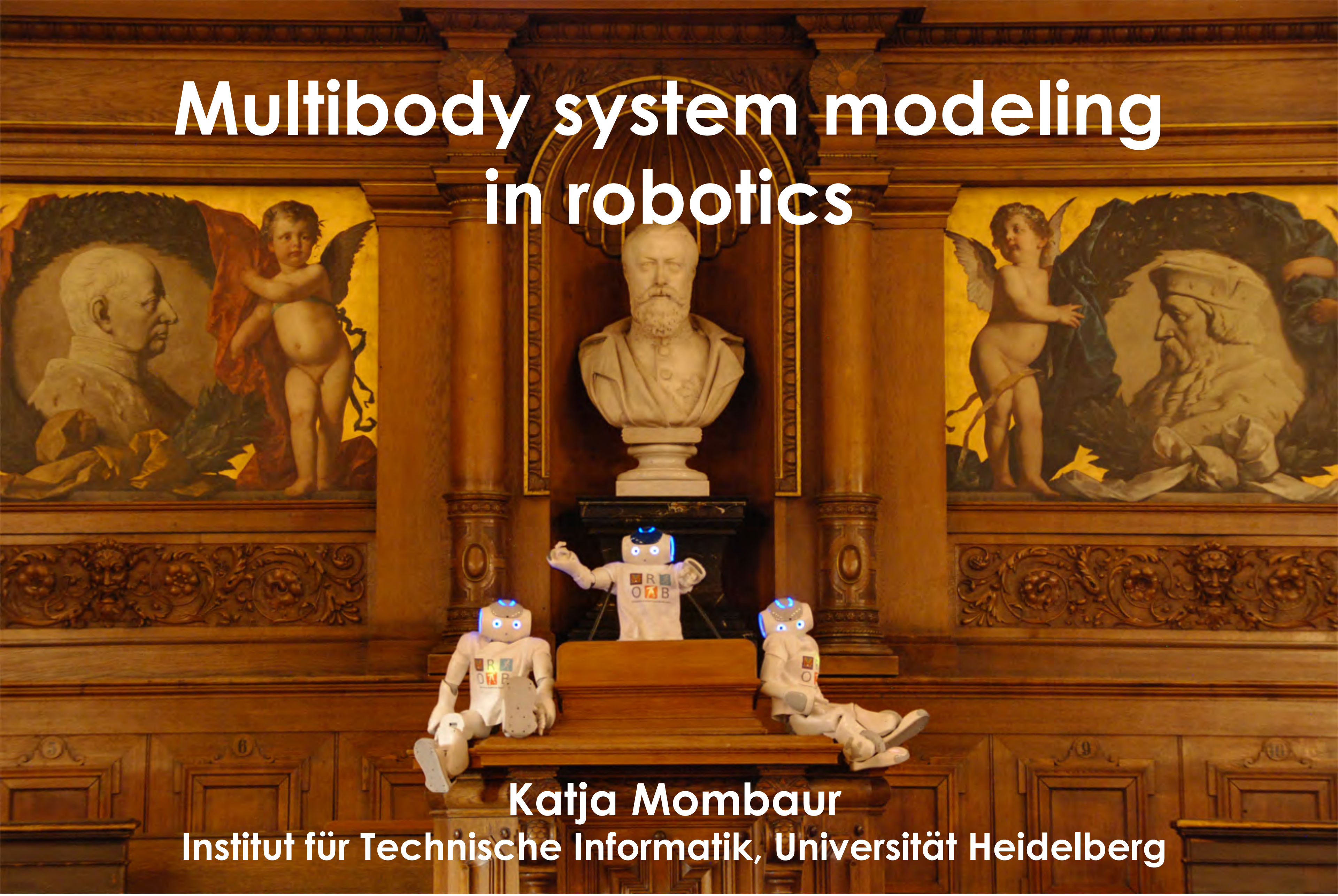
Multibody system modeling in robotics



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Robotics 2 - 6 May 2019

Summary of last lecture

- Kinematics of multibody systems for robot modeling
- Dynamics of multibody systems for robot modeling
- Redundant coordinate formulations of multibody systems resulting in DAE models

Reality

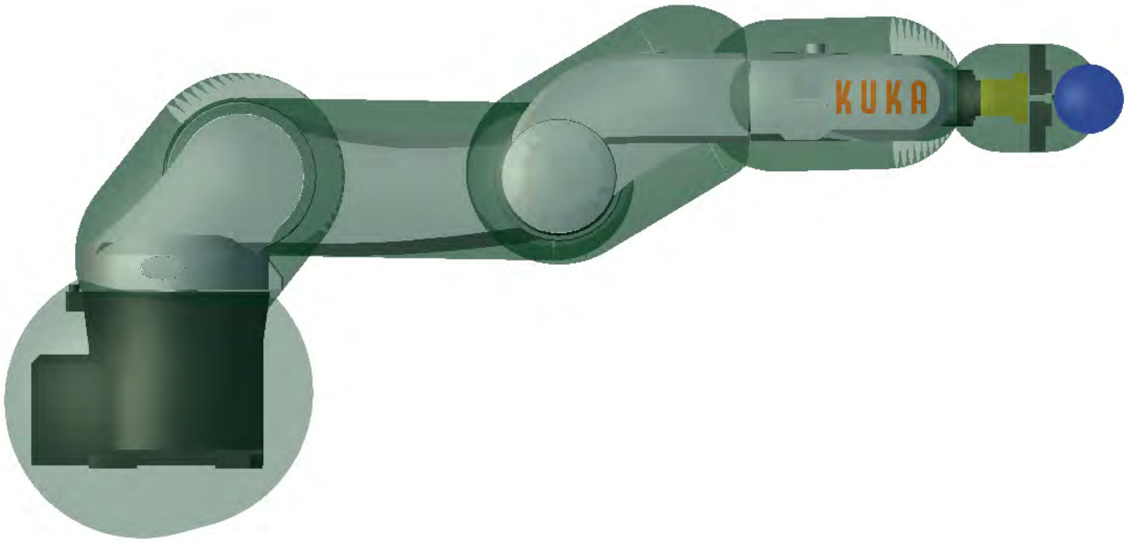
Model

$$
\begin{aligned}
\ddot{r}_{x,1} &= \ddot{r}_{x,H} + c_T(\cos\phi_1\ddot{\phi}_1 - \sin\phi_1\dot{\phi}_1^2)\\
\ddot{r}_{y,1} &= \ddot{r}_{y,H} + c_T(\sin\phi_1\ddot{\phi}_1 + \cos\phi_1\dot{\phi}_1^2)\\
\\
\ddot{r}_{x,2} &= \ddot{r}_{x,H} + l_T(\cos\phi_1\ddot{\phi}_1 - \sin\phi_1\dot{\phi}_1^2) + c_S(\cos\phi_2\ddot{\phi}_2 - \sin\phi_2\dot{\phi}_2^2) - w_S(\sin\phi_2\ddot{\phi}_2 - \cos\phi_2\dot{\phi}_2^2)\\
\ddot{r}_{y,2} &= \ddot{r}_{y,H} + l_T(\sin\phi_1\ddot{\phi}_1 + \cos\phi_1\dot{\phi}_1^2) + c_S(\sin\phi_2\ddot{\phi}_2 + \cos\phi_2\dot{\phi}_2^2) + w_S(\cos\phi_2\ddot{\phi}_2 - \sin\phi_2\dot{\phi}_2^2)\\
\\
\ddot{r}_{x,3} &= \ddot{r}_{x,H} + c_T(+\cos\phi_3\ddot{\phi}_3 - \sin\phi_3\dot{\phi}_3^2)\\
\ddot{r}_{y,3} &= \ddot{r}_{y,H} + c_T(+\sin\phi_3\ddot{\phi}_3 + \cos\phi_3\dot{\phi}_3^2)\\
\\
\ddot{r}_{x,4} &= \ddot{r}_{x,H} + l_T(\cos\phi_3\ddot{\phi}_3 - \sin\phi_3\dot{\phi}_3^2) + c_S(\cos\phi_4\ddot{\phi}_4 - \sin\phi_4\dot{\phi}_4^2) - w_S(\sin\phi_4\ddot{\phi}_4 - \cos\phi_4\dot{\phi}_4^2)\\
\ddot{r}_{y,4} &= \ddot{r}_{y,H} + l_T(\sin\phi_3\ddot{\phi}_3 + \cos\phi_3\dot{\phi}_3^2) + c_S(\sin\phi_4\ddot{\phi}_4 + \cos\phi_4\dot{\phi}_4^2) + w_S(\cos\phi_4\ddot{\phi}_4 - \sin\phi_4\dot{\phi}_4^2)\\
\\
\phi_4 &\equiv \phi_3\\
\dot{\phi}_4 &\equiv \dot{\phi}_3\\
\ddot{\phi}_4 &\equiv \ddot{\phi}_3\\
\\
(\ddot{\phi}_2 &\equiv \ddot{\phi}_1 \quad \text{during second phase})
\end{aligned}
$$

