Unramified Milnor-Witt K-theory and the Scissors congruence group

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1 Introduction

1.1 Goal

Definition 1.1.1 (Scissors congruence group)**.** Let $F_k :=$ the category of fields over a fixed base field k. We have the functor $P : F \to Ab$, defined pointwise as
$P(F) := \frac{\mathbb{Z}\{[u];u\in F^*-\{1\}\}}{<[x]-[y]+[\frac{y}{x}]-[\frac{1-x^1}{1-y^-1}]+[\frac{1-x}{1-y}]>}$.

This group appears in Hutchinson's (See 1) computation of K_2 of a field F. The properties of this group and the intriguing 5-term relation has been explored a lot starting extensively from Suslin (See 3). The approaches so far rely on explicit algebraic computations. This project aims to explore this 5-term relation in the light of A^1-homotopy theory.

The motivation behind this pursuit comes from works by Morel and others (See 2) on Milnor K- theory and its generalisation, Milnor-Witt K-theory. These two are functors defiined on F. But they have been extended to functors on Sm_k, as unramified sheaves of graded abelian groups which are moreover strongly A^1- invariant (See 5.1.11). This latter notion, that exists in the world of A^1-homotopy theory enables us to give meanings to the relations appearing in $K_*^M(F)$ or $K_*^{MW}(F)$ (See 3.3.6). Hence, it was expected that if P were extended as a A^1-invariant sheaf on Sm_k, we could understand the 5-term relation better.

1.2 Notes for the reader

- There is a naive way to extend P as a sheaf on Sm_k (See 4.0.2). We show in Section 4 (See 4.0.3), that this is not a strongly A^1-invariant sheaf.
- To show this, we first study the theory of unramified sheaves of sets (or groups) on Sm_k in Section 5. This is a recipe to extend a "nice" functor on F_k to a "nice" functor on Sm_k. Several axioms are noted that imply strong A^1-invariance.

- In theorem (See 3.3.5) of Section 3, we shall see a way to compare strongly A^1-invariant sheaves of groups on Sm_k, where the Milnor-Witt K-groups K_n^{MW} turn out to be the free strongly A^1-invariant sheaf generated by the sheaf $\mathbb{G}_m^{\wedge n}$ sending $X \mapsto (O(X)^*)^{\wedge n}$. As a consequence of the result for $P(F)$ (see 4.0.1), we also show in corollary (see 4.0.4), that the sheaf $F \mapsto F^* \wedge F^*$ can not be extended as a strongly A^1-invariant sheaf on Sm_k.

The questions that still remain are:

- Whether P can be extended as an unramified sheaf in a way similar to K_n^{MW} by defining it for DVR's as the kernel of a certain residue map? Residue maps like this exist (see 3) but the kernel is not in general independent of the choice of uniformizing parameter.
- Suppose we can extend it to an unramified sheaf on Sm_k, whether it is A^1-invariant? For instance, $\mathbb{Z}(\mathbb{G}_m)$ is A^1-invariant but not strongly A^1-invariant and P is a quotient of it. (See 4.0.2)

The theory of unramified sheaves and properties of K_*^{MW} in this text follow 2 closely. Section 4 is an original contribution by the author. Apart from that several proofs in have been elaborated in Sections 3 and 5. One may straightaway go to Section 4 while referring necessary results from Section 3 and 5.

1.3 Acknowledgment

I am highly grateful to my supervisor Utsav Choudhury for suggesting me these questions about the Scissors congruence group in the light of A^1-homotopy theory.

2 Notations

- $F_k :=$ Category of fields over k of finite transcedence degree
- O_v denotes the valuation ring of F whenever v is a valuation on F; m_v denote its maximal ideal
- $Sm_k :=$ Category of smooth finite type $k-$schemes
- $Sm_k^{'} :=$ Category of essentially smooth $k-$schemes. It is a noetherian $k-$scheme which is the inverse limit of a left filtering system with each transition map being an etale affine morphism between smooth $k-$schemes
- $Set :=$ Category of sets
- $Ab :=$ Category of abelian groups
- $\Delta^{op}C :=$ Category of simplicial objects in C
- $H_\bullet(k)$ denotes the pointed A^1-homotopy category of smooth $k-$schemes.
- We will denote by ν, the catgeory Sm_k occasionally

3 Unramified Milnor-Witt K-theory

3.1 Milnor-Witt K-theory of fields

Definition 3.1.1 (Milnor-Witt K-groups)**.** Consider the graded associative ring generated by the symbols $[u]$ for each $u \in F^*$ and one symbol η of degree (-1) with the following relations:

- (Steinberg Relation) $\forall a \in F^* - \{1\}, [a][1-a] = 0$
- $\forall (a,b) \in (F^*)^2, [ab] = [a] + [b] + \eta[a][b]$
- $[u]\eta = \eta[u], \forall u \in F^*$
- $h := \eta[-1] + 2$, then $\eta h = 0$.

Denote this ring by $K_*^{MW}(F)$ and its n^{th} degree part by the abelian group $K_n^{MW}(F)$.

Proposition 3.1.2. *Let $\tilde{K_n}^{MW}(F)$ denote the abelian group generated by the symbols of $[\eta^m, u_1, ..., u_r], r = n+m, u_i \in F^*, m \in \mathbb{N}, r \in \mathbb{N}, n \in \mathbb{Z}$ subject to:*

- $[\eta^m, u_1, ..., u_r] = 0$ *if for some i, $u_i + u_{i+1} = 1$*
- $[\eta^m, ..., u_{i-1}, ab, u_{i+1}, ...]$
  $=[\eta^m, ..., u_{i-1}, a, u_{i+1}, ...] + [\eta^m, ..., u_{i-1}, b, u_{i+1}, ...] + [\eta^{m+1}, ..., u_{i-1}, a, b, u_{i+1}, ...]$
- $[\eta^{m+2}, ..., u_{i-1}, -1, u_{i+1}, ...] + 2[\eta^{m+1}, ..., u_{i-1}, u_{i+1}, ...] = 0.$
  Then, the map $[\eta^m, u_1, ..., u_r] \to \eta^m[u_1]...[u_r]$ induces an isomorphism $\tilde{K_n}(F) \cong K_n^{MW}(F)$.

Proof. Note that the above map makes sense as all the relations in $\tilde{K_n}(F)$ map to 0 in $K_n^{MW}(F)$, by definition. Let R be the associated graded ring generated by the symbols $[u], u \in F^*$ of degree 1 and η of degree (-1) such that $\eta[u] = [u]\eta, \forall [u]$. Let I be the ideal in R generated by the relations as in the proposition. Then, clearly $K_n^{MW}(F) = \frac{R_n}{R_n \cap I}$. Define a map from R_n to $\tilde{K_n}^{MW}(F)$ sending $\eta^m[u_1]...[u_r]$ to $[\eta^m, u_1, ..., u_r]$. By definition, R_n is generated by such elements. Any element in $R_n \cap I$ is generated by elements of the form $xiy, i \in I, x, y \in R$ such that i is one of the relations as in the proposition and degree(xiy) is n. These map to 0 in $\tilde{K_n}^{MW}(F)$, hence induces a map $K_n^{MW}(F) \to \tilde{K_n}^{MW}(F)$ which is clearly the inverse to the given map. $\square$

Remark 3.1.3. For $a \in F^*$, denote by $< a >:= 1 + \eta[a] \in K_0^{MW}(F)$. So, $h = 1+ < -1 >$. Define $\epsilon := - < -1 > \in K_0^{MW}(F)$. Then, $\epsilon\eta = \eta$.

1. $[ab] = [a] + [b] + \eta[a][b] = [a] + (1 + \eta[a])[b] = [a](1 + \eta[b]) + [b]$
   So, $[ab] = [a]+ < a > [b] = [a] < b > +[b]$.
2. $< ab >= 1 + \eta[ab] = 1 + \eta[a] + \eta[b] + \eta^2[a][b] = (1 + \eta[a])(1 + \eta[b])$ as $\eta[u] = [u]\eta$.
   So, $< ab >=< a >< b >$.
3. By 1., $[ab] = [ba] = [a] + [b] < a >= [a]+ < a > [b] \implies < a > [b] = [b] < a >$.
   So, $K_0^{MW}(F) \subset Z(K_*^{MW}(F))$, as elements $\eta^m[u_1]...[u_m]$ generate $K_0^{MW}(F)$.
4. $\eta h = 0 \implies 0 = \eta h[1] = (< 1 > -1)(< -1 > +1) =< 1 > -1 \implies < 1 >= 1$.
   $\implies \eta[1] = 0$ and $[1] = [1]+ < 1 > [1] = 2[1]$ (by 1.).
   So, $< 1 >= 1 \in K_0^{MW}(F)$ and $[1] = 0 \in K_1^{MW}(F)$.
5. $< a >< a^{-1} >=< 1 >= 1$, so $< a >$ is a unit in $K_0^{MW}(F)$.
6. $[\frac{a}{b}]+ < \frac{a}{b} > [b] = [a]$, by 1.
7. By definition, $K_n^{MW}(F)$ for $n \geq 1$, is generated by the products of the form $\eta^m[u_1]...[u_{n+m}]$. But $\eta[a][b] = [ab] - [a] - [b]$. So, we get rid of the power m in η till it is 0. Note that we can do this only because $n + m > m$ as $n \geq 1$.
   So, $K_n^{MW}(F)$ for $m \geq 1$ is generated by the products of the form $[u_1]...[u_n]$.
8. If $n < 0$ in 7., then $\eta^m[u_1]...[u_r] = \eta^{-n}\eta^r[u_1]...[u_r]$ and $\eta[a] =< a > -1$.
   So, $K_n^{MW}(F)$ for $n < 0$ is generated by $\eta^{-n} < u >$, and the product with η, $K_n^{MW}(F) \to K_{n-1}^{MW}(F)$ is surjective for $n \leq 0$.
9. Define n_ϵ as mh, if $n = 2m$ and $mh + 1$, if $n = 2m + 1$, elements of $K_0^{MW}(F), n \geq 0$ and $n_\epsilon := (-n)_\epsilon, n < 0$. $[a^n] = [a^{n-1}] + [a] + \eta[a^{n-1}][a]$ and $[a^{-1}] = -([a] + \eta[a][a])$.
   So, inductively we have for $n \in \mathbb{Z}$, $[a^n] = n_\epsilon[a]$.

Lemma 3.1.4. *For $a \in F^*$,*

- $[a][-a] = 0$.
- $< a > + < a^{-1} >= h$.
- $< a^2 >= 1$.
- $[a][b] = \epsilon[b][a]$.

- $[a][a] = [a][-1] = \epsilon[a][-1] = [-1][a] = \epsilon[-1][a]$.

Proof. If $a = 1, [a] = 0$ and we are done. So, let $a \neq 1 \implies (-a) = \frac{1-a}{1-a^{-1}}$.
$\implies [-a] = [1-a] - < -a > [1-a^{-1}] \implies [a][-a] = [a][1-a] - < -a > [a][1-a^{-1}] = 0 - < -a > [a][1-a^{-1}] = < -a >< a > [a^{-1}][1-a^{-1}] = 0$ using 6.
$[-a] = [-1] + < -1 > [a]$. Multiplying by $[a]$, we get $0 = [a][-1] + < -1 > [a][a] \implies [a][a] = - < -1 > [a][-1] = \epsilon[a][-1]$. Note that by 2., $\epsilon^2 = 1$ and $[-1] + < -1 > [-1] = [1] = 0 \implies \epsilon[-1] = [-1] \implies \epsilon[a][-1] = [a][-1]$. Similarly, $[a][a] = [-1][a] = \epsilon[a][-1] = \epsilon[-1][a]$.
To show $< a^2 >= 1$, it is enough to show that $\eta[a^2] = 0$. But $[a^2] = (2 + \eta[-1])[a]$ as $[a][a] = [-1][a]$. So, $\eta[a^2] = 0$.
Now, $[ab][-ab] = 0 \implies ([a] + < a > [b])([-a] + < -a > [b]) = 0$
$\implies 0 = [a][-a] + < -a > [a][b] + < a > [b][-a] + < -a^2 > [b][b]$.
$\implies < a > ([b][-a] + < -1 > [a][b]) + < -1 > [-1][b] = 0$. Further simplifying by $[b][-1] = \epsilon[-1][b]$ and $[-a] = [a] + < a > [-1]$, we get $< a > ([b][a] + < -1 > [a][b]) = 0$. As $< a >$ is a unit, we have $[b][a] = \epsilon[a][b]$. □

Definition 3.1.5 (Grothendieck-Witt Ring)**.** It is defined to be the isomorphism classes of non-degenerate symmetric bi-linear forms over F, denoted by $GW(F)$. Let $< u > \in GW(F)$ denote the quadratic form $F^2 \to F; (x, y) \mapsto uxy$.

