j = 1, 2$.*

Theorem 8.0.10 (Serre's Conjecture)**.** *If R is a PID, every projective module over $R[t]$ is free.*

Proof. The following is a sketch of Quillen's proof of Serre's conjecture.

1. **Local Horrock's theorem:** Let S be the multiplicative set of monic polynomials in $R[t]$. $R < t >:= S^{-1}R[t]$. If P is a projective $R[t]$-module such that $S^{-1}P$ is $R < t >$-free, then P is $R[t]$-free. (See 7)

2. $Q(P) := \{f \in R : P_f \cong M \bigotimes_{R_f} R_f[t]$ for some R_f-module $M\}$. Then, $Q(P)$ is an ideal in R for P finitely presented. Note that this implies if $P \cong M \bigotimes_{R_m} R_m[t]$ for some R_m-module M for all maximal ideals m of R, then $Q(P) = R$, i.e., $P \cong N \bigotimes_R R[t]$ for some R-module N.
3. **Affine Horrock's theorem:** If P is a projective $R[t]$-module and $P < t > := S^{-1}P \cong M \bigotimes_R R < t >$ for some finitely generated projective R-module M, then $P \cong N \bigotimes_R R[t]$ for some finitely generated projective R-module N.
   This is true because: the hypothesis implies that $P_m < t > \cong M_m \bigotimes_{R_m} R_m < t >$ and as R_m is local P_m is $R_m[t]$-free for all maximal ideal m of R. By 2, we are done.
4. If P is a projective $R[t]$-module, then $P < t >$ is a projective module over the PID $R < t >$ (See 7), hence free by Proposition 8.0.5. By 3, $P \cong M \bigotimes_R R[t]$ for some finitely generated projective R-module M. But again by Proposition 8.0.5, M is R-free, hence P is $R[t]$-free.

□

9 Results from homological algebra; Spectral sequence

Theorem 9.0.1. *If C, D be two chain complexes and we have maps $f_i : C_i \to D_i, i \leq r$. if C_i is projective for $i > r$ and $H_i(C) = 0, i \geq r$, then the f_i's extend to a chain map f which is unique upto homtopy. (See 18)*

The following is a useful consequence of this theorem:

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, dotted] & F_0 \arrow[r, "\epsilon"'] \arrow[d, dotted] & M \arrow[r] \arrow[d, "Id"] & 0 \arrow[d, "Id"] \\ \cdots \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r, "\epsilon'"'] & M \arrow[r] & 0 \end{tikzcd}$$

In this scenario, we have two projective resolutions of M, then we get an augmentation preserving map $f : F \to F'$ such that $\epsilon' \circ f = \epsilon$ and it is unique upto homotopy. Ans this f is a homotopy equivalence.

Definition 9.0.2 (Group homology of G)**.** Let M be a $G-$module. Consider a projective $\mathbb{Z}G-$resolution P_* of $\mathbb{Z}$ where $\mathbb{Z}$ is treated as a trivial G-module. Then, $H_*(G, M)$ is defined to be the homology of the complex:

$$\cdots \longrightarrow P_2 \bigotimes\nolimits_{\mathbb{Z}G} M \longrightarrow P_1 \bigotimes\nolimits_{\mathbb{Z}G} M \longrightarrow P_0 \bigotimes\nolimits_{\mathbb{Z}G} M \longrightarrow 0$$

Definition 9.0.3 (Bar/Standard resolution)**.** Let F_n be the free abelian group on the $(n+1)$-tuples $(g_0, ..., g_n); g_i \in G$ and consider the boundary map: $(g_0, ..., g_n) \mapsto \sum_{i=0}^{n} (g_0, ..., \hat{g}_i, ..., g_n)$ and the augmentation map sending (g) to 1 for all $g \in G$. Then, this is a free resolution for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Theorem 9.0.4. *$H_0(G, M) = M/ < g.m - m >$, $H_1(G, \mathbb{Z}) = G/[G, G]$ and if G is abelian there is an isomorphism from $H_2(G)$ to $\bigwedge^2 G$ which on the chain level in terms of the bar resolution looks like $g \wedge h \mapsto [(gh, h, 1) - (gh, g, 1)] \otimes 1$. Also, let G be a perfect group, then $H_2(G)$ is isomorphic to the kernel of the universal central extension of G.*
If $\mathbb{Z}$ is treated as a trivial module, then $P_n \bigotimes_{\mathbb{Z}G} \mathbb{Z}$ is simply the free abelian group on the symbols $[g_0|...|g_n]$ where this symbol represents the equivalence class of $(g_0, ..., g_n)$ under the right action of $\mathbb{Z}G$.

