COMP4680 Notes

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0 Introduction

A mathematical optimisation problem requires us to minimize an objective function $f_0(x)$, subject to constraint functions $f_i(x) \leq b_i$ for $i = 1, \ldots, m$.

A solution, or optimal point, x^*, has the smallest value of f_0 among all vectors that satisfy the constraints.

There are three main types of problems which can be solved to different extents:

0.1 Least-squares Problems

Least-squares problems attempt to minimize $\|Ax - b\|_2^2$ for some matrix A and vector b.

There exists an analytical solution $x^* = (A^T A)^{-1} A^T b$, there are reliable and efficient algorithms with computation time $O(n^2 k)$ when $A \in \mathbb{R}^{k \times n}$, less if A satisfies certain structures.

0.2 Convex Optimization Problems

Convex optimization problems are optimization problems where both the objective and constraint functions are convex, and is a superset of least-squares problems.

There is no analytical solution, but there are algorithms with computation time $\max(O(n^3), O(n^2 m), F)$ where F is the cost of evaluating f_i's and their second derivatives.

0.3 Nonconvex Optimization Problems

There is no general way to solve nonconvex optimization problems: they all involve some kind of compromise.

We may use local optimization methods (nonlinear programming), which is fast and finds a local minima around an initial guess, but may not be the global minima.

Or we may use global optimization methods, which finds the global solution but requires exponential time complexity.

1 Preliminaries

1.1 Sets

A set, denoted as $S = \{a_1, \ldots, a_n\}$, is a collection of distinct objects.

Some common notations:

- $a \in S$ denotes a is an element of S
- $S \subseteq T$ denotes S is a subset of T, that is, every element of S is also an element of T
- $S \cup T$ denotes the union of S and T, that is, the set of all elements that are in S or T
- $S \cap T$ denotes the intersection of S and T, that is, the set of all elements that are in both S and T
- $S \times T$ denotes the Cartesian product of S and T, that is, the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$
- $S \setminus T$ denotes the set difference of S and T, that is, the set of all elements that are in S but not in T

Some common sets:

- $\mathbb{R}$ is the set of real numbers
- $\mathbb{R}^n$ is the set of n-dimensional real vectors
- $\mathbb{R}^{m \times n}$ is the set of $m \times n$ real matrices
- $\mathbb{C}$ is the set of complex numbers
- $\mathbb{Z}$ is the set of integera
- $\mathbb{R}_+$ is the set of nonnegative real numbers
- $\mathbb{R}_{++}$ is the set of positive real numbers
- $\emptyset$ is the empty set
- $[a, b]$ is the closed interval from a to b (i.e. $\{x \in \mathbb{R} \mid a \leq x \leq b\}$)
- (a, b) is the open interval from a to b (i.e. $\{x \in \mathbb{R} \mid a < x < b\}$)
- $[a, b)$ and $(a, b]$ are half-open intervals, defined similarly

Open and Closed Sets

A subset $S \subseteq \mathbb{R}$ is **open** if for every $x \in S$, there exists $\epsilon > 0$ such that if $\|y - x\|_2 < \epsilon$, then $y \in S$.

A subset $S \subseteq \mathbb{R}$ is **closed** if its complement $\mathbb{R} \setminus S$ is open.

A subset $S \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ such that $\|a - b\|_2 \leq M$ for all $a, b \in S$.

Infimum and Supremum

The **infimum** of a set $S \subseteq \mathbb{R}$, written as $\inf(S)$, is the largest $y \in \mathbb{R}$ such that $y \leq x$ for all $x \in S$. If no such y exists, we say $\inf(S) = -\infty$.

The **supremum** of a set $S \subseteq \mathbb{R}$, written as $\sup(S)$, is the smallest $y \in \mathbb{R}$ such that $y \geq x$ for all $x \in S$. If no such y exists, we say $\sup(S) = \infty$.

We define $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$.

1.2 Functions

A function $f : A \to B$ is a mapping from its **domain** A to its **codomain** B.

If $U \subseteq A$ and $V \subseteq B$, we define the **image** of U under f as $f(U) = \{f(x) \mid x \in U\} \subseteq B$, and the **preimage** of V under f as $f^{-1}(V) = \{x \in A \mid f(x) \in V\} \subseteq A$.

