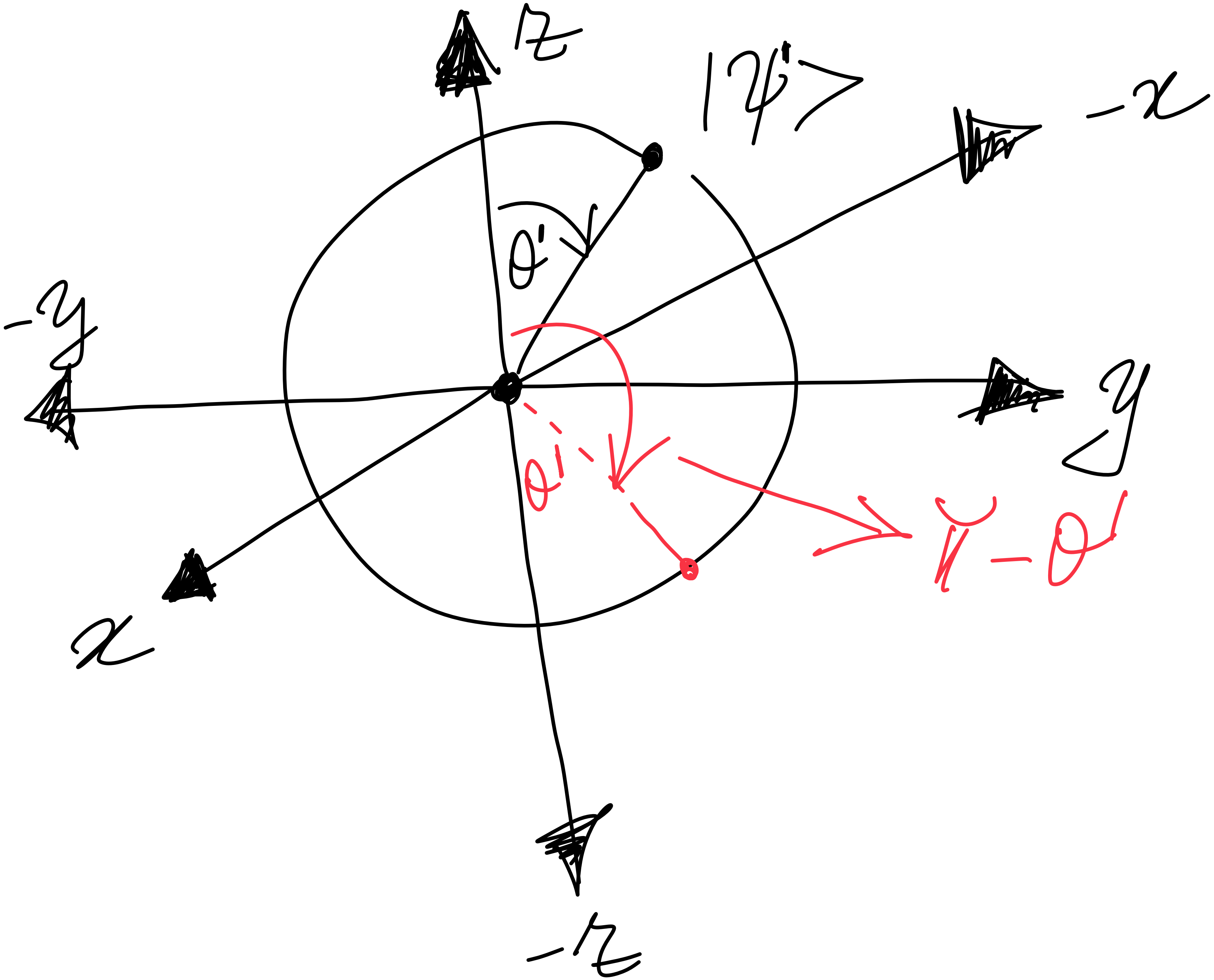
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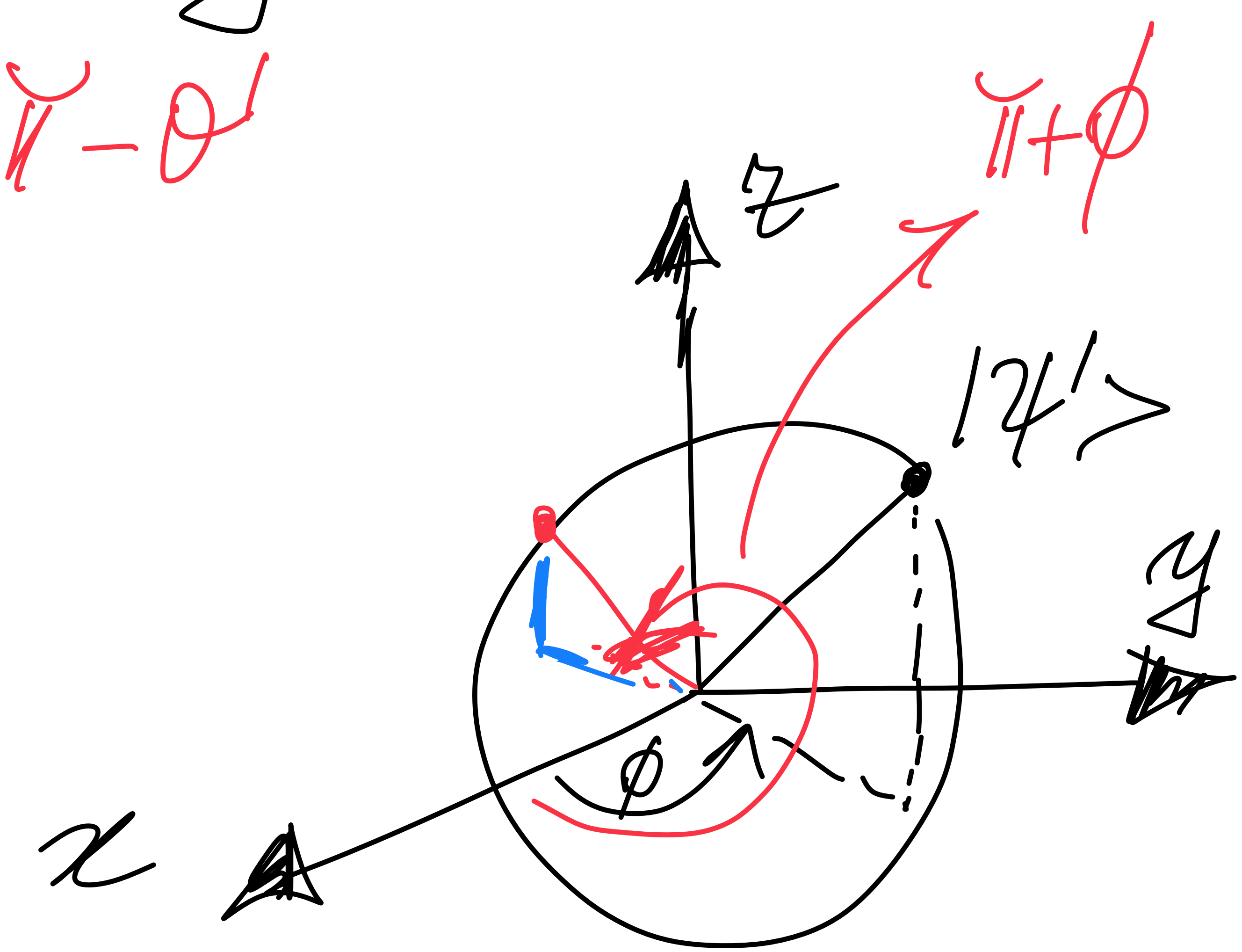
z
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INTRODUCTION TO QUANTUM COMPUTING AND PHYSICAL BASICS OF COMPUTING

