Density of Critical Points and Integer Partitioning

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Abstract: Analysis and numerical experiments on the density of critical points in finite gap solutions for the NLS equation. The density of critical points is posed as the distribution of a partition problem.

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Section 1 Problem Statement

Consider a sample space $\Omega \coloneqq \{-1, 0, 1\}$ with probability mass function $\rho_\bullet : \Omega \to \{0, 1\}$ such that $\rho_{-1} = \rho_1$. Define a sequence of finite partitions of $[0, 1]$

$$x^{(\bullet)} : \mathbb{N} \to \mathcal{P}[0, 1] \quad \text{where} \quad x^{(n)} \coloneqq \left\{x_0^{(n)}, \dots, x_n^{(n)}\right\}. \tag{1}$$

We aim to analyze the distribution of the discrete random variable

$$\mathbf{s}_n \coloneqq \sum_{k=1}^{n} \mathbf{w}_k x_k^{(n)} \quad \text{as} \quad n \to \infty \quad \text{where} \quad \mathbf{w}_k \in \Omega \tag{2}$$

since this coincides with the density of critical points in finite gap solutions for the NLS equation.

1.1 Integer Partitioning

To make the problem more tractable, we consider uniform partitions of the form $x^{(n)} \coloneqq \{k/n\}_{k=0}^{n}$. Then the only possible values of $\mathbf{s}_n$ are of the form m/n where $m \in \mathbb{Z}$ and $|m| \leq n(n+1)/2$. Observe

$$\mathbf{s}_n = \sum_{k=1}^{n} \frac{k}{n} \mathbf{w}_k = \frac{m}{n} \quad \Leftrightarrow \quad \mathbf{r}_n \coloneqq \sum_{k=1}^{n} k \mathbf{w}_k = m \tag{3}$$

and hence the distribution of $\mathbf{s}_n$ is equivalent to that of a probabilistic generalization of an integer partition problem $\mathbf{r}_n$.

Section 2 Recurrence Relations

Fortunately, we can calculate the distribution of $\mathbf{r}_n$ efficiently by observing that the last term of $\mathbf{r}_{n+1}$ can be expanded to three cases, each reducing to $\mathbf{r}_n$.

2.1 **THM** Define a sequence of functions $\{R_n : \mathbb{Z} \to \mathbb{R}\}_{n=0}^{\infty}$ recursively by

(0) $R_0(0) \coloneqq 1$ and $R_0(m) \coloneqq 0$ for all $m \in \mathbb{Z} \setminus \{0\}$, and
(+) $R_{n+1}(m) \coloneqq \sum_{w \in \Omega} \rho_w \cdot R_n(m - w(n+1))$ for all $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$.

Then $P(\mathbf{s}_n = m/n) = P(\mathbf{r}_n = m) = R_n(m)$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

By Theorem 2.1, R_n is a discrete probability distribution. Moreover by definition of $\mathbf{r}_n$, it follows that R_n is even with compact support over $[^{-n(n+1)}/_2, {}^{n(n+1)}/_2] \cap \mathbb{Z}$. Given that $\rho_{-1} = \rho_1$, the recursive definition (+) can be "simplified" to

$$R_{n+1}(m) = \rho_0 \cdot R_n(m) + \rho_1 \cdot [R_n(m-n-1) + R_n(m+n+1)]\,. \tag{4}$$

2.1 Generating Functions

Using the theory of formal Laurent series, we can reframe R_n as a generating function. Define for each $n \in \mathbb{N}_0$

$$f_n(z) \coloneqq \sum_{m\in\mathbb{Z}} R_n(m) z^m. \tag{5}$$

Applying Eq. (4) to f_{n+1} then expanding and index shifting, we obtain the recurrence relation

$$f_{n+1}(z) = \left[\rho_0 + \rho_1\left(z^{-n-1} + z^{n+1}\right)\right] \cdot f_n(z) \quad \text{for all} \quad n \in \mathbb{N}_0 \tag{6}$$

with base case $f_0 \equiv 1$. In fact, this recurrence relation over functions admits an explicit solution.

2.2 **THM** Let $n \in \mathbb{N}_0$. Then

$$f_n(z) = \prod_{k=1}^{n} \left[\rho_0 + \rho_1\left(z^{-k} + z^k\right)\right] \tag{7}$$

with the convention $\prod_{k=1}^{0}[\dots] \coloneqq 1$.

