Optimization, Robotics & Biomechanics (ORB)
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Robotics 1 (WS 2018/2019)

Exercise Sheet 9

Presentation during exercises in calendar week 3

Exercise 9.1 – Manipulability Ellipsoids

Anstrengung/Leistung/ Durchsatz

Ellipsoids can be used to analyze or geometrically visualize the direction of 'effort' of a function, i.e. how easily the output changes for different input-direction changes. For that purpose points on a n-dimensional sphere are mapped through the function. The resulting set of points can then be analyzed. In the case of linear mappings the resulting set describes an ellipsoid.

Some examples where ellipsoids are used for analysis purposes include:

- Manipulability ellipsoids (see lecture notes)
- Force ellipsoids (see lecture notes)
- Mapping acceleration spheres $\|\ddot{q}\|_2 = 1$ to torque/mass ellipsoids
- Interpreting covariance matrices as confidence regions

Rough Mathematical Background

An arbitrarily oriented ellipsoid centered at the coordinate origin can be described as

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad A = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$ $\to a = \sqrt{\lambda_1}$

$$\left\{x \in \mathbb{R}^n \mid 1 = x^T A^{-1} x\right\}$$ $\to$ Ellipsengleichung

with a symmetric, positive definite (eigenvalues > 0) matrix $A \in \mathbb{R}^n$. The eigenvectors of A scaled by their corresponding eigenvalues define the semi-axis of the ellipsoid.

Applying a linear mapping $y = Bx$ to all points of the ellipsoid results in another ellipsoid, given by

$$\left\{y \in \mathbb{R}^m \mid 1 = y^T \big(\underbrace{BAB^T}_{\tilde{A}}\big)^{-1} y\right\} \tag{1}$$

Info: More information about the mathematical background of the relationship of eigenvectors, eigenvalues and their interpretation as ellipsoids can be found here:
`https://en.wikipedia.org/wiki/Singular_value_decomposition`

Mapping a unit circle of joint velocities $\|\dot{\theta}\|_2 = 1$ through the linear relation between joint velocities and end-effector velocities $v_e = J(\theta)\dot{\theta}$, we obtain the *manipulability ellipsoid*. From equation (1) we get that the resulting ellipsoids semi-axes are defined by eigenvectors and eigenvalues of JJ^T. It geometrically describes the directions in which the end-effector moves with least effort or with greatest effort.

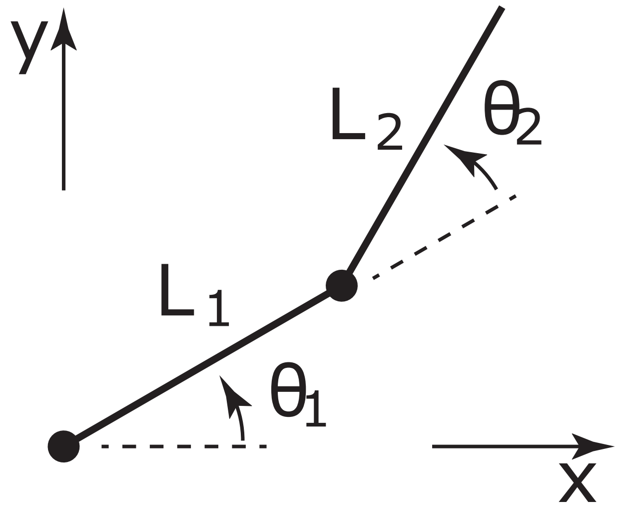


Figure 1: The 2R robot arm.