Proposition 3.1.6. *$GW(F)$ has the presentation with:*

- *Generators:* $\{< u >; u \in F^*\}$
- *Relations:* $< uv^2 >=< u >; < u > + < -u >= 1 + < -1 >;$
  $< u > + < v >=< u + v > + < uv(u + v) >$ *if* $u + v = 0$.

Proposition 3.1.7. *By the second relation above, the subgroup generated by $h := 1 + < -1 >$ is actually an ideal. Define $W(F) := GW(F)/h$, the Witt ring of F. The following square* *is Cartesian:*

$$\begin{tikzcd} GW(F) \arrow[r, "rank"'] \arrow[d] & \mathbb{Z} \arrow[d] \\ W(F) \arrow[r, "rank"'] & \frac{\mathbb{Z}}{2\mathbb{Z}} \end{tikzcd}$$

$I(F) := Ker(rank : W(F) \to \frac{\mathbb{Z}}{2\mathbb{Z}})$*, called the fundamental ideal of* F*.*

Proposition 3.1.8. *By the properties of* $< a > \in K_0^{MW}(F)$*,* $< uv^2 >=< u >; < u > + < -u >= 1+ < -1 >; < u > + < v >=< u + v > + < uv(u + v) >$*. So, we have a map* $GW(F) \to K_0^{MW}(F)$ *which turns out be an isomorphism.*

Proof. (See 2) □

Lemma 3.1.9. *The above map gives a* $\mathbb{Z}[F^*/(F^*)^2]$*-module structure on* $K_*^{MW}(F)$*.*

Proof. Since the map in the above proposition is an isomorphism, $< u > .\alpha$ makes sense as $K_*^{MW}(F)$ is a $K_0^{MW}(F)$-module. But by the first relation of $GW(F)$, $< uv^2 >=< u >$, extending linearly we get a $\mathbb{Z}[F^*/(F^*)^2]$-module structure on $K_*^{MW}(F)$. □

Lemma 3.1.10. *If* F *is a field where every unit is a square, i.e.,* F *is quadratically closed, then* $K_*^{MW}(F) \to K_*^{M}(F)$ *is an isomorphism in degree* ≥ 0*.* $K_*^{MW}(F) \to K_*^{W}(F)$ *is an isomorphism in degree* < 0*.*

Proof. $< -1 >=< a^2 >= 1$ for some $a \in F^*$. So, $2\eta = 0 \implies \eta[a^2] = 2\eta[a] = 0 \implies \eta[u] = 0, \forall u \in F^*$. Hence, $[ab] = [a] + [b]$ and the lemma follows. □

3.2 Residue map and its consequences

Theorem 3.2.1. *Let F be a field with discrete valuation v, and uniformising parameter π. Then, $\exists!$ morphisms of graded groups $\partial_{\pi,v} : K_*^{MW}(F) \to K_{*-1}^{MW}(k(v))$ which commutes with the multiplication by η map such that:*

- $\partial_{\pi,v}([\pi][u_2]...[u_n]) = [\bar{u_2}]...[\bar{u_n}]$
- $\partial_{\pi,v}([u_1][u_2]...[u_n]) = 0.$

for $u_i \in O_v^*$.

Proof. **Existence:** Define the map $\Theta_\pi : F^* \to K_*^{MW}(F)[\zeta]$ by $\pi^n u \mapsto [\bar{u}] + n_\epsilon < \bar{u} > \zeta$ where $K_*^{MW}(F)[\zeta]$ is defined to be the quotient of the polynomial ring in 1 variable over $K_*^{MW}(F)$, i.e., $K_*^{MW}(F)[T]$, by the relation $T^2 - [-1]T$.

Note that this map makes sense as any element of F^* is uniquely represented as $\pi^n u$ for unique $n \in \mathbb{Z}$ and $u \in O_v^*$. Sending the units of F as above and $\eta \to \eta$, we claim Θ_π induces a map from $K_*^{MW}(F)$.

1. Let $\pi^n u \in F^*$. If $n > 0, 1 - \pi^n u \in O_v^*$ and $1 \bar{-} \pi^n u = 1 \implies \theta_\pi(1 - \pi^n u) = [1] = 0$. So, $\theta_\pi(\pi^n u)\theta_\pi(1 - \pi^n u) = 0$. If $n = 0$, then $1 - u \in \pi^m v$. If $m > 0$, we are done as before. If $m = 0$, $\theta_\pi(u)\theta_\pi(1-u) = [\bar{u}][1-\bar{u}] = 0$ in $K_*^{MW}(k(v))[\zeta]$. If $n < 0$, $1 - \pi^n u = \pi^n(-u)(1 - \pi^{-n}u^{-1})$ where $(-u)(1 - \pi^{-n}u^{-1}) \in O_v^*$. Expanding $\theta_\pi(\pi^n u)\theta_\pi(1 - \pi^n u)$, we get $[\bar{u}][-\bar{u}] + n_\epsilon < \bar{u} > \zeta[-\bar{u}][\zeta] + n_\epsilon < -\bar{u} > [\bar{u}][\zeta] + n_\epsilon^2 < -1 > \zeta^2$. By lemma (see ref), $[\bar{u}][-\bar{u}] = 0$. Observe that $n_\epsilon^2[-1] = n_\epsilon(n_\epsilon[-1]) = n_\epsilon[(-1)^n] = [(-1)^{n^2}] = [(-1)^n]$ as $n^2 \equiv n (\mathrm{mod}\ 2)$. So, the previous expression becomes
   $n_\epsilon(< -\bar{u} > ([\bar{u}] - [-\bar{u}]) + < -1 > [-1])\zeta$.
   Now, $[\bar{u}] - [-\bar{u}] = [\bar{u}] - [\bar{u}] - [-1] - \eta[\bar{u}][-1] = - < \bar{u} > [-1]$. So, the expression inside the bracket is 0. $\implies \theta_\pi(\pi^n u)\theta_\pi(1 - \pi^n u) = 0$.
2. Let $a := \pi^n u; b := \pi^m v$.
   $\theta_\pi([ab] - [a] - [b] - \eta[a][b]) = [\bar{u}\bar{v}] - [\bar{u}] - [\bar{v}] - \eta[\bar{u}][\bar{v}] + \zeta((n+m)_\epsilon < \bar{u} >< \bar{v} > -(n_\epsilon + m_\epsilon) < \bar{u} >< \bar{v} > +(n_\epsilon m_\epsilon - < -1 > n_\epsilon m_\epsilon) < \bar{u} >< \bar{v} >)$. Note that $(n+m)_\epsilon = n_\epsilon + m_\epsilon + \eta n_\epsilon m_\epsilon[-1]$. We show it for $n, m \geq 0$ and the other cases follow similarly. If $n = 2k, m = 2l$, then $(n+m)_\epsilon = (k+l)h = n_\epsilon + m_\epsilon + \eta n_\epsilon m_\epsilon[-1]$ as

$\eta n_\epsilon m_\epsilon[-1] = \eta khm_\epsilon[-1] = 0$. If $n = 2k, m = 2l+1$, then $(n+m)_\epsilon = (k+l)h+1; n_\epsilon + m_\epsilon + \eta n_\epsilon m_\epsilon[-1] = kh + lh + 1 + \eta khm_\epsilon[-1] = (k+l)h+1$. If $n = 2k+1, m = 2l+1$, then $(n+m)_\epsilon = (k+l+1)h; n_\epsilon + m_\epsilon + \eta n_\epsilon m_\epsilon[-1] = kh+1+lh+1+\eta(kh+1)(lh+1)[-1] = (k+l)h+2+\eta[-1] = (k+l+1)h$ as $h = 2+\eta[-1]$. So, $\theta_\pi([ab]-[a]-[b]-\eta[a][b]) = 0$.

3. As $[-1] \mapsto [-1]$ and $\eta \mapsto \eta$ under θ_π, relation (3) and (4) hold.

Uniqueness: Note that $\theta_\pi : K_*^{MW}(F) \to K_*^{MW}(k(v))[\zeta]$ is a morphism of graded rings. But $K_*^{MW}(k(v))[\zeta]$ is a free $K_*^{MW}(k(v))$-module of rank 2.

$\implies$ $\theta_\pi(\alpha) = s_v^\pi(\alpha) + \partial_v^\pi(\alpha)\zeta$ for unique $s_v^\pi(\alpha), \partial_v^\pi(\alpha) \in K_*^{MW}(k(v))$. Note that from the definition of θ_π,

- s_v^π is a morphism of rings. This is the unique map $K_*^{MW}(F) \to K_*^{MW}(k(v))$ with $\eta \mapsto \eta$ and $[\pi^n u] \mapsto [\bar{u}]$.
- $\theta_\pi([\pi][u_2]...[u_n]) = ([\bar{u_2}]...[\bar{u_n}])\zeta$. Note that $\partial_v^\pi([\pi]) = 1$. If ρ is any other uniformizing parameter, i.e., $\rho = u\pi$; so $\partial_v^\pi(\rho) = [\bar{u}], u \in O_v^*$. Clearly, $\partial_v^\pi([u_1]...[u_n]) = 0$.

Now, given these above two properties of ∂_v^π, and the fact that $K_n^{MW}(F)$ is generated by the products of the form $[u_1]...[u_n]; u_i \in F^*$, we are done. For a simple example, let $x = [u_1][u_2]; u_1 = \pi^n v_1; u_2 = \pi^m v_2; n, m \in \mathbb{Z}; v_i \in O_v^*$.

Without loss of generality, we can assume $n = m = 1$ as otherwise, $[a][a] = [a][-1]; [a^{-1}] = -([a] + \eta[a][a])$. So, $x = [\pi v_1][\pi v_2] = ([\pi] + [v_1] + \eta[\pi][v_1])[u_2] = [\pi][\pi v_2] + [v_1][\pi v_2] + \eta[\pi][v_1][\pi v_2]$. Again using $[a][a] = [a][-1]; [ab] = [a] + [b] + \eta[a][b]$, the image of this element under ∂_v^π is completely determined by the two properties. $\square$

Remark 3.2.2.

- $\partial_v^\pi([-\pi]\alpha) =< -1 > s_v^\pi(\alpha); \alpha \in K_*^{MW}(F)$
- $\partial_v^\pi([u]\alpha) = - < -1 > [\bar{u}]\partial_v^\pi(\alpha); u \in O_v^*$
- $\partial_v^\pi(< u > \alpha) =< \bar{u} > \partial_v^\pi(\alpha); u \in O_v^*$

These follow at once from the definition of θ_π and working with the generators of $K_n^{MW}(F)$.

Proposition 3.2.3. *Let $E \subset F$ be a field extension with discrete valuation v on F restricting to w on E with valuation rings O_v and O_w respectively. Let π be a uniformizing parameter of v and ρ of w with $\rho = u\pi^e; u \in O_v^*$, i.e., the ramification index is e.*

Then, $\partial_v^\pi(\alpha) = e_\epsilon < \bar{u} > \partial_w^\rho(\alpha); \alpha \in K_*^{MW}(E)$. *Here,* $\alpha \in K_*^{MW}(E)$ *is seen as an element of* $K_*^{MW}(F)$. *Similarly,* $\partial_w^\rho(\alpha) \in K_*^{MW}(k(w))$ *is seen as an element of* $K_*^{MW}(k(v))$.