Section 3 Asymptotics

From here on we will consider the special case $\rho_0 = 1/2$ (i.e. $\rho_1 = 1/4$). Through some algebra we obtain

$$f_n(z) = \frac{1}{4^n}\prod_{k=1}^{n}\left[z^{-k} + 2 + z^k\right] = \frac{g_n(z)}{2^{2n+1}} \quad \text{where} \quad g_n(z) \coloneqq \prod_{k=-n}^{n}\left[1 + z^k\right]. \tag{8}$$

Note $g_n(z) = \sum_{m\in\mathbb{Z}} A_n(m) z^m$ where $A_n(m)$ is the number of ways m can be partitioned into $\sum_{k=-n}^{n} k\varepsilon_k$ with $\varepsilon_k \in \{0,1\}$. R.C. Entringer's paper "Representation of m as $\sum_{k=-N}^{N}\varepsilon_k k$" proved the surprising result

$$A_n(tn) \sim (3/\pi)^{1/2}\, 2^{2n+1} n^{-3/2} \quad \text{as} \quad n \to \infty \quad \text{where} \quad t \in \mathbb{N}_0 \quad \text{is fixed.} \tag{9}$$

Since $R_n(m) = 1/2^{2n+1} A_n(m)$, it also follows that

$$R_n(tn) \sim (3/\pi)^{1/2} n^{-3/2} \quad \text{as} \quad n \to \infty. \tag{10}$$

Thus we aim to choose a relationship between m and n that grows strictly faster than linear but stays bounded above by quadratic growth so as to keep within the support of R_n.

3.1 Residue Theory

Since f_n is a finite product of Laurent polynomials, it is also a Laurent polynomial and hence an object of study in complex analysis. Indeed, we can recover the coefficient $R_n(m)$ of z^m for each $m \in \mathbb{Z}$ using Cauchy's integral formula.

3.1 **THM** Let $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. Then

$$R_n(m) = \frac{1}{\pi}\int_0^{\pi} \cos(mt) \prod_{k=1}^{n} \cos^2\left(\frac{kt}{2}\right) dt. \tag{11}$$

Note that Eq. (11) is an oscillatory integral in argument m parameterized by n. The integrand is especially peculiar in that it rapidly vanishes on $[\pi/n, \pi]$.

3.2 **CONJ** Let $m \gg n$. Then the leading asymptotic behavior of $R_n(m)$ is given by

$$R_n(m) \sim \frac{1}{\pi}\int_0^{\pi/n} \cos(mt) \prod_{k=1}^{n} \cos^2\left(\frac{kt}{2}\right) dt \quad \text{as} \quad n \to \infty. \tag{12}$$

Recalling the observation made from Eq. (10), we relate $m = \alpha n^2$ where $\alpha \in (0, 1/2]$ since this grows faster than linear while staying within the support of R_n. Supposing Conjecture 3.2 true, we can substitute $\xi \coloneqq nt$ into Eq. (12) to normalize the integration bounds.

$$R_n(\alpha n^2) \sim \frac{1}{n\pi} \int_0^\pi \cos(\alpha n \xi) \exp\left\{ \sum_{k=1}^n \ln \cos^2\left(\frac{k\xi}{n}\right) \right\} d\xi \quad \text{as} \quad n \to \infty. \tag{13}$$

Note the sum in Eq. (13) is nonpositive and concave down on $\xi \in [0, \pi]$. Consequently, the exponential contributes far less to the integral past its stationary point at $\xi = 0$.

3.3 **CONJ** Suppose Conjecture 3.2 is true. Then the leading asymptotic behavior of $R_n(\alpha n^2)$, where $\alpha \in (0, 1/2]$, is given by

$$R_n(\alpha n^2) \sim \frac{1}{n\pi} \int_0^\pi \cos(\alpha n \xi) \exp\left\{ \frac{n}{\xi} \int_0^\xi \ln \cos^2 \frac{x}{2}\, dx \right\} d\xi \quad \text{as} \quad n \to \infty. \tag{14}$$

Note the nested integral comes from the observation that the sum in Eq. (13) takes the form of a right Riemann sum as $n \to \infty$ if we scale by $1/n$.