1.3 Vector Spaces

A vector space V is a set with two operations, vector addition and scalar multiplication, that satisfy the following axioms:

- $x + y = y + x$ (commutativity of vector addition)
- $(x + y) + z = x + (y + z)$ (associativity of vector addition)
- $x + \mathbf{0} = x$ (additive identity)
- $\forall x \in V, \exists y \in V$ such that $x + y = 0$, we write y as $-x$ (additive inverse)
- $\alpha(x + y) = \alpha x + \alpha y$ (right distributivity)
- $(\alpha + \beta)x = \alpha x + \beta x$ (left distributivity)
- $1x = x$ (multiplicative identity)

We define the **zero vector** as a vector with all elements equal to 0, and the **ones vector** as a vector with all elements equal to 1.

Euclidean Norm

The Euclidean norm of a vector $\mathbf{v} = (v_1, \dots, v_n)$ is

$$\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + x_n^2}$$

$\|\mathbf{v}\|_2$ measures the length of $\mathbf{v}$.

The norm satisfies:

- $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$

- $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality)
- $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positive definiteness)

There are other norms such as $\| \quad \|_1$ and $\| \quad \|_\infty$.

Inner Products

The inner product of two vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The inner product satisfies:

- $\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- $\langle \mathbf{u}_1 + \mathbf{u}_2, \mathbf{v} \rangle = \langle \mathbf{u}_1, \mathbf{v} \rangle + \langle \mathbf{u}_2, \mathbf{v} \rangle$
- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|_2^2$

Subspaces

A subspace of a vector space is a subset of the vector space that is also a vector space.

Independence

A set of vectors $v_1, \ldots, v_n$ is (linearly) independent if and only if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$.

Conversely, if a set of vectors is linearly dependent, we can write one of the vectors as a linear combination of the others.

Bases

The set of vectors $\{v_1, \ldots, v_n\}$ form a basis of a vector space V if

- they are linearly independent
- they span V, that is, every vector in V can be written as a linear combination of the vectors in the set

Equivalently, $\{v_1, \ldots, v_n\}$ form a basis for V if every $v \in V$ can be uniquely expressed as $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

We define the **dimension** of a vector space V to be the number of vectors in any basis of V.

The standard basis of $\mathbb{R}^n$ is the set of vectors $\{e_1, \ldots, e_n\}$ where e_i is the vector with a 1 in the i^{th} position and 0 elsewhere.

1.4 Matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array of real numbers with m rows and n columns.

We write A_{ij} for the entry in the i^{th} row and j^{th} column of A.

A $n \times 1$ matrix is called a (column) **vector**, and a $1 \times n$ matrix is called a row **vector**.

We say a matrix is **diagonal** if its nonzero entries are all on the main diagonal (top left to bottom right).

The **zero matrix**, denoted $\mathbf{0}_{m \times n}$, is the matrix with all entries equal to zero.

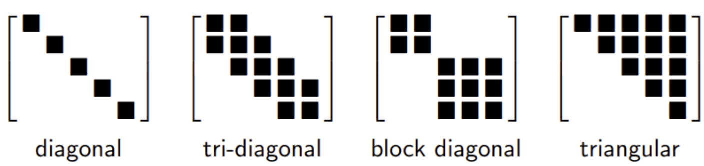
The **identity matrix**, denoted $\boldsymbol{I}_n$, is the $n \times n$ matrix with ones on the main diagonal and zeros elsewhere.

Special Types of Matrices

A matrix is **triangular** if all its entries above or below the main diagonal are zero. In particular, we refer to a matrix as **upper triangular** if all its entries below the main diagonal are zero, and **lower triangular** if all its entries above the main diagonal are zero.

A matrix is **block diagonal** if it is diagonal and each diagonal entry is itself a matrix.

A matrix is **tri-diagonal** if it has nonzero entries only on the main diagonal and the diagonals immediately above and below the main diagonal.