Proof. The following square is commutative:

$$\begin{array}{ccc} K_*^{MW}(F) & \xrightarrow{\theta_\pi} & K_*^{MW}(k(v))[\zeta] \\ \uparrow & & \uparrow \psi \\ K_*^{MW}(E) & \xrightarrow[\theta_\rho]{} & K_*^{MW}(k(w))[\zeta] \end{array}$$

where ψ is defined by $[a] \mapsto [a]$ and $\zeta \mapsto [\bar{u}] + e_\epsilon < \bar{u} > \zeta$. Let $w\rho^n \in E; n \in \mathbb{Z}, w \in O_w^* \subset O_v^*$. So, $\theta_\rho(w\rho^n) = [\bar{w}] + n_\epsilon < \bar{w} > \zeta$. Let $d := \theta_\pi(w\rho^n) = \theta_\pi(wu^n\pi^{ne}) = [w\bar{u}^n] + (ne)_\epsilon < w\bar{u}^n > \zeta$. Using $[a^n] = n_\epsilon[a]$, we get $d = [\bar{w}] + n_\epsilon[\bar{u}] + n_\epsilon\eta[\bar{w}][\bar{u}] + n_\epsilon e_\epsilon < \bar{w} > (1 + \eta n_\epsilon[\bar{u}])\zeta = [\bar{w}] + n_\epsilon < \bar{w} > [\bar{u}] + n_\epsilon e_\epsilon < \bar{w} > (1 + \eta n_\epsilon[\bar{u}])\zeta = [\bar{w}] + n_\epsilon[\bar{u}] < \bar{w} > + n_\epsilon e_\epsilon < \bar{w} > \zeta + n_\epsilon^2 e_\epsilon(< \bar{u} > -1)\zeta < \bar{w} >$. So, we get $d = [\bar{w}] + n_\epsilon < \bar{w} > [\bar{u}] + n_\epsilon^2 e_\epsilon < \bar{u} >< \bar{w} > \zeta - n_\epsilon^2 e_\epsilon < \bar{w} > \zeta + n_\epsilon e_\epsilon < \bar{w} > \zeta$. Now, observe that $e_\epsilon n_\epsilon(1 - n_\epsilon) < \bar{u} >= e_\epsilon n_\epsilon(1 - n_\epsilon)(1 + \eta[\bar{u}]) = e_\epsilon n_\epsilon(1 - n_\epsilon) + e_\epsilon n_\epsilon(1 - n_\epsilon)\eta[\bar{u}]$. If $n_\epsilon = kh$, then $n_\epsilon\eta = 0$. If $n_\epsilon = kh + 1$, then $1 - n_\epsilon = -kh \implies (1 - n_\epsilon)\eta = 0$. So, the second term is 0.

$\implies e_\epsilon n_\epsilon(1 - n_\epsilon) < \bar{u} >= e_\epsilon n_\epsilon(1 - n_\epsilon)$

$\implies e_\epsilon n_\epsilon(1 - n_\epsilon) < \bar{u} >< \bar{w} >= e_\epsilon n_\epsilon(1 - n_\epsilon) < \bar{w} >$

$\implies e_\epsilon n_\epsilon < \bar{u} >< \bar{w} > - e_\epsilon n_\epsilon^2 < \bar{u} >< \bar{w} >= e_\epsilon n_\epsilon < \bar{w} > - e_\epsilon n_\epsilon^2 < \bar{w} >$

So, $d = [\bar{w}] + n_\epsilon[\bar{u}] < \bar{w} > + e_\epsilon n_\epsilon < \bar{w} >< \bar{u} > \zeta$. Hence the diagram commutes. To get the relation as in the preparation, we just take the second coordinates of θ_π and θ_ρ. □

Theorem 3.2.4. *First observe that if we define* $\boldsymbol{K}_n^{MW}(O_v) := Ker(\partial_v^\pi) \subset K_n^{MW}(F)$*, by the previous proposition it is independent of the choice of uniformising parameter* π*. Also, the definition of residue map implies* $\boldsymbol{K}_*^{MW}(O_v)$ *is a graded ring.*
As a ring $\boldsymbol{K}_*^{MW}(O_v)$ *is generated by* η *and* $[u] \in K_1^{MW}(F); u \in O_v^*$.

Proof. Let Q_* denote the graded abelian group obtained by quotienting $K_*^{MW}(F)$ by the subring A_* generated by $\eta, [u]; u \in O_v^*$. Let π be a uniformising parameter and ∂_v^π : $K_*^{MW}(F) \to K_*^{MW}(k(v))$. Note that, by definition of ∂_v^π, $A_* \mapsto 0$ in $K_*^{MW}(k(v))$. So, we have $\partial_v^\pi : Q_* \to K_*^{MW}(k(v))$.

$K_*^{MW}(k(v))$**-module structure on** Q_***:**

Let $E_* := \bigoplus_{n\in\mathbb{Z}} E_n$ where $E_n := \{f \in End(Q_*) : f(Q_n) \subset Q_{m+n}; \forall m \in \mathbb{Z}\}$. E_* is called the graded ring of endomorphisms of Q_*.

Let $\bar{a} \in k(v)^*$, for $a \in O_v^*$. Suppose $\bar{a} = \bar{b}$ for some $b \in O_v^*$; $\implies$ $b = \beta a$ where $\beta \equiv 1 \pmod{\pi}$. This is because $a - b = \pi^k u$ and $b \in O_v^*, k \geq 1$. $\implies$ $b = a(1 - \pi^k u a^{-1})$ where $\beta = 1 - \pi^k u a^{-1} \equiv 1 \pmod{\pi}$ and $[b] = [a] + [\beta] + \eta[\beta][a]$. So, it is sufficient to check that $[\beta][d] \in A_*$ for $d \in F^*$. Writing $d = \pi^n u; n \in \mathbb{Z}, u \in O_v^*$ and using relations in $K_*^{MW}(F)$, it is enough to show that products of the form $[1 - \pi^n v][\pi] \in A_*; n > 0, v \in O_v^*$.

If $n = 1$, $[1 - \pi v][\pi v] = 0 \implies [\pi v] = [\pi](1 + \eta[v]) + [v] \implies [1 - \pi v][\pi](1 + \eta[v]) \in A_*$ but $1 + \eta[v] =< v >$, a unit in $K_*^{MW}(F)$ and $< v^2 >= 1$.

If $n \geq 2$, $1 - \pi^n v = (1 - \pi) + \pi(1 - \pi^{n-1} v) = (1 - \pi)(1 + \pi(\frac{1-\pi^{n-1}}{1-\pi})) = (1 - \pi)(1 - \pi w)$ for $w \in O_v^*$. So, $[1 - \pi^n v][\pi] = [1 - \pi][\pi] + [1\pi w][\pi] + \eta[1 - \pi][1 - \pi w][\pi] = [1 - \pi w][\pi] \in A_*$ by $n = 1$ case.

Similarly, $\eta \in E_{-1}$.

Since the module action is defined by lifting elements and multiplying in $K_*^{MW}(F)$, they satisfy the Milnor-Witt relations. So, we have a $K_*^{MW}(k(v))$-module structure on Q_*. Let $[\pi] \in Q_1 = K_1^{MW}(F)/A_1$. $K_{*-1}^{MW(k(v))} \xrightarrow{f} Q_*$ defined by $\alpha \mapsto \alpha[\pi]$ is clearly a section of $\partial_v^\pi : Q_* \to K_{*-1}^{MW}(k(v))$. Now, if we show f is onto, we are done as then $\mathrm{Ker}(\partial_v^\pi) \subset A_* \subset \mathrm{Ker}(\partial_v^\pi)$. So, $\mathbf{K}_*^{MW}(O_v)$ has the required generators. But any element of $K_*^{MW}(F)$ is a sum of elements of the form $\eta^m[\pi][u_2]...[u_n]$ and $\eta^m[u_1]...[u_n]; u_i \in O_v^*$ and the latter type of elements are in A_* while the former elements are in $Im(f) = Q_*$, hence we are done.

$\square$

Theorem 3.2.5. *There is a split short exact sequence of $K_*^{MW}(F)$-modules:*

$$0 \longrightarrow K_n^{MW}(F) \longrightarrow K_n^{MW}(F(T)) \xrightarrow{\sum \partial_{(P)}^P} \bigoplus_P K_{n-1}^{MW}(F[T]/P) \longrightarrow 0$$

where P runs over monic irreducibles in $F[T]$.

Proof. (See 2) □

Theorem 3.2.6. *$K_n^{MW}, n \in \mathbb{Z}$ is a strongly A^1-invariant sheaf of abelian groups on Sm_k.*

Proof. Note that for all $n \in \mathbb{Z}, K_n^{MW}(F)$ is a $K_0^{MW}(F)-$module, which by (see 3.1.9) gives a $\mathbb{Z}[F^*/(F^*)^2]-$module structure on it that is clearly functorial. Also, we have the product $F^* \times K_n^{MW}(F) \to K_{n+1}^{MW}(F)$ induced by the grading in $K_*^{MW}(F)$ which is also functorial in F. So, we have (D4)(i) and (D4)(ii),
The residue maps ∂_v^π gives (D4)(iii), by (see 3.2.3) as we are taking ramification index 1.
(B0), (B1) and (B2) clearly follow from our previous results on $K_n^{MW}(F)$.
By (see 3.2.3), (B3) follows. (HA)(i) follows from (see 3.2.5). (HA)(ii) follows from our definition of ∂_v^π.
The axioms (B4) and (B5) are also satisfied (see 2).
By (see 5.3.11), K_n^{MW} is an unramified sheaf of abelian groups on Sm_k that is also strongly A^1-invariant. □

3.3 Universality of $\mathbf{K}_n^{MW}$

Definition 3.3.1. Let $n \geq 1$ and $(\mathbb{G}_m)^{\wedge n}$ be the element of $Shv(\nu_{Nis})$ associated to the presheaf $S : X \to (O^*(X))^{\wedge n}$. Hence, it can be treated as an element of $H_\bullet(k)$, pointed by 1.

Proposition 3.3.2. *$S \in Preshv(\nu)$ as in the above definition is an unramified sheaf of pointed sets.*

Proof. As per definition of unramified presheaves and by the properties of the structure sheaf for a scheme, it is a Zariski sheaf. It is enough to check axiom (A1) to show it's a Nisnevich sheaf, as (A2) is clear from the properties of structure sheaf. For (A1), let $i : E \subset F$ be a separable extension in F_k, v a discrete valuation on F that restricts to w on E with ramification index 1. $S(O_w) \to S(O_v)$ is clear by choosing suitable models as in (see 5.2.9).

The second part also follows by noting that for a family of pointed sets $E_\alpha \in E$ where E is also a pointed set, we have $\cap_\alpha (E_\alpha)^{\wedge n} = (\cap_\alpha E_\alpha)^{\wedge n}$. □

Definition 3.3.3. Fix an irreducible $X \in Sm_k$ with function field F. As X is irreducible, $(O(X)^*)^{\wedge n} \subset (F^*)^{\wedge n}$, where for any pointed set (A, a), $A^{\wedge n} := \frac{A^n}{A \vee ... \vee A}$ where the $A \vee A := \frac{A \coprod A}{A \times \{a\} \sim \{a\} \times A}$. So, we have a map $(O(X)^*)^{\wedge n} \to K_n^{MW}(F)$ such that $(u_1, ..., u_n) \mapsto [u_1]...[u_n]$. But $[u_1]...[u_n] \in \mathbf{K}_n^{MW}(X)$ by definition (see 3.2.1).