Multiphase problems, impacts and other constraints in robot MBS models

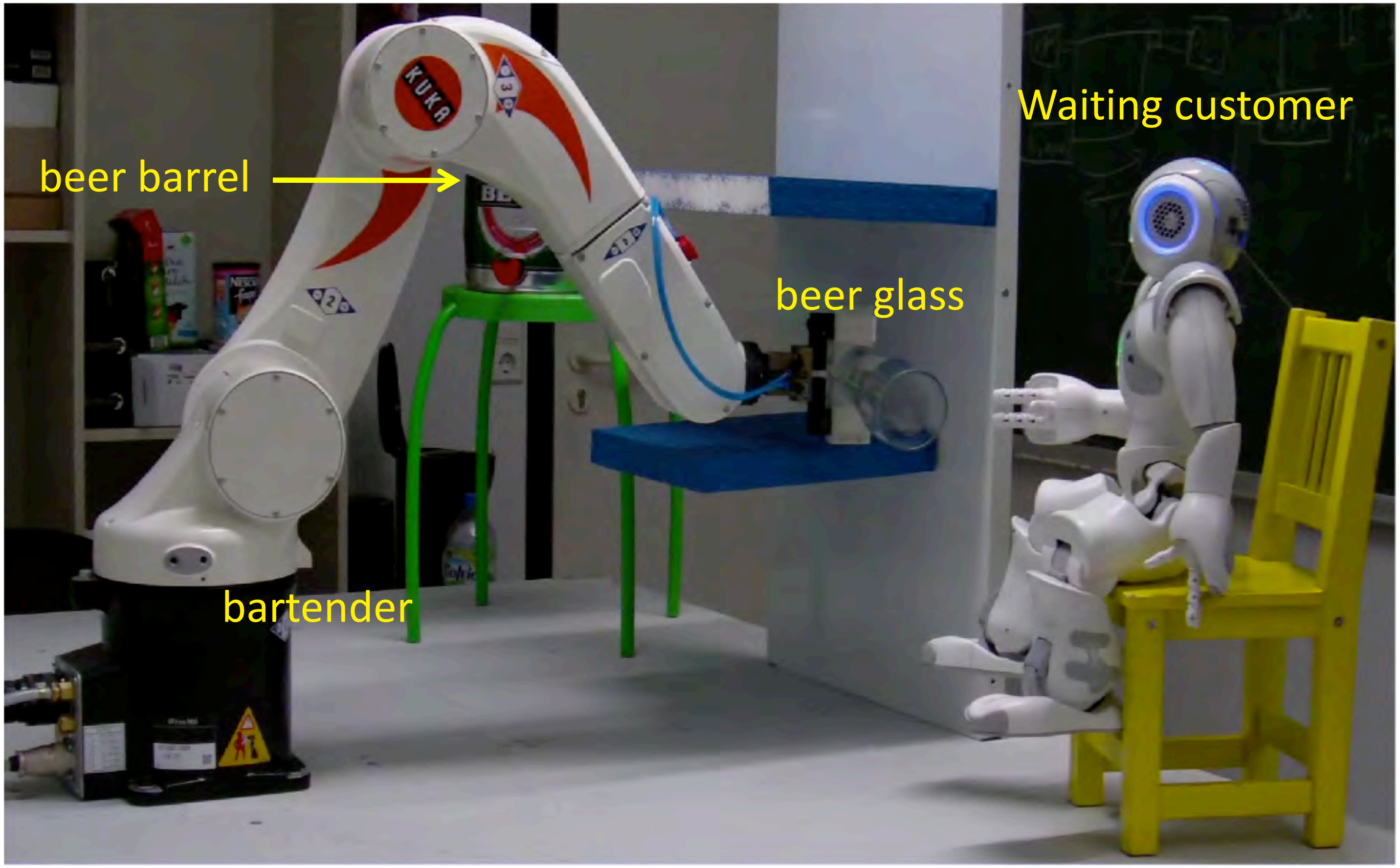
Inequality constraints for MBS: avoiding auto collision

- **Avoiding auto collision:**
  - simple models of body geometry are needed, e.g. capsules
    otherwise the complexity will increase dramatically
  - Autocollision of neighboring segments can be avoided by joint angle limits



The KUKA Bartender robot

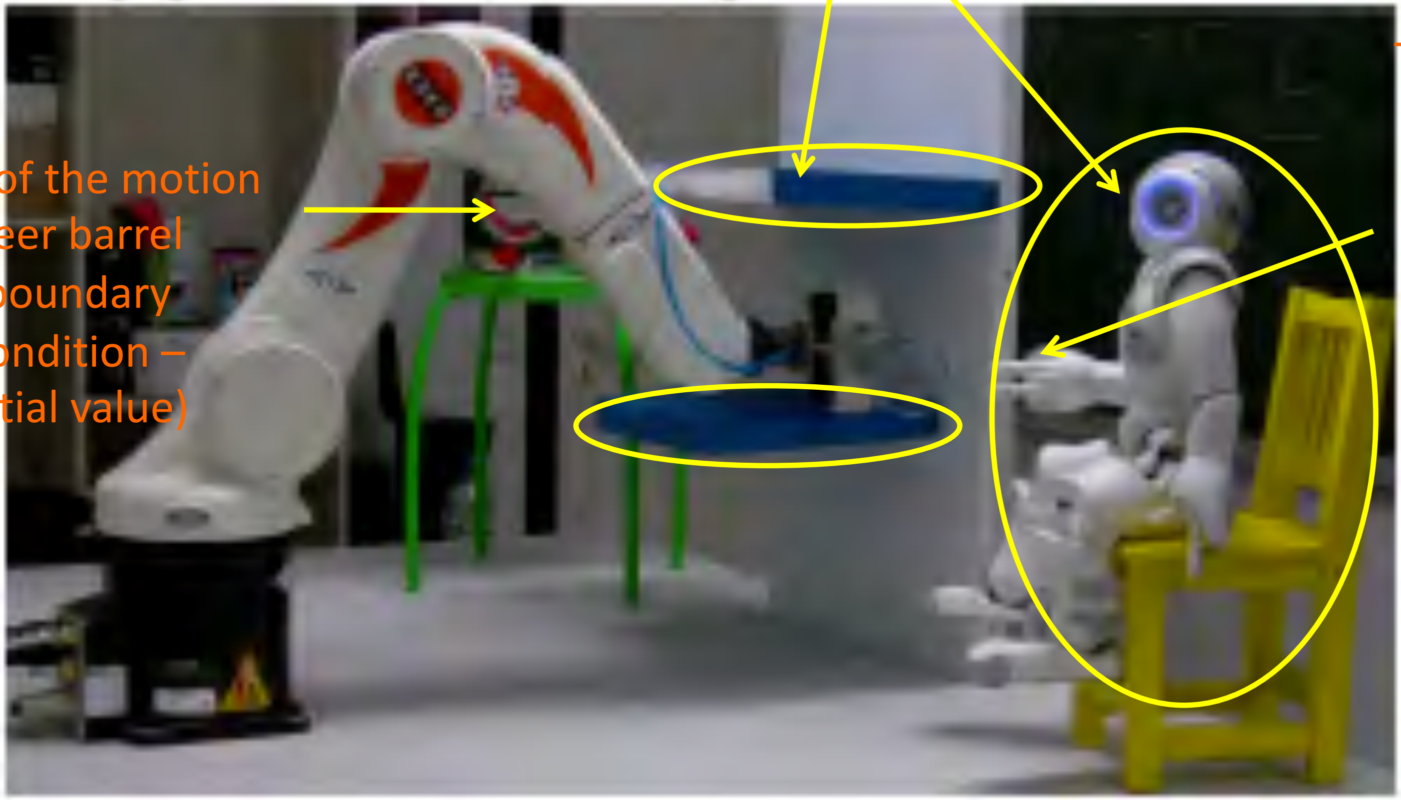
The robot is supposed to serve beer to a customer,
i.e. move the glass from A to B without hitting anything



Modeling the environment and the start and end conditions

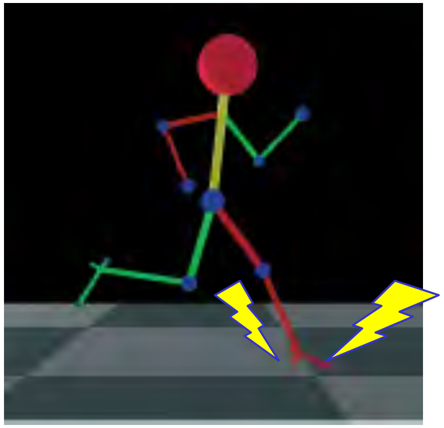
Collision between robot arm/glass and
Shelve / wall / Nao must be avoided (inequality constraints)

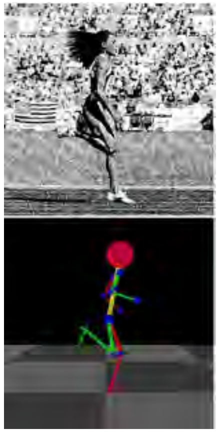
Start of the motion
beer barrel
(boundary
condition –
initial value)



Target of the
motion:
Nao's
hand
(boundary
condition –
final value)