Revisit the 2 segment arm from exercise sheet 8 given in figure 1. Again, let $L_1 = L_2 = 1$. From the lecture you know, that the Jacobian of this system is

$$J(\theta_1, \theta_2) = \begin{bmatrix} -\sin(\theta_1) - \sin(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \cos(\theta_1) + \cos(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} \tag{2}$$

Given three robot configurations

$$(\theta_1, \theta_2) \in \{(-10^\circ, 20^\circ), (135^\circ, 90^\circ), (190^\circ, 160^\circ)\}.$$

a) • Draw the arm in all configurations, as well as their manipulability ellipses centered at the endpoint of the arm.
Hint: Calculate the eigenvalues and eigenvectors of JJ^T for all three robot configurations first. Then, draw the ellipses main axis. Choose a suitable scaling for your ellipses to make your plots clearer!

b) • Let $\lambda_{\min}$ be the minimum and $\lambda_{\max}$ be the maximum eigenvalue of JJ^T. Calculate the following manipulability measures for all three configurations:

  – Ratio of longest and shortest semi-axis of the ellipsoid

$$\mu_1(JJ^T) = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}$$

  – Condition number

$$\mu_2(JJ^T) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

  – Volume of the ellipse

$$\mu_3(JJ^T) = \sqrt{\lambda_{\min} \cdot \lambda_{\max}} = \sqrt{\det(JJ^T)}$$

c) • Does the ratio of the length of the major axis of the manipulability ellipse and the length of the minor axis depend on θ_1? On θ_2? Explain your answers.

$$J(\theta_1, \theta_2) = \begin{bmatrix} -\sin(\theta_1) - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \cos(\theta_1) + \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

configurations

$$(\theta_1, \theta_2) \in \{(-10°, 20°), (135°, 90°), (190°, 160°)\}.$$

$$J(-10,20) = \begin{pmatrix} 0 & -0{,}17 \\ 1{,}97 & 0{,}98 \end{pmatrix} = J_1 \rightarrow A_1 = J_1 J_1^T = \begin{pmatrix} 0{,}289 & -0{,}17 \\ -0{,}17 & 4{,}8 \end{pmatrix}$$

$$J(135,90) = \begin{pmatrix} 0 & -0{,}71 \\ -1{,}41 & -0{,}71 \end{pmatrix} = J_2 \rightarrow A_2 = J_2 J_2^T = \begin{pmatrix} 0.5 & 0{,}5 \\ 0{,}5 & 2{,}5 \end{pmatrix}$$

$$J(190,160) = \begin{pmatrix} 0{,}35 & 0{,}17 \\ 0 & 0{,}98 \end{pmatrix} = J_3 \rightarrow A_3 = \begin{pmatrix} 0{,}15 & 0{,}17 \\ 0{,}17 & 0{,}96 \end{pmatrix}$$

wolfram alpha:

A_1

Input:

eigenvectors $\begin{pmatrix} 0.289 & -0.17 \\ -0.17 & 4.8 \end{pmatrix}$

Results:

$v_1 \approx (-0.0376323, 1)$

$v_2 \approx (26.5729, 1)$

Corresponding eigenvalues:

$\sqrt{\lambda_1 \approx 4.8064} \approx 2{,}2 = \lambda_{max}$

$\sqrt{\lambda_2 \approx 0.282603} \approx 0{,}5 = \lambda_{min}$

A_2

Input:

eigenvectors $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}$

Results:

$v_1 \approx (0.236068, 1)$

$v_2 \approx (-4.23607, 1)$

Corresponding eigenvalues:

$\sqrt{\lambda_1 \approx 2.61803} \approx 1{,}62$ max

$\sqrt{\lambda_2 \approx 0.381966} \approx 0{,}62$ min

A_3

Input:

eigenvectors $\begin{pmatrix} 0.15 & 0.17 \\ 0.17 & 0.96 \end{pmatrix}$

Results:

$v_1 \approx (0.201366, 1)$

$v_2 \approx (-4.96607, 1)$

Corresponding eigenvalues:

$\sqrt{\lambda_1 \approx 0.994232} \approx 1$ max

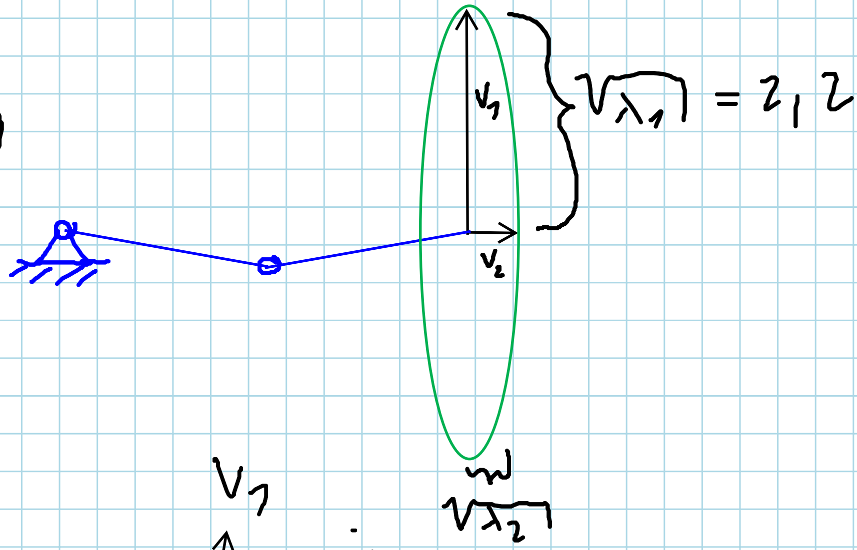
$\sqrt{\lambda_2 \approx 0.115768} \approx 0{,}3$ min

b)

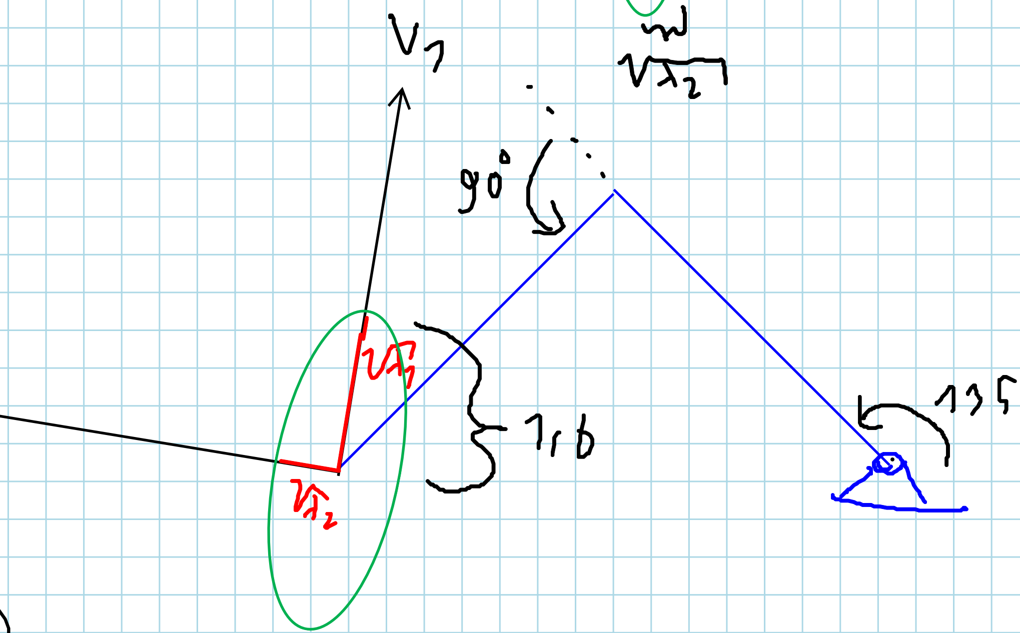
1. $\mu_1 = 2{,}1$ $\quad \mu_2 = 4{,}4$ $\quad \mu_3 = 1{,}05$
2. $\mu_1 = 1{,}62$ $\quad \mu_2 = 2{,}6$ $\quad \mu_3 = 1$
3. $\mu_1 = 1{,}83$ $\quad \mu_2 = 3{,}33$ $\quad \mu_3 = 0{,}55$

d)

1



2



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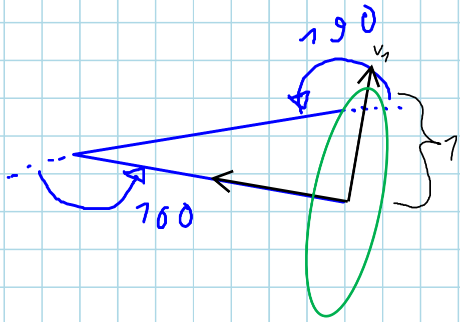


c)

yes it does. if theta2=0 -> we have a singularity and the minor axis of the ellipsoid will be zero, wile the major axis will be about infinite.

if theta = (90,90) we will get a good mobility and the ellipsoid will actually be a sphere.