Quantum Mechanics [Basics of the Basics]

Ulya Karpuzcu

Axioms

→ Applies for any dim.
→ Any math. entity satisfying these $\equiv |\cdot\rangle$

① $|A\rangle + |B\rangle = |C\rangle$

② $(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$

③ $|A\rangle + |B\rangle = |B\rangle + |A\rangle$

④ $|A\rangle + 0 = |A\rangle$

⑤ $|A\rangle + (-|A\rangle) = 0$

$w \in \mathbb{C}$

$z \in \mathbb{C}$

⑥ $z|A\rangle = |zA\rangle = |B\rangle$

⑦ $z(|A\rangle + |B\rangle) = z|A\rangle + z|B\rangle$

$(z+w)|A\rangle = z|A\rangle + w|A\rangle$

⇒ linear.

Complex Conjugate Equivalent $\Rightarrow \langle \cdot |$

$$|A\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \underbrace{\langle A| = [a^* \quad b^*]}_{\text{All 4 axioms apply}}$$

$$(z \cdot |A\rangle)^* = \langle A| \cdot z^*$$

Inner Product : $\langle A|A\rangle =$

$$= [a^* \quad b^*] \begin{bmatrix} a \\ b \end{bmatrix} = a^* a + b^* b$$

Axioms for inner product

$$\langle C | (|A\rangle + |B\rangle) = \langle C|B\rangle + \langle C|A\rangle$$

$$\langle A|B\rangle = \langle B|A\rangle^*$$

"Normal" : $\langle A|A\rangle = 1$

"Orthogonal" : $\langle A|B\rangle = 0$

Orthonormal Basis: $|i\rangle$

$|i\rangle \quad i = 1, 2, 3, \ldots, N$

$|A\rangle = \sum_i c_i |i\rangle \quad N = 2^n$ ← #qbits

Span N-dim. complex space

Orthonormal Basis: $|i\rangle, |j\rangle$

$|i\rangle \quad i = 1, 2, 3, \ldots, N$

$N = 2^{n}$ → # qbits

$|A\rangle = \sum_i c_i |i\rangle$

Span N-dim. complex space

$|j\rangle$ with $j = 1, 2, 3, \ldots, N$

$\langle j|A\rangle = \sum_i c_i \underbrace{\langle j|i\rangle}$ $= 1$ if $i=j$, 0 if $j \neq i$

$= c_j$

$$\langle j | A \rangle = c_j \qquad i, j = 1, 2, 3, \ldots, N$$

$$|A\rangle = \sum_i \underbrace{\langle i | A \rangle}_{c_i} |i\rangle$$

Example: $N = 4$

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$|1\rangle, |2\rangle, |3\rangle, |4\rangle$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example: Spin Orientation

$\uparrow$ up $\equiv |0\rangle$ or $|u\rangle$

$\downarrow$ down $\equiv |1\rangle$ or $|d\rangle$

$\underbrace{\qquad\qquad}_{\text{Computational basis}}$

$$\langle u|d\rangle = 0$$
$$\langle d|u\rangle = 0$$

Orthogonality: if in "up" state cannot be in "down" state and vice versa

$$|A\rangle = \underbrace{\alpha_u}_{\langle u|A\rangle}|u\rangle + \underbrace{\alpha_d}_{\langle d|A\rangle}|d\rangle$$

Inner Product

$\Downarrow$

"Overlap"

$$|A\rangle = \underbrace{\alpha_u}_{\langle u|A\rangle}|u\rangle + \underbrace{\alpha_d}_{\langle d|A\rangle}|d\rangle$$

$$P_u = \alpha_u^* \alpha_u = \underbrace{\langle u|A\rangle^*}_{\langle A|u\rangle}\langle u|A\rangle$$

$$P_d = \alpha_d^* \alpha_d$$

$$= \langle A|d\rangle\langle d|A\rangle$$

$$\langle A|A\rangle = 1 \qquad P_u + P_d = 1$$

Basics

- States = vectors in a complex vector space
- Physical observables described by linear operators
  - Observables = what we can measure
  - Operators must be linear and Hermitian

Observables

- What we can measure
- Examples
  - Coordinates of a point
  - Energy of a system
  - ...
- "M(achine)" analogy

$$M|A >= |B >$$

Linear Operators

$$M|A>=|B>$$

1. Should result in a unique output for every (input) vector in the space

2. For any complex z: $Mz|A>=z|B>$

3. $M(|A>+|B>)=M|A>+M|B>$

Linear Operators

$M|A >= |B >$

$$|A >= \sum_j \alpha_j |j > \qquad |B >= \sum_j \beta_j |j >$$

$$\sum_j M|j > \alpha_j = \sum_j \beta_j |j >$$

Linear Operators

$M|A>=|B>$

$$|A>=\sum_j \alpha_j|j> \qquad |B>=\sum_j \beta_j|j>$$

$$\sum_j M|j>\alpha_j=\sum_j \beta_j|j> \qquad \Big| \cdot <k| \Big.$$

$$\sum_j <k|M|j>\alpha_j=\sum_j \beta_j <k|j>$$

Linear Operators

$M|A> = |B>$

$$|A> = \sum_j \alpha_j |j> \qquad |B> = \sum_j \beta_j |j>$$

$$\sum_j M|j> \alpha_j = \sum_j \beta_j |j>$$

$$\sum_j < k|M|j> \alpha_j = \sum_j \beta_j < k|j>$$

$< k|j> = 0$ if j and k not equal (1 otherwise)

$$\sum_j < k|M|j> \alpha_j = \beta_k \qquad \sum_j m_{kj} \alpha_j = \beta_k$$

m_{kj} matrix elements of M

Linear Operators

$M|A>=|B>$

$$\sum_j M|j> \alpha_j = \sum_j \beta_j |j>$$

$$\sum_j <k|M|j> \alpha_j = \sum_j \beta_j <k|j>$$

$<k|j>= 0$ if j and k not equal (1 otherwise)

$$\sum_j <k|M|j> \alpha_j = \beta_k$$

$$\sum_j m_{kj}\alpha_j = \beta_k$$

m_{kj} matrix elements of M

arranged by an NxN matrix

Eigenvectors and Eigenvalues

- A linear operator acting on a vector generally changes the direction of the vector
- Any linear operator preserves the direction of specific input vectors
  - Eigenvectors

$$M|\lambda >= \lambda|\lambda >$$

- λ: both a complex number and a ket
- Constant multiplier: eigenvalue (generally complex)

Linear Operators

Can also act on bra-vectors

$$< B| = [\beta_1^* \beta_2^* ... \beta_n^*]$$

$$< B|M$$

Hermitian Conjugation

Complex conjugation for operators

$$M|A> = |B>$$

$$\sum_i m_{ji}\alpha_i = \beta_j$$

$z \to z^*$

$|A> \to <A|$

$M \to ?$

M^t

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Hermitian Conjugation

Complex conjugation for operators

$$M|A> = |B>$$

$$\sum_i m_{ji}\alpha_i = \beta_j$$

complex conjugate?