Discontinuities / discrete phases





- States after contact can be formulated by:
  - The constraints to be satisfied after impact (constraints of the new phase)
  - Conservation of angular momentum (across the impact) about the impact point and a series of other points

$$\begin{pmatrix} M(q,p) & G(q,p)^T \\ G(q,p) & 0 \end{pmatrix} \begin{pmatrix} v_+ \\ \Lambda \end{pmatrix} = \begin{pmatrix} M(q)v_- \\ 0 \end{pmatrix}$$

Box constraints for the HeiCub robot

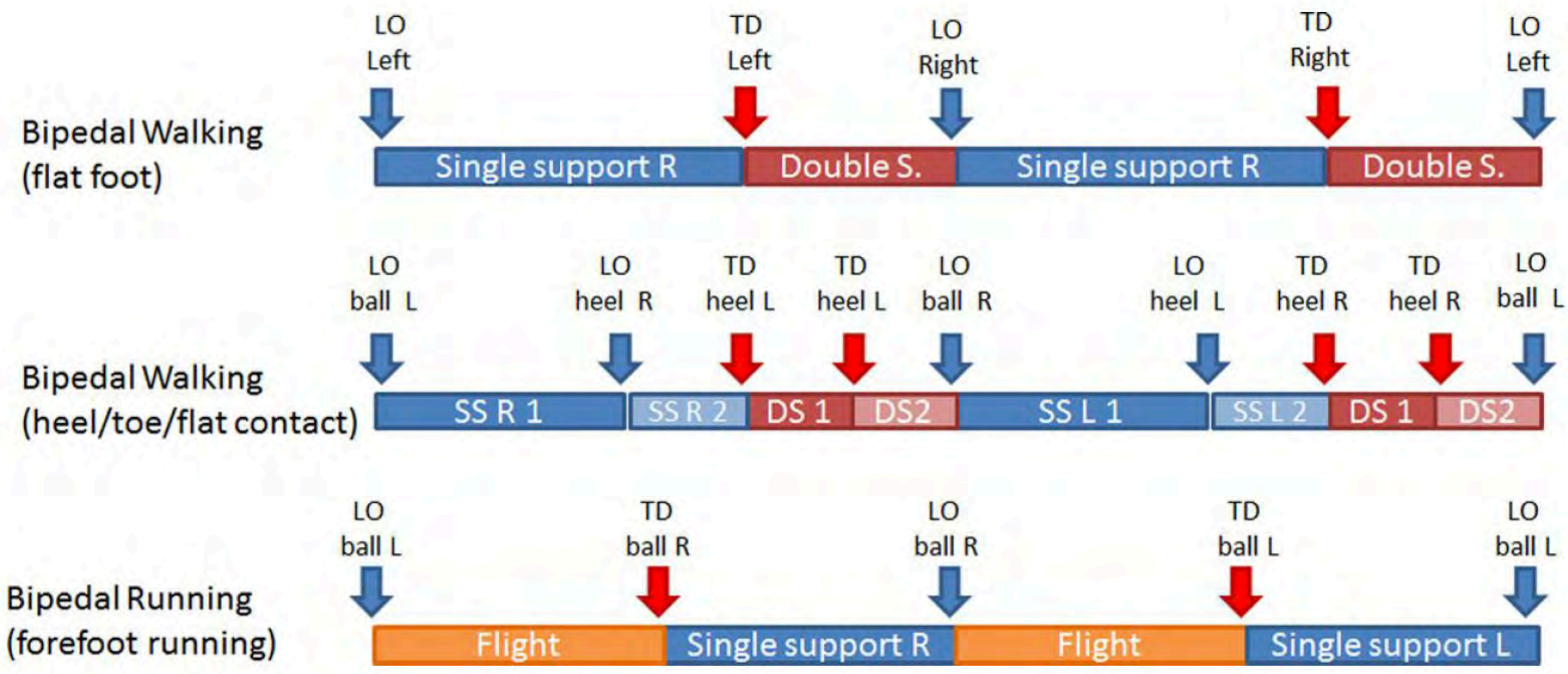
- Kinematic and dynamic constraints

Table 1. HeiCub joint limits.

Joint	Range Limits (deg)	Velocity Limits (deg/s)	Torque Limits (Nm/s)
l_hip_pitch, r_hip_pitch	$[-33, 100]$	$[-100, 100]$	$[-50, 50]$
l_hip_roll, r_hip_roll	$[-19, 90]$	$[-150, 150]$	$[-50, 50]$
l_hip_yaw, r_hip_yaw	$[-75, 75]$	$[-150, 150]$	$[-50, 50]$
l_knee, r_knee	$[-100, 0]$	$[-150, 150]$	$[-50, 50]$
l_ankle_pitch, r_ankle_pitch	$[-36, 27]$	$[-150, 150]$	$[-50, 50]$
l_ankle_roll, r_ankle_roll	$[-24, 24]$	$[-150, 150]$	$[-50, 50]$
torso_pitch	$[-20, 60]$	$[-150, 150]$	$[-50, 50]$
torso_roll	$[-26, 26]$	$[-150, 150]$	$[-50, 50]$
torso_yaw	$[-50, 50]$	$[-150, 150]$	$[-50, 50]$

Phase orders for biological and robotic walking and running

- Continuous phases:



- Red arrows denote touchdown events at which usually discontinuities (=discrete phases) occur

A full walking sequence of the HeiCub (and similar humanoid robots)

- A full walking sequence with a starting step, periodic steps & a stopping step

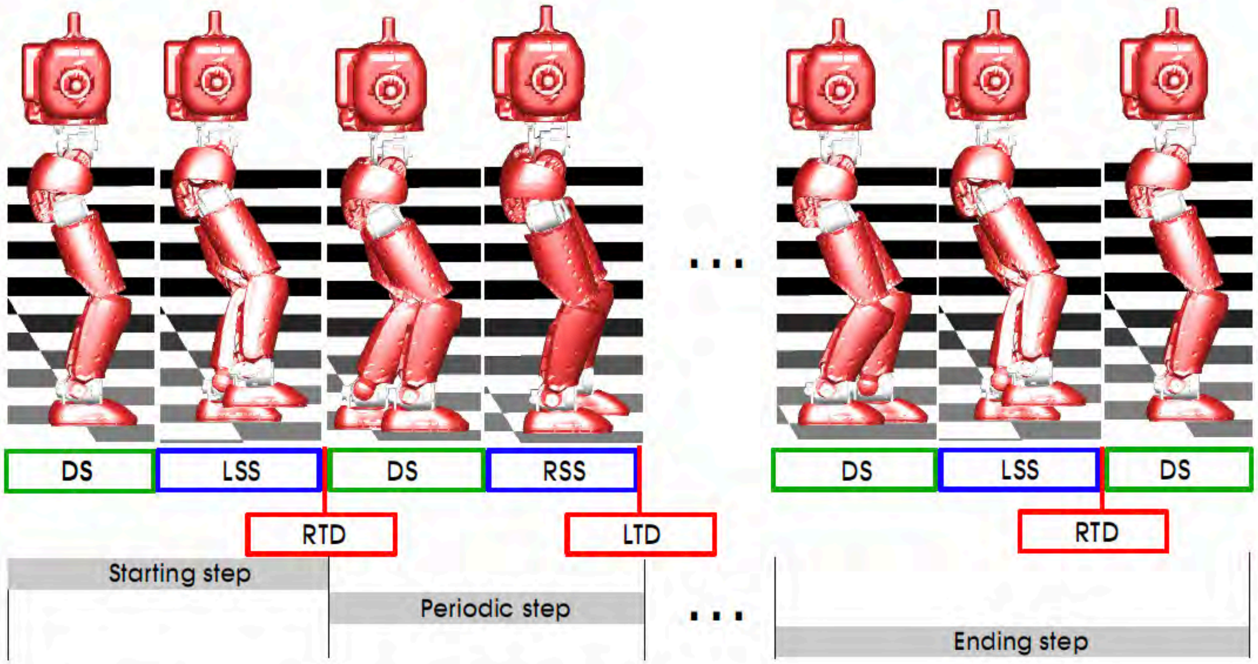
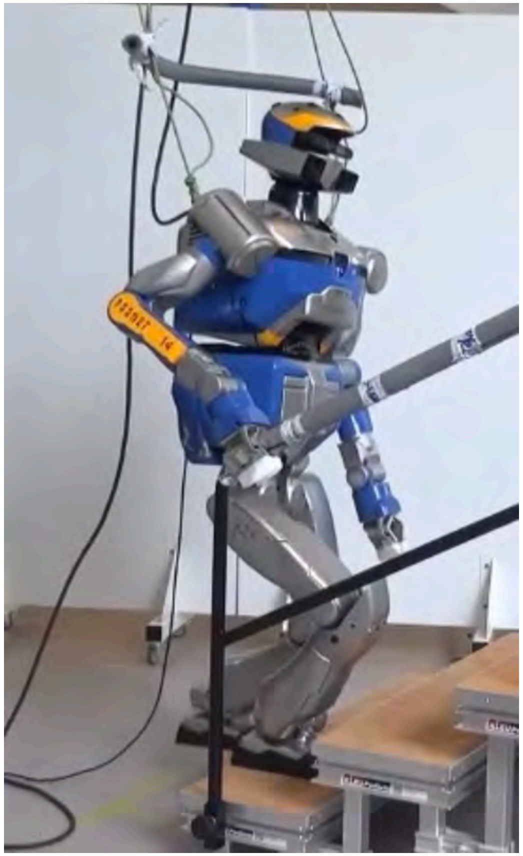