Proposition 3.3.4. *The map $\sigma_n : (\mathbb{G}_m)^{\wedge n} \to \boldsymbol{K}_n^{MW}$ (called the canonical symbol map) is a morphism of sheaves on Sm_k.*

Proof. It is defined on irreducible schemes and generalised as in (See 5.2.1). By Corollary (see 5.2.11), we just need to show that the following square commutes, as $\mathbf{K}_n^{MW}$ is an unramified sheaf:

$$\begin{tikzcd} \mathbb{G}_m^{\wedge n}(O_v) \arrow[r, "s_v"'] \arrow[d, "\sigma"'] & \mathbb{G}_m^{\wedge n}(k(v)) \arrow[d, "\sigma"] \\ K_n^{MW}(O_v) \arrow[r, "s_v"'] & K_n^{MW}(k(v)) \end{tikzcd}$$

Now, $\mathbb{G}_m^{\wedge n}(O_v) = (O_v^*)^{\wedge n}$ and the map s_v is just going mod the maximal ideal of O_v in each coordinate. Since the s_v on the bottom row maps $[u_1]...[u_n] \to [\bar{u_1}]...[\bar{u_n}]$,(see 3.2.1) the diagram commutes. □

Theorem 3.3.5. *Let $n \geq 1$. The morphism σ_n is the universal morphism from $(\mathbb{G}_m)^{\wedge n}$ to a strongly A^1-invariant sheaf of abelian groups. That is, given a morphism of pointed sheaves $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ where M is a strongly A^1-invariant sheaf of abelian groups, there exists a unique morphism of sheaves of abelian groups (pointed by 0) Φ such the following diagram commutes:*

$$\begin{tikzcd} (\mathbb{G}_m)^{\wedge n} \arrow[d, "\sigma_n"] \arrow[dr, "\phi"] & \\ \boldsymbol{K}_n^{MW} \arrow[r, "\Phi"'] & M \end{tikzcd}$$

Theorem 3.3.6. *Suppose M is a strongly A^1-invariant sheaf of abelian groups on Sm_k. Let $n \geq 1$ be an integer, and let $\phi : \mathbb{G}_m^{\wedge n} \to M$ be a morphism of pointed sheaves. Then, for any $F \in F_k$, there is a unique morphism $\Phi_n(F) : K_n^{MW}(F) \to M(F)$ such that for any $(u_1, ..., u_n) \in (F^*)^n$, $\Phi_n(F)([u_1, ..., u_n]) = \phi(u_1, ..., u_n)$.*

Proof. **Existence:** We claim that it is enough to prove this for the base field k. To see this let $E \in F_k$ be the direct limit of the family E_α, each of finite type over k. Since k is perfect, we have each E_α finite separable over k, hence $Spec(E_\alpha) \to Spec(k)$ is smooth of finite type, and their inverse limit is $Spec(E) \in Sm^{'}(k)$. So, we have the pullback $f_\alpha^{-1} : Sm_k \to Sm_{E_\alpha}$. Sm_{E_α} is endowed with Nisnevich topology such that we have the functor $(f_\alpha)_* : Shv(Sm_{E_\alpha}) \to Shv(Sm_k)$ given by $F \mapsto F \circ f_\alpha^{-1}$. It has a left adjoint $f_\alpha^* : Shv(Sm_k) \to Shv(Sm_{E_\alpha})$, such that if a sheaf is represented by $X \in Sm_k$, then it is mapped to the sheaf $f_\alpha^*(X)$ represented by $f_\alpha^{-1}(X)$. Pulling back the map $\phi : (\mathbb{G}_{m,k})^{\wedge n} \to M$, we get a map of pointed sheaves on Sm_{E_α}, $f_\alpha^*\phi : (\mathbb{G}_{m,E_\alpha})^{\wedge n} \to f_\alpha^* M$.

Clearly $\mathbb{G}_{m,k}$ is represented by k^* and hence by definiton, $\mathbb{G}_{m,E_\alpha} = f_\alpha^*(\mathbb{G}_{m,k})$. This passed onto the smash product as $f_\alpha^*(X \wedge Y) = f_\alpha^*(X) \wedge f_\alpha^*(Y)$ for $X, Y \in Shv(Sm_k)$.
By Lemma 5.1.2 (1) in (See 4), we have the bijection $\varinjlim_\alpha Hom_{Shv(Sm_{E\alpha})}(f_\alpha^*(X), f_\alpha^*(F)) \to Hom_{Shv(Sm_E)}(f^*(X), f^*(F))$ for $X \in Sm_k, F \in Shv(Sm_k)$.

In our case, taking the pullback with respect to $k \subset E$, we get the map of pointed sheaves $(\mathbb{G}_m)^{\wedge n} \to f^*(M)$, where again by (See 4), $f^*(M)$ is strongly A^1-invariant element of $Shv(Sm_E)$. But by definition, evaluating $f^*(M)$ at E we get $M(E)$ by 5.1 (See 4). So, it is enough to prove the statement for $F = k$.

Since M is a strongly A^1-invariant sheaf of abelian groups, by (See 2), we have the bijection between the sets $Hom_{Shv(\nu_{Nis})}(((\mathbb{G}_m)^{\wedge n}, 1), (M, 0)) \leftrightarrow Hom_{H_\bullet(k)}(\Sigma((\mathbb{G}_m)^{\wedge n}), K(M, 1)) \leftrightarrow M_{-n}(k)$.

For any $(u_1, ..., u_r) \in (k^*)^r, r \in \mathbb{N}$, we have pointed morphisms $[u_i] : S^0 \to \mathbb{G}_m$ determined by u_i. Taking the smash product of these morphisms and then the suspension, we get $\Sigma([u_1, ..., u_r]) : \Sigma S^0 \to \Sigma((\mathbb{G}_m)^{\wedge r})$.

By (See 2), we have for $X, Y \in H_\bullet(k), \Sigma(X) \vee \Sigma(Y) \vee \Sigma(X \wedge Y) \cong \Sigma(X \times Y)$ and the product map $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ induces the map $\Sigma(\mu) =< Id_{\Sigma(\mathbb{G}_m)}, Id_{\Sigma(\mathbb{G}_m)}, \eta >$: $\Sigma(\mathbb{G}_m) \vee \Sigma(\mathbb{G}_m) \vee \Sigma((\mathbb{G}_m)^{\wedge 2}) \to \Sigma(\mathbb{G}_m)$.

Now, let $I = \{1, 2, ..., r\} = I_1 \coprod ... \coprod I_n$ be a partition of this finite set. We can define analogously a map $\eta_{I_1,...,I_n} : \Sigma((\mathbb{G}_m)^{\wedge r} \to \Sigma((\mathbb{G}_m)^{\wedge n})$, by passing to the summand $\Pi_{j=1}^n \mathbb{G}_m^{|I_j|} \to \mathbb{G}_m^{|I|}$, where each map $\mathbb{G}_m^{|I_j|} \to \mathbb{G}_m$ is the multiplication map. Suppose we have another partition of I, the claim is that the induced maps in both cases are homotopic, i.e., same as elements of $Hom_{H_\bullet(k)}(\Sigma((\mathbb{G}_m)^{\wedge r}), \Sigma((\mathbb{G}_m)^{\wedge n}))$. We illustrate it for the maps $\eta_{12}, \eta_{23} : \Sigma((\mathbb{G}_m)^{\wedge 3}) \to \Sigma((\mathbb{G}_m)^{\wedge 2})$ obtained from μ_{12}, μ_{23} respectively. $\mu_{12} : \mathbb{G}_m^3 \to \mathbb{G}_m^2; (x, y, z) \mapsto (xy, z)$ and similarly for μ_{23}.

Lemma 3.3.7. *The maps $\mathbb{G}_m \wedge \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$: $(x \wedge y \mapsto y \wedge x), Id \wedge (x \mapsto x^{-1})$ and $(x \mapsto x^{-1}) \wedge Id$ are all same as elements of $Hom_{H_\bullet(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$.*

Proof. (See 5) □

Now, let τ_{ij} denote the switch map for indices i and j. Clearly, $\mu_{23} : (x, y, z) \xrightarrow{\tau_{12}} (y, x, z) \xrightarrow{\tau_{23}} (y, z, x) \xrightarrow{\mu_{12}} (yz, x) \xrightarrow{\tau_{12}} (x, yz)$. Let i_j denote the map inverting the i^{th} coordinate, then $\mu_{12} : (x, y, z) \xrightarrow{i_1} (x^{-1}, y, z) \xrightarrow{i_2} (x^{-1}, y^{-1}, z) \xrightarrow{\mu_{12}} ((xy)^{-1}, z) \xrightarrow{i_1} (xy, z)$. Using the above lemma, we get the two maps η_{12}, η_{23} are homotopic.

The pointed morphism $[ab] : S^0 \to \mathbb{G}_m$ factors as $S^0 \xrightarrow{[a][b]} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m$. Taking the suspension, we get $\Sigma([ab]) = \Sigma([a]) \vee \Sigma([b]) \vee \eta([a][b])$ in the group $Hom_{H_\bullet(k)}(\Sigma(S^0), \Sigma(\mathbb{G}_m))$ where the group operation is $\vee$.

The last relation in the definition of K_n^{MW} follows from similar arguments as in lemma 3.3.7. To prove the Steinberg relation, note that the morphism $[a, 1-a] : S^{(0)} \to (\mathbb{G}_m)^{\wedge 2}$ factors in $H_\bullet(k)$ through $\tilde{\Sigma}(\mathbb{A}^1 - \{0,1\}) \xrightarrow{f} \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ as the morphism $Spec(k) \to \mathbb{G}_m \wedge \mathbb{G}_m$ factors through $\mathbb{A}^1 - \{0,1\}$. From a result in 5, we have, f is a trivial morphism in $H_\bullet(k)$. So, we have proven that the mapping $[\eta^m, u_1, ..., u_r] \to \eta^m \Sigma([u_1, ..., u_n]) \in M(k)$ factors through the relations in $K_n^{MW}(k)$.

Uniqueness: Uniqueness follows as $K_n^{MW}(F)$ is generated by the products $[u_1]...[u_n]$ if $n \geq 1$ and by the commutativity of the diagram in question. $\square$

Proof of Theorem 3.3.5. By Lemma 3.45 in (See 2), if M is A^1-invariant sheaf of pointed sets on Sm_k, then, for any smooth irreducible scheme X with function field F, the map $M(X) \to M(F)$ is injective.

Now, by (See 5.2.11), as $\mathbf{K}_n^{MW}$ is unramified, and that M is also unramified, to give a morphism of sheaves Φ :$\mathbf{K}_n^{MW} \to M$ it is sufficient to give natural transformation $K_n^{MW}|_{F_k} \to M|_{F_k}$ such that:

- For any discrete valuation v on $F \in F_k$, the image of $\mathbf{K}_n^{MW}(O_v)$ through Φ is inside $M(O_v)$.
- The following square commutes:

$$\begin{array}{ccc} \mathbf{K}_n^{MW}(O_v) & \xrightarrow{s_v} & \mathbf{K}_n^{MW}(k(v)) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ M(O_v) & \longrightarrow & M(k(v)) \end{array}$$

From Theorem (See 3.2.4), $\mathbf{K}_n^{MW}(O_v)$ is generated by the symbols of the form $[u_1]...[u_n], u_i \in O_v^*$. For any such symbol, we have a smooth model X of O_v and a morphism $X \to (\mathbb{G}_m)^{\wedge n}$ which induces $[u_1]...[u_n]$ when composed with $(\mathbb{G}_m)^{\wedge n} \to \mathbf{K}_n^{MW}$. Note that here

$X \in Shv(Sm_k)$ by the Yoneda embedding. So, composing with ϕ we get an element in $M(X) \subset M(O_v)$ which is also the image under Φ of $[u_1]...[u_n]$. So, we have shown $\mathbf{K}_n^{MW}(O_v) \to M(O_v)$.

For the second property, choose irreducible $X \in Sm_k$ with function field F, Y irreducible closed in X of codimension 1 such that $O_{X,Y} = O_v \subset F$. So, the u_i's come from the map $Y \to X \to \mathbb{G}_m$. $\implies$ $\Phi([u_1]...[u_n]) \in M(O_v)$ comes from $Y \to X \to (\mathbb{G}_m)^{\wedge n} \to M$. Since $k(v)$ is the function field of Y, the required diagram commutes. $\square$

4 Scissors Congruence group

The main theorem of this section is the following:

Theorem 4.0.1. *Let F be a field such that $char(F) = 0$ and every element of F^* is a square, then $\nexists \Phi : K_1^{MW}(F) \to P(F)$ such the following diagram commutes where $\phi : \mathbb{G}_m(F) \to P(F)$ is the canonical map sending $u \to [u], u \in F^* - \{1\}$ and $1 \to 0$.*

$$\begin{tikzcd}
\mathbb{G}_m(F) = F^* \arrow[d, "\sigma"] \arrow[dr, "\phi"] & \\
K_1^{MW}(F) \arrow[r, "\Phi"] & P(F)
\end{tikzcd}$$

Proof. Suppose to the contrary such a Φ exists. Then, by definition of σ as in (See 3.2.1), $\Phi([u]) = [u], u \in F^* - \{1\}$. In $K_1^{MW}(F)$, $[u(1-u)] = [u] + [1-u] \implies [u(1-u)] = [u] + [1-u] \in P(F), \forall u \in F^* - \{1\}$.