Antwort com tutor:

lustiger weise hängts nicht von theta1 ab, weil thata1 nicht bestimmt ob wir uns ner Singularität nähern. wenn theta2=0 singu. wenn theta2=180 -> singu!!!

Exercise 9.2 – Inertia Matrices

Consider a satellite-like object as shown in figure 2: two spheres and two cubes are interconnected by two massless rods. The masses of the spheres and the cubes are given by $m_{\text{sphere}} = 2 \cdot m_{\text{cube}}$.

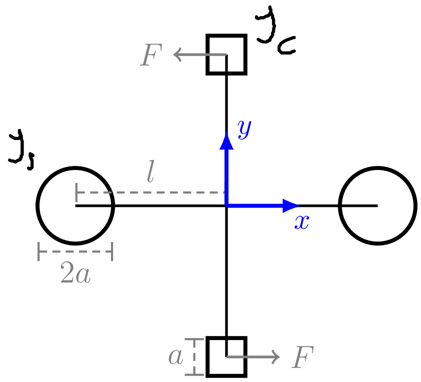


Figure 2: Satellite

a) Calculate the intertia tensor Θ for the satellite. For convience, the main axes of inertia coincide with the coordinate axes. Therefore the resulting inertia tensor has only entries $\neq 0$ along the diagonal:

$$\Theta = \begin{bmatrix} \Theta_{xx} & 0 & 0 \\ 0 & \Theta_{yy} & 0 \\ 0 & 0 & \Theta_{zz} \end{bmatrix} \tag{3}$$

Look up the formulae for the moments of inertia for a cube and a sphere. Use the Parallel Axis Therorem (Steinerscher Satz) to calculate the moment of inertia Θ_{zz} for a rotation around the z axis. Calculate Θ_{xx} and Θ_{yy} similarly.

Hint: Approximate values for $m_{\text{cube}} = 1\,\text{kg}$, $l = 2\,\text{m}$ and $a = 0.5\,\text{m}$ are:

$$\Theta_{xx} \approx 8.48\,\text{kg} \cdot \text{m}^2$$
$$\Theta_{yy} \approx 16.48\,\text{kg} \cdot \text{m}^2$$
$$\Theta_{zz} \approx 24.48\,\text{kg} \cdot \text{m}^2$$

b) The cubes on the satellite are thrusters producing each a force F. This results in a torque T around the z axis which accelerates the satellite around the z axis.

How long do the thrusters have to fire, to reach a rotational given velocity $\dot{\varphi}_z$?

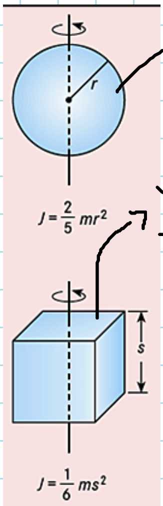
Hint: For an end velocity $\dot{\varphi}_z = 1s^{-1} \approx {}^{360°}/_{4\,\text{s}}$ and a force of $F = 10\,\text{N}$ per thruster, they have to burn for $t \approx 1.22\,\text{s}$

c) Thrusters are suboptimal to rotate a satellite as their reservoir will empty each time they are used. The preferred method is a small electric motor accelerating a so called reaction wheel. According to Newton's third axiom (actio = re-actio) the motor torque that accelerates the reaction wheel also acts in the opposite direction on the satellite and thus rotating it.

Assume we can control the motor torque T to accelerate the satellite with the same torque for the same time t as the thrusters did. What is the reaction wheel's velocity $\dot{\varphi}_{\text{rw}}$ with a given inertia Θ_{rw} when the satellite has reached the velocity $\dot{\varphi}_z$?