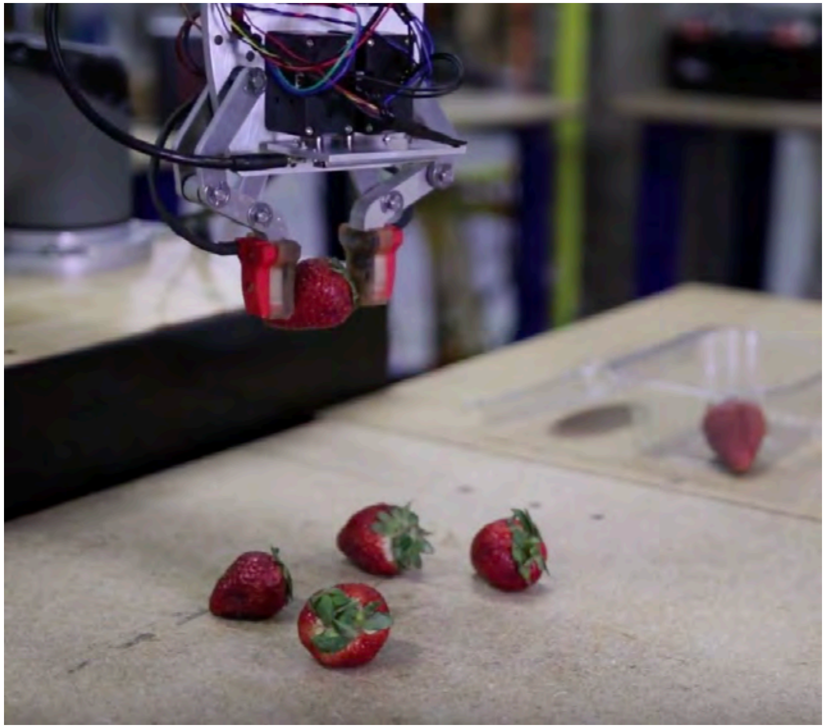
Figure 2. Walking phases of HeiCub. DS = Double Support, LSS = Left Single Support, RSS = Right Single Support, LTD = Left Touch Down, RTD = Right Touch Down. The whole sequence can be seen as three sub-sequences. The periodic step can be repeated for a desired number of times that do not need to be further modeled.

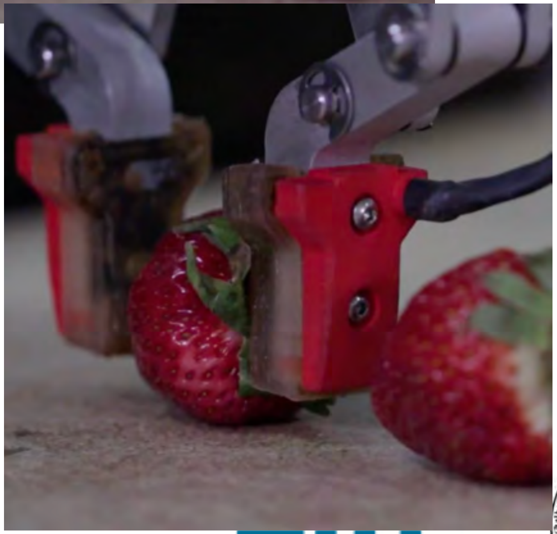
Phases also change in function of other contacts



Also in the context of robot manipulators there are phases

- Equations of motion change when there is contact with the environment or with objects





Examples for general objective functions for motions (not restricted to locomotion)

- Minimization of energy consumption
- Minimization of joint torques
- Minimization of (absolute) mechanical work
- Maximization of efficiency
- Minimization of total time of a motion, phase times etc.
- Maximization or minimization of characteristic velocities
- Maximization of smoothness of motion (minimization of joint accelerations or jerks)
- Minimization of interaction forces with environment / humans / other robots
- Maximization of manipulability
- Minimization of deviations from periodicity
- Minimization of deviations from desired positions, angles, velocities etc.

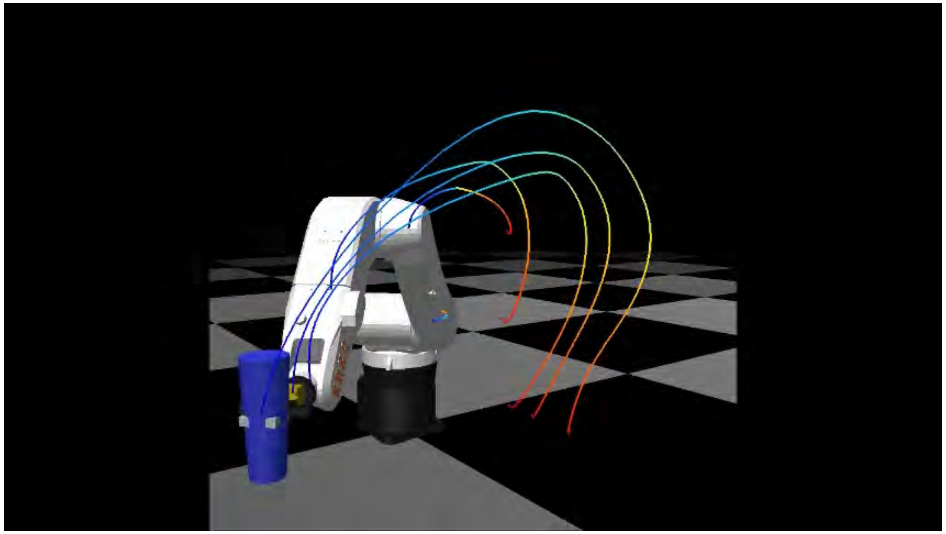
Examples for objective functions specific to locomotion problems (or similar wholebody problems)

- Minimization / maximization of step width / step length
- Minimization / maximization of step frequency
- Minimization of cost of transport, defined as energy used (electrical energy or mechanical energy / work) divided by weight and distance traveled
- Minimization of angular momentum about COM (about different axes)
- Maximization of stability related criteria such as ZMP criteria (based on full model or simplified model) or capture point criteria based on a point-wise evaluation of the system
- Maximization of stability based on criteria that require evaluations of the entire trajectories / limit cycles such as Lyapunov stability criteria
- Minimization of head motions (in particular rotational head motions) to stabilize gaze.

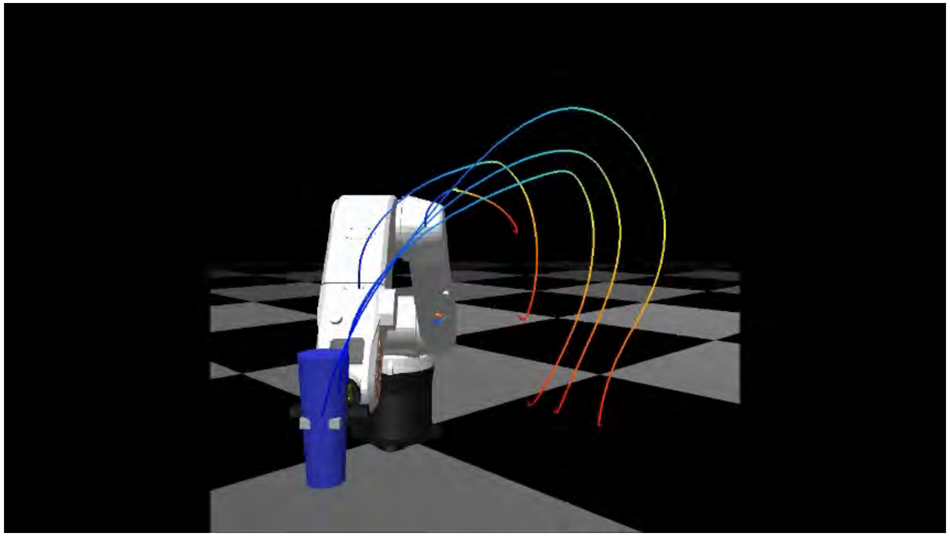
Examples for objective functions specific to manipulation, pointing or grasping

- Minimization of end point variability
- Maximization of end point velocity
- Minimization / maximization of end point acceleration
- Minimization of end point jerk

