Density of Critical Points and Integer Partitioning

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Overview

Problem Setup

General

The density of critical points in finite gap solutions for the NLS equation can be reduced to the following combinatoric problem:

Define a sample space $\Omega := \{-1, 0, 1\}$ with probability mass function $\rho\colon \Omega \to [0, 1]$ such that $\rho(-1) = \rho(1)$. Let $\{x_0, \dots, x_n\}$ be a partition of $[0, 1]$ and $(\omega_1, \dots, \omega_n) \in \Omega^n$ a vector of random variables where $n \in \mathbb{N}$. Find the distribution of $s := \sum_{k=1}^{n} \omega_k x_k$. I.e., calculate

$$P(s = x) = \sum_{\vec{\omega} \in Y} \prod_{k=1}^{n} \rho(\omega_k) \quad \text{where} \quad \begin{array}{l} x \in \mathbb{R} \text{ and} \\ Y = \left\{\vec{\omega} \in \Omega^n \mid \sum_{k=1}^{n} \omega_k x_k = x\right\}. \end{array}$$

To make the problem tractable, we restrict to uniform partitions $\{0, 1/n, \dots, n/n\}$ of $[0, 1]$. Then the only possible sums are rationals of the form m/n where $m \in \mathbb{N}$. Define $r_n := \sum_{k=1}^{n} \omega_k k$. Then

$$P(s = m/n) = P(r_n = m)$$

since the solution set for both equations coincide. This transforms our problem into a probabilistic generalization of the integer partition problem.

Since there are three choices for how each term in the sum r_n is added, we can reduce the problem by expanding each case. Starting from the end,

$$P(r_{n+1} = m) = \sum_{\omega \in \Omega} \rho(\omega) P\big(r_n = m - \omega(n+1)\big)$$
$$= \rho(0) P(r_n = m) + \rho(1) \left[P(r_n = m - (n+1)) + P(r_n = m + n + 1) \right].$$

This multivariate recurrence on m and n ends at the trivial cases,

$$P(r_1 = m) = \begin{cases} \rho(m) & \text{if } m \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, denote $R_n(m) := P(r_n = m)$ where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then the recurrence property can be stated by

$$R_{n+1}(m) = \rho(0)R_n(m) + \rho(1)\left[R_n(m-n-1) + R_n(m+n+1)\right].$$

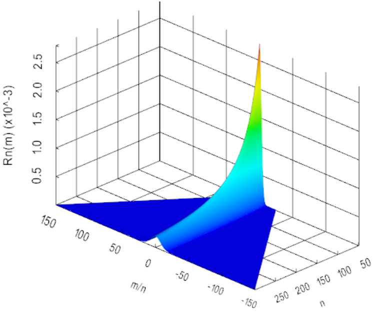
Using induction, we can prove some basic properties:

1. $R_n(m) = R_n(-m) \geq 0$ for every $m \in \mathbb{Z}$.
2. $R_n(m) = 0$ iff $|m| > {}^{n(n+1)}/_{2}$.
3. $\sum_{m \in \mathbb{Z}} R_n(m) = 1$.

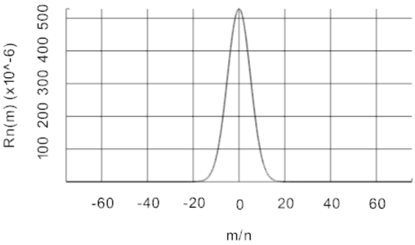
For the special case, $\rho(0) = {}^1/_2$, we also obtain

$$R_n\left(\pm\frac{n(n+1)}{2}\right) = \frac{1}{4^n}$$

Recall $P(s = {}^{m}/_{n}) = R_n(m)$. For the special case, $\rho(0) = {}^{1}/_{2}$, all probabilities are dyadic rationals and can be computed exactly using the recurrence property.



(i) Cases $n = 50$ to 300



(ii) Case $n = 150$

Define the formal laurent series $f_n(z) := \sum_{k\in\mathbb{Z}} R_n(k)z^k$. Applying the recurrence property, we obtain

$$f_{n+1}(z) = \left[\rho(0) + \rho(1)\left(z^{-n-1} + z^{n+1}\right)\right] f_n(z).$$

Using induction, we can prove that

$$f_n(z) = \prod_{k=1}^{n} \left[\rho(0) + \rho(1)\left(z^{-k} + z^{k}\right)\right]$$

which describes a laurent polynomial. For what follows, we will only consider the special case $\rho(0) = 1/2$. The generating function then becomes

$$f_n(z) = \frac{1}{4^n} \prod_{k=1}^{n} \left[z^{-k} + 2 + z^{k}\right].$$

Consider a similar combinatoric problem. Let $A_n(m)$ be the number of ways m can be partitioned into $\sum_{k=-n}^{n} \varepsilon_k k$ where $\varepsilon_k \in \{0, 1\}$. The corresponding generating function is

$$g_n(z) := \sum_{k \in \mathbb{Z}} A_n(k) z^k = \prod_{k=-n}^{n} \left[1 + z^k\right].$$

Consequently, we have the relation

$$f_n(z) = \frac{1}{4^n} \prod_{k=1}^{n} \left[z^{-k} + 2 + z^k\right] = \frac{1}{2^{2n+1}} \prod_{k=-n}^{n} \left[1 + z^k\right] = \frac{g_n(z)}{2^{2n+1}}.$$

Thus $R_n(m) = \frac{1}{2^{2n+1}} A_n(m)$.