Observe the following relations in $P(F)$:

1. $[x] - [y] + [\frac{y}{x}] - [\frac{1-x^1}{1-y^{-1}}] + [\frac{1-x}{1-y}]$ and
2. $[1-y] - [1-x] + [\frac{1-x}{1-y}] - [\frac{1-x^1}{1-y^{-1}}] + [\frac{y}{x}]$, replacing x by $(1-y)$ and y by $(1-x)$.

Subtracting (2) from (1), we get $[x] + [1-x] = [y] + [1-y], \forall x, y \in F^* - \{1\}$.

So, $[u(1-u)] = [v(1-v)], \forall u, v \in F^* - \{1\}$. Since every element of F is a square, every quadratic equation in F has a solution in F, i.e., given $z \in F^* - \{1\}, \exists u \in F^* - \{1\} :$ $u(1-u) = z$. So, $[x] = [y] \in P(F), \forall x, y \in F^* - \{1\}$. But since we have a 5-term relation as in (1) above in $P(F)$, we have $[x] = 0 \in P(F), \forall x \in F^* - \{1\} \implies P(F) = 0$.

Now, recall from (1) the exact sequence obtained while computing $K_2^M(F)$ using spectral sequence for equivariant homology:

$$P(F) \xrightarrow{z \mapsto z \wedge (1-z)} F^* \wedge F^* \xrightarrow{a \wedge b \mapsto \{a,b\}} K_2^M(F) \longrightarrow 0$$

If $P(F) = 0$ we have $F^* \wedge F^* \cong K_2^M(F)$ where $F^* \wedge F^* := \frac{F^* \bigotimes_{\mathbb{Z}} F^*}{<a \otimes b + b \otimes a>}$. By the above exact sequence, we then have $a \wedge (1-a) = 0, \forall a \in F^* - \{1\}$. Since $char(F) = 0$, we have that

$-2, 3 \in F^*$ are $\mathbb{Z}$-linearly independent as $(-2)^a \neq 3^b$ in $\mathbb{Q}, \forall a, b \in \mathbb{Z}$. Let $G =< -2, 3 >$, the subgroup of F^* generated by 2 and 3. So, G is free abelian of rank 2. Consider the following commutative diagram:

$$\begin{tikzcd}
\bigwedge^2 G = G \wedge G \arrow[r, "i"] \arrow[d] & F^* \wedge F^* = \bigwedge^2 F^* \arrow[d] \\
\bigwedge^2 G \bigotimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[d, "\cong"'] & \bigwedge^2 F^* \bigotimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "\cong"'] \\
\bigwedge^2_{\mathbb{Q}}(G \bigotimes_{\mathbb{Z}} \mathbb{Q}) \arrow[r, hook] & \bigwedge^2_{\mathbb{Q}}(F^* \bigotimes_{\mathbb{Z}} \mathbb{Q})
\end{tikzcd}$$

The map on the bottom is injective as $G \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is a vector subspace of $F^* \bigotimes_{\mathbb{Z}} \mathbb{Q}$. Now, $\bigwedge^2_{\mathbb{Q}}(G \bigotimes_{\mathbb{Z}} \mathbb{Q})$ is a $\mathbb{Q}$-vector space spanned by $((-2) \otimes 1) \wedge (3 \otimes 1)$ which is 0 in $\bigwedge^2_{\mathbb{Q}}(F^* \bigotimes_{\mathbb{Z}} \mathbb{Q})$ by our assumption. A contradiction!

So, no such Φ exists. $\square$

Remark 4.0.2.

- By (see 3.1.10), if F is quadratically closed, $K_1^{MW}(F) \cong K_1^M(F)$ and in the latter we have $[xy] = [x] + [y]; \forall x, y \in F^*$. Applying this to the 5-term relation in $P(F)$, we get $[x] = [y]; \forall x, y \in F^*$ which implies the above the theorem, as then $P(F) = 0$.
- From (see 6), $\mathbb{Z}(\mathbb{G}_m)$ is a A^1-invariant sheaf of abelian groups which is not strongly A^1-invariant, where $\mathbb{Z}(\mathbb{G}_m)(X) = \mathbb{Z}(O(X)^* - \{1\})$, the free abelian group generated by elements of $O(X)^* - \{1\}$. Similarly, if we define $P(X)$ by the quotient naively, we get a sheaf (after sheafifying). So, $P \in Shv(\nu_\tau)$.
- Note that since direct limit is exact, we have for a field $F \in F_k$, this new definition of $P(F)$ agrees with the old one since $\mathbb{Z}(\mathbb{G}_m)(F) = \mathbb{Z}(F^* - \{1\})$. Again by (see 6), the inclusion as generators of $\mathbb{G}_m$ in $\mathbb{Z}(\mathbb{G}_m)$ sending $1 \in \mathbb{G}_m$ to 0 is a morphism of pointed sheaves. With our new definition of P we get a map of pointed sheaves $\mathbb{G}_m \to P$ obtained by composing $\mathbb{G}_m \to \mathbb{Z}(\mathbb{G}_m) \to P$.
- Suppose that $P \in Shv(\nu_\tau)$ is strongly A^1-invariant, then by theorem (see 3.3.6), we

have the following commutative diagram:

$$\begin{tikzcd} \mathbb{G}_m(F) = F^* \arrow[d, "\sigma"'] \arrow[dr] & \\ K_1^{MW}(F) \arrow[r] & P(F) \end{tikzcd}$$

But for quadratically closed fields of characteristic 0, we have by theorem (see 4.0.1), a contradiction. So, we have the following theorem.

Theorem 4.0.3. *For k, quadratically closed of characteristic 0, P can not be extended (naively) as a strongly A^1-invariant sheaf of abelian groups on Sm_k.*

Corollary 4.0.4. *In the naive way as before, $W(F) := F^* \wedge F^*$ can not be extended as a strongly A^1-invariant sheaf of abelian groups on Sm_k when k is of characteristic 0 .*

Proof. Suppose it were, again as in case of $P(F)$, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{G}_m^{\wedge 2}(F) = (F^*)^{\wedge 2} \arrow[d, "\sigma"'] \arrow[dr, "(a,b)\mapsto a\wedge b"] & \\ K_2^{MW}(F) \arrow[r, "{[a][b]\mapsto a\wedge b}"'] & W(F) = F^* \wedge F^* \end{tikzcd}$$

Note that our map from $\mathbb{G}_m^{\wedge 2}$ is defined as $a \wedge 1 = 0 = 1 \wedge a \in F^* \wedge F^*; \forall a \in F^*$. $0 = [a][1-a] \mapsto a \wedge (1-a)$

But then again as in theorem (see 4.0.1) for k of characteristics 0, $3 \wedge (-2) \neq 0 \in F^* \wedge F^*$. A contradiction. $\square$

5 Results from A^1-homotopy theory

In this section, we collect results from A^1-homotopy theory that has been cited several times in the previous sections. Everywhere in the section, k is a perfect field and Sm_k denotes the category of smooth k-schemes of finite type. Let ν denote this category. For any scheme X and $x \in X$, $(O_{X,x}, m_x)$ denotes the local ring at x with residue field $k(x)$. In case the point is of codimension 1 in X, we will occasionally denote the residue field by $k(v)$ where v is the discrete valuation at x on $K := K(X)$, the function field of X.

5.1 Preliminaries

Definition 5.1.1 (Nisnevich Topology on Sm_k)**.** Let $\{U_\alpha \to X\}_\alpha$ be a finite family of etale morphisms in ν. It is called a Nisnevich covering if for each point $x \in X$, there exists some $y_\alpha \in U_\alpha$ such that $f_\alpha(y_\alpha) = x$ and $k(y_\alpha) \cong k(x)$ via f_α. The Grothendieck topology generated by this covering is called the Nisnevich topology on ν.

Remark 5.1.2.

- A covering $\{U_\alpha \to X\}_\alpha$ is called an etale covering if X is the union of the images of U_α's and the f_α's are etale.
- It is called a Zariski covering if f_α's are open immersions and their images cover X.
- Let $Preshv(\nu)$ be the category of presheaves of sets on Sm_k. Let τ denote any one of three topologies Nis, Et, Zar. We call an F in $Preshv(\nu)$ a sheaf in τ-topology if for any covering family in the τ-topology as in the definition, the set $F(X)$ is the equalizer of the two maps on the right:

$$F(X) \longrightarrow \Pi_\alpha F(U_\alpha) \quad \rightrightarrows \quad \Pi_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

- Denote by $Shv(\nu_\tau)$ be the full subcategory of sheaves in τ-topology in $Preshv(\nu)$. For any $X \in \nu$, we have the element of $Shv(\nu_{Et})$ defined by $Y \mapsto Hom_\nu(Y, X)$.This is a fully faithful embedding. Clearly, from the definition and above remarks, $\nu \subset Shv(\nu_{Et}) \subset Shv(\nu_{Nis}) \subset Shv(\nu_{Zar}) \subset Preshv(\nu)$.

Definition 5.1.3 (Distinguished square)**.** A distinguished square in ν is a cartesian square such that p is an etale morphism and i is an open immersion and $p^{-1}(X-U) \to (X-U)$ is an isomorphism when both are considered with reduced structures.

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd}$$

Lemma 5.1.4. *Let $F \in Preshv(\nu)$. Then, $F \in Shv(\nu_{Nis})$ iff for any distinguished square as above the map $F(X) \to F(U) \times_{F(W)} F(V)$ is bijective, i.e., the following square is cartesian:*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(W) \end{tikzcd}$$

Lemma 5.1.5 (Associated sheaf)**.** *For any τ, we have a left adjoint $a_\tau : Preshv(\nu) \to Shv(\nu_\tau)$ to the inclusion, i.e, $Hom_{Shv(\nu_\tau)}(a_\tau(F), G)$ is in a natural bijection with $Hom_{Preshv(\nu)}(F, G)$. This sheaf is obtained by the usual sheafification of Grothendieck topologies.*

Definition 5.1.6 (Simplicial Presheaves)**.** Let Δ denote the category of ordered set $[n] := \{0, 1, ..., n\}$ and order-preserving set maps. We have two distinguished maps $d^i : [n-1] \to [n]$ (forgetting i) and $s^i : [n+1] \to [n]$ (repeating i). Denote by $\Delta^{op}Shv(\nu_\tau)$ the category of functors from $\Delta^{op} \to Shv(\nu_\tau)$. This is called the category of simplicial sheaves on ν_τ.

Remark 5.1.7.

- If $S \in Shv(\nu_\tau)$ (for example $\mathbb{G}_m$), then it is seen as an element of $\Delta^{op}Shv(\nu_\tau)$ by treatig it as a simplex whose every degree is the constant sheaf S and all morphisms are identity.
- For any set E, the presheaf that assigns to each $X \in Sm_k$, the set E is denoted by E. This is also a sheaf in Et and Nis. This gives a functor $Set \to Shv(\nu_\tau)$ which extends to a functor $\Delta^{op}Set \to \Delta^{op}Shv(\nu_\tau)$. This is a fully faithful embedding. For a simplicial set K, denote its associated simplicial sheaf by K.

- For each $n \geq 0$ let Δ^n be the standard simplex. $S^1 := \Delta^1/\Delta^0$, where the quotient takes place in $\Delta^{op}Set$.
- Similarly, for $X, Y \in \Delta^{op}Shv(\nu_\tau)$, $X \vee Y$ and $X \wedge Y$ makes sense. (Wedge, like quotient, is the colimit of certain diagrams which exist in $\Delta^{op}Shv(\nu_\tau)$.
- $\Sigma(X) := X \wedge S^1$ is called the suspension.