Optimized motions for the KUKA robot

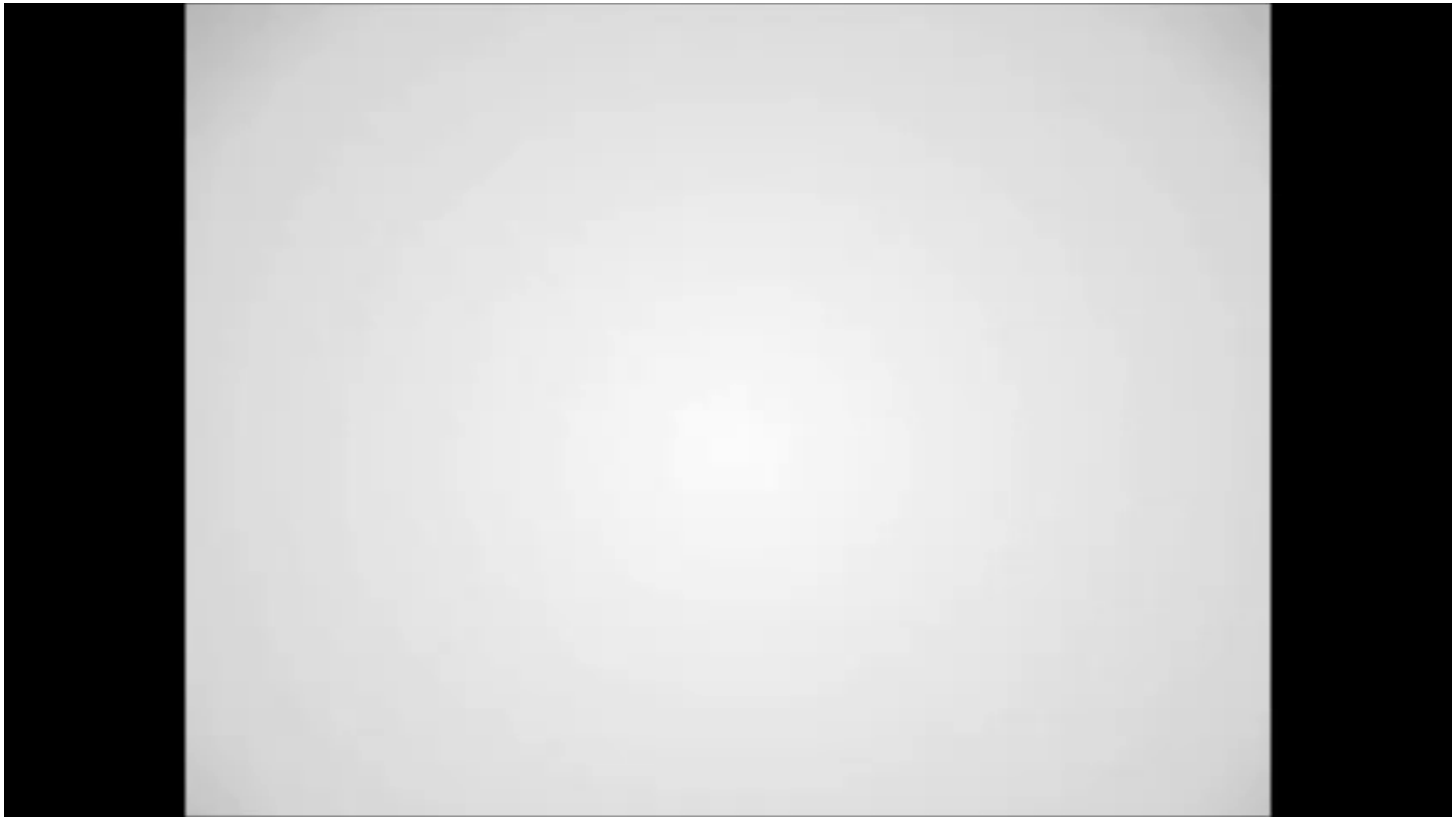


Combined objective
- Min time
- Min lateral accelerations in glass

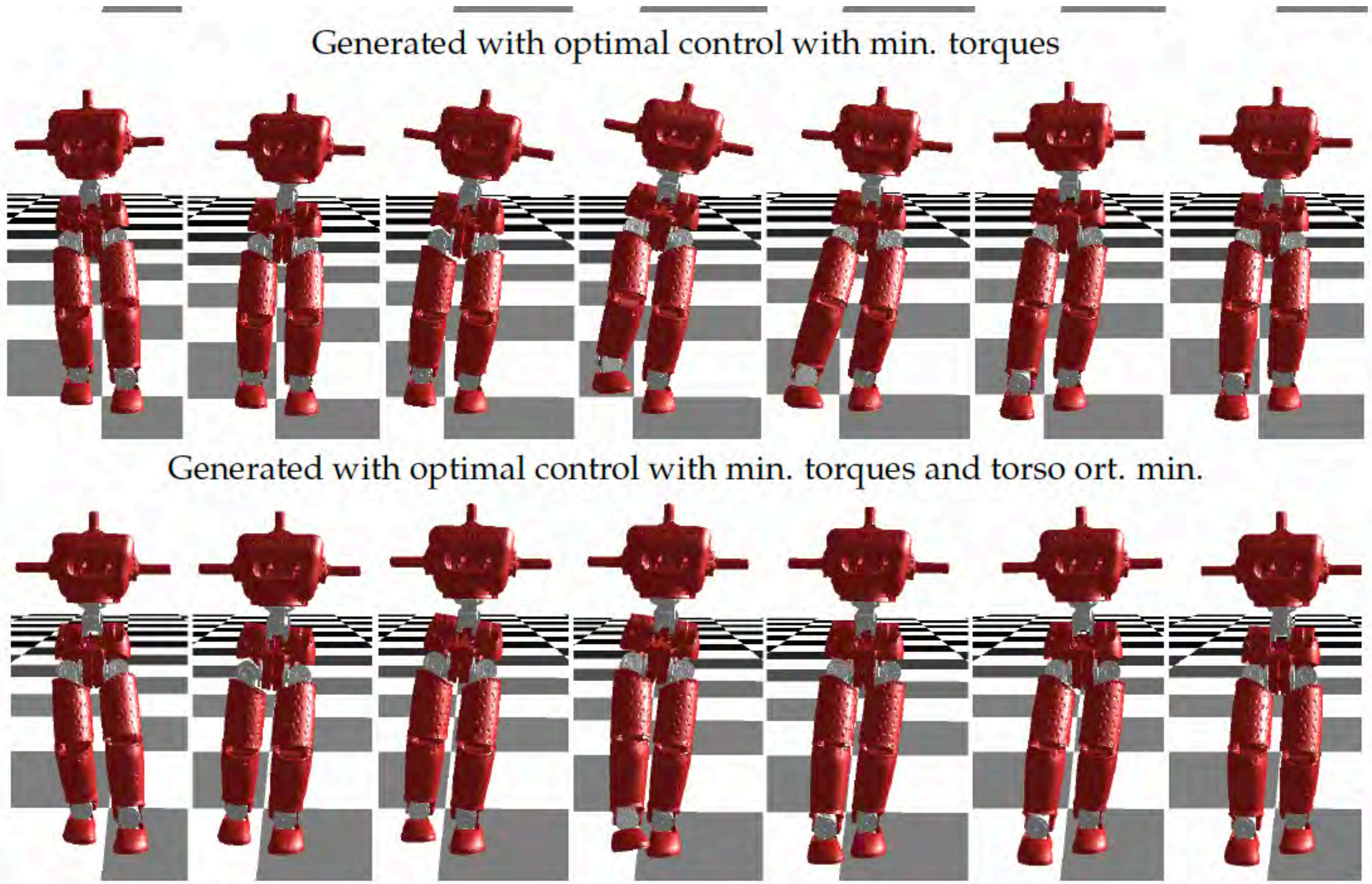


Add. constraint on longitudinal acceleration in glass

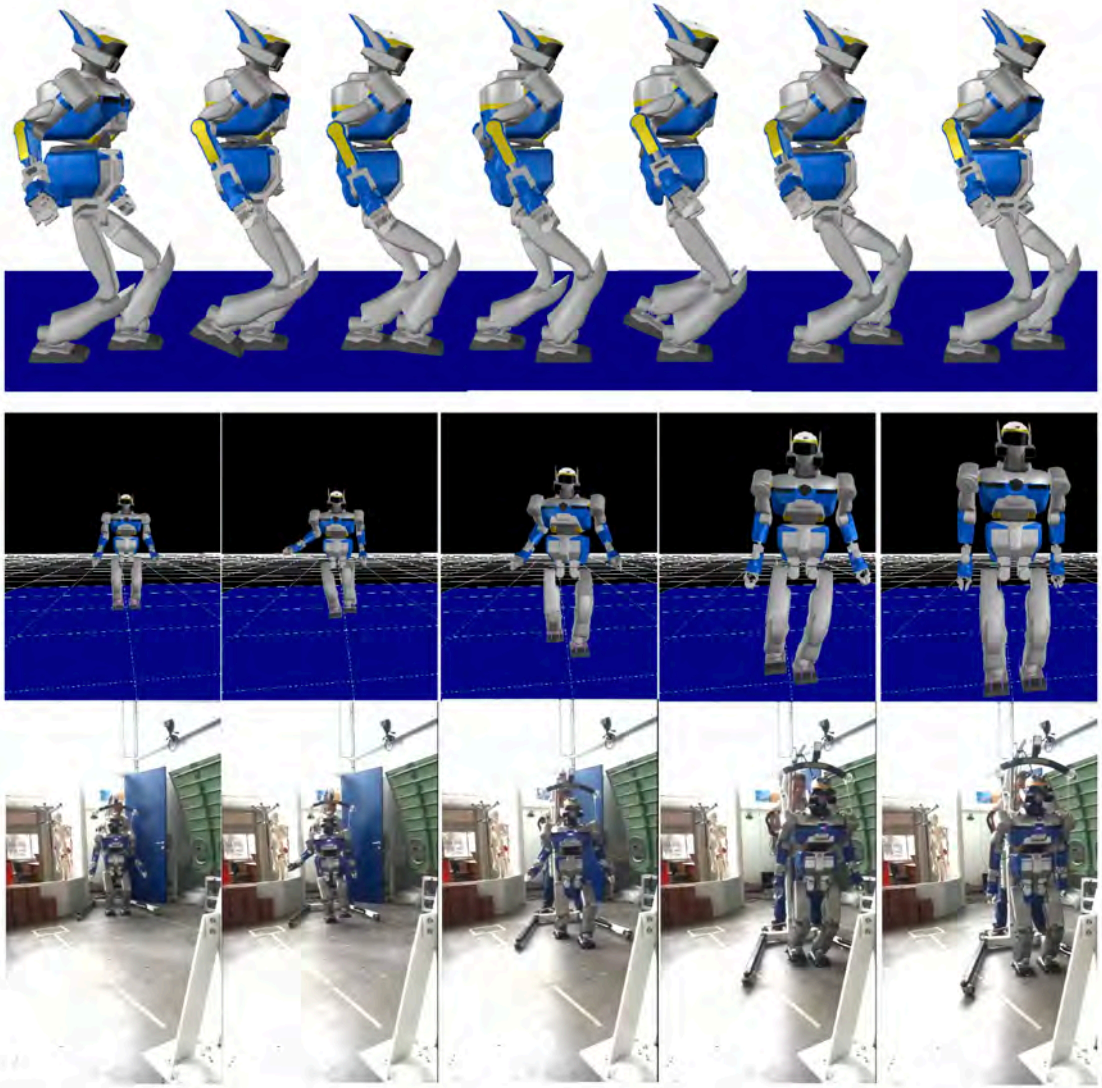
KUKA in action



Optimized motions for HeiCub

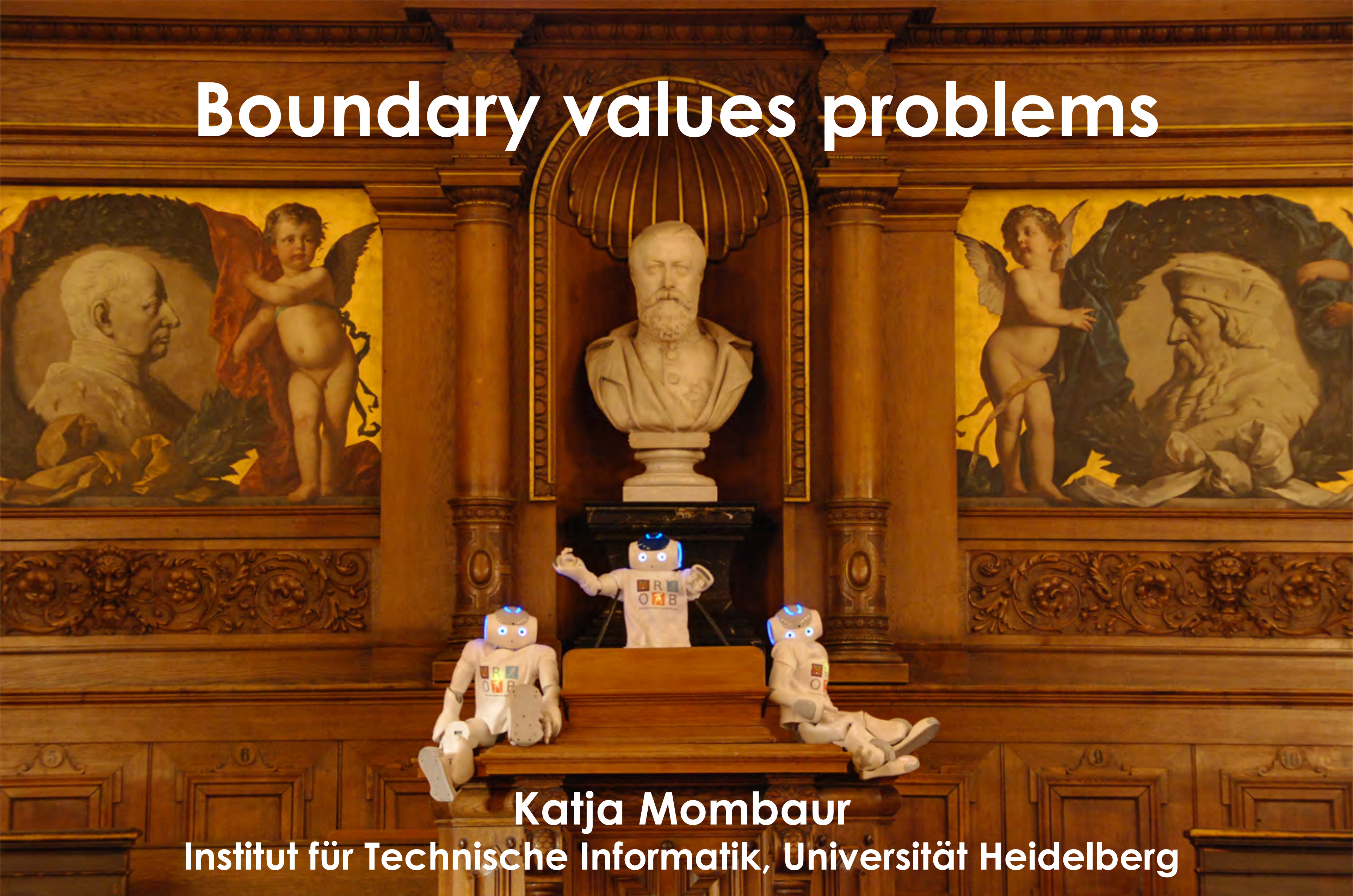


Optimized motions for HRP-2



Minimization of joint torques

Boundary values problems



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Robotics 2 - 6 May 2019

Overview

- What are boundary value problems?