$$\sum_i m^*_{ji}\alpha^*_i = \beta^*_j$$

Hermitian Conjugation

Complex conjugation for operators

$$M|A>=|B>$$

$$\sum_i m_{ji}\alpha_i = \beta_j$$

`complex conjugate?`

$$\sum_i m_{ji}^* \alpha_i^* = \beta_j^*$$

`matrix form using kets?`

Hermitian Conjugation

Complex conjugation for operators

$$M|A> = |B> \rightarrow \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ & \vdots & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{bmatrix}$$

$$\sum_i m_{ji}\alpha_i = \beta_j$$

complex conjugate?

$$\sum_i m_{ji}^* \alpha_i^* = \beta_j^*$$

matrix form using kets?

$$< A|M^\dagger = [\alpha_1^* \ \alpha_2^* \ \alpha_3^*] \begin{bmatrix} m_{11}^* & m_{21}^* & m_{31}^* \\ m_{12}^* & m_{22}^* & m_{32}^* \\ m_{13}^* & m_{23}^* & m_{33}^* \end{bmatrix}$$

Hermitian Conjugation

changing an equation from ket-form to bra-form:

1. Interchange rows and column (i.e., transpose matrix)
2. Complex conjugate each matrix element

Hermitian Conjugation

$M^\dagger$ Hermitian conjugate (complex conjugate of transpose)

$$M|A>=|B>$$

$<A|M^\dagger =< B|$ where $M^\dagger = [M^T]^*$

Hermitian Operators

- Any measurement renders real numbers
- Observables are represented by linear operators
  - which are equal to their own Hermitian conjugates
  - called "Hermitian operators"
  - special properties

$$M^\dagger = M$$

$$m_{ji} = m_{ij}^*$$

Hermitian Operators

$$M^\dagger = M$$

- Have real eigenvalues

$L|\lambda >= \lambda|\lambda >$ `assume L is Hermitian`

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda> = \lambda|\lambda>$ assume L is Hermitian

by definition of Hermitian conjugation

Cheatsheet: Hermitian Conjugation

$M^\dagger$ Hermitian conjugate (complex conjugate of transpose)

$$M|A >= |B >$$

$$< A|M^\dagger =< B| \quad \texttt{where} \quad M^\dagger = [M^T]^*$$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda >= \lambda|\lambda >$ assume L is Hermitian

by definition of Hermitian conjugation

$< \lambda|L^\dagger =< \lambda|\lambda^*$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda >= \lambda|\lambda >$ assume L is Hermitian

by definition of Hermitian conjugation

$$< \lambda|L^\dagger =< \lambda|\lambda^*$$

$< \lambda|L =< \lambda|\lambda^*$ as $L = L^\dagger$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda> = \lambda|\lambda>$ `assume L is Hermitian` `multiply by` $<\lambda|$

`by definition of Hermitian conjugation`

$$<\lambda|L^\dagger = <\lambda|\lambda^*$$

$<\lambda|L = <\lambda|\lambda^*$ `as` $L = L^\dagger$ `multiply by` $|\lambda>$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda >= \lambda|\lambda >$ assume L is Hermitian multiply by $< \lambda|$

by definition of Hermitian conjugation

$$< \lambda|L^\dagger =< \lambda|\lambda^*$$

$< \lambda|L =< \lambda|\lambda^*$ as $L = L^\dagger$ multiply by $|\lambda >$

$$< \lambda|L|\lambda >= \lambda < \lambda|\lambda >$$

$$< \lambda|L|\lambda >= \lambda^* < \lambda|\lambda >$$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda >= \lambda|\lambda >$ assume L is Hermitian multiply by $< \lambda|$

by definition of Hermitian conjugation

$< \lambda|L^\dagger =< \lambda|\lambda^*$

$< \lambda|L =< \lambda|\lambda^*$ as $L = L^\dagger$ multiply by $|\lambda >$

$$< \lambda|L|\lambda >= \lambda < \lambda|\lambda >$$

$$< \lambda|L|\lambda >= \lambda^* < \lambda|\lambda >$$

for both equations to be true $\lambda = \lambda^*$

Hermitian Operators $M^\dagger = M$

- Have real eigenvalues

$L|\lambda> = \lambda|\lambda>$ assume L is Hermitian multiply by $<\lambda|$

by definition of Hermitian conjugation

$<\lambda|L^\dagger = <\lambda|\lambda^*$

$<\lambda|L = <\lambda|\lambda^*$ as $L = L^\dagger$ multiply by $|\lambda>$

$<\lambda|L|\lambda> = \lambda <\lambda|\lambda>$

$<\lambda|L|\lambda> = \lambda^* <\lambda|\lambda>$

for both equations to be true $\lambda = \lambda^*$ QED

Hermitian Operators & Orthonormal Basis

- **Observables are represented by Hermitian operators**
- **Eigenvectors of a Hermitian operator form a complete set**
  - Linear combinations of eigenvectors can represent any vector that the operator can generate
- **The eigenvectors corresponding to different eigenvalues are orthogonal**
- **Even for equal eigenvalues, corresponding eigenvectors can be chosen orthogonal**
  - Different eigenvectors having the same eigenvalue: degeneracy

Eigenvectors of a Hermitian operator form an orthonormal basis!

Hermitian Operators & Orthonormal Basis

- **The eigenvectors corresponding to different eigenvalues are orthogonal**

$$L|\lambda_1 >= \lambda_1|\lambda_1 >$$

$$L|\lambda_2 >= \lambda_2|\lambda_2 >$$

Hermitian Operators & Orthonormal Basis

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L is Hermitian $\implies$ flip first to bra-form

$$<\lambda_1|L = \lambda_1 <\lambda_1|$$

Hermitian Operators & Orthonormal Basis

- **The eigenvectors corresponding to different eigenvalues are orthogonal**

$$L|\lambda_1 >= \lambda_1|\lambda_1 >$$

$$L|\lambda_2 >= \lambda_2|\lambda_2 > \qquad \texttt{multiply by } < \lambda_1|$$

$$\texttt{L is Hermitian} \implies \texttt{flip first to bra-form}$$

$$< \lambda_1|L = \lambda_1 < \lambda_1| \qquad \texttt{multiply by } |\lambda_2 >$$

Hermitian Operators & Orthonormal Basis

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$$<\lambda_1|L = \lambda_1 <\lambda_1| \qquad \texttt{multiply by} \; |\lambda_2>$$

$$<\lambda_1|L|\lambda_2> = \lambda_1 <\lambda_1|\lambda_2>$$

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Hermitian Operators & Orthonormal Basis