Matrix Transpose

Transpose, denotes as ^{T}, flips a matrix over is main diagonal, i.e. if A is an $m \times n$ matrix then A^T is an $n \times m$ matrix. It satisfies the following properties:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$

If a matrix A satisfies $A = A^T$ we say A is **symmetric**.

If a matrix A satisfies $A = -A^T$ we say A is **anti-symmetric**.

Every square matrix A can be written as the sum of a symmetric part and an anti-symmetric part:

$$A = \underbrace{\frac{1}{2}\left(A + A^T\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(A - A^T\right)}_{\text{anti-symmetric}}$$

Notation for Symmetric Matrices

We write

- $\mathbb{S}^n$ for the set of symmetric $n \times n$ matrices
- $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \preceq 0\}$ for the positive semi-definite $n \times n$ matrices

$$X \in \mathbb{S}^n_+ \Leftrightarrow z^T X z \geq 0 \text{ for all } z.$$

- $\mathbb{S}^n_{++} = \{X \in \mathbb{S}^n \mid X \prec 0\}$ for the positive definite $n \times n$ matrices

Matrix Addition

Two matrices of the same size can be added together: we simply add the corresponding elements in each matrix.

Matrix Multiplication

The product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is an $m \times p$ matrix with elements

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Matrix multiplication satisfies:

- $(AB)C = A(BC)$ (associativity)
- $A(B + C) = AB + AC$ (left distributivity)
- $(A + B)C = AC + BC$ (right distributivity)

but matrix multiplication is not commutative: $AB \neq BA$ generally.

Null Space

The null space of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

$\mathcal{N}(A)$ can be interpreted as

- the set of all vectors mapped to zero by $y = Ax$
- the set of all vectors orthogonal to the rows of A

Range Space

The range space of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

$\mathcal{R}(A)$ can be interpreted as

- the set of all vectors that can be "hit" by $y = Ax$
- the span of the columns of A
- the set of all vectors y such that $Ax = y$ has a solution

Orthogonal Complement

The orthogonal complement of $V \subseteq \mathbb{R}^n$ is defined as

$$V^\perp = \left\{x \mid z^T x = 0 \text{ for all } z \in V\right\}.$$

We have $V \oplus V^\perp = \mathbb{R}^n$.
A result from the Fundamental Theorem of Linear Algebra states that $\mathcal{N}(A) = \mathcal{R}(A^T)^\perp$.

Rank

The rank of a matrix $A \in \mathbb{R}^{m \times n}$ is

$$\text{rank}(A) = \dim \mathcal{R}(A).$$

- $\text{rank}(A) = \text{rank}(A^T)$
- $\text{rank}(A)$ is the maximum number of independent columns (or rows) of A. Hence $\text{rank}(A) \leq \min\{m, n\}$.
- $\text{rank}(A) + \dim \mathcal{N}(A) = n$ (rank-nullity)

We say a matrix A is **full rank** if $\text{rank}(A) = \min\{m, n\}$.
The rank of the product of two matrices satisfies

$$\text{rank}(AB) \leq \min\left\{\text{rank}(A), \text{rank}(B)\right\}.$$

If $A \in \mathbb{R}^{m \times n}$ has rank r then A can be factored as BC with $B \in \mathbb{R}^{m \times r}$ and $C \in \mathbb{R}^{r \times n}$.

Trace

The trace of a square matrix $A \in \mathbb{R}^{n\times n}$ is the sum of its diagonal entries, i.e.

$$\mathrm{tr}(A) = \sum_{j=1}^{n} A_{jj}.$$

Trace satisfies the following properties:

- $\mathrm{tr}(A) = \mathrm{tr}(A^T)$
- $\mathrm{tr}(\alpha A + \beta B) = \alpha \mathrm{tr}(A) + \beta \mathrm{tr}(B)$
- if AB is square then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Determinant

The determinant of a square matrix $A \in \mathbb{R}^{n\times n}$ is a function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ that satisfies the following properties:

- $\det \boldsymbol{I} = 1$
- $\det \alpha A = \alpha^n \det A$
- swapping any two rows/columns changes the sign of the determinant
- $\det AB = \det A \det B$

We can interpret the determinant as the volume of the parallelepiped spanned by the rows (or columns) of A.