Definition 5.1.8 (Points in $Shv(\nu_\tau)$)**.** For $\tau = Nis$ or Zar, a τ-point x is a morphism $x : Spec(K) \to X$ in $Sm_k = \nu$, where residue field of the image of $Spec(K)$ is K.

Definition 5.1.9 (Neighbourhoods)**.** For a τ-point $x \in X$, the neighbourhood of x ($Neib^x_\tau$) is the category of pairs $f : U \to X; x : Spec(K) \to U$ such that f is etale and U is irreducible and we have some y, τ-point of U with same residue field K which lifts x.

Remark 5.1.10.
- For $x : Spec(K) \to X$, a τ-point in ν. The fibre of any $F \in Preshv(\nu)$ at x is defined to be $F_x := \varinjlim_{(U \to X, y) \in Neib^x_\tau} F(U)$. For example, for the affine line $\mathbb{A}^1$, the fibre is $O_{X,x}$.
- The canonical map $F_x \to a_\tau(F)_x$ is a bijection.
- A morphism in $Shv(\nu_\tau)$ is an isomorphism iff it induces bijection at the fibres.

Definition 5.1.11 (A^1-invariance)**.** For an $S \in \nu_{Nis}$, we have the following definitions:

- A^1-invariant if for any $X \in \nu$, the map $S(X) \to S(X \times \mathbb{A}^1)$ induced by projection, is a bijection.
- Let it be a sheaf of groups. It is called strongly A^1-invariant if for $i = 0, 1$, $H^i_{Nis}(X; S) \to H^i_{Nis}(X \times \mathbb{A}^1; S)$, induced by projection, is a bijection.

5.2 Unramified Sheaf of sets

Definition 5.2.1. An unramified presheaf of sets S on Sm_k is a presheaf of sets such that:

1. If $X \in Sm_k$ has irreducible components X_α, then the induced map $S(X) \to \Pi_\alpha S(X_\alpha)$ is bijective.
2. If U is an open subscheme of $X \in Sm_k$ that is dense in each irreducible component of X, then $S(X) \to S(U)$ is injective.

3. For any irreducible $X \in Sm_k$ and $x \in X$, define $S(O_{X,x}) := \varinjlim_{x \in U, U \in Nb^x_{Zar}} S(U)$, $S(F) := S(O_{X,x_0})$, where x_0 is the generic point of X. Then, by (2), $S(X) \subset S(O_{X,x}) \subset S(F), \forall x \in X^{(1)}$. We demand that the map $S(X) \to \cap_{x \in X^{(1)}} \subset S(F)$ is a bijection.

Proposition 5.2.2. *Any unramified presheaf as above is automatically a Zariski sheaf.*

Proof. Replacing each term in the diagram and using (3) from the above definition we need to show the following diagram is exact: $\cap_{x \in X^{(1)}} S(O_{X,x}) \to \Pi_\alpha \cap_{x \in U_\alpha^{(1)}} S(O_{U_\alpha,x}) \rightrightarrows \Pi_{\alpha,\beta} \cap_{x \in U_{\alpha,\beta}^{(1)}} S(O_{U_{\alpha,\beta},x})$. But $x \in X^{(1)}$ iff $x \in U_\alpha^{(1)}$ for some α iff $x \in U_{\alpha,\beta}^{(1)}$ for some β as $U_{\alpha,\beta} = U_\alpha \times_X U_\beta = U_\alpha \cap U_\beta$ and $O_{X,x} = O_{U_\alpha,x} = O_{U_{\alpha,\beta},x}$ imply the required exactness. □

Remark 5.2.3.
- It is true that for any X essentially smooth over k (see 2) and irreducible with function field F, condition (3) in the above definition holds.
- As we will see next, any strictly A^1-invariant sheaf on Sm_k is unramified; and any strongly A^1-invariant sheaf is strictly A^1-invariant.
- Some examples of unramified sheaves that existed well before this definition are Rost's cycle modules (See 2) and the sheaf associated to the Witt groups (See 2).

Proposition 5.2.4. *Strictly (Strongly) A^1-invariant $\implies$ Unramfified*

Definition 5.2.5 (Unramified $\tilde{F_k}$-datum)**.** We have the following data:

(D1) A continuous functor $S : F_k \to Set$. By continuous, we mean $S(F)$ is the direct limit of $S(F_\alpha)$'s where F_α's run over subfields of F of finite type (finitely generated) over k.

(D2) For any F and any discrete valuation v on F, a subset $S(O_v) \subset S(F)$

Satisfying the following axioms:

(A1) If $i : E \subset F$ is a separable extension in F_k and v, a discrete valuation on F that restricts to a discrete valuation w on E with ramification index 1, then $S(i)$ maps $S(O_w)$ into $S(O_v)$. Moreover, if the induced extension $\bar{i} : k(w) \to k(v)$ is an isomorphism, then the following square is cartesian:

$$\begin{tikzcd} S(O_w) \arrow[r] \arrow[d, hook] & S(O_v) \arrow[d, hook, "p"] \\ S(E) \arrow[r, "S(i)"'] & S(F) \end{tikzcd}$$

(A2) Let $X \in Sm_k$ irreducible with function field F. If $x \in S(F)$, then x lies in all but a finite number of $S(O_x)$'s, where x runs over the set $X^{(1)}$.

Theorem 5.2.6. *The category of unramified sheaves on $\tilde{Sm_k}$ is equivalent to the category of unramfied $\tilde{F_k}$-datum.*

Proof. As seen in the definition, given an unramified sheaf S on $\tilde{Sm_k}$, we can take a smooth model in Sm_k for $F \in \tilde{F_k}$ and evaluate S at it as in definition (see 5.2.1). If v is a discrete valuation of F, there exists $X \in Sm_k, x \in X^{(1)}$ such that function field of X is F and v comes from x. Now, as argued in definition, by using (2), we have $S(O_v) \subset S(F)$. To prove (A1), we have models X and Y in Sm_k irreducible with function fields E and F respectively and a smooth map $f : X \to Y$ mapping the generic point of X to that of Y. This induces the map $S(E) \to S(F)$. We can modify this such that the point $x \in X$ giving w maps to the point $y \in Y$ giving v via f. Then, for any open subscheme U of Y containing y, we have the square:

$$\begin{tikzcd} S(U) \arrow[r] \arrow[d] & S(f^{-1}(U)) \arrow[d] \\ S(E) \arrow[r] & S(F) \end{tikzcd}$$

Taking the colimit over this diagram, we get $S(O_w)$ maps into $S(O_v)$. Now, the following is an elementary distinguished square over $Spec(O_w)$:

$$\begin{tikzcd} Spec(F) \arrow[r] \arrow[d] & Spec(O_v) \arrow[d] \\ Spec(E) \arrow[r] & Spec(O_w) \end{tikzcd}$$

which is the colimit of the following diagram where V is a smooth model for $Spec(F)$ and X is a smooth model for $Spec(E)$.

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd}$$

By 5.1.4, the required square in (A1) is cartesian. To prove (A2), note that any $s \in S(F)$ by definition, comes from an element in $S(U)$, where U is an open subscheme of X. But as X is irreducible, $U^c \cap X^{(1)}$ is a finite set. So, the element s lies in all $S(O_{X,x}), x \in U$ and the rest of the x's are finite.

For the reverse map, given any $\tilde{F}_k$-datum S, and any $X \in Sm_k$, define $S(X) := \cap_{x \in X^{(1)}} S(O_{X,x}) \subset S(F)$. Extend it to all $X \in Sm_k$ such that (1) is true in definition. Now given a smooth morphism $f : Y \to X$, we can assume Y and X are irreducible with function field E and F respectively. Since image of f is open, we can assume it is dominant. So, if $x \in X^1$, $f^{-1}(x)$ has finitely many irreducible components and the generic points of those components are of codimension 1 in Y. Using (A1) and the definition of $S(X)$, we have the desired map $S(f)$. From second part of (A1) and (see 5.1.4), this gives a sheaf in the Nisnevich topology and is inverse to the earlier functor. □

Definition 5.2.7 (Unramified F_k-datum)**.** It is an unramified $\tilde{F}_k$-datum along with: (D3) For any $F \in F_k$ and a discrete valuation v on F, a map $s_v : S(O_v) \to S(k(v))$, called the specialization map associated to v, such that the following axioms are satisfied:

(A3)(i) If $i : E \subset F$ is an extension in F_k, v, a discrete valuation on F that restricts to a discrete valuation w on E, then $S(i)$ maps $S(O_w)$ to $S(O_v)$ and the following diagram is commutative:

$$\begin{tikzcd} S(O_w) \arrow[r] \arrow[d, hook] & S(O_v) \arrow[d, hook] \\ S(E) \arrow[r] & S(F) \end{tikzcd}$$

(A3)(ii) If v as above restricts to 0 on E, then $Image(S(i)) \subset S(O_v)$. Here, $j : E \subset k(v)$ is a field extension. We demand that $S(E) \to S(O_v) \xrightarrow{s_v} S(k(v))$ is equal to $S(j)$.

(A4)(i) For any X, essentially smooth scheme, local of dimension 2 with closed point $z \in X^{(2)}$, and for any point $y_0 \in X^{(1)}$ with $\bar{y_0}$ essentially smooth scheme, then $s_{y_0} : S(O_{y_0}) \to S(k(y_0))$ maps $\cap_{y \in X^{(1)}} S(O_y)$ into $S(O_{\bar{y_0},z}) \subset S(k(y_0))$.

(A4)(ii) The composition $\cap_{y \in X^{(1)}} S(O_y) \to S(O_{\bar{y_0},z}) \to S(k(z))$, doesn't depend on the choice of y_0 such that y_0 is essentially smooth over k.

Theorem 5.2.8. *The category of unramified sheaves on Sm_k is equivalent to the category of unramfied F_k-datum.*

Proof. Given an unramified sheaf S on Sm_k, we have unramified $\tilde{F_k}$-data. If v is a discrete valuation on $F \in F_k$ with residue field $k(v)$ separable over k, then by choosing smooth models for the closed immersion $Spec(k(v)) \to Spec(O_v)$, we get the specialisation map s_v. We have a smooth X and x a codimension 1 point in X with closure Z. As $k(v)$ is separable over k, we may assume Z is smooth. As $S(O_v)$ is the direct limit of $S(U)$ over all U open (affine say) neighbourhoods U of x iin X, taking the direct limit over $S(U) \to S(U \cap Z)$, we get a map $S(O_v) \to S(k(v))$. Note that $k(v)$ is the function field of Z. To check (A3)(i), we can assume X and Y are irreducible and $f : X \to Y$ a smooth map mapping generic points to each other and v to w. So, (A3)(i) follows as we have the commutative square:

$$\begin{array}{ccc} \bar{v} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{w} & \longrightarrow & Y \end{array}$$

(A3)(ii) follows by choosing similar models where v maps to the generic point of Y (as the vaulation restricts to 0).

To show (A4)(i), note that we have:

$$\begin{array}{ccc} \bar{y_0} \cap U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \bar{y_0} & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} S(X) & \longrightarrow & S(O_{y_0}) \\ \downarrow & & \downarrow \\ S(\bar{y_0}) & \longrightarrow & S(k(y_0)) \end{array}$$

Replacing X by any open subscheme U containing z, and $\bar{y_0}$ by $U \cap \bar{y_0}$, we get:

$$\begin{tikzcd}
S(O_z) \arrow[r] \arrow[d] & S(O_{y_0}) \arrow[d] \\
S(O_{\bar{y_0},z}) \arrow[r] & S(k(y_0))
\end{tikzcd}$$

So, we get (A4)(i). For the second part, we have:

$$\begin{tikzcd}
\bar{z} \cap U \arrow[r] \arrow[d] & \bar{y_0} \cap U \arrow[r] \arrow[d] & U \arrow[d] \\
\bar{z} \arrow[r] & \bar{y_0} \arrow[r] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
S(X) \arrow[r] \arrow[d] & S(\bar{y_0}) \arrow[r] \arrow[d] & S(\bar{z}) \arrow[d] \\
S(U) \arrow[r] & S(\bar{y_0} \cap U) \arrow[r] & S(k(z))
\end{tikzcd}$$

Now, (A4)(ii) follows as every open set containing z contains all codimension 1 points y such that $z \in \bar{y}$.