- Why are we interested in boundary value problems?
- Some theory about boundary value problems
- Solution methods:
  - Single Shooting
  - Multiple Shooting
  - Collocation

What are boundary value problems?

Boundary value problems (BVP) – standard forms

- **2 point BVP**

$$\boxed{\begin{aligned} \dot{x} &= f(t,x) \\ r(x(t_0), x(t_{end})) &= 0 \end{aligned}} \qquad (1)$$

- **Multipoint BVP:**

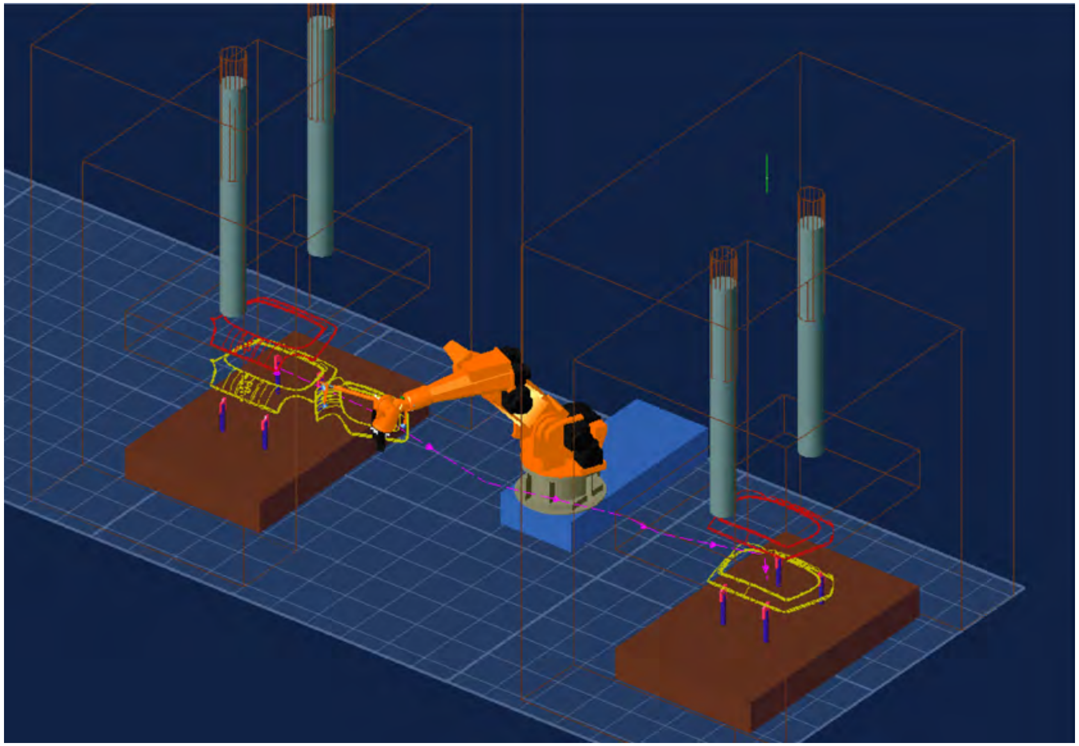
$$\boxed{\begin{aligned} \dot{x} &= f(t,x) \\ r(x(t_0), x(t_1), \ldots, x(t_{end})) &= 0 \end{aligned}} \qquad (2)$$