Matrix Inverse

The inverse of a square matrix $A \in \mathbb{R}^{n\times n}$ is a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = \boldsymbol{I}$$

A matrix is **invertible** (i.e. has an inverse) if and only if $\det A \neq 0$. This is equivalent to:

- the columns/rows of A form a basis for $\mathbb{R}^n$
- $y = Ax$ has a unique solution for all $x \in \mathbb{R}^n$
- A is full-rank (i.e. $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = \mathbb{R}^n$)
- $\det A^T A = \det AA^T \neq 0$

Cauchy-Schwarz Inequality

For any vectors $x, y \in \mathbb{R}^n$, we have that

$$\left|x^T y\right| \leq \|x\|_2 \|y\|_2.$$

The angle between vectors in $\mathbb{R}^n$ is given by

$$\theta = \cos^{-1}\left(\frac{x^T y}{\|x\|_2 \|y\|_2}\right).$$

- If x and y are aligned then $x^T y =$

Eigenvalues and Eigenvectors

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ if

$$\det(\lambda I - A) = 0.$$

Equivalently, there exists a non-zero $v \in \mathbb{C}^n$ such that $(\lambda I - A)v = 0$, or $Av = \lambda v$. Any such v here is called an eigenvector of A, associated with eigenvalue λ.

The eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n \times n}$ are real. Moreover, there exists a set of orthogonal eigenvectors $q_1, \ldots, q_n$ such that $Aq_i = \lambda_i q_i$ and $q_i^T q_j = 0$ if $i \neq j$.

In matrix form, there is an orthonormal Q such that $A = Q\Lambda Q^T$.

Norm Matrices

A matrix norm is a function $\| \quad \| : \mathbb{R}^{m \times n} \to \mathbb{R}$ that, similar to vector norms, satisfy linearity, positive definiteness, and the triangle inequality.

- Induced norms: $\|A\| = \sup\left\{\|Ax\| \mid x \in \mathbb{R}^n, \|x\| \leq 1\right\}$
- Frobenius norm: $\|A\|_F = \sqrt{\left\{\sum_{ij} a_{ij}^2\right\}}$
- Nuclear norm: $\|A\|_* = \sum_i \sigma_i(A) = \mathrm{tr}(\sqrt{A^T A})$

Square matrices also satisfy the sub-multiplicative property:

$$\|AB\| \leq \|A\|\|B\|.$$

1.5 Matrix Factorization

LU Factorization

Every nonsingular matrix $A \in \mathbb{R}^{n \times n}$ can be factored as

$$A = PLU$$

where P is a permutation matrix, L is unit lower triangular, and U is upper triangular and non-singular.

Cholesky Factorization

Every symetric positive definite matrix $A \in \mathbb{R}^{n\times n}$ can be factored as

$$A = LL^T$$

where L is lower triangular and non-singular with positive diagonal elements.

Singular Value Decomposition

Any matrix A can be decomposed as

$$A = U\Sigma V^T$$

where $A \in \mathbb{R}^{m\times n}$ has rank r, $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{n\times r}$ which satisfy $U^TU = I$ and $V^TV = I$, and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Since $A^TA = V\Sigma^2V^T$ we have v_i are the eigenvectors of A^TA. Similarly, u_i are the eigenvectors of AA^T.

We can use SVD to interpret a linear map $y = Ax$ as follows:

- we compute coefficients of x along the input directions $v_1, \dots, v_r$
- scale the coefficients by σ_i
- re-constitute along the output directions $u_1, \dots, u_r$

Here, v_1 is the most sensitive input direction, and u_1 is the highest gain output direction.