- **The eigenvectors corresponding to different eigenvalues are orthogonal**

$$L|\lambda_1 >= \lambda_1|\lambda_1 >$$

$$L|\lambda_2 >= \lambda_2|\lambda_2 >$$ multiply by $< \lambda_1|$

L is Hermitian $\Longrightarrow$ flip first to bra-form

$$< \lambda_1|L = \lambda_1 < \lambda_1|$$ multiply by $|\lambda_2 >$

$$< \lambda_1|L|\lambda_2 >= \lambda_1 < \lambda_1|\lambda_2 >$$

subtract

$$< \lambda_1|L|\lambda_2 >= \lambda_2 < \lambda_1|\lambda_2 >$$

Hermitian Operators & Orthonormal Basis

- **The eigenvectors corresponding to different eigenvalues are orthogonal**

$$L|\lambda_1> = \lambda_1|\lambda_1>$$

$$L|\lambda_2> = \lambda_2|\lambda_2> \qquad \texttt{multiply by } <\lambda_1|$$

L is Hermitian $\implies$ flip first to bra-form

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$$<\lambda_1|L|\lambda_2> = \lambda_1 <\lambda_1|\lambda_2>$$

subtract

$$<\lambda_1|L|\lambda_2> = \lambda_2 <\lambda_1|\lambda_2>$$

$$0 = (\lambda_1 - \lambda_2) <\lambda_1|\lambda_2>$$

Hermitian Operators & Orthonormal Basis

- **The eigenvectors corresponding to different eigenvalues are orthogonal**

$$L|\lambda_1 >= \lambda_1|\lambda_1 >$$

$$L|\lambda_2 >= \lambda_2|\lambda_2 >$$

$$0 = (\lambda_1 - \lambda_2) < \lambda_1|\lambda_2 >$$

for $\lambda_1 \neq \lambda_2$ inner product $< \lambda_1|\lambda_2 >$ must be 0

the two eigenvectors must be orthonormal

Hermitian Operators & Orthonormal Basis

- **Even for equal eigenvalues, corresponding eigenvectors can be chosen orthogonal**

Hermitian Operators & Orthonormal Basis

- **Even for equal eigenvalues, corresponding eigenvectors can be chosen orthogonal**

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consider an arbitrary linear combination

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consider an arbitrary linear combination

$$|A>= \alpha|\lambda_1> +\beta|\lambda_2>$$

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operating on both sides with L

Hermitian Operators & Orthonormal Basis

- **Even for equal eigenvalues, corresponding eigenvectors can be chosen orthogonal**

$$L|\lambda_1> = \lambda|\lambda_1>$$

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consider an arbitrary linear combination

$$|A> = \alpha|\lambda_1> + \beta|\lambda_2>$$

operating on both sides with L

$$L|A> = \alpha L|\lambda_1> + \beta L|\lambda_2>$$

Hermitian Operators & Orthonormal Basis

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$$L|A>=\alpha L|\lambda_1>+\beta L|\lambda_2>$$

$$L|A>=\alpha\lambda|\lambda_1>+\beta\lambda|\lambda_2>$$

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operating on both sides with L

$$L|A >= \alpha L|\lambda_1 > +\beta L|\lambda_2 >$$

$$L|A >= \alpha\lambda|\lambda_1 > +\beta\lambda|\lambda_2 >$$

$$L|A >= \lambda(\alpha|\lambda_1 > +\beta|\lambda_2 >) = \lambda|A >$$

Gram-Schmidt Procedure

- What if a linearly independent set of eigenvectors do not form an orthonormal set?
- Usually happens in a system with degenerate states
  - Distinct states with same eigenvalues
- It is possible to construct orthonormal vectors, example for two vectors:
  1. Divide first vector by its length, to render a unit vector parallel to the first
     - forms first vector in the orthonormal set
  2. Project the second into the direction of the first (inner product)
  3. Subtract the result from 2. from the second vector
  4. Normalize the outcome from 3. (divide by its own length)

Principles

- Observable or measurable quantities are represented by linear operators L

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- Possible results of a measurement:
  - the eigenvalues of the operator representing the observable
  - State for which the measurement result is unambiguously a specific eigenvalue:
    - the corresponding eigenvector
  - If the system is in the eigenstate
    - the measurement result is guaranteed to be the corresponding eigenvalue

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  - the probability of observing the eigenvalue λi is

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$$P(\lambda_i) =< A|\lambda_i >< \lambda_i|A >$$

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$$P(\lambda_i) = \langle A|\lambda_i \rangle \langle \lambda_i|A \rangle$$

$$\langle \lambda_i | A \rangle$$

$$|A\rangle = \sum_i c_i \, |\lambda_i\rangle$$

Principles

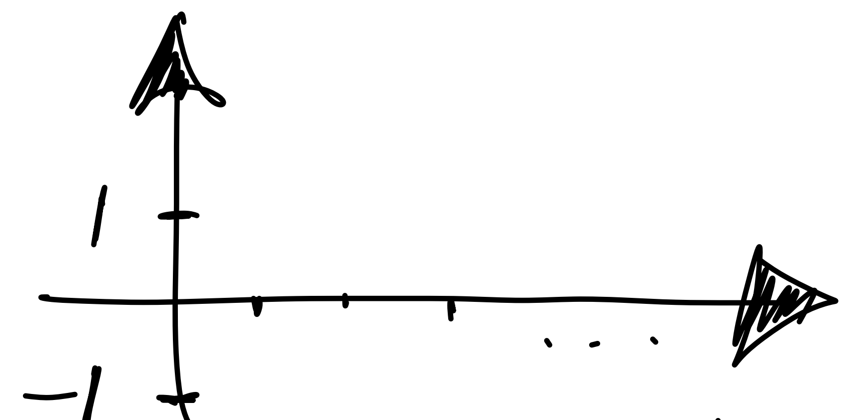
- **Observable or measurable quantities are represented by linear operators L**
  - Observable = measurable
  - Operator: states + (their) eigenvalues (i.e., possible measurement results)
- Example: (measurable) components of a spin $\boldsymbol{\sigma}$x, $\boldsymbol{\sigma}$y, $\boldsymbol{\sigma}$z
  - Each measurement renders +1 or -1
  - Result of a measurement is generally statistically uncertain
    - Except for particular states
    - E.g., measuring $\boldsymbol{\sigma}$z for |u> always renders +1; for |d>, -1
  - Each observable is associated with a specific linear operator
    - in the 2D space of states describing spin

$|u\rangle \to \boxed{\sigma_z} \to ?$ $\quad +1$

$|d\rangle \to \boxed{\sigma_z} \to ?$ $\quad -1$

$|r\rangle \to \boxed{\sigma_z} \to ?$

$(+x)$



$N \to \infty$

$$\underbrace{\#1 = \#-1}_{avg \to 0}$$

Principles

- **Possible results of a measurement:**
  - **the eigenvalues of the operator representing the observable**
  - State for which the measurement result is unambiguously a specific eigenvalue:
    - the corresponding eigenvector
  - If the system is in the eigenstate
    - the measurement result is guaranteed to be the corresponding eigenvalue

- The measurement result is **always** a real number from a set of possible results
  - Spin example: +1, -1
  - Each component of the spin operator must have eigenvalues equal to +1 and -1

Principles

- **Unambiguously distinguishable states are represented by orthogonal vectors**

- Two states are physically distinct if there is a measurement to tell them apart
- Spin example: |u> and |d> are distinguishable by measuring σz
  - Similarly for |l> and |r> σx, and |i> and |o> for σy
- Assume that we don't know the initial state, but it is either of |u> and |r>
  - No measurement can tell these possibilities apart
  - Even if σz measurement renders +1 the initial state might have been |r>
  - |u> and |d> are physically distinguishable, |u> and |r> are not
- Inner product of two states can serve as a proxy for distinguishability ("overlap")
  - The principle requires physically distinct states to be represented by
    - orthogonal state vectors = vectors with no overlap
    - <u|d> = 0 where <u|r> =

