On The Distribution of a Combinatorics Problem

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1 Problem Setup

Let $(\Omega, \mathscr{F}, P)$ be a probability space over the sample space $\Omega = \{-1, 0, 1\}$ with $P(0) = 1/2$ and $P(-1) = P(1) = 1/4$. Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[0,1]$ and $\vec{\omega}_n = (\omega_1, \omega_2, \ldots, \omega_n)^T \in \Omega^n$ be a vector of random variables. Determine the probability distribution of $s_n = \sum_{k=1}^n \omega_k x_k$ as $n \to \infty$.

1.1 Problem Reduction

Since P_n forms a regular partition, $x_k = k/n$ for each index $0 \le k \le n$. Consequently, $s_n = \sum_{k=1}^n \omega_k \frac{k}{n} = \frac{1}{n} A_n \vec{\omega}_n$ where $A_n = (1, 2, \ldots, n)$. Observe $A_n \vec{\omega}_n \in \mathbb{Z}$. Thus $s_n \in \{m/n \mid m \in \mathbb{Z}\}$. Fix $m \in \mathbb{Z}$. Then,

$$P\left(s_n = \frac{m}{n}\right) = P(A_n \vec{\omega}_n = m). \tag{1}$$

Therefore the problem reduces to finding the solution set of $m = A_n \vec{\omega}_n$ for each $m \in \mathbb{Z}$ as $n \to \infty$.

2 Observations

2.1 Recurrence Relations

Finding all $\vec{\omega}_n$ that solves the matrix equation $A_n \vec{\omega}_n = m$ for given $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ is computationally expensive and does not suggest much on the shape of the probability distribution of s_n itself. To rectify these issues, this section establishes relationships among the probabilities as well as a collection of simple properties.

PROP 1 Let $n \in \mathbb{N}$. Then for all $m \in \mathbb{Z}$,

$$P\left(s_{n+1} = \frac{m}{n+1}\right) = \frac{1}{4} P\left(s_n = \frac{m+n+1}{n}\right) + \frac{1}{2} P\left(s_n = \frac{m}{n}\right) + \frac{1}{4} P\left(s_n = \frac{m-n-1}{n}\right). \tag{2}$$

P–f Fix $m \in \mathbb{Z}$. Let $X_{n+1}(m)$ be the set of solutions to $A_{n+1}\vec{\omega}_{n+1} = m$. Define $X_{n+1}^y(m) = \{\vec{x} \in X_{n+1}(m) \mid x_{n+1} = y\}$. Then,

$$P\left(s_{n+1} = \frac{m}{n+1}\right) = P(A_{n+1}\vec{\omega}_{n+1} = m) = \sum_{\vec{x} \in X_{n+1}(m)} P(\vec{\omega}_{n+1} = \vec{x}) = \sum_{\vec{x} \in X_{n+1}^{-1}(m)} P(\vec{\omega}_{n+1} = \vec{x}) + \sum_{\vec{x} \in X_{n+1}^{0}(m)} P(\vec{\omega}_{n+1} = \vec{x}) + \sum_{\vec{x} \in X_{n+1}^{1}(m)} P(\vec{\omega}_{n+1} = \vec{x}).$$

We claim the last three sums equate to $\frac{1}{4}P(s_n = m+n+1/n)$, $\frac{1}{2}P(s_n = m/n)$, and $\frac{1}{4}P(s_n = m-n-1/n)$, respectively. Consider the sum for $X_{n+1}^{-1}(m)$. Define

$$\phi : X_{n+1}^{-1}(m) \to X_n(m+n+1) \quad \text{by} \quad \phi(\vec{x}) = (x_1, x_2, \ldots, x_n)^T.$$

Note ϕ is injective because the $\vec{x}$'s remain distinct after removing x_{n+1} since it is shared by definition. Moreover, ϕ is surjective since each vector in $X_n(m+n+1)$ can be extended to a vector in $X_{n+1}^{-1}(m)$ by appending -1. Thus ϕ is bijective. Then, (note all summations below range over $X_{n+1}^{-1}(m)$)

$$\sum_{\vec{x}} P(\vec{\omega}_{n+1} = \vec{x}) = \sum_{\vec{x}} \prod_{k=1}^{n+1} P(\omega_k = x_k) = \sum_{\vec{x}} \frac{1}{4} \prod_{k=1}^{n} P(\omega_k = x_k) = \frac{1}{4} \sum_{\vec{x}} P(\vec{\omega}_n = \phi(\vec{x})) = \frac{1}{4} P\left(s_n = \frac{m+n+1}{n}\right).$$

Similar arguments lead to constructions like the one above for the other two sums which prove their respective equalities. Given this, $P(s_{n+1} = m/n+1)$ immediately reduces to Eq. (2). ■

PROP 2 $P(s_1=0)=1/2$, $P(s_1=1)=P(s_1=-1)=1/4$, and $P(s_1=m)=0$ if $|m|>1$.

P–f $s_1=\frac{1}{1}A_1\vec{\omega}_1=\omega_1$. Therefore $P(s_1=m)=P(\omega_1=m)$ for each $m\in\mathbb{Z}$ by Eq. (1). Since $\omega_1\in\Omega$ is a random variable, the proposition follows by definition of the probability space. ∎

THM 3 For each $n\in\mathbb{N}\cup\{0\}$, define a recurrence relation $R_n:\mathbb{Z}\to\mathbb{R}$ with base case as follows.

1. $R_0(0)=1$ and $R_0(m)=0$ for all $m\neq 0$.
2. $R_{n+1}(m)=\frac{1}{4}R_n(m+n+1)+\frac{1}{2}R_n(m)+\frac{1}{4}R_n(m-n-1)$ for all $m\in\mathbb{Z}$ and $n\in\mathbb{N}\cup\{0\}$.

Then for each $n\in\mathbb{N}$, $P(s_n={}^{m}/{}_{n})=R_n(m)$ for all $m\in\mathbb{Z}$.

P–f Substituting $P(s_n={}^{m}/{}_{n})$ into the recurrence formula for $R_n(m)$, we obtain

$$P(s_{n+1}={}^{m}/{}_{n+1})=\frac{1}{4}P(s_n={}^{m+n+1}/{}_{n})+\frac{1}{2}P(s_n={}^{m}/{}_{n})+\frac{1}{4}P(s_n={}^{m-n-1}/{}_{n})\qquad\forall m\in\mathbb{Z},n\in\mathbb{N}.$$

This agrees with Proposition 1 when $n\in\mathbb{N}$. Thus it suffices to show that $P(s_1=m)=R_1(m)$ for all $m\in\mathbb{Z}$ to prove the theorem. From (i) and (ii), it follows that $R_1(0)=1/2$ and $R_1(1)=R_1(-1)=1/4$. Suppose $m>1$. Then $m,m+1,m-1>0$. Thus for all $m>1$,

$$R_1(m)=\frac{1}{2}R_0(m)+\frac{1}{4}R_0(m+1)+\frac{1}{4}R_0(m-1)=0.$$

Similar argument proves $R_1(m)=0$ for all $m<1$. Summarizing the above calculations, we obtain by Proposition 2

$$P(s_1=0)=R_1(0)=1/2,\qquad P(s_1=m)=R_1(m)=1/4\quad\text{if } |m|=1,\text{ and}\qquad P(s_1=m)=R_1(m)=0\quad\text{if } |m|>1.$$ ∎

Theorem 3 reframes the problem as a multivariable recurrence problem. For simplicity, we will now work with $R_n(m)$ noting that it is equivalent to $P(s_n={}^{m}/{}_{n})$ when $n\in\mathbb{N}$.

COR 3.1 Let $n\in\mathbb{N}\cup\{0\}$. Then $R_n(m)\geq 0$ for all $m\in\mathbb{Z}$.

P–f The corollary holds for $n=0$ by base case. Suppose for some $k\in\mathbb{N}\cup\{0\}$, $R_k(m)\geq 0$ for all $m\in\mathbb{Z}$. Then

$$R_{k+1}(m)=\frac{1}{2}R_k(m)+\frac{1}{4}R_k(m+k+1)+\frac{1}{4}R_k(m-k-1)\geq 0+0+0=0.$$

Therefore by induction on n, the corollary holds for all $n\in\mathbb{N}\cup\{0\}$. ∎

COR 3.2 Let $n\in\mathbb{N}\cup\{0\}$. Then $R_n(m)=R_n(-m)$ for all $m\in\mathbb{Z}$.

P–f The corollary holds for $n=0$ by base case. Suppose for some $k\in\mathbb{N}\cup\{0\}$, $R_k(m)=R_k(-m)$ for all $m\in\mathbb{Z}$. Then,

$$\begin{aligned}R_{k+1}(m)&=\frac{1}{2}R_k(m)+\frac{1}{4}R_k(m+k+1)+\frac{1}{4}R_k(m-k-1)\\&=\frac{1}{2}R_k(-m)+\frac{1}{4}R_k(-m-k-1)+\frac{1}{4}R_k(-m+k+1)=R_{k+1}(-m)\qquad\forall m\in\mathbb{Z}.\end{aligned}$$

Therefore by induction on n, the corollary holds for all $n\in\mathbb{N}\cup\{0\}$. ∎

PROP 4 Let $n\in\mathbb{N}\cup\{0\}$. Then $R_n(m)=0$ if and only if $|m|>\frac{n(n+1)}{2}$.

P–f The proposition holds for $n=0$ by base case. Suppose for some $k\in\mathbb{N}\cup\{0\}$, $R_k(m)=0$ iff $|m|>\frac{k(k+1)}{2}$. Note proof of the case $m\geq 0$ implies the case $m<0$ by Corollary 3.2. Then,

(⇐) Suppose $m>\frac{(k+1)(k+2)}{2}>0$. Then $m>\frac{k(k+1)}{2}+k+1$ which implies $m,m+k+1,m-k-1>\frac{k(k+1)}{2}$. Thus $R_{k+1}(m)=\frac{1}{2}R_k(m)+\frac{1}{4}R_k(m+k+1)+\frac{1}{4}R_k(m-k-1)=0$.

(⇒) Consider the contrapositive. Suppose $0\leq m\leq\frac{(k+1)(k+2)}{2}$. Then $0\leq m\leq\frac{k(k+1)}{2}+k+1$ which reduces to $m-k-1\leq\frac{K(K+1)}{2}$. Thus $R_{k+1}(m)=\frac{1}{2}R_k(m)+\frac{1}{4}R_k(m+k+1)+\frac{1}{4}R_k(m-k-1)>0$ by Corollary 3.1.

Therefore by induction on n, the biconditional proposition holds for all $n\in\mathbb{N}\cup\{0\}$. ∎

PROP 5 Let $n \in \mathbb{N} \cup \{0\}$. Then $R_n\left(\pm\frac{n(n+1)}{2}\right) = 4^{-n}$.

P–f The proposition holds for $n = 0$ by base case. Suppose for some $k \in \mathbb{N} \cup \{0\}$, $R_k\left(\pm\frac{k(k+1)}{2}\right) = 4^{-k}$. Then,

$$R_{k+1}\left(\frac{(k+1)(k+2)}{2}\right) = \frac{1}{2}R_k\left(\frac{(k+1)(k+2)}{2}\right) + \frac{1}{4}R_k\left(\frac{(k+1)(k+4)}{2}\right) + \frac{1}{4}R_k\left(\frac{k(k+1)}{2}\right) = \frac{1}{4}\left(\frac{1}{4}\right)^k = \left(\frac{1}{4}\right)^{k+1}.$$

By Corollary 3.2, $R_{k+1}\left(\pm\frac{(k+1)(k+2)}{2}\right) = 4^{-(k+1)}$. Therefore by induction on n, this holds for all $n \in \mathbb{N} \cup \{0\}$. ■

2.2 Generating Functions

Since the problem is now of solving a multivariable recurrence, the theory of generating functions naturally applies. The methods used in this section are justified in the book "generatingfunctionology" by Herbert S. Wilf. From here on out (unless otherwise stated) sums over an isolated index means taking the sum over $\mathbb{Z}$. Moreover, if $b < a$ then $\sum_{k=a}^{b} c_k = 0$ and $\prod_{k=a}^{b} c_k = 1$ regardless of c_k. For each $n \in \mathbb{N} \cup \{0\}$ define the *formal laurent series*,

$$f_n(z) = \sum_m R_n(m)z^m. \tag{3}$$

LEM 6 Let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{Z}$. Then $\sum_m R_n(m+\alpha)z^m = z^{-\alpha} f_n(z)$.

P–f Observe $z^\alpha \sum_m R_n(m+\alpha)z^m = \sum_m R_n(m+\alpha)z^{m+\alpha}$. Since the sum is over $\mathbb{Z}$, shifting indices by an integer leaves the sum the same. Thus $\sum_m R_n(m+\alpha)z^{m+\alpha} = \sum_m R_n(m)z^m = f_n(z)$. Therefore $\sum_m R_n(m+\alpha)z^m = z^{-\alpha} f_n(z)$. ■

PROP 7 For all $n \in \mathbb{N} \cup \{0\}$,

$$f_{n+1}(z) = \frac{z^{-n-1} + 2 + z^{n+1}}{4} f_n(z). \tag{4}$$

P–f Given the recurrence relation in Theorem 3, we obtain

$$\sum_m R_{n+1}(m)z^m = \frac{1}{4}\sum_m R_n(m+n+1)z^m + \frac{1}{2}\sum_m R_n(m)z^m + \frac{1}{4}\sum_m R_n(m-n-1)z^m.$$

Applying Lemma 6, the equation above reduces to

$$f_{n+1}(z) = \frac{1}{4}z^{-n-1}f_n(z) + \frac{1}{2}f_n(z) + \frac{1}{4}z^{n+1}f_n(z) = \frac{z^{-n-1}+2+z^{n+1}}{4}f_n(z).$$ ■

THM 8 Let $n \in \mathbb{N} \cup \{0\}$. Then

$$f_n(z) = \frac{1}{4^n}\prod_{k=1}^{n}\left(z^{-k} + 2 + z^k\right). \tag{5}$$

P–f Note the base case in Theorem 3 implies $f_0(y) = \sum_m R_0(m)z^m = R_0(0)z^0 = 1$. Thus the theorem holds for $n = 0$. Suppose for some $k \in \mathbb{N} \cup \{0\}$, $f_k(z) = \frac{1}{4^k}\prod_{i=1}^{n}\left(z^{-i} + 2 + z^i\right)$. Then by Proposition 7,

$$f_{k+1}(z) = \frac{z^{-k-1}+2+z^{k+1}}{4}f_k(z) = \frac{z^{-k-1}+2+z^{k+1}}{4}\cdot\frac{1}{4^k}\prod_{i=1}^{k}\left(z^{-i}+2+z^i\right) = \frac{1}{4^{k+1}}\prod_{i=1}^{k+1}\left(z^{-i}+2+z^i\right).$$

Therefore by induction on n, the theorem holds for all $n \in \mathbb{N} \cup \{0\}$. ■

Theorem 8 states that for each $n \in \mathbb{N} \cup \{0\}$, the corresponding formal laurent series $f_n(z)$ is described by a finite product of laurent polynomials and hence is itself a laurent polynomial.

COR 8.1 Let $n \in \mathbb{N} \cup \{0\}$. Then

$$R_n(m) = \frac{1}{\pi}\int_0^{\pi} \cos m\theta \prod_{k=1}^{n} \cos^2\left(\frac{k\theta}{2}\right) d\theta \qquad \forall m \in \mathbb{Z}. \tag{6}$$

P–f Fix $m \in \mathbb{Z}$. Consider the unit circle C about the origin. Since f_n is a laurent polynomial, we obtain

$$R_n(m) = \frac{1}{2\pi i}\oint_C \frac{f_n(z)}{z^{m+1}}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_n(e^{i\theta})}{e^{im\theta}}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-im\theta}}{4^n}\prod_{k=1}^{n}\left(2 + e^{ik\theta} + e^{-ik\theta}\right) d\theta$$

from Cauchy's formula and definition of f_n. Thus substituting in Euler's formula,

$$R_n(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos m\theta - i\sin m\theta}{4^n}\prod_{k=1}^{n}(2+2\cos k\theta)\, d\theta = \frac{1}{2^{n+1}\pi}\int_{-\pi}^{\pi}(\cos m\theta - i\sin m\theta)\prod_{k=1}^{n}(1+\cos k\theta)\, d\theta.$$

Observe the complex component is odd and hence vanishes while the real component is even and hence is twice the same integral taken over half the domain. These two observations reduce the integral above to Eq. (6). ∎

2.3 Bell Polynomials

Although Corollary 8.1 provides an exact integral formula for the probability distribution of s_n for all $n \in \mathbb{N}$, explicitly calculating the integral is a complicated endeavor. This section attempts a different approach involving differentiation. Our main object of study will be the following. For each $n \in \mathbb{N} \cup \{0\}$ define the function,

$$g_n(z) = z^{n(n+1)/2} f_n(z) = \sum_m R_n\left(m - \frac{n(n+1)}{2}\right) z^m = \frac{1}{4^n}\prod_{k=1}^{n} z^k(z^{-k} + 2 + z^k) = \frac{1}{4^n}\prod_{k=1}^{n}(1+z^k)^2 \tag{7}$$

which expands to the other identities by Lemma 6 and Theorem 8 and hence describes a polynomial of degree $n(n+1)$ where the coefficients represent the probability distribution of s_n starting at $R_n(-\frac{n(n+1)}{2})$ for z^0.

PROP 9 Let $n \in \mathbb{N} \cup \{0\}$. Then

$$g_n'(z) = g_n(z)\sum_{k=1}^{n}\frac{2kz^{k-1}}{1+z^k}. \tag{8}$$

P–f Consider $\ln g_n$ restricted to a domain where g_n is nonzero. Then $\ln g_n = -n\ln 4 + \sum_{k=1}^{n}\ln\left(1+z^k\right)^2$. Taking the derivative of $\ln g_n$ with respect to z, we obtain the equivalent expression of Eq. (8)

$$\frac{d}{dz}\ln g_n = \frac{g_n'}{g_n} = \sum_{k=1}^{n}\frac{2(1+z^k)kz^{k-1}}{(1+z^k)^2}.$$

∎

LEM 10 Let $n \in \mathbb{N} \cup \{0\}$. Then

$$\left.\frac{d^m}{dz^m}\sum_{k=1}^{n}\frac{2kz^{k-1}}{1+z^k}\right|_{z=0} = 2(m)!\sum_{k=1}^{n} -k(-1)^{m+1/k}\,[k \mid m+1] \qquad \forall m \in \mathbb{N} \cup \{0\} \tag{9}$$

where the Iverson bracket, [condition], is 1 if condition is true and 0 otherwise.

P–f Fix $m \in \mathbb{N} \cup \{0\}$. Recall that $1/(1+z)$ corresponds to the Taylor series $\sum_{n=0}^{\infty}(-1)^n z^n$ for $|z| < 1$. Thus for fixed $k \in \mathbb{N}$, $2kz^{k-1}/(1+z^k)$ corresponds to $\sum_{n=0}^{\infty} 2k(-1)^n z^{k(n+1)-1}$ for $|z| < 1$. By definition of a Taylor series,

$$\begin{aligned}\left.\frac{1}{m!}\frac{d^m}{dz^m}\frac{2kz^{k-1}}{1+z^k}\right|_{z=0} &= 2k(-1)^n\,[\exists! n \in \mathbb{N}.\ k(n+1)-1 = m] \\ &= 2k(-1)^{(m+1/k)-1}\,[k \mid m+1]\end{aligned}$$

which follows from the fact that $k(n+1) = m+1$ means exactly that $k \mid m+1$. The above expression therefore reduces to Eq. (9) after summing both sides from $k = 1$ to n. ∎

LEM 11 Let f and g be holomorphic functions on an open disc and suppose $f' = fg$. Then

$$f^{(n)} = f B_n\left(g, g', \dots, g^{(n-1)}\right) \qquad \forall n \in \mathbb{N} \tag{10}$$

where $B_n(x_1, x_2, \dots, x_n)$ is the nth complete exponential Bell polynomial recursively defined by $B_0 = 1$ and

$$B_{n+1}(x_1, \dots, x_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(x_1, \dots, x_{n-i}) x_{i+1}. \tag{11}$$

P–f Given that g is holomorphic on a simply connected domain, g has antiderivative G. Note $f' = fg$ has general solution $c_0 e^G$. Thus, there exists constant $c \in \mathbb{C}$ such that $f = ce^G$ by the uniqueness of holomorphic solutions to linear ODEs. Since f is a composition of e^z and G, it falls under the special exponential case of Faà di Bruno's formula. That is,

$$f^{(n)} = ce^G B_n(G', G'', \dots, G^{(n)}) = f B_n(g, g', \dots, g^{(n-1)}) \qquad \forall n \in \mathbb{N}.$$

■

THM 12 Let $n \in \mathbb{N} \cup \{0\}$. Define the sequence $\{\nu_h\}_{h=1}^{\infty}$ by

$$\nu_h = 2(h-1)! \sum_{k=1}^{n} -k(-1)^{h/k}\,[k \mid h]. \tag{12}$$

Then

$$R_n\left(\frac{n(n+1)}{2} - m\right) = \frac{1}{4^n m!} B_m(\nu_1, \nu_2, \dots, \nu_m) \qquad \forall m \in \mathbb{N} \cup \{0\}. \tag{13}$$

P–f Fix $m \in \mathbb{N} \cup \{0\}$. By Corollary 3.2 and Proposition 4, $g_n(z) = \sum_{h=0}^{n(n+1)} R_n({}^{n(n+1)}\!/_2 - h) z^h$. Since g_n describes a polynomial, g_n is entirely holomorphic. Moreover, the coefficient of z^m is precisely

$$R_n\left(\frac{n(n+1)}{2} - m\right) = \frac{g_n^{(m)}(0)}{m!}.$$

By Proposition 9, $g_n'(z) = g(x) \sum_{k=1}^{n} 2kz^{k-1}(1+z^k)^{-1}$ where $\sum_{k=1}^{n} 2kz^{k-1}(1+z^k)^{-1}$ is clearly holomorphic along a neighborhood of 0. Thus by Lemma 11,

$$g_n^{(m)}(z) = g_n(z) B_m\left(\sum_{k=1}^{n} \frac{2kz^{k-1}}{1+z^k}, \frac{d}{dz}\sum_{k=1}^{n} \frac{2kz^{k-1}}{1+z^k}, \dots, \frac{d^{m-1}}{dz^{m-1}} \sum_{k=1}^{n} \frac{2kz^{k-1}}{1+z^k}\right)$$

along a neighborhood of 0. Evaluating at $z = 0$, we obtain by Lemma 10,

$$g_n^{(m)}(0) = \frac{1}{4^n} B_m(\nu_1, \nu_2, \dots, \nu_m).$$

■

CONJ 13 Let $n \in \mathbb{N}$. If $1 \le h \le n$, then $\nu_h = 2(h-1)!$A000593(h) where A000593(h) is the sum of odd divisors of h.

> Conjecture 13 has been numerically checked for values up to $n = 40$. This relationship seems to always break for $h > n$ since at least the last divisor h will always be missing from the sum. Moreover, sequence A000593 appears related to partition functions. Specifically, it is "the total number of parts in all partitions of h into an odd number of equal parts."

2.4 q-Pochhammer Symbols

While there is plenty of literature on Bell polynomials to continue analysis, it may also be useful to explore other representations of $R_n(m)$. This section attempts to bring the results of the theory of partitions to the problem of determining the probability distribution of s_n.

PROP 14 Let $n \in \mathbb{N} \cup \{0\}$. Then

$$f_n(z) = \frac{1}{4^n} z^{-n(n+1)/2}(-z;z)_n^2 \tag{14}$$

where $(a;q)_n$ is the q-Pochhammer symbol defined by $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for each $n \in \mathbb{N} \cup \{0\}$.

P–f Directly from Eq. (7), we obtain $z^{n(n+1)/2} f_n(z) = \frac{1}{4^n}(-z;z)_n^2$. ■

2.5 Numerical Computations

Using the recurrence formulation described in Theorem 3, we can iteratively calculate the probability distribution of s_n for each $n \in \mathbb{N}$. More importantly, Theorem 15 suggests that the probability distribution may be approximately normal. To empirically verify for small n, we need to calculate suitable parameters $\alpha, \beta \in \mathbb{R}$ for the gaussian equation $\alpha e^{-\beta x^2}$ that will minimize the mean squared error over all nonzero-probability $m \in \mathbb{Z}$. Note $\ln\left(\alpha e^{-\beta x^2}\right) = \ln a - \beta x^2$ so the problem reduces to the linear regression problem

$$M\vec{b} = \begin{bmatrix} 1 & -0^2 \\ 1 & -\left(\frac{1}{n}\right)^2 \\ \vdots & \vdots \\ 1 & -\left(\frac{n+1}{2}\right)^2 \end{bmatrix} \cdot \begin{bmatrix} \ln \alpha \\ \beta \end{bmatrix} = \operatorname{proj}_M \begin{bmatrix} \ln R_n(0) \\ \ln R_n(1) \\ \vdots \\ \ln R_n\left(\frac{n(n+1)}{2}\right) \end{bmatrix} = \operatorname{proj}_M \vec{y}.$$

However, as will be demonstrated, most points lie at the tail of the distribution which may skew the estimate to be unusable for points close to the mean. To avoid this, we will fit the regression to the first n nonnegative points in the distribution. The following figures were produced in Python 3.7 using NumPy and Matplotlib.

(a) Case $n = 4$

(b) Case $n = 60$

Figure 1: Graphs of $P(s_n = x)$ and corresponding gaussian equations over a horizontal x-axis.

2.6 Incomplete

THM 15 Let $n \in \mathbb{N} \cup \{0\}$. Then $R_n(m)$ is a unimodal function of m with maximum attained at $m = 0$.

P–f Since $R_n(m)$ is symmetric by Corollary 3.2, it suffices to show that $R_n(m)$ is a nonincreasing function of $m \geq 0$. The lemma in "Representation of m as Sum_{k=-n..n} epsilon_k k" by R. C. Entringer states that the coefficients of $\prod_{k=-n}^{n}(1+z^k)$ are nonincreasing for $m \geq 0$ for fixed $n \in \mathbb{N} \cup \{0\}$. Given that $R_n(m) = \frac{1}{4^n \cdot 2}[z^m]\left\{\prod_{k=-n}^{n}(1+z^k)\right\}$ for $m \in \mathbb{Z}$, $R_n(m)$ is a nonincreasing function of $m \geq 0$ by the referenced lemma. ■