- Other special types of boundary conditions

  - **Linear 2 point boundary condition** $\quad Ax(t_0) + Bx(t_{end}) = c_1$
  - **Linear separated boundary cond.** $\quad A_i x(t_i) = c_i \quad i = 1, .., k$
  - **General separated boundary cond.** $\quad r(x(t_i)) = c_i \quad i = 1, .., k$

Example for a boundary value problem

- An industrial robot transporting parts from one place to another in a production chain

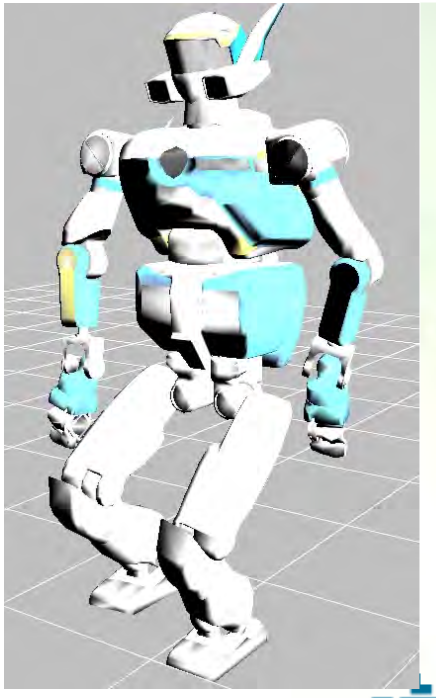


Examples for periodic boundary value problems

$$r(x(t_0), x(t_{end})) = x(t_0) - x(t_{end}) = 0$$

Figure eight motions of a kite

Periodic walking motions of a robot



Why are we interested in boundary value problems in this lecture?

Importance of BVP for optimization

- BVP appear as underlying problems in many types of optimization problems
  - Optimal control
  - Parameter estimation
  - Optimum experimental design
  - Real time optimization

BVP in optimal control problems

- BVP is a subproblem of the optimal control problem

$$\min_{y,u,q} \int_{t_0}^{t_f} L(t, y(t), u(t), p)dt + \phi(t_f, y(t_f), p)$$

$$\begin{aligned} \text{s.t.} \quad \dot{y}(t) &= f(t, y(t), u(t), p), \\ g(t, y(t), u(t), p) &\geq 0, \\ r_{\text{eq}}(y(t_0), \ldots, y(t_f), p) &= 0, \\ r_{\text{ineq}}(y(t_0), \ldots, y(t_f), p) &\geq 0. \end{aligned}$$

Multipoint BVP

For the sake of completeness: BVP in parameter estimation problems*

- BVP is a subproblem of the parameter estimation problem

$$\min_{y,p} \frac{1}{2} \sum_{i=1}^{M} \left(\frac{\eta_i - h_i(t_i, y(t_i), p)}{\sigma_i} \right)^2$$

$$\text{s.t. } \dot{y}(t) = f(t, y(t), p)$$

$$y(t_0) = y_0(p)$$

$$\sum_{i=1}^{K} r_i(y(t_i), p) = 0$$

Multipoint BVP

* Not subject of this lecture

For the sake of completeness: BVP in optimum experimental design problems*

- BVP is a subproblem of the optimum experimental design problem

$$\min_{y, y_v, \xi} \phi(C)$$

$$C = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} J_1^T J_1 & J_2^T \\ J_2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$J_1 = \left(\frac{\sqrt{w_i}}{\sigma_i} \left(\frac{\partial h_i}{\partial y} y_v(t_i) + \frac{\partial h_i}{\partial v} \right) \right)_{i=1,\dots,M}, \quad J_2 = \left(\frac{\partial F_{2i}}{\partial v} y_v(t_i) + \frac{\partial F_{2i}}{\partial v} \right)_{i=1,\dots,M}$$

y satisfies the model equations

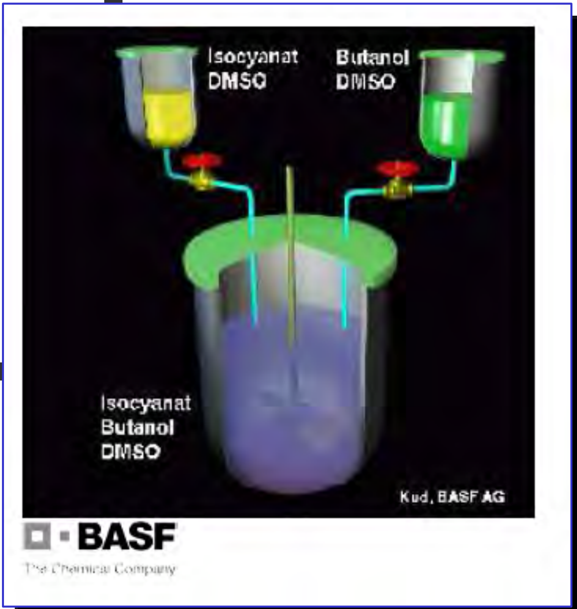
Multipoint BVP

y_v satisfies derivatives of the model equations

State and control constraints:

$$0 \leq \psi(y(t), p, q, u(t), w)$$

$$w \in \{0, 1\}^M$$



* Not subject of this lecture

Special cases of boundary value problems

1. **BVP with free parameters:** $\dot{x} = f(t, x, p)$

   can be transformed to standard form $\dot{y} = \tilde{f}(t, y)$

   by introducing a new state vector $y = \begin{pmatrix} x \\ \tilde{p} \end{pmatrix}$

   with new differential equation $\dot{\tilde{p}} = 0$ and initial condition $\tilde{p}(0) = p$

2. **BVP with free end time** t_{end}

   Time transf $t \in [t_t, t_{end}] \longrightarrow \tau \in [0, 1], \tau = \dfrac{t - t_0}{t_{end} - t_0}$

   $\longrightarrow t(\tau) = t_0 + \tau(t_{end} - t_0), \bar{x}(\tau) = x(t(\tau))$

   It follows

$$\frac{d\bar{x}}{d\tau} = \dot{x}(t)\frac{dt}{d\tau} = f(t, \bar{x}) \cdot (t_{end} - t_0)$$

$$r(\bar{x}(0), \bar{x}(1), t_{end}) = 0,$$

was thus transformed to BVP with free parameter, see above, which can be transformed to standard form

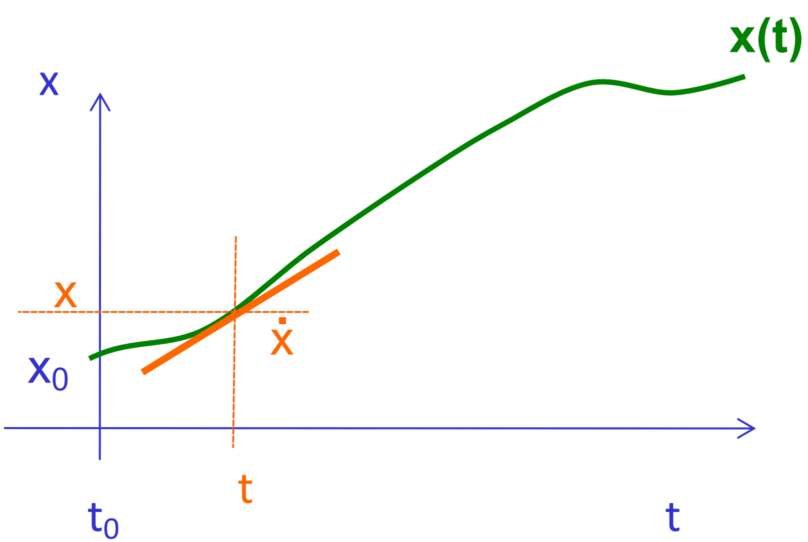
Some theory &

Formulation of boundary value problems

Excursion: Initial value problems & integrators

- Initial value problem $\dot{x} = f(t, x) \qquad x(0) = x_0$

- We are looking for the numerical solution $x(t)$



Programs that perform this task are called numerical integrators

See course NUMERIK 1

You will also use integrators in the exercises of this course

Excursion: Initial value problems

- **Initial value problems (IVP)** $\dot{x} = f(t, x) \qquad x(0) = x_0$

- **Variational Differential Equation**

$$\frac{d}{dt}\left(\frac{\partial x}{\partial x_0}\right) = f_x(t, x)\frac{\partial x}{\partial x_0} \qquad \text{mit} \qquad f_x = \frac{\partial f}{\partial x}$$

with $G := \frac{\partial x}{\partial x_0}$ Sensitivity matrix, transfer matrix

follows

$$\begin{aligned} \dot{G}(t; t_0) &= f_x(t, x)G(t; t_0) \\ G(t_0; t_0) &= I \end{aligned}$$