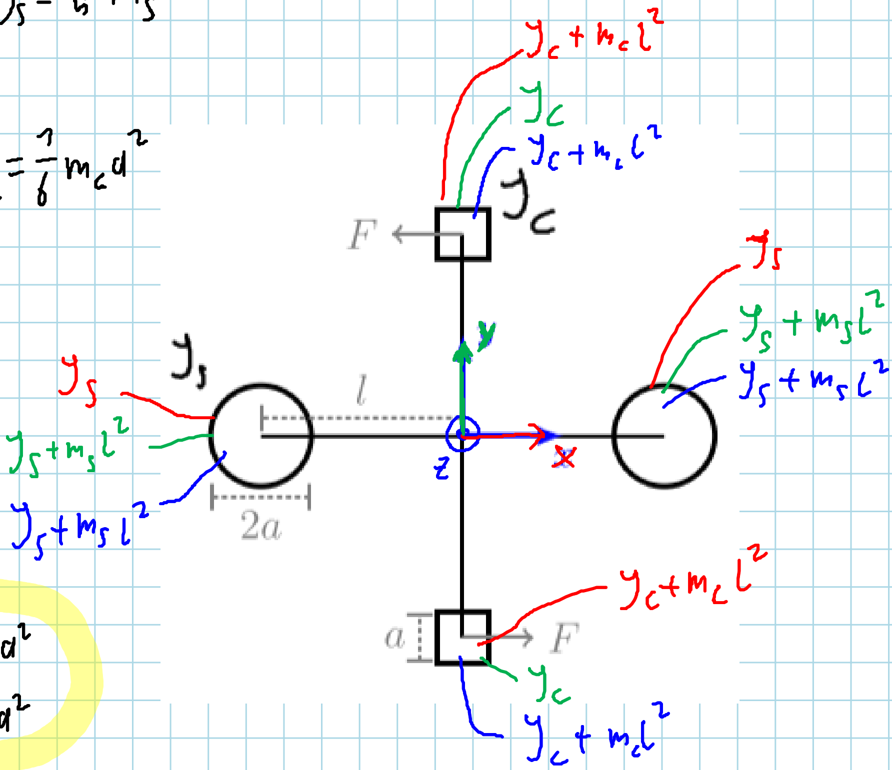
Hint: With $\Theta_{\text{rw}} = 1\,\text{kg} \cdot \text{m}^2$ the velocity is $\dot{\varphi}_{\text{rw}} = 24.48\,\text{s}^{-1}$

d)



$J_s = \frac{2}{5} m_s a^2$

$J_c = \frac{1}{6} m_c a^2$



$m_s = 2 m_c$

$$J_s = \frac{2}{5} m_s a^2 = \frac{4}{5} m_c a^2$$

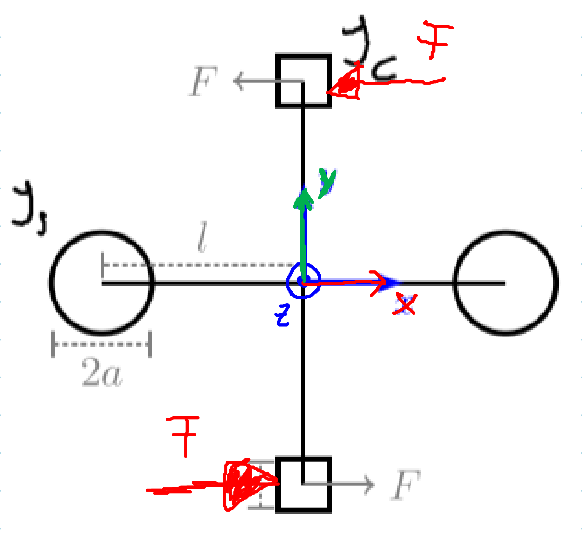
$$J_c = \frac{1}{6} m_c a^2 = \frac{1}{12} m_s a^2$$

$$\Theta_{zz} = \sum = 2 J_s + 2 J_c + 2 m_s l^2 + 2 m_c l^2$$
$$= 2\left[\frac{4}{5} m_c a^2 + \frac{1}{6} m_c a^2 + 2 m_c l^2 + m_c l^2\right]$$
$$= 2 m_c \left[a^2\left(\frac{4}{5} + \frac{1}{6}\right) + 3 l^2\right]$$