Definition 9.0.5 (Spectral Sequences)**.** A spectral sequence consists of the following data:

- A collection of R-modules $E^r_{pq}, r \geq 0; , p, q \in \mathbb{Z}$
- R-linear maps $d^r_{pq} : E^r_{pr} \to E^r_{p-r,q+r-1}$
- and fixed isomorphisms between E^{r+1}_{pq} and $Ker(d^r_{pq})/Im(d^r_{p+r,q-(r-1)})$.

Theorem 9.0.6.

- *If $E^r_{pq} = 0, (p, q) \notin \mathbb{Z} \times \mathbb{Z}$, then $\exists s : E^t_{pq} = E^s_{pq}, \forall t \geq s$. This term is denoted by E^∞_{pq}.*
- *We say $\{E^r_{pq}\} \implies H_{p+q}$, where H_n is a sequence of filtered R-modules; iff $E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$. So, $\bigoplus_{p+q=n} E^\infty_{pq} \cong \bigoplus_p F_pH_{p+q}/F_{p-1}H_{p+q} =: gr_pH_{p+q}$.*
- *If R- is a field and the H_n's are finite dimensional vector spaces and the filtrations are finite, then $H_{p+q} \cong gr_pH_{p+q}$.*
- *Given a filtered chain complex which is finite at each dimension and such that the filtration is compatible with the boundary map.*

  Then, we have the following sequence of R-modules:

  $B^0_{pq} \subset B^1_{pq} \subset ... \subset B^\infty_{pq} \subset Z^\infty_{pq} \subset ... \subset Z^1_{pq} \subset Z^0_{pq}$

  where $Z^0_{pq} = F^pH_{p+q}$, $E^r_{pq} = Z^r_{pq}/B^r_{pq}$ and $B^0_{pq} = Image(F^pH_{p+q})$. Also, there is an isomorphism between E^∞_{pq} and $F^pH_{p+q}/F^{p-1}H_{p+q}$ where the isomorphism is simply sending the class of x in the domain to the class of x in the co-domain.

  Theorem 9.0.7. *Take a double complex as in the following diagram where $d_h^2 = 0 = d_v^2 = d_h \circ d_v + d_v \circ d_h$. We form the total complex by $Tot_n(C) = \bigoplus_{p+q=n} C_{p,q}$. This can be filtered in two ways:*

  - $F_iTot_n(C) := \bigoplus_{p \geq i} C_{p,n-p}$.
  - $F_iTot_n(C) := \bigoplus_{p \geq i} C_{n-p,p}$.

  WLOG consider the second filtration. Here, we have $E^0_{pq} = C_{pq}$. Let d^0_{pq} be the vertical map d_v. Then, $E^1_{pq} = H_p(C, d_v)$. We have maps $d_h : E^1_{pq} \to E^1_{p-1,q}$. To get a map between the E^2 terms, take $z \in C_{pq}$ represent the class of an element in E^2_{pq}. Then, there exists $y \in C_{p-1,q+1} : d_v(y) = d_h(z)$. Then, d^2_{pq} maps the class of z to the class of $d_h(y)$. Similarly, to get d^3_{pq}, then there exists $x \in C_{p-2,q+2} : d_v(x) = d_h(y)$. Then, d^3_{pq} maps it to the class of $d_h(x) \in C_{p-3,q+2}$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-3,q+3} \arrow[l] \arrow[d] & C_{p-2,q+3} \arrow[l] \arrow[d] & C_{p-1,q+3} \arrow[l] \arrow[d] & C_{p,q+3} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-3,q+2} \arrow[l] \arrow[d] & C_{p-2,q+2} \arrow[l] \arrow[d] & C_{p-1,q+2} \arrow[l] \arrow[d] & C_{p,q+2} \arrow[l] \arrow[d, "d_v"] & \cdots \arrow[l] \\
\cdots & C_{p-3,q+1} \arrow[l] \arrow[d] & C_{p-2,q+1} \arrow[l] \arrow[d] & C_{p-1,q+1} \arrow[l] \arrow[d] & C_{p,q+1} \arrow[l, "d_h"'] \arrow[d, "d_v"] & \cdots \arrow[l] \\
\cdots & C_{p-3,q} \arrow[l] \arrow[d] & C_{p-2,q} \arrow[l] \arrow[d] & C_{p-1,q} \arrow[l] \arrow[d] & C_{p,q} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Similarly, we can consider the other filtration. Note that, if $C_{p,q} = F_q \bigotimes_{\mathbb{Z}G} C_p$ *where* C_* *is a chain of* G*-modules. If suppose,* C_* *is acyclic. Then, the first spectral sequence has the following* E^2 *page.* $E^2_{pq} = 0$ *if* $q > 0$*. If suppose* F_* *is a projective resolution of the trivial module* $\mathbb{Z}$*, then,* $E^2_{p,0} = H_p(G, H_0(C_*))$ *and is this remains till infinity, hence is isomorphic to the 2nd homology of the total complex.*

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