Lemma 5.2.9. *Given an unramified F_k-datum S, there is a unique way to extend the unramified sheaf of sets $S : \tilde{Sm}_k^{op} \to Set$ to a sheaf $S : (Sm_k)^{op} \to Set$ such that for any discrete valuation v on $F \in F_k$ with separable residue field, the map $S(O_v) \to S(k(v))$ induced by the sheaf structure is the specialization map $s_v : S(O_v) \to S(k(v))$. This sheaf is automatically unramified.*

Proof. We first define a restriction map $s(i) : S(X) \to S(Y)$ for a closed immersion $i : Y \to X$ in Sm_k of codimension 1. If $Y = \coprod_\alpha Y_\alpha$ be the decomposition of Y into irreducible components. Then, $S(Y) = \Pi_\alpha S(Y_\alpha)$ and $s(i)$ is the product of $s(i_\alpha) : S(X) \to S(Y_\alpha)$. Hence, we may assume without loss of generality that Y and X (as image of irreducible is irreducible) are irreducible. Now, we show the existence of $s(i) : S(X) \to S(Y)$ such that the following diagram commutes where $y \in Y$ is the generic point of Y.

$$\begin{tikzcd}
S(X) \arrow[r, "s(i)"] \arrow[d, hook] & S(Y) \arrow[d, hook] \\
S(O_{X,y}) \arrow[r, "s_y"'] & S(k(y))
\end{tikzcd}$$

If such a map exists, then by commutativity of the previous diagram and by definition of unramified sheaves, s_y will map $S(X)$ inside $S(O_{Y,z}); \forall z \in Y^{(1)}$. So, to get the above map it is sufficient to prove that for any $z \in Y^{(1)}$, the image of $S(X)$ through s_y is contained in $S(O_{Y,z})$. Note that z has codimension 2 in X; so by (A4)(i), s_y maps $\cap_{x\in X^{(1)}} S(O_{X,x}) \subset \cap_{y\in X_z^{(1)}} S(O_y)$ into $S(O_{Y,z})$.

Lemma 5.2.10. *Suppose $i : Z \to X$ is a closed immersion in Sm_k of codimension $d > 0$. Suppose there is a factorisation of i into a composition of codimension 1 closed immersions, with Y_i closed subschemes of X and each smooth over k:*

$$Z \xrightarrow{j_1} Y_1 \xrightarrow{j_2} Y_2 \longrightarrow \cdots \xrightarrow{j_d} Y_d = X$$

Then, upon applying S, the composition doesnt depend on the choice of the above factorisation of i:

$$S(X) \xrightarrow{S(j_d)} \cdots \longrightarrow S(Y_2) \xrightarrow{S(j_2)} S(Y_1) \xrightarrow{S(j_1)} S(Z)$$

Denote this composition by $S(i)$.

Proof. We prove this by induction on d. For $d = 1$, the claim is obvious. So, let $d \geq 2$. As in the arguments earlier, since S is unramified, we can reduce to the case when Z is irreducible with generic point z. Similarly, we can also assume X is irreducible in Sm_k. Now, by the commutativity of the following diagram and the fact that $S(X) \to S(U)$ is injective for U open subscheme of X irreducible in Sm_k, we can reduce to the case of an open subscheme of X containing z, in particular, $Spec(O_{X,z})$:

$$\begin{tikzcd}
S(X) \arrow[r, "S(j_d)"] \arrow[d, hook] & \cdots \arrow[r] & S(Y_2) \arrow[r, "S(j_2)"] \arrow[d, hook] & S(Y_1) \arrow[r, "S(j_1)"] \arrow[d, hook] & S(Z) \arrow[d, hook] \\
S(U) \arrow[r] & \cdots \arrow[r] & S(Y_2 \cap U) \arrow[r] & S(Y_1 \cap U) \arrow[r] & S(Z \cap U)
\end{tikzcd}$$

Note that as $k(z)$ is separable over k (as z is the generic point of $Z \in Sm_k$, which corresponds to $Spec(k(z))$), by (A4), $S(X) = \cap_{y\in X^{(1)}} S(O_y) \to S(O_{\bar{y_0},z}) \to S(k(z))$ is independent of the choice of y_0. This proves the claim for $d = 2$.

Now, for the open set around z, $Spec(O_{X,z})$, we know that as Z is irreducible smooth over k, $O_{X,z}$ is a regular local ring of dimension d as Z is of codimension d in X. So, there exists a sequence of elements $(x_1, ..., x_d) \in m_{X,z} \leq O_{X,z}$. The following flag induces $Z \cap U \to Y_1 \cap U \to Y_2 \cap U \to ... \to X \cap U$:

$$Spec(A/(x_1, ..., x_d)) \longrightarrow Spec(A/(x_2, ..., x_d)) \longrightarrow \cdots \longrightarrow Spec(A/(x_d)) \longrightarrow Spec(A)$$

where $A := O_{X,z}$. After S acts, we have to show that the resulting sequence is independent of the choice of generators $(x_1, ..., x_d)$. Each such choice of parameters comes from a $k(z)$-vector space $m_{X,z}/m^2_{X,z}$. Any two bases differ by an element of $Gl_d(k(z))$ which lifts to a matrix $M \in M_d(A)$. If we permute x_i and x_{i+1}, by case $d = 2$, the composition $S(A) \to S(k(z))$ doesn't change after permutation.
Since A is local, the lift $M \in Gl_d(A)$. Multiplying by a unit of A to some element x_i doesn't change the flag. So, without loss of generality, we may assume that $det(M) = 1$. Since A is local, $Sl_d(A) = E_d(A)$; so M splits as a product of elementary matrices in A. Since we have handled permutations, we just need to show that the sequence $(x_1 + ax_2, x_2, ..., x_n)$ induces the same map $S(A) \to S(k(z))$. But this is trivial as both sequences induce the same flag. □

Now, let $i : Z \to X$ be a closed immersion in Sm_k. As above, X can be covered by open sets U such that the induced closed immersion $U \cap Z \to U$ admits a factorization as in the previous lemma. We have $s_U : S(U) \to S(U \cap Z)$. Applying the previous lemma to the intersection of these U's, we get that the s_U's are compatible. So, we get $S(i) : S(X) \to S(Z)$. Now, for any $f \in Hom_{Sm_k}(Y, X)$, it can be factored as $Y \to Y \times_k X \to X$ where the first map is a closed immersion and the second map is a smooth projection. Applying S, we get $S(f) : S(X) \to S(Y \times_k X) \to S(Y)$. If we have a smooth morphism $\pi : X' \to X$ and closed immersion $i : Z \to X$ in Sm_k. Let $p_{X'} : Z \times_X X' \to X'$ and $p_Z : Z \times_X X' \to Z$. Then, the following diagram is commutative:

$$\begin{tikzcd}
S(X) \arrow[r, "S(\pi)"] \arrow[d, "S(i)"'] & S(X') \arrow[d, "S(p_{X'})"] \\
S(Z) \arrow[r, "S(p_z)"'] & S(Z \times_X X')
\end{tikzcd}$$

Note that we can reduce to the case using the proof of previous lemma that the closed immersion is of codimension 1 and both X and Z are irreducible. But then the commutativity of the diagram follows from (A3)(i). To prove the functoriality in Sm_k, let $Z \to Y \to X$ in Sm_k. We have the commutative diagram:

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] & Z \times_k Y \arrow[r, hook] \arrow[d] & Z \times_k Y \times_k X \arrow[d] \\
Z \arrow[r] \arrow[d] & Y \arrow[r, hook] \arrow[d] & Y \times_k X \arrow[d] \\
Z \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

Applying S gives us a commutative diagram. It is unramified as a presheaf on Sm_k as the definitions depend on its restriction to $\tilde{Sm}_k$. □

□

Corollary 5.2.11. *Let S and G be sheaves of sets on Sm_k with S unramified and G satisfying the first two properties of definition (see 5.2.1); then to give a morphism ϕ between G and S is equivalent to giving a natural transformation $G|_{F_k} \to S|_{F_k}$ such that:*

- *for any discrete valuation v on $F \in F_k$, the image of $G(O_v)$ under ϕ is contained in $S(O_v)$.*
- *If the residue field is separable over k, then the induced square commutes:*

$$\begin{tikzcd}
G(O_v) \arrow[r, "s_v"] \arrow[d, "\phi"'] & G(k(v)) \arrow[d, "\phi"] \\
S(O_v) \arrow[r] & S(k(v))
\end{tikzcd}$$

Proof. If we have a morphism of sheaves over Sm_k, these two properties are clear. Conversely, suppose we have the two properties, then $G(X) \to S(X)$ exists by definition of G and S on Sm_k and the first property. By the second property, if $Z \to X$ is a closed immersion of codimension 1, we have:

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d] & G(Z) \arrow[d] \\ S(X) \arrow[r] & S(Z) \end{tikzcd}$$

Now, following the proof of the previous two lemmas, we have by S being unramified, a morphism of sheaves. $\square$

5.3 Unramfied Sheaf of groups

Definition 5.3.1. Let G be an unramified sheaf of groups on Sm_k (or $\tilde{Sm}_k$). For any discrete valuation v on $F \in F_k$, let $H^1_v(O_v; G) := G(F)/G(O_v)$, a left $G(F)$-set pointed by $G(O_v)$. Generalizing this let $y \in X^{(1)}, X \in Sm_k$, $H^1_y(X; G) := H^1_y(O_{X,y}; G)$. Define the weak product $\Pi'_{y\in X^{(1)}} H^1_y(X; G) \subset \Pi_{y\in X^{(1)}} H^1_y(X; G)$ by the set of "tuples" in $\Pi_{y\in X^{(1)}} H^1_y(X; G)$ such that all but finitely many of the coordinates are the base point of $H^1_y(X; G)$. By the axiom (A2) of unramified datum on F_k, if X is irreducible with function field F, the induced action of $G(F)$ on $\Pi_{y\in X^{(1)}} H^1_y(X; G)$ preserves the weak product. Clearly, the isotropy group of this action of $G(F)$ on the base point of the weak product is $G(X) = \cap_{y\in X^{(1)}} H^1_v(X; G)$.

Definition 5.3.2. Let $1 \to H \subset G \rightrightarrows E \to F$ be a sequence with G acting on a set E (double arrow denote the left action) which is pointed as a set; H is a subgroup of G and the map $E \to F$ is a G-equivariant map of sets where F is endowed with a trivial action. This sequence is called exact if the isotropy group of the base point of E is H and the kernel of the pointed map (pre-image of the image of the base point of E) between E and F is equal to the orbit under G of the base point of E.

We say it is exact in the strong sense if moreover the map $E \to F$ induces an injection into F of the left quotient set $G\ E$. Hence, in this language the following sequence is exact:
$1 \to G(X) \to G(F) \rightrightarrows \Pi'_{y\in X^{(1)}} H^1_y(X; G)$.

Definition 5.3.3. For any point $z \in X^{(2)}; X \in Sm_k$, $H^2_z(X; G) :=$ Orbit of the weak product under the left action of $G(F)$, where $F \in F_k$ is the function field of $X_z := Spec(O_{X,z})$. (Note that this is the function field of if X is irreducible). Hence, for X essentially smooth with function field F we have a $G(F)$-equivariant map $\Pi'_{y\in X^{(1)}} H^1_y(X;G) \to \Pi'_{y\in X_z^{(1)}} H^1_y(X;G) \to H^2_z(X;G)$. So, we have a $G(F)$-equivariant map $\Pi'_{y\in X^{(1)}} H^1_y(X;G) \to \Pi_{z\in X^{(2)}} H^2_z(X;G)$ and it's not clear whether the image of the weak product in LHS lies in the weak product contained in the RHS. So, we impose that axiom:

(A2') For any irreducible essentially smooth k-scheme X, the image of the above map is contained in the weak product.