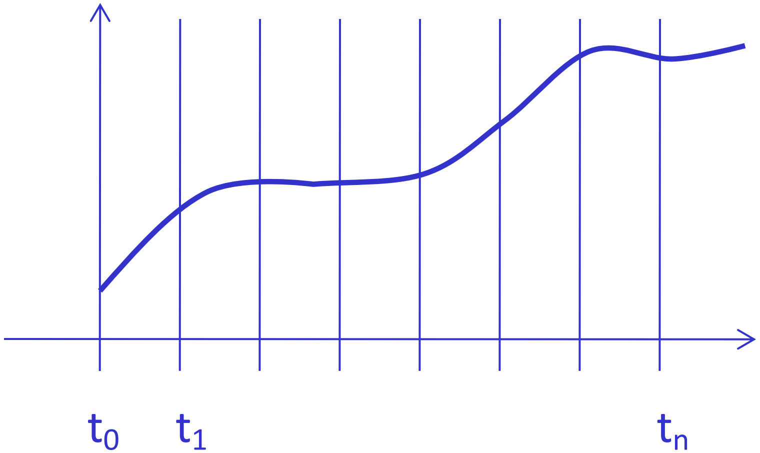
The solution of **the variational differential equation exists if the solution of the IVP exists and is differentiable**

IVP – Variational differential equation

- Properties of the sensitivity matrix

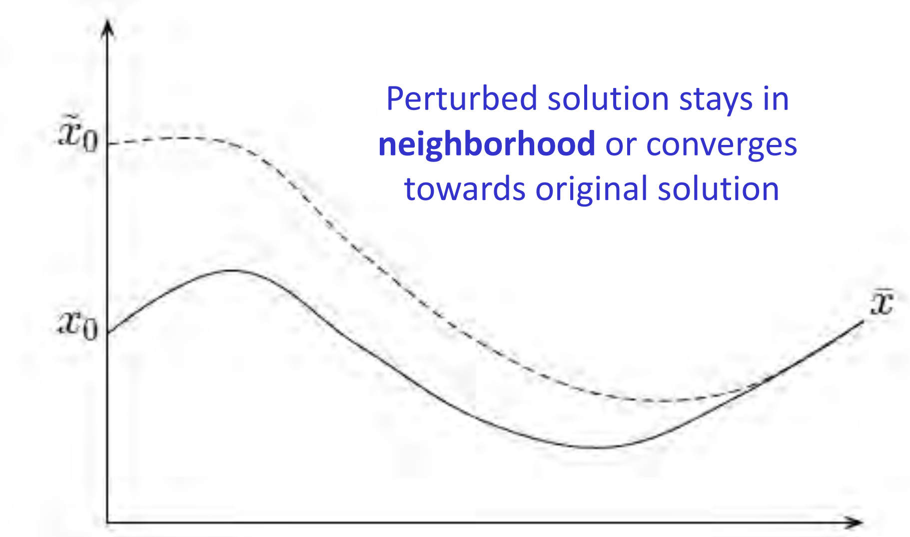
  - $G(t_n;t_0) = G(t_n;t_{n-1}) \cdot G(t_{n-1};t_{n-2}) \cdots G(t_2;t_1) \cdot G(t_1;t_0)$
  - $G(t_1;t_0) = G^{-1}(t_0;t_1)$ if G is regular

with $G(t_i;t_{i-1}) = \dfrac{\partial x(t_i)}{\partial x(t_{i-1})}$

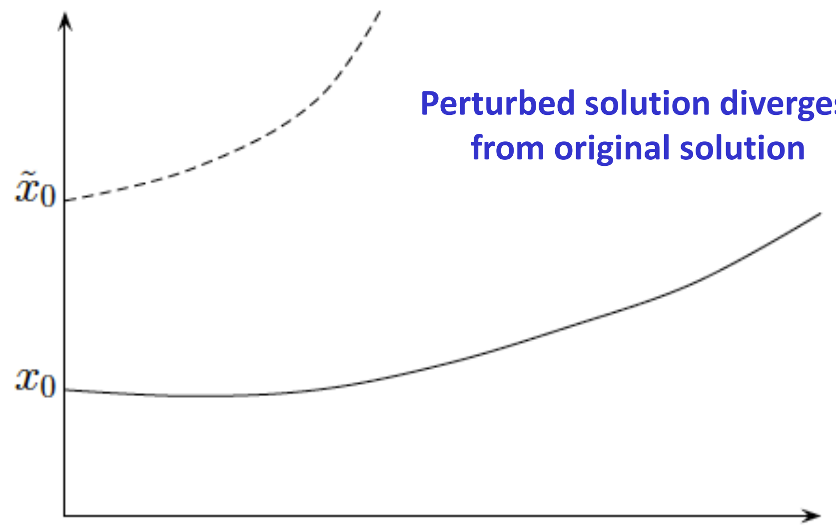


Stability of the solution of IVPs

- How does the IVP solution change if the initial value is slighly perturbed?



stable solution



unstable solution

- This type of stability is a property of the differential equation and the particular solution. It should not be confounded with the stability properties of integrators or other algorithms (see previous talk)

Bondary value problems

- In contrast to initial value problems (IVP) **Statements about Existence and uniqueness are not trivial** for boundary value problems. In general, no globally valid statement can be made about existence and uniqueness.

- Boundary value problems can have one, no, or infinitely many solutions.

BVP – An example for different numbers of solutions

ODE:

$$\ddot{x} + x = 0 \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{v}_x \end{pmatrix} = \begin{pmatrix} v_x \\ -x \end{pmatrix}$$

$$\Rightarrow \quad x(t) = a_1 \sin t + a_2 \cos t$$

a. **Boundary condition 1:**

$$\begin{aligned} s.t.\ x(0) &= 0 \\ x(\frac{\pi}{2}) &= 1 \end{aligned}$$

unique solution:

$$a_1 = 1, a_2 = 0$$

$$x(t) = \sin t$$

b. **Boundary condition 2**

$$\begin{aligned} s.t.\ x(0) &= 0 \\ x(\pi) &= 0 \end{aligned}$$

infinitely many solutions

$x(t) = a_1 \sin t,\ a_1$ arbitrary

c. **Boundary condition 3**

$$\begin{aligned} s.t.\ x(0) &= 0 \\ x(\pi) &= 1 \end{aligned}$$

no solution

Existence and uniqueness of solution of linear BVP

- Linear 2 point BVP

$$\begin{aligned} \dot{x}(t) = C(t) \cdot x(t) + \varphi(t) &= f(t,x) \\ \text{s.t.} \qquad Ax(t_0) + Bx(t_{end}) &= r \end{aligned} \qquad A, B, C \in \mathbb{R}^{n \times n}$$

- This problem has got a unique solution, if the matrix

$$E = A + BG(t_{end}; t_0)$$

is regular, and where G is the solution of the variational differential equation:

$$\begin{aligned} \dot{G}(t;t_0) &= C(t)G(t;t_0) \\ G(t_0;t_0) &= I \end{aligned}$$

Back to our example

$$\ddot{x} + x = 0 \qquad \left.\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{array}\right\} \rightarrow \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{C} x$$

$$\dot{G} = C \cdot G \quad \text{mit} \quad G(0;0) = I \quad \Longrightarrow \quad G = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Case a)

$$\left.\begin{array}{l} x(0) = 0 \\ x(\frac{\pi}{2}) = 1 \end{array}\right\} \qquad \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{B} \begin{pmatrix} x_1(\frac{\pi}{2}) \\ x_2(\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$E := A + B \; \underbrace{G(\frac{\pi}{2};0)}_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

➡ **E regular**

unique solution

Nonlinear BVPs

$$\begin{aligned} \dot{x} &= f(t, x(t)) \\ r(x(t_0), x(t_{end})) &= 0 \end{aligned} \qquad f \in C^2 \qquad (1)$$

- There are no general statements about global existence and uniqueness (only for special examples and conditions)
- We consider the corresponding parameterized IVP:

$$\begin{aligned} \dot{x} &= f(t, x), \qquad t \geq t_0 \\ x(t_0) &= s \end{aligned}$$

- **Theorem (Ascher et al)**:
  The number of solutions of the nonlinear BVP (1) is equal to the number of different roots s* of the equation

$$g(s) = r(s, x(t_{end}; s)) = 0$$

Local uniqueness of nonlinear BVP

- If BVP (1) has a solution $\bar{x}(t)$, this solution is locally unique if the matrix

$$E := A + BG(t_{end}; t_0)$$

is regular with

$$A = \frac{\partial r}{\partial x_0}(\bar{x}(t_0), \bar{x}(t_{end})), \qquad B = \frac{\partial r}{\partial x_{end}}(\bar{x}(t_0), \bar{x}(t_{end}))$$

- and G is solution of the variational DE:

$$\dot{G}(t; t_0) = f_x(t, \bar{x}(t)) \cdot G(t; t_0)$$

Condition of boundary value problems

- For well conditioned BVP, a small change in the right hand side or in the boundary conditions should only result in a small change of the solution

- The term "Condition of a BVP" corresponds to the "Stability of an IVP"

- Problem of dichotomy: rapidly increasing as well as rapidly decreasing modes must be avoided (a rapid decrease corresponds to a rapid increase for reverse time, i.e. a large influence of perturbations in the end value)

Thank you very much for your attention!