$|u\rangle$ $\qquad \boxed{\sigma_z}$

$|d\rangle$

$$\langle u|d\rangle = 0$$

$$|u\rangle \to \boxed{\sigma_z} \to +1$$

$$|d\rangle \to \boxed{\sigma_z} \to -1$$

Initial state unknown ($|u\rangle$ or $|r\rangle$)

$$|u\rangle \rightarrow \boxed{\sigma_z} \rightarrow +1$$

$$|r\rangle \rightarrow \boxed{\sigma_z} \begin{array}{l} \nearrow +1 \\ \searrow -1 \end{array}$$

Similar for $\boxed{\sigma_x}$

$$\Rightarrow \langle u|r\rangle \neq 0$$

Principles

- **If |A> is the state-vector of a system and the observable L is measured**
  - **the probability of observing the eigenvalue λ_i is**

$$P(\lambda_i) = < A|\lambda_i >< \lambda_i|A >$$

$$= [\alpha_1^* \ \alpha_2^* \ \alpha_3^*] \begin{bmatrix} \lambda_{i1} \\ \lambda_{i2} \\ \lambda_{i3} \end{bmatrix} [\lambda_{i1}^* \ \lambda_{i2}^* \ \lambda_{i3}^*] \begin{bmatrix} \alpha 1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = |< A|\lambda_i >|^2$$

- Probability = (overlap)^2

Principles

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  - State for which the measurement result is unambiguously a specific eigenvalue:
    - the corresponding eigenvector
  - If the system is in the eigenstate
    - the measurement result is guaranteed to be the corresponding eigenvalue
- Unambiguously distinguishable states are represented by orthogonal vectors.
- If |A> is the state-vector of a system and the observable L is measured
  - the probability of observing the eigenvalue λ_i is

$$P(\lambda_i) = < A|\lambda_i >< \lambda_i|A >$$

$\Rightarrow$ Operators that represent observables are Hermitian

- Measurement result must be a real number, hence eigenvalues of L
  - Eigenvectors (states) representing unambiguously distinguishable results
    - must have different eigenvalues
    - must be orthogonal

Spin Operators

- Each component is represented by a linear operator, including **σ**z
- Eigenvectors of **σ**z are |u> and |d>, eigenvalues +1 and -1

$$\sigma_z|u>=|u>$$

$$\sigma_z|d>=-|d>$$

- States |u> and |d> are orthogonal to each other $<u|d>=0$

$$\begin{bmatrix}\sigma_{z,11} & \sigma_{z,12}\\ \sigma_{z,21} & \sigma_{z,22}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}\sigma_{z,11} & \sigma_{z,12}\\ \sigma_{z,21} & \sigma_{z,22}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=-\begin{bmatrix}0\\1\end{bmatrix}$$

Spin Operators

- Each component is represented by a linear operator, including **σ**z
- Eigenvectors of **σ**z are |u> and |d>, eigenvalues +1 and -1

$$\sigma_z|u>=|u>$$

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$|u>$ $|d>$

- There is only one matrix that satisfies these equations:

$$\sigma_z=\begin{bmatrix}\sigma_{z,11} & \sigma_{z,12}\\ \sigma_{z,21} & \sigma_{z,22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Spin Operators

- From experimental data: measurement of an observable
  - unambiguously renders +1 or -1 for (the so called) |u> and |d> states
- From principles
  - |u> and |d> are orthogonal, and are eigenvectors of $\boldsymbol{\sigma}$z
    - $\boldsymbol{\sigma}$z: linear operator representing this observable
  - Corresponding eigenvalues are measured values +1 and -1
- Similar derivation applies for $\boldsymbol{\sigma}$x and $\boldsymbol{\sigma}$y

Spin Operators

- Eigenvectors of $\boldsymbol{\sigma}$x are |r> and |l>, eigenvalues +1 and -1
- |r> and |l> are linear super-positions of |u> and |d>

$$\sigma_x|r>=|r>$$

$$\sigma_x|l>=-|l>$$

$$|r>=\frac{1}{\sqrt{2}}|u>+\frac{1}{\sqrt{2}}|d>$$

$$|l>=\frac{1}{\sqrt{2}}|u>-\frac{1}{\sqrt{2}}|d>$$

$$\begin{bmatrix}\sigma_{x,11} & \sigma_{x,12}\\ \sigma_{x,21} & \sigma_{x,22}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$$

$<r|l>=0$

$$\begin{bmatrix}\sigma_{x,11} & \sigma_{x,12}\\ \sigma_{x,21} & \sigma_{x,22}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2}\\ +1/\sqrt{2}\end{bmatrix}$$

$$\sigma_z \rightarrow \begin{array}{l} |\lambda_1\rangle \\ |\lambda_2\rangle \end{array}$$

$$|A\rangle = \underbrace{c_{\lambda_1}}_{\langle \lambda_1 | A \rangle} |\lambda_1\rangle + \underbrace{c_{\lambda_2}}_{\langle \lambda_1 | A \rangle} |\lambda_2\rangle$$

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Spin Operators

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$$\sigma_y|o>=-|o>$$

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$$|o>=\frac{1}{\sqrt{2}}|u>-\frac{i}{\sqrt{2}}|d>$$

$$\langle i|o\rangle = 0$$

$$\begin{bmatrix}\sigma_{y,11} & \sigma_{y,12}\\ \sigma_{y,21} & \sigma_{y,22}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{bmatrix}$$

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Spin Operators

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Pauli Matrices

$$\sigma_z = \begin{bmatrix} \sigma_{z,11} & \sigma_{z,12} \\ \sigma_{z,21} & \sigma_{z,22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Putting It All Together

- Operators are used to calculate eigenvalues and eigenvectors
- Operators act on states, not on actual physical systems
- Operator acting on a state vector generates another state vector
- Measuring an observable is **not** the same as
  - operating with the corresponding operator L on the state
  - e.g., from initial state |A>, measurement cannot render L|A>

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- Example:

$$\sigma_z|u>=|u>$$

$$\sigma_z|d>=-|d>$$

If the system is prepared in state |d>

- measurement certainly renders -1
- post measurement state becomes -|d>
- |d> and -|d> are practically the same states

Putting It All Together

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- Example:

$$\sigma_z|u>=|u>$$

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If the system is prepared in |r>, measurement renders, with equal probability,

- either +1 and a post measurement state |u>
- or -1 and a post measurement state -|d>

$$|r>=\frac{1}{\sqrt{2}}|u>+\frac{1}{\sqrt{2}}|d>$$

$$\sigma_z|r>=\frac{1}{\sqrt{2}}\sigma_z|u>+\frac{1}{\sqrt{2}}\sigma_z|d>$$

$$=\frac{1}{\sqrt{2}}|u>-\frac{1}{\sqrt{2}}|d>$$

Spin component along any axis?

$$\hat{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \qquad \sigma_n = \sigma\hat{n} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

$$\sigma_n = \sigma\hat{n} = n_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + n_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{bmatrix}$$

Ex:

$$\hat{n} = \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} n_x \\ n_y \\ n_z \end{matrix} \Rightarrow \sigma_n = \sigma_z$$