Matrix Calculus

We can compute partial derivatives of a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$\frac{\partial f(x)}{\partial x_{ij}} = \lim_{\alpha\to 0} \frac{f(x + \alpha e_i e_j^T) - f(x)}{\alpha}.$$

We can also compute the gradient (Jacobian) of f as

$$\nabla_A f(A) = \begin{pmatrix} \frac{\partial f}{\partial A_{11}} & \frac{\partial f}{\partial A_{12}} & \cdots & \frac{\partial f}{\partial A_{1n}} \\ \frac{\partial f}{\partial A_{21}} & \frac{\partial f}{\partial A_{22}} & \cdots & \frac{\partial f}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial A_{m1}} & \frac{\partial f}{\partial A_{m2}} & \cdots & \frac{\partial f}{\partial A_{mn}} \end{pmatrix}$$

Partial derivatives are linear:

- $\nabla_A(f + g) = \nabla_A f + \nabla_A g$
- $\nabla_A(tf) = t\nabla_A f$

Chain rule and product rule also extend to matrix calculus.

In vector calculus, the **Hessian** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the matrix of second-order partial derivatives of f, i.e.

$$\nabla_x^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

1.6 Probability Theory

A probability distribution is a function that maps outcomes of an experiment to probabilities:

- for discrete variables we have **probability mass functions**
- for continuous variables we have **probability density functions**

The **mean** or **expected value** of a random variable is the sum of possible values weighted by their probabilities:

$$\mathbb{E}[X] = \int_x x P(X = x) \, \mathrm{d}x$$

The **variance** of a random variable X is $\mathbb{E}[(X - \mathbb{E}[X])^2]$.

1.7 Geometric Concepts

Lines

A line through two points x_1 and x_2 has the equation $x = \theta x_1 + (1 - \theta) x_2$.

The **line segment** between x_1 and x_2 is the set of points $x = \theta x_1 + (1 - \theta) x_2$ for $0 \leq \theta \leq 1$.

Affine Sets

An affine set contains the line through any two distinct points in the set: if $x_1, x_2 \in S$ then $\theta x_1 + (1 - \theta) x_2 \in S$.

Every affine set can be expressed as the solution set of a system of linear equations.

Convex Sets

A convex set contains the line segment between any two distinct points in the set: if $x_1, x_2 \in S$ then $\theta x_1 + (1 - \theta) x_2 \in S$ for $0 \leq \theta \leq 1$.

Common examples:

- nonnegative orthant: $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0\}$
- positive semidefinite matrices: $\mathbb{S}_+^n = \left\{X \in \mathbb{R}^{n \times n} \mid z^T X z \geq 0, z \in \mathbb{R}^n\right\}$

Convex Combinations and Hulls

A convex combination of $x_1, \ldots, x_k$ is any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

where $\theta_1 + \theta_2 + \ldots + \theta_k = 1$ and $\theta_i \geq 0$ for all i.

The convex hull of a set S, $\text{conv}(S)$, is the set of all convex combinations of points in S.

Convex Cones

A **conic combination** of points x_1 and x_2 is any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1, \theta_2 \geq 0$.

A cone is a set containing all non-negative multiples of its points (i.e. if $x \in C$ then $\alpha x \in C$ for all $\alpha \geq 0$).

A convex cone is a set containing all conic combinations of its points.

Hyperplanes and Halfspaces

A hyperplane is a set of the form $\left\{x \mid a^T x = b\right\}$ with $a \neq 0$.

A halfspace is a set of the form $\left\{x \mid a^T x \leq b\right\}$ with $a \neq 0$.

In the 3D case, a plane is a hyperplane while a halfspace is everything on one side of the plane.

Hyperplanes are affine and convex, and halfspaces are convex.

Euclidean Balls and Ellipsoids

A Euclidean ball with center x and radius r is a set $B(x, r) = \{y \mid \|y - x\|_2 \leq r\}$.

An ellipsoid is a set of the form

$$\left\{y \mid (y - x)^T P^{-1} (y - x) \leq 1\right\}$$

with $P \in \mathbb{S}^n_{++}$ (symmetric positive definite).

Alternatively, we can represent a ball as

$$B(x, r) = \{x + ru \mid \|u\|_2 \leq 1\}$$

and an ellipsoid as

$$\{x + Au \mid \|u\|_2 \leq 1\}$$

with A a square, nonsinguar matrix.

Norm Balls and Cones

A norm ball with center x and radius r is the set $\{y \mid \|y - x\| \leq r\}$.

A norm cone is the set $\{(x, t) \mid \|x\| \leq t\}$. Norm balls and cones are convex.