With some renaming and application of Euler's formula, we can present Eq. (14) more succinctly.

$$R_n(\alpha n^2) \sim \frac{1}{2n\pi} \int_{-\pi}^\pi e^{n(\psi(\xi) + i\alpha\xi)}\, d\xi \quad \text{where} \quad \psi(z) \coloneqq \frac{1}{z} \int_0^z \ln \cos^2 \frac{x}{2}\, dx. \tag{15}$$

3.2 Digression on ψ

Using Wolfram Alpha, we can express ψ in analytic form.

$$\psi(z) = \frac{1}{z} \left[2i \operatorname{Li}_2(-e^{ix}) + \frac{ix^2}{2} - 2x \ln\left(1 + e^{ix}\right) + x \ln \cos^2 \frac{x}{2} \right]_{x=0}^{z} \tag{16}$$

where Li_n is the polylogarithm, also known as Jonquière's function, defined by

$$\operatorname{Li}_n(z) \coloneqq \sum_{k=1}^\infty \frac{z^k}{k^n}. \tag{17}$$

Note $\operatorname{Li}_2(-1) = {-\pi^2}/12$. The polylogarithm has some especially nice properties including the fact

$$\frac{d}{dz} \operatorname{Li}_n(z) = \frac{1}{z} \operatorname{Li}_{n-1}(z). \tag{18}$$

We can visualize the phase function in Eq. (15) over $\mathbb{C}$. Consider the case $\alpha = 1/4$. The following is the real (left) and imaginary (right) graphs of $\psi(z) + i\alpha z$ as well as its projections onto $\mathfrak{Re}\, z$ (Fig. 1) and $\mathfrak{Im}\, z$ (Fig. 2) respectively.

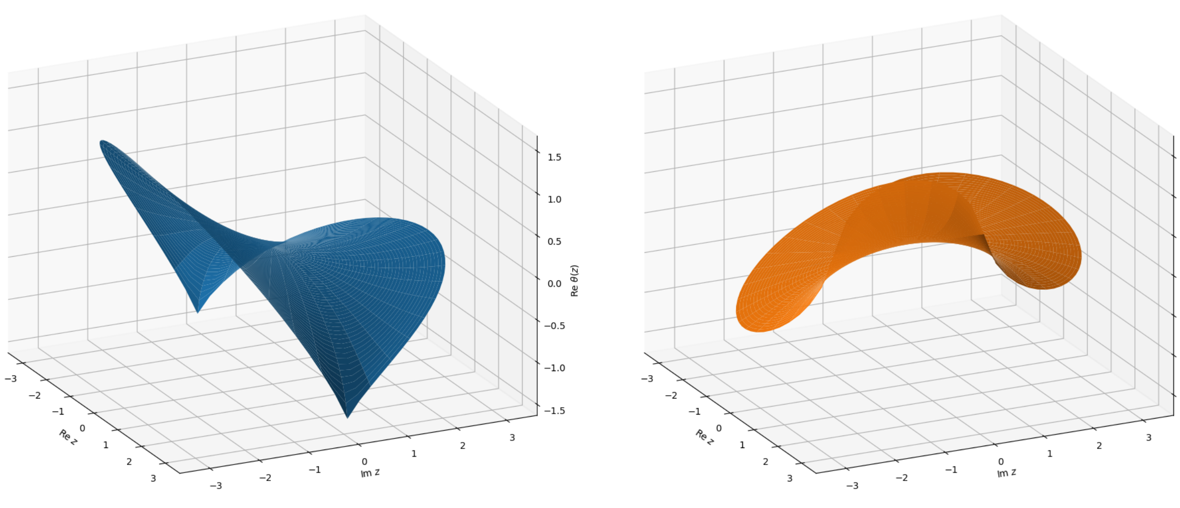

Figure 1: Real and imaginary graphs of $\psi(z) + i\alpha z$.

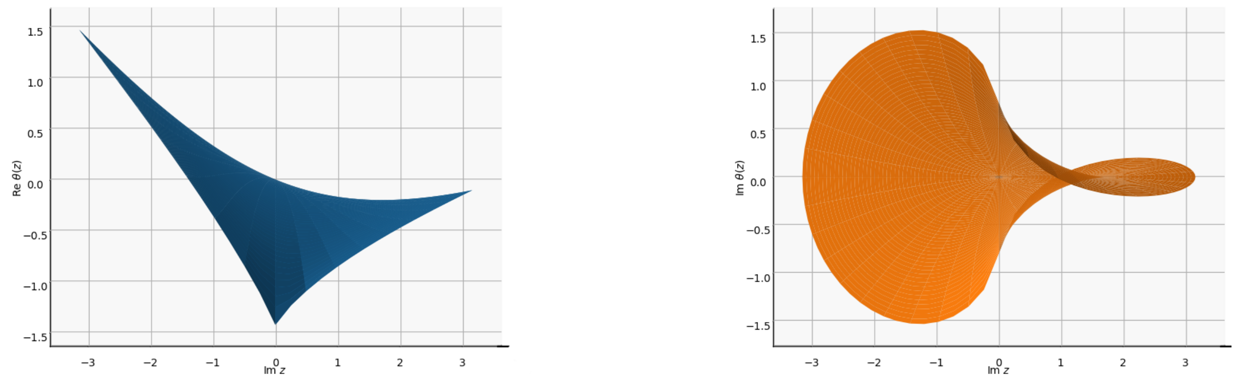

Figure 3: Real and imaginary graphs of $\psi(z) + i\alpha z$ projected onto $\mathfrak{Im}\, z$.