In R.C. Entringer's paper "Representation of m as $\sum_{k=-n}^{n} \varepsilon_k k$", he proved the surprising asymptotic

$$A_n(tn) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \qquad \text{where } t \in \mathbb{Z}.$$

Translating back to our problem,

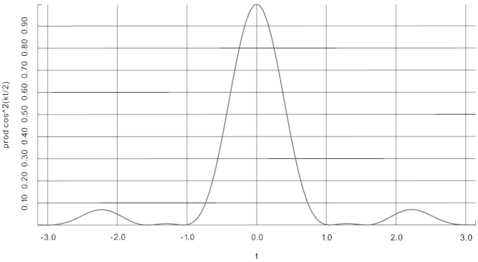
$$P(s = t) \sim \left(\frac{3}{n^3 \pi}\right)^{1/2} \qquad \text{as } n \to \infty.$$

(That is, we are increasing the partition number n).

Since $f_n(z)$ is a laurent polynomial, we can determine the coefficient $R_n(m)$ of z^m for each $m \in \mathbb{Z}$ using residues. Namely, let C be the unit circle about the origin. Then

$$R_n(m) = \frac{1}{2\pi i} \oint_C \frac{f_n(z)}{z^{m+1}}\, dz = \frac{1}{\pi} \int_0^\pi \cos(m\theta) \prod_{k=1}^{n} \cos^2\left(\frac{k\theta}{2}\right) d\theta.$$

This oscillatory integral has a peculiar integrand. Taking $m = 0$,



Suppose $m >> n$ but $m \leq {}^{n(n+1)}/_{2}$. Based on the graph, the leading asymptotic in n appears to be given on $[0, {}^{\pi}/_{n}]$.

$$R_n(m) \sim \frac{1}{\pi} \int_0^{\pi/n} \cos(m\theta) \prod_{k=1}^{n} \cos^2\left(\frac{k\theta}{2}\right) d\theta$$
$$= \frac{1}{\pi} \int_0^{\pi/n} \cos(m\theta) e^{\sum_{k=1}^{n} \ln \cos^2({}^{k\theta}/_{2})} d\theta$$

Since each $\ln \cos^2$ is concave on the interval, keeping the quadratic term in their taylor expansion appears accurate. Continuing,

$$R_n(m) \sim \frac{1}{\pi} \int_0^{\pi/n} \cos(m\theta) e^{\sum_{k=1}^{n} -{}^{k^2\theta^2}/_{4}} d\theta$$
$$= \frac{1}{\pi} \int_0^{\pi/n} \cos(m\theta) e^{-\beta_n \theta^2} d\theta \quad \text{where} \quad \beta_n = \frac{n(n+1)(2n+1)}{24}.$$

Since the integrand is super-expontentially decaying, the leading asymptotic appears to still be given on $[0, \pi/n]$.

$$\begin{aligned}
R_n(m) &\sim \frac{1}{\pi}\int_0^{\pi/n} \cos(m\theta) e^{-\beta_n \theta^2}\, d\theta \\
&\sim \frac{1}{\pi}\int_0^{\infty} \cos(m\theta) e^{-\beta_n \theta^2}\, d\theta \\
&\sim \frac{1}{2\pi}\sqrt{\frac{\pi}{\beta_n}} e^{-m^2/4\beta_n} \\
&= \frac{1}{2\pi}\sqrt{\frac{24\pi}{n(n+1)(2n+1)}} \exp\left\{\frac{-6m^2}{n(n+1)(2n+1)}\right\}
\end{aligned}$$

A different approach to solving the integral could be a change of variables. Let $\overline{\theta} = n\theta/2$ and fix ${}^{m}\!/_{n} = q \in \mathbb{Q}$. Then

$$\begin{aligned} R_n(m) &= \frac{1}{\pi}\int_0^{\pi} \cos(m\theta) \prod_{k=1}^{n} \cos^2\left(\frac{k\theta}{2}\right) d\theta \\ &= \frac{2}{n\pi}\int_0^{n\pi/2} \cos\left(2q\overline{\theta}\right) \prod_{k=1}^{n} \cos^2\left(\frac{k\overline{\theta}}{n}\right) d\overline{\theta} \\ &= \frac{2}{n\pi}\int_0^{n\pi/2} \cos\left(2q\overline{\theta}\right) e^{n\sum_{k=1}^{n}\frac{1}{n}\ln\cos^2\left(\frac{k\overline{\theta}}{n}\right)} d\overline{\theta}. \end{aligned}$$

The periodicity of $\ln\cos^2$ suggests that the Riemann sum could be replaced with its integral without altering the leading asymptotics.

$$R_n(m) \sim \frac{2}{n\pi}\int_0^{n\pi/2} \cos\left(2q\overline{\theta}\right) e^{n\int_0^{\overline{\theta}} \ln\cos^2(x)\,dx} d\overline{\theta}.$$

Observations

Log Probability