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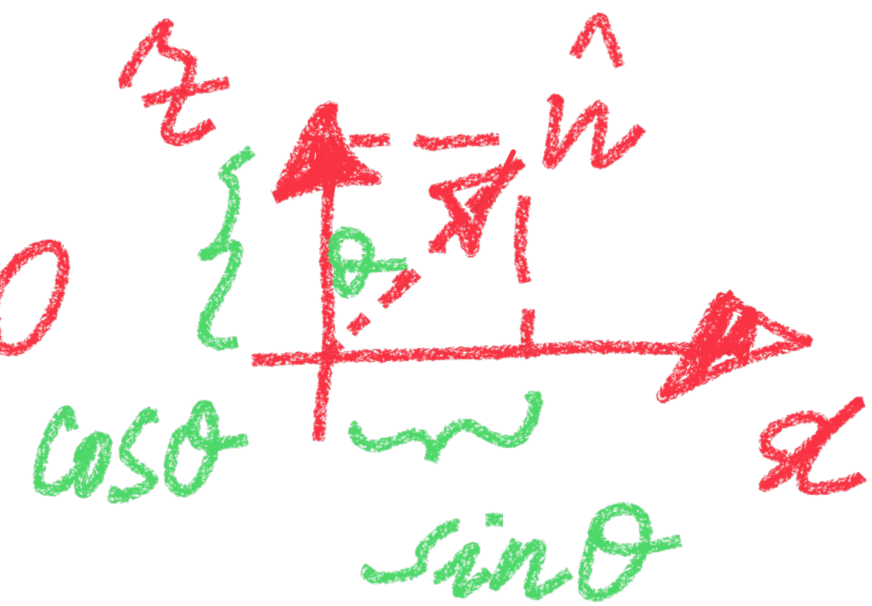
$n_y = 0$



- Assume $\hat{n}$ lies in the x-z plane:

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$\sin\frac{\theta}{2}$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

Assume that the system is prepared in |u>, we measure observable $\boldsymbol{\sigma}$n.

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$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$P(+1)$

$+ \cos^2\theta/2 = 1$

Expected value of a measurement

- Average correspondent
- What if we prepared a large number of qubits in |u> and measured **σ**n?
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Ex : $|r\rangle \rightarrow \sigma_z \begin{cases} +1 \\ -1 \end{cases}$ avg : 0

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vs. $\hat{n} \cdot \hat{z}$

$$\begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Given any state $|A >= \alpha_u|u > +\alpha_d|d >$

There exists a direction p such that $\sigma_p|A >= |A >$

How do states change with time?

- The state at time t: U(t) acting on system state at time 0
  - $|\psi(t)>$ is determined by $|\psi(0)>$
  - U: time development operator of the system

$$|\psi(t)> = U(t)|\psi(0)>$$

S

- State vector evolves in a deterministic manner
- Measurement is still of statistical nature
- Quantum evolution of states allows computation of probabilities of measurements

$|\psi(t)>= U(t)|\psi(0)>$

- U(t) is required to be a linear operator
  - State-space is a vector-space

Linear Operators $M|A>=|B>$

1. Should result in a unique output for every (input) vector in the space

2. For any complex z: $Mz|A>=z|B>$

3. $M(|A>+|B>)=M|A>+M|B>$

$|\psi(t)>= U(t)|\psi(0)>$

- U(t) is required to be a linear operator
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- Also required is "conservation of distinction"
  - Two states are distinguishable if they are orthogonal
    - Two different basis vectors represent two distinct states
    - There is a precise measurement that can tell them apart

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$<\psi(t)|\phi(t)>= 0$ for all values of t

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Flip $|\psi(t)>=U(t)|\psi(0)>$ and substitute for $|\boldsymbol{\psi}(t)>$, $|\cancel{\boldsymbol{\psi}(0)}>$:

$<\psi(t)|=<\psi(0)|U^{\dagger}(t)$

$U(t)\cdot|\phi(0)>$

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Unitarity

$$< \psi(0)|U^\dagger(t)U(t)|\phi(0) >= 0$$

Unitarity

- Consider an orthonormal basis

$$< \psi(0)|U^\dagger(t)U(t)|\phi(0) >= 0$$

$$< i|j >= \delta_{ij}$$

$i = j \Rightarrow \delta_{ij} = 1$

$\neq \qquad\qquad 0$

u d

+ −

0 1

⋮

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$U^\dagger(t)U(t)$ acts as I between members of a basis set

$= I$

Unitarity

$$U^\dagger(t)U(t) = I$$ if U is unitary

- Time evolution is unitary
- One more principle of QM
  - The evolution of state vectors with time is unitary

Principles

- Observable or measurable quantities are represented by linear operators L
- Possible results of a measurement:
  - the eigenvalues of the operator representing the observable.
  - State for which the measurement result is unambiguously a specific eigenvalue:
    - the corresponding eigenvector
  - If the system is in the eigenstate
    - the measurement result is guaranteed to be the corresponding eigenvalue
- Unambiguously distinguishable states are represented by orthogonal vectors.
- If |A> is the state-vector of a system and the observable L is measured
  - the probability of observing the eigenvalue λi is

$$P(\lambda_i) = < A|\lambda_i >< \lambda_i|A >$$

- Evolution of state-vectors with time is unitary

Hamiltonian

$U^\dagger(t)\,U(t) = I$

$U^\dagger(\epsilon)U(\epsilon) = I$ unitarity $t = \epsilon \to 0$

$U(\epsilon) = \underbrace{I - i\epsilon H}_{I - \underbrace{\Delta}_{\to 0}}$ continuity (state vector changes smoothly)

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$$U^\dagger(\epsilon) = I + i\epsilon H^\dagger$$

$$(I + i\epsilon H^\dagger)(I - i\epsilon H) = I$$

$$H^\dagger - H = 0$$

$\cancel{I^2} - \cancel{i^2 \epsilon^2 H^\dagger H} - i\epsilon H + i\epsilon H^\dagger = \cancel{I}$ $\quad \epsilon \to 0$

$\Rightarrow i\epsilon(-H + H^\dagger) = 0$

$\Rightarrow H^\dagger = H$

Hamiltonian

$$U^\dagger(\epsilon)U(\epsilon) = I$$ unitarity

$$U(\epsilon) = I - i\epsilon H$$ continuity (state vector changes smoothly)

$$U^\dagger(\epsilon) = I + i\epsilon H^\dagger$$

$$(I + i\epsilon H^\dagger)(I - i\epsilon H) = I$$

$$H^\dagger - H = 0$$ from unitarity condition

⇒ H is Hermitian, hence is observable

H: Quantum Hamiltonian

eigenvalues ≡ measurem. of energy of a q-system

Hamiltonian

$$|\psi(t)> = U(t)|\psi(0)> \qquad U(\epsilon) = I - i\epsilon H$$

$t = \mathcal{E}$

$$|\psi(\epsilon)> = |\psi(0)> - i\epsilon H|\psi(0)>$$

$$\frac{|\psi(\epsilon)> - |\psi(0)>}{\epsilon} = -iH|\psi(0)>$$

$$\epsilon \rightarrow 0$$

$$\frac{\partial|\psi>}{\partial t} = -iH|\psi>$$

Hamiltonian

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$$|\psi(\epsilon)>= |\psi(0)> - i\epsilon H|\psi(0)>$$