Polyhedra

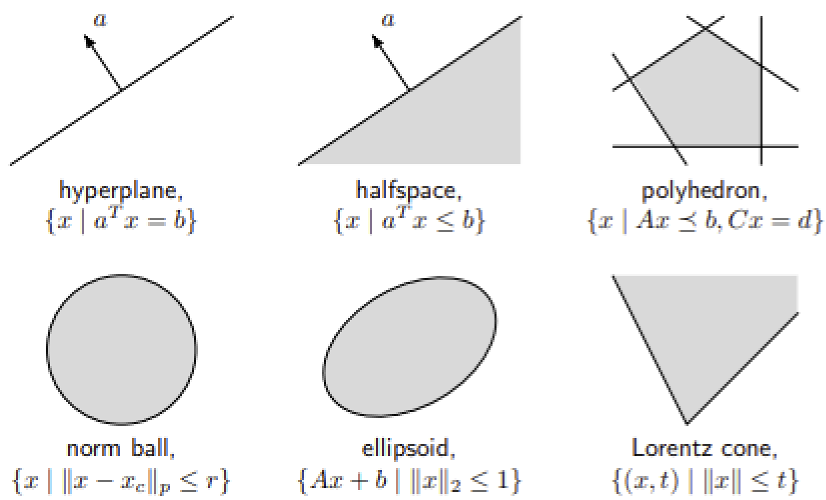
A polyhedron is the solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, Cx = d$$

where $\preceq$ is componentwise inequality.

So a polyhedron is the intersection of a finite number of halfspaces and hyperplanes. Polyhedra are convex sets.

Summary of Convex Sets



Property of Convex Sets

We can either show a set C is convex by applying the definition, or obtain C using the following properties:

- the intersection of any number of convex sets is convex
- the image of a convex set under an affine map is convex (recall affine maps are of the form $f(x) = Ax + b$)
- the preimage of a convex set under an affine map is convex
- perspective functions preserve convexity: these are functions of the form

$$P(x, t) = x/t, t > 0$$

- linear-fractional functions preserve convexity: these are functions of the form

$$f(x) = \frac{Ax + b}{c^T x + d}, c^T x + d > 0$$

Generalized Inequalities

A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Examples include,

- the nonnegative orthant
- positive semidefinite cone $\mathbb{S}_+^n$
- nonnegative polynomials on $[0, 1]$

$$K = \left\{x \in \mathbb{R}^n \mid x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n \geq 0, t \in [0, 1]\right\}$$

A generalized inequality is a relation of the form

$$x \preceq_K y \Leftrightarrow y - x \in K$$

and

$$x \prec_K y \Leftrightarrow y - x \in \operatorname{int}(K)$$

where $\operatorname{int}(K)$ denotes the interior of K.

- In the case of the nonnegative orthant $K = \mathbb{R}_+^n$, we have componentwise inequality $x \preceq y \Leftrightarrow x_i \leq y_i$ for all i.
- In the case of the positive semidefinite cone $K = \mathbb{S}_+^n$, we have $X \preceq Y \Leftrightarrow Y - X \in \mathbb{S}_+^n$.

In these cases, we omit the subscript K and write $x \preceq y$ and $X \preceq Y$.

Minimum and Minimal Elements

As $\preceq$ is not a linear order, we have to define

- $x \in S$ is the minimum element of S with respect to $\preceq$ if $x \preceq y$ for all $y \in S$
- $x \in S$ is the minimal element of S with respect to $\preceq$ if there is no $y \in S$ such that $y \prec x$

1.7.1 Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, there exists $a \neq 0, b$ such that

$$a^T x \leq b \text{ for } x \in C, \quad , a^T x \geq b \text{ for } x \in D$$

That is, there exists a hyperplane separating any two convex sets C and D. (Strict separation requires additional assumptions, e.g. C is closed, D is singleton)

Supporting Hyperplane Theorem

A supporting hyperplane to a set C at a boundary point x_0 is

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T \leq a^T x_0$ for all $x \in C$.

This can be thought of as a tangent hyperplane.

The theorem states that there exists a supporting hyperplane at any point on the boundary of a convex set.