So, we have the following diagram; a complex $C^*(X;G)$ of groups, actions and pointed sets:

$$1 \longrightarrow G(X) \hookrightarrow G(F) \quad \rightrightarrows \quad \Pi'_{y\in X^{(1)}} H^1_y(X;G) \longrightarrow \Pi_{z\in X^{(1)}} H^2_z(X;G)$$

Define for $X \in Sm_k$:

- $G^{(0)}(X) := \Pi'_{x\in X^{(0)}} G(k(x))$
- $G^{(1)}(X) := \Pi'_{x\in X^{(1)}} H^1_y(X;G)$
- $G^{(2)}(X) := \Pi'_{x\in X^{(2)}} H^2_z(X;G)$

Lemma 5.3.4. *The presheaf $X \mapsto G^{(i)}(X); i \leq 2$ can be extended to an unramified presheaf of groups on $\tilde{Sm}_k$. So, they are Zariski sheaves. However, $G^{(0)}$ is also a Nisnevich sheaf.*

Definition 5.3.5. We add two more axioms for G which will aid in showing strong A^1-invariance.

(A5)(i) For separable field extension $E \subset F$ in F_k and any discrete valuation v on F, restricting to w on E with ramification index 1 and such that $\bar{i} : k(w) \to k(v)$ is an isomorphism, then commutative square of groups induces a bijection $H^1_v(O_v;G) \cong H^1_w(O_w;G)$:

$$\begin{tikzcd} G(O_w) \arrow[r, hook] \arrow[d] & G(E) \arrow[d] \\ G(O_v) \arrow[r, hook] & G(F) \end{tikzcd}$$

(A5)(ii) For any etale morphism $X^{'} \to X$ between smooth local k-schemes of dimension 2, with closed point z and $z^{'}$ respectively, inducing an isomorphism on the residue fields $k(z) \cong k(z^{'})$, then the pointed map $H^2_z(X;G) \to H^2_{z^{'}}(X^{'};G)$ has trivial kernel.

(A6) For any localisation $U := X_u$ of a smooth k-scheme at some point u of codimension ≤ 1, the following complex is exact:

$$1 \longrightarrow G(\mathbb{A}^1_U) \hookrightarrow G^{(0)}(\mathbb{A}^1_U) \rightrightarrows G^{(1)}(\mathbb{A}^1_U) \longrightarrow G^{(2)}(\mathbb{A}^1_U)$$

and the morphism $G(U) \to G(\mathbb{A}^1_U)$ is an isomorphism.

Theorem 5.3.6 (Strong A^1-invariance)**.** *Let G be an unramified sheaf of groups on Sm_k that satisfies (A2'), (A5) and (A6). Then, it is strongly A^1-invariant.*

Next, we add some axioms which will imply axioms (A4) in some particular cases of $\tilde{F}_k$-data.

Definition 5.3.7. Let $M_* : F_k \to Ab_*$ be a functor to the category of $\mathbb{Z}$-graded abelian groups. We assume the following data (D4) and axioms:

(D4)(i) For any $F \in F_k$, a $\mathbb{Z}[F^*/(F^*)^2]$-module structure on $M_*(F)$, denoted by $(u, \alpha) \mapsto <u> \alpha \in M_n(F), u \in F^*, \alpha \in M_n(F)$ and it is functorial in F_k.

(D4)(ii) For any $F \in F_k, n \in \mathbb{Z}$, a map $F^* \times M_{n-1}(F) \to M_n(F) : (u, \alpha) \mapsto [u]\alpha$ functorial in F_k.

(D4)(iii) For any discrete valuation v on $F \in F_k$ and uniformizing parameter π, a graded epimorphism of degree (-1): $\partial^{\pi}_v : M_*(F) \to M_{*-1}(k(v))$ which is also functorial with respect to $E \subset F$ such that v restricts to a discrete valuation on E of ramification index 1, choosing π in E.

(B0) For $(u, v) \in (F^*)^2, \alpha \in M_n(F)$, we have $[uv]\alpha = [u]\alpha + <u> [v]\alpha; [u][v]\alpha = -<-1> [v][u]\alpha$.

(B1) For a k-smooth integral domain A with field of fractions F, for any $\alpha \in M_n(F)$, then for all but finitely many points $x \in Spec(A)^{(1)}$, we have for any uniformizing parameter π for x, $\partial^{\pi}_x(\alpha) \neq 0$.

(B2) For any discrete valuation v on $F \in F_k$ with uniformizing parameter π, one has $\partial_v^\pi([u]\alpha) = [\bar{u}]\partial_v^\pi(\alpha) \in M_n(k(v)); \partial_v^\pi(< u > \alpha) =< \bar{u} > \partial_v^\pi(\alpha) \in M_{(n-1)}(k(v))$, for $u \in (O_v)^*, \alpha \in M_n(F)$.

(B3) For field extension $E \subset F \in F_k$ and for any discrete valuation v that restricts to w on E, with ramification index e, let $\pi \in O_v$ be a uniformizing parameter for v and $\rho \in O_w$ be a uniformizing parameter for w. That is $\rho = u\pi^e, u \in (O_v)^*$. Then, one has for $\alpha \in M_*(E), \partial_v^\pi(\alpha|_F) = e_\epsilon < \bar{u} > (\partial_w^\rho(\alpha))|_{k(v)} \in M_*(k(v))$, where $n \in \mathbb{Z}; n_\epsilon = \Sigma_{i=1}^n < (-1)^{(i-1)} >$. Note that from this as in (See ref), the kernel of the surjective homomorphism ∂_v^π is independent of the choice of uniformizing element π. Denote that kernel by $M_*(O_v) \subset M_*(F)$. Note that now clearly, axiom (A2) of $\tilde{F}_k$-data is equivalent to (B1) here.

Lemma 5.3.8. *If M_* satisfies axioms (B1), (B2) and (B3). Then, it satisfies (in each degree) the axioms for an unramified $\tilde{F}_k$-abelian group datum. Moreover, it satisfied axiom (A5)(i).*

Proof. As observed above, since it satisfies (B3), (B1) implies (A2) of unramified $\tilde{F}_k$-datum, by covering an irreducible $X \in Sm_k$ via finitely many open affine $Spec(A)$'s where the A's are k-smooth integral domains. (D1) and (D2) are clear from the definition of M_*. To prove (A1), let $E \subset F$ be a separable extension in F_k, v, a discrete valuation on F restricting to one on E, say w of ramification index 1. We have by the functoriality in (D4)(iii):

$$\begin{array}{ccc} M_*(E) & \xrightarrow{\partial_w^\pi} & M_{*-1}(k(w)) \\ \downarrow & & \downarrow \\ M_*(F) & \xrightarrow{\partial_v^\pi} & M_{*-1}(k(v)) \end{array}$$

Then, by the commutativity of the diagram, clearly $M_*(O_w) \to M_*(O_v)$. Now, suppose that the induced map $k(w) \to k(v)$ is an isomorphism, then again by the commutativity of the previous diagram, where the right vertical map $M_*(k(w)) \to M_*(k(v))$ is an isomorphism. So, if we have $g \in M_*(E)$ such that its image lies in $M_*(O_v)$, then $g \in M_*(O_w)$. This shows that the following square is cartesian:

$$\begin{array}{ccc} M_*(O_w) & \longrightarrow & M_*(O_v) \\ \big\downarrow & & \big\downarrow \\ M_*(E) & \longrightarrow & M_*(F) \end{array}$$

This proves (A1) and that for each n, M_* gives an unramified $\tilde{F_k}$-datum.

To prove (A5), we simply need to show $H^1_v(O_v; M_*) \cong H^1_w(O_w; M_*)$. Clearly, we have the induced map between these groups by (A1). Again as the previous square is cartesian, the kernel of the map is 0 and it is surjective because of the isomorphism $M_{*-1}(k(w)) \to M_{*-1}(k(v))$. This proves (A5)(i). □

Lemma 5.3.9. *Suppose M_* satisfies (B0), (B1), (B2) and (B3). So, by above lemma each M_n is a sheaf of abelian groups in $\tilde{Sm_k}$. By (See ref), $H^1_v(O_v, M_n) = M_n(F)/M_n(O_v)$ and let ∂_v be the projection from $M_n(F)$ to $H^1_v(O_v, M_n)$. So, choosing an uniformizing parameter π we get an isomorphism $\theta_\pi : M_{(n-1)}(k(v)) \xrightarrow{\cong} H^1_v(O_v, M_n)$ and $\partial_v = \theta_\pi \circ \partial^\pi_v$. Similarly, define $s^\pi_v : M_*(F) \to M_*(k(v)); \alpha \mapsto \partial^\pi_v([\pi]\alpha)$, where v is a discrete valuation on F. Then, s^π_v is independent of the choice of π.*

Proof. From (B0), for any unit $u \in O^*_v$, uniformizing parameter π and $\alpha \in M_n(F)$: $[u\pi]\alpha = [u]\alpha + < u > [\pi]\alpha$. If $\alpha \in M_*(O_v)$, $s^{u\pi}_v(\alpha) = \partial^{u\pi}_v([u\pi]\alpha) = \partial^{u\pi}_v([u]\alpha) + \partial^{u\pi}_v(< u > [\pi]\alpha) = \partial^{u\pi}_v(< u > [\pi]\alpha)$ as by (B2), $\partial^{u\pi}_v([u]\alpha) = [\bar{u}]\partial^{u\pi}_v(\alpha) = [\bar{u}]0 = 0$. (B2) also implies, $\partial^{u\pi}_v(< u > [\pi]\alpha) =< \bar{u} > \partial^{u\pi}_v([\pi]\alpha)$. By (B3), the RHS is equal to $< \bar{u} >< \bar{u} > \partial^{u\pi}_v([\pi]\alpha) = \partial^{u\pi}_v([\pi]\alpha)$. So, the claim is proven. □

Lemma 5.3.10. *Denote by s_v this map which is independent of π. So, M_* has datum (D3).*

We demand two more axioms:

(HA)(i) For any $F \in F_k$, the following is a short exact sequence:

$$0 \longrightarrow M_*(F) \longrightarrow M_*(F(T)) \xrightarrow{\sum \partial^P_{(P)}} \bigoplus_{P \in \mathbb{A}^1_F} M_{*-1}(F[T]/P) \longrightarrow 0$$

where P runs over all monic irreducibles in $F[T]$.

(HA)(ii) For any $\alpha \in M_*(F)$, $\partial^T_{(T)}([T]\alpha|_{F(T)}) = \alpha$. Note that this implies $M_*(F) \to M_*(\mathbb{A}^1_F)$ is an isomorphism and $H^1_{Zar}(\mathbb{A}^1_F; M) = 0$.

Suppose M_* satisfies (B0), (B1), (B2), (B3), (HA)(i) and (HA)(ii), then (A1)(ii) (second part), (A3)(i) and (A3)(ii) hold.

Proof. For the second part of (A1)(ii), let π be a uniformizing parameter of O_w which is also a uniformizing parameter of O_v (as the ramification index is 1). By (D4)(iii), the following is a commutative diagram:

$$\begin{tikzcd}
M_*(F) \arrow[r, "\partial^\pi_v"] & M_{*-1}(k(v)) \\
M_*(E) \arrow[u] \arrow[r, "\partial^\pi_w"'] & M_{*-1}(k(w)) \arrow[u]
\end{tikzcd}$$

By (D4)(i), the morphism $M_*(E) \to M_*(F)$ preserves the product by π. For (A3)(i), $E \subset O_v \subset F$. Let π be a uniformizing parameter of v. Consider the extension $E(T) \subset F; T \mapsto \pi$. The restriction of v is the valuation defined by T on $E[T]$, with ramification index 1. So, we can reduce to the case $E \subset F; v = (T)$ and our claim follows from (HA)(i) and (HA)(ii). $\square$

Theorem 5.3.11. *Let M_* be a functor $F_k \to Ab_*$ with data (D4)(i), (D4)(ii), (D4)(iii) satisfying the axioms (B0), (B1), (B2), (B3), (HA)(i), (HA)(ii), (B4) and (B5). Then, for each $n \in \mathbb{Z}$ with the s_v's , M_n is an unramified F_k-abelian group datum. So, it defines an unramified sheaf of abelian groups on Sm_k. This M_n is also strongly A^1-invariant.*

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