$$\frac{|\psi(\epsilon)> - |\psi(0)>}{\epsilon} = -iH|\psi(0)>$$

$$\epsilon \to 0$$

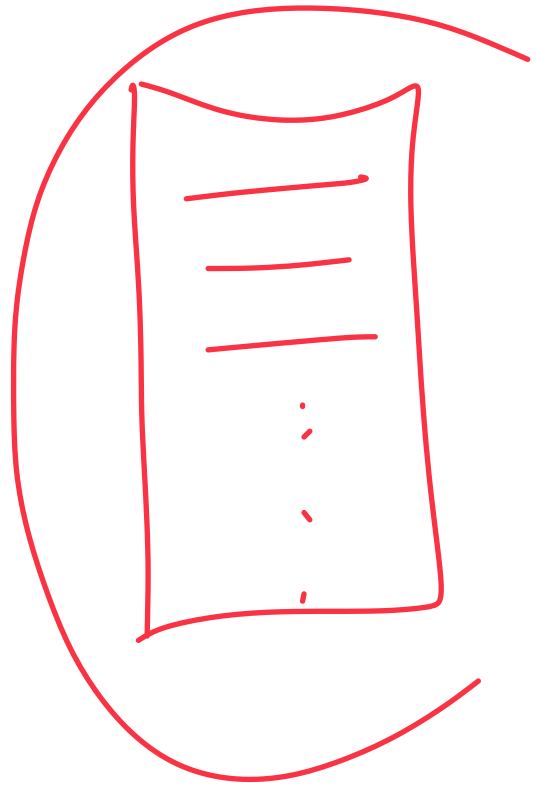
Plank's const.

$$\frac{\partial|\psi>}{\partial t} = -\underbrace{iH}_{w}|\psi> \qquad \text{dimension fix} \qquad \overset{M}{\hbar}\frac{\partial|\psi>}{\partial t} = -iH|\psi>$$

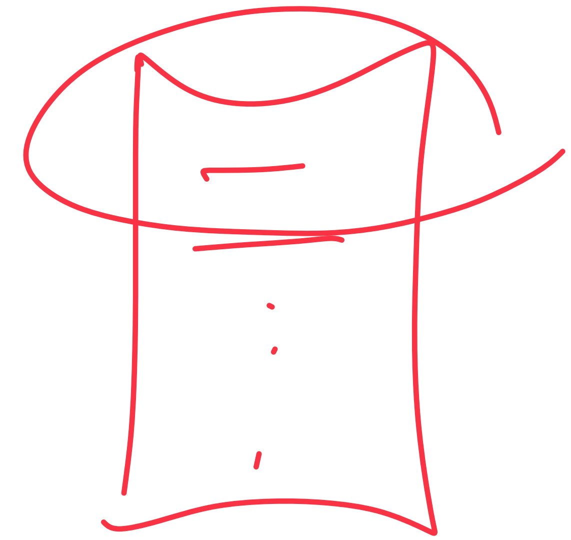
s^{-1}

$$\text{Joule} = \frac{kgm^2}{s}$$

$\mathcal{U}(t)$:



vs.



Expectation $< L >= \sum_i \lambda_i P(\lambda_i)$

$$|A >= \sum_i \alpha_i |\lambda_i >$$

$$< A|L|A >=?$$

$$L|A >= \sum_i \alpha_i L|\lambda_i >$$

Let L operate on both sides ...

$$L|A >= \sum_i \alpha_i \lambda_i |\lambda_i >$$

|λi> are L's eigenvectors ...

$$< A|L|A >= \sum_i (\alpha_i^* \alpha_i) \lambda_i$$

take inner product with <A|,
use orthonormality of eigenvectors ...

$$< A| = \sum_i \alpha_i^* < \lambda_i|$$

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$$< A|L|A >=?$$

$$< L >$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Ex: $|A> = |r> \Rightarrow <r|\sigma_z|r> = ?$

$$\sigma_z|r >= \underbrace{\frac{1}{\sqrt{2}}\sigma_z|u >}_{1/\sqrt{2}\,|u>} + \underbrace{\frac{1}{\sqrt{2}}\sigma_z|d >}_{-1/\sqrt{2}\,|d>}$$

Classical Equivalent

$|r\rangle \equiv$ unit vector in $+x$ direction

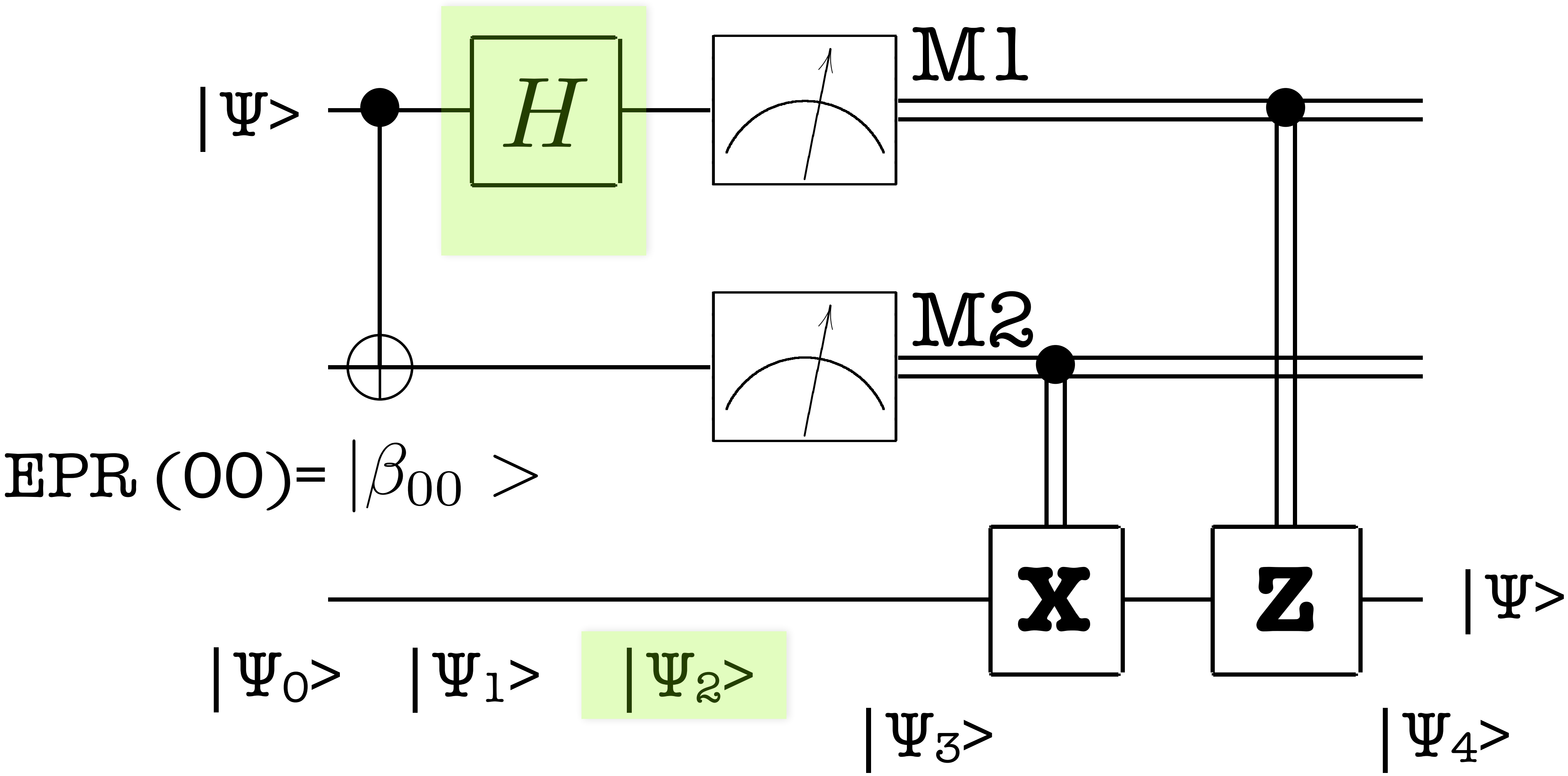
$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \langle r | N_z | r \rangle$$