$$\Theta_{xx} = \sum = 2 J_s + 2(J_c + m_c l^2) = 2\left[J_s + J_c + m_c l^2\right]$$
$$= 2\left[\frac{4}{5} m_c a^2 + \frac{1}{6} m_c a^2 + m_c l^2\right]$$

$$\Theta_{yy} = \sum = 2\left[J_c + J_s + m_s{}^2\right]$$

b)



$$\Theta_{zz} \cdot \ddot{\varphi}_z = \sum M_z = 2 F \cdot l$$

$$\Theta_{zz} \cdot \ddot{\varphi}_z = 2 F l$$

$$\ddot{\varphi}_z = \frac{2 F l}{\Theta_{zz}} \quad \Big| \int$$

$$\dot{\varphi}_z = \frac{2 F l}{\Theta_{zz}} t_e$$

$$\hookrightarrow t_e = \frac{\dot{\varphi}_z \Theta_{zz}}{2 F l}$$

Hint: For an end velocity $\dot{\varphi}_z = 1\,\mathrm{s}^{-1} \approx 360°/4\,\mathrm{s}$ and a force of $F = 10\,\mathrm{N}$ per thruster, they have to burn for $t \approx 1.22\,\mathrm{s} \cdot 2$

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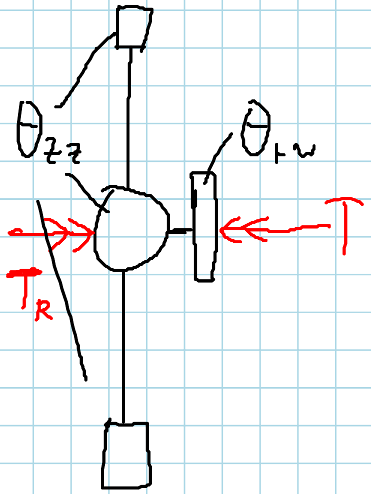
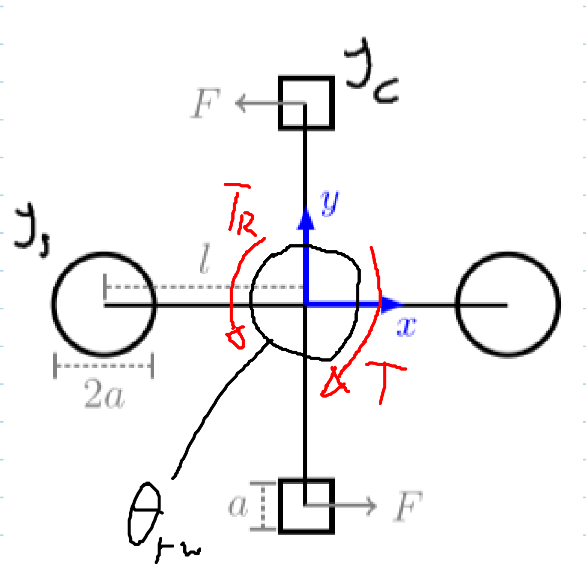
Hint: With $\Theta_{\mathrm{rw}} = 1\,\mathrm{kg}\cdot\mathrm{m}^2$ the velocity is $\dot{\varphi}_{\mathrm{rw}} = 24.48\,\mathrm{s}^{-1}$





Satellit: $\Theta_{zz} \cdot \ddot{\varphi}_z = T_R$

Motor masse $\Rightarrow$ $\Theta_{rw} \cdot \ddot{\varphi}_{rw} = T$

Actio = Reactio

$$T_R = T$$

$$\Theta_{zz} \dot{\varphi}_z = \Theta_{rw} \cdot \dot{\varphi}_{rw} \quad \Bigg| \quad \dot{\varphi}_z = \frac{2Fl}{\Theta_{zz}} t_e$$

$$2Fl t_e = \Theta_{rw} \cdot \dot{\varphi}_{rw}$$

$$\Rightarrow \dot{\varphi}_{rw} = \frac{2Flt_e}{\Theta_{rw}}$$