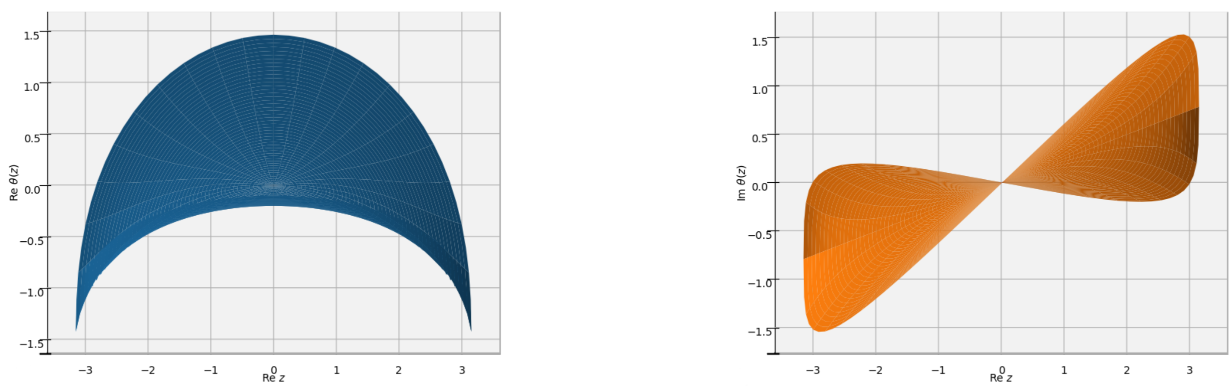

Figure 2: Real and imaginary graphs of $\psi(z) + i\alpha z$ projected onto $\mathfrak{Re}\, z$.

3.3 Method of Steepest Descent

Recall the following theorem for approximating oscillatory integrals.

3.4 **THM** **Method of Steepest Descent**. Consider the oscillatory integral

$$I(\lambda) = \int_a^b g(z) e^{\lambda f(z)}\, dz \quad \text{as} \quad \lambda \to \infty$$

where f and g are analytic functions. Suppose we can deform the contour of integration to pass through a unique stationary point z_0 of f. Then the oscillatory integral is given by the asymptotic

$$I(\lambda) \sim g(z_0) e^{\lambda f(z_0)} e^{i\phi} \left(\frac{2\pi}{\lambda |f''(z_0)|} \right)^{1/2} \quad \text{as} \quad \lambda \to \infty \quad \text{where} \quad \phi = \frac{\pi - \arg f''(z_0)}{2}.$$

Continuing with Eq. (15), define $\theta(z) \coloneqq \psi(z) + i\alpha z$. Using product rule and substitutions, we obtain derivatives

$$z\psi'(z) + \psi(z) = \ln \cos^2 \frac{z}{2}, \tag{19}$$

$$z\theta'(z) + \theta(z) = \ln \cos^2 \frac{z}{2} + 2i\alpha z, \tag{20}$$

$$z\theta''(z) + 2\theta'(z) = -\tan \frac{z}{2} + 2i\alpha. \tag{21}$$

Based on the numerics in Section 3.2, we have the following.

3.5 **CONJ** $\forall\alpha \in (0, 1/2]$. $\theta(z)$ has a unique equilibrium point on the positive imaginary axis of z. That is, $\exists x_0 \in (0,\infty)$ such that $\theta'(ix_0) = 0$ and $\theta''(ix_0) < 0$.

Evaluating at the saddle point asserted in Conjecture 3.5, we have

$$\theta(ix_0) = \ln\cosh^2\frac{x_0}{2} - 2\alpha x_0, \tag{22}$$

$$\theta''(ix_0) = \frac{1}{x_0}\left(2\alpha - \tanh\frac{x_0}{2}\right). \tag{23}$$

Therefore, we can apply the saddle point method to Eq. (15) to obtain the asymptotic

$$R_n(\alpha n^2) \sim n^{-3/2}e^{n\theta(ix_0)}e^{i\phi}\frac{1}{\sqrt{2\pi|\theta''(ix_0)|}} \quad \text{where} \quad \phi = \frac{\pi - \arg\theta''(ix_0)}{2} = 0 \tag{24}$$

$$\sim n^{-3/2}e^{n\left(\ln\cosh^2(x_0/2) - 2\alpha x_0\right)}\left(2\pi\left|\frac{1}{x_0}\left(2\alpha - \tanh\frac{x_0}{2}\right)\right|\right)^{-1/2}. \tag{25}$$

3.4 Calculating Stationary Point

Recall the following theorem for inverting analytic functions.

3.6 **THM** **Lagrange Inversion Theorem**. Suppose $z = f(w)$ where f is analytic at $w = w_0$ and $f'(w_0) \neq 0$. Then f has an inverse g analytic at $z = f(w_0)$ given by the power series

$$g(z) = w_0 + \sum_{k=1}^{\infty}\frac{g_k}{k!}(z - f(w_0))^k \quad \text{where} \quad g_k = \lim_{w\to w_0}\partial_w^{k-1}\left[\frac{w - w_0}{f(w) - f(w_0)}\right]^k,$$

i.e. $g(z) = w$ in a neighborhood of $z = f(w_0)$.

Expanding Eq. (22) then identifying Eq. (19), we obtain

$$\psi(ix_0) - \alpha x_0 = \ln\cosh^2\frac{x_0}{2} - 2\alpha x_0 \quad \Longrightarrow \quad \alpha = i\psi'(ix_0). \tag{26}$$

Note ψ' is analytic at $z = 0$ with $\psi''(0) \neq 0$ since $\psi(z) = -z^2/12 + \mathcal{O}(z^4)$. Thus, we can apply the Lagrange Inversion Theorem to obtain a formula for x_0 for sufficiently small α.

$$x_0 = \sum_{k=1}^{\infty}\frac{g_k}{k!}\alpha^k \quad \text{where} \quad g_k = \lim_{x\to 0}\partial_x^{k-1}\left[\frac{x}{i\psi'(ix)}\right]^k. \tag{27}$$

Calculating the first couple of terms, we obtain the formula

$$x_0 = 6\alpha + \frac{324}{5}\alpha^3 + \frac{128304}{35}\alpha^5 + \mathcal{O}(\alpha^7) \quad \text{where} \quad \alpha \to 0. \tag{28}$$

Plugging into Eq. (25) and expanding coefficients, we obtain the final asymptotic

$$R_n(\alpha n^2) \sim n^{-3/2}e^{n\left(-3\alpha^2 + \frac{513}{10}\alpha^4 + \mathcal{O}(\alpha^6)\right)}\left(2\pi\left|-\frac{1}{6} - \frac{21}{10}\alpha^2 - \frac{24111}{175}\alpha^4 + \mathcal{O}(\alpha^6)\right|\right)^{-1/2} \quad \text{as} \quad n \to \infty \tag{29}$$

or more succinctly,

$$R_n(\alpha n^2) \sim n^{-3/2}e^{-3n\alpha^2}\left(\frac{\pi}{3} + \frac{21\pi}{5}\alpha^2\right)^{-1/2} \quad \text{as} \quad n \to \infty. \tag{30}$$

3.5 Large Scale Asymptotics

Expanding Eq. (26), we obtain the expression

$$\alpha = \frac{1}{x_0}\ln\cosh^2\frac{x_0}{2} - \frac{1}{x_0^2}\int_0^{x_0}\ln\cosh^2\frac{t}{2}\,dt \tag{31}$$

which can be rewritten as

$$\frac{\alpha}{2}x_0^2 = x_0 \ln\cosh\frac{x_0}{2} - \int_0^{x_0} \ln\cosh\frac{t}{2}\,dt. \tag{32}$$

Expanding definitions on the right, we obtain

$$\frac{\alpha}{2}x_0^2 = x_0\left(\frac{x_0}{2} + \ln(1+e^{-x_0}) - \ln 2\right) - \int_0^{x_0}\left(\frac{t}{2} + \ln\left(1+e^{-t}\right) - \ln 2\right)\,dt \tag{33}$$

$$= \frac{1}{4}x_0^2 + x_0\ln(1+e^{-x_0}) - \int_0^{x_0} \ln\left(1+e^{-t}\right)dt. \tag{34}$$

Consequently, we end up with

$$(1-2\alpha)x_0^2 = 4\int_0^{x_0} \ln\left(1+e^{-t}\right)dt - 4x_0\ln(1+e^{-x_0}) \tag{35}$$

$$\approx 4\int_0^{x_0} \ln\left(1+e^{-t}\right)dt - 4x_0\ln(1+e^{-x_0}) \tag{36}$$