$$\equiv$$

$$|B\rangle = e^{i\theta}|A\rangle$$

$$\langle A|L|A\rangle \overset{?}{=} \underbrace{\langle B|}_{e^{-i\theta}\langle A|} L \underbrace{|B\rangle}_{e^{i\theta}|A\rangle}$$

Quantum Teleportation



$$|\psi_2 >= \frac{1}{2}[|00 > (\alpha|0 > +\beta|1 >) + |01 > (\alpha|1 > +\beta|0 >)$$
$$+|10 > (\alpha|0 > -\beta|1 >) + |11 > (\alpha|1 > -\beta|0 >)]$$

Tensor Products

$$|ud\rangle = |u\rangle \otimes |d\rangle$$

$$|u\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |d\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|ud\rangle = \begin{bmatrix} 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big(\underset{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}{\sigma_z} \otimes \underset{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{\sigma_x}\Big)|ud\rangle \overset{?}{=} \underbrace{\sigma_z|u\rangle}_{|u\rangle} \otimes \underbrace{\sigma_x|d\rangle}_{|u\rangle}$$

$$\sigma_z \otimes \sigma_x = \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = M$$

$$M\,|ud\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \underset{?}{=} |uu\rangle = \begin{bmatrix} 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

vs.

$$|ud\rangle = |uu\rangle$$

$\sigma_z \quad \sigma_x$

$$\sigma_z |u\rangle = |u\rangle$$

$$\sigma_x |d\rangle = |u\rangle$$

$$(A \otimes B)(|a\rangle \otimes |b\rangle) =$$

$$A|a\rangle \otimes B|b\rangle \rightarrow \text{Axiom}$$

$$|u\,d\rangle \rightarrow \tilde{\sigma}_x$$

$$\downarrow$$

$$I \qquad I \otimes \tilde{\sigma}_x$$

$$|A\rangle = \alpha_u |u\rangle + \alpha_d |d\rangle$$

$$|B\rangle = \beta_u |u\rangle + \beta_d |d\rangle$$

$$\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$$

$$\beta_u^* \beta_u + \beta_d^* \beta_d = 1$$

$$= \begin{bmatrix} \alpha_u \begin{bmatrix} \beta_u \\ \beta_d \end{bmatrix} \\ \alpha_d \begin{bmatrix} \beta_u \\ \beta_d \end{bmatrix} \end{bmatrix}$$

$$|AB\rangle = \left.\begin{array}{l} \alpha_{uu} |uu\rangle + \\ \alpha_{ud} |ud\rangle + \\ \alpha_{du} |du\rangle + \\ \alpha_{dd} |dd\rangle \end{array}\right\}$$

Outer Product vs. Inner Prod.

$$|\psi\rangle \quad \langle\phi| \qquad \text{vs.} \qquad \langle\phi|\psi\rangle$$

$$n\times 1 \quad 1\times n \qquad\qquad 1\times 1$$

$$\underbrace{|\psi\rangle\cdot\langle\phi|}_{n\times n}\cdot|A\rangle$$

$$= |\psi\rangle \overbrace{\langle\phi|A\rangle}^{\text{scaled by } \in \mathbb{C}}$$

Outer Product vs. Inner Prod.

$$\underset{n\times 1}{|\psi\rangle}\ \underset{1\times n}{\langle\phi|} \quad \text{vs.} \quad \underset{1\times 1}{\langle\phi|\psi\rangle}$$

$$\langle B|\cdot\underbrace{|\psi\rangle\cdot\langle\phi|}_{n\times n}$$

$$= \underbrace{\langle B|\psi\rangle}\ \underbrace{\langle\phi|}\ \text{scaled by} \in \mathbb{C}$$

Outer Products cont.

Projection operator:

$$\underbrace{|\psi\rangle\langle\psi| \cdot |A\rangle}_{|\psi\rangle\langle\psi|A\rangle}$$

Project $|A\rangle$ in the direction of $|\psi\rangle$

$|\psi\rangle\langle\psi|$: Observable

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 0$

$|\lambda_1\rangle = |\psi\rangle$ $\qquad$ $|\lambda_2\rangle = \underbrace{|\perp\rangle}$

anything orthogonal to $|\psi\rangle$: $\langle\perp|\psi\rangle = 0$

$|\psi\rangle\langle\psi|$: Observable

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 0$

$|\lambda_1\rangle = |\psi\rangle$ $\qquad$ $|\lambda_2\rangle = \underbrace{|\downarrow\rangle}$

anything orthogonal to $|\psi\rangle$: $\underbrace{\langle\downarrow|\psi\rangle}_{=0}$

$$|\psi\rangle\langle\psi| \cdot |\lambda_i\rangle = \lambda_i |\lambda_i\rangle$$
$$i = 1, 2$$

$|\psi\rangle\langle\psi|$: Observable

$\lambda_1 = 1$

$|\lambda_1\rangle = |\psi\rangle$

$$|\psi\rangle\langle\psi| \cdot \underbrace{|\lambda_1\rangle}_{|\psi\rangle} = \underbrace{\lambda_1}_{1} \underbrace{|\lambda_1\rangle}_{|\psi\rangle}$$

$$\Rightarrow |\psi\rangle \cdot 1 = 1 \cdot |\psi\rangle$$

$$|\psi\rangle\langle\psi| \cdot |\lambda_i\rangle = \lambda_i |\lambda_i\rangle$$

$i = 1, 2$

$|\psi\rangle\langle\psi|$: Observable

$$\overbrace{|\psi\rangle\langle\psi|\cdot|\lambda_2\rangle}^{0}$$

$$= \lambda_2 \cdot |\lambda_2\rangle$$

$$\Rightarrow 0 = \underbrace{0 \cdot |\downarrow\rangle}_{0}$$

$\lambda_2 = 0$

$|\lambda_2\rangle = \underbrace{|\downarrow\rangle}_{\text{anything orthogonal to } |\psi\rangle}$

anything orthogonal to $|\psi\rangle$: $\underbrace{\langle\downarrow|\psi\rangle}_{=0}$

$$|\psi\rangle\langle\psi|\cdot|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

$$i = 1, 2$$

Bibliography

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INTRODUCTION TO QUANTUM COMPUTING AND PHYSICAL BASICS OF COMPUTING

Quantum Mechanics [Basics